1

OF/DE

1

MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS 1963-A

32153

National Library
of CanadaBibliothèque nationale
du CanadaCANADIAN THESES
ON MICROFICHETHÈSES CANADIENNES
SUR MICROFICHENAME OF AUTHOR/NOM DE L'AUTEUR ARTHUR HING-CHIU CHANTITLE OF THESIS/TITRE DE LA THÈSE ON THE INCREMENTS OF MULTI-PARAMETER
GAUSSIAN PROCESSESUNIVERSITY/UNIVERSITÉ CARLETON UNIVERSITYDEGREE FOR WHICH THESIS WAS PRESENTED/
GRADE POUR LEQUEL CETTE THÈSE FUT PRÉSENTÉE Ph.D.YEAR THIS DEGREE CONFERRED/ANNÉE D'OBTENTION DE CE DEGRÉ 1977NAME OF SUPERVISOR/NOM DU DIRECTEUR DE THÈSE MIKLÓS CSÖRGŐ

Permission is hereby granted to the NATIONAL LIBRARY OF
CANADA to microfilm this thesis and to lend or sell copies
of the film.

L'autorisation est, par la présente, accordée à la BIBLIOTHÈ-
QUE NATIONALE DU CANADA de microfilmer cette thèse et
de prêter ou de vendre des exemplaires du film.

The author reserves other publication rights, and neither the
thesis nor extensive extracts from it may be printed or other-
wise reproduced without the author's written permission.

L'auteur se réserve les autres droits de publication; ni la
thèse ni de longs extraits de celle-ci ne doivent être imprimés
ou autrement reproduits sans l'autorisation écrite de l'auteur.

DATED/DATE 13/5/1977 SIGNED/SIGNÉ A. H. ChanPERMANENT ADDRESS/RÉSIDENCE FIXE 26-3415 UPLANDS DRIVE
OTTAWA, ONTARIO
CANADA

National Library of Canada

Cataloguing Branch
Canadian Theses Division

Ottawa, Canada
K1A 0N4

Bibliothèque nationale du Canada

Direction du catalogage
Division des thèses canadiennes

NOTICE

The quality of this microfiche is heavily dependent upon the quality of the original thesis submitted for microfilming. Every effort has been made to ensure the highest quality of reproduction possible.

If pages are missing, contact the university which granted the degree.

Some pages may have indistinct print especially if the original pages were typed with a poor typewriter ribbon or if the university sent us a poor photocopy.

Previously copyrighted materials (journal articles, published tests, etc.) are not filmed.

Reproduction in full or in part of this film is governed by the Canadian Copyright Act, R.S.C. 1970, c. C-30. Please read the authorization forms which accompany this thesis.

**THIS DISSERTATION
HAS BEEN MICROFILMED
EXACTLY AS RECEIVED**

AVIS

La qualité de cette microfiche dépend grandement de la qualité de la thèse soumise au microfilmage. Nous avons tout fait pour assurer une qualité supérieure de reproduction.

S'il manque des pages, veuillez communiquer avec l'université qui a conféré le grade.

La qualité d'impression de certaines pages peut laisser à désirer, surtout si les pages originales ont été dactylographiées à l'aide d'un ruban usé ou si l'université nous a fait parvenir une photocopie de mauvaise qualité.

Les documents qui font déjà l'objet d'un droit d'auteur (articles de revue, examens publiés, etc.) ne sont pas microfilmés.

La reproduction, même partielle, de ce microfilm est soumise à la Loi canadienne sur le droit d'auteur, SRC 1970, c. C-30. Veuillez prendre connaissance des formules d'autorisation qui accompagnent cette thèse.

**LA THÈSE A ÉTÉ
MICROFILMÉE TELLE QUE
NOUS L'AVONS REÇUE**

ON THE INCREMENTS OF MULTI-PARAMETER
GAUSSIAN PROCESSES

by

Arthur Hing-Chiu Chan, B.Sc., M.Sc.

A thesis submitted to the Faculty of
Graduate Studies and Research in partial fulfillment
of the requirements for the degree of

Doctor of Philosophy

Department of Mathematics

Carleton University

Ottawa, Canada

May 1977

© Arthur Hing-Chiu Chan 1977

The undersigned recommend to the Faculty of Graduate
Studies and Research acceptance of the thesis

"On the Increments of Multi-parameter Gaussian Processes"

submitted by Arthur Hing-Chiu Chan, B.Sc., M.Sc., in partial
fulfillment of the requirements for the degree of Doctor of
Philosophy.

.....
Dr. M. Csörgő, Thesis Supervisor

.....
Dr. D. A. Dawson, Chairman, Department of Mathematics

.....
Dr. P. Révész, External Examiner

Carleton University

May 1977

ABSTRACT

The law of iterated logarithm and the Erdős-Rényi law of large numbers for the standard Wiener process can be combined into one statement. Namely

$$\lim_{T \rightarrow \infty} \sup_{0 \leq t \leq T - a_T} \frac{|W(t + a_T) - W(t)|}{\sqrt{2a_T [\log \frac{T}{a_T} + \log \log T]}} = 1 \text{ w.p. 1.}$$

where W is the standard Wiener process and $a_T = T$ or $a_T = \log T$. The former condition $a_T = T$ gives the statement of the law of iterated logarithm while the latter $a_T = \log T$ the Erdős-Rényi law of large numbers. Recently, Csörgő and Révész proved that if $0 < a_T \leq T$ and a_T satisfies certain mild conditions, then the above limit still holds. Their conditions are also satisfied by $a_T = \log T$ and $a_T = T$. Hence, Csörgő and Révész "fill" the gap between the law of iterated logarithm and the Erdős-Rényi law of large numbers.

The analogous results for the multi-parameter Wiener process are unknown, except that the law of iterated logarithm for the multi-parameter Wiener process was proved by R. Pyke in 1972. In this dissertation, we first prove the Erdős-Rényi law of large numbers for the multi-parameter Wiener process. This result also gives a more general form of the Erdős-Rényi law of large numbers even in the standard Wiener case. Then, like Csörgő and Révész for the standard Wiener process, we can combine these two theorems into one

statement, that is we prove a multi-parameter version of the result of Csörgő and Révész. We use the same technique to prove a multi-parameter version of the P. Lévy modulus of continuity and then incorporate it into our Csörgő-Révész type theorem for the multi-parameter Wiener process. Finally, we prove some analogous results for the Kiefer process.

Acknowledgement

I would like to thank Professor Miklós Csörgő, my thesis supervisor, who enriched my knowledge in Mathematics and shared his ideas unselfishly with me, for his guidance and constant encouragement during all these years I worked under his supervision.

The financial support from the National Research Council of Canada is gratefully appreciated.

Mrs. G. Murray, who typed this thesis, deserves a great deal of credit for her fast and accurate work.

Lastly, but not least, I wish to thank my wife, Katherine, whose love and understanding made it possible for me to finish my study and be happy.

TABLE OF CONTENTS

I.	INTRODUCTION AND PRELIMINARIES	1
II.	THE INCREMENTS OF A STANDARD WIENER PROCESS	7
	1. Definitions and some basic properties.	7
	2. The P. Lévy modulus of continuity.	11
	3. The law of iterated logarithm.	11
	4. The Erdős-Rényi law of large numbers.	12
	5. The Csörgő-Révész theorem on increments for a Wiener process.	16
III.	THE INCREMENTS OF A MULTI-PARAMETER WIENER PROCESS	20
	1. Definitions and some basic properties.	20
	2. The P. Lévy modulus of continuity for $W^d(\cdot)$.	27
	3. The L.I.L. and the Erdős-Rényi L.L.N. for $W^d(\cdot)$.	28
	4. The Csörgő-Révész theorem on increments for $W^d(\cdot)$.	34
	5. Some further discussions.	49
IV.	ON THE INCREMENTS OF A KIEFER PROCESS	57
	1. The increments of the Wiener part of a Kiefer process.	59
	2. The Δ increments of a Kiefer process.	66
	3. Further discussions on the increments of a Kiefer process.	70
V.	SUMMARY	75
	REFERENCES	77

I. Introduction and preliminaries.

Let $U(\underline{x})$, $\underline{x} \in [0,1]^d$, be the uniform distribution on the d -dimensional, $d \geq 1$, unit cube. Let, $\underline{X}_1, \underline{X}_2, \dots, \underline{X}_n, \dots$ be a sequence of independent random vectors with common distribution U . It is shown, in [7], that the empirical process

$$\alpha_n(\underline{x}) = \sqrt{n}(U_n(\underline{x}) - U(\underline{x})), \quad \underline{x} \in [0,1]^d,$$

where $U_n(\underline{x}) = \frac{1}{n} \# \{ \underline{X}_i : 1 \leq i \leq n, \underline{X}_i \leq \underline{x} \}$, can be approximated by a sequence of d -parameter Brownian bridges and a $(d+1)$ -parameter Kiefer process (in the above, $\underline{X}_i \leq \underline{x}$ means $\underline{X}_i$ less than or equal to $\underline{x}$ in each coordinate and $\# \{-\}$ stands for the number of elements in the set $\{-\}$). These processes can be defined via a multi-parameter Wiener process.

In establishing the best rate of approximation in the case of $d = 1$, [17], certain increment properties of the standard Wiener process, namely the law of iterated logarithm and the Erdős-Rényi law of large numbers played an important role. These two increment properties for the standard Wiener process can be combined into one theorem as follows: Let $W(\cdot)$ be the standard Wiener process. Then

$$\lim_{T \rightarrow \infty} \sup_{0 \leq t \leq T - a_T} \sup_{0 \leq s \leq a_T} \frac{|W(t+s) - W(t)|}{\sqrt{2a_T [\log \frac{T}{a_T} + \log \log T]}} = 1 \text{ w.p.1.}$$

where $a_T = \log T$ or $a_T = T$. If $a_T = \log T$, then it is the

Erdős-Rényi law of large numbers and if $a_T = T$, then it is the law of iterated logarithm. The immediate question that whether or not the above limit will hold for $\log T \leq a_T \leq T$ is answered positively by Csörgő and Révész [9] under certain mild conditions on a_T , filling the gap between the law of iterated logarithm and the law of large numbers of Erdős and Rényi [13].

For the case of $d > 1$, not much is known about the increment properties of the d -parameter Wiener process. The law of iterated logarithm for the d -parameter Wiener process has been proved by R. Pyke [28] (cf. Wichura [32]). In this thesis, we first extend the Erdős-Rényi law of large numbers for the d -parameter Wiener process. Then we show that these two laws can be combined into one statement as in the one dimensional case and prove a d -parameter version of the Csörgő-Révész theorem. These results are then used to study certain increment properties of a $(d+1)$ -parameter Kiefer process.

In this first chapter we give some basic definitions and results which are needed in the sequel. The second chapter deals with the increment properties of the standard Wiener process. In Chapter 3 we extend the increment properties for the standard Wiener process to the d -parameter case. Finally, in Chapter 4, we study certain increment properties of the $(d+1)$ -parameter Kiefer process.

Preliminaries. Basics of probability theory and the theory of stochastic processes (for references see [1], [5], [11], and [29]) are assumed.

Here, in this section, we introduce some notations and give some basic results that are needed in the following Chapters.

Let $(\Omega, \mathcal{A}, P)$ always denote a probability space. A stochastic process $X(\underline{t}, \omega)$, $\underline{t} \in T \subseteq \mathbb{R}^d$ and $\omega \in \Omega$, is a function

$$X(\cdot, \cdot) : T \times \Omega \rightarrow \mathbb{R}$$

such that, for each fixed $\underline{t} \in T$, $X(\underline{t}, \cdot)$ is measurable with respect to the σ -algebra $\mathcal{A}$, i.e. a random variable. We sometimes call X a stochastic process that lives on T and is defined on the underlying probability space $(\Omega, \mathcal{A}, P)$. The family of real-valued functions $X(\cdot, \omega)$, $\omega \in \Omega$, is called the sample paths family of the stochastic process X .

Definition 1.1. A stochastic process $X(\underline{t}, \omega)$, $\underline{t} \in T \subseteq \mathbb{R}^d$ and $\omega \in \Omega$, is said to have continuous sample paths with probability one (w.p.1.) if there is a null set N , i.e. $P(N) = 0$, such that $X(\cdot, \omega)$ is continuous for every $\omega \notin N$.

Hereafter, we shall suppress the ω in the notation and write $X(\underline{t})$ for $X(\underline{t}, \omega)$. Also, we write $\{X(\underline{t}) < \lambda\}$ for $\{\omega : X(\underline{t}, \omega) < \lambda\}$ etc..

The probability $P\{X(\underline{t}) < \lambda\}$, $\underline{t} \in T$ and $-\infty < \lambda < \infty$, is called the distribution of $X(\underline{t})$.

Definition 1.2. A stochastic process $X(\underline{t})$, $\underline{t} \in T$, is called a Gaussian process if $x(\underline{t}_1), \dots, x(\underline{t}_n)$ has a joint Gaussian distribution for every $\underline{t}_1, \dots, \underline{t}_n$ in T and for every finite $n \geq 1$.

Definition 1.3. A stochastic process $X(\underline{t})$, $\underline{t} \in T$, is called separable if there exists a countable dense set $D \subseteq T$ ($T \subseteq \mathbb{R}^d$ has the relative topology of $\mathbb{R}^d$) such that X is uniquely determined by $X(\underline{t})$, $\underline{t} \in D$. In the latter case D is called a separant.

A vigorous treatment of separability can be found in [11]. The following result on separability is going to be frequently used in this thesis.

Proposition 1.4. Suppose that $X(\underline{t})$, $\underline{t} \in T$, is a separable process with separant D . Then

$$\sup_{\underline{t} \in T} X(\underline{t}) \stackrel{D}{=} \sup_{\underline{t} \in D} X(\underline{t}),$$

i.e.
$$P\left\{ \sup_{\underline{t} \in T} X(\underline{t}) < \lambda \right\} = P\left\{ \sup_{\underline{t} \in D} X(\underline{t}) < \lambda \right\}$$

for every $\lambda \in (-\infty, \infty)$.

The separant of a separable process is not unique; in fact we have:

Proposition 1.6. If $X(\underline{t})$, $\underline{t} \in T$, is a separable process which has continuous sample paths w.p.1., then any countable dense set $D \subseteq T$ is a separant.

For a proof, see Proposition III.4.2 on p.88 of [23].

Definition 1.7. Let X and Y be two separable processes that live on the same $T \subseteq \mathbb{R}^d$ and are defined on the same probability space $(\Omega, \mathcal{A}, P)$. We write

$$X \stackrel{\Delta}{=} Y \quad \text{or} \quad X(\underline{t}) \stackrel{\Delta}{=} Y(\underline{t}), \quad \underline{t} \in T, \text{ jointly,}$$

if the finite joint distributions agree, that is

$$P\{X(\underline{t}_1) \leq a_1, \dots, X(\underline{t}_n) \leq a_n\} = P\{Y(\underline{t}_1) \leq a_1, \dots, Y(\underline{t}_n) \leq a_n\}$$

for every finite sequence $\underline{t}_1, \dots, \underline{t}_n \in T$ and $a_1, \dots, a_n \in \mathbb{R}$.

If X is a separable Gaussian process which has continuous sample paths w.p.1. (in fact, a Gaussian process which has continuous sample paths w.p.1. must be separable), then the finite joint distributions are determined by the covariance structure of X . In turn, it determines the distribution of X on a countable dense set and thus the process itself. Hence,

Proposition 1.7. A separable Gaussian process is uniquely determined by its covariance structure.

Notations. We always use $\phi(\cdot)$ for the standard normal distribution, i.e.

$$\phi(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\lambda} e^{-x^2/2} dx$$

for every $-\infty < \lambda < \infty$.

An elementary inequality that will be used in the proofs in the subsequent chapters is the following:

$$\frac{1}{\lambda + \frac{1}{\lambda}} e^{-\lambda^2/2} \leq \int_{\lambda}^{\infty} e^{-x^2/2} dx \leq \frac{1}{\lambda} e^{-\lambda^2/2}$$

for every $\lambda > 0$.

II. The increments of a standard Wiener process.

In this chapter we examine the increments properties of a standard Wiener process (W.P.). After giving the definitions and some basic properties in §2.1, we discuss the P. Lévy modulus of continuity of a W.P.. In §2.3 and §2.4 we study the law of iterated logarithm (L.I.L.) and the Erdős-Rényi law of large numbers (E-R.L.L.N.) of a W.P.. Finally, in §2.5 we show that the L.I.L. and the E-R.L.L.N. can be combined into one theorem on increments of a W.P. with different conditions on the increments. These conditions are two extremes of a family of increment sequences and thus leave a considerable gap between them. Hence we bring in the Theorem of Csörgő and Révész that "fills" this gap.

2.1. Definitions and some basic properties.

Definition 2.1. A Wiener process $W(t)$, $t \geq 0$ is a separable Gaussian process such that

$$(2.1.1) \quad W(\cdot) \text{ is continuous w.p.1. ;}$$

$$(2.1.2) \quad W(0) = 0 \text{ w.p.1. and, if } t > 0 ,$$

$$P\{W(t) < \lambda\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\lambda/\sqrt{t}} \exp\left\{-\frac{x^2}{2}\right\} dx = \Phi\left(\frac{\lambda}{\sqrt{t}}\right), \quad \lambda \in (-\infty, \infty);$$

$$(2.1.3) \quad E(W(s)W(t)) = s \wedge t = \min\{s, t\}, \text{ for all } s, t \geq 0;$$

(2.1.4) if $0 = t_0 < t_1 < t_2 < \dots < t_n < \infty$, then the Gaussian random variables $W(t_i) - W(t_{i-1})$ are mutually independent with mean zero and variance $t_i - t_{i-1}$.

