1

National Library
of Canada

Bibliothèque nationale
du Canada

CANADIAN THESES
ON MICROFICHE

THÈSES CANADIENNES
SUR MICROFICHE

0-315-01655-8

49512

NAME OF AUTHOR/NOM DE L'AUTEUR JUNG HONG KIM
TITLE OF THESIS/TITRE DE LA THÈSE RADIATIVE TAU DECAY AND PION STRUCTURE FORM FACTORS
UNIVERSITY/UNIVERSITÉ CARLETON UNIVERSITY
DEGREE FOR WHICH THESIS WAS PRESENTED/
GRADE POUR LEQUEL CETTE THÈSE FUT PRÉSENTÉE PH. D.
YEAR THIS DEGREE CONFERRED/ANNÉE D'OBTENTION DE CE DEGRÉ 1980
NAME OF SUPERVISOR/NOM DU DIRECTEUR DE THÈSE LAZER RESNICK

Permission is hereby granted to the NATIONAL LIBRARY OF
CANADA to microfilm this thesis and to lend or sell copies
of the film.

L'autorisation est, par la présente, accordée à la BIBLIOTHÈ-
QUE NATIONALE DU CANADA de microfilmer cette thèse et
de prêter ou de vendre des exemplaires du film.

The author reserves other publication rights, and neither the
thesis nor extensive extracts from it may be printed or other-
wise reproduced without the author's written permission.

L'auteur se réserve les autres droits de publication; ni la
thèse ni de longs extraits de celle-ci ne doivent être imprimés
ou autrement reproduits sans l'autorisation écrite de l'auteur.

DATED/DATE Dec. 17, 1979 SIGNED/SIGNÉ J. H. Kim

PERMANENT ADDRESS/RÉSIDENCE FIXE PHYSICS DEPARTMENT
JEONBUK NATIONAL UNIVERSITY
JEONJU, JEONBUK 520, KOREA

National Library of Canada
Collections Development Branch

Canadian Theses on
Microfiche Service

Bibliothèque nationale du Canada
Direction du développement des collections

Service des thèses canadiennes
sur microfiche

NOTICE

The quality of this microfiche is heavily dependent upon the quality of the original thesis submitted for microfilming. Every effort has been made to ensure the highest quality of reproduction possible.

If pages are missing, contact the university which granted the degree.

Some pages may have indistinct print especially if the original pages were typed with a poor typewriter ribbon or if the university sent us a poor photocopy.

Previously copyrighted materials (journal articles, published tests, etc.) are not filmed.

Reproduction in full or in part of this film is governed by the Canadian Copyright Act, R.S.C. 1970, c. C-30. Please read the authorization forms which accompany this thesis.

**THIS DISSERTATION
HAS BEEN MICROFILMED
EXACTLY AS RECEIVED**

AVIS

La qualité de cette microfiche dépend grandement de la qualité de la thèse soumise au microfilmage. Nous avons tout fait pour assurer une qualité supérieure de reproduction.

S'il manque des pages, veuillez communiquer avec l'université qui a conféré le grade.

La qualité d'impression de certaines pages peut laisser à désirer, surtout si les pages originales ont été dactylographiées, à l'aide d'un ruban usé ou si l'université nous a fait parvenir une photocopie de mauvaise qualité.

Les documents qui font déjà l'objet d'un droit d'auteur (articles de revue, examens publiés, etc.) ne sont pas microfilmés.

La reproduction, même partielle, de ce microfilm est soumise à la Loi canadienne sur le droit d'auteur, SRC 1970, c. C-30. Veuillez prendre connaissance des formules d'autorisation qui accompagnent cette thèse.

**LA THÈSE A ÉTÉ
MICROFILMÉE TELLE QUE
NOUS L'AVONS REÇUE**

"RADIATIVE TAU DECAY
AND
PION STRUCTURE FORM FACTORS"

by

JUNG HONG KIM

A thesis submitted to the Faculty of Graduate
Studies in partial fulfillment of the requirements for
the Degree of Doctor of Philosophy.

Carleton University

Ottawa, Ontario

Canada

1979

The undersigned hereby recommend to the Faculty
of Graduate Studies acceptance of the thesis, "Radiative
Tau Decay and Pion Structure Form Factors" submitted by
Jong Hong Kim, in partial fulfillment of the requirements
for the Degree of Doctor of Philosophy.

Laya Resnick

.....
Thesis Supervisor

M. K. Sundaresan

.....
Chairman, Department of Physics

Carleton University

1979

TO MY PARENTS AND MY WIFE

1

ABSTRACT

In the framework of the standard model the radiative $\tau^- \rightarrow \nu_\tau \pi^- \gamma$ decay process is calculated for the first time in terms of the pion structure form factors, which also describe the radiative $\pi^- \rightarrow e^- \bar{\nu}_e \gamma$ decay. The phenomenological behaviours of the two form factors are derived from the VMD with and without isospin rotation, and applied to the radiative τ^- decay to estimate the expected spectrum and counting rates. We also predict the branching ratio for the $A_1 \rightarrow \pi^- \gamma$ decay and discuss the possibility of measuring the form factors separately.

The intrinsically interesting structure form factors of the pion are calculated with the coloured and fractionally charged quarks in a one-loop approximation, modified by a $\pi q \bar{q}$ vertex function simulating bound state effects. The numerical evaluations are performed to obtain $\gamma_\pi(0) = .84$ and $[\langle r_\pi^2 \rangle]^{1/2} = .677$ fm. We also discuss the gauge invariance problem for the radiative $\pi \rightarrow e \bar{\nu}_e \gamma$ decay by making a simple prescription satisfying the soft-photon techniques and the simple gauge projection operator.

ACKNOWLEDGEMENTS

It is a great pleasure to thank Professor Lazer Resnick, my Research Supervisor, whose constant guidance enabled me to complete this work.

I wish to thank Professor M.K. Sundaresan for his understanding and advice. I wish to thank Dr. A. Sen for helpful discussions. Thanks also goes to Mrs. Debbie Stone for the typing.

TABLE OF CONTENTS

ABSTRACT	1
ACKNOWLEDGEMENTS	11
CHAPTER I INTRODUCTION AND HISTORY	1
CHAPTER II THE $\tau^- \rightarrow \nu_\tau \pi^- \gamma$ DECAY	30
1. Introduction	30
2. Decay Width of the $\tau^- \rightarrow \nu_\tau \pi^-$ Process	32
3. Decay Width of the $\tau^- \rightarrow \nu_\tau \pi^- \gamma$ Process	36
4. Isospin Rotation and VMD	60
5. Numerical Results and Discussions	83
CHAPTER III GAUGE INVARIANT AMPLITUDE FOR THE $\pi^- \rightarrow e^- \bar{\nu}_e \gamma$ DECAY	100
1. Introduction	100
2. Pion as a Pointlike Particle	101
3. The $\pi q \bar{q}$ Vertex Function	106
4. Invariant Amplitude for the $\pi^- \rightarrow e^- \bar{\nu}_e \gamma$ Decay	111
5. Gauge Projection Operator	123
CHAPTER IV STRUCTURE FORM FACTORS AND MEAN CHARGE RADIUS OF THE PION	128
1. Introduction	128
2. Structure Form Factors and Their Ratio	129
3. Pion Decay Constant and Isovector Form Factor	138
4. Electromagnetic Form Factor and Mean Charge Radius	142
5. Numerical Results and Comments	152
CHAPTER V CONCLUSIONS	162

REFERENCES

CHAPTER I	168
CHAPTER II	174
CHAPTER III	178
CHAPTER IV	180
CHAPTER V	182

FIGURES

Fig. 1	5
Fig. 2	5
Fig. 3	31
Fig. 4	37
Fig. 5	44
Fig. 6	57a
Fig. 7	61
Fig. 8	67
Fig. 9	80
Fig. 10	101
Fig. 11	102
Fig. 12	107
Fig. 13	112
Fig. 14	112
Fig. 15	113
Fig. 16	140
Fig. 17	143

TABLES

TABLE 1	19
TABLE 2	93
TABLE 3	93
TABLE 4	94
TABLE 5	94
TABLE 6	95
TABLE 7	95
TABLE 8	96
TABLE 9	96
TABLE 10	97
TABLE 11	97
TABLE 12	98
TABLE 13	98
TABLE 14	99
TABLE 15	99
TABLE 16	159

CHAPTER I

INTRODUCTION AND HISTORY

The radiative electronic decay of the charged pions, $\pi \rightarrow e \nu_e \gamma$, has been of great interest as a source of information on the hadronic structure of pion due to the joint effect of weak and electromagnetic interactions. The radiative leptonic (electronic or muonic) decay proceeds via the ordinary inner bremsstrahlung (IB) and structure-dependent (SD) interactions under the vector-axial vector (V-A) currents. The muonic decay mode, however, is not so much of interest because the amplitude is dominated by the IB contribution which does not give any information on the hadronic structure.

The structure-dependent vector (SDV) contribution may be obtained from the isovector part of the $\pi^0 \rightarrow 2\gamma$ decay or from the vector meson dominance (VMD) approach. Thus, the problem of the $\pi \rightarrow e \nu_e \gamma$ decay is essentially to determine the amount of structure-dependent axial vector (SDA) contribution, i.e. the parameter γ_π which contains all the information on the hadronic structure of the pion. The parameter γ_π is defined as a ratio of the SDA form factor to the SDV form factor of the charged pions.

During the last twenty-eight years, theoretical analyses have been given by many authors¹⁻³⁶⁾ for the $\pi \rightarrow e\nu_e\gamma$ decay as well as the $K \rightarrow e\nu_e\gamma$ decay, and experiments have been performed for the $\pi \rightarrow e\nu_e\gamma$ process³⁷⁻⁴⁰⁾, the $K \rightarrow e\nu_e\gamma$ process^{41,42)} and also for the $\pi \rightarrow e\nu_e e^+e^-$ process⁴³⁾. The theoretical analyses can be classified into three types based on the models developed in a chronological order:

- (1) the first dynamical nucleon-loop approach (1952-1959)
- (2) the phenomenological approach (1960-1964; 1967-1974)
- (3) the second dynamical approach with the quantum chiral theory, σ -model and the relativistic quark model (1975-1979-?)

In order to understand the radiative leptonic decays more clearly we now review briefly these three approaches by introducing our conventions and by comparing with the experiments.

According to the Universal Fermi theory⁴⁴⁾ and Tomono-Wheeler triangle scheme (symmetric coupling scheme)⁴⁵⁾, Ruderman and Finkelstein⁴⁶⁾ calculated the ratio of the decay rates of the $\pi \rightarrow \mu\nu_\mu$ decay and the $\pi \rightarrow e\nu_e$ decay by assuming that a π -meson disappears

with the production of a pair of virtual nucleons and they annihilate with the creation of a (μ, ν_μ) or (e, ν_e) pair. They showed that the symmetric coupling scheme is tenable only if the pion is pseudoscalar and if the β -decay coupling is at least partially axial vector. In 1952, Ruderman¹⁾ subsequently considered for the first time the radiative electronic decay of pion. He calculated only the virtual nucleon loop corresponding to Fig. 2c and obtained the branching ratio $B_Y^\pi = \Gamma(\pi \rightarrow e \nu \gamma) / \Gamma(\pi \rightarrow \mu \nu) = 2.4 \times 10^{-3}$, an unreasonably large value compared with the experiment, because he neglected the Feynman diagrams of Figs. 2a and 2b.

Other authors^{2,3)} (1956) included all the three diagrams in Fig. 2 and expanded their methods to the general cases of the radiative meson decays into the leptons. Particularly, in Ref. [3], Treiman and Wyld found $B_Y^\pi = 4 \times 10^{-8}$ (compare to the recent value of $B_Y^{\pi, \text{exp}} = (2.15 \pm 0.050) \times 10^{-8}$) by assuming $G_1 + G_2 = -f_\pi$ from the gauge invariance condition and the Feynman convergence method⁴⁷⁾, where G_1, G_2 represent h_1, h_2 in their papers and are defined below in Eq. (I-10).

In 1958, according to the conserved vector current (CVC) hypothesis and the vector-axial vector (V-A) hypothesis,⁴⁸⁾ Vaks and Ioffe⁴⁾ investigated the

4

radiative pion decay and established an exact relation between the decay rates of $\pi \rightarrow e(\mu)\nu\gamma$ and $\pi^0 \rightarrow 2\gamma$ decays through the isovector coupling. They have found that, in the total decay rate of the $\pi \rightarrow e\nu\gamma$ decay, the main role is played by the decay through the axial vector coupling, and they pointed out that the experimental observation of a decay $\pi \rightarrow e\nu\gamma$ by the vector coupling is of special interest, because it would offer the possibility to check the V-A hypothesis as to the presence of direct interaction of π -meson with an electron-neutrino field and the theorem of the connection between the $\pi \rightarrow e(\mu)\nu\gamma$ decay and the $\pi^0 \rightarrow 2\gamma$ decay through the vector coupling (isovector).

For the $\pi \rightarrow \mu\nu\gamma$ a similar result was calculated⁵⁾ to the lowest order in the frame work of the V-A hypothesis, and it was suggested that the decay through the axial vector interaction can be related to that through the vector interaction by choosing $|g_V| = 1$.

We now discuss the second part, the phenomenological analyses. The phenomenological analyses can also be divided in two parts: the first one (1960-1964) is based just on the CVC and V-A hypothesis in a gauge invariant way; the second part (1967-1974), based on the VMD⁴⁹⁾, sum rules, dispersion relations, the partially

Fig. 1 Feynman diagrams for the $\pi \rightarrow e(\mu)\nu$ decay
(to lowest order in G_F)

Fig. 2 Feynman diagrams for the $\pi \rightarrow e(\mu)\nu\gamma$ decay
(to lowest order in G_F)

conserved axial vector current (PCAC) hypothesis and the current algebra in addition to the CVC and V-A hypothesis. The first phenomenological analyses have been done in Refs. [6-9].

In 1960, Bludman and Young⁶⁾ amplified the previous results^{4,5)} by constructing the phenomenological amplitude for the $\pi \rightarrow e \bar{\nu}_e \gamma$ in a gauge invariant way, and discussed the possibility of distinguishing the structure-dependent effects from the less interesting structure-independent effects, and supplemented the electron and photon spectra presented already^{4,5)}, which may be more easily observed experimentally. They, however, supposed that Figs. 2a and 2b, when defined by gauge invariance, give rise to the IB part and Fig. 2c gives the SD part, and they are separately gauge invariant. It will be shown later that this assumption is not correct. The intermediate vector boson effects were discussed in detail for $K \rightarrow e \bar{\nu}_e \gamma$ as well as $\pi \rightarrow e \bar{\nu}_e \gamma$ by Neville⁷⁾ (1961). In order to get some information on the SDA form factor Brown and Bludman⁹⁾ (1964) reinvestigated the $\pi^- \rightarrow e^- \bar{\nu}_e \gamma$ mode in detail, and concluded that the experiment by Depommier et al.³⁷⁾ established the presence of such axial vector structure contributing to the decay, but one could not assign this

comment to a weakly-interacting vector meson rather than to a strong interaction structure. They also discussed the applicability of the VMD.⁴⁹⁾

In order to extract information about the parameter γ_π from experimental studies for the $\pi \rightarrow e \bar{\nu}_e \gamma$ decay, it is first necessary to obtain the amplitude completely. Here we show how the total amplitude for Fig. 2 has been established from the standard phenomenological method, which has been used in the experiments for the $\pi(K) \rightarrow e(\mu) \bar{\nu} \gamma$ decays.

The S-matrix for the radiative electronic decay $\pi^-(K) \rightarrow e^-(p_e) + \bar{\nu}_e(p_\nu) + \gamma(k, \epsilon)$ is given by⁵⁰⁾

$$\begin{aligned}
 S_{fi} &= -i \int \langle f | H_{INT}(x) | i \rangle d^4x \\
 &= \frac{-i}{(2\pi)^2} \left(\frac{m_e m_\nu}{4E_\pi E_e E_\nu E_\gamma} \right)^{1/2} \delta(K - p_e - k - p_\nu) M_{fi} \quad (I-1)
 \end{aligned}$$

where $K(E_\pi)$, $p_e(E_e)$, $p_\nu(E_\nu)$ and $k(E_\gamma)$ are the four momenta (energies) of pion, electron, neutrino and photon, respectively, and m_e , m_ν are the masses of leptons. ϵ is the photon polarization vector. The invariant amplitude, M_{fi} , is expressed as

$$M_{fi} = \frac{G_F}{\sqrt{2}} \cos \theta_c \langle f | J_V^+ L^V | i \rangle \quad (I-2)$$

where we have assumed the Cabibbo angle $\theta_c = \theta_v = \theta_A$, G_F is the Universal Fermi constant, the strangeness conserving weak hadronic current J_V^+ is the sum of the vector and axial vector currents:

$$J_V^+ = V_V^+ - A_V^+ \quad (I-3)$$

and the weak leptonic current L^V is given by

$$L^V = \bar{u}(p_e) \gamma^V (1 - \gamma_5) v(p_\nu) \quad (I-4)$$

The amplitude M_{fi} can be rewritten as

$$M_{fi} = M_a + M_b + M_c$$

which correspond to Figs. 2a, 2b and 2c, respectively.

The amplitudes M_a , M_b are calculated directly by the general method, and M_c can be obtained by the gauge invariance condition $M_{fi}(\epsilon \rightarrow k) = 0$. We will also show that they are not separately gauge invariant.

We know that the vacuum state $|0\rangle$ is a scalar, even under parity, while the π -meson is a pseudo-scalar, and hence $\langle 0 | V_\nu^\dagger(0) | \pi^-(K) \rangle$ is an axial vector quantity, which cannot be constructed from the only four-vector, the π^- momentum, K . Therefore, the vector parts of Fig. 1, Figs. 2a and 2b do not contribute to the pion decay. On the other hand, $\langle 0 | A_\nu^\dagger(0) | \pi^-(K) \rangle$ is a vector, and we write

$$\langle 0 | A_\nu^\dagger(0) | \pi^-(K) \rangle = i f_\pi K_\nu \quad (\text{I-5})$$

where f_π is the pion decay constant. Actually f_π is a function of the momentum squared and $f_\pi(K^2) = f_\pi$ only for $K^2 = m_\pi^2$.

From Fig. 2b the amplitude M_b is calculated as

$$M_b = -ie \frac{G_F}{\sqrt{2}} f_\pi \cos\theta_c K_\nu \bar{u}(p_e) \left(\frac{p_e \cdot \epsilon}{p_e \cdot k} + \frac{\not{\epsilon} \not{k}}{2 p_e \cdot k} \right) \gamma^\nu (1 - \gamma_5) v(p_\nu) \quad (\text{I-6a})$$

$$= -ie \frac{G_F}{\sqrt{2}} f_\pi \cos\theta_c \bar{u}(p_e) \left[1 + \frac{m_e}{m_e} \left(\frac{p_e \cdot \epsilon}{p_e \cdot k} + \frac{\not{\epsilon} \not{k}}{2 p_e \cdot k} \right) \right] (1 - \gamma_5) v(p_\nu) \quad (\text{I-6b})$$

where we have used the transversality condition $\epsilon \cdot k = 0$ and e is the protonic charge. The amplitude M_a for Fig. 2a is obtained

$$M_a = +ie \frac{G_F}{\sqrt{2}} f_\pi \cos \theta_c \frac{K \cdot \epsilon}{K \cdot k} (K-k)_\nu L^\nu \quad (I-7a)$$

$$= +ie \frac{G_F}{\sqrt{2}} f_\pi \cos \theta_c m_e \bar{u}(p_e) \frac{K \cdot \epsilon}{K \cdot k} (1-\gamma_5) v(p_\nu) \quad (I-7b)$$

where we have used the first term of the expansion of $f_\pi(s)$ with respect to k , based on the soft-photon techniques⁶⁴, and $s = (K-k)^2$. The second term of the expansion of $f_\pi(s)$ is assumed to be included in the structure part and finally be cancelled by the gauge invariance condition (see Sec. III-1). The sum of two amplitudes M_a and M_b gives the IB contribution, but it is not gauge invariant because of the presence of the first term in the square bracket of Eq. (I-6b) (or the second term in the bracket of Eq. (I-7a)). This term is cancelled by a term from the amplitude of Fig. 2c as shown below.

The amplitude for Fig. 2c is given by

$$\begin{aligned}
 M_c &= \frac{G_F}{\sqrt{2}} \cos \theta_c \epsilon^{\mu_L \nu} \int e^{ik \cdot x} \langle 0 | T \{ j_\mu^{\text{em}}(x) J_\nu^+(0) \} | \pi^-(K) \rangle d^4 x \\
 &= \frac{G_F}{\sqrt{2}} \cos \theta_c (M_{\mu\nu}^V + M_{\mu\nu}^A) \epsilon^{\mu_L \nu}
 \end{aligned}
 \tag{I-8}$$

where

$$M_{\mu\nu}^V = \int e^{ik \cdot x} \langle 0 | j_\mu^{\text{em}}(x) V_\nu^+(0) | \pi^-(K) \rangle d^4 x$$

and

$$M_{\mu\nu}^A = - \int e^{ik \cdot x} \langle 0 | j_\mu^{\text{em}}(x) A_\nu^+(0) | \pi^-(K) \rangle d^4 x$$

depend only on the two four-momenta k and $q (q=K-k)$.

Obviously $M_{\mu\nu}^V$ must transform as a pseudo-tensor.

Considerations of Lorentz invariance lead to the expression

$$M_{\mu\nu}^V = e v(s=q^2, k^2) \epsilon_{\mu\nu\alpha\beta} K^\alpha k^\beta \tag{I-9}$$

$v(s, k^2)$ is an unknown SDV form factor that describes the contribution of the vector current. This form factor is a function of two independent invariants k^2 (for the real photon $k^2=0$) and $q^2=s$ that is the square of the

momentum transfer to the leptons. The most general expression for the tensor $M_{\mu\nu}^A$ which can be constructed from the four-momenta k and q (or K) has the form¹⁷⁾

$$M_{\mu\nu}^A = -ie(G_1 g_{\mu\nu} + G_2 K_\mu k_\nu + G_3 K_\mu K_\nu + G_4 k_\mu K_\nu + G_5 k_\mu k_\nu). \quad (I-10)$$

Making use of the transversality condition $\epsilon \cdot k = 0$ and the gauge invariance condition $M_{\mu\nu}(\epsilon \rightarrow k) = 0$ we obtain (see Sec. II-3 for the details)

$$M_{\mu\nu}^A = -ie\{f_\pi g_{\mu\nu} + a(s, k^2)(K \cdot k g_{\mu\nu} - K_\mu k_\nu)\} \quad (I-11)$$

where $a(s, k^2)$ is an unknown SDA form factor which describes the axial vector current contribution and is also a function of k^2 and s . As noted previously, the first term in the square bracket of Eq. (I-11) is cancelled by the first term in the square bracket of Eq. (I-6b). Thus, the invariant total amplitude for the $\pi^-(K) \rightarrow e^-(p_e) + \bar{\nu}_e(p_\nu) + \gamma(k, \epsilon)$ decay is found

$$M_{fi} = M(\pi^- e \bar{\nu}_e \gamma) = M_{IB} + M_{SDV} + M_{SDA} \quad (I-12)$$

$$M_{IB} = +ie \frac{G_F}{\sqrt{2}} \cos\theta_c m_e \bar{u}(p_e) \left(\frac{K \cdot \epsilon}{K \cdot k} - \frac{p_e \cdot \epsilon}{p_e \cdot k} - \frac{\not{\epsilon} \not{k}}{2p_e \cdot k} \right) (1 - \gamma_5) v(p_\nu) \quad (I-13)$$

$$M_{SDV} = +e \frac{G_F}{\sqrt{2}} \cos \theta_c v(s,0) \epsilon_{\mu\nu\alpha\beta} \epsilon^{\mu L \nu} K^{\alpha} k^{\beta} \quad (I-14)$$

$$M_{SDA} = -ie \frac{G_F}{\sqrt{2}} \cos \theta_c a(s,0) (K \cdot k g_{\mu\nu} - K_{\mu} k_{\nu}) \epsilon^{\mu L \nu} \quad (I-15)$$

where M_{IB} , M_{SDV} and M_{SDA} are the separately gauge invariant amplitudes corresponding to the IB, SDV and SDA contributions, respectively. $v(s,0)$ and $a(s,0)$ are real, assuming time reversibility.

Since the IB amplitude does not depend on any unknown form factor the IB part of the differential decay rate can be explicitly determined with the photon energy cut that is characterized by the experimental apparatuses. From Eqs. (I-12) - (I-15) one can obtain the total differential decay rate with respect to photon energy E_{γ} and electron energy E_e in the pion rest frame:

$$\frac{d^2\Gamma}{dx dy} = \frac{d^2\Gamma_{IB}}{dx dy} + \frac{d^2\Gamma_{SD}}{dx dy} + \frac{d^2\Gamma_{INT}}{dx dy} \quad (I-16)$$

$$\frac{d^2\Gamma_{IB}}{dx dy} = \frac{G_F^2 f_{\pi}^2 \cos^2 \theta_c m_{\pi}^3}{(4\pi)^2} \left(\frac{m_e}{m_{\pi}} \right)^2 \left\{ \frac{1-y+\Delta^2}{x^2(x+y-1-\Delta^2)} (x^2+2(1-x)(1-\Delta^2)) \right. \\ \left. + \frac{2x\Delta^2(1-\Delta^2)}{x+y-1-\Delta^2} \right\} \quad (I-17)$$

$$\frac{d^2 \Gamma_{SD}}{dx dy} = \frac{G_F^2 \cos^2 \theta_c m_\pi^7 \alpha |v(s, 0)|^2}{64 \pi^2} \{ (1 + \gamma_\pi)^2 (x + y - 1 - \Delta^2) [(x + y - 1)(1 - x) + \Delta^2] + (1 - \gamma_\pi)^2 (1 - y + \Delta^2) [(1 - x)(1 - y) + \Delta^2] \} \quad (I-18)$$

$$\frac{d^2 \Gamma_{INT}}{dx dy} = \frac{G_F^2 \cos^2 \theta_c m_\pi^5 \alpha |v(s, 0)|}{(4 \pi)^2} \left(\frac{m_e}{m_\pi} \right)^2 \left\{ \frac{1 - y + \Delta^2}{x(x + y - 1 - \Delta^2)} \right. \\ \times \{ (1 + \gamma_\pi) [(1 - x)(1 - x - y) + \Delta^2] + (1 - \gamma_\pi) [x^2 - (1 - x)(1 - x - y) - \Delta^2] \} \} \quad (I-19)$$

where $\Delta = m_e/m_\pi$, $\gamma_\pi(s) = a(s, 0)/v(s, 0)$, and $x \equiv 2E_\gamma/m_\pi$, $y \equiv 2E_e/m_\pi$ have the range

$$1 - \frac{y}{2} - \frac{1}{2} (y^2 - 4\Delta^2)^{1/2} \leq x \leq 1 - \frac{y}{2} + \frac{1}{2} (y^2 - 4\Delta^2)^{1/2} \\ 2\Delta \leq y \leq 1 + \Delta^2 \text{ or } 1 + \Delta^2 \frac{x(1 - x - \Delta^2)}{1 - x} \leq y \leq 1 + \Delta^2 \quad (I-20)$$

which are determined from energy and momentum conservation in the phase space. Eqs. (I-17) - (I-19) are the same as those in Refs. [9, 21].

The non radiative decay width $\Gamma(\pi \rightarrow e \nu_e)$ is calculated directly from Fig. 1 such that

$$\Gamma(\pi \rightarrow e \nu_e) = \frac{G_F^2 f_\pi^2 \cos^2 \theta_c m_\pi^3}{8\pi} \left(\frac{m_e}{m_\pi} \right)^2 (1 - \Delta^2)^2 \quad (\text{I-21})$$

which also gives $f_\pi \cos \theta_c = .9448 m_\pi$ with $\sin \theta_c = 0.229^{51}$.

As mentioned previously, if the CVC hypothesis is assumed, the modulus of $v(s=0,0)$ can be related to the $v_0^{I=1}(0,0)$ for the $\pi^0 \rightarrow 2\gamma$ decay. That is, the neutral pion decay width $\Gamma(\pi^0 \rightarrow 2\gamma)$ can be calculated in terms of the isovector form factor $v_0^{I=1}(0,0)$ which is related to the SDV form factor $v(s=0,0)$ by means of the Vaks - Ioffe relation.^{4,52)} The amplitude for $\pi^0(K) \rightarrow \gamma(k_1, \epsilon_1) + \gamma(k_2, \epsilon_2)$ due to the axial vector anomaly⁵³⁾ is given by

$$M_{(\pi^0 \rightarrow 2\gamma)} = 2e^2 v_0^{I=1}(0,0) \epsilon_{\mu\nu\alpha\beta} \epsilon_1^\mu \epsilon_2^\nu v_{k_1 k_2}^{\alpha\beta} \quad (\text{I-22})$$

where ϵ_1 and ϵ_2 are the polarization vectors of the photons which correspond to momenta k_1 and k_2 , and K is

the pion momentum. Note that the normalization factors are included in the phase factor as in Eq. (I-1) for the $\pi^- \rightarrow e^- \bar{\nu}_e \gamma$ decay. The decay width is then

$$\begin{aligned} \Gamma_{\pi^0 \rightarrow 2\gamma} &= \frac{1}{T_{\pi^0 \rightarrow 2\gamma}} = \pi \alpha^2 m_\pi^3 |v^{I=1}(0,0)|^2 \\ &= \frac{\pi}{2} \alpha^2 m_\pi^3 |v(s=0,0)|^2 \end{aligned} \quad (I-23)$$

where we have used the Vaks-Ioffe relations

$$v_0^1(0,0) = v_0^0(0,0), \quad v(s=0,0) = \sqrt{2} v_0^1(0,0) \quad (I-24)$$

under the isospin symmetry. Therefore, the SDV form factor $v(s,0)$ for $s=0$ is determined from the measured mean life of the neutral pion.

Both $v(s,0)$ and $a(s,0)$ depend on the square of the momentum transfer s . For the radiative pion decay, this dependency on s is negligible^{9,37)}, and it is about 4-3 % in the range $m_e^2 (=0) < s < m_\pi^2$ in our quark-loop model as seen in Sec. IV-5. For the radiative leptonic kaon decay it will be quite large²⁷⁾. Thus, it is a quite interesting problem to look at the behaviours of the SD form factors for $s \neq 0$. We will discuss the

radiative heavy lepton (τ) decay, $\tau^- \rightarrow \nu_\tau \pi^- \gamma$, as a good example for the $s \neq 0$ cases in Chapter II.

We can see from Eqs. (I-17) - (I-19) that for the muonic decay mode, with muon in replace of electron, the IB term dominates completely over the SD and interference (INT) terms. For the electron mode, the INT terms are small compared with the SD terms, except in the region where the denominator is small (small x and y , or angle between $\vec{k}$ and $\vec{p}_e$ small); in this region the IB part will also become large and will dominate over both the INT and SD parts. Thus, for the electronic mode, the INT term may be neglected^{7,9)}. In what follows, only the SD terms will be considered, and the approximation $\Delta = m_e/m_\pi \approx 0$ will be made. Making use of Eq. (I-21) and (I-23) and neglecting any momentum dependence ($s=0$) we rewrite Eqs. (I-17) - (I-18) as

$$\frac{d^2 r_{IB}}{dx dy} = \frac{\alpha}{2\pi} \Gamma_{(\pi \rightarrow e \nu)} \frac{(1-y) \{ (x-1)^2 + 1 \}}{x^2 (x+y-1)} \quad (I-17a)$$

$$\begin{aligned} \frac{d^2 r_{SD}}{dx dy} &= \frac{G_F^2 \cos^2 \theta_c m_\pi^7 |v(0,0)|^2}{(8\pi)^2} \{ (1+\gamma_\pi)^2 (1-x)(x+y-1)^2 \\ &\quad + (1-\gamma_\pi)^2 (1-x)(1-y)^2 \} \\ &= \frac{G_F^2 \cos^2 \theta_c m_\pi^4 \Gamma_{\pi^0 \rightarrow 2\gamma}}{4\alpha(2\pi)^3} \{ (1+\gamma_\pi)^2 (1-x)(x+y-1)^2 + (1-\gamma_\pi)^2 (1-x)(1-y)^2 \} \end{aligned} \quad (I-18b)$$

which are the expressions used in the experiments for the $\pi \rightarrow e \nu_e \gamma$ decay. These have also been used for the $K \rightarrow e \nu_e \gamma$ ^{41,42)} by replacing $m_\pi \rightarrow m_K$, $\cos\theta_c \rightarrow \sin\theta_c$ and $\Gamma_{K \rightarrow e \nu}$.

The first experiment on the $\pi^+ \rightarrow e^+ \nu_e \gamma$ was performed by Depommier, et al.³⁷⁾ at CERN in 1963. They determined from 143 observed events

1) $\gamma_\pi = .4$ or -2.1 (error not cited) (I-25a)
for $T_{\pi^0} = 1.05 \times 10^{-16}$ sec.⁵⁸⁾ So far there are another four experimental data:

2) $\gamma_\pi = 1.6 \pm .7$ or $-3.1 \pm .7$ ³⁸⁾ for $T_{\pi^0} = 1.8 \times 10^{-16}$ sec.⁵⁷⁾ (I-25b)

3) $\gamma_\pi = .15 \pm .11$ or $-2.07 \pm .11$ ³⁹⁾ for $T_{\pi^0} = .84 \times 10^{-16}$ sec.⁵⁵⁾ (I-25c)

4) $\gamma_\pi = 1.0^{+1.05}_{-1.5}$ or $-2.15^{+1.65}_{-1.0}$ ⁴³⁾ for $T_{\pi^0} = .84 \times 10^{-16}$ sec.⁵⁵⁾ (I-25d)

5) $\gamma_\pi = .44 \pm .12$ or $-2.36 \pm .12$ ⁴⁰⁾ for $T_{\pi^0} = .828 \times 10^{-16}$ sec.⁵⁴⁾ (I-25e)

Unfortunately, comparison of these various values for γ_{π} is complicated by the fact that the experimental values for γ_{π} depend in a non-trivial way on the π^0 lifetime through the vector form factor $v(s=0,0)$. What the experimenters actually measured is the decay rate $\pi \rightarrow e \nu_e \gamma$, which is quadratic in both $v(0,0)$ and $a(0,0)$ and also contains the interference terms proportional to the product $v(0,0) a(0,0)$. Thus, one obtains a quadratic equation for γ_{π} whose coefficients depend on $v(0,0)$ as shown in Eqs. (I-18) and (I-18a).

The π^0 life-time is difficult to measure and various experiments have given inconsistent results as shown below in Table 1.

