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**Preliminary Test Approach Estimation:
Regression Model With Spherically Symmetric
Errors**

by

S.M. Mehdi Tabatabaey, B.Sc., M.Sc.

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfilment of
the requirements for the degree of
Doctor of Philosophy

Department of Mathematics and Statistics
Ottawa-Carleton Institute of Mathematics and Statistics

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Spherically Symmetric Errors**

submitted by
S.M. Mehdi Tabatabaey , B.Sc., M.Sc.
in partial fulfilment of the requirements
for the degree of Doctor of Philosophy

Professor V. Dlab, Chair
Department of Mathematics and Statistics

Professor A.K.Md.E. Saleh, Thesis Supervisor

Professor Chien-Pai Han, External Examiner

This thesis is dedicated to my parents and my wife.

ABSTRACT

In this thesis, we follow the preliminary test approach to estimation of regression and variance parameters, in a multiple regression model with errors distributed according to a law belonging to the subclass of spherically symmetrically compound normal distributions. In particular, we consider the multivariate t -distribution as error distribution. This is motivated by increasing criticism of normal theory with its often unrealistic assumptions of independence and identity, as well as being non-robust. This criticism has led researchers to find a better alternative among the class of symmetric distributions. The usefulness of multivariate t -model opposed to normal model are (i) it is very sensitive to deviations from its assumption; (ii) it places too much weight on "outliers"; (iii) it fails to use the sample beyond the first two moments and (iv) it appeals to the central limit theorem at most approximately, not exactly. Thus for the model

$$y = X\beta + e, \quad e \sim Mt(\nu : 0, \sigma_e^2 I_n)$$

under uncertain prior information given by $H_0 : H\beta = h$, we consider the properties of five estimators of β and four estimators of σ_e^2 under squared loss error. Therefore, we consider the following estimators of β :

(I) Unrestricted Estimator (UE) of β as

$$\tilde{\beta} = C^{-1}X'y, \quad C = X'X,$$

(II) Restricted Estimator (RE) of β as

$$\hat{\beta} = \tilde{\beta} - C^{-1}H'(HC^{-1}H')^{-1}(H\tilde{\beta} - h),$$

(III) Preliminary Test Estimator (PTE) of β as

$$\hat{\beta}^{PT} = \tilde{\beta} - I(\mathcal{L}_n \leq \mathcal{L}_{n,\alpha})(\tilde{\beta} - \hat{\beta}),$$

where $\mathcal{L}_n$ is the optimal test statistic for testing: $H_0 : H\beta = h$ against $H_A : H\beta \neq h$ and $\mathcal{L}_{n,\alpha}$ is the α -level critical values of the distribution of $\mathcal{L}_n$ under H_0 , $I(A)$ is the indicator of the set A .

(IV) Shrinkage Estimator (SE) of β as

$$\hat{\beta}^S = \tilde{\beta} - d\mathcal{L}_n^{-1}(\tilde{\beta} - \hat{\beta}), \quad 0 < d < 1$$

and finally

(V) The Positive-rule Shrinkage Estimator (PRSE) of β as

$$\hat{\beta}^{S+} = \hat{\beta}^S - (1 - d\mathcal{L}_n^{-1})I(\mathcal{L}_n \leq d)(\tilde{\beta} - \hat{\beta}).$$

We show that under quadratic loss and MSE (mean-squared error) criteria, the performance property of these estimators is robust in the class of multivariate t -errors which is determined by the degrees of freedom and the dimension of the $\mathbf{H}$ -matrix.

We extend the study of the estimation of β when the uncertain prior information is stochastic i.e., $\mathbf{r} = \mathbf{R}\beta + \mathbf{v}$ where $\mathbf{v}$ has the same distribution as $\mathbf{e}$ and verify the same robustness property of the five estimators.

Thirdly, in addition to the uncertain prior information $\mathbf{H}\beta = \mathbf{h}$ (non-stochastic) we consider the case $\mathbf{C}$ being ill-conditioned and study the performance properties of the ridge-type estimators under quadratic loss and MSE criterion.

Finally, we consider the estimation of the variance σ_e^2 of the error distribution. Here, we define four relevant estimators and show that the Stein-type estimators (which is also a preliminary test type estimator) dominate the usual estimators as well as preliminary test estimators. Further, the performance property of the estimators is robust under the multivariate t -model.

Main contribution in this thesis relate to the study of the properties of the shrinkage and positive-rule shrinkage estimators of β as well as the Stein-type estimator of σ_e^2 under multivariate t -errors together with their robustness properties.

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Chapter 1

Introduction

In problems of statistical inference, the use of prior information on some (all) of the parameters in a statistical model usually leads to an improved inference procedure for other (all) parameters of interest. The prior information may be known or uncertain. The known prior information is generally incorporated in the model in the form of a constraint, giving rise to restricted models. The analysis of such restricted models leads to an improved statistical procedure than the unrestricted model case when such constraints hold. The estimators resulting from restricted (unrestricted) model is known as the restricted (unrestricted) estimators of the parameters of the model. The validity and efficiency of the restricted model analysis retains its properties over the restricted parameter space induced by the constraint, while the same holds for the unrestricted model analysis over the entire parameter space. Therefore, the results of an analysis of the restricted and unrestricted models need to weigh loss of efficiency against the validity of the constraints in order to choose between two extreme inference techniques. Choosing restricted estimation may be unjustified when a full confidence may not be put in the prior information.

When we encounter problems with uncertain prior information in a statistical model, we may impose some prior information (which may be available from some extraneous considerations) on the model. However, there may be reasons to suspect their validity in recommending the restricted model analysis. This uncertain prior information considered in the form of a constraint may be regarded as "nuisance parameter" in the statistical inference of the full model. To eliminate this uncertainty of the prior information in the model one could naturally

follow "Fisher's recipe" of elimination of the nuisance parameter by a preliminary test on the validity of the uncertain prior information in the form of a parametric constraint and then choose either the restricted or the unrestricted inference depending on the fate of the preliminary test. This theme brings home a compromise inference procedure between two extremes. (It could be looked upon as a happy marriage between two extremes.) This line of thought was pioneered by Bancroft (1944) in the study of the consequences of incorporating a preliminary test in pooling several estimates of variance in an analysis of variance table. This idea of preliminary test estimation has been developed further in a series of papers by Bancroft (1944), Bancroft and Han [(1968, 1983)] (see Bancroft and Han 1980 for an excellent review) among others. In 1956, Charles Stein discovered an unexpected but surprising result (in the form of a counter example) that the sample mean was an inadmissible estimator of the mean of a normal distribution under squared error loss in three or higher dimension. Afterwards, James and Stein (1961) provided an explicit estimator dominating the sample mean. This work was a shock to the whole statistical world and profoundly affected the course of statistical theory. The James-Stein work attacked the utility of all good properties of statistical estimators; least square, maximum likelihood, unbiasedness, invariance and the minimax principle. The usual estimator of sample mean is uniformly minimum covariance unbiased, maximum likelihood, best invariant, and minimax, and yet its performance in terms of expected squared error loss is demonstratively inferior to that of James-Stein estimator. In the last fifty years or so, statistical science has witnessed a fundamental growth of the literature in this fruitful area of research. See Berger (1985) for a detail account of some of these development, mostly related to the classical multinormal and some specific types of exponential families of distributions. For further development in multivariate Stein-type estimation see Ghosh and Lin (1986).

Estimators which uniformly improve on standard estimators are usually called Stein-type estimators as a gesture of honor to Professor Charles Stein. Basically, Stein-type estimators may be obtained via a decision theoretic approach for the normal theory models and an empirical Bayes approach as proposed by Efron and Morris (1973). However, Casella (1985) pointed out that the Stein-type estimators need appropriate test statistics for testing out the adequacy of the uncertain prior information that is incorporated in the actual formation of the estimators. Stein-type estimators adjust the unrestricted estimators by an amount of the difference (= the restricted estimator - the unrestricted estimator) multiplied by a shrinkage factor times the reciprocal of the relevant test-statistics (for testing the uncertain prior information). Generally, these test-statistics measures the normalized distance between the restricted and unrestricted estimators and follow a non-central chi-squared (or non-central F) distribution with appropriate degrees of freedom. The risks of Stein-type estimators (including other estimators) depend on the non-centrality parameter of the chi-square (or F) distribution.

On this ground, Saleh and Sen (1978-86) took the frequentist path of improving standard estimators regarding non-parametric problems on location and linear regression parameters by developing Stein-type estimators. The path they have chosen actually combined the idea of preliminary test approach and the concept of shrinkage of the James-Stein-type estimation. See Saleh and Sen (1984) for an overview. In this thesis, we shall follow this route to develop estimator for the linear model

$$y = X\beta + e \quad (1.1)$$

where y is an $(n \times 1)$ vector of observations, X is an $(n \times p)$ matrix of full rank p , $\beta = (\beta_1, \dots, \beta_p)'$ is the vector of p parameters and $e = (e_1, \dots, e_n)'$ is an $(n \times 1)$

random error vector distributed according to a law belonging to the class of spherical compound normal distributions. Particularly, we consider the multivariate t -distribution as error distribution. This is due to increasing criticism of normal theory with its often unrealistic assumptions of independence and identity as well as being non-robust. This has led researchers to find a better alternative among the class of symmetric distributions. In many cases, the theoretical advantages and mathematical conveniences are negligible compared to the price paid in terms of loss of efficiency and precision under the normal theory. The concern was voiced by Fisher (1956, p.133) in the following words, "It is a noteworthy peculiarity of inductive inference that comparatively slight differences in mathematical specification of a problem may have logically important effects on the inference possible". Not surprisingly, Fisher (1960, p.37) analyzed Darwin's data under normal theory as well as under another symmetric distribution. Fraser (1979) illustrated the fact that the results derived under the t -family are essentially valid for the normal model but not the vice-versa. Prucha and Kelejian (1984) categorically discussed the usefulness of the Student's t -model opposed to the normal model as the latter (i) is very sensitive to deviations from its assumption (ii) places too much weight on "outliers", (iii) fails to use the sample beyond its first two moments and (iv) appeals to central limit theorem at most approximately, not exactly. Zellner (1976) first provided the classical as well as Bayesian analyses for the regression model with multivariate t -errors. In this thesis, we follow the preliminary test approach estimation of β as well as the variance σ_e^2 when the errors follow a multivariate t -distribution under uncertain prior information, $H\beta = h$. For this, we need a preliminary test of H_0 . Let $\mathcal{L}_n$ be such an optimal test for testing

$$H_0 : H\beta = h \quad \text{Vs} \quad H_A : H\beta \neq h. \quad (1.2)$$

We may then define the following estimators of β .

- (1) **Unrestricted Estimator (UE).** Let $\hat{\beta}$ denote the estimator of β using the full model.
- (2) **Restricted Estimator (RE).** Let $\tilde{\beta}$ denote the restricted estimator of β under the restriction $H\beta = h$. Then the restricted estimator (RE) performs better than the Unrestricted Estimator (UE) when $H\beta = h$ holds. However, as $H\beta$ differs from h , the restricted estimator (RE) may be considerably biased, inefficient and inconsistent while the performance of the unrestricted estimator (UE) remains steady over such departure. For this reason, it is desirable to develop estimator which is a compromise between the unrestricted estimator (UE) and the restricted estimator (RE) under uncertain prior information $H\beta = h$. We do this by using the preliminary test (PT) approach.
- (3) **Preliminary Test Estimator (PTE).** Let $\mathcal{L}_{n,\alpha}$ be the α -level of the critical value of $\mathcal{L}_n$, then the PTE of β is defined by

$$\hat{\beta}^{PT} = \hat{\beta} + [1 - I(\mathcal{L}_n \leq \mathcal{L}_{n,\alpha})](\tilde{\beta} - \hat{\beta}) \quad (1.3)$$

where $I(A)$ is the indicator function of the set A . This type of estimator was proposed by Bancroft (1944). In this case the estimator $\hat{\beta}$ or $\tilde{\beta}$ is chosen according as H_0 is accepted or rejected. The performance of preliminary test estimator (PTE) depends on the size of the test α ($0 < \alpha < 1$). Though its performance is reasonable compared with the unrestricted estimator (UE) and the restricted estimator (RE), the discontinuous nature of the indicator function loses its total grip on the performance. Hence, we go to another choice of a function of $\mathcal{L}_n$.

- (4) **Shrinkage Estimator (SE).** A simple form of a shrinkage estimator of β may be defined as

$$\begin{aligned}\hat{\beta}^S &= \hat{\beta} + (1 - d\mathcal{L}_n^{-1})(\tilde{\beta} - \hat{\beta}) \\ &= \tilde{\beta} - d\mathcal{L}_n^{-1}(\tilde{\beta} - \hat{\beta})\end{aligned}\tag{1.4}$$

where d is a positive constant generally called the shrinkage constant and depends on the dimensional of β and the matrix $\mathbf{H}$. In this case, the SE is based on the classical Stein-rule estimator. Note that the shrinkage estimator (SE) is a smooth version of the preliminary test estimator (PTE). The two compromised estimators are similar looking where the test statistic $\mathcal{L}_n$ adjust the estimators to departure from H_0 . When $\mathcal{L}_n$ is large, the preliminary test estimator, the shrinkage estimator and the unrestricted estimators are essentially equivalent, while for small values of $\mathcal{L}_n$ there is a difference.

- (5) **Positive-Rule Shrinkage Estimator (PRSE).** A simple form of a positive-rule shrinkage estimator of β may be defined as

$$\begin{aligned}\hat{\beta}^{S+} &= \hat{\beta} + (1 - d\mathcal{L}_n^{-1})I(\mathcal{L}_n > d)(\tilde{\beta} - \hat{\beta}) \\ &= \hat{\beta}^S - (1 - d\mathcal{L}_n^{-1})I(\mathcal{L}_n \leq d)(\tilde{\beta} - \hat{\beta})\end{aligned}\tag{1.5}$$

where $I(A)$ is the indicator function of the set A and d is a positive constant generally called the shrinkage constant a d depends on the dimensional of β and the matrix $\mathbf{H}$. The performance of positive-rule shrinkage estimator (PRSE) depends on the indicator which depends on the shrinkage constant d ($0 < d < 1$). In this case, the PRSE is based on the James and Stein positive-rule estimator. Note that the positive-rule shrinkage estimator (PRSE) dominates the shrinkage estimator (SE). PRSE can never change the sign of maximum likelihood estimates and for this reason has been called the

Stein positive-rule estimator. Thus, one studies critically the performance characteristics of the five estimators using the following criterion of decision:

Consider the loss function

$$L(\beta^*; \mathbf{W}) = (\beta^* - \beta)' \mathbf{W} (\beta^* - \beta) \quad (1.6)$$

which is a quadratic form with a positive definite matrix $\mathbf{W}$. Then, the risk function for the decision rule β^* is defined by

$$\begin{aligned} R(\beta^*; \mathbf{W}) &= E_{\beta} (\beta^* - \beta)' \mathbf{W} (\beta^* - \beta) \\ &= \text{tr}(\mathbf{W}\mathbf{M}) \end{aligned} \quad (1.7)$$

where

$$\mathbf{M} = E[(\beta^* - \beta)(\beta^* - \beta)']$$

is the mean squared error matrix for the decision rule β^* .

In assessing the performance of various estimators represented by β^* , one may study the risks of the estimators or MSE-matrices. In this thesis we shall do so for the estimators of β defined earlier. Relative performance pictures of the estimators are given in Chapter 2.

In Chapter 3, we consider the model (1.1) with the same error distribution but the uncertain prior information is now stochastic in nature, namely,

$$\mathbf{r} = \mathbf{R}\beta + \mathbf{v}, \quad (1.8)$$

where $\mathbf{v}$ has the distribution similar to $\mathbf{e}$. We develop estimation of β under the above uncertain stochastic constraint.

Next, we develop the estimation of β under the uncertain prior information $H\beta = h$ except that now $C = X'X$ is ill-conditioned. This leads to the combination of ridge-regression principle and preliminary test approach. Thus, we define five new ridge type estimators and present the performance characteristics in Chapter 4.

Finally, we consider the related problem of the study of the estimation of variance of the error distribution under uncertain prior information $H_0 : H\beta = h$ as specified earlier. For this again, we follow the preliminary test approach of estimation. Accordingly, we consider the following estimators

(i) Unrestricted unbiased estimators (UUE) of σ_e^2

$$\tilde{\sigma}_e^2 = S_e^2 = \frac{(y - X\tilde{\beta})'(y - X\tilde{\beta})}{m}, \quad m = n - p \quad (1.9)$$

(ii) Restricted unbiased estimator (RUE)

$$\hat{\sigma}_e^2 = S_e^{*2} = \frac{(y - X\hat{\beta})'(y - X\hat{\beta})}{m + q}, \quad m = n - p \quad (1.10)$$

(iii) Preliminary test estimator (PTE) of σ_e^2

$$\hat{\sigma}_e^{PT} = \hat{\sigma}_e^2 + [1 - I(\mathcal{L}_n \leq F_\alpha)](\tilde{\sigma}_e^2 - \hat{\sigma}_e^2), \quad (1.11)$$

(iv) The Stein-type estimator (SE) of σ_e^2

$$\hat{\sigma}_e^{2S} = \frac{m}{m+2} I\left(\mathcal{L}_n > \frac{m}{m+2}\right) S_e^2 + \frac{m(1 + \frac{q}{m}\mathcal{L}_n)}{m+q+2} I\left(\mathcal{L}_n \leq \frac{m}{m+2}\right) S_e^2, \quad m = n - p. \quad (1.12)$$

Notice that both PTE and SE are preliminary test type estimator where PTE depends on α while SE does not.

To study the performances of these estimators we consider the quadratic loss function

$$L(\sigma_e^{*2}; \sigma_e^2) = \frac{1}{\sigma_e^4} (\sigma_e^{*2} - \sigma_e^2)^2 \quad (1.13)$$

and risk

$$R(\sigma^{*2}, \sigma_e^2) = \frac{1}{\sigma_e^4} E(\sigma_e^{*2} - \sigma_e^2)^2 \quad (1.14)$$

Relative performance picture of the estimators are given in details in Chapter 5.

This thesis generalizes the estimation scope of the regression model with normal errors to a sub-class of spherically symmetric distribution models establishing the robustness of performance properties of various estimators described earlier extending the contents of the book by Judge and Bock (1978) which deals with normal models only.

Chapter 2

Estimation of Regression Parameters Under Uncertain Prior Sub-Space Restriction with Spherically Symmetric Distributions

2.1 Introduction

Consider the linear model

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{e} \quad (2.1)$$

where $\mathbf{y}$ is an $(n \times 1)$ vector of observations, $\mathbf{X}$ is an $(n \times p)$ matrix of full rank p , $\boldsymbol{\beta} = (\beta_1, \dots, \beta_p)'$ is the vector of p parameters and $\mathbf{e} = (e_1, \dots, e_n)'$ is the $(n \times 1)$ error vector distributed according to a law belonging to the class of spherical compound normal distributions with $E(\mathbf{e}) = \mathbf{0}$ and $E(\mathbf{e}\mathbf{e}') = \sigma_e^2 \mathbf{I}_n$, $\mathbf{I}_n$ is the n -dimensional identity matrix, where σ_e^2 is the common variance of e_i 's ($i = 1, \dots, n$). This class is a subclass of the family of spherically symmetric distributions (SSD's) which can be expressed as a variance mixture of normal distributions, that is,

$$f(\mathbf{e}) = \int_0^\infty f_n(\mathbf{e}|\tau)g(\tau)d\tau \quad (2.2)$$

where $f(\mathbf{e})$ is the probability density function (pdf) of $\mathbf{e}$, $f_n(\mathbf{e}|\tau)$ is the pdf of $N_n(\mathbf{0}, \tau^2 \mathbf{I}_n)$ and $g(\tau)$ is the pdf of τ with support $[0, \infty)$. In this case, $\sigma_e^2 = E(\tau^2)$ and we write $\mathbf{e} \sim SSD_n(\mathbf{0}, E(\tau^2)\mathbf{I}_n)$. Well-known members of SSD class are the normal and multivariate student's t -distributions, the latter includes the Cauchy distribution.

Multivariate t -distribution (Mt) arises, if $g(\tau)$ is an inverted Gamma (IG) density with, say, scale σ^2 and degrees of freedom ν , denoted by $IG(\nu, \sigma^2)$ and is given by

$$g(\tau) = 2[\Gamma(\nu/2)]^{-1} \left(\frac{\nu\sigma^2}{2} \right)^{\nu/2} \tau^{-(\nu+1)} e^{-\frac{\nu\sigma^2}{2\tau^2}}, 0 < \nu, \sigma, \tau < \infty \quad (2.3)$$

Then, the multivariate t -distribution is given by the pdf

$$f(\mathbf{e}) = \frac{\Gamma\left(\frac{n+\nu}{2}\right)}{(\pi\nu)^{n/2}\Gamma\left(\frac{\nu}{2}\right)\sigma^n} \left(1 + \frac{\mathbf{e}'\mathbf{e}}{\nu\sigma^2}\right)^{-\frac{(n+\nu)}{2}}; \nu, \sigma > 0, -\infty < e_i < \infty \quad (2.4)$$

The mean of $\mathbf{e}$ is $\mathbf{0}$ and $E(\mathbf{e}\mathbf{e}') = \frac{\nu}{\nu-2}\sigma^2\mathbf{I}_n = \sigma_e^2\mathbf{I}_n$ for $\nu > 2$. The marginal distributions are univariate student t -distributions. For $\nu = 1$, the pdf is Cauchy and as $\nu \rightarrow \infty$, the pdf approaches normality.

Our primary object of this chapter is to estimate β , when the error distribution belongs to (2.4) and it is suspected that β may be restricted to the subspace defined by

$$H_0 : \mathbf{H}\beta = \mathbf{h} \quad (2.5)$$

where $\mathbf{H}$ is a $(q \times p)$ known matrix of rank q and $\mathbf{h}$ is a q vector with known elements. If H_0 is true then one may obtain estimators of β under (2.5) and with error distribution given by (2.4). However, if H_0 is suspicious, then one may carry out a test on H_0 using the statistic $\mathcal{L}_n$, defined by

$$\mathcal{L}_n = \frac{(\mathbf{H}\tilde{\beta} - \mathbf{h})'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}(\mathbf{H}\tilde{\beta} - \mathbf{h})}{qS_e^2} \quad (2.6)$$

with

$$S_e^2 = \frac{(\mathbf{y} - \mathbf{X}\tilde{\beta})'(\mathbf{y} - \mathbf{X}\tilde{\beta})}{n - p} \quad (2.7)$$

It may be shown that $\mathcal{L}_n$ is a likelihood ratio test (LRT). A proof of this statement will be given later in this chapter.

With this background at hand we shall proceed with the discussion of various estimators of β and σ_e^2 under multivariate t -errors in the linear model (2.1).

2.2. Estimation of β and σ_e^2 and testing of hypotheses

2.2.1. Estimation of β and σ_e^2

First, we note that the maximum likelihood estimator (MLE) of β under (2.4) is

$$\tilde{\beta} = (X'X)^{-1}X'y = C^{-1}X'y, \text{ where } C = X'X \quad (2.8)$$

which is also the least squares estimator (LSE) of β . See for example Zellner (1976) and Singh (1991). Similarly, the MLE of σ^2 is

$$\hat{\sigma}^2 = \frac{1}{n}(y - X\tilde{\beta})'(y - X\tilde{\beta}) \quad (2.9)$$

Theorem 2.1. Suppose $\nu > 2$. Then $S_e^2 = \frac{(y - X\tilde{\beta})'(y - X\tilde{\beta})}{n-p}$ is an unbiased estimator of $\sigma_e^2 = \frac{\nu}{\nu-2}\sigma^2$, $\nu > 2$.

Proof. We have the following identity

$$(y - X\tilde{\beta})'(y - X\tilde{\beta}) = (\beta - \tilde{\beta})'X'X(\beta - \tilde{\beta}) + e'e + 2(\beta - \tilde{\beta})'X'e \quad (2.10)$$

Taking expectation on both sides of (2.10), we get

$$E[(y - X\tilde{\beta})'(y - X\tilde{\beta})] = p\sigma_e^2 + n\sigma_e^2 - 2p\sigma_e^2 = (n - p)\sigma_e^2 \quad (2.11)$$

Thus, the unbiased estimator of σ_e^2 is

$$\hat{\sigma}_e^2 = S_e^2 = (n - p)^{-1}(y - X\tilde{\beta})'(y - X\tilde{\beta}). \quad (2.12)$$

From (2.8) and (2.12), and for $\nu > 1$,

$$E(\tilde{\beta}) = \beta, \text{ since } E(e) = 0 \quad (2.13)$$

Also, for $\nu > 2$, the covariance matrix for $\tilde{\beta}$ is

$$\text{Var}(\tilde{\beta}) = C^{-1}\sigma_e^2, \text{ since } \text{var}(e) = E(ee') = \sigma_e^2 I_n \quad (2.14)$$

Note that as $\nu \rightarrow \infty$, $\mathbf{e} \rightarrow N_n(\mathbf{0}, \sigma^2 I_n)$ and $\text{Var}(\tilde{\beta}) \rightarrow \mathbf{C}^{-1}\sigma^2$. Thus, for small and moderate values of ν , the variances of the elements of $\tilde{\beta}$ are inflated relative to those for large values of ν , in which case the multivariate student t -distribution of $\tilde{\beta}$ is close to $N_p(\beta, \mathbf{C}^{-1}\sigma^2)$.

The restricted estimator (RE) of β under $\mathbf{H}\beta = \mathbf{h}$ is given by

$$\hat{\beta} = \tilde{\beta} - \mathbf{C}^{-1}\mathbf{H}'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}(\mathbf{H}\tilde{\beta} - \mathbf{h}), \quad \mathbf{C} = \mathbf{X}'\mathbf{X} \quad (2.15)$$

From (2.13) and (2.14), we get

$$\begin{aligned} E(\hat{\beta}) &= \beta - \mathbf{C}^{-1}\mathbf{H}'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}(\mathbf{H}\beta - \mathbf{h}) \\ &= \beta, \text{ under } \mathbf{H}\beta = \mathbf{h}. \end{aligned} \quad (2.16)$$

Also, the covariance matrix for $\hat{\beta}$ is given by

$$\text{Var}(\hat{\beta}) = [\mathbf{C}^{-1} - \mathbf{C}^{-1}\mathbf{H}'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}\mathbf{H}\mathbf{C}^{-1}]\sigma_e^2 \quad (2.17)$$

Theorem 2.2. Suppose $\nu > 2$. Then Under $\mathbf{H}\beta = \mathbf{h}$, $S_e^{*2} = (n - p + q)^{-1}(\mathbf{y} - \mathbf{X}\hat{\beta})'(\mathbf{y} - \mathbf{X}\hat{\beta})$ is an unbiased estimator of σ_e^2 .

Proof. We have the following identity

$$(\mathbf{y} - \mathbf{X}\hat{\beta})'(\mathbf{y} - \mathbf{X}\hat{\beta}) = (\mathbf{y} - \mathbf{X}\tilde{\beta})'(\mathbf{y} - \mathbf{X}\tilde{\beta}) + (\mathbf{H}\tilde{\beta} - \mathbf{h})'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}(\mathbf{H}\tilde{\beta} - \mathbf{h}). \quad (2.18)$$

Taking expectation on both sides of (2.18), we get

$$\begin{aligned} E_\tau \left\{ \tau^2 E \left[\frac{(\mathbf{y} - \mathbf{X}\hat{\beta})'(\mathbf{y} - \mathbf{X}\hat{\beta})}{\tau^2} \middle| \tau \right] \right\} &= E_\tau \left\{ \tau^2 E \left[\frac{(\mathbf{y} - \mathbf{X}\tilde{\beta})'(\mathbf{y} - \mathbf{X}\tilde{\beta})}{\tau^2} \middle| \tau \right] \right\} \\ &\quad + E_\tau \left\{ \tau^2 E \left[\frac{(\mathbf{H}\tilde{\beta} - \mathbf{h})'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}(\mathbf{H}\tilde{\beta} - \mathbf{h})}{\tau^2} \middle| \tau \right] \right\} \\ &= (n - p + q + \Delta^*)\sigma_e^2, \end{aligned}$$

where $\Delta^* = \theta/\sigma_e^2$ and $\theta = (\mathbf{H}\beta - \mathbf{h})'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}(\mathbf{H}\beta - \mathbf{h})$. Therefore,

$$\hat{\sigma}_e^2 = S_e^{*2} = (n - p + q)^{-1}(\mathbf{y} - \mathbf{X}\hat{\beta})'(\mathbf{y} - \mathbf{X}\hat{\beta}) \quad (2.19)$$

is unbiased only when the restriction $\mathbf{H}\beta = \mathbf{h}$ holds.

2.2.2. Test of Hypothesis

In this section, we obtain the test statistic for the hypothesis $H_0 : \mathbf{H}\beta = \mathbf{h}$ and derive its distribution under alternative $H_A : \mathbf{H}\beta \neq \mathbf{h}$.

Theorem 2.3. Suppose $\nu > 2$. If $\mathbf{e} \sim Mt(\nu : \mathbf{0}, \sigma_e^2 \mathbf{I}_n)$, then the likelihood ratio test statistic for $H_0 : \mathbf{H}\beta = \mathbf{h}$ is given by

$$\mathcal{L}_n = \frac{(\mathbf{H}\hat{\beta} - \mathbf{h})'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}(\mathbf{H}\hat{\beta} - \mathbf{h})}{qS_e^2} \quad (2.20)$$

Proof. Using (2.4), (2.9), (2.15) and (2.18), and also the definition of the likelihood ratio test,

$$\lambda = L_0/L_1 \quad (2.21)$$

where L_i , $i = 0, 1$, is the largest value which the likelihood function takes in region Ω_i , $i = 0, 1$, where $\Omega_1 = \{(\beta, \sigma_e); \beta \in R^p, \sigma_e > 0\}$, and $\Omega_0 = \{(\beta, \sigma_e); \mathbf{H}\beta = \mathbf{h}, \sigma_e > 0\}$. Then,

$$\begin{aligned} L_0 &= \frac{\Gamma\left(\frac{n+\nu}{2}\right)}{(\pi\nu)^{n/2}\Gamma\left(\frac{\nu}{2}\right)(\hat{\sigma}_e^2)^{n/2}} \left[1 + \frac{(\mathbf{y} - \mathbf{X}\hat{\beta})'(\mathbf{y} - \mathbf{X}\hat{\beta})}{\nu\hat{\sigma}_e^2}\right]^{-\left(\frac{n+\nu}{2}\right)} \\ &= \frac{\Gamma\left(\frac{n+\nu}{2}\right)}{(\pi\nu)^{n/2}\Gamma\left(\frac{\nu}{2}\right)(\hat{\sigma}_e^2)^{n/2}} \left[1 + \frac{n-p+q}{\nu}\right]^{-\left(\frac{n+\nu}{2}\right)} \end{aligned}$$

and

$$\begin{aligned} L_1 &= \frac{\Gamma\left(\frac{n+\nu}{2}\right)}{(\pi\nu)^{n/2}\Gamma\left(\frac{\nu}{2}\right)(\hat{\sigma}_e^2)^{n/2}} \left[1 + \frac{(\mathbf{y} - \mathbf{X}\hat{\beta})'(\mathbf{y} - \mathbf{X}\hat{\beta})}{\nu\hat{\sigma}_e^2}\right]^{-\left(\frac{n+\nu}{2}\right)} \\ &= \frac{\Gamma\left(\frac{n+\nu}{2}\right)}{(\pi\nu)^{n/2}\Gamma\left(\frac{\nu}{2}\right)(\hat{\sigma}_e^2)^{n/2}} \left[1 + \frac{n-p}{\nu}\right]^{-\left(\frac{n+\nu}{2}\right)}, \end{aligned}$$

giving

$$\lambda = L_0/L_1 = a \left[\frac{(y - X\hat{\beta})'(y - X\hat{\beta})}{(y - X\hat{\beta})'(y - X\hat{\beta}) + (H\hat{\beta} - h)'(HC^{-1}H')^{-1}(H\hat{\beta} - h)} \right]^{n/2}$$

$$= a \left(\frac{1}{1 + \frac{q}{(n-p)}\mathcal{L}_n} \right)^{n/2}$$

where $a = \left(\frac{n-p+q}{n-p} \right)^{n/2} \left(\frac{(n-p+\nu)}{(n-p+q+\nu)} \right)^{\frac{n+p}{2}}$. Therefore,

$$\lambda^{2/n} = a^{2/n} \left(\frac{1}{1 + \frac{q}{(n-p)}\mathcal{L}_n} \right) \quad (2.22)$$

λ is a decreasing function of $\mathcal{L}_n$. Thus $\lambda < \lambda_0$ is equivalent to $\mathcal{L}_n > F_\alpha$ for certain constant F_α . That is, the likelihood ratio test statistic is equivalent to the test statistic

$$\mathcal{L}_n = \frac{(H\hat{\beta} - h)'(HC^{-1}H')^{-1}(H\hat{\beta} - h)}{qS_e^2}$$

Therefore, the theorem is proved.

Theorem 2.4. Suppose $\nu > 2$. If $e \sim Mt(\nu : 0, \sigma_e^2 \mathbf{I}_n)$, then

$$\mathcal{L}_n = \frac{(H\hat{\beta} - h)'(HC^{-1}H')^{-1}(H\hat{\beta} - h)}{qS_e^2}$$

has the following pdf,

$$g_{q,n-p}(\mathcal{L}_n; \Delta^*, \nu)$$

$$= \sum_{r=0}^{\infty} \left(\frac{q}{n-p} \right)^{\frac{q}{2}+r} \frac{\Gamma\left(\frac{q}{2}+r\right)\Gamma\left(\frac{n-p+q}{2}+r\right)\left(\frac{\Delta^*}{\nu-2}\right)^r \mathcal{L}_n^{\frac{q}{2}+r-1}}{\Gamma(r+1)\Gamma\left(\frac{q}{2}\right)\Gamma\left(\frac{q}{2}+r\right)\Gamma\left(\frac{n-p}{2}\right)\left(1+\frac{\Delta^*}{\nu-2}\right)^{\nu/2+r}\left(1+\frac{q}{n-p}\mathcal{L}_n\right)^{\frac{n-p+q}{2}+r}} \quad (2.23)$$

Proof. Let $\tau \sim IG(\sigma^2, \nu)$ and $e|\tau \sim N_n(0, \tau^2 \mathbf{I}_n)$. Then $Y = \frac{n\hat{\beta}^2}{\tau^2} = \frac{(y-X\hat{\beta})'(y-X\hat{\beta})}{\tau^2} | \tau \sim \chi_{n-p}^2(0)$ and

$$X = \frac{(H\hat{\beta} - h)'(HC^{-1}H')^{-1}(H\hat{\beta} - h)}{\tau^2} | \tau \sim \chi_q^2(\Delta)$$

are conditionally independent with

$$\Delta = \frac{(\mathbf{H}\boldsymbol{\beta} - \mathbf{h})'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}(\mathbf{H}\boldsymbol{\beta} - \mathbf{h})}{\tau^2} = \frac{\theta}{\tau^2}.$$

Thus,

$$\mathcal{L}_n = \frac{(\mathbf{H}\tilde{\boldsymbol{\beta}} - \mathbf{h})'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}(\mathbf{H}\tilde{\boldsymbol{\beta}} - \mathbf{h})}{qS_e^2} = \frac{n-p}{q} \frac{X}{Y} \Big|_{\tau} \sim f_{q,n-p}(\Delta).$$

Therefore, we have

$$\begin{aligned} g_{q,n-p}(\mathcal{L}_n; \Delta^*, \nu) \\ &= \sum_{r=0}^{\infty} \left(\frac{q}{n-p} \right)^{q/2+r} \frac{\Gamma\left(\frac{n-p+q}{2} + r\right) \mathcal{L}_n^{q/2+r-1} \int_0^{\infty} e^{-\theta/2\tau^2} \left(\frac{\theta}{2\tau^2} \right)^r g(\tau) d\tau}{\Gamma(r+1)\Gamma(q/2+r)\Gamma\left(\frac{n-p}{2}\right) \left(1 + \frac{q}{n-p}\mathcal{L}_n\right)^{\frac{n-p+q}{2}+r}} \\ &= \sum_{r=0}^{\infty} \left(\frac{q}{n-p} \right)^{q/2+r} \frac{\Gamma\left(\frac{q}{2} + r\right) \Gamma\left(\frac{n-p+q}{2} + r\right) \left(\frac{\Delta^*}{\nu-2}\right)^r \mathcal{L}_n^{q/2+r-1}}{\Gamma(r+1)\Gamma\left(\frac{q}{2}\right) \Gamma\left(\frac{q}{2} + r\right) \Gamma\left(\frac{n-p}{2}\right) \left(1 + \frac{\Delta^*}{\nu-2}\right)^{\nu/2+r} \left(1 + \frac{q}{n-p}\mathcal{L}_n\right)^{\frac{n-p+q}{2}+r}} \end{aligned}$$

where

$$\Delta^* = \frac{(\mathbf{H}\boldsymbol{\beta} - \mathbf{h})'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}(\mathbf{H}\boldsymbol{\beta} - \mathbf{h})}{\sigma_e^2} = \theta/\sigma_e^2, \quad \sigma_e^2 = \frac{\nu}{\nu-2}\sigma^2, \quad \nu > 2.$$

Hence the theorem is proved.

Corollary 2.4.1. The exact distribution of $\mathcal{L}_n$ under H_0 is

$$f_{q,n-p}(\mathcal{L}_n) = \left(\frac{q}{n-p} \right)^{q/2} \frac{\Gamma\left(\frac{n-p+q}{2}\right)}{\Gamma(q/2)\Gamma\left(\frac{n-p}{2}\right)} \frac{\mathcal{L}_n^{q/2-1}}{\left(1 + \frac{q}{n-p}\mathcal{L}_n\right)^{\frac{n-p+q}{2}}} \quad (2.24)$$

which is a central F distribution with $(q, n-p)$ degrees of freedom.

Proof. This corollary follows from Theorem 2.4 when Δ^* is zero.

Note: It is important to observe that even though the elements of $\mathbf{e}$ have the nonnormal pdf in (2.4) and are not independent, tests based on the usual F -statistic remain valid (i.e. $\mathcal{L}_n$). Under H_0 , $\mathcal{L}_n$ follows a central F -distribution

with $(q, n - p)$ degrees of freedom (2.24) while under $H_A : \mathbf{H}\beta \neq \mathbf{h}$, the pdf of $\mathcal{L}_n$ is equation (2.23). In this connection see also Sutradhar (1988) for the power of the F -test. The pdf in (2.23) will be frequently used to compute the risk of the various estimators we are going to propose.

2.2.3 Other Estimators of β

Following Saleh and Han (1990) and Ali and Saleh (1991), we define the preliminary test estimator (PTE) of β as a convex combination of $\tilde{\beta}$ and $\hat{\beta}$

$$\hat{\beta}^{PT} = \hat{\beta} + [1 - I(\mathcal{L}_n \leq F_\alpha)](\tilde{\beta} - \hat{\beta}), \quad (2.25)$$

where $I(A)$ is the indicator of the set A and F_α is the upper 100α percentile of the central F distribution with $(q, n - p)DF$.

The preliminary test estimator has the disadvantage that it depends on α ($0 < \alpha < 1$), the level of significance and also it yields the extreme results, namely, $\tilde{\beta}$ and $\hat{\beta}$ depending on the outcome of the test. What about the intermediate values between $\tilde{\beta}$ and $\hat{\beta}$? The answer is resolved by the Stein-type estimator.