The existence of the Wiener process is proved by N. Wiener (see [32] or [1]). From the proposition 1.7. and the conditions (2.1.3) and (2.1.1), the other conditions are redundant. Also, we have

Proposition 2.2. Let $W(t)$, $t \geq 0$ be a Wiener process. Then, for each $T > 0$, we have

$$(2.2.1) \quad \sqrt{T} W(t/T) \stackrel{D}{=} W(t), \quad 0 \leq t \leq T, \text{ jointly}$$

and hence

$$(2.2.2) \quad \sup_{0 \leq t \leq T} W(t) \stackrel{D}{=} \sqrt{T} \sup_{0 \leq t \leq 1} W(t).$$

Next, we shall define a Brownian bridge. As we have mentioned before a Brownian bridge has the same covariance structure as the empirical process $\alpha_n(x) = \sqrt{n} (F_n(x) - x)$, where $F_n(x)$ is the empirical distribution defined from the first n samples from a uniform distribution; $\alpha_n(x)$ has the covariance structure

$$E(\alpha_n(x) \cdot \alpha_n(y)) = x \wedge y - xy$$

for all $0 \leq x, y \leq 1$. We define

Definition 2.3. A Brownian bridge $B(t)$, $0 \leq t \leq 1$, is a separable Gaussian process such that

$$(2.3.1) \quad B(0) = B(1) = 0 \text{ w.p.1 ; and if } 0 < t < 1$$

$$P\{B(t) \leq \lambda\} = \Phi(\lambda/\sqrt{t(1-t)}) , \quad -\infty < \lambda < \infty ;$$

$$(2.3.2) \quad E(B(s) B(t)) = s \wedge t - st .$$

Thus, we can write

$$(2.3.3) \quad B(t) = W(t) - tW(1) , \quad 0 \leq t \leq 1 ,$$

where $W(\cdot)$ is a Wiener process as in Definition 2.1..

As a result of the special covariance structure of the two Gaussian processes involved, we have

Proposition 2.4. Let $B(\cdot)$ and $W(\cdot)$ be a Brownian bridge and a Wiener process respectively. Then

$$(2.4.1) \quad (1+t) B\left(\frac{t}{1+t}\right) \stackrel{D}{=} W(t) , \quad 0 \leq t < \infty , \text{ or}$$

$$(2.4.2) \quad B(t) \stackrel{D}{=} (1-t) W\left(\frac{t}{1-t}\right) , \quad 0 \leq t < 1 .$$

One of the very first problems in dealing with properties of a stochastic process is to find the distribution of the supremum over an interval. In the following we state, without proofs, the distribution of $\sup_{0 \leq t \leq 1} W(t)$ and $\sup_{0 \leq t \leq 1} B(t)$. Proofs of these results can be found in [10].

Theorem 2.5. For each $\lambda > 0$, we have

$$(2.5.1) \quad P\left\{ \sup_{0 \leq t \leq 1} W(t) \leq \lambda \right\} = 1 - 2\Phi(-\lambda)$$

and hence, by Proposition 2.2, we have

$$(2.5.2) \quad P\left\{ \sup_{0 \leq t \leq T} W(t) \leq \lambda \right\} = 1 - 2\Phi\left(-\frac{\lambda}{\sqrt{T}}\right),$$

for every $T > 0$.

An ingenious proof of a similar result for $B(t)$ is given by J. L. Doob in [10]. We state his result here:

Theorem 2.6. For each $\lambda > 0$, we have

$$(2.6.1) \quad P\left\{ \sup_{0 \leq t \leq 1} B(t) \leq \lambda \right\} = 1 - \exp\{-2\lambda^2\}$$

and

$$(2.6.2) \quad P\left\{ \sup_{0 \leq t \leq 1} |B(t)| \leq \lambda \right\} = 1 - 2 \sum_{n=1}^{\infty} (-1)^{n+1} \exp\{-2n^2\lambda^2\}.$$

For the distribution of $\sup_{0 \leq t \leq 1} |W(t)|$, the form is less simple.

Theorem 2.7. One has

$$P\left\{ \sup_{0 \leq t \leq 1} |W(t)| > \lambda \right\} = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \exp\left\{-\frac{(2n+1)^2 \pi^2}{8\lambda^2}\right\}.$$

For a proof, see [1] p. 80.

2.2. The P. Lévy modulus of continuity.

As we have seen the sample paths of a W.P. are continuous. However they are also highly irregular. The following theorem, proved by A. Dvoretzky, P. Erdős and S. Kakutani [12] manifests this irregularity.

Theorem 2.8. With probability one, the sample paths of the Wiener process are nowhere differentiable.

It is equally interesting to see how far they are from being differentiable. The result that answers this question is the P. Lévy modulus of continuity.

Theorem 2.9. One has, with probability one,

$$(2.9.1) \quad \overline{\lim}_{h \rightarrow 0} \sup_{0 < t < 1-h} \frac{|W(t+h) - W(h)|}{\sqrt{2h \log \frac{1}{h}}} = 1.$$

Remark. It is shown by Orey and Taylor [25] that the $\overline{\lim}$ in (2.9.1) can be, in fact, replaced by $\lim$. The proof of Theorem 2.9 can also be found in many standard texts on stochastic processes.

2.3. The law of iterated logarithm.

The P. Lévy modulus of continuity deals with the small increments properties of the Wiener process. In particular, the increment considered goes to zero. The classical law of iterated

logarithm deals with behaviour at $t = \infty$. First we give the Gaussian version of Khintchine's result [14].

Theorem 2.10. One has, with probability one,

$$(2.10.1) \quad \overline{\lim}_{n \rightarrow \infty} \frac{W(n)}{\sqrt{2n \log \log n}} = 1.$$

Remark. The original result of Khintchine deals with sums of i.i.d.r.v.'s with arbitrary distribution and expectation zero. Clearly, $W(n) = \sum_{k=1}^n W(k) - W(k-1)$ is a sum of i.i.d.r.v.'s.

The following is a stronger version of Theorem 2.10.:

Theorem 2.11. One has, with probability one,

$$(2.11.1) \quad \overline{\lim}_{T \rightarrow \infty} \sup_{0 \leq t \leq T} \frac{W(t)}{\sqrt{2T \log \log T}} = 1.$$

and hence

$$(2.11.2) \quad \overline{\lim}_{h \rightarrow 0} \sup_{0 \leq t \leq 1-h} \frac{W(t)}{\sqrt{2h \log \log \frac{1}{h}}} = 1.$$

We shall give the proof of these two theorems, since Theorem 2.11 is a direct consequence of Theorem 1.6.

2.4. The Erdős-Rényi law of large numbers.

In their paper [13] Erdős and Rényi proved a new law of large numbers, a non-distribution free result, contrary to the classical

distribution free law of large numbers. Their result concerns the problem of maximal moving average of a sequence of i.i.d.r.v.'s. For example, if $\{X_n\}$ is a head-tail sequence, then what is the longest run, say, of a head sequence in n trials. Let X_n be a sequence of independent random variables with common distribution $F(\cdot)$ whose moment generating function exists in the neighbourhood of zero. The Erdős-Rényi law of large numbers asserts that, with probability one,

$$\lim_{N \rightarrow \infty} \max_{0 \leq n \leq N - a_N} \frac{S_{n+a_N} - S_n}{a_N} = \alpha,$$

where $a_N = [c \log N]$, $[x]$ = the greatest integer less or equal to x , and $\alpha = \alpha(c)$ is uniquely determined by the moment generating function of $F(\cdot)$. For example, if F is discrete and $F(1) = F(-1) = \frac{1}{2}$, α is defined by

$$\frac{1}{c} = 1 - h\left(\frac{1+\alpha}{2}\right)$$

with $h(x) = x \log_2 \frac{1}{x} + (1-x) \log_2 \frac{1}{1-x}$, $0 < x < 1$; and if F is continuous, α is defined by

$$e^{-\frac{1}{c}} = \min_t \phi(t) e^{-\alpha t}$$

where $\phi(t) = E(e^{X_1 t})$ and assumed to be finite in the neighbourhood of $t = 0$.

For sums of i.i.d. normal zero-one r.v.s. $S_n = W(n)$, the Erdős-Rényi law of large numbers has the following special form:

Theorem 2.12. One has, with probability one,

$$(2.12.1) \quad \lim_{N \rightarrow \infty} \max_{0 \leq n \leq N - [c \ln N]} \frac{W(n + [c \ln N]) - W(n)}{[c \ln N]} = \sqrt{\frac{2}{c}},$$

for every $c > 0$.

It is quite easy to see that, for every $\epsilon > 0$,

$$\begin{aligned} & P\left\{ \sup_{n \leq t \leq n+1} W(t + [c \ln N]) - W(t) > \left(\sqrt{\frac{2}{c}} + \epsilon\right) [c \ln N] \right\} \\ &= P\left\{ \sup_{0 \leq t \leq 1} W(t + [c \ln N]) - W(t) > \left(\sqrt{\frac{2}{c}} + \epsilon\right) [c \ln N] \right\} \\ &= P\left\{ \sup_{0 \leq t \leq 1} W(t) > \left(\sqrt{\frac{2}{c}} + \epsilon\right) \sqrt{[c \ln N]} \right\} \\ &= 2P\left\{ W(1) > \left(\sqrt{\frac{2}{c}} + \epsilon\right) \sqrt{[c \ln N]} \right\} \\ &\leq \sqrt{\frac{2}{\pi}} \frac{1}{\left(\sqrt{\frac{2}{c}} + \epsilon\right) \sqrt{[c \ln N]}} \exp\left\{ - \left(\sqrt{\frac{2}{c}} + \epsilon\right)^2 [c \ln N] \right\} \\ &\leq A \frac{1}{N^{1+\delta}}, \end{aligned}$$

for each $n \geq 0$ and some appropriate constants $A > 0$ and $\delta > 0$.

The Borel-Cantelli lemma and the above inequalities render that

$$\lim_{N \rightarrow \infty} \sup_{n \leq t \leq n+1} \frac{W(t + [c \ln N]) - W(t)}{[c \ln N]} \leq \sqrt{\frac{2}{c}} \text{ w.p.l. ,}$$

for every $n \geq 0$. Consequently, we have

Corollary 2.13. One has, with probability one,

$$(2.13.1) \quad \lim_{N \rightarrow \infty} \sup_{0 \leq t \leq N - [c \log n]} \frac{W(t + [c \log N]) - W(t)}{[c \log N]} = \sqrt{\frac{2}{c}} .$$

Remark. It is also equally trivial to show that

$$(2.13.2) \quad \lim_{T \rightarrow \infty} \sup_{0 \leq t \leq T - [c \log T]} \frac{W(t + [c \log T]) - W(t)}{[c \log T]} = \sqrt{\frac{2}{c}} \text{ w.p.l.}$$

for every $c > 0$.

In [3], Chan improved Theorem 2.12 in the following way:

Theorem 2.14. Let a_N be a sequence of integers such that $a_N/N^\delta \rightarrow 0$, for every $\delta > 0$. Then

$$(2.14.1) \quad \lim_{N \rightarrow \infty} \max_{0 \leq n \leq N - a_N} \frac{W(n + a_N) - W(n)}{\sqrt{a_N} [c \log N]} = \sqrt{\frac{2}{c}} \text{ w.p.l. ,}$$

for each $c > 0$. If, in addition, we have $a_N \rightarrow \infty$ as $N \rightarrow \infty$, then

$$(2.14.2) \quad \lim_{N \rightarrow \infty} \sup_{0 \leq t \leq N - a_N} \sup_{0 \leq s \leq a_N} \frac{W(t+s) - W(t)}{\sqrt{a_N} [c \log N]} = \sqrt{\frac{2}{c}} \text{ w.p.l.}$$

Proof of the above theorem will not be given here. But we

will give a complete proof of a multiparameter version of this theorem which will also contain this result.

2.5. A theorem of Csörgö and Révész on increments of a Wiener process.

Let us consider the following observation: Suppose that a_T is a real valued function of T . Then one can easily induce from Theorems 2.11 and 2.12 that

$$\lim_{T \rightarrow \infty} \sup_{0 \leq t \leq T - a_T} \sup_{0 \leq s \leq a_T} \frac{W(t+s) - W(t)}{\sqrt{2a_T [\log \frac{T}{a_T} + \log \log T]}} = 1 \text{ w.p.1.}$$

if $a_T = T$ or $a_T = [\log T]$. Obviously, there is a considerable gap between $\log T$ and T . In [9] Csörgö and Révész proved that, under certain mild conditions on a_T , the above equality is always true for all $0 < a_T \leq T$. Hence, they fill the gap between the L.I.L. and the E-R.L.L.N..

In order to state their theorem we first introduce

Definition 2.15. A monotonically non-decreasing positive valued function a_T is called a β_T -function if

$$(2.15.1) \quad 0 < a_T \leq T,$$

$$(2.15.2) \quad \frac{a_{k+1}}{a_k} \leq \theta \text{ for all } \theta > 1 \text{ and for all } k > 0;$$

$$(2.15.3) \quad a_T/T \text{ is monotonically non-increasing.}$$

Remark. We note that (2.15.3) implies (2.15.2). To see this, suppose (2.15.2) does not hold. Then there is a $\theta > 1$ such that

$$\frac{a_{k+1}}{\theta} / \frac{a_k}{\theta} > \theta$$

for infinitely many k 's. That means

$$a_k / \theta < a_{k+1} / \theta$$

or

$$a_k / \theta^k < a_{k+1} / \theta^{k+1}$$

for infinitely many k 's which, in turn, implies that a_T/T is no non-increasing. This, however, contradicts (2.15.3). Nevertheless we shall keep this condition in the definition for convenience later on.

Theorem 2.16. Suppose that a_T , $T \geq 0$, is a β_T -function. Then, one has, with probability one

$$(2.16.1) \quad \lim_{T \rightarrow \infty} \sup_{0 \leq t \leq T-a_T} \beta_T |W(t+a_T) - W(t)| = 1$$

and

$$(2.16.2) \quad \lim_{T \rightarrow \infty} \sup_{0 \leq t \leq T-a_T} \sup_{0 \leq s \leq a_T} \beta_T |W(t+s) - W(t)| = 1,$$

where $\beta_T = (2a_T [\log \frac{T}{a_T} + \log \log T])^{-1/2}$.

If in addition we have

$$(2.16.3) \quad \lim_{T \rightarrow \infty} \frac{\log T/a_T}{\log \log T} = \infty,$$

then $\overline{\lim}$ in (2.16.1) and (2.16.2) can be replaced by $\lim$.

We shall not give the proof of Theorem 2.16 now, for an analogous result for the multi-parameter Wiener process, which also subsumes this result, will be proved. Some immediate corollaries are the following:

Corollary 2.17. One has, with probability one,

$$(2.17.1) \quad \lim_{T \rightarrow \infty} \beta_T |W(T + a_T) - W(t)| = 1,$$

where a_T is a β_T -function and β_T is defined as in Theorem 2.16.

The above is a simple generalization of a theorem of Lai [19] who proved this result under somewhat stronger conditions on a_T .

Corollary 2.18. One has, with probability one,

$$(2.18.1) \quad \lim_{T \rightarrow \infty} \sup_{0 \leq t \leq T-1} \frac{|W(t+1) - W(t)|}{(2 \log T)^{1/2}} = 1.$$

Of course, Theorem 2.14 also implies that

$$\lim_{N \rightarrow \infty} \frac{X_n^{(n)}}{[2 \log n]} = 1 \quad \text{w.p.1.},$$

where $X_n^{(n)}$ is the n -th order statistic of a sequence of i.i.d. normal zero-one random variables, i.e. $X_n^{(n)} = \max_{0 \leq i \leq n} X_i$.

In [17], Komlós-Major-Tusnády proved that if $S_n = \sum_{i=1}^n X_i$ is the partial sum of a sequence of $\{X_n\}$ of i.i.d.r.v.'s with $EX_1 = 0$, $EX_1^2 = 1$ and $Ee^{tx_1} < \infty$ in some neighborhood of zero, then

$$|S_n - W(n)| = O(\log n) \text{ w.p.l.}$$

as $n \rightarrow \infty$. Hence, if a_T is β_T -function such that $\log T/a_T \rightarrow 0$ as $T \rightarrow \infty$, then similar results hold for S_n . More precisely, we have

Corollary 2.19. Let a_T be a β_T -function such that $\log T/a_T \rightarrow 0$ as $T \rightarrow \infty$. Suppose that S_n is defined as above. Then we have

$$(2.19.1) \quad \lim_{N \rightarrow \infty} \max_{0 \leq n \leq N-a_N} \beta_N |S_{n+a_N} - S_n| = 1 \text{ w.p.l.},$$

and

$$(2.19.2) \quad \lim_{N \rightarrow \infty} \beta_N |S_{N+a_N} - S_N| = 1 \text{ w.p.l.},$$

where β_T is defined as in Theorem 2.16.

III. The increments of a multi-parameter Wiener process.

Let us denote by $[0, \infty)^d$ ($d \geq 1$) the d -fold of the positive half real line; i.e. the set of all $\underline{x} = (x_1, \dots, x_d)$ such that $x_i \geq 0$, for all $1 \leq i \leq d$. A generalization of the standard Wiener process of Definition 2.1 to the parameter space $\underline{t} \in [0, \infty)^d$ is intuitively simple, but not at all trivial. Many properties of this kind of a process are still unknown. In this chapter, we extend increments properties of the standard Wiener process to a d -parameter Wiener process. We first give some definitions and discuss a number of basic properties in §3.1. Then we state a multi-parameter version of the P. Lévy modulus of continuity, in §3.2, which was proved by R. Pyke and by S. Orey and W. E. Pruitt [24] independently. In §3.3, we give the extension of the law of iterated logarithm and the Erdős-Rényi law of large numbers for the d -parameter Wiener process. The former is a result of R. Pyke [28] and the latter is a new result. Finally, in §3.4, we extend the Csörgő-Révész theorem on increments to the multi-parameter case which "fills" the gap between the L.I.L. and the E.-R.L.L.N. of the multi-parameter Wiener process.

1

3.1. Definitions and some basic properties.

First some notations which are relevant in the sequel.

Notations. Let $\underline{x} = (x_1, \dots, x_d)$ and $\underline{y} = (y_1, \dots, y_d)$ be in $[0, \infty)^d$.

(N1) We write $\underline{x} \leq \underline{y}$ ($\underline{x} < \underline{y}$) iff $x_i \leq y_i$ ($x_i < y_i$) for all i ;

(N2) $|\underline{x}| = x_1 \dots x_d$;

(N3) $\underline{x} \pm \underline{y} = (x_1 \pm y_1, \dots, x_d \pm y_d)$, $\underline{x} \underline{y} = (x_1 y_1, \dots, x_d y_d)$;

(N4) $\langle \underline{x}, \underline{y} \rangle = \{ \underline{z} : \underline{x} \leq \underline{z} \leq \underline{y} \}$ if $\underline{x} \leq \underline{y}$; and

the interior of $\langle \underline{x}, \underline{y} \rangle$ is defined by

$$\text{int } \langle \underline{x}, \underline{y} \rangle = \{ \underline{z} : \underline{x} < \underline{z} < \underline{y} \};$$

(N5) $|\langle \underline{x}, \underline{y} \rangle| = |\text{int } \langle \underline{x}, \underline{y} \rangle| = \prod_{i=1}^d (y_i - x_i)$;

(N6) $F_{\underline{x}}^{(i)}(\underline{y}) = (y_1, \dots, y_{i-1}, x_i, y_{i+1}, \dots, y_d)$, $1 \leq i \leq d$.