Table 1: Dependence of $v(0,0)$ on T_{π^0}

		1976- ⁵⁴⁾	1971- ⁵⁵⁾	1967- ⁵⁶⁾	1964- ⁵⁷⁾	1963- ⁵⁸⁾	1961- ⁵⁹⁾
$T_{\pi^0} (x10^{-16} \text{ sec})$		.828	.84	.89	1.80	1.05	2
$\Gamma_{\pi^0} (x10^{-6} \text{ MeV})$		7.95	7.84	7.40	3.66	6.27	3.29
$v(0,0)$	m_{π}	2.61	2.59	2.52	1.80	2.32	1.68
$(x10^{-2} m_{\pi}^{-1})$	m_{π^0}	2.65	2.63	2.56	1.81	2.36	1.71

We are using the recent average value $T_{\pi^0} = (.828 \pm .057) \times 10^{-16}$ sec., computed by the Particle Data Group in 1978⁵⁴⁾. They say that they have increased the stated errors of the experiments used in averaging by a scale factor of 1.8 in order to eliminate this inconsistency.

Thus, it is understandable that authors at various times in the past have used very different values of the π^0 lifetime. To obtain the second result with 11 observed events in Ref. [38], Booth, et al. used $T_{\pi^0} = 1.8 \times 10^{-16}$ sec, and they said that with this value of T_{π^0} the Depommier experiment gives $\gamma_{\pi} = .9$ or -2.6 in agreement with their own results in Eq. (I-25b), but unfortunately they did not tell the reader how to compute what their own experiment gives for other values of T_{π^0} .

The fourth result has been determined from 12 observed events for the $\pi^+ \rightarrow e^+ \nu_e e^+ e^-$ decay at JINR⁴³⁾ (1976) using $T_{\pi^0} = .89 \times 10^{-16}$ sec⁵⁶⁾. The negative value of γ_{π} in this experiment is in agreement with other experiments, and the positive value is not, but is consistent with some theoretical predictions $|\gamma| \approx 1.2$ for $\delta = -1^{12-16,29)$. This experiment also did not give enough information to recompute γ_{π} with any value of T_{π^0} . Thus, the second and fourth results are considered to be unreliable.

Stetz, et al.^{39,40)} at LBL have obtained the third and fifth results in 1974 and 1978, respectively. With their first result they claimed that they would like to draw attention to the prediction $\gamma_\pi = 0$ of the static quark model¹⁰⁾ which is in agreement with their results. Their second result, however, is not consistent with the prediction $\gamma_\pi = 0$ obtained in the static quark model or $SU(2) \times SU(2)$ σ -model³³⁾.

To recompute γ_π with any value of T_{π^0} we obtain the general expressions

$$143 = N(E_e, E_\gamma > 48 \text{ MeV}) = 33 + \left(\frac{1.05 \times 10^{-16}}{T_{\pi^0}} \right) [54(1+\gamma_\pi)^2 + 4(1-\gamma_\pi)^2] \quad (\text{I-26})$$

from the first Depommier experiment³⁷⁾ (1963) and

$$226.3 = N(E_e > 56 \text{ MeV}, E_\gamma > 42 \text{ MeV}) = 27.5 + \left(\frac{.828 \times 10^{-16} \text{ s}}{T_{\pi^0}} \right) [95.2(1+\gamma_\pi)^2 + 2(1-\gamma_\pi)^2] \quad (\text{I-27})$$

from the fifth Stetz experiment⁴⁰⁾ (1978). For

$T_{\pi^0} = .828 \times 10^{-16} \text{ sec}$, Eqs. (I-26) and (I-27) give

$$\gamma_{\pi} \approx .25 \text{ or } -1.97 \quad (I-28)$$

$$\gamma_{\pi} = .44 \pm .12 \text{ or } -2.36 \pm .12 \quad (I-25e)$$

These values should be compared to our prediction

$$\gamma_{\pi}(0) = .84$$

According to our definitions of $v(s,0)$ and $a(s,0)$ for the $\pi^+ \rightarrow e^+ \nu_e \gamma$ decay³²⁾, the sign of $\gamma_{\pi}(s)$ is positive. Since the vector and axial-vector hadronic currents change under the charge conjugation transformation such that $C V_{\mu} C^{-1} = -V_{\mu}$ and $C A_{\mu} C^{-1} = A_{\mu}$, the parameter $\gamma_{\pi}(s)$ for the $\pi^- \rightarrow e^- \bar{\nu}_e \gamma$ decay has negative sign. Thus our sign conventions for $\gamma_{\pi}(s)$ are opposite to those of Moreno and Pestieau.

We now go back to look at the second era of phenomenological analyses. These approaches were started from the soft-pion theorem obtained by Das, Mathur and Okubo¹¹⁾ (DMO). Das, Mathur and Okubo found the SDA form factor

$$a(s=0,0) = \frac{1}{f_{\pi}} \int \frac{dm^2}{m^4} \{ \rho_A(m^2) - \rho_V(m^2) \} + \frac{f_{\pi}}{3} \langle r_{\pi}^2 \rangle : \quad (I-29)$$

from Eq. (I-10) by using the current algebra and PCAC in the limit $K \rightarrow 0$. Further, saturating the sum rule Eq. (I-29) by ρ and A_1 poles and using the Weinberg sum rules⁶⁰⁾ with KSRF relation⁶¹⁾, they obtained

$$a(s=0,0) = \frac{1}{2} \frac{f_{\pi}}{m_{\rho}^2} \quad (I-30)$$

which gives $|\gamma_\pi| \approx .59$ for $T_{\pi^0} = .828 \times 10^{-16}$ sec and $m_\rho = 776$ MeV.⁵³⁾ Fayyazuddin, Riazuddin,¹⁴⁾ and Vaishya¹⁶⁾ calculated the similar results

$$a(s=0,0) \approx -\frac{\delta f_\pi}{2m_\rho^2} \quad \text{and} \quad |\gamma_\pi| \approx .59|\delta| \quad (\text{I-31})$$

from the hard pion techniques and subtracted dispersion relation, the soft-pion theorem of DMO in the framework of the hard pion techniques, respectively. Their results are identical to those of DMO for $\delta = -1$.

The method of hard pions is a combination of $SU(2) \times SU(2)$ current algebra, PCAC and ρ -dominance developed in Refs. [62,63]. The SDA form factor $a(s=0,0)$ obtained from the hard pion techniques is all summarized by

$$a(s=0,0) \approx -\frac{\delta f_\pi}{m_\rho^2} \quad (\text{I-32})$$

where δ is an unknown dimensionless parameter introduced by Schnitzer and Weinberg⁶²⁾ in the A_1 - ρ - π vertex. The value of δ is confined by the experimental data:

$-1 \leq \delta \leq 0$. Eq. (I-32) gives the parameter γ_π

$$|\gamma_\pi| \approx 1.2$$

(I-33)

for $\delta = -1$, which was also calculated by an unsubtracted dispersion relation.^{14,15)} The method of Low's low-energy theorem⁶⁴⁾ was applied to give similar results to those of the hard-pion methods, corresponding to $\delta \approx -1$. If we set $\delta \approx -\frac{1}{2}$ Eq. (I-32) gives $|\gamma_\pi| \approx .59$ which is identical to the result of the DMO's soft-pion theorem.

In 1974, from the DMO's soft-pion theorem Eq. (I-29) Scheck and Wulfschleger²⁰⁾ found the SDA form factor

$$a(s=0,0) = -\frac{3}{2} \frac{f_\pi}{m_\rho^2} + \frac{f_\pi}{3} \langle r_\pi^2 \rangle \quad (\text{I-34})$$

by using Low's low energy theorem⁶⁴⁾ and current algebra, which was also obtained by Berezinskii¹⁹⁾ (1969) by determining the dependence on the pion momentum of the $A_1-\pi-\gamma$ vertex on the assumption of $A_1-\rho-\pi$ dominance. Scheck and Wulfschleger showed that the relation⁶¹⁾ between $\langle r_\pi^2 \rangle$ and parameter δ :

$$\frac{\langle r_\pi^2 \rangle}{6} = \frac{3-\delta}{4m_\rho^2} \quad (\text{I-35})$$

enable us to obtain Eq. (I-32) from Eq. (I-34).

Berezinski¹⁹⁾ calculated the parameter γ_π for $(\langle r_\pi^2 \rangle)^{1/2} = .80 \pm .10$ fm from Eq. (I-34);

$$|\gamma_\pi| = 2.2^{+1.1}_{-1.0}$$

and Vainstein¹⁸⁾ obtained $|\gamma_\pi| = 1.8$ from the first term in the right hand side of Eq. (I-34). However, G₀ and Leutwyler²⁹⁾ have shown recently that $|\gamma_\pi| \leq 1$ by reexamining the PCAC sum rule - DMO's soft-pion sum rule - Eq. (I-29), which is consistent with our value.

The second dynamical models have been developed after 1974. They are the nucleon (and baryon) loop model based on the quantum chiral theory³⁰⁾, SU(2) x SU(2) quark σ -model^{33,35)} or SU(3) x SU(3) σ -model³⁴⁾ and relativistic quark-loop models^{31,32,36)}. Volkov and Perushin analyzed all pion decay processes using a chiral quantum field theory, and obtained a good description with $\gamma_\pi \approx .41$ when only nucleons in the loop are included. This value increases when the other members of the baryon octet are included.

The standard relativistic quark-loop model^{31,32)} gives $\gamma_\pi \leq 1$ which is in agreement with G₀-Leutwyler's

result²⁹⁾. Moreno³³⁾ extended the relativistic quark model to $SU(2) \times SU(2)$ quark σ -model, recovering PCAC and current algebra. But he used the uncoloured quarks and fixed the value of the σ -particle mass m_σ . He found for heavy quarks, $\gamma_\pi \geq 0$, and then showed, using a cut-off based confinement procedure, that the sign of γ_π can be reversed for the light quarks. It is not clear to us. An extension to an $SU(3) \times SU(3)$ or σ -model has been considered by Decker³⁴⁾, and he found $\gamma_\pi \approx 1.2$ for the coloured quarks.

Recently, Pascual and Tarrach³⁵⁾ studied in the one-loop approximation the pion decays and the pion electromagnetic form factor at low q^2 in the framework of a σ -model with the coloured heavy quarks. They showed that γ_π is a monotonically decreasing function of m_σ which starts at $\gamma_\pi = .86$ for $m_\sigma = .2m_\pi$, drops to $\gamma_\pi = .72$ for $m_\sigma = 6m_\pi$ and then decreases very slowly towards the asymptotic value $\gamma_\pi \approx .7$. A single-loop approximation with a constant $\pi q\bar{q}$ vertex has been considered by Gerasimov³⁶⁾, and he obtained

$$\langle r_\pi^2 \rangle = .34 \text{ fm}^2 ; m_\sigma = 240 \text{ MeV}$$

$$\frac{f_{\pi^0 q\bar{q}}^2}{4\pi} = \frac{\pi}{6} ; \gamma_\pi = 1$$

(I-36)

So far we have reviewed the whole theoretical and experimental history for the radiative electronic decay of the pion, and we now understand that the theoretical and experimental situation is far from clear, even for $s=0$.

The main purpose of this thesis is

1) to do a consistent calculation of $v(s,0)$, $a(s,0)$, f_π and the pion mean charge radius $[\langle r_\pi^2 \rangle]^{1/2}$ using the coloured and fractionally charged quarks in a one-loop approximation, modified by a $\pi q\bar{q}$ vertex function simulating bound state effects,

2) to calculate for the first time the radiative $\tau^- \rightarrow \nu_\tau \pi^- \gamma$ decay process in terms of $v(s,0)$ and $a(s,0)$, using the VMD to estimate the expected spectrum and counting rates.

The remainder of this thesis is arranged as follows:

In Chapter II we calculate the decay rate of the $\tau^- \rightarrow \nu_\tau \pi^- \gamma$ process in terms of the two form factors $v(s,0)$ and $a(s,0)$, which also describe the radiative pion decay, by using the standard model⁶⁵⁾, and discuss the possibility of measuring the form factors

separately. The behaviours of $v(s,0)$ and $a(s,0)$ are derived in the phenomenological way from the isospin rotation processes and the VMD, and applied to the $\tau^- \rightarrow \nu_\tau \pi^- \gamma$ decay to estimate the decay rate⁶⁶⁾.

In Chapter III we study the gauge invariance problem for the charged pion assuming the pion to be a pointlike elementary particle and a $q\bar{q}$ bound state. When we introduce the Drell-Lee ansatz type function to the $\pi q\bar{q}$ vertex, it makes all quark loops convergent but, violates gauge invariance. Thus, we make a simple prescription to obtain the gauge invariant amplitude for the radiative leptonic pion decay, which can also be derived by the soft-photon techniques⁶⁴⁾ and a simple gauge projection operator method⁶⁷⁾.

In Chapter IV in the one-loop approximation we calculate the form factors $v(s,0)$, $a(s,0)$ and the ratio $\gamma_\pi(s) = a(s,0)/v(s,0)$ in terms of the three free parameters: the quark mass m , the vertex mass parameter M and the $\pi^0 q\bar{q}$ coupling constant. Three parameters are determined from the well-known processes: the $\pi^- \rightarrow \mu^- \bar{\nu}_\mu$ decay, the $\pi^0 \rightarrow 2\gamma$ decay and the normalization condition of the pion electromagnetic form factor $F_\pi(0) = 1$. In order to test our quark-loop model, the pion mean

squared charge radius $\langle r_{\pi}^2 \rangle$ is calculated. The numerical values are computed from the Gaussian quadrature method.

Finally, Chapter V contains a summary of our results and our conclusions.

CHAPTER II

THE $\tau^- \rightarrow \nu_\tau \pi^- \gamma$ DECAY

1. Introduction

The existence of the τ -heavy lepton is by now well established^{1,2)}, and much has been learned about its properties since it has been discovered⁴⁾ and subsequently confirmed⁵⁾ from the anomalous lepton production (e, μ) in e^+e^- annihilations. The presently accepted interpretation of the τ -lepton is that it is a sequential spin-1/2 heavy lepton with a mass 1782 MeV/c², accompanied by its own neutrino ν_τ and that its interactions conserve a new τ -lepton number via the standard V-A current with universal coupling $g = g_{e\nu_e W} = g_{\mu\nu_\mu W} = g_{\tau\nu_\tau W}$. Here W denotes the intermediate vector boson (IVB).

The τ -lepton has been observed to decay into the pure leptonic states $\nu_\tau \mu \bar{\nu}_\mu$ and $\nu_\tau e \bar{\nu}_e$ as well as into the semi-hadronic channels $\nu_\tau \pi(K)$, $\nu_\tau \rho$, $\nu_\tau A_1$, etc., as shown in Fig. 3. The observed branching ratios are in excellent agreement^{1,2)} with the theoretical predictions of the standard model.⁶⁾

In this chapter, in order to see the effects of $v(s,0)$ and $a(s,0)$ for s large, we investigate a semi-hadronic radiative decay $\tau^-(p_\tau) \rightarrow \nu_\tau(p_\nu) + \pi^-(K) + \gamma(k)$ in terms of the SDV and SDA form factors - $v(s,0)$ and

Fig. 3 Feynman diagrams for the non-radiative τ^- decay models.

$a(s,0)$ - of the charged pions, which also describe the radiative electronic pion decay $\pi^- \rightarrow e^- \bar{\nu}_e \gamma$ as discussed in Chapter I. However, $v(s,0)$ and $a(s,0)$ in the present case depend strongly on $s = (p_\tau - p_\nu)^2 = (K + k)^2$, which varies over the kinematic range $m_\pi^2 \leq s \leq m_\tau^2$. In the π^- radiative decay, the argument of the form factors varies between m_e^2 and m_π^2 , a very small range, and they have been well approximated by $v(0,0)$ and $a(0,0)$.

To calculate the decay rate for the $\tau^- \rightarrow \nu_\tau \pi^- \gamma$ process, we apply the standard model⁶⁾ under the foregoing assumption of lepton universality. The standard model is

defined⁶⁾ as follows:

a) τ^- and ν_τ have their own lepton number different from that of $e^-(\mu^-)$ and $\nu_e(\nu_\mu)$ and also from that of $e^+(\mu^+)$ and $\bar{\nu}_e(\bar{\nu}_\mu)$.

b) ν_τ is massless ($m_{\nu_\tau} < 250 \text{ MeV}/c^2$ with 95% C.L.^{1,2)}

c) V-A coupling between W and $\tau \nu_\tau$ with the full strength g.

All the known leptons (e^-, ν_e), (μ^-, ν_μ) and (τ^-, ν_τ) seem to share this property.

Also, we discuss the possibility of measuring the form factors separately. The behaviours of $v(s,0)$ and $a(s,0)$ can be derived in the phenomenological way from the isorotation hypotheses and the VMD, and are applied to the $\tau^- \rightarrow \nu_\tau \pi^- \gamma$ decay to estimate the decay rate.

2. Decay width for the $\tau^- \rightarrow \nu_\tau \pi^-$ process.

Before we investigate the radiative decay made, we first calculate the decay rate for the non-radiative $\tau^-(p_\tau) \rightarrow \nu_\tau(p_\nu) + \pi^-(K)$ process as shown in Fig. 3c. from the knowledge of $\pi^- \rightarrow \mu^- \bar{\nu}_\mu$ by using the standard model.

The S-matrix for this decay is given by⁷⁾

$$S_{fi} = - \frac{1}{(2\pi)^{9/2}} \left\{ \frac{m_\tau m_\nu}{E_\tau E_\nu 2E_\pi} \right\}^{1/2} (2\pi)^4 \delta^4(p_\tau - p_\nu - K) M_{fi} \quad (\text{II-1})$$

where $p_\tau(E_\tau)$, $p_\nu(E_\nu)$ and $K(E_\pi)$ are the four-momenta (energies) of τ^- , ν_τ and pion, respectively, and m_τ , m_ν are the masses of τ^- , ν_τ . The invariant amplitude M_{fi} to order in G_F is given by

$$\begin{aligned} M_{fi} &= \frac{G_F}{\sqrt{2}} \cos\theta_c L^\mu \langle \pi^-(K) | J_\mu(0) | 0 \rangle \\ &= \frac{1}{\sqrt{2}} \frac{G_F}{f_\pi} \cos\theta_c K_\mu L^\mu \end{aligned} \quad (\text{II-2})$$

with the definitions

$$\begin{aligned} \langle \pi^-(K) | V_\mu(0) | 0 \rangle &= 0 \\ \langle \pi^-(K) | A_\mu(0) | 0 \rangle &= -i f_\pi K_\mu \end{aligned} \quad (\text{II-3})$$

where $L^\mu = \bar{u}(p_\nu) \gamma^\mu (1-\gamma_5) u(p_\tau)$ is the heavy leptonic weak current and $J_\mu(0)$ is the hadronic weak current defined in Eq. (I-3)

If the polarizations are not observed, we must average over initial and sum over final spins. One can obtain from Eqs. (II-1) and (II-2) the decay width $\Gamma_{\nu\pi}$.

$$\begin{aligned}
 \Gamma_{\nu\pi} &= \sum_{\text{spin } i, \text{ spin } f} \int |S_{fi}|^2 \frac{(2\pi)^3}{VT} d^3K d^3p_\nu \\
 &= \frac{1}{2(2\pi)^2} \frac{m_\tau m_{\nu\tau}}{E_\tau} \int \frac{d^3K}{2E_\pi} \frac{d^3p_\nu}{E_\nu} \delta^4(p_\tau - K - p_\nu) \sum_{\text{spin } i, f} |M_{fi}|^2
 \end{aligned}
 \quad (\text{II-4})$$

where VT is the reaction volume of space-time, and

$$\sum_{\text{spin } i, f} |M_{fi}|^2 = \frac{G_F^2 f_\pi^2 \cos^2 \theta_c}{m_\tau m_{\nu\tau}} (2 p_\tau \cdot K K \cdot p_\nu - p_\tau \cdot p_\nu K^2)
 \quad (\text{II-5})$$

Since, in the rest frame of τ^- -lepton $p_\tau = (m_\tau, \vec{0})$,

$$E_\pi = m_\tau (1 + \Delta_\pi^2 - \Delta_\nu^2) / 2$$

$$E_\nu = m_\tau (1 - \Delta_\pi^2 + \Delta_\nu^2) / 2 \quad (\text{II-6})$$

$$K \cdot p_\nu = m_\tau (1 - \Delta_\pi^2 - \Delta_\nu^2) / 2$$

and

$$\int \frac{d^3K}{2E_\pi} \frac{d^3p_\nu}{2E_\nu} \delta^4(p_\tau - K - p_\nu) = \frac{\pi}{2} [(1 - \Delta_\pi^2 - \Delta_\nu^2)^2 - 4\Delta_\pi \Delta_\nu]^{1/2}
 \quad (\text{II-7})$$

The non-radiative decay width in the rest frame of τ^- then is obtained

$$\Gamma_{\nu\pi} = \frac{G_F^2 f_\pi^2 \cos^2 \theta_c m_\tau^3}{16\pi} (1 - \Delta_\pi^2 - \Delta_\nu^2)^2 (1 - \Delta_\nu^2 - \frac{2\Delta_\pi^2 \Delta_\nu^2}{1 - \Delta_\pi^2 - \Delta_\nu^2}) (1 - \frac{4\Delta_\pi^2 \Delta_\nu^2}{1 - \Delta_\pi^2 - \Delta_\nu^2})^{1/2} \quad (\text{II-8})$$

where $\Delta_\pi \equiv m_\pi/m_\tau$ and $\Delta_\nu \equiv m_{\nu_\tau}/m_\tau$.

It is worthwhile to make a precise measurement of $\tau^- \rightarrow \nu_\tau \pi^-$ process because the energy spectrum of π^- near the edge of phase space (near the maximum point) can provide information on the ν_τ mass. If we neglect the ν_τ mass Eq. (II-8) becomes^{6,8)}

$$\Gamma_{\nu\pi} = \frac{G_F^2 f_\pi^2 \cos^2 \theta_c}{16\pi} m_\tau^3 (1 - \Delta_\pi^2)^2 \quad (\text{II-8a})$$

which give the τ^- life time

$T_\tau \approx 2.3 \times 10^{-13}$ sec. by using $B_{\nu\pi}^\tau = \Gamma(\tau \rightarrow \nu_\tau \pi)/\Gamma(\tau \rightarrow \text{all}) = .083^{1,2}$. This value increases about 6% at the upper limit of ν_τ mass from Eq. (II-8). Recently the PLUTO collaborators⁹⁾ measured the upper limit on the τ^- life time by using events assigned to $\tau^- \tau^+$ production in e^+e^- annihilation in the centre of mass energy 3.9 to 5 GeV. Their result is $T_\tau < 9.0 \times 10^{-12}$ sec. at 95% confidence level.

3. Decay width of the $\tau^- \rightarrow \nu_\tau \pi^- \gamma$ process.

For the radiative $\tau^-(p_\tau) \rightarrow \nu_\tau(p_\nu) + \pi^-(k) + \gamma(k)$ decay there are four possible Feynman diagrams as shown below in Fig. 4. To the lowest order in G_F ($G_F = g^2/m_W^2$) the diagram of Fig. 4d is neglected. Thus, the S-matrix for this decay is given by, as the same form in Eq. (I-1),

$$S_{fi} = \frac{-1}{(2\pi)^2} \left(\frac{m_\tau m_\nu}{E_\tau E_\nu 2E_\pi 2E_\gamma} \right)^{1/2} \delta^4(p_\tau - p_\nu - k - k) M_{fi} \quad (\text{II-9})$$

where E_γ is the energy corresponding to photon momentum k . The invariant total amplitude can be written as

$$M_{fi} = \frac{G_F}{\sqrt{2}} \cos\theta_c \langle f | J_\nu L^\nu | i \rangle$$

$$= M_{IB} + M_{SDV} + M_{SDA} \quad (\text{II-10})$$

where we have assumed the Cabibbo angle $\theta_c = \theta_V = \theta_A$, and M_{IB} , M_{SDV} and M_{SDA} are the IB, SDV and SDA amplitudes, respectively, which will be shown to be separately gauge invariant.

The amplitudes for Feynman diagrams of Figs. 4a, 4b and 4c are obtained by the photon transversality.

Fig. 4 Feynman diagrams for the radiative $\tau^- \rightarrow \nu_\tau \pi^- \gamma$ decay.

$\epsilon \cdot k = 0$ and the gauge invariance condition, as studied in Chapter I. ϵ is the polarization vector associated with the photon four-momentum k . The amplitude for Fig. 4a, in which the photon is emitted from the τ^- -lepton, is

$$M_a = \frac{G_F}{\sqrt{2}} \cos \theta_c \bar{u}(p_\nu) \gamma^\nu (1 - \gamma_5) \frac{1}{\not{p}_\tau - \not{k} - m_\tau + i\epsilon} (-ie\epsilon) u(p_\tau) \langle \pi^-(K) | J_\nu(0) | 0 \rangle$$

$$= ie \frac{G_F}{\sqrt{2}} f_\pi \cos \theta_c K_\nu \bar{u}(p_\nu) \gamma^\nu (1 - \gamma_5) \frac{(\not{p}_\tau - \not{k} + m_\tau) \not{\epsilon}}{(p_\tau - k)^2 - m_\tau^2} u(p_\tau) \quad (\text{II-11a})$$

where we have used Eq. (II-3). The small imaginary part is inserted in the denominator of τ^- -lepton propagator to avoid the singularity. Actually this is not necessary because there is no dynamical singularity in the off-shell, except at the kinematic limit for $k \rightarrow 0$, which gives the infrared divergence in k^{-1} order. Making use of the relations

$$\{\gamma_\mu, \gamma_\nu\} = 2 \delta_{\mu\nu} \quad (\text{II-12})$$

$$\{\gamma_5, \gamma_\nu\} = 0$$

$$(\not{p}_\tau - m_\tau) u(p_\tau) = 0, \quad \bar{u}(p_\nu)(\not{p}_\nu - m_\nu) = 0 \quad (\text{II-13})$$

we obtain

$$\begin{aligned} M_a = & ie \frac{G_F}{\sqrt{2}} f_\pi \cos\theta_c \bar{u}(p_\nu) \{ m_\nu \left(-\frac{p_\tau \cdot \epsilon}{p_\tau \cdot k} + \frac{k \epsilon}{2p_\tau \cdot k} \right) (1 - \gamma_5) \\ & + (1 + \gamma_5) \left(-\not{\epsilon} + m_\tau \left(\frac{p_\tau \cdot \epsilon}{p_\tau \cdot k} - \frac{k \epsilon}{2p_\tau \cdot k} \right) \right) u(p_\tau) \} \end{aligned} \quad (\text{II-13b})$$

Similarly, the amplitude for Fig. 4b, in which the emitted photon comes from the charged pion line, is obtained as follows:

$$M_b = \frac{G_F}{\sqrt{2}} \cos \theta_c L^\nu \langle \pi^-(K+k) | J_\nu(0) | 0 \rangle = \frac{1}{(K+k)^2 - m_\pi^2 + i\epsilon} (-ie)(2K+k) \cdot \epsilon \quad (II-14)$$

$$= -ie \frac{G_F}{\sqrt{2}} f_\pi \cos \theta_c \frac{K \cdot \epsilon}{K \cdot k} (K+k)_\nu L^\nu \quad (II-14a)$$

$$= -ie \frac{G_F}{\sqrt{2}} f_\pi \cos \theta_c \frac{K \cdot \epsilon}{K \cdot k} \bar{u}(p_\nu) [-m_\nu (1-\gamma_5) + m_\tau (1+\gamma_5)] u(p_\tau) \quad (II-14b)$$

Finally, the amplitude for Feynman diagram of Fig. 4c can be written as

$$M_c = \frac{G_F}{\sqrt{2}} \cos \theta_c L^\nu \langle \gamma(K, \epsilon) \pi^-(K) | J_\nu(0) | 0 \rangle \quad (II-15)$$

$$= \frac{G_F}{\sqrt{2}} \cos \theta_c \epsilon^\mu L^\nu \int e^{ik \cdot x} \langle \pi^-(K) | T \{ j_\mu^{\text{em}}(x) J_\nu(0) \} | 0 \rangle d^4x$$

$$= \frac{G_F}{\sqrt{2}} \cos \theta_c \epsilon^\mu L^\nu (M_{\mu\nu}^V + M_{\mu\nu}^A) \quad (II-15b)$$

where

$$M_{\mu\nu}^V = \int e^{ik \cdot x} \langle \pi^-(K) | T \{ j_\mu^{\text{em}}(x) V_\nu(0) \} | 0 \rangle d^4x,$$

$$M_{\mu\nu}^A = - \int e^{ik \cdot x} \langle \pi^-(K) | T \{ j_\mu^{\text{em}}(x) A_\nu(0) \} | 0 \rangle d^4x$$

depend only on the two four-momenta k and $q' = K + k$.

The most general expressions as written in Eq. (I-9)

and (I-10) are given by

$$M_{\mu\nu}^V = -e v(s,0) \epsilon_{\mu\nu\alpha\beta} K^\alpha k^\beta \quad (\text{II-16})$$

$$M_{\mu\nu}^A = ie[G_1 g_{\mu\nu} + G_2 K_\mu k_\nu + G_3 K_\mu k_\nu + G_4 k_\mu K_\nu + G_5 k_\mu k_\nu] \quad (\text{II-17})$$

The SDV form factor $v(s,0)$ and five unknown form factors

$G_i (i = 1, \dots, 5)$ are the functions of $s = q'^2 = (K+k)^2$

the square of momentum transfer, and k^2 ($k^2 = 0$ for the

real photon). By means of the transversality condition

$\epsilon \cdot k = 0$, Eq. (II-17) becomes

$$M_{\mu\nu}^A = ie[G_1 g_{\mu\nu} + G_2 K_\mu k_\nu + G_3 K_\mu k_\nu] \quad (\text{II-17a})$$

then, the gauge invariance condition, $m_{f1}(\epsilon \rightarrow k) = 0$,

enables us to reduce the three unknown form factors to

only one. From Eqs. (II-11c), (II-14b), (II-16)

and (II-17a) we have

$$\begin{aligned} M_{f1}(\epsilon \rightarrow k) &= [m_a + m_b + m_c]_{\epsilon \rightarrow k} \\ &= ie \frac{G_F}{\sqrt{2}} \cos \theta_c L^\nu [(-f_\pi + G_1 + G_2 K \cdot k) k_\nu + G_3 K \cdot k K_\nu] = 0 \end{aligned} \quad (\text{II-18})$$

Since the photon momentum k and the pion momentum K are independent, condition (II-18) can be satisfied if and only if

$$G_1 + G_2 K \cdot k = f_\pi \quad \text{and} \quad G_3 = 0 \quad (\text{II-19})$$

The form factor G_1 , however, becomes $G_1 = f_\pi$ for $K \cdot k = 0$, and thus the general form of this form factor can be written as¹⁰⁾

$$G_1 = f_\pi + a(s,0) K \cdot k$$

$$G_2 = -a(s,0) \quad (\text{II-20})$$

which give

$$M_{\mu\nu}^A = ie[f_\pi g_{\mu\nu} + a(s,0) (K \cdot k g_{\mu\nu} - K_\mu k_\nu)] \quad (\text{II-21})$$

Therefore, we obtain the invariant total amplitude:

$$M_{f1} = M (\tau^- + \nu_\tau \pi^- \gamma)$$

$$= M_{IB} + M_{SDV} + M_{SDA} \quad (\text{II-22})$$

$$\begin{aligned}
M_{IB} = & ie \frac{G_F}{\sqrt{2}} f_\pi \cos\theta_c m_\nu \bar{u}(p_\nu) \left(\frac{K \cdot \epsilon}{K \cdot k} - \frac{p_\tau \cdot \epsilon}{p_\tau \cdot k} + \frac{k \cdot \epsilon}{2p_\tau \cdot k} \right) (1 - \gamma_5) u(p_\tau) \\
& + ie \frac{G_F}{\sqrt{2}} f_\pi \cos\theta_c m_\tau \bar{u}(p_\nu) (1 + \gamma_5) \left(\frac{p_\tau \cdot \epsilon}{p_\tau \cdot k} - \frac{K \cdot \epsilon}{K \cdot k} - \frac{k \cdot \epsilon}{2p_\tau \cdot k} \right) u(p_\tau)
\end{aligned}
\tag{II-23}$$

$$M_{SDV} = - e \frac{G_F}{\sqrt{2}} \cos\theta_c v(s,0) \epsilon_{\mu\nu\alpha\beta} \epsilon^\mu L^\nu K^\alpha k^\beta \tag{II-24a}$$

$$= - e \frac{G_F}{\sqrt{2}} \cos\theta_c v(s,0) \tilde{F}_{\mu\nu} K^\mu L^\nu \tag{II-24b}$$

$$M_{SDA} = ie \frac{G_F}{\sqrt{2}} \cos\theta_c a(s,0) (K \cdot k g_{\mu\nu} - K_\mu k_\nu) \epsilon^\mu L^\nu \tag{II-25a}$$

$$= ie \frac{G_F}{\sqrt{2}} \cos\theta_c a(s,0) F_{\mu\nu} K^\mu L^\nu \tag{II-25b}$$

where $F_{\mu\nu} = k_\mu \epsilon_\nu - \epsilon_\mu k_\nu$ is the electromagnetic field tensor and $\tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}$ is its dual, and two form factors $v(s,0)$, $a(s,0)$ are real, assuming T-invariance. This is the Lorentz and gauge invariant total amplitude which we wanted for the $\tau^- \rightarrow \nu_\tau \pi^- \gamma$ decay. M_{IB} , M_{SDV} and M_{SDA} now are separately gauge invariant. If we neglect the ν_τ mass the first term in R.H.S. of Eq. (II-23) vanishes.

We now calculate the decay width from Eqs.

(II-9), (II-22) - (II-25) in the rest frame of τ^- lepton.