For the Stein-type shrinkage estimator (SE) of β , we define

$$\begin{aligned} \hat{\beta}^S &= \hat{\beta} + (1 - d\mathcal{L}_n^{-1})(\tilde{\beta} - \hat{\beta}) \\ &= \tilde{\beta} - d\mathcal{L}_n^{-1}(\tilde{\beta} - \hat{\beta}), \end{aligned} \quad (2.26)$$

where

$$d = \frac{(q - 2)(n - p)}{q(n - p + 2)} \quad \text{and} \quad q \geq 3. \quad (2.27)$$

Note that (2.26) is not a convex combination. Both (2.25) and (2.26) involve the statistic $\mathcal{L}_n$ which adjusts the estimator for departure from H_0 . For large values of $\mathcal{L}_n$ both (2.25) and (2.26) yield UE , while for small values of $\mathcal{L}_n$ they may act differently. The SE has the disadvantage that it has strange behavior for

small $\mathcal{L}_n$. Also, the shrinkage factor $(1 - d\mathcal{L}_n^{-1})$ becomes negative for $\mathcal{L}_n < d$. This encourages one to find an alternative estimator. Hence, we define a better estimator, namely, the positive-rule shrinkage estimator (PRSE) of β , as follows

$$\begin{aligned}\hat{\beta}^{S+} &= \hat{\beta} + (1 - d\mathcal{L}_n^{-1})I(\mathcal{L}_n > d)(\tilde{\beta} - \hat{\beta}) \\ &= \hat{\beta}^S - (1 - d\mathcal{L}_n^{-1})I(\mathcal{L}_n \leq d)(\tilde{\beta} - \hat{\beta}).\end{aligned}\quad (2.28)$$

Note that this estimator is a convex combination of $\tilde{\beta}$ and $\hat{\beta}$. Thus it can never change the sign of maximum likelihood estimates and for this reason has been called the Stein positive-part or the Stein positive-rule shrinkage estimator. In the next section, we will provide the bias, mean squared error (MSE) matrix and quadratic risk expression of the estimators when the errors follow a multivariate student- t (Mt) distribution.

2.3 Bias, Quadratic Risk and MSE Matrix of the Estimators

In this section, we give the expressions for the bias, quadratic risk and MSE matrix of the estimators $\tilde{\beta}, \hat{\beta}, \hat{\beta}^{PT}, \hat{\beta}^S$ and $\hat{\beta}^{S+}$. If the loss-function is $(\beta^* - \beta)'W(\beta^* - \beta)$ for a given non-singular matrix W using the estimator β^* , then the risk is defined by

$$\text{Risk}(R) = E[(\beta^* - \beta)'W(\beta^* - \beta)] = \text{tr}(WM), \quad (2.29)$$

where M is the mean-squared error matrix of β^* . We determine the bias, risk and MSE matrix expressions of the estimators using t -errors.

2.3.1 Biases of the Estimators

In this sub-section we present expressions for the biases of all the five estimators.

Theorem 2.5. *Bias of the unrestricted estimator (UE), restricted estimator (RE), preliminary test estimator (PTE), shrinkage estimator (SE) and positive-rule shrinkage estimator (PRSE) are given by (2.30) – (2.34) respectively:*

$$\text{i) } \mathbf{b}_1 = E(\tilde{\beta} - \beta) = 0 \quad (2.30)$$

$$\text{ii) } \mathbf{b}_2 = E(\hat{\beta} - \beta) = -\delta \quad (2.31)$$

$$\text{iii) } \mathbf{b}_3 = E(\hat{\beta}^{PT} - \beta) = -\delta G_{q+2, n-p}^{(2)}(x', \Delta^*) \quad (2.32)$$

$$\text{iv) } \mathbf{b}_4 = E(\hat{\beta}^S - \beta) = -qd\delta E^{(2)}[\chi_{q+2}^{-2}(\Delta^*)] \quad (2.33)$$

$$\begin{aligned} \text{v) } \mathbf{b}_5 = E(\hat{\beta}^{S+} - \beta) = & \delta \left\{ \frac{qd}{q+2} E^{(2)} \left[F_{q+2, n-p}^{-1}(\Delta^*) I \left(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2} \right) \right] \right. \\ & \left. - \frac{qd}{q+2} E^{(2)} \left[F_{q+2, n-p}^{-1}(\Delta^*) \right] - G_{q+2, n-p}^{(2)}(x, \Delta^*) \right\} \end{aligned} \quad (2.34)$$

where

$$\delta = \mathbf{C}^{-1} \mathbf{H}' (\mathbf{H} \mathbf{C}^{-1} \mathbf{H}')^{-1} (\mathbf{H} \beta - \mathbf{h}), \quad \mathbf{C} = \mathbf{X}' \mathbf{X}. \quad (2.35)$$

$$\begin{aligned} & G_{q+2i, n-p}^{(j)}(x', \Delta^*) \\ &= \sum_{r=0}^{\infty} \frac{\Gamma\left(\frac{q}{2} + r + j - 2\right)}{\Gamma(r+1) \Gamma\left(\frac{q}{2} + j - 2\right)} \frac{(\Delta^*/(\nu-2))^r I_x \left[\frac{1}{2}(q+2i) + r, \frac{1}{2}(n-p) \right]}{(1 + \Delta^*/(\nu-2))^{\nu/2+r+j-2}}, \end{aligned} \quad (2.36)$$

$$x' = \frac{qF_{\alpha}}{n-p+qF_{\alpha}}, \quad \Delta^* = \theta/\sigma_e^2, \quad \theta = (\mathbf{H}\beta - \mathbf{h})' (\mathbf{H} \mathbf{C}^{-1} \mathbf{H}')^{-1} (\mathbf{H}\beta - \mathbf{h}).$$

Also,

$$\begin{aligned} E^{(j)}[\chi_{q+s}^{-2}(\Delta^*)] &= \sum_{r=0}^{\infty} \frac{\Gamma\left(\frac{q}{2} + r + j - 2\right)}{\Gamma(r+1) \Gamma\left(\frac{q}{2} + j - 2\right)} \frac{(\Delta^*/(\nu-2))^r (q+s-2+2r)^{-1}}{(1 + \Delta^*/(\nu-2))^{\nu/2+r+j-2}}, \\ E^{(j)}[\chi_{q+s}^{-2}(0)] &= (q+s-2)^{-1}, \quad j = 1, 2 \end{aligned} \quad (2.37)$$

and

$$\begin{aligned}
& E^{(j)} \left[F_{q+s, n-p}^{-1}(\Delta^*) I \left(F_{q+s, n-p}(\Delta^*) \leq \frac{qd}{q+s} \right) \right] \\
&= \sum_{r=0}^{\infty} \frac{\Gamma\left(\frac{k}{2} + r + j - 2\right)}{\Gamma(r+1)\Gamma\left(\frac{k}{2} + j - 2\right)} \frac{(\frac{\Delta^*}{\nu-2})^r (q+s)(q+s-2+2r)^{-1}}{(1 + \Delta^*/(\nu-2))^{\nu/2+r+j-2}} \\
&\quad \times I_x \left[\frac{1}{2}(q+s-2+2r), \frac{1}{2}(n-p+2) \right], \\
& E^{(j)} \left[F_{q+s, n-p}^{-1}(0) I \left(F_{q+s, n-p}(0) \leq \frac{qd}{q+s} \right) \right] \\
&= (q+s)(q+s-2)^{-1} I_x \left[\frac{1}{2}(q+s-2), \frac{1}{2}(n-p+2) \right],
\end{aligned} \tag{2.38}$$

where

$$x = \frac{qd}{n-p+qd}, \quad \Delta^* = \theta/\sigma_e^2, \quad \sigma_e^2 = \frac{\nu}{\nu-2}\sigma^2, \quad \nu > 2.$$

Proof. i) Using (2.13),

$$\mathbf{b}_1 = E(\tilde{\beta} - \beta) = E(\tilde{\beta}) - \beta = \mathbf{0}$$

ii) Using (2.16),

$$\mathbf{b}_2 = E(\hat{\beta} - \beta) = E(\hat{\beta}) - \beta = \beta - \mathbf{C}^{-1}\mathbf{H}'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}(\mathbf{H}\beta - \mathbf{h}) - \beta$$

$$\mathbf{b}_2 = -\delta, \quad \delta = \mathbf{C}^{-1}\mathbf{H}'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}(\mathbf{H}\beta - \mathbf{h}).$$

iii) Using (2.25) and (2.15),

$$\begin{aligned}
\mathbf{b}_3 &= E(\hat{\beta}^{PT} - \beta) = E[\tilde{\beta} - I(\mathcal{L}_n \leq F_\alpha)(\tilde{\beta} - \hat{\beta}) - \beta] \\
&= E(\tilde{\beta} - \beta) - E[I(\mathcal{L}_n \leq F_\alpha)(\tilde{\beta} - \hat{\beta})] \\
&= -\mathbf{C}^{-1}\mathbf{H}'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1/2} E \left\{ [I(\mathcal{L}_n \leq F_\alpha)(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1/2}(\mathbf{H}\tilde{\beta} - \mathbf{h})] \right\}
\end{aligned}$$

Now let $\mathbf{u} = (\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1/2}(\mathbf{H}\tilde{\beta} - \mathbf{h})$, then

$$\mathbf{u}/\tau | \tau \sim N_q \left[\frac{(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1/2}(\mathbf{H}\beta - \mathbf{h})}{\tau}, I_q \right]$$

and

$$\frac{\mathbf{u}'\mathbf{u}}{\tau^2}|\tau \sim \chi_q^2(\Delta), \Delta = \theta/\tau^2, \theta = (\mathbf{H}\beta - \mathbf{h})'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}(\mathbf{H}\beta - \mathbf{h}).$$

So, we can write

$$\mathbf{b}_3 = -\mathbf{C}^{-1}\mathbf{H}'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1/2}E_\tau\left\{\tau E\left[I\left(\frac{\mathbf{u}'\mathbf{u}}{qS_\epsilon^2} \leq F_\alpha\right)\frac{\mathbf{u}}{\tau}|\tau\right]\right\}$$

Using Theorem 4.1 of Judge and Bock (1978),

$$\begin{aligned}\mathbf{b}_3 &= -\mathbf{C}^{-1}(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1/2}(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1/2}(\mathbf{H}\beta - \mathbf{h})G_{q+2, n-p}^{(2)}(x', \Delta^*) \\ &= -\delta G_{q+2, n-p}^{(2)}(x', \Delta^*),\end{aligned}$$

where δ and $G_{q+2, n-p}^{(2)}(x', \Delta^*)$ are defined in (2.35) and (2.36). If $\Delta^* = 0$ in (2.36), then

$$G_{q+2i, n-p}^{(j)}(x', 0) = I_{x'}\left[\frac{1}{2}(q+2i), \frac{1}{2}(n-p)\right] = F_{q+2i, n-p}(\ell_\alpha^{*(i)}, 0) = 1 - \alpha, \quad (2.39)$$

where $F(\ell_\alpha^{*(i)}, 0)$ is the cumulative central F -distribution with $(q+2i, n-p)DF$, $I_{x'}\left[\frac{1}{2}(q+2i), \frac{1}{2}(n-p)\right]$ is a usual incomplete beta distribution function and $\ell_\alpha^{*(i)} = \frac{qF_\alpha}{q+2i}$.

iv) Using (2.15) and (2.26),

$$\begin{aligned}\mathbf{b}_4 &= E(\hat{\beta}^S - \beta) = E[\tilde{\beta} - d\ell_n^{-1}(\tilde{\beta} - \hat{\beta}) - \beta] \\ &= -dE[\mathcal{L}_n^{-1}(\tilde{\beta} - \hat{\beta})] \\ &= -d\mathbf{C}^{-1}\mathbf{H}'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1/2}E_\tau\left\{qE(S_\epsilon^2|\tau)E\left[\frac{(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1/2}(\mathbf{H}\tilde{\beta} - \mathbf{h})}{(\mathbf{H}\tilde{\beta} - \mathbf{h})'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}(\mathbf{H}\tilde{\beta} - \mathbf{h})}|\tau\right]\right\}\end{aligned}$$

Since S_ϵ^2 and $(\mathbf{H}\tilde{\beta} - \mathbf{h})$ are conditionally independent, we have

$$\mathbf{b}_4 = -d\mathbf{C}^{-1}\mathbf{H}'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1/2}E_\tau\left\{\tau qE\left(\frac{\mathbf{u}/\tau}{\mathbf{u}'\mathbf{u}/\tau^2}|\tau\right)\right\},$$

since $E(S_e^2|\tau) = \tau^2$. Therefore,

$$\mathbf{b}_4 = -qd\delta E^{(2)}[\chi_{q+2}^{-2}(\Delta^*)],$$

where $E^{(2)}[\chi_{q+2}^{-2}(\Delta^*)]$ is defined in (2.37).

v) Using (2.28),

$$\begin{aligned}\mathbf{b}_5 &= E(\hat{\beta}^{S+} - \beta) = E[\hat{\beta}^S - (1 - d\mathcal{L}_n^{-1})I(\mathcal{L}_n \leq d)(\tilde{\beta} - \hat{\beta}) - \beta] \\ &= E(\hat{\beta}^S - \beta) - E[I(\mathcal{L}_n \leq d)(\tilde{\beta} - \hat{\beta})] + dE[\mathcal{L}_n^{-1}I(\mathcal{L}_n \leq d)(\tilde{\beta} - \hat{\beta})]\end{aligned}$$

where

$$E(\hat{\beta}^S - \beta) = -qd\delta E^{(2)}[\chi_{q+2}^{-2}(\Delta^*)],$$

$$E[I(\mathcal{L}_n \leq d)(\tilde{\beta} - \hat{\beta})] = \delta G_{q+2, n-p}^{(2)}(x, \Delta^*), \quad x = \frac{qd}{n-p+qd},$$

and

$$\begin{aligned}&E[\mathcal{L}_n^{-1}I(\mathcal{L}_n \leq d)(\tilde{\beta} - \hat{\beta})] \\ &= \mathbf{C}^{-1}\mathbf{H}'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1/2}E_\tau\left\{\tau E\left[\left(\frac{\mathbf{u}'\mathbf{u}}{qS_e^2}\right)^{-1}I\left(\frac{\mathbf{u}'\mathbf{u}}{qS_e^2} \leq d\right)\frac{\mathbf{u}}{\tau}|\tau\right]\right\} \\ &= \delta\left(\frac{q}{q+2}\right)E^{(2)}\left[F_{q+2, n-p}^{-1}(\Delta^*)I\left(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2}\right)\right].\end{aligned}$$

Substituting these results in $\mathbf{b}_5$ and using the following identity

$$E^{(j)}[\chi_{q+s}^{-2}(\Delta^*)] = \frac{1}{q+s}E^{(j)}[F_{q+s, n-p}^{-1}(\Delta^*)], \quad (2.40)$$

we have

$$\begin{aligned}\mathbf{b}_5 &= E(\hat{\beta}^{S+} - \beta) = \delta\left\{\frac{qd}{q+2}E^{(2)}\left[F_{q+2, n-p}^{-1}(\Delta^*)I\left(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2}\right)\right]\right. \\ &\quad \left.- \frac{qd}{q+2}E^{(2)}\left[F_{q+2, n-p}^{-1}(\Delta^*)\right] - G_{q+2, n-p}^{(2)}(x, \Delta^*)\right\}\end{aligned}$$

where $E^{(2)}\left[F_{q+2, n-p}^{-1}(\Delta^*)I\left(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2}\right)\right]$ is defined in (2.38).

Note that for $\alpha = 0$, the bias of $\hat{\beta}^{PT}$ coincides with the bias of the restricted estimator, $\hat{\beta}$, while for $\alpha = 1$, it coincides with that of $\tilde{\beta}$, the unrestricted estimator. Also, as the non-centrality parameter $\Delta^* \rightarrow \infty$, $\mathbf{b}_1 = \mathbf{b}_3 = \mathbf{b}_4 = \mathbf{b}_5 = 0$ while $\mathbf{b}_2$ becomes unbounded. However, under $H_0 : \mathbf{H}\beta = \mathbf{h}$, $\Delta^* = 0$, hence all the estimators are unbiased:

$$\mathbf{b}_1 = \mathbf{b}_2 = \mathbf{b}_3 = \mathbf{b}_4 = \mathbf{b}_5 = 0. \quad (2.41)$$

2.3.2 Risks of the Estimators

For the risks of the estimators, we consider quadratic loss function in (2.29) where β^* can be any one of the five estimators and $\mathbf{W}$ is a known positive definite matrix. Direct computation following Judge and Bock (1978) lead to the following theorem.

Theorem 2.6. *risks of UE, RE, PTE, SE and PRSE are given by (2.42) – (2.46) respectively:*

$$\text{i) } R_1(\tilde{\beta}; \mathbf{W}) = \sigma_e^2 \text{tr}(\mathbf{C}^{-1} \mathbf{W}) \quad (2.42)$$

$$\text{ii) } R_2(\hat{\beta}; \mathbf{W}) = \sigma_e^2 \text{tr}(\mathbf{C}^{-1} \mathbf{W}) - \sigma_e^2 \text{tr}(\mathbf{A}_{11}) + \eta_1' \mathbf{A}_{11} \eta_1 \quad (2.43)$$

$$\begin{aligned} \text{iii) } R_3(\hat{\beta}^{PT}; \mathbf{W}) &= \sigma_e^2 \text{tr}(\mathbf{C}^{-1} \mathbf{W}) - \sigma_e^2 \text{tr}(\mathbf{A}_{11}) G_{q+2, n-p}^{(1)}(x', \Delta^*) \\ &\quad + \eta_1' \mathbf{A}_{11} \eta_1 \left[2G_{q+2, n-p}^{(2)}(x', \Delta^*) - G_{q+4, n-p}^{(2)}(x', \Delta^*) \right] \end{aligned} \quad (2.44)$$

$$\begin{aligned} \text{iv) } R_4(\hat{\beta}^S; \mathbf{W}) &= \sigma_e^2 \text{tr}(\mathbf{C}^{-1} \mathbf{W}) - dq \text{tr}(\mathbf{A}_{11}) \sigma_e^2 \left\{ (q-2) E^{(1)}[\chi_{q+2}^{-4}(\Delta^*)] \right. \\ &\quad \left. + \left[1 - \frac{(q+2)\eta_1' \mathbf{A}_{11} \eta_1}{2\sigma_e^2 \Delta^* \text{tr}(\mathbf{A}_{11})} \right] (2\Delta^*) E^{(2)}[\chi_{q+4}^{-4}(\Delta^*)] \right\} \end{aligned} \quad (2.45)$$

$$\begin{aligned} \text{v) } R_5(\hat{\beta}^{S+}; \mathbf{W}) &= R_4(\hat{\beta}^S; \mathbf{W}) \\ &\quad - \sigma_e^2 \left\{ \text{tr}(\mathbf{A}_{11}) E^{(1)} \left[\left(1 - \frac{qd}{q+2} F_{q+2, n-p}^{-1}(\Delta^*) \right)^2 I \left(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2} \right) \right] \right. \\ &\quad \left. + \frac{\eta_1' \mathbf{A}_{11} \eta_1}{\sigma_e^2} E^{(2)} \left[\left(1 - \frac{qd}{q+4} F_{q+4, n-p}^{-1}(\Delta^*) \right)^2 I \left(F_{q+4, n-p}(\Delta^*) \leq \frac{qd}{q+4} \right) \right] \right\} \\ &\quad - 2\eta_1' \mathbf{A}_{11} \eta_1 E^{(2)} \left[\left(\frac{qd}{q+2} F_{q+2, n-p}^{-1}(\Delta^*) - 1 \right) I \left(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2} \right) \right] \end{aligned} \quad (2.46)$$

where

$$\text{tr}(\mathbf{A}_{11}) = \text{tr}[(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{W}\mathbf{C}^{-1}\mathbf{H}'], \quad (2.47)$$

$$\boldsymbol{\eta}'_1 \mathbf{A}_{11} \boldsymbol{\eta}_1 = (\mathbf{H}\boldsymbol{\beta} - \mathbf{h})'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{W}\mathbf{C}^{-1}\mathbf{H}'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}(\mathbf{H}\boldsymbol{\beta} - \mathbf{h}), \quad (2.48)$$

$$\begin{aligned} E^{(j)}[\chi_{q+s}^{-4}(\Delta^*)] \\ = \sum_{r=0}^{\infty} \frac{\Gamma\left(\frac{q}{2} + r + j - 2\right)}{\Gamma(r+1)\Gamma\left(\frac{q}{2} + j - 2\right)} \frac{(\Delta^*/(\nu-2))^r (q+s-2+2r)^{-1} (q+s-4+2r)^{-1}}{(1 + \Delta^*/(\nu-2))^{\frac{q}{2}+r+j-2}}, \\ E^{(j)}[\chi_{q+s}^{-4}(0)] = (q+s-2)^{-1} (q+s-4)^{-1} \end{aligned} \quad (2.49)$$

and

$$\begin{aligned} E^{(j)}\left[F_{q+s,n-p}^{-2}(\Delta^*) I\left(F_{q+s,n-p}(\Delta^*) \leq \frac{qd}{q+s}\right)\right] \\ = \sum_{r=0}^{\infty} \frac{\Gamma\left(\frac{q}{2} + r + j - 2\right)(q+s)^2}{\Gamma(r+1)\Gamma\left(\frac{q}{2} + j - 2\right)(n-p)} \frac{(\Delta^*/(\nu-2))^r (q+s-2+2r)^{-1} (q+s-4+2r)^{-1}}{(1 + \Delta^*/(\nu-2))^{\nu/2+r+j-2}} \\ \times I_x\left[\frac{1}{2}(q+s-4) + r, \frac{1}{2}(n-p+4)\right], \\ E^{(j)}\left[F_{q+s,n-p}^{-2}(0) I\left(F_{q+s,n-p}(0) \leq \frac{qd}{q+s}\right)\right] = \frac{(q+s)^2}{(n-p)} (q+s-2)^{-1} (q+s-4)^{-1} \\ \times I_x\left[\frac{1}{2}(q+s-4), \frac{1}{2}(n-p+4)\right]. \end{aligned} \quad (2.50)$$

Proof. i) Using (2.14),

$$\begin{aligned} R_1(\hat{\boldsymbol{\beta}}; \mathbf{W}) &= E[(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})' \mathbf{W}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})] \\ &= \text{tr}\{\mathbf{W} E[(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})']\} \\ &= \text{tr}(\mathbf{W}\mathbf{C}^{-1}\sigma_e^2) \\ &= \sigma_e^2 \text{tr}(\mathbf{C}^{-1}\mathbf{W}) \end{aligned}$$

ii) Using (2.17) and (2.31)

$$\begin{aligned}
R_2(\hat{\beta}; \mathbf{W}) &= E[(\hat{\beta} - \beta)' \mathbf{W}(\hat{\beta} - \beta)] = \text{tr}[\mathbf{W} E[(\hat{\beta} - \beta)(\hat{\beta} - \beta)']] \\
&= \text{tr}\{\mathbf{W}[\mathbf{C}^{-1}(\mathbf{I}_p - \mathbf{H}'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}\mathbf{H}\mathbf{C}^{-1})\sigma_e^2 + \delta\delta']\} \\
&= \sigma_e^2 \text{tr}(\mathbf{C}^{-1}\mathbf{W}) - \sigma_e^2 \text{tr}\{\mathbf{W}[\mathbf{C}^{-1}\mathbf{H}'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}\mathbf{H}\mathbf{C}^{-1}]\} + \delta'\mathbf{W}\delta
\end{aligned}$$

Note that $\mathbf{C}^{-1/2}\mathbf{H}'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}\mathbf{H}\mathbf{C}^{-1/2}$ is a symmetric idempotent matrix of rank $q \leq p$. Therefore, there exists an orthonormal matrix $\mathbf{Q}$ (see Judge and Bock (1978)) such that

$$\mathbf{Q}\mathbf{C}^{-1/2}\mathbf{H}'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}\mathbf{H}\mathbf{C}^{-1/2}\mathbf{Q}' = \begin{bmatrix} \mathbf{I}_q & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \quad (2.51)$$

$$\mathbf{Q}\mathbf{C}^{-1/2}\mathbf{W}\mathbf{C}^{-1/2}\mathbf{Q}' = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} = \mathbf{A} \quad (2.52)$$

The matrices $\mathbf{A}_{11}$ and $\mathbf{A}_{22}$ are of order q and $p - q$ respectively. Now, we define the random variable

$$\mathbf{w} = \mathbf{Q}\mathbf{C}^{1/2}\tilde{\beta} - \mathbf{Q}\mathbf{C}^{-1/2}\mathbf{H}'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}\mathbf{h} \quad (2.53)$$

so

$$\mathbf{w}|\tau \sim N_p(\delta, \tau^2\mathbf{I}_p),$$

where

$$\eta = E(\mathbf{w}|\tau) = \mathbf{Q}\mathbf{C}^{1/2}\beta - \mathbf{Q}\mathbf{C}^{-1/2}\mathbf{H}'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}\mathbf{h} \quad (2.54)$$

partitioning the vectors $\mathbf{w} = (\mathbf{w}'_1, \mathbf{w}'_2)'$ and $\eta = (\eta'_1, \eta'_2)'$ where $\mathbf{w}_1|\tau$ and $\mathbf{w}_2|\tau$ are conditionally independent sub-vector of order q and $p - q$ respectively, we obtain

$$\tilde{\beta} - \beta = \mathbf{C}^{-1/2}\mathbf{Q}'(\mathbf{w} - \eta) \quad (2.55)$$

Using (2.51), (2.53) and (2.54),

$$\mathcal{L}_n = \frac{\mathbf{w}_1' \mathbf{w}_1}{q S_e^2}, \Delta = \theta / \tau^2, \Delta^* = \theta / \sigma_e^2, \eta_1' \eta_1 = \theta = (\mathbf{H}\beta - \mathbf{h})' (\mathbf{H}\mathbf{C}^{-1} \mathbf{H}')^{-1} (\mathbf{H}\beta - \mathbf{h}) \quad (2.56)$$

Now, we can write

$$\begin{aligned} & \text{tr} \{ \mathbf{W} [\mathbf{C}^{-1} \mathbf{H}' (\mathbf{H}\mathbf{C}^{-1} \mathbf{H}')^{-1} \mathbf{H}\mathbf{C}^{-1}] \} \\ &= \text{tr} \{ (\mathbf{Q}\mathbf{C}^{-1/2} \mathbf{W}\mathbf{C}^{-1/2} \mathbf{Q}') [\mathbf{Q}\mathbf{C}^{-1/2} \mathbf{H}' (\mathbf{H}\mathbf{C}^{-1} \mathbf{H}')^{-1} \mathbf{H}\mathbf{C}^{-1/2} \mathbf{Q}'] \} \\ &= \text{tr} \left\{ \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right\} \\ &= \text{tr}(\mathbf{A}_{11}) \end{aligned}$$

since $\text{tr}(\mathbf{BD}) = \text{tr}(\mathbf{DB})$ and $\mathbf{Q}'\mathbf{Q} = \mathbf{I}_r$. Using (2.54),

$$\delta' \mathbf{W} \delta = (\mathbf{H}\beta - \mathbf{h})' (\mathbf{H}\mathbf{C}^{-1} \mathbf{H}')^{-1} \mathbf{H}\mathbf{C}^{-1} \mathbf{W}\mathbf{C}^{-1} \mathbf{H}' (\mathbf{H}\mathbf{C}^{-1} \mathbf{H}')^{-1} (\mathbf{H}\beta - \mathbf{h}) = \eta_1' \mathbf{A}_{11} \eta_1.$$

Thus, we have

$$R_2(\hat{\beta}; \mathbf{W}) = \sigma_e^2 \text{tr}(\mathbf{C}^{-1} \mathbf{W}) - \sigma_e^2 \text{tr}(\mathbf{A}_{11}) + \eta_1' \mathbf{A}_{11} \eta_1.$$

iii) Using (2.25),

$$\begin{aligned} R_3(\hat{\beta}^{PT}; \mathbf{W}) &= E[(\hat{\beta}^{PT} - \beta)' \mathbf{W} (\hat{\beta}^{PT} - \beta)] \\ &= E\left\{ [(\tilde{\beta} - \beta) - I(\mathcal{L}_n \leq F_\alpha)(\tilde{\beta} - \hat{\beta})]' \mathbf{W} [(\tilde{\beta} - \beta) - I(\mathcal{L}_n \leq F_\alpha)(\tilde{\beta} - \hat{\beta})] \right\} \\ &= E[(\tilde{\beta} - \beta)' \mathbf{W} (\tilde{\beta} - \beta)] - 2E[I(\mathcal{L}_n \leq F_\alpha)(\tilde{\beta} - \beta)' \mathbf{W} (\tilde{\beta} - \hat{\beta})] \\ &\quad + E[I(\mathcal{L}_n \leq F_\alpha)(\tilde{\beta} - \hat{\beta})' \mathbf{W} (\tilde{\beta} - \hat{\beta})]. \end{aligned}$$

Using (2.47)–(2.48) and (2.51)–(2.55), then

$$\begin{aligned} R_3(\hat{\beta}^{PT}; \mathbf{W}) &= \sigma_e^2 \text{tr}(\mathbf{C}^{-1} \mathbf{W}) - E_\tau \left\{ \tau^2 E[I(\mathcal{L}_n \leq F_\alpha) \frac{\mathbf{w}_1'}{\tau} \mathbf{A}_{11} \frac{\mathbf{w}_1}{\tau} | \tau] \right\} \\ &\quad - 2E_\tau \left\{ \tau^2 E[I(\mathcal{L}_n \leq F_\alpha) \frac{\mathbf{w}_2'}{\tau} \mathbf{A}_{21} \frac{\mathbf{w}_1}{\tau} | \tau] \right\} \\ &\quad + 2\eta_1' \mathbf{A}_{11} E_\tau \left\{ \tau E[I(\mathcal{L}_n \leq F_\alpha) \frac{\mathbf{w}_1}{\tau} | \tau] \right\} \\ &\quad + 2\eta_2' \mathbf{A}_{21} E_\tau \left\{ \tau E[I(\mathcal{L}_n \leq F_\alpha) \frac{\mathbf{w}_1}{\tau} | \tau] \right\}. \end{aligned}$$

Using Theorems 4.1 and 4.2 of Judge and Bock (1978),

$$R_3(\hat{\beta}^{PT}; \mathbf{W}) = \sigma_e^2 \text{tr}(\mathbf{C}^{-1} \mathbf{W}) - \sigma_e^2 \text{tr}(\mathbf{A}_{11}) G_{q+2, n-p}^{(1)}(x', \Delta^*) \\ + \eta_1' \mathbf{A}_{11} \eta_1 \left\{ 2G_{q+2, n-p}^{(2)}(x', \Delta^*) - G_{q+4, n-p}^{(2)}(x', \Delta^*) \right\}.$$

iv) Using (2.26), and (2.27),

$$R_4(\hat{\beta}^S; \mathbf{W}) = E[(\hat{\beta}^S - \beta)' \mathbf{W} (\hat{\beta}^S - \beta)] \\ = E\left\{ [(\tilde{\beta} - \beta) - d\mathcal{L}_n^{-1}(\tilde{\beta} - \hat{\beta})]' \mathbf{W} [(\tilde{\beta} - \beta) - d\mathcal{L}_n^{-1}(\tilde{\beta} - \hat{\beta})] \right\} \\ = E[(\tilde{\beta} - \beta)' \mathbf{W} (\tilde{\beta} - \beta)] - 2dE[\mathcal{L}_n^{-1}(\tilde{\beta} - \beta)' \mathbf{W} (\tilde{\beta} - \hat{\beta})] \\ + d^2 E[\mathcal{L}_n^{-2}(\tilde{\beta} - \hat{\beta})' \mathbf{W} (\tilde{\beta} - \hat{\beta})].$$

Using (2.47)–(2.48) and (2.51)–(2.55), then we get

$$R_4(\hat{\beta}^S; \mathbf{W}) \\ = \sigma_e^2 \text{tr}(\mathbf{C}^{-1} \mathbf{W}) - 2dE_r \left\{ E[\mathcal{L}_n^{-1}(\mathbf{w}_1' \mathbf{A}_{11} \mathbf{w}_1 - \eta_1' \mathbf{A}_{11} \mathbf{w}_1 + \mathbf{w}_2' \mathbf{A}_{21} \mathbf{w}_1 - \eta_2' \mathbf{A}_{21} \mathbf{w}_1) | \tau] \right\} \\ + d^2 E_r \left\{ E[\mathcal{L}_n^{-2}(\mathbf{w}_1' \mathbf{A}_{11} \mathbf{w}_1) | \tau] \right\} \quad (2.57) \\ = \sigma_e^2 \text{tr}(\mathbf{C}^{-1} \mathbf{W}) - dq \text{tr}(\mathbf{A}_{11}) \sigma_e^2 \left\{ 2E^{(1)}[\chi_{q+2}^{-2}(\Delta^*)] - (q-2)E^{(1)}[\chi_{q+2}^{-4}(\Delta^*)] \right\} \\ + \eta_1' \mathbf{A}_{11} \eta_1 dq \left\{ (q-2)E^{(2)}[\chi_{q+4}^{-4}(\Delta^*)] + 2[E^{(2)}(\chi_{q+2}^{-2}(\Delta^*)) - E^{(2)}(\chi_{q+4}^{-2}(\Delta^*))] \right\}$$

Now, we use the following identities to obtain the final form of the risk $\hat{\beta}^S$.

$$E^{(2)}[\chi_p^{-2}(\Delta^*)] - E^{(2)}[\chi_{p+2}^{-2}(\Delta^*)] = 2E^{(2)}[\chi_{p+2}^{-4}(\Delta^*)] \quad \forall p > 2, \\ E^{(1)}[\chi_{p+2}^{-2}(\Delta^*)] - (p-2)E^{(1)}[\chi_{p+2}^{-4}(\Delta^*)] = \Delta^* E^{(2)}[\chi_{p+4}^{-4}(\Delta^*)]. \quad (2.58)$$

Then (2.57) reduces to the following form

$$R_4(\hat{\beta}^S; \mathbf{W}) = \sigma_e^2 \text{tr}(\mathbf{C}^{-1} \mathbf{W}) - dq \text{tr}(\mathbf{A}_{11}) \sigma_e^2 \left\{ (q-2)E^{(1)}[\chi_{q+2}^{-4}(\Delta^*)] \right. \\ \left. + \left[1 - \frac{(q+2)\eta_1' \mathbf{A}_{11} \eta_1}{2\sigma_e^2 \Delta^* \text{tr}(\mathbf{A}_{11})} \right] (2\Delta^*) E^{(2)}[\chi_{q+4}^{-4}(\Delta^*)] \right\},$$

where $E^{(1)}[\chi_{q+2}^{-4}(\Delta^*)]$ and $E^{(2)}[\chi_{q+4}^{-4}(\Delta^*)]$ are defined in (2.49). Finally, the risk of the positive-rule shrinkage estimator (PRSE) $\hat{\beta}^{S+}$ is given in (v).

v) Using (2.28),

$$\begin{aligned} R_5(\hat{\beta}^{S+}; \mathbf{W}) &= E[(\hat{\beta}^{S+} - \beta)' \mathbf{W} (\hat{\beta}^{S+} - \beta)] \\ &= E\{[(\hat{\beta}^S - \beta) - (1 - d\mathcal{L}_n^{-1})I(\mathcal{L}_n \leq d)(\tilde{\beta} - \hat{\beta})]' \mathbf{W} [(\hat{\beta}^S - \beta) \\ &\quad - (1 - d\mathcal{L}_n^{-1})I(\mathcal{L}_n \leq d)(\tilde{\beta} - \hat{\beta})]\} \\ &= R_4(\hat{\beta}^S; \mathbf{W}) + E[(1 - d\mathcal{L}_n^{-1})^2 I(\mathcal{L}_n \leq d)(\tilde{\beta} - \hat{\beta})' \mathbf{W} (\tilde{\beta} - \hat{\beta})] \\ &\quad - 2E[(\hat{\beta}^S - \beta)' \mathbf{W} (1 - d\mathcal{L}_n^{-1})I(\mathcal{L}_n \leq d)(\tilde{\beta} - \hat{\beta})]. \end{aligned}$$

But, using (2.26),

$$\begin{aligned} &E[(\hat{\beta}^S - \beta)' \mathbf{W} (1 - d\mathcal{L}_n^{-1})I(\mathcal{L}_n \leq d)(\tilde{\beta} - \hat{\beta})] \\ &= E\{[(\tilde{\beta} - \beta) + (1 - d\mathcal{L}_n^{-1})(\tilde{\beta} - \hat{\beta})]' \mathbf{W} (1 - d\mathcal{L}_n^{-1})I(\mathcal{L}_n \leq d)(\tilde{\beta} - \hat{\beta})\} \\ &= E\{(\tilde{\beta} - \beta)' \mathbf{W} [(1 - d\mathcal{L}_n^{-1})I(\mathcal{L}_n \leq d)(\tilde{\beta} - \hat{\beta})]\} \\ &\quad + E[(1 - d\mathcal{L}_n^{-1})^2 I(\mathcal{L}_n \leq d)(\tilde{\beta} - \hat{\beta})' \mathbf{W} (\tilde{\beta} - \hat{\beta})]. \end{aligned}$$

Therefore,

$$\begin{aligned} R_5(\hat{\beta}^{S+}; \mathbf{W}) &= R_4(\hat{\beta}^S; \mathbf{W}) - E[(1 - d\mathcal{L}_n^{-1})^2 I(\mathcal{L}_n \leq d)(\tilde{\beta} - \hat{\beta})' \mathbf{W} (\tilde{\beta} - \hat{\beta})] \\ &\quad - 2E\{(\tilde{\beta} - \beta)' \mathbf{W} [(1 - d\mathcal{L}_n^{-1})I(\mathcal{L}_n \leq d)(\tilde{\beta} - \hat{\beta})]\}. \end{aligned}$$

Using (2.47)–(2.48), (2.51)–(2.55) and theorems 4.1 and 4.2 of Judge and Bock (1978), we get

$$\begin{aligned} R_5(\hat{\beta}^{S+}; \mathbf{W}) &= R_4(\hat{\beta}^S; \mathbf{W}) - \sigma_e^2 \left\{ \text{tr}(\mathbf{A}_{11}) E^{(1)} \left[\left(1 - \frac{qd}{q+2} F_{q+2, n-p}^{-1}(\Delta^*) \right)^2 \right. \right. \\ &\quad \times \left. \left. I\left(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2}\right) \right] \right. \\ &\quad \left. + \frac{\eta_1' \mathbf{A}_{11} \eta_1}{\sigma_e^2} E^{(2)} \left[\left(1 - \frac{qd}{q+4} F_{q+4, n-p}^{-1}(\Delta^*) \right)^2 I\left(F_{q+4, n-p}(\Delta^*) \leq \frac{qd}{q+4}\right) \right] \right\} \\ &\quad - 2\eta_1' \mathbf{A}_{11} \eta_1 E^{(2)} \left[\left(\frac{qd}{q+2} F_{q+2, n-p}^{-1}(\Delta^*) - 1 \right) I\left(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2}\right) \right], \end{aligned}$$

where

$$E^{(j)} \left[F_{q+s, n-p}^{-1}(\Delta^*) I \left(F_{q+s, n-p}(\Delta^*) \leq \frac{qd}{q+s} \right) \right],$$

$$E^{(j)} \left[F_{q+s, n-p}^{-2}(\Delta^*) I \left(F_{q+s, n-p}(\Delta^*) \leq \frac{qd}{q+s} \right) \right],$$

$j = 1, 2$, are defined in (2.38) and (2.50).

2.3.3 MSE Matrices of the Estimators

In this subsection we present the mean squared error (MSE) matrices of the five estimators. Let $M_1(\tilde{\beta})$, $M_2(\hat{\beta})$, $M_3(\hat{\beta}^{PT})$, $M_4(\hat{\beta}^S)$ and $M_5(\hat{\beta}^{S+})$ denote mean squared error matrix of $\tilde{\beta}$, $\hat{\beta}$, $\hat{\beta}^{PT}$, $\hat{\beta}^S$ and $\hat{\beta}^{S+}$ respectively.