Definition 3.1. A d -parameter Wiener process $W^d(\underline{x})$, $\underline{x} \in [0, \infty)^d$, is a real separable Gaussian process defined on some underlying probability space, say $(\Omega, \mathcal{A}, P)$, such that

(3.1.1) $W^d(\underline{x}) = 0$ w.p.1. if $|\underline{x}| = 0$, and, if $|\underline{x}| > 0$,

$$P\{W^d(\underline{x}) < \lambda\} = \Phi(\lambda |\underline{x}|^{-1/2}), \quad -\infty < \lambda < \infty;$$

(3.1.2) $W^d(\cdot)$ is continuous w.p.1.;

(3.1.3) for every $\underline{x} \leq \underline{y}$, we define the increment of $\langle \underline{x}, \underline{y} \rangle$ by

$$\begin{aligned} W^d(\langle \underline{x}, \underline{y} \rangle) &= W^d(\underline{y}) + \sum_{i=1}^d (-1) W^d(F_{\underline{x}}^{(i)}(\underline{y})) \\ &\quad + \sum_{i < j} (-1)^2 W^d(F_{\underline{x}}^{(i)} \circ F_{\underline{x}}^{(j)}(\underline{y})) \\ &\quad \vdots \end{aligned}$$

$$+ \sum_{i=1}^d (-1)^{d-1} W^d(F_Y^{(i)}(\underline{x})) \\ + (-1)^d W^d(\underline{x}) .$$

Then $W^d(\langle \underline{x}, \underline{y} \rangle)$ and $W^d(\langle \underline{u}, \underline{v} \rangle)$ are independent if $\text{int } \langle \underline{x}, \underline{y} \rangle \cap \text{int } \langle \underline{u}, \underline{v} \rangle = \emptyset$ and

$$E[W^d(\langle \underline{x}, \underline{y} \rangle) W^d(\langle \underline{u}, \underline{v} \rangle)] = |\langle \underline{x}, \underline{y} \rangle \cap \langle \underline{u}, \underline{v} \rangle|$$

if their interiors are not disjoint.

Remark. The existence of this kind of a process was first proved by Chentson [4], in 1956, for the case when $d = 2$. Then J. Yeh [34] showed that the related Wiener measure exists on $C[0,1]^2$. Subsequently, in 1970, W. J. Park [27] proved the existence of $W^d(\underline{x})$ for all $d > 1$.

It is quite obvious that when $d = 1$, $W^1(\underline{x}) = W(x)$ and $W^1(\langle \underline{x}, \underline{y} \rangle) = W(y) - W(x)$.

Now, suppose that $\underline{S} = (s_1, \dots, s_d) > \underline{0} = (0, \dots, 0)$. Then $|\underline{S}|^{-1} W^d(\underline{S}^{-1} \underline{x})$ satisfies all the conditions of Definition 3.1., where $\underline{S}^{-1} = (s_1^{-1}, \dots, s_d^{-1})$, i.e. $|\underline{S}|^{-1} W^d(\underline{S}^{-1} \underline{x}) = W^d(\underline{x})$, $0 \leq \underline{x} \leq \underline{S}$. Hence, we have

Proposition 3.2. For each $\underline{S} > \underline{0}$, we have

$$P\left\{ \sup_{0 \leq \underline{x} \leq \underline{S}} W^d(\underline{x}) < \lambda \right\} = P\left\{ \sup_{0 \leq \underline{x} \leq \underline{1}} W^d(\underline{x}) < \frac{\lambda}{\sqrt{|\underline{S}|}} \right\} ,$$

for every $\lambda \geq 0$.

An analogous result of Theorem 2.5. for $W^d(\cdot)$, that is the exact value for the probability in (3.2.1), is not known. However, there are some bounds for this probability which will be useful in the sequel. The following is due to Orey and Pruitt [24].

Proposition 3.3. Let I be an interval in $[0, \infty)^d$, i.e. $I = \langle \underline{x}, \underline{y} \rangle$ for some $\underline{x} < \underline{y}$ in $[0, \infty)^d$. If $J = \langle \underline{u}, \underline{v} \rangle \subseteq I$ is an interval, then we write $W^d(J) = W^d(\langle \underline{u}, \underline{v} \rangle)$. For each $\lambda > 0$, one has

$$(3.3.1) \quad P\left\{ \sup_{J \subseteq I} W^d(J) > \lambda \right\} \leq 4^d P\{W^d(I) > \lambda\} ;$$

Hence, we have for every $\lambda > 0$

$$(3.3.2) \quad P\left\{ \sup_{J \subseteq I} W^d(J) > \lambda \right\} \leq 4^d \Phi(-\lambda/|I|^{1/2}) ,$$

where $|I| = |\langle \underline{x}, \underline{y} \rangle| = \prod_{i=1}^d (y_i - x_i)$.

For the case when $d = 2$ some sharper bounds are also available. Namely we have

Proposition 3.4. For each $\lambda > 0$, we have

$$(3.4.1) \quad P\left\{ \sup_{0 < \underline{x} < 1} W^2(\underline{x}) > \lambda \right\} \geq 3\Phi(-\lambda) - e^{4\lambda^2} \Phi(-3\lambda)$$

and,

$$(3.4.2) \quad P\left\{ \sup_{0 < \underline{x} < 1} W^2(\underline{x}) > \lambda \right\} \leq \Phi(\epsilon)^{-1} [3\Phi(-\lambda-\epsilon) - e^{4(\lambda-\epsilon)^2} \Phi(-3(\lambda-\epsilon))] ,$$

for every $\epsilon > 0$.

The inequality (3.4.1) is due to Paranjape and Park [26]. and (3.4.2) is due to Chan [2]. We shall not give the proofs, because they will not be of particular interest in this thesis.

Let us now give a Central limit theorem for a matrix array of i.i.d.r.v.'s. Let $\underline{k} = (k_1, \dots, k_d)$ be a d-tuple of integers and $\underline{n} = (n, n, \dots, n)$, $n \geq 1$. Suppose that $\{X_{\underline{k}}\}$ is a sequence of i.i.d.r.v.'s, indexed by the matrix array $\underline{k}$ with common distribution $F(\underline{x}) = x_1 \dots x_d$ for all $\underline{x} \in \langle 0, 1 \rangle^d = [0, 1]^d$. Define $S_{\underline{n}} = \sum_{\underline{k} \leq \underline{n}} X_{\underline{k}}$ and $[\underline{k} \cdot \underline{t}] = ([k_1 \cdot t_1], \dots, [k_d \cdot t_d])$ for all $\underline{k}$ and $\underline{t} \in \langle 0, 1 \rangle$. The following theorem is due to Wichura [32].

Theorem 3.5. Let $X_{\underline{k}}$ and $S_{\underline{n}}$ be defined as above. Then, let $Z_{\underline{n}}(\underline{t}) = |\underline{n}|^{-1} S_{[\underline{n} \cdot \underline{t}]}$, we have

$$(3.5.1) \quad Z_{\underline{n}} \xrightarrow{\mathcal{D}} W^d,$$

where $\xrightarrow{\mathcal{D}}$ means that the sequence $Z_{\underline{n}}$ converges in distribution to W^d (c.f. [1]).

Definition 3.6. A d-parameter Brownian bridge $B^d(\underline{t})$, $\underline{t} \in \langle 0, 1 \rangle$ is a real separable Gaussian process which satisfies the following conditions:

$$(3.6.1) \quad B^d(\underline{t}) = 0 \text{ w.p.1., if } |\underline{t}|(1 - |\underline{t}|) = 0;$$

$$(3.6.2) \quad P(B^d(\underline{t}) < \lambda) = \Phi(\lambda / \sqrt{|\underline{t}|(1 - |\underline{t}|)}) \text{, if } |\underline{t}|(1 - |\underline{t}|) > 0,$$

for every $\lambda \in (-\infty, \infty)$;

$$(3.6.3) \quad E[B^d(\underline{s})B^d(\underline{t})] = (s_1 \wedge t_1) \dots (s_d \wedge t_d) - s_1 \dots s_d t_1 \dots t_d,$$

for every $\underline{s}$ and $\underline{t}$ in $<0, 1>$.

Hence, we can write

$$(3.6.4) \quad B^d(\underline{t}) = W^d(\underline{t}) - |\underline{t}| W^d(1), \quad 0 \leq \underline{t} \leq 1.$$

Let $\alpha_n(\underline{t})$ denote the empirical process of a random sample from the uniform distribution over $[0,1]^d$, that is

$$\alpha_n(\underline{t}) = \sqrt{n} (F_n(\underline{t}) - |\underline{t}|), \quad 0 \leq \underline{t} \leq 1,$$

where $F_n(\underline{t}) = \#\{X_i \leq \underline{t}, 1 \leq i \leq n\}/n$ and $\{X_i\}$ is a sequence of i.i.d. random vectors with common d -dimensional uniform distribution on $[0,1]^d$. When $d = 1$, Brillinger [1] showed that one can construct a sequence of independent Brownian bridges $\{B_n\}$ which are almost surely near to $\alpha_n(\underline{x})$. Then, in 1975, Komlós, Major and Tusnády [18] proved that the best possible rate of such a construction is

$$\sup_{x \in [0,1]} |\alpha_n(x) - B_n(x)| = O(n^{-1/4} \log n) \text{ w.p.1.,}$$

for large n . Note that the underlying probability space of X_i 's is assumed to be "rich enough" so that the construction is possible. Preceding [18], Csörgő and Révész [7] proved a $d > 1$ version of the above result as follows:

Theorem 3.7. Suppose that $\{X_i\}$ is a sequence of i.i.d.r.v.'s with common uniform distribution on $[0,1]^d$ on some underlying probability space $(\Omega, \mathcal{A}, P)$. Suppose that $(\Omega, \mathcal{A}, P)$ is "rich enough". Then one can define a sequence $\{B_n^d\}$ of Brownian bridges on $(\Omega, \mathcal{A}, P)$ such that

$$(3.7.1) \quad \sup_{\underline{x} \in [0,1]^d} |\alpha_n(\underline{x}) - B_n^d(\underline{x})| = O(n^{\frac{-1}{2(d+2)}} (\log n)^{\frac{3}{2}}) \text{ w.p.1.,}$$

for large n .

Kiefer [15], in 1961, proved the following result:

Proposition 3.8. Let α_n be defined as above. Then, for each $\epsilon > 0$, there is a constant $C(\epsilon, d) > 0$, where d is the dimension of the domain of α_n , such that

$$(3.8.1) \quad P\left\{ \sup_{\underline{x} \in [0,1]^d} |\alpha_n(\underline{x})| \geq \lambda \right\} \leq C(\epsilon, d) e^{-(2-\epsilon)\lambda^2},$$

for every $\lambda > 0$.

Hence, combining Theorem 3.7 and Proposition 3.8, we get

Proposition 3.9. For every $\epsilon > 0$ and $d > 1$, there exists a constant $C(\epsilon, d) > 0$, such that

$$(3.9.1) \quad P\left\{ \sup_{\underline{x} \in [0,1]^d} |B_n^d(\underline{x})| > \lambda \right\} \leq C(\epsilon, d) e^{-(2-\epsilon)\lambda^2},$$

for every $\lambda > 0$.

Again, the exact value of the probability in (3.9.1) is not known. Moreover, the problem is more complex than in the one parameter case. For the case of $d=1$, one can use Proposition 2.4 to derive Theorem 2.6. However, there is no such transformation even for $d=2$. The problems of finding the exact value of (3.9.1) and that of (3.3.1) do seem to be independent.

3.2. The P. Lévy modulus of continuity for $W^d(\underline{x})$.

An analogous result of Theorem 2.9 for the process $W^d(\cdot)$ was proved by R. Pyke [28] and by S. Orey and W. E. Pruitt [24]. We state a version of their result now.

Theorem 3.10. Let $I = \langle 0, 1 \rangle \subseteq [0, \infty)^d$ be the d -cube unit interval. Then one has with probability one,

$$(3.10.1) \quad \lim_{|\underline{h}| \rightarrow 0} \sup_{0 < \underline{x} < 1 - \underline{h}} \frac{|W^d(\langle \underline{x}, \underline{x} + \underline{h} \rangle)|}{\sqrt{2|\underline{h}| \log |\underline{h}|^{-1}}} = 1,$$

where $\underline{h} = (h_1, \dots, h_d)$.

We shall not give the proof of the above theorem here. However, we will derive this result from a result in §3.4. Quite obviously, the partial derivatives of the sample paths of $W^d(\underline{x})$ do not exist anywhere with probability one.

3.3. The L.I.L. and the Erdős-Rényi increments for $W^d(\cdot)$.

An extension of the law of iterated logarithm for the process $W^d(\cdot)$ was proved by Pyke [28]. His result is a Strassen type result, i.e. the sequence

$$Z_n(\underline{x}) = \frac{W^d(\underline{n} \cdot \underline{x})}{\sqrt{2n^d \log \log n}}, \quad n \geq 3,$$

is relatively compact, with its limit points coinciding with the set of all absolutely continuous (w.r.t. the Lebesgue measure) square integrable Radon-Nikodym derivatives on $[0,1]^d$.

In this thesis, we are interested only in the following form of the L.I.L.

Theorem 3.11. Let $\underline{T} = (T, \dots, T) \in [0, \infty)^d$. Then

$$(3.11.1) \quad \lim_{T \rightarrow \infty} \sup_{0 < \underline{x} < \underline{T}} \frac{W(\underline{x})}{\sqrt{2|\underline{T}| \log \log T}} = 1$$

with probability one.

Theorem 3.11, of course, is a direct consequence of Pyke's result. However, we shall not bring in the proof here, for we shall discuss it in §3.4.

Now, let us look at the Erdős-Rényi increments of W^d .

The following theorem is an extension of Theorem 2.12 and is proved by Chan [3].

Theorem 3.12. Let $\underline{a}_N = (a_{1N}, a_{2N}, \dots, a_{dN})$ be a sequence in $[0, \infty)^d$ with integer coordinates. If $\{a_N\}$ satisfies

$$(3.12.1) \quad \lim_{N \rightarrow \infty} \frac{|\underline{a}_N|}{N^\delta} = 0,$$

for every $\delta > 0$, then we have

$$(3.12.2) \quad \lim_{N \rightarrow \infty} \max_{0 < n < N - \underline{a}_N} \frac{w^d(\langle \underline{n}, \underline{n} + \underline{a}_N \rangle)}{\sqrt{2d} |\underline{a}_N| [c \ln N]} = \sqrt{\frac{1}{c}} \text{ w.p.l.,}$$

for every $c > 0$. If we have, in addition to (3.12.1),

$$(3.12.3) \quad \lim_{N \rightarrow \infty} a_{iN} = \infty, \quad 1 \leq i \leq d,$$

then

$$(3.12.4) \quad \lim_{N \rightarrow \infty} \sup_{0 < \underline{y} < \underline{a}_N} \sup_{0 < \underline{x} < N - \underline{y}} \frac{w^d(\langle \underline{x}, \underline{x} + \underline{y} \rangle)}{\sqrt{2d} |\underline{a}_N| [c \log N]} = \sqrt{\frac{1}{c}} \text{ w.p.l.,}$$

for every $c > 0$.

Proof. Define, for each $\underline{N} = (N, \dots, N)$ and $\underline{k} = (k_1, \dots, k_d) \leq \underline{N}$,

$$o(\underline{N}, \underline{k}) = \max_{0 < \underline{n} < \underline{N} - \underline{k}} \frac{w^d(\langle \underline{n}, \underline{n} + \underline{k} \rangle)}{\sqrt{2d} |\underline{k}| [\log N]}$$

and

$$o^*(\underline{N}, \underline{k}) = \max_{0 < \underline{n} < \underline{N} - \underline{k}} \sup_{\underline{J} \leq \langle \underline{n}, \underline{n} + \underline{k} \rangle} \frac{w^d(\underline{J})}{\sqrt{2d} |\underline{k}| [\log N]}.$$

Let $\epsilon > 0$ be arbitrary and $\delta > 0$ such that $\delta < 2\epsilon + \epsilon^2$.

Observe that, for each $\delta > 0$, the number of all $\underline{k}$ such that $1 \leq |\underline{k}| \leq N^\delta$ will not exceed $N^{\delta d}$. By Proposition 3.3 and the observation that $\int_0^\infty e^{-u^2/2} \leq \frac{1}{\lambda} e^{-\lambda^2/2}$, $\lambda > 0$, we have, for some $A > 0$,

$$\begin{aligned} & P\left\{ \max_{1 \leq |\underline{k}| \leq N^\delta} O^*(N, \underline{k}) > 1 + \epsilon \right\} \\ & \leq \sum_{1 \leq |\underline{k}| \leq N^\delta} \sum_{\underline{n} \leq \underline{N} - \underline{k}} 4^d P\{W^d(\langle \underline{n}, \underline{n} + \underline{k} \rangle) > (1 + \epsilon) \sqrt{2d|\underline{k}|[\log N]}\} \\ & \leq AN^{-(1+\epsilon)^2 d} + (1 + \delta)d \\ & \leq AN^{-\gamma}, \end{aligned}$$

where $\gamma = (1 + \epsilon)^2 d - (1 + \delta)d > 0$. Let E_N denote the event that

$\max_{1 \leq |\underline{k}| \leq N^\delta} O^*(N, \underline{k}) > 1 + \epsilon$. Then $E_N \cap \overline{E}_{N+1} = \emptyset$ if $[\log N] = [\log N+1]$.

But $[\log N] \neq [\log N+1]$ when and only when $N = [e^k]$ for some k , uniquely determined by N . Let $Q = \{N : [\log N] \neq [\log N+1]\}$.

