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Two-Dimensional BEM Analysis of Cracked Fibre-Metal Laminates With Circular Cut-Outs

by

BOGDAN E. CUDZILO, B.Eng

a thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfillment of
the requirements for the degree of
Master of Engineering

Ottawa-Carleton Institute for
Mechanical and Aerospace Engineering

Department of
Mechanical and Aerospace Engineering
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Cut-Outs**
submitted by
Bogdan E. Cudzilo, B.Eng.
in partial fulfillment of the requirements for
the degree of Master of Engineering

Thesis Supervisor

Chairman, Department of Mechanical and Aerospace Engineering

Carleton University
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ABSTRACT

A two-dimensional numerical study employing the boundary element method (BEM) of crack-face bridging for cracked fibre-metal laminates (FML) with notches is presented in this thesis. Fibre-metal laminates are a class of structural materials which consist of fibre-reinforced prepreg layers sandwiched by metal layers. The fracture mechanics prediction of crack growth in these perforated laminates involves the interaction of several parameters with the stress field resulting from the presence of the cut-out. The parameters include the geometry and crack size, the extent of fibre-bridging, the delamination between the prepreg and metal layers and debonding of the fibre-matrix interface. The present work investigates the effects of a stress concentration on the stress intensity factor of a bridged crack in fibre-metal laminates. Two different FML's are considered, namely, ARALL2 and GLARE2. Three-dimensional finite element method (FEM) analyses are also carried out to confirm the validity of the 2-D BEM models. FML's with circular cut-outs will contain high bridging stresses near the cut-out resulting in fibre failure there. This causes a reduction of the extent of fibre-bridging of the crack. Results of the study show that the 2-D BEM models employed can indeed be used to provide a quick and reasonable estimate of the stress intensity factor for a bridged crack in a FML with a circular cut-out.

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NOMENCLATURE

a	- crack length
a_{ij}	- elastic compliance matrix
A	- cross-sectional area of the fibre-epoxy laminae
A_T	- total cross-sectional area of the laminate
A_{ji}	- complex constants
ARALL	- <i>A</i> ramid <i>R</i> einforced <i>A</i> luminum <i>L</i> aminate
Δa	- an incremental crack extension
b	- fibre-bridging zone length
b_i	- body force vector per unit volume
BEM	- boundary element method
BIE	- boundary integral equation
c	- effective crack length
C	- compliance
C_b	- compliance of crack bridging system
C_b^0	- steady-state compliance
C'_b	- effective compliance of the crack-face bridge
C_e	- compliance without crack bridging
COD	- crack opening displacement
d	- delamination zone length
e_{ij}	- engineering strains
E	- Young's modulus of an isotropic material

E_{11}	- Young's modulus in the 1 st material principal axis direction
E_{22}	- Young's modulus in the 2 nd material principal axis direction
E_f	- modulus of elasticity of fiber
E_{la}	- Young's modulus of laminates in fiber direction
E_m	- modulus of elasticity of matrix
f_i, f_i'	- body forces
F	- Airy's stress function
F_{ad}	- stiffness of the adhesive
F_{Al}	- stiffness of the metal layers
F_{fm}	- stiffness of the fiber-adhesive layers
F_{la}	- stiffness of the laminate
G	- shear modulus of an isotropic material
G_{ad}	- shear modulus of the adhesive
G_{ij}	- shear modulus of an anisotropic material
GLARE	- <i>Glass Reinforced Aluminum Laminate</i>
H	- plate height
K_a	- applied stress intensity factor,
K_b	- stress intensity factor due to fiber bridging
K_c	- fracture toughness of the material
K_I	- mode-I stress intensity factor
K_{II}	- mode-II stress intensity factor
l	- crack-tip element length
l_b	- length of the bridging zone

l_b^0	- steady-state bridging zone length
l_d	- debonding distance
m	- power index for fibre stress which increases towards crack-tip
M	- number of elements in the mesh
n	- power index for fibre stress which decreases towards crack-tip
n	- unit outward normal
n_d	- power index
n_j	- unit outward normal components
N	- number of nodes
N_L	- number of interfaces in the specimen
N^c	- shape functions
p	- closure pressure
p, P	- source point
P	- external load
P_b	- crack-bridging load
P_e	- spring force at the load line
q	- power index
q, Q	- field point
r	- distance between p and q (or P and Q)
R	- fiber radius
R	- BEM domain
R	- hole radius
s_j	- material property parameters

S	- BEM boundary
S_{ϵ}	- boundary region around source point
t	- fibre-layer thickness
t_{ad}	- thickness of adhesive in interface
t_i, t_i'	- surface tractions
$\bar{t}_j$	- computed traction coefficients
T	- total laminate thickness
T_{ji}	- fundamental solution for tractions
u	- crack opening displacement
u^*	- opening displacement at the furthest fiber from the crack tip
u_b	- crack face opening displacement at the bridging load line
u_i	- displacements
U_{ji}	- fundamental solution for displacements
v	- crack opening displacements
v^*	- half-crack opening displacement
$\bar{v}^*$	- half-crack opening displacement without fibre-bridging
V_f	- volume fraction of fiber
V_m	- volume fraction of matrix
W	- plate width
x	- distance from the crack tip or from the crack-centre
x_i	- Cartesian coordinate
z_j	- complex variable
δ_{adh}	- shear deformation of the adhesive

δ_{ij}	- Kronecker delta
ε^*	- strain in last intact fiber furthest from the crack tip
$\eta_{ij,k}$	- coefficients of mutual influence
μ	- shear modulus
μ_j	- roots of the characteristic equation
ν	- Poisson's ratio of isotropic material
ν_{ij}	- Poisson's ratio of anisotropic material
σ	- applied stress
$\sigma_{br,i}$	- bridging stress
σ_B	- bridging stress distribution
σ_B^*	- bridging stress at the furthest fiber from the crack tip
σ_{ij}	- the stress components
σ_∞	- uniform far-field stress in the direction of the fibers
τ	- interfacial friction stress at fiber/matrix interface
ζ	- intrinsic co-ordinates

CHAPTER 1

INTRODUCTION

Cut-outs in aerospace structures are often unavoidable. Cracks frequently initiate and grow from the intense energy regions around holes and cut-outs when the structure is under mechanical loads. Under cyclic loading, a toughening mechanism that slows fatigue crack growth rates is highly desirable to increase the service life of the component. One extrinsic toughening mechanism which reduces the extent of crack opening in fibre-reinforced materials is crack-face fibre-bridging. This mechanism slows crack growth and may even lead to crack arrest. The effect of the crack-tip singular behaviour and its interaction with a hole in a laminate, on the fibre-bridging mechanism, is investigated in this study.

A new class of lightweight, fatigue resistant materials called fibre-metal laminates (FML), developed by the Delft University of Technology in Holland, combine high strength and fatigue properties of composites with good machinability and impact resistance of metals. The commercially available ARALL (*Aramid Reinforced Aluminum Laminate*), GLARE (*Glass Reinforced Aluminum Laminate*), and CARALL (*Carbon Reinforced Aluminum Laminate*) consist of alternating layers of thin aluminum alloy sheets bonded by a structural adhesive impregnated by high strength fibres. The

use of these fatigue-critical materials in lower wing skins, fuselage, exterior doors, and horizontal and vertical stabilizers can result in weight savings of up to 30%, see e.g. Marissen (1988), compared to monolithic aluminum.

Much of the experimental and analytical works on fibre-metal laminates consist mainly of fatigue crack propagation studies. Marissen (1988) has analytically quantified the relationship between bridging stress and delamination zone size with the assumption of a uniform bridging stress profile. Cain and Tan (1997) have analyzed statically loaded crack problems with a range of bridging profiles and showed that the uniform stress profile assumption can only be applied to a central crack geometry. They used a two-dimensional simplified boundary element analysis to model a truly three-dimensional problem. Sheng (1998) expanded the work of Cain and Tan to include BEM modeling of symmetric cracks emanating from a hole centred in a plate, as well as the influence of auxiliary holes on the bridging stress and stress intensity factors. There have been very few numerical studies to date on the influence of a stress field, for example, caused by the presence of a cut-out, on the fibre-bridging mechanism in fibre-metal laminates. The present study is expanded from the two above-mentioned previous works to analyze cracks emanating from notches in several geometries with the variation of the fatigue crack size, delamination shape, and the bridging stress profile. Also, an extension of the analysis for ARALL is made to GLARE to investigate if the results for one fibre-metal laminate can be generally applied to another. Finally, the effects of reducing the bridging zone length on the bridging stress and stress intensity factor are also presented.

Early work in the modelling of fibre-bridging has been carried out for ceramic matrix composites, metal matrix composites and polymeric matrix composites. The

fibre-bridging process in these composites is briefly reviewed in Chapter 2, along with an outline of the numerous ongoing research in this area. Several commonly used methods to determine crack-face bridging stresses and stress intensity factors are the shear-lag model, the compliance-spring model, and the stress intensity approach. A brief description of these models and the various parameters which influence fibre-bridging are also presented. The applicability of each model to analyse fibre-metal laminates is discussed and a suitable method is chosen. The fibre-bridging mechanism in fibre-metal laminates is three-dimensional, but this may be time-consuming and costly to deal with. As a result, a two-dimensional study using power-law expressions for the bridging stress distribution and other simplifying assumptions is presented.

For the numerical solution of problems containing stress concentrations and regions of high stress gradients, such as those near cracks, the boundary integral equation (BIE) method, commonly known as the boundary element method (BEM), is widely recognized as a very efficient computational technique. Chapter 3 briefly reviews the formulation of the direct BEM for two-dimensional, isotropic and anisotropic elastostatics. Also discussed is the application of the boundary element method to linear-elastic fracture mechanics (LEFM) in obtaining expressions for the crack-tip stress intensity factors. Finally, several test problems with cracks and cracks emanating from cut-outs are analyzed and the numerically obtained results are compared with those available in literature.

The main physical problems for the various cases investigated in this study are defined in Chapter 4. Numerical results of the bridging stresses and stress intensity factors for a fully bridged ARALL and GLARE laminate are presented and discussed.

More specifically, the results obtained from the two-dimensional boundary element method are compared with those obtained from the three-dimensional finite element method, and a discussion of the effects of the various parameters on the bridging stress and stress intensity factor is presented. A comparison is also made between the results for ARALL and GLARE, as well as the applicability of the two-dimensional simplified analysis in analyzing a three-dimensional, notched fibre-metal laminate.

A partially bridged fibre-metal laminate of ARALL and GLARE is analyzed in Chapter 5. In the partial bridging cases studied, the bridging zone length is reduced to 75% and 50% of the crack length following the assumption that the fibres near the hole have failed. Again, numerical results for partial bridging from the two-dimensional boundary element method analysis are presented and compared with those obtained by the three-dimensional finite element method. A comparison is made between the case of full bridging, partial bridging, and no bridging of the crack, and the benefit of fibre-bridging in fibre-metal laminates is clearly shown.

A summary of the results of the boundary and finite element analyses and the conclusions obtained from this study are presented in Chapter 6. Some suggestions for future research in the area of fibre-bridging in fibre-metal laminates are also made in this final Chapter.

CHAPTER 2

FIBER-BRIDGING OF CRACKS IN COMPOSITES

In this Chapter, the various toughening mechanisms that increase the fracture toughness of a material are briefly described. The fiber-bridging process in composites is also described, followed by a review of some of the commonly used methods of analysis in determining crack-face bridging effects. They include the shear-lag model, the compliance-spring model, and the stress intensity approach. A more detailed discussion of the fiber-bridging process in fiber-metal laminates is presented along with the description of the major damage mechanisms in these materials.

2.0 Toughening and Crack Shielding Mechanisms

Upon the initiation of a crack in any material, there are intrinsic and extrinsic toughening mechanisms that act to impede further crack growth and shield the crack tip. In metals, the common toughening mechanisms result from micro-structural properties and changes associated with deformation. To increase the fracture toughness of metals, the inherent microstructure can be modified to obtain the proper balance of strength and intrinsic toughness.

In composites, extrinsic toughening mechanisms such as crack deflection, zone shielding, contact shielding, and combined zone and contact shielding act to lower the crack driving force and increase the toughness of the material. Figure 2.1 shows some of the extrinsic toughening mechanisms in composites and many other materials. There are a number of toughening mechanisms that shield the crack tip upon crack extension at a given time, but the influence of the dominant mechanism has been investigated with possible consideration of the lesser mechanisms. In particulate-reinforced composites, crack deflection can occur around particles and the energy expended lowers the crack growth rate. In fiber-reinforced composites, the primary mechanism is ligament or fiber toughening.

2.1 Fiber-Bridging Process in Composites

The toughness of ceramics, cementitious materials, and metals can be increased by the use of fibers which bridge cracks and provide resistance to fracture. The fiber-bridging process in Ceramic Matrix Composites (CMC), Metal Matrix Composites (MMC) and Polymeric Matrix Composites (PMC) consists of crack initiation in the ductile matrix material with the stronger fibers remaining intact and transmitting the applied force in the crack wake. Figure 2.2 shows a typical bridged crack in which different mechanisms act in the crack wake. If the resulting stress in the fibers is lower than the ultimate stress of the fiber, the crack is bridged by the intact fibers. This bridging action reduces the stress intensity factor for the crack and increases the necessary applied load for additional crack growth. The crack in the matrix material can propagate around the bridging fibers with the fibers exerting tractions on the matrix.

These tractions are due to frictional sliding between the fiber and matrix and from pullout of the fibers from the matrix. The toughness of the fiber/matrix interface must be low relative to that of the fibers such that the matrix crack induces fiber debonding rather than fiber cracking.

Crack propagation leads to three distinct modes of failure, Ritchie, *et al.* (1989): longitudinal crack growth or Mode II failure, Mode I non-catastrophic failure, and Mode I catastrophic failure. Longitudinal crack growth occurs when the fiber/matrix interface is very weak and the crack propagates at 90° to the notch along the fiber/matrix interface. Mode I non-catastrophic failure occurs when the interface is stronger with only partial debonding of the fibers and the matrix crack is free to propagate, leaving the fibers intact to bridge the specimen. Mode I catastrophic failure occurs when the fiber/matrix interface is so well bonded that fiber cracking occurs and the result is immediate fracture. The conditions that determine the type of mode are complex and are controlled by fiber size, volume fraction, and the fiber/matrix interface strength.

The tractions resulting from fiber pullout and frictional sliding transmit a force through the fibers during crack extension. This transmitted force is a function of position as well as the crack opening displacement. If this force is known, the stress intensity at the crack tip as well as the crack growth rate can be calculated. Unfortunately, this force is not known *a priori* in the fracture mechanics analysis of fibre-bridged composites.

2.2 Analysis Models for Fiber-Bridging

Several models have been proposed to analyze fiber bridging in composites. The goal of each method is to obtain a stress-displacement relationship during crack growth to

predict the critical stresses and stress intensity factors. The underlying assumptions in each model are based on fracture mechanics principles. The shear lag model, compliance spring model, and the stress intensity approach are predominantly utilized and will be reviewed in this section.

2.2.1 Shear Lag Model

The effect of fiber bridging on the crack driving force in the matrix has been analyzed using a shear lag model. The basis of the model involves the stress analysis of a single fiber embedded in a cylindrical matrix subjected to axial loading. It is assumed that the sliding of the matrix over the fibers is resisted by frictional forces only. Figure 2.3 shows a single fiber embedded in a cylindrical matrix.

Research has concentrated on finding an accurate representation of the interfacial friction stress, τ at the fiber/matrix interface. The cases of bonded and partially bonded fibers have been studied in order to improve predictions on the stress distribution in the fiber-bridging region of the crack. For the bonded case, the fiber/matrix interface sliding is resisted by friction forces or the interface is chemically bonded such that the fiber/matrix deformation is equal. For the partially bonded case, the interface debonds a certain length while the rest is fully bonded by the interfacial friction stress. The earliest work by Aveston and Kelly (1973), using a simple shear lag analysis, was based on determining the magnitude of the interfacial friction stress, τ and its role in load transfer between the fibers and the matrix.

The relationship between the magnitude of traction and the crack opening displacement (COD) can be determined with the required knowledge of the load in each

individual fiber in the bridging zone. Using a continuum approximation, this procedure is equivalent to applying a closing pressure to the crack surface. Marshall, *et al.* (1985) have determined the relationship between the closure pressure, p , and the crack opening displacement (COD), u , as

$$p = 2 \left[u \tau V_f^2 E_f (1 + \eta) / R \right]^{1/2} \quad (2.1)$$

where $\eta = E_f V_f / E_m V_m$, R is the fiber radius, τ is the frictional sliding stress at the interface, V_f and V_m are the volume fractions of fiber and matrix, respectively, and E_f and E_m are the modulus of elasticity of fiber and matrix, respectively. This pressure is a function of position, x along the crack flanks and the COD, u . The variation of τ along the length of the bridging zone has been assumed constant due to steady-state crack growth observed in this region for a number of studies.

The determination of the traction in the bridging zone depends on the type of crack growth. For large cracks, the crack opening, u , asymptotically approaches the equilibrium separation of the failed matrix and the net force in the intact bridging fibers exactly balances the applied force (see Figure 2.4). This growth is referred to as steady-state growth and the stress needed to extend the crack is independent of the total crack length. For short cracks, the stress required to extend the crack is dependent on the entire crack and this is referred to as unsteady-state growth.

Numerous studies employing the shear lag model have been carried out on concrete, cements, ceramic matrix composites and metal matrix composites. Budiansky, *et al.* (1986) used the shear lag model to study critical conditions for the onset of widespread matrix cracking with unbonded and weakly bonded fibers with the use of

energy relations to estimate stress distributions. The work of Marshall and Oliver (1987) focused on the role of the interface in fracture processes. The analysis of pure frictional sliding and one where interface debonding and interface sliding combine suggested that the frictional stress is uniform along the fiber and is not influenced by irregularities of fiber shape or thickness.

The shear-lag model has been extended to fatigue loading by Marshall and Oliver (1987), Cox and Marshall (1991), and Nguyen and Yang (1994). The roles of interfacial strength, interfacial friction stress, fiber volume fraction, and fiber and matrix moduli in determining the mode of failure have been deduced by Cox and Marshall (1991) for uniaxially loaded fibers. It was found that longitudinal failure is favored by weak interfaces, high fiber volume fraction, high interfacial friction, and the presence of notches or bending. Mode I non-catastrophic failure is favored by large fiber radius and matrix modulus, and small friction stress and fiber modulus. They suggested that the force-displacement relation, $p(u)$ should be regarded as an engineering material property.

Nguyen and Yang (1994) concluded that the debonding distance, l_d , (see Figure 2.5) is the important and defining variable rather than the fiber/matrix interfacial strength, τ . The crack bridging model incorporated the mechanism of fiber crack-tip shielding in addition to fibre-bridging to calculate the stress intensity.

More complex relations between fiber tractions and the COD have been proposed by Hutchinson and Jensen (1990). Debonding between the fiber and matrix was viewed as a Mode II interface fracture. Models of debonding and fiber pullout with either constant friction or Coulomb friction were used to obtain fiber/matrix interfacial

properties. In Coulomb friction, a coefficient of friction between the fiber and the matrix is introduced.

The use of an improved shear lag formulation by Danchaivijit and Shetty (1993) allowed the calculation of the critical stress for extension of matrix cracks by using an equation applicable to fibers in the steady state and non-steady state regions of the crack. Assuming a constant interfacial shear stress, τ , the crack closure stress was modified as,

$$p = \frac{\eta \sigma_{\infty}}{2(1+\eta)} \left\{ \left[1 + \frac{16(1+\eta)^2 E_f V_f^2 \tau u}{\eta^2 \sigma_{\infty}^2 R} \right]^{\frac{1}{2}} + 1 \right\} \quad (2.2)$$

where σ_{∞} is the uniform far-field stress in the direction of the fibers.

The models of Marshall, *et al.* (1985) enforced a zero bridging stress at the crack tip, whereas the work by Danchaivijit and Shetty (1993), Nguyen and Yang (1994), and Buchanan, *et al.* (1997) enforced a non-zero bridging stress there. All these models show an increase in bridging stress with an increase in distance away from the crack tip.

The relations from the shear lag model were used in a finite element analysis of crack bridging under fatigue loading by Kim (1994). An optimum range of τ is expected to exist where the most resistance to crack propagation is attained and, as a consequence, the non-uniform distribution of τ over the bridging distance should be obtained.

A number of authors have used stress functions that satisfy equilibrium equations in their work to better predict the bridging effects due to bridging fibers. Quek and Yue (1994) developed a method based on assumed stress functions to derive stress distributions in the fiber and matrix. They studied the pull-out of a single fiber embedded in a surrounding matrix and included radial and hoop stresses to obtain a more accurate

predictions on the interfacial shear stress distribution. Mathematical functions for cracks emanating from notches for both monotonic and cyclic loading have been presented by Cox and Lo (1992b). Using the shear lag model, the crack growth influenced by the bridging effects can be predicted by these functions for any crack size, specimen size, or notch size.

Fiber bridging in a finite orthotropic plate containing an internal crack has been studied by Ji and Ishikawa (1997). The boundary collocation method (BCM) was combined with the shear lag model to propose a new method for solving crack problems under self-consistent loading conditions (i.e. the applied stress and the traction forces on the crack surfaces are inter-related).

Unlike the fracture mechanics analysis using continuum models, Herrmann and Hillberry (1997) used the shear lag theory to model a composite laminae divided into a finite number of cells. This alternative approach for determining COD and bridging pressure can use different bridging relations for each cell and the system can be solved fairly rapidly by an iteration process. It is however only applicable to unidirectional composite systems that do not exhibit matrix splitting. The authors noted a number of ceramic matrix composites and titanium matrix composites that do not exhibit matrix splitting. This assumption cannot apply to composites where the matrix is much softer, such as the polymer-based matrix composites.

Buchanan, *et al.* (1997) developed a procedure to deduce the bridging stress distribution from limited COD data. They used the weight function method and verified the work by finite element analysis. It was concluded that the bridging stress distribution increases with COD and is non-zero at the crack tip. Hwu and Derby (1999) included the

influence of metal plasticity near the crack tip in their shear-lag formulation for metal/ceramic laminates. Chiang (2000) analyzed the effects of crack-wake interfacial debonding coupled with frictional shear stress in the debonded region on the matrix cracking stress for fiber-reinforced brittle matrix composites. The debonding length was predicted by an interfacial debonding criterion developed for the analysis. For a perfectly bonded interface, the fiber and the matrix deform elastically above the crack plane and no slippage occurs as matrix crack propagates.

The shear lag theory can be used to predict the bridging stress distribution from the bridged fibers in the crack wake but one of the major drawbacks is that the accuracy of the prediction relies on interfacial properties which are difficult to determine. The effects of neighbouring fibers are not accounted in the shear lag models, except for the work of Herrmann and Hillberry (1997) where the single cell model was used to implement the modeling of broken fibers. For a large problem with many fibers, this can be quite tedious indeed.

2.2.2 Compliance-Spring Model

The compliance spring model can determine the bridging stress and the bridging stress distribution along the bridging zone with the advantage that no information of the crack opening displacement is needed. The model is based on measuring the compliance change during incremental crack extension. The approach consists of measuring the compliance, C , during step-wise re-notching. The compliance is one of the most commonly measured parameters during fracture mechanics experiments and many variations of the compliance spring model have been used. Hu and Wittman (1991) used

the compliance technique to obtain crack tip bridging stresses in ceramic materials. A mathematical function was defined from the measured compliances but it is purely based on bridging, ignoring a mix of bridging and other micro-cracking mechanisms.

An equivalent spring system for an elastic body with crack bridging can be used to calculate the compliance of the whole bridging system, $C(a)$. A typical spring model, as shown in Figure 2.6, consists of two springs and ideally stiff rods with a free joint at a distance, a , away from an applied load line. The crack face opening displacements and the loading were assumed to vary linearly and given by,

$$P = P_e + \frac{l_b}{2a} P_b \quad (2.3)$$

where l_b is the length of the bridging zone, P is the external load, P_e is spring force at the load line, and P_b is the crack-bridging load, and:

$$u_b = \frac{l_b}{2a} u \quad (2.4)$$

where u and u_b is the crack face opening displacement at the applied load line and the bridging load line, respectively.

The compliance expression is then given by Lutz and Sakai (1993),

$$\frac{1}{C(a)} = \frac{1}{C_e(a)} + \frac{1}{C_b(l_b)} \left(\frac{l_b}{2a} \right)^2 \quad (2.5)$$

where $C_e(a)$ and $C_b(l_b)$ are the compliance of the elastic test specimen without crack bridging and the compliance of the crack bridging system, respectively.

Re-writing Eq. (2.5) gives the compliance of the composite as:

$$C(a) = \frac{C_a(a)}{1 + \frac{C_e(a)}{C'_b}} \quad (2.6)$$

and C'_b is the effective compliance of the crack-face bridge and is given by:

$$C'_b = \frac{C_b^0 (l_b^0 / \Delta a)^m}{(\Delta a / 2a)^2} \quad (2.7)$$

for transient bridging, and:

$$C'_b = \frac{C_b^0}{(l_b^0 / 2a)^2} \quad (2.8)$$

for steady-state bridging.

Here, Δa is an incremental crack extension, l_b^0 is the steady-state bridging zone length, C_b^0 is the steady-state compliance, and m is a power index which is equal to zero for a uniform distribution.

2.2.3 Stress Intensity Approach

The stress intensity approach has been used due to its simplicity in obtaining the crack tip stresses and stress intensity factor. The work of Marshall and Cox (1987) dealt with analyzing matrix cracking in which fiber failure accompanies matrix cracking and finding a relation for the critical fiber strength at which the change in failure mechanism occurs. Mai and Lawn (1987) used the stress intensity approach as a basis to derive a T-curve (R-curve) model based on bridging tractions at the crack interface behind an advancing crack tip. Marshall and Cox (1988) presented a derivation of a general solution

for steady-state matrix cracking stress using a J-integral to calculate the stress intensity factor. Cotterell and Mai (1988) examined the effect of more exact modeling of the fracture process zone (FPZ) on crack growth resistance of fiber-reinforced cementitious composites. The FPZ at the crack tip was replaced by a crack with a distribution of closing forces across the faces to predict crack growth resistance curves.