Theorem 2.7. Mean squared error matrices of UE, RE, PTE, SE and PRSE are given by (2.59) – (2.63) respectively:

$$\text{i) } M_1(\tilde{\beta}) = E[(\tilde{\beta} - \beta)(\tilde{\beta} - \beta)'] = \sigma_e^2 C^{-1} \quad (2.59)$$

$$\begin{aligned} \text{ii) } M_2(\hat{\beta}) &= E[(\hat{\beta} - \beta)(\hat{\beta} - \beta)'] \\ &= \sigma_e^2 C^{-1} - \sigma_e^2 C^{-1} H' (H C^{-1} H')^{-1} H C^{-1} + \delta \delta' \end{aligned} \quad (2.60)$$

$$\begin{aligned} \text{iii) } M_3(\hat{\beta}^{PT}) &= E[(\hat{\beta}^{PT} - \beta)(\hat{\beta}^{PT} - \beta)'] \\ &= \sigma_e^2 C^{-1} - \sigma_e^2 C^{-1} H' (H C^{-1} H')^{-1} H C^{-1} G_{q+2, n-p}^{(1)}(x', \Delta^*) \\ &\quad + \delta \delta' [2G_{q+2, n-p}^{(2)}(x', \Delta^*) - G_{q+4, n-p}^{(2)}(x', \Delta^*)] \end{aligned} \quad (2.61)$$

$$\begin{aligned} \text{iv) } M_4(\hat{\beta}^S) &= E[(\hat{\beta}^S - \beta)(\hat{\beta}^S - \beta)'] \\ &= \sigma_e^2 C^{-1} - dq \sigma_e^2 C^{-1} H' (H C^{-1} H')^{-1} H C^{-1} \left\{ (q-2) E^{(1)} [\chi_{q+2}^{-4}(\Delta^*)] \right. \\ &\quad \left. + 2 \Delta^* E^{(2)} [\chi_{q+4}^{-4}(\Delta^*)] \right\} \\ &\quad + dq(q+2) \delta \delta' E^{(2)} [\chi_{q+4}^{-4}(\Delta^*)] \end{aligned} \quad (2.62)$$

$$\begin{aligned} \text{v) } M_5(\hat{\beta}^{S+}) &= E[(\hat{\beta}^{S+} - \beta)(\hat{\beta}^{S+} - \beta)'] \\ &= M_4(\hat{\beta}^S) - \sigma_e^2 \left\{ C^{-1} H' (H C^{-1} H')^{-1} H C^{-1} E^{(1)} \left[\left(1 - \frac{qd}{q+2} F_{q+2, n-p}^{-1}(\Delta^*) \right)^2 \right. \right. \\ &\quad \left. \left. \times I \left(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2} \right) \right] \right. \\ &\quad \left. + \frac{\delta \delta'}{\sigma_e^2} E^{(2)} \left[\left(1 - \frac{qd}{q+4} F_{q+4, n-p}^{-1}(\Delta^*) \right)^2 I \left(F_{q+4, n-p}(\Delta^*) \leq \frac{qd}{q+4} \right) \right] \right. \\ &\quad \left. - 2 \delta \delta' E^{(2)} \left[\left(\frac{qd}{q+2} F_{q+2, n-p}^{-1}(\Delta^*) - 1 \right) I \left(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2} \right) \right] \right\}, \end{aligned} \quad (2.63)$$

where $\delta, G_{q+s, n-p}^{(j)}(x', \Delta^*), E^{(j)} \left[F_{q+s, n-p}^{-1}(\Delta^*) I \left(F_{q+s, n-p}(\Delta^*) \leq \frac{qd}{q+s} \right) \right]$ and $E^{(j)} \left[F_{q+s, n-p}^{-2}(\Delta^*) I \left(F_{q+s, n-p}(\Delta^*) \leq \frac{qd}{q+s} \right) \right]$, $j = 1, 2$, are defined in (2.35), (2.36), (2.38) and (2.49).

Proof. i) Using (2.14),

$$M_1(\tilde{\beta}) = E[(\tilde{\beta} - \beta)(\tilde{\beta} - \beta)'] = \text{var}(\tilde{\beta}) = \sigma_e^2 C^{-1}$$

ii) Using (2.17) and (2.31),

$$M_2(\hat{\beta}) = E[(\hat{\beta} - \beta)(\hat{\beta} - \beta)'] = \sigma_e^2 C^{-1} - \sigma_e^2 C^{-1} H' (H C^{-1} H')^{-1} H C^{-1} + \delta \delta'$$

iii) Using (2.25) and Theorem 3 from Appendix B of Judge and Bock (1978),

$$\begin{aligned} M_3(\hat{\beta}^{PT}) &= E[(\hat{\beta}^{PT} - \beta)(\hat{\beta}^{PT} - \beta)'] \\ &= \sigma_e^2 C^{-1} - \sigma_e^2 C^{-1} H' (H C^{-1} H')^{-1} H C^{-1} G_{q+2, n-p}^{(1)}(x', \Delta^*) \\ &\quad + \delta \delta' [2G_{q+2, n-p}^{(2)}(x', \Delta^*) - G_{q+4, n-p}^{(2)}(x', \Delta^*)] \end{aligned}$$

iv) Using (2.26), (2.27) and Theorem 3 from Appendix B of Judge and Bock (1978),

$$\begin{aligned} M_4(\hat{\beta}^S) &= E[(\hat{\beta}^S - \beta)(\hat{\beta}^S - \beta)'] \\ &= \sigma_e^2 C^{-1} - dq \sigma_e^2 C^{-1} H' (H C^{-1} H')^{-1} H C^{-1} \left\{ (q-2) E^{(1)} [\chi_{q+2}^{-4}(\Delta^*)] \right. \\ &\quad \left. + 2\Delta^* E^{(2)} [\chi_{q+4}^{-4}(\Delta^*)] \right\} + dq(q+2) \delta \delta' E^{(2)} [\chi_{q+4}^{-4}(\Delta^*)] \end{aligned}$$

v) Using (2.28) and Theorem 3 from Appendix B of Judge and Bock (1978),

$$\begin{aligned} M_5(\hat{\beta}^{S+}) &= E[(\hat{\beta}^{S+} - \beta)(\hat{\beta}^{S+} - \beta)'] \\ &= M_4(\hat{\beta}^S) - \sigma_e^2 \left\{ C^{-1} H' (H C^{-1} H')^{-1} H C^{-1} E^{(1)} \left[\left(1 - \frac{qd}{q+2} F_{q+2, n-p}^{-1}(\Delta^*) \right)^2 \right. \right. \\ &\quad \left. \left. \times I \left(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2} \right) \right] \right. \\ &\quad \left. + \frac{\delta \delta'}{\sigma_e^2} E^{(2)} \left[\left(1 - \frac{qd}{q+4} F_{q+4, n-p}^{-1}(\Delta^*) \right)^2 I \left(F_{q+4, n-p}(\Delta^*) \leq \frac{qd}{q+4} \right) \right] \right\} \\ &\quad - 2\delta \delta' E^{(2)} \left[\left(\frac{qd}{q+2} F_{q+2, n-p}^{-1}(\Delta^*) - 1 \right) I \left(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2} \right) \right]. \end{aligned}$$

Thus the theorem is proved.

2.4 Relative Performance of the Estimators

2.4.1 Risk Analysis of the Estimators

In this subsection, we provide the risk analysis of the estimators with the general loss function and impose conditions on the W matrix so that the shrinkage estimator and positive-rule shrinkage estimator dominates the usual estimator. We study the relative performance of the estimators under the null-hypothesis as well as under the alternative hypothesis. It is clear that the risk of $\tilde{\beta}$, UE, is constant while the risk of $\hat{\beta}$, (RE), varies depending on $\eta_1' A_{11} \eta_1$.

We get from Courant Theorem [see e.g., Mardia (1979)] that

$$ch_{\min}(A_{11}) \leq \frac{\eta_1' A_{11} \eta_1}{\eta_1' \eta_1} \leq ch_{\max}(A_{11}) \quad (2.64)$$

or

$$\sigma_e^2 \Delta^* ch_{\min}(A_{11}) \leq \eta_1' A_{11} \eta_1 \leq \sigma_e^2 \Delta^* ch_{\max}(A_{11}) \quad (2.65)$$

where $ch_{\min}(A_{11})$ and $ch_{\max}(A_{11})$ are the minimum and maximum eigenvalues of A_{11} , and $\Delta^* = \frac{\eta_1' \eta_1}{\sigma_e^2}$. Then, one may easily see that

$$\begin{aligned} R_1(\tilde{\beta}; W) - \sigma_e^2 tr(A_{11}) + \sigma_e^2 \Delta^* ch_{\min}(A_{11}) &\leq R_2(\hat{\beta}; W) \leq R_1(\tilde{\beta}; W) \\ &- \sigma_e^2 tr(A_{11}) + \sigma_e^2 \Delta^* ch_{\max}(A_{11}) \end{aligned} \quad (2.66)$$

When $\Delta^* = 0$, bounds are equal. Thus, (2.66) means that $\tilde{\beta}$ performs better than $\hat{\beta}$ whenever

$$\Delta^* \leq \frac{tr(A_{11})}{ch_{\max}(A_{11})}, \quad (2.67)$$

while $\tilde{\beta}$ performs better than $\hat{\beta}$ whenever

$$\Delta^* \geq \frac{tr(A_{11})}{ch_{\min}(A_{11})} \quad (2.68)$$

For $\mathbf{W} = \mathbf{C}$, we note that $\hat{\beta}$ performs better than $\tilde{\beta}$ in the interval $[0, q\sigma_e^2)$ and worse outside this interval. Next, we compare $\hat{\beta}^{PT}$ versus $\tilde{\beta}$. The risk difference in this case is as follows: i.e.

$$R_1(\hat{\beta}; \mathbf{W}) - R_3(\hat{\beta}^{PT}; \mathbf{W}) = \sigma_e^2 \text{tr}(\mathbf{A}_{11}) G_{q+2, n-p}^{(1)}(x', \Delta^*) - \eta_1' \mathbf{A}_{11} \eta_1 [2G_{q+2, n-p}^{(2)}(x', \Delta^*) - G_{q+4, n-p}^{(2)}(x', \Delta^*)] \quad (2.69)$$

Thus, we see the RHS of (2.69) is nonnegative (≥ 0), whenever

$$\Delta^* \leq \frac{\text{tr}(\mathbf{A}_{11})}{ch_{\max}(\mathbf{A}_{11})} \frac{G_{q+2, n-p}^{(1)}(x', \Delta^*)}{[2G_{q+2, n-p}^{(2)}(x', \Delta^*) - G_{q+4, n-p}^{(2)}(x', \Delta^*)]}. \quad (2.70)$$

In this range $\hat{\beta}^{PT}$ performs better than $\tilde{\beta}$ while $\tilde{\beta}$ performs better whenever

$$\Delta^* \geq \frac{\text{tr}(\mathbf{A}_{11})}{ch_{\min}(\mathbf{A}_{11})} \frac{G_{q+2, n-p}^{(1)}(x', \Delta^*)}{[2G_{q+2, n-p}^{(2)}(x', \Delta^*) - G_{q+4, n-p}^{(2)}(x', \Delta^*)]}. \quad (2.71)$$

For $\mathbf{W} = \mathbf{C}$ we note that $\text{tr}(\mathbf{A}_{11}) = q$ and the required intervals follows (2.70) and (2.71). Under H_0 , we note that $\hat{\beta}^{PT}$ is superior to $\tilde{\beta}$ since the difference is positive for all α . We can describe the graph of $R_3(\hat{\beta}^{PT}; \mathbf{W})$ as follows: It begins with a value

$$\sigma_e^2 \text{tr}(\mathbf{C}^{-1} \mathbf{W}) - \sigma_e^2 \text{tr}(\mathbf{A}_{11}) G_{q+2, n-p}^{(1)}(x', \Delta^*) \quad (2.72)$$

at $\Delta^* = 0$, then increases crossing the risk of $\tilde{\beta}$ to a maximum then drops gradually towards $\sigma_e^2 \text{tr}(\mathbf{C}^{-1} \mathbf{W})$ as $\Delta^* \rightarrow \infty$.

By setting $\alpha = 0$ and using (2.39), we note that

$$R_2(\hat{\beta}; \mathbf{W}) = R_3(\hat{\beta}^{PT}; \mathbf{W}) \quad (2.73)$$

while for $\alpha = 1$,

$$R_1(\hat{\beta}; \mathbf{W}) = R_3(\hat{\beta}^{PT}; \mathbf{W}). \quad (2.74)$$

Now we compare $\hat{\beta}$ and $\hat{\beta}^{PT}$. Both are superior to $\tilde{\beta}$ under the null-hypothesis $H_0 : \mathbf{H}\beta = \mathbf{h}$. In general, the risk difference is given by

$$\begin{aligned} R_2(\hat{\beta}; \mathbf{W}) - R_3(\hat{\beta}^{PT}; \mathbf{W}) &= -\sigma_e^2 \text{tr}(\mathbf{A}_{11}) [1 - G_{q+2, n-p}^{(1)}(x', \Delta^*)] \\ &\quad + \eta_1' \mathbf{A}_{11} \eta_1 [1 - 2G_{q+2, n-p}^{(2)}(x', \Delta^*) + G_{q+4, n-p}^{(2)}(x', \Delta^*)]. \end{aligned} \quad (2.75)$$

Thus, we obtain that the risk difference is nonpositive (≤ 0) whenever

$$\Delta^* \leq \frac{\text{tr}(\mathbf{A}_{11}) [1 - G_{q+2, n-p}^{(1)}(x', \Delta^*)]}{ch_{\max}(\mathbf{A}_{11}) [1 - 2G_{q+2, n-p}^{(2)}(x', \Delta^*) + G_{q+4, n-p}^{(2)}(x', \Delta^*)]}, \quad (2.76)$$

and nonnegative (≥ 0) whenever

$$\Delta^* \geq \frac{\text{tr}(\mathbf{A}_{11}) [1 - G_{q+2, n-p}^{(1)}(x', \Delta^*)]}{ch_{\min}(\mathbf{A}_{11}) [1 - 2G_{q+2, n-p}^{(2)}(x', \Delta^*) + G_{q+4, n-p}^{(2)}(x', \Delta^*)]}. \quad (2.77)$$

Under H_0 , the risk of the three estimators may be ordered as

$$\hat{\beta} \succ \hat{\beta}^{PT} \succ \tilde{\beta} \quad (2.78)$$

where the notation $\succ$ means dominates.

Now, we compare the risks of $\tilde{\beta}$ and $\hat{\beta}^S$. The risk difference is given by

$$\begin{aligned} R_1(\tilde{\beta}; \mathbf{W}) - R_4(\hat{\beta}^S; \mathbf{W}) &= \sigma_e^2 dq \text{tr}(\mathbf{A}_{11}) \left\{ (q-2) E^{(1)}[\chi_{q+2}^{-4}(\Delta^*)] \right. \\ &\quad \left. + \left[1 - \frac{(q+2)\eta_1' \mathbf{A}_{11} \eta_1}{2\sigma_e^2 \Delta^* \text{tr}(\mathbf{A}_{11})} \right] (2\Delta^*) E^{(2)}[\chi_{q+4}^{-4}(\Delta^*)] \right\}. \end{aligned} \quad (2.79)$$

The risk difference is positive for all $\mathcal{A}$ such that

$$\left\{ \mathcal{A} : \frac{\text{tr}(\mathbf{A}_{11})}{ch_{\max}(\mathbf{A}_{11})} \geq \frac{q+2}{2} \right\} \quad (2.80)$$

Thus, $\hat{\beta}^S$ uniformly dominates $\tilde{\beta}$. Further, as $\Delta^* \rightarrow \infty$, the risk difference tends to 0 from below. To compare $\hat{\beta}$ and $\hat{\beta}^S$ we may write

$$R_4(\hat{\beta}^S; \mathbf{W}) = R_2(\hat{\beta}; \mathbf{W}) + \sigma_e^2 \text{tr}(\mathbf{A}_{11}) - \eta_1' \mathbf{A}_{11} \eta_1 - \sigma_e^2 d q \text{tr}(\mathbf{A}_{11}) \left\{ (q-2)E^{(1)}[\chi_{q+2}^{-4}(\Delta^*)] \right. \\ \left. + \left[1 + \frac{(q+2)\eta_1' \mathbf{A}_{11} \eta_1}{2\sigma_e^2 \Delta^* \text{tr}(\mathbf{A}_{11})} \right] (2\Delta^*)E^{(2)}[\chi_{q+4}^{-4}(\Delta^*)] \right\}. \quad (2.81)$$

Under H_0 , this becomes

$$R_4(\hat{\beta}^S; \mathbf{W}) = R_2(\hat{\beta}; \mathbf{W}) + \sigma_e^2(1-d)\text{tr}(\mathbf{A}_{11}) \geq R_2(\hat{\beta}; \mathbf{W}) \quad (2.82)$$

while

$$R_2(\hat{\beta}; \mathbf{W}) = R_1(\tilde{\beta}; \mathbf{W}) - \sigma_e^2 \text{tr}(\mathbf{A}_{11}) \leq R_1(\tilde{\beta}; \mathbf{W}). \quad (2.83)$$

Thus, $\hat{\beta}$ performs better than $\hat{\beta}^S$ under H_0 . However, as η_1 moves away from 0, $\eta_1' \mathbf{A}_{11} \eta_1$ increases and the risk of $\hat{\beta}$ becomes unbounded while the risk of $\hat{\beta}^S$ remains below the risk of $\tilde{\beta}$ and merges with it as $\Delta^* \rightarrow \infty$. Thus, $\hat{\beta}^S$ dominates $\hat{\beta}$ outside an interval around the origin.

Now, we compare $\hat{\beta}^S$ and $\hat{\beta}^{PT}$. First, under H_0 ,

$$R_4(\hat{\beta}^S; \mathbf{W}) = R_3(\hat{\beta}^{PT}; \mathbf{W}) + \sigma_e^2 \text{tr}(\mathbf{A}_{11}) [G_{q+2, n-p}^{(1)}(x', 0) - d] \geq R_3(\hat{\beta}^{PT}; \mathbf{W}), \quad (2.84)$$

where

$$G_{q+2, n-p}^{(1)}(x', 0) = I_{x'}\left[\frac{1}{2}(q+2), \frac{1}{2}(n-p)\right] = F_{q+2, n-p}(\ell_\alpha^{*(1)}, 0) = 1 - \alpha, \ell_\alpha^{*(1)} = \frac{qF_\alpha}{q+2},$$

for all α such that $\ell_\alpha^{*(1)}$ satisfies the inequality

$$\left\{ \alpha : \ell_\alpha^{*(1)} > F_{q+2, n-p}^{-1}(d, 0) \right\}. \quad (2.85)$$

The risk of $\hat{\beta}^S$ is smaller than that of the risk of $\hat{\beta}^{PT}$ when the critical value $\ell_\alpha^{*(1)}$ satisfies the opposite inequality in (2.85). This means the estimator $\hat{\beta}^S$ does not always dominate $\hat{\beta}^{PT}$ under H_0 and we can order the risks under H_0 as follows:

$$\hat{\beta} \succ \hat{\beta}^{PT} \succ \hat{\beta}^S \succ \tilde{\beta} \quad (2.86)$$

when α satisfies (2.85).

The picture changes as Δ^* moves away from 0. As $\Delta^* \rightarrow \infty$, the risk of $\hat{\beta}^{PT}$ and $\hat{\beta}^S$ converge to the risk of $\tilde{\beta}$. But for reasonable Δ^* near 0, the risk of $\hat{\beta}^{PT}$ is smaller than the risk of $\hat{\beta}^S$ for α satisfying (2.85) and for $q \geq 3$. Thus, none of the estimators dominate each other.

Finally, we compare the risks of $\hat{\beta}^{S+}$ and $\hat{\beta}^S$. The risk difference is given by

$$\begin{aligned} R_5(\hat{\beta}^{S+}; \mathbf{W}) - R_4(\hat{\beta}^S; \mathbf{W}) = & -\sigma_e^2 \left\{ \text{tr}(\mathbf{A}_{11}) E^{(1)} \left[\left(1 - \frac{qd}{q+2} F_{q+2, n-p}^{-1}(\Delta^*) \right)^2 \right. \right. \\ & \times I \left(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2} \right) \Big] \\ & + \frac{\eta_1' \mathbf{A}_{11} \eta_1}{\sigma_e^2} E^{(2)} \left[\left(1 - \frac{qd}{q+4} F_{q+4, n-p}^{-1}(\Delta^*) \right)^2 I \left(F_{q+4, n-p}(\Delta^*) \leq \frac{qd}{q+4} \right) \right] \Big\} \\ & - 2\eta_1' \mathbf{A}_{11} \eta_1 E^{(2)} \left[\left(\frac{qd}{q+2} F_{q+2, n-p}^{-1}(\Delta^*) - 1 \right) I \left(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2} \right) \right] \end{aligned} \quad (2.87)$$

The RHS of (2.87) is negative since the expectation of a positive random variable is positive, using the definition of an indicator function, we get

$$\left(0 < F_{q+2, n-p}(\Delta^*) < \frac{qd}{q+2} \right) \leftrightarrow \left(\frac{qd}{q+2} F_{q+2, n-p}^{-1}(\Delta^*) - 1 \right) \geq 0 \quad (2.88)$$

Thus,

$$E^{(2)} \left[\left(\frac{qd}{q+2} F_{q+2, n-p}^{-1}(\Delta^*) - 1 \right) I \left(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2} \right) \right] \geq 0.$$

Therefore, for all β , $R_5(\hat{\beta}^{S+}; W) \leq R_4(\hat{\beta}^S; W)$. so the positive-rule shrinkage estimator (PRSE) not only confirms the inadmissibility of the shrinkage estimator (SE), but also demonstrates a simple superior estimator. To compare $\tilde{\beta}$ and $\hat{\beta}^{S+}$, we note that,

$$\frac{R_5(\hat{\beta}^{S+}; W)}{R_1(\tilde{\beta}; W)} \leq 1 \quad \forall \eta_1, q \geq 3 \quad (2.89)$$

since (2.80) and $R_5(\hat{\beta}^{S+}; W) \leq R_4(\hat{\beta}^S; W)$ for all β . Thus, $\hat{\beta}^{S+}$ uniformly dominates $\tilde{\beta}$. Further, the risk of $\hat{\beta}^{S+}$ remains below the risk of $\tilde{\beta}$ and merges with it as $\Delta^* \rightarrow \infty$. To compare $\hat{\beta}$ and $\hat{\beta}^{S+}$, first consider the case under H_0 i.e., $\eta_1 = 0$. In this case

$$\begin{aligned} R_5(\hat{\beta}^{S+}; W) &= R_2(\tilde{\beta}; W) + \sigma_e^2 \text{tr}(A_{11}) \left\{ (1-d) - E^{(1)} \left[\left(1 - \frac{qd}{q+2} F_{q+2, n-p}^{-1}(0) \right)^2 \right. \right. \\ &\quad \left. \left. \times I(F_{q+2, n-p}(0) \leq \frac{qd}{q+2}) \right] \right\} \\ &\geq R_2(\tilde{\beta}; W) \end{aligned} \quad (2.90)$$

since

$$\begin{aligned} &E^{(1)} \left[\left(1 - \frac{qd}{q+2} F_{q+2, n-p}^{-1}(0) \right)^2 I(F_{q+2, n-p}(0) \leq \frac{qd}{q+2}) \right] \\ &\leq E^{(1)} \left[\left(1 - \frac{qd}{q+2} F_{q+2, n-p}^{-1}(0) \right)^2 \right] = 1-d \end{aligned} \quad (2.91)$$

Thus $\hat{\beta}$ performs better than $\hat{\beta}^{S+}$ under H_0 . However, as η_1 moves away from 0, $\eta_1' A_{11} \eta_1$ increases and the risk of $\hat{\beta}$ becomes unbounded while the risk of $\hat{\beta}^{S+}$ remains below the risk of $\tilde{\beta}$ and merges with it as $\Delta^* \rightarrow \infty$. Thus $\hat{\beta}^{S+}$ dominates $\hat{\beta}$ outside an interval around the origin.

Now, we compare $\hat{\beta}^{S+}$ and $\hat{\beta}^{PT}$. First, consider the case when H_0 holds, i.e., $\eta_1 = 0$. In this case

$$\begin{aligned} R_5(\hat{\beta}^{S+}; W) &= R_3(\hat{\beta}^{PT}; W) + \sigma_e^2 \text{tr}(A_{11}) \left\{ G_{q+2, n-p}^{(1)}(x', 0) - d \right. \\ &\quad \left. - E^{(1)} \left[\left(1 - \frac{qd}{q+2} F_{q+2, n-p}^{-1}(0) \right)^2 I(F_{q+2, n-p}(0) \leq \frac{qd}{q+2}) \right] \right\} \\ &\geq R_3(\hat{\beta}^{PT}; W) \end{aligned} \quad (2.92)$$

for all α satisfying the condition

$$\left\{ \alpha : F_{q+2, n-p}(\ell_{\alpha}^{*(1)}, 0) \geq d + E^{(1)} \left[\left(1 - \frac{qd}{q+2} F_{q+2, n-p}^{-1}(0) \right)^2 I \left(F_{q+2, n-p}(0) \leq \frac{qd}{q+2} \right) \right] \right\} \quad (2.93)$$

The risk of $\hat{\beta}^{S+}$ is smaller than that of the risk of $\hat{\beta}^{PT}$ when the critical value $\ell_{\alpha}^{*(1)} = \frac{qF_0}{q+2}$ satisfies the opposite inequality in (2.93). Thus, $\hat{\beta}^{S+}$ does not always dominate $\hat{\beta}^{PT}$ when the null-hypothesis H_0 holds. Thus, we may order the dominance picture under (2.93) as follows:

$$\hat{\beta} \succ \hat{\beta}^{PT} \succ \hat{\beta}^{S+} \succ \hat{\beta}^S \succ \tilde{\beta} \quad (2.94)$$

The position of the preliminary test estimator may shift from "in between" $R_2(\hat{\beta}; W)$ and $R_5(\hat{\beta}^{S+}; W)$ to "in between" $R_4(\hat{\beta}^S; W)$ and $R_1(\tilde{\beta}; W)$. Thus, ordering of the risks under the null-hypothesis may change as follows:

$$\hat{\beta} \succ \hat{\beta}^{S+} \succ \hat{\beta}^S \succ \hat{\beta}^{PT} \succ \tilde{\beta} \quad (2.95)$$

The picture changes as η_1 moves away from 0. First, note that $\tilde{\beta}$ has the constant risk $tr(C^{-1}W)$ while the risk of $\hat{\beta}$ depends on η_1 . Consequently, the risk of $\hat{\beta}$ becomes unbounded as η_1 moves away from 0. Also, as $\Delta^* \rightarrow \infty$ the risk of $\hat{\beta}^{PT}$ and $\hat{\beta}^{S+}$ converge to the risk of $\tilde{\beta}$. But for reasonable η_1 near 0, the risk of $\hat{\beta}^{PT}$ is smaller than the risk of $\hat{\beta}^{S+}$ for α satisfying (2.93) and for $q \geq 3$. Thus, neither $\hat{\beta}^{PT}$ nor $\hat{\beta}^{S+}$ dominates the other except that they share common property that as $\Delta^* \rightarrow \infty$ the risk of both becomes $tr(C^{-1}W)$. However, the risk of $\hat{\beta}^{S+}$ is below the risk of $\tilde{\beta}$ while the risk of $\hat{\beta}^{PT}$ exceeds the risk of $\tilde{\beta}$ at some intermediate values of Δ^* depends on α .

Some graphical display of relative risks of the various estimators follow.

FIGURE 1. Relative risk functions for the unrestricted, restricted and pre-test estimators of β when $e \sim N(0, \sigma^2 I_n)$: $n = 30, p = 5, q_1 = 3$.

FIGURE 2. Relative risk functions for the unrestricted, restricted, and pre-test estimators of β when $e \sim Mt(0, \sigma^2/(v-2)I_n)$: $n = 30, p = 5, q_1 = 3, v = 5$.

FIGURE 3. Relative risk functions for the unrestricted, restricted, and pre-test estimators of β when $e \sim Mt(0, v\sigma^2/(v-2)I_n)$: $n = 30$, $p = 5$, $q_k = 3$, $v = 10$.

FIGURE 4. Relative risk functions for the unrestricted, restricted, and pre-test estimators of β when $e \sim Mt(0, v\sigma^2/(v-2)I_n)$: $n = 30$, $p = 5$, $q_k = 3$, $v = 100$.

2.4.2 MSE Analysis of the Estimator

In this subsection we focus on MSE analysis of the estimators and determine their dominance properties.

For the MSE comparison of $\tilde{\beta}$ and $\hat{\beta}$, we consider the MSE-difference

$$M_1(\tilde{\beta}) - M_2(\hat{\beta}) = \sigma_e^2 C^{-1} H' (H C^{-1} H')^{-1} H C^{-1} - \delta \delta' \quad (2.96)$$

This MSE-difference is positive semi-definite whenever for a given non-zero vector $b = (b_1, \dots, b_p)'$, we have $b' [M_1(\tilde{\beta}) - M_2(\hat{\beta})] b \geq 0$. That is to say,

$$\sigma_e^2 b' C^{-1} H' (H C^{-1} H')^{-1} H C^{-1} b \geq b' \delta \delta' b \quad (2.97)$$

Hence for all $(q \times 1)$ non-zero vectors b , we have

$$\frac{b' \delta \delta' b}{\sigma_e^2 b' C^{-1} b} \leq \frac{b' [C^{-1} H' (H C^{-1} H')^{-1} H C^{-1}] b}{b' C^{-1} b} \quad \forall b \quad (2.98)$$

since $b' C^{-1} b > 0$. Namely,

$$\max_b \frac{b' \delta \delta' b}{\sigma_e^2 b' C^{-1} b} \leq \max_b \frac{b' [C^{-1} H' (H C^{-1} H')^{-1} H C^{-1}] b}{b' C^{-1} b} \quad (2.99)$$

But

$$\max_b \frac{b' \delta \delta' b}{\sigma_e^2 b' C^{-1} b} = \frac{\delta' C \delta}{\sigma_e^2} = \frac{\theta}{\sigma_e^2} = \Delta^*$$

and

$$\max_b \frac{b' [C^{-1} H' (H C^{-1} H')^{-1} H C^{-1}] b}{b' C^{-1} b} = ch_{\max} [H' (H C^{-1} H')^{-1} H C^{-1}] = 1$$

since $H' (H C^{-1} H')^{-1} H C^{-1}$ is an idempotent matrix (see Searle, S.R., 1971).

Thus, (2.96) is positive semi-definite if and only if

$$\Delta^* \leq 1 \quad (2.100)$$

Therefore under condition (2.100) the restricted estimator (RE) $\hat{\beta}$, performs better than the unrestricted estimator (UE) $\tilde{\beta}$, otherwise $\tilde{\beta}$ is superior. Comparison of $\tilde{\beta}$ and $\hat{\beta}^{PT}$: The MSE difference

$$M_1(\tilde{\beta}) - M_3(\hat{\beta}^{PT}) = \sigma_e^2 C^{-1} H' (H C^{-1} H')^{-1} H C^{-1} G_{q+2, n-p}^{(1)}(x', \Delta^*) - \delta \delta' [2G_{q+2, n-p}^{(2)}(x', \Delta^*) - G_{q+4, n-p}^{(2)}(x', \Delta^*)] \quad (2.101)$$

This MSE difference is positive semi-definite whenever for any nonzero vector b , we have

$$\sigma_e^2 b' [C^{-1} H' (H C^{-1} H')^{-1} H C^{-1}] b G_{q+2, n-p}^{(1)}(x', \Delta^*) - b' \delta \delta' b [2G_{q+2, n-p}^{(2)}(x', \Delta^*) - G_{q+4, n-p}^{(2)}(x', \Delta^*)] \geq 0 \quad (2.102)$$

namely,

$$\frac{b' \delta \delta' b}{\sigma_e^2 b' C^{-1} b} \leq \frac{G_{q+2, n-p}^{(1)}(x', \Delta^*)}{[2G_{q+2, n-p}^{(2)}(x', \Delta^*) - G_{q+4, n-p}^{(2)}(x', \Delta^*)]} \times \frac{b' [C^{-1} H' (H C^{-1} H')^{-1} H C^{-1}] b}{b' C^{-1} b}, \forall b \quad (2.103)$$

Hence,

$$\max_b \frac{b' \delta \delta' b}{\sigma_e^2 b' C^{-1} b} \leq \frac{G_{q+2, n-p}^{(1)}(x', \Delta^*)}{[2G_{q+2, n-p}^{(2)}(x', \Delta^*) - G_{q+4, n-p}^{(2)}(x', \Delta^*)]} \times \max_b \frac{b' [C^{-1} H' (H C^{-1} H')^{-1} H C^{-1}] b}{b' C^{-1} b} \quad (2.104)$$

which implies that the MSE difference is positive semi-definite whenever

$$\Delta^* \leq \frac{G_{q+2, n-p}^{(1)}(x', \Delta^*)}{[2G_{q+2, n-p}^{(2)}(x', \Delta^*) - G_{q+4, n-p}^{(2)}(x', \Delta^*)]} \quad (2.105)$$

Thus, $\hat{\beta}^{PT}$ is superior to $\tilde{\beta}$ in this range of Δ^* , otherwise $\tilde{\beta}$ is superior. Notice the difference between (2.70) and (2.105).

To compare $\hat{\beta}$ and $\hat{\beta}^{PT}$, in general, the MSE difference is given by

$$\begin{aligned} M_2(\hat{\beta}) - M_2(\hat{\beta}^{PT}) = & -\sigma_e^2 C^{-1} H' (H C^{-1} H')^{-1} H C^{-1} [1 - G_{q+2, n-p}^{(1)}(x', \Delta^*)] \\ & + \delta \delta' [1 - 2G_{q+2, n-p}^{(2)}(x', \Delta^*) + G_{q+4, n-p}^{(2)}(x', \Delta^*)] \end{aligned} \quad (2.106)$$

The MSE difference is negative semi-definite whenever for any nonzero vector b we have

$$\begin{aligned} & -\sigma_e^2 b' C^{-1} H' (H C^{-1} H')^{-1} H C^{-1} b [G_{q+2, n-p}^{(1)}(x', \Delta^*)] \\ & + b' \delta \delta' b [1 - 2G_{q+2, n-p}^{(2)}(x', \Delta^*) + G_{q+4, n-p}^{(2)}(x', \Delta^*)] \leq 0 \end{aligned} \quad (2.107)$$

which implies that the MSE difference is negative semi-definite whenever

$$\Delta^* \leq \frac{1 - G_{q+2, n-p}^{(1)}(x', \Delta^*)}{[1 - 2G_{q+2, n-p}^{(2)}(x', \Delta^*) + G_{q+4, n-p}^{(2)}(x', \Delta^*)]} \quad (2.108)$$

Thus, $\hat{\beta}$ is superior to $\hat{\beta}^{PT}$ in this range of Δ^* , otherwise $\hat{\beta}^{PT}$ is superior.

Comparison of $\tilde{\beta}$ and $\hat{\beta}^S$: Here the MSE difference is given by

$$\begin{aligned} M_1(\tilde{\beta}) - M_4(\hat{\beta}^S) = & dq\sigma_e^2 C^{-1} H' (H C^{-1} H')^{-1} H C^{-1} \left\{ (q-2)E^{(1)}[\chi_{q+2}^{-4}(\Delta^*)] \right. \\ & \left. + 2\Delta^* E^{(2)}[\chi_{q+4}^{-4}(\Delta^*)] \right\} \\ & - dq(q+2)\delta\delta' E^{(2)}[\chi_{q+4}^{-4}(\Delta^*)] \end{aligned} \quad (2.109)$$

This difference is positive semi-definite whenever for any nonzero vector b , we have

$$\begin{aligned} & dq\sigma_e^2 b' [C^{-1} H' (H C^{-1} H')^{-1} H C^{-1} b \left\{ (q-2)E^{(1)}[\chi_{q+2}^{-4}(\Delta^*)] + 2\Delta^* E^{(2)}[\chi_{q+4}^{-4}(\Delta^*)] \right\} \\ & - dq(q+2)b' \delta \delta' b E^{(2)}[\chi_{q+4}^{-4}(\Delta^*)]] \geq 0 \end{aligned} \quad (2.110)$$

This implies that (after simplification)

$$q\Delta^* E^{(2)}[\chi_{q+4}^{-4}(\Delta^*)] \leq (q-2)E^{(1)}[\chi_{q+2}^{-4}(\Delta^*)] \quad (2.111)$$

which holds for all $\Delta^* \in (0, \infty)$. Thus $\hat{\beta}^S$ is uniformly better than $\hat{\beta}$. Comparison of $\hat{\beta}$ and $\hat{\beta}^S$: the MSE difference is

$$\begin{aligned} M_2(\hat{\beta}) - M_4(\hat{\beta}^S) = & -\sigma_e^2 \mathbf{C}^{-1} \mathbf{H}' (\mathbf{H} \mathbf{C}^{-1} \mathbf{H}')^{-1} \mathbf{H} \mathbf{C}^{-1} \left\{ 1 - dq(q-2)E^{(1)}[\chi_{q+2}^{-4}(\Delta^*)] \right. \\ & \left. - 2dq\Delta^* E^{(2)}[\chi_{q+4}^{-4}(\Delta^*)] \right\} \\ & + \delta\delta' \left\{ 1 - dq(q+2)E^{(2)}[\chi_{q+4}^{-4}(\Delta^*)] \right\} \end{aligned} \quad (2.112)$$

Under H_0 , this becomes

$$M_2(\hat{\beta}) - M_4(\hat{\beta}^S) = -(1-d)\sigma_e^2 \mathbf{C}^{-1} \mathbf{H}' (\mathbf{H} \mathbf{C}^{-1} \mathbf{H}')^{-1} \mathbf{H} \mathbf{C}^{-1} \quad (2.113)$$

This difference is negative semi-definite, therefore $\hat{\beta}$ performs better than $\hat{\beta}^S$ under H_0 . Comparison of $\hat{\beta}^S$ and $\hat{\beta}^{PT}$: the MSE difference

$$\begin{aligned} M_4(\hat{\beta}^S) - M_3(\hat{\beta}^{PT}) &= \sigma_e^2 \mathbf{C}^{-1} \mathbf{H}' (\mathbf{H} \mathbf{C}^{-1} \mathbf{H}')^{-1} \mathbf{H} \mathbf{C}^{-1} \left\{ G_{q+2, n-p}^{(1)}(x', \Delta^*) \right. \\ &\quad \left. - dq(q-2)E^{(1)}[\chi_{q+2}^{-4}(\Delta^*)] - 2dq\Delta^* E^{(2)}[\chi_{q+4}^{-4}(\Delta^*)] \right\} \\ &\quad - \delta\delta' \left\{ 2G_{q+2, n-p}^{(2)}(x', \Delta^*) - G_{q+4, n-p}^{(2)}(x', \Delta^*) - dq(q+2)E^{(2)}[\chi_{q+4}^{-4}(\Delta^*)] \right\} \end{aligned} \quad (2.114)$$

Similarly, the MSE analysis tells us that (2.114) is positive semi-definite if for all nonzero vector $\mathbf{b}$ we have

$$\begin{aligned} \sigma_e^2 \mathbf{b}' \mathbf{C}^{-1} \mathbf{H}' (\mathbf{H} \mathbf{C}^{-1} \mathbf{H}')^{-1} \mathbf{H} \mathbf{C}^{-1} \mathbf{b} \left\{ G_{q+2, n-p}^{(1)}(x', \Delta^*) - dq(q-2)E^{(1)}[\chi_{q+2}^{-4}(\Delta^*)] \right. \\ \left. - 2dq\Delta^* E^{(2)}[\chi_{q+4}^{-4}(\Delta^*)] \right\} \\ - \mathbf{b}' \delta\delta' \mathbf{b} \left\{ 2G_{q+2, n-p}^{(2)}(x', \Delta^*) - G_{q+4, n-p}^{(2)}(x', \Delta^*) - dq(q+2)E^{(2)}[\chi_{q+4}^{-4}(\Delta^*)] \right\} \geq 0 \end{aligned} \quad (2.115)$$

which implies that (2.115) holds whenever

$$\begin{aligned} & \Delta^* \left\{ 2G_{q+2, n-p}^{(2)}(x', \Delta^*) - G_{q+4, n-p}^{(2)}(x', \Delta^*) - dq^2 E^{(2)}[\chi_{q+4}^{-4}(\Delta^*)] \right\} \\ & \leq \left\{ G_{q+2, n-p}^{(1)}(x', \Delta^*) - dq(q-2)E^{(1)}[\chi_{q+2}^{-4}(\Delta^*)] \right\} \end{aligned} \quad (2.116)$$

Hence $\hat{\beta}^{PT}$ is superior to $\hat{\beta}^S$ whenever (2.116) holds and inferior otherwise. Note that under H_0 , $\hat{\beta}^{PT}$ is superior to $\hat{\beta}^S$ whenever (2.85) holds. Comparison of $\hat{\beta}^{S+}$ and $\hat{\beta}^S$: the MSE difference

$$\begin{aligned} & M_5(\hat{\beta}^{S+}) - M_4(\hat{\beta}^S) \\ & = -\sigma_e^2 \left\{ C^{-1}H'(HC^{-1}H')^{-1}HC^{-1}E^{(1)} \left[\left(1 - \frac{qd}{q+2} F_{q+2, n-p}^{-1}(\Delta^*) \right)^2 \right. \right. \\ & \quad \times I \left(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2} \right) \Big] \\ & \quad + \frac{\delta\delta'}{\sigma_e^2} E^{(2)} \left[\left(1 - \frac{qd}{q+4} F_{q+4, n-p}^{-1}(\Delta^*) \right)^2 I \left(F_{q+4, n-p}(\Delta^*) \leq \frac{qd}{q+4} \right) \right] \Big\} \\ & \quad - 2\delta\delta' E^{(2)} \left[\left(\frac{qd}{q+2} F_{q+2, n-p}^{-1}(\Delta^*) - 1 \right) I \left(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2} \right) \right] \end{aligned} \quad (2.117)$$

This difference is negative semi-definite since the expectation of a positive random variable is positive. Therefore, for all β , $\hat{\beta}^{S+}$ is superior to $\hat{\beta}^S$.