The decay width for the $\tau^- \rightarrow \nu_\tau \pi^- \gamma$ process is given by

$$\begin{aligned} \Gamma_{\nu\pi\gamma} &= \frac{1}{2(2\pi)^5} m_{\nu_\tau} \int \frac{d^3K}{2E_\pi} \frac{d^3k}{2E_\gamma} \frac{d^3p_\nu}{E_\nu} \delta^4(p_\tau - K - k - p_\nu) \sum_{\text{spin } i,f} |M_{fi}|^2 \\ &= m_{\nu_\tau} \int \sum_{\text{spin } i,f} |M_{fi}|^2 d\text{Lips}(m_\tau; K, k, p_\nu). \quad (\text{II-26}) \end{aligned}$$

where $d\text{Lips}(m_\tau; K, k, p_\nu)$ is the restricted Lorentz invariant phase space element in the CM frame of τ^- -lepton, and is given by¹¹⁾

$$d\text{Lips}(m_\tau; K, k, p_\nu) = (2\pi)^{-5} \delta^3(\vec{K} + \vec{k} + \vec{p}_\nu) \delta(m_\tau - E_\pi - E_\gamma - E_\nu)$$

$$\times \frac{d^3K}{2E_\pi} \frac{d^3k}{2E_\gamma} \frac{d^3p_\nu}{2E_\nu} \quad (\text{II-27a})$$

$$= (2\pi)^{-5} \delta(m_\tau - E_\pi - E_\gamma - E_\nu) \frac{d^3K}{2E_\pi} \frac{d^3k}{2E_\gamma} \frac{1}{2E_\nu}$$

$$(\text{II-27b})$$

with

$$E_\nu = [m_{\nu_\tau}^2 + (\vec{K} + \vec{k})^2]^{1/2} \quad (\text{II-28})$$

Eq. (II-27b) can be cast in a more convenient form by choosing the z-axis in the momentum space of photon to lie along $\vec{K}$. The polar angle coordinate of $\vec{k}$ then becomes the angle θ_{Kk} between $\vec{K}$ and $\vec{k}$ as shown in Fig. 5 and so we can write.

$$d^3k = |\vec{k}|^2 dk d(\cos\theta_{Kk}) d\phi_k$$

$$d^3K d^3k = |\vec{K}|^2 |\vec{k}|^2 dK dk d(\cos\theta_{Kk}) d\Omega'$$

where

$$d\Omega' = d(\cos\theta_K) d\phi_K d\phi_k$$

Fig. 5 The angles θ_K and ϕ_K give the spatial orientation of pion. The angles θ_{Kk} and ϕ_k give the relative orientation of pion and photon.

Since Eq. (II-28) gives

$$E_V^2 = m_{V\tau}^2 + |\vec{K}|^2 + |\vec{k}|^2 + 2|\vec{K}| |\vec{k}| \cos\theta_{Kk} \quad (\text{II-29})$$

we can set

$$E_V dE_V = |\vec{K}| |\vec{k}| d(\cos\theta_{Kk})$$

for fixed $|\vec{K}|$ and $|\vec{k}|$. Thus, since we can also set

$$E_\pi dE_\pi = |\vec{K}| dK \text{ and } E_Y dE_Y = |\vec{k}| dk, \text{ we find that}$$

Eq. (II-27b) becomes

$$d\text{Lips}(m_\tau; K, k, p_V) = \frac{(2\pi)^{-5}}{8} d\Omega' dE_K dE_Y dE_V \delta(m_\tau - E_\pi - E_Y - E_V) \quad (\text{II-30})$$

The decay width for this decay can now be found from Equation (II-26) that, after integration over Ω' ,

$$\Gamma_{V\pi\gamma} = \frac{m_{V\tau}}{4(2\pi)^3} \int dE_\pi dE_Y dE_V \delta(m_\tau - E_\pi - E_Y - E_V) \sum_{\text{spin}} |M_{f1}|^2 \quad (\text{II-30a})$$

We note that $dE_V \delta(m_\tau - E_\pi - E_Y - E_V)$ can be effectively equated to one, provided that we remember that E_V will

then be fixed and equal to $m_T - E_\pi - E_Y$. Thus, we can rewrite Eq. (II-30a) as

$$\Gamma_{\pi\pi Y} = \frac{m_T}{4(2\pi)^3} \int dE_\pi dE_Y \sum_{\text{spin}} |M_{f1}|^2 \quad (\text{II-31})$$

We consider the region in the (E_π, E_Y) plane that corresponds to physically allowed values of E_π and E_Y . It is determined by the condition

$$E_V^- \leq m_T - E_\pi - E_Y \leq E_V^+ \quad (\text{II-32})$$

which follows from the equality $E_V = m_T - E_\pi - E_Y$ and the fact that E_V has well defined maximum and minimum values

$$E_V^\pm = [m_T^2 + (|\vec{K}| \pm |\vec{k}|)^2]^{1/2} \quad (\text{II-32a})$$

for given $|\vec{K}|$ and $|\vec{k}|$. This condition defines a closed finite region bounded by a single smooth curve given by the equation

$$f(m_T; E_\pi, E_Y, m_\pi^2, m_Y^2) = 0 \quad (\text{II-33})$$

where the function f is defined by

$$\begin{aligned}
 f = & -8m_\tau (E_\pi^2 E_Y + E_\pi E_Y^2) + 4m_\tau^2 E_\pi^2 + 4E_Y^2 (m_\tau^2 + m_\pi^2) \\
 & + 4E_\pi E_Y (3m_\tau^2 - m_\pi^2 - m_{\nu\tau}^2) - 4m_\tau E_\pi (m_\tau^2 + m_\pi^2 - m_{\nu\tau}^2) \\
 & - 4m_\tau E_Y (m_\tau^2 + m_\pi^2 - m_{\nu\tau}^2) + (m_\tau^2 + m_\pi^2 - m_{\nu\tau}^2)^2
 \end{aligned} \quad (\text{II-34})$$

Events on the boundary curve of Eq. (II-33) correspond to $E_\nu^\pm$; that is, to events in which the three particles have colinear momenta so that $\cos\theta_{kk} = \pm 1$. If we solve Eq. (II-33) we obtain the energy ranges of E_π and E_Y .

$$\begin{aligned}
 m_\tau - E_\pi - [E_\pi^2 - m_\pi^2]^{1/2} & \leq 2E_Y \leq m_\tau - E_\pi + [E_\pi^2 - m_\pi^2]^{1/2} \\
 m_\tau [1 + \Delta_\pi^2 - \frac{2E_Y}{m_\tau} + \frac{2m_\pi^2 E_Y}{m_\tau^2 (m_\tau - 2E_Y)}] & \leq 2E_\pi \leq m_\tau (1 + \Delta_\pi^2)
 \end{aligned} \quad (\text{II-35})$$

Thus, as long as Eq. (II-32) does hold the differential energy spectrum can be given by

$$\frac{d^2\Gamma_{\nu\pi Y}}{dE_\pi dE_Y} = \frac{m_{\nu\tau}}{4(2\pi)^3} \sum_{\text{spin}} |M_{fi}|^2 \quad (\text{II-36})$$

Let us now calculate $\sum_{\text{spin}} |M_{fi}|^2$ by neglecting

the v_τ mass. However, we have to use $\sum_{\text{spin}} u(p_v) \bar{u}(p_v) =$

$(\not{p}_v + m_{v_\tau})/2m_{v_\tau} = \not{p}_v/2m_{v_\tau}$, the summation of projection operator over spin for v_τ , to eliminate m_{v_τ} in numerator of Eq. (II-36). The total amplitude consists of three terms m_{IB} , M_{SDV} and M_{SDA} and thus, $\sum_{\text{spin}} |M_{fi}|^2$ has six

terms. In order to discuss the possibility of determination of $v(s,0)$ and $a(s,d)$ separately we calculate each term separately as follows:

$$\begin{aligned} \sum_{\text{sp}} |M_{IB}|^2 &= C f_\pi^2 m_\tau^2 \sum_{\text{sp}} [\bar{u}(p_v)(1 + \gamma_5) \left(\frac{p_\tau \cdot \epsilon}{p_\tau \cdot k} - \frac{k \cdot \epsilon}{K \cdot k} - \frac{k \not{\epsilon}}{2p_\tau \cdot k} \right) u(p_\tau)]^2 \\ &= C f_\pi^2 \frac{m_\tau}{2m_{v_\tau}} \sum_{\text{pol}} \text{Tr} \left[\left(\frac{p_\tau \cdot \epsilon}{p_\tau \cdot k} - \frac{k \cdot \epsilon}{K \cdot k} - \frac{k \not{\epsilon}}{2p_\tau \cdot k} \right) \not{p}_\tau (1 - \gamma_5) \right. \\ &\quad \left. \times \left(\frac{p_\tau \cdot \epsilon}{p_\tau \cdot k} - \frac{k \cdot \epsilon}{K \cdot k} - \frac{k \not{\epsilon}}{2p_\tau \cdot k} \right) \not{p}_v \right] \\ &= 2C f_\pi^2 \frac{m_\tau}{m_{v_\tau}} \sum_{\text{pol}} \left[\left(\frac{p_\tau \cdot \epsilon}{p_\tau \cdot k} - \frac{k \cdot \epsilon}{K \cdot k} \right)^2 p_\tau \cdot p_v + \frac{1}{p_\tau \cdot k} \left(\frac{p_\tau \cdot \epsilon}{p_\tau \cdot k} - \frac{k \cdot \epsilon}{K \cdot k} \right) \right. \\ &\quad \left. \times (p_\tau \cdot k p_v \cdot \epsilon - p_\tau \cdot \epsilon p_v \cdot k) + \frac{p_v \cdot k}{p_\tau \cdot k} \right] \end{aligned}$$

$$\begin{aligned}
&= 2C_F^2 \frac{m_\tau}{m_{\nu_\tau}} \{ p_\tau \cdot p_\nu \left[\frac{2p_\tau \cdot k}{p_\tau \cdot k \, K \cdot k} - \frac{p_\tau^2}{(p_\tau \cdot k)^2} - \frac{K^2}{(k \cdot k)^2} \right] \right. \\
&\quad + \left[\frac{p_\tau^2 p_\nu \cdot k}{(p_\tau \cdot k)^2} - \frac{p_\tau \cdot p_\nu}{p_\tau \cdot k} + \frac{p_\nu \cdot K}{p_\tau \cdot k} - \frac{p_\tau \cdot K p_\nu \cdot k}{p_\tau \cdot k \, K \cdot k} \right. \\
&\quad \left. \left. + \frac{p_\nu \cdot k}{p_\tau \cdot k} \right] \right\} \quad (\text{II-37})
\end{aligned}$$

$$\begin{aligned}
\sum_{\text{sp}} (M_{\text{SDV}})^2 &= C_V^2(s,0) \sum_{\text{sp}} [\epsilon_{\mu\nu\alpha\beta} K^\mu k^\alpha \epsilon^{\beta} \bar{u}(p_\nu) \gamma^\nu (1 - \gamma_5) u(p_\tau)]^2 \\
&= C_V^2(s,0) \frac{1}{2m_\tau m_{\nu_\tau}} [-\epsilon_{\beta\nu\alpha} \epsilon_{\beta'\mu'\nu'\alpha'} K^\mu k^\alpha K^{\mu'} k^{\alpha'} \\
&\quad \text{Tr}(\gamma^\nu \not{p}_\tau \gamma^{\nu'} \not{p}_\nu - \gamma_5 \gamma^\nu \gamma^{\nu'} \not{p}_\tau \not{p}_\nu)] \\
&= 2C_V^2(s,0) \frac{1}{m_\tau m_{\nu_\tau}} [K^2 k_{\nu\nu'} + g_{\nu\nu'} (K \cdot k)^2 \\
&\quad - K \cdot k (K_\nu k_{\nu'} + k_\nu K_{\nu'})] \\
&\quad [p_\tau^\nu p_\nu^{\nu'} + p_\tau^{\nu'} p_\nu^\nu - g^{\nu\nu'} p_\tau \cdot p_\nu] \\
&= 4C_V^2(s,0) \frac{1}{m_\tau m_{\nu_\tau}} [K \cdot k (p_\tau \cdot k p_\nu \cdot K + p_\tau \cdot K p_\nu \cdot k) \\
&\quad - K^2 p_\tau \cdot k p_\nu \cdot k] \quad (\text{II-38})
\end{aligned}$$

$$\begin{aligned}
\sum_{sp} |M_{SDA}|^2 &= Ca^2(s,0) \sum_{sp} [F_{\mu\nu} K^\mu \bar{u}(p_\nu) \gamma^\nu (1 - \gamma_5) u(p_\tau)]^2 \\
&= Ca^2(s,0) \frac{1}{4m_\tau m_{\nu_\tau}} \sum_{pol} \{ (K \cdot k \epsilon_\nu - K \cdot \epsilon k_\nu) (K \cdot k \epsilon_{\nu'} - K \cdot \epsilon_{\nu'}) \\
&\quad \times (p_\tau^\nu p_{\nu'}^\nu + p_\tau^{\nu'} p_\nu^\nu - g^{\nu\nu'} p_\tau \cdot p_\nu) \} \\
&= 4Ca^2(s,0) \frac{1}{m_\tau m_{\nu_\tau}} [K \cdot k (p_\tau \cdot K p_\nu \cdot k + p_\tau \cdot k p_\nu \cdot K) \\
&\quad - K^2 p_\tau \cdot k p_\nu \cdot k] \quad (II-39)
\end{aligned}$$

$$\begin{aligned}
\sum_{sp} M_{IB} \bar{M}_{SDV} &= -Cf_\pi v(s,0) m_\tau \sum_{sp} \{ i \bar{u}(p_\nu) (1 + \gamma_5) (\frac{p_\tau \cdot \epsilon}{p_\tau \cdot k} - \frac{K \cdot \epsilon}{K \cdot k} - \frac{K \cdot \epsilon}{2p_\tau \cdot k}) \\
&\quad \times u(p_\tau) \epsilon_{\mu\nu\alpha\beta} K^\mu k^\alpha \epsilon^\beta L^\nu \} \\
&= -Cf_\pi v(s,0) \frac{m_\tau}{2m_{\nu_\tau}} \sum_{pol} \{ i \epsilon_{\mu\nu\alpha\beta} K^\mu k^\alpha \epsilon^\beta (\frac{p_\tau \cdot \epsilon}{p_\tau \cdot k} - \frac{K \cdot \epsilon}{K \cdot k}) \\
&\quad - \frac{1}{2p_\tau \cdot k} (p_\nu \cdot k \epsilon^\nu - p_\nu \cdot \epsilon k^\nu) \\
&\quad - \frac{1}{2p_\tau \cdot k} \epsilon^{\nu\lambda\sigma\delta} k_\lambda \epsilon_\sigma p_{\nu\delta} \}
\end{aligned}$$

$$= -Cf_{\pi} v(s, 0) \frac{m_{\tau}}{m_{\nu_{\tau}}} \frac{1}{p_{\tau} \cdot k} \sum \text{pol}_{\mu\nu\alpha\beta}^{\epsilon} \epsilon^{\nu\gamma\sigma\delta} K^{\mu} k^{\alpha} k_{\lambda} p_{\nu\delta} \epsilon^{\beta} \epsilon_{\sigma}$$

$$= -2Cf_{\pi} v(s, 0) \frac{m_{\tau}}{m_{\nu_{\tau}}} \frac{p_{\nu} \cdot k}{p_{\tau} \cdot k} K \cdot k$$

$$= \sum_{sp} M_{SDV} \bar{M}_{IB}$$

$$\sum_{sp} \{M_{IB} \bar{M}_{SDV} + M_{SDV} \bar{M}_{IB}\} = 2 \sum_{sp} M_{IB} \bar{M}_{SDV} = -4Cf_{\pi} v(s, 0) \frac{m_{\tau}}{m_{\nu_{\tau}}} \frac{p_{\nu} \cdot k}{p_{\tau} \cdot k} K \cdot k$$

(II-40)

$$\sum_{sp} M_{IB} \bar{M}_{SDA} = Cf_{\pi} a(s, 0) m_{\tau} \sum_{sp} \{\bar{u}(p_{\nu})(1 + \gamma_5) \left(\frac{p_{\tau} \cdot \epsilon}{p_{\tau} \cdot k} - \frac{K \cdot \epsilon}{K \cdot k} - \frac{K \epsilon}{2p_{\tau} \cdot k} \right) \right.$$

$$\left. \times u(p_{\tau}) F_{\mu\nu} K^{\mu} L^{\nu} \right\}$$

$$= 2Cf_{\pi} a(s, 0) \frac{m_{\tau}}{m_{\nu_{\tau}}} \sum_{sp} \{ (K \cdot k \hat{p}_{\nu} \cdot \epsilon - K \cdot \epsilon p_{\nu} \cdot k) \}$$

$$\times \left(\frac{p_{\tau} \cdot \epsilon}{p_{\tau} \cdot k} - \frac{K \cdot \epsilon}{K \cdot k} - \frac{K \cdot k p_{\nu} \cdot k \epsilon \cdot \epsilon}{2p_{\tau} \cdot k} \right)$$

$$= 2Cf_{\pi} a(s,0) \frac{m_{\tau}}{m_{\nu_{\tau}}} \left[\frac{p_{\tau} \cdot K}{p_{\tau} \cdot k} p_{\nu} \cdot k - \frac{K \cdot k}{p_{\tau} \cdot k} p_{\tau} \cdot p_{\nu} + p_{\nu} \cdot K \right. \\ \left. - \frac{K^2}{K \cdot k} p_{\nu} \cdot k + \frac{K \cdot k}{p_{\tau} \cdot k} p_{\nu} \cdot k \right]$$

$$= \sum_{sp} M_{SDA} \bar{M}_{IB}$$

$$\sum_{sp} \{M_{IB} \bar{M}_{SPA} + M_{SDA} \bar{M}_{IB}\} = 2 \sum_{sp} M_{IB} \bar{M}_{SDA}$$

$$= 4Cf_{\pi} a(s,0) \frac{m_{\tau}}{m_{\nu_{\tau}}} \left[\frac{p_{\tau} \cdot K}{p_{\tau} \cdot k} p_{\nu} \cdot k - \frac{K \cdot k}{p_{\tau} \cdot k} p_{\tau} \cdot p_{\nu} + p_{\nu} \cdot K \right. \\ \left. - \frac{K^2}{K \cdot k} p_{\nu} \cdot k + \frac{K \cdot k}{p_{\tau} \cdot k} p_{\nu} \cdot k \right] \quad (II-41)$$

$$\sum_{sp} M_{SDA} \bar{M}_{SDV} = -Ca(s,0) \dot{v}(s,0) \sum_{sp} \{1 F_{\mu\nu} K^{\mu} L^{\nu} \epsilon_{\mu'\nu'\alpha'\beta'} K^{\mu'} k^{\alpha'} \epsilon^{\beta'} \bar{L}^{\nu}\}$$

$$= -2Ca(s,0) \dot{v}(s,0) \frac{1}{m_{\tau} m_{\nu_{\tau}}} \sum_{pol} \{K \cdot k \epsilon_{\mu'\nu'\alpha'\beta'} \epsilon^{\nu\nu'\lambda\sigma}$$

$$\{K^{\mu} k^{\alpha} p_{\tau\lambda} p_{\nu\sigma} \epsilon^{\beta'} \epsilon_{\nu'}\}$$

$$= 4Ca(s,0) v(s,0) \frac{K \cdot k}{m_\tau m_{\nu\tau}} (p_\tau \cdot k p_\nu \cdot K - p_\tau \cdot K p_\nu \cdot k)$$

$$= \sum_{sp} M_{SDV} \bar{M}_{SDA}$$

$$\sum_{sp} \{M_{SDA} \bar{M}_{SDV} + M_{SDV} \bar{M}_{SDA}\} = 2 \sum_{sp} M_{SDV} \bar{M}_{SDA}$$

$$= 8Ca(s',0) v(s,a) \frac{K \cdot k}{m_\tau m_{\nu\tau}} (p_\tau \cdot k p_\nu \cdot K - p_\tau \cdot K p_\nu \cdot k)$$

(II-42)

where $C = \frac{1}{2} e^2 G_F^2 \cos^2 \theta_c$ and we have used the summation schemes

$$g_{\alpha\beta} g^{\alpha\beta} = 4$$

$$\sum_{pol} \epsilon \cdot \epsilon = -2$$

$$\sum_{pol} a \cdot \epsilon b \cdot \epsilon = -a \cdot b$$

(II-43)

provided that $k \cdot a = 0$ and $k \cdot b = 0$.

It is convenient to work with dimensionless

variables

$$x \equiv 2E_Y/m_\tau \quad \text{and} \quad y \equiv 2E_\pi/m_\tau \quad (\text{II-44})$$

and hence we obtain for the invariant variables in terms of x and y

$$\begin{aligned} p_\tau \cdot k &= m_\tau^2 x/2 & p_\nu \cdot k &= m_\tau^2 (1 + \Delta_\pi^2 - y)/2 \\ p_\tau \cdot K &= m_\tau^2 y/2 & p_\nu \cdot K &= m_\tau^2 (1 - x - \Delta_\pi^2)/2 \\ p_\tau \cdot p_\nu &= m_\tau^2 (2-x-y)/2 & K \cdot k &= m_\tau^2 (x + y - 1 - \Delta_\pi^2)/2 \end{aligned} \quad (\text{II-45})$$

and

$$S = (K + k)^2 = m_\tau^2 (x + y - 1) \quad (\text{II-46})$$

by using the four-momentum conservation in the rest frame of τ^- -lepton and by neglecting the ν_τ mass. If we substitute the relations in Eqs. (II-44) and (II-45) into Eqs. (II-36) - (II-42) and rearrange the equations, we finally obtain the differential spectrum with respect to x and y .

$$\begin{aligned}
\frac{d^2 \Gamma_{v\pi\gamma}}{dx dy} = & \frac{\alpha G_F^2 \cos^2 \theta_c}{32\pi^2} m_\tau^3 \left\{ f_\pi^2 F_{BB}(x,y) + \frac{m_\tau^4}{4} [v^2(s,0) + a^2(s,0)] F_{VV}(x,y) \right. \\
& + f_\pi m_\tau^2 [a(s,0) F_{BA} - v(s,0) F_{BV}] \\
& \left. + \frac{m_\tau^4}{2} v(s,0) a(s,0) F_{VA}(x,y) \right\}
\end{aligned}
\tag{II-47}$$

where

$$F_{BB}(x,y) = \frac{1-\Delta_\pi^2-x}{x} \left[\frac{1+\Delta_\pi^2+y}{x+y-1-\Delta_\pi^2} - \frac{2}{x} \right] + \frac{y(1+\Delta_\pi^2-y)}{x(x+y-1-\Delta_\pi^2)} - \frac{2\Delta_\pi^2(2-x-y)}{(x+y-1-\Delta_\pi^2)^2}
\tag{II-48}$$

$$F_{BV}(x,y) = (x+y-1-\Delta_\pi^2) \frac{1+\Delta_\pi^2-y}{x}
\tag{II-49}$$

$$F_{BA}(x,y) = \frac{1+\Delta_\pi^2-y}{x} [y+(1-\Delta_\pi^2-x) - \frac{2\Delta_\pi^2 x}{x+y-1-\Delta_\pi^2}]
\tag{II-50}$$

$$F_{VV}(x,y) = F_{AA}(x,y)$$

$$= (x+y-1-\Delta_\pi^2) [y(1+\Delta_\pi^2-y) + x(1-\Delta_\pi^2-x)]$$

$$- 2x \Delta_\pi^2 (1+\Delta_\pi^2-y)
\tag{II-51}$$

$$F_{VA}(x,y) = (x+y-1-\Delta_{\pi}^2) [x(1-\Delta_{\pi}^2-x) - y(1+\Delta_{\pi}^2-y)] \quad (II-52)$$

with the kinematically allowed region

$$1-y/2 - (y^2 - 4\Delta_{\pi}^2)^{1/2}/2 \leq x \leq 1 - y/2 + (y^2 - \Delta_{\pi}^2)^{1/2}/2 \quad (II-53)$$

$$1+\Delta_{\pi}^2-x + \frac{\Delta_{\pi}^2 x}{1-x} \leq y \leq 1 + \Delta_{\pi}^2 \quad \text{or} \quad 2\Delta_{\pi} \leq y \leq 1 + \Delta_{\pi}^2$$

Eq. (II-53) is obtained from Eq. (II-35), and the subscripts of phase space functions BB, BV, BA, VV, AA and VA are the pure bremsstrahlung, bremsstrahlung-vector interference term, etc. The phase space functions of Eqs. (II-48) -- (II-52) represent the density of each term inside the phase space bounded by Eq. (II-53) as shown in Fig. 6. Fig. 6 shows that the pure bremsstrahlung function $F_{BB}(x,y)$ dominates over all other functions in the low regions of s .

Eq. (II-47) can be expressed as a dimensionless branching ratio with respect to the decay width of the $\tau^- \rightarrow \nu_{\tau} \pi^-$ process which was calculated in the previous sections (see Eq. (II-8a)).

Fig. 6 Dalitz plot for $\tau^- \rightarrow \nu_\tau \pi^- \gamma$ decay in the τ^- rest frame. $x \equiv 2E_\gamma/m_\tau$, $y \equiv 2E_\pi/m_\tau$. The boundary is determined by $2\Delta_\pi < +1\Delta_\pi^2$ and $1 - \frac{y-1}{2}(y^2 - 4\Delta_\pi^2)^{1/2} \leq x \leq 1 - \frac{y+1}{2}(y^2 - 4\Delta_\pi^2)^{1/2}$

$$\frac{d^2 R_{v\pi\gamma}}{dx dy} = \frac{1}{f_{v\pi}} \frac{d^2 \Gamma_{v\pi\gamma}}{dx dy}$$

$$= \frac{\alpha}{2\pi(1-\Delta_\pi^2)^2} F_{BB}(x,y) + \frac{\alpha m_\tau^4}{8\pi(1-\Delta_\pi^2)^2 f_\pi^2}$$

$$[v^2(s,0) + a^2(s,0)] F_{VV}(x,y)$$

$$+ \frac{\alpha}{2\pi(1-\Delta_\pi^2)^2} \frac{m_\tau^2}{f_\pi} [a(s,0) F_{BA}(x,y) - v(s,0) F_{BV}]$$

$$+ \frac{\alpha m_\tau^4}{4\pi(1-\Delta_\pi^2)^2 f_\pi^2} v(s,0) a(s,0) F_{VA}(x,y)$$

$$= \frac{\alpha}{2\pi(1-\Delta_\pi^2)^2} F_{BB}(x,y) + D \left\{ \left[\frac{v(s,0)}{v(0,0)} \right]^2 + \left[\frac{a(s,0)}{v(0,0)} \right]^2 \right\} F_{VV}(x,y)$$

$$+ 2 D \left[\frac{a(s,0)}{v(0,0)} \right] \left[\frac{v(s,0)}{v(0,0)} \right] F_{VA}(x,y)$$

$$+ 2 B \left\{ \left[\frac{a(s,0)}{v(0,0)} \right] F_{BA}(x,y) - \left[\frac{v(s,0)}{v(0,0)} \right] \right.$$

$$\left. \times F_{BV}(x,y) \right\}$$

(II-54)

where

$$D = \frac{\Gamma_{\pi^0} m_\tau}{4\pi^2 \alpha (1-\Delta_\pi^2)^2 f_\pi^2} \left(\frac{m_\tau}{m_{\pi^0}}\right)^3 = 6.714 \times 10^{-3},$$

$$B = \frac{1}{4\pi(1-\Delta_\pi^2)^2} \left(\frac{2\Gamma_{\pi^0}}{\pi m_{\pi^0}}\right)^{1/2} \left(\frac{m_\tau^2}{m_{\pi^0} f_\pi}\right) = 2.811 \times 10^{-3}$$

and we have used

$$v(0,0) = \left(\frac{2\Gamma_{\pi^0}}{\alpha^2 \pi m_{\pi^0}^3}\right)^{1/2} = 0.0265 m_{\pi^0}^{-1}$$

to make the dimensionless form factor ratios. This is the desired expression to calculate the decay rate.

It is apparent that $d^2R/dx dy$ depends on the two unknown form factors $v(s,0)$ and $a(s,0)$. However, since $s = m_\tau^2(x+y-1)$, v and a are functions of $x+y$ only, and are constant on the straight lines $x+y = \text{constant}$ in the (x,y) plane. Consequently, two measurements at different (x,y) on the same line allow v and a to be determined for that particular s . Because of the quadratic dependence, the solution of the equations will involve ambiguities; these can be resolved if more than two points are measured on the same line. For example, five independent measurements give five linear equations for

the unknown v^2 , a^2 , a , v , v and a . v and a can be found unambiguously and the three remaining equations are redundant and provide a consistency check. The procedure is repeated for different values of $x+y$ to obtain the s dependence.

It is interesting to compare this situation to the radiative pion decay. In the latter, the bremsstrahlung contribution could be suppressed by going to the electronic mode (the electron mass supplies the suppression) and hard photons. Here, there is no small lepton mass, and all six terms must be considered. For the pion decay, the s dependence can be neglected, and $v(0,0)$ was directly related to the $\pi^0 \rightarrow 2\gamma$ decay. Thus, the radiative pion decay can be parametrized by the single unknown number $\gamma_\pi = a(0,0)/v(0,0)$. In an interesting recent publication¹³⁾, the question of isospin breaking has been raised¹⁴⁾, and the suggestion made for also measuring directly $v(0,0)$ as well as $a(0,0)$ in the radiative pion decay. For the radiative τ decay as seen above, the decay is described by the two unknown functions $v(s,0)$ and $a(s,0)$.

We see from Fig. 6 that $F_{BB}(x,y)$ dominates the structure part functions except for the upper region; thus, if the value of $|v(s,0)/v(0,0)|$ is nearly equal to one, our calculation shows that it is quite

difficult to get information on the hadronic structure of pion from this $\tau^- \rightarrow \nu_\tau \pi^- \gamma$ decay. To get some information we will discuss the isospin rotated process of $\pi^0 \rightarrow 2\gamma$ and the vector meson dominance in the next section. We will also show that $|v(s,0)/v(0,0)|$ can have a surprisingly large value from the VDM.

4. Isospin Rotation and Vector Meson Dominance

In this section we obtain the vector vertex function corresponding to the SDV form factor $v(s,0)$ by using isospin rotation and vector meson dominance (VMD) assuming isospin symmetry. For $a(s,0)$ we will use those results of the second phenomenological analysis discussed in Chapter I.

For the radiative $\tau^- \rightarrow \nu_\tau \pi^- \gamma$ decay, the vector form factor $v(s,0)$ can be related by an isospin rotation to a physically accessible reaction. This is the $e^+e^- \rightarrow \pi^0 \gamma$ process at the same value of s , where s here is the centre of mass energy squared. This reaction can be described by the form factor $v_0(s,0)$, defined by

$$\langle \pi^0(K) \gamma(k, \epsilon) | j_\mu^{\text{em}}(0) | 0 \rangle = \epsilon^\nu \int e^{ik \cdot x}$$

$$x \langle \pi^0(K) | T \{ j_\nu^{\text{em}}(x) j_\mu^{\text{em}}(0) \} | 0 \rangle d^4x$$

$$= -e v_0(s,0) \tilde{F}_{\mu\nu} K^\nu \quad (\text{II-55})$$

from Lorentz invariance and the parity conservation.

To find the possibility of determining $v_0(s,0)$ let us calculate the spin averaged differential and total cross section for the $e^+e^- \rightarrow \pi^0 \gamma$ process. The Lorentz invariant amplitude of Fig. 7 for this process can be given from Eq. (II-55) by

$$\begin{aligned} M(e^+e^- \rightarrow \pi^0 \gamma) &= \bar{v}(p_+) (-ie\gamma_\mu) u(p_-) \left[\frac{-1}{q^2 + i\epsilon} g^{\mu\nu} \right] [-e^2 v_0(s,0) \tilde{F}_{\nu\alpha} K^\alpha] \\ &= \frac{e^3}{s} v_0(s,0) \bar{v}(p_+) \gamma_\mu u(p_-) g^{\mu\nu} \epsilon_{\nu\alpha\beta\delta} K^\alpha k^\beta \epsilon^\delta \end{aligned} \quad (\text{II-56})$$

where p_+ , p_- the four-momenta of electron and positron, respectively, and $s = q^2 = (p_+ + p_-)$ is the centre of mass energy squared.

Fig. 7 The isorotation process of the $\pi^0 \rightarrow 2\gamma$ decay is the $e^+e^- \rightarrow \gamma + \pi^0 \gamma$ reaction.

Then, we can get easily

$$\begin{aligned}
 \sum_{\text{spin}} |M_{fi}|^2 &= \frac{e^6 v_0^2(s,0)}{4 m_e^2 s^2} \sum_{\text{pol}} \{(-1) \text{Tr} \{ \gamma_\mu (\not{p}_- + m_e) \gamma^{\mu'} (\not{p}_+ - m_e) \\
 &\quad \times g^{\mu\nu} g_{\mu'\nu'} \epsilon_{\nu\alpha\beta\delta} \epsilon^{\nu'\alpha'\beta'\delta'} \\
 &\quad \times K^\alpha k^\beta \epsilon^\delta \epsilon_\delta k_\beta K_{\alpha'} \} \\
 &= \frac{2e^6}{m_e^2 s^2} v_0^2(s,0) |q^2 p_- \cdot k p_+ \cdot k - q \cdot k (p_- \cdot q p_+ \cdot k \\
 &\quad + p_+ \cdot q p_- \cdot k) - (q \cdot k)^2 m_e^2| \\
 &= \frac{e^6 v_0^2(s,0)}{2m_e^2} |2(p_+ \cdot k)(1 - \frac{m_\pi^2}{s} - \frac{2p_+ \cdot k}{s}) \\
 &\quad - (1 - \frac{m_\pi^2}{s})^2 (\frac{s}{2} + m_e^2)| \\
 &= \frac{e^6 v_0^2(s,0)}{2m_e^2} \{ \frac{4 E^2 E_Y^2 (1-\beta \cos\theta)^2}{s} - 2(1 - \frac{m_\pi^2}{s}) \\
 &\quad \times E E_Y (1-\beta \cos\theta) + (1 - \frac{m_\pi^2}{s})^2 (\frac{s}{2} + m_e^2) \}
 \end{aligned}$$

(II-57)

where θ is the angle between electron and photon and β is the speed of the electron or positron with energy E in the CM frame. In the first step of the above calculations we have replaced K by q for convenience by the four-momentum conservation. The spin averaged differential cross section then is given by

$$d\sigma (e^+e^- \rightarrow \pi^0 \gamma) = \frac{m_e^2}{4(2\pi)^2 E^2 \beta_{\text{rel}}} \int \frac{d^3K}{2E_K} \frac{d^3k}{2E_\gamma} \delta^4(q-K-k) \sum_{\text{spin}} |M_{fi}|^2 \quad (\text{II-58})$$

where β_{rel} is the relative velocity between electron and positron. Making use of the relation

$$E^2 \beta_{\text{rel}} = 2E^2 \left(1 - \frac{4m_e^4}{4E^4}\right)^{1/2} = s/2$$

and integrating over the phase space, we obtain

$$\begin{aligned} d\sigma (e^+e^- \rightarrow \pi^0 \gamma) &= \frac{\pi \alpha^3}{2s} v_0^2(s,0) \left(1 - \frac{m_{\pi^0}^2}{s}\right)^3 \int d\Omega [E^2(1 - \cos\theta)^2 \\ &\quad - \frac{s}{2} (1 - \cos\theta) + \frac{s}{2}] \\ &= \frac{\pi \alpha^3}{8} v_0^2(s,0) \left(1 - \frac{m_{\pi^0}^2}{s}\right)^3 \int d\Omega (1 + \cos^2\theta) \end{aligned}$$

where we have assumed $\beta \approx 1$.