Then

$$\begin{aligned} \sum_N P(E_N \cap \overline{E}_{N+1}) &= \sum_{N \in Q} P(E_N \cap \overline{E}_{N+1}) \\ &\leq \sum_{N \in Q} P(E_N) \\ &\leq \sum_k A[e^k]^{-\gamma} \\ &< \infty. \end{aligned}$$

Since $P(E_N) \leq AN^{-\gamma} \rightarrow 0$ as $N \rightarrow \infty$, it follows that

$$\begin{aligned} P(E_N \text{ i.o.}) &= \lim_{N \rightarrow \infty} P\left(\bigcup_{m \geq N} E_m\right) \\ &= \lim_{N \rightarrow \infty} P\left\{E_N \cup \left(\bigcup_m (E_{m+1} \cap \overline{E}_{m+2})\right)\right\} \\ &\leq \lim_{N \rightarrow \infty} \left\{P(E_N) + \sum_{N < m} P(E_m \cap \overline{E}_{m+1})\right\} \\ &= 0. \end{aligned}$$

This means that,

$$\overline{\lim}_{N \rightarrow \infty} \max_{1 \leq |k| \leq N^\delta} \theta^*(N, k) \leq 1 + \epsilon \quad \text{w.p.l.}$$

for every $\epsilon > 0$. Since $|a_N|/N^\delta \rightarrow 0$, it renders that

$$\overline{\lim}_{N \rightarrow \infty} \theta(N, \underline{a}_N) \leq 1 \quad \text{w.p.l.,}$$

$$\text{or } \overline{\lim}_{N \rightarrow \infty} \max_{0 < n < N - \underline{a}_N} \frac{W^d(\langle \underline{n}, \underline{n} + \underline{a}_N \rangle)}{\sqrt{2d} |\underline{a}_N| [\log N]} \leq 1 \quad \text{w.p.l.}$$

Now, suppose that we also have (3.12.3). Define, for each $k \leq N$,

$$S(N, k) = \sup_{0 < y < k} \sup_{0 < x < N - y} \frac{W^d(\langle x, x + y \rangle)}{\sqrt{2d} |k| [\log N]}.$$

Observe that, for each $0 \leq y \leq \underline{a}_N$ and $0 \leq x \leq N - y$, there is an $\underline{n} \leq N$ and a $k \leq \underline{a}_N + 1$ such that

$$\langle \underline{x}, \underline{x} + \underline{y} \rangle \subseteq \langle \underline{n}, \underline{n} + \underline{k} \rangle .$$

Thus, for large N such that $|\underline{a}_N + 1| \leq N^\delta$, we have

$$S(N, \underline{a}_N + 1) \leq \max_{1 \leq |\underline{k}| \leq N^\delta} \theta^*(N, \underline{k}) ,$$

while

$$S(N, \underline{a}_N) \leq \left(\frac{|\underline{a}_N + 1|}{|\underline{a}_N|} \right)^{\frac{1}{2}} S(N, \underline{a}_N + 1) .$$

Since we have proved that $\lim_{N \rightarrow \infty} \max_{1 \leq |\underline{k}| \leq N^\delta} \theta^*(N, \underline{k}) \leq 1$ w.p.1.

and (3.12.3) implies that $|\underline{a}_N + 1|/|\underline{a}_N| \rightarrow 1$ as $N \rightarrow \infty$, we have

$$\begin{aligned} & \lim_{N \rightarrow \infty} S(N, \underline{a}_N) \\ &= \lim_{N \rightarrow \infty} \sup_{0 \leq \underline{y} \leq \underline{a}_N} \sup_{0 \leq \underline{x} \leq N - \underline{y}} \frac{w^d(\langle \underline{x}, \underline{x} + \underline{y} \rangle)}{\sqrt{2d} |\underline{a}_N| [\log N]} \\ & \leq 1 \end{aligned}$$

with probability one.

Conversely, let $\varepsilon > 0$ be arbitrary. Define

$\underline{k}_N = (k_{1N}, \dots, k_{dN})$ where $k_{iN} = \left\lfloor \frac{N}{a_{iN}} \right\rfloor$, $1 \leq i \leq d$. Then $|\underline{k}_N| \geq N^d / 2^d |\underline{a}_N|$. Let $\sigma > 0$ be small enough such that $\alpha = (2\varepsilon - \varepsilon^2)d - \sigma > 0$. Then, if N is large enough such that $\sqrt{(1-\varepsilon)^2 2d[\log N]} / 1 + (1-\varepsilon)^2 2d[\log N] > N^{-\sigma}$, we have

$$P\{\theta(N, \underline{a}_N) \leq 1 - \varepsilon\}$$

$$\leq P\left\{ \frac{w^d(\langle n \underline{a}_N, (n+1) \underline{a}_N \rangle)}{\sqrt{2d} |\underline{a}_N| [\log N]} \leq 1 - \epsilon, \quad 0 \leq n \leq N - \underline{a}_N \right\}$$

$$\leq \prod_{0 \leq n \leq N - \underline{a}_N} P\left\{ \frac{w^d(\langle n \underline{a}_N, (n+1) \underline{a}_N \rangle)}{\sqrt{2d} |\underline{a}_N| [\log N]} \leq 1 - \epsilon \right\}$$

$$\leq \left(P\left\{ w^d(\underline{1}) \leq \sqrt{(1-\epsilon)^2 2d [\log N]} \right\} \right)^{|\underline{k}_N|}$$

$$\leq \left(1 - \frac{N^{-(1-\epsilon)^2 d - \sigma}}{\sqrt{2\pi}} \right)^{N^{d/2} |\underline{a}_N|}$$

$$\leq e^{-BN^\alpha}$$

where $\alpha = (2\epsilon - \epsilon^2)d - \sigma > 0$. The Borel-Cantelli Lemma now implies that

$$\lim_{N \rightarrow \infty} \theta(N, \underline{a}_N) \geq 1 \text{ w.p.1. ,}$$

and thus

$$\lim_{N \rightarrow \infty} S(N, \underline{a}_N) \geq \lim_{N \rightarrow \infty} \theta(N, \underline{a}_N) \geq 1$$

with probability one. This proves (3.12.2) and (3.12.4) for the case when $c = 1$. For all general $c > 0$, the result follows from the observation that $[c \log N]/[\log N] \rightarrow c$ as $N \rightarrow \infty$. This completes our proof.

Of course, Theorem 3.12 does include the case when $d = 1$.

Thus in case of $d = 1$, we have Theorem 2.14 and hence also the result of Theorem 2.12.

Let us now give some consequences of Theorem 3.12.

Corollary 3.13. Let $c > 0$ be fixed. Suppose that $\lim_{N \rightarrow \infty} \frac{|a_N|}{[c \log N]} = 1$.

Then

$$(3.13.1) \quad \lim_{N \rightarrow \infty} \max_{0 < n < N - a_N} \frac{W^d(\langle \underline{n}, \underline{n+a_N} \rangle)}{|a_N|} = \sqrt{\frac{2d}{c}} \text{ w.p.1.}$$

Corollary 3.14. Suppose that a_N satisfies (3.12.1) and (3.12.3).

Then

$$(3.14.1) \quad \lim_{N \rightarrow \infty} \sup_{0 < x < N - a_N} \frac{W^d(\langle \underline{x}, \underline{x+a_N} \rangle)}{\sqrt{2d} |a_N| [c \log N]} = \sqrt{\frac{1}{c}} \text{ w.p.1.,}$$

for every $c > 0$.

3.4. The Csörgő-Révész Theorem on increments for $W^d(\cdot)$.

Now we can put Theorems 3.11 and 3.12 together into the following form

$$(*) \quad \lim_{N \rightarrow \infty} \sup_{0 < y < a_N} \sup_{0 < x < N - a_N} \beta_N |W^d(\langle \underline{x}, \underline{x+y} \rangle)| = 1 \text{ w.p.1.,}$$

where $\beta_N = (2 |a_N| [\log(|N|/|a_N|) + \log \log N])^{-1/2}$ and $a_N = N$ or a_N satisfies (3.12.1) and (3.12.3). Then, the same question, as in §2.5, that whether or not (*) will hold for other a_N is the main question in this section.

Let us first define:

Definition 3.15. A d - β_T -function $\underline{a}_T = (a_{t_1}^{(1)}, \dots, a_{t_d}^{(d)})$ is a function from $[0, \infty)^d$ into itself such that each $a_t^{(i)}$, $1 \leq i \leq d$, is a β_T -function (see Definition 2.15) and for every pair of i and j , $1 \leq i, j \leq d$, the limit

$$\lim_{T \rightarrow \infty} \frac{a_T^{(i)}}{a_T^{(j)}} = \rho_{ij}$$

exists (it can be infinite).

Let us first state our main result in this section.

Theorem 3.16. For each $T \geq 0$, let $\underline{T} = (T_1, \dots, T_d)$. Suppose that $\underline{a}$ is a d - β_T -function. Then,

$$(3.16.1) \quad \overline{\lim}_{T \rightarrow \infty} \sup_{0 < \underline{x} < \underline{T} - \underline{a}_{\underline{T}}} \beta_T |W^d(\underline{x}, \underline{x} + \underline{a}_{\underline{T}})| = 1 \text{ w.p.1.,}$$

and

$$(3.16.2) \quad \overline{\lim}_{T \rightarrow \infty} \sup_{0 < \underline{x} < \underline{T} - \underline{a}_{\underline{T}}} \sup_{J \subseteq \langle \underline{x}, \underline{x} + \underline{a}_{\underline{T}} \rangle} \beta_T |W^d(J)| = 1 \text{ w.p.1.,}$$

where

$$\beta_T = (2|a_{\underline{T}}| [\log |\underline{T}| / |a_{\underline{T}}| + \log \log T])^{-1}.$$

If we also have

$$(3.16.3) \quad \lim_{T \rightarrow \infty} \frac{\log \frac{|\underline{T}|}{|a_{\underline{T}}|}}{\log \log T} = \infty,$$

then

$$(3.16.4) \quad \lim_{T \rightarrow \infty} \sup_{0 < x < T - a_T} \beta_T |W^d(\underline{x}, \underline{x} + a_T)| = 1 \text{ w.p.1}$$

and

$$(3.16.5) \quad \lim_{T \rightarrow \infty} \sup_{0 < x < T - a_T} \sup_{J \subseteq \langle \underline{x}, \underline{x} + a_T \rangle} \beta_T |W^d(J)| = 1 \text{ w.p.1}.$$

Remark. If we put $a_T = (T, \dots, T)$ then the above Theorem is true and it simply reduced to Theorem 3.11, a multiparameter version of L.I.L. On the other hand, if a_N satisfies the conditions (3.12.1) and (3.12.3) also, then

$$\beta_N \left(\frac{2d}{c} |a_N| [c \log N] \right)^{\frac{1}{2}} \rightarrow 1$$

as $N \rightarrow \infty$. This proves Theorem 3.12, with some restriction on a_N . In fact, Theorem 3.16. subsumes the result of Theorem 2.16 because a $1 - \beta_T$ -function is by definition a β_T -function.

Before we prove Theorem 3.16., let us prove some lemmas which are relevant in the sequel.

Lemma 3.17. For each small $\varepsilon > 0$, say $\varepsilon \leq \sqrt{4/5}$, there exists a constant $c = c(\varepsilon, d) > 0$ such that

$$(3.17.1) \quad P \left\{ \sup_{0 < t < 1} \sup_{J \subseteq \langle \underline{t}, \underline{t} + \underline{h} \rangle} |W^d(J)| > \lambda |\underline{h}|^{\frac{1}{2}} \right\} \leq C |\underline{h}|^{-1} e^{-\lambda^2/2 + \varepsilon},$$

for every $\lambda > 0$ and $0 < \underline{h} \leq 1$.

Proof. Let $\delta > 0$ be such that $(1+\delta)^d = (1 + \frac{\varepsilon}{2})$. Suppose that $\underline{h} = (h_1, \dots, h_d)$. Let R_i be the smallest integer such that $R_i^{-1} \leq \delta^2 h_i / 2$, for each $1 \leq i \leq d$, and let m_i be the smallest integer such that $h_i \leq m_i / R_i$. Then, for each x_i , $0 \leq x_i \leq 1$, there is an n_i , $0 \leq n_i \leq R_i$ such that

$$(x_i, x_i+h) \subseteq (n_i/R_i, (n_i/R_i) + (m_i+1/R_i)) .$$

Let $\underline{m} = (m_1, \dots, m_d)$, $\underline{R} = (R_1, \dots, R_d)$ and $\underline{R}^{-1} = (R_1^{-1}, \dots, R_d^{-1})$. Then, for each $\underline{x} \in [0,1]^d$, there is an $\underline{n}$, $0 \leq \underline{n} \leq \underline{R}$ such that

$$\langle \underline{x}, \underline{x}+\underline{h} \rangle \subseteq \langle \underline{n} \underline{R}^{-1}, \underline{n} \underline{R}^{-1} + (\underline{m}+1) \underline{R}^{-1} \rangle .$$

This implies that

$$\sup_{0 \leq x \leq 1} \sup_{0 \leq s \leq h} |W^d(\langle \underline{x}, \underline{x}+\underline{s} \rangle)| \leq \max_{0 \leq \underline{n} \leq \underline{R}} \sup_{0 \leq \underline{s} \leq \underline{m}+1} |W^d(\langle \underline{n} \underline{R}^{-1}, \underline{n} \underline{R}^{-1} + \underline{s} \rangle)| .$$

Since $h_i \leq m_i / R_i < h_i + 1 / R_i$, $h_i \leq m_i + 1 / R_i \leq h_i + 2 / R_i \leq h_i (1 + \delta^2) \leq h_i (1 + \delta)^2$, we have $R_i h_i / (m_i + 1) > (1 + \delta)^{-2}$, and thus

$$\frac{|\underline{R}| |\underline{h}|}{|\underline{m} + 1|} = \prod_{i=1}^d \left(\frac{R_i h_i}{m_i + 1} \right) \geq (1 + \delta)^{-2d} = (1 + \frac{\varepsilon}{2})^{-2} .$$

Hence, by Proposition 3.3 and that $\int_{\lambda}^{\infty} e^{-x^2/2} dx \leq \frac{1}{\lambda} e^{-\lambda^2/2}$,

$\lambda > 0$, we have

$$\begin{aligned}
 & P\left\{ \max_{0 \leq s \leq m+1} |W^d(\langle \underline{n} \underline{R}^{-1}, \underline{n} \underline{R}^{-1} + \underline{s} \rangle)| > \lambda |\underline{h}|^{-1} \right\} \\
 & \leq 4^d P\left\{ |W^d(\langle \underline{R}^{-1}, \underline{n} \underline{R}^{-1} + (\underline{m}+1)\underline{R}^{-1} \rangle)| > \lambda |\underline{h}|^{-1} \right\} \\
 & \leq 4^{d+1} \Phi\left(-\lambda \left(\frac{|\underline{h}| |\underline{R}|}{|\underline{m}+1|}\right)^{1/2}\right) \\
 & \leq 4^{d+1} \Phi\left(-\frac{\lambda}{(1 + \frac{\varepsilon}{2})}\right) \\
 & \leq A e^{-\lambda^2/2+\varepsilon} \quad (\lambda > 1) ,
 \end{aligned}$$

for some $A = A(d) > 0$. Consequently,

$$\begin{aligned}
 & P\left\{ \sup_{0 \leq x \leq 1} \sup_{J \subseteq \langle \underline{x}, \underline{x}+\underline{h} \rangle} |W^d(J)| > \lambda |\underline{h}|^{1/2} \right\} \\
 & \leq A |\underline{R} + \underline{1}| e^{-\lambda^2/2+\varepsilon} \\
 & \leq C |\underline{h}|^{-1} e^{-\lambda^2/2+\varepsilon}
 \end{aligned}$$

for some appropriate $C = C(\varepsilon, d) > 0$. (This can be done as long as each $R_i > 2$. But since $\frac{1}{R_i} \leq \frac{h_i \delta^2}{2} < \frac{1}{2}$, $R_i > 2$ always.) This completes the proof.

Since $\{W^d(\underline{x}), 0 \leq x \leq \underline{s}\} \stackrel{D}{=} \{|\underline{s}|^{1/2} W^d(\underline{x} \underline{s}^{-1}), 0 \leq x \leq \underline{s}\}$ for each $\underline{s} = (s_1, \dots, s_d)$ and $\underline{s}^{-1} = (s_1^{-1}, \dots, s_d^{-1})$, we have:

Lemma 3.18. For each $\varepsilon > 0$, there is a $C = C(\varepsilon, d) > 0$ such that

$$(3.18.1) \quad P\left\{ \sup_{0 < x < \delta} \sup_{J \subseteq \langle x, x+h \rangle} |W^d(J)| > \lambda |h|^{1/2} \right\} \leq C \frac{|s|}{|h|} e^{-\frac{\lambda^2}{2+\epsilon}}$$

for each $\epsilon > 0$ and each $0 \leq h \leq s = (s_1, s_2, \dots, s_d)$.

Now, we proceed to prove Theorem 3.16.