Comparison of $\tilde{\beta}$ and $\hat{\beta}^{S+}$. It is easy to see that $\hat{\beta}^{S+}$ is uniformly better than $\tilde{\beta}$ since (2.111) and (2.117).

Comparison of $\hat{\beta}$ and $\hat{\beta}^{S+}$: consider the case under H_0 . In this case the MSE difference

$$\begin{aligned} & M_5(\hat{\beta}^{S+}) - M_3(\hat{\beta}) \\ & = \sigma_e^2 C^{-1}H'(HC^{-1}H')^{-1}HC^{-1} \left\{ 1 - d - E^{(1)} \left[\left(1 - \frac{qd}{q+2} F_{q+2, n-p}^{-1}(0) \right)^2 \right. \right. \\ & \quad \times I \left(F_{q+2, n-p}(0) \leq \frac{qd}{q+2} \right) \Big] \Big\} \end{aligned} \quad (2.118)$$

This difference is positive semi-definite since (2.91). Thus $\hat{\beta}$ is superior to $\hat{\beta}^{S+}$ under H_0 . Comparison of $\hat{\beta}^{S+}$ and $\hat{\beta}^{PT}$: consider the case when H_0 holds. In this case the MSE difference

$$\begin{aligned} M_5(\hat{\beta}^{S+}) - M_3(\hat{\beta}^{PT}) &= \sigma_e^2 C^{-1} H' (H C^{-1} H')^{-1} H C^{-1} \left\{ G_{q+2, n-p}^{(1)}(x', 0) - d \right. \\ &\quad \left. - E^{(1)} \left[\left(1 - \frac{qd}{q+2} F_{q+2, n-p}^{-1}(0) \right)^2 I \left(F_{q+2, n-p}(0) \leq \frac{qd}{q+2} \right) \right] \right\} \end{aligned} \quad (2.119)$$

This MSE difference is positive semi-definite whenever for any nonzero vector b we have

$$\begin{aligned} \sigma_e^2 b' C^{-1} H' (H C^{-1} H')^{-1} H C^{-1} b &\left\{ G_{q+2, n-p}^{(1)}(x', 0) - d - E^{(1)} \left[\left(1 - \frac{qd}{q+2} F_{q+2, n-p}^{-1}(0) \right)^2 \right. \right. \\ &\quad \left. \left. \times I \left(F_{q+2, n-p}(0) \leq \frac{qd}{q+2} \right) \right] \right\} \geq 0 \end{aligned} \quad (2.120)$$

which implies that the MSE difference is positive semi-definite whenever (2.93) holds. Thus $\hat{\beta}^{PT}$ is superior to $\hat{\beta}^{S+}$ in (2.93), otherwise $\hat{\beta}^{S+}$ is superior. Therefore, $\hat{\beta}^{S+}$ does not always dominate $\hat{\beta}^{PT}$ when $H_0 : H\beta = h$ holds. Hence we can order the MSE matrices as

$$M_1(\tilde{\beta}) \geq M_4(\hat{\beta}^S) \geq M_5(\hat{\beta}^{S+}) \geq M_3(\hat{\beta}^{PT}) \geq M_2(\hat{\beta}) \quad (2.121)$$

where $\geq$ stands for $M_1(\tilde{\beta}) - M_4(\hat{\beta}^S) \geq 0$, $M_4(\hat{\beta}^S) - M_5(\hat{\beta}^{S+}) \geq 0$, $M_5(\hat{\beta}^{S+}) - M_3(\hat{\beta}^{PT}) \geq 0$ and $M_3(\hat{\beta}^{PT}) - M_2(\hat{\beta}) \geq 0$.

The position of the preliminary test estimator may be changed to the following:

$$M_1(\tilde{\beta}) \geq M_3(\hat{\beta}^{PT}) \geq M_4(\hat{\beta}^S) \geq M_5(\hat{\beta}^{S+}) \geq M_2(\hat{\beta}). \quad (2.122)$$

2.5 Conclusion.

In this chapter, we studied five estimators of β when the null-hypothesis $H_0 : H\beta = h$ may hold. First, note that the application of $\hat{\beta}^{S+}$ or $\hat{\beta}^S$ is constrained by the requirement $q \geq 3$ while $\hat{\beta}^{PT}$ does not need such a constraint but needs the determination of the size of the preliminary test and the extent of departure of Δ^* from 0. Note that the maximal savings in risk for the shrinkage estimator in the normal case is $\frac{(q-2)(n-p)}{p(n-p+2)}$. In the multivariate t -case with ν D.F. the maximal savings in risk is $\frac{(q-2)(n-p)}{p(n-p+2)} \frac{\nu-2}{\nu}$. Thus, for ν moderately large, the risk savings compares favourably with the normal case. Secondly risk saving for the PTE in the normal case is $\frac{1}{p} G_{q+2, n-p}^{(1)}(x', 0)$ while for the multivariate t -case it is $\frac{1}{p} G_{q+2, n-p}^{(1)}(x', 0) \left(\frac{\nu-2}{\nu} \right)$ which converge to the normal case as $\nu \rightarrow \infty$. Thus, the performance property of the five estimators is robust in the class of t -distribution which is determined by the degrees of freedom ν . Also we need the restriction $\nu > 4$. Thus, when $q \geq 3$, one uses $\hat{\beta}^{S+}$ otherwise $\hat{\beta}^{PT}$ with some optimum size α . Therefore, from the point of robust efficiency both preliminary test estimator (PTE) and positive-rule shrinkage estimator (PRSE) may be advocated leaning towards $\hat{\beta}^{S+}$ since the size of Δ^* is generally unknown.

Chapter 3

Estimation of Regression Parameters With Stochastic Constraints When the Errors are Spherically Symmetric Distributions

3.1 Introduction

In Chapter 2 we have discussed five estimators of regression coefficients for the general multiple linear model when there is a suspicion that the parameters belong to a non-stochastic sub-space. It is often that this type of uncertain prior information rarely occur. Some uncertainty about the prior information is the norm for many practical situations. In this chapter, we consider a stochastic subspace model (see Theil, H. and Goldberger, A.S. (1961) and Theil (1963)) instead of the non-stochastic one in the earlier chapter and study the properties of five estimators that may be possible to define similar to Chapter 2. With this idea in our mind we again consider the basic linear model

$$y_1 = X\beta + e \quad (3.1)$$

where e follows the multivariate student- t distribution given by the pdf

$$f_1(e) = \frac{\Gamma\left(\frac{\nu+n}{2}\right)}{(\pi\nu)^n/2\Gamma\left(\frac{\nu}{2}\right)\sigma^n} \left(1 + \frac{e'e}{\nu\sigma^2}\right)^{-\frac{(\nu+n)}{2}} \quad \nu, \sigma > 0, -\infty < e_i < \infty \quad (3.2)$$

Our primary object of this chapter is to estimate β , when the error distribution belongs to (3.2) and it is suspected that β may be restricted to the subspace defined by

$$r = R\beta + v, \quad (3.3)$$

where r is a $(q \times 1)$ observable vector, R is a $(q \times p)$ matrix of known constants of rank q and v is a $(q \times 1)$ random vector which is distributed as a non-normal distribution with pdf $f_2(v)$ defined by

$$f_2(v) = \int_0^\infty f_q(e|\tau)g(\tau)d\tau, \quad (3.4)$$

where $f_2(\mathbf{v})$ is the probability density function (pdf) of $\mathbf{v}$, $f_q(\mathbf{e}|\tau)$ is the pdf of $N_q(\boldsymbol{\psi}, \tau^2 \boldsymbol{\Omega})$ and $g(\tau)$ is inverted Gamma density with scale σ^2 and degrees of freedom ν denoted by $IG(\nu, \sigma^2)$. Thus

$$f_2(\mathbf{v}) = \frac{\Gamma\left(\frac{q+\nu}{2}\right) |\boldsymbol{\Omega}|^{-1/2}}{(\pi\nu)^{q/2} \Gamma\left(\frac{\nu}{2}\right) \sigma^q} \left[1 + \frac{(\mathbf{v} - \boldsymbol{\psi})' \boldsymbol{\Omega}^{-1} (\mathbf{v} - \boldsymbol{\psi})}{\nu \sigma^2}\right]^{-\frac{q+\nu}{2}}, \quad \sigma, \nu > 0, -\infty < u_i < \infty \quad (3.5)$$

where

$$E(\mathbf{v}) = \boldsymbol{\psi} \text{ and } E(\mathbf{v}\mathbf{v}') = \sigma_e^2 \boldsymbol{\Omega} + \boldsymbol{\psi}\boldsymbol{\psi}', \quad \sigma_e^2 = \frac{\nu}{\nu-2} \sigma^2.$$

We assume $\mathbf{v}|\tau$ and $\mathbf{e}|\tau$ are independent. We combine the stochastic prior and sample information, then we get the linear statistical model

$$\begin{bmatrix} \mathbf{y}_1 \\ \mathbf{r} \end{bmatrix} = \begin{bmatrix} \mathbf{X} \\ \mathbf{R} \end{bmatrix} \boldsymbol{\beta} + \begin{bmatrix} \mathbf{e} \\ \mathbf{v} \end{bmatrix}, \quad (3.6)$$

where

$$\begin{bmatrix} \mathbf{e} \\ \mathbf{v} \end{bmatrix} | \tau \sim N_{n+q} \left\{ \begin{bmatrix} \mathbf{0} \\ \boldsymbol{\psi} \end{bmatrix}, \tau^2 \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Omega} \end{bmatrix} \right\} \quad (3.7)$$

subject to

$$[-\mathbf{R}, \mathbf{I}_q] \begin{bmatrix} \boldsymbol{\beta} \\ \mathbf{R}\boldsymbol{\beta} + \boldsymbol{\psi} \end{bmatrix} = \boldsymbol{\psi} = \mathbf{0}. \quad (3.8)$$

Rewrite the model as

$$\mathbf{y} = \mathbf{Z}\boldsymbol{\phi} + \boldsymbol{\epsilon}, \quad (3.9)$$

subject to the exact restriction

$$\mathbf{H}\boldsymbol{\phi} = \boldsymbol{\psi} = \mathbf{0} \quad (3.10)$$

where

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \boldsymbol{\Omega}^{-1/2} \mathbf{r} \end{bmatrix}, \quad \mathbf{Z} = \begin{bmatrix} \mathbf{X} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Omega}^{-1/2} \end{bmatrix},$$

$$\phi = \begin{bmatrix} \beta \\ \mathbf{R}\beta + \psi \end{bmatrix}, \quad \mathbf{H} = [-\mathbf{R}, \mathbf{I}_q]$$

and

$$\epsilon = \begin{bmatrix} \mathbf{e} \\ \Omega^{-1/2}(\mathbf{v} - \psi) \end{bmatrix} \Big| \tau \sim N_{n+q}(\mathbf{0}, \tau^2 \mathbf{I}_{n+q}). \quad (3.11)$$

Therefore,

$$f(\epsilon) = \frac{\Gamma\left(\frac{n+q+\nu}{2}\right)}{(\pi\nu)^{\frac{n+q}{2}} \Gamma\left(\frac{\nu}{2}\right) \sigma^{(n+q)}} \left(1 + \frac{\epsilon' \epsilon}{\nu \sigma^2}\right)^{-\frac{(n+q+\nu)}{2}}, \quad \nu, \sigma > 0, -\infty < \epsilon_i < \infty \quad (3.12)$$

If H_0 is true then one may obtain estimators of β under (3.10) and with error distribution given by (3.12). However, if H_0 is suspicious, then one may carry out a test on H_0 using the statistic $\mathcal{L}_n$, defined by

$$\mathcal{L}_n = \frac{\tilde{\phi}' \mathbf{H}' [\mathbf{H}(\mathbf{Z}' \mathbf{Z})^{-1} \mathbf{H}']^{-1} \mathbf{H} \tilde{\phi}}{q S_e^2} \quad (3.13)$$

with

$$S_e^2 = \frac{(\mathbf{y} - \mathbf{Z} \tilde{\phi})' (\mathbf{y} - \mathbf{Z} \tilde{\phi})}{(n - p)} \quad (3.14)$$

It is easy to see that

$$(\mathbf{y} - \mathbf{Z} \tilde{\phi})' (\mathbf{y} - \mathbf{Z} \tilde{\phi}) = (\mathbf{y}_1 - \mathbf{X} \tilde{\beta})' (\mathbf{y}_1 - \mathbf{X} \tilde{\beta})$$

and

$$\begin{aligned} (\tilde{\phi} - \hat{\phi})' \mathbf{Z}' \mathbf{Z} (\tilde{\phi} - \hat{\phi}) &= \tilde{\phi}' \mathbf{H}' [\mathbf{H}(\mathbf{Z}' \mathbf{Z})^{-1} \mathbf{H}']^{-1} \mathbf{H} \tilde{\phi} \\ &= (\mathbf{R} \tilde{\beta} - \mathbf{r})' (\mathbf{R} \mathbf{C}^{-1} \mathbf{R}' + \Omega)^{-1} (\mathbf{R} \tilde{\beta} - \mathbf{r}), \quad \mathbf{C} = \mathbf{X}' \mathbf{X}. \end{aligned} \quad (3.15)$$

With this background at hand we shall proceed with the discussion of various estimators of β and σ_e^2 under multivariate t -errors in the linear model (3.9).

3.2 The Proposed Estimators

Following Theorems 2.1-2.4, the unrestricted estimator (UE) of ϕ under (3.12) is

$$\tilde{\phi} = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}\mathbf{y} = \begin{bmatrix} \tilde{\phi}_1 \\ \tilde{\phi}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{C}^{-1}\mathbf{X}'\mathbf{y}_1 \\ \mathbf{r} \end{bmatrix} = \begin{bmatrix} \tilde{\beta} \\ \mathbf{r} \end{bmatrix} \quad (3.16)$$

The restricted estimator (RE) due to restriction (3.10) is given by

$$\begin{aligned} \hat{\phi} = \begin{bmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \end{bmatrix} &= \tilde{\phi} - (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{H}'[\mathbf{H}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{H}]^{-1}\mathbf{H}\tilde{\phi} \\ &= \begin{bmatrix} \tilde{\beta} \\ \mathbf{r} \end{bmatrix} - \begin{bmatrix} \mathbf{C}^{-1}\mathbf{R}'(\mathbf{RC}^{-1}\mathbf{R}' + \mathbf{\Omega})^{-1}(\mathbf{R}\tilde{\beta} - \mathbf{r}) \\ -\mathbf{\Omega}(\mathbf{RC}^{-1}\mathbf{R}' + \mathbf{\Omega})^{-1}(\mathbf{R}\tilde{\beta} - \mathbf{r}) \end{bmatrix} = \begin{bmatrix} \hat{\beta} \\ \mathbf{R}\hat{\beta} \end{bmatrix} \end{aligned} \quad (3.17)$$

The usual preliminary test estimator (PTE) of ϕ is defined by

$$\hat{\phi}^{PT} = \hat{\phi} + [1 - I(\mathcal{L}_n \leq F_\alpha)](\tilde{\phi} - \hat{\phi}), \quad (3.18)$$

where $I(A)$ is the indicator function of the set A , F_α is the upper 100α percentile of the central F distribution with $(q, n-p)$ DF and under $\mathbf{H}\phi \neq \psi \neq \mathbf{0}$ the pdf of $\mathcal{L}_n$ is given by

$$\begin{aligned} &g_{q,n-p}(\mathcal{L}_n; \Delta^*, \nu) \\ &= \sum_{r=0}^{\infty} \left(\frac{q}{n-p}\right)^{q/2+r} \frac{\Gamma\left(\frac{q}{2} + r\right) \Gamma\left(\frac{n-p+q}{2} + r\right) \left(\frac{\Delta^*}{\nu-2}\right)^r \mathcal{L}_n^{q/2+r-1}}{\Gamma(r+1) \Gamma(\nu/2) \Gamma\left(\frac{q}{2} + r\right) \Gamma\left(\frac{n-p}{2}\right) \left(1 + \frac{\Delta^*}{\nu-2}\right)^{\frac{q}{2}+r} \left(1 + \frac{q}{n-p} \mathcal{L}_n\right)^{\frac{n-p+q}{2}+r}} \end{aligned} \quad (3.19)$$

with

$$\Delta^* = \frac{\theta_1}{\sigma_e^2}, \quad \theta_1 = \psi'(\mathbf{RC}^{-1}\mathbf{R}' + \mathbf{\Omega})^{-1}\psi, \quad \psi = E(\mathbf{r}) - \mathbf{R}\beta. \quad (3.20)$$

The shrinkage estimator (SE) of ϕ is given by

$$\begin{aligned} \hat{\phi}^S &= \hat{\phi} + (1 - d\mathcal{L}_n^{-1})(\tilde{\phi} - \hat{\phi}) \\ &= \tilde{\phi} - d\mathcal{L}_n^{-1}(\tilde{\phi} - \hat{\phi}) \end{aligned} \quad (3.21)$$

where

$$d = \frac{(q-2)(n-p)}{q(n-p+2)} \quad \text{and } q \geq 3. \quad (3.22)$$

The positive-rule shrinkage estimator (PRSE) of ϕ is given by

$$\begin{aligned} \hat{\phi}^{S+} &= \hat{\phi} + (1 - d\mathcal{L}_n^{-1})I(\mathcal{L}_n > d)(\tilde{\phi} - \hat{\phi}) \\ &= \hat{\phi}^S - (1 - d\mathcal{L}_n^{-1})I(\mathcal{L}_n \leq d)(\tilde{\phi} - \hat{\phi}) \end{aligned} \quad (3.23)$$

In the next section, we give the expressions for the bias, quadratic risk and mean squared error (MSE) matrix of the estimators when the errors follow a multivariate student- t (Mt) distribution.

3.3 Bias, Quadratic Risk and MSE Matrix of the Estimators

3.3.1 Biases of the Estimators

Theorem 3.1 Bias of the unrestricted estimator (UE), restricted estimator (RE), preliminary test estimator (PTE), shrinkage estimator (SE) and positive-rule shrinkage estimator (PRSE) are given by (3.24)–(3.28) respectively:

$$b_1 = E(\tilde{\phi} - \phi) = 0 \quad (3.24)$$

$$b_2 = E(\hat{\phi} - \phi) = -(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{H}'[\mathbf{H}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{H}']^{-1}\mathbf{H}\phi \quad (3.25)$$

$$b_3 = E(\hat{\phi}^{PT} - \phi) = -(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{H}'[\mathbf{H}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{H}']^{-1}\mathbf{H}\phi G_{q+2, n-p}^{(2)}(x', \Delta^*) \quad (3.26)$$

$$b_4 = E(\hat{\phi}^S - \phi) = -qd(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{H}'[\mathbf{H}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{H}']^{-1}\mathbf{H}\phi E^{(2)}[\chi_{q+2}^{-2}(\Delta^*)] \quad (3.27)$$

$$\begin{aligned} b_5 &= E(\hat{\phi}^{S+} - \phi) = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{H}'[\mathbf{H}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{H}']^{-1}\mathbf{H}\phi \\ &\quad \times \left\{ \frac{qd}{q+2} E^{(2)} \left[F_{q+2, n-p}^{-1}(\Delta^*) I \left(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2} \right) \right] \right. \\ &\quad \left. - \frac{qd}{q+2} E^{(2)} \left[F_{q+2, n-p}^{-1}(\Delta^*) \right] - G_{q+2, n-p}^{(2)}(x, \Delta^*) \right\} \end{aligned} \quad (3.28)$$

where

$$G_{q+2, n-p}^{(2)}(\cdot, \Delta^*), E^{(2)}[\chi_{q+2}^{-2}(\Delta^*)], E^{(2)}\left[F_{q+2, n-p}^{-1}(\Delta^*)I\left(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2}\right)\right], \\ E^{(2)}\left[F_{q+2, n-p}^{-1}(\Delta^*)\right], x \text{ and } x' \text{ are defined in (2.36)–(2.38) and (2.40).}$$

Proof. Following Theorem 2.5, then the result follows.

3.3.2 Risks of the Estimators

For the risk of the estimator, we consider quadratic loss function

$$L(\phi^*; \phi) = (\phi^* - \phi)'Z'Z(\phi^* - \phi) \quad (3.29)$$

where ϕ^* can be any one of the five estimators. Thus we have the risk of ϕ^* as

$$R(\phi^*; \phi) = E[(\phi^* - \phi)'Z'Z(\phi^* - \phi)]. \quad (3.30)$$

Direct computation and following Judge and Bock (1978) lead to the following theorem.

Theorem 3.2. *Risks of UE, RE, PTE, SE and PRSE are given by (3.31) – (3.35) respectively:*

$$\text{i) } R_1(\tilde{\phi}; \phi) = (p+q)\sigma_e^2 \quad (3.31)$$

$$\text{ii) } R_2(\hat{\phi}; \phi) = p\sigma_e^2 + \eta_1' \eta_1 \quad (3.32)$$

$$\text{iii) } R_3(\hat{\phi}^{PT}; \phi) = (p+q)\sigma_e^2 - q\sigma_e^2 G_{q+2, n-p}^{(1)}(x', \Delta^*) \\ + \eta_1' \eta_1 \left[2G_{q+2, n-p}^{(2)}(x', \Delta^*) - G_{q+4, n-p}^{(2)}(x', \Delta^*) \right] \quad (3.33)$$

$$\text{iv) } R_4(\hat{\phi}^S; \phi) = (p+q)\sigma_e^2 - dq^2\sigma_e^2 \left\{ (q-2)E^{(1)}[\chi_{q+2}^{-4}(\Delta^*)] \right. \\ \left. + \frac{(q-2)}{q}\Delta^* E^{(2)}[\chi_{q+4}^{-4}(\Delta^*)] \right\} \quad (3.34)$$

$$\text{v) } R_5(\hat{\phi}^{S+}; \phi) = R_4(\hat{\phi}^S; \phi) - \sigma_e^2 \left\{ qE^{(1)} \left[\left(1 - \frac{qd}{q+2} F_{q+2, n-p}^{-1}(\Delta^*) \right)^2 I\left(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2}\right) \right] \right. \\ \left. + \Delta^* E^{(2)} \left[\left(1 - \frac{qd}{q+4} F_{q+4, n-p}^{-1}(\Delta^*) \right)^2 I\left(F_{q+4, n-p}(\Delta^*) \leq \frac{qd}{q+4}\right) \right] \right\} \\ - 2\eta_1' \eta_1 E^{(2)} \left[\left(\frac{qd}{q+2} F_{q+2, n-p}^{-1}(\Delta^*) - 1 \right) I\left(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2}\right) \right], \quad (3.35)$$

where $G_{q+s, n-p}^{(j)}(x', \Delta^*)$, $E^{(j)}[\chi_{q+s}^{-1}(\Delta^*)]$, $E^{(j)}\left[F_{q+2, n-p}^{-1}(\Delta^*)I\left(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2}\right)\right]$, and $E^{(j)}\left[F_{q+s, n-p}^{-2}(\Delta^*)I\left(F_{q+s, n-p}(\Delta^*) \leq \frac{qd}{q+s}\right)\right]$, $j = 1, 2, s = 1, 2$, are defined in (2.49), (2.50), (2.36) and (2.38).

Proof. In Theorem 2.6, let $W = C = Z'Z$ and

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} I_q & 0 \\ 0 & I_p \end{pmatrix},$$

then we get

$$\begin{aligned} \text{i)} \quad R_1(\tilde{\phi}; \phi) &= E[(\tilde{\phi} - \phi)' Z' Z (\tilde{\phi} - \phi)] \\ &= \text{tr}\{Z' Z E[(\tilde{\phi} - \phi)(\tilde{\phi} - \phi)']\} \\ &= (p+q)\sigma_e^2 \end{aligned}$$

since $\text{var}(\tilde{\phi}) = \sigma_e^2(Z'Z)^{-1}$.

ii)

$$\begin{aligned} R_2(\hat{\phi}; \phi) &= E[(\hat{\phi} - \phi)' Z' Z (\hat{\phi} - \phi)] \\ &= \text{tr}\{Z' Z E[(\hat{\phi} - \phi)(\hat{\phi} - \phi)']\} \\ &= \text{tr}\{Z' Z [(Z'Z)^{-1}(I_{p+q} - H'[H(Z'Z)^{-1}H']^{-1}H(Z'Z)^{-1})\sigma_e^2 + b_2b_2']\} \\ &= (p+q)\sigma_e^2 - \sigma_e^2 \text{tr}\{(Z'Z)^{-1/2}H'[H(Z'Z)^{-1}H']^{-1}H(Z'Z)^{-1/2}\} + b_2'Z'Zb_2 \end{aligned}$$

Note that $(Z'Z)^{-1/2}H'[H(Z'Z)^{-1}H']^{-1}H(Z'Z)^{-1/2}$ is a symmetric idempotent matrix of rank q . Therefore, there exists an orthonormal matrix Q such that

$$Q(Z'Z)^{-1/2}H'[H(Z'Z)^{-1}H']^{-1}H(Z'Z)^{-1/2}Q' = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix} \quad (3.36)$$

$$\begin{aligned} Q(Z'Z)^{-1/2}(Z'Z)(Z'Z)^{-1/2}Q' &= QQ' = \begin{bmatrix} I_q & 0 \\ 0 & I_p \end{bmatrix} \\ &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = A \end{aligned} \quad (3.37)$$

Hence, $A_{11} = I_q$, $A_{22} = I_p$. Now, we define the random variable

$$\mathbf{w} = \mathbf{Q}(\mathbf{Z}'\mathbf{Z})^{\frac{1}{2}}\tilde{\phi} \quad (3.38)$$

so

$$\mathbf{w}|\tau \sim N_{p+q}(\eta, \tau^2 \mathbf{I}_{p+q})$$

where

$$\eta = E(\mathbf{w}|\tau) = \mathbf{Q}(\mathbf{Z}'\mathbf{Z})^{1/2}\phi \quad (3.39)$$

Partitioning the vectors $\mathbf{w} = (\mathbf{w}'_1, \mathbf{w}'_2)'$ and $\eta = (\eta'_1, \eta'_2)'$ where $\mathbf{w}_1|\tau$ and $\mathbf{w}_2|\tau$ are independent subvector of order q and p respectively. Using (3.36), (3.38) and (3.39), we get

$$\mathcal{L}_n = \frac{\mathbf{w}'_1 \mathbf{w}_1}{qS_e^2}, \Delta^* = \frac{\theta_1}{\sigma_e^2}, \theta_1 = \eta'_1 \eta_1 = \phi' \mathbf{H}' [\mathbf{H}(\mathbf{Z}'\mathbf{Z})^{-1} \mathbf{H}']^{-1} \mathbf{H} \phi \quad (3.40)$$

Now, we can write

$$\begin{aligned} & \text{tr}\{(\mathbf{Z}'\mathbf{Z})^{-1/2} \mathbf{H}' [\mathbf{H}(\mathbf{Z}'\mathbf{Z})^{-1} \mathbf{H}']^{-1} \mathbf{H} (\mathbf{Z}'\mathbf{Z})^{-1/2}\} \\ &= \text{tr}\{\mathbf{Q}(\mathbf{Z}'\mathbf{Z})^{-1/2} \mathbf{H}' [\mathbf{H}(\mathbf{Z}'\mathbf{Z})^{-1} \mathbf{H}']^{-1} \mathbf{H} (\mathbf{Z}'\mathbf{Z})^{-1/2} \mathbf{Q}'\} = q \end{aligned}$$

since $\text{tr}(\mathbf{BD}) = \text{tr}(\mathbf{DB})$. Using (3.25)

$$\begin{aligned} \mathbf{b}'_2 \mathbf{Z}' \mathbf{Z} \mathbf{b}_2 &= \phi' \mathbf{H}' [\mathbf{H}(\mathbf{Z}'\mathbf{Z})^{-1} \mathbf{H}']^{-1} \mathbf{H} (\mathbf{Z}'\mathbf{Z})^{-1} (\mathbf{Z}'\mathbf{Z}) (\mathbf{Z}'\mathbf{Z})^{-1} \mathbf{H}' [\mathbf{H}(\mathbf{Z}'\mathbf{Z})^{-1} \mathbf{H}']^{-1} \mathbf{H} \phi \\ &= \phi' \mathbf{H}' [\mathbf{H}(\mathbf{Z}'\mathbf{Z})^{-1} \mathbf{H}']^{-1} \mathbf{H} \phi = \eta'_1 \eta_1 \end{aligned}$$

Thus, we have

$$R_2(\hat{\phi}; \phi) = p\sigma_e^2 + \eta'_1 \eta_1$$

iii) Following Theorem 2.6 and using (3.18), we get

$$\begin{aligned} R_3(\hat{\phi}^{PT}; \phi) &= (p+q)\sigma_e^2 - q\sigma_e^2 G_{q+2, n-p}^{(1)}(x', \Delta^*) \\ &\quad + \eta'_1 \eta_1 [2G_{q+2, n-p}^{(2)}(x', \Delta^*) - G_{q+4, n-p}^{(2)}(x', \Delta^*)]. \end{aligned}$$

iv) Following Theorem 2.6 and using (2.58),

$$\begin{aligned} R_4(\hat{\phi}^S; \phi) &= E[(\hat{\phi}^S - \phi)' Z' Z (\hat{\phi}^S - \phi)] \\ &= (p+q)\sigma_e^2 - dq^2\sigma_e^2 \left\{ (q-2)E^{(2)}[\chi_{q+2}^{-4}(\Delta^*)] \right. \\ &\quad \left. + \frac{(q-2)}{q}\Delta^* E^{(2)}[\chi_{q+4}^{-4}(\Delta^*)] \right\}. \end{aligned}$$

v) Following Theorem 2.6 and using (3.23),

$$\begin{aligned} R_5(\hat{\phi}^{S+}; \phi) &= E[(\hat{\phi}^{S+} - \phi)' Z' Z (\hat{\phi}^{S+} - \phi)] \\ &= R_4(\hat{\phi}^S; \phi) - \sigma_e^2 \left\{ qE^{(1)} \left[\left(1 - \frac{qd}{q+2} F_{q+2, n-p}^{-1}(\Delta^*)\right)^2 I(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2}) \right] \right. \\ &\quad \left. + \Delta^* E^{(2)} \left[\left(1 - \frac{qd}{q+4} F_{q+4, n-p}^{-1}(\Delta^*)\right)^2 I(F_{q+4, n-p}(\Delta^*) \leq \frac{qd}{q+4}) \right] \right\} \\ &\quad - 2\eta_1' \eta_1 E^{(2)} \left[\left(\frac{qd}{q+2} F_{q+2}^{-1}(\Delta^*) - 1 \right) I(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2}) \right], \end{aligned}$$

where $\Delta^* = \frac{\eta_1' \eta_1}{\sigma_e^2} = .$

3.3.3 MSE matrices of the Estimators

In this subsection we present the mean squared error (MSE) matrices of the five estimators. Let $M_1(\tilde{\phi})$, $M_2(\hat{\phi})$, $M_3(\hat{\phi}^{PT})$, $M_4(\hat{\phi}^S)$ and $M_5(\hat{\phi}^{S+})$ denote mean squared error matrix of $\tilde{\phi}$, $\hat{\phi}$, $\hat{\phi}^{PT}$, $\hat{\phi}^S$ and $\hat{\phi}^{S+}$ respectively.

Theorem 3.3 Mean squared error matrices of UE, RE, PTE, SE and PRSE are given by (3.41) – (3.45) respectively:

$$\text{i) } M_1(\tilde{\phi}) = E[(\tilde{\phi} - \phi)(\tilde{\phi} - \phi)'] = \sigma_e^2(Z'Z)^{-1} \quad (3.41)$$

$$\begin{aligned} \text{ii) } M_2(\hat{\phi}) &= E[\hat{\phi} - \phi)(\hat{\phi} - \phi)'] \\ &= \sigma_e^2(Z'Z)^{-1} - \sigma_e^2(Z'Z)^{-1} H' [H(Z'Z)^{-1} H']^{-1} H(Z'Z)^{-1} + b_2 b_2' \quad (3.42) \end{aligned}$$

where b_2 is defined in (3.25)

$$\begin{aligned}
 \text{iii) } M_3(\hat{\phi}^{PT}) &= E[(\hat{\phi}^{PT} - \phi)(\hat{\phi}^{PT} - \phi)'] \\
 &= \sigma_e^2(Z'Z)^{-1} - \sigma_e^2(Z'Z)^{-1}H'[H(Z'Z)^{-1}H']^{-1}H(Z'Z)^{-1}G_{q+2,n-p}^{(1)}(x', \Delta^*) \\
 &\quad + b_2b_2' \left[2G_{q+2,n-p}^{(2)}(x', \Delta^*) - G_{q+4,n-p}^{(2)}(x', \Delta^*) \right]. \quad (3.43)
 \end{aligned}$$

$$\begin{aligned}
 \text{iv) } M_4(\hat{\phi}^S) &= E[(\hat{\phi}^S - \phi)(\hat{\phi}^S - \phi)'] \\
 &= \sigma_e^2(Z'Z)^{-1} - dq\sigma_e^2(Z'Z)^{-1}H'[H(Z'Z)^{-1}H']^{-1}H(Z'Z)^{-1} \left\{ (q-2)E^{(1)}[\chi_{q+2}^{-4}(\Delta^*)] \right. \\
 &\quad \left. + 2\Delta^*E^{(2)}[\chi_{q+4}^{-4}(\Delta^*)] \right\} + dq(q+2)b_2b_2'E^{(2)}[\chi_{q+4}^{-4}(\Delta^*)] \quad (3.44)
 \end{aligned}$$

$$\begin{aligned}
 \text{v) } M_5(\hat{\phi}^{S+}) &= E[(\hat{\phi}^{S+} - \phi)(\hat{\phi}^{S+} - \phi)'] \\
 &= M_4(\hat{\phi}^S) - \sigma_e^2 \left\{ (Z'Z)^{-1}H'[H(Z'Z)^{-1}H']^{-1}H(Z'Z)^{-1} \right. \\
 &\quad \times E^{(1)} \left[\left(1 - \frac{qd}{q+2} F_{q+2,n-p}^{-1}(\Delta^*) \right)^2 I \left(F_{q+2,n-p}(\Delta^*) \leq \frac{qd}{q+2} \right) \right] \\
 &\quad + \frac{b_2b_2'}{\sigma_e^2} E^{(2)} \left[\left(1 - \frac{qd}{q+4} F_{q+4,n-p}^{-1}(\Delta^*) \right)^2 I \left(F_{q+4,n-p}(\Delta^*) \leq \frac{qd}{q+4} \right) \right] \Big\} \\
 &\quad - 2b_2b_2'E^{(2)} \left[\left(\frac{qd}{q+2} F_{q+2,n-p}^{-1}(\Delta^*) - 1 \right) I \left(F_{q+2,n-p}(\Delta^*) \leq \frac{qd}{q+2} \right) \right] \quad (3.45)
 \end{aligned}$$

Proof. Following Theorem 2.7, the result follows.