The stress intensity factor at the crack tip can be expressed by,

$$K_{tip} = K_a - K_b \quad (2.9)$$

For crack extension to occur,

$$K_{tip} \geq K_c \quad (2.10)$$

Here, K_a is the applied stress intensity factor, K_b is the stress intensity factor due to fiber bridging, and K_c is the fracture toughness of the material. For a particular geometry, K_a and K_c can be obtained from published results and polynomial functions obtained from previous tests. K_b , on the other hand, depends on bridging tractions and the crack opening displacement (see Figure 2.7).

The bridging stress distribution for fully and partially bridged cracks emanating from notches has been studied by many researchers using the stress intensity method, e.g. Marshall and Cox (1991a) (1991b), Cox (1991), and Cox and Lo (1992b). It was noted that fully bridged cracks occur at low values of applied stress while partially bridged cracks occur at high-applied stress. Lee (1990) used the stress intensity approach to analyze fully and partially bridged cracks near a free surface in CMC's. An iteration scheme was used to solve for a nonlinear bridging force. It was found that the loss of bridging due to fiber breakage increases the stress intensity factor, and that fiber breakage is initially unstable but becomes stable thereafter.

An energy approach can also be used instead of the stress intensity approach. The work of Jain and Mai (1994) compared the two methods during the development of the bridging zone. It was noted that the energy approach underestimates the effect of bridging on crack growth resistance but the difference between the two methods may not be large. They argued that the stress intensity approach is more accurate and simpler to use.

2.3 Bridging Mechanisms in Fiber-Metal Laminates

Fiber-metal laminates, such as ARALL and GLARE, are hybrid composites that have fracture characteristics similar to conventional composites but with differences in the fracture process. Figure 2.8 shows a 2/1 lay-up (2 layers of metal and one layer of fibre lamina) of a fibre-metal laminate. The crack in fiber-metal laminates initiates and propagates in the metal layers while the composite lamina remains intact. With greater loads, the matrix in the composite layer will begin to crack and eventually the fibers will start breaking. The fibers and matrix transfer the load from the cracked metal layer and thus bridge the crack by restraining further crack opening. There is also load transfer from the metal layer to the fiber layer through a shear stress in the adhesive between the metal and fiber layers. This causes fatigue deformation of the adhesive and results in delamination between the metal and fiber layers. Therefore, there are two inter-related damage systems in such laminates as ARALL and GLARE, namely, crack growth in the metal layers accompanied by delamination between the fiber and metal layers resulting from adhesive shear deformation.

2.3.1 Adhesive Shear Deformation

The function of the adhesive is to impregnate the fibers and bond them to the metal layers. When the laminate is loaded, the adhesive is deformed (see Figure 2.9) due to a local shear stress system. There is a load transfer from the crack bridging fibers to the metal sheets by the shear stresses in the adhesive layer and the resulting shear deformation allows some crack opening and fatigue crack growth (see Figure 2.10). Under fatigue loading, the cyclic load transfer causes cyclic adhesive shear stresses to initiate fatigue delamination of the adhesive between the fibers and the metal sheets. Plasticity may occur in the adhesive but it has been found that the magnitude is negligible for fatigue crack growth where cyclic deformations remain mainly elastic and thus the effect on the stress intensity factor in the metal layer is negligible. In some cases, however, the plasticity may cause compressive stresses to do pre-damage in the fibers, Marissen (1989).

The effect of the adhesive shear on the bridging stresses has been investigated in recent work by Guo and Wu (1999), Guo and Wu (1998), and Takamatsu, *et al.* (1999). The shear deformation of the adhesive was calculated from the interlaminar shear stress of a double-cracked lap shear (DCLS) specimen and given by,

$$\delta_{adh}(x_i) = \frac{2\sigma_{br,i}}{E_{la}} \sqrt{\frac{F_{la}F_{Al}}{N_L F_{ad}F_{fm}}} \quad (2.11)$$

where

$$\begin{aligned} F_{Al} &= E_{Al}t_{Al} \\ F_{fm} &= E_{fm}t_{fm} \\ F_{la} &= F_{Al} + F_{fm} \\ F_{ad} &= G_{ad}/t_{ad} \end{aligned} \quad (2.12)$$

$\sigma_{br,i}$ = bridging stress at a distance x_i

E_{la} = Young's modulus of laminates in fiber direction

F_{la} , F_{Al} , F_{ad} , F_{fm} = Stiffness of the laminate, metal layers, adhesive, fiber-adhesive layers, respectively

N_L = number of interfaces in the specimen

x_i = distance from the crack center

G_{ad} = shear modulus of the adhesive

t_{ad} = thickness of adhesive in an interface

This equation can also be used to calculate the shear deformation of the adhesive in a centre-cracked tension (CCT) specimen. This adhesive deformation is a necessary component for the bridging mechanism in fiber-metal laminates but its effect on the bridging stress has been found to be relatively small. The plasticity effects on the adhesive shear deformation are small as well and can be neglected in most cases, see Marissen (1991). The characterization of the shear stress-strain relationship of the adhesive under an elastic-plastic state is difficult and more studies are needed.

2.3.2 Delamination Zone

The delamination between the fiber and metal layers is an important mechanism in the crack bridging of fiber-metal laminates. Fiber bridging in fiber-metal laminates is different from crack-face bridging in conventional composites because the delamination has a large influence on the bridging stress. Delamination causes stress redistribution ahead and behind the crack tip that permits fibers to remain intact in the wake of the crack tip, Wilson and Wilson (1991). The delamination shape and size are related to the crack length and magnitude of the bridging stress.

There is disagreement as to the location of delamination initiation and which location contributes most to the bridging stress. Ritchie, *et al.* (1989) argued that the

failure of the individual fiber/epoxy interfaces is the primary source of the delamination process while stating that others claim that it occurs within the epoxy near the prepreg/metal interface.

An increase in delamination causes a decreasing fiber effectiveness and increasing stress intensity and crack growth rate in the metal layer, Marissen (1994). The fiber bridging stress is transferred to the cracked metal layer through the interface so a higher bridging stress will result in a higher driving force for delamination and consequently a higher delamination growth rate, Lin and Kao (1996). But with an increasing delamination, the bridging stress and adhesive shear stress are reduced and the resulting low bridging fiber stress will slow delamination growth but increase crack growth, Marissen (1991). Thus, it can be seen that the delamination zone is strongly dependent on the bridging stress.

The shape of the delamination zone was observed to be generally elliptical. Using ultrasonic scanning of specimens, several studies revealed an elliptical delamination shape, Lin and Kao (1996), Marissen (1991), and Takamatsu, *et al.* (1999). A triangular shape was considered by Guo and Wu (1998) in their work with CCT specimens made of GLARE. In another study, Guo and Wu (1999) stated that the delamination shapes are irregular during fatigue and not elliptical in most cases but closer to a triangle. A general delamination shape is shown in Figure 2.10 where a crack from a notch is bridged in the delaminated area.

2.3.3 Power-Law Expressions for Bridging Stress

In order to obtain the critical parameters that characterize the fiber-bridging process in fiber-metal laminates, it is essential to find the bridging stress distribution on the crack faces. Analytical expressions were proposed by Foote, *et al.* (1986) which were based on their work using strain-softening materials. An empirical power law relationship for the bridging stress distribution as a function of crack opening displacement was given by,

$$\sigma_B(u) = \sigma_B^* \left[1 - \frac{u(x)}{u^*} \right]^{m_i} \quad (2.13)$$

Here, u^* and σ_B^* are the opening displacement and the bridging stress, respectively, at the fiber furthest from the crack tip, and m_i is a power index. This gives a distribution of increasing crack-face bridging stress toward the crack tip.

Suzuki and Sakai (1994) later used a relationship which gave a decreasing bridging stress toward the crack tip with another power index, n_d . The equation for this bridging stress is given as,

$$\sigma_B(u) = \sigma_B^* \left[\frac{u(x)}{u^*} \right]^{n_d} \quad (2.14)$$

The range of the power index m_i (or n_d) is from zero to infinity and covers all types of engineering materials. The case when m_i (or n_d) = 0 describes, for example, the behaviour of elastic perfectly plastic solids, and m_i (or n_d) = ∞ is appropriate for truly brittle materials with no softening region. Materials with frictional pullout as the major

softening mechanism are characterized with a power index, m_i (or n_d) = 1. The range of $1 < m_i$ (or n_d) $< \infty$ describes the behaviour of materials like ceramics and steel fiber reinforced ceramics in which fiber pull-out is not the only softening mechanism. For $0 < m_i$ (or n_d) < 1 , the stress-displacement relationship is typical of the fracture of ductile metals and polymers in plane stress.

As it is convenient to express the bridging stress distribution as a function of distance along the crack in the form of a power law, the relationship between the crack opening displacement and distance may also be similarly assumed. The crack opening displacement in terms of the bridging zone length, b , can be written as,

$$u(x) = u^* \left(\frac{x}{b} \right)^q \quad (2.15)$$

The power index, q , corresponds to a value of $\frac{1}{2}$ used in equations (2.1) and (2.2). Cain and Tan (1997) modified equation (2.13) to define σ_B^* in the same manner as equation (2.11) such that,

$$\sigma_B(u) = \sigma_B^* \left[1 + \left(1 - \frac{u(x)}{u^*} \right)^{m_i} \right] \quad (2.16)$$

The use of Equations (2.14) and (2.16) with (2.15) yields new expressions with combined power indices (m and n) such that the respective expressions for σ_B as a function of distance x from the crack tip become:

$$\sigma_B(x) = \sigma_B^* \left(\frac{x}{b} \right)^n \quad (2.17)$$

for stresses decreasing towards the crack-tip, and:

$$\sigma_B(x) = \sigma_B^* \left[1 + \left(1 - \frac{x}{b} \right)^m \right] \quad (2.18)$$

for stresses increasing towards the crack-tip (see Figure 2.11).

The bridging stress distribution can be obtained once the magnitude of the stress in the last intact fiber away from the crack tip is known. This stress depends on the extend of debond between the fiber which it acts through, and the matrix. It can be determined by considering the strain in this fiber, Cain and Tan (1997). This strain, ε^* , is related to the half-crack opening displacement v^* and the delamination length between the layers, d , at that position by:

$$\varepsilon^* = \frac{v^*}{d} \quad (2.19)$$

It can be seen that the bridging stress distribution in fibre-metal laminates is a three-dimensional problem. However, it can be treated as a two-dimensional one in the plane of the laminate by homogenization of mechanical properties through the thickness of the laminate. Although the bridging stress only acts on the fiber layer crack faces, it was assumed that it acts as a crack-closure stress all through the thickness, Cain and Tan (1997). The net force in the two-dimensional model must be the same as in the three-dimensional model and thus in the two-dimensional analysis, the stress in the last intact fiber farthest from the crack tip is:

$$\sigma_B^* = E_{11} \left(\frac{A}{A_T} \right) \varepsilon^* \quad (2.20)$$

E_{11} is the Young's modulus in the fiber direction of the fiber laminae, A is the cross-sectional area of the laminae, and A_T is the total cross-sectional area of the whole laminate. Combining equations (2.19) and (2.20) yields the expression:

$$\sigma_B^* = E_{11} \left(\frac{A}{A_T} \right) \left(\frac{v^*}{d} \right) \quad (2.21)$$

It is shown here that in order to obtain the bridging stress distribution along the crack length, the half-crack opening displacement and the bridging stress have to be known at the last intact fiber from the crack tip. Hence, the procedure to obtain σ_B^* is an iterative one. The maximum half crack opening displacement with no crack bridging, v^* is first obtained using a numerical approach such as the finite element or the boundary element method. Using equation (2.21), the stress in the farthest intact fiber from the crack tip, σ_B^* is then calculated and used in equations (2.17) and (2.18) to find the bridging stress distribution. A new v^* is found in a subsequent numerical stress analysis. This procedure is then repeated until the values of v^* and σ_B^* converge to set tolerances. An example of this iterative procedure is shown in Appendix A.

For a uniform crack bridging stress distribution, where $n = m = 0$, the distribution can be calculated directly if the half crack opening displacement without any crack bridging is known. If the half-crack opening displacement at the crack mouth is,

$$v_m^* = \bar{v}^* \quad (2.22)$$

By using Bueckner's superposition principle, Cain and Tan (1997) obtained the half crack opening displacement for a uniformly distributed crack bridging stress under an applied stress σ as,

$$v_m^* = \left(\frac{\sigma - \sigma_B}{\sigma} \right) \bar{v}^* \quad (2.23)$$

Then using equation (2.21), the uniform bridging stress distribution becomes,

$$\sigma_B = \frac{E_{11} \left(\frac{A}{A_r} \right) \left(\frac{\bar{v}^*}{d} \right)}{1 + \frac{E_{11}}{\sigma} \left(\frac{A}{A_r} \right) \left(\frac{\bar{v}^*}{d} \right)} \quad (2.24)$$

CHAPTER 3

REVIEW OF THE BOUNDARY INTEGRAL EQUATION METHOD

The boundary integral equation (BIE) method, also commonly known as the boundary element method (BEM), is an alternative to the finite element method (FEM) for the numerical solution of many engineering problems. Boundary element analysis is efficient in treating problems with stress concentrations and regions with high stress gradients such as those found near cracks. The use of fewer elements than in FEM is possible because only the boundary of the solution domain is discretized. As a result, the BIE method has become widely used in the study of fracture and fatigue of engineering components.

This Chapter reviews the formulation of the direct boundary integral equation for problems of two-dimensional, linear, isotropic and anisotropic elastostatics. The BIE method is then applied to linear-elastic fracture mechanics (LEFM) to obtain expressions for the crack-tip stress intensity factors. Quarter-point crack-tip elements are discussed and several examples are presented to demonstrate the accuracy of the boundary element method.

3.1 The Boundary Integral Equation Method for Isotropic Elasticity

The analytical formulation of the direct BIE method starts with the differential equations of equilibrium and transforms them into integral equations by the use of a fundamental solution and a reciprocal work theorem. For convenience, the indicial notation will be used here. In this notation, subscript indices (1, 2, 3) are used to replace the component variables in the (x, y, z) co-ordinate system such that (x, y, z) becomes (x₁, x₂, x₃).

The BIE method is developed by first considering an isotropic, linear elastic body R , with boundary S , in equilibrium in a two-dimensional co-ordinate system shown in Figure 3.1, for which the differential equation is:

$$\sigma_{ij,j} + b_i = 0 \quad i, j = 1, 2 \quad (3.1)$$

where σ_{ij} are the stress components and b_i are the body force vector per unit volume components, and the comma index implies differentiation.

The engineering strains, ε_{ij} , in terms of displacements, u are:

$$\varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \quad i, j = 1, 2 \quad (3.2)$$

Hooke's law constitutive equation for isotropic elasticity is:

$$\varepsilon_{ij} = \frac{1}{2\mu} \left[\sigma_{ij} - \frac{\nu^*}{1 + \nu^*} \sigma_{kk} \delta_{ij} \right] \quad i, j = 1, 2 \quad (3.3)$$

where μ is the shear modulus, ν is the Poisson's ratio of the material, δ_{ij} is the Kronecker delta, and

$$\nu^* = \begin{cases} \nu & \text{for plane stress} \\ \frac{\nu}{1-\nu} & \text{for plane strain} \end{cases} \quad (3.4)$$

$$\mu = \frac{E}{2(1+\nu)} \quad (3.5)$$

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \quad (3.6)$$

Equating ε_{ij} 's in equations (3.2) and (3.3) and substituting into equation (3.1)

gives the Navier's equations of equilibrium in terms of displacements as:

$$\mu \left[u_{i,jj} - \frac{1}{1-2\nu'} u_{j,ji} \right] + f_i = 0 \quad i, j = 1, 2 \quad (3.7)$$

where

$$\nu' = \begin{cases} \nu & \text{for plane strain} \\ \frac{\nu}{1+\nu} & \text{for plane stress} \end{cases} \quad (3.8)$$

Equations (3.7) are then transformed into integral equations with the use of a fundamental solution and the reciprocal work theorem.

The fundamental solution of Navier's equations for displacement is:

$$U_{ji}(p, q) = \frac{1}{8\pi\mu(1-\nu')} \left[(3-4\nu') \delta_{ij} \log_e \frac{1}{r} + r_i r_j \right] \quad (3.9)$$

where $U_{ji}(p, q)$ is the displacement in the x_i -direction at $q(x)$ due to the application of a unit concentrated load in the x_j -direction at $p(x)$, and r is the distance between p and q .

The corresponding solution for tractions is:

$$T_{ji}(p, q) = \frac{-(1-2\nu')}{4\pi(1-\nu')} \frac{1}{r} \left\{ \frac{dr}{dn} \left[\delta_{ij} + \frac{2}{1-2\nu'} \right] r_i r_j \right\} \quad (3.10)$$

where n is a unit outward normal at $q(x)$ and $T_{ji}(p, q)$ is the traction in the x_i -direction at $q(x)$ due to the application of a unit concentrated load in the x_j -direction at $p(x)$.

The Betti-Rayleigh's reciprocal work theorem relates the equilibrium states of an elastic body under different applied loads and is the basis of the direct formulation of the BIE method for elasticity. The theorem states that if an elastic body is subjected to two systems of body and surface forces, then the work by the first system t_i, f_i acting through the displacements u_i' due to the second system of forces is equal to the work done by the second system t_i', f_i' acting through the displacements u_i due to the first system of forces. In the absence of body forces, the reciprocal work theorem is expressed as:

$$\int_S t_i u_i' dS = \int_S t_i' u_i dS \quad (3.11)$$

where t_i and u_i are the surface tractions and displacements, respectively.

It is assumed that the primed state corresponds to the fundamental solution to Navier's equations and the unprimed state is unknown. Following the usual limiting process, the boundary integral equation for elastostatics can then be written as, (see, e.g. Tan (1987)):

$$\begin{aligned} C_{ji}(P) u_i(P) + \int_S u_i(Q) T_{ji}(P, Q) dS(Q) = \\ \int_S t_i(Q) U_{ji}(P, Q) dS(Q) \end{aligned} \quad (3.12)$$

where

$$C_{ji}(P) = \lim_{\epsilon \rightarrow 0} \int_{S_\epsilon} T_{ji}(P, Q) dS(Q) \quad (3.13)$$

$C_{ji}(P)$ depends on the local geometry of the surface at point P , and S_ϵ is the boundary region around P . Equation (3.11) is an integral constraint equation relating the boundary tractions to the boundary displacements. These equations are, in general, intractable to solve analytically and therefore numerical methods must be employed.

3.1.1 Numerical Solution of the BIE

The BIE equations are solved numerically by dividing the boundary S , of the solution domain into a series of M line elements as shown in Figure 3.2. The geometry of the elements and the variation of the BIE functions over the elements are assumed to be quadratic. The formulation for the numerical procedure has been presented by Tan and Gao (1992a) (1992b), and Tan, *et al.* (1992). The quadratic isoparametric elements, as shown in Figure 3.3, are each defined by three equally spaced nodes with intrinsic coordinates $\zeta = -1$, $\zeta = 0$, $\zeta = +1$, respectively.

The shape functions for these elements are:

$$\begin{aligned} N^{(1)}(\zeta) &= -\frac{1}{2}\zeta(1-\zeta) \\ N^{(2)}(\zeta) &= 1-\zeta^2 \\ N^{(3)}(\zeta) &= \frac{1}{2}\zeta(1+\zeta) \end{aligned} \quad -1 \leq \zeta \leq +1 \quad (3.14)$$

The superscripts on the shape functions $N(\zeta)$ represent the local nodes of the element.

The geometry, displacements and tractions can then be expressed in terms of the shape functions and nodal values as:

$$x_j(\zeta) = N^c(\zeta) x_j^c \quad (3.15)$$

$$u_j(\zeta) = N^c(\zeta) u_j^c \quad (3.16)$$

$$t_j(\zeta) = N^c(\zeta) t_j^c \quad c = 1, 3 \quad (3.17)$$

Substituting Eqs. (3.15) and (3.7) into Eq. (3.12), the discretised form of the boundary integral equations becomes:

$$\begin{aligned} C_{ji}(P^a) u_i(P^a) + \sum_{b=1}^M \sum_{c=1}^3 u_i(P^{d(b,c)}) \int_S T_{ji}(P^a, Q) N^c(\zeta) |J(\zeta)| dS \\ = \sum_{b=1}^M \sum_{c=1}^3 t_i(P^{d(b,c)}) \int_S U_{ji}(P^a, Q) N^c(\zeta) |J(\zeta)| dS \\ P^a = 1, N \end{aligned} \quad (3.18)$$

This gives a total of $4M$ linear algebraic equations for the unknown displacements and tractions at the boundary nodes of the solution domain where,

$$\begin{aligned} M &= \text{number of elements in the mesh} \\ N &= \text{number of nodes} = 2M \\ p^{d(b,c)} &= c^{\text{th}} \text{ node of the } b^{\text{th}} \text{ element} = 1, N \\ b &= 1, M \\ c &= 1, 3 \\ |J(\zeta)| &= \frac{dS}{d\zeta} = \text{Jacobian of transformation} \end{aligned}$$

Standard matrix methods can then be used to solve for the unknown displacements and tractions on the boundary, S .

3.2 The Boundary Integral Equation Method for Anisotropic Elasticity

The derivation of the BIE for anisotropic elasticity follows similar steps as that for the isotropic case. The difference lies in the stress-strain relations and the form of the fundamental solution, $U_{ji}(P, Q)$ and $T_{ji}(P, Q)$. The stress-strain relations for a two-dimensional anisotropic material in plane stress are:

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{16} \\ a_{12} & a_{22} & a_{26} \\ a_{16} & a_{26} & a_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} \quad (3.19)$$

Here a_{ij} are the elastic compliances related to the engineering constants by, see, e.g.

Hoskin and Baker (1986):

$$a_{11} = \frac{1}{E_{11}} \quad (3.20)$$

$$a_{22} = \frac{1}{E_{22}} \quad (3.21)$$

$$a_{12} = -\frac{\nu_{12}}{E_{11}} = -\frac{\nu_{21}}{E_{22}} \quad (3.22)$$

$$a_{16} = \frac{\eta_{12,1}}{E_{11}} = \frac{\eta_{1,12}}{G_{12}} \quad (3.23)$$

$$a_{26} = \frac{\eta_{12,2}}{E_{22}} = \frac{\eta_{2,12}}{G_{12}} \quad (3.24)$$

E_{11} and E_{22} are Young's moduli in the principal material axis directions, ν_{ij} are Poisson's ratios, G_{12} is the shear modulus, and $\eta_{ij,i}$ and $\eta_{i,ij}$ are the coefficients of mutual influence.

The coefficients of mutual influence, $\eta_{ij,i}$ and $\eta_{i,ij}$, are both zero for the case of orthotropy.

The Airy's stress function $F(x_1, x_2)$ may be used to obtain the stresses in two-dimensions, see, e.g. Cruse (1988):

$$\sigma_{11} = \frac{\partial^2 F}{\partial x_2^2} \quad \sigma_{22} = \frac{\partial^2 F}{\partial x_1^2} \quad \sigma_{12} = \frac{\partial^2 F}{\partial x_1 \partial x_2} \quad (3.25)$$

The compatibility equation in terms of strains is:

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} = \frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} \quad (3.26)$$

The governing equation for $F(x_1, x_2)$ becomes:

$$\begin{aligned} a_{22} \frac{\partial^4 F}{\partial x_1^4} - 2a_{26} \frac{\partial^4 F}{\partial x_1^3 \partial x_2} + (2a_{12} + a_{66}) \frac{\partial^4 F}{\partial x_1^2 \partial x_2^2} \\ - 2a_{16} \frac{\partial^4 F}{\partial x_1 \partial x_2^3} + a_{11} \frac{\partial^4 F}{\partial x_2^4} = 0 \end{aligned} \quad (3.27)$$

The solution to the above equation may be obtained in terms of complex variable z_j , where

$$z_j = x_1 + \mu_j x_2 \quad (3.28)$$

and

$$\begin{aligned} \mu_j &= \alpha_j + i\beta_j \\ \bar{\mu}_j &= \alpha_j - i\beta_j \quad j = 1, 2 \end{aligned} \quad (3.29)$$

Also, $i = \sqrt{-1}$, α_j and β_j are real constants, and $\beta_j > 0$ from thermodynamic considerations. Substituting equations (3.28) and (3.29) into (3.27) gives the characteristic equation for μ derived by Lekhnitskii (1963) as:

$$a_{11}\mu^4 - a_{16}\mu^3 + (2a_{12} + a_{66})\mu^2 - a_{26}\mu + a_{22} = 0 \quad (3.30)$$

The roots of this equation are distinct and never real as long as the material is non-isotropic. These roots are required for the fundamental solution for anisotropic elasticity. The fundamental solutions for displacements and tractions, respectively, are now given by, see, e.g. Tan and Gao (1992b):

$$U_{ji} = 2 \operatorname{Re} [r_{i1} A_{j1} \log_e(z_1) + r_{i2} A_{j2} \log_e(z_2)] \quad (3.31)$$

$$T_{j1} = 2n_1 \operatorname{Re} \left[\mu_1^2 \left(\frac{A_{j1}}{z_1} \right) + \mu_2^2 \left(\frac{A_{j2}}{z_2} \right) \right] - 2n_2 \operatorname{Re} \left[\mu_1 \left(\frac{A_{j1}}{z_1} \right) + \mu_1 \left(\frac{A_{j2}}{z_2} \right) \right] \quad (3.32)$$

$$T_{j2} = -2n_1 \operatorname{Re} \left[\mu_1 \left(\frac{A_{j1}}{z_1} \right) + \mu_2 \left(\frac{A_{j2}}{z_2} \right) \right] + 2n_2 \operatorname{Re} \left[\frac{A_{j1}}{z_1} + \frac{A_{j2}}{z_2} \right] \quad i, j = 1, 2 \quad (3.33)$$

where

$$r_{1j} = a_{11}\mu_j^2 + a_{12} - a_{16}\mu_j \quad (3.34a)$$

$$r_{2j} = a_{12}\mu_j + \frac{a_{22}}{\mu_j} - a_{26} \quad (3.34b)$$

and n_j are the unit outward normal components at Q with respect to the x_1 - x_2 co-ordinate system. The complex constants A_{ji} may be obtained from the following equations:

$$\begin{bmatrix} \operatorname{Im}\{B_1\} & \operatorname{Re}\{B_1\} & \operatorname{Im}\{B_2\} & \operatorname{Re}\{B_2\} \end{bmatrix} \begin{bmatrix} \operatorname{Re}\{A_{j1}\} \\ \operatorname{Im}\{A_{j1}\} \\ \operatorname{Re}\{A_{j2}\} \\ \operatorname{Im}\{A_{j2}\} \end{bmatrix} = \begin{bmatrix} -1/4\pi \delta_{j2} \\ 1/4\pi \delta_{j1} \\ 0 \\ 0 \end{bmatrix} \quad (3.35)$$

where

$$\{B_i\} = [i \quad \mu_i \quad r_{1i} \quad r_{2i}]^T \quad (3.36)$$

3.3 Linear-Elastic Fracture Mechanics Analysis by the BIE Method

The stress intensity factor characterizes the crack tip stress field in a linear elastic material. The aim of fracture mechanics analysis is to calculate this parameter from the measured displacements and stresses near the crack tip which can then be compared to

the fracture toughness of the material to establish structural integrity. Figure 3.4 defines the coordinate system.