Proof of Theorem 3.16. Let $\underline{T} = (T, \dots, T)$ and

$$A(\underline{T}) = \sup_{0 < t < T - a_{\underline{T}}} \sup_{J \subseteq \langle t, t+a_{\underline{T}} \rangle} \beta_{\underline{T}} |W^d(J)|.$$

Then, by Lemma 3.18, we have

$$\begin{aligned} P\{A(\underline{T}) \geq 1+\epsilon\} &\leq C \frac{|\underline{T}|}{|a_{\underline{T}}|} \exp \left\{ -(1+\epsilon) \left[\ln \frac{|\underline{T}|}{|a_{\underline{T}}|} + \log \log T \right] \right\} \\ &\leq C \left(\frac{|a_{\underline{T}}|}{|\underline{T}|} \right)^\epsilon \frac{1}{(\log T)^{1+\epsilon}}. \end{aligned}$$

Let $T_N = \theta^N$ ($\theta > 1$). Then, if $a_{\underline{T}}$ is a d - $\beta_{\underline{T}}$ -function, we have

$$\sum_N P\{A(T_N) \geq 1 + \epsilon\} < \infty$$

for every $\epsilon > 0$ and $\theta > 1$. Thus, by the Borel-Cantelli lemma, we have

$$\lim_{N \rightarrow \infty} A(T_N) \leq 1 \text{ w.p. 1.}$$

Observe that

$$(3.16.6) \quad 1 \leq \frac{\beta_{T_N}}{\beta_{T_{N+1}}} \leq \theta^{\frac{d}{2}},$$

for θ close to 1 and N large enough. Since

$$\beta_{T_N}^{-1} \leq \beta_T^{-1} \leq \beta_{T_{N+1}}^{-1}$$

and

$$\beta_{T_N}^{-1} A(T_N) \leq \beta_T^{-1} A(T) \leq \beta_{T_{N+1}}^{-1} A(T_{N+1}),$$

if $T_N < T < T_{N+1}$, it follows from (3.16.6) that, for large T ,

$$A(T) \leq \frac{\beta_T}{\beta_{T_{N+1}}} A(T_{N+1}) \leq \theta^{\frac{d}{2}}$$

with probability one. Taking $\theta = (1 + \epsilon)^{\frac{2}{d}}$, we get

$$\overline{\lim}_{T \rightarrow \infty} A(T) \leq 1 + \epsilon \quad \text{w.p.1}$$

for each $\epsilon > 0$. Evidently,

$$(3.16.7) \quad \overline{\lim}_{T \rightarrow \infty} A(T) \leq 1 \quad \text{w.p.1.}$$

We shall prove (3.16.4) and (3.16.5) first. Suppose that in addition to a_T being a d - β_T -function, we also have (3.16.3). Observe that

$$\begin{aligned} & P\left\{ \sup_{0 < x < T - a_T} \beta_T |W^d(\langle x, x + a_T \rangle)| \leq 1 - \epsilon \right\} \\ & \leq P\left\{ \sup_{0 < k < \left\lfloor \frac{T}{a_T} \right\rfloor - 1} \beta_T |W^d(\langle k a_T, (k+1) a_T \rangle)| \leq 1 - \epsilon \right\} \\ & \leq P\left\{ |W^d(1)| \leq (1 - \epsilon) \sqrt{2 \left[\log \frac{T}{a_T} + \log \log T \right]} \right\}^{\left\lfloor \frac{T}{a_T} \right\rfloor} \end{aligned}$$

$$\begin{aligned}
 & \leq (1 - A \frac{1}{[\log \frac{|T|}{|a_T|} + \log \log T]} \exp\{-(1-\epsilon)^2 [\log \frac{|T|}{|a_T|} + \log \log T]\})^{\left[\frac{|T|}{|a_T|}\right]} \\
 & \leq \left\{1 - A([\log \frac{|T|}{|a_T|} + \log \log T])^{-1} \left(\frac{|T|}{|a_T|}\right)^{-(1-\epsilon)^2}\right\}^{\left[\frac{|T|}{|a_T|}\right]} \\
 & \leq \exp\left\{-\left(\frac{|T|}{|a_T|}\right)^{2\epsilon - \epsilon^2} \left(\frac{1}{\log T}\right)^{(1-\epsilon)^2} \left(\frac{A}{\log \frac{|T|}{|a_T|} + \log \log T}\right)\right\}, \\
 & \leq \exp\left\{-\left(\frac{|T|}{|a_T|}\right)^{\epsilon'} \left(\frac{1}{\log T}\right)^{1-\epsilon'}\right\}
 \end{aligned}$$

for $\epsilon' = 2\epsilon - \epsilon^2$. Since (3.16.3) implies that $\sum \exp\left\{-\left(\frac{|T|}{|a_T|}\right)^{\epsilon'} \left(\frac{1}{\log T}\right)^{1-\epsilon'}\right\} < \infty$ for every $\epsilon > 0$, by the Borel-Cantelli Lemma, we have,

$$(3.16.8) \quad \lim_{T \rightarrow \infty} \sup_{0 < x < T - a_T} \beta_T |W(x, x + a_T)| \geq 1 \text{ w.p. 1.}$$

This proves (3.16.4) and (3.16.5).

To prove (3.16.1) and (3.16.2), let us first consider the case that $\lim_{T \rightarrow \infty} |a_T|/|T| < 1$. Then, in this case, there must be an $a_T^{(i)}$ such that $\lim_{T \rightarrow \infty} a_T^{(i)}/T < 1$. Without loss of generality, we can assume that

$$\lim_{T \rightarrow \infty} \frac{a_T^{(i)}}{T} = c_i < 1, \quad 1 \leq i \leq r$$

for some r , $1 \leq r \leq d$, and $\lim_{T \rightarrow \infty} a_T^{(i)}/T = 1$, for $r+1 \leq j \leq d$.

Since $a_T^{(j)}/T$ is non-increasing, we then must have $a_T^{(j)} = T$, for $r+1 \leq j \leq 1$. Assume first, that $a_T^{(i)}/a_T^{(j)}$, $1 \leq i, j \leq r$ are also monotone. Then, we have that $c_1 \geq c_i$, $1 \leq i \leq r$ and that $a_T^{(1)} \geq a_T^{(i)}$ for large T . Define

$$f(T) = T - a_T^{(1)}, \quad T \geq 0,$$

and

$$T_k = \inf \{T : T_{k-1} = f(T)\}, \quad k \geq 2.$$

with $T_1 > 0$ such that $f(T_1) > 0$. Let us denote by Z_k the collection of all intervals J in the form $J = \langle \underline{x}, \underline{y} \rangle$, with

$$\underline{x} = (T_k - n_1 a_{T_k}^{(1)}, \dots, T_k - n_r a_{T_k}^{(r)}, 0, \dots, 0)$$

and

$$\underline{y} = \underline{x} + (a_{T_k}^{(1)}, \dots, a_{T_k}^{(d)})$$

where $1 \leq n_i \leq [T_k/a_{T_k}^{(i)}]$, $1 \leq i \leq r$, with at least one of the $n_i = 1$. Then all the J 's in Z_k have disjoint interiors; also the interior of every J in Z_k is disjoint from the interior of every I in Z_{k+1} . Therefore, the events

$$B_k = \max_{J \in Z_k} \beta_{T_k} |W^d(J)|, \quad k \geq 1$$

are independent. Now, to prove (3.16.1) and (3.16.2), it suffices to show that $\sum P\{B_k > 1-\epsilon\} = \infty$. First, we observe that

$$T_k = T_1 \left(\prod_{n=2}^k (1 - a_{T_n}^{(1)}/T_n) \right)^{-1}, \quad k \geq 2.$$

From the elementary inequality $e^{-x} \geq 1 - x \geq e^{-\frac{x}{1-x}}$, $0 < x < 1$, we get

$$\begin{aligned} T_k &\leq T_1 \exp \left\{ \sum_{n=2}^k \frac{a_{T_n}^{(1)}/T_n}{(1-a_{T_n}^{(1)}/T_n)} \right\} \\ &\leq T_1 \exp \left\{ \frac{1}{T_1 - a_{T_1}^{(1)}} \sum_{n=2}^k a_{T_n}^{(1)}/T_n \right\}, \end{aligned}$$

and hence

$$(T_1 - a_{T_1}^{(1)}) (\log T_k - \log T_1) \leq \sum_{n=2}^k \frac{a_{T_n}^{(1)}}{T_n}.$$

Since $a_T^{(j)}/T \rightarrow 1$ as $k \rightarrow \infty$, for large k , $\#Z_k$, the number of intervals in Z_k , is larger than $(T_k^{d-1}/|a_{T_k}|) a_{T_k}^{(1)} \geq T_k^{r-1} a_{T_k}^{(1)}/a_{T_k}^{(1)} \dots a_{T_k}^{(r)}$. For appropriate constants A_1 and A_0 we have, for every arbitrary $\epsilon > 0$,

$$P\{B_k < 1 - \epsilon\}$$

$$= (P\{W^d(1) < (1-\epsilon) \sqrt{2 \log \left(\frac{|T_k|}{|a_{T_k}|} \log T_k \right)}\})^{\#Z_k}$$

$$\begin{aligned} &\leq \left\{ 1 - A_1 \left(\frac{a_{T_k}^{(1)} \dots a_{T_k}^{(r)} T_k \dots T_k}{T_k^d \log T_k} \right)^{1-2\epsilon} \right\} T_k^{r-1} a_{T_k}^{(1)}/a_{T_k}^{(1)} \dots a_{T_k}^{(r)} \\ &\leq \exp \left\{ - A_1 \frac{(a_{T_k}^{(1)} \dots a_{T_k}^{(r)})^{-2\epsilon} a_{T_k}^{(1)}}{T_k^{1-2\epsilon r} (\log T_k)^{1-2\epsilon}} \right\} \end{aligned}$$

$$\begin{aligned}
 & \leq \exp \left\{ - A_1 \left(\frac{a_{T_k}^{(1)}}{T_k} \right)^{1-2\epsilon r} \left(\frac{1}{\log T_k} \right)^{1-2\epsilon} \right\} \\
 & \leq 1 - A_0 \left(\frac{a_{T_k}^{(1)}}{T_k} \right)^{1-2\epsilon r} \left(\frac{1}{\log T_k} \right)^{1-2\epsilon} \\
 & \leq 1 - A_0 \left(\frac{a_{T_k}^{(1)}}{T_k} \right) \left(\frac{1}{\log T_k} \right)^{1-2\epsilon} .
 \end{aligned}$$

Hence

$$\begin{aligned}
 \sum_{n=2}^K P\{B_n > 1-\epsilon\} & \geq \sum_{n=2}^K A_0 \left(\frac{a_{T_n}^{(1)}}{T_n} \right) \left(\frac{1}{\log T_n} \right)^{1-2\epsilon} \\
 & \geq A_0 \left(\frac{1}{\log T_k} \right)^{1-2\epsilon} \sum_{n=2}^K \left(\frac{a_{T_n}^{(1)}}{T_n} \right) \\
 & \geq A_0 \frac{(\log T_k - \log T_1)(T_1 - a_{T_1}^{(1)})}{(\log T_k)^{1-2\epsilon}} \rightarrow \infty .
 \end{aligned}$$

Now, suppose the monotonicity condition for $a_T^{(i)}/a_T^{(j)}$ is not assumed. Then, as a consequence of the definition of a $d\text{-}\beta_T$ -function, we can assume that $\lim a_T^{(1)}/a_T^{(i)} \geq 1$, $1 \leq i \leq r$, (it can be infinite) and $a_T^{(j)} = T$, $r+1 \leq j \leq l$, as before. Let Z_k be defined the same way. Define also

$$\hat{Z}_k = \{J \cap \bar{L}_k : J \in Z_k\} \quad \text{and} \quad \check{Z}_k = \{J \cap L_k : J \in Z_k\} ,$$

where

$$L_k = \{x : 0 \leq x_i \leq T_{k-1}, 1 \leq i \leq r, \text{ and } 0 \leq x_j \leq T_k, r+1 \leq j \leq d\}.$$

Since $J = (J \cap \bar{L}_k) \cup (J \cap L_k)$, for every $J \in Z_k$, we have

$$\hat{B}_k - \check{B}_k \leq B_k \leq \hat{B}_k + \check{B}_k, \quad k \geq 2,$$

where B_k is defined as above, and

$$\begin{aligned} \hat{B}_k &= \sup \{ \beta_{T_k} |W^d(J)| : J \in \hat{Z}_k \}, \\ \check{B}_k &= \sup \{ \beta_{T_k} |W^d(J)| : J \in \check{Z}_k \}. \end{aligned}$$

Also, $\lim_{T \rightarrow \infty} a_T^{(i)}/a_T \geq 1$, $1 \leq i \leq r$, implies that $\# \hat{Z}_k = \# Z_k$ for large k , and $\# \check{Z}_k \leq \# Z_k \leq \frac{|T_k|}{|a_{T_k}|}$. Let $m_k = \min \{|J| : J \in \hat{Z}_k\}$

and $M_k = \max \{|J| : J \in \check{Z}_k\}$. Since $\lim_{T \rightarrow \infty} a_T^{(1)}/a_T \geq 1$, $1 \leq i \leq r$, we have $m_k/|a_{T_k}| \rightarrow 1$, and $M_k/|a_{T_k}| \rightarrow 0$ as $k \rightarrow \infty$. Since now,

$\hat{B}_k$ are independent, repeating the same proof as for the B_k 's, and using the observation that $m_k/|a_{T_k}| \rightarrow 1$ as $k \rightarrow \infty$, we have that $\lim \hat{B}_k \geq 1$ with probability one. We sketch the proof. Observe that

$$P\{\hat{B}_k < 1-\epsilon\}$$

$$\leq \prod_{J \in \hat{Z}_k} P\{W^d(\underline{1}) < (1-\epsilon) \sqrt{2 \frac{|a_{T_k}|}{|J|} [\log \frac{|T_k|}{|a_{T_k}|} + \log \log T_k]}\}$$

$$\begin{aligned}
 & \leq \prod_{j \in \hat{Z}_k} P\{W^d(\underline{1}) < (1-\epsilon) \sqrt{2 \frac{|a_{T_k}|}{m_k} \left[\log \frac{|T_k| \log T_k}{|a_{T_k}|} \right]}\} \\
 & \leq (P\{W^d(\underline{1}) < (1-\epsilon) \sqrt{2(1+\epsilon)^2 \left[\log \frac{|T_k| \log T_k}{|a_{T_k}|} \right]}\})^{\#Z_k} \\
 & \leq 1 - A_0 \left(\frac{a_{T_k}^{(1)}}{T_k} \right) \left(\frac{1}{\log T_k} \right)^{1-2\epsilon^4}
 \end{aligned}$$

for k large enough such that $|a_{T_k}|/m_k \leq (1+\epsilon)^2$, and $A_0 > 0$ is some appropriate constant. Thus, likewise before for the B_k , we have

$$\sum P\{\hat{B}_n > 1 - \epsilon\} = \infty.$$

Now, to prove the statements of (3.16.1) and (3.16.2), it suffices to show that $\lim_{k \rightarrow \infty} \check{B}_k = 0$ w.p.1., since for each ω , $B_k \geq \hat{B}_k - \check{B}_k$. Let $\epsilon > 0$ be arbitrary. Then

$$\begin{aligned}
 & P\{\check{B}_k > \epsilon\} \\
 & \leq (\#Z_k) P\{|W^d(\underline{1})| > \epsilon \sqrt{2 \frac{|a_{T_k}|}{m_k} \left[\log \frac{|T_k| \log T_k}{|a_{T_k}|} \right]}\} \\
 & \leq 2(\#Z_k) \exp \left\{ -\epsilon^2 \frac{|a_{T_k}|}{m_k} \left[\log \frac{|T_k| \log T_k}{|a_{T_k}|} \right] \right\} \\
 & \leq 2(\#Z_k) \left(\frac{|a_{T_k}|}{|T_k| \log T_k} \right)^{\epsilon^2 |a_{T_k}|/m_k}
 \end{aligned}$$

$$\begin{aligned} & \leq 2 \left(\frac{|T_k|}{|a_{T_k}|} \right) \left(\frac{|a_{T_k}|}{|T_k|} \right)^{\epsilon^2 |a_{T_k}|/M_k} \\ & \leq 2 \left(\frac{|a_{T_k}|}{|T_k|} \right)^{(\epsilon^2 |a_{T_k}|/M_k) - 1} \end{aligned}$$

Since $|a_{T_k}|/M_k \rightarrow \infty$ and $|a_{T_k}|/|T_k| \rightarrow c < 1$, we have

$$\sum P\{\tilde{B}_k > \epsilon\} < \infty.$$

This is sufficient to conclude that $\lim \tilde{B}_k = 0$ w.p.1.

Let us now consider the case when $a_T^{(i)} = T$ for every $i = 1, 2, \dots, d$. Define $T_k = k^k$ and $\underline{T}_k = (T_k, \dots, T_k)$ for $k = 1, 2, \dots$. Let $\Gamma_1^{(k)}$, for each $1 \leq i \leq d$, be the hyperplane that goes through the point $(0, \dots, T_k, \dots, 0)$, where the i th coordinate is T_k and the rest are 0, and is parallel to the $x_1 \dots x_{i-1} x_{i+1} \dots x_d$ plane. Then $\Gamma_1^{(k)}$ divides the interval $<0, T_{k+1}>$ into two intervals with disjoint interiors. Likewise, the hyperplane $\Gamma_2^{(k)}$ divides these two intervals into four intervals. So, by induction, we have 2^d many intervals with disjoint interiors. Amongst them, there is the interval $<\underline{T}_k, \underline{T}_{k+1}>$; let us call the rest $J_1, \dots, J_{2^d-1}$. Observe that $a_{T_{k+1}}^{(i)} > T_{k+1} - T_k$, for all k large enough. Thus we have

$$E(W^d(J_m))^2 = |J_m| \leq (T_{k+1} - T_k)^{d-1} T_k$$

for every $m = 1, 2, \dots, 2^d - 1$. Since $|a_{T_{k+1}}|/|T_{k+1}| \rightarrow 1$, for large k , we have

$$\begin{aligned} & P\left\{ \max_m \beta_{T_{k+1}} |W^d(J_m)| > \delta \right\} \\ & \leq 2^d \max P\left\{ \beta_{T_{k+1}} |W^d(J_m)| > \delta \right\} \\ & \leq 2^d P\left\{ |W^d(\underline{1})| > \delta \left(2 \left[\log \frac{|T_{k+1}|}{|a_{T_{k+1}}|} + \log \log T_{k+1} \right] \right. \right. \\ & \quad \left. \left. \times \frac{|a_{T_{k+1}}|}{(T_{k+1} - T_k)^{d-1} T_k} \right) \right\} \\ & \leq A_2 \exp \left\{ - \delta |a_{T_{k+1}}| / (T_{k+1} - T_k)^{d-1} T_k \right\}, \end{aligned}$$

for every $\delta > 0$ and some appropriate constant $A_2 = A_2(d) > 0$.

Since $a_T^{(i)}/T = 1$, for each i , we have $|a_{T_{k+1}}|/(T_{k+1} - T_k)^{d-1} T_k \rightarrow \infty$.