Thus, the spin averaged differential cross section is

$$\frac{d\sigma}{d\Omega} (e^+e^- \rightarrow \pi^0 \gamma) = \frac{\pi\alpha^3}{8} v_0^2(s,0) \left(1 - \frac{m_{\pi^0}^2}{s}\right)^3 (1 + \cos^2\theta) \quad (\text{II-59a})$$

$$= \frac{\alpha \Gamma_{\pi^0}}{2m_{\pi^0}^3} \left(1 - \frac{m_{\pi^0}^2}{s}\right)^3 (1 + \cos^2\theta) \left[\frac{v_0(s,0)}{v_0(0,0)}\right]^2 \quad (\text{II-59b})$$

and the total cross section is

$$\sigma (e^+e^- \rightarrow \pi^0 \gamma) = \frac{2}{3} \pi^2 \alpha^3 v_0^2(s,0) \left(1 - \frac{m_{\pi^0}^2}{s}\right)^3 \quad (\text{II-60a})$$

$$= \frac{8\pi\alpha}{3m_{\pi^0}^3} \Gamma_{\pi^0} \left(1 - \frac{m_{\pi^0}^2}{s}\right)^3 \left[\frac{v_0(s,0)}{v_0(0,0)}\right]^2 \quad (\text{II-60b})$$

$$= 6.96 \times 10^{-35} \left(1 - \frac{m_{\pi^0}^2}{s}\right)^3 \left[\frac{v_0(s,0)}{v_0(0,0)}\right]^2 \text{ cm}^2 \quad (\text{II-60c})$$

where we have used $v_0(0,0) = (4\Gamma_{\pi^0}/\alpha^2 \pi m_{\pi^0}^3)^{1/2}$ to make the dimensionless form factor ratio. Eqs. (II-59) and (II-60) allow $v_0(s,0)$ to be measured directly, except for sign.

The form factor $v_0(s,0)$ can be split into two parts depending on the isospin character of the photon;

$$v_0(s,0) = v_0^1(s,0) + v_0^0(s,0) \quad (\text{II-61})$$

The superscript denotes the total isospin and the subscript represents also its third component. $v(s,0)$ corresponds of course to isospin 1 and its third component ± 1 , and the isospin symmetry implies that

$$v(s,0) = \sqrt{2} v_0^1(s,0) \quad (\text{II-62})$$

assuming that it is also true for any s value. In the $\pi^0 \rightarrow 2\gamma$ decay, we have

$$v_0^1(0,0) = v_0^0(0,0) = \frac{1}{2} v_0(0,0) \quad (\text{II-63})$$

which is used as a normalization condition.

If isospin symmetry is assumed and $v_0^1(s,0)$ is obtained from $v(s,0)/\sqrt{2}$, then $v_0^0(s,0)$ can be extracted. These might then be compared with theoretical models, for example the vector meson dominance (VMD) model, where ρ , ρ' are expected to contribute to $v_0^1(s,0)$ and ω , ϕ to $v_0^0(s,0)$. However, this is the reversed case. Actually we desire to determine $v(s,0)$. Although the behaviour of $v_0(s,0)$ can be measured from Eq. (II-59) or Eq. (II-60) by experiment at the

energy range $m_\pi^2 \leq s \leq m_\tau^2$, we cannot determine $v(s,0)$ by simply using the isospin symmetry. The reason is that $v_0(s,0)$ comes from two parts $v_0^1(s,0)$ and $v_0^0(s,0)$, and we do not know how much $v_0^1(s,0)$ and $v_0^0(s,0)$ contribute to $v_0(s,0)$ for $s \neq 0$ separately. This means that we first have to know $v_0^1(s,0)$ or $v_0^0(s,0)$ (or both $v_0^1(s,0)$ and $v_0^0(s,0)$).

In the VMD the reaction $e^+e^- \rightarrow \pi^0\gamma$ is dominated in the neighbourhood of $s = m_V^2$ by the vector meson contribution according to the chain $e^+e^- \rightarrow V \rightarrow \pi^0\gamma$ reaction described by the diagram in Fig. 8a.

The presence of the vector-meson would lead to a pole contribution to the vertex at the relevant value of $s = m_V^2$, where m_V is the mass of the vector meson.

The ω , ϕ can be strongly coupled to the same state, for example, the $\pi^+\pi^-\pi^0$ state. But all the interferences between the ω - and ϕ - contributions are always extremely small¹⁵⁾ and can be neglected. More generally the

V-meson contribution is important only in a range of energy approximately defined by $|\sqrt{s} - m_V| \leq \Gamma_V$ and it is clear in particular that the two domains

$|\sqrt{s} - m_\omega| \leq \Gamma_\omega$ and $|\sqrt{s} - m_\phi| \leq \Gamma_\phi$ do not overlap. For the ρ , ρ' the situation is similar, but is quite different for the ρ , ω , where the mass m_ω belongs to

Fig. 8 Feynman diagrams on the vector meson dominances.

the range $|\sqrt{s} - m_\rho| \leq \Gamma_\rho$. For the final state $\pi^0 \gamma$, the G-parity is not defined, and thus the ρ - and ω - contributions occur on the same footing and the $\rho - \omega$ interference may be important¹⁶⁾. However, this interference can be neglected^{15,17)}. Here we neglect the $\rho - \omega$ interference in this thesis for simplicity.

From a theoretical point of view Gell-Mann and Zachariasen¹⁸⁾ formulated the vector meson dominance model (VMD) by postulating dispersion relations in the photon mass variable and saturating them by vector meson poles. In the field theory the formulation relies on the possibility of using as electromagnetic current operator the vector meson field operator; this is called the current-field identity¹⁹⁾ which can be obtained in a Lagrangian field theory from the equation of motion.

According to the Gell-Mann-Nishijima relation $Q = I_3 + \frac{1}{2} Y$, one writes the hadronic part of the electromagnetic current $j^\mu(x)$:

$$j^\mu(x) = j_{I_3}^\mu(x) + \frac{1}{2} j_Y^\mu(x) = j_3^\mu(x) + \frac{1}{\sqrt{3}} j_8^\mu(x) \quad (\text{II-64})$$

where 3 and 8 labels are the usual SU(3) generators. The current-field identities then allow us to express $J^\nu(x)$ in terms of the vector meson fields having the same

quantum numbers (i.e. isovector and isoscalar vector meson, in the corresponding SU(3) combination);

$$j^\mu(x) = \sum_V g_{V\gamma} V^\mu(x) = \sum_V \frac{m_V^2}{f_V} V^\mu(x) \\ = g_{\rho\gamma} \rho^\mu(x) + g_{\omega\gamma} \omega^\mu(x) + g_{\phi\gamma} \phi^\mu(x) + \dots \quad (\text{II-65})$$

and thus the interaction Lagrangian is given by

$$L_{\text{INT}}(x) = e j^\mu(x) A_\mu^{\text{em}}(x) \quad (\text{II-66})$$

Here ... denotes the possible higher vector mesons (the extended or the generalized VMD model)²⁰, $A_\mu^{\text{em}}(x)$ is the electromagnetic field and $g_{V\gamma}$ (or f_V) is the $V - \gamma$ coupling constant. $V^\mu(x)$ is the vector meson field. For example with standard reduction formulas one gets for any $A \rightarrow B + \gamma$ reaction

$$M(A \rightarrow \gamma B) = \langle B | j^\mu(0) | A \rangle = \sum_V \frac{m_V^2}{f_V} \langle B | V^\mu(0) | A \rangle \\ = \sum_V \frac{m_V^2}{f_V} \Delta_V(q^2) \langle B | J^\mu(0) | A \rangle \quad (\text{II-67})$$

where $q^2 = (p_A - p_B)^2$, $(\Box + m_V^2) V^\mu(x) = J_V^\mu(x)$ and $\Delta_V(q^2)$

is the vector meson propagator. In the finite width approximation the vector meson propagator is

$$\Delta_V(q^2) = \frac{1}{m_V^2 - q^2 - i m_V \Gamma_V}$$

As mentioned earlier, if we assume $v_0^1(s,0)$ to be dominated by the ρ -, ρ' - meson contributions and $v_0^0(0,0)$ by the ω -, ϕ - meson contributions we can use the VMD model^{18,21)} to calculate them in the finite width approximation as indicated in Fig. 8. The direct $V - \gamma$ coupling is written as

$$e \frac{m_V^2}{f_V} = e g_{V\gamma}$$

as in Eq. (II-65), and the coupling constant for the radiative vector meson vertex $f_{V\pi^0\gamma}$ is defined by the Lorentz invariant matrix element

$$\begin{aligned} M_{(V \rightarrow \pi^0 \gamma)} &= \langle \pi^0(K) \gamma(k, \epsilon) | T | V(q, \epsilon_V) \rangle \\ &= e f_{V\pi^0\gamma} \epsilon_{\mu\nu\alpha\beta} q^\mu \epsilon_V^\nu k^\alpha \epsilon^\beta \end{aligned} \quad (\text{II-68})$$

using parity conservation, ϵ , ϵ_V are the polarization vectors of photon and vector meson, respectively. Thus,

the $V \rightarrow \pi^0 \gamma$ decay width can be calculated in a straightforward way with the result:

$$\Gamma(V \rightarrow \pi^0 \gamma) = \frac{\alpha}{24} m_V^3 \left(1 - \frac{m_{\pi^0}^2}{s}\right)^3 f_{V\pi^0\gamma}^2 \quad (\text{II-69})$$

which has been used to determine $f_{V\pi^0\gamma}$ from the measured value of $\Gamma(V \rightarrow \pi^0 \gamma)$. From the data²²⁾ for ω -, ϕ - meson we compute

$$f_{\omega\pi^0\gamma} = .349 m_{\pi^0}^{-1} = .361 m_{\pi}^{-1} = 2.58 \times 10^{-3} \text{ MeV}^{-1} \quad (\text{II-70})$$

$$f_{\phi\pi^0\gamma} = .0185 m_{\pi^0}^{-1} = .0191 m_{\pi}^{-1} = 1.370 \times 10^{-4} \text{ MeV}^{-1}$$

Unfortunately, we do not have any data for the $\rho \rightarrow \pi^0 \gamma$ decay width.

The direct $V - \gamma$ coupling f_V also can easily be calculated from the $V \rightarrow e^+ e^-$ reaction by using the current-field identity. The invariant amplitude for this reaction is simply expressed by

$$M(V \rightarrow e^+ e^-) = - \frac{e^2 m_V^2}{2 f_V} \epsilon_V^\nu \bar{u}(p_-) \gamma_\nu v(p_+) \quad (\text{II-71})$$

and thus, we obtain the decay width

$$\begin{aligned}
 \Gamma(V \rightarrow e^+e^-) &= \frac{4\pi\alpha^2}{3} \frac{m_V}{f_V^2} \left(1 - \frac{4m_e^2}{m_V^2}\right) \left(1 + \frac{2m_e^2}{m_V^2}\right) \\
 &= \frac{4\pi\alpha^2}{3} \frac{m_V}{f_V^2}
 \end{aligned}
 \tag{II-72}$$

This has been used to determine f_V from the experimental data. We compute its value for each vector meson from the data²²⁾:

$$f_\rho = 5.098$$

$$f_\omega = 15.084 \tag{II-73}$$

$$f_\phi = 13.380$$

Their inverse ratios are, except for signs,

$$\frac{1}{f_\rho} : \frac{1}{f_\omega} : \frac{1}{f_\phi} = 2.96 : 1 : 1.13 \tag{II-74}$$

compared to the SU(3) quark model result²³⁾

$$\frac{1}{f_\rho} : \frac{1}{f_\omega} : \frac{1}{f_\phi} = 3 : 1 : -\sqrt{2} \tag{II-75}$$

which defines the relative sign of f_ϕ as negative.

We now look again at the Feynman diagram of Fig. 8a for the $e^+e^- \rightarrow V + \pi^0\gamma$ reaction. By means of the current-field identities in Eq. (II-65), we can calculate the total cross section in terms of two coupling constants f_V and $f_{V\pi^0\gamma}$. The Lorentz invariant amplitude can be given in the finite width approximation by

$$M(e^+e^- \rightarrow V + \pi^0\gamma) = \frac{e^3}{s} \sum_V \frac{f_{V\pi^0\gamma}}{f_V} \frac{m_V^2}{m_V^2 - s - i m_V \Gamma_V} \bar{v}(p_+) \gamma_\mu u(p_-) g^{\mu\nu} \epsilon_{\nu\alpha\beta\delta} q^\alpha k^\beta \epsilon^\delta \quad (\text{II-76})$$

where V denotes all the possible vector mesons including the higher ones. The total cross section for the $e^+e^- \rightarrow V + \pi^0\gamma$ process is obtained by comparing with Eq. (II-56) and Eq. (II-60b)

$$\sigma(e^+e^- \rightarrow V + \pi^0\gamma) = \frac{2\pi^2\alpha^3}{3} \left(1 - \frac{m_{\pi^0}^2}{s}\right)^3 \left| \sum_V \frac{f_{V\pi^0\gamma}}{f_V} \frac{m_V^2}{[(m_V^2 - s)^2 + m_V^2 \Gamma_V^2]^{1/2}} \right|^2 \quad (\text{II-77})$$

which can be factorized by using Eqs. (II-69) and (II-72)

as

$$\sigma(e^+e^- \rightarrow V \rightarrow \pi^0 \gamma) \Big|_{s=m_V^2} = \frac{12\pi}{m_V^2} \frac{\Gamma(V \rightarrow e^+e^-)}{\Gamma(V \rightarrow \text{all})} \frac{\Gamma(V \rightarrow \pi^0 \gamma)}{\Gamma(V \rightarrow \text{all})} \quad (\text{II-78})$$

$$= \frac{8\pi\alpha}{3 m_\pi^3} \Gamma_{\pi^0} \left(1 - \frac{m_\pi^2}{m_V^2}\right)^3 \left[\frac{v_0(m_V^2, 0)}{v_0(0, 0)}\right]^2 \quad (\text{II-79})$$

From the experimental data²²⁾ we find that

$$v_0^0(m_\omega^2, 0)/v_0^0(0, 0) \approx 47.74$$

and

$$v_0^0(m_\phi^2, 0)/v_0^0(0, 0) \approx 8.49.$$

These values imply that $v_0^0(s, 0)$ depends strongly on the s . Unfortunately, we do not have any experimental data for $\Gamma(\rho \rightarrow \pi^0 \gamma)$ and $\sigma(e^+e^- \rightarrow \rho(\rho') \rightarrow \pi^0 \gamma)$. However, the fact that the isovector part $v_0^1(s, 0)$ also has a similar behaviour may be expected.

Comparing Eqs. (II-76) and (II-77) with Eqs. (II-56) and (II-60a) gives immediately the relation between the vector form factor $v_0(s, 0)$ and the vector coupling $f_V, f_{V\pi^0\gamma}$:

$$v_0(s,0) = \sum_V \frac{f_{V\pi^0\gamma}}{f_V} \frac{m_V^2}{m_V^2 - s - i m_V \Gamma_V} \quad (\text{II-80})$$

in the finite width approximation. Thus, the isoscalar and isovector form factors $v_0^0(s,0)$ and $v_0^1(s,0)$ can be separated as the Breit-Wigner form in the VMD model.

$$v_0^0(s,0) = \frac{f_{\omega\pi^0\gamma}}{f_\omega} [(1 - s/m_\omega^2)^2 + \Gamma_\omega^2/m_\omega^2]^{-1/2} + \frac{f_{\phi\pi^0\gamma}}{f_\phi} [(1 - s/m_\phi^2)^2 + \Gamma_\phi^2/m_\phi^2]^{-1/2} \quad (\text{II-81})$$

$$v_0^1(s,0) = \frac{f_{\rho\pi^0\gamma}}{f_\rho} [(1 - s/m_\rho^2)^2 + \Gamma_\rho^2/m_\rho^2]^{-1/2} + \frac{f_{\rho'\pi^0\gamma}}{f_{\rho'}} [(1 - s/m_{\rho'}^2)^2 + \Gamma_{\rho'}^2/m_{\rho'}^2]^{-1/2} \quad (\text{II-82})$$

For $s = 0$ Eq. (II-81) and (II-82) become

$$v_0^0(0,0) = \frac{f_{\omega\pi^0\gamma}}{f_\omega} [1 + \Gamma_\omega^2/m_\omega^2]^{-1/2} + \frac{f_{\phi\pi^0\gamma}}{f_\phi} [1 + \Gamma_\phi^2/m_\phi^2]^{-1/2} \\ = \frac{f_{\omega\pi^0\gamma}}{f_\omega} + \frac{f_{\phi\pi^0\gamma}}{f_\phi} \quad (\text{II-83})$$

$$\begin{aligned}
 v_0^{1,0}(0,0) &= \frac{f_{\rho\pi^0\gamma}}{f_\rho} [1 + r_\rho^2/m_\rho^2]^{-1/2} + \frac{f_{\rho'\pi^0\gamma}}{f_{\rho'}} [1 + r_{\rho'}^2/m_{\rho'}^2]^{-1/2} \\
 &= \frac{f_{\rho\pi^0\gamma}}{f_\rho} + \frac{f_{\rho'\pi^0\gamma}}{f_{\rho'}} \quad (\text{II-84})
 \end{aligned}$$

The normalization condition (II-63) thus gives a relation

$$\left| \frac{f_{\omega\pi^0\gamma}}{f_\omega} + \frac{f_{\phi\pi^0\gamma}}{f_\phi} \right| = \left| \frac{f_{\rho\pi^0\gamma}}{f_\rho} + \frac{f_{\rho'\pi^0\gamma}}{f_{\rho'}} \right| \quad (\text{II-85})$$

This relation also can be derived in terms of the coupling constant $f_{\rho\pi^0\omega}$; from the Gell-Mann-Sharp-Wagner (GSW) model²⁴⁾ as shown graphically in Fig. 8b, 8c, and 8d.

In the GSW model, we obtain the relations

$$\begin{aligned}
 f_{\rho\pi^0\omega} &= f_\omega f_{\rho\pi^0\gamma} \\
 &= f_\rho f_{\omega\pi^0\gamma} \quad (\text{II-86})
 \end{aligned}$$

and

$$\begin{aligned}
 [v_0^{1,0}(0,0)]_{\text{GSW}} &= \frac{f_{\rho\pi^0\omega}}{f_\rho f_\omega} = \frac{f_{\omega\pi^0\gamma}}{f_\omega} = \frac{f_{\rho\pi^0\gamma}}{f_\rho} \\
 &= .0231 m_{\pi^0}^{-1}
 \end{aligned}$$

which is in reasonable agreement with $v_0^{1,0}(0,0) = 0.0188 m_\pi^{-1}$.

If we assume that

$$\frac{f_{\omega\pi^0\gamma}}{f_\omega} \approx \frac{f_{\rho\pi^0\gamma}}{f_\rho}, \text{ and } \frac{f_{\phi\pi^0\gamma}}{f_\phi} \approx \frac{f_{\rho'\pi^0\gamma}}{f_{\rho'}} \quad (\text{II-87})$$

$v_0^1(s,0)$ can be rewritten as

$$v_0^1(s,0) \approx \frac{f_{\omega\pi^0\gamma}}{f_\omega} [(1 - s/m_\rho^2)^2 + \Gamma_{\rho'/\rho}^2/m_{\rho'}^2]^{1/2} + \frac{f_{\phi\pi^0\gamma}}{f_\phi} [(1 - s/m_{\rho'}^2)^2 + \Gamma_{\rho'/\rho'}^2/m_{\rho'}^2]^{-1/2} \quad (\text{II-88a})$$

$$\approx 2.310 \times 10^{-2} m_\pi^{-1} [(1 - s/m_\rho^2)^2 + \Gamma_{\rho'/\rho}^2/m_{\rho'}^2]^{-1/2}$$

$$- .393 \times 10^{-2} m_\pi^{-1} [(1 - s/m_{\rho'}^2)^2 + \Gamma_{\rho'/\rho'}^2/m_{\rho'}^2]^{-1/2}$$

$$+ \Gamma_{\rho'/\rho'}^2/m_{\rho'}^2]^{-1/2} \quad (\text{II-88b})$$

or

$$\frac{v_0^1(s,0)}{v_0^1(0,0)} \approx 1.229 [(1 - s/m_\rho^2)^2 + \Gamma_{\rho'/\rho}^2/m_{\rho'}^2]^{-1/2}$$

$$- .226 [(1 - s/m_{\rho'}^2)^2 + \Gamma_{\rho'/\rho'}^2/m_{\rho'}^2]^{-1/2}$$

$$(\text{II-89})$$

which has a maximum value ~ 5.87 at $s = m_\rho^2$.

Eq. (II-80) may be used to compute $v_0^1(s,0)$ and $v(s,0)$ for any value of s .

We now remember that we have already assumed that $v_0^1(s,0)$ to be dominated by the $\pi^+\pi^-$, $I = 1$, $J = 1$ contribution. It may be possible to write $v_0^1(s,0)$, with the correct phase and singularities, as proportional¹⁶⁾ to the charged pion electromagnetic form factor, $F_\pi(s)$, normalized to unity for $s = 0$. Since $F_\pi(s)$ dominates in the ρ region, we can then take into account the relation¹⁶⁾

$$v_0^1(s,0) = v_0^1(0,0) F_\pi(s) \quad (\text{II-90})$$

where $F_\pi(s)$ can be derived from the $e^+e^- \rightarrow V \rightarrow \pi^+\pi^-$ reaction¹⁷⁾, and has the form,

$$F_\pi(s) = \sum_{V=\rho,\rho'} \frac{f_{V\pi\pi}}{f_V} [(1 - s/m_V^2)^2 + \Gamma_V^2/m_V^2]^{-1/2} \quad (\text{II-91})$$

Therefore we have

$$\begin{aligned} \frac{v_0^1(s,0)}{v_0^1(0,0)} &= \frac{f_{\rho\pi\pi}}{f_\rho} [(1 - s/m_\rho^2)^2 + \Gamma_\rho^2/m_\rho^2]^{-1/2} \\ &\quad + \frac{f_{\rho'\pi\pi}}{f_{\rho'}} [(1 - s/m_{\rho'}^2)^2 + \Gamma_{\rho'}^2/m_{\rho'}^2]^{-1/2} \end{aligned} \quad (\text{II-92})$$

The $\rho(\rho'')$ - $\pi\pi$ coupling constants can be determined from the experimental data by using the expression

$$\Gamma(V \rightarrow \pi^+\pi^-) = \frac{f_{V\pi\pi}^2}{48\pi} m_V \left(1 - \frac{4m_\pi^2}{m_V^2}\right)^{3/2} \quad (\text{II-93})$$

and their values are $f_{\rho\pi\pi} = 6.090$ and $f_{\rho''\pi\pi} = 4.44 \sim 10.39$. According to the normalization condition, we obtain that

$$\frac{f_{\rho''\pi\pi}}{f_{\rho\pi\pi}} = -0.175 \quad (\text{II-94})$$

$$\frac{\Gamma(\rho'' \rightarrow e^+e^-)}{\Gamma(\rho'' \rightarrow \text{all})} = (3.3 \sim 16.7) \times 10^{-7} \quad (\text{prediction})$$

and

$$\frac{v_0^1(s,0)}{v_0^1(0,0)} = 1.195 \left[\left(1 - s/m_\rho^2\right)^2 + \Gamma_\rho^2/m_\rho^2 \right]^{-1/2} - 0.175 \left[\left(1 - s/m_{\rho''}^2\right)^2 + \Gamma_{\rho''}^2 \right]^{-1/2} \quad (\text{II-95})$$

which has a maximum ~ 5.76 at $s \sim m_\rho^2$. This result is very close to Eq. (II-89).

On the other hand the $\tau^- \rightarrow \nu_\tau \pi^- \gamma$ decay process may be described in the VMD model without the isospin rotation, as shown below diagrammatically in Figs. 9a and 9b, assuming $\rho(\rho')$ - and A_1 - dominance.

Fig. 9 The $\tau^- \rightarrow \nu_\tau \pi^- \gamma$ decay through the VMD and the vector meson decays.

Figs. 9a and 9b give us the vector and axial vector form factors $v(s,0)$ and $a(s,0)$ which have the forms

$$v(s,0) = \sum_{V=\rho,\rho'} \frac{f_{V\pi\gamma}}{f_V} [(1 - s/m_V^2)^2 + \Gamma_V^2/m_V^2]^{-1/2} \quad (\text{II-96})$$

$$a(s,0) = \sum_A \frac{f_{A\pi\gamma}}{f_A} [(1 - s/m_A^2)^2 + \Gamma_A^2/m_A^2]^{-1/2} \quad (\text{II-97})$$

where A denotes all the possible axial-vector particles, but one has observed only $A_1(1100)$ so far. By the normalization condition for $v(s,0)$

$$\begin{aligned} v(0,0) &= \frac{f_{\rho\pi\gamma}}{f_\rho} [(1 + \Gamma_\rho^2/m_\rho^2)^{-1/2}] + \frac{f_{\rho'\pi\gamma}}{f_{\rho'}} [1 + \Gamma_{\rho'}^2/m_{\rho'}^2]^{-1/2} \\ &= 2.61 \times 10^{-2} m_\pi^{-1} \end{aligned} \quad (\text{II-98})$$

we obtain

$$\begin{aligned} v(s,0) &= 1.47 \times 10^{-2} m_\pi^{-1} [(1 - s/m_\rho^2)^2 + \Gamma_\rho^2/m_\rho^2]^{-1/2} \\ &\quad + 1.19 \times 10^{-2} m_\pi^{-1} [(1 - s/m_{\rho'}^2)^2 + \Gamma_{\rho'}^2/m_{\rho'}^2]^{-1/2} \end{aligned} \quad (\text{II-99})$$

and

$$\frac{v(s,0)}{v(0,0)} = .563 [(1 - s/m_\rho^2)^2 + \Gamma_\rho^2/m_\rho^2]^{-1/2} + .456 [(1 - s/m_{\rho'}^2)^2 + \Gamma_{\rho'}^2/m_{\rho'}^2]^{-1/2} \quad (\text{II-100})$$

where we have used $\Gamma_{(\rho \rightarrow \pi^- \gamma)}/\Gamma_{(\rho \rightarrow \text{all})} = .024 \%^{22}$ to compute $f_{\rho \pi^- \gamma}$ (large error 30. %). Eq. (II-100) also can be used to evaluate the $\tau^- \rightarrow \nu_\tau \pi^- \gamma$ decay rate.

For the $a(s,0)$, we obtain from Eqs. (II-97) and (I -32)

$$\begin{aligned} a(s,0) &= - \frac{2f_\pi \delta}{m_{A_1}^2} [(1 - s/m_{A_1}^2)^2 + \Gamma_{A_1}^2/m_{A_1}^2]^{-1/2} \\ &= - \delta \frac{f_\pi}{m_\rho^2} [(1 - s/2m_\rho^2)^2 + \Gamma_{A_1}^2/2m_\rho^2]^{-1/2} \end{aligned} \quad (\text{II-101})$$

where we have used the Weinberg's first sum rule $m_{A_1} = \sqrt{2} m_\rho$. Thus, the ratio to the vector form factor $v(0,0)$ is

$$\begin{aligned} \frac{a(s,0)}{v(0,0)} &= -2 \frac{\delta f_\pi}{v(0,0) m_\rho^2} [(2 - s/m_\rho^2)^2 + \Gamma_{A_1}^2/m_\rho^2]^{-1/2} \\ &= -2.230\delta [(2 - s/m_\rho^2)^2 + .2989]^{-1/2} \end{aligned} \quad (\text{II-102})$$

which gives

$$\frac{a(0,0)}{v(0,0)} \approx 1.13$$

5. Numerical Results and Discussions

To estimate the numerical results for the $\tau^- \rightarrow \nu_\tau \gamma$ decay rate we rewrite Eq. (II-54) by using the results obtained in the previous section as

$$\frac{d^2 R_{\nu\tau\gamma}}{dx dy} = R_{BB}'' + R_{BV}'' + R_{BA}'' + R_{VV}'' + R_{VA}'' + R_{AA}'' \quad (\text{II-103})$$

$$R_{BB}'' = \frac{d^2 R_{BB}}{dx dy} = 1.176 \times 10^{-3} F_{BB}(x, y)$$

$$R_{BV}'' = \frac{d^2 R_{BV}}{dx dy} = -5.622 \times 10^{-3} F^1(s) F_{BV}(x, y)$$

$$R_{BA}'' = \frac{d^2 R_{BA}}{dx dy} = 5.622 \times 10^{-3} A(s) F_{BA}(x, y)$$

$$R_{VV}'' = \frac{d^2 R_{VV}}{dx dy} = 6.714 \times 10^{-3} [F^1(s)]^2 F_{VV}(x, y)$$

$$R_{VA}'' = \frac{d^2 R_{VA}}{dx dy} = 1.343 \times 10^{-2} F^1(s) A(s) F_{VA}(x, y)$$

$$R_{AA}'' = \frac{d^2 R_{AA}}{dx dy} = 6.714 \times 10^{-3} [A(s)]^2 F_{AA}(x, y) \quad (\text{II-104})$$

where

$$F^i(s) = \left[\frac{v(s,0)}{v(0,0)} \right]^i \quad (i = 1, 2, 3)$$

from Eqs. (II-89), (II-95) and (II-100)

$$\begin{aligned} F^1(s) &= 1.229 [(1 - s/m_p^2)^2 + \Gamma_p^2/m_p^2]^{-1/2} \\ &\quad - .226 [(1 - s/m_{p,\pi}^2)^2 + \Gamma_{p,\pi}^2/m_{p,\pi}^2]^{-1/2} \\ &= 1.229 [(1 - s/m_p^2)^2 + .0339]^{-1/2} \\ &\quad - .226 [(1 - s/m_{p,\pi}^2)^2 + .0352]^{-1/2} \end{aligned}$$

(II-105)

$$\begin{aligned} F^2(s) &= 1.195 [(1 - s/m_p^2)^2 + .0399]^{-1/2} \\ &\quad - .175 [(1 - s/m_{p,\pi}^2)^2 + .0352]^{-1/2} \end{aligned}$$

(II-106)

$$\begin{aligned} F^3(s) &= .563 [(1 - s/m_p^2)^2 + .0399]^{-1/2} \\ &\quad + .456 [(1 - s/m_{p,\pi}^2)^2 + .0352]^{-1/2} \end{aligned}$$

(II-107)

and

$$A(s) = \frac{A(s,0)}{v(0,0)} = -2.230 \delta [(2 - s/m_0^2)^2 + .2989]^{-1/2} \quad (\text{III-108})$$

from Eq. (II-102).

We have used the experimental data for the Review of Particle Properties by the Particle Data Group for 1978²².

The integrated values are

$$R_{\nu\pi\gamma} = \iint \left(\frac{d^2 R_{\nu\pi\gamma}}{dx dy} \right) dx dy$$

$$= R_{BB} + R_{BV} + R_{BA} + R_{VV} + R_{VA} + R_{AA} \quad (\text{III-109})$$

When we integrate we choose two possible values of δ :

$\delta = -1$ and $\delta = -1/2$ which are more favoured in the second phenomenological analyses. Also we cut the photon energy at 100 MeV and 200 MeV in integration. All experiments performed for τ -decay have used these energy cuts.

Actually it is very difficult to detect the photons having less than 100 MeV energy.²⁵

We have already shown that $F_{BB}(x,y)$ dominates over all the other phase space functions $F_{BV}(x,y)$, $F_{BA}(x,y)$, $F_{VV}(x,y)$, $F_{VA}(x,y)$ and $F_{AA}(x,y)$ within the

phase space, except for a small region as shown in Fig. 6. Therefore only the region where $F^1(s)$ and $A(s)$ are large enough to make the structure terms comparable with $F_{BB}(x,y)$ can be important for seeing the structure-dependent effect.

Eqs. (II-105) - (II-107) show that $[F^1(s)]^2 = [F^2(s)]^2 = 30$ and $[F^3(s)]^2 = 12$ near the ρ corresponding to $x + y = 1.19$, and thus R_{VV}'' can be considerably larger than R_{BB}'' in the ρ -band region for x not too small. For example, at $x = .80$, $y = .39$, $R_{VV}'' = 11 \times 10^{-3}$, $R_{BB}'' = 3 \times 10^{-3}$, $R_{BV}'' = -4 \times 10^{-3}$, $R_{BA}'' = 4 \times 10^{-3}$, $R_{VA}'' = -2 \times 10^{-3}$ and $R_{AA}'' = 2 \times 10^{-3}$.

If we integrate over the ρ -band ($s \pm \Delta s = (m_\rho \pm \Gamma_\rho)^2$ or $1.1214 \leq x + y \leq 1.2730$ say) we estimate the decay rates $R_{VV} = 9.45 \times 10^{-4}$, $R_{BB} = 8.44 \times 10^{-4}$, $R_{AA} = 2.56 \times 10^{-4}$, $R_{BA} = 6.96 \times 10^{-4}$, $R_{VA} = .1 \times 10^{-6}$ and $R_{V\pi\gamma} = 2.44 \times 10^{-3}$ with $F^2(s)$, $\delta = -1$, and $E_\gamma \geq 100$ MeV as shown in Table 2. For $F^1(s)$, $F^3(s)$ and $\delta = -1/2$ the computed results are also shown in Table 2. Table 3 shows the estimated results with photon energy cut $E_\gamma \geq 200$ MeV, and shows that the total decay rate $R_{V\pi\gamma}$ decreased ~ 6 %. In the two tables we see that the pure bremsstrahlung contribution R_{BB} depends on the photon energy cut more strongly than the other contributions do. $F^1(s)$ and $F^2(s)$ give almost the

2

same result as expected. Despite its quite different coupling constant ratios $F^3(s)$ also gives an almost consistent result in total decay rate $R_{V\pi\gamma}$. Tables 2, 3 show that R_{VV} contribution in $F^3(s)$ is decreased by $\sim 60\%$, which gives the difference between $F^3(s)$ and $F^1(s)$ or $F^2(s)$.

With $B_{V\pi}^{\tau} = \Gamma(\tau^- \rightarrow \nu_{\tau} \pi^-) / \Gamma(\tau^- \rightarrow \text{all}) = .083^{1;2)}$ the total branching ratios $B_{V\pi\gamma}^{\tau}$ are 2.03×10^{-4} , 2.02×10^{-4} and 1.64×10^{-4} for $F^1(s)$, $F^2(s)$ and $F^3(s)$, respectively. Assuming the accompanying τ^+ would be detected by a signal of e^+ or μ^+ with $B_{e(\mu)}^{\tau} \approx 2 \times .18 \approx .36$, we estimate an effective detectable $B_{V\pi\gamma}^{\tau}$ of $\sim .73 \times 10^{-5}$ for $F^1(s)$ and $F^2(s)$, $\sim 5.9 \times 10^{-5}$ for $F^3(s)$ or about $3 \sim 4$ detectable radiative events in the ρ -band for every $\approx 6.4 \times 10^4$ τ -pairs produced.