The stress state in terms of the stress intensity factors in anisotropic elasticity is the same as in isotropic elasticity; the stresses are given by:

$$\begin{aligned}\sigma_{yy} &= K_I / \sqrt{2\pi r} \\ \sigma_{xy} &= K_{II} / \sqrt{2\pi r}\end{aligned}\quad (3.37)$$

The displacements on the crack faces ($\theta = 180^\circ$ in Figure 3.4) in terms of the stress intensity factors are, see, e.g. Tan and Gao (1992a):

$$\begin{Bmatrix} u_x \\ u_y \end{Bmatrix} = \sqrt{\frac{2r}{\pi}} \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{Bmatrix} K_I \\ K_{II} \end{Bmatrix} \quad (3.38)$$

where

$$D_{11} = \text{Im} \left[\frac{s_2 p_1 - s_1 p_2}{s_1 - s_2} \right] \quad (3.39)$$

$$D_{12} = \text{Im} \left[\frac{p_1 - p_2}{s_1 - s_2} \right] \quad (3.40)$$

$$D_{21} = \text{Im} \left[\frac{s_2 q_1 - s_1 q_2}{s_1 - s_2} \right] \quad (3.41)$$

$$D_{22} = \text{Im} \left[\frac{q_1 - q_2}{s_1 - s_2} \right] \quad (3.42)$$

The material property parameters, s_j , given in Tan and Gao (1992a), are related to the roots of the characteristic equation, μ_j by:

$$s_j = \frac{\mu_j \cos \phi + \sin \phi}{\cos \phi - \mu_j \sin \phi} \quad j = 1, 2 \quad (3.43)$$

and p_j and q_j are:

$$p_j = a'_{11} s_j^2 + a'_{12} s_j \quad (3.44)$$

$$q_j = \frac{a'_{12}s_j^2 + a'_{22} - a'_{26}s_j}{s_j} \quad j = 1, 2 \quad (3.45)$$

The local compliances are obtained from the following transformation from the global x_1 - x_2 co-ordinate system to the local Cartesian system by:

$$a'_{11} = a_{11} \cos^4 \phi + (2a_{12} + a_{66}) \sin^2 \phi \cos^2 \phi + a_{22} \sin^4 \phi + (a_{16} \cos^2 \phi + a_{26} \sin^2 \phi) \sin 2\phi \quad (3.46)$$

$$a'_{22} = a_{11} \sin^4 \phi + (2a_{12} + a_{66}) \sin^2 \phi \cos^2 \phi + a_{22} \cos^4 \phi - (a_{16} \cos^2 \phi + a_{26} \sin^2 \phi) \sin 2\phi \quad (3.47)$$

$$a'_{12} = a_{12} + (a_{11} + a_{22} - 2a_{12} - a_{66}) \sin^2 \phi \cos^2 \phi + \frac{1}{2}(a_{26} - a_{16}) \sin 2\phi \cos 2\phi \quad (3.48)$$

$$a'_{66} = a_{66} + 2(a_{11} + a_{22} - 2a_{12} - a_{66}) \sin^2 2\phi + 2(a_{26} - a_{16}) \sin 2\phi \cos 2\phi \quad (3.49)$$

$$a'_{16} = \left[a_{22} \sin^2 \phi - a_{11} \cos^2 \phi + \frac{1}{2}(2a_{12} + a_{66}) \cos 2\phi \right] \sin 2\phi + (a_{16} \cos^2 \phi)(\cos^2 \phi + 3 \sin^2 \phi) + (a_{16} \sin^2 \phi)(3 \cos^2 \phi - \sin^2 \phi) \quad (3.50)$$

$$a'_{26} = \left[a_{22} \sin^2 \phi - a_{11} \cos^2 \phi - \frac{1}{2}(2a_{12} + a_{66}) \cos 2\phi \right] \sin 2\phi + (a_{16} \sin^2 \phi)(\cos^2 \phi - \sin^2 \phi) + (a_{26} \cos^2 \phi)(\cos^2 \phi - 3 \sin^2 \phi) \quad (3.51)$$

3.3.1 Quarter-Point Crack-Tip Element in Plane Orthotropic Bodies

The expressions for the stresses and displacements in the vicinity of the crack-tip are proportional to $1/\sqrt{r}$ and $\sqrt{r}$, respectively. In order to obtain the proper variation of the displacements, the mid-point node of the element adjacent to the crack tip can be shifted to the quarter-point position as shown in Figure 3.5. This results in the proper $\sqrt{r}$

variation for the displacement field. In order to obtain the variation of the tractions, the nodal shape functions associated with nodal tractions have to be multiplied by $1/\sqrt{r}$, where l is the length of the element. Details of this formulation have been presented by Tan and Gao (1992a), Martinez and Dominguez (1984), and Cruse and Wilson (1977).

The displacement variation in this quarter-point element can be expressed in terms of nodal values as:

$$u_j = u_j^A + (-u_j^C + 4u_j^B - 3u_j^A) \left(\frac{r}{l} \right)^{1/2} + (2u_j^C - 4u_j^B + 2u_j^A) \left(\frac{r}{l} \right) \quad (3.52)$$

where the superscripts in u_j denote the nodes as shown in Figure 3.1. The analytical expressions for K_I and K_{II} based on the nodal displacements may be obtained by equating the coefficients of $\sqrt{r}$ in equations (3.38) and (3.52). If the crack lies parallel to the global x_I -axis, the expressions may be written as:

$$\begin{bmatrix} K_I \\ K_{II} \end{bmatrix} = \frac{\sqrt{\pi/2l}}{\text{Det}[D]} \begin{bmatrix} D_{22} & -D_{12} \\ -D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} -u_1^C + 4u_1^B - 3u_1^A \\ -u_2^C + 4u_2^B - 3u_2^A \end{bmatrix} \quad (3.53)$$

$$\text{where } \text{Det}[D] = (D_{11}D_{22} - D_{21}D_{12}) \quad (3.54)$$

This equation is commonly referred to as the ‘displacement formula’ in BEM analysis.

The computed traction values $\bar{t}_j$ are related to the physical traction values t_j at the nodes, again denoted by the superscripts, by:

$$\bar{t}_j^E = t_j^E \quad (3.55)$$

$$\bar{t}_j^D = \frac{1}{2} t_j^D \quad (3.56)$$

$$\bar{t}_j^A = \lim_{r \rightarrow 0} t_j^A \sqrt{\frac{r}{l}} \quad (3.57)$$

If the crack lies parallel to the global x_I -axis, then the tractions with the incorporated singularity at the crack-tip are related to the stress intensity factors by:

$$\bar{t}_1^A = \lim_{r \rightarrow 0} t_1^A \sqrt{\frac{r}{l}} = \lim_{r \rightarrow 0} \sigma_{12} \sqrt{\frac{r}{l}} = \frac{K_{II}}{\sqrt{2\pi l}} \quad (3.58a)$$

$$\bar{t}_2^A = \lim_{r \rightarrow 0} t_2^A \sqrt{\frac{r}{l}} = \lim_{r \rightarrow 0} \sigma_{22} \sqrt{\frac{r}{l}} = \frac{K_I}{\sqrt{2\pi l}} \quad (3.58b)$$

or

$$K_I = \bar{t}_2^A \sqrt{2\pi l} \quad (3.59a)$$

$$K_{II} = \bar{t}_1^A \sqrt{2\pi l} \quad (3.59b)$$

This equation is referred to as the ‘traction formula’ in BEM analysis.

It is evident that the displacement formula involves additional computational effort as compared with the traction formula. Tan and Gao (1992a) also showed that the stress intensity factors obtained using the displacement formulas are significantly more sensitive to the size of the crack-tip elements used. The traction formula is more widely used and the stress intensity factors can be obtained directly from the computed nodal tractions.

Although these equations have been derived for a case where the crack lies parallel to the global x_I -axis, they remain valid for an arbitrarily oriented crack provided that the x_I - and x_2 -components of the displacements and tractions are replaced by the x - and y - components in the local co-ordinate system.

3.4 Numerical Examples

To demonstrate the effectiveness of the boundary element method in fracture mechanics, several crack problems were analyzed. The test problems consisted of various geometries with isotropic and orthotropic material properties with the objective of obtaining the crack-tip stress intensity factors. The BIE results were compared with the corresponding results available in literature.

3.4.1 Test Problems in Isotropy

Four sample problems in material isotropy were analyzed. The first three problems were of a flat plate with central, edge, and double edge cracks shown in Figures 3.7 to 3.9. The fourth problem was that of a flat plate with two cracks emanating from a hole in the centre of the plate shown in Figure 3.10. Advantage of symmetry was taken in modeling the four problems such that only one quarter of the plate was modeled with quarter-point elements being employed at the crack tip.

3.4.1.1 Cracks in a Plate

With reference to Figures 3.7 to 3.9 for the case of the plate without the hole, the crack sizes were varied from $a/W = 0.1$ to 0.5 for the central crack and double edge crack, and $a/2W = 0.1$ to 0.5 for the edge crack. The height-to-width ratio, H/W , for the central crack and double edge crack was chosen to be unity, whereas $H/W = 2$ for the edge crack problem. A typical boundary element mesh used is shown in Figure 3.11. The solutions for stress intensity factors presented in Rooke and Cartwright (1976) were used for

comparison with the BIE results. The literature solutions were obtained using boundary collocation, conformal mapping techniques, and interpolation techniques and it is claimed to be accurate to within 1% to 5% in error.

The normalized stress intensity factors, $K_I/\sigma\sqrt{\pi a}$ from the BIE analysis are compared with the solutions given by Rooke and Cartwright (1976) in Tables 3.1 and 3.2. It can be seen that the BIE method produced results which are in very good agreement to those in the literature.

3.4.1.2 Cracks in a Plate with a Hole

With reference to Figure 3.10, for the problem of the plate with a hole, the crack sizes were varied from $c/W = 0.2$ to 0.5 . The height-to-width ratio, H/W , was taken to equal to two. The hole radius-to-width ratio, R/W was set equal to 0.1 and a typical boundary element mesh is shown in Figure 3.12. The stress intensity factors for a central hole in a plate with cracks emanating from the two sides of the hole have also been presented in Rooke and Cartwright (1976). The boundary collocation method was used to obtain the theoretical results, with accuracy to within 1%. Table 3.3 lists the normalised stress intensity factors, $K_I/\sigma\sqrt{\pi c}$ from the BIE analysis compared to those presented in Rooke and Cartwright (1976). The results are again in excellent agreement with each other.

3.4.2 Test Problems in Orthotropy

Two sample problems have been analyzed with material orthotropy. The first problem is that of a rectangular plate with a central crack subjected to uniform tension in the fiber direction, as shown in Figure 3.13. The second problem is that of a rectangular plate in tension with two cracks emanating from the sides of a hole, as shown in Figure 3.14. The stress intensity factors in an orthotropic material are not only a function of the geometry and applied loads, they also depend on the elastic properties of the material and the orientation of the principal material axes.

3.4.2.1 Cracks in a Plate

The case of a central crack in a plate with an orthotropic material property was analyzed by Bowie and Freese (1972). The theoretical results were obtained numerically by using purely imaginary constants, β_1 and β_2 , where:

$$\beta_1 \beta_2 = \left(\frac{E_1}{E_2} \right)^{1/2} \quad (3.60)$$

$$\beta_1 + \beta_2 = \sqrt{2} \left\{ \left(\frac{E_1}{E_2} \right)^{1/2} + \frac{E_1}{2\mu_{12}} - \nu_{12} \right\}^{1/2} \quad (3.61)$$

Here, E_1 and E_2 are the Young's moduli, ν_{12} is the Poisson's ratio, and μ_{12} is the shear modulus. The direction parallels and perpendicular to the fibers are denoted by the subscripts 1 and 2, respectively. The material properties are defined in the x_1 - and x_2 -axes as shown in Figure 3.13.

The material used in the BIE analysis was S-Glass/Epoxy with the following properties:

$$\begin{aligned} E_1 &= 55 \text{ GPa} \\ E_2 &= 16 \text{ GPa} \\ \mu_{12} &= 7.6 \text{ GPa} \\ \nu_{12} &= 0.28 \end{aligned}$$

The material axes were oriented 90° to those in Figure 3.13 and the material constants were transformed accordingly. Figure 3.11 shows a typical boundary element mesh used. Table 3.4 shows the normalized stress intensity factors, $K_I/\sigma\sqrt{\pi a}$ from the BIE analysis compared to those from the literature. Agreement between the two sets of results for all crack sizes treated can be seen to be very good indeed.

3.4.2.2 Cracks in a Plate with a Hole

The analysis of two radial cracks emanating from a hole in a rectangular plate has been carried out using FEM by Lakshminarayana, *et al.* (1979). The plate geometry is shown in Figure 3.14 and the geometric parameters considered were:

$$h/W = 1.5 \quad R/W = 0.25 \quad c/W = 0.7$$

Figure 3.15 shows a typical BEM mesh used. The corresponding FEM mesh that was used for the FE analysis by Lakshminarayana, *et al.* (1979) is shown in Figure 3.16. Three unidirectional composite materials were chosen for the purpose of comparison. The fiber direction was oriented 90° with respect to the crack plane. The materials and their properties were:

Glass/Epoxy	Boron/Epoxy	Graphite/Epoxy
$E_1 = 51.7 \text{ GPa}$	$E_1 = 117.2 \text{ GPa}$	$E_1 = 206.8 \text{ GPa}$
$E_2 = 17.2 \text{ GPa}$	$E_2 = 11.7 \text{ GPa}$	$E_2 = 5.17 \text{ GPa}$
$\mu_{12} = 10.3 \text{ GPa}$	$\mu_{12} = 4.48 \text{ GPa}$	$\mu_{12} = 3.10 \text{ GPa}$
$\nu_{12} = 0.25$	$\nu_{12} = 0.21$	$\nu_{12} = 0.25$

The normalized stress intensity factors obtained from the BIE analysis are listed in Table 3.5. Also shown are the values obtained by Lakshminarayana, *et al.* (1979).

The BIE can be seen to be again in very good agreement with the results of these authors.

3.5 Concluding Remarks

It is evident from the above test examples that the BIE method can produce accurate stress intensity factor solutions with relatively simple mesh designs involving only the boundary of the numerical solution domain. As shown in Figures 3.15 and 3.16, the ease of modelling is most clearly demonstrated in the last example when compared with the FEM.

CHAPTER 4

ANALYSIS OF FULLY BRIDGED CRACKED FIBRE-METAL LAMINATES WITH NOTCHES

4.0 Introduction

Fibre-metal laminates, such as ARALL and GLARE, are ideally suited for aerospace applications as their toughness-to-weight ratios are superior to those of monolithic alloys. To fully understand the bridging effects of these materials, a three-dimensional study is necessary. This is intractable analytically and is expensive and time consuming numerically when repeated analyses are required. This Chapter presents the results of the simplified two-dimensional analysis using the boundary element method (BEM) as described in Chapters 2 and 3, to obtain estimates of the bridging stresses across the crack flanks and the corresponding fracture parameters. These results are compared with the results obtained using three-dimensional FEM analysis.

4.1 Problem Definition and Scope

Three different rectangular fibre-metal laminate plates with cracks emanating from the edge of a circular cut-out were analysed by the boundary element method (BEM) in two-dimensions and the finite element method (FEM) in three-dimensions.

The rectangular plates were subjected to a remote uniform tension, σ , in the fibre direction. The three different crack problems are shown in Figures 4.1 to 4.3. They were (i) a plate with a semi-circular hole at one edge with a single crack emanating from it, (ii) a plate with a semi-circular hole at both edges with a crack emanating from each, and (iii) a plate with a central circular hole with two symmetric cracks emanating from it. The orientation of the plane of the cracks was chosen to be perpendicular to the fibre direction.

The geometry of the rectangular plates in the x_1 - x_2 plane was defined by the height-to-width ratio, $H/2W = 2$ for case (i), and $H/W = 2$ for cases (ii) and (iii). The radius-to-width ratio of the circular cut-out for case (i) was chosen as $R/2W = 0.1$, and $R/W = 0.1$ for cases (ii) and (iii). The relative crack lengths considered for case (i) were $c/2W = 0.2, 0.3, 0.4, 0.5$, and for cases (ii) and (iii), where $c/W = 0.2, 0.3, 0.4, 0.5$. The height to fibre-reinforced laminae thickness ratio, H/t , and the total laminate to fibre-reinforced laminae thickness ratio, T/t were taken to be 400 and 4, respectively.

As has been indicated previously, throughout this study, the material deformation is assumed to be elastic and that linear elastic fracture mechanics (LEFM) is applicable. Also, the thickness of the adhesive between the aluminum layer and the fibre layer is assumed to be small and have negligible influence on the material properties of the laminate. Furthermore, it is also assumed that there were no residual stresses in the laminate.

4.1.1 Two-Dimensional Analysis

A simplified study in two dimensions requires the treatment of the problem in the x_1 - x_2 plane as well as the x_1 - x_3 plane. However, in ARALL laminates without notches, Cain and Tan (1997) showed that the analysis in the x_1 - x_3 plane significantly underestimates the magnitude of the largest bridging stress in the fibres. This was attributed to bending about the crack-tip in the x_1 - x_2 plane, which could not be accounted for in the analysis for the x_1 - x_3 plane. Therefore, the two-dimensional analysis in this study was only applied to the x_1 - x_2 plane of the laminate.

The problem to be treated in the x_1 - x_2 plane involved several parameters such as: the fatigue crack length, a , the equivalent crack length, c , the delamination zone size, d , and the power law indices, n and m of Eqns. (2.17) and (2.18). In the case of partial bridging, the bridging zone length, b is also varied depending on the amount of bridging fibres present. For full bridging of the crack faces, b is equal to the crack length, a .

The two-dimensional numerical stress analysis was carried out using the boundary element method (BEM) for anisotropic elasticity employing quadratic isoparametric elements. The bridging stress distribution was assumed to be a continuous closing pressure on the crack faces. An iterative scheme was employed because the bridging stress and the crack opening profile are interrelated, but both are not known *a priori*. The bridging stress profiles were varied using equations (2.17) and (2.18), with both of the power-law indices n and m having values of 0, 0.5 and 1.0. The crack was assumed to have propagated through both the metal layers and the fibre layer, and plane stress conditions were assumed here since the laminate is very thin through the thickness. One-quarter of each plate was modelled due to geometric and material symmetry about two

axes. A typical BIE mesh is shown in Figure 4.4, where it can be seen that special quarter-point traction-singular nodes were used for elements adjacent to the crack-tip.

4.1.2 Three-Dimensional Analysis

The finite element method (FEM) was used for the three-dimensional stress analysis, with the commercial software packages: SDRC I-DEAS 7.0 used for the mesh generation, and MSC/Nastran 70.7 used as the FE solver. The elements used for the FEM analysis were 20-noded quadratic iso-parametric solid bricks with the mid-side nodes of the elements adjacent to the crack-tip shifted to the quarter-point position. The nodes of the elements at the delamination boundary were not shifted to quarter-point due to the difficulties in meshing the region. This is not expected to greatly affect the bridging stresses or the mode I stress intensity factor at the crack-tip in the aluminum layer.

The 3-D analysis of a fully bridged fibre-metal laminate consisted of a crack propagating in the metal layers only with the fibre layer remaining crack free. Each plate in the 3-D analysis was modelled as one-eighth of the whole plate due to geometric and material symmetry in three planes. A typical 3-D FEM mesh is shown in Figure 4.5.

4.1.3 Delamination Shape and Size

Two delamination zone shapes were considered in this study. One shape was that of a quarter-circle and the other shape was that of a quarter-ellipse with a major-minor axis ratio of 2-to-1, around each crack flank. These shapes were centred at the crack mouth of each crack, which corresponds to one or more edges of the hole. The

delamination zone size, d , was taken at the last intact fibre away from the crack-tip in the bridged zone. For a fully bridged crack, the edge of the hole represents the last intact fibre away from the crack tip and hence d was set equal to the crack length, a , for the circular delamination shape and $\frac{1}{2}a$ for the elliptical delamination shape.

Experimentally observed delamination shapes are generally elliptical so a circular shape represents a more severe situation than would likely be encountered. Nevertheless, it is of interest to obtain the effect of an increased delamination size on the bridging stress and the stress intensity factor.

4.1.4 Material Properties

Two fibre-metal laminate materials, ARALL2-2/1 and GLARE2-2/1, were used in this study. ARALL2 and GLARE2 are hybrid composites consisting of alternating aluminum 2024-T3 layers with aramid-epoxy layers and glass-epoxy layers, respectively. The stacking is defined by the lay-up, which in this case is 2/1 and signifies two layers of aluminum sandwiching one layer of fibre-epoxy.

The material properties of each layer are shown in Table 4.1 and the indices 1, 2, and 3 denote the fibre, transverse and through-the-thickness directions. In the 3-D FEM analysis, the aluminum layer was treated as isotropic and the fibre layers were treated as orthotropic with the following material property relations in the thickness direction, Gibson (1994):

$$\begin{aligned}
 E_{33} &= E_{22} & G_{13} &= G_{12} & G_{23} &= \frac{E_{22}}{2(1+\nu_{32})} \\
 \nu_{21} &= \nu_{31} & \nu_{23} &= \nu_{32} & -\frac{\nu_{21}}{E_{22}} &= -\frac{\nu_{12}}{E_{11}}
 \end{aligned}
 \tag{4.1}$$

For the 2-D BEM modelling, the material properties were homogenized through the thickness of the laminate using classical laminate plate theory, Gibson (1994). The resulting homogenized material properties for ARALL and GLARE that are used in the two-dimensional BEM analysis are listed in Table 4.2. The thickness of the aluminum layer was chosen as 0.3 mm and the thickness of the aramid-epoxy and glass-epoxy layers was chosen as 0.2 mm. These values for the layer thickness are within the range of the actual manufactured thickness, Krishnakumar (1993).

4.2 Results and Discussion

The bridging stress and the stress intensity factor results for an edge crack, double-edge crack, and central crack are presented in this section. These results are for a fully bridged laminate with circular and elliptical delamination shapes and two materials, ARALL and GLARE, considered. The bridging stress and stress intensity factor results obtained by a BEM analysis in two-dimensions are compared to those obtained from a three-dimensional FEM analysis. A comparison of the results is also made between the two delamination shapes and the two materials.

It should be noted that the discussion in the following sections is for the ARALL material only; a comparison to GLARE will be treated in separate sub-sections. The reason for this is because the results for GLARE show almost the same variations of the corresponding bridging stresses and stress intensity factors as for ARALL. This will be further elaborated upon in sub-sections 4.2.1.2 and 4.2.2.3.

4.2.1 FEM Analysis

4.2.1.1 Bridging Stress Distribution

The bridging stress, σ_B , was taken along the mid-plane of the fibre layer in the finite element analysis and normalized with respect to the remote applied tensile stress, σ . The variations of the normalized bridging stress, σ_B/σ , with relative distance $x/a = -1.0$ to $x/a = 1.0$ from the crack-tip, along the centreline of the fibre layer for an edge crack, double-edge crack and central crack are shown in Figures 4.6 to 4.11. These geometries have circular and elliptical delamination shapes and varying normalized crack lengths, c/W , from 0.2 to 0.5 for the double-edge and central crack and $c/2W$ of 0.2 to 0.5 for the edge crack.

The normalized bridging stress, σ_B/σ , for all crack geometries increases as x/a approaches 0 and 1.0, corresponding to the crack-tip and crack mouth, respectively. For example, for a central crack with a circular delamination and crack length of $c/W = 0.5$, the normalized bridging stress at $x/a = 0$ is 4.24. This stress decreases to about 2.05 at $x/a = 0.53$ and increases to 10.14 at $x/a = 1.0$. The increase of the normalized bridging stress at $x/a = 0$ is likely due to the enforced singularity of the quarter-point crack-tip elements in the aluminum layer adjacent to the mid-point elements in the fibre-reinforced layer of the mesh. The large increase of the normalized bridging stress, σ_B/σ , at $x/a = 1.0$, which represents the notch root, is due to the effect of stress concentration near the notch.

The results in Figures 4.6 to 4.8 show that for all crack lengths, plate geometries, and delamination zone shapes, the normalized bridging stress, σ_B/σ , at the notch root (or

crack mouth), represented by $x/a = 1.0$, is very large compared to the normalized bridging stress anywhere else along the crack line. The highest normalized bridging stress, σ_B/σ , for example, is for the edge crack with a circular delamination shape and crack length, $c/2W$ of 0.5, where it reaches $\sigma_B/\sigma = 18.92$. The trends in these figures show that the fibres near the hole could be highly strained and would likely fail when their ultimate tensile stress is exceeded. The rapid increase of the normalized bridging stress, σ_B/σ , near the hole is generally confined to a small region, typically 5 to 10 % of the total crack length, where the stress concentration effects are the highest. Therefore, the presence of a notch causes the load through the fibres to be concentrated near the notch, which in turn greatly increases the bridging stresses in those fibres. This therefore adds to the bridging stress already caused by the opening of the crack in the aluminum layer.