This renders

$$\sum P\left\{ \max_m \beta_{T_{k+1}} |W^d(J_m)| > \delta \right\} < \infty,$$

and that $\max_m \beta_{T_{k+1}} |W^d(J_m)| \rightarrow 0$ w.p.1., as $k \rightarrow \infty$. On the other hand, the observation that, for every $\varepsilon > 0$, we have

$$\begin{aligned} & (2(1-\varepsilon)(T_{k+1} - T_k)^d \log \log (T_{k+1} - T_k))^{1/2} \\ & \geq (1 - \frac{1}{2}\varepsilon) (2|a_{T_{k+1}}| \left[\log \frac{T_{k+1}^d}{|a_{T_{k+1}}|} + \log \log T_{k+1} \right])^{1/2} \end{aligned}$$

implies that

$$\begin{aligned} & P\{\beta_{T_{k+1}} W^d(\langle \underline{T}_k, \underline{T}_{k+1} \rangle) > 1 - \frac{1}{2} \epsilon\} \\ & \geq P\{W^d(\langle \underline{T}_k, \underline{T}_{k+1} \rangle) > (2(1-\epsilon) (T_{k+1} - T_k)^d \log \log (T_{k+1} - T_k))^{\frac{1}{d}}\} \\ & \geq \frac{A_3}{k^{1-\frac{1}{2}\epsilon}} \end{aligned}$$

for some appropriate constant $A_3 > 0$. The independence of the $\langle \underline{T}_k, \underline{T}_{k+1} \rangle$ and the Borel-Cantelli Lemma imply

$$\lim_{k \rightarrow \infty} \beta_{T_{k+1}} |W^d(\langle \underline{T}_k, \underline{T}_{k+1} \rangle)| \geq 1 \text{ w.p.1.}$$

But since $|W^d(\langle \underline{T}_{k+1} - a_{T_{k+1}}^{(i)}, \underline{T}_{k+1} \rangle)| \geq |W^d(\langle \underline{T}_k, \underline{T}_{k+1} \rangle)| - \sum_{m=1}^{2^d-1} |W^d(J_m)|$,

this proves the case when $a_T^{(i)}/T \rightarrow 1$, for each i . This also completes the proof of Theorem 3.16.

3.5. Some further discussions on large and small increments.

Recall that $W^d(\underline{x}) = |\underline{S}|^{-\frac{1}{d}} W^d(\underline{S} \underline{x})$, $0 \leq \underline{x} \leq \underline{1}$, for every $\underline{S} = (s_1, \dots, s_d) > \underline{0}$. Then, letting $\underline{h} = (h_1, \dots, h_d)$, we have

$$P\left\{ \sup_{0 \leq t \leq 1-\underline{h}} \sup_{J \subseteq \langle \underline{t}, \underline{t}+\underline{h} \rangle} \alpha_{\underline{h}} |W^d(J)| > 1-\epsilon \right\} \leq C |\underline{h}|^{-\epsilon}$$

where $\alpha_{\underline{h}} = (2|\underline{h}| [\log |\underline{h}|^{-1} + \log \log \frac{1}{|\underline{h}|}])^{-\frac{1}{d}}$, by using the same

argument as in the proof of Theorem 3.16. This renders

$$\overline{\lim}_{h \rightarrow 0} \sup_{0 \leq t \leq 1-h} \sup_{J \subseteq \langle t, t+h \rangle} \frac{|W^d(J)|}{\sqrt{2|h|} \log |h|^{-1}} \leq 1 \text{ w.p.1.}$$

And similarly, we can show that

$$\underline{\lim}_{|h| \rightarrow 0} \sup_{0 \leq t \leq 1-h} \sup_{J \subseteq \langle t, t+h \rangle} \frac{|W^d(J)|}{\sqrt{2|h|} \log |h|^{-1}} \geq 1 \text{ w.p.1.}$$

These two inequalities combined will yield Theorem 3.10. In this section, we shall combine the P. Lévy modulus of continuity and the Csörgő-Révész theorem on increments into one theorem, i.e. we will let part of the coordinates in the increments go to zero and let the rest be non-decreasing. The next two theorems summarize the meaning of this.

Let us consider $W^2(\underline{x}) = W^2(x_1, x_2)$, a 2-parameter Wiener process, where $0 \leq x_1 \leq 1$ and $x_2 \geq 0$. We will use J to denote an interval in $[0, 1] \times [0, \infty)$.

Theorem 3.19. Let $\underline{a}_T = (a_T^{(1)}, a_T^{(2)})$, $\underline{T} = (T, T)$ and $T \geq 0$, be a $2\text{-}\beta_T$ -function such that $a_T^{(1)}/T \rightarrow 0$ as $T \rightarrow \infty$. Define $b_T = a_T^{(1)}/T$, $c_T = a_T^{(2)}$ and $\hat{\underline{a}}_T = (b_T, c_T)$. Let $\alpha_T = (2|\hat{\underline{a}}_T|, [\log(T/|\hat{\underline{a}}_T|) + \log \log T])^{-1/2}$. Then

$$(3.19.1) \quad \overline{\lim}_{T \rightarrow \infty} \sup_{0 \leq \underline{x} \leq (1, T) - \hat{\underline{a}}_T} \alpha_T |W^2(\langle \underline{x}, \underline{x} + \hat{\underline{a}}_T \rangle)| = 1 \text{ w.p.1.}$$

and

$$(3.19.2) \quad \overline{\lim}_{T \rightarrow \infty} \sup_{0 < \underline{x} < (1, T) - \hat{a}_T} \sup_{J \subseteq \langle \underline{x}, \underline{x} + \hat{a}_T \rangle} \alpha_T |W^2(J)| = 1 \text{ w.p.1.}$$

If at least one $a_T^{(i)}$ also satisfies (3.16.3) of Theorem 3.16., then the $\overline{\lim}$ in (3.19.1) and (3.19.2) can be replaced by $\lim$.

An obvious generalization of the above theorem of course is the case when we consider the d -parameter Wiener process $W^d(\underline{x}) = W(x_1, \dots, x_d)$ ($d > 2$), where $(x_1, \dots, x_r) \in [0, 1]^r$ and $(x_{r+1}, \dots, x_d) \in [0, \infty)^{d-r}$, for some $1 \leq r \leq d$, where $[0, \infty)^0$ is taken to be the empty set. Let $e_T = (1, \dots, 1, T, \dots, T)$, where the first r coordinates are 1 and the rest are T . Then we have:

Theorem 3.20. Let $a_T = (a_T^{(1)}, \dots, a_T^{(d)})$ be a d - β_T -function such that $a_T^{(i)}/T \rightarrow 0$ ($1 \leq i \leq r$) as $T \rightarrow \infty$. Define $b_T^{(i)} = a_T^{(i)}/T$, $1 \leq i \leq r$, and $c_T^{(j)} = a_T^{(j)}$, $r+1 \leq j \leq d$, and $\hat{a}_T = (b_T^{(1)}, \dots, b_T^{(r)}, c_T^{(r+1)}, \dots, c_T^{(d)})$. Then, with $\alpha_T = (2|\hat{a}_T| [\log T^{d-r}/|\hat{a}_T| + \log \log T])^{-1/2}$, we have

$$(3.20.1) \quad \overline{\lim}_{T \rightarrow \infty} \sup_{0 < \underline{x} < e_T - \hat{a}_T} \alpha_T |W^d(\langle \underline{x}, \underline{x} + \hat{a}_T \rangle)| = 1 \text{ w.p.1.}$$

and

$$(3.20.2) \quad \overline{\lim}_{T \rightarrow \infty} \sup_{0 < \underline{x} < e_T - \hat{a}_T} \sup_{J \subseteq \langle \underline{x}, \underline{x} + \hat{a}_T \rangle} \alpha_T |W^d(J)| = 1 \text{ w.p.1.}$$

If at least one $a_T^{(i)}$, $1 \leq i \leq d$, also satisfies (2.16.3) of Theorem 2.16. then the $\overline{\lim}$ can be replaced by $\lim$.

We shall only prove Theorem 3.19 since in the light of Theorem 3.16, the proof of Theorem 3.20 is essentially the same.

On the surface, Theorem 3.19 may not look much like P. Lévy's modulus of continuity. However, the next theorem manifests its likeness to it.

Theorem 3.21. Let $W^2(t_1, t_2)$, $0 \leq t_1 \leq 1$ and $0 \leq t_2 < \infty$, be a 2-parameter Wiener process. Suppose that c_T is a β_T -function. Let $\hat{h} = (h, \frac{1}{h})$. Then

$$(3.21.1) \quad \lim_{h \rightarrow 0} \sup_{0 < t < (1, 1/h) - \hat{h}} \alpha_{\frac{1}{h}} |W^2(\langle t, t + \hat{h} \rangle)| = 1 \text{ w.p.1.}$$

and

$$(3.21.2) \quad \lim_{h \rightarrow 0} \sup_{0 < t < (1, 1/h) - \hat{h}} \sup_{J \subseteq \langle t, t + \hat{h} \rangle} \alpha_{\frac{1}{h}} |W^2(J)| = 1 \text{ w.p.1.}$$

$$\text{where } \alpha_{\frac{1}{h}} = (2|\hat{h}| [\log \frac{1}{h|\hat{h}}] + \log \log \frac{1}{h})^{-1/2},$$

Obviously, Theorem 3.21 can be also generalized to higher dimensional parameter spaces. That is we also have:

Theorem 3.22. Let $W^d(\underline{t})$, $t_i \in [0, 1]$ for $1 \leq i \leq r \leq d$ and $t_j \in [0, \infty)$ for $r+1 \leq j \leq d$, be a d -parameter Wiener process. Suppose that $a_T^{(j)}$, $r+1 \leq j \leq d$, are β_T -functions and $(1, \dots, 1, a_T^{(d+1)}, \dots, a_T^{(d)})$ is a d - β_T -function. Let

$$\hat{h} = (h, \dots, h, \frac{1}{h}, \dots, \frac{1}{h})^{(r+1, \dots, d)} \text{ and } \underline{e}_h = (1, \dots, 1, \frac{1}{h}, \dots, \frac{1}{h}). \text{ Then}$$

$$(3.22.1) \quad \lim_{h \rightarrow 0} \sup_{0 < t < \hat{a}_h - \hat{h}} \alpha_{\frac{1}{h}} |W^d(\langle \underline{t}, t+\hat{h} \rangle)| = 1 \text{ w.p.1}$$

and

$$(3.22.2) \quad \lim_{h \rightarrow 0} \sup_{0 < t < \hat{a}_h - \hat{h}} \sup_{J \subseteq \langle \underline{t}, t+\hat{h} \rangle} \alpha_{\frac{1}{h}} |W^d(J)| = 1 \text{ w.p.1}$$

where $\alpha_{\frac{1}{h}} = (2|\hat{h}| [\log(h^{r-d}|\hat{h}|^{-1}) + \log \log \frac{1}{h}])^{-\frac{1}{2}}$.

Remark. In the above two theorems, we only have the case of $\lim$. This is so, for when we take $a_T^{(1)} = 1$, then it satisfies (3.16.3) and thus the case of $\overline{\lim}$ is redundant.

The proof of Theorem 3.19 follows. The proofs of Theorems 3.20, 3.21 and 3.22 are similar to that of Theorem 3.19. First, we prove the following lemma.

Lemma 3.23. Let $\underline{a}_T$, $\hat{\underline{a}}_T$ and α_T be defined as in Theorem 3.19. Let us put

$$(3.23.1) \quad V(T) = \sup_{0 < x < (1,T) - \hat{\underline{a}}_T} \sup_{J \subseteq \langle \underline{x}, x+\hat{\delta}_T \rangle} |W^2(J)|.$$

Then, we have

$$(3.23.2) \quad \overline{\lim}_{T \rightarrow \infty} \alpha_T V(T) \leq 1 \text{ w.p.1.}$$

Proof. By Lemma 3.18,

$$\begin{aligned}
 & P\{\alpha_T V(T) > 1 + \varepsilon\} \\
 & \leq C \frac{T}{|a_T|} \exp\left\{-\frac{2(1+\varepsilon)^2}{2+2\varepsilon} \log\left(\frac{T}{|a_T|} \log T\right)\right\} \\
 & \leq C \left(\frac{|a_T|}{T^2}\right)^\varepsilon \left(\frac{1}{\log T}\right)^{1+\varepsilon} \\
 & = C \left(\frac{|a_T|}{T^2}\right)^\varepsilon \left(\frac{1}{\log T}\right)^{1+\varepsilon}.
 \end{aligned}$$

Let $\theta = 1 + \varepsilon$ and $T_k = \theta^k$. Then we have

$$\sum_k P\{\alpha_{T_k} V(T_k) > 1 + \varepsilon\} < \infty$$

and thus, by the Borel-Cantelli Lemma, for every $\varepsilon > 0$,

$$\lim_{k \rightarrow \infty} \alpha_{T_k} V(T_k) \leq 1 + \varepsilon \quad \text{w.p.1.}$$

Since $\frac{\alpha_T}{\alpha_{T_{k+1}}} \leq \theta$ and $V(T) < V(T_{k+1})$ if $T_k < T < T_{k+1}$,

$$\alpha_T V(T) \leq \frac{\alpha_T}{\alpha_{T_{k+1}}} \alpha_{T_{k+1}} V(T_{k+1}) \leq (1 + \varepsilon)^2$$

for every $\varepsilon > 0$ and for k large. Since $\varepsilon > 0$ is arbitrary, this proves our Lemma.

Proof of Theorem 3.19. Define

$$\Gamma(T) = \sup_{0 < x < (1, T) - \hat{a}_T} \alpha_T |W^2(\underline{x}, \underline{x} + \hat{a}_T)|.$$

To prove (3.19.1) and (3.19.2) of the first part of Theorem 3.19, as

consequence of Lemma 3.23., it suffices to show that

$$\overline{\lim}_{T \rightarrow \infty} \Gamma(T) \geq 1 \quad \text{w.p.1.}$$

For each T , let Z_T be the collection of intervals J of the form $J = \langle \underline{x}, \underline{y} \rangle$ where

$$\underline{x} = (1 - n_1 a_T^{(1)}/T, T - n_2 a_T^{(2)})$$

$$\underline{y} = \underline{x} + (a_T^{(1)}/T, a_T^{(2)})$$

where $n_i = 1, 2, \dots, [T/a_T^{(i)}] - 1$, $i = 1, 2$ and $n_1 \wedge n_2 = 1$.

Choosing T_k appropriately as in the proof of Theorem 3.16 and also defining Z_k as in the proof of Theorem 3.16, we have

$$P\left\{ \max_{J \in Z_{T_k}} \alpha_{T_k} |W^2(J)| > 1 - \epsilon \right\} = P\left\{ \max_{I \in Z_k} \alpha_{T_k} |W^2(I)| > 1 - \epsilon \right\} = P\{B_k > 1 - \epsilon\},$$

where the $W^2(\cdot)$ in the second term stands for the Wiener process on $[0, \infty)^2$. Consequently, $\sum P\left\{ \max_{J \in Z_{T_k}} \alpha_{T_k} |W^2(J)| > 1 - \epsilon \right\} = \infty$ which, in turn, renders that

$$\overline{\lim}_{T \rightarrow \infty} \Gamma(T) \geq \overline{\lim}_{k \rightarrow \infty} \max_{J \in Z_{T_k}} \alpha_{T_k} |W^2(J)| \geq 1 \quad \text{w.p.1.}$$

The second part of Theorem 3.19 follows from observing that if $a_T^{(1)}$ satisfies (3.16.3) of Theorem 3.16, then

$$\sum \exp \left\{ - \left(\frac{n^2}{|a_n|} \right) \left(\frac{1}{\log n} \right)^{1-\epsilon} \right\} < \infty.$$

This, of course, means

$$\sum P\{\Gamma(n) < 1 - \epsilon\} < \infty,$$

and the Borel-Cantelli lemma gives us the conclusion of part two of Theorem 3.19, and hence the result.

IV. On the increment of a Kiefer process.

Roughly speaking, a $(d+1)$ -parameter Kiefer process is a Gaussian process such that if we fix the first d parameter, it is a 1-parameter Wiener process, and if we fix the $(d+1)^{\text{st}}$ parameter, it is a d -parameter Brownian Bridge. A precise definition follows:

Definition 4.1. A $(d+1)$ -parameter Kiefer process $K(\underline{x}, t)$, $\underline{x} \in [0,1]^d$ and $t \geq 0$, is a separable Gaussian process with covariance structure

$$(4.1.1) \quad EK(\underline{x}, t) K(\underline{y}, s) = t \wedge s (\prod_{i=1}^d x_i \wedge y_i - x_1 \dots x_d y_1 \dots y_d) .$$

for all $(\underline{x}, t), (\underline{y}, s) \in [0,1]^d \times [0, \infty)$.

We can also write

$$(4.1.2) \quad K(\underline{x}, t) = W^{d+1}(\underline{x}, t) - |\underline{x}| W^{d+1}(\underline{1}, t)$$

for all $(\underline{x}, t) \in [0,1]^d \times [0, \infty)$, where $W^{d+1}(\underline{x}, t) = W(x_1, x_2, \dots, x_d, t)$ is a $(d+1)$ -parameter Wiener process. In this case, if I is an interval in $[0,1]^d \times [0, \infty)$, say

$$I = \langle \underline{x}, \underline{y} \rangle \times \langle s, t \rangle ,$$

a Δ increment of K will be written as $\Delta K(I)$ and defined as follows:

$$\begin{aligned} (4.1.3) \quad \Delta K(I) &= K(\langle \underline{x}, \underline{y} \rangle, t) - K(\langle \underline{x}, \underline{y} \rangle, s) \\ &= W^{d+1}(\langle \underline{x}, \underline{y} \rangle, t) - |\underline{y} - \underline{x}| W^{d+1}(\underline{1}, t) \\ &\quad - W^{d+1}(\langle \underline{x}, \underline{y} \rangle, s) + |\underline{y} - \underline{x}| W^{d+1}(\underline{1}, s) , \end{aligned}$$

where $w^{d+1}(\underline{x}, \underline{y}, t)$ is defined in a similar way as

$w^{d+1}(\underline{x}, \underline{y})$ is in

$$\begin{aligned} w^{d+1}(\underline{x}, \underline{y}, t) &= w^{d+1}(\underline{y}, t) + (-1) \sum_{i=1}^d w^{d+1}(y_1, \dots, x_i, \dots, y_d, t) \\ &\quad \vdots \\ &\quad + (-1)^d w^{d+1}(\underline{x}, t). \end{aligned}$$

Kiefer processes play a prominent role in a number of strong invariance principles for the multivariate empirical process. The following theorem sums up its importance.

Theorem 4.2. One can construct a probability space $(\Omega, \mathcal{A}, P)$

with a sequence of independent random variables uniformly distributed over I^d , a Kiefer process $K(\cdot, \cdot)$ such that

$$(4.2.1) \quad \sup_{\underline{x}} |n^{1/2} \alpha_n(\underline{x}) - K(\underline{x}, n)| \stackrel{a.s.}{=} o(r(n)),$$

with $r(n) = \log^2 n$, if $d = 1$; and $r(n) = n^{\frac{d+1}{2(d+2)}} \log^2 n$, for all $d > 1$.