If we consider only the pure vector part R_{VV} in this band we obtain

$$B_{VV}^1 = [\Gamma(\tau \rightarrow \nu_{\tau} \rho \rightarrow \nu_{\tau} \pi^- \gamma) / \Gamma(\tau^- \rightarrow \text{all})]^1: \quad i = 1, 2, 3$$

corresponding to

$F^1(s)$, $F^2(s)$ and $F^3(s)$;

$$B_{VV}^1 = [1.083] \times [9.73 \times 10^{-4}] = 8.08 \times 10^{-5}$$

$$B_{VV}^2 = [1.083] \times [9.45 \times 10^{-4}] = 7.84 \times 10^{-5}$$

$$B_{VV}^3 = [0.83] \times [3.80 \times 10^{-4}] = 3.15 \times 10^{-5}$$

The experimental data^{1,2,22)} give

$$\begin{aligned} B_{VV}^{\text{EXP}} &= [\Gamma(\tau^- \rightarrow \nu_\tau^- \rho^-) / \Gamma(\tau^- \rightarrow \text{all})] \times [\Gamma(\rho^- \rightarrow \pi^- \gamma) / \Gamma(\rho \rightarrow \text{all})] \\ &= [1.24] \times [2.4 \times 10^{-4}] = 5.76 \times 10^{-5} \end{aligned}$$

which is in agreement with our estimates.

However, the new world average values³⁾

$$B_{V\pi}^{\tau} = .098$$

$$B_{V\rho}^{\tau} = .215$$

seem to give more confidence to the $F^3(s)$ case which was obtained from the direct vector meson dominance. In this case we estimate

$$B_{VV}^3 = 3.72 \times 10^{-5}$$

which now should be compared with.

$$B_{VV}^{\text{EXP}} = [1.215] \times [2.4 \times 10^{-4}] = 5.16 \times 10^{-5}$$

When we derived the expression for $F^3(s)$ we have used $B_{\pi^- \gamma}^{\rho} = \Gamma(\rho \rightarrow \pi^- \gamma) / \Gamma(\rho \rightarrow \text{all}) = .024 \%$ ($\Gamma(\rho \rightarrow \pi^- \gamma) = 37.2 \text{ KeV}$)²²⁾ which is approximately 2 - 3 times smaller than anticipated, using quark or higher symmetry models²⁶⁾. This is a long standing problem. Recently Kamal and Kane²⁷⁾ suggested that the measurement of vector meson radiative decays in the Primakoff-effect experiments on nuclei should be reanalyzed including isovector hadronic exchange. Thus, if we take $\Gamma(\rho \rightarrow \pi^- \gamma) = 70 \text{ KeV}$ and 100 KeV ²⁹⁾, Eq. (II-100) for $F^3(s)$ becomes

$$F^3(s) = .773 [(1 - s/m_\rho^2)^2 + .0399]^{-1/2} + .247 [(1 - s/m_\rho^2)^2 + .0352]^{-1/2} \quad (\text{II-110})$$

$$F^3(s) = .923 [(1 - s/m_\rho^2)^2 + .0399]^{-1/2} + .096 [(1 - s/m_\rho^2)^2 + .0352]^{-1/2} \quad (\text{II-111})$$

which yield the results shown in Tables 12 - 15.

With $B_{\nu\pi}^{\tau} = .098$ and the values for R_{VV} in Tables 12 and 14, we estimate for $\delta = -1$ $B_{VV}^3 = 5.28 \times 10^{-5}$, 6.56×10^{-5} for $\Gamma(\rho \rightarrow \pi^- \gamma) = 70 \text{ KeV}$ and 100 KeV , respectively. These estimates now should be compared

with the assumed experimental values

$$B_{VV}^{\text{EXP}} (\text{assumed}) = [0.215] \times [2.4 \times 10^{-4} \times 70/37.2] = 9.71 \times 10^{-5}$$

$$B_{VV}^{\text{EXP}} (\text{assumed}) = [0.215] \times [2.4 \times 10^{-10} \times 100/37.2] = 13.90 \times 10^{-5}$$

and do not give good agreement comparing with previous estimates.

Similarly, since $[a(s,0)/v(0,0)]^2 \approx 16$ near the A_1 , R_{AA} may be significantly larger than R_{BB} in the A_1 -band ($s \pm \Delta s = (m_{A_1} \pm \Gamma_{A_1})^2$ or $1.202 \leq x + y \leq 1.617$). Integrating over the A_1 -band gives us the results shown in Tables 4 and 5. Two results in both tables are almost the same because $x_{\min}$ in this band corresponds to ~ 200 MeV photon energy cut. Table 4 shows that the estimated total decay rates $R_{\nu\pi\gamma}$ in the A_1 -band are almost equal to 4.113×10^{-3} for $F^1(s)$, $F^2(s)$ and $F^3(s)$. Particularly, for R_{AA} and R_{BA} the three results for $F^1(s)$, $F^2(s)$ and $F^3(s)$ are exactly all the same in any band (i.e. ρ^- , A_1^- , $\rho-A_1$, and above ρ -band), even in the overall phase space.

In the A_1 -band we obtained $R_{AA} = 1.93 \times 10^{-3}$, $R_{BB} = 4.73 \times 10^{-4}$ and $R_{\nu\pi\gamma} = 4.113 \times 10^{-3}$; we estimate one detectable radiative event for every $\sim 8.3 \times 10^3$

(with $B_{\nu\pi}^{\tau} = .083$) or $\sim 7.0 \times 10^{-3}$ (with $B_{\nu\pi}^{\tau} = .098$)

τ -pairs produced. Since $\Gamma(\tau^- \rightarrow \nu_{\tau} A_1)/\Gamma(\tau^- \rightarrow \text{all}) = 0.108$ has been seen³⁾, we can also predict the branching ratio

$\Gamma(A_1^- \rightarrow \pi^- \nu)/\Gamma(A_1^- \rightarrow \text{all}) \approx 1.75 \times 10^{-3}$ for $\delta = -1$ with $B_{\nu\pi}^{\tau} = .098$. For $\delta = -1/2$ this value becomes four times smaller. The estimated results in the ρ - A_1 band ($1.1214 \leq x+y \leq 1.617$) and above the ρ -band ($1.124 \leq x+y \leq 2.0$) are shown in Tables 6 - 9.

At $s = 15 \text{ GeV}^2$ (just above τ -threshold),

$$\sigma(e^+e^- \rightarrow \tau^+\tau^-) = \frac{4\pi\alpha^2}{3s} \left(1 + \frac{2m_{\tau}^2}{s}\right) \left(1 - \frac{4m_{\tau}^2}{s}\right)^{1/2} \quad (112)$$

gives

$$\sigma(e^+e^- \rightarrow \tau^+\tau^-) \approx 3.3 \times 10^{-33} \text{ cm}^2$$

A luminosity of $10^{31} \text{ cm}^{-2} \text{ sec}^{-1}$ and 15 days running time will produce 4.3×10^4 τ -pairs or 3 detectable radiative events in the ρ -band and 5 events in the A_1 band. For the overlapping ρ - A_1 band ~ 7 detectable events can be expected.

When we now integrate overall the phase space ($1.0 \leq x + y \leq 2.0$) with the photon energy cuts of 100 MeV and 200 MeV, we get $R_{\nu\pi\gamma}^{\tau} \approx 1.38 \times 10^{-2}$ with $E_{\gamma} \geq 100 \text{ MeV}$, $R_{\nu\pi\gamma}^{\tau} \approx 9.54 \times 10^{-3}$ with $E_{\gamma} \geq 200 \text{ MeV}$ (see Table 10, 11), this implies that we need about.

20 hours running time with a luminosity of $10^{31} \text{ cm}^{-2} \text{ sec}^{-1}$ to produce one detectable radiative event in the $\tau^- \rightarrow \nu_\tau \pi^- \gamma$ process. The detailed results of integration over all the phase space are shown in Table 10 and 11.

To measure separately $v(s,0)$ and $a(s,0)$ as functions of s would still require correspondingly greater luminosity or running time. By considering the facts that $R_{\nu\pi\gamma}$ is large and that W. Bacino³⁰⁾ et al. have recently observed a clear signal of the $\tau^- \rightarrow \nu_\tau \pi^-$ decay based on a sample of 41 events of the type, $e^+e^- \rightarrow e^+X^-$ ($X \neq e$) and no observed photons with $E_\gamma > 100 \text{ MeV}$, it should be clear that a first generation experiment, in which $\tau^- \rightarrow \nu_\tau \pi^- \gamma$ is simply detected, should be feasible now with existing beams and detectors. A second generation experiment, in which $v(s,0)$ and $a(s,0)$ are measured near the ρ or A_1 bands, might just be feasible if present facilities are pushed to the limit, and if the VMD mode estimates are valid. Since the higher region $1.617 \leq x + y \leq 2.0$ contributes $< 2\%$ to the total branching ratio, the measurement of $v(s,0)$ and $a(s,0)$ for large s (approaching the kinematic limit $m_\tau^2 \approx 3.2 \text{ GeV}^2$) is probably not possible with present luminosities, although there is probably an enhancement at $s = 2.6 \text{ GeV}^2$ due to $\rho(1600)$.

Table 2: Decay Rates in ρ -band ($1.1214 \leq x+y \leq 1.2730$) with $E_Y \geq 100$ MeV.

(x10 ⁻³)	$\delta = -1$			$\delta = -1/2$		
	Isospin Rotation		Direct Way	Isospin Rotation		Direct Way
	F ¹ (s)	F ² (s)	F ³ (s)	F ¹ (s)	F ² (s)	F ³ (s)
R _{BB}	.844	.844	.844	.844	.844	.844
R _{BV}	-.307	-.302	-.194	-.307	-.302	-.194
R _{BA}	.696	.696	.696	.348	.348	.348
R _{VV}	.973	.945	.380	.973	.945	.380
R _{VA}	.1x10 ⁻³	.1x10 ⁻³	.8x10 ⁻⁴	.4x10 ⁻⁴	.4x10 ⁻⁴	.4x10 ⁻⁴
R _{AA}	.256	.256	.256	.064	.064	.064
R _{VπY}	2.463	2.437	1.981	1.920	1.899	1.440

Table 3: Decay Rates in ρ -band with $E_Y \geq 200$ MeV

(x10 ⁻³)	$\delta = -1$			$\delta = -1/2$		
	Isospin Rotation		Direct Way	Isospin Rotation		Direct Way
	F ¹ (s)	F ² (s)	F ³ (s)	F ¹ (s)	F ² (s)	F ³ (s)
R _{BB}	.764	.764	.764	.764	.764	.764
R _{BV}	-.303	-.299	-.192	-.303	-.299	-.192
R _{BA}	.677	.677	.677	.339	.339	.339
R _{VV}	.950	.923	.371	.950	.923	.371
R _{VA}	-.012	-.012	-.007	-.006	-.006	-.004
R _{AA}	.235	.235	.235	.063	.063	.063
R _{VπY}	2.328	2.302	1.866	1.810	1.780	1.340

Table 4: Decay Rates in A_1 -band ($1.2015 \leq x+y \leq 1.6172$)
with $E_\gamma \geq 100$ MeV.

(x10 ⁻³)	$\delta = -1$			$\delta = -1/2$		
	Isospin Rotation		Direct Way	Isospin Rotation		Direct Way
	F ¹ (s)	F ² (s)	F ³ (s)	F ¹ (s)	F ² (s)	F ³ (s)
R _{BB}	.475	.475	.475	.475	.475	.475
R _{BV}	-.236	-.246	-.318	-.236	-.246	-.318
R _{BA}	1.410	1.410	1.410	.706	.706	.706
R _{VV}	.536	.536	.570	.536	.536	.570
R _{VA}	.008	.008	.005	.004	.004	.002
R _{AA}	1.930	1.930	1.930	.483	.483	.483
R _{Vπγ}	4.135	4.125	4.085	1.965	1.955	1.924

Table 5: Decay Rates in A_1 -band with $E_\gamma \geq 200$ MeV

(x10 ⁻³)	$\delta = -1$			$\delta = -1/2$		
	Isospin Rotation		Direct Way	Isospin Rotation		Direct Way
	F ¹ (s)	F ² (s)	F ³ (s)	F ¹ (s)	F ² (s)	F ³ (s)
R _{BB}	.473	.473	.473	.473	.473	.473
R _{BV}	-.236	-.246	-.317	-.236	-.246	-.318
R _{BA}	1.410	1.410	1.410	.706	.706	.706
R _{VV}	.533	.534	.569	.533	.534	.570
R _{VA}	.006	.006	.004	.003	.003	.002
R _{AA}	1.930	1.930	1.930	.483	.483	.483
R _{Vπγ}	4.123	4.113	4.063	1.963	1.953	1.922

Table 6: Decay Rates in ρ - A_1 -band ($1.1214 \leq x+y \leq 1.6172$)
with $E_\gamma \geq 100$ -MeV

(x10 ⁻³)	$\delta = -1$			$\delta = -1/2$		
	Isospin Rotation		Direct Way	Isospin Rotation		Direct Way
	F ¹ (s)	F ² (s)	F ³ (s)	F ¹ (s)	F ² (s)	F ³ (s)
R _{BB}	1.090	1.090	1.090	1.090	1.090	1.090
R _{BA}	-.413	-.421	-.425	-.413	-.421	-.425
R _{BA}	1.770	1.770	1.770	.885	.885	.885
R _{VV}	1.100	1.080	.771	1.100	1.080	.771
R _{VA}	-.4x10 ⁻⁴	-.3x10 ⁻⁴	.1x10 ⁻³	-.1x10 ⁻⁴	-.9x10 ⁻⁵	.7x10 ⁻⁴
R _{AA}	2.010	2.010	2.010	.502	.502	.502
R _{V$\pi$$\gamma$}	5.560	5.530	5.220	3.160	3.14	2.830

Table 7: Decay Rates in ρ - A_1 -band with $E_\gamma \geq 100$ MeV

(x10 ⁻³)	$\delta = -1$			$\delta = -1$		
	Isospin Rotation		Direct Way	Isospin Rotation		Direct Way
	F ¹ (s)	F ² (s)	F ³ (s)	F ¹ (s)	F ² (s)	F ³ (s)
R _{BB}	1.010	1.010	1.010	1.010	1.010	1.010
R _{BA}	-.410	-.417	-.423	-.410	-.417	-.423
R _{BA}	1.750	1.750	1.750	.876	.876	.876
R _{VV}	1.070	1.060	.763	1.070	1.060	.763
R _{VA}	-.012	-.012	-.007	-.006	-.006	-.004
R _{AA}	2.000	2.000	2.000	.501	.501	.501
R _{V$\pi$$\gamma$}	5.420	5.390	5.100	3.05	3.02	2.730

Table 8: Decay Rates above and Including the ρ -band
($1.1214 \leq x+y \leq 2.0$) with $E_\gamma \geq 100$ MeV.

(x10 ⁻³)	$\delta = -1$			$\delta = -1/2$		
	Isospin Rotation		Direct Way	Isospin Rotation		Direct Way
	F ¹ (s)	F ² (s)	F ³ (s)	F ¹ (s)	F ² (s)	F ³ (s)
R _{BB}	1.110	1.110	1.110	1.110	1.110	1.110
R _{BV}	-.391	-.407	-.515	-.391	-.407	-.515
R _{BA}	1.830	1.830	1.830	.917	.917	.917
R _{VV}	1.120	1.090	1.130	1.120	1.090	1.130
R _{VA}	-.2x10 ⁻⁴	-.2x10 ⁻⁴	-.6x10 ⁻⁴	-.5x10 ⁻⁵	-.5x10 ⁻⁵	-.2x10 ⁻⁴
R _{AA}	2.120	2.120	2.120	.529	.529	.529
R _{Vπγ}	5.790	5.740	5.670	3.286	3.240	3.162

Table 9: Decay Rates Above ρ -band with $E_\gamma \geq 200$ MeV

(x10 ⁻³)	$\delta = -1$			$\delta = -1/2$		
	Isospin Rotation		Direct Way	Isospin Rotation		Direct Way
	F ¹ (s)	F ² (s)	F ³ (s)	F ¹ (s)	F ² (s)	F ³ (s)
R _{BB}	1.030	1.030	1.030	1.030	1.030	1.030
R _{BV}	-.388	-.404	-.513	-.388	-.404	-.513
R _{BA}	1.820	1.820	1.820	.908	.908	.908
R _{VV}	1.100	1.070	1.120	1.100	1.070	1.120
R _{VA}	-.012	-.012	-.007	-.006	-.006	-.004
R _{AA}	2.110	2.110	2.110	.528	.528	.528
R _{Vπγ}	5.660	5.610	5.550	3.171	3.125	3.066

Table 10: Decay Rates in the Phase Space ($1.0 \leq x+y \leq 2.0$)
with $E_\gamma \geq 100$ MeV

(x10 ⁻³)	$\delta = -1$			$\delta = -1/2$		
	Isospin Rotation		Direct Way	Isospin Rotation		Direct Way
	F ¹ (s)	F ² (s)	F ³ (s)	F ¹ (s)	F ² (s)	F ³ (s)
R _{BB}	8.720	8.720	8.720	8.720	8.720	8.720
R _{BV}	-.436	-.452	-.550	-.436	-.452	-.550
R _{BA}	2.300	2.300	2.300	1.150	1.150	1.150
R _{VV}	1.170	1.140	1.150	1.170	1.140	1.150
R _{VA}	-.4x10 ⁻³	-.4x10 ⁻³	-.3x10 ⁻³	-.2x10 ⁻³	-.2x10 ⁻³	-.2x10 ⁻³
R _{AA}	2.180	2.180	2.180	.545	.545	.545
R _{VπY}	13.930	13.890	13.810	11.143	11.097	11.019

Table 11: Decay Rates in the Phase Space with $E_\gamma \geq 200$ MeV

(x10 ⁻³)	$\delta = -1$			$\delta = -1/2$		
	Isospin Rotation		Direct Way	Isospin Rotation		Direct Way
	F ¹ (s)	F ² (s)	F ³ (s)	F ¹ (s)	F ² (s)	F ³ (s)
R _{BB}	4.590	4.590	4.590	4.590	4.590	4.590
R _{BV}	-.429	-.445	-.545	-.429	-.445	-.545
R _{BA}	2.190	2.190	2.190	1.090	1.090	1.090
R _{VV}	1.140	1.110	1.140	1.140	1.110	1.140
R _{VA}	-.014	-.014	-.009	-.007	-.007	-.004
R _{AA}	2.180	2.180	2.180	.544	.544	.544
R _{VπY}	9.650	9.600	9.540	6.935	6.890	6.820

Table 12: Decay Rates for the $F^3(s)$ Case with
 $\Gamma(\rho \rightarrow \pi^- \gamma) = 70 \text{ KeV}$ and $E_\gamma \geq 100 \text{ MeV}$

(x10 ⁻³)	$\delta = -1$			$\delta = -1/2$		
	All Space	ρ -Band	A_1 -Band	All Space	ρ -Band	A_1 -Band
R_{BB}	8.720	.844	.475	8.720	.844	.475
R_{BV}	-.518	-.230	-.301	-.518	-.230	-.301
R_{BA}	2.300	.696	1.430	1.150	.348	.715
R_{VV}	.976	.539	.516	.976	.539	.516
R_{VA}	$-.4 \times 10^{-3}$	$-.9 \times 10^{-4}$	$-.7 \times 10^{-6}$	$-.2 \times 10^{-3}$	$.4 \times 10^{-4}$	$.6 \times 10^{-5}$
R_{AA}	2.180	.256	1.970	.545	$.641 \times 10^{-1}$	.493
$R_{\nu\pi\gamma}$	13.660	2.104	4.095	10.871	1.564	1.899

Table 13: Decay Rates for the $F^3(s)$ Case with
 $\Gamma(\rho \rightarrow \pi^- \gamma) = 70 \text{ KeV}$ and $E_\gamma \geq 200 \text{ MeV}$

(x10 ⁻³)	$\delta = -1$			$\delta = -1/2$		
	All space	ρ -Band	A_1 -Band	All Space	ρ -band	A_1 -Band
R_{BB}	4.590	.764	.474	4.590	.764	.474
R_{BV}	-.512	-.228	-.301	-.512	-.228	-.301
R_{BA}	2.190	.677	1.430	1.090	.339	.715
R_{VV}	.961	.526	.514	.961	.526	.514
R_{VA}	-.011	-.009	$-.1 \times 10^{-5}$	.005	-.004	$-.7 \times 10^{-3}$
R_{AA}	2.180	.253	1.970	.544	$.632 \times 10^{-1}$	.493
$R_{\nu\pi\gamma}$	9.400	1.985	4.094	6.672	1.460	1.898

Table 14: Decay Rates for the $F^3(s)$ case with
 $\Gamma(\rho \rightarrow \pi^- \gamma) = 100 \text{ KeV}$ and $E_\gamma > 100 \text{ MeV}$

(x10 ⁻³)	$\delta = -1$			$\delta = -1/2$		
	All space	ρ -band	A ₁ -band	All Space	ρ -band	A-band
R _{BB}	8.720	.844	.475	8.720	.844	.475
R _{BV}	-.494	-.256	-.284	-.494	-.256	-.284
R _{BA}	2.300	.696	1.430	1.150	.348	.715
R _{VV}	.951	.669	.502	.951	.669	.502
R _{VA}	-.4x10 ⁻³	.9x10 ⁻⁴	-.6x10 ⁻⁵	-.2x10 ⁻³	.4x10 ⁻⁴	.3x10 ⁻⁵
R _{AA}	2.180	.256	1.970	.545	.641x10 ⁻¹	.493
R _{Vπγ}	13.650	2.210	4.105	10.875	1.670	1.902

Table 15: Decay Rates for the $F^3(s)$ Case with $\Gamma(\rho \rightarrow \pi^- \gamma) = 100 \text{ KeV}$
and $E_\gamma \geq 200 \text{ MeV}$

(x10 ⁻³)	$\delta = -1$			$\delta = -1/2$		
	All Space	ρ -band	A ₁ -band	All Space	ρ -band	A ₁ -band
R _{BB}	4.590	.764	.474	4.590	.764	.474
R _{BV}	-.488	-.253	-.284	-.488	-.253	-.284
R _{BA}	2.190	.677	1.430	1.090	.339	.715
R _{VV}	.933	.654	.501	.933	.654	.501
R _{VA}	-.012	-.010	-.002	-.006	-.005	-.8x10 ⁻³
R _{Vπγ}	9.390	2.085	4.094	6.667	1.560	1.898

CHAPTER III

GAUGE INVARIANT AMPLITUDE FOR THE $\pi^- \rightarrow e^- \bar{\nu}_e \gamma$ DECAY

1. Introduction

In this chapter we will discuss the gauge invariant amplitude for the radiative leptonic pion decay by assuming the pion to be a (1) structureless elementary particle and (2) quark-antiquark ($q\bar{q}$) bound state. For the point-like pion, the gauge invariant amplitude for the $\pi^- \rightarrow e^- \bar{\nu}_e \gamma$ can be easily obtained by the gauge condition using the soft-photon techniques.^{1,2)} When we assume the pion to be a $q\bar{q}$ bound state and introduce the $\pi q\bar{q}$ vertex function, this vertex function violates the Ward-Takahashi (W-T) identity unless it is a constant. We use the Drell-Lee ansatz³⁾ as the $\pi q\bar{q}$ vertex function.

The main purpose of this chapter is to make a simple prescription for gauge invariance and to obtain a gauge invariant amplitude for the $\pi^- \rightarrow e^- \bar{\nu}_e \gamma$ decay in the one-loop approximation with the fractionally-charged coloured quarks. It will also be shown that the same invariant amplitude can be derived by using a gauge projection operator⁴⁾.

2. Pion as a Point-like Particle

Before we treat the pion as a $q\bar{q}$ bound state, we consider the structureless elementary pion. The weak interaction is mediated by the IVB, which couples to the pion and the leptons. The pion decays to the leptons with the phenomenological amplitude, to lowest order in G_F ,

$$M(\pi \rightarrow e\nu_e) = i \frac{G_F}{\sqrt{2}} f_\pi \cos\theta_c K_\nu L^\nu \quad (\text{III-1})$$

as shown in Fig. 10.

Fig. 10. The non-radiative leptonic decay of a point-like pion.

For the radiative leptonic decay, we have three possible Feynman diagrams as shown in Fig. 11. Low has

shown¹⁾ that the amplitude for any collision process describing emission of a photon with four-momentum k has terms of order k^{-1} and of order zero in k , which depends only on the non-radiative amplitude. In this section we obtain the gauge invariant amplitude for the structureless pion decay by using the soft-photon techniques^{1,2)}. Fig. 11c is necessary to obtain the gauge invariant amplitude. Figs. 11a and 11b contain the pole terms which are singular like k^{-1} as $k \rightarrow 0$, and also contain the non-gauge invariant term which is compensated by a term from Fig. 11c.

Fig. 11 The radiative leptonic decay of a point-like pion.

With our earlier definitions, we obtain the amplitude for Fig. 11a

$$M_a = ie \frac{G_F}{\sqrt{2}} \cos \theta_c \frac{K \cdot \epsilon}{K \cdot k} F_\pi(m_\pi^2, s, 0) f_\pi(s) (K-k)_\nu L^\nu \quad (\text{III-2a})$$

where $s = (K-k)^2$, and $f_\pi(s)$ is a function of the momentum squared rather than a constant and $f_\pi \equiv f_\pi(m_\pi^2)$. The electromagnetic form factor $F_\pi(m_\pi^2, s, 0)$ is defined from the relation

$$\langle \pi^-(K') | j_\mu^{\text{em}} | \pi^-(K) \rangle = ie \{ (K+K')_\mu F_\pi(K^2, K'^2, k^2) + (K-K')_\mu F'_\pi(K^2, K'^2, k^2) \} \quad (\text{III-3})$$

where $k^2 = (K-K')^2$. Because of the W-T identity, $F_\pi(m_\pi^2, m_\pi^2, 0) = 1$. The Feynman diagram of Fig. 11a is regarded as the pion pole diagram, and hence $F_\pi(m_\pi^2, s, 0)$ is assumed not to contribute to the gauge invariant amplitude. We remember that we calculated the phenomenological invariant amplitude based on the soft-photon techniques in Chapters I and II. For Fig. 11b we have

$$M_b = -ie \frac{G_F}{\sqrt{2}} \cos \theta_c f_\pi K_\nu \bar{u}(p_e) \left[\frac{p_e \cdot \epsilon}{p_e \cdot k} + \frac{\not{\epsilon} \not{k}}{2 p_e \cdot k} \right] \gamma^\nu (1 - \gamma_5) v(p_\nu)$$

(III-4)

The gauge invariance condition gives us

$$[M_a + M_b]_{\epsilon \rightarrow k} = - [M_c]_{\epsilon \rightarrow k} \quad (\text{III-5})$$

This relation must be satisfied for each term of the expansions of M_a , M_b and M_c in k . We assume that M_c tends to be a constant as $k \rightarrow 0$. Since the terms in k^{-1} come only from the contributions M_a and M_b , the term of order zero in k is given by the gauge condition. In accordance with the above observations, we replace the form factor $F_\pi(m_\pi^2, s, 0)$ in M_a by unity, and by expanding f_π with respect to k , we obtain

$$M_a = ie \frac{G_F}{\sqrt{2}} \cos \theta_c \left[\frac{K \cdot \epsilon}{K \cdot k} f_\pi - 2K \cdot \epsilon \frac{\partial f_\pi(s)}{\partial s} \right]_{s=m_\pi^2} + O(k) (K-k)_\nu L^\nu \quad (\text{III-2b})$$

From Eqs. (III-4), (III-5) and (III-2b), we can get

$$M_c = ie \frac{G_F}{\sqrt{2}} \cos \theta_c \left\{ f_\pi \epsilon_\nu + 2K \cdot \epsilon \frac{\partial f_\pi(s)}{\partial s} \right\}_{s=m_\pi^2} (K-k)_\nu + O(k) L^\nu$$

(III-6)

Thus, the total invariant amplitude is

$$\begin{aligned}
 M_{\text{POINT}}(\pi^- \rightarrow e^- \bar{\nu}_e \gamma) &= -ie \frac{G_F}{\sqrt{2}} f_\pi \cos\theta_c K_\nu \bar{u}(p_e) \left[\frac{p_e \cdot \epsilon}{p_e \cdot k} - \frac{K \cdot \epsilon}{K \cdot k} + \frac{2K}{2p_e \cdot k} \right] \gamma^\nu (1-\gamma_5) v(p_\nu) \\
 &\quad - ie \frac{G_F}{\sqrt{2}} f_\pi \cos\theta_c \frac{\epsilon_{\mu\nu\lambda\sigma} K^\lambda p_e^\sigma}{K \cdot k} [K_\mu k_\nu - K \cdot k g_{\mu\nu}] \quad (\text{III-7a})
 \end{aligned}$$

or

$$\begin{aligned}
 M_{\text{POINT}}(\pi^- \rightarrow e^- \bar{\nu}_e \gamma) &= -ie \frac{G_F}{\sqrt{2}} f_\pi \cos\theta_c m_e \bar{u}(p_e) \left[\frac{p_e \cdot \epsilon}{p_e \cdot k} - \frac{K \cdot \epsilon}{K \cdot k} + \frac{2K}{2p_e \cdot k} \right] (1-\gamma_5) v(p_\nu) \\
 &\quad \quad \quad (\text{III-7b})
 \end{aligned}$$

which implies that if we neglect the structure effects, except for the contact term in Eq. (I-10), Eq. (I-12) becomes Eq. (III-7b). Here, we understand that this contact term arises by considering the simple model where the axial vector current for the $\pi^- \rightarrow e^- \bar{\nu}_e$ decay is taken to be $A_\mu^{\Delta Y=0}(x) = f_\pi \partial_\mu \phi_-(x)$, where $\phi_-(x)$ is the pion field and $A_\mu^{\Delta Y=0}(x)$ is the strangeness-conserving axial vector current. In the presence of electromagnetic interaction, this quantity becomes $f_\pi \partial_\mu \phi_-(x) + ief_\pi A_\mu^{\text{em}}(x)\phi_-(x)$ by the

minimal gauge invariant substitution $\partial_\mu \rightarrow \partial_\mu + ieA_\mu^{\text{em}}(x)$, and then the contribution to the $\pi^- \rightarrow e^- \bar{\nu}_e \gamma$ decay arising from the second of these terms corresponds exactly to the contact term. Here $A_\mu^{\text{em}}(x)$ is the electromagnetic field.

3. The π - q - $\bar{q}$ Vertex Function

If the pion is a $q\bar{q}$ bound state it is natural to consider the bound state wave function $\chi_K(p)$ satisfying a relativistic two body bound state equation, for example, the Bethe-Salpeter (B-S) equation.⁵⁾ In constructing the explicit bound state wave function for the physical pion, there is a certain degree of arbitrariness with respect to both the nature of its constituents and that of the binding forces. Apart from the mathematical complexity of solving the B-S equation, the covariant potential is not known.

The B-S equation for the $q\bar{q}$ bound state in the ladder approximation is given by⁶⁾

$$\overset{-1}{S(p+K)} \overset{-1}{\chi_K(p)} \overset{-1}{S(p)} = \frac{-i\lambda}{(2\pi)^4} \int V(p-p') \Gamma \chi_K(p') \Gamma d^4p' \quad (\text{III-8})$$

and is graphically shown in Fig. 12.

Fig. 12 The Bethe-Salpeter equation in the ladder approximation.

$S(p)$, $S(p+K)$ are the quark propagators, and p is the quark loop momentum, and m is the mass of the quarks ($m = m_u = m_d$). $V(q)$ is the covariant potential. The subscript K indicates that the total four-momentum K_ν acts as a constant parameter for the integral equation. In Eq. (III-8) the exchanged particle is scalar, pseudoscalar, or vector according to $\Gamma = 1$, γ_5 or γ_μ . Thus, the bound state vertex function $\Gamma_K(p)$ can be written as

$$\Gamma_K(p) = S(p+K) \chi_K(p) S(p) = -i \frac{\lambda}{(2\pi)^4} \int V(p-p') \Gamma S(p'+K) \Gamma_K(p') S(p) \Gamma d^4p'$$

(III-9)

The most general form of $\Gamma_K(p)$ is given by⁷⁾

$$\Gamma_K(p) = \gamma_5 (A + K \cdot p B \not{p} + C \not{K} + D[\not{p}, \not{K}]) \quad (\text{III-10})$$

where A, B, C and D are Lorentz invariant functions.

For simplicity we take the first term only, and define

the $\pi q \bar{q}$ coupling constant f_1 as

$$\Gamma_K(p) = \gamma_5 f_1 \not{p} \quad (\text{III-11})$$

where $1 = +, -, 0$ correspond to π^+ , π^- and π^0 , respectively, and

$$f_{\pm} = \sqrt{2} f_0 \quad (\text{III-12})$$

Functions $\Gamma_K(p)$ and $\chi_K(p)$ depend implicitly on the total four-momentum K. For a given $V(q)$, by setting $K^2 = m_\pi^2$, Eq. (III-8) or (III-9) can be regarded as an eigenvalue equation for the coupling constant λ .

The asymptotic behaviour of $\Gamma_K(p)$ at large p can be obtained from Eqs. (III-8) and (III-9) by taking $V(q)$ outside the integral, provided that

$$\int \chi_K(p') d^4 p' = \chi_K(x_\mu = 0) < \infty \quad (\text{III-13})$$

The covariant potential responsible for the binding is assumed to be of the general form

$$V(q) = \int \frac{\sigma(\mu^2)}{q^2 - \mu^2 + i\epsilon} d\mu^2 \quad (\text{III-14})$$

where q denotes the square of the four-momentum transfer and μ is the mass parameter for the exchanged particle.