For the central crack and double edge crack in Figures 4.9 and 4.10, the normalized bridging stress distribution, σ_B/σ , decreases with increasing crack length, c/W . This is opposite to the trend in fibre-metal laminates without notches for these two cracked geometries, where the bridging stress distribution increases with increasing crack length, Cain and Tan (1997). The decrease in the normalized bridging stress observed here can be explained by considering the influence of the stress concentration caused by the notch. For relatively small crack lengths compared to the size of the notch, the bridging stresses caused by the presence of the notch are higher than the bridging stresses caused by the crack alone, and the effect of the stress concentration slowly diminishes with increasing distance from the notch. For example, for a double-edge crack with an elliptical delamination the normalized bridging stresses, σ_B/σ , are highest for $c/W = 0.2$ and lowest for $c/W = 0.5$. The same trend is observed for a double-edge crack with

circular delamination and a central crack with both delamination shapes. It is expected that the bridging stress distribution would begin to increase with increasing crack length far away from the notch and this is confirmed in Figures 4.9 and 4.10, where an inflexion point can be seen and the normalized bridging stress distribution, σ_B/σ , near the crack-tip begins to increase with increasing crack length, c/W , since the effect of the crack on the stresses becomes dominant over that due to the notch.

For the edge crack in Figure 4.11, the bridging stress distribution is different from that for the double-edge and central cracks. The normalized bridging stress distribution, σ_B/σ , for the edge crack increases with increasing crack length, $c/2W$ of 0.3 to 0.5 for both delamination shapes. This can be attributed to the effect of bending about the crack tip as a result of the unsymmetrical geometry caused by the presence of the crack and notch. The bending causes higher strains in the fibres with increasing distance from the crack tip, x , and hence the normalized bridging stress, σ_B/σ increase as the crack length, $c/2W$ increases. In this case, the effects on the bridging stress of the crack opening plus bending dominate over the effects on the bridging stress caused by the stress concentration near the notch. Also, there does not appear to be an inflexion of the bridging stress distributions for either the circular delamination or the elliptical delamination shape in the $x/a = 0$ to $x/a = 1.0$ region as was observed for the central and double-edge crack geometries. This signifies, in this case, that the bending effect of the edge crack geometry is greater than the effect of the stress concentration caused by the notch for the whole crack length, resulting in the observed distribution.

The claim of a dominant stress concentration effect over the whole crack length cannot be made for every delamination shape because the size of the delamination shape

is another factor influencing the bridging stress distribution. For the circular delamination shape, the normalized bridging stress distribution, σ_B/σ , away from the crack tip, for $c/2W = 0.2$ is almost equal to the distribution for $c/2W = 0.4$. For the elliptical delamination shape, the normalized bridging stress distribution, σ_B/σ , away from the crack tip, for $c/2W = 0.2$ is higher than that for $c/2W = 0.5$. The reason for this can be explained by considering the effects of bending and stress concentration effects. The contribution of bending on bridging stress is much lower for shorter crack lengths, but the effects of stress concentration near the notch are much greater for shorter crack lengths. Therefore, the bending effect on the bridging stress distribution is sufficient to overcome the stress concentration effect, but only for relatively large crack lengths compared to the hole size. The bending effect on the magnitude of the bridging stress also appears to depend on the delamination shape.

The effect of the delamination zone size on the bridging stress is shown in Figures 4.6 to 4.11. For the edge crack, double-edge crack, and central crack, the normalized bridging stress, σ_B/σ , is higher for an elliptical delamination than for a circular delamination for all crack lengths, c/W or $c/2W$. As a comparison, the central crack with $c/W = 0.5$ and an elliptical delamination has a normalized bridging stress, σ_B/σ , equal to 4.24 at $x/a = 0$. For the same problem with a circular delamination, $\sigma_B/\sigma = 3.64$ at $x/a = 0$. The delamination zone shape and the bridging stress influence each other such that an increase in the delamination zone size decreases the bridging stress, and a decrease in the bridging stress increases the delamination zone size. As the delamination zone size increases, the amount of applied load transferred from the aluminum layers to the fibre layer decreases due to a smaller area for shear stress transfer between the metal and fibre

layers. Thus, an increasing delamination causes a decrease in the stress in the fibres, and hence, a lower bridging stress, as shown in the obtained results.

The normalized bridging stress distribution, σ_B/σ , for the central crack and the double-edge crack increases more rapidly for the elliptical delamination than for the circular delamination, with increasing distance away from the crack-tip, x/a . For the edge crack with an elliptical delamination, the normalized bridging stress, σ_B/σ , for $c/2W = 0.2$ is the highest compared to the rest of the crack lengths as mentioned previously. The normalized bridging stress increases more rapidly for the elliptical delamination than for the circular delamination, with distance away from the crack tip, x . Therefore, for a smaller delamination size, the stress in the fibres near the notch is greater and the influence of the stress concentration is more pronounced.

The normalized bridging stress, σ_B/σ , at $x/a = 1.0$ for all geometries is generally larger for an elliptical delamination than for a circular delamination. As an example, a central crack with $c/W = 0.2$, $\sigma_B/\sigma = 9.24$ for an elliptical delamination and only 7.85 for a circular delamination.

In the 3-D FEM analysis, the stress intensity factors, K_I , were calculated using the displacement extrapolation technique relating the stress intensity factor, K_I , to the half-crack vertical displacement, v , of the crack face by, Guinea, *et al.* (2000):

$$K_I^* = \frac{H\sqrt{2\pi}}{4} \left(\frac{v}{\sqrt{r}} \right) \quad (4.2)$$

where, r = distance from the crack-tip, and

$$H = \begin{cases} E & \text{plane stress} \\ \frac{E}{(1-\nu^2)} & \text{plane strain} \end{cases} \quad (4.3)$$

E = Young's Modulus

ν = Poisson's Ratio

This was a well-established technique in FEM fracture mechanics analysis before the development of special formulas for the determination of stress intensity factors. The reason for using this is because, although MSC/Nastran v70.7 has an isotropic, 3-D crack-tip element, it contains a program bug which precludes the direct evaluation of these values when isotropic and anisotropic material properties are defined for different elements in the FE analysis.

4.2.1.2 ARALL vs. GLARE

The variations of the normalized bridging stress, σ_B/σ , in the FEM analysis for the central, double-edge, and edge cracks with circular and elliptical delamination shapes were compared between ARALL and GLARE. Figures 4.12 to 4.14 show the variations of the percentage difference in bridging stress of GLARE from ARALL with the normalized distance from the crack-tip for the three geometries, crack lengths, and two delamination shapes. For the purpose of comparison of the magnitude of the bridging stress, a negative percentage signifies that the bridging stress for GLARE is smaller than the bridging stress for ARALL. As was mentioned above, the trend of the bridging stress for GLARE is the same as that for ARALL, and as a result, only a difference in the magnitude of the bridging stress was used as a comparison of this stress between the two materials.

The results for the central crack and double edge crack problems show that the difference in the magnitude of bridging stress between ARALL and GLARE is no more than 10% for the same relative crack size, except at very close to the hole. In general, the bridging stress in GLARE is less than in ARALL with the exception of the vicinity of the crack-tip where the influence of the induced singularity dominates.

The results for the edge crack show a greater increase in bridging stress for ARALL than GLARE near the un-notched edge of the plate for the elliptical delamination. This may be attributed to the higher elastic modulus of the fibres in ARALL as compared to GLARE. This leads to a greater difference in the bridging stress near the un-notched edge of the plate because GLARE will be more compliant than ARALL, resulting in lower stresses in that region.

The curves in all the figures show an increasing percentage difference as the hole is approached, signifying that the bridging stress for GLARE is greater than for ARALL. This can be explained by noting that for GLARE the bridging stress increases gradually toward the hole, whereas in ARALL the bridging stress increases very rapidly only near the hole. The curves are then seen to rapidly increase with a positive percentage difference at the root of the hole, represented by $x/a = 1.0$. This signifies that the fibres in GLARE near the hole have a lower bridging stress than those in ARALL. In general, the difference at $x/a = 1.0$ is about 30-35% for all crack problems and indicates that the fibres in GLARE have a better chance of remaining intact than those in ARALL.

4.2.2 BEM Analysis

4.2.2.1 Bridging Stresses

In the analysis, the bridging stress at the crack mouth, σ_B^* was again normalized with respect to the remote applied stress, σ for the three physical problems considered. The computed normalized bridging stress, σ_B^*/σ , for ARALL, is plotted in Figures 4.15 to 4.17 as a function of the normalized crack length, c/W or $c/2W$. These figures also show the variation of σ_B^*/σ with the fibre-bridging power-law indices, n and m of 0 to 1.0, and circular and elliptical delamination zone shapes. The corresponding results for GLARE are plotted in Figures 4.18 to 4.20 as a comparison, but the discussion here is again only based on the results of ARALL. Additionally, the tabulated results from the BIE analysis can be found in Tables 4.3 to 4.5.

For the central crack and the double-edge crack problems, the normalized bridging stress, σ_B^*/σ , decreases with increasing crack size, c/W , for both delamination shapes. For example, for a central crack with an elliptical delamination and $n = 0.5$, at $c/W = 0.2$, $\sigma_B^*/\sigma = 4.55$, and at $c/W = 0.5$, $\sigma_B^*/\sigma = 3.40$. This decrease is attributed to the fact that the stress concentration near the hole has a larger influence on smaller crack lengths, and hence σ_B^*/σ is highest for $c/W = 0.2$, and decreases as c/W increases. This same trend was observed in the results for the three dimensional FEM analysis for the bridging stress distributions.

The normalized bridging stress, σ_B^*/σ , for the central and double-edge cracks increase with increasing power-law index n and m . For a double-edge crack with a circular delamination and $c/W = 0.5$, σ_B^*/σ for $n = 0$ is 1.80, while for $n = 1.0$ it is equal

to 2.38. The normalized bridging stress for $m = 0.5$ and 1.0 is generally between the bridging stress for $n = 0$ and 0.5, for both delamination shapes and all crack lengths. The highest normalized bridging stress is for $n = 1.0$, which represents a linear bridging stress distribution with decreasing stresses toward the crack-tip.

For an edge crack problem, the normalized bridging stress, σ_B^*/σ , decreases from $c/2W = 0.2$ to $c/2W = 0.3$ and then increases as the crack length increases from $c/2W = 0.3$ to $c/2W = 0.5$. This trend is again the same as the one predicted by the three dimensional FEM analysis for the normalized bridging stress distributions. As an example, for an elliptical delamination and $n = 0.5$, the normalized bridging stress, σ_B^*/σ is 5.43 at $c/2W = 0.2$ and decreases to 5.09 at $c/2W = 0.3$ and then increases to 5.25 and 5.56 at $c/2W = 0.4$ and 0.5, respectively.

The normalized bridging stress, σ_B^*/σ , for the edge crack increases as the power-law index, m and n increases, for both delamination shapes. The highest normalized bridging stress for the edge crack is also represented by the index $n = 1.0$. It should be noted that for the edge crack a constant bridging stress represented by $n = m = 0$ does not show a decrease around $c/2W = 0.3$ but only increases from $c/W = 0.2$ to 0.5.

For all geometries, the delamination zone shape influences the bridging stresses such that the normalized bridging stress, σ_B^*/σ , is higher for an elliptical delamination as compared to a circular delamination. This is the same trend as seen in the FEM analysis in the previous section. Comparing an edge crack with $c/2W = 0.2$ and $n = 1.0$, $\sigma_B^*/\sigma = 5.872$ for an elliptical delamination, while for a circular delamination, $\sigma_B^*/\sigma = 3.776$.

In general, the 3-D FEM results lie between a constant distribution given by $n = 0$, and a linear distribution given by $n = 1.0$. The 2-D BEM results also lie between these

two bounds of power-law index, being closer to $n = 0.5$. The bridging stresses generally decrease toward the crack-tip and it can be concluded that the profiles are that of a 'strain-hardening' material. Therefore, the power-law index range of $0 \leq n \leq 1.0$, as obtained by the 2-D BEM analysis and confirmed by the 3-D FEM analysis, describes the variation of bridging stresses in fibre-metal laminates with notches. Now, it remains to be seen if the results for the stress intensity factors are in agreement with this determined power-law index range.

4.2.2.2 Stress Intensity Factors

The mode I stress intensity factor, K_I , was calculated using the traction formula, equation 3.60, in the two-dimensional boundary element method (BEM) analysis. This stress intensity factor was normalized by $\sigma \sqrt{\pi c}$, and plotted with respect to the normalized crack length, c/W or $c/2W$ in Figures 4.21 to 4.23 for ARALL, and Figures 4.24 to 4.26 for GLARE. These figures show the normalized stress intensity factors, $K_I / \sigma \sqrt{\pi c}$, for the edge crack, double-edge crack, and central crack with circular and elliptical delamination zone shapes and power-law indices, n and m , equal to 0, 0.5, and 1.0. Additionally, the tabulated results from the BIE analysis and the FEM analysis can be found in Tables 4.6 to 4.11.

The normalized stress intensity factor, $K_I / \sigma \sqrt{\pi c}$, increases gradually with increasing crack length, c/W for the central crack and the double-edge crack geometries for both delamination shapes and power-law indices, n and m , equal to 0.5 and 1.0. The normalized stress intensity factor slightly decreases from $c/W = 0.2$ and then increases up

to $c/W = 0.5$ for the assumed constant bridging stress distribution represented by $n = m = 0$. The normalized stress intensity factor, $K_I / \sigma \sqrt{\pi c}$, also increases with increasing power-law index n , for both the circular and elliptical delamination shapes. For both delamination shapes, the stress intensity factors for the power-law index $m = 0.5$ and 1.0 are almost the same in magnitude and significantly lower than those for the other power-law indices.

The normalized stress intensity factor, $K_I / \sigma \sqrt{\pi c}$, for the edge crack, also increases as the crack length, $c/2W$ increases. This increase is more rapid than that for the central and double-edge cracks. The normalized stress intensity factors also increase as the power-law index n increases, with the exception of an elliptical delamination shape at crack sizes of $c/2W > 0.3$, where the constant distribution represented by $n = m = 0$ is higher than the one for $n = 0.5$. In the case of an elliptical delamination shape, the magnitude of the normalized stress intensity factor obtained using $m = 0.5$ and 1.0 is almost zero for all crack lengths. Also, the magnitude of the normalized stress intensity factor is significantly smaller for $m = 0.5$ and 1.0 as compared to that for the power law index n .

The BEM results in Figures 4.21 to 4.26 show that the delamination zone shape has an influence on the stress intensity factor for the crack in the metal layer. A smaller delamination zone size results in lower crack-tip stress intensity factors. This can be seen, for example, in the edge crack problem with $n = 0.5$ and $c/W = 0.5$ where

$K_I / \sigma \sqrt{\pi c} = 0.858$ for an elliptical delamination and $K_I / \sigma \sqrt{\pi c} = 1.265$ for a circular delamination.

The 2-D BEM results suggest that the bridging stress distribution represented by $m = 0.5$ and $m = 1.0$ is not applicable in the two dimensional analysis because it cannot predict accurate values of stress intensity factor. A possible explanation for this is that the stresses in the fibres are higher with decreasing distance from the hole due to the stress concentration there, but the m index assumes a decreasing stress profile towards the hole. As a result, the stress intensity factors are very small for the power-law index, $m > 0$, since they are calculated based on the bridging stress calculated from this profile. It should be noted that in the two dimensional BEM analysis, the crack is assumed to have propagated through the fibre and metal layers and the laminate is modelled as a 2-D equivalent system to the true 3-D problem. The 2-D tractions are thus applied to the crack faces of the fibre layer and the metal layer, as opposed to an un-cracked fibre layer and a cracked metal layer in 3-D.

The normalized stress intensity factors from a three-dimensional FEM analysis are plotted with the two-dimensional BEM normalized stress intensity factors for comparison purposes. The results show that the normalized stress intensity factors from FEM lie between the power law index $n = 0$ and $n = 1.0$ for all crack geometries and delamination shapes. The FEM results are generally closer to $n = 0.5$ and thus this index best describes the variation of stress intensity factors with crack length, c/W or $c/2W$. The stress intensity factors obtained by the FEM analysis therefore confirm the range of those obtained by the two-dimensional BEM analysis.

4.2.2.3 ARALL vs. GLARE

The variations of the normalized bridging stress, σ_B^*/σ , and the normalized stress intensity factors, $K_I/\sigma\sqrt{\pi c}$, obtained by the 2-D BEM analysis for GLARE are the same as those discussed in the previous section for ARALL. Tables 4.3 to 4.5 show the percentage difference of normalized bridging stress, σ_B^*/σ , between ARALL and GLARE for all the cases treated. A negative percentage signifies that the value for GLARE is lower than that for ARALL.

The BEM results show that the magnitude of the bridging stress obtained using the two-dimensional analysis is lower for GLARE for all power-law indices, delamination shapes, and geometries. In general, the difference in bridging stress is approximately 10% with the highest difference being 11.7% for the central crack with $n = 1.0$ and a circular delamination.

Table 4.12 shows the percentage difference of normalized stress intensity factors, $K_I/\sigma\sqrt{\pi c}$ between ARALL and GLARE for all crack lengths, power-law indices, and delamination shapes analysed by the 2-D BIE method. It can be seen that for the power-law index n , the results for GLARE only vary by an average of 5% from the results of ARALL. The variation for the power-law index m is much greater but as discussed in the previous sub-section, this index is not applicable for the problems studied here. Table 4.13 shows the corresponding percentage difference of $K_I/\sigma\sqrt{\pi c}$ between ARALL and GLARE from a 3-D FEM analysis. Here, the stress intensity factors are almost the same with a maximum difference of only 4.5%.

It can therefore be concluded that the crack-tip stress intensity factors in the aluminum are approximately equal for ARALL and GLARE. On the other hand, the bridging stresses for GLARE are lower than for ARALL as a result of the glass-epoxy layer being able to undergo larger strain due to a lower stiffness in the fibre direction.

4.3 Concluding Remarks

The fibre-bridging power-law index range of $0 < n < 1.0$ describes the variation of the stress intensity factors and the bridging stresses in the cases studied here. Therefore, it can be concluded that a three-dimensional fibre-metal laminate can be modelled using a two-dimensional BEM analysis yielding a fibre-bridging index range which adequately describes the variations and magnitudes of the bridging stresses and stress intensity factors. The two-dimensional analysis is significantly less time consuming and less labour intensive than the corresponding three-dimensional analysis. Also, based on the results of the analysis of the bridging stresses near the hole, the high magnitude of the stress in the fibres will likely lead to partial bridging in notched fibre-laminates..

CHAPTER 5

PARTIAL BRIDGING OF NOTCHED FIBRE-METAL LAMINATES

It was shown in the previous Chapter for the full bridging analysis of cracks emanating from a hole, that the stresses in bridging fibres near the hole are very high. This could lead to fibre failure in that region and result in partial bridging of the crack. It is therefore of interest to obtain the bridging stresses and stress intensity factors for such situations and compare these values with the full bridging case.

5.1 Problem Definition and Scope

Two cases of partial bridging were analysed in this study. The first case was that of 75% of the crack length being bridged by fibres, and the second case was that of 50% of the crack length being bridged by fibres. This implies that 25% and 50% of the fibres have failed in the first and second cases, respectively. Also, an unbridged laminate was analysed by FEM and BEM to quantitatively determine the benefit of fibre bridging in reducing the crack-tip stress intensity factors.

The same three rectangular plates considered in Chapter 4 were analysed here. The loading conditions and geometry were the same as defined in Section 4.1 and shown in Figures 4.1 to 4.3.

5.1.1 Two-Dimensional Analysis

The bridging length, b , was varied depending on the amount of bridging present. For the 75 % bridging case, b was set equal to $0.75a$, and for 50 % bridging b was equal to $0.5a$. The mesh in the two-dimensional BIE analysis was the same as the one used for the full bridging case for a given cracked plate, with the elements in the 25% and 50% un-bridged regions of the crack now being traction free.

5.1.2 Three-Dimensional Analysis

The meshes used in the three-dimensional FE analysis of partial bridging were the same as those used for full bridging in Chapter 4. The nodes of the elements in the unbridged region in the fibre layer were unrestrained to represent the failed fibres.

5.1.3 Delamination Shape and Size

As before, two delamination shapes, quarter-circular and quarter-elliptical with a 2-1 major-minor axis ratio, were used in the partial bridging analysis. This assumes that the delamination shape present in full bridging will remain the same for partial bridging. The delamination zone size, d , was taken at the last intact fibre away from the crack-tip in the bridged zone. For the 75% and 50% bridging cases, d was determined from an equation of a circle for the circular delamination, and an equation of an ellipse with a 2-1 major-minor axis ratio for the elliptical delamination.

5.1.4 Material Properties

The same two fibre-metal laminate materials as before, namely, ARALL 2-2/1 and GLARE 2-2/1, were considered in the partial bridging analysis. Their material properties are shown in Tables 4.1 and 4.3.

5.2 Results and Discussion

The bridging stress and the stress intensity factor results for an edge crack, double edge crack, and central crack with 75% and 50% partial bridging are presented in this section. The results of an unbridged fibre-metal laminate for all the cracked geometries are also presented. A comparison of the bridging stresses and stress intensity factors is made between the case of full bridging and that of partial bridging or no bridging at all. The bridging stress and stress intensity factor results obtained by a BEM analysis are also compared to those obtained from the 3-D FEM analysis.

5.2.1 FEM Analysis

5.2.1.1 Bridging Stress Distribution

The bridging stress, σ_B , was taken along the centreline of the fibre layer and normalized with respect to the remote applied stress, σ . The variations of normalized bridging stress, σ_B/σ , with relative distance, x/a , from the crack tip are plotted in Figures 5.1 to 5.6. These figures show the results for a central, double edge, and edge crack with 75% and 50% partial bridging zones, and relative normalized crack lengths, c/W or $c/2W$.

The material properties for the results presented correspond to those of ARALL. The normalized bridging stress, σ_B/σ , for all crack geometries increases very rapidly near $x/a = 0.75$ and $x/a = 0.50$ for the 75% and 50% partial bridging cases, respectively. This is due to the singularity at that location as a result of a crack now present in the fibre layer. The normalized bridging stress also increases near the crack tip, corresponding to $x/a = 0$. As discussed in Section 4.2.1.1, this is likely due to the enforced singularity of the quarter-point crack-tip elements in the aluminum layer adjacent to the mid-point elements in the fibre-reinforced layer in the numerical model.

In the 75% and 50% partial bridging cases, the normalized bridging stress for the central crack and double edge crack decreases as the relative crack length, c/W increases. For the edge crack, the normalized bridging stress increases as $c/2W$ increases. These trends show that the influence of the stress concentration on the bridging stresses is the same as discussed in the previous chapter for the case of full bridging. More specifically, the stress concentration near the notch increases the bridging stresses along the crack and the effect is more pronounced with shorter crack lengths. The trends for the edge crack problem differ than those for the other geometries due to the bending effect arising from its geometry. The bending effect is the dominating factor for larger crack lengths. As a result, for the edge crack, the bridging stresses increase with increasing crack length.

For both partial bridging cases studied here, the normalized bridging stress is higher for the elliptical delamination than for the circular delamination. Again, this trend is the same as that seen in fully bridged laminates in the previous chapter and this is due to a greater amount of stress transferred from the metal layers to the fibre layer for a smaller delamination size.

The variations of percentage difference between partial bridging and full bridging is shown in Figures 5.7 to 5.12. These figures show that the normalized bridging stress increases as the relative bridging zone length, b decreases. As an example, for the edge crack at $x/a = 0$, with an elliptical delamination and $c/2W = 0.5$, $\sigma_B/\sigma = 4.37$ for full bridging, $\sigma_B/\sigma = 5.39$ for 75% bridging, and $\sigma_B/\sigma = 7.46$ for 50% bridging.

For the central crack and double edge crack with 75% partial bridging, the differences of the bridging stress between the full bridging case are approximately in the range of 5% to 7%, and 7% to 13% for the circular delamination and elliptical delamination, respectively. For the 50% partial bridging, the differences between the full bridging case increase to about 11% to 14%, and 17% to 26% for the circular delamination and elliptical delamination, respectively. The bridging stresses for the edge crack problem with 75% bridging has a percentage difference from the full bridging case of approximately 7% to 20% for circular delamination, and 10% to 25% for elliptical delamination. At 50% bridging, the corresponding ranges of differences increase to 15% to 35% and 20% to 50%.

It can be seen from these results that a decrease in the amount of fibres bridging the crack increases the bridging stress in the remaining fibres. With less fibres bridging the crack, the remaining fibres will have higher bridging stresses for an equal magnitude of applied stress.

5.2.1.2 Stress Intensity Factors

Table 5.13 shows the normalized stress intensity factors, $K_I / \sigma \sqrt{\pi c}$, at the crack-tip for the 75% and 50% bridged fibre-metal laminates. These results are for ARALL

central, double edge, and edge cracks with circular and elliptical delamination. The percentage difference of $K_I / \sigma \sqrt{\pi c}$ between the full bridging and each partial bridging case of the different crack configurations are shown in Table 5.14. These results show that the normalized stress intensity factor increases as the bridging length is decreased as to be expected. A decrease in the amount of fibres bridging the crack causes the stress intensity factor in the aluminum layer to increase because the crack is able to open up more for a decreased bridging length.

5.2.1.3 ARALL vs. GLARE

From the comparison of the normalized bridging stress, σ_B/σ , between ARALL and GLARE in the previous chapter for full bridging, σ_B/σ was found to be lower for GLARE. The difference in the magnitude of σ_B/σ was no more than 10% away from the hole. This was attributed to the lower material property stiffness of GLARE. Therefore, the bridging stress results for the partial bridging cases are also expected to lie within 10% difference between the two fibre-metal laminates since the material property is the only variable that was varied between ARALL and GLARE in the partial bridging analyses for a given cracked configuration.

5.2.2 2-D BEM Analysis

5.2.2.1 Bridging Stress Distribution

The normalized bridging stress, σ_B^*/σ , distribution for ARALL as a function of normalized crack length, c/W or $c/2W$, is plotted in Figures 5.13 to 5.15 for each of the

cases treated. The corresponding results for GLARE are plotted in Figures 5.16 to 5.18 for comparison. These figures show the variations of σ_B^*/σ for 75% and 50% partial bridging cases with varying fibre-bridging power-law indices, n and m , and circular and elliptical delamination zone shapes. The numerical results for these are also listed in Tables 5.1 to 5.3.