In the above theorem, the case $d = 1$ is due to Komlós-Major-Tusnády [18] (also cf. Kiefer [16]) and, for the case $d > 1$, it is due to Csörgő-Révész [8].

In this chapter, we shall study some increment properties of a $(d+1)$ -parameter Kiefer process. In §4.1., we examine the increment properties of the Wiener part of a Kiefer process (the $d+1^{\text{st}}$ parameter). These results are due to Csörgő and Chan [6].

In §4.2, we discuss the Δ increment properties of a $(d+1)$ -parameter Kiefer process.

4.1. The increments of the Wiener part of a Kiefer process.

A $(d+1)$ -parameter Kiefer process $K(\underline{x}, t)$ becomes a Wiener process, if we fix $\underline{x}$. In this section, we study the increments of the Wiener part while taking the supremum over all $\underline{x} \in \langle 0, 1 \rangle$.

Theorem 4.3. Let a_T be a β_T -function. Then

$$(4.3.1) \quad \lim_{T \rightarrow \infty} \sup_{0 \leq t \leq T-a_T} \sup_{\underline{x} \in [0,1]^d} \tilde{\alpha}_T |K(\underline{x}, t+a_T) - K(\underline{x}, t)| = 1 \text{ w.p.1}$$

and

$$(4.3.2) \quad \lim_{T \rightarrow \infty} \sup_{0 \leq t \leq T-a_T} \sup_{0 \leq s \leq a_T} \sup_{\underline{x} \in [0,1]^d} \tilde{\alpha}_T |K(\underline{x}, t+s) - K(\underline{x}, t)| = 1, \text{ w.p.1}$$

where $\tilde{\alpha}_T = (\frac{1}{2} a_T [\log \frac{T}{a_T} + \log \log T])^{-1/2}$. If we also have (2.16.3), i.e.

$$(4.3.3) \quad \frac{\log \frac{T}{a_T}}{\log \log T} \rightarrow \infty \text{ as } T \rightarrow \infty,$$

then

$$(4.3.4) \quad \lim_{T \rightarrow \infty} \sup_{0 \leq t \leq T-a_T} \sup_{\underline{x} \in [0,1]^d} \tilde{\alpha}_T |K(\underline{x}, t+a_T) - K(\underline{x}, t)| = 1 \text{ w.p.1,}$$

and

$$(4.3.5) \quad \lim_{T \rightarrow \infty} \sup_{0 \leq t \leq T-a_T} \sup_{0 \leq s \leq a_T} \sup_{\underline{x} \in [0,1]^d} \widetilde{a}_T |K(\underline{x}, t+s) - K(\underline{x}, t)| = 1 \text{ w.p.1.}$$

If we put $a_T = T$, then we have:

Corollary 4.4. One has

$$(4.4.1) \quad \lim_{T \rightarrow \infty} \sup_{0 \leq t \leq T} \sup_{0 \leq x \leq 1} \frac{|K(\underline{x}, t)|}{\sqrt{2T \log \log T}} = 1,$$

with probability one.

And, if we put $a_T = [\log T]$, then we have:

Corollary 4.5. One has

$$(4.5.1) \quad \lim_{T \rightarrow \infty} \sup_{0 \leq t \leq T - [\log T]} \sup_{0 \leq x \leq 1} \frac{|K(\underline{x}, t + [\log T]) - K(\underline{x}, t)|}{[\log T]} = 1$$

with probability one.

Before we prove Theorem 4.3, let us first prove some lemmas.

Lemma 4.6. For each $\varepsilon > 0$, there is a constant $C_1 = C_1(\varepsilon, d)$ such that

$$(4.6.1) \quad P\left\{\sup_{\underline{x} \in [0,1]^d} (K(\underline{x}, t+h) - K(\underline{x}, t)) > \lambda\right\} \leq C_1 e^{-(2-\varepsilon)\lambda^2/h},$$

for every $\lambda > 0$, $h > 0$ and $t \geq 0$. Consequently,

$$(4.6.2) \quad P\left\{\sup_{\underline{x} \in [0,1]^d} |K(\underline{x}, t+h) - K(\underline{x}, t)| > \lambda \sqrt{h}\right\} \leq C e^{-(2-\varepsilon)\lambda^2},$$

where $C = 2C_1$.

Proof. Observe that for each pair of positive numbers t and h , we have

$$K(\underline{x}, t+h) - K(\underline{x}, t) \stackrel{D}{=} K(\underline{x}, h) \stackrel{D}{=} \sqrt{h} \cdot B(\underline{x})$$

jointly for all $\underline{x} \in [0,1]^d$, where $B(\underline{x})$ is a d -parameter Brownian bridge, as in Definition 3.6. Then (4.6.1) and (4.6.2) follow from Proposition 3.9. This also completes our proof of Lemma 4.6.

Lemma 4.7. For each $\varepsilon > 0$, there is a constant $C_2 = C_2(\varepsilon, d) > 0$ such that:

$$(4.7.1) \quad P\left\{ \sup_{0 \leq s \leq h} \sup_{\underline{x} \in [0,1]^d} |K(\underline{x}, t+s) - K(\underline{x}, t)| > \lambda \sqrt{h} \right\} \leq C_2 e^{-(2-\varepsilon)\lambda^2}$$

for every $h > 0$, $\lambda > 0$ and $t \geq 0$.

Proof. Since $K(\underline{x}, t)$ is stationary, independent increment process in t , it suffices to show that

$$(4.7.2) \quad P\left\{ \sup_{0 \leq s \leq h} \sup_{\underline{x} \in [0,1]^d} |K(\underline{x}, s)| > \lambda \sqrt{h} \right\} \leq C_2 e^{-(2-\varepsilon)\lambda^2}$$

Also, $K(\underline{x}, t)$ is a separable process. Hence, the countable dense set $\mathcal{D} = \mathcal{U}_n$, where

$$\mathcal{U}_n = \left\{ \left(\frac{m_1}{2^n}, \dots, \frac{m_d}{2^n}, \frac{hm_{d+1}}{2^n} \right) : 0 \leq m_i \leq 2^n, 1 \leq i \leq d+1 \right\}$$

completely determines $K(\cdot, \cdot)$ on $[0,1]^d \times [0,h]$. For each fixed n , let us put $m = 1 + m_1 + m_2 2^n + \dots + m_{d+1} 2^{nd}$, for $0 \leq m_i \leq 2^n$ and $1 \leq i \leq d+1$. Define

$$K_m = K \left(\frac{m_1}{2^n}, \dots, \frac{m_d}{2^n}, \frac{hm_{d+1}}{2^n} \right), K'_m = K \left(\frac{m_1}{2^n}, \dots, \frac{m_d}{2^n}, h \right),$$

and

$$E_m = \{K_m > \lambda; K'_m \leq \lambda, \text{ for } m' < m\}, \quad m = 1, 2, \dots, 2^{n(d+1)}.$$

Clearly, E_m 's are disjoint and

$$\bigcup_m E_m = \left\{ \max_{(x,t) \in \mathcal{D}_n} K(\underline{x}, t) > \lambda \right\}.$$

Consider the event

$$B_n = \{K_m > \lambda, m_{d+1} = 2^n\} = \{K'_m > \lambda\}.$$

Then, because $K'_m - K_m$ is symmetric and is independent of E_m ,

$$P\{E_m B_n\} \leq P\{E_m, K'_m - K_m < 0\} = \frac{1}{2} P\{E_m\}.$$

Thus

$$P\{E_m\} \leq 2P\{E_m B_n\}. \quad \text{Since } UE_m \supset B_n,$$

we have $P\{UE_m\} \leq 2P\{B_n\}$. Taking $n \rightarrow \infty$, we get, for every $\lambda > 0$

$$P\left\{ \sup_{0 \leq s \leq h} \sup_{\underline{x} \in [0,1]^d} K(\underline{x}, s) > \lambda \right\} \leq 2P\left\{ \sup_{\underline{x} \in [0,1]^d} K(\underline{x}, h) > \lambda \right\}.$$

Now, (4.7.2) follows from Lemma 4.6. This completes our proof of Lemma 4.7.

Lemma 4.8. For each small $\epsilon > 0$, there is a $C = C(\epsilon, d) > 0$

such that

$$(4.8.1) \quad P\left\{ \sup_{0 \leq t \leq T} \sup_{0 \leq s \leq h} \sup_{\underline{x} \in [0,1]^d} |K(\underline{x}, t+s) - K(\underline{x}, t)| > \lambda h^{\frac{1}{2}} \right\} \leq C \frac{T}{h} e^{-(2-\epsilon)\lambda^2}$$

for every $\lambda > 0$ and $h > 0$.

Proof: Since $K(\underline{x}, t) \stackrel{D}{=} \sqrt{t} K(\underline{x}, \frac{t}{T})$ jointly in $\underline{x} \in [0, 1]^d$ and

$0 \leq t \leq T$, we need only to verify (4.8.1) in the case when $T = 1$.

Let R be the smallest integer such that $\frac{1}{R} \leq \frac{\varepsilon^2 h}{4}$ and m be the

smallest integer such that $h \leq \frac{m}{k} \leq h + \frac{1}{k} \leq h(1 + \frac{\varepsilon^2}{4}) \leq h(1 + \frac{\varepsilon}{2})$.

Then

$$\sup_{0 \leq t \leq 1} \sup_{0 \leq s \leq h} \sup_{\underline{x} \in [0, 1]^d} |K(\underline{x}, t+s) - K(\underline{x}, t)| \leq \max_{0 \leq n \leq R} \sup_{1 \leq s \leq \frac{m}{R}} \sup_{\underline{x} \in [0, 1]^d}$$

$$|K(\underline{x}, \frac{n}{R} + s) - K(\underline{x}, \frac{n}{R})|.$$

Let $\varepsilon' = \frac{\varepsilon^2}{2}$. Since $\frac{Rh}{m} > \frac{1}{1 + \frac{\varepsilon}{2}}$, we have

$$P\left\{ \sup_{0 \leq t \leq 1} \sup_{0 \leq s \leq h} \sup_{\underline{x} \in [0, 1]^d} |K(\underline{x}, t+s) - K(\underline{x}, t)| > \lambda h^{\frac{1}{2}} \right\}$$

$$\leq \sum_{n=0}^R P\left\{ \sup_{0 \leq s \leq \frac{m}{R}} \sup_{\underline{x} \in [0, 1]^d} |K(\underline{x}, \frac{n}{R} + s) - K(\underline{x}, \frac{n}{R})| > \lambda h^{\frac{1}{2}} \right\}$$

$$\leq (R+1) C_2(\varepsilon', d) \exp \left\{ -(2-\varepsilon') \lambda^2 \left[\frac{Rh}{m} \right] \right\}$$

$$\leq (R+1) C_2 \exp \left\{ - \frac{(2-\varepsilon')}{(1 + \frac{\varepsilon}{2})} \lambda^2 \right\}$$

$$\leq C h^{-1} \exp \left\{ -(2-\varepsilon) \lambda^2 \right\}.$$

The last inequality holds whenever $R > 2$. This completes our proof of Lemma 4.8.

Now, we can prove Theorem 4.3.

Proof of Theorem 4.3. Let $\varepsilon > 0$ be arbitrary and define

$$A(T) = \sup_{0 \leq t \leq T-a_T} \sup_{0 \leq s \leq T-a_T} \sup_{\underline{x} \in [0,1]^d} \tilde{\alpha}_T |K(\underline{x}, t+s) - K(\underline{x}, t)|.$$

Choose $0 < \varepsilon' < 2(1 - \frac{1}{(1+\varepsilon)^2})$. Then $(2-\varepsilon')(1+\varepsilon)^2 > 2$ and let

$$(2-\varepsilon')(1+\varepsilon)^2 = 2 + 2\delta, \quad \delta > 0. \quad \text{Thus,}$$

$$\begin{aligned} P\{A(T) > 1+\varepsilon\} &\leq C(\varepsilon', d) \frac{T}{a_T} \exp \left\{ - \log \left(\frac{T \log T}{a_T} \right)^{1+\delta} \right\} \\ &\leq C(\varepsilon', d) \left(\frac{a_T}{T} \right)^\delta \left(\frac{1}{\log T} \right)^{1+\delta}. \end{aligned}$$

Put $\theta = 1 + \varepsilon$ and $T_N = \theta^N$. Then

$$\sum P\{A(T_N) > 1 + \varepsilon\} < \infty$$

and $\overline{\lim} A(T_N) \leq 1 + \varepsilon$ w.p.1.

Observe that $1 \leq \frac{\tilde{\alpha}_{T_{N+1}}}{\tilde{\alpha}_{T_N}} \leq \theta$,

for large N . Since $\tilde{\alpha}_{T_N}^{-1} \leq \tilde{\alpha}_T^{-1} \leq \tilde{\alpha}_{T_{N+1}}^{-1}$ and

$$\tilde{\alpha}_{T_N}^{-1} A(T_N) \leq \tilde{\alpha}_T^{-1} A(T) \leq \tilde{\alpha}_{T_{N+1}}^{-1} A(T_{N+1}),$$

for $T_N \leq T \leq T_{N+1}$, it follows from the above observation that

$$A(T) \leq \frac{\tilde{\alpha}_T}{\tilde{\alpha}_{T_{N+1}}} A(T_{N+1}) \leq (1 + \epsilon)^2,$$

with probability one. This way we get

$$\overline{\lim}_{T \rightarrow \infty} A(T) \leq (1 + \epsilon)^2 \text{ w.p.1.}$$

for every $\epsilon > 0$. This is precisely

$$\overline{\lim}_{T \rightarrow \infty} A(T) \leq 1 \text{ w.p.1.}$$

On the other hand, observe that

$$K(\frac{1}{2}, 1, \dots, 1, t) \stackrel{D}{=} \frac{1}{2} W(t), \quad 0 \leq t < \infty.$$

Also $\frac{1}{2} \beta_T = \tilde{\alpha}_T$, where β_T is defined as in Theorem 2.16. It follows from Theorem 2.16 that

$$\overline{\lim}_{T \rightarrow \infty} \sup_{0 \leq t \leq T - a_T} \tilde{\alpha}_T |K(\frac{1}{2}, 1, \dots, 1, t + a_T) - K(\frac{1}{2}, 1, \dots, 1, t)| \geq 1 \text{ w.p.1.,}$$

and also

$$\lim_{T \rightarrow \infty} \sup_{0 \leq t \leq T - a_T} \tilde{\alpha}_T |K(\frac{1}{2}, 1, \dots, 1, t + a_T) - K(\frac{1}{2}, 1, \dots, 1, t)| \geq 1 \text{ w.p.1.,}$$

whenever (4.3.3) is satisfied. This also completes the proof of Theorem 4.3.

Using a similar argument as in §3.5, we can prove

Theorem 4.9. Let $K(\cdot, \cdot)$ be a Kiefer process as in Theorem 4.3.

Then

$$\lim_{h \rightarrow 0} \sup_{0 \leq t \leq 1-h} \sup_{\underline{x} \in [0,1]^d} \frac{|K(\underline{x}, t+h) - K(\underline{x}, t)|}{\sqrt{h \log h^{-1}}} = \frac{1}{\sqrt{2}} \quad \text{w.p.l.}$$

The above theorem shows that the process $Y(t)$ defined by

$$Y(t) = \sup_{0 \leq x \leq 1} K(\underline{x}, t), \quad t \geq 0,$$

is almost surely a positive valued process whose sample paths are continuous but nowhere differentiable.

4.2. The Δ increments of a Kiefer process.

Recall that a Δ increment (see (4.1.3) of Definition 4.1) of a $(d+1)$ -parameter Kiefer process is an increment over the whole parameter domain, unlike that of the Wiener part in the previous section. In Theorems 3.21 and 3.22, we investigated the increments of $W^2(\underline{x})$ and respectively $W^d(\underline{x})$ such that part of the coordinates of the increments go to zero while the other part is simply non-decreasing. In this section we show that the results of Theorems 3.21 and 3.22 remain true if we substitute K for $W^2(\cdot)$ and $W^d(\cdot)$. More precisely, we have:

Theorem 4.10. Suppose that $K(\underline{x}, t)$, $\underline{x} \in [0,1]^2$ and $t \geq 0$,

is a 2-parameter Kiefer process and that C_T is a β_T -function.

Define

$$(4.10.1) \quad A(h) = \sup_{0 \leq x \leq 1-h} \sup_{0 \leq t \leq T-\frac{1}{h}} \sup_{0 \leq y \leq h} \sup_{0 \leq s \leq \frac{1}{h}} |K(\langle x, x+y \rangle, t+s) - K(\langle x, x+y \rangle, s)|$$

Then, we have

$$(4.10.2) \quad \lim_{h \rightarrow 0} \alpha_{\frac{1}{h}} A(h) = 1 \text{ w.p.l.,}$$

$$\text{where } \alpha_{\frac{1}{h}} = (2hc_{\frac{1}{h}} [\log (h^2 c_{\frac{1}{h}})^{-1} + \log \log h^{-1}])^{-1}.$$

Proof. Define

$$(4.10.3) \quad B(h) = \sup_{0 \leq x \leq 1-h} \sup_{0 \leq t \leq T-c_{\frac{1}{h}}} \sup_{0 \leq y \leq h} \sup_{0 \leq s \leq c_{\frac{1}{h}}} |W^2(\langle x, x+y \rangle, t+s) - W^2(\langle x, x+y \rangle, t)|$$

$$(4.10.4) \quad C(h) = \sup_{0 \leq t \leq T-c_{\frac{1}{h}}} \sup_{0 \leq y \leq h} \sup_{0 \leq s \leq c_{\frac{1}{h}}} |y| |W^2(1, t+s) - W^2(1, t)|.$$

Then we have

$$B(h) - C(h) \leq A(h) \leq B(h) + C(h).$$

By Theorem 3.21., we have $\lim_{h \rightarrow 0} \alpha_{\frac{1}{h}} B(h) = 1$ w.p.l. Thus, to

prove Theorem 4.10, it suffices to show that $\lim_{h \rightarrow 0} \alpha_{\frac{1}{h}} C(h) = 0$ w.p.l.