The integral $\int \sigma(\mu^2) d\mu^2$ is assumed to be convergent; thus

$$V(q) \sim 0 \left(\frac{1}{q^2} \right) \text{ as } q^2 \rightarrow \infty \quad (\text{III-15})$$

As the relative momentum $2p+K$ between the two "bare" particles (i.e. quarks) in the bound state approaches infinity, it has been shown^{3,8,9} from Eqs. (III-8) and (III-13) that the vertex function $\Gamma_K(p)$ of the B-S equation has a similar asymptotic behaviour; i.e.

$$\Gamma_K(p) \sim 0 \left(\frac{1}{p^2} \right) \text{ as } p^2 \rightarrow \infty \quad (\text{III-16})$$

This asymptotic behaviour has also been found and used for the boson-boson bound state, boson-fermion state and fermion-fermion state by investigating the

asymptotic electromagnetic form factors^{3,8-13)} and asymptotic scaling laws.^{3,14)}

Assuming that the physical nucleon is simply a two-body bound state composed of a "bare" nucleon of spin- $\frac{1}{2}$ and a "bare" meson of spin-0, Drell and Lee³⁾ made a simple ansatz for the vertex function $\phi(p)$:

$$\phi(p) = \text{const} \frac{1}{p^2 - M^2} \quad (\text{III-17})$$

from the bound state solution¹⁵⁾ for the covariant potential $V(q) = -\frac{1}{q^2}$. Then, they could establish the scaling property in deep-inelastic electron scattering. Eq. (III-17) is the Drell-Lee ansatz. For the physical pion assumed to be a $q\bar{q}$ bound state; we take a similar form for $\phi(p)$;

$$\phi(p) = \frac{-M'^2}{p^2 - M'^2} \quad (\text{III-18})$$

where M' is the unknown mass parameter to be determined.

We use this Drell-Lee ansatz for our dynamical quark-loop. Hence, $\phi(p) \rightarrow 1$ as $M' \rightarrow \infty$

4. Invariant Amplitude for The $\pi^- \rightarrow e^- \bar{\nu}_e \gamma$ Decay

In the quark-loop model the IVB couples to the quarks and the leptons. In this thesis we work with the one-loop approximation with the fractionally-charged colored quarks. In this approximation the Feynman diagrams for the $\pi^- \rightarrow e^- \bar{\nu}_e$ process and the $\pi^- \rightarrow e^- \bar{\nu}_e \gamma$ process in Figs. 10 and 11 become, by introducing the quark loop and the $\pi q \bar{q}$ vertex $\Gamma_K(p)$, as shown in Figs. 13 and 14. In order to keep the problem as simple as possible, we take two replacements in Figs. 13 and 14. The full quark propagator is replaced by a free quark propagator $S(p) = (\not{p} - m + i\epsilon)^{-1}$ and the full electromagnetic vertex function of the quark is replaced by the bare vertex γ_μ .

For the non-radiative pion decay in Fig. 13 we obtain

$$M_{(\pi^- \rightarrow e^- \bar{\nu}_e)} = \frac{G_F}{\sqrt{2}} \cos \theta_c M_V(K) L^\nu \quad (\text{III-19})$$

where

$$M_V(a) = 3\sqrt{2} f_0 \int \frac{d^4 p}{(2\pi)^4} \phi(p + \frac{a}{2}) \text{Tr} \{ \gamma_5 (\not{p} - m)^{-1} \gamma_\nu (1 - \gamma_5) (\not{p} + \not{a} - m)^{-1} \}$$

(III-20)

Fig. 13 The non-radiative decay of the pion, assumed to be a $q\bar{q}$ bound state.

Fig. 14 The radiative decay of the pion, assumed to be a $q\bar{q}$ bound state.

and factor 3 is the color factor. Comparing with Eq. (III-1), we find a relation

$$f_{\pi} K_V = -i M_V(K) \quad (\text{III-21})$$

For a constant vertex, for example, $\phi(p+\frac{K}{2}) = 1$, we can easily obtain the manifestly gauge invariant amplitude for the radiative leptonic decay by using the W-T identity

$$k^{\mu} M_{\mu\nu} = -e [M_{\nu}(K-k) - M_{\nu}(K)] \quad (\text{III-22})$$

For the non-constant vertex $\phi(p)$ we can not obtain a gauge invariant amplitude because the vertex $\phi(p)$ spoils the W-T identity. In order to see how the W-T identity breaks down by introducing the vertex function let us calculate the amplitude for the diagrams in which the photon is emitted from the free quark and antiquark lines, as shown in Fig. 15.

Fig. 15 Diagrams in which the photons are emitted from the quark lines.

The amplitudes for these diagrams are given by

$$M_{\mu\nu}^q = 3\sqrt{2}e e_q f_0 \int \frac{d^4 p}{(2\pi)^4} \phi(p+\frac{K}{2}) \text{Tr} \{ \gamma_5 (\not{p}-m)^{-1} \gamma_\nu (1-\gamma_5) (\not{p}+K-\not{k}-m)^{-1} \gamma_\mu (\not{p}+K-m)^{-1} \} \quad (\text{III-23a})$$

$$M_{\mu\nu}^{\bar{q}} = -3\sqrt{2}e e_{\bar{q}} f_0 \int \frac{d^4 p}{(2\pi)^4} \phi(p-\frac{K}{2}) \text{Tr} \{ \gamma_5 (\not{p}-K-m)^{-1} \gamma_\mu (\not{p}-K+\not{k}-m)^{-1} \gamma_\nu (1-\gamma_5) (\not{p}-m)^{-1} \} \quad (\text{III-23b})$$

It is understood that the mass m includes a small imaginary part $-i\epsilon$. Using the relation

$$(\not{p}-K-m)^{-1} K (\not{p}-m)^{-1} = (\not{p}-K-m)^{-1} - (\not{p}-m)^{-1}$$

we obtain

$$\begin{aligned} k^\mu M_{\mu\nu} &= k^\mu (M_{\mu\nu}^q + M_{\mu\nu}^{\bar{q}}) \\ &= -3\sqrt{2}ef_0(e_q + e_{\bar{q}}) \int \frac{d^4 q}{(2\pi)^4} \phi(p+\frac{K}{2}) \text{Tr} \{ \gamma_5 (\not{p}-m)^{-1} \gamma_\nu (1-\gamma_5) (\not{p}+K-m)^{-1} \\ &\quad + 3\sqrt{2}ef_0 \int \frac{d^4 p}{(2\pi)^4} [e_q \phi(p+\frac{K}{2}) + e_{\bar{q}} \phi(p+\frac{K}{2}-k)] \text{Tr} \{ \gamma_5 (\not{p}-m)^{-1} \gamma_\nu (1-\gamma_5) \\ &\quad \times (\not{p}+K-\not{k}-m)^{-1} \} \end{aligned} \quad (\text{III-24})$$

Here we have shifted the integral origin by $p \rightarrow p+K$ in the first term and by $p \rightarrow p+K-k$ in the second term of $k^\mu M_{\nu\mu}^{\bar{q}}$, and $e_q = -\frac{1}{3}$, $e_{\bar{q}} = -\frac{2}{3}$ are the fractional charge quantities for the negative pion. We rewrite Eq. (III-24) as

$$\begin{aligned}
 k^\mu M_{\nu\mu} = & -e [M_\nu(K-k) - M_\nu(K)] \\
 & + 3\sqrt{2}f_0 \int \frac{d^4p}{(2\pi)^4} \text{Tr} [\gamma_5 (\not{p}-m)^{-1} \gamma_\nu (1-\gamma_5) (\not{p}+K-k-m)^{-1}] \\
 & \times [e_q \phi(p+\frac{K}{2}) + e_{\bar{q}} \phi(p+\frac{K}{2}-k) + \phi(p+\frac{K-k}{2})] \quad (\text{III-25})
 \end{aligned}$$

As mentioned before, for the constant vertex, the second term of the right-hand side in Eq. (III-25) vanishes, and we obtain the W-T identity. For $\phi(p) = 1$ case the exact gauge invariance to the second order in G_F has been shown.¹⁶⁾ Eq. (III-25) shows clearly that the $\pi q\bar{q}$ vertex function spoils gauge invariance. Moreover, we know that if the exact covariant potential and the vertex function are known, the total amplitude for the $\pi \rightarrow e\nu\gamma$ decay will be gauge invariant by summing all the ladder diagrams. In the soft photon limit $k \rightarrow 0$, Eq. (III-25) gives the W-T identity. In other words, in the limit $k \rightarrow 0$, we can obtain the

gauge invariant amplitude even for the one-loop approximation with the $\pi q\bar{q}$ vertex function. From these points of view, we make a simple prescription for gauge invariance by adding an extra $-k/2$ to $\phi(p+\frac{K-k}{2})$ i.e.

$$\phi(p+\frac{K-k}{2}) \xrightarrow{\text{add } (-k/2)} \phi(p+\frac{K}{2}-k) \quad (\text{III-26})$$

Then, the second term in Eq. (III-25) vanishes after momentum integration by shifting the origin by $p \rightarrow p-K+k$. This simple prescription gives us the desired gauge invariant amplitude for the $\pi \rightarrow e \nu_e \gamma$ decay, which also can be obtained by using the soft photon techniques^{1,2)} and the gauge projection operator.⁴⁾

We now calculate the total amplitude for the $\pi^-(K) \rightarrow e^-(p_e) + \bar{\nu}_e(p_\nu) + \gamma(k)$ to the lowest order in G_F in our simple quark-loop by using the simple prescription. The Lorentz invariant amplitudes M_a , M_b , M_{c_1} and M_{c_2} corresponding to Figs. 14a, 14b, 14c1 and 14c2 are obtained as follows.

$$M_a = e \frac{G_F}{\sqrt{2}} \cos\theta_c \frac{K \cdot \epsilon}{K \cdot k} M_\nu(K-k) L^\nu \quad (\text{III-27})$$

$$M_b = -e \frac{G_F}{\sqrt{2}} \bar{u}(p_e) \left(\frac{p_e \cdot \epsilon}{p_e \cdot k} + \frac{k}{2p_e \cdot k} \gamma^\nu (1-\gamma_5) \right) v(p_\nu) M_\nu(K) \cos\theta_c \quad (\text{III-28})$$

$$M_{c_1} = \frac{G_F}{\sqrt{2}} \cos\theta_c \epsilon^{\mu L \nu} M_{\mu\nu}^q \quad (\text{III-29a})$$

$$M_{c_2} = \frac{G_F}{\sqrt{2}} \cos\theta_c \epsilon^{\mu L \nu} M_{\mu\nu}^{\bar{q}} \quad (\text{III-29b})$$

where $M_\nu(a)$, $M_{\mu\nu}^q$ and $M_{\mu\nu}^{\bar{q}}$ are given in Eqs: (III-20) and (III-23a,b). The traces are calculated to be

$$\text{Tr}[\gamma_5(\not{p}+m)\gamma_\nu(1-\gamma_5)(\not{p}+\not{K}+m)] = 4ma_\nu \quad (\text{III-30})$$

$$\begin{aligned} & \text{Tr}[\gamma_5(\not{p}+m)\gamma_\nu(1-\gamma_5)(\not{p}+\not{K}-\not{K}+m)\gamma_\mu(\not{p}+\not{K}+m)] \\ &= 4im\epsilon_{\mu\nu\alpha\beta}K^\alpha k^\beta \\ &+ 4m\{(K \cdot k g_{\mu\nu} - K_\mu k_\nu) - g_{\mu\nu}[(p+K)^2 - m^2] + 2(p_\mu + K_\mu)K_\nu + k_\mu K_\nu\} \end{aligned} \quad (\text{III-31a})$$

$$\begin{aligned} & \text{Tr}[\gamma_5(\not{p}-\not{K}+m)\gamma_\mu(\not{p}-\not{K}+\not{K}+m)\gamma_\nu(1-\gamma_5)(\not{p}+m)] \\ &= 4im\epsilon_{\mu\nu\alpha\beta}K^\alpha k^\beta \\ &- 4m\{(K \cdot k g_{\mu\nu} - K_\mu k_\nu) - g_{\mu\nu}[(p-K)^2 - m^2] + 2(p_\mu - K_\mu)K_\nu + k_\mu K_\nu\} \end{aligned} \quad (\text{III-31b})$$

Thus, Eqs. (III-27) - (III-29b) can be rewritten as

$$M_a = C \frac{K \cdot \epsilon}{K \cdot k} (K-k)_\nu L^\nu I_{32} \quad (\text{III-32})$$

$$M_b = -\alpha \{ \bar{u}(p_e) \left[\frac{p_e \cdot \epsilon}{p_e \cdot k} + \frac{dk}{2p_e \cdot k} \gamma^\nu (1-\gamma_5) v(p_\nu) K_\nu \right] I_{31} \quad (\text{III-33})$$

$$M_{c1} = C e_q \epsilon^\mu L^\nu \{ [i\epsilon_{\mu\nu\alpha\beta} K^\alpha k^\beta + (K \cdot k g_{\mu\nu} - K_\mu k_\nu)] I_{41} \\ + 2K_\nu I_\mu^{41} - g_{\mu\nu} I_{33} \}$$

$$M_{c2} = -C e_{\bar{q}} \epsilon^\mu L^\nu \{ [i\epsilon_{\mu\nu\alpha\beta} K^\alpha k^\beta - (K \cdot k g_{\mu\nu} - K_\mu k_\nu)] I_{42} \\ - 2K_\nu I_\mu^{42} + g_{\mu\nu} I_{34} \}$$

where $C = 12eG_F f_0 m \cos\theta_c$, and

$$I_{31} = \int \frac{d^4 p}{(2\pi)^4} \phi(p+\frac{K}{2}) \{ (p^2 - m^2) [(p+K)^2 - m^2] \}^{-1} \quad (\text{III-34})$$

$$I_{32} = \int \frac{d^4 p}{(2\pi)^4} \phi(p+\frac{K-k}{2}) \{ (p^2 - m^2) [(p+K-k)^2 - m^2] \}^{-1}$$

$$I_{33} = \int \frac{d^4 p}{(2\pi)^4} \phi(p+\frac{K}{2}) \{ (p^2 - m^2) [(p+K-k)^2 - m^2] \}^{-1}$$

$$I_{34} = \int \frac{d^4 p}{(2\pi)^4} \phi(p - \frac{K}{2}) \{ (p^2 - m^2) [(p-K+k)^2 - m^2] \}^{-1}$$

$$I_{41} = \int \frac{d^4 p}{(2\pi)^4} \phi(p + \frac{K}{2}) \{ (p^2 - m^2) [(p+K)^2 - m^2] [(p+K-k)^2 - m^2] \}^{-1}$$

(III-35)

$$I_{42} = \int \frac{d^4 p}{(2\pi)^4} \phi(p - \frac{K}{2}) \{ (p^2 - m^2) [(p-K)^2 - m^2] [(p-K+k)^2 - m^2] \}^{-1}$$

$$I_{\mu}^{41} = \int \frac{d^4 p}{(2\pi)^4} (p+K)_{\mu} \phi(p + \frac{K}{2}) \{ (p^2 - m^2) [(p+K)^2 - m^2] [(p+K-k)^2 - m^2] \}^{-1}$$

(III-36)

$$I_{\mu}^{42} = \int \frac{d^4 p}{(2\pi)^4} (p-K)_{\mu} \phi(p - \frac{K}{2}) \{ (p^2 - m^2) [(p-K)^2 - m^2] [(p-K+k)^2 - m^2] \}^{-1}$$

If we replace K, k by $-K, -k$ in I_{34}, I_{42} and I_{μ}^{42} , they become I_{33}, I_{41} and I_{μ}^{41} , respectively, and hence $I_{34} = I_{33}, I_{42} = I_{41}$ and $I_{\mu}^{42} = I_{\mu}^{41}$ by symmetry property of integration.

Combining M_{c1} and M_{c2} we obtain

$$M_c = -C \epsilon^{\mu\nu} L^{\nu} \{ [-\frac{i}{3} \epsilon_{\mu\nu\alpha\beta} K^{\alpha} k^{\beta} + (K \cdot k g_{\mu\nu} - K_{\mu} k_{\nu})] I_{41} + 2K_{\nu} I_{\mu}^{41} - g_{\mu\nu} I_{33} \}$$

(III-37)

Therefore, the total amplitude is obtained by summing,

M_a , M_b and M_c , so that

$$\begin{aligned}
 M = & -C K_\nu \bar{u}(p_e) \left[\left(\frac{p_e \cdot \epsilon}{p_e \cdot k} + \frac{\not{\epsilon} \not{k}}{2 p_e \cdot k} \right) I_{31} - \frac{K \cdot \epsilon}{K \cdot k} I_{32} \right] \gamma^\nu (1 - \gamma_5) v(p_\nu) \\
 & - C \epsilon^{\mu L \nu} \left[-\frac{i}{3} \epsilon_{\mu \nu \alpha \beta} K^\alpha k^\beta + (K \cdot k g_{\mu \nu} - K_\mu k_\nu) \right] I_{41} \\
 & - C L^\nu \left[\frac{K \cdot \epsilon}{K \cdot k} K_\nu I_{32} - \epsilon^\mu g_{\mu \nu} I_{33} + 2 \epsilon^\mu K_\nu I_{41} \right]
 \end{aligned}$$

or

$$\begin{aligned}
 M = & -C \epsilon^{\mu L \nu} \left[-\frac{i}{3} \epsilon_{\mu \nu \alpha \beta} K^\alpha k^\beta + (K \cdot k g_{\mu \nu} - K_\mu k_\nu) \right] I_{41} \\
 & - C K_\nu \bar{u}(p_e) \left[\frac{p_e \cdot \epsilon}{p_e \cdot k} - \frac{K \cdot \epsilon}{K \cdot k} + \frac{\not{\epsilon} \not{k}}{2 p_e \cdot k} \right] \gamma^\nu (1 - \gamma_5) v(p_\nu) I_{31} \\
 & + C \epsilon^{\mu L \nu} \frac{1}{K \cdot k} [K \cdot k g_{\mu \nu} - K_\mu k_\nu] I_{33} \\
 & + C \frac{K \cdot \epsilon}{K \cdot k} (K - k)_\nu L^\nu [I_{32} - I_{33}] - C \epsilon^{\mu L \nu} \frac{K_\mu}{K \cdot k} (I_{31} - I_{33}) \\
 & + 2 I_{41} K_\nu.
 \end{aligned} \tag{III-38}$$

This is the total amplitude for the $\pi^- \rightarrow e^- \bar{\nu}_e \gamma$ decay in the one-loop approximation. The first three terms are

manifestly gauge invariant, but the last two terms are not. Making use of the following general expansion valid for any operator A, B:

$$\frac{1}{A \pm B} = \frac{1}{A} \mp \frac{1}{A} B \frac{1}{A \pm B} \quad (\text{III-39})$$

we can expand I_{33} , so that

$$\begin{aligned} I_{33} &= \int \frac{d^4 p}{(2\pi)^4} \phi(p + \frac{K}{2}) \{ (p^2 - m^2) [(p+K-k)^2 - m^2] \}^{-1} \\ &= \int \frac{d^4 p}{(2\pi)^4} \phi(p + \frac{K}{2}) \{ (p^2 - m^2) [(p+K-k)^2 - m^2] \}^{-1} \\ &\quad \times \{ 1 + 2k \cdot (p+K) [(p+K-k)^2 - m^2]^{-1} \} \\ &= I_{31} + 2K \cdot k I_{411} \end{aligned}$$

where

$$I_{411} = \int \frac{d^4 p}{(2\pi)^4} (1 + \frac{p \cdot k}{K \cdot k}) \phi(p + \frac{K}{2}) \{ (p^2 - m^2) [(p+K)^2 - m^2] [(p+K-k)^2 - m^2] \}^{-1}$$

Thus, the last two terms in Eq. (III-38) becomes

$$M' = C \frac{K \cdot \epsilon}{K \cdot k} (K-k)_\nu L^\nu [I_{32} - I_{33}] + 2C \epsilon^{\mu\nu} K_\nu (K_\mu I_{411} - I_\mu^{41})$$

(III-40)

Since $K_\mu I_{411} = I_\mu^{41}$ after integration over p , the second term in Eq. (III-40) vanishes. Now the non-gauge invariant term is $\frac{K \cdot \epsilon}{K \cdot k} (K-k)_\nu L^\nu [I_{32} - I_{33}]$ which does not vanish unless $I_{32} = I_{33}$. If we apply our prescription to I_{32} and shift the origin by $p \rightarrow p - K + k$, it becomes I_{34} which is equal to I_{33} by the symmetry properties of integrals. Hence, the first term in Eq. (III-40) also vanishes.

With simple prescription we obtain the gauge invariant amplitude

$$M(\pi^- \rightarrow e^- \bar{\nu}_e \gamma) = -C K_\nu \bar{u}(p_e) \left[\frac{p_e \cdot \epsilon}{p_e \cdot k} - \frac{K \cdot \epsilon}{K \cdot k} + \frac{2K \cdot \epsilon}{2p_e \cdot k} \right] \gamma^\nu (1 - \gamma_5) v(p_\nu) I_{31}$$

$$-C \epsilon^{\mu\nu} L^\nu \frac{1}{K \cdot k} (K_\mu k_\nu - K \cdot k g_{\mu\nu}) I_{31}$$

$$+ \frac{1}{3} C \epsilon_{\mu\nu\alpha\beta} \epsilon^{\mu\nu} K^\alpha k^\beta I_{41}$$

$$-C \epsilon^{\mu\nu} (K \cdot k g_{\mu\nu} - K_\mu k_\nu) [I_{41} - 2I_{411}] \quad (\text{III-41})$$

where $C = 12eG_F f_0 m \cos\theta_c$.

According to the soft-photon techniques^{1,2)}, we expand I_{32} in Eq. (III-38) with respect to k , and in addition to Eq. (III-41), we obtain a non-gauge invariant term

$$C \epsilon^{\mu\nu} L^{\nu} K_{\mu} (K-k)_{\nu} \frac{k}{K \cdot k} \int \frac{d^4 p}{(2\pi)^4} \left[\frac{\partial \phi(p + \frac{K-k}{2})}{\partial k} \right] \Big|_{k=0} \{ (p^2 - m^2) [(p+K-k)^2 - m^2] \}^{-1}$$

which does not appear in our prescription because of $I_{32} \rightarrow I_{34} = I_{33}$ as mentioned above.

6. Gauge Projection Operator

In this section we will show that the gauge invariant amplitude obtained in Eq. (III-41) by simple prescription can be derived by the gauge projection operator, $I_{\alpha\beta}$. This simple projection operator has been introduced by Bardeen and Tung⁴⁾ to construct the gauge invariant amplitudes for any photon processes. All known soft-photon theorems on photon processes can be derived directly by using this simple projection operator. Fearing, Fischbach, and Smith¹⁷⁾ have used this projection operator and soft-photon techniques, and obtained a gauge invariant amplitude for the $K^0 \rightarrow \pi^+ l^- \nu_e \gamma$.

The gauge projection operator $I_{\alpha\beta}$ is defined by

$$I_{\alpha\beta} = g_{\alpha\beta} - \frac{q_\alpha k_\beta}{q \cdot k} \quad (\text{III-42})$$

where q is any four-momentum of the external charged particle in the three point-interactions. In our case $q = K$ or $K-k$. $I_{\alpha\beta}$ has the following basic properties:

$$1) \quad I_{\alpha\beta} M^\beta = M_\alpha$$

if M^α satisfies the gauge condition $k_\alpha M^\alpha = 0$

$$2) \quad k^\alpha I_{\alpha\beta} = 0$$

$$3) \quad q^\beta I_{\alpha\beta} = 0$$

These properties establish $I_{\alpha\beta}$ as a projection operator for gauge invariance. The third property implies that the terms in the amplitude proportional to q^β are immediately eliminated. The more general gauge projection operator was also introduced by Yennie, Frautschi and Suura¹⁸⁾ to extract the infrared factors

from the gauge invariant amplitudes for both the real and virtual photon processes.

We now apply this projection operator

$$I_{\alpha\beta} = g_{\alpha\beta} - \frac{K_\alpha k_\beta}{K \cdot k} \quad (\text{III-43})$$

on the photon vertices of Feynman diagrams in Fig. 14, and obtain

$$\tilde{M}_a = e \frac{G_F}{\sqrt{2}} \frac{\epsilon^\alpha}{2K \cdot k} (2K-k)^\beta \left[g_{\alpha\beta} - \frac{K_\alpha k_\beta}{K \cdot k} \right] M_\nu(K-k)$$

$$= 0$$

$$\tilde{M}_b = -e \frac{G_F}{\sqrt{2}} \{ \bar{u}(p_e) \frac{\epsilon^\alpha \gamma^\beta}{q p_e \cdot k} [g_{\alpha\beta} - \frac{K_\alpha k_\beta}{K \cdot k}] (\not{p}_e + k + m_e) \gamma^\nu (1-\gamma_5) v(p_\nu) \} M_\nu(K)$$

$$= -e \frac{G_F}{\sqrt{2}} \{ \bar{u}(p_e) \frac{\epsilon^\alpha}{2p_e \cdot k} [\gamma_\alpha (\not{p}_e + k + m_e) - \frac{K_\alpha k}{K \cdot k} (\not{p}_e + k + m_e)]$$

$$\gamma^\nu (1-\gamma_5) v(p_\nu) \} M_\nu(K)$$

$$= -e K_\nu \bar{u}(p_e) \left[\frac{p_e \cdot \epsilon}{p_e \cdot k} - \frac{K \cdot \epsilon}{K \cdot k} + \frac{\not{\epsilon} \not{K}}{2p_e \cdot k} \right] \gamma^\nu (1-\gamma_5) v(p_\nu) I_{31}$$

$$\tilde{M}_{c1} = e e_q G_F f_0 \int \frac{d^4 p}{(2\pi)^4} \phi(p + \frac{K}{2}) \text{Tr} \{ \gamma_5 (\not{p}-m)^{-1} \gamma_\nu (1-\gamma_5) (\not{p}+K-\not{k}-m)^{-1} \\ [\not{\epsilon} - \frac{K \cdot \epsilon}{K \cdot R} \not{k}] (\not{p}+K-m)^{-1} \} L^\nu$$

$$= \frac{G_F}{\sqrt{2}} \epsilon^{\mu\nu} \{ M_{\mu\nu}^q - \frac{e e_q f_0}{K \cdot k} K_\mu \int \frac{d^4 p}{(2\pi)^4} \phi(p + \frac{K}{2}) \text{Tr} \{ \gamma_5 (\not{p}-m)^{-1} \\ \gamma_\nu (1-\gamma_5) (\not{p}+K-\not{k}-m)^{-1} \} \}$$

$$+ \frac{e e_q f_0}{K \cdot k} K_\mu \int \frac{d^4 p}{(2\pi)^4} \phi(p + \frac{K}{2}) \text{Tr} \{ \gamma_5 (\not{p}-m)^{-1} \gamma_\nu (1-\gamma_5) (\not{p}+K-m)^{-1} \} \}$$

$$\tilde{M}_{c2} = \frac{G_F}{\sqrt{2}} \epsilon^{\mu\nu} \{ M_{\mu\nu}^{\bar{q}} + \frac{e f_0 e_{\bar{q}}}{K \cdot k} K_\mu \int \frac{d^4 p}{(2\pi)^4} \phi(p - \frac{K}{2}) \text{Tr} \{ \gamma_5 (\not{p}-K-m)^{-1} \\ \gamma_\nu (1-\gamma_5) (\not{p}-m)^{-1} \} \}$$

$$- \frac{e f_0 e_{\bar{q}}}{K \cdot k} K_\mu \int \frac{d^4 p}{(2\pi)^4} \phi(p - \frac{K}{2}) \text{Tr} \{ \gamma_5 (\not{p}-K+\not{k}+m)^{-1} \gamma_\nu (1-\gamma_5) (\not{p}-m)^{-1} \} \}$$

where $C = 12eG_F f_0 m \cos\theta_c$, $\tilde{M}$ distinguishes from M , and $M_{\mu\nu}^q$ and $M_{\mu\nu}^{\bar{q}}$ are given in Eqs. (III-23a) and (III-23b). Combining the four amplitudes gives the total amplitude:

$$\bar{M}(\pi^- \rightarrow e^- \bar{\nu}_e \gamma) = -C K_\nu \bar{u}(p_e) \left[\frac{p_e \cdot \varepsilon}{K \cdot k} - \frac{K \cdot \varepsilon}{K \cdot k} + \frac{2K}{2p_e \cdot k} \gamma^\nu (1-\gamma_5) v(p_\nu) \right] I_{31}$$

$$- C \varepsilon^{\mu L \nu} \left[-\frac{1}{3} \varepsilon_{\mu\nu\alpha\beta} K^\alpha k^\beta + (K \cdot k g_{\mu\nu} - K_\mu k_\nu) \right] I_{41}$$

$$- C \varepsilon^{\mu L \nu} \frac{1}{K \cdot k} (K_\mu k_\nu - K \cdot k g_{\mu\nu}) I_{33} + K_\mu K_\nu (I_{31} - I_{33})$$

$$+ 2K \cdot k K_\nu I_{\mu}^{41}$$

$$- C K_\mu \bar{u}(p_e) \left[\frac{p_e \cdot \varepsilon}{K \cdot k} - \frac{K \cdot \varepsilon}{K \cdot k} + \frac{2K}{2p_e \cdot k} \gamma^\nu (1-\gamma_5) v(p_\nu) \right] I_{31}$$

$$- C \varepsilon^{\mu L \nu} \left[-\frac{1}{3} \varepsilon_{\mu\nu\alpha\beta} K^\alpha k^\beta I_{41} + (K \cdot k g_{\mu\nu} - K_\mu k_\nu) (I_{41} - \frac{1}{K \cdot k} I_{33}) \right]$$

$$- 2C \varepsilon^{\mu L \nu} K_\nu (I_{\mu}^{41} - I_{411})$$

(III-44)

After momentum integration the last term vanishes, and the first two terms are manifestly gauge invariant and are identical to Eq. (III-41) obtained by our simple prescription.

CHAPTER IV

PION STRUCTURE FORM FACTORS AND MEAN CHARGE RADIUS

1. Introduction

The gauge invariant amplitude for the radiative pion decay was obtained in the one-loop approximation in Chapter III. In this chapter we calculate the SDV form factor $v(s,0)$, the SDA form factor $a(s,0)$ and the ratio $\gamma_\pi(s)$. In our quark-loop model there are three unknown parameters; the quark mass m ($m=m_u=m_d$), the $\pi q\bar{q}$ coupling constant f_0 and the mass parameter M . These parameters are determined from the well-known processes, for example, the $\pi^- \rightarrow \mu^- \bar{\nu}_\mu$ decay, the $\pi^0 \rightarrow 2\gamma$ decay and the normalization condition of the electromagnetic pion form factor $F_\pi(0) = 1$. In our one-loop approximation, we find the three relations for f_π , $v_0^1(0,0)$ and $F_\pi(0) = 1$ in terms of m , M , f_0 and the Feynman parameters. All the integrals are absolutely convergent in the Feynman parameter space, because we assumed that $m > m_\pi$. The exact analytical calculations of the integrals are difficult, and we determine the three unknown parameters by numerical methods. Thus, our results include some amounts of error.

We adopt the Gaussian quadrature method for the numerical integrations. After we determine the parameters m , M and f_0 , we test our quark-loop model by calculating the pion mean squared charge radius $\langle r_\pi^2 \rangle$, and calculate $v(s,0)$ and $a(s,0)$, and estimate the ratio $\gamma_\pi(s) = a(s,0)/v(s,0) \approx .84$. Our results as compared with others, and some comments are also given at the end of this chapter.

2. Weak Structure Form Factors and Their Ratio

From Eqs. (III-20) and (III-21) we obtain, after the trace calculation,

$$f_\pi = -12\sqrt{2} i f_0 m I_{31} \quad (\text{IV-1})$$

The integral I_{31} was defined in Eq. (III-34). We substitute Eq. (IV-1) into Eq. (III-7a) to obtain

$$\begin{aligned} \text{POINT} \\ M_{(\pi^- \rightarrow e^- \bar{\nu}_e \gamma)} = -C K_\nu \bar{u}(p_e) \left\{ \frac{p_e \cdot \epsilon}{p_e \cdot k} - \frac{K \cdot \epsilon}{K \cdot k} + \frac{2K}{2p_e \cdot k} \right\} \gamma^\nu (1-\gamma_5) v(p_\nu) I_{31} \\ - C \frac{\epsilon^{\mu L \nu}}{K \cdot k} [K_\mu k_\nu - K \cdot k g_{\mu\nu}] I_{31} \end{aligned} \quad (\text{IV-2})$$

which gives the pure pointlike interaction effects. The pure structure-dependent amplitude M_{SD} can be obtained^{1,2)}

by extracting Eq. (IV-2) from the bound state total amplitude in Eq. (III-41). We then separate this into the SDV and SDA parts to get

$$M_{SD} = M_{SDV} + M_{SDA} \quad (IV-3)$$

$$\begin{aligned} M_{SDV} &= \frac{C}{3} \epsilon_{\mu\nu\alpha\beta} \epsilon^{\mu L \nu} K^{\alpha} k^{\beta} I_{41} \\ &= \frac{C}{3} \tilde{F}_{\mu\nu} K^{\mu} L^{\nu} I_{41} \end{aligned} \quad (IV-4)$$

$$\begin{aligned} M_{SDA} &= -C \epsilon^{\mu L \nu} (K \cdot k g_{\mu\nu} - K_{\mu} k_{\nu}) (I_{41} - 2I_{411}) \\ &= -C F_{\mu\nu} K^{\mu} L^{\nu} (I_{41} - 2I_{411}) \end{aligned} \quad (IV-5)$$

$$C = 12eG_F f_0 m \cos\theta_c,$$

where $F_{\mu\nu} = k_{\mu} \epsilon_{\nu} - \epsilon_{\mu} k_{\nu}$ is the electromagnetic field tensor and $\tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}$ is its dual.

If we compare with Eqs. (I-14) and (I-15) we can get

$$v(s,0) = + i 4\sqrt{2} f_0 m I_{41} \quad (IV-6)$$

$$a(s,0) = - i 12\sqrt{2} f_0 m (I_{41} - 2I_{411}) \quad (IV-7)$$

We now evaluate the integrals I_{41} and I_{411} by putting the Drell-Lee ansatz vertex in Eq. (III-18).