The variations of normalized bridging stress, σ_B^*/σ , with crack length, c/W (or $c/2W$) are the same for all the partial bridging cases as they were for the full bridging case discussed in Chapter 4. However, the magnitudes of the bridging stresses increase as the bridging length, b , is decreased. The percentage differences of the stress between the 75% and 50% partial bridging cases and that for full bridging are tabulated in Tables 5.7 to 5.9, for the central, double edge, and edge cracks. The trend in percentage difference of the normalized bridging stress for the partial bridging cases compared to full bridging is shown graphically in Figure 5.26 for the ARALL central, double edge, and edge crack problems with an elliptical delamination and power-law index, $n = 0.5$.

The two-dimensional BIE results also show an increase of the normalized bridging stress, σ_B^*/σ , with a decreasing amount of fibres bridging the crack. The largest percent difference of 49.4% is for the edge crack with 50% bridging, circular delamination, $c/2W = 0.5$, and power-law index $m = 0.5$.

5.2.2.2 Stress Intensity Factors

The normalized crack-tip stress intensity factor, $K_I / \sigma \sqrt{\pi c}$, obtained from the BIE analysis are plotted with respect to the normalized crack length, c/W or $c/2W$, in Figures 5.19 to 5.21, for the ARALL central, double edge, and edge cracks, respectively.

The corresponding results for GLARE are shown in Figures 5.22 to 5.24. In these figures, the normalized crack-tip stress intensity factor results from the three-dimensional FEM analysis are also included for comparison. Additionally, the numerical results from the BEM analysis can be found in Tables 5.4 to 5.6.

The normalized stress intensity factor, $K_I / \sigma \sqrt{\pi c}$, generally increases with increasing power-law index n and m , and crack length, c/W or $c/2W$, for the cracked geometries considered with the partial bridging. The delamination zone shape also influences the magnitude of the normalized stress intensity factor; a smaller delamination size results in lower crack-tip stress intensity factors. This can be seen in Tables 5.4 to 5.6 and Figures 5.19 to 5.24, where the normalized stress intensity factors are lower for the elliptical delamination than for the circular delamination for a given crack size.

It can also be seen for these results that as the length of the bridging zone, b , decreases, the normalized stress intensity factors generally increase. This is to be expected, for the crack is able to open up more when the amount of bridging fibres decreases. Comparing, for example, a central crack with $n = 0.5$, elliptical delamination and $c/W = 0.5$, $K_I / \sigma \sqrt{\pi c} = 0.730$ for 100% bridging, $K_I / \sigma \sqrt{\pi c} = 0.777$ for 75% bridging and $K_I / \sigma \sqrt{\pi c} = 0.845$ for 50% bridging. Compared to 100% bridging, this corresponds to an increase of 6.0% and 13.6% for the 75% and 50% partial bridging cases, respectively. Figure 5.27 shows the percentage difference of the normalized stress intensity factor, $K_I / \sigma \sqrt{\pi c}$, between the partial bridging cases and full bridging for the central, double edge, and edge crack problems with an elliptical delamination. This figure also shows the trends of the normalized stress intensity factors for 75% and 50% bridging between the results obtained from the BIE and FEM analyses.

Based on the percentage differences of normalized stress intensity factors shown in Tables 5.10 and 5.11 (when comparing the solutions for the partially bridged cases with the corresponding fully bridged cases) for a circular delamination with n (and m) > 0 , the normalized stress intensity factors for the central crack and double edge crack actually decrease with decreasing bridging length. From Table 5.12, it can be seen that there is also a decrease in the stress intensity factor for $m > 0$. This is possibly due to a very large increase in the bridging stress as a result of a greater deformation of the fibres with the reduction of the extent of bridging. This in turn leads to an increase of the stress intensity factor. Thus, it appears that a circular delamination would actually slow or even arrest crack growth as the length of the bridging zone is decreased. Experimentally, a circular delamination has never been reported in the literature. Nevertheless, it is interesting to conduct such a parametric study to determine the variations in the stress intensity factors for partial bridging. It should also be noted that for an elliptical delamination, which is the experimentally observed delamination shape, the BEM analysis in Tables 5.10 to 5.12 shows an increase in the stress intensity factor with decreasing fibre bridging.

The normalized stress intensity factors, $K_I / \sigma \sqrt{\pi c}$, for an unbridged ARALL central crack, double edge crack, and edge crack problem are shown in Figure 5.25. For the previous example of the central crack, with $n = 0.5$, elliptical delamination and $c/W = 0.5$, the normalized stress intensity factor for the unbridged case is 1.21, which is 39.6% higher than the corresponding value for full bridging. Figure 5.25 also contains the normalized stress intensity factors from the FEM analysis calculated in the aluminum layer and the fibre-epoxy layer. It is expected that the stress intensity factors in the fibre

layer should be less than those in the aluminum layer since the restraint of fibres on crack opening is greater than for aluminum. This is confirmed in these figures where the crack-tip normalized stress intensity factors are much lower in the fibre layer than the aluminum layer. The two-dimensional BEM analysis results for the normalized stress intensity factor fall between those of the three-dimensional FEM analysis, and accurately predict the trend of $K_I / \sigma \sqrt{\pi c}$ with relative crack length, c/W or $c/2W$.

The numerical values of the normalized stress intensity factors, $K_I / \sigma \sqrt{\pi c}$, for an unbridged central crack, double edge crack, and edge crack are shown in Table 5.15, 5.16, and 5.17, respectively. These tabulated results contain the normalized stress intensity factors from the BEM and FEM analyses, and the percentage difference between the two methods. For the stress intensity factors calculated in the aluminum layer, the BEM results are approximately 10% lower than the FEM results. For the stress intensity factors calculated in the fibre-epoxy layer, the BEM results are approximately 25 to 30% higher than the FEM results. It can therefore be seen that the two-dimensional BEM analysis predicts the crack-tip stress intensity factors in the aluminum layer to within 10% of the actual three-dimensional result.

5.2.2.3 ARALL vs. GLARE

As was mentioned in Section 5.1.2.3, the bridging stress results using the two-dimensional BEM for the GLARE partial bridging cases are also expected to lie within 10% of those obtained for ARALL since the material property is the only variable that was varied between ARALL and GLARE. From Tables 5.15 to 5.17 for an unbridged laminate, the normalized stress intensity factors for GLARE are almost equal to those for

ARALL in the aluminum layer. This is consistent with the findings in Section 4.2.2.3 of the previous chapter. The corresponding stress intensity factors calculated in the fibre-epoxy layer are expected to be lower for GLARE than for ARALL because the bridging stress in the GLARE laminate is lower.

5.3 Concluding Remarks

High bridging stresses in the fibres near a hole in a cracked fibre-metal laminate under tension will likely lead to partial bridging of the crack. With partial bridging of the crack, the remaining intact fibres bridging the crack will experience an increase in bridging stress and the crack-tip stress intensity factor in the metal layer. The influence of the stress concentration caused by the presence of the hole is still the same for partial bridging as was seen in full bridging, where it is most pronounced at short crack lengths relative to the hole size. The analysis of an unbridged crack in fibre-metal laminates demonstrated the benefit of fibre bridging in reducing the crack-tip stress intensity factor, and hence slowing the crack growth.

CHAPTER 6

SUMMARY AND CONCLUSIONS

The various toughening mechanisms that increase the fracture toughness of a material have been briefly described. The role of the fibre-bridging process in shielding the crack and impeding crack growth in composites has been discussed. Commonly used models for stress analysis, such as the shear-lag model, compliance-spring model, and the stress intensity approach, along with numerous studies employing these models, have also been briefly reviewed in this study. The bridging mechanism in fibre-metal laminates has been described and because of its simplicity, the stress intensity approach with power-law expressions for bridging stresses has been chosen to analyse fibre-metal laminates. The three dimensional bridging mechanism in fibre-metal laminates has been reduced to a two-dimensional one with the assumption of homogenised material properties through the thickness, and the bridging stresses as a closing pressure on the crack flanks. Since these bridging stresses are initially unknown, this two-dimensional, mathematically non-linear problem has to be solved iteratively and a numerical technique to solve for these unknown bridging stresses has been presented.

The two-dimensional analysis was carried out using the boundary element method (BEM) and a review of this method for isotropic and anisotropic linear elasticity has been

presented. The principles of linear-elastic fracture mechanics (LEFM) have been briefly introduced and the procedure for calculating stress intensity factors using the tractions obtained from BEM has been described. The efficiency and accuracy of BEM has been demonstrated by the analysis of several isotropic and orthotropic example test problems containing cracks.

A three-dimensional analysis of three rectangular fibre-metal laminate plates with cracks emanating from the edge of a circular cut-out and under remote uniform tension in the fibre direction, was performed using FEM. Depending on the location of these cut-outs, these problems were referred to as the central, double edge, and edge crack problems. The applicability of a two-dimensional analysis using BEM was presented along with the discussion of the influence of varying crack size, geometry, bridging stress profile and delamination zone size on the bridging stress and stress intensity factors. A comparison was then made between the results obtained from the 2-D BEM analysis and the 3-D FEM analysis.

The results of the BEM and FEM analysis for the case of full bridging can be summarized as follows:

- A high normalized bridging stress, σ_B/σ , at the crack mouth, was obtained indicating high strain in the fibres there and therefore the likelihood of fibre failure near the edge of the hole.
- The high normalized bridging stress is generally confined to a 5% to 10% region of the total crack length.

- For the central and double edge crack problems, σ_B/σ decreases with increasing relative crack length, c/W , due to the effect of the stress concentration near the hole.
- For the edge crack problem, σ_B/σ increases with $c/2W$. In this case, bending about the crack-tip has a greater effect on the bridging stress than the stress concentration.
- An increasing delamination size decreases the magnitude of the bridging stresses but an increase of σ_B/σ with increasing distance from the crack tip, x/a , is more rapid for an elliptical delamination than for a circular delamination;
- In the BEM analysis, the normalized bridging stress, σ_B^*/σ , at the last intact fibre away from the crack-tip increases with increasing power-law index, n and m , and is highest for $n = 1.0$.
- The normalized bridging stress, σ_B/σ is about 10% lower for GLARE as compared to ARALL away from edge of the hole, and about 30% to 35% lower for GLARE at the crack mouth (edge of the hole).
- The normalized stress intensity factor, $K_I/\sigma(\pi c)^{1/2}$, increases gradually with increasing crack length, c/W or $c/2W$, for the central and double edge crack problems, and more rapidly for the edge crack problem.
- The normalized stress intensity factors also increase with increasing power-law index, n and m , but the power-law index m does not accurately predict stress intensity factors and hence cannot be applied to the analysis of notched fibre-metal laminates with cracks.

- The magnitude of the normalized stress intensity factors, $K_I/\sigma(\pi c)^{1/2}$, is lower for an elliptical delamination than for a circular delamination, and hence a decrease for a smaller delamination size.
- From both the FEM and BEM analyses, the normalized stress intensity factors vary about 5% between ARALL and GLARE and can therefore, for practical purposes, be taken to be equal.
- The FEM results lie between $0 \leq n \leq 1.0$ of the BEM results, but are generally closer to $n = 0.5$, which describes a decreasing σ_B^*/σ towards crack tip indicating the behaviour of a 'strain-hardening' material.

The effect of the extent of fibre-bridging on the bridging mechanism has been examined in this study. This investigation considered 75% and 50% of the crack being bridged by fibres and the results from the FEM and BEM analysis were presented. The conclusions from the partial bridging analysis are itemised below.

- The normalized bridging stress, σ_B/σ , again decreases with increasing crack length, c/W , for the central and double edge crack problems and increases with increasing $c/2W$ for the edge crack problem.
- The normalized bridging stress, σ_B/σ increases with a decreasing bridging length, b , but the normalized stress intensity factors, $K_I/\sigma(\pi c)^{1/2}$, increase with a decreasing bridging length due to more opening of the crack faces as a result of less fibres bridging the crack.
- The normalized bridging stress, σ_B/σ , for 75% and 50% bridging was found to be 5% to 13% and 11% to 26% higher than full bridging, respectively, for the central

and double edge crack problems. For the edge crack problem, the corresponding differences are about 7% to 25% and 15% to 50% higher than full bridging case.

The benefit of fibre-bridging in fibre-metal laminates was quantified with the FEM and BEM analysis of an unbridged crack. The stress intensity factors were found to be much higher when there is no bridging of the crack as compared to full bridging, and even some partial bridging.

The study has clearly demonstrated that the simplified 2-D BEM analysis can provide reasonable estimates of the stress intensity factor for cracks in fibre-metal laminates with stress concentrations. It was less time consuming and much easier to obtain the estimates than a corresponding 3-D finite element analysis.

5.1 Recommendations

- (a) The present study investigated the effects of a stress field on the bridging stresses and stress intensity factors in fibre-metal laminates. There have been very few reports on the influence of residual stresses on the fibre-bridging mechanism. The curing and pre- and post-stretching of fibre-metal laminates induces favourable residual stresses to improve fatigue properties and thus the quantitative effect of these residual stresses would complement this report.
- (b) An important mechanism in re-distributing stresses in fibre-metal laminates is the load transfer in the adhesive between the metal and the fibre layers. The interdependency of this mechanism with crack length, delamination size, and fibre-matrix debonding should be treated in more detail.

- (c) It has been observed experimentally that the delamination shape is generally elliptic, but for partial bridging the delamination shape may change. A study of the effects of bridging length and fibre-bridging stress on the delamination zone should be considered.

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Crack Type	a/W	$\left(\frac{K_I}{\sigma \sqrt{\pi a}} \right)_{Rooke}$	$\left(\frac{K_I}{\sigma \sqrt{\pi a}} \right)_{BIE}$	% Difference
Central	0.1	1.0250	1.0262	0.12
	0.2	1.0625	1.0682	0.54
	0.3	1.1250	1.1377	1.13
	0.4	1.2000	1.2314	2.62
	0.5	1.3250	1.3522	2.05
Double Edge	0.1	1.1300	1.1493	1.71
	0.2	1.1600	1.1898	2.57
	0.3	1.1900	1.2437	4.51
	0.4	1.2300	1.2970	5.45
	0.5	1.2800	1.3489	5.38

Table 3.1: Comparison of BIE normalized stress intensity factors for an isotropic plate with values from Rooke and Cartwright (1976) for central and double edge cracks.

Crack Type	a/2W	$\left(\frac{K_I}{\sigma \sqrt{\pi a}} \right)_{Rooke}$	$\left(\frac{K_I}{\sigma \sqrt{\pi a}} \right)_{BIE}$	% Difference
Edge	0.1	1.1843	1.2019	1.49
	0.2	1.3730	1.3812	0.60
	0.3	1.6653	1.6759	0.64
	0.4	2.1134	2.1252	0.56
	0.5	2.8425	2.8330	-0.33

Table 3.2: Comparison of BIE normalized stress intensity factors for an isotropic plate with values from Rooke and Cartwright (1976) for an edge crack.

c/W	$\frac{K_I}{\sigma \sqrt{\pi a}}$	$\left(\frac{K_I}{\sigma \sqrt{\pi a}} \right)_{BIE}$	% Difference
0.2	1.0730	1.0761	0.29
0.3	1.0830	1.0889	0.54
0.4	1.1280	1.1342	0.55
0.5	1.2030	1.2129	0.82

Table 3.3: Comparison of BIE normalized stress intensity factors for an isotropic plate containing two symmetric cracks emanating from a hole with values from Rooke and Cartwright (1976).

a/W	$\frac{K_I}{\sigma \sqrt{\pi a}}$	$\left(\frac{K_I}{\sigma \sqrt{\pi a}} \right)_{BIE}$	% Difference
0.1	1.0200	1.0403	1.99
0.2	1.1000	1.1205	1.86
0.3	1.2100	1.2444	2.84
0.4	1.3700	1.4036	2.45
0.5	1.5700	1.5993	1.87

Table 3.4: Comparison of BIE normalized stress intensity factors for an orthotropic plate with a central crack, with values obtained by Bowie and Freese (1972).

Material	$\frac{K_I}{\sigma \sqrt{\pi a}}$	$\left(\frac{K_I}{\sigma \sqrt{\pi a}} \right)_{BIE}$	% Difference
Glass/Epoxy	1.7175	1.7184	0.05
Boron/Epoxy	2.0505	2.0743	1.16
Graphite/Epoxy	2.5904	2.7311	5.43

Table 3.5: Comparison of BIE normalized stress intensity factors for cracks emanating from a hole in an orthotropic plate, with values from Lakshminarayana *et al.* (1979).

Material Property		Aluminum ¹	Aramid-Epoxy ¹	Glass-Epoxy ²
Young's Modulus (Gpa)	E_{11}	71.7	62.7	55
	E_{22}	71.7	4.56	16
	E_{33}	71.7	4.56	16
Shear Modulus (GPa)	G_{12}	27.6	1.63	7.6
	G_{13}	27.6	1.63	7.6
	G_{23}	27.6	1.53	5.4
Poisson's Ratio	ν_{12}	0.30	0.355	0.280
	ν_{13}	0.30	0.355	0.280
	ν_{23}	0.30	0.493	0.493

Table 4.1: Material Properties for Aluminum, Aramid-Epoxy, and Glass-Epoxy. ¹Yeh (1988) and ²Schwartz (1997).

Material Property		ARALL2 - 2/1	GLARE2 - 2/1
Young's Modulus (GPa)	E_{11}	69.5	67.5
	E_{22}	55.9	58.3
Shear Modulus (GPa)	G_{12}	20.7	22.6
Poisson's Ratio	ν_{12}	0.30	0.30

Table 4.2: Material Properties for ARALL2 – 2/1 and GLARE2 – 2/1.

		<i>Elliptical Delamination</i>			<i>Circular Delamination</i>		
Index	c/W	ARALL	GLARE	% diff	ARALL	GLARE	% diff
		σ_b^*/σ	σ_b^*/σ		σ_b^*/σ	σ_b^*/σ	
n = m = 0	0.2	2.684	2.556	-5.0	2.020	1.876	-7.7
	0.3	2.508	2.372	-5.7	1.828	1.688	-8.3
	0.4	2.452	2.312	-6.1	1.768	1.624	-8.9
	0.5	2.448	2.308	-6.1	1.764	1.624	-8.6
n = 0.5	0.2	4.548	4.176	-8.9	2.920	2.624	-11.3
	0.3	3.792	3.484	-8.8	2.428	2.180	-11.4
	0.4	3.504	3.224	-8.7	2.256	2.028	-11.2
	0.5	3.396	3.132	-8.4	2.208	1.992	-10.8
n = 1.0	0.2	4.856	4.436	-9.5	3.044	2.724	-11.7
	0.3	4.060	3.708	-9.5	2.532	2.268	-11.6
	0.4	3.768	3.444	-9.4	2.360	2.116	-11.5
	0.5	3.668	3.360	-9.2	2.320	2.084	-11.3
m = 0.5	0.2	3.364	3.152	-6.7	2.380	2.180	-9.2
	0.3	2.776	2.604	-6.6	1.968	1.800	-9.3
	0.4	2.524	2.376	-6.2	1.808	1.656	-9.2
	0.5	2.404	2.268	-6.0	1.740	1.600	-8.7
m = 1.0	0.2	3.600	3.360	-7.1	2.496	2.276	-9.7
	0.3	2.976	2.784	-6.9	2.064	1.884	-9.6
	0.4	2.712	2.540	-6.8	1.896	1.736	-9.2
	0.5	2.584	2.428	-6.4	1.832	1.680	-9.0

Table 4.3: Normalized bridging stress in the last intact fibre, σ_b^*/σ , for different crack sizes, c/W, and power-law indices: *Central crack* problem for ARALL2 and GLARE2 with Circular and Elliptical delamination.

		<i>Elliptical Delamination</i>			<i>Circular Delamination</i>		
Index	c/W	ARALL	GLARE	% diff	ARALL	GLARE	% diff
		σ_b^*/σ	σ_b^*/σ		σ_b^*/σ	σ_b^*/σ	
n = m = 0	0.2	2.712	2.588	-4.8	2.052	1.912	-7.3
	0.3	2.560	2.424	-5.6	1.880	1.740	-8.0
	0.4	2.496	2.364	-5.6	1.816	1.676	-8.4
	0.5	2.480	2.340	-6.0	1.796	1.656	-8.5
n = 0.5	0.2	4.664	4.288	-8.8	3.004	2.704	-11.1
	0.3	3.924	3.612	-8.6	2.528	2.276	-11.1
	0.4	3.612	3.336	-8.3	2.340	2.112	-10.8
	0.5	3.460	3.200	-8.1	2.260	2.040	-10.8
n = 1.0	0.2	4.988	4.564	-9.3	3.136	2.812	-11.5
	0.3	4.212	3.856	-9.2	2.640	2.372	-11.3
	0.4	3.892	3.568	-9.1	2.456	2.204	-11.4
	0.5	3.740	3.436	-8.8	2.376	2.136	-11.2
m = 0.5	0.2	3.428	3.220	-6.5	2.440	2.236	-9.1
	0.3	2.844	2.676	-6.3	2.032	1.864	-9.0
	0.4	2.588	2.440	-6.1	1.860	1.716	-8.4
	0.5	2.448	2.312	-5.9	1.780	1.640	-8.5
m = 1.0	0.2	3.676	3.436	-7.0	2.560	2.340	-9.4
	0.3	3.056	2.864	-6.7	2.136	1.956	-9.2
	0.4	2.784	2.616	-6.4	1.964	1.800	-9.1
	0.5	2.636	2.480	-6.3	1.876	1.724	-8.8

Table 4.4: Normalized bridging stress in the last intact fibre, σ_b^*/σ , for different crack sizes, c/W, and power-law indices: *Double edge crack problem for ARALL2 and GLARE2 with circular and elliptical delamination.*

		<i>Elliptical Delamination</i>			<i>Circular Delamination</i>		
Index	c/2W	ARALL	GLARE	% diff	ARALL	GLARE	% diff
		σ_b^*/σ	σ_b^*/σ		σ_b^*/σ	σ_b^*/σ	
n = m = 0	0.2	2.904	2.784	-4.3	2.280	2.136	-6.7
	0.3	2.960	2.844	-4.1	2.348	2.208	-6.3
	0.4	3.140	3.040	-3.3	2.588	2.452	-5.5
	0.5	3.364	3.284	-2.4	2.904	2.788	-4.2
n = 0.5	0.2	5.432	5.020	-8.2	3.592	3.244	-10.7
	0.3	5.088	4.752	-7.1	3.516	3.204	-9.7
	0.4	5.248	4.968	-5.6	3.864	3.568	-8.3
	0.5	5.564	5.344	-4.1	4.408	4.140	-6.5
n = 1.0	0.2	5.872	5.392	-8.9	3.776	3.396	-11.2
	0.3	5.608	5.200	-7.8	3.756	3.404	-10.3
	0.4	5.912	5.560	-6.3	4.212	3.864	-9.0
	0.5	6.412	6.120	-4.8	4.924	4.592	-7.2
m = 0.5	0.2	3.848	3.632	-5.9	2.820	2.600	-8.5
	0.3	3.364	3.212	-4.7	2.596	2.420	-7.3
	0.4	3.236	3.128	-3.5	2.648	2.508	-5.6
	0.5	3.216	3.140	-2.4	2.792	2.684	-4.0
m = 1.0	0.2	4.140	3.896	-6.3	2.980	2.732	-9.1
	0.3	3.660	3.484	-5.1	2.772	2.572	-7.8
	0.4	3.556	3.424	-3.9	2.860	2.696	-6.1
	0.5	3.568	3.476	-2.6	3.052	2.924	-4.4

Table 4.5: Normalized bridging stress in the last intact fibre, σ_b^*/σ , for different crack sizes, c/2W, and power-law indices: *Edge crack* problem for ARALL2 and GLARE2 with circular and elliptical delamination.

		<i>Elliptical Delamination</i>					
		(ARALL) 2-D	(ARALL) 3-D		(GLARE) 2-D	(GLARE) 3-D	
Index	c/W	$K_I/\sigma(\pi c)^{1/2}$	$K_I/\sigma(\pi c)^{1/2}$	%diff	$K_I/\sigma(\pi c)^{1/2}$	$K_I/\sigma(\pi c)^{1/2}$	%diff
n = m = 0	0.2	0.605	0.683	11.4	0.629	0.681	7.6
	0.3	0.566	0.642	11.8	0.595	0.658	9.6
	0.4	0.564	0.640	11.9	0.598	0.662	9.7
	0.5	0.582	0.650	10.5	0.619	0.678	8.7
n = 0.5	0.2	0.660	0.683	3.4	0.695	0.681	-2.1
	0.3	0.670	0.642	-4.4	0.705	0.658	-7.1
	0.4	0.695	0.640	-8.6	0.731	0.662	-10.4
	0.5	0.730	0.650	-12.3	0.768	0.678	-13.3
n = 1.0	0.2	0.773	0.683	-13.2	0.801	0.681	-17.6
	0.3	0.781	0.642	-21.7	0.809	0.658	-22.9
	0.4	0.806	0.640	-25.9	0.835	0.662	-26.1
	0.5	0.845	0.650	-30.0	0.877	0.678	-29.4
m = 0.5	0.2	0.041	0.683	94.0	0.107	0.681	84.3
	0.3	0.072	0.642	88.8	0.136	0.658	79.3
	0.4	0.109	0.640	83.0	0.170	0.662	74.3
	0.5	0.138	0.650	78.8	0.200	0.678	70.5
m = 1.0	0.2	0.043	0.683	93.7	0.112	0.681	83.6
	0.3	0.075	0.642	88.3	0.141	0.658	78.6
	0.4	0.112	0.640	82.5	0.177	0.662	73.3
	0.5	0.143	0.650	78.0	0.208	0.678	69.3

Table 4.6: Normalized crack-tip stress intensity factors, $K_I/\sigma(\pi c)^{1/2}$, for different relative crack sizes, c/W, and power-law indices: *Central crack problem for ARALL2 and GLARE2 with elliptical delamination.*

		Circular Delamination					
Index	c/W	(ARALL) 2-D	(ARALL) 3-D		(GLARE) 2-D	(GLARE) 3-D	
		$K_I/\sigma(\pi c)^{1/2}$	$K_I/\sigma(\pi c)^{1/2}$	% diff	$K_I/\sigma(\pi c)^{1/2}$	$K_I/\sigma(\pi c)^{1/2}$	% diff
n = m = 0	0.2	0.720	0.848	15.1	0.747	0.849	12.0
	0.3	0.707	0.824	14.2	0.737	0.831	11.3
	0.4	0.722	0.836	13.6	0.757	0.846	10.5
	0.5	0.757	0.864	12.4	0.794	0.878	9.6
n = 0.5	0.2	0.807	0.848	4.8	0.835	0.849	1.6
	0.3	0.820	0.824	0.5	0.848	0.831	-2.0
	0.4	0.850	0.836	-1.7	0.880	0.846	-4.0
	0.5	0.897	0.864	-3.8	0.929	0.878	-5.8
n = 1.0	0.2	0.884	0.848	-4.2	0.906	0.849	-6.7
	0.3	0.896	0.824	-8.7	0.917	0.831	-10.3
	0.4	0.927	0.836	-10.9	0.950	0.846	-12.3
	0.5	0.978	0.864	-13.2	1.004	0.878	-14.4
m = 0.5	0.2	0.342	0.848	59.7	0.405	0.849	52.3
	0.3	0.367	0.824	55.5	0.430	0.831	48.3
	0.4	0.399	0.836	52.3	0.461	0.846	45.5
	0.5	0.433	0.864	49.9	0.497	0.878	43.4
m = 1.0	0.2	0.358	0.848	57.8	0.422	0.849	50.3
	0.3	0.385	0.824	53.3	0.447	0.831	46.2
	0.4	0.418	0.836	50.0	0.479	0.846	43.4
	0.5	0.452	0.864	47.7	0.517	0.878	41.1

Table 4.7: Normalized crack-tip stress intensity factors, $K_I/\sigma(\pi c)^{1/2}$, for different relative crack sizes, c/W , and power-law indices: *Central crack* problem for ARALL2 and GLARE2 with circular delamination.