3.4 Relative Performance of the Estimators

3.4.1 Risk Analysis of the Estimators

In this subsection, we study the relative performance of the estimators under the null hypothesis as well as under the alternative hypothesis. Firstly, we note that

$$\frac{R_2(\hat{\phi}; \phi)}{R_1(\hat{\phi}; \phi)} \begin{pmatrix} > \\ = \\ < \end{pmatrix} 1 \text{ according as } \Delta^* \begin{pmatrix} > \\ = \\ < \end{pmatrix} q \quad (3.46)$$

Secondly,

$$\frac{R_3(\hat{\phi}^{PT}; \phi)}{R_1(\tilde{\phi}; \phi)} \begin{pmatrix} > \\ = \\ < \end{pmatrix} 1 \text{ according as}$$

$$\Delta^* \left[2G_{q+2, n-p}^{(2)}(x', \Delta^*) - G_{q+4, n-p}^{(2)}(x', \Delta^*) \right] \begin{pmatrix} > \\ = \\ < \end{pmatrix} qG_{q+2, n-p}^{(1)}(x', \Delta^*) \quad (3.47)$$

and

$$\frac{R_3(\hat{\phi}^{PT}; \phi)}{R_2(\hat{\phi}; \phi)} \begin{pmatrix} > \\ = \\ < \end{pmatrix} 1 \text{ according as}$$

$$q \left[1 - G_{q+2, n-p}^{(1)}(x', \Delta^*) \right] \begin{pmatrix} > \\ = \\ < \end{pmatrix} \Delta^* \left\{ 1 - \left[2G_{q+2, n-p}^{(2)}(x', \Delta^*) - G_{q+4, n-p}^{(2)}(x', \Delta^*) \right] \right\} \quad (3.48)$$

Thirdly,

$$\frac{R_4(\hat{\phi}^S; \phi)}{R_1(\tilde{\phi}; \phi)} \leq 1 \quad \forall \eta_1, q \geq 3 \quad (3.49)$$

Thus the risk of $\hat{\phi}^S$ is uniformly smaller than $\tilde{\phi}$ and attains the upper limit $R_1(\tilde{\phi}; \phi)$ as $\Delta^* \rightarrow \infty$. This shows inadmissibility of $\tilde{\phi}$ relative to $\hat{\phi}^S$. To compare $\hat{\phi}$ and $\hat{\phi}^S$, we may write

$$R_4(\hat{\phi}^S; \phi) = R_2(\hat{\phi}; \phi) + q\sigma_e^2 - \eta_1' \eta_1 - \sigma_e^2 dq^2 \left\{ (q-2)E^{(1)}[\chi_{q+2}^{-4}(\Delta^*)] + \frac{(q-2)}{q} \Delta^* E^{(2)}[\chi_{q+4}^{-4}(\Delta^*)] \right\}. \quad (3.50)$$

Under restriction $H\phi = \psi = 0$, this becomes

$$R_4(\hat{\phi}^S; \phi) = R_2(\hat{\phi}; \phi) + \frac{2(n-p+q)}{n-p+2} > R_2(\hat{\phi}; \phi) \quad (3.51)$$

since $q \geq 3$. Hence, the risk of $\hat{\phi}^S$ under restriction is greater than the risk of $\hat{\phi}$. However, as η_1 moves away from 0, $\eta_1' \eta_1$ increases and the risk of $\hat{\phi}$ becomes unbounded while the risk of $\hat{\phi}^S$ remains below the risk of $\tilde{\phi}$ and merges with it as $\Delta^* \rightarrow \infty$. Thus $\hat{\phi}^S$ does not dominate $\hat{\phi}$ near $\Delta^* = 0$. However, outside a small

interval around $\eta_1 = 0$, $\hat{\phi}^S$ dominates $\hat{\phi}$. Now, we compare $\hat{\phi}^S$ and $\hat{\phi}^{PT}$. First, consider the case when the restriction holds. In this case

$$\begin{aligned} R_4(\hat{\phi}^S; \phi) &= R_3(\hat{\phi}^{PT}; \phi) + q\sigma_c^2 [G_{q+2, n-p}^{(1)}(x', 0) - d] \\ &\geq R_3(\hat{\phi}^{PT}; \phi) \end{aligned} \quad (3.52)$$

where

$$G_{q+2, n-p}^{(1)}(x', 0) = I_x \left[\frac{1}{2}(q+2), \frac{1}{2}(n-p) \right] = F_{q+2, n-p}(\ell_\alpha^{*(1)}, 0) = 1 - \alpha, \ell_\alpha^{*(1)} = \frac{qF_\alpha}{q+2},$$

for all α such that $\ell_\alpha^{*(1)}$ satisfies the inequality

$$\left\{ \alpha : \ell_\alpha^{*(1)} > F_{q+2, n-p}^{-1}(d, 0) \right\} \quad (3.53)$$

The risk of $\hat{\phi}^S$ is smaller than risk of $\hat{\phi}^{PT}$ when the critical value $\ell_\alpha^{*(1)}$ satisfies the opposite inequality in (3.53). Thus, $\hat{\phi}^S$ does not always dominate $\hat{\phi}^{PT}$ when the restriction holds. Therefore, we may order the dominance picture under (3.53) as follows:

$$\hat{\phi} \succ \hat{\phi}^{PT} \succ \hat{\phi}^S \succ \tilde{\phi} \quad (3.54)$$

The picture changes as Δ^* moves away from 0. As $\Delta^* \rightarrow \infty$, the risk of $\hat{\phi}^{PT}$ and $\hat{\phi}^S$ converge to the risk of $\tilde{\phi}$.

Finally, we compare the risks of $\hat{\phi}^{S+}$ and $\hat{\phi}^S$. The risk difference is given by

$$\begin{aligned} &R_5(\hat{\phi}^{S+}; \phi) - R_4(\hat{\phi}^S; \phi) \\ &= -\sigma_c^2 \left\{ qE^{(1)} \left[\left(1 - \frac{qd}{q+2} F_{q+2, n-p}^{-1}(\Delta^*) \right)^2 I \left(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2} \right) \right] \right. \\ &\quad \left. + \Delta^* E^{(2)} \left[\left(1 - \frac{qd}{q+4} F_{q+4, n-p}^{-1}(\Delta^*) \right)^2 I \left(F_{q+4, n-p}(\Delta^*) \leq \frac{qd}{q+4} \right) \right] \right\} \\ &\quad - 2\eta_1' \eta_1 E^{(2)} \left[\left(\frac{qd}{q+2} F_{q+2, n-p}^{-1}(\Delta^*) - 1 \right) I \left(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2} \right) \right] \end{aligned} \quad (3.55)$$

The right hand side of (3.55) is negative since the expectation of positive random variable is positive. Thus for all ϕ , $R_5(\hat{\phi}^{S+}; \phi) \leq R_4(\hat{\phi}^S; \phi)$. Therefore the positive-rule shrinkage estimator not only confirms the inadmissibility of the shrinkage estimator, but also demonstrates a simple superior estimator. We note that

$$\frac{R_5(\hat{\phi}^{S+}; \phi)}{R_1(\hat{\phi}; \phi)} \leq 1 \quad \forall \eta_1, q \geq 3 \quad (3.56)$$

since (3.49) and $R_5(\hat{\phi}^{S+}; \phi) \leq R_4(\hat{\phi}^S; \phi)$ for all ϕ . Thus, $\hat{\phi}^{S+}$ uniformly dominates $\hat{\phi}$. To compare $\hat{\phi}$ and $\hat{\phi}^{S+}$, we may write under $H\phi = \psi = 0$

$$\begin{aligned} & R_5(\hat{\phi}^{S+}; \phi) \\ &= R_2(\hat{\phi}; \phi) + q\sigma_e^2 \left\{ 1 - d - E^{(1)} \left[\left(1 - \frac{qd}{q+2} F_{q+2, n-p}^{-1}(0) \right)^2 I \left(F_{q+2, n-p}(0) \leq \frac{qd}{q+2} \right) \right] \right\} \\ &\geq R_2(\hat{\phi}; \phi) \end{aligned} \quad (3.57)$$

since (2.91). Hence, the risk of $\hat{\phi}^{S+}$ under restriction is greater than the risk of $\hat{\phi}$. However, as η_1 moves away from 0, $\eta_1' \eta_1$ increases and the risk of $\hat{\phi}$ becomes unbounded while the risk of $\hat{\phi}^{S+}$ remains below the risk of $\hat{\phi}$ and merges with it as $\Delta^* \rightarrow \infty$. Thus $\hat{\phi}^{S+}$ dominates $\hat{\phi}$ outside an interval around the origin. Now, we compare $\hat{\phi}^{S+}$ and $\hat{\phi}^{PT}$. First, under restriction,

$$\begin{aligned} R_5(\hat{\phi}^{S+}; \phi) &= R_3(\hat{\phi}^{PT}; \phi) + q\sigma_e^2 \left\{ G_{q+2, n-p}^{(1)}(x', 0) - d \right. \\ &\quad \left. - E^{(1)} \left[\left(1 - \frac{qd}{q+2} F_{q+2, n-p}^{-1}(0) \right)^2 I \left(F_{q+2, n-p}(0) \leq \frac{qd}{q+2} \right) \right] \right\} \\ &\geq R_3(\hat{\phi}^{PT}; \phi) \end{aligned} \quad (3.58)$$

for all α satisfying the condition

$$\left\{ \alpha : F_{q+2, n-p}(\ell_\alpha^{s(1)}, 0) \geq d + E^{(1)} \left[\left(1 - \frac{qd}{q+2} F_{q+2, n-p}^{-1}(0) \right)^2 I \left(F_{q+2, n-p}(0) \leq \frac{qd}{q+2} \right) \right] \right\}. \quad (3.59)$$

The risk of $\hat{\phi}^{S+}$ is smaller than risk of $\hat{\phi}^{PT}$ when the critical value $\ell_{\alpha}^{*(1)} = \frac{qF_{\alpha}}{q+2}$ satisfies opposite inequality in (3.59). Thus, $\hat{\phi}^{S+}$ does not always dominate $\hat{\phi}^{PT}$ when the restriction holds. Therefore, we may order the dominance picture under (3.59) as follows:

$$\hat{\phi} \succ \hat{\phi}^{PT} \succ \hat{\phi}^{S+} \succ \hat{\phi}^S \succ \tilde{\phi} \quad (3.60)$$

The picture changes as η_1 moves away from 0. As $\Delta^* \rightarrow \infty$, the risk of $\hat{\phi}^{PT}$ and $\hat{\phi}^{S+}$ converge to the risk of $\tilde{\phi}$. But for reasonable Δ^* near 0, the risk of $\hat{\phi}^{PT}$ is smaller than the risk of $\hat{\phi}^{S+}$ for α satisfying (3.59) and for $q \geq 3$. Thus, neither $\hat{\phi}^{PT}$ nor $\hat{\phi}^{S+}$ dominates the other except that they share common property that as $\Delta^* \rightarrow \infty$ the risk of both becomes $(p+q)\sigma_e^2$. However, the risk of $\hat{\phi}^{S+}$ is below the risk of $\tilde{\phi}$ while the risk of $\hat{\phi}^{PT}$ exceeds the risk of $\tilde{\phi}$ at some intermediate values of Δ^* depends on α .

3.4.2 MSE Analysis of the Estimator

In this subsection we focus on MSE analysis of the estimators and determine their dominance properties. For the MSE comparison of $\tilde{\phi}$ and $\hat{\phi}$, we consider the MSE difference

$$M_1(\tilde{\phi}) - M_2(\hat{\phi}) = \sigma_e^2(Z'Z)^{-1}H'[H(Z'Z)^{-1}H']^{-1}H(Z'Z)^{-1} - b_2b_2' \quad (3.61)$$

Using (2.97)-(2.100), we conclude that (3.61) is positive semi-definite if and only if

$$\Delta^* \leq 1 \quad (3.62)$$

since

$$b_2'Z'Zb_2 = \phi'H'[H(Z'Z)^{-1}H']^{-1}H\phi = \eta_1'\eta_1.$$

Therefore under condition (3.62) the restricted estimator $\hat{\phi}$, performs better than the unrestricted estimator $\tilde{\phi}$, otherwise $\tilde{\phi}$ is superior. Comparison of $\hat{\phi}$ and $\hat{\phi}^{PT}$: the MSE difference

$$\begin{aligned} M_1(\tilde{\phi}) - M_3(\hat{\phi}^{PT}) &= \sigma_e^2 (\mathbf{Z}'\mathbf{Z})^{-1} \mathbf{H}' [\mathbf{H}(\mathbf{Z}'\mathbf{Z})^{-1} \mathbf{H}']^{-1} \mathbf{H}(\mathbf{Z}'\mathbf{Z})^{-1} G_{q+2, n-p}^{(1)}(x', \Delta^*) \\ &\quad - \mathbf{b}_2 \mathbf{b}_2' \left[2G_{q+2, n-p}^{(2)}(x', \Delta^*) - G_{q+4, n-p}^{(2)}(x', \Delta^*) \right] \end{aligned} \quad (3.63)$$

Using (2.102)–(2.104), implies that the MSE difference is positive semi-definite whenever

$$\Delta^* \leq \frac{G_{q+2, n-p}^{(1)}(x', \Delta^*)}{\left[2G_{q+2, n-p}^{(2)}(x', \Delta^*) - G_{q+4, n-p}^{(2)}(x', \Delta^*) \right]}. \quad (3.64)$$

Thus, $\hat{\phi}^{PT}$ is superior to $\tilde{\phi}$ in this range of Δ^* , otherwise $\tilde{\phi}$ is superior. To compare $\hat{\phi}$ and $\hat{\phi}^{PT}$, the MSE difference is

$$\begin{aligned} M_2(\hat{\phi}) - M_3(\hat{\phi}^{PT}) &= -\sigma_e^2 \mathbf{Z}'\mathbf{Z})^{-1} \mathbf{H}' [\mathbf{H}(\mathbf{Z}'\mathbf{Z})^{-1} \mathbf{H}']^{-1} \mathbf{H}(\mathbf{Z}'\mathbf{Z})^{-1} \left[1 - G_{q+2, n-p}^{(1)}(x', \Delta^*) \right] \\ &\quad + \mathbf{b}_2 \mathbf{b}_2' \left[1 - 2G_{q+2, n-p}^{(2)}(x', \Delta^*) + G_{q+4, n-p}^{(2)}(x', \Delta^*) \right] \end{aligned} \quad (3.65)$$

This MSE difference is negative semi-definite whenever

$$\Delta^* \leq \frac{1 - G_{q+2, n-p}^{(1)}(x', \Delta^*)}{\left[1 - 2G_{q+2, n-p}^{(2)}(x', \Delta^*) + G_{q+4, n-p}^{(2)}(x', \Delta^*) \right]}. \quad (3.66)$$

Thus, $\hat{\phi}$ is superior to $\hat{\phi}^{PT}$ in this range of Δ^* , otherwise $\hat{\phi}^{PT}$ is superior. Comparison of $\tilde{\phi}$ and $\hat{\phi}^S$: the MSE difference

$$\begin{aligned} M_1(\tilde{\phi}) - M_4(\hat{\phi}^S) &= dq\sigma_e^2 (\mathbf{Z}'\mathbf{Z})^{-1} \mathbf{H}' [\mathbf{H}(\mathbf{Z}'\mathbf{Z})^{-1} \mathbf{H}']^{-1} \mathbf{H}(\mathbf{Z}'\mathbf{Z})^{-1} \left\{ (q-2)E^{(1)}[\chi_{q+2}^{-4}(\Delta^*)] \right. \\ &\quad \left. + 2\Delta^* E^{(2)}[\chi_{q+4}^{-4}(\Delta^*)] \right\} - dq(q+2)\mathbf{b}_2 \mathbf{b}_2' E^{(2)}[\chi_{q+4}^{-4}(\Delta^*)]. \end{aligned} \quad (3.67)$$

This difference is positive semi-definite whenever

$$q\Delta^*E^{(2)}[\chi_{q+4}^{-4}(\Delta^*)] \leq (q-2)E^{(1)}[\chi_{q+2}^{-4}(\Delta^*)] \quad (3.68)$$

which holds for all $\Delta^* \in (0, \infty)$. Thus $\hat{\phi}^S$ is uniformly better than $\tilde{\beta}$. To compare $\hat{\phi}$ and $\hat{\phi}^S$, the MSE difference is

$$\begin{aligned} M_2(\hat{\phi}) - M_4(\hat{\phi}^S) &= -\sigma_e^2(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{H}'[\mathbf{H}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{H}']^{-1}\mathbf{H}(\mathbf{Z}'\mathbf{Z})^{-1} \left\{ 1 - dq(q-2)E^{(1)}[\chi_{q+2}^{-4}(\Delta^*)] \right. \\ &\quad \left. - 2dq\Delta^*E^{(2)}[\chi_{q+4}^{-4}(\Delta^*)] \right\} + \mathbf{b}_2\mathbf{b}_2' \left\{ 1 - dq(q+2)E^{(2)}[\chi_{q+4}^{-4}(\Delta^*)] \right\} \end{aligned} \quad (3.69)$$

Under restriction, this becomes

$$M_2(\hat{\phi}) - M_4(\hat{\phi}^S) = -(1-d)\sigma_e^2(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{H}'[\mathbf{H}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{H}']^{-1}\mathbf{H}(\mathbf{Z}'\mathbf{Z})^{-1} \quad (3.70)$$

This difference is negative semi-definite, therefore $\hat{\phi}$ performs better than $\hat{\phi}^S$ under restriction. To compare $\hat{\phi}^S$ and $\hat{\phi}^{PT}$, the MSE difference

$$\begin{aligned} M_4(\hat{\phi}^S) - M_3(\hat{\phi}^{PT}) &= \sigma_e^2(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{H}'[\mathbf{H}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{H}']^{-1}\mathbf{H}(\mathbf{Z}'\mathbf{Z})^{-1} \left\{ G_{q+2,n-p}^{(1)}(x', \Delta^*) \right. \\ &\quad \left. - dq(q-2)E^{(1)}[\chi_{q+2}^{-4}(\Delta^*)] - dq\Delta^*E^{(2)}[\chi_{q+4}^{-4}(\Delta^*)] \right\} \\ &\quad - \mathbf{b}_2\mathbf{b}_2' \left\{ 2G_{q+2,n-p}^{(2)}(x', \Delta^*) - G_{q+4,n-p}^{(2)}(x', \Delta^*) - 2dq(q+2)E^{(2)}[\chi_{q+4}^{-4}(\Delta^*)] \right\}. \end{aligned} \quad (3.71)$$

This difference is positive semi-definite whenever

$$\begin{aligned} \Delta^* \left\{ 2G_{q+2,n-p}^{(2)}(x', \Delta^*) - G_{q+4,n-p}^{(2)}(x', \Delta^*) - dq^2E^{(2)}[\chi_{q+4}^{-4}(\Delta^*)] \right\} \\ \leq \left\{ G_{q+2,n-p}^{(1)}(x', \Delta^*) - dq(q-2)E^{(1)}[\chi_{q+2}^{-4}(\Delta^*)] \right\} \end{aligned} \quad (3.72)$$

Hence $\hat{\phi}^{PT}$ is superior to $\hat{\phi}^S$ whenever (3.7₋) holds and inferior otherwise. Comparison of $\hat{\phi}^{S+}$ and $\hat{\phi}^S$: the MSE difference

$$\begin{aligned} M_5(\hat{\phi}^{S+}) - M_4(\hat{\phi}^S) = & -\sigma_e^2 \left\{ (\mathbf{Z}'\mathbf{Z})^{-1} \mathbf{H}' [\mathbf{H}(\mathbf{Z}'\mathbf{Z})^{-1} \mathbf{H}]^{-1} \mathbf{H}(\mathbf{Z}'\mathbf{Z})^{-1} \right. \\ & \times E^{(1)} \left[\left(1 - \frac{qd}{q+2} F_{q+2, n-p}^{-1}(\Delta^*) \right)^2 I \left(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2} \right) \right] \\ & + \frac{\mathbf{b}_2 \mathbf{b}_2'}{\sigma_e^2} E^{(2)} \left[\left(1 - \frac{qd}{q+4} F_{q+4, n-p}^{-1}(\Delta^*) \right)^2 I \left(F_{q+4, n-p}(\Delta^*) \leq \frac{qd}{q+4} \right) \right] \Big\} \\ & - 2\mathbf{b}_2 \mathbf{b}_2' E^{(2)} \left[\left(\frac{qd}{q+2} F_{q+2, n-p}^{-1}(\Delta^*) - 1 \right) I \left(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2} \right) \right] \end{aligned} \quad (3.73)$$

This difference is negative semi-definite since the expectation of a positive random variable is positive. Therefore, for all ϕ , $\hat{\phi}^{S+}$ is superior to $\hat{\phi}^S$. To compare $\hat{\phi}$ and $\hat{\phi}^{S+}$. It is easy to see that $\hat{\phi}^{S+}$ is uniformly better than $\hat{\phi}$ since (3.68) and (3.73). To compare $\hat{\phi}$ and $\hat{\phi}^{S+}$, consider the case under restriction. In this case the MSE difference

$$\begin{aligned} M_5(\hat{\phi}^{S+}) - M_2(\hat{\phi}) = & \sigma_e^2 (\mathbf{Z}'\mathbf{Z})^{-1} \mathbf{H}' [\mathbf{H}(\mathbf{Z}'\mathbf{Z})^{-1} \mathbf{H}]^{-1} \mathbf{H}(\mathbf{Z}'\mathbf{Z})^{-1} \left\{ 1 - d \right. \\ & \left. - E^{(1)} \left[\left(1 - \frac{qd}{q+2} F_{q+2, n-p}^{-1}(0) \right)^2 I \left(F_{q+2, n-p}(0) \leq \frac{qd}{q+2} \right) \right] \right\} \end{aligned} \quad (3.74)$$

This difference is positive semi-definite since (2.91). Thus $\hat{\phi}$ is superior to $\hat{\phi}^{S+}$ under restriction. To compare $\hat{\phi}^{S+}$ and $\hat{\phi}^{PT}$, consider the case when restriction holds. In this case the MSE difference

$$\begin{aligned} M_5(\hat{\phi}^{S+}) - M_2(\hat{\phi}^{PT}) = & \sigma_e^2 (\mathbf{Z}'\mathbf{Z})^{-1} \mathbf{H}' [\mathbf{H}(\mathbf{Z}'\mathbf{Z})^{-1} \mathbf{H}]^{-1} \mathbf{H}(\mathbf{Z}'\mathbf{Z})^{-1} \left\{ G_{q+2, n-p}^{(1)}(x', 0) - d \right. \\ & \left. - E^{(1)} \left[\left(1 - \frac{qd}{q+2} F_{q+2, n-p}^{-1}(0) \right)^2 I \left(F_{q+2, n-p}(0) \leq \frac{qd}{q+2} \right) \right] \right\} \end{aligned} \quad (3.75)$$

This MSE difference is positive semi-definite whenever (3.59) holds. Thus $\hat{\phi}^{PT}$ is superior to $\hat{\phi}^{S+}$ in (3.59), otherwise $\hat{\phi}^{S+}$ is superior. Therefore, $\hat{\phi}^{S+}$ does not

always dominate $\hat{\phi}^{PT}$ when the restriction holds. Hence we can order the MSE matrices as

$$\mathbf{M}_1(\tilde{\phi}) \geq \mathbf{M}_4(\hat{\phi}^S) \geq \mathbf{M}_5(\hat{\phi}^{S+}) \geq \mathbf{M}_3(\hat{\phi}^{PT}) \geq \mathbf{M}_2(\hat{\phi}) \quad (3.76)$$

where $\geq$ stands for $\mathbf{M}_1(\tilde{\phi}) - \mathbf{M}_4(\hat{\phi}^S) \geq 0$, $\mathbf{M}_4(\hat{\phi}^S) - \mathbf{M}_5(\hat{\phi}^{S+}) \geq 0$, $\mathbf{M}_5(\hat{\phi}^{S+}) - \mathbf{M}_3(\hat{\phi}^{PT}) \geq 0$ and $\mathbf{M}_3(\hat{\phi}^{PT}) - \mathbf{M}_2(\hat{\phi}) \geq 0$.

The position of the preliminary test estimator may be changed to the following

$$\mathbf{M}_1(\tilde{\phi}) \geq \mathbf{M}_3(\hat{\phi}^{PT}) \geq \mathbf{M}_4(\hat{\phi}^S) \geq \mathbf{M}_5(\hat{\phi}^{S+}) \geq \mathbf{M}_2(\hat{\phi}).$$

3.5 Conclusion

In this chapter, we studied five estimators of ϕ when the restriction is suspicious. First, note that the application of $\hat{\phi}^{S+}$ or $\hat{\phi}^S$ is constrained by the requirement $q \geq 3$ while $\hat{\phi}^{PT}$ does not need such a constraint but needs the determination of the size of the preliminary test and the extent of departure of Δ^* from 0. The performance property of the five estimators is robust in the class of t -distribution which is determined by the degrees of freedom ν . Also we need the condition $\nu > 4$. Thus, when $q \geq 3$, one uses $\hat{\phi}^{S+}$ otherwise $\hat{\phi}^{PT}$ with some optimum size α . Therefore, from the point of robust efficiency both preliminary test estimator and positive-rule shrinkage estimator may be advocated leaning towards $\hat{\phi}^{S+}$ since the size of Δ^* is generally unknown.

Chapter 4

Ridge Regression Under Uncertain Prior Information with Spherically Symmetric Distributions

4.1 Introduction

In the last two chapters we have analyzed five estimators of regression coefficients for the general linear model when there is suspicion that the parameters belong to a non-stochastic and stochastic subspace. In this chapter we consider five estimators of regression coefficient under uncertain prior information when the matrix $C = X'X$ moves from a well-conditioned to an ill-conditioned one. In this case, the unrestricted estimator is sensitive to a number of "errors", namely, there is an "explosion" of the sampling variance of the estimators and meaningful statistical inference becomes impossible for practitioners. To overcome this problem, Hoerl and Kennard (1970) suggested the use of $C_{(k)} = X'X + kI_p$, ($k \geq 0$) rather than $C = X'X$, in the estimation of β , thereby developing the idea of ridge estimation of β given by $\hat{\beta}_{(k)} = C_{(k)}^{-1}X'y$. With this background we consider the general linear model

$$y = X\beta + e \quad (4.1)$$

where y is an $(n \times 1)$ vector of observations, X is an $(n \times p)$ matrix of rank p , β is a $(p \times 1)$ vector and unknown and e is an $(n \times 1)$ vector random variable which is distributed as $Mt(0, \sigma_e^2 I_n)$. We are primarily interested in the estimation of β when the error distribution is given by (2.4) and it is a priori suspected that β may be restricted to the subspace $H_0 : H\beta = h$, where H is a $(q \times p)$ known matrix of rank $q(< p)$ and h is a $(q \times 1)$ vector of known constants.

Our primary object of this chapter is to estimate β , when a priori $H_0 : H\beta = h$ is suspected, with a new estimator. We combine the idea of preliminary test

estimator and Stein-rule estimator with the ridge regression approach of estimation to obtain better estimators of β . In this chapter, we consider this completely new approach to the estimation of β and study the properties of such estimators when the errors follow the distribution (2.4).

4.2 The Proposed Estimators

We consider the ridge regression version of the usual five estimators. Accordingly, the unrestricted ridge regression estimator (URRE) is defined by

$$\begin{aligned}\tilde{\beta}_{(k)} &= \mathbf{W}\tilde{\beta}, \quad \mathbf{W} = (\mathbf{I}_p + k\mathbf{C}^{-1})^{-1}, \quad k \geq 0, \\ &= \mathbf{WC}^{-1}\mathbf{X}'\mathbf{y} = (\mathbf{C} + k\mathbf{I}_p)^{-1}\mathbf{X}'\mathbf{y} = \mathbf{C}_{(k)}^{-1}\mathbf{X}'\mathbf{y}\end{aligned}\quad (4.2)$$

This is the well-known Hoerl and Kennard (1970) estimate. Next, we consider the restricted ridge regression estimator (RRRE) defined by

$$\hat{\beta}_{(k)} = \mathbf{W}\hat{\beta} = \mathbf{W}\tilde{\beta} - \mathbf{WC}^{-1}\mathbf{H}'(\mathbf{HC}^{-1}\mathbf{H}')^{-1}(\mathbf{H}\tilde{\beta} - \mathbf{h}) \quad (4.3)$$

This estimator has been considered by Sarkar (1992). Thirdly, we consider the preliminary test ridge regression estimator (PTRRE) defined by

$$\hat{\beta}_{(k)}^{PT} = \mathbf{W}\hat{\beta}^{PT} = \mathbf{W}\hat{\beta} + [1 - I(\mathcal{L}_n \leq F_\alpha)]\mathbf{W}(\tilde{\beta} - \hat{\beta}) \quad (4.4)$$

where $\mathcal{L}_n$ and F_α are defined in (2.6) and (2.25). Also, the pdf of $\mathcal{L}_n$ is defined in (2.23) and (2.24). This estimator has been considered by Saleh and Kibria (1993). Fourthly, we consider the shrinkage ridge regression estimator (SRRE) defined by

$$\hat{\beta}_{(k)}^S = \mathbf{W}\hat{\beta}^S = \mathbf{W}\hat{\beta} + (1 - d\mathcal{L}_n^{-1})\mathbf{W}(\tilde{\beta} - \hat{\beta}) \quad (4.5)$$

where d is defined in (2.27). Finally, we consider the positive-rule shrinkage ridge regression estimator (PRSRRE) defined by

$$\hat{\beta}_{(k)}^{S+} = \mathbf{W}\hat{\beta}^{S+} = \mathbf{W}\hat{\beta}^S - (1 - d\mathcal{L}_n^{-1})I(\mathcal{L}_n \leq d)\mathbf{W}(\tilde{\beta} - \hat{\beta}) \quad (4.6)$$

Note that $\hat{\beta}_{(k)}^{PT}$ is a compromised estimator of β when we suspected $H\beta = h$ and C is ill-conditioned giving rise to multicollinearity.

4.3 Bias, Quadratic Risk and MSE Matrix of the Estimators

In this section, we provide the expressions for the bias, quadratic risk and mean squared error matrix of the estimators $\tilde{\beta}_{(k)}$, $\hat{\beta}_{(k)}$, $\hat{\beta}_{(k)}^{PT}$, $\hat{\beta}_{(k)}^S$, and $\hat{\beta}_{(k)}^{S+}$.

4.3.1 Biases of the Estimators

In this subsection we present expressions for the biases of all five estimators.

Theorem 4.1 *Bias of the unrestricted ridge regression estimator (URRE), restricted ridge regression estimator (RRRE), preliminary test ridge regression estimator (PTRRE), shrinkage ridge regression estimator (SRRE) and positive-rule shrinkage ridge regression estimator (PRSRRE) are given by (4.7) – (4.11) respectively:*

$$\begin{aligned} \text{i) } b_1(k) &= E(\tilde{\beta}_{(k)} - \beta) = (W - I_p)\beta \\ &= -kC_{(k)}^{-1}\beta, \quad C_{(k)} = C + kI_p \end{aligned} \quad (4.7)$$

$$\begin{aligned} \text{ii) } b_2(k) &= E(\hat{\beta}_{(k)} - \beta) \\ &= -W\delta - kC_{(k)}^{-1}\beta \end{aligned} \quad (4.8)$$

$$\begin{aligned} \text{iii) } b_3(k) &= E(\hat{\beta}_{(k)}^{PT} - \beta) \\ &= -W\delta G_{q+2, n-p}^{(2)}(x', \Delta^*) - kC_{(k)}^{-1}\beta \end{aligned} \quad (4.9)$$

$$\begin{aligned} \text{iv) } b_4(k) &= E(\hat{\beta}_{(k)}^S - \beta) \\ &= -qdW\delta E^{(2)}[\chi_{q+2}^{-2}(\Delta^*)] - kC_{(k)}^{-1}\beta \end{aligned} \quad (4.10)$$

$$\begin{aligned} \text{v) } b_5(k) &= E(\hat{\beta}_{(k)}^{S+} - \beta) \\ &= W\delta \left\{ \frac{qd}{q+2} E^{(2)} \left[F_{q+2, n-p}^{-1}(\Delta^*) I \left(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2} \right) \right] \right. \\ &\quad \left. - \frac{qd}{q+2} E^{(2)} \left[F_{q+2, n-p}^{-1}(\Delta^*) \right] - G_{q+2, n-p}^{(2)}(x', \Delta^*) \right\} - kC_{(k)}^{-1}\beta \end{aligned} \quad (4.11)$$

where $x, x', \delta, \Delta^*, G_{q+2, n-p}^{(2)}(\cdot, \Delta^*), E^{(2)}[\chi_{q+2}^{-2}(\Delta^*)], E^{(2)}[F_{q+2, n-p}^{-1}(\Delta^*)]$ and $E^{(2)}\left[F_{q+2, n-p}^{-1}(\Delta^*)I\left(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2}\right)\right]$ are defined in (2.35) – (2.38).

Proof. i) Using (4.2),

$$\mathbf{b}_1(k) = E[\tilde{\beta}_{(k)} - \beta] = E(\mathbf{W}\tilde{\beta} - \beta) = (\mathbf{W} - \mathbf{I}_p)\beta$$

From $\mathbf{W} = (\mathbf{I}_p + k\mathbf{C}^{-1})^{-1}$, $k \geq 0$, we get

$$\begin{aligned} \mathbf{W}^{-1} &= \mathbf{I}_p + k\mathbf{C}^{-1} \leftrightarrow k\mathbf{C}^{-1} = \mathbf{W}^{-1} - \mathbf{I}_p \leftrightarrow \\ k\mathbf{W}\mathbf{C}^{-1} &= \mathbf{I}_p - \mathbf{W} \leftrightarrow k(\mathbf{I}_p + k\mathbf{C}^{-1})^{-1}\mathbf{C}^{-1} = \mathbf{I}_p - \mathbf{W} \leftrightarrow \\ k(\mathbf{C} + k\mathbf{I}_p)^{-1} &= \mathbf{I}_p - \mathbf{W} \leftrightarrow k\mathbf{C}_{(k)}^{-1} = \mathbf{I}_p - \mathbf{W}. \end{aligned}$$

Thus $\mathbf{b}_1(k) = -k\mathbf{C}_{(k)}^{-1}\beta$. Following Theorem 2.5, we have

ii) Using (4.3),

$$\begin{aligned} \mathbf{b}_2(k) &= E(\hat{\beta}_{(k)} - \beta) \\ &= E(\tilde{\beta}_{(k)} - \beta) - \mathbf{W}\mathbf{C}^{-1}\mathbf{H}'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}(\mathbf{H}\beta - \mathbf{h}) \\ &= -\mathbf{W}\delta - k\mathbf{C}_{(k)}^{-1}\beta. \end{aligned}$$

iii) Using (4.4),

$$\begin{aligned} \mathbf{b}_3(k) &= E(\hat{\beta}_{(k)}^{PT} - \beta) \\ &= E(\tilde{\beta}_{(k)} - \beta) - \mathbf{W}\delta G_{q+2, n-p}^{(2)}(x', \Delta^*) \\ &= -\mathbf{W}\delta G_{q+2, n-p}^{(2)}(x', \Delta^*) - k\mathbf{C}_{(k)}^{-1}\beta \end{aligned}$$

iv) Using (4.5),

$$\begin{aligned} \mathbf{b}_4(k) &= E(\hat{\beta}_{(k)}^S - \beta) \\ &= E(\tilde{\beta}_{(k)} - \beta) - qd\mathbf{W}\delta E^{(2)}[\chi_{q+2}^{-2}(\Delta^*)] \\ &= -qd\mathbf{W}\delta E^{(2)}[\chi_{q+2}^{-2}(\Delta^*)] - k\mathbf{C}_{(k)}^{-1}\beta. \end{aligned}$$

v) Using (4.6),

$$\begin{aligned}
 b_5(k) &= E(\hat{\beta}_{(k)}^{S+} - \beta) \\
 &= E(\hat{\beta}_{(k)}^S - \beta) - W\delta G_{q+2, n-p}^{(2)}(x', \Delta^*) \\
 &\quad + \frac{qd}{q+2} W\delta E^{(2)} \left[F_{q+2, n-p}^{-1}(\Delta^*) I \left(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2} \right) \right] \\
 &= W\delta \left\{ \frac{qd}{q+2} E^{(2)} \left[F_{q+2, n-p}^{-1}(\Delta^*) I \left(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2} \right) \right] \right. \\
 &\quad \left. - \frac{qd}{q+2} E^{(2)} \left[F_{q+2, n-p}^{-1}(\Delta^*) \right] - G_{q+2, n-p}^{(2)}(x', \Delta^*) \right\} - kC_{(k)}^{-1}\beta.
 \end{aligned}$$

4.3.2 MSE Matrices and Risks of the Estimators

For the risks of the estimators, we consider

$$\begin{aligned}
 \text{Risk (R)} &= E[(\beta_{(k)}^* - \beta)'(\beta_{(k)}^* - \beta)] \\
 &= \text{tr}[M(\beta_{(k)}^*)],
 \end{aligned} \tag{4.12}$$

where

$$M(\beta_{(k)}^*) = E[(\beta_{(k)}^* - \beta)(\beta_{(k)}^* - \beta)'], \tag{4.13}$$

is the mean-squared error matrix of $\beta_{(k)}^*$ and $\beta_{(k)}^*$ can be any one of the five estimators. Direct computation and following Theorem 2.6 lead to the following theorem.

Theorem 4.2 *MSE matrices and risks of URRE, RRRE, PTRRE, SRRE and P-SURE are given by (4.14)–(4.23) respectively:*

$$M_1(\tilde{\beta}_{(k)}) = \sigma_e^2 WC^{-1}W + k^2 C_{(k)}^{-1} \beta \beta' C_{(k)}^{-1} \tag{4.14}$$

$$R_1(\tilde{\beta}_{(k)}; \beta) = \sigma_e^2 \text{tr}(WC^{-1}W) + k^2 \beta' C_{(k)}^{-2} \beta \tag{4.15}$$

$$\begin{aligned}
 M_2(\hat{\beta}_{(k)}) &= \sigma_e^2 WC^{-1}W - \sigma_e^2 [WC^{-1}H'(HC^{-1}H')^{-1}HC^{-1}W] \\
 &\quad + [W\delta + kC_{(k)}^{-1}\beta][W\delta + kC_{(k)}^{-1}\beta]'
 \end{aligned} \tag{4.16}$$

$$R_2(\hat{\beta}_{(k)}; \beta) = \sigma_e^2 \text{tr}(\mathbf{W}\mathbf{C}^{-1}\mathbf{W}) - \sigma_e^2 \text{tr}[\mathbf{W}\mathbf{C}^{-1}\mathbf{H}'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{W}] \\ + \delta'\mathbf{W}^2\delta + k^2\beta'\mathbf{C}_{(k)}^{-2}\beta + 2k\delta'\mathbf{W}\mathbf{C}_{(k)}^{-1}\beta \quad (4.17)$$

$$\mathbf{M}_3(\hat{\beta}_{(k)}^{PT}) = \sigma_e^2 \mathbf{W}\mathbf{C}^{-1}\mathbf{W} - \sigma_e^2 [\mathbf{W}\mathbf{C}^{-1}\mathbf{H}'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{W}]G_{q+2, n-p}^{(1)}(x', \Delta^*) \\ + \mathbf{W}\delta\delta'\mathbf{W} [2G_{q+2, n-p}^{(2)}(x', \Delta^*) - G_{q+4, n-p}^{(2)}(x', \Delta^*)] + k^2\mathbf{C}_{(k)}^{-1}\beta\beta'\mathbf{C}_{(k)}^{-1} \\ + k[\mathbf{W}\delta\beta'\mathbf{C}_{(k)}^{-1} + \mathbf{C}_{(k)}^{-1}\beta\delta'\mathbf{W}]G_{q+2, n-p}^{(2)}(x', \Delta^*) \quad (4.18)$$

$$R_3(\hat{\beta}_{(k)}^{PT}; \beta) = \sigma_e^2 \text{tr}(\mathbf{W}\mathbf{C}^{-1}\mathbf{W}) - \sigma_e^2 \text{tr}[\mathbf{W}\mathbf{C}^{-1}\mathbf{H}'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{W}]G_{q+2, n-p}^{(1)}(x', \Delta^*) \\ + \delta'\mathbf{W}^2\delta [2G_{q+2, n-p}^{(2)}(x', \Delta^*) - G_{q+4, n-p}^{(2)}(x', \Delta^*)] \\ + 2k\delta'\mathbf{W}\mathbf{C}_{(k)}^{-1}\beta G_{q+2, n-p}^{(2)}(x', \Delta^*) + k^2\beta'\mathbf{C}_{(k)}^{-2}\beta. \quad (4.19)$$

$$\mathbf{M}_4(\hat{\beta}_{(k)}^S) = \sigma_e^2 \mathbf{W}\mathbf{C}^{-1}\mathbf{W} - dq\sigma_e^2 \mathbf{W}\mathbf{C}^{-1}\mathbf{H}'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{W} \\ \times \left\{ (q-2)E^{(1)}[\chi_{q+2}^{-4}(\Delta^*)] + 2\Delta^*E^{(2)}[\chi_{q+4}^{-4}(\Delta^*)] \right\} \\ + dq(q+2)\mathbf{W}\delta\delta'\mathbf{W}E^{(2)}[\chi_{q+4}^{-4}(\Delta^*)] \\ + qdk[\mathbf{W}\delta\beta'\mathbf{C}_{(k)}^{-1} + \mathbf{C}_{(k)}^{-1}\beta\delta'\mathbf{W}]E^{(2)}[\chi_{q+2}^{-2}(\Delta^*)] + k^2\mathbf{C}_{(k)}^{-1}\beta\beta'\mathbf{C}_{(k)}^{-1} \quad (4.20)$$

$$R_4(\hat{\beta}_{(k)}^S; \beta) \\ = \sigma_e^2 \text{tr}(\mathbf{W}\mathbf{C}^{-1}\mathbf{W}) - dq\sigma_e^2 \text{tr}[\mathbf{W}\mathbf{C}^{-1}\mathbf{H}'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{W}] \left\{ (q-2)E^{(1)}[\chi_{q+2}^{-4}(\Delta^*)] \right. \\ \left. + \left[1 - \frac{(q+2)\delta'\mathbf{W}^2\delta}{2\sigma_e^2\Delta^* \text{tr}[\mathbf{W}\mathbf{C}^{-1}\mathbf{H}'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{W}]} \right] (2\Delta^*)E^{(2)}[\chi_{q+4}^{-4}(\Delta^*)] \right\} \\ + 2qdk\delta'\mathbf{W}\mathbf{C}_{(k)}^{-1}\beta E^{(2)}[\chi_{q+2}^{-2}(\Delta^*)] + k^2\beta'\mathbf{C}_{(k)}^{-2}\beta \quad (4.21)$$

$$\mathbf{M}_5(\hat{\beta}_{(k)}^{S+}) \\ = \mathbf{M}_4(\hat{\beta}_{(k)}^S) - \sigma_e^2 \left\{ \mathbf{W}\mathbf{C}^{-1}\mathbf{H}'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{W}E^{(1)} \left[\left(1 - \frac{qd}{q+2}F_{q+2, n-p}^{-1}(\Delta^*) \right)^2 \right. \right. \\ \left. \times I\left(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2}\right) \right] \\ + \frac{\mathbf{W}\delta\delta'\mathbf{W}}{\sigma_e^2} E^{(2)} \left[\left(1 - \frac{qd}{q+4}F_{q+4, n-p}^{-1}(\Delta^*) \right)^2 I\left(F_{q+4, n-p}(\Delta^*) \leq \frac{qd}{q+4}\right) \right] \right\} \\ - 2\mathbf{W}\delta\delta'\mathbf{W}E^{(2)} \left[\left(\frac{qd}{q+2}F_{q+2, n-p}^{-1}(\Delta^*) - 1 \right) I\left(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2}\right) \right], \\ - k[\mathbf{W}\delta\beta'\mathbf{C}_{(k)}^{-1} + \mathbf{C}_{(k)}^{-1}\beta\delta'\mathbf{W}]E^{(2)} \left[\left(\frac{qd}{q+2}F_{q+2, n-p}^{-1}(\Delta^*) - 1 \right) I\left(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2}\right) \right], \quad (4.22)$$

$$\begin{aligned}
R_5(\hat{\beta}_{(k)}^{S+}; \beta) &= R_4(\hat{\beta}_{(k)}^S; \beta) - \sigma_e^2 \left\{ \text{tr}[\mathbf{W}\mathbf{C}^{-1}\mathbf{H}'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{W}] \right. \\
&\quad \times E^{(1)} \left[\left(1 - \frac{qd}{q+2} F_{q+2, n-p}^{-1}(\Delta^*) \right)^2 I \left(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2} \right) \right] \\
&\quad + \frac{\delta' \mathbf{W}^2 \delta}{\sigma_e^2} E^{(2)} \left[\left(1 - \frac{qd}{q+4} F_{q+4, n-p}^{-1}(\Delta^*) \right)^2 I \left(F_{q+4, n-p}(\Delta^*) \leq \frac{qd}{q+4} \right) \right] \Big\} \\
&\quad - 2\delta' \mathbf{W}^2 \delta E^{(2)} \left[\left(\frac{qd}{q+2} F_{q+2, n-p}^{-1}(\Delta^*) - 1 \right) I \left(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2} \right) \right] \\
&\quad - 2k\delta' \mathbf{W}\mathbf{C}_{(k)}^{-1} \beta E^{(2)} \left[\left(\frac{qd}{q+2} F_{q+2, n-p}^{-1}(\Delta^*) - 1 \right) I \left(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2} \right) \right]
\end{aligned} \tag{4.23}$$

Using theorems 2.6 and 2.7, the result follows.