That is,

$$I_{41}^V(s) = -M^2 \int \frac{d^4 p}{(2\pi)^4} \{ (p^2 - m^2) [(p+K)^2 - m^2] [(p+K-k)^2 - m^2] [(p+\frac{K}{2})^2 - M^2] \}^{-1} \quad (IV-8a)$$

$$\begin{aligned} I_{41}(s) &= I_{41} - 2I_{411} \\ &= -M^2 \int \frac{d^4 p}{(2\pi)^4} [1 - 2(1 + \frac{p \cdot k}{K \cdot k})] \{ (p^2 - m^2) [(p+K)^2 - m^2] [(p+K-k)^2 - m^2] [(p+\frac{K}{2})^2 - M^2] \}^{-1} \end{aligned} \quad (IV-8b)$$

where $M = \frac{1}{2}M$

These integrals can be evaluated by the Feynman parameter techniques.^{3,4)} The Feynman parameter integration formula is given by

$$\begin{aligned} \frac{1}{a_1^{\alpha_1} \dots a_n^{\alpha_n}} &= \frac{\Gamma(\alpha_1 + \dots + \alpha_n)}{\Gamma(\alpha_1) \dots \Gamma(\alpha_n)} \int_0^1 dz_1 \dots \int_0^1 dz_n \frac{\delta(1 - \sum_{i=1}^n z_i) z_1^{\alpha_1-1} \dots z_n^{\alpha_n-1}}{(\sum_{i=1}^n a_i z_i)^N} \\ &= \frac{\Gamma(\alpha_1 + \dots + \alpha_n)}{\Gamma(\alpha_1) \dots \Gamma(\alpha_n)} \int_0^1 dx_1 \int_0^{x_1} dx_2 \dots \int_0^{x_{n-2}} dx_{n-1} \frac{x_{n-1}^{\alpha_1-1} (x_{n-2} - x_{n-1})^{\alpha_2-1} \dots (1 - x_1)^{\alpha_n-1}}{[a_1 x_{n-1} + a_2 (x_{n-2} - x_{n-1}) + \dots + a_n (1 - x_1)]^N} \end{aligned} \quad (IV-9)$$

where

$$N = \alpha_1 + \alpha_2 + \dots + \alpha_n, \quad [z]_n = z_1 + z_2 + \dots + z_n$$

$$z_1 = x_{n-1}$$

$$z_2 = x_{n-2} - x_{n-1}$$

$$z_{n-1} = x_1 - x_2$$

$$z_n = 1 - x_1$$

(IV-10)

Thus, by using Eq. (IV-9) we can write Eqs. (IV-8a, 8b) as

$$I_{41}^V(s) = -\Gamma(4)M^2 \int \frac{d^4 p}{(2\pi)^4} \int_0^1 dz_1 \dots \int_0^1 dz_4 \frac{\delta(1-[z]_4)}{[D]^4}$$

$$I_{41}^A(s) = \Gamma(4)M^2 \int \frac{d^4 p}{(2\pi)^4} \int_0^1 dz_1 \dots \int_0^1 dz_4 \frac{\delta(1-[z]_4)}{[D]^4} \left(1 + 2 \frac{p \cdot k}{K \cdot k}\right)$$

where the denominator D is given by

$$D = \sum_{i=1}^4 (p_i^2 - m_i^2) z_i$$

$$= (p + KZ - kz_3)^2 + \frac{m_\pi^2}{4} [4Z(1-Z) - z_4] - 2K \cdot kz_3(1-Z) - m^2(z_1 + z_2 + z_3) - M^2 z_4$$

$$= (p + KZ - kz_3)^2 + \frac{m_\pi^2}{4} [4(1-z)(Z - z_3) - 3z_4] + sz_3(1-z) - m^2(z_1 + z_2 + z_3) - M^2 z_4$$

with

$$Z = z_2 + z_3 + \frac{z_4}{2} \quad \text{and} \quad 2K \cdot k = m_\pi^2 - s$$

The momentum integrals are convergent, and we can interchange orders of integrations and first perform the momentum integral by shifting the origin in p-space by $p \rightarrow p - KZ + kz_3$. Then, the denominator involves only the p^2 and in the numerators we need only consider scalar function $f(p^2)$ because the integrals having numerators of any other forms become

$$\int \frac{p^\mu f(p^2)}{(p^2 + \beta)^\alpha} d^4 p = 0$$

$$\int \frac{p^\mu p^\nu f(p^2)}{(p^2 + \beta)^\alpha} d^4 p = \frac{1}{4} g^{\mu\nu} \int \frac{p^2 f(p^2)}{(p^2 + \beta)^\alpha} d^4 p \quad (\text{IV-11})$$

where $p^\mu p^\nu = \frac{1}{4} g^{\mu\nu} p^2$ in 4-dimension

Since the term $\frac{p \cdot k}{K \cdot k}$ in the numerator of $I_{41}^A(s)$ becomes

$$\frac{p \cdot k}{K \cdot k} \rightarrow -Z + \frac{p \cdot k}{K \cdot k}$$

with shifting by $p \rightarrow p - KZ + z_3$, the integral involving $p \cdot k$, after shifting the origin, vanishes. The remaining integrals are

$$I_{41}^V(s) = -\Gamma(4)M^2 \int_0^1 dz_1 \dots \int_0^1 dz_4 \delta(1-[z]_4) \int \frac{d^4 p}{(2\pi)^4} \frac{1}{(p^2 + \beta_1)^4}$$

$$I_{41}^A(s) = \Gamma(4)M^2 \int_0^1 dz_1 \dots \int_0^1 dz_4 (1-2Z)\delta(1-[z]_4) \int \frac{d^4 p}{(2\pi)^4} \frac{1}{(p^2 + \beta_1)^4}$$

where

$$\beta_1 = \frac{m_\pi^2}{4} [4(1-Z)(Z-z_3)-z_4] + sz_3(1-Z) - m^2(z_1+z_2+z_3) - M^2 z_4$$

(IV-12)

Using the formula⁵⁾

$$\int \frac{d^n p}{(2\pi)^n} \frac{1}{(p^2 + \beta)^\alpha} = \frac{1}{(4\pi)^{n/2}} \frac{\Gamma(\alpha - n/2)}{\Gamma(\alpha)} \beta^{\frac{n}{2} - \alpha} \quad \text{(IV-13)}$$

we obtain

$$I_{41}^V(s) = \frac{iM^2}{(4\pi)^2} \int_0^1 dz_1 \dots \int_0^1 dz_4 \delta(1-[z]_4) \beta_1^{-2}$$

$$I_{41}^A(s) = \frac{iM^2}{(4\pi)^2} \int_0^1 dz_1 \dots \int_0^1 dz_4 (1-2Z) \delta(1-[z]_4) \beta_1^{-2}$$

For convenience for calculation we change the integration variables by $z_1 = x_3 = z$, $z_2 = x_2 - x_3 = y - z$, $z_3 = x_1 - x_2 = x - y$ and $z_4 = 1 - x_1 = 1 - x$, as in Eq. (IV-10), and then, the 4-dimensional integrals reduce immediately to the 3-dimensional integrals.

After making the denominators dimensionless, we obtain the form factors $v(s,0)$ and $a(s,0)$, i.e.

$$v(s,0) = \frac{\sqrt{2}}{4\pi^2} \frac{f_0}{m_\pi} \Delta_q \Delta_M^2 \int_0^1 dx \int_0^x dy \int_0^y dz \beta_1'^{-2} \quad (\text{IV-14})$$

$$a(s,0) = -\frac{3\sqrt{2}}{4\pi^2} \frac{f_0}{m_\pi} \Delta_q \Delta_M^2 \int_0^1 dx \int_0^x dy \int_0^y dz (x-2z) \beta_1'^{-2} \quad (\text{IV-15})$$

where

$$\beta_1' = \Delta_M^2 + (\Delta_q^2 - \Delta_M^2)x - [z(y-z) + (1-x)(2y-x)/4] - \frac{1}{2}\Delta_s^2(x-y)(1-x+2z) \quad (\text{IV-16})$$

$$\Delta_Q = \frac{m}{m_\pi} ; \Delta_M = \frac{M}{m_\pi} ; \Delta_S^2 = \frac{s}{m_\pi^2} \quad (\text{IV-16})$$

Eqs. (IV-14) and (IV-15) are the results for the vector and axial vector form factors of the pion from one-loop approximation with Drell-Lee ansatz vertex. It is quite difficult to evaluate the integrals analytically in the Feynman parameter space without any approximation. Only the numerical solution can be done, and the numerical evaluation is performed in Sec. IV-5.

Assuming that $m^2, M^2 \gg m_\pi^2/4 > m_\pi^2/4[+z(y-z)+(1-x)(2y-x)]$, we can simplify the above expression like

$$v(s,0) \approx v(0,0) \approx \frac{\sqrt{2}}{8\pi^2} \frac{f_0}{m} \left(\frac{m}{M}\right)^2 \int_0^1 dx \frac{x^2}{\left[\left(1+\frac{m^2}{M^2}-1\right)x\right]^2}$$

$$a(s,0) \approx a(0,0) \approx -\frac{\sqrt{2}}{8\pi^2} \frac{f_0}{m} \left(\frac{m}{M}\right)^2 \int_0^1 dx \frac{x^3}{\left[\left(1+\frac{m^2}{M^2}-1\right)x\right]^2}$$

which give

$$v(0,0) \approx \begin{cases} \frac{\sqrt{2}}{24\pi^2} \frac{f_0}{m} & \text{for } m=M \quad (\text{IV-17a}) \\ \frac{\sqrt{2}}{8\pi^2} \frac{f_0}{m} \eta(1-\eta)^{-2} [1+\eta^{-1}+2(1-\eta)^{-1} \ln \eta] & \text{for } m \neq M \quad (\text{IV-17b}) \end{cases}$$

$$a(0,0) \approx \begin{cases} -\frac{\sqrt{2}}{32\pi^2} \frac{f_0}{m} & \text{for } m=M \quad (\text{IV-18a}) \\ -\frac{\sqrt{2}}{8\pi^2} \frac{f_0}{m} \eta(1-\eta)^{-2} \left\{ \frac{1}{2}(1-\eta^{-1}) + \frac{3}{2}(1-\eta)^{-1} [1+\eta^{-1}+2(1-\eta)^{-1} \ln \eta] \right\} & \text{for } m \neq M \quad (\text{IV-18b}) \end{cases}$$

where $\eta = (\frac{m}{M})^2$. For the assumption with $m^2 = M^2 \gg m_\pi^2/4$, the ratio of SDA form factor $a(0,0)$ to the SDV form factor $v(0,0)$ is

$$\gamma_\pi(0) \approx -0.75 \quad (\text{IV-19})$$

which will be compared to the exact numerical computation result in Sec. IV-5.

3. Pion Decay Constant and Isovector Form Factor

The pion decay constant f_π is calculated in our one-loop approximation for the non-radiative pion decay using the relation in Eq. (IV-1)

$$f_\pi = -12\sqrt{2} i f_0^m I_{31} \\ = + 12\sqrt{2} i f_0^m M^2 \int \frac{d^4 p}{(2\pi)^4} \{ (p^2 - m^2) [(p+K)^2 - m^2] [(p+\frac{K}{2})^2 - M^2] \}^{-1}$$

If we apply the same techniques used in the previous sections, we easily obtain

$$f_\pi = \frac{3\sqrt{2}}{4\pi^2} f_0^m \Delta_M^2 \int_0^1 dx \int_0^x dy \beta_2'^{-1} \quad (\text{IV-20})$$

with

$$\beta_2' = \Delta_M^2 + (\Delta_q^2 - \Delta_M^2)x - \frac{1}{4}[x - (2y-x)^2]$$

Assuming that $m^2, M^2 \gg \frac{m_\pi^2}{4} > \frac{m_\pi^2}{4}[x - (2y-x)^2]$, we get

$$f_{\pi} \approx \frac{3\sqrt{2}}{4\pi^2} f_0^m \int_0^1 dx \frac{x}{1+(n-1)x}$$

by neglecting the third term in the denominator β_2' . Then, the integration is easily done, and we obtain

$$f_{\pi} \approx \begin{cases} \frac{3\sqrt{2}}{8\pi^2} f_0^m & \text{for } m=M \quad (\text{IV-21a}) \end{cases}$$

$$\begin{cases} \frac{3\sqrt{2}}{4\pi^2} f_0^{m(1-\eta)^{-1}} [-1-(1-\eta)^{-1} \ln \eta] & \text{for } m \neq M \quad (\text{IV-21b}) \end{cases}$$

For the $\pi^0 \rightarrow 2\gamma$ decay, there are only two possible Feynman diagrams in the lowest perturbation theory as shown in Fig. 16, and the two diagrams give the same contribution by Bose symmetry. Purely kinematic considerations tell us that the amplitude for this process is

$$M_{(\pi^0 \rightarrow 2\gamma)} = 2e^2 v_0^1(0,0) \epsilon_{\mu\nu\alpha\beta} \epsilon_1^\mu \epsilon_2^\nu k_1^\alpha k_2^\beta \quad (\text{IV-22})$$

with (k_1, ϵ_1) and (k_2, ϵ_2) the momenta and polarization four-vectors of the two photons. The decay rate is given by

$$\Gamma_{\pi^0} = \pi \alpha^2 m_{\pi^0}^3 |v_0^1(0,0)|^2 \quad (\text{IV-23})$$

Fig. 16 Feynman diagrams for the $\pi^0 \rightarrow 2\gamma$ decay.

In the quark-loop model, there are also two possible Feynman diagrams for each $u\bar{u}$, $d\bar{d}$ bound state. The total amplitude then is given by

$$M_{(\pi^0 \rightarrow 2\gamma)} = 2i\epsilon_1^\mu \epsilon_2^\nu (M_{\mu\nu}^{u\bar{u}} - M_{\mu\nu}^{d\bar{d}}) \quad (\text{IV-24})$$

with

$$M_{\mu\nu}^{u\bar{u}} = -3if_0 e^2 e_{qu}^2 \int \frac{d^4 p}{(2\pi)^4} \phi(p+\frac{K}{2}) \text{Tr}[\gamma_5 (\not{p}-m_u)^{-1} \gamma_\mu (\not{p}+k_2-m_u)^{-1} \gamma_\nu (\not{p}+K-m_u)^{-1}]$$

$$M_{\mu\nu}^{d\bar{d}} = -3if_0 e^2 e_{qd}^2 \int \frac{d^4 p}{(2\pi)^4} \phi(p+\frac{K}{2}) \text{Tr}[\gamma_5 (\not{p}-m_d)^{-1} \gamma_\mu (\not{p}+k_2-m_d)^{-1} \gamma_\nu (\not{p}+K-m_d)^{-1}]$$

where $K = k_1 + k_2$, 3 is the colour factor, and we have assumed that $f_0 = f_{\pi^0 u\bar{u}} = f_{\pi^0 d\bar{d}}$.

Since the traces are calculated, to be

$$\text{Tr}[\gamma_5(\not{p}+m)\gamma_\mu(\not{p}+k_2+m)\gamma_\nu(\not{p}+K+m)] \\ = 4im\epsilon_{\mu\nu\alpha\beta}k_1^\alpha k_2^\beta$$

we have, for the equal quark masses ($m = m_u = m_d$),

$$M(\pi^0 \rightarrow 2\gamma^2) = -24ie^2 f_0 m M^2 (e_{qu}^2 - e_{qd}^2) \epsilon_{\mu\nu\alpha\beta} \epsilon_1^\mu \epsilon_2^\nu k_1^\alpha k_2^\beta$$

$$\times \frac{d^4 p}{(2\pi)^4} \{[(p+K)^2 - m^2](p^2 - m^2)[(p+k_2)^2 - m^2][(p+\frac{K}{2})^2 - M^2]\}^{-1}$$

If we replace k_2 by $K - k_1$, the momentum integral above is the same as in $I_{41}^V(s)$. Since $k_2^2 = (K - k_1)^2 = 0$ in this case, the denominator in Eq. (IV-14) changes into

$$\beta_3^1 = \Delta_M^2 + (\Delta_q^2 - \Delta_M^2)x - \zeta[z(y-z) + (1-x)(2y-x)/4] \quad (\text{IV-25})$$

Here $\zeta = (\frac{m_{\pi^0}}{m_\pi})^2$ is introduced to unify the notations, and if we neglect the electromagnetic mass differences ζ

becomes unity. Comparing with Eq. (IV-22), we find

$$v_0^1(0,0) = \frac{1}{4\pi^2} \frac{f_0}{m_\pi} \Delta_q \Delta_M^2 \int_0^1 dx \int_0^x dy \int_0^y dz \beta_3^{-2} \quad (\text{IV-26})$$

which shows that $v(0,0) = \sqrt{2} v_0^1(0,0)$ in comparing with Eq. (IV-14).

Assuming that $m^2, M^2 \gg m_\pi^2/4$, we get, as in Eq. (IV-17),

$$v_0^1(0,0) = \begin{cases} \frac{1}{24\pi^2} \frac{f_0}{m} & \text{for } m=M \\ \frac{1}{8\pi^2} \frac{f_0}{m} \eta(1-\eta)^{-2} [1+\eta^{-1} + 2(1-\eta)^{-1} \ln \eta] & \text{for } m \neq M \end{cases} \quad (\text{IV-27a})$$

$$(\text{IV-27b})$$

4. Electromagnetic Form Factor and Mean Charge Radius

The pion electromagnetic form factor $F_\pi(k^2)$ was defined in Chapter III from the relation

$$\begin{aligned} \Gamma_\mu^{\text{em}} &= \langle \pi^-(K') | j_\mu^{\text{em}} | \pi^-(K) \rangle \\ &= +ie \{ (K+K')_\mu F_\pi(K^2, K'^2, k^2) + (K-K')_\mu F'_\pi(K^2, K'^2, k^2) \} \end{aligned}$$

(III-3)

where $k^2 = (K-K')^2$. The second term does not contribute, because of the transversality condition, and since the pions are on-shell, Eq. (III-3) can be rewritten as.

$$\Gamma_{\mu}^{\text{em}}(k^2) = +2ieK_{\mu} F_{\pi}(k^2) \quad (\text{IV-28})$$

Here $F_{\pi}(m_{\pi}^2, m_{\pi}^2, k^2) = F_{\pi}(k^2)$ is only function of k^2 .

Fig. 17 The pion electromagnetic form factor.

In the quark-loop model, $\Gamma_{\mu}^{\text{em}}(k^2)$ is shown graphically in Figure 17, and is expressed by

$$\begin{aligned} \Gamma_{\mu}^{\text{em}}(k^2) = & 3(\sqrt{2}f_0)^2 ee_q \int \frac{d^4 p}{(2\pi)^4} \phi(p+\frac{K}{2}) \phi(p+\frac{K-k}{2}) \text{Tr}[\gamma_5 (\not{p}-m)^{-1} \\ & \gamma_5 (\not{p}+K'-m)^{-1} \gamma_{\mu} (\not{p}+K-m)^{-1}] \\ & - 3(\sqrt{2}f_0)^2 ee_{\bar{q}} \int \frac{d^4 p}{(2\pi)^4} \phi(p-\frac{K}{2}) \phi(p-\frac{K-k}{2}) \text{Tr}[\gamma_5 (\not{p}-K-m)^{-1} \\ & \gamma_{\mu} (\not{p}-K'-m)^{-1} \gamma_5 (\not{p}-m)^{-1}] \end{aligned}$$

The trace calculations, are

$$\begin{aligned}
 & \text{Tr}[\gamma_5(\not{p}+m)\gamma_5(\not{p}+K'+m)\gamma_\mu(\not{p}+K+m)] \\
 &= 4\{(m^2-p^2+K\cdot K')p_\mu + [2(m^2-p^2-p\cdot K)+p\cdot k]K_\mu - (m^2-p^2-p\cdot K)k_\mu\} \\
 &= 4N_{(1)}
 \end{aligned}$$

$$\begin{aligned}
 & \text{Tr}[\gamma_5(\not{p}-K+m)\gamma_\mu(\not{p}-K+m)\gamma_5(\not{p}+m)] \\
 &= 4\{(m^2-p^2+K\cdot K')p_\mu - [2(m^2-p^2+p\cdot K)-p\cdot k]K_\mu + (m^2-p^2+p\cdot K)k_\mu\} \\
 &= 4N_{(2)}
 \end{aligned}$$

It is understood that the last terms involving k_μ in both $N_{(1)}$ and $N_{(2)}$ do not contribute because of $k\cdot\epsilon = 0$. Then, $\Gamma_\mu^{\text{em}}(k^2)$ becomes

$$\begin{aligned}
 \Gamma_\mu^{\text{em}}(k^2) &= 24f_0^2 eM^4 \int \frac{d^4 p}{(2\pi)^4} \\
 &\times \{e_q N_{(1)} [(p^2-m^2) [(p+K)^2-m^2] [(p+K')^2-m^2] [(p+\frac{K}{2})^2-m^2] \\
 &\quad [(p+\frac{K'}{2})^2-m^2]]^{-1} \\
 &- e_{\bar{q}} N_{(2)} [(p^2-m^2) [(p-K)^2-m^2] [(p-K')^2-m^2] [(p-\frac{K}{2})^2-m^2] \\
 &\quad [(p-\frac{K'}{2})^2-m^2]]^{-1}\} \quad (\text{IV-29})
 \end{aligned}$$

We know that $F_\pi(0) = 1$, and the terms including $K \cdot k$ vanish for $k^2 = 0$. For $k^2 = 0$, Eq. (IV-29) reduces to a simple form, such that

$$\begin{aligned} \Gamma_\mu^{\text{em}}(0) = & 24f_0^2 eM^4 \int \frac{d^4 p}{(2\pi)^4} \\ & \times \{e_q N_{(1)} \{(p^2 - m^2) [(p+K)^2 - m^2]^2 [(p+\frac{K}{2})^2 - m^2]^2\}^{-1} \\ & - e_{\bar{q}} N_{(2)} \{(p^2 - m^2) [(p-K)^2 - m^2]^2 [(p-\frac{K}{2})^2 - m^2]^2\}^{-1}\} \end{aligned}$$

The momentum integrations can also be done by applying the same techniques used in Sec. IV-2. Using Eq. (IV-9) we get.

$$\begin{aligned} \Gamma_\mu^{\text{em}}(0) = & 24\Gamma(5) f_0^2 eM^4 \int_0^1 dz_1 \dots \int_0^1 dz_3 \delta(1 - [z]_3) \times \\ & \times \int \frac{d^4 p}{(2\pi)^4} \left\{ \frac{e_q N_{(1)}}{[D_{(1)}]^5} - \frac{e_{\bar{q}} N_{(2)}}{[D_{(2)}]^5} \right\} z_2 z_3 \end{aligned}$$

with

$$D_{(1)} = (p + Kz')^2 + \frac{m_\pi^2}{4} \{4z'(1-z') - z_3\} - m^2(z_1 + z_2) - M^2 z_3$$

$$D_{(2)} = (p - KZ')^2 + \frac{m_\pi^2}{4} [4Z'(1-Z') - z_3] - m^2(z_1 + z_2) - m^2 z_3$$

where $Z' = z_2 + \frac{z_3}{2}$. If we shift the integral origin by $p \rightarrow p + KZ'$, the numerators become

$$N_{(1)} = [m^2 - (p - KZ')^2 + m_\pi^2] (p_\mu - K_\mu Z') + \{2[m^2 - (p - KZ')^2 - (p - KZ') \cdot K] - (p - KZ') \cdot k\} K_\mu$$

$$= \{m^2(2 - Z') + m_\pi^2 Z'(Z' - 1)^2 + (\frac{3}{2}Z' - 2)p^2\} K_\mu$$

$$N_{(2)} = [m^2 - (p + KZ')^2 + m_\pi^2] (p_\mu + K_\mu Z') - \{2[m^2 - (p + KZ')^2 + (p + KZ') \cdot K] K_\mu - (p + KZ') \cdot k\} K_\mu$$

$$= -N_{(1)}$$

where we have neglected the terms including p_μ , $p \cdot K$ and $p \cdot k$ after shifting the origin, whose integrations vanish by the symmetry properties, as mentioned earlier. The terms involving $K \cdot k$ also vanish.

After momentum integrations using Eq. (IV-13), we obtain

$$r_{\mu}^{\text{em}}(0) = i \frac{3e}{\pi^2} f_0^2 M^4 (e_q + e_{\bar{q}}) \int_0^1 dz_1 \dots \int_0^1 dz_3 \delta(1-z_3) z_2 z_3$$

$$\times \left\{ \frac{m^2(2-Z') + m_{\pi}^2 Z'(1-Z')^2}{\beta_4^3} + \frac{\frac{3}{2}Z' - 2}{\beta_4^2} \right\} K_{\mu}$$

Here $\beta_4 = \frac{m_{\pi}^2}{4} [4Z'(1-Z') - z_3] - m^2(z_1 + z_2) - M^2 z_3$, therefore, we find, by comparing with Eq. (IV-28),

$$F_{\pi}(0) = \frac{3}{2\pi^2} f_0^2 \Delta_M^4 \int_0^1 dz_1 \dots \int_0^1 dz_3 \delta(1-z_3) z_2 z_3$$

$$\times \left\{ \frac{\Delta_q^2(2-Z') + Z'(1-Z')^2}{\beta_4'^3} + \frac{2 - \frac{3}{2}Z'}{\beta_4'^2} \right\} \quad (\text{IV-30a})$$

$$= 1$$

$$\text{where } \beta_4' = \Delta_M^2 + (\Delta_q^2 - \Delta_M^2)(z_1 + z_2) - [Z'(1-Z') - z_3/4]$$

Eq. (IV-30a) is used as the normalization condition.

Changing the integration variables by $z_1 = y$, $z_2 = x-y$ and $z_3 = 1-x$, we rewrite Eq. (IV-30a) as

$$1 = \frac{3}{2\pi^2} f_0^2 \Delta_M^4 \int_0^1 dx \int_0^x dy (1-x)(x-y) \left\{ \frac{\frac{\Delta_Q^2}{2}(3-x+2y) + \frac{1}{8}(1+x-2y)(1-x+2y)^2}{\beta_4'^3} + \frac{2 - \frac{3}{4}(1+x-2y)}{\beta_4'^2} \right\} \quad (\text{IV-30b})$$

with

$$\beta_4' = \Delta_M^2 + (\Delta_Q^2 - \Delta_M^2)x - \frac{1}{4}[x - (x-2y)^2]$$

Assuming that $m^2, M^2 \gg m_\pi^2/4$, we have

$$1 = \frac{3}{2\pi^2} f_0^2 \eta \int_0^1 dx \left\{ \frac{1}{12}(9x^2 - 10x^3 + x^4) [1 + (\eta-1)x]^{-3} + \frac{\eta-1}{8} (5x^2 - 6x^3 + x^4) [1 + (\eta-1)x]^{-2} \right\}$$

which gives

$$1 = \frac{5}{32\pi^2} f_0^2 \quad \text{or} \quad \frac{f_0^2}{4\pi} \approx 1.6\pi \quad (\text{IV-31})$$

for $m=M$, and

$$\begin{aligned}
1 = & \frac{1}{16\pi^2} f_0^2 \eta \xi^{-2} \{ 17\eta^{-2} - 27\eta^{-1} + \xi^{-1} (27 - 50\xi^{-1} + \xi^{-2} + 4\eta^{-1}) \\
& - \eta^{-2} \xi^{-1} (59 - 59\xi^{-1} + 10\xi^{-2}) - 6(3 - 10\xi^{-1} + 2\xi^{-2}) \xi^{-1} \ln \eta \\
& + 3\eta^{-1} (5 - 9\xi^{-1} + 2\xi^{-2}) (1 + \eta^{-1} + 2\xi^{-1} \ln \eta) \} \quad (\text{IV-32})
\end{aligned}$$

for $m \neq M$, where

$$\eta = \left(\frac{m}{M}\right)^2 \quad \text{and} \quad \xi = 1 - \eta.$$

To calculate the pion mean squared charge radius $\langle r_\pi^2 \rangle$, we have to evaluate the integrals for $k^2 \neq 0$ in Eq. (IV-29). If we apply Eq. (IV-9) to Eq. (IV-29), the denominators become

$$D_{(1)}^1 = (p + KZ'' - kZ_3)^2 + \beta_5$$

$$D_{(2)}^1 = (p - KZ'' + kZ_3)^2 + \beta_5$$

$$\beta_5 = \frac{m_\pi^2}{4} [4Z''(1-Z'')(z_4+z_5)] + k^2 Z_3 (Z'' - Z_3) - m^2 (z_1+z_2+z_3) - M^2 (z_4+z_5)$$

$$Z'' = z_2 + z_3 + \frac{z_4+z_5}{2} \quad ; \quad Z_3 = z_3 + \frac{z_5}{2}$$

thus, if the integral origins are shifted by $p \rightarrow p + \bar{p} K Z'' + k Z_3$,
the numerators are also changed into

$$N_{(1)} = [m^2(2-Z'') + m_\pi^2 Z''(1-Z'')^2 + k^2 Z_3(2-Z'')(Z''-Z_3) + (\frac{3}{2}Z''-2)p^2] K_\mu$$

$$N_{(2)} = -N_{(1)}$$

With aid of Eq. (IV-13) we perform the momentum integration
over p , and obtain

$$F_\pi(k^2) = \frac{3}{2\pi^2} f_0^2 M^4 \int_0^1 dz_1 \dots \int_0^1 dz_5 \delta(1-[z]_5)$$

$$\times \left\{ \frac{m^2(2-Z'') + m_\pi^2 Z''(1-Z'')^2 + k^2 Z_3(2-Z'')(Z''-Z_3)}{\beta_5'^3} \right.$$

$$\left. + \frac{2 - \frac{3}{2} Z''}{\beta_5'^2} \right\}$$

(IV-33)

where

$$\beta_5' = M^2 + (m^2 - M^2)(z_1 + z_2 + z_3) - \frac{m_\pi^2}{4} [4Z''(1-Z'') - (z_4 + z_5)] - k^2 Z_3(Z'' - Z_3)$$

The electromagnetic form factor $F_\pi(k^2)$ is related to the pion mean squared charge radius $\langle r_\pi^2 \rangle$ by⁶⁾

$$F_\pi(k^2) \approx 1 + \frac{k^2}{6} \langle r_\pi^2 \rangle \quad (\text{IV-34})$$

The mean squared charge radius $\langle r_\pi^2 \rangle$ can be calculated from Eq. (IV-33) in our one-loop approximation, and the result is

$$\begin{aligned} \langle r_\pi^2 \rangle &= 6 \frac{dF_\pi(k^2)}{dk^2} \Big|_{k^2=0} \\ &= \frac{9f_0^2}{\pi^2} M^4 \int_0^1 dz_1 \dots \int_0^1 dz_5 \delta(1-[z]_5) Z_3(Z''-Z_3) \\ &\quad \times \left\{ \frac{6-4Z''}{\beta_5^{''3}} + \frac{3[m^2(2-Z'') + m_\pi^2 Z''(1-Z'')^2]}{\beta_5^{''4}} \right\} \end{aligned}$$

with $\beta_5^{''} = \beta_5'(k^2=0)$

It is convenient to evaluate the Feynman integrals by changing the variables $z_1, z_2, \dots, z_5$ by $z_1=t, z_2=z-t, z_3=y-z, z_4=x-y$ and $z_5=1-x$, and then, we obtain

$$\begin{aligned}
\langle r_{\pi}^2 \rangle = & \frac{9}{4\pi^2} \frac{f_0^2}{m_{\pi}^2} \Delta_M^4 \int_0^1 dx \int_0^x dy \int_0^y dz \int_0^z dt (1-x+2y-2z)(x-y+2z-2t) \\
& \times \left\{ \frac{6-2(1-y-2t)}{\beta_5'''^3} + \frac{3\frac{\Delta^2}{2}(3-y+2t) + \frac{1}{8}(1+y-2t)(1-y+2t)^2}{\beta_5'''^4} \right\}
\end{aligned}$$

(IV-35)

where

$$\beta_5''' = \Delta_M^2 (\Delta_q^2 - \Delta_M^2) y - \frac{1}{4} [1 + 3y - 4t - (1+y-2t)^2]$$

5. Numerical Results and Comments

For the assumption with $m^2, M^2 \gg m_{\pi}^2/4$ in the previous sections, we obtained, for the $m = M$ case,

$$v(s,0) \approx v(0,0) \approx \frac{\sqrt{2} f_0}{24\pi^2 m} \quad (\text{IV-17a})$$

$$a(s,0) \approx a(0,0) \approx -\frac{\sqrt{2} f_0}{32\pi^2 m} \quad (\text{IV-18a})$$

which give immediately

$$\gamma_{\pi}(s) \approx \gamma_{\pi}(0) \approx -0.75 \quad (\text{IV-19})$$

Also we found that

$$f_{\pi} \approx \frac{3\sqrt{2}}{8\pi^2} f_0^m \quad (\text{IV-21a})$$

$$v_0^1 \approx \frac{f_0}{24\pi^2 m} \quad (\text{IV-27a})$$

$$\frac{f_0^2}{4\pi} \approx 1.6\pi \quad (\text{IV-31})$$

Using these relations we can determine the values of m , M and f_0 as follows:

$$f_0 \approx 7.95 ; m = M \approx 301 \text{ MeV} : \text{Eqs. (IV-31) and (IV-21a)}$$

$$f_0 \approx 7.95 ; m = M \approx 242 \text{ MeV} : \text{Eqs. (IV-31) and (IV-27a)(IV-36)}$$

$$f_0 \approx 8.86 ; m = M \approx 269 \text{ MeV} : \text{Eqs. (IV-21a) and (IV-27a)}$$

where we have used the experimental data^{7,8)}

$$f_{\pi} = .945 m_{\pi}$$

$$\sin \theta_c = .229$$

$$v_0^1(0,0) = 1.876 \times 10^{-2} m_{\pi}^{-1}$$

For the $m \neq M$ case with $m^2, M^2 \gg m_{\pi}^2$, we have found that

$$f_{\pi} \approx \frac{3\sqrt{2}}{4\pi^2} f_0 m \xi^{-1} (-1 - \xi^{-1} \ln \eta) \quad (\text{IV-21b})$$

$$v_0^1(0,0) = \frac{f_0}{8\pi^2 m} \eta \xi^{-1} (1 + \eta^{-1} + 2\xi^{-1} \ln \eta) \quad (\text{IV-27b})$$

$$1 \approx \frac{1}{16\pi^2} f_0^2 \eta \xi^{-2} \{ 21\eta^{-2} - 27\eta^{-1} + \xi^{-1} (27 - 50\xi^{-1} + 10\xi^{-2} + 4\eta^{-1})$$

$$- \eta^{-2} \xi^{-1} (63 - 58\xi^{-1} + 10\xi^{-2}) - 6\xi^{-1} (3 - 10\xi^{-1} + 2\xi^{-2}) \ln \eta$$

$$+ 3\eta^{-1} (5 - 9\xi^{-1} + 2\xi^{-2}) (1 + \eta^{-1} + 2\xi^{-1} \ln \eta) \}$$

(IV-32)

where $\eta = \left(\frac{m}{M}\right)^2$ and $\xi = 1 - \eta$. From these relations we compute

$$\eta = .738$$

$$m = 266 \text{ MeV}; M = 310 \text{ MeV}$$

(IV-37)

$$f_0 = 7.58$$

which are very close to those for the $m = M$ case.

Putting these values into the SDV and SDA form factor expressions with the same approximation

$$v(0,0) = \frac{\sqrt{2} f_0}{8\pi^2 m} \eta \xi^{-2} (1 + \eta^{-1} + 2\xi^{-1} \ln \eta)$$

$$a(0,0) = -\frac{\sqrt{2} f_0}{8\pi^2 m} \eta \xi^{-2} \left(\frac{1}{2}(1 - \eta^{-1}) + \frac{3}{2} \xi^{-1} (1 + \eta^{-1} + 2\xi^{-1} \ln \eta) \right)$$

we obtain

$$\gamma_\pi(0) \approx -.773$$

(IV-38)

This is also close to that in Eq. (IV-19).