		Elliptical Delamination					
Index	c/W	(ARALL) 2-D	(ARALL) 3-D	% diff	(GLARE) 2-D	(GLARE) 3-D	% diff
		$K_I/\sigma(\pi c)^{1/2}$	$K_I/\sigma(\pi c)^{1/2}$		$K_I/\sigma(\pi c)^{1/2}$	$K_I/\sigma(\pi c)^{1/2}$	
n = m = 0	0.2	0.630	0.703	10.4	0.655	0.706	7.2
	0.3	0.575	0.655	12.2	0.606	0.670	9.6
	0.4	0.557	0.644	13.5	0.589	0.664	11.3
	0.5	0.557	0.647	13.9	0.593	0.669	11.4
n = 0.5	0.2	0.676	0.703	3.8	0.714	0.706	-1.1
	0.3	0.674	0.655	-2.9	0.711	0.670	-6.1
	0.4	0.684	0.644	-6.2	0.720	0.664	-8.4
	0.5	0.703	0.647	-8.7	0.740	0.669	-10.6
n = 1.0	0.2	0.794	0.703	-12.9	0.823	0.706	-16.6
	0.3	0.789	0.655	-20.5	0.819	0.670	-22.2
	0.4	0.797	0.644	-23.8	0.827	0.664	-24.5
	0.5	0.819	0.647	-26.6	0.849	0.669	-26.9
m = 0.5	0.2	0.042	0.703	94.0	0.109	0.706	84.6
	0.3	0.060	0.655	90.8	0.124	0.670	81.5
	0.4	0.086	0.644	86.6	0.146	0.664	78.0
	0.5	0.110	0.647	83.0	0.169	0.669	74.7
m = 1.0	0.2	0.043	0.703	93.9	0.115	0.706	83.7
	0.3	0.062	0.655	90.5	0.129	0.670	80.7
	0.4	0.087	0.644	86.5	0.151	0.664	77.3
	0.5	0.112	0.647	82.7	0.175	0.669	73.8

Table 4.8: Normalized crack-tip stress intensity factors, $K_I/\sigma(\pi c)^{1/2}$, for different relative crack sizes, c/W , and power-law indices: *Double edge crack problem for ARALL2 and GLARE2 with elliptical delamination.*

		Circular Delamination					
Index	c/W	(ARALL) 2-D	(ARALL) 3-D		(GLARE) 2-D	(GLARE) 3-D	
		$K_I/\sigma(\pi c)^{1/2}$	$K_I/\sigma(\pi c)^{1/2}$	% diff	$K_I/\sigma(\pi c)^{1/2}$	$K_I/\sigma(\pi c)^{1/2}$	% diff
n = m = 0	0.2	0.746	0.881	15.3	0.774	0.875	11.5
	0.3	0.718	0.853	15.8	0.750	0.858	12.6
	0.4	0.714	0.850	16.0	0.748	0.858	12.8
	0.5	0.728	0.862	15.5	0.764	0.875	12.7
n = 0.5	0.2	0.830	0.881	5.8	0.861	0.875	1.6
	0.3	0.831	0.853	2.6	0.862	0.858	-0.5
	0.4	0.843	0.850	0.8	0.873	0.858	-1.7
	0.5	0.868	0.862	-0.7	0.899	0.875	-2.7
n = 1.0	0.2	0.911	0.881	-3.4	0.934	0.875	-6.7
	0.3	0.911	0.853	-6.8	0.934	0.858	-8.9
	0.4	0.922	0.850	-8.5	0.945	0.858	-10.1
	0.5	0.949	0.862	-10.1	0.974	0.875	-11.3
m = 0.5	0.2	0.349	0.881	60.4	0.415	0.875	52.6
	0.3	0.361	0.853	57.7	0.425	0.858	50.5
	0.4	0.381	0.850	55.2	0.440	0.858	48.7
	0.5	0.401	0.862	53.5	0.463	0.875	47.1
m = 1.0	0.2	0.367	0.881	58.3	0.433	0.875	50.5
	0.3	0.379	0.853	55.6	0.443	0.858	48.4
	0.4	0.396	0.850	53.4	0.459	0.858	46.5
	0.5	0.419	0.862	51.4	0.481	0.875	45.0

Table 4.9: Normalized crack-tip stress intensity factors, $K_I/\sigma(\pi c)^{1/2}$, for different relative crack sizes, c/W , and power-law indices: *Double edge crack* problem for ARALL2 and GLARE2 with circular delamination.

		Elliptical Delamination					
Index	c/2W	(ARALL) 2-D	(ARALL) 3-D	% diff	(GLARE) 2-D	(GLARE) 3-D	% diff
		$K_I/\sigma(\pi c)^{1/2}$	$K_I/\sigma(\pi c)^{1/2}$		$K_I/\sigma(\pi c)^{1/2}$	$K_I/\sigma(\pi c)^{1/2}$	
n = m = 0	0.2	0.787	0.800	1.6	0.814	0.802	-1.5
	0.3	0.821	0.785	-4.6	0.856	0.808	-5.9
	0.4	0.903	0.798	-13.2	0.944	0.827	-14.1
	0.5	1.031	0.802	-28.6	1.076	0.840	-28.1
n = 0.5	0.2	0.770	0.800	3.8	0.818	0.802	-2.0
	0.3	0.811	0.785	-3.3	0.869	0.808	-7.5
	0.4	0.845	0.798	-5.9	0.915	0.827	-10.6
	0.5	0.858	0.802	-7.0	0.937	0.840	-11.5
n = 1.0	0.2	0.915	0.800	-14.4	0.954	0.802	-19.0
	0.3	0.979	0.785	-24.7	1.031	0.808	-27.6
	0.4	1.042	0.798	-30.6	1.108	0.827	-34.0
	0.5	1.082	0.802	-34.9	1.164	0.840	-38.6
m = 0.5	0.2	0.034	0.800	95.8	0.110	0.802	86.3
	0.3	0.031	0.785	96.1	0.106	0.808	86.9
	0.4	0.025	0.798	96.9	0.096	0.827	88.4
	0.5	0.003	0.802	99.6	0.071	0.840	91.5
m = 1.0	0.2	0.037	0.800	95.4	0.116	0.802	85.5
	0.3	0.032	0.785	95.9	0.111	0.808	86.3
	0.4	0.021	0.798	97.4	0.099	0.827	88.0
	0.5	-0.008	0.802	101.0	0.066	0.840	92.1

Table 4.10: Normalized crack-tip stress intensity factors, $K_I/\sigma(\pi c)^{1/2}$, for different relative crack sizes, $c/2W$, and power-law indices: *Edge crack* problem for ARALL2 and GLARE2 with elliptical delamination.

		Circular Delamination					
Index	c/2W	(ARALL) 2-D	(ARALL) 3-D	% diff	(GLARE) 2-D	(GLARE) 3-D	% diff
		$K_I/\sigma(\pi c)^{1/2}$	$K_I/\sigma(\pi c)^{1/2}$		$K_I/\sigma(\pi c)^{1/2}$	$K_I/\sigma(\pi c)^{1/2}$	
n = m = 0	0.2	0.911	1.017	10.4	0.942	1.019	7.6
	0.3	0.994	1.069	7.0	1.036	1.077	3.8
	0.4	1.115	1.154	3.4	1.170	1.162	-0.7
	0.5	1.275	1.223	-4.3	1.340	1.242	-7.9
n = 0.5	0.2	0.971	1.017	4.5	1.012	1.019	0.7
	0.3	1.072	1.069	-0.3	1.127	1.077	-4.6
	0.4	1.178	1.154	-2.1	1.252	1.162	-7.7
	0.5	1.265	1.223	-3.4	1.363	1.242	-9.7
n = 1.0	0.2	1.075	1.017	-5.7	1.107	1.019	-8.6
	0.3	1.204	1.069	-12.6	1.248	1.077	-15.9
	0.4	1.349	1.154	-16.9	1.414	1.162	-21.7
	0.5	1.485	1.223	-21.4	1.579	1.242	-27.1
m = 0.5	0.2	0.389	1.017	61.8	0.467	1.019	54.2
	0.3	0.402	1.069	62.4	0.488	1.077	54.7
	0.4	0.403	1.154	65.1	0.495	1.162	57.4
	0.5	0.374	1.223	69.4	0.471	1.242	62.1
m = 1.0	0.2	0.409	1.017	59.8	0.490	1.019	51.9
	0.3	0.426	1.069	60.1	0.517	1.077	52.0
	0.4	0.429	1.154	62.8	0.527	1.162	54.6
	0.5	0.400	1.223	67.3	0.504	1.242	59.4

Table 4.11: Normalized crack-tip stress intensity factors, $K_I/\sigma(\pi c)^{1/2}$, for different relative crack sizes, $c/2W$, and power-law indices: *Edge crack problem for ARALL2 and GLARE2 with circular delamination.*

		Elliptical Delamination			Circular Delamination		
		Central Crack	Double-Edge Crack	Edge Crack	Central Crack	Double-Edge Crack	Edge Crack
Index	c/W	%diff	%diff	%diff	%diff	%diff	%diff
n = m = 0	0.2	3.8	3.8	3.3	3.6	3.6	3.3
	0.3	4.9	5.1	4.1	4.1	4.3	4.1
	0.4	5.7	5.4	4.3	4.6	4.5	4.7
	0.5	6.0	6.1	4.2	4.7	4.7	4.9
n = 0.5	0.2	5.0	5.3	5.9	3.4	3.6	4.1
	0.3	5.0	5.2	6.7	3.3	3.6	4.9
	0.4	4.9	5.0	7.7	3.4	3.4	5.9
	0.5	4.9	5.0	8.4	3.4	3.4	7.2
n = 1.0	0.2	3.5	3.5	4.1	2.4	2.5	2.9
	0.3	3.5	3.7	5.0	2.3	2.5	3.5
	0.4	3.5	3.6	6.0	2.4	2.4	4.6
	0.5	3.6	3.5	7.0	2.6	2.6	6.0
m = 0.5	0.2	61.7	61.5	69.1	15.6	15.9	16.7
	0.3	47.1	51.6	70.8	14.7	15.1	17.6
	0.4	35.9	41.1	74.0	13.4	13.4	18.6
	0.5	31.0	34.9	95.8	12.9	13.4	20.6
m = 1.0	0.2	61.6	62.6	68.1	15.2	15.2	16.5
	0.3	46.8	51.9	71.2	13.9	14.4	17.6
	0.4	36.7	42.4	78.8	12.7	13.7	18.6
	0.5	31.3	36.0	112.1	12.6	12.9	20.6

Table 4.12: Percentage difference of normalized stress intensity factors, $K_I/\sigma(\pi c)^{1/2}$, between ARALL and GLARE obtained from BIE analysis for different relative crack sizes, c/W, and power-law indices: *Central, double edge, and edge crack* problems with circular and elliptical delamination.

		Elliptical Delamination			Circular Delamination		
		Central Crack	Double-Edge Crack	Edge Crack	Central Crack	Double-Edge Crack	Edge Crack
c/W or c/2W		%diff	%diff	%diff	%diff	%diff	%diff
0.2		-0.3	0.4	0.2	0.1	-0.7	0.2
0.3		2.4	2.2	2.8	0.8	0.6	0.7
0.4		3.3	3.0	3.5	1.2	0.9	0.7
0.5		4.1	3.3	4.5	1.6	1.5	1.5

Table 4.13: Percentage difference of normalized stress intensity factors, $K_I/\sigma(\pi c)^{1/2}$, between ARALL and GLARE obtained from FEM analysis for different relative crack sizes, c/W and c/2W: *Central, double edge, and edge crack* problems with circular and elliptical delamination.

c/W or $c/2W$	Elliptical Delamination			Circular Delamination		
	Central Crack $K_I/\sigma(\pi c)^{1/2}$	Double-Edge Crack $K_I/\sigma(\pi c)^{1/2}$	Edge Crack $K_I/\sigma(\pi c)^{1/2}$	Central Crack $K_I/\sigma(\pi c)^{1/2}$	Double-Edge Crack $K_I/\sigma(\pi c)^{1/2}$	Edge Crack $K_I/\sigma(\pi c)^{1/2}$
0.2	0.738	0.764	0.885	0.894	0.935	1.095
0.3	0.700	0.715	0.905	0.879	0.918	1.205
0.4	0.697	0.706	0.954	0.895	0.915	1.350
0.5	0.711	0.708	1.000	0.928	0.929	1.500

(a)

c/W or $c/2W$	Elliptical Delamination			Circular Delamination		
	Central Crack $K_I/\sigma(\pi c)^{1/2}$	Double-Edge Crack $K_I/\sigma(\pi c)^{1/2}$	Edge Crack $K_I/\sigma(\pi c)^{1/2}$	Central Crack $K_I/\sigma(\pi c)^{1/2}$	Double-Edge Crack $K_I/\sigma(\pi c)^{1/2}$	Edge Crack $K_I/\sigma(\pi c)^{1/2}$
0.2	0.835	0.865	1.031	0.967	1.000	1.200
0.3	0.799	0.821	1.110	0.946	0.983	1.353
0.4	0.793	0.804	1.233	0.983	1.000	1.630
0.5	0.825	0.821	1.400	1.021	1.012	1.877

(b)

Table 5.1: Normalized crack-tip stress intensity factors, $K_I/\sigma(\pi c)^{1/2}$, obtained from FEM analysis for an ARALL laminate for different relative crack lengths, c/W and $c/2W$: *Central, double edge, and edge crack* problems with (a) 75% partial bridging, and (b) 50% partial bridging.

Elliptical Delamination						
c/W or c/2W	Central Crack		Double Edge Crack		Edge Crack	
	75% % diff	50% % diff	75% % diff	50% % diff	75% % diff	50% % diff
0.2	7.5	18.2	8.0	18.7	9.6	22.4
0.3	8.3	19.6	8.4	20.2	13.3	29.3
0.4	8.2	19.3	8.8	19.9	16.4	35.3
0.5	8.6	21.2	8.6	21.2	19.8	42.7

(a)

Circular Delamination						
c/W or c/2W	Central Crack		Double Edge Crack		Edge Crack	
	75% % diff	50% % diff	75% % diff	50% % diff	75% % diff	50% % diff
0.2	5.1	12.3	5.8	11.9	7.1	15.3
0.3	6.3	12.9	7.1	13.2	11.3	21.0
0.4	6.6	15.0	7.1	15.0	14.5	29.2
0.5	6.9	15.4	7.2	14.8	18.5	34.8

(b)

Table 5.2: Percentage difference of normalized stress intensity factors, $K_I/\sigma(\pi c)^{1/2}$, between partial bridging and full bridging obtained from FEM analysis, for different relative crack lengths, c/W and c/2W: Central, double edge, and edge crack problems with (a) elliptical delamination and (b) circular delamination.

		Elliptical Delamination			Circular Delamination		
Index	c/W	ARALL σ_b^*/σ	GLARE σ_b^*/σ	% diff	ARALL σ_b^*/σ	GLARE σ_b^*/σ	% diff
n = m = 0	0.2	2.580	2.448	-5.4	2.160	2.016	-7.1
	0.3	2.408	2.268	-6.2	1.980	1.832	-8.1
	0.4	2.360	2.216	-6.5	1.928	1.780	-8.3
	0.5	2.368	2.224	-6.5	1.936	1.788	-8.3
n = 0.5	0.2	4.708	4.276	-10.1	3.472	3.116	-11.4
	0.3	3.888	3.532	-10.1	2.876	2.580	-11.5
	0.4	3.620	3.292	-10.0	2.688	2.416	-11.3
	0.5	3.552	3.240	-9.6	2.660	2.392	-11.2
n = 1.0	0.2	4.952	4.480	-10.5	3.604	3.220	-11.9
	0.3	4.096	3.704	-10.6	2.988	2.668	-12.0
	0.4	3.824	3.464	-10.4	2.800	2.504	-11.8
	0.5	3.768	3.420	-10.2	2.780	2.488	-11.7
m = 0.5	0.2	3.696	3.428	-7.8	2.892	2.640	-9.5
	0.3	3.032	2.812	-7.8	2.376	2.172	-9.4
	0.4	2.788	2.592	-7.6	2.204	2.016	-9.3
	0.5	2.700	2.516	-7.3	2.148	1.972	-8.9
m = 1.0	0.2	3.900	3.604	-8.2	3.016	2.740	-10.1
	0.3	3.200	2.960	-8.1	2.480	2.260	-9.7
	0.4	2.952	2.732	-8.1	2.304	2.100	-9.7
	0.5	2.864	2.656	-7.8	2.252	2.056	-9.5

(a)

		Elliptical Delamination			Circular Delamination		
Index	c/W	ARALL σ_b^*/σ	GLARE σ_b^*/σ	% diff	ARALL σ_b^*/σ	GLARE σ_b^*/σ	% diff
n = m = 0	0.2	2.508	2.372	-5.7	2.372	2.232	-6.3
	0.3	2.352	2.208	-6.5	2.212	2.064	-7.2
	0.4	2.316	2.168	-6.8	2.172	2.028	-7.1
	0.5	2.332	2.188	-6.6	2.192	2.044	-7.2
n = 0.5	0.2	4.812	4.328	-11.2	4.332	3.884	-11.5
	0.3	4.044	3.636	-11.2	3.644	3.264	-11.6
	0.4	3.848	3.464	-11.1	3.472	3.112	-11.6
	0.5	3.852	3.476	-10.8	3.480	3.128	-11.3
n = 1.0	0.2	5.000	4.480	-11.6	4.484	4.004	-12.0
	0.3	4.208	3.768	-11.7	3.776	3.372	-12.0
	0.4	4.008	3.596	-11.5	3.604	3.216	-12.1
	0.5	4.024	3.616	-11.3	3.620	3.240	-11.7
m = 0.5	0.2	3.984	3.648	-9.2	3.652	3.324	-9.9
	0.3	3.336	3.052	-9.3	3.056	2.784	-9.8
	0.4	3.148	2.888	-9.0	2.892	2.640	-9.5
	0.5	3.120	2.868	-8.8	2.872	2.628	-9.3
m = 1.0	0.2	4.156	3.792	-9.6	3.792	3.444	-10.1
	0.3	3.480	3.176	-9.6	3.180	2.888	-10.1
	0.4	3.288	3.004	-9.5	3.012	2.736	-10.1
	0.5	3.264	2.988	-9.2	2.996	2.728	-9.8

(b)

Table 5.3: Normalized bridging stress in the last intact fibre, σ_b^*/σ , for different crack lengths, c/W, and power-law indices: *Central crack* problem for ARALL2 and GLARE2 with circular and elliptical delamination containing (a) 75% partial bridging, and (b) 50% partial bridging.

		Elliptical Delamination			Circular Delamination		
Index	c/W	ARALL σ_b^*/σ	GLARE σ_b^*/σ	% diff	ARALL σ_b^*/σ	GLARE σ_b^*/σ	% diff
n = m = 0	0.2	2.616	2.484	-5.3	2.200	2.056	-7.0
	0.3	2.464	2.324	-6.0	2.032	1.892	-7.4
	0.4	2.408	2.268	-6.2	1.976	1.832	-7.9
	0.5	2.392	2.252	-6.2	1.960	1.816	-7.9
n = 0.5	0.2	4.856	4.420	-9.9	3.592	3.228	-11.3
	0.3	4.048	3.696	-9.5	3.008	2.708	-11.1
	0.4	3.752	3.424	-9.6	2.796	2.520	-11.0
	0.5	3.628	3.316	-9.4	2.716	2.452	-10.8
n = 1.0	0.2	5.112	4.632	-10.4	3.732	3.340	-11.7
	0.3	4.276	3.880	-10.2	3.128	2.804	-11.6
	0.4	3.968	3.604	-10.1	2.916	2.616	-11.5
	0.5	3.844	3.496	-10.0	2.840	2.548	-11.5
m = 0.5	0.2	3.796	3.524	-7.7	2.976	2.720	-9.4
	0.3	3.136	2.920	-7.4	2.472	2.264	-9.2
	0.4	2.880	2.684	-7.3	2.284	2.096	-9.0
	0.5	2.760	2.576	-7.1	2.200	2.020	-8.9
m = 1.0	0.2	4.008	3.708	-8.1	3.104	2.828	-9.8
	0.3	3.320	3.076	-7.9	2.584	2.360	-9.5
	0.4	3.052	2.832	-7.8	2.388	2.184	-9.3
	0.5	2.928	2.720	-7.6	2.304	2.108	-9.3

(a)

		Elliptical Delamination			Circular Delamination		
Index	c/W	ARALL σ_b^*/σ	GLARE σ_b^*/σ	% diff	ARALL σ_b^*/σ	GLARE σ_b^*/σ	% diff
n = m = 0	0.2	2.548	2.412	-5.6	2.412	2.272	-6.2
	0.3	2.396	2.256	-6.2	2.256	2.116	-6.6
	0.4	2.348	2.204	-6.5	2.204	2.060	-7.0
	0.5	2.336	2.192	-6.6	2.196	2.052	-7.0
n = 0.5	0.2	4.984	4.492	-11.0	4.488	4.032	-11.3
	0.3	4.212	3.800	-10.8	3.800	3.416	-11.2
	0.4	3.960	3.572	-10.9	3.572	3.212	-11.2
	0.5	3.880	3.500	-10.9	3.504	3.152	-11.2
n = 1.0	0.2	5.180	4.652	-11.3	4.648	4.160	-11.7
	0.3	4.388	3.940	-11.4	3.940	3.528	-11.7
	0.4	4.128	3.708	-11.3	3.712	3.320	-11.8
	0.5	4.048	3.640	-11.2	3.644	3.260	-11.8
m = 0.5	0.2	4.116	3.776	-9.0	3.772	3.444	-9.5
	0.3	3.460	3.180	-8.8	3.176	2.904	-9.4
	0.4	3.236	2.976	-8.7	2.976	2.720	-9.4
	0.5	3.148	2.896	-8.7	2.900	2.652	-9.4
m = 1.0	0.2	4.296	3.924	-9.5	3.924	3.568	-10.0
	0.3	3.616	3.308	-9.3	3.308	3.012	-9.8
	0.4	3.380	3.096	-9.2	3.096	2.820	-9.8
	0.5	3.296	3.020	-9.1	3.020	2.752	-9.7

(b)

Table 5.4: Normalized bridging stress in the last intact fibre, σ_b^*/σ , for different crack lengths, c/W, and power-law indices: *Double edge crack* problem for ARALL2 and GLARE2 with circular and elliptical delamination containing (a) 75% partial bridging, and (b) 50% partial bridging.