4.4 Performance of the Estimators

It is clear that $\mathbf{C}$ is a positive-definite matrix. So there exists an orthogonal matrix Γ such that $\mathbf{C} = \Gamma\Lambda\Gamma'$ and $\Lambda = \Gamma'\mathbf{C}\Gamma = \text{Diag}(\lambda_1, \dots, \lambda_p)$ where

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_p > 0 \tag{4.24}$$

are the eigenvalues of $\mathbf{C}$ (see Anderson, T.W., 1984). It is easy to see that the eigenvalues of $\mathbf{W} = (\mathbf{I}_p + k\mathbf{C}^{-1})^{-1}$ and $\mathbf{C}_{(k)} = \mathbf{C} + k\mathbf{I}_p$ are $\frac{\lambda_i}{\lambda_i + k}$ and $\lambda_i + k$, $i = 1, \dots, p$, respectively. We note that the eigenvectors of $\mathbf{C}$, $\mathbf{W}$ and $\mathbf{C}_{(k)}$ are all the same. With this background we get the following identities:

$$\begin{aligned}
\beta' \mathbf{C}_{(k)}^{-2} \beta &= \beta' \Gamma(\Lambda + k\mathbf{I}_p)^{-2} \Gamma' \beta = \alpha' (\Lambda + k\mathbf{I}_p)^{-2} \alpha \\
&= \sum_{i=1}^p \frac{\alpha_i^2}{(\lambda_i + k)^2}, \quad \alpha = \Gamma' \beta
\end{aligned} \tag{4.25}$$

$$\text{tr}(\mathbf{W}\mathbf{C}^{-1}\mathbf{W}) = \text{tr}(\mathbf{C}^{-1}\mathbf{W}^2) = \sum_{i=1}^p \frac{\lambda_i}{(\lambda_i + k)^2} \tag{4.26}$$

$$\begin{aligned}
& \text{tr}[\mathbf{W}\mathbf{C}^{-1}\mathbf{H}'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}\mathbf{H}\mathbf{C}^{-1}\mathbf{W}] \\
&= \text{tr}[\mathbf{\Gamma}(\mathbf{\Lambda} + k\mathbf{I}_p)^{-1}\mathbf{\Gamma}'\mathbf{H}'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}\mathbf{H}\mathbf{\Gamma}(\mathbf{\Lambda} + k\mathbf{I}_p)^{-1}\mathbf{\Gamma}'] \\
&= \text{tr}\left\{[\mathbf{\Gamma}'\mathbf{H}'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}\mathbf{\Gamma}](\mathbf{\Lambda} + k\mathbf{I}_p)^{-2}\right\} \\
&= \sum_{i=1}^p \frac{h_{ii}^*}{(\lambda_i + k)^2}, \tag{4.27}
\end{aligned}$$

where $h_{ii}^* \geq 0$ is the i th diagonal element of $\mathbf{\Gamma}'\mathbf{H}'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}\mathbf{H}\mathbf{\Gamma} = \mathbf{H}^*$. Let us now compare the performance of the ridge estimator with the usual one.

4.4.1. Comparison between $\tilde{\beta}_{(k)}$ and $\tilde{\beta}$

Theorem 4.3 *There always exists a $k > 0$ in range $0 < k < k^* = \frac{\sigma_e^2}{\alpha_{\max}^2}$ such that the unrestricted ridge regression estimator has smaller risk value than the unrestricted estimator.*

Proof. Using (4.15), (4.25) and (4.26), we obtain

$$\begin{aligned}
R_1(\tilde{\beta}_{(k)}; \beta) &= \sigma_e^2 \sum_{i=1}^p \frac{\lambda_i}{(\lambda_i + k)^2} + k^2 \sum_{i=1}^p \frac{\alpha_i^2}{(\lambda_i + k)^2} \\
&= \sigma_e^2 \sum_{i=1}^p \frac{\lambda_i}{(\lambda_i + k)^2} + \sum_{i=1}^p \frac{\alpha_i^2}{\left(1 + \frac{\lambda_i}{k}\right)^2}. \tag{4.28}
\end{aligned}$$

It is obvious that for $k = 0$, the first and second terms are equal $\sigma_e^2 \sum_{i=1}^p \frac{1}{\lambda_i}$ and zero respectively. The first term is a continuous, monotonically decreasing function of k and its derivative w.r.t k approaches $-\infty$ as $k \rightarrow 0^+$ and $\lambda_p \rightarrow 0$. The second term is also a continuous, monotonically increasing function of k and its derivative tends to zero as $k \rightarrow 0^+$. We note that the second term approaches $\beta'\beta$ as $k \rightarrow \infty$. Differentiating w.r.t k we get

$$\frac{\partial R_1(\tilde{\beta}_{(k)}; \beta)}{\partial k} = 2 \sum_{i=1}^p \frac{\lambda_i}{(\lambda_i + k)^3} (k\alpha_i^2 - \sigma_e^2). \tag{4.29}$$

Thus, a sufficient condition for (4.29) to be negative is that $0 < k < k^*$ where

$$k^* = \frac{\sigma_e^2}{\max_{1 \leq i \leq p} \alpha_i^2} = \frac{\sigma_e^2}{\alpha_{\max}^2}, \quad \alpha_{\max} = \text{largest element of } \alpha \quad (4.30)$$

Hence, the theorem is proved.

4.4.2 Comparison Between $\hat{\beta}_{(k)}$ and $\hat{\beta}$

We consider two cases, namely, under H_0 and under H_A .

Theorem 4.4 *There always exists a $k > 0$ in range $0 < k < k_1^*$ such that $\hat{\beta}_{(k)}$ has smaller risk value than $\hat{\beta}$ under $H_0 : H\beta = h$.*

Proof. Using (4.17) and (4.25)–(4.27), we get

$$\begin{aligned} R_2(\hat{\beta}_{(k)}; \beta) &= \sigma_e^2 \sum_{i=1}^p \frac{\lambda_i}{(\lambda_i + k)^2} - \sigma_e^2 \sum_{i=1}^p \frac{h_{ii}^*}{(\lambda_i + k)^2} + k \sum_{i=1}^p \frac{\alpha_i^2}{(\lambda_i + k)^2} \\ &= \sigma_e^2 \sum_{i=1}^p \frac{\lambda_i - h_{ii}^*}{(\lambda_i + k)^2} + \sum_{i=1}^p \frac{\alpha_i^2}{\left(1 + \frac{\lambda_i}{k}\right)^2} \end{aligned} \quad (4.31)$$

following Theorem 4.3 and differentiating w.r.t. k , we find

$$\frac{\partial R_2(\hat{\beta}_{(k)}; \beta)}{\partial k} = 2 \sum_{i=1}^p \frac{[k\alpha_i^2\lambda_i - \sigma_e^2(\lambda_i - h_{ii}^*)]}{(\lambda_i + k)^2} \quad (4.32)$$

It is clear from (4.32) that a sufficient condition for the first-order derivative of $R_2(\hat{\beta}_{(k)}; \beta)$ to be negative is that $0 < k < k_1^*$ where

$$k_1^* = \frac{\sigma_e^2 \min_{1 \leq i \leq p} (\lambda_i - h_{ii}^*)}{\max_{1 \leq i \leq p} \alpha_i^2 \lambda_i}. \quad (4.33)$$

It is obvious that $k_1^* \leq k^*$ since $\frac{\lambda_i - h_{ii}^*}{\lambda_i} \leq 1$ for all $h_{ii}^* \geq 0$ and $\lambda_i > 0$, $i = 1, \dots, p$.

Hence, we conclude that the range of values of k for the dominance of $\hat{\beta}_{(k)}$ over $\hat{\beta}$ is smaller than the dominance of $\hat{\beta}_{(k)}$ over $\tilde{\beta}$.

Corollary 4.1 The restricted ridge regression estimator has smaller risk value than the unrestricted ridge regression estimator under $H_0 : \mathbf{H}\beta = \mathbf{h}$.

Proof. From (4.28) and (4.31), the result follows.

Theorem 4.5 The restricted ridge regression estimator is superior to the restricted estimator for $0 < k < k_2^*$ under $H_A : \mathbf{H}\beta \neq \mathbf{h}$.

Proof. Using (4.17) and (4.25)–(4.27), we obtain

$$R_2(\hat{\beta}_{(k)}; \beta) = \sum_{i=1}^p \frac{1}{(\lambda_i + k)^2} [\sigma_e^2(\lambda_i - h_{ii}^*) + k^2 \alpha_i^2 + \lambda_i^2 \delta_i^{*2} + 2k \alpha_i \lambda_i \delta_i^*] \quad (4.34)$$

where $\delta^* = \Gamma' \delta$. Now, differentiating the above expression in (4.34) with respect to k , we find

$$\begin{aligned} \frac{\partial R_2(\hat{\beta}_{(k)}; \beta)}{\partial k} &= 2 \sum_{i=1}^p \frac{1}{(\lambda_i + k)^2} \left\{ k \lambda_i \alpha_i (\alpha_i - \delta_i^*) \right. \\ &\quad \left. - [\sigma_e^2(\lambda_i - h_{ii}^*) - \lambda_i^2 \delta_i^* (\alpha_i - \delta_i^*)] \right\} \end{aligned} \quad (4.35)$$

Hence, a sufficient condition for (4.35) to be negative is that $0 < k < k_2^*$ where

$$k_2^* = \frac{\min_{1 \leq i \leq p} [\sigma_e^2(\lambda_i - h_{ii}^*) - \lambda_i^2 \delta_i^* (\alpha_i - \delta_i^*)]}{\max_{1 \leq i \leq p} \lambda_i \alpha_i (\alpha_i - \delta_i^*)}. \quad (4.36)$$

Hence, the proof is complete.

Corollary 4.2 a sufficient condition for the restricted ridge regression to have smaller risk value than the unrestricted ridge estimator is that there exists a value k such that $0 < k < k_1$ where k_1 is given by

$$k_1 = \frac{\min_{1 \leq i \leq p} (\sigma_e^2 h_{ii}^* - \lambda_i^2 \delta_i^{*2})}{\max_{1 \leq i \leq p} (2 \alpha_i \lambda_i \delta_i^*)}. \quad (4.37)$$

Proof. Using (4.28) and (4.34), we find

$$\begin{aligned} R_1(\hat{\beta}_{(k)}; \beta) - R_2(\hat{\beta}_{(k)}; \beta) \\ = \sum_{i=1}^p \frac{(\sigma_e^2 h_{ii}^* - \lambda_i^2 \delta_i^{*2} - 2k\alpha_i \lambda_i \delta_i^*)}{(\lambda_i + k)^2}. \end{aligned} \quad (4.38)$$

This difference is positive for all $k < k_1$, which is given in (4.37).

4.4.3 Comparison between $\hat{\beta}^{PT}$ and $\hat{\beta}_{(k)}^{PT}$

We consider two cases, namely, under H_0 and under H_A .

Case 1. Under $H_0 : \mathbf{H}\beta = \mathbf{h}$ (i.e. $\Delta^* = 0$).

Theorem 4.6 *There always exists a $k > 0$ in the region $0 < k < k_3^*$ such that $\hat{\beta}_{(k)}^{PT}$ has smaller risk value than $\hat{\beta}^{PT}$ under $H_0 : \mathbf{H}\beta = \mathbf{h}$.*

Proof. Using (4.19) and (4.25)–(4.27), we find

$$R_3(\hat{\beta}_{(k)}^{PT}; \beta) = \sigma_e^2 \sum_{i=1}^p \frac{\lambda_i - h_{ii}^* G_{q+2, n-p}^{(1)}(x', 0)}{(\lambda_i + k)^2} + k^2 \sum_{i=1}^p \frac{\alpha_i^2}{(\lambda_i + k)^2}. \quad (4.39)$$

Differentiating w.r.t. k , we get

$$\frac{\partial R_3(\hat{\beta}_{(k)}^{PT}; \beta)}{\partial k} = 2 \sum_{i=1}^p \frac{1}{(\lambda_i + k)^3} \left\{ k\alpha_i^2 \lambda_i - \sigma_e^2 \left[\lambda_i - h_{ii}^* G_{q+2, n-p}^{(1)}(x', 0) \right] \right\} \quad (4.40)$$

Thus a sufficient condition for (4.40) to be negative is that $0 < k < k_3^*$ where

$$k_3^* = \frac{\sigma_e^2 \min_{1 \leq i \leq p} \left[\lambda_i - h_{ii}^* G_{q+2, n-p}^{(1)}(x', 0) \right]}{\max_{1 \leq i \leq p} \alpha_i^2 \lambda_i}. \quad (4.41)$$

Hence, the proof is complete.

We note that for $\alpha = 0, 1$, we obtain theorems 4.4 and 4.3 respectively.

Corollary 4.3 *The PTRRE has smaller risk value than URRE under $H_0 : \mathbf{H}\beta = \mathbf{h}$.*

Proof. Using (4.28) and (4.39), the result follows.

Case 2. Under $H_A : \mathbf{H}\beta \neq \mathbf{h}$ (i.e. $\Delta^* \neq 0$).

Theorem 4.7 *There always exists a $k > 0$ in the region $0 < k < k_4^*$ such that $\hat{\beta}_{(k)}^{PT}$ has smaller risk value than $\hat{\beta}^{PT}$ under $H_A : \mathbf{H}\beta \neq \mathbf{h}$.*

Proof. Using (4.19) and (4.25)–(4.27), the risk expression is given by

$$\begin{aligned} R_3(\hat{\beta}_{(k)}^{PT}; \beta) &= \sum_{i=1}^p \frac{1}{(\lambda_i + k)^2} \left\{ \sigma_e^2 \left[\lambda_i - h_{ii}^* G_{q+2, n-p}^{(1)}(x', \Delta^*) \right] \right. \\ &\quad + \lambda_i^2 \delta_i^{*2} \left[2G_{q+2, n-p}^{(2)}(x', \Delta^*) - G_{q+4, n-p}^{(2)}(x', \Delta^*) \right] \\ &\quad \left. + k^2 \alpha_i^2 + 2k \alpha_i \lambda_i \delta_i^* G_{q+2, n-p}^{(2)}(x', \Delta^*) \right\}. \end{aligned} \quad (4.42)$$

Differentiating with respect to k , we obtain

$$\begin{aligned} \frac{\partial R_3(\hat{\beta}_{(k)}^{PT}; \beta)}{\partial k} &= 2 \sum_{i=1}^p \frac{1}{(\lambda_i + k)^3} \left\{ k \alpha_i \lambda_i \left[\alpha_i - \delta_i^* G_{q+2, n-p}^{(2)}(x', \Delta^*) \right] \right. \\ &\quad - \left[\sigma_e^2 \left(\lambda_i - h_{ii}^* G_{q+2, n-p}^{(1)}(x', \Delta^*) \right) + \lambda_i^2 \delta_i^{*2} \left(2G_{q+2, n-p}^{(2)}(x', \Delta^*) - G_{q+4, n-p}^{(2)}(x', \Delta^*) \right) \right. \\ &\quad \left. \left. - \alpha_i \lambda_i^2 \delta_i^* G_{q+2, n-p}^{(2)}(x', \Delta^*) \right] \right\}. \end{aligned} \quad (4.43)$$

Hence, a sufficient condition for (4.43) to be negative is that $0 < k < k_4^*$ where

$$\begin{aligned} k_4^* &= \min_{1 \leq i \leq p} \left[\sigma_e^2 \left(\lambda_i - h_{ii}^* G_{q+2, n-p}^{(1)}(x', \Delta^*) \right) \right. \\ &\quad + \lambda_i^2 \delta_i^{*2} \left(2G_{q+2, n-p}^{(2)}(x', \Delta^*) - G_{q+4, n-p}^{(2)}(x', \Delta^*) \right) \\ &\quad \left. - \alpha_i \lambda_i^2 \delta_i^* G_{q+2, n-p}^{(2)}(x', \Delta^*) \right] / \max_{1 \leq i \leq p} \alpha_i \lambda_i \left(\alpha_i - \delta_i^* G_{q+2, n-p}^{(2)}(x', \Delta^*) \right). \end{aligned} \quad (4.44)$$

Thus, $\hat{\beta}_{(k)}^{PT}$ has risk value less than the risk of $\hat{\beta}^{PT}$ for $0 < k < k_4^*$.

4.4.3.1 Comparison of PTRRE with URRE and RRRE

We shall now proceed to compare the performance of $\hat{\beta}_{(k)}^{PT}$ against that of $\tilde{\beta}_{(k)}$.

Theorem 4.8 *A sufficient condition for PTRRE to have risk value less than or equal to that of URRE is that there exists a value of k such that $0 < k < k_2$ where k_2 is given by (4.46).*

Proof. Using (4.28) and (4.42), we obtain

$$\begin{aligned} R_1(\tilde{\beta}_{(k)}; \beta) - R_3(\hat{\beta}_{(k)}^{PT}; \beta) \\ = \sum_{i=1}^p \frac{1}{(\lambda_i + k)^2} \left\{ \sigma_e^2 h_{ii}^* G_{q+2, n-p}^{(1)}(x', \Delta^*) \right. \\ \left. - \lambda_i^2 \delta_i^{*2} \left[2G_{q+2, n-p}^{(2)}(x', \Delta^*) - G_{q+4, n-p}^{(2)}(x', \Delta^*) \right] - 2k\alpha_i \lambda_i \delta_i^* G_{q+2, n-p}^{(2)}(x', \Delta^*) \right\}. \end{aligned} \quad (4.45)$$

This difference is nonnegative (≥ 0), whenever

$$k_2 = \frac{\min_{1 \leq i \leq p} \left\{ \sigma_e^2 h_{ii}^* G_{q+2, n-p}^{(1)}(x', \Delta^*) - \lambda_i^2 \delta_i^{*2} \left[2G_{q+2, n-p}^{(2)}(x', \Delta^*) - G_{q+4, n-p}^{(2)}(x', \Delta^*) \right] \right\}}{\max_{1 \leq i \leq p} \left[2\alpha_i \lambda_i \delta_i^* G_{q+2, n-p}^{(2)}(x', \Delta^*) \right]}. \quad (4.46)$$

Thus, the theorem is proved.

Similarly, we compare PTRRE to RRRE.

Theorem 4.9 *A sufficient condition for PTRRE to have risk value less than or equal to that of RRRE is that there exists a value of k such that $0 < k_3 < k$ where k_3 is given by (4.48).*

Proof. Using (4.34) and (4.42), we find

$$\begin{aligned}
& R_3(\hat{\beta}_{(k)}^{PT}; \beta) - R_2(\hat{\beta}_{(k)}; \beta) \\
&= \sum_{i=1}^p \frac{1}{(\lambda_i + k)^2} \left\{ \sigma_e^2 h_{ii}^* \left[1 - G_{q+2, n-p}^{(1)}(x', \Delta^*) \right] \right. \\
&\quad - \lambda_i^2 \delta_i^{*2} \left[1 - 2G_{q+2, n-p}^{(2)}(x', \Delta^*) + G_{q+4, n-p}^{(2)}(x', \Delta^*) \right] \\
&\quad \left. - 2k\alpha_i \lambda_i \delta_i^* \left[1 - G_{q+2, n-p}^{(2)}(x', \Delta^*) \right] \right\}. \tag{4.47}
\end{aligned}$$

This difference is negative (< 0), whenever

$$\begin{aligned}
k_3 = \max_{1 \leq i \leq p} \left\{ \sigma_e^2 h_{ii}^* \left[1 - G_{q+2, n-p}^{(1)}(x', \Delta^*) \right] - \lambda_i^2 \delta_i^{*2} \left[1 - 2G_{q+2, n-p}^{(2)}(x', \Delta^*) \right. \right. \\
\left. \left. + G_{q+4, n-p}^{(2)}(x', \Delta^*) \right] \right\} / \min_{1 \leq i \leq p} \left\{ 2\alpha_i \lambda_i \delta_i^* \left[1 - G_{q+2, n-p}^{(2)}(x', \Delta^*) \right] \right\}. \tag{4.48}
\end{aligned}$$

Hence, the theorem is proved.

We note that under $H_0 : \mathbf{H}\beta = \mathbf{h}$, we have

$$R_3(\hat{\beta}_{(k)}^{PT}; \beta) - R_2(\hat{\beta}_{(k)}; \beta) = \sigma_e^2 \left[1 - G_{q+2, n-p}^{(1)}(x', 0) \right] \sum_{i=1}^p \frac{h_{ii}^*}{(\lambda_i + k)^2} \geq 0$$

since

$$G_{q+2, n-p}^{(1)}(x', 0) = F_{q+2, n-p}(\ell_\alpha^{*(1)}, 0) = 1 - \alpha, \quad \ell_\alpha^{*(1)} = \frac{qF_\alpha}{q+2}$$

and $\delta_i^* = 0$, $i = 1, \dots, p$. Thus $\hat{\beta}_{(k)}$ is superior to $\hat{\beta}_{(k)}^{PT}$ under $H_0 : \mathbf{H}\beta = \mathbf{h}$.

4.4.4 Comparison between $\hat{\beta}^S$ and $\hat{\beta}_{(k)}^S$

Theorem 4.10 *There always exists a $k > 0$ in the region $0 < k < k_3^*$ such that $\hat{\beta}_{(k)}^S$ has smaller risk value than $\hat{\beta}^S$ under $H_0 : \mathbf{H}\beta = \mathbf{h}$.*

Proof. Using (4.21) and (4.25)–(4.27), we obtain

$$R_4(\hat{\beta}_{(k)}^S; \beta) = \sum_{i=1}^p \frac{1}{(\lambda_i + k)^2} \left\{ \sigma_e^2 (\lambda_i - d h_{ii}^*) + k^2 \alpha_i^2 \right\}. \tag{4.49}$$

Differentiating with respect to k , we find

$$\frac{\partial R_4(\hat{\beta}_{(k)}^S; \beta)}{\partial k} = 2 \sum_{i=1}^p \frac{1}{(\lambda_i + k)^3} \{k\alpha_i^2 \lambda_i - \sigma_e^2(\lambda_i - dh_{ii}^*)\}. \quad (4.50)$$

Thus, a sufficient condition for (4.50) to be negative is that $0 < k < k_5^*$ where

$$k_5^* = \frac{\sigma_e^2 \min_{1 \leq i \leq p} (\lambda_i - dh_{ii}^*)}{\max_{1 \leq i \leq p} \alpha_i^2 \lambda_i}. \quad (4.51)$$

Hence, the proof is complete.

Theorem 4.11 *There always exists a $k > 0$ in the region $0 < k < k_6^*$ such that $\hat{\beta}_{(k)}^S$ has smaller risk value than $\hat{\beta}^S$ under $H_A : \mathbf{H}\beta \neq \mathbf{h}$.*

Proof. Using (4.21) and (4.25)–(4.27), the risk expression is given by

$$\begin{aligned} R_4(\hat{\beta}_{(k)}^S; \beta) &= \sum_{i=1}^p \frac{1}{(\lambda_i + k)^2} \left\{ \sigma_e^2 \{ \lambda_i - dqh_{ii}^* [(q-2)E^{(1)}(\chi_{q+2}^{-4}(\Delta^*)) \right. \\ &\quad \left. + \left(1 - \frac{(q+2)\lambda_i^2 \delta_i^{*2}}{2\sigma_e^2 \Delta^* h_{ii}^*} \right) (2\Delta^*) E^{(2)}(\chi_{q+4}^{-4}(\Delta^*)) \right] \} \\ &\quad \left. + 2qdk\alpha_i \lambda_i \delta_i^* E^{(2)}(\chi_{q+2}^{-2}(\Delta^*)) + k^2 \alpha_i^2 \right\}. \end{aligned} \quad (4.52)$$

Differentiating with respect to k , we obtain

$$\begin{aligned} \frac{\partial R_4(\hat{\beta}_{(k)}^S; \beta)}{\partial k} &= 2 \sum_{i=1}^p \frac{1}{(\lambda_i + k)^3} \left\{ k\lambda_i \alpha_i \left[\alpha_i - qd\delta_i^* E^{(2)}(\chi_{q+2}^{-2}(\Delta^*)) \right] \right. \\ &\quad \left. - \sigma_e^2 \{ \lambda_i - dqh_{ii}^* [(q-2)E^{(1)}(\chi_{q+2}^{-4}(\Delta^*)) + \left(1 - \frac{(q+2)\lambda_i^2 \delta_i^{*2}}{2\sigma_e^2 \Delta^* h_{ii}^*} \right) (2\Delta^*) E^{(2)}(\chi_{q+4}^{-4}(\Delta^*)) \right] \right\} \\ &\quad \left. - qd\alpha_i \lambda_i^2 \delta_i^* E^{(2)}(\chi_{q+2}^{-2}(\Delta^*)) \right\} \end{aligned} \quad (4.53)$$

Hence, a sufficient condition for (4.53) to be negative is that $0 < k < k_6^*$ where

$$\begin{aligned} k_6^* &= \min_{1 \leq i \leq p} \left\{ \sigma_e^2 \{ \lambda_i - dqh_{ii}^* [(q-2)E^{(1)}(\chi_{q+2}^{-4}(\Delta^*)) \right. \\ &\quad \left. + \left(1 - \frac{(q+2)\lambda_i^2 \delta_i^{*2}}{2\sigma_e^2 \Delta^* h_{ii}^*} \right) (2\Delta^*) E^{(2)}(\chi_{q+4}^{-4}(\Delta^*)) \right] \} \right. \\ &\quad \left. + qd\alpha_i \lambda_i^2 \delta_i^* E^{(2)}(\chi_{q+2}^{-2}(\Delta^*)) \right\} / \max_{1 \leq i \leq p} \alpha_i \lambda_i \left[\alpha_i - qd\delta_i^* E^{(2)}(\chi_{q+2}^{-2}(\Delta^*)) \right]. \end{aligned} \quad (4.54)$$

Thus, $\hat{\beta}_{(k)}^S$ has risk value less than the risk of $\hat{\beta}^S$ for $0 < k < k_6^*$.

4.4.4.1 Comparison of SRRE with URRE, RRRE and PTRRE

Theorem 4.12 A sufficient condition for SRRE to have risk value less than or equal to that of URRE is that there exists a value of k such that $0 < k < k_4$ where k_4 is given by (4.56).

Proof. Using (4.28) and (4.52), we find

$$\begin{aligned} & R_1(\hat{\beta}_{(k)}; \beta) - R_4(\hat{\beta}_{(k)}^S; \beta) \\ &= \sum_{i=1}^p \frac{1}{(\lambda_i + k)^2} \left\{ dq\sigma_e^2 h_{ii}^* \left[(q-2)E^{(1)}(\chi_{q+2}^{-4}(\Delta^*)) \right. \right. \\ & \quad \left. \left. + \left(1 - \frac{(q+2)\lambda_i^2 \delta_i^{*2}}{2\sigma_e^2 \Delta^* h_{ii}^*} \right) (2\Delta^*)E^{(2)}(\chi_{q+4}^{-4}(\Delta^*)) \right] \right. \\ & \quad \left. - 2qdk\alpha_i \lambda_i \delta_i^* E^{(2)}(\chi_{q+2}^{-2}(\Delta^*)) \right\}. \end{aligned} \quad (4.55)$$

This difference is non-negative (≥ 0), whenever

$$k_4 = \frac{\sigma_e^2 \min_{1 \leq i \leq p} \left\{ h_{ii}^* \left[(q-2)E^{(1)}(\chi_{q+2}^{-4}(\Delta^*)) + \left(1 - \frac{(q+2)\lambda_i^2 \delta_i^{*2}}{2\sigma_e^2 \Delta^* h_{ii}^*} \right) (2\Delta^*)E^{(2)}(\chi_{q+4}^{-4}(\Delta^*)) \right] \right\}}{\max_{1 \leq i \leq p} \left[2\alpha_i \lambda_i \delta_i^* E^{(2)}(\chi_{q+2}^{-2}(\Delta^*)) \right]}. \quad (4.56)$$

Hence, the theorem is proved.

Theorem 4.13 A sufficient condition for SRRE to have smaller risk value than RRRE is that $0 < k_5 < k$ where k_5 is given by (4.58).

Proof. Using (4.52) and (4.34), we obtain

$$\begin{aligned} & R_4(\hat{\beta}_{(k)}^S; \beta) - R_2(\hat{\beta}_{(k)}; \beta) \\ &= \sum_{i=1}^p \frac{1}{(\lambda_i + k)^2} \left\{ \sigma_e^2 h_{ii}^* \left\{ 1 - dq \left[(q-2)E^{(1)}(\chi_{q+2}^{-4}(\Delta^*)) \right. \right. \right. \\ & \quad \left. \left. + \left(1 - \frac{(q+2)\lambda_i^2 \delta_i^{*2}}{2\sigma_e^2 \Delta^* h_{ii}^*} \right) (2\Delta^*)E^{(2)}(\chi_{q+4}^{-4}(\Delta^*)) \right] \right\} - \lambda_i^2 \delta_i^{*2} \right. \\ & \quad \left. - 2k\alpha_i \lambda_i \delta_i^* \left[1 - qdE^{(2)}(\chi_{q+2}^{-2}(\Delta^*)) \right] \right\}. \end{aligned} \quad (4.57)$$

This difference is nonpositive (≤ 0), whenever

$$k_5 = \max_{1 \leq i \leq p} \left\{ \sigma_e^2 h_{ii}^* \left\{ 1 - dq \left[(q-2)E^{(1)}(\chi_{q+2}^{-4}(\Delta^*)) \right. \right. \right. \\ \left. \left. \left. + \left(1 - \frac{(q+2)\lambda_i^2 \delta_i^{*2}}{2\sigma_e^2 \Delta^* h_{ii}^*} \right) (2\Delta^*)E^{(2)}(\chi_{q+4}^{-4}(\Delta^*)) \right] \right\} - \lambda_i^2 \delta_i^{*2} \right\} \\ \left/ \min_{1 \leq i \leq p} \left\{ 2\alpha_i \lambda_i \delta_i^* \left[1 - qdE^{(2)}(\chi_{q+2}^{-2}(\Delta^*)) \right] \right\} \right\} \quad (4.58)$$

Thus, the theorem is proved.

We note that under $H_0 : \mathbf{H}\beta = \mathbf{h}$, we have

$$R_4(\hat{\beta}_{(k)}^S; \beta) - R_2(\hat{\beta}_{(k)}; \beta) = \sum_{i=1}^p \frac{(1-d)\sigma_e^2 h_{ii}^*}{(\lambda_i + k)^2} \geq 0$$

Thus $\hat{\beta}_{(k)}$ is superior to $\hat{\beta}_{(k)}^S$ under $H_0 : \mathbf{H}\beta = \mathbf{h}$.

Theorem 4.14 *A sufficient condition for SRRE to have smaller risk value than PTRRE is that $0 < k_6 < k$ where k_6 is given by (4.60).*

Proof. Using (4.52) and (4.42), we get

$$R_4(\hat{\beta}_{(k)}^S; \beta) - R_3(\hat{\beta}_{(k)}^{PT}; \beta) \\ = \sum_{i=1}^p \frac{1}{(\lambda_i + k)^2} \left\{ \sigma_e^2 h_{ii}^* \left\{ G_{q+2, n-p}^{(1)}(x', \Delta^*) - dq \left[(q-2)E^{(1)}(\chi_{q+2}^{-4}(\Delta^*)) \right. \right. \right. \\ \left. \left. \left. + \left(1 - \frac{(q+2)\lambda_i^2 \delta_i^{*2}}{2\sigma_e^2 \Delta^* h_{ii}^*} \right) (2\Delta^*)E^{(2)}(\chi_{q+4}^{-4}(\Delta^*)) \right] \right\} - \lambda_i^2 \delta_i^{*2} \left[2G_{q+2, n-p}^{(2)}(x', \Delta^*) - G_{q+4, n-p}^{(2)}(x', \Delta^*) \right] \right. \\ \left. - 2k\alpha_i \delta_i^* \lambda_i \left[G_{q+2, n-p}^{(2)}(x', \Delta^*) - qdE^{(2)}(\chi_{q+2}^{-2}(\Delta^*)) \right] \right\} \quad (4.59)$$

This difference is nonpositive (≤ 0), whenever

$$k_6 = \max_{1 \leq i \leq p} \left\{ \sigma_e^2 h_{ii}^* \left\{ G_{q+2, n-p}^{(1)}(x', \Delta^*) - dq \left[(q-2)E^{(1)}(\chi_{q+2}^{-4}(\Delta^*)) \right. \right. \right. \\ \left. \left. \left. + \left(1 - \frac{(q+2)\lambda_i^2 \delta_i^{*2}}{2\sigma_e^2 \Delta^* h_{ii}^*} \right) (2\Delta^*)E^{(2)}(\chi_{q+4}^{-4}(\Delta^*)) \right] \right\} \right. \\ \left. - \lambda_i^2 \delta_i^{*2} \left[2G_{q+2, n-p}^{(2)}(x', \Delta^*) - G_{q+4, n-p}^{(2)}(x', \Delta^*) \right] \right\} \\ \left/ \min_{1 \leq i \leq p} \left\{ 2\alpha_i \lambda_i \delta_i^* \left[G_{q+2, n-p}^{(2)}(x', \Delta^*) - qdE^{(2)}(\chi_{q+2}^{-2}(\Delta^*)) \right] \right\} \right\}. \quad (4.60)$$

Thus, the proof is complete.

We note that under $H_0 : \mathbf{H}\beta = \mathbf{h}$, we have

$$R_4(\hat{\beta}_{(k)}^S; \beta) - R_3(\hat{\beta}_{(k)}^{PT}; \beta) = \sum_{i=1}^p \frac{\sigma_e^2 [G_{q+2, n-p}^{(1)}(x', 0) - d] h_{ii}^*}{(\lambda_i + k)^2},$$

where $G_{q+2, n-p}^{(1)}(x', 0) = F_{q+2, n-p}(\ell_\alpha^{*(1)}, 0) = 1 - \alpha$, $\ell_\alpha^{*(1)} = \frac{qF_\alpha}{q+2}$. Thus, the risk of $\hat{\beta}_{(k)}^S$ is smaller than that of the risk of $\hat{\beta}_{(k)}^{PT}$ when the critical value $\ell_\alpha^{*(1)}$ satisfies the opposite inequality in (2.85), otherwise $\hat{\beta}_{(k)}^{PT}$ is smaller than $\hat{\beta}_{(k)}^S$.

4.4.5 Comparison between $\hat{\beta}^{S+}$ and $\hat{\beta}_{(k)}^{S+}$

Theorem 4.15 *There always exists a $k > 0$ in the region $0 < k < k_7^*$ such that $\hat{\beta}_{(k)}^{S+}$ has smaller risk value than $\hat{\beta}^{S+}$ under $H : \mathbf{H}\beta = \mathbf{h}$.*

Proof. Using (4.23), (4.49) and (4.25)–(4.27), the risk expression is given by

$$\begin{aligned} R_5(\hat{\beta}_{(k)}^{S+}; \beta) &= \sum_{i=1}^p \frac{1}{(\lambda_i + k)^2} \left\{ \sigma_e^2 (\lambda_i - dh_{ii}^*) + k^2 \alpha_i^2 \right. \\ &\quad \left. - \sigma_e^2 h_{ii}^* E^{(1)} \left[\left(1 - \frac{qd}{q+2} F_{q+2, n-p}^{-1}(0) \right)^2 I \left(F(0) \leq \frac{qd}{q+2} \right) \right] \right\} \end{aligned} \quad (4.61)$$

Differentiating with respect to k , we obtain

$$\begin{aligned} \frac{\partial R_5(\hat{\beta}_{(k)}^{S+}; \beta)}{\partial k} &= 2 \sum_{i=1}^p \frac{1}{(\lambda_i + k)^3} \left\{ k \alpha_i^2 \lambda_i \right. \\ &\quad \left. - \sigma_e^2 \left\{ \lambda_i - dh_{ii}^* - h_{ii}^* E^{(1)} \left[\left(1 - \frac{qd}{q+2} F_{q+2, n-p}^{-1}(0) \right)^2 I \left(F_{q+2, n-p}(0) \leq \frac{qd}{q+2} \right) \right] \right\} \right\} \end{aligned} \quad (4.62)$$

Thus, a sufficient condition for (4.62) to be negative is that $0 < k < k_7^*$ where

$$k_7^* = \frac{\sigma_e^2 \min_{1 \leq i \leq p} \left\{ \lambda_i - dh_{ii}^* - h_{ii}^* E^{(1)} \left[\left(1 - \frac{qd}{q+2} F_{q+2, n-p}^{-1}(0) \right)^2 I \left(F_{q+2, n-p}(0) \leq \frac{qd}{q+2} \right) \right] \right\}}{\max_{1 \leq i \leq p} \alpha_i^2 \lambda_i}. \quad (4.63)$$

Hence, the proof is complete.