To see this, let

$$F(t) = \sup_{0 \leq t \leq T-c_T} \sup_{0 \leq s \leq c_T} |W^2(1, t+s) - W^2(1, t)|$$

and

$$\beta_T = (2 c_T [\log \frac{T}{c_T} + \log \log T])^{-1}.$$

Then, by Theorem 2.16, and by the fact that $W^2(1,t) \stackrel{D}{=} W(t)$, $t \geq 0$, we have

$$\overline{\lim}_{T \rightarrow \infty} \beta_T F(T) \leq 1 \quad \text{w.p.1.}$$

or

$$\overline{\lim}_{h \rightarrow 0} \beta_{\frac{1}{h}} F\left(\frac{1}{h}\right) \leq 1 \quad \text{w.p.1.}$$

Observe that $\alpha_T \leq \sqrt{T} \beta_T$, or

$$\alpha_{\frac{1}{h}} \leq \sqrt{\frac{1}{h}} \beta_{\frac{1}{h}}.$$

Thus,

$$\begin{aligned} \alpha_{\frac{1}{h}} C(h) &\leq \sqrt{\frac{1}{h}} \beta_{\frac{1}{h}} F\left(\frac{1}{h}\right) \sup_{0 < y < h} |y| \\ &\leq \sqrt{h} \beta_{\frac{1}{h}} F\left(\frac{1}{h}\right). \end{aligned}$$

This renders that $\alpha_{\frac{1}{h}} C(h) \rightarrow 0$ w.p.1. as $h \rightarrow 0$. This also completes our proof.

The above theorem can be easily generalized to a $(d+1)$ -parameter Kiefer process. First we prove

Lemma 4.11. For $h > 0$, let

$$E(h) = \sup_{0 < y < h} \sup_{0 < t < T - a_T} \sup_{0 < s < a_T} |y| |W^{d+1}(\underline{1}, t+s) - W^{d+1}(\underline{1}, t)|$$

where $\underline{h} = (h, \dots, h)$ and $T = \frac{1}{h}$. Let α_T be defined by

$$a_T = (2 \frac{a_T}{T^d} [\log (\frac{T^{d+1}}{a_T}) + \log \log T])^{-1}.$$

Then

$$(4.11.1) \quad \overline{\lim}_{h \rightarrow 0} \frac{a_1}{h} E(h) = 0 \quad \text{w.p.l.},$$

provided that a_T is a β_T -function.

Proof: Let $\beta_T = (2a_T [\log \frac{T}{a_T} + \log \log T])^{-1}$. Then $a_T \leq T^{\frac{d}{2}} \beta_T$. Define, for each T ,

$$F(T) = \sup_{0 \leq t \leq T-a_T} \sup_{0 \leq s \leq a_T} |W^{d+1}(\underline{1}, t+s) - W^{d+1}(\underline{1}, t)|.$$

By Theorem 2.16. we have

$$\overline{\lim}_{T \rightarrow \infty} \beta_T F(T) \leq 1 \quad \text{w.p.l.}$$

$$\text{Since } \frac{a_1}{h} E(h) \leq \frac{|h|}{h} \frac{a_1}{h} F\left(\frac{1}{h}\right) \leq \frac{|h|}{h} \left(\frac{1}{h}\right)^{d/2} \beta_{\frac{1}{h}} F\left(\frac{1}{h}\right) \leq |h|^{\frac{1}{2}} \beta_{\frac{1}{h}} F\left(\frac{1}{h}\right),$$

we have

$$\overline{\lim}_{h \rightarrow 0} \frac{a_1}{h} E(h) = 0 \quad \text{w.p.l.},$$

and Lemma 4.11. is proved.

Using Theorem 3.22 and Lemma 4.11 with a similar argument as in the proof of Theorem 4.10, we can prove that

Theorem 4.12. Let $\underline{h} = (h, \dots, h) > \underline{0} \in [0, 1]^d$ and C_T is a β_T -function. Define

$$(4.12.1) \quad D(T) = \sup_{\underline{0} < \underline{x} < \underline{1} - \underline{h}} \sup_{\underline{0} < \underline{t} < T - C_T} \sup_{\underline{0} < \underline{y} < \underline{h}} \sup_{\underline{0} < \underline{s} < C_T} |K(\langle \underline{x}, \underline{x} + \underline{y} \rangle, t+s) - K(\langle \underline{x}, \underline{x} + \underline{y} \rangle, t)|$$

where $T = \frac{1}{h}$. Let $\alpha_{\frac{1}{h}} = (2h^d C_{\frac{1}{h}} [\log (h^d C_{\frac{1}{h}})^{-1} + \log \log \frac{1}{h}])^{-1}$.

Then we have

$$(4.12.2) \quad \lim_{h \rightarrow 0} \alpha_{\frac{1}{h}} D(h) = 1 \text{ w.p.1.}$$

Proof. Observe that

$$B(h) - E(h) \leq D(h) \leq B(h) + E(h)$$

where $E(h)$ is defined as in Lemma 4.11 and

$$B(h) = \sup_{\underline{0} < \underline{x} < \underline{1} - \underline{h}} \sup_{\underline{0} < \underline{y} < \underline{h}} \sup_{\underline{0} < \underline{t} < \frac{1}{h} - C_{\frac{1}{h}}} \sup_{\underline{0} < \underline{s} < C_{\frac{1}{h}}} |W^{d+1}(\langle \underline{x}, \underline{x} + \underline{y} \rangle, t+s) - W^{d+1}(\langle \underline{x}, \underline{x} + \underline{y} \rangle, t)|.$$

Theorem 3.22. and Lemma 4.11. imply that $\lim_{h \rightarrow 0} \alpha_{\frac{1}{h}} B(h) = 1$ w.p.1.

and $\lim_{h \rightarrow 0} \alpha_{\frac{1}{h}} E(h) = 0$ w.p.1. respectively. This completes our proof.

4.3. Further discussions on the increments of a Kiefer process.

We observe that the $\alpha_{\frac{1}{h}}$ in Theorems 4.10 and 4.12 can be replaced by

$$\gamma_h = (2v(h) [\log \frac{h^{-1}}{v(h)} + \log \log h^{-1}])^{-1/2}, \text{ where } v(h) = h^d (1-h^d) c_{\frac{1}{h}},$$

$d \geq 1$, and then one would have the variance of the considered increments for the Kiefer process exactly the same way as that of the Wiener process in Theorems 3.19 and 3.20.

Statements fashioned after Theorems 3.19 and 3.20 are also true for the Kiefer process. In fact, in this latter context, we can also consider a Kiefer process whose t -parameter is also multi-dimensional. For example, such a Kiefer process could be defined by

$$(**) K(\underline{x}, \underline{t}) = W^{d+r}(\underline{x}_1, \dots, \underline{x}_d, t_1, \dots, t_r) = x_1 x_2 \dots x_d W^{d+r}(1, \dots, 1, t_1, \dots, t_r).$$

Then, mutatis mutandis, Theorem 3.20 holds for $K(\underline{x}, \underline{t})$ defined as above and this, in turn, results in a corresponding new version of Theorem 4.12.

As to the question of constructing a Kiefer process as that of (**), we can have the following procedure: Let the time parameter set T^r consist of all lattice points with positive coordinates in the d -dimensional Euclidean space R^r . Let

$x_{t_1}, \dots, x_{t_r}$ ($t_1 = 1, 2, \dots, t_r = 1, 2, \dots$) be independent random variables, uniformly distributed over the unit cube I^d of the d -dimensional

Euclidean space R^d . Let $\underline{N} = (n_1, \dots, n_r) \in T^r$ and $\underline{x} \in I^d$.

Define the empirical distribution function

$$F_{|\underline{N}|}(\underline{x}) = \frac{\# \underline{x}_{t_1, \dots, t_r} \leq \underline{x} : t_1 \leq n_1, \dots, t_r \leq n_r}{|\underline{N}|}$$

and the empirical process

$$\alpha_{|\underline{N}|}(\underline{x}) = |\underline{N}|^{1/2} (F_{|\underline{N}|}(\underline{x}) - |\underline{x}|).$$

Using now the method of proof of [7] and [8] one can prove the following

Theorem 4.13. Given a monotone sequence $\underline{N}_1 < \underline{N}_2 < \dots$, $\underline{N}_1 \in T^r$, one can construct a probability space with a set $\underline{X}_{\underline{N}}$ of random variables uniformly distributed over I^d and a Kiefer process $K(\cdot, \cdot)$ as in (**) such that

$$(4.13.1) \quad \sup_{\underline{x} \in I^d} \left| |\underline{N}_k|^{1/2} \alpha_{|\underline{N}_k|}(\underline{x}) - K(\underline{x}_1, \dots, \underline{x}_d, n_{k1}, \dots, n_{kr}) \right| \\ = o(f(|\underline{N}_k|))$$

with probability one as $k \rightarrow \infty$, where $f(|\underline{N}_k|) = |\underline{N}_k|^{\frac{d+1}{2(d+2)}} \log^2 |\underline{N}_k|$ for any $d = 1, 2, \dots$.

For this kind of a Kiefer process statements similar to Theorems 3.19 and 3.20 hold. We state one here.

Theorem 4.14. Let $\underline{a}_T = (a_T^{(1)}, \dots, a_T^{(d)}, a_T)$ be a $(d+1)$ - β_T -function and let the $a_T^{(i)}/T \rightarrow 0$, as $T \rightarrow \infty$, $1 \leq i \leq d$. Define

$$\hat{\underline{a}}_T = \left(\frac{a_T^{(1)}}{T}, \dots, \frac{a_T^{(d)}}{T} \right) \quad \text{and} \quad |\hat{\underline{a}}_T| = \frac{a_T^{(1)} \dots a_T^{(d)}}{T^d}. \quad \text{Let}$$

$$\gamma_T = (2a_T |\hat{a}_T| (1 - |\hat{a}_T|) [\log \frac{T}{a_T} \frac{1}{|\hat{a}_T| (1 - |\hat{a}_T|)} + \log \log T])^{-1/2}.$$

Then we have

$$(4.14.1) \quad \overline{\lim}_{T \rightarrow \infty} \sup_{0 \leq t \leq T - a_T} \sup_{0 \leq x \leq 1 - \hat{a}_T} \sup_{J \subseteq \langle x, x + \hat{a}_T \rangle} \gamma_T |K(J, t + a_T) - K(J, t)| = 1$$

and

$$(4.14.2) \quad \overline{\lim}_{T \rightarrow \infty} \sup_{0 \leq t \leq T - a_T} \sup_{0 \leq s \leq a_T} \sup_{0 \leq x \leq 1 - \hat{a}_T} \sup_{J \subseteq \langle x, x + \hat{a}_T \rangle} \gamma_T |K(J, t + s) - K(J, t)| = 1$$

with probability one. If at least one of $a_T^{(1)}, \dots, a_T^{(d)}, a_T$ also satisfies (2.16.3) of Theorem 2.16, then the $\overline{\lim}$ in (4.14.1) and (4.14.2) can be replaced by $\lim$.

Remark. Theorems 4.3 and 4.14 are also useful to study the multivariate empirical process when combined with a theorem like Theorem 4.13 above.

Naturally, one could replace the $(1 - |\hat{a}_T|)$ -factors by 1 in the definition of γ_T of Theorem 4.14, without affecting its statements. It is more informative, however, to define γ_T the way we did above, for $a_T |\hat{a}_T| (1 - |\hat{a}_T|)$ is the variance of the considered increment of the $(d+1)$ -parameter Kiefer process involved.

It is also of interest to note that though Theorem 4.14 is somewhat more general and perhaps more interesting than Theorem 4.3 one cannot get the latter directly from the former via simply letting $\hat{a}_T = 1$, for then $\gamma_T = \infty$ and Theorem 4.14 becomes meaningless. In

this context, the factor $|\hat{a}_T|(1 - |\hat{a}_T|)$ in the definition of γ_T is helpful, however, in heuristically explaining the factor $\frac{1}{2}$ in the definition of α_T of Theorem 4.3. Thus as they stand, Theorem 4.3 and 4.14 are two separate theorems and it would, perhaps, be of some interest to find out whether they could be combined into one.

SUMMARY

We have studied increment properties of a multi-parameter Wiener process and those of a Kiefer process. In Chapter 3, we studied the increments over rectangles and all possible rates. Likewise, in Chapter 4, the same was investigated for the Kiefer process. However, there are still several problems of some interest related to these increment properties. We state a few here.

1. Recall that $W^d(\cdot)$ is the limiting process of the partial sum process $Z_n(\cdot)$ (cf. Theorem 3.5 of Chapter 3) from a d -dimensional uniform distribution U . Suppose that $\tilde{Z}_n(\cdot)$ is the empirical distribution process from a distribution F on R^d and $F \neq U$.

Révész in [30], for $d = 2$, and for $d > 1$, Csörgő and Révész in [5] show that $\tilde{Z}_n$ converges to a Gaussian process whose covariance structure is a function of F . This is quite different from the one-parameter case where $K(\cdot)$ does not depend on the distribution. For this kind of a Gaussian process it is very difficult even to define increments. An interesting problem will be to define increments for this kind of process and then to prove results similar to those as in Chapters 3 and 4.

2. It would be of great interest to prove a Strassen type theorem for the relevant part of Theorem 3.16, or even for that of Theorem 2.16.

3. The condition (3.16.3) of Theorem 3.16 is sufficient for changing the $\overline{\lim}$ to $\lim$. Is this also necessary? If not, what is the weakest condition?

REFERENCES

- [1] Billingsley, P. (1968). "Convergence of Probability Measures."
John Wiley, New York.
- [2] Chan, Arthur H. C. (1975). Some lower bounds for the supremum
of the Yeh-Wiener process over a rectangular region. J.
Appl. Prob., 12, 824-830.
- [3] Chan, Arthur H. C. (1976). Erdős-Rényi type modulus of continuity
for Brownian sheets. To appear in Studia Sci. Math. Hungar.
- [4] Chentson, N. N. (1956). Wiener random fields depending on
several parameters. Dokl. Akad. Nauk SSSR (N.S.) 106,
607-609.
- [5] Chung, K. L. (1974). "A Course in Probability Theory, Second
Edition, Academic Press, New York.
- [6] Csörgö, M. and Chan, A. H. C. (1976). On the Erdős-Rényi
increments and P. Lévy modulus of continuity of a Kiefer
process. To appear in the Springer-Verlag Empirical Process
Lecture Note - Volume.
- [7] Csörgö, M. and Révész, P. (1975). A new method to prove
Strassen type laws of invariance principle II. Z.
Wahrscheinlichkeitstheorie verw. Geb. 31 261-269.
- [8] _____ (1976). A strong approximation of the multivariate
empirical process. To appear in Studia Sci. Math. Hungar.
- [9] _____ (1976). How big are the increments of a Wiener process.
To appear.

- [10] Doob, J. L. (1949). Heuristic approach to the Kolmogorov-Smirnov theorems. Ann. Math. Statist. 20, 393-403.
- [11] _____ (1953). "Stochastic Process", John Wiley, New York.
- [12] Dvoretzky, A., Erdős, P. and Kakutani, S. (1961). Nonincreasing everywhere of the Brownian motion process, 4th BSMSP 2, 103-116.
- [13] Erdős, P. and Rényi, A. (1970). On a new law of large numbers. J. d'Analyse Mathématique 13, 103-111.
- [14] Khintchine, A. (1924). Über einen satz der Wahrscheinlichkeitsrechnung. Fundamenta Math. 6, 9-20.
- [15] Kiefer, J. (1961). On large deviations of vector chance variables and a law of iterated logarithm. Pacific J. Math. 11, 649-660.
- [16] _____ (1972). Skorohod embedding of multivariate RV's and the sample DF. Z. Wahrscheinlichkeitstheorie verw. Geb. 24, 1-35.
- [17] Komlós, J., Major, P. and Tusnády, G. (1975). An approximation of partial sums of independent RV's, and the sample DF. I. Z. Wahrscheinlichkeitstheorie verw. Geb. 32, 111-132.
- [18] _____ (1975). An approximation of partial sums of independent RV's, and the sample DF. II. Z. Wahrscheinlichkeitstheorie verw. Geb. 34, 33-58.

- [19] Lai, T. L. (1974). Limiting theorems for delayed sums.
Ann. Probability 3, 432-440.
- [20] Lévy, P. (1937). Theorie de l'addition des variables
aleatoires independantes, Gauthier-Villars, Paris,
- [21] Major, P. (1976). The approximation of partial sums of
independent r.v.'s. To appear.
- [22] Muller, D. W. (1970). On Glivenko-Cantelli convergence.
Z. Wahrscheinlichkeitstheorie verw. Geb. 16, 195-210.
- [23] Neveu, J. (1965). "Mathematical Foundations of Calculus of
Probability" (translated by A. Feinstein). Holden-Day,
San Francisco.
- [24] Orey, S. and Pruitt, W. E. (1973). Sample functions of the
N-parameter Wiener process. Ann. of Prob., 1, 138-163.
- [25] Orey, S. and Taylor, S. J. (1974). How often on a Brownian
path does the law of iterated logarithm fail? Proc. London
Math. Soc. (3), 28, 174-192.
- [26] Paranjape, S. R. and Park, C. (1973). Distribution of the
supremum of the two-parameter Yeh-Wiener process on the
boundary. J. Appl. Prob. 10, 875-880.
- [27] Park, W. J. (1970). A multi-parameter Gaussian process.
Ann. Math. Statist. 41, 1582-1595.
- [28] Pyke, J. (1973). Partial Sums of Matrix Arrays, and Brownian
sheets. Stochastic Analysis. Edited by Kendall and Harding,
John Wiley, New York.

[29] Rényi, A. (1970). ^uFoundations of Probability.ⁿ Holden-Day, San Francisco.

[30] Révész, P. (1976). Three theorems on the multivariate empirical process. To appear in the Springer-Verlag Empirical Process-Lecture Note-volume.

[31] Révész, P. (1976). On strong approximation of multidimensional empirical process. Ann. of Prob., 4, 729-743.

[32] Wichura, M. J. (1973). Some Strassen-type laws of the iterated logarithm for multiparameter stochastic processes with independent increments. Ann. of Prob., 1, 272-296.

[33] Wiener, N. (1923). Differential space, J. Math. Physics. MIT 2, 131-174.

[34] Yeh, J. (1960). Wiener measure in the space of functions of two variables. Trans. Am. Math. Soc. 95, 433-450.

END

04

111

77

FIN