		Elliptical Delamination			Circular Delamination		
Index	c/2W	ARALL σ_b^*/σ	GLARE σ_b^*/σ	% diff	ARALL σ_b^*/σ	GLARE σ_b^*/σ	% diff
n = m = 0	0.2	2.812	2.688	-4.6	2.416	2.276	-6.2
	0.3	2.868	2.744	-4.5	2.480	2.344	-5.8
	0.4	3.052	2.944	-3.7	2.700	2.568	-5.1
	0.5	3.280	3.192	-2.8	2.988	2.872	-4.0
n = 0.5	0.2	5.744	5.252	-9.4	4.308	3.884	-10.9
	0.3	5.484	5.060	-8.4	4.228	3.840	-10.1
	0.4	5.904	5.512	-7.1	4.716	4.332	-8.9
	0.5	6.620	6.268	-5.6	5.520	5.148	-7.2
n = 1.0	0.2	6.096	5.540	-10.0	4.504	4.040	-11.5
	0.3	5.900	5.408	-9.1	4.468	4.040	-10.6
	0.4	6.456	5.988	-7.8	5.060	4.624	-9.4
	0.5	7.380	6.944	-6.3	6.040	5.596	-7.9
m = 0.5	0.2	4.360	4.068	-7.2	3.480	3.200	-8.7
	0.3	3.952	3.728	-6.0	3.252	3.020	-7.7
	0.4	4.008	3.824	-4.8	3.420	3.216	-6.3
	0.5	4.224	4.080	-3.5	3.748	3.572	-4.9
m = 1.0	0.2	4.628	4.300	-7.6	3.648	3.340	-9.2
	0.3	4.232	3.972	-6.5	3.440	3.180	-8.2
	0.4	4.332	4.116	-5.2	3.656	3.420	-6.9
	0.5	4.612	4.440	-3.9	4.052	3.848	-5.3

(a)

		Elliptical Delamination			Circular Delamination		
Index	c/2W	ARALL σ_b^*/σ	GLARE σ_b^*/σ	% diff	ARALL σ_b^*/σ	GLARE σ_b^*/σ	% diff
n = m = 0	0.2	2.744	2.612	-5.1	2.616	2.480	-5.5
	0.3	2.796	2.668	-4.8	2.672	2.536	-5.4
	0.4	2.976	2.860	-4.1	2.860	2.740	-4.4
	0.5	3.204	3.104	-3.2	3.108	3.000	-3.6
n = 0.5	0.2	6.008	5.424	-10.8	5.428	4.884	-11.1
	0.3	5.984	5.436	-10.1	5.440	4.920	-10.6
	0.4	6.800	6.232	-9.1	6.236	5.684	-9.7
	0.5	8.188	7.588	-7.9	7.588	6.996	-8.5
n = 1.0	0.2	6.272	5.640	-11.2	5.644	5.056	-11.6
	0.3	6.300	5.696	-10.6	5.700	5.132	-11.1
	0.4	7.240	6.596	-9.8	6.604	5.988	-10.3
	0.5	8.840	8.144	-8.5	8.148	7.468	-9.1
m = 0.5	0.2	4.868	4.480	-8.7	4.484	4.104	-9.3
	0.3	4.684	4.344	-7.8	4.344	4.008	-8.4
	0.4	5.104	4.776	-6.9	4.776	4.448	-7.4
	0.5	5.824	5.516	-5.6	5.516	5.196	-6.2
m = 1.0	0.2	5.100	4.672	-9.2	4.676	4.264	-9.7
	0.3	4.932	4.556	-8.3	4.560	4.188	-8.9
	0.4	5.408	5.044	-7.2	5.044	4.676	-7.9
	0.5	6.232	5.876	-6.1	5.880	5.516	-6.6

(b)

Table 5.5: Normalized bridging stress in the last intact fibre, σ_b^*/σ , for different crack lengths, c/2W, and power-law indices: *Edge crack* problem for ARALL2 and GLARE2 with circular and elliptical delamination containing (a) 75% partial bridging, and (b) 50% partial bridging.

		Elliptical Delamination		Circular Delamination	
Index	c/W	75% % diff	50% % diff	75% % diff	50% % diff
n = m = 0	0.2	-4.0	-7.0	6.5	14.8
	0.3	-4.2	-6.6	7.7	17.4
	0.4	-3.9	-5.9	8.3	18.6
	0.5	-3.4	-5.0	8.9	19.5
n = 0.5	0.2	3.4	5.5	15.9	32.6
	0.3	2.5	6.2	15.6	33.4
	0.4	3.2	8.9	16.1	35.0
	0.5	4.4	11.8	17.0	36.6
n = 1.0	0.2	1.9	2.9	15.5	32.1
	0.3	0.9	3.5	15.3	32.9
	0.4	1.5	6.0	15.7	34.5
	0.5	2.7	8.8	16.5	35.9
m = 0.5	0.2	9.0	15.6	17.7	34.8
	0.3	8.4	16.8	17.2	35.6
	0.4	9.5	19.8	18.0	37.5
	0.5	11.0	22.9	19.0	39.4
m = 1.0	0.2	7.7	13.4	17.2	34.2
	0.3	7.0	14.5	16.8	35.1
	0.4	8.1	17.5	17.7	37.1
	0.5	9.8	20.8	18.7	38.9

Table 5.6: Percentage difference of normalized bridging stress, σ_B^*/σ , between partial bridging and full bridging for different relative crack lengths, c/W, and power-law indices: *Central crack* with ARALL2 and circular and elliptical delamination.

		Elliptical Delamination		Circular Delamination	
Index	c/W	75% % diff	50% % diff	75% % diff	50% % diff
n = m = 0	0.2	-3.7	-6.4	6.7	14.9
	0.3	-3.9	-6.8	7.5	16.7
	0.4	-3.7	-6.3	8.1	17.6
	0.5	-3.7	-6.2	8.4	18.2
n = 0.5	0.2	4.0	6.4	16.4	33.1
	0.3	3.1	6.8	16.0	33.5
	0.4	3.7	8.8	16.3	34.5
	0.5	4.6	10.8	16.8	35.5
n = 1.0	0.2	2.4	3.7	16.0	32.5
	0.3	1.5	4.0	15.6	33.0
	0.4	1.9	5.7	15.8	33.8
	0.5	2.7	7.6	16.3	34.8
m = 0.5	0.2	9.7	16.7	18.0	35.3
	0.3	9.3	17.8	17.8	36.0
	0.4	10.1	20.0	18.6	37.5
	0.5	11.3	22.2	19.1	38.6
m = 1.0	0.2	8.3	14.4	17.5	34.8
	0.3	8.0	15.5	17.3	35.4
	0.4	8.8	17.6	17.8	36.6
	0.5	10.0	20.0	18.6	37.9

Table 5.7: Percentage difference of normalized bridging stress, σ_B^*/σ , between partial bridging and full bridging for different relative crack lengths, c/W, and power-law indices: *Double edge crack* with ARALL2 and circular and elliptical delamination.

		<i>Elliptical Delamination</i>		<i>Circular Delamination</i>	
Index	c/2W	75% % diff	50% % diff	75% % diff	50% % diff
n = m = 0	0.2	-3.3	-5.8	5.6	12.8
	0.3	-3.2	-5.9	5.3	12.1
	0.4	-2.9	-5.5	4.1	9.5
	0.5	-2.6	-5.0	2.8	6.6
n = 0.5	0.2	5.4	9.6	16.6	33.8
	0.3	7.2	15.0	16.8	35.4
	0.4	11.1	22.8	18.1	38.0
	0.5	16.0	32.0	20.1	41.9
n = 1.0	0.2	3.7	6.4	16.2	33.1
	0.3	4.9	11.0	15.9	34.1
	0.4	8.4	18.3	16.8	36.2
	0.5	13.1	27.5	18.5	39.6
m = 0.5	0.2	11.7	21.0	19.0	37.1
	0.3	14.9	28.2	20.2	40.2
	0.4	19.3	36.6	22.6	44.6
	0.5	23.9	44.8	25.5	49.4
m = 1.0	0.2	10.5	18.8	18.3	36.3
	0.3	13.5	25.8	19.4	39.2
	0.4	17.9	34.2	21.8	43.3
	0.5	22.6	42.7	24.7	48.1

Table 5.8: Percentage difference of normalized bridging stress, σ_B^*/σ , between partial bridging from full bridging for different relative crack lengths, c/W, and power-law indices: *Edge crack* with ARALL2 and circular and elliptical delamination.

		<i>Elliptical Delamination</i>		<i>Circular Delamination</i>	
Index	c/W	(ARALL)	(GLARE)	(ARALL)	(GLARE)
		$K_I/\sigma(\pi c)^{1/2}$	$K_I/\sigma(\pi c)^{1/2}$	$K_I/\sigma(\pi c)^{1/2}$	$K_I/\sigma(\pi c)^{1/2}$
n = m = 0	0.2	0.683	0.705	0.747	0.770
	0.3	0.651	0.677	0.728	0.755
	0.4	0.662	0.691	0.748	0.778
	0.5	0.692	0.725	0.786	0.820
n = 0.5	0.2	0.696	0.733	0.795	0.826
	0.3	0.709	0.744	0.806	0.836
	0.4	0.739	0.776	0.840	0.870
	0.5	0.777	0.817	0.885	0.920
n = 1.0	0.2	0.798	0.826	0.873	0.897
	0.3	0.808	0.836	0.883	0.905
	0.4	0.839	0.868	0.917	0.941
	0.5	0.882	0.914	0.968	0.995
m = 0.5	0.2	0.095	0.167	0.308	0.376
	0.3	0.134	0.204	0.340	0.404
	0.4	0.169	0.238	0.371	0.436
	0.5	0.194	0.264	0.401	0.469
m = 1.0	0.2	0.108	0.183	0.327	0.397
	0.3	0.149	0.220	0.359	0.424
	0.4	0.183	0.254	0.391	0.457
	0.5	0.208	0.282	0.422	0.492

(a)

		<i>Elliptical Delamination</i>		<i>Circular Delamination</i>	
Index	c/W	(ARALL)	(GLARE)	(ARALL)	(GLARE)
		$K_I/\sigma(\pi c)^{1/2}$	$K_I/\sigma(\pi c)^{1/2}$	$K_I/\sigma(\pi c)^{1/2}$	$K_I/\sigma(\pi c)^{1/2}$
n = m = 0	0.2	0.766	0.784	0.782	0.802
	0.3	0.743	0.765	0.763	0.786
	0.4	0.765	0.789	0.787	0.811
	0.5	0.810	0.836	0.834	0.861
n = 0.5	0.2	0.760	0.794	0.791	0.823
	0.3	0.770	0.803	0.801	0.831
	0.4	0.801	0.835	0.833	0.865
	0.5	0.845	0.882	0.880	0.915
n = 1.0	0.2	0.847	0.873	0.871	0.895
	0.3	0.856	0.881	0.879	0.902
	0.4	0.889	0.915	0.913	0.938
	0.5	0.938	0.968	0.965	0.993
m = 0.5	0.2	0.221	0.294	0.292	0.364
	0.3	0.245	0.317	0.315	0.384
	0.4	0.269	0.341	0.338	0.409
	0.5	0.290	0.366	0.363	0.436
m = 1.0	0.2	0.241	0.315	0.314	0.385
	0.3	0.265	0.338	0.336	0.405
	0.4	0.289	0.363	0.360	0.432
	0.5	0.312	0.389	0.385	0.461

(b)

Table 5.9: Normalized crack-tip stress intensity factors, $K_I/\sigma(\pi c)^{1/2}$, for different relative crack lengths, c/W, and power-law indices: *Central crack problem for ARALL2 and GLARE2 with circular and elliptical delamination containing (a) 75% partial bridging, and (b) 50% partial bridging.*

		Elliptical Delamination		Circular Delamination	
Index	c/W	(ARALL)	(GLARE)	(ARALL)	(GLARE)
		$K_I/\sigma(\pi c)^{1/2}$	$K_I/\sigma(\pi c)^{1/2}$	$K_I/\sigma(\pi c)^{1/2}$	$K_I/\sigma(\pi c)^{1/2}$
n = m = 0	0.2	0.710	0.733	0.774	0.799
	0.3	0.664	0.692	0.742	0.770
	0.4	0.656	0.685	0.742	0.772
	0.5	0.667	0.698	0.759	0.791
n = 0.5	0.2	0.714	0.753	0.818	0.851
	0.3	0.714	0.752	0.817	0.849
	0.4	0.728	0.765	0.831	0.863
	0.5	0.749	0.787	0.857	0.889
n = 1.0	0.2	0.820	0.851	0.899	0.925
	0.3	0.818	0.848	0.897	0.922
	0.4	0.831	0.860	0.912	0.936
	0.5	0.855	0.885	0.939	0.965
m = 0.5	0.2	0.093	0.169	0.313	0.385
	0.3	0.119	0.189	0.329	0.397
	0.4	0.141	0.210	0.347	0.413
	0.5	0.160	0.228	0.366	0.433
m = 1.0	0.2	0.108	0.185	0.334	0.406
	0.3	0.132	0.205	0.349	0.417
	0.4	0.154	0.226	0.368	0.435
	0.5	0.173	0.245	0.387	0.455

(a)

		Elliptical Delamination		Circular Delamination	
Index	c/W	(ARALL)	(GLARE)	(ARALL)	(GLARE)
		$K_I/\sigma(\pi c)^{1/2}$	$K_I/\sigma(\pi c)^{1/2}$	$K_I/\sigma(\pi c)^{1/2}$	$K_I/\sigma(\pi c)^{1/2}$
N = m = 0	0.2	0.796	0.816	0.813	0.834
	0.3	0.763	0.786	0.783	0.807
	0.4	0.764	0.789	0.787	0.811
	0.5	0.785	0.810	0.809	0.834
n = 0.5	0.2	0.782	0.818	0.815	0.849
	0.3	0.781	0.817	0.814	0.847
	0.4	0.796	0.831	0.829	0.862
	0.5	0.820	0.856	0.854	0.888
n = 1.0	0.2	0.873	0.901	0.898	0.924
	0.3	0.871	0.899	0.896	0.922
	0.4	0.886	0.913	0.911	0.936
	0.5	0.913	0.941	0.939	0.966
m = 0.5	0.2	0.223	0.299	0.297	0.371
	0.3	0.236	0.309	0.308	0.379
	0.4	0.250	0.323	0.321	0.393
	0.5	0.265	0.339	0.337	0.410
m = 1.0	0.2	0.244	0.322	0.319	0.394
	0.3	0.256	0.332	0.329	0.402
	0.4	0.271	0.345	0.344	0.416
	0.5	0.286	0.361	0.360	0.433

(b)

Table 5.10: Normalized crack-tip stress intensity factors, $K_I/\sigma(\pi c)^{1/2}$, for different relative crack lengths, c/W, and power-law indices: Double edge crack problem for ARALL2 and GLARE2 with circular and elliptical delamination containing (a) 75% partial bridging, and (b) 50% partial bridging.

		Elliptical Delamination		Circular Delamination	
Index	c/2W	(ARALL)	(GLARE)	(ARALL)	(GLARE)
		$K_I/\sigma(\pi c)^{1/2}$	$K_I/\sigma(\pi c)^{1/2}$	$K_I/\sigma(\pi c)^{1/2}$	$K_I/\sigma(\pi c)^{1/2}$
n = m = 0	0.2	0.893	0.917	0.960	0.986
	0.3	0.989	1.021	1.079	1.113
	0.4	1.183	1.219	1.290	1.333
	0.5	1.494	1.534	1.612	1.663
n = 0.5	0.2	0.829	0.878	0.964	1.007
	0.3	0.905	0.965	1.076	1.132
	0.4	0.989	1.065	1.214	1.289
	0.5	1.050	1.147	1.343	1.447
n = 1.0	0.2	0.961	1.001	1.068	1.101
	0.3	1.062	1.113	1.205	1.251
	0.4	1.181	1.251	1.381	1.447
	0.5	1.286	1.380	1.565	1.660
m = 0.5	0.2	0.095	0.182	0.351	0.436
	0.3	0.103	0.193	0.378	0.471
	0.4	0.089	0.184	0.385	0.491
	0.5	0.040	0.137	0.354	0.472
m = 1.0	0.2	0.111	0.202	0.378	0.464
	0.3	0.119	0.214	0.406	0.502
	0.4	0.104	0.205	0.417	0.528
	0.5	0.051	0.156	0.387	0.512

(a)

		Elliptical Delamination		Circular Delamination	
Index	c/2W	(ARALL)	(GLARE)	(ARALL)	(GLARE)
		$K_I/\sigma(\pi c)^{1/2}$	$K_I/\sigma(\pi c)^{1/2}$	$K_I/\sigma(\pi c)^{1/2}$	$K_I/\sigma(\pi c)^{1/2}$
N = m = 0	0.2	1.004	1.025	1.022	1.042
	0.3	1.169	1.194	1.191	1.217
	0.4	1.469	1.497	1.493	1.523
	0.5	1.954	1.985	1.980	2.014
n = 0.5	0.2	0.930	0.976	0.973	1.016
	0.3	1.055	1.113	1.110	1.165
	0.4	1.221	1.299	1.295	1.370
	0.5	1.404	1.512	1.508	1.615
n = 1.0	0.2	1.044	1.080	1.077	1.110
	0.3	1.195	1.242	1.239	1.283
	0.4	1.401	1.468	1.463	1.527
	0.5	1.646	1.743	1.738	1.833
m = 0.5	0.2	0.253	0.344	0.341	0.430
	0.3	0.274	0.376	0.374	0.475
	0.4	0.276	0.396	0.394	0.514
	0.5	0.240	0.379	0.376	0.521
m = 1.0	0.2	0.278	0.372	0.369	0.459
	0.3	0.303	0.408	0.405	0.509
	0.4	0.312	0.435	0.432	0.557
	0.5	0.278	0.426	0.421	0.573

(b)

Table 5.11: Normalized crack-tip stress intensity factors, $K_I/\sigma(\pi c)^{1/2}$, for different relative crack lengths, c/W, and power-law indices: *Edge crack* problem for ARALL2 and GLARE2 with circular and elliptical delamination containing (a) 75% partial bridging, and (b) 50% partial bridging.

		Elliptical Delamination			Circular Delamination		
		75% % diff	50% % diff	0% % diff	75% % diff	50% % diff	0% % diff
Index	c/W						
n = m = 0	0.2	11.4	21.0	43.5	3.6	7.9	32.8
	0.3	13.1	23.8	47.9	2.9	7.3	34.9
	0.4	14.8	26.3	50.1	3.5	8.3	36.2
	0.5	15.9	28.1	51.8	3.7	9.2	37.3
n = 0.5	0.2	5.2	13.2	38.4	-1.5	-2.0	24.6
	0.3	5.5	13.0	38.3	-1.7	-2.4	24.5
	0.4	6.0	13.2	38.5	-1.2	-2.0	24.8
	0.5	6.0	13.6	39.6	-1.4	-1.9	25.7
n = 1.0	0.2	3.1	8.7	27.8	-1.3	-1.5	17.5
	0.3	3.3	8.8	28.1	-1.5	-1.9	17.5
	0.4	3.9	9.3	28.7	-1.1	-1.5	18.0
	0.5	4.2	9.9	30.0	-1.0	-1.3	19.0
m = 0.5	0.2	56.8	81.4	96.2	-11.0	-17.1	68.1
	0.3	46.3	70.6	93.4	-7.9	-16.5	66.2
	0.4	35.5	59.5	90.4	-7.5	-18.0	64.7
	0.5	28.9	52.4	88.6	-8.0	-19.3	64.2
m = 1.0	0.2	60.2	82.2	96.0	-9.5	-14.0	66.6
	0.3	49.7	71.7	93.1	-7.2	-14.6	64.5
	0.4	38.8	61.2	90.1	-6.9	-16.1	63.0
	0.5	31.3	54.2	88.2	-7.1	-17.4	62.6

Table 5.12: Percentage difference of normalized stress intensity factors, $K_I/\sigma(\pi c)^{1/2}$, between partial bridging and full bridging obtained from BIE analysis for different relative crack lengths, c/W, and power-law indices: *Central crack* with circular and elliptical delamination.

		Elliptical Delamination			Circular Delamination		
		75% % diff	50% % diff	0% % diff	75% % diff	50% % diff	0% % diff
Index	c/W						
n = m = 0	0.2	11.3	20.9	43.1	3.6	8.2	32.7
	0.3	13.4	24.6	48.4	3.2	8.3	35.6
	0.4	15.1	27.1	50.9	3.8	9.3	37.1
	0.5	16.5	29.0	52.7	4.1	10.0	38.1
n = 0.5	0.2	5.3	13.6	39.0	-1.5	-1.8	25.1
	0.3	5.6	13.7	39.6	-1.7	-2.1	25.5
	0.4	6.0	14.1	39.7	-1.4	-1.7	25.7
	0.5	6.1	14.3	40.3	-1.3	-1.6	26.3
n = 1.0	0.2	3.2	9.0	28.3	-1.3	-1.4	17.8
	0.3	3.5	9.4	29.2	-1.6	-1.7	18.3
	0.4	4.1	10.0	29.8	-1.1	-1.2	18.8
	0.5	4.2	10.3	30.4	-1.1	-1.1	19.4
m = 0.5	0.2	54.8	81.2	96.2	-11.5	-17.5	68.5
	0.3	49.6	74.6	94.6	-9.7	-17.2	67.6
	0.4	39.0	65.6	92.4	-9.8	-18.7	66.4
	0.5	31.3	58.5	90.7	-9.6	-19.0	65.9
m = 1.0	0.2	60.2	82.4	96.1	-9.9	-15.0	66.9
	0.3	53.0	75.8	94.4	-8.6	-15.2	66.0
	0.4	43.5	67.9	92.3	-7.6	-15.1	65.1
	0.5	35.3	60.8	90.5	-8.3	-16.4	64.4

Table 5.13: Percentage difference of normalized stress intensity factors, $K_I/\sigma(\pi c)^{1/2}$, between partial bridging and full bridging obtained from BIE analysis for different relative crack lengths, c/W, and power-law indices: *Double edge crack* with circular and elliptical delamination.

		<i>Elliptical Delamination</i>			<i>Circular Delamination</i>		
Index	c/2W	75% % diff	50% % diff	0% % diff	75% % diff	50% % diff	0% % diff
n = m = 0	0.2	11.9	21.6	42.3	5.1	10.9	33.2
	0.3	17.0	29.8	50.5	7.9	16.5	40.0
	0.4	23.7	38.5	57.2	13.6	25.3	47.1
	0.5	31.0	47.2	63.4	20.9	35.6	54.8
n = 0.5	0.2	7.1	17.2	43.5	-0.7	0.2	28.8
	0.3	10.4	23.1	51.1	0.4	3.4	35.3
	0.4	14.6	30.8	59.9	3.0	9.0	44.1
	0.5	18.3	38.9	69.6	5.8	16.1	55.1
n = 1.0	0.2	4.8	12.4	32.9	-0.7	0.2	21.2
	0.3	7.8	18.1	40.9	0.1	2.8	27.3
	0.4	11.8	25.6	50.6	2.3	7.8	36.0
	0.5	15.9	34.3	61.6	5.1	14.6	47.3
m = 0.5	0.2	64.2	86.6	97.5	-10.8	-14.1	71.5
	0.3	69.9	88.7	98.1	-6.3	-7.5	75.7
	0.4	71.9	90.9	98.8	-4.7	-2.3	80.9
	0.5	92.5	98.8	99.9	-5.6	0.5	86.7
m = 1.0	0.2	66.7	86.7	97.3	-8.2	-10.8	70.0
	0.3	73.1	89.4	98.1	-4.9	-5.2	74.3
	0.4	79.8	93.3	99.0	-2.9	0.7	79.6
	0.5	115.7	102.9	100.3	-3.4	5.0	85.8

Table 5.14: Percentage difference of normalized stress intensity factors, $K_I/\sigma(\pi c)^{1/2}$, between partial bridging and full bridging obtained from BIE analysis for different relative crack lengths, c/2W, and power-law indices: *Edge crack with circular and elliptical delamination.*

Circular Delamination						
c/W	ARALL 2D $K_I/\sigma(\pi c)^{1/2}$	ARALL 3D $K_I/\sigma(\pi c)^{1/2}$	% diff	GLARE 2D $K_I/\sigma(\pi c)^{1/2}$	GLARE 3D $K_I/\sigma(\pi c)^{1/2}$	% diff
0.2	1.071	1.182	9.4	1.073	1.175	8.7
0.3	1.086	1.196	9.2	1.087	1.192	8.8
0.4	1.131	1.250	9.5	1.132	1.246	9.1
0.5	1.208	1.332	9.3	1.210	1.329	9.0
Elliptical Delamination						
c/W	ARALL 2D $K_I/\sigma(\pi c)^{1/2}$	ARALL 3D $K_I/\sigma(\pi c)^{1/2}$	% diff	GLARE 2D $K_I/\sigma(\pi c)^{1/2}$	GLARE 3D $K_I/\sigma(\pi c)^{1/2}$	% diff
0.2	1.071	1.197	10.5	1.073	1.184	9.4
0.3	1.086	1.216	10.7	1.087	1.206	9.9
0.4	1.131	1.272	11.1	1.132	1.261	10.2
0.5	1.208	1.359	11.1	1.210	1.344	10.0

(a)

Circular Delamination						
c/W	ARALL 2D $K_I/\sigma(\pi c)^{1/2}$	ARALL 3D $K_I/\sigma(\pi c)^{1/2}$	% diff	GLARE 2D $K_I/\sigma(\pi c)^{1/2}$	GLARE 3D $K_I/\sigma(\pi c)^{1/2}$	% diff
0.2	1.071	0.867	-23.5	1.073	0.790	-35.8
0.3	1.086	0.889	-22.2	1.087	0.800	-35.9
0.4	1.131	0.920	-22.9	1.132	0.815	-38.9
0.5	1.208	0.967	-24.9	1.210	0.870	-39.1
Elliptical Delamination						
c/W	ARALL 2D $K_I/\sigma(\pi c)^{1/2}$	ARALL 3D $K_I/\sigma(\pi c)^{1/2}$	% diff	GLARE 2D $K_I/\sigma(\pi c)^{1/2}$	GLARE 3D $K_I/\sigma(\pi c)^{1/2}$	% diff
0.2	1.071	0.875	-22.4	1.073	0.790	-35.8
0.3	1.086	0.868	-25.1	1.087	0.800	-35.9
0.4	1.131	0.900	-25.7	1.132	0.810	-39.8
0.5	1.208	0.950	-27.2	1.210	0.857	-41.2

(b)

Table 5.15: Normalized crack-tip stress intensity factors, $K_I/\sigma(\pi c)^{1/2}$, calculated in (a) the aluminum layer, and (b) fibre-epoxy layer; for different relative crack lengths, c/W . Unbridged ARALL *central crack* problem with circular and elliptical delamination.