Theorem 4.16 *There always exists a $k > 0$ in the region $0 < k < k_8^*$ such that $\hat{\beta}_{(k)}^{S+}$ has smaller risk value than $\hat{\beta}^{S+}$ under $H_A : \mathbf{H}\beta \neq \mathbf{h}$.*

Proof. = Using (4.23) and (4.25)–(4.17), we obtain

$$\begin{aligned}
 R_5(\hat{\beta}_{(k)}^{S+}; \beta) &= R_4(\hat{\beta}_{(k)}^S; \beta) \\
 &- \sum_{i=1}^p \frac{1}{(\lambda_i + k)^2} \left\{ \sigma_c^2 \{ h_{ii}^* E^{(1)} \left[\left(1 - \frac{qd}{q+2} F_{q+2, n-p}^{-1}(\Delta^*) \right)^2 I(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2}) \right] \right. \\
 &+ \frac{\lambda_i^2 \delta_i^{*2}}{\sigma_c^2} E^{(2)} \left[\left(1 - \frac{qd}{q+4} F_{q+4, n-p}^{-1}(\Delta^*) \right)^2 I(F_{q+4, n-p}(\Delta^*) \leq \frac{qd}{q+4}) \right] \} \\
 &+ 2\lambda_i^2 \delta_i^{*2} E^{(2)} \left[\left(\frac{qd}{q+2} F_{q+2, n-p}^{-1}(\Delta^*) - 1 \right) I(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2}) \right] \\
 &+ 2k\alpha_i \lambda_i \delta_i^* E^{(2)} \left[\left(\frac{qd}{q+2} F_{q+2, n-p}^{-1}(\Delta^*) - 1 \right) I(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2}) \right] \}, \tag{4.64}
 \end{aligned}$$

where $R_4(\hat{\beta}_{(k)}^S; \beta)$ is given by (4.52). Differentiating with respect to k , we find

$$\begin{aligned}
 \frac{\partial R_5(\hat{\beta}_{(k)}^{S+}; \beta)}{\partial k} &= \frac{\partial R_4(\hat{\beta}_{(k)}^S; \beta)}{\partial k} \\
 &+ 2 \sum_{i=1}^p \frac{1}{(\lambda_i + k)^3} \left\{ k\alpha_i \lambda_i \delta_i^* E^{(2)} \left[\left(\frac{qd}{q+2} F_{q+2, n-p}^{-1}(\Delta^*) - 1 \right) I(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2}) \right] \right. \\
 &+ \sigma_c^2 \{ h_{ii}^* E^{(1)} \left[\left(1 - \frac{qd}{q+2} F_{q+2, n-p}^{-1}(\Delta^*) \right)^2 I(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2}) \right] \\
 &+ \frac{\lambda_i^2 \delta_i^{*2}}{\sigma_c^2} E^{(2)} \left[\left(1 - \frac{qd}{q+4} F_{q+4, n-p}^{-1}(\Delta^*) \right)^2 I(F_{q+4, n-p}(\Delta^*) \leq \frac{qd}{q+4}) \right] \} \\
 &- (\alpha_i - 2\delta_i^*) \lambda_i^2 \delta_i^* E^{(2)} \left[\left(\frac{qd}{q+2} F_{q+2, n-p}^{-1}(\Delta^*) - 1 \right) I(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2}) \right] \}, \tag{4.65}
 \end{aligned}$$

where $\frac{\partial R_4(\hat{\beta}_{(k)}^S; \beta)}{\partial k}$ is given by (4.53). Hence, a sufficient condition for (4.65) to be negative is that $0 < k < k_8^*$ where

$$k_8^* = \frac{f_1(\alpha, \Delta^*)}{f_2(\alpha, \Delta^*)} \tag{4.66}$$

with

$$\begin{aligned}
f_1(\alpha, \Delta^*) &= \min_{1 \leq i \leq p} \left\{ \sigma_c^2 \left\{ \lambda_i - dq h_{ii}^* \left[(q-2) E^{(1)}(\chi_{q+2}^{-4}(\Delta^*)) \right. \right. \right. \\
&\quad + \left(1 - \frac{(q+2) \lambda_i^2 \delta_i^{*2}}{2 \sigma_c^2 \Delta^* h_{ii}^*} \right) (2 \Delta^*) E^{(2)}(\chi_{q+4}^{-4}(\Delta^*)) \\
&\quad - h_{ii}^* E^{(1)} \left[\left(1 - \frac{qd}{q+2} F_{q+2, n-p}^{-1}(\Delta^*) \right)^2 I \left(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2} \right) \right] \\
&\quad - \frac{\lambda_i^2 \delta_i^{*2}}{\sigma_c^2} E^{(2)} \left[\left(1 - \frac{qd}{q+4} F_{q+4, n-p}^{-1}(\Delta^*) \right)^2 I \left(F_{q+4, n-p}(\Delta^*) \leq \frac{qd}{q+4} \right) \right] \Big\} \\
&\quad + (\alpha_i - 2 \delta_i^*) \lambda_i^2 \delta_i^* E^{(2)} \left[\left(\frac{qd}{q+2} F_{q+2, n-p}^{-1}(\Delta^*) - 1 \right) I \left(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2} \right) \right] \\
&\quad + dq \alpha_i \lambda_i^2 \delta_i^* E^{(2)}(\chi_{q+2}^{-2}(\Delta^*)) \Big\}
\end{aligned}$$

and

$$\begin{aligned}
f_2(\alpha, \Delta^*) &= \max_{1 \leq i \leq p} \alpha_i \lambda_i \left\{ \alpha_i + \delta_i^* E^{(2)} \left[\left(\frac{qd}{q+2} F_{q+2, n-p}^{-1}(\Delta^*) - 1 \right) I \left(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2} \right) \right] \right. \\
&\quad \left. - qd \delta_i^* E^{(2)}(\chi_{q+2}^{-2}(\Delta^*)) \right\}.
\end{aligned}$$

4.4.5.1 Comparison of PRSRRE with SRRE, URRE, RRRE and PTRRE

Theorem 4.17 *PRSRRE has smaller risk value than SRRE for all $k > 0$.*

Proof. Using (4.64) and (2.88) the result follows.

Theorem 4.18 *A sufficient condition for PRSRRE to have risk value less than or equal to that of URRE is that k_4 where k_4 is given by (4.56).*

Proof. Using (4.56) and (4.64) the result follows.

Theorem 4.19 *A sufficient condition for PRSRRE to have smaller risk value than RRRE is that $0 < k_7 < k$ where k_7 is given by (4.68).*

Proof. Using (4.57) and (4.64), we find

$$\begin{aligned}
& R_5(\hat{\beta}_{(k)}^{S+}; \beta) - R_2(\hat{\beta}_{(k)}; \beta) \\
&= \sum_{i=1}^p \frac{1}{(\lambda_i + k)^2} \left\{ \sigma_e^2 h_{ii}^* \left\{ 1 - dq \left[(q-2) E^{(1)}(\chi_{q+2}^{-4}(\Delta^*)) \right. \right. \right. \\
&\quad \left. \left. + \left(1 - \frac{(q+2)\lambda_i^2 \delta_i^{*2}}{2\sigma_e^2 \Delta^* h_{ii}^*} \right) (2\Delta^*) E^{(2)}(\chi_{q+4}^{-4}(\Delta^*)) \right] \right\} - \lambda_i^2 \delta_i^{*2} \\
&\quad - \sigma_e^2 \left\{ h_{ii}^* E^{(1)} \left[\left(1 - \frac{qd}{q+2} F_{q+2, n-p}^{-1}(\Delta^*) \right)^2 I \left(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2} \right) \right] \right. \\
&\quad \left. + \frac{\lambda_i^2 \delta_i^{*2}}{\sigma_e^2} E^{(2)} \left[\left(1 - \frac{qd}{q+4} F_{q+4, n-p}^{-1}(\Delta^*) \right)^2 I \left(F_{q+4, n-p}(\Delta^*) \leq \frac{qd}{q+4} \right) \right] \right\} \\
&\quad - 2\lambda_i^2 \delta_i^{*2} E^{(2)} \left[\left(\frac{qd}{q+2} F_{q+2, n-p}^{-1}(\Delta^*) - 1 \right) I \left(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2} \right) \right] \\
&\quad - 2k\alpha_i \lambda_i \delta_i^* \left\{ 1 - qd E^{(2)}(\chi_{q+2}^{-2}(\Delta^*)) \right. \\
&\quad \left. + E^{(2)} \left[\left(\frac{qd}{q+2} F_{q+2, n-p}^{-1}(\Delta^*) - 1 \right) I \left(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2} \right) \right] \right\} \right\}. \quad (4.67)
\end{aligned}$$

This difference is nonpositive (≤ 0), whenever

$$k_7 = \frac{f_3(\alpha, \Delta^*)}{f_4(\alpha, \Delta^*)} \quad (4.68)$$

with

$$\begin{aligned}
f_3(\alpha, \Delta^*) &= \max_{1 \leq i \leq p} \left\{ \sigma_e^2 h_{ii}^* \left\{ 1 - dq \left[(q-2) E^{(1)}(\chi_{q+2}^{-4}(\Delta^*)) \right. \right. \right. \\
&\quad \left. \left. + \left(1 - \frac{(q+2)\lambda_i^2 \delta_i^{*2}}{2\sigma_e^2 \Delta^* h_{ii}^*} \right) (2\Delta^*) E^{(2)}(\chi_{q+4}^{-4}(\Delta^*)) \right] \right\} - \lambda_i^2 \delta_i^{*2} \\
&\quad - \sigma_e^2 \left\{ h_{ii}^* E^{(1)} \left[\left(1 - \frac{qd}{q+2} F_{q+2, n-p}^{-1}(\Delta^*) \right)^2 I \left(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2} \right) \right] \right. \\
&\quad \left. + \frac{\lambda_i^2 \delta_i^{*2}}{\sigma_e^2} E^{(2)} \left[\left(1 - \frac{qd}{q+4} F_{q+4, n-p}^{-1}(\Delta^*) \right)^2 I \left(F_{q+4, n-p}(\Delta^*) \leq \frac{qd}{q+4} \right) \right] \right\} \\
&\quad - 2\lambda_i^2 \delta_i^{*2} E^{(2)} \left[\left(\frac{qd}{q+2} F_{q+2, n-p}^{-1}(\Delta^*) - 1 \right) I \left(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2} \right) \right] \right\}
\end{aligned}$$

and

$$\begin{aligned}
f_4(\alpha, \Delta^*) &= \min_{1 \leq i \leq p} \left\{ 2\alpha_i \lambda_i \delta_i^* \left\{ 1 - qd E^{(2)}(\chi_{q+2}^{-2}(\Delta^*)) \right. \right. \\
&\quad \left. \left. + E^{(2)} \left[\left(\frac{qd}{q+2} F_{q+2, n-p}^{-1}(\Delta^*) - 1 \right) I \left(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2} \right) \right] \right\} \right\}.
\end{aligned}$$

Theorem 4.20 A sufficient condition for PRSRRE to have smaller risk value than PTRRE is that $0 < k_8 < k$ where k_8 is given by (4.70).

Proof. Using (4.59) and (4.64), we obtain

$$\begin{aligned}
& R_5(\hat{\beta}_{(k)}^{S+}) - R_3(\hat{\beta}_{(k)}^{PT}; \beta) \\
&= \sum_{i=1}^p \frac{1}{(\lambda_i + k)^2} \left\{ \sigma_e^2 h_{ii} \left[G_{q+2, n-p}^{(1)}(x', \Delta^*) - dq \left[(q-2) E^{(1)}(\chi_{q+2}^{-4}(\Delta^*)) \right. \right. \right. \\
&\quad \left. \left. + \left(1 - \frac{(q+2)\lambda_i^2 \delta_i^{*2}}{2\sigma_e^2 \Delta^* h_{ii}^*} \right) (2\Delta^*) E^{(2)}(\chi_{q+4}^{-4}(\Delta^*)) \right] \right\} \\
&\quad - \lambda_i^2 \delta_i^{*2} \left[2G_{q+2, n-p}^{(2)}(x', \Delta^*) - G_{q+4, n-p}^{(2)}(x', \Delta^*) \right] \\
&\quad - \sigma_e^2 \left\{ h_{ii}^* E^{(1)} \left[\left(1 - \frac{qd}{q+2} F_{q+2, n-p}^{-1}(\Delta^*) \right)^2 I \left(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2} \right) \right] \right. \\
&\quad \left. + \frac{\lambda_i^2 \delta_i^{*2}}{\sigma_e^2} E^{(2)} \left[\left(1 - \frac{qd}{q+4} F_{q+4, n-p}^{-1}(\Delta^*) \right)^2 I \left(F_{q+4, n-p}(\Delta^*) \leq \frac{qd}{q+4} \right) \right] \right\} \\
&\quad - 2\lambda_i^2 \delta_i^{*2} E^{(2)} \left[\left(\frac{qd}{q+2} F_{q+2, n-p}^{-1}(\Delta^*) - 1 \right) I \left(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2} \right) \right] \\
&\quad - 2k\alpha_i \lambda_i \delta_i^* \left\{ G_{q+2, n-p}^{(2)}(x', \Delta^*) - qd E^{(2)}(\chi_{q+2}^{-2}(\Delta^*)) \right. \\
&\quad \left. + E^{(2)} \left[\left(\frac{qd}{q+2} F_{q+2, n-p}^{-1}(\Delta^*) - 1 \right) I \left(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2} \right) \right] \right\} \}.
\end{aligned} \tag{4.69}$$

This difference is nonpositive (≤ 0), whenever

$$k_8 = \frac{f_5(\alpha, \Delta^*)}{f_6(\alpha, \Delta^*)} \tag{4.70}$$

with

$$\begin{aligned}
f_5(\alpha, \Delta^*) &= \max_{1 \leq i \leq p} \left\{ \sigma_e^2 h_{ii}^* \left[G_{q+2, n-p}^{(1)}(x', \Delta^*) - dq \left[(q-2) E^{(1)}(\chi_{q+2}^{-4}(\Delta^*)) \right. \right. \right. \\
&\quad \left. \left. + \left(1 - \frac{(q+2)\lambda_i^2 \delta_i^{*2}}{2\sigma_e^2 \Delta^* h_{ii}^*} \right) (2\Delta^*) E^{(2)}(\chi_{q+4}^{-4}(\Delta^*)) \right] \right\} \\
&\quad - \lambda_i^2 \delta_i^{*2} \left[2G_{q+2, n-p}^{(2)}(x', \Delta^*) - G_{q+4, n-p}^{(2)}(x', \Delta^*) \right] \\
&\quad - \sigma_e^2 \left\{ h_{ii}^* E^{(1)} \left[\left(1 - \frac{qd}{q+2} F_{q+2, n-p}^{-1}(\Delta^*) \right)^2 I \left(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2} \right) \right] \right. \\
&\quad \left. + \frac{\lambda_i^2 \delta_i^{*2}}{\sigma_e^2} E^{(2)} \left[\left(1 - \frac{qd}{q+4} F_{q+4, n-p}^{-1}(\Delta^*) \right)^2 I \left(F_{q+4, n-p}(\Delta^*) \leq \frac{qd}{q+4} \right) \right] \right\} \\
&\quad - 2\lambda_i^2 \delta_i^{*2} E^{(2)} \left[\left(\frac{qd}{q+2} F_{q+2, n-p}^{-1}(\Delta^*) - 1 \right) I \left(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2} \right) \right] \},
\end{aligned}$$

and

$$f_6(\alpha, \Delta^*) = \min_{1 \leq i \leq p} \left\{ 2\alpha_i \lambda_i \delta_i^* \{ G_{q+2, n-p}^{(2)}(x', \Delta^*) - qdE^{(2)}(\lambda_{q+2}^{-2}(\Delta^*)) \right. \\ \left. + E^{(2)} \left[\left(\frac{qd}{q+2} F_{q+2, n-p}^{-1}(\Delta^*) - 1 \right) I \left(F_{q+2, n-p}(\Delta^*) \leq \frac{qd}{q+2} \right) \right] \right\}.$$

Remarks. (i) We note that under $H_0 : \mathbf{H}\beta = \mathbf{h}$, Theorem 4.19 simplifies to the following:

$$R_3(\hat{\beta}_{(k)}^{S+}; \beta) - R_2(\hat{\beta}_{(k)}; \beta) = \sum_{i=1}^p \frac{1}{(\lambda_i + k)^2} \left\{ \sigma_e^2 (1-d) h_{ii}^* \right. \\ \left. - \sigma_e^2 h_{ii}^* E^{(1)} \left[\left(1 - \frac{qd}{q+2} F_{q+2, n-p}^{-1}(0) \right)^2 I \left(F_{q+2, n-p}(0) \leq \frac{qd}{q+2} \right) \right] \right\} \geq 0$$

since (2.91). Thus $\hat{\beta}_{(k)}$ is superior to $\hat{\beta}_{(k)}^{S+}$ under $H_0 : \mathbf{H}\beta = \mathbf{h}$.

(ii) We note that under $H_0 : \mathbf{H}\beta = \mathbf{h}$, Theorem 4.20 simplifies to the following:

$$R_3(\hat{\beta}_{(k)}^{S+}; \beta) - R_3(\hat{\beta}_{(k)}^{PT}; \beta) = \sum_{i=1}^p \frac{1}{(\lambda_i + k)^2} \left\{ \sigma_e^2 h_{ii}^* G_{q+2, n-p}^{(1)}(x', 0) - d \right. \\ \left. - \sigma_e^2 h_{ii}^* E^{(1)} \left[\left(1 - \frac{qd}{q+2} F_{q+2, n-p}^{-1}(0) \right)^2 I \left(F_{q+2, n-p}(0) \leq \frac{qd}{q+2} \right) \right] \right\} \geq 0$$

since (2.93). Thus, the risk of $\hat{\beta}_{(k)}^{S+}$ is smaller than that of the risk of $\hat{\beta}_{(k)}^{PT}$ when the critical value satisfied the opposite inequality in (2.93), otherwise $\hat{\beta}_{(k)}^{PT}$ is smaller than $\hat{\beta}_{(k)}^{S+}$.

4.5 Conclusion

In this chapter, we have combined the idea of preliminary test estimator and Stein-rule estimator with the ridge regression approach of estimation to obtain better estimator of regression parameter in a multiple regression model. We have studied this completely new approach for the five estimators of β when the null-hypothesis $H_0 : \mathbf{H}\beta = \mathbf{h}$ is suspicious and $\mathbf{C}$ is ill-conditioned. The URRE, RRRE, PTRRE, SRRE and PRSRRE have been studied by their risks and MSE matrices. The risk criterion has been effectively determined a sufficient condition for the superiority of these estimators over traditional estimators. We find there exists a k such that the new estimators dominate over the other traditional estimators. The property of the five estimators is robust in the class of t -distribution which is determined by the degrees of freedom $\nu (\nu > 4)$.

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Chapter 5

The Improved Estimators of Variance in the Linear Model Under Uncertain Prior Sub-Space Restriction with Spherically Symmetric Distributions

5.1 Introduction

In the previous chapters we have discussed five estimators of regression coefficients for the general linear model when there is a suspicion that the parameters belong to:

- (1) a non-stochastic subspace

$$\mathbf{H}\boldsymbol{\beta} = \mathbf{h}$$

- (2) a stochastic subspace

$$\mathbf{r} = \mathbf{R}\boldsymbol{\beta} + \mathbf{v}$$

where $\mathbf{v}$ is a random vector.

- (3) a non-stochastic subspace

$$\mathbf{H}\boldsymbol{\beta} = \mathbf{h}$$

when the matrix $\mathbf{C} = \mathbf{X}'\mathbf{X}$ is ill conditioned. In this chapter, we consider four estimators of the variance in the linear model under uncertain prior subspace information $\mathbf{H}\boldsymbol{\beta} = \mathbf{h}$ with spherically symmetric distributions. We organize this chapter as follows. In Section 5.2, we discuss four estimators of the variance τ^2 when the errors are normal. In Section 5.3, we analyze four estimators of the variance σ_e^2 when the errors are multivariate t -distribution.

5.2 The Improved Estimators of Variance in the Linear Model Under Uncertain Prior Information with Normal Distributions

We again consider the basic linear model

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{e}, \quad (5.1)$$

where $\mathbf{y}$ is an $(n \times 1)$ vector of observations, $\mathbf{X}$ is an $(n \times p)$ matrix of rank p , $\boldsymbol{\beta}$ is a $(p \times 1)$ vector of unknown parameters and $\mathbf{e}$ is an $(n \times 1)$ vector random variable which is distributed as $N_n(\mathbf{0}, \tau^2 \mathbf{I}_n)$. We are primarily interested in the estimation of τ^2 when the error distribution is normal and it is a priori suspected that $H_0 : \mathbf{H}\boldsymbol{\beta} = \mathbf{h}$, where $\mathbf{H}$ is a $(q \times p)$ known matrix of rank $q (< p)$ and $\mathbf{h}$ is a $(q \times 1)$ vector of known constants. With this background at hand we shall proceed with the discussion of various estimators of τ^2 under normal distribution in the linear model (5.1).

5.2.1 The Proposed Estimators

We consider four estimators of τ^2 as follows:

- (i) The unrestricted unbiased estimator (UUE) of τ^2 is defined by

$$\tilde{\sigma}_e^2 = S_e^2 = \frac{(\mathbf{y} - \mathbf{X}\tilde{\boldsymbol{\beta}})'(\mathbf{y} - \mathbf{X}\tilde{\boldsymbol{\beta}})}{m}, \quad m = n - p \quad (5.2)$$

where $\tilde{\boldsymbol{\beta}}$ and $\tilde{\sigma}_e^2$ are given by (2.8) and (2.12).

- (ii) The restricted unbiased estimator (RUE) of τ^2 is defined by

$$\hat{\sigma}_e^2 = S_e^{*2} = \frac{(\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}})'(\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}})}{m + q}, \quad m = n - p \quad (5.3)$$

where $\hat{\boldsymbol{\beta}}$ and $\hat{\sigma}_e^2$ are given by (2.15) and (2.19).

(iii) The preliminary test estimator (PTE) of τ^2 is defined by

$$\begin{aligned}\hat{\sigma}_e^{2PT} &= \hat{\sigma}_e^2 I(\mathcal{L}_n > F_\alpha) + \hat{\sigma}_e^2 I(\mathcal{L}_n \leq F_\alpha) \\ &= \hat{\sigma}_e^2 + [1 - I(\mathcal{L}_n \leq F_\alpha)](\hat{\sigma}_e^2 - \hat{\sigma}_e^2),\end{aligned}\quad (5.4)$$

where $\mathcal{L}_n$ and F_α are given by (2.20) and (2.24)–(2.25). This estimator has been considered by Giles (1991). Here the PTE depends on α . Therefore, we look for an estimator which does not depend on α . So, we propose the following estimator:

(iv) The Stein-type estimator (SE) of τ^2 is defined by

$$\hat{\sigma}_e^{2S} = \frac{m}{m+2} I\left(\mathcal{L}_n > \frac{m}{m+2}\right) S_e^2 + \frac{m\left(1 + \frac{q}{m} \mathcal{L}_n\right)}{m+q+2} I\left(\mathcal{L}_n \leq \frac{m}{m+2}\right) S_e^2, \quad m = n-p \quad (5.5)$$

In the next subsection, we give the expressions for the bias and risk of the estimators $\hat{\sigma}_e^2$, $\hat{\sigma}_e^{2PT}$ and $\hat{\sigma}_e^{2S}$ when the errors follow a normal distribution.

5.2.2 Bias and Risk of the Estimators

We provide the biases of four estimators of τ^2 when the error terms are normal.

5.2.2.1 Biases of the Estimators

Theorem 5.1 *bias of the unrestricted unbiased estimator (UUE), restricted unbiased estimator (RUE), preliminary test estimator (PTE) and Stein-type estimator (SE) are given by (5.6)–(5.9) respectively:*

$$\text{i) } B_1 = E[(\hat{\sigma}_e^2 - \tau^2)] = 0, \quad (5.6)$$

$$\text{ii) } B_2 = E[(\hat{\sigma}_e^{2PT} - \tau^2)] = \frac{\Delta}{m+q} \tau^2, \quad (5.7)$$

$$\begin{aligned}\text{iii) } B_3 &= E[(\hat{\sigma}_e^{2S} - \tau^2)] \\ &= \frac{q\tau^2}{m(m+q)} \left\{ E\left[\chi_m^2 F_{q,m}(\Delta) I(F_{q,m}(\Delta) \leq F_\alpha)\right] - E\left[\chi_m^2 I(F_{q,m}(\Delta) \leq F_\alpha)\right] \right\},\end{aligned}\quad (5.8)$$

$$\begin{aligned}\text{iv) } B_4 &= \tau^2 \left\{ \frac{q}{m(m+q+2)} E\left[\chi_m^2 F_{q,m}(\Delta) I\left(F_{q,m}(\Delta) \leq \frac{m}{m+2}\right)\right] \right. \\ &\quad \left. - \frac{q}{(m+2)(m+q+2)} E\left[\chi_m^2 I\left(F_{q,m}(\Delta) \leq \frac{m}{m+2}\right)\right] - \frac{2}{m+2} \right\},\end{aligned}\quad (5.9)$$

where

$$\Delta = \theta/\tau^2, \quad \theta = (\mathbf{H}\boldsymbol{\beta} - \mathbf{h})'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}(\mathbf{H}\boldsymbol{\beta} - \mathbf{h}) \quad (5.10)$$

and

$$\begin{aligned} & E\left[(\chi_m^2)^{k_1} [F_{q,m}(\Delta)]^{k_2} I\left(F_{q,m}(\Delta) \leq \frac{m}{m+2}\right)\right] \\ &= \sum_{r=0}^{\infty} 2^{k_1} \left(\frac{m}{q}\right)^{k_2} \frac{e^{-\frac{q}{2}} \left(\frac{\Delta}{2}\right)^r}{\Gamma(r+1)} \frac{\Gamma\left(\frac{q}{2} + k_2 + r\right) \Gamma\left(\frac{m}{2} + k_1 - k_2\right)}{\Gamma\left(\frac{q}{2} + r\right) \Gamma\left(\frac{m}{2}\right)} \\ &\times I_{x_1}\left(\frac{q}{2} + k_2 + r, \frac{m}{2} + k_1 - k_2\right), \end{aligned} \quad (5.11)$$

$$\begin{aligned} & E\left[(\chi_m^2)^{k_1} [F_{q,m}(0)]^{k_2} I\left(F_{q,m}(0) \leq \frac{m}{m+2}\right)\right] \\ &= 2^{k_1} \left(\frac{m}{q}\right)^{k_2} \frac{\Gamma\left(\frac{q}{2} + k_2\right) \Gamma\left(\frac{m}{2} + k_1 - k_2\right)}{\Gamma\left(\frac{q}{2}\right) \Gamma\left(\frac{m}{2}\right)} I_{x_1}\left(\frac{q}{2} + k_2, \frac{m}{2} + k_1 - k_2\right), \end{aligned} \quad (5.12)$$

with $x_1 = \frac{q}{m+2+q}$ and $I_x(\cdot, \cdot)$ is a usual incomplete beta distribution function.

Proof. (i) We have the following identity

$$(\mathbf{y} - \mathbf{X}\tilde{\boldsymbol{\beta}})'(\mathbf{y} - \mathbf{X}\tilde{\boldsymbol{\beta}}) = \mathbf{e}'(\mathbf{I}_n - \mathbf{X}\mathbf{C}^{-1}\mathbf{X}')\mathbf{e}. \quad (5.13)$$

Therefore,

$$\frac{\mathbf{e}'(\mathbf{I}_n - \mathbf{X}\mathbf{C}^{-1}\mathbf{X}')\mathbf{e}}{\tau^2} = \frac{mS_e^2}{\tau^2} \sim \chi_m^2(0),$$

since

$$\mathbf{e} \sim N_n(\mathbf{0}, \tau^2 \mathbf{I}_n) \text{ and } \mathbf{I}_n - \mathbf{X}\mathbf{C}^{-1}\mathbf{X}'$$

is a symmetric and idempotent matrix (see Mardia, 1979). Thus, we get

$$B_1 = E(\hat{\sigma}_e^2) - \tau^2 = \tau^2 = 0.$$

(ii) Using Theorem 2.2 in special case, we get

$$B_2 = E[(\hat{\sigma}_\epsilon^2 - \tau^2)] = \frac{\Delta}{m+q} \tau^2, \quad \Delta = \frac{\theta}{\tau^2}.$$

(iii) Using (2.18) and (5.4), we obtain

$$\bar{\sigma}_\epsilon^2 - \hat{\sigma}_\epsilon^2 = \frac{q}{m+q} S_\epsilon^2 - \frac{q}{m+q} \mathcal{L}_n S_\epsilon^2. \quad (5.14)$$

Now, we can write

$$\begin{aligned} B_3 &= E(\hat{\sigma}_\epsilon^{2PT}) - \tau^2 \\ &= E(\bar{\sigma}_\epsilon^2 - \tau^2) - E[I(\mathcal{L}_n \leq F_\alpha)(\bar{\sigma}_\epsilon^2 - \hat{\sigma}_\epsilon^2)] \\ &= \frac{q}{m+q} E\left[\left(\frac{\tau^2}{m}\right) \left(\frac{m S_\epsilon^2}{\tau^2}\right) \mathcal{L}_n I(\mathcal{L}_n \leq F_\alpha)\right] \\ &\quad - \frac{q}{m+q} E\left[\left(\frac{\tau^2}{m}\right) \left(\frac{m S_\epsilon^2}{\tau^2}\right) I(\mathcal{L}_n \leq F_\alpha)\right]. \end{aligned}$$

Hence

$$B_3 = \frac{q\tau^2}{m(m+q)} \left\{ E[\chi_m^2 F_{q,m}(\Delta) I(F_{q,m}(\Delta) \leq F_\alpha)] - E[\chi_m^2 I(F_{q,m}(\Delta) \leq F_\alpha)] \right\}.$$

(iv) Using (5.5) and (5.11),

$$\begin{aligned} B_4 &= E(\bar{\sigma}_\epsilon^{2S}) - \tau^2 = \frac{\tau^2}{m+2} E\left[\left(\frac{m S_\epsilon^2}{\tau^2}\right) I(\mathcal{L}_n > \frac{m}{m+2})\right] \\ &\quad + \frac{\tau^2}{m+q+2} E\left[\left(\frac{m S_\epsilon^2}{\tau^2}\right) \left(1 + \frac{q}{m} \mathcal{L}_n\right) I(\mathcal{L}_n \leq \frac{m}{m+2})\right] - \tau^4. \end{aligned}$$

Thus

$$\begin{aligned} B_4 &= \tau^2 \left\{ \frac{q}{m(m+q+2)} E\left[\chi_m^2 F_{q,m}(\Delta) I\left(F_{q,m}(\Delta) \leq \frac{m}{m+2}\right)\right] \right. \\ &\quad \left. - \frac{q}{(m+2)(m+q+2)} E\left[\chi_m^2 I\left(F_{q,m}(\Delta) \leq \frac{m}{m+2}\right)\right] - \frac{2}{m+2} \right\}. \end{aligned}$$

5.2.2.2 Risks of the Estimators

In this subsection we present the risks of the four estimators $\tilde{\sigma}_e^2$, $\hat{\sigma}_e^2$, $\hat{\sigma}_e^{2PT}$ and $\hat{\sigma}_3^{2S}$.

Theorem 5.2 *Risks of the unrestricted unbiased estimator (UUE), restricted unbiased estimator (RUE), preliminary test estimator (PTE) and Stein-type estimator (SE) are given by (5.15) – (5.18) respectively:*

$$(i) R_1(\tilde{\sigma}_e^2; \tau^2) = \frac{2\tau^4}{m} \quad (5.15)$$

$$(ii) R_2(\hat{\sigma}_e^2; \tau^2) = \frac{[2(m+q) + \Delta(\Delta+4)]}{(m+q)^2} \tau^4 \quad (5.16)$$

$$\begin{aligned} (iii) R_3(\hat{\sigma}_e^{2PT}; \tau^2) = & \tau^4 \left\{ \frac{q^2}{m^2(m+q)^2} E[(\chi_m^2)^2 [F_{q,m}(\Delta)]^2 I(F_{q,m}(\Delta) \leq F_\alpha)] \right. \\ & + \frac{2q}{m(m+q)^2} E[(\chi_m^2)^2 F_{q,m}(\Delta) I(F_{q,m}(\Delta) \leq F_\alpha)] \\ & - \frac{q(2m+q)}{m^2(m+q)^2} E[(\chi_m^2)^2 I(F_{q,m}(\Delta) \leq F_\alpha)] \\ & - \frac{2q}{m(m+q)} E[\chi_m^2 F_{q,m}(\Delta) I(F_{q,m}(\Delta) \leq F_\alpha)] \\ & \left. + \frac{2q}{m(m+q)} E[\chi_m^2 I(F_{q,m}(\Delta) \leq F_\alpha)] + \frac{2}{m} \right\}. \quad (5.17) \end{aligned}$$

$$\begin{aligned} (iv) R_4(\hat{\sigma}_e^{2S}; \tau^2) = & \tau^4 \left\{ \frac{q^2}{m^2(m+q+2)^2} E[(\chi_m^2)^2 [F_{q,m}(\Delta)]^2 I(F_{q,m}(\Delta) \leq \frac{m}{m+2})] \right. \\ & + \frac{2q}{m(m+q+2)^2} E[(\chi_m^2)^2 F_{q,m}(\Delta) I(F_{q,m}(\Delta) \leq \frac{m}{m+2})] \\ & - \frac{q(2m+q+4)}{(m+2)^2(m+q+2)^2} E[(\chi_m^2)^2 I(F_{q,m}(\Delta) \leq \frac{m}{m+2})] \\ & + \frac{2q}{(m+2)(m+q+2)} E[\chi_m^2 I(F_{q,m}(\Delta) \leq \frac{m}{m+2})] \\ & \left. - \frac{2q}{m(m+q+2)} E[\chi_m^2 F_{q,m}(\Delta) I(F_{q,m}(\Delta) \leq \frac{m}{m+2})] + \frac{2}{m+2} \right\}. \quad (5.18) \end{aligned}$$

where Δ and $E\left[(\chi_m^2)^{k_1} [F_{q,m}(\Delta)]^{k_2} I\left(F_{q,m}(\Delta) \leq \frac{m}{m+2}\right)\right]$ are given by (5.10) – (5.11) respectively.

Proof. (i) Using (5.13)

$$R_1(\hat{\sigma}_e^2; \tau^2) = E[(\hat{\sigma}_e^2 - \tau^2)^2] = \text{var}(S_e^2) = \frac{2\tau^4}{m}.$$

(ii) Using (2.18), we obtain

$$\begin{aligned} \text{Var}\left[\frac{(\mathbf{y} - \mathbf{X}\hat{\beta})'(\mathbf{y} - \mathbf{X}\hat{\beta})}{\tau^2}\right] &= \text{var}\left[\frac{(\mathbf{y} - \mathbf{X}\tilde{\beta})'(\mathbf{y} - \mathbf{X}\tilde{\beta})}{\tau^2}\right] \\ &+ \text{var}\left[\frac{(\mathbf{H}\tilde{\beta} - \mathbf{h})'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}(\mathbf{H}\tilde{\beta} - \mathbf{h})}{\tau^2}\right] = 2m + 2q + 4\Delta, \end{aligned}$$

since the two terms on the r.h.s. are independent. Thus

$$\text{var}(\hat{\sigma}_e^2) = \frac{(2m + 2q + 4\Delta)}{(m + q)^2} \tau^4$$

and

$$\begin{aligned} R_2(\hat{\sigma}_e^2; \tau^2) &= \text{var}(\hat{\sigma}_e^2) + B_2^2 \\ &= \frac{[2(m + q) + \Delta(\Delta + 4)]}{(m + q)^2} \tau^4. \end{aligned}$$

(iii) Using (5.4) and (5.11)

$$\begin{aligned} R_3(\hat{\sigma}_e^{2PT}; \tau^2) &= E[(\hat{\sigma}_e^{2PT} - \tau^2)^2] = E[(\hat{\sigma}_e^{2PT})^2] - 2\tau^2 B_3 - \tau^4 \\ &= \frac{\tau^4}{m^2} E\left[\left(\frac{mS_e^2}{\tau^2}\right)^2\right] + \frac{q^2\tau^4}{m^2(m+q)^2} E\left[(\chi_m^2)^2 (1 - F_{q,m}(\Delta))^2 I(F_{q,m}(\Delta) \leq F_\alpha)\right] \\ &- \frac{2q\tau^4}{m^2(m+q)} E\left[(\chi_m^2)^2 (1 - F_{q,m}(\Delta)) I(F_{q,m}(\Delta) \leq F_\alpha)\right] \\ &+ \frac{2q\tau^4}{m(m+q)} E\left[\chi_m^2 (1 - F_{q,m}(\Delta)) I(F_{q,m}(\Delta) \leq F_\alpha)\right] - \tau^4. \end{aligned}$$

After simplifying we get (5.17).

(iv) Using (5.5) and (5.11)

$$\begin{aligned}
R_4(\hat{\sigma}_e^{2S}; \tau^2) &= E[(\hat{\sigma}_e^{2S} - \tau^2)^2] = E[(\hat{\sigma}_e^{2S})^2] - 2\tau^2 B_4 - \tau^4 = \frac{\tau^4}{(m+2)^2} E\left[\left(\frac{mS_e^2}{\tau^2}\right)^2\right] \\
&+ \frac{q^2\tau^4}{m^2(m+q+2)^2} E\left[\left(\frac{m}{m+2} - \mathcal{L}_n\right)^2 \left(\frac{mS_e^2}{\tau^2}\right)^2 I\left(\mathcal{L}_n \leq \frac{m}{m+2}\right)\right] \\
&- \frac{2q\tau^4}{m(m+2)(m+q+2)} E\left[\left(\frac{mS_e^2}{\tau^2}\right)^2 \left(\frac{m}{m+2} - \mathcal{L}_n\right) I\left(\mathcal{L}_n \leq \frac{m}{m+2}\right)\right] \\
&+ \frac{2q\tau^4}{m(m+q+2)} E\left[\left(\frac{mS_e^2}{\tau^2}\right) \left(\frac{m}{m+2} - \mathcal{L}_n\right) I\left(\mathcal{L}_n \leq \frac{m}{m+2}\right)\right] - \tau^4.
\end{aligned}$$

After simplifying (5.18) follows.