We now compute the exact numerical values for m , M and f_0 from Eqs. (IV-20), (IV-26) and (IV-30) by performing the numerical integrations using the Gaussian quadrature method⁹⁾. For the numerical integrations let us rewrite those equations as

$$f_{01} = \frac{4\pi^2 f_\pi}{3\sqrt{2} m_\pi \Delta_q \Delta_M^2 I_1} = \frac{8.793}{\Delta_q \Delta_M^2 I_1} \quad (\text{IV-39a})$$

$$f_{02} = \frac{4\pi^2 m v_0^1(0,0)}{\Delta_q \Delta_M^2 I_2} = \frac{.741}{\Delta_q \Delta_M^2 I_2} \quad (\text{IV-39b})$$

$$f_{03} = \left(\frac{2\pi^2}{3\Delta_M^2 I_3} \right)^{1/2} = \frac{2.565}{\Delta_M^2 I_3} \quad (\text{IV-39c})$$

where

$$I_1 = \int_0^1 dx \int_0^x dy \beta_6^{-1}$$

$$I_2 = \int_0^1 dx \int_0^x dy \int_0^y dz \{ \Delta_M^2 + (\Delta_q^2 - \Delta_M^2)x - \zeta \cdot [z(y-z) + (1-x)(2y-x)/4] \}^{-2}$$

$$I_3 = \int_0^1 dx \int_0^x dy (1-x)(x-y) \left\{ 12 - \frac{3}{4} (1+x-2y) \beta_6^{-2} \right.$$

$$\left. + \left[\frac{\Delta_q^2}{2} (3-x+2y) + \frac{1}{8} (1+x-2y)(1-x+2y)^2 \right] \beta_6^{-3} \right\}$$

$$B_6 = \Delta_M^2 + (\Delta_q^2 - \Delta_M^2)x - \frac{1}{4} [x - (2y - x)^2]$$

$$\zeta = \left(\frac{m_{\pi^0}}{m_\pi} \right)^2 = .935$$

and we have changed the subscript of f_0 by f_{01} , f_{02} and f_{03} to distinguish their origins. f_{01} , f_{02} and f_{03} should be the same.

The integrals I_1 , I_2 and I_3 are functions of the Feynman parameters m , M and the pion mass m_π . Since $m > m_\pi$ in our quark model, all the integrals are absolutely convergent in Feynman parameter space. Using the Gaussian quadrature method with 10, 20, 40 points we compute the integrals in the range of $.5 < (\Delta_q, \Delta_M) < 14$, and with aid of the least square fitting we find only one solution at $\Delta_q = 1.741$ and $\Delta_M = 3.525$. Thus, we determine the values of three parameters m , M (or M') and f_0 as follows:

$$m = 243 \text{ MeV}$$

$$M = 492 \text{ MeV}$$

$$f_0 = 4.28 \text{ or } \frac{f_0^2}{4\pi} = 1.45$$

The coupling constant is quite different from that in Eqs. (IV-38) obtained from the approximation with $m \neq M$ and $m^2, M^2 \gg m_\pi^2$.

Before we evaluate $v(s,0)$, $a(s,0)$ and $\gamma_\pi(s)$, let us first calculate the pion mean squared charge radius $\langle r_\pi^2 \rangle$ by using the above values in order to test our one-loop approximation. If we compute Eq. (IV-35) by the same method we obtain

$$\langle r_\pi^2 \rangle = .459 \text{ fm}^2 \text{ or } [\langle r_\pi^2 \rangle]^{1/2} = .677 \text{ fm}$$

which is in good agreement with the recent value .695 fm obtained from the inelastic fits of all data by Gensini¹⁰⁾. This shows that the quark-loop model with Drell-Lee ansatz vertex is not unreasonable. Being encouraged by this fact, we now proceed to evaluate the SDV form factor $v(s,0)$ and the SDA form factor $a(s,0)$, and estimate their ratio $\gamma_\pi(s) = a(s,0)/v(s,0)$. We write our expressions for $v(s,0)$ and $a(s,0)$ again here.

$$v(s;0) = \frac{\sqrt{2} f_0}{4\pi^2 m_\pi} \Delta_Q \Delta_M^2 \int_0^1 dx \int_0^x dy \int_0^y dz \beta_1^2 \quad (\text{IV-14})$$

$$a(s,0) = -\frac{3\sqrt{2} f_0}{4\pi^2 m_\pi} \Delta_q \Delta_M^2 \int_0^1 dx \int_0^x dy \int_0^y dz (x-2z) \beta_1'^2 \quad (\text{IV-15})$$

where

$$\beta_1' = \Delta_M^2 + (\Delta_q^2 - \Delta_M^2)x - [z(y-z) + (1-x)(2y-x)/4] - \frac{1}{2} \Delta_s^2 (x-y)(1-x+2z)$$

$$\Delta_M^2 = \left(\frac{M}{m_\pi}\right)^2, \quad \Delta_q^2 = \left(\frac{m}{m_\pi}\right)^2, \quad \Delta_s^2 = \frac{s}{m_\pi^2}$$

We remember that the range of s is $m_e^2 \leq s \leq m_\pi^2$. So the integrals are absolutely convergent in our quark-loop.

Using the same computing method as used earlier, we obtain the results as shown below in Table 16.

Table 16 The estimated values of $v(s,0)$, $a(s,0)$ and $\gamma_\pi(s)$

$s(x10^{-2}\text{GeV}^2)$	$v(s,0)(x10^{-2}m_\pi^{-1})$	$a(s,0)(x10^{-2}m_\pi^{-1})$	$\gamma_\pi(s)$
0.00	2.661	- 2.229	- .838
.01	2.662	- 2.229	- .838
.16	2.690	- 2.235	- .837
.49	2.688	- 2.246	- .836
1.21	2.729	- 2.273	- .833
1.95	2.772	- 2.300	- .830

In summary we obtain

$$|\gamma_\pi(0)| = .84$$

$$m = 243 \text{ MeV}$$

$$M = 492 \text{ MeV (or } M' = 984 \text{ MeV)}$$

$$\frac{f_0^2}{4\pi} = 1.45$$

$$[\langle r_\pi^2 \rangle]^{1/2} = .677 \text{ fm} \quad (\text{IV-40})$$

These values should be compared with other theoretical and experimental results. The quark mass is somewhat less than the standard value $m \approx 336 \text{ MeV}^{11)}$, and the $\pi^0 q\bar{q}$ coupling is weaker than the $\pi^0 - N\bar{N}$ coupling: $\frac{g^2}{4\pi} \approx 14.6$. As seen already, the pion mean charge radius gives a good agreement with data.

The parameter $\gamma_\pi(0)$ is in approximate agreement with $|\gamma_\pi| \leq 1^{1,12,13)}$, but is not in agreement with the data.

In Ref. [13], Gerasimov calculated $\gamma_\pi(0) = 1$ and $\frac{f_0^2}{4\pi} = \frac{\pi}{6}$ with $m = 240 \text{ MeV}$ from the one-loop approximation with the constant $\pi q\bar{q}$ vertex (i.e. $\phi(p)=1$).

The value of $f_0^2/4\pi$ is 2.8 times less than ours. He obtained those values by deriving some relations for

$\langle r_\pi^2 \rangle$, m , $v(0,0)$ and $a(0,0)$ from the Cabibbo-Radicati sum rule, axial vector anomaly, Goldberger-Treiman relation and the spectral representations of $\rho_1^V(s)$ and $\rho_1^A(s)$ using the low-energy theorem in Eq. (I-29).

In Table 16, we find that when s increases 0 to m_π^2 , the values of $v(s,0)$ and $|a(s,0)|$ also increase $\sim 4\%$ and $\sim 3\%$ respectively, but the ratio $|\gamma_\pi(s)|$ decreases $\sim 1\%$. Hence, it is possible to approximate $v(s,0) \approx v(0,0)$, $a(s,0) \approx a(0,0)$ and $\gamma_\pi(s) \approx \gamma_\pi(0)$ for the pion decay, as the experimenters do. As shown already for the $\tau^- \rightarrow \nu_\tau \pi^- \gamma$ decay in Chapter II, there is a possibility of the rapid increases in the values of $v(s,0)$ and $|a(s,0)|$ for large s , particularly in some typical regions corresponding to the vector meson dominances. Carron and Schult¹⁴⁾ estimated for the Kaon decay that $v_K(0,0) = .12 m_K^{-1}$ and $v_K(m_K^2,0) = .17 m_K^{-1}$, which implies that $v_K(s,0)$ increases $\sim 42\%$ as s changes 0.0 to m_K^2 . So we can expect that the values of $v(s,0)$ and $a(s,0)$ will increase quite rapidly for large s , and we cannot neglect those effects for the radiative heavy pseudoscalar meson decays, for example, $K \rightarrow e \nu_e \gamma$ and $F(D) \rightarrow e \nu_e \gamma$. However, the ratio $\gamma_P(s)$ may not be changed even for large s , because both form factors increase. So it may be possible to set $\gamma_P(s) \approx \gamma_P(0)$, where the subscript P denotes the pseudoscalar mesons.

CHAPTER V

CONCLUSIONS

The radiative $\tau^- \rightarrow \nu_\tau \pi^- \gamma$ decay process should give further information on the pion structure form factors $v(s,0)$ and $a(s,0)$ for large s . The present theoretical calculations should also be interesting from the point of view that such experiments may be performed in the near future. We calculated in the framework of the standard model¹⁾ the $\tau^- \rightarrow \nu_\tau \pi^- \gamma$ decay in terms of those form factors. With and without the iso-rotation we obtained the expressions describing the behaviours of $v(s,0)$ and $a(s,0)$ using the VMD, and estimated the expected spectrum and counting rates. Those expressions show that, in the regions corresponding to the vector meson resonances, the vector or axial vector contributions are considerably larger than the IB and INT contributions.

Because of those large contributions and the large kinematic factor we obtained the surprisingly large branching ratio $B_{\nu_\tau \pi \gamma}^\tau = 1.35 \times 10^{-3}$ with the photon energy cut of 100 MeV and $\delta = -1$. This implies that we need about 20 hours running time with a luminosity of $10^{31} \text{ cm}^{-2} \text{ sec}^{-1}$ to produce one detectable radiative event in the $\tau^- \rightarrow \nu_\tau \pi^- \gamma$ process. Our estimate for the

pure vector contribution in the ρ -band (ρ' contribution is negligible) is consistent with existing data, even if there is a long standing problem about the $\rho^- \rightarrow \pi^- \gamma$ decay. For the A_1 -band our calculation enables us to predict the branching ratio $B_{\pi^- \gamma}^{A_1} = \Gamma(A_1^- \rightarrow \pi^- \gamma) / \Gamma(A_1^- \rightarrow \text{all}) \approx 1.75 \times 10^{-3}$ with $\delta = -1$.

If the experimentalists are able to observe the radiative events in the ρ , and A_1 bands, then we can determine $v(s,0)$ and $a(s,0)$ at $s \approx m_\rho^2, m_{A_1}^2$. From this we can also determine δ . If VMD estimates are valid, we can expect to see how $v(s,0)$ and $a(s,0)$ behave for any s . However, to measure separately $v(s,0)$ and $a(s,0)$ as functions of s would still require correspondingly greater luminosity or running time.

The form factors $v(s,0)$ and $a(s,0)$ of the pion are of intrinsic interest, because they represent a description of the pion structure by the joint effect of weak and electromagnetic interactions. Thus we performed a consistent calculation of $v(s,0)$ and $a(s,0)$ as well as f_π and $\langle r_\pi^2 \rangle$ using the coloured and fractionally charged quarks in a one-loop approximation, modified by a $\pi q \bar{q}$ vertex function simulating bound state effects. We worked with the Drell-Lee ansatz vertex²⁾ which is a scalar function of the relative momentum between the

quark and the anti-quark.

This vertex function makes all the quark loops convergent, but spoils gauge invariance.* So, to obtain a gauge invariant amplitude for the radiative $\pi^- \rightarrow e^- \bar{\nu}_e \gamma$ decay we make a simple prescription, Eq. (III-26).

It looks artificial, but is supported by the soft-photon techniques³⁾ and a simple gauge projection operator.⁴⁾

From the obtained gauge invariant amplitude we subtracted the pointlike effect and obtained Eqs. (IV-14) and (IV-15) for $v(s,0)$ and $a(s,0)$, respectively. From the well-known processes: the $\pi^- \rightarrow \mu^- \bar{\nu}_\mu$ decay, the $\pi^0 \rightarrow 2\gamma$ decay and the normalization condition of the pion electromagnetic form factor $F_\pi(0) = 1$, we determined the three parameters

$$m = 243 \text{ MeV}$$

$$M = 492 \text{ MeV (or } M' = 984 \text{ MeV)}$$

$$f_0 = 4.28$$

and estimated the mean charge radius $[\langle r_\pi^2 \rangle]^{1/2} = .677 \text{ fm}$ and the parameter $|\gamma_\pi(0)| = .84$. The charge radius gives a confirmation that the one-loop approximation with the Drell-Lee ansatz vertex is not unreasonable. The value

* However, the pion electromagnetic form factor is still gauge invariant by the symmetry properties of the vertex functions and the integrals.

of $\gamma_\pi(0)$ is not in agreement with existing data as shown in Chapters I and IV, and only is consistent with other previous theoretical prediction.⁵⁾ Our result is in good agreement with Pascual-Tarrach's result.⁵⁾ They combined the standard quark-loop and σ -model to eliminate the logarithmically divergent loop integrals, and obtained $\gamma_\pi(0) = .86 \sim .7$ for $m_\sigma = .2m_\pi \sim 6m_\pi$ using the quark mass $m = 336$ MeV.⁶⁾

We showed clearly in Table 16 that the dependence of $v(s,0)$ and $a(s,0)$ on s are negligible for pion decay, because of small s . From the present calculations and other results⁷⁾ we can expect that the values of $|v(s,0)|$ and $|a(s,0)|$ will increase quite rapidly for large s , and hence we cannot neglect those effects for the radiative heavy pseudoscalar meson decays, for example, $K \rightarrow e \nu_e \gamma$ and $D(F) \rightarrow e \nu_e \gamma$ decays.

The present method may be extended to the kaon decay. However, the theoretical investigation of the form factors of Kaon decay is a much more complicated problem than for pion decay for two reasons. First, the quark mass difference must be considered. Recently, Bernabéu et al.,⁹⁾ and Moreno and Montemayor¹⁰⁾ claimed

that there is a large isospin breaking due to the quark mass difference.¹¹⁾ Secondly, in Kaon decay does not exist such a simple and reliable method as is the Vaks-Ioffe relation⁸⁾ for pion decay.

We can see from Eqs. (I-17) - (I-19) that the SD parts of the $D(F) \rightarrow e \nu_e \gamma$ decay dominate completely over the IB and INT parts. Thus, the $D(F) \rightarrow e \nu_e \gamma$ decay may be an excellent area to study the pure SD effects. The difficulties for application of our model to the $D(F)$ meson decay arise not only from the large quark mass difference, but also from the fact that $s > m$.

In our model there is an extra mass parameter M (or M') originating from the $\pi q \bar{q}$ vertex function. It is quite important to understand the properties of this parameter, because it can give information on the bound state wave function. We know that, in the electron loop integrals for the vacuum polarization correction to the electromagnetic process, the large mass parameter (cut-off mass) is introduced to make the integrals convergent. The mass parameter $M(M')$ also acts like the same cut-off parameter, even if its origin is quite different. It is interesting to note that the Drell-Lee ansatz vertex function is the same as the Feynman

convergence factor.¹²⁾ The latter, however, is independent of the photon momentum, k , and thus, does not effect gauge invariance. The Feynman convergence factor was used in the nucleon-loop calculations¹³⁾ by setting $M' = m_N$. Our mass parameter plays the double roles: as a cut-off parameter and a vertex mass parameter. In order to understand the mass parameter M (or M') and the vertex function, we need further study.

Our dynamical value of $\gamma_\pi(0) \approx \gamma_\pi(s)$ could be used to our phenomenological calculations of the radiative $\tau^- \rightarrow \nu_\tau \pi^- \gamma$ decay process in Chapter II. Our quark-loop model may also be applied to this process.

REFERENCES TO CHAPTER I

1. M. Ruderman, Phys. Rev. 85, 157 (1952)
2. S.A. Bludman and M.A. Ruderman, Phys. Rev. 101, 910 (1956)
3. S.B. Treiman and H.W. Wyld Jr., Phys. Rev. 101, 1552 (1956)
4. V.G. Vaks and B.L. Ioffe, Nuovo Cimento 10, 342 (1958)
5. N. Cabibbo, Nuovo Cimento 11, 87 (1959)
6. S.A. Bludman and J.A. Young, Phys. Rev. 118, 602 (1960)
7. D.E. Neville, Phys. Rev. 124, 2037 (1961)
8. V.F. Muller, Z. Physik 172, 224 (1963)
9. S.G. Brown and S.A. Bludman, Phys. Rev. 136, B1160 (1964)
10. W. Kummer and J. Kuti, Nuovo Cimento 45A, 242 (1966)
11. T. Das, V.S. Mathur and S. Okubo, Phys. Rev. Lett. 19, 859 (1967)
12. P. De Baenst and J. Pestieau, Nuovo Cimento 53A, 407 (1968)
13. S.G. Brown and G.B. West, Phys. Rev. 168, 1605 (1968)
14. Riazuddin and Fayyazuddin, Phys. Rev. 171, 1428 (1968)
15. V.S. Berezinskiĭ, Sov. J. Nucl. Phys. (Yad. Fiz.) 7, 745 (1968)
16. J.S. Vaishya, Phys. Rev. 173, 1757 (1968)

17. R.E. Marshak, Riazuddin and C.P. Ryan, "Theory of Weak Interactions in Particle Physics", Wiley-Interscience, Toronto (1969)
18. A.I. Vainshtein, Sov. Phys. JETP Lett. 6, 266 (1967)
19. V.S. Berezinskii, Sov. J. Nucl. Phys. (Yad. Fiz.) 8, 700 (1969)
20. F. Scheck and A. Wulfschlegel, Nucl. Phys. B67, 504 (1973)
21. M.G. Smoes, Nucl. Phys. B20, 237 (1970)
22. D. Yu Bardin and S.M. Bilen'kii, Sov. J. Nucl. Phys. (Yad. Fiz.) 16, 311 (1973)
23. D. Yu Bardin et al., Sov. J. Nucl. Phys. (Yad. Fiz.) 14, 239 (1972)
24. M.V. Terent'ev, Sov. J. Nucl. Phys. (Yad. Fiz.) 16, 87 (1973)
25. A.Q. Sarker, Phys. Rev. 173, 1749 (1968)
26. R. Rockmore, Phys. Rev. 177, 2573 (1969)
27. N.J. Carron and R.L. Schult, Phys. Rev. D1, 3171 (1970)
28. E. Gava, F. Legovini and N. Paver, Lett. Nuovo Cimento 14, 41 (1975)
29. T.H. Go and H. Leutwyler, Nucl. Phys. B118, 493 (1977)

30. M.K. Volkov and V.N. Pervushin, Sov. J. Nucl. Phys. (Yad. Fiz.) 22, 189 (1975)
31. M. Moreno and J. Pestieau, Phys. Rev. D13, 175 (1976)
32. L. Resnick and J.H. Kim, Can. J. Phys. 54, 158 (1976)
33. M. Moreno, Phys. Rev. D16, 720 (1977)
34. R. Decker, Phys. Lett. 65B, 153 (1976)
35. P. Pascual and R. Tarrach, Nucl. Phys. B146, 509 (1978)
36. S.B. Gerasimov, Sov. J. Nucl. Phys. (Yad. Fiz.) 29, 259 (1979)
37. P. Depommier, J. Heintze, C. Rubbia and V. Soergel, Phys. Lett. 7, 285 (1969)
38. P.S.L. Booth et al., Proc. Phys. Soc. 86, 1317 (1965)
39. A. Stetz, J. Carroll, D. Ortendahl, V. Perez-Mendez, G. Igo, N. Chirapatpimol and M.A. Nasser, Phys. Rev. Lett. 33, 1455 (1974)
40. A. Stetz, J. Carroll, D. Ortendahl, V. Perez-Mendez, G. Igo, N. Chirapatpimol and M.A. Nasser, Nucl. Phys. B138, 285 (1978)
41. J. Heintze et al., Nucl. Phys. B149, 101 (1979)
42. K.S. Heard et al., Phys. Lett. 55B, 324 (1975)

43. S.M. Karenchenko et al., Sov. Phys. JETP 44, 35 (1976)
44. E. Fermi, Nuovo Cimento 11, 1 (1934); also Z. Physik 88, 161 (1934)
45. J. Tiomno and J.A. Wheeler, Rev. Mod. Phys. 21, 144 (1949)
O. Klein, Nature 161, 897 (1948)
T.D. Lee, M. Rosenbluth and C.N. Yang, Phys. Rev. 75, 905 (1949)
46. M. Ruderman and R. Finkelstein, Phys. Rev. 76, 1458 (1949)
47. R.P. Feynman, Phys. Rev. 76, 769 (1949)
48. R.P. Feynman and M. Gell-Mann, Phys. Rev. 109, 193 (1958)
E.C.G. Sudarshan and R.E. Marshak, Phys. Rev. 109, 1860 (1958)
J.J. Sakurai, Nuovo Cimento 7, 649 (1958)
49. J.J. Sakurai, Ann. Phys. 11, 1 (1960)
M. Gell-Mann and F. Zachariasen, Phys. Rev. 124, 953 (1961)
M. Gell-Mann, D. Sharp and W.G. Wagner, Phys. Rev. Lett. 8, 261 (1962)
50. Our notations are those of J.D. Bjorken and S.D. Drell, "Relativistic Quantum Mechanics", McGraw-Hill, Toronto (1964)

51. M.M. Nagels et al., Nucl. Phys. B147, 189 (1979)
52. V.F. Müller, Z. Physik 173, 438 (1963)
P. De Baenst and J. Pestieau, Nuovo Cimento 53A,
137 (1968)
53. S.L. Adler, Phys. Rev. 177, 2426 (1969); also in
Lectures on Elementary Particle Physics, edited by
S. Deser et al., MIT, Mass. (1970)
J.S. Bell and R. Jackiw, Nuovo Cimento 60A, 47 (1969)
R. Jackiw, in Lectures on Current Algebra and its
Applications by S.B. Treiman, R. Jackiw and D.J. Gross,
Princeton Univ. Press (1972)
54. Particle Data Group, Phys. Lett. 75B, 1 (1978);
Rev. Mod. Phys. 48, S1 (1976)
55. Particle Data Group, Phys. Lett. 50B, 1 (1974);
Rev. Mod. Phys. 45, S1 (1973);
56. Particle Data Group, Rev. Mod. Phys. 42, 87 (1970);
41, 109 (1969)
A.H. Rosenfeld et al., Rev. Mod. Phys. 40, 77 (1968);
39, 1 (1967)
57. A.H. Rosenfeld et al., Rev. Mod. Phys. 36, 977
(1964)
58. G. Von Dardel et al., Phys. Lett. 4, 51 (1963)
59. R.G. Glasser et al., Phys. Rev. 123, 1014 (1961)

60. S. Weinberg, Phys. Rev. Lett. 18, 507 (1967)
T. Das, V.S. Mathur and S. Okubo, Phys. Rev. Lett.
18, 761 (1967)
61. K. Kawarabayashi and M. Suzuki, Phys. Rev. Lett.
16, 255 (1966)
Riazuddin and Fayyazuddin, Phys. Rev. 147, 1071 (1966)
62. H.J. Schnitzer and S. Weinberg, Phys. Rev. 164, 1828
(1967)
63. S.G. Brown and G.B. West, Phys. Rev. Lett. 19,
812 (1967); Phys. Rev. 168, 1605 (1968)
T. Das, V.S. Mathur and S. Okubo, Phys. Rev. Lett.
19, 1067 (1967)
64. F.E. Low, Phys. Rev. 110, 974 (1958)
S.L. Adler and Y. Dothan, Phys. Rev. 151, 1267 (1967)
T.H. Burnett and N.M. Kroll, Phys. Rev. Lett. 20,
86 (1968)
65. Y.S. Tsai, Phys. Rev. D4, 2821 (1971); SLAC-PUB-2105
April 1978); SLAC-PUB-2242 (December 1978)
H.B. Thacker and J.J. Sakurai, Phys. Lett. 36B, 103
(1971)
66. J.H. Kim and L. Resnick, to be published in Phys.
Rev. D
67. W.A. Bardeen and Wu-Ki Tung, Phys. Rev. 173, 1423
(1968)

REFERENCES TO CHAPTER II

1. G.J. Feldman, SLAC Preprints SLAC-PUB-2224 (October 1978) and SLAC-PUB-2311 (April 1979)
2. M.L. Perl, SLAC Preprint SLAC-PUB-2219 (November 1978)
3. G. Flügge, DESY Preprint DESY 79/37 (July 1979)
4. M.L. Perl et al., Phys. Rev. Lett. 35, 1489 (1975);
38, 117 (1976)
5. G.J. Feldman et al., Phys. Lett. 63B, 466 (1976);
M.L. Perl et al., Phys. Lett. 70B, 487 (1977)
PLUTO Coll., J. Burmester et al., Phys. Lett. 68B,
297, 301 (1977)
DASP Coll., R. Brandelik et al., Phys. Lett. 70B, 125
(1977)
A. Barbaro-Galtieri et al., Phys. Rev. Lett. 39, 1058
(1977)
6. Y.S. Tsai, Phys. Rev. D4, 2821 (1971); SLAC-PUB-2105
(April 1978) SLAC-PUB-2242 (December 1978)
H.B. Thacker and J.J. Sakurai, Phys. Lett. 36B, 103
(1971)
7. We use the conventions of J.D. Bjorken and S.D. Drell,
"Relativistic Quantum Mechanics", McGraw Hill,
Toronto (1964)

8. J.D. Bjorken and C.H. Llewellyn Smith, Phys. Rev. D7, 887 (1973)
T. Hagiwara, S.Y. Pi and A.I. Sanda, Ann. Phys. 106, 134 (1977)
F.J. Gilman and D.H. Miller, Phys. Rev. D17, 1846 (1978)
9. PLUTO Coll., G. Alexander et al., Phys. Lett. 81B, 84 (1979)
10. R.E. Marshak, Riazuddin and C.P. Ryan, "Theory of Weak Interactions in Particle Physics", Wiley-Interscience, Toronto (1969)
11. L. Copley, in Lecture Notes on Elementary Particle Physics (unpublished)
12. S. Weinberg, Phys. Rev. Lett. 18, 507 (1967)
13. J. Bernabeu, R. Tarrach and F.J. Yndurain, Phys. Lett. 79B, 464 (1978)
14. However, isospin symmetry violation can remain even if the current quark masses have large splitting.
See D.J. Gross, S.B. Treiman and F. Wilczek, Phys. Rev. D19, 2188 (1979); J. Kripfganz and C. Leroy, McGill Univ. Preprint (1979)
15. M. Gourdin, Phys. Reports 11C, 29 (1974)
F.M. Renard, Springer Tracks in Mod. Phys. 63, 98 (1972)

16. E. Cremmer and M. Gourdin, Nucl. Phys. B10, 179 (1969)
17. G.J. Feldman and M.L. Perl, Phys. Reports 19C, 233 (1975)
18. M. Gell-Mann and F. Zachariasen, Phys. Rev. 124 953 (1961)
19. N.M. Kroll, T.D. Lee and B. Zumino, Phys. Rev. 157, 1376 (1967)
T.D. Lee and B. Zumino, Phys. Rev. 163, 1667 (1967)
20. K. Fujikawa, Phys. Rev. D4, 2794 (1971)
J.J. Sakurai and D. Schildknecht, Phys. Lett. 40B, 121 (1972)
21. J.J. Sakurai, Ann. Phys. 11, 1 (1960)
22. Particle Data Group, Phys. Lett. 75B, 1 (1978)
23. R.L. Thews, Phys. Rev. D14, 3021 (1976)
24. M. Gell-Mann, D. Sharp and W.G. Wagner, Phys. Rev. Lett. 8, 261 (1962)
25. P. Darriulat et al., Nucl. Inst. and Meth. 129, 105 (1975)
26. R. Torgerson, Phys. Rev. D10, 2951 (1974)
W.C. Carithers et al., Phys. Rev. Lett. 35, 349 (1975)
27. A.N. Kamal and G.L. Kane, Phys. Rev. Lett. 43, 551 (1979)

28. A.N. Kamal, Phys. Rev. D18, 3512 (1978)

29. G. Grunberg and F.M. Renard, Nuovo Cimento 33A, 617
(1976)

Etim-Etim and M. Greco, Nuovo Cimento 42A, 124 (1977)

30. W. Baçino et al., Phys. Rev. Lett. 42, 6 (1979)

REFERENCES TO CHAPTER III

1. F.E. Low, Phys. Rev. 110, 974 (1958)
2. S.L. Adler and Y. Dothan, Phys. Rev. 151, 1267 (1966)
J. Pestieau, Phys. Rev. 160, 1555 (1967)
T.H. Burnett and N.M. Kroll, Phys. Rev. Lett. 20, 86 (1968)
3. S.D. Drell and T.D. Lee, Phys. Rev. D5, 1738 (1972)
4. W.A. Bardeen and Wu-Ki Tung, Phys. Rev. 173, 1423 (1968)
5. There exists a substantial amount of literature on the Bethe-Salpeter equation. Among the early ones are
E. Salpeter and H. Bethe, Phys. Rev. 84, 1232 (1951)
M. Gell-Mann and F. Low, Phys. Rev. 84, 350 (1951)
G.C. Wick, Phys. Rev. 96, 1124 (1954)
R. Cutkosky, Phys. Rev. 96, 1135 (1954)
S. Mandelstam, Proc. Roy. Soc. (London) A233, 248 (1955)
For a review, see
N. Nakanishi, Suppl. Prog. Theor. Phys. 43, 1 (1969)
For further references, see
C.H. Llewellyn Smith, Ann. Phys. 53, 521 (1969)
M.K. Sundareshan and P.J.S. Watson, Ann. Phys. 59, 375 (1970)

- M. Bohm, H. Joos and M. Krammer, Acta. Phys. Austriaca Suppl. XI, 3 (1973) (Schladming Lectures 1973)
6. We use the conventions of J.D. Bjorken and S.D. Drell, "Relativistic Quantum Mechanics" McGraw-Hill, Toronto (1964)
 7. C.H. Llewellyn Smith, Ref. 5
 8. J.S. Ball and F. Zachariasen, Phys. Rev. 170, 1541 (1968)
 9. D. Amati, L. Caneschi and R. Jengo, Nuovo Cimento 58A, 783 (1968)
 10. M. Ciafaloni and P. Menotti, Phys. Rev. 173, 1575 (1968)
 11. M. Ciafaloni, Phys. Rev. 176, 1898 (1968)
 12. P. Menotti, Phys. Rev. D9, 2767 (1974)
 13. C. Alabiso and G. Schierholz, Phys. Rev. D10, 960 (1974)
 14. T. Appelquist and E. Poggio, Phys. Rev. D10, 3280 (1974)
 15. G.C. Wick, Ref. 5
R. Cutkosky, Ref. 5
 16. L. Resnick and J.H. Kim, Can. J. Phys. 54, 158 (1976)
 17. H.W. Fearing, E. Fischbach and J. Smith, Phys. Rev. D2, 542 (1970)
 18. D.R. Yennie, S.C. Frautschi and H. Suura, Ann. Phys. 13 379 (1961)

REFERENCES TO CHAPTER IV

1. P. De Baenst and J. Pestieau, Nuvov Cimento 53A 407 (1968)
2. L. Resnick and J.H. Kim, Can. J. Phys. 54, 158 (1976)
3. R.P. Feynman, Phys. Rev. 76, 769 (1949)
4. L.D. Landau and E.M. Lifshitz, "Relativistic Quantum Theory", Pergamon Press, Toronto, (1974)
5. G.'t Hooft and M. Veltman, Nucl. Phys. B44, 189 (1972)
6. J. Bersntein, "Elementary Particles and Their Currents", W.H. Freeman and Company, San Francisco (1968)
7. M.M. Nagels et al., Nucl. Phys. B147, 189 (1979)
8. Particle Data Group, Phys. Lett. 75B, 1 (1978)
9. A.H. Stroud and Don Secrest, "Gaussian Quadrature Formulas", Prentice-Hall, INC. (1966)
10. P.M. Gensini, SLAC Preprint SLAC-PUB-1952 (1977)
11. A De Rújula, H. Georgi and S.L. Glashow, Phys. Rev. D12, 147 (1975)
12. P. Pasqual and R. Tarrach, Nucl. Phys. B146, 509 (1978)
13. S.B. Gerasimov, Sov. J. Nucl. Phys. (Yad. Fiz.) 29, 513 (1979)

14. N.J. Carron and R.L. Schult, Phys. Rev. D₁, 3171
(1970)

REFERENCES TO CHAPTER V

1. S. Tsai, Phys. Rev. D4, 2821 (1971); SLAC-PUB-2105 (1978); SLAC-PUB-2242 (1978)
H.B. Thacker and J.J. Sakurai, Phys. Lett. 36B, 103 (1971)
2. S.D. Drell and T.D. Lee, Phys. Rev. D5, 1738 (1972)
3. F.E. Low, Phys. Rev. 110, 974 (1958)
S.L. Adler and Y. Dothan, Phys. Rev. 151, 1267 (1967)
4. W.A. Bardeen and Wu-Ki Tung, Phys. Rev. 173, 1423 (1968)
D.R. Yennie, S.C. Frautschi and H. Suura, Ann. Phys. 13, 349 (1961)
5. P. Pascual and R. Tarrach, Nucl. Phys. B146, 509 (1978) and references therein.
6. A. De Rújula, H. Georgi and S.L. Glashow, Phys. Rev. D12, 147 (1975)
7. N.J. Carron and R.L. Schult, Phys. Rev. D1, 3171 (1970)
8. V.G. Vaks and B.L. Ioffe, Nuovo Cimento 10, 342 (1958)
9. J. Bernabeau, R. Tarrach and F.J. Ynduráin, Phys. Lett. 79B, 464 (1978)
10. R. Montemayor and M. Moreno, Phys. Rev. D20, 250 (1979)

11. However, isospin symmetry violation can remain small even if the current quark masses have large splitting. See D.J. Gross, S.B. Treiman and F. Wilczek, Phys. Rev. D 19, 2188 (1979); J. Kripfganz and C. Leroy, McGill Univ. Preprint (1979)
12. R.P. Feynman, Phys. Rev. 76, 769 (1949)
13. S.B. Treiman and H.W. Wyld Jr., Phys. Rev. 101, 1552 (1956)

3 3

OF / DE

END

1 8 1 2 8 1

FIN