Circular Delamination						
c/W	ARALL 2D $K_I/\sigma(\pi c)^{1/2}$	ARALL 3D $K_I/\sigma(\pi c)^{1/2}$	% diff	GLARE 2D $K_I/\sigma(\pi c)^{1/2}$	GLARE 3D $K_I/\sigma(\pi c)^{1/2}$	% diff
0.2	1.108	1.227	9.7	1.112	1.223	9.1
0.3	1.115	1.235	9.7	1.118	1.232	9.3
0.4	1.135	1.258	9.8	1.137	1.256	9.5
0.5	1.177	1.300	9.5	1.179	1.297	9.1
Elliptical Delamination						
c/W	ARALL 2D $K_I/\sigma(\pi c)^{1/2}$	ARALL 3D $K_I/\sigma(\pi c)^{1/2}$	% diff	GLARE 2D $K_I/\sigma(\pi c)^{1/2}$	GLARE 3D $K_I/\sigma(\pi c)^{1/2}$	% diff
0.2	1.108	1.244	10.9	1.112	1.231	9.7
0.3	1.115	1.254	11.1	1.118	1.244	10.1
0.4	1.135	1.278	11.2	1.137	1.267	10.3
0.5	1.177	1.322	11.0	1.179	1.312	10.1

(a)

Circular Delamination						
c/W	ARALL 2D $K_I/\sigma(\pi c)^{1/2}$	ARALL 3D $K_I/\sigma(\pi c)^{1/2}$	% diff	GLARE 2D $K_I/\sigma(\pi c)^{1/2}$	GLARE 3D $K_I/\sigma(\pi c)^{1/2}$	% diff
0.2	1.108	0.875	-26.6	1.112	0.790	-40.8
0.3	1.115	0.875	-27.4	1.118	0.780	-43.3
0.4	1.135	0.860	-32.0	1.137	0.800	-42.1
0.5	1.177	0.873	-34.8	1.179	0.810	-45.6
Elliptical Delamination						
c/W	ARALL 2D $K_I/\sigma(\pi c)^{1/2}$	ARALL 3D $K_I/\sigma(\pi c)^{1/2}$	% diff	GLARE 2D $K_I/\sigma(\pi c)^{1/2}$	GLARE 3D $K_I/\sigma(\pi c)^{1/2}$	% diff
0.2	1.108	0.843	-31.4	1.112	0.775	-43.5
0.3	1.115	0.825	-35.2	1.118	0.800	-39.8
0.4	1.135	0.860	-32.0	1.137	0.800	-42.1
0.5	1.177	0.870	-35.3	1.179	0.800	-47.4

(b)

Table 5.16: Normalized crack-tip stress intensity factors, $K_I/\sigma(\pi c)^{1/2}$, calculated in (a) the aluminum layer, and (b) fibre-epoxy layer; for different relative crack lengths, c/W . Unbridged ARALL *double edge crack* problem with circular and elliptical delamination.

Circular Delamination						
$c/2W$	ARALL 2D $K_I/\sigma(\pi c)^{1/2}$	ARALL 3D $K_I/\sigma(\pi c)^{1/2}$	% diff	GLARE 2D $K_I/\sigma(\pi c)^{1/2}$	GLARE 3D $K_I/\sigma(\pi c)^{1/2}$	% diff
0.2	1.364	1.517	10.1	1.367	1.504	9.1
0.3	1.657	1.849	10.4	1.660	1.838	9.7
0.4	2.108	2.367	10.9	2.112	2.347	10.0
0.5	2.820	3.200	11.9	2.825	3.167	10.8
Elliptical Delamination						
$c/2W$	ARALL 2D $K_I/\sigma(\pi c)^{1/2}$	ARALL 3D $K_I/\sigma(\pi c)^{1/2}$	% diff	GLARE 2D $K_I/\sigma(\pi c)^{1/2}$	GLARE 3D $K_I/\sigma(\pi c)^{1/2}$	% diff
0.2	1.364	1.533	11.0	1.367	1.517	9.9
0.3	1.657	1.874	11.6	1.660	1.856	10.6
0.4	2.108	2.400	12.2	2.112	2.367	10.8
0.5	2.820	3.229	12.7	2.825	3.171	10.9

(a)

Circular Delamination						
$c/2W$	ARALL 2D $K_I/\sigma(\pi c)^{1/2}$	ARALL 3D $K_I/\sigma(\pi c)^{1/2}$	% diff	GLARE 2D $K_I/\sigma(\pi c)^{1/2}$	GLARE 3D $K_I/\sigma(\pi c)^{1/2}$	% diff
0.2	1.364	1.075	-26.9	1.367	0.920	-48.6
0.3	1.657	1.283	-29.2	1.660	1.142	-45.4
0.4	2.108	1.667	-26.5	2.112	1.500	-40.8
0.5	2.820	2.285	-23.4	2.825	2.050	-37.8
Elliptical Delamination						
$c/2W$	ARALL 2D $K_I/\sigma(\pi c)^{1/2}$	ARALL 3D $K_I/\sigma(\pi c)^{1/2}$	% diff	GLARE 2D $K_I/\sigma(\pi c)^{1/2}$	GLARE 3D $K_I/\sigma(\pi c)^{1/2}$	% diff
0.2	1.364	1.050	-29.9	1.367	0.960	-42.4
0.3	1.657	1.250	-32.6	1.660	1.157	-43.5
0.4	2.108	1.625	-29.7	2.112	1.500	-40.8
0.5	2.820	2.285	-23.4	2.825	2.000	-41.3

(b)

Table 5.17: Normalized crack-tip stress intensity factors, $K_I/\sigma(\pi c)^{1/2}$, calculated in (a) the aluminum layer, and (b) fibre-epoxy layer; for different relative crack lengths, $c/2W$. Unbridged ARALL edge crack problem with circular and elliptical delamination.

Crack Deflection and Meandering

Zone Shielding

- transformation toughening
- microcrack toughening
- crack wake plasticity
- crack field void formation
- residual stress fields
- crack tip dislocation shielding

Contact Shielding

- wedging:
 - corrosion debris-induced crack closure
 - crack surface roughness-induced closure
- bridging:
 - ligament or fiber toughening
- sliding:
 - sliding crack surface interference
- wedging + bridging:
 - fluid pressure-induced crack closure

Combined Zone and Contact Shielding

- plasticity-induced crack closure
- phase transformation-induced closure

Figure 2.1: Extrinsic Toughening Mechanisms, Ritchie, *et al.* (1989).

Figure 2.2: Crack Bridging with Delamination

Figure 2.3: Fiber in a Cylindrical Matrix used in the Shear Lag Model, Bao and Song (1993).

Figure 2.4: Steady-state Bridged Crack, Marshall, *et al.* (1985).

Figure 2.5: Longitudinal Debonding of Fiber from the Matrix, Nguyen and Yang (1994).

Figure 2.6: Compliance Equivalent Spring System of Crack Bridging, Lutz and Sakai (1993).

Figure 2.7: Superposition of Stress Intensity Factors for a Bridged Crack, Foote, *et al.* (1986).

Figure 2.8: 2/1 Lay-up of a Fiber-Metal Laminate, Marissen (1988).

Figure 2.9: Adhesive Shear Deformation upon Crack Opening, Marissen (1989).

Figure 2.10: General Delamination Shape due to a Crack from a Notch, Marissen (1991).

Figure 2.11: Power-Law Bridging Stress Distributions

Figure 3.1: Linear elastic body R , with boundary S in a two-dimensional co-ordinate system.

Figure 3.2: Solution domain R , with boundary S , divided into M elements.

Figure 3.3: Quadratic one-dimensional element.

Figure 3.4: Co-ordinate system and material orientation in LEFM analysis.

Figure 3.5: Quarter-point crack-tip element of length, l .

Figure 3.6: Node symbols for the crack-tip elements.

Figure 3.7: Plate with central crack under uniform tension for the isotropic test case.

Figure 3.8: Plate with double edge cracks under uniform tension for the isotropic test case.

Figure 3.9: Plate with single edge crack under uniform tension for the isotropic test case.

Figure 3.10: Plate with circular hole and two radial cracks emanating from the hole for the isotropic test case.

Figure 3.11: Typical boundary element mesh for the isotropic and orthotropic test cases.

Figure 3.12: Typical boundary element mesh used for the isotropic test case with a hole.

Figure 3.13: Plate with central crack under uniform tension for the orthotropic test case.

Figure 3.14: Plate with circular hole with two radial cracks for the orthotropic test case.

Figure 3.15: Boundary element mesh used in the test case for the problem of cracks emanating from a hole in an orthotropic plate.

Figure 3.16: Finite element mesh used in the test case for the problem of crack emanating from a hole in an orthotropic plate, Lakshminarayana, *et al.* (1979).

Figure 4.1: An edge crack emanating from a semi-circular cut-out of a fibre-metal laminate plate.

Figure 4.2: The double edge crack problem of cracks emanating from semi-circular cut-outs of a fibre-metal laminate plate.

Figure 4.3: The central crack problem of cracks emanating from a circular hole in a fibre-metal laminate plate.

Figure 4.4: Typical BIE mesh for crack problems in the x_1 - x_2 plane of the laminate, $c/W = 0.5$.

Figure 4.5: Typical 3-D FEM mesh for a fully bridged laminate with $c/W = 0.5$, $d/H = 0.5$. No. of elements = 7716, no. of nodes = 45555.

(a)

(b)

Figure 4.6: Variations of normalized bridging stress, σ_B/σ , with relative distance from the crack-tip, x/a , along the centreline of the aramid-epoxy layer, for the fully bridged *central crack* problem with (a) circular delamination, and (b) elliptical delamination. (FEM analysis)

(a)

(b)

Figure 4.7: Variations of normalized bridging stress, σ_B/σ , with relative distance from the crack-tip, x/a , along the centreline of the aramid-epoxy layer, for the fully bridged *double edge crack* problem with (a) circular delamination, and (b) elliptical delamination. (FEM analysis)

(a)

(b)

Figure 4.8: Variations of normalized bridging stress, σ_B/σ , with relative distance from the crack-tip, x/a , along the centreline of the aramid-epoxy layer, for the fully bridged *edge crack* problem with (a) circular delamination, and (b) elliptical delamination. (FEM analysis)

Figure 4.9: Variations of normalized bridging stress, σ_B/σ , with relative distance from the crack-tip, $x/a \geq 0$, along the centreline of the aramid-epoxy layer, for the fully bridged *central* crack problem with (a) circular delamination, and (b) elliptical delamination. (FEM analysis)

(a)

(b)

Figure 4.10: Variations of normalized bridging stress, σ_B/σ , with relative distance from the crack-tip, $x/a \geq 0$, along the centreline of the aramid-epoxy layer, for the fully bridged *double edge crack* problem with (a) circular delamination, and (b) elliptical delamination. (FEM analysis)

(a)

(b)

Figure 4.11: Variations of normalized bridging stress, σ_B/σ , with relative distance from the crack-tip, $x/a \geq 0$, along the centreline of the aramid-epoxy layer, for the fully bridged *edge crack* problem with (a) circular delamination, and (b) elliptical delamination. (FEM analysis)

(a)

(b)

Figure 4.12: Percentage difference between ARALL and GLARE, for the variations of the normalized bridging stress, σ_B/σ , with relative distance from the crack-tip, x/a , along the centreline of the fibre-epoxy layer. Fully bridged *central crack* problem with (a) circular delamination, and (b) elliptical delamination.

(a)

(b)

Figure 4.13: Percentage difference between ARALL and GLARE, for the variations of the normalized bridging stress, σ_B/σ , with relative distance from the crack-tip, x/a , along the centreline of the fibre-epoxy layer. Fully bridged *double edge crack* problem with (a) circular delamination, and (b) elliptical delamination.

(a)

(b)

Figure 4.14: Percentage difference between ARALL and GLARE, for the variations of the normalized bridging stress, σ_B/σ , with relative distance from the crack-tip, x/a , along the centreline of the fibre-epoxy layer. Fully bridged *edge crack* problem with (a) circular delamination, and (b) elliptical delamination.

(a)

(b)

Figure 4.15: Variations of normalized bridging stress, σ_B^*/σ , with relative crack length, c/W , for the ARALL central crack problem with (a) circular delamination, and (b) elliptical delamination.

(a)

(b)

Figure 4.16: Variations of normalized bridging stress, σ_B^*/σ_0 , with relative crack length, c/W , for the ARALL *double edge crack* problem with (a) circular delamination, and (b) elliptical delamination.

(a)

(b)

Figure 4.17: Variations of normalized bridging stress, σ_B^*/σ_0 , with relative crack length, $c/2W$, for the ARALL edge crack problem with (a) circular delamination, and (b) elliptical delamination.

(a)

(b)

Figure 4.18: Variations of normalized bridging stress, σ_B^*/σ_0 , with relative crack length, c/W , for the GLARE *central crack* problem with (a) circular delamination, and (b) elliptical delamination.

(a)

(b)

Figure 4.19: Variations of normalized bridging stress, σ_B^*/σ_0 , with relative crack length, c/W , for the GLARE double edge crack problem with (a) circular delamination, and (b) elliptical delamination.

(a)

(b)

Figure 4.20: Variations of normalized bridging stress, σ_B^*/σ_0 , with relative crack length, $c/2W$, for the GLARE *edge crack* problem with (a) circular delamination, and (b) elliptical delamination.

(a)

(b)

Figure 4.21: Variations of normalized crack-tip stress intensity factor, $K_I/\sigma(\pi c)^{1/2}$, with relative crack length, c/W , for the ARALL central crack problem with (a) circular delamination, and (b) elliptical delamination.

(a)

(b)

Figure 4.22: Variations of normalized crack-tip stress intensity factor, $K_I/\sigma(\pi c)^{1/2}$, with relative crack length, c/W , for the ARALL *double edge crack* problem with (a) circular delamination, and (b) elliptical delamination.

(a)

(b)

Figure 4.23: Variations of normalized crack-tip stress intensity factor, $K_I/\sigma(\pi c)^{1/2}$, with relative crack length, $c/2W$, for the ARALL edge crack problem with (a) circular delamination, and (b) elliptical delamination.

(a)

(b)

Figure 4.24: Variations of normalized crack-tip stress intensity factor, $K_I/\sigma(\pi c)^{1/2}$, with relative crack length, c/W , for the GLARE *central crack* problem with (a) circular delamination, and (b) elliptical delamination.

(a)

(b)

Figure 4.25: Variations of normalized crack-tip stress intensity factor, $K_I/\sigma(\pi c)^{1/2}$, with relative crack length, c/W , for the GLARE *double edge crack* problem with (a) circular delamination, and (b) elliptical delamination.

(a)

(b)

Figure 4.26: Variations of normalized crack-tip stress intensity factor, $K_I/\sigma(\pi c)^{1/2}$, with relative crack length, $c/2W$, for the GLARE *edge crack* problem with (a) circular delamination, and (b) elliptical delamination.

(a)

(b)

Figure 5.1: Variations of normalized bridging stress, σ_B/σ , with relative distance from the crack tip, $x/a \geq 0$, along the centreline of the aramid-epoxy layer, for the 75% partially bridged *central crack* problem with (a) circular delamination, and (b) elliptical delamination. (FEM Analysis)

Figure 5.2: Variations of normalized bridging stress, σ_B/σ , with relative distance from the crack tip, $x/a \geq 0$, along the centreline of the aramid-epoxy layer, for the 75% partially bridged *double edge crack* problem with (a) circular delamination, and (b) elliptical delamination. (FEM Analysis)

(a)

(b)

Figure 5.3: Variations of normalized bridging stress, σ_B/σ , with relative distance from the crack tip, $x/a \geq 0$, along the centreline of the aramid-epoxy layer, for the 75% partially bridged *edge crack* problem with (a) circular delamination, and (b) elliptical delamination. (FEM Analysis)

(a)

(b)

Figure 5.4: Variations of normalized bridging stress, σ_B/σ , with relative distance from the crack tip, $x/a \geq 0$, along the centreline of the aramid-epoxy layer, for the 50% partially bridged *central crack* problem with (a) circular delamination, and (b) elliptical delamination. (FEM Analysis)

(a)

(b)

Figure 5.5: Variations of normalized bridging stress, σ_B/σ , with relative distance from the crack tip, $x/a \geq 0$, along the centreline of the aramid-epoxy layer, for the 50% partially bridged *double edge crack* problem with (a) circular delamination, and (b) elliptical delamination. (FEM Analysis)

(a)

(b)

Figure 5.6: Variations of normalized bridging stress, σ_B/σ , with relative distance from the crack tip, $x/a \geq 0$, along the centreline of the aramid-epoxy layer, for the 50% partially bridged *edge crack* problem with (a) circular delamination, and (b) elliptical delamination. (FEM Analysis)

(a)

(b)

Figure 5.7: Percentage difference between 75% and 100% bridging, for the variations of the normalized bridging stress, σ_B/σ , with relative distance from the crack-tip, x/a , along the centreline of the aramid-epoxy layer. *Central crack* problem with (a) circular delamination, and (b) elliptical delamination.

(a)

(b)

Figure 5.8: Percentage difference between 75% and 100% bridging, for the variations of the normalized bridging stress, σ_B/σ , with relative distance from the crack-tip, x/a , along the centreline of the aramid-epoxy layer. *Double Edge crack problem with (a) circular delamination, and (b) elliptical delamination.*

(a)

(b)

Figure 5.9: Percentage difference between 75% and 100% bridging, for the variations of the normalized bridging stress, σ_B/σ , with relative distance from the crack-tip, x/a , along the centreline of the aramid-epoxy layer. *Edge crack* problem with (a) circular delamination, and (b) elliptical delamination.

(a)

(b)

Figure 5.10: Percentage difference between 50% and 100% bridging, for the variations of the normalized bridging stress, σ_B/σ , with relative distance from the crack-tip, x/a , along the centreline of the aramid-epoxy layer. Central crack problem with (a) circular delamination, and (b) elliptical delamination.

(a)

(b)

Figure 5.11: Percentage difference between 50% and 100% bridging, for the variations of the normalized bridging stress, σ_B/σ , with relative distance from the crack-tip, x/a , along the centreline of the aramid-epoxy layer. Double edge crack problem with (a) circular delamination, and (b) elliptical delamination.

(a)

(b)

Figure 5.12: Percentage difference between 50% and 100% bridging, for the variations of the normalized bridging stress, σ_B/σ , with relative distance from the crack-tip, x/a , along the centreline of the aramid-epoxy layer. *Edge crack* problem with (a) circular delamination, and (b) elliptical delamination.

(a)

(b)

Figure 5.13: Variations of normalized bridging stress, σ_B^*/σ_0 , with relative crack length, c/W , for the ARALL *central crack* problem containing circular and elliptical delamination with (a) 75% partial bridging, and (b) 50% partial bridging.

(a)

(b)

Figure 5.14: Variations of normalized bridging stress, σ_B^*/σ_0 , with relative crack length, c/W , for the ARALL double edge crack problem containing circular and elliptical delamination with (a) 75% partial bridging, and (b) 50% partial bridging.

Figure 5.15: Variations of normalized bridging stress, σ_B^*/σ , with relative crack length, $c/2W$, for the ARALL edge crack problem containing circular and elliptical delamination with (a) 75% partial bridging, and (b) 50% partial bridging.

Figure 5.16: Variations of normalized bridging stress, σ_B^*/σ_0 , with relative crack length, c/W , for the GLARE *central crack* problem containing circular and elliptical delamination with (a) 75% partial bridging, and (b) 50% partial bridging.

(a)

(b)

Figure 5.17: Variations of normalized bridging stress, σ_B^*/σ_0 , with relative crack length, c/W , for the GLARE double edge crack problem containing circular and elliptical delamination with (a) 75% partial bridging, and (b) 50% partial bridging.

(a)

(b)

Figure 5.18: Variations of normalized bridging stress, σ_B^*/σ_0 , with relative crack length, $c/2W$, for the GLARE2 edge crack problem containing circular and elliptical delamination with (a) 75% partial bridging, and (b) 50% partial bridging.

(a)

(b)

Figure 5.19: Variations of normalized crack-tip stress intensity factor, $K_I/\sigma(\pi c)^{1/2}$, with relative crack length, c/W , for the ARALL *central crack* problem containing circular and elliptical delamination with (a) 75% partial bridging, and (b) 50% partial bridging.

(a)

(b)

Figure 5.20: Variations of normalized crack-tip stress intensity factor, $K_I/\sigma(\pi c)^{1/2}$, with relative crack length, c/W , for the ARALL double edge crack problem containing circular and elliptical delamination with (a) 75% partial bridging, and (b) 50% partial bridging.

(a)

(b)

Figure 5.21: Variations of normalized crack-tip stress intensity factor, $K_I/\sigma(\pi c)^{1/2}$, with relative crack length, $c/2W$, for the ARALL edge crack problem containing circular and elliptical delamination with (a) 75% partial bridging, and (b) 50% partial bridging.

(a)

(b)

Figure 5.22: Variations of normalized crack-tip stress intensity factor, $K_I/\sigma(\pi c)^{1/2}$, with relative crack length, c/W , for the GLARE *central crack* problem containing circular and elliptical delamination with (a) 75% partial bridging, and (b) 50% partial bridging.

(a)

(b)

Figure 5.23: Variations of normalized crack-tip stress intensity factor, $K_I/\sigma(\pi c)^{1/2}$, with relative crack length, c/W , for the GLARE *double edge crack* problem containing circular and elliptical delamination with (a) 75% partial bridging, and (b) 50% partial bridging.

(a)

(b)

Figure 5.24: Variations of normalized crack-tip stress intensity factor, $K_I/\sigma(\pi c)^{1/2}$, with relative crack length, $c/2W$, for the GLARE edge crack problem containing circular and elliptical delamination with (a) 75% partial bridging, and (b) 50% partial bridging.

Figure 5.25: Variations of the normalized stress intensity factor, $K_I/\sigma(\pi c)^{1/2}$, with relative crack length, c/W or $c/2W$, for the ARALL (a) *central crack* problem, (b) *double edge crack* problem, and (c) *edge crack* problem. Results from the BEM and the FEM for an unbridged fibre-metal laminate.

Figure 5.26: Percentage difference between the partial bridging cases and full bridging of the normalized bridging stress, σ_B^*/σ , for the *central*, *double edge*, and *edge crack* problem. Power-law index, $n = 0.5$ and an elliptical delamination. (BEM Analysis)

(a)

(b)

Figure 5.27: Percentage difference between the partial bridging cases and full bridging of the normalized stress intensity factor, $K_I/\sigma(\pi c)^{1/2}$, for the *central*, *double edge*, and *edge crack* problem with an elliptical delamination; (a) BEM analysis, and (b) FEM analysis.

APPENDIX A

AN EXAMPLE OF THE NUMERICAL DETERMINATION OF σ_B^*

An example of the iterative process to determine the bridging stress in the last intact fibre from the crack-tip is presented here. A fully bridged edge crack problem was chosen with a relative crack length, $c/2W = 0.5$, circular delamination, and power-law index, $n = 0.5$. The bridging stress profile is taken to be of the form of equation 2.17. For the analysis, the applied stress, σ , is 1 MPa and $c = 10$ mm, which gives $d = 10$ mm for a circular delamination. The Young's modulus in the fibre direction, E_{II} , is 55 GPa.

Initialization:

- 1) Arbitrarily set the initial value of σ_B^* .
- 2) Calculate the crack-face bridging stresses to be applied at nodal points from equation 2.17 (or 2.18 for the m index).
- 3) Carry out the BEM analysis and obtain the crack opening displacement, v^* in the last intact fibre away from the crack-tip.
- 4) Compute the corresponding value of σ_B^* from equation 2.21.

Subsequent Iterations:

Once the bridging stress in the last intact fibre, σ_B^* , is known from the initialization step, steps 2 to 4 are repeated until the values of σ_B^* (or v^*) converge to set tolerances.

The following numerical example is presented for the problem described above. The results from these iterations are listed in Tables A.1 and A.2 for the crack opening displacements and bridging stresses, respectively. In these tables, the crack mouth is represented by node 1 and the crack tip by node 9.

Iteration #1:

- 1) Set $\sigma_B^* = 1.0$ MPa.
- 2) Calculate $\sigma_B(x)$.
- 3) Carry out the BEM analysis and compute v^* . For this case:

$$v^* = 0.31356 \times 10^{-3} \text{ mm}$$

- 4) Compute the corresponding using equation 2.21:

$$\begin{aligned} \sigma_B^* &= E_{11} \left(\frac{A}{A_T} \right) \left(\frac{v^*}{d} \right) \\ &= (55 \times 10^9) \left(\frac{1}{4} \right) \left(\frac{0.31356 \times 10^{-6}}{10 \times 10^{-3}} \right) \\ &= 1.078 \text{ MPa} \end{aligned}$$

Iteration #2:

To speed up the iterative process, for example, an average of the previous bridging stresses can be used for the next iteration.

$$\sigma_B^* = \frac{1}{2}(1.000 + 1.078) = 1.039 \text{ MPa}$$

With this new bridging stress, repeat steps 2 to 4 above. The results for this second BEM run give the crack opening displacement as,

$$v^* = 0.29955 \times 10^{-3} \text{ mm}$$

and the corresponding σ_B^* is calculated using equation 2.21 to be 1.030 MPa.

Iteration #3:

An average of the previous bridging stresses is,

$$\sigma_B^* = \frac{1}{2}(1.039 + 1.030) = 1.035 \text{ MPa}$$

Steps 2 to 4 are repeated and the third BEM run gives the crack opening displacement as,

$$v^* = 0.30111 \times 10^{-3} \text{ mm}$$

and the corresponding σ_B^* is calculated using equation 2.21 to be 1.035 MPa.

For this case, the final bridging stress is found to be 1.035 MPa. To obtain the corresponding three-dimensional equivalent bridging stress, this value is multiplied by the area ratio A_T/A , equal to 4 in this case, to give 4.140 MPa.

Node	x/a	σ_B (Mpa)		
		Iteration # 1	Iteration # 2	Iteration # 3
1	1.000	1.000	1.039	1.035
2	0.825	0.908	0.944	0.940
3	0.650	0.806	0.838	0.834
4	0.500	0.707	0.735	0.732
5	0.350	0.592	0.615	0.612
6	0.225	0.474	0.493	0.491
7	0.100	0.316	0.329	0.327
8	0.025	0.158	0.164	0.164
9	0	0	0	0

Table A.1: Nodal crack opening displacements (COD) on the crack face for all iterations.

Node	x/a	COD (10^{-3} mm)		
		Iteration # 1	Iteration # 2	Iteration # 3
1	1.000	0.3136	0.2996	0.3011
2	0.825	0.2703	0.2577	0.2591
3	0.650	0.2314	0.2203	0.2216
4	0.500	0.1986	0.1248	0.1903
5	0.350	0.1639	0.1563	0.1572
6	0.225	0.1306	0.1248	0.1254
7	0.100	0.0872	0.0834	0.0839
8	0.025	0.0439	0.0421	0.0423
9	0	0	0	0

Table A.2: Nodal bridging stresses, σ_B , on the crack face for all iterations.