Remarks (1) The Stein-type estimator can be written as follows:

$$\hat{\sigma}_e^{2S} = m\phi_S(\mathcal{L}_n)S_e^2 \quad (5.19)$$

where

$$\phi_S(\mathcal{L}_n) = \min\left(\frac{1}{m+2}, \frac{1 + \frac{q}{m}\mathcal{L}_n}{m+q+2}\right)$$

(2) The preliminary test estimator for $F_\alpha = 1$, can be written as follows

$$\hat{\sigma}_e^{2PT} = m\phi_{PT}(\mathcal{L}_n)S_e^2 \quad (5.20)$$

where

$$\phi_{PT}(\mathcal{L}_n) = \min\left(\frac{1}{m}, \frac{1 + \frac{q}{m}\mathcal{L}_n}{m+q}\right).$$

5.2.3 Analysis of the Risks of the Estimators

In this subsection, we study the relative performance of the estimators of τ^2 . First we note that the risk of UUE of τ^2 is $\frac{2\tau^2}{m}$ which is constant but that of RUE depends on Δ . Under H_0 , the risk of RUE is given by $\frac{2\tau^4}{m+q}$ which is less than

$\frac{2\tau^2}{m}$. Thus, RUE dominates UUE under H_0 . However, the range of values of Δ for which RUE dominates UUE is given by the positive solution of the equation

$$m\Delta^2 + 4m\Delta - 2(m+q)q = 0.$$

Let

$$\Delta_0 = -2 + 2\sqrt{1 + \frac{q(m+q)}{2m}}$$

represents the positive solution of the equation then, in the interval $[0, \Delta_0]$, RUE dominates UUE and outside $[0, \Delta_0]$ UUE dominates RUE. We note further, that as $\Delta \rightarrow \infty$ the risk of RUE is unbounded while risk of UUE is fixed as noted earlier.

Now, we compare the Stein-type estimator (SE) of τ^2 against UUE. We present the following domination theorem on it using Stein technique.

Theorem 5.3 If $e \sim N_n(0, \tau^2 I_n)$, then $\hat{\sigma}_e^{2S} = m\phi_S(\mathcal{L}_n)S_e^2$ dominates $\frac{mS_e^2}{m+2}$ with respect to the loss

$$L(\hat{\sigma}_e^{2S}; \tau^2) = \frac{1}{\tau^4}(\hat{\sigma}_e^{2S} - \tau^2)^2. \quad (5.21)$$

Proof. Consider the risk of the estimator $\hat{\sigma}_e^{2S}$ with respect to the loss (5.21),

$$\begin{aligned} & \frac{E[(m\phi(\mathcal{L}_n)S_e^2 - \tau^2)^2]}{\tau^4} \\ &= E\left\{\left[\phi(\mathcal{L}_n)\left(\frac{mS_e^2}{\tau^2}\right) - 1\right]^2\right\} \\ &= E\left[\phi^2(\mathcal{L}_n)\left(\frac{mS_e^2}{\tau^2}\right)^2 - 2\phi(\mathcal{L}_n)\left(\frac{mS_e^2}{\tau^2}\right) + 1\right] \\ &= E_{\mathcal{L}_n}\left\{E\left[\left(\phi^2(\mathcal{L}_n)\left(\frac{mS_e^2}{\tau^2}\right)^2 - 2\phi(\mathcal{L}_n)\left(\frac{mS_e^2}{\tau^2}\right) + 1\right)|\mathcal{L}_n\right]\right\} \\ &= E_{\mathcal{L}_n}\left\{\phi^2(\mathcal{L}_n)E\left[\left(\frac{mS_e^2}{\tau^2}\right)^2|\mathcal{L}_n\right] - 2\phi(\mathcal{L}_n)E\left[\left(\frac{mS_e^2}{\tau^2}\right)|\mathcal{L}_n\right] + 1\right\} \quad (5.22) \end{aligned}$$

where $\phi(\mathcal{L}_n)$ is a real-valued function. Consider the term inside the curly brackets of (5.22). For fixed Δ and for each $\mathcal{L}_n$ this is a quadratic in $\phi(\cdot)$ with minimum at

$$\phi^*(\mathcal{L}_n) = \frac{E\left[\left(\frac{mS_e^2}{\tau^2}\right) | \mathcal{L}_n\right]}{E\left[\left(\frac{mS_e^2}{\tau^2}\right)^2 | \mathcal{L}_n\right]}, \quad (5.23)$$

which is a function of $\mathcal{L}_n$ and Δ (see Figure 1).

Fig. 1 For fixed $\mathcal{L}_n$, the quadratic

$$\phi^2(\mathcal{L}_n)E\left[\left(\frac{mS_e^2}{\tau^2}\right)^2 | \mathcal{L}_n\right] - 2\phi(\mathcal{L}_n)E\left[\left(\frac{mS_e^2}{\tau^2}\right) | \mathcal{L}_n\right] + 1,$$

the value of ϕ^* is the minimum and ϕ_s yields Steins' estimator.

The maximum of $\phi^*(\mathcal{L}_n)$ over Δ is attained at $\Delta = 0$ (see Stein, 1964 and Maatta and Casella, 1990), which eliminates the need to work with noncentral chi-squares.

Using

$$\begin{aligned} & E[(\chi_m^2)^{k_1} | F_{q,m}(\Delta) = \mathcal{L}_n] \\ &= \frac{1}{g_1[\mathcal{L}_n]} \sum_{r=0}^{\infty} 2^{k_1} \left(\frac{q}{m}\right)^{\frac{q}{2}+r} \frac{e^{-\Delta/2} (\Delta/2)^r}{\Gamma(r+1)} \frac{\Gamma\left(\frac{m+q}{2} + k_1 + r\right)}{\Gamma\left(\frac{q}{2} + r\right) \Gamma\left(\frac{m}{2}\right)} \frac{\mathcal{L}_n^{\frac{q}{2}+r-1}}{\left(1 + \frac{q}{m} \mathcal{L}_n\right)^{\frac{m+q}{2}+k_1+r}} \end{aligned} \quad (5.24)$$

where

$$g_1(\mathcal{L}_n) = \sum_{r=0}^{\infty} \left(\frac{q}{m}\right)^{\frac{m}{2}+r} \frac{e^{-\frac{\Delta}{2}}(\Delta/2)^r}{\Gamma(r+1)} \frac{\Gamma\left(\frac{m+q}{2}+r\right)}{\Gamma\left(\frac{q}{2}+r\right)\Gamma\left(\frac{m}{2}\right)} \frac{\mathcal{L}_n^{\frac{q}{2}+r-1}}{\left(1+\frac{q}{m}\mathcal{L}_n\right)^{\frac{m+q}{2}+r}} \quad (5.25)$$

and

$$E[(\chi_m^2)^{k_1} | F_{q,m}(0) = \mathcal{L}_n] = 2^{k_1} \frac{\Gamma\left(\frac{m+q}{2} + k_1\right)}{\Gamma\left(\frac{m+q}{2}\right)} \frac{1}{\left(1 + \frac{q}{m}\mathcal{L}_n\right)^{k_1}}, \quad (5.26)$$

we obtain,

$$\begin{aligned} \phi_0(\mathcal{L}_n) &= \frac{E\left[\left(\frac{mS_c^2}{\tau^2}\right) | \mathcal{L}_n\right]}{E\left[\left(\frac{mS_c^2}{\tau^2}\right)^2 | \mathcal{L}_n\right]} = \max_{\Delta} \phi^*(\mathcal{L}_n), \quad (\text{with } \Delta = 0) \\ &= \frac{E[\chi_m^2 | F_{q,m}(0)]}{E[(\chi_m^2)^2 | F_{q,m}(0)]} \\ &= \frac{1 + \frac{q}{m}\mathcal{L}_n}{m + q + 2} \end{aligned} \quad (5.27)$$

If $\mathcal{L}_n < \frac{m}{m+2}$, then $\frac{1+\frac{q}{m}\mathcal{L}_n}{m+q+2} < \frac{1}{m+2}$ which also implies that

$$\phi^*(\mathcal{L}_n) \leq \phi_0(\mathcal{L}_n) \leq \frac{1}{m+2}, \quad \text{for all } \Delta, \quad (5.28)$$

that is, $\phi_0(\mathcal{L}_n)$ is closer to the minimizing value than $\frac{1}{m+2}$. So it is obvious that for each Δ , $\mathcal{L}_n$,

$$E\left\{\left[\phi_S(\mathcal{L}_n)\left(\frac{mS_c^2}{\tau^2}\right) - 1\right]^2 | \mathcal{L}_n\right\} \leq E\left\{\left[\frac{1}{m+2}\left(\frac{mS_c^2}{\tau^2}\right) - 1\right]^2 | \mathcal{L}_n\right\}. \quad (5.29)$$

Thus $m\phi_S(\mathcal{L}_n)S_c^2$ is better than $\frac{mS_c^2}{m+2}$ and the theorem is proved.

Now, consider the PTE. We note that as $\alpha \rightarrow 1$, PTE = UUE while for $\alpha \rightarrow 0$ and $\Delta^* = 0$, PTE = RUE. The graph of the risk of PTE depends on α . Giles (1991) have shown that *minimum risk* of PTE is obtained for $F_{\alpha^*} = 1$ i.e., $\alpha^* = 1 - G_{q,m}^{-1}(1, 0)$. Therefore, we have the following corollary.

Corollary 5.1 Risk of Stein-type estimator for τ^2 dominates the risk of PTE for $\alpha^* = 1 - G_{q,m}^{-1}(1, 0)$.

Proof. Follows from (5.28), (5.29) and (5.20). Further, under H_0 , when $\mathcal{L}_n \rightarrow 1$ the risk of the SE is given by $\frac{2\tau^4}{m+q+2} < \frac{2\tau^4}{m+q} =$ the risk of RUE and when $\mathcal{L}_n \rightarrow 0$, the risk of SE leans towards $\frac{2\tau^4}{m+2} < \frac{2\tau^4}{m} =$ risk of UUE.

This analysis concludes that we should always use the Stein-type estimator when the sub-space restriction $\mathbf{H}\boldsymbol{\beta} = \mathbf{h}$ is uncertain.

5.3 The Improved Estimators of Variance in the Linear Model Under Uncertain Prior Information with Multivariate t -Errors

In this section we consider the general linear model (5.1) when the errors have multivariate t -distribution with pdf given by (2.4). Namely, $\mathbf{e} \sim Mt(\mathbf{0}, \sigma_e^2 \mathbf{I}_n)$, $\sigma_e^2 = \frac{\nu}{\nu-2} \sigma^2$. We are interested in the estimation of σ_e^2 under multivariate t -errors in the linear model (5.1) and it is a priori suspected that $H_0 : \mathbf{H}\boldsymbol{\beta} = \mathbf{h}$, where $\mathbf{H}$ is a $(q \times p)$ known matrix of rank of $q(< p)$ and $\mathbf{h}$ is a $(q \times 1)$ vector of known constants.

5.3.1 The Proposed Estimators

We consider four estimators of σ_e^2 exactly the same as we defined in (5.2)–(5.5). In the next subsection, we give the expressions for the bias and risk of the estimators $\hat{\sigma}_e^2$, $\hat{\sigma}_e^{2*}$, $\hat{\sigma}_e^{PT}$ and $\hat{\sigma}_e^{2S}$ when the errors follow a univariate t -distribution.

5.3.2 Bias and Risk of the Estimators

We provide four estimators of σ_e^2 when the error is a multivariate t -distribution.

5.3.2.1 Biases of the estimators

Theorem 5.4 *Bias of the unrestricted unbiased estimator (UUE), restricted unbiased estimator (RUE), preliminary test estimator (PTE) and Stein-type estimator are given by (5.30)–(5.33) respectively:*

$$(i). B_1 = E(\hat{\sigma}_e^2 - \sigma_e^2) = 0 \quad (5.30)$$

$$(ii) B_2 = E(\hat{\sigma}_e^2 - \sigma_e^2) = \frac{\Delta^*}{m+q} \sigma_e^2 \quad (5.31)$$

$$(iii) B_3 = \frac{q\sigma_e^2}{m(m+q)} \left\{ E^{(1)} \left[\chi_m^2 F_{q,m}(\Delta^*) I(F_{q,m}(\Delta^*) \leq F_\alpha) \right] - E^{(1)} \left[\chi_m^2 I(F_{q,m}(\Delta^*) \leq F_\alpha) \right] \right\} \quad (5.32)$$

$$(iv) B_4 = \sigma_e^2 \left\{ \frac{q}{m(m+q+2)} E^{(1)} \left[\chi_m^2 F_{q,m}(\Delta^*) I(F_{q,m}(\Delta^*) \leq \frac{m}{m+2}) \right] - \frac{q}{(m+2)(m+q+2)} E^{(1)} \left[\chi_m^2 I(F_{q,m}(\Delta^*) \leq \frac{m}{m+2}) \right] - \frac{2}{m+2} \right\} \quad (5.33)$$

where

$$\Delta^* = \frac{\theta}{\sigma_e^2}, \quad \theta = (\mathbf{H}\beta - \mathbf{h})'(\mathbf{H}\mathbf{C}^{-1}\mathbf{H}')^{-1}(\mathbf{H}\beta - \mathbf{h}) \quad (5.34)$$

and

$$\begin{aligned} & E^{(j)} \left[(\chi_m^2)^{k_1} [F_{q,m}(\Delta^*)]^{k_2} I(F_{q,m}(\Delta^*) \leq \frac{m}{m+2}) \right] \\ &= \sum_{r=0}^{\infty} 2^{k_1} \left(\frac{m}{q} \right)^{k_2} \frac{\Gamma\left(\frac{q}{2} + r - j\right) \Gamma\left(\frac{q}{2} + k_2 + r\right) \Gamma\left(\frac{m}{2} + k_1 - k_2\right) \left(\frac{\Delta^*}{\nu-2}\right)^r}{\Gamma(r+1) \Gamma\left(\frac{q}{2} - j\right) \Gamma\left(\frac{q}{2} + r\right) \Gamma\left(\frac{m}{2}\right) \left(1 + \frac{\Delta^*}{\nu-2}\right)^{\frac{q}{2} + r - j}} \\ & \times I_{x_1}\left(\frac{q}{2} + k_2 + r, \frac{m}{2} + k_1 - k_2\right), \quad j = 1, 2, \end{aligned} \quad (5.35)$$

$$\begin{aligned} & E^{(j)} \left[(\chi_m^2)^{k_1} [F_{q,m}(0)]^{k_2} I(F_{q,m}(0) \leq \frac{m}{m+2}) \right] \\ &= 2^{k_1} \left(\frac{m}{q} \right)^{k_2} \frac{\Gamma(q/2 + k_2) \Gamma\left(\frac{m}{2} + k_1 - k_2\right)}{\Gamma(q/2) \Gamma\left(\frac{m}{2}\right)} I_{x_1}(q/2 + k_2, \frac{m}{2} + k_1 - k_2), \quad j = 1, 2, \end{aligned} \quad (5.36)$$

with $x_1 = \frac{q}{m+2+q}$ and $I_x(\cdot, \cdot)$ is a usual incomplete beta distribution function.

Proof. (i) Using (5.2),

$$\begin{aligned} B_1 &= E(\bar{\sigma}_e^2 - \sigma_e^2) = E_r[E(S_e^2|\tau)] - \sigma_e^2 \\ &= E(\tau^2) - \sigma_e^2 = 0, \end{aligned}$$

where

$$E(\tau^{2j}) = \frac{\left(\frac{\nu}{2}\sigma^2\right)^j \Gamma\left(\frac{\nu}{2} - j\right)}{\Gamma\left(\frac{\nu}{2}\right)}, \quad \nu > 2j \text{ and } \sigma_e^2 = \frac{\nu}{\nu-2}\sigma^2, \quad \nu > 2. \quad (5.37)$$

(ii) Using Theorem 2.2, we get

$$B_2 = E(\hat{\sigma}_e^2 - \sigma_e^2) = \frac{\Delta^*}{m+q}\sigma_e^2.$$

(iii) Using (5.4), (5.14) and (5.35)

$$\begin{aligned} B_3 &= E\left[(\hat{\sigma}_e^{2PT} - \sigma_e^2)\right] \\ &= E(\bar{\sigma}_e^2 - \sigma_e^2) - E[I(\mathcal{L}_n \leq F_\alpha)(\bar{\sigma}_e^2 - \hat{\sigma}_e^2)] \end{aligned}$$

$$\begin{aligned} B_3 &= \frac{q}{m+q} E_r \left\{ \left(\frac{\tau^2}{m} \right) E \left[\left(\frac{mS_e^2}{\tau^2} \right) \mathcal{L}_n I(\mathcal{L}_n \leq F_\alpha) | \tau \right] \right\} \\ &\quad - \frac{q}{m+q} E_r \left\{ \left(\frac{\tau^2}{m} \right) E \left[\left(\frac{mS_e^2}{\tau^2} \right) I(\mathcal{L}_n \leq F_\alpha) | \tau \right] \right\}. \end{aligned}$$

Hence,

$$\begin{aligned} B_3 &= \frac{q\sigma_e^2}{m(m+q)} \left\{ E^{(1)} \left[\chi_m^2 F_{q,m}(\Delta^*) I(F_{q,m}(\Delta^*) \leq F_\alpha) \right] \right. \\ &\quad \left. - E \left[\chi_m^2 I(F_{q,m}(\Delta^*) \leq F_\alpha) \right] \right\}. \end{aligned}$$

(iv) Using (5.5) and (5.35)

$$\begin{aligned} B_4 &= E(\hat{\sigma}_e^{2S} - \sigma_e^2) \\ &= \frac{1}{m+2} E_\tau \left\{ \tau^2 E \left[\left(\frac{mS_e^2}{\tau^2} \right) I \left(\mathcal{L}_n > \frac{m}{m_2} \right) | \tau \right] \right\} \\ &\quad + \frac{1}{m+q+2} E_\tau \left\{ \tau^2 E \left[\left(\frac{mS_e^2}{\tau^2} \right) \left(1 + \frac{q}{m} \mathcal{L}_n \right) I \left(\mathcal{L}_n \leq \frac{m}{m+2} \right) | \tau \right] \right\} - \sigma_e^2 \end{aligned}$$

Thus,

$$\begin{aligned} B_4 &= \sigma_e^2 \left\{ \frac{q}{m(m+q+2)} E^{(1)} \left[\chi_m^2 F_{q,m}(\Delta^*) I \left(F_{q,m}(\Delta^*) \leq \frac{m}{m+2} \right) \right] \right. \\ &\quad \left. - \frac{q}{(m+2)(m+q+2)} E^{(1)} \left[\chi_m^2 F_{q,m}(\Delta^*) I \left(F_{q,m}(\Delta^*) \leq \frac{m}{m+2} \right) \right] - \frac{2}{m+2} \right\}. \end{aligned}$$

Hence the theorem is proved.

5.3.2.2 Risk of the Estimators

In this subsection we present the risks of the four estimators with multivariate t -error terms.

Theorem 5.5 *risk of the unrestricted unbiased estimator (UUE), restricted unbiased estimator (RUE), preliminary test estimator (PTE) and Stein-type estimator (SE) are given by (5.38) – (5.41) respectively:*

$$(i) R_1(\hat{\sigma}_e^2; \sigma_e^2) = \frac{2(m+\nu-2)}{m(\nu-4)} \sigma_e^4, \quad \nu > 4 \quad (5.38)$$

$$(ii) R_2(\hat{\sigma}_e^2; \sigma_e^2) = \frac{[2(m+q)(m+q+\nu-2) + (\nu-4)\Delta^*(\Delta^*+4)]}{(\nu-4)(m+q)^2} \quad (5.39)$$

$$\begin{aligned} (iii) R_3(\hat{\sigma}_e^{2PT}; \sigma_e^2) &= \sigma_e^4 \left\{ \frac{q^2(\nu-2)}{m^2(m+q)^2(\nu-4)} E^{(2)} \left[(\chi_m^2)^2 [F_{q,m}(\Delta^*)]^2 I(F_{q,m}(\Delta^*) \leq F_\alpha) \right] \right. \\ &\quad + \frac{2q(\nu-2)}{m(m+q)^2(\nu-4)} E^{(2)} \left[(\chi_m^2)^2 F_{q,m}(\Delta^*) I(F_{q,m}(\Delta^*) \leq F_\alpha) \right] \\ &\quad - \frac{q(2m+q)(\nu-2)}{m^2(m+q)^2(\nu-4)} E^{(2)} \left[(\chi_m^2)^2 I(F_{q,m}(\Delta^*) \leq F_\alpha) \right] \\ &\quad - \frac{2q}{m(m+q)} E^{(1)} \left[\chi_m^2 F_{q,m}(\Delta^*) I(F_{q,m}(\Delta^*) \leq F_\alpha) \right] \\ &\quad \left. + \frac{2q}{m(m+q)} E^{(1)} \left[\chi_m^2 I(F_{q,m}(\Delta^*) \leq F_\alpha) \right] + \frac{2(m+\nu-2)}{m(\nu-4)} \right\}. \quad (5.40) \end{aligned}$$

(iv) $R_4(\hat{\sigma}_\epsilon^{2S}; \sigma_\epsilon^2)$

$$\begin{aligned}
&= \sigma_\epsilon^4 \left\{ \frac{q^2(\nu-2)}{m^2(m+q+2)^2(\nu-4)} E^{(2)} \left[(\chi_m^2)^2 [F_{q,m}(\Delta^*)]^2 I \left(F_{q,m}(\Delta^*) \leq \frac{m}{m+2} \right) \right] \right. \\
&+ \frac{2q(\nu-2)}{m(m+q+2)^2(\nu-4)} E^{(2)} \left[(\chi_m^2)^2 F_{q,m}(\Delta^*) I \left(F_{q,m}(\Delta^*) \leq \frac{m}{m+2} \right) \right] \\
&- \frac{q(2m+q+4)(\nu-2)}{(m+2)^2(m+q+2)^2(\nu-4)} E^{(2)} \left[(\chi_m^2)^2 I \left(F_{q,m}(\Delta^*) \leq \frac{m}{m+2} \right) \right] \\
&- \frac{2q}{m(m+q+2)} E^{(1)} \left[\chi_m^2 F_{q,m}(\Delta^*) I \left(F_{q,m}(\Delta^*) \leq \frac{m}{m+2} \right) \right] \\
&\left. + \frac{2q}{(m+2)(m+q+2)} E^{(1)} \left[\chi_m^2 I \left(F_{q,m}(\Delta^*) \leq \frac{m}{m+2} \right) \right] + \frac{2(m+\nu-4)}{(m+2)(\nu-4)} \right\}, \tag{5.41}
\end{aligned}$$

where $E^{(j)} \left[(\chi_m^2)^{k_1} [F_{q,m}(\Delta^*)]^{k_2} I \left(F_{q,m}(\Delta^*) \leq \frac{m}{m+2} \right) \right]$ is given by (5.35).

Proof. (i) Using (5.2), (2.18) and (5.37), we obtain

$$\begin{aligned}
R_1(\hat{\sigma}_\epsilon^2; \sigma_\epsilon^2) &= E[(\hat{\sigma}_\epsilon^2 - \sigma_\epsilon^2)^2] = \text{var}(S_\epsilon^2) \\
&= E_\tau \left\{ \frac{\tau^4}{m^2} E \left[\left(\frac{mS_\epsilon^2}{\tau^2} \right)^2 \middle| \tau \right] \right\} \\
&\quad - \left\{ E_\tau \left\{ E \left[\frac{\tau^2}{m} E \left[\left(\frac{mS_\epsilon^2}{\tau^2} \right) \middle| \tau \right] \right] \right\} \right\}^2
\end{aligned}$$

Thus,

$$R_1(\hat{\sigma}_\epsilon^2; \sigma_\epsilon^2) = \frac{2(m+\nu-2)}{m(\nu-4)} \sigma_\epsilon^4, \quad \nu-4 > 0$$

(ii) Using (5.3) and (5.37)

$$R_2(\hat{\sigma}_\epsilon^2; \sigma_\epsilon^2) = E[(\hat{\sigma}_\epsilon^2 - \sigma_\epsilon^2)^2] = E(\hat{\sigma}_\epsilon^4) - 2\sigma_\epsilon^2 E(\hat{\sigma}_\epsilon^2) + \sigma_\epsilon^4$$

after expression and simplifying, we get (5.39).

(iii) Using (5.4), we obtain

$$\begin{aligned}
R_3(\hat{\sigma}_\epsilon^{2PT}; \sigma_\epsilon^2) &= E[(\hat{\sigma}_\epsilon^{2PT} - \sigma_\epsilon^2)^2] = E[(\hat{\sigma}_\epsilon^{2PT})^2] - 2\sigma_\epsilon^2 B_3 - \sigma_\epsilon^4 \\
&= E_\tau \left\{ \frac{\tau^4}{m^2} E \left[\left(\frac{mS_\epsilon^2}{\tau^2} \right)^2 \middle| \tau \right] \right\} \\
&+ \frac{q^2}{m^2(m+q)^2} E_\tau \left\{ \tau^4 E \left[(\chi_m^2)^2 (1 - F_{q,m}(\Delta))^2 I(F_{q,m}(\Delta) \leq F_\alpha) \middle| \tau \right] \right\} \\
&- \frac{2q}{m^2(m+q)} E_\tau \left\{ \tau^4 E \left[(\chi_m^2)^2 (1 - F_{q,m}(\Delta)) I(F_{q,m}(\Delta) \leq F_\alpha) \middle| \tau \right] \right\} \\
&+ \frac{2q}{m(m+q)} E_\tau \left\{ \tau^4 E \left[(\chi_m^2) (1 - F_{q,m}(\Delta)) I(F_{q,m}(\Delta) \leq F_\alpha) \middle| \tau \right] \right\} - \sigma_\epsilon^4
\end{aligned}$$

after simplifying we get (5.40).

(iv) Using (5.5),

$$R_4(\hat{\sigma}_\epsilon^{2S}; \sigma_\epsilon^2) = E[(\hat{\sigma}_\epsilon^{2S} - \sigma_\epsilon^2)^2] = E[(\hat{\sigma}_\epsilon^{2S})^2] - 2\sigma_\epsilon^2 B_4 \sigma_\epsilon^4 - \sigma_\epsilon^4$$

$$\begin{aligned} R_4(\hat{\sigma}_\epsilon^{2S}; \sigma_\epsilon^2) &= \frac{1}{(m+2)^2} E_\tau \left\{ \tau^4 E \left[\left(\frac{mS_\epsilon^2}{\tau^2} \right)^2 \middle| \tau \right] \right\} \\ &+ \frac{q^2}{m^2(m+q+2)^2} E_\tau \left\{ \tau^4 E \left[\left(\frac{m}{m+2} - \mathcal{L}_n \right)^2 \left(\frac{mS_\epsilon^2}{\tau^2} \right)^2 I \left(\mathcal{L}_n \leq \frac{m}{m+2} \right) \middle| \tau \right] \right\} \\ &- \frac{2q}{m(m+2)(m+q+2)} E_\tau \left\{ \tau^4 E \left[\left(\frac{mS_\epsilon^2}{\tau^2} \right)^2 \left(\frac{m}{m+2} - \mathcal{L}_n \right) I \left(\mathcal{L}_n \leq \frac{m}{m+2} \right) \middle| \tau \right] \right\} \\ &+ \frac{2q}{m(m+q+2)} E_\tau \left\{ \tau^4 E \left[\left(\frac{mS_\epsilon^2}{\tau^2} \right) \left(\frac{m}{m+2} - \mathcal{L}_n \right) I \left(\mathcal{L}_n \leq \frac{m}{m+2} \right) \middle| \tau \right] \right\} - \sigma_\epsilon^4 \end{aligned}$$

after simplifying (5.41) follows.

5.3.3 Analysis the Risk of the Estimators

In this subsection, we study the relative performance of the estimators of σ_ϵ^2 . First we note that the risk of UUE of σ_ϵ^2 is $\frac{2(m+\nu-2)}{m(\nu-4)}\sigma_\epsilon^4$ which is constant but that of RUE depends on Δ^* . Under $H_0 : \mathbf{H}\beta = \mathbf{h}$, the risk of RUE is given by $\frac{2(m+q+\nu-2)}{(m+q)(\nu-4)}\sigma_\epsilon^4$ which is less than $\frac{2(m+\nu-2)}{m(\nu-4)}\sigma_\epsilon^4$. Thus, RUE dominates UUE under H_0 . However, the range of values of Δ^* for which RUE dominates UUE is given by the positive solution of the equation

$$m(\nu-4)\Delta^{*2} + 4m(\nu-4)\Delta^* - 2q(m+q)(\nu-2) = 0.$$

Let $\Delta_0^* = -2 + 2\sqrt{1 + \frac{q(m+q)(\nu-2)}{2m(\nu-4)}}$ represents the positive solution of the equation, then in the interval $[0, \Delta_0^*]$ RUE dominates UUE and outside $[0, \Delta_0^*]$ the risk of UUE dominates RUE. We note further, that as $\Delta \rightarrow \infty$ the risk of RUE is unbounded while the risk of UUE is fixed as noted earlier.

Now, we compare the Stein-type estimator (SE) of σ_e^2 against UUE. We present the following domination theorem on it using Stein technique.

Theorem 5.6 *If $\mathbf{e} \sim Mt(\nu : 0, \sigma_e^2 \mathbf{I}_n)$, then $\hat{\sigma}_e^{2S} = m\phi_S(\mathcal{L}_n)S_e^2$ dominates $\frac{mS_e^2}{m+2}$ with respect to the loss*

$$L(\hat{\sigma}_e^{2S}; \sigma_e^2) \frac{1}{\sigma_e^4} (\hat{\sigma}_e^{2S} - \sigma_e^2)^2. \quad (5.42)$$

Proof. Consider the risk of the estimator $\hat{\sigma}_e^{2S}$ w.r.t. the loss of (5.42),

$$\begin{aligned} & \frac{1}{\sigma_e^4} E \left[(m\phi(\mathcal{L}_n)S_e^2 - \sigma_e^2)^2 \right] \\ &= E_{\mathcal{L}_n} \left\{ \phi^2(\mathcal{L}_n) E \left[\left(\frac{\tau^2}{\sigma_e^2} \right)^2 \left(\frac{mS_e^2}{\tau^2} \right)^2 \middle| \mathcal{L}_n \right] \right. \\ & \quad \left. - 2\phi(\mathcal{L}_n) E \left[\left(\frac{\tau^2}{\sigma_e^2} \right) \left(\frac{mS_e^2}{\tau^2} \right) \middle| \mathcal{L}_n \right] + 1 \right\} \end{aligned} \quad (5.43)$$

where $\phi(\mathcal{L}_n)$ is a real-valued function. Consider the term inside the curly brackets of (5.43). For fixed Δ^* and for each $\mathcal{L}_n$, this is a quadratic in $\phi(\cdot)$ with minimum at

$$\phi^*(\mathcal{L}_n) = \frac{E \left[\left(\frac{\tau^2}{\sigma_e^2} \right) \left(\frac{mS_e^2}{\tau^2} \right) \middle| \mathcal{L}_n \right]}{E \left[\left(\frac{\tau^2}{\sigma_e^2} \right)^2 \left(\frac{mS_e^2}{\tau^2} \right)^2 \middle| \mathcal{L}_n \right]} \quad (5.44)$$

which is a function of $\mathcal{L}_n$ and Δ^* (see Figure 2).

Fig. 2 For fixed $\mathcal{L}_n$, the quadratic

$$\phi^2(\mathcal{L}_n) E \left[\left(\frac{\tau^2}{\sigma_e^2} \right)^2 \left(\frac{mS_e^2}{\tau^2} \right)^2 \middle| \mathcal{L}_n \right] - 2\phi(\mathcal{L}_n) E \left[\left(\frac{\tau^2}{\sigma_e^2} \right) \left(\frac{mS_e^2}{\tau^2} \right) \middle| \mathcal{L}_n \right] + 1,$$

the value of ϕ^* is the minimum and ϕ_S yields Stein's estimator.

The maximum of $\phi^*(\mathcal{L}_n)$ over Δ^* is attained at $\Delta^* = 0$ (see Stein, 1964 and Maatta and Casella, 1990). Using

$$\begin{aligned}
& E\left\{(\tau^2)^j(\chi_m^2)^{k_1} | F_{q,m}(\Delta) = \mathcal{L}_n\right\} \\
&= \frac{1}{g_1(\mathcal{L}_n)} \sum_{r=0}^{\infty} 2^{k_1} (q/m)^{\frac{q}{2}+r} \left(\frac{\nu-2}{2}\right)^j (\sigma_e^2)^j \frac{\Gamma(\frac{\nu}{2}+r-j)\Gamma(\frac{m+q}{2}+k_1+r)}{\Gamma(r+1)\Gamma(m/2)\Gamma(\nu/2)\Gamma(q/2+r)} \\
&\times \frac{\left(\frac{\Delta^*}{\nu-2}\right)^r}{(1+\Delta^*/(\nu-2))^{\nu/2+r-j}} \frac{\mathcal{L}_n^{\frac{q}{2}+r-1}}{(1+\frac{q}{m}\mathcal{L}_n)^{\frac{m+q}{2}+k_1+r}},
\end{aligned} \tag{5.45}$$

where

$$\begin{aligned}
g_1(\mathcal{L}_n) &= \sum_{r=0}^{\infty} (q/m)^{q/2+r} \\
&\times \frac{\Gamma(\frac{\nu}{2}+r)\Gamma(\frac{m+q}{2}+r)\left(\frac{\Delta^*}{\nu-2}\right)^r \mathcal{L}_n^{q/2+r-1}}{\Gamma(r+1)\Gamma(\nu/2)\Gamma(q/2+r)\Gamma(m/2)(1+\Delta^*/(\nu-2))^{\nu/2+r}(1+\frac{q}{m}\mathcal{L}_n)^{\frac{m+q}{2}+r}},
\end{aligned} \tag{5.46}$$

and

$$\begin{aligned}
& E\left[(\tau^2)^j(\chi_m^2)^{k_1} | F_{q,m}(0) = \mathcal{L}_n\right] \\
&= 2^{k_1} \left(\frac{\nu-2}{2}\right)^j (\sigma_e^2)^j \frac{\Gamma(\frac{\nu}{2}-j)}{\Gamma(\frac{\nu}{2})} \frac{\Gamma(\frac{m+q}{2}+k_1)}{\Gamma(\frac{m+q}{2})} \frac{1}{(1+\frac{q}{m}\mathcal{L}_n)^{k_1}}.
\end{aligned} \tag{5.47}$$

Then we obtain

$$\begin{aligned}
\phi_0(\mathcal{L}_n) &= \max_{\Delta^*} \phi^*(\mathcal{L}_n) \\
&= \frac{E\left[\left(\frac{\tau^2}{\sigma_e^2}\right)\left(\frac{mS_e^2}{\tau^2}\right) | \mathcal{L}_n\right]}{E\left[\left(\frac{\tau^2}{\sigma_e^2}\right)^2 \left(\frac{mS_e^2}{\tau^2}\right)^2 | \mathcal{L}_n\right]}, \quad (\text{with } \Delta^* = 0) \\
&= \frac{1+\frac{q}{m}\mathcal{L}_n}{m+q+2} \left(\frac{\nu-4}{\nu-2}\right).
\end{aligned}$$

If $\mathcal{L}_n < \frac{m}{m+2}$, then $\frac{1+\frac{2}{m}\mathcal{L}_n}{m+q+2} < \frac{1}{m+2}$ which also implies that

$$\phi^*(\mathcal{L}_n) \leq \phi_0(\mathcal{L}_n) \leq \frac{1}{m+2}, \quad (5.48)$$

for all Δ^* , that is, $\phi_0(\mathcal{L}_n)$ is closer to the minimizing value than $\frac{1}{m+2}$. So it is obvious that for each Δ^* , $\mathcal{L}_n$,

$$\begin{aligned} & \frac{1}{\sigma_\epsilon^4} E \left\{ \left[\phi_S(\mathcal{L}_n) m S_\epsilon^2 - \sigma_\epsilon^2 \right]^2 | \mathcal{L}_n \right\} \\ & \leq \frac{1}{\sigma_\epsilon^4} E \left\{ \left[\frac{m S_\epsilon^2}{m+2} - \sigma_\epsilon^2 \right]^2 | \mathcal{L}_n \right\}. \end{aligned} \quad (5.49)$$

So $m\phi_S(\mathcal{L}_n)S_\epsilon^2$ is better than $\frac{mS_\epsilon^2}{m+2}$ and the theorem is proved.

Now, consider the PTE. We note that as $\alpha \rightarrow 1$ PTE = UUE while for $\alpha \rightarrow 0$ and $\Delta^* = 0$, PTE = RUE. The graph of the risk of PTE depends on α . Giles (1991) have shown that *minimum risk* of PTE is obtained for $F_{\alpha^*} = 1$ i.e., $\alpha^* = 1 - G_{q,m}^{-1}(1, 0)$. Therefore, we have the following corollary.

Corollary 5.2 Risk of Stein-type estimator for σ_ϵ^2 dominantes the risk of PTE for $\alpha^* = 1 - G_{q,m}^{-1}(1, 0)$.

Proof. Follows from (5.47), (5.48) and (5.20). Further, under H_0 , when $\mathcal{L}_n \rightarrow 1$, the risk of the SE is given by

$$\frac{2(m+q+\nu-4)}{(m+q+2)(\nu-4)} \sigma_\epsilon^4 < \frac{2(m+q+\nu-2)}{(m+q)(\nu-4)} \sigma_2^4 = \text{the risk of RUE}$$

and when $\mathcal{L}_n \rightarrow 0$, the risk of SE leans towards

$$\frac{2(m+\nu-4)}{(m+2)(\nu-4)} \sigma_\epsilon^4 = R\left(\frac{mS_\epsilon^2}{m+2}; \sigma_\epsilon^2\right) < R_1(\tilde{\sigma}_\epsilon^2; \sigma_\epsilon^2) = \frac{2(m+\nu-2)}{m(\nu-4)} \sigma_\epsilon^4$$

This analysis concludes that we should always use the Stein-type estimator when the subspace restriction $\mathbf{H}\beta = \mathbf{h}$ is uncertain.

Some graphical display of relative risks of the various estimators follow.

FIGURE 5. Relative risk functions for the unrestricted, restricted, and pre-test estimators of the error variance when $e \sim N(0, \tau^2 I_n)$: $n = 30$, $p = 5$, $q_k = 3$.

FIGURE 6. Relative risk functions for the unrestricted, restricted, and pre-test estimators of the error variance when $e \sim Mt(0, \nu\sigma^2/(\nu-2)I_n)$: $n = 30$, $p = 5$, $q_k = 3$, $\nu = 5$.

5.4 Conclusion

In this chapter, we studied four estimators of τ^2 when the errors are normal and four estimators of σ_e^2 when the error is a multivariate t -distribution when the null-hypothesis $H_0 : \mathbf{H}\beta = \mathbf{h}$ is suspicious. First we note that the risk of UUE of $\tau^2(\sigma_e^2)$ is constant but that of RUE depend on $\Delta(\Delta^*)$. The risk of RUE dominates UUE under H_0 . We note further, that as $\Delta \rightarrow \infty$ ($\Delta^* \rightarrow \infty$) the risk of RUE is unbounded while the risk of UUE are fixed as noted earlier. Secondly, we note that as $\alpha \rightarrow 1$, PTE = UUE while for $\alpha \rightarrow 0$ and $\Delta^* = 0$, PTE = RUE. The graph of the risk of PTE depends on α and it is minimum at $F_{\alpha^*} = 1$ i.e. $\alpha^* = 1 - G_{q,m}^{-1}(1, 0)$ (see Giles 1991). finally the Stein-type estimator dominates the preliminary test estimator uniformly. The performance property of the four estimators is robust in the class of t -distribution which is determined by the degrees of freedom ν ($\nu > 4$).

Chapter 6

Future Directions

The study of various estimators of β and σ_e^2 in the last five chapters raises many questions. We present some of them to be followed in due course.

(1) The positive-rule shrinkage estimation of β is not admissible. Can we find some admissible estimators of β , when $H\beta = h$ may or may not hold?

(2) In the study of ridge-regression problem we gave sufficient conditions for existence of the constant k . The question naturally arises how to estimate k ? What is its effect on the final estimators of β ?

(3) We proposed a Stein-type estimator of σ_e^2 . This estimator is not admissible. Can we proceed to Brewster-Zidek methodology to obtain an admissible estimator which is generalized Bayes estimator?

(4) Open problem is the study of recentered confidence sets using the improvement estimators.

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