Dynamics of Elastic Nonlinear Rotating Composite Beams with Embedded Actuators

By

Mehrdaad Ghorashi, P.Eng.

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Department of Mechanical and Aerospace Engineering
Carleton University

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Abstract

A comprehensive study of the nonlinear dynamics of composite beams is presented. The study consists of static and dynamic solutions with and without active elements. The static solution provides the initial conditions for the dynamic analysis. The dynamic problems considered include the analyses of clamped (hingeless) and articulated (hinged) accelerating rotating beams. Numerical solutions for the steady state and transient responses have been obtained. It is shown that the transient solution of the nonlinear formulation of accelerating rotating beam converges to the steady state solution obtained by the shooting method. The effect of perturbing the steady state solution has also been calculated and the results are shown to be compatible with those of the accelerating beam analysis. Next, the coupled flap-lag rigid body dynamics of a rotating articulated beam with hinge offset and subjected to aerodynamic forces is formulated. The solution to this rigid-body problem is then used, together with the finite difference method, in order to produce the nonlinear elasto-dynamic solution of an accelerating articulated beam. Next, the static and dynamic responses of nonlinear composite beams with embedded Anisotropic Piezo-composite Actuators (APA) are presented. The effect of activating actuators at various directions on the steady state force and moments generated in a rotating composite beam has been presented. With similar results for the transient response, this analysis can be used in controlling the response of adaptive rotating beams.

To the Memory of
my beloved Mother, Touraandokht Khoshravaan
and
my dear Brother, Mahdaad

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Nomenclature

A	= cross-sectional area of the undeformed beam in x_2 - x_3 plane
B	= deformed reference frame
b	= undeformed reference frame
C	= finite rotation tensor
D	= material matrix
D	= matrix of electric displacements
d_{ijk}	= piezoelectric moduli
E	= electric field
E_i	= Young's moduli
e	= hinge offset
e_{ijk}	= permutation symbol
e_I	= $[1 \ 0 \ 0]^T$
F_i	= elements of the column matrix of internal forces
f	= applied forces per unit length
G_{ij}	= shear moduli
g	= determinant of the metric tensor in curvilinear coordinates
g	= current tip boundary value
H	= sectional angular momenta

i_2, i_3	= cross-sectional mass moment
i_{23}	= cross-sectional product of inertia
K	= deformed beam curvature vector
k	= undeformed beam curvature vector
L	= length of the beam
L	= lift per unit length
M_i	= elements of the column matrix of internal moments
m	= applied moments per unit length
N	= number of nodes
P	= sectional linear momenta
R	= rotor radius
S	= stiffness matrix
$S(x_2, x_3)$	= matrix of the FEM shape functions
t	= time
U	= strain energy per unit length
u_i	= displacement field in b
V	= column matrix of the nodal values of the warping displacements
V	= velocity field in B
v	= velocity field in b
w_i	= warping displacement field
x_i	= global system of coordinates
x_I	= axis along the beam

$\bar{x}_2$ and $\bar{x}_3$	= offsets from the reference line of the cross-sectional mass center
x_G	= spanwise position of the mass center
α	= magnitude of the rotation used in the Rodrigues parameters
α	= assumed initial conditions at root
β	= tip boundary condition
β	= flap angle
Γ	= strain tensor = $\begin{bmatrix} \Gamma_{11} & 2\Gamma_{12} & 2\Gamma_{13} & \Gamma_{22} & 2\Gamma_{23} & \Gamma_{33} \end{bmatrix}^T$
γ	= $\begin{bmatrix} \gamma_{11} & 2\gamma_{12} & 2\gamma_{13} \end{bmatrix}^T$
γ	= Lock number
$\bar{\gamma}_{11}$	= extension of the reference line
Δ	= 3×3 identity matrix
$\delta\bar{q}$	= virtual displacement vector
$\delta\bar{\psi}$	= virtual rotation vector
$\bar{\epsilon}$	= generalized strains of the classical theory = $\begin{bmatrix} \bar{\gamma}_{11} & \bar{\kappa}_1 & \bar{\kappa}_2 & \bar{\kappa}_3 \end{bmatrix}^T$
ϵ^T	= matrix of dielectric permittivity at constant strain
θ	= Rodrigues parameters = $\begin{bmatrix} \theta_1 & \theta_2 & \theta_3 \end{bmatrix}^T$; $\theta_i = 2e_i \tan(\alpha/2)$
κ_1	= elastic twist
κ_i	= elastic bending curvatures (i=2,3)
μ	= mass per unit length
ν_{ij}	= Poisson's ratios

ρ	= mass density
σ_{ij}	= stress tensor components = $[\sigma_{11} \quad \sigma_{12} \quad \sigma_{13} \quad \sigma_{22} \quad \sigma_{23} \quad \sigma_{33}]^T$
ψ	= kernel matrix
Ω	= angular velocity in B basis
ω	= angular velocity in b basis
ξ	= lead-lag angle
$(\bar{\bullet})$	= perturbations in space
$(\dot{\bullet})$	= perturbations in time
$(\bullet)'$	= $\frac{\partial(\bullet)}{\partial x_1}$ = derivative w.r.t. the undeformed reference line
$(\dot{\bullet})$	= $\frac{d(\bullet)}{dt}$ = absolute time derivative
$(\delta \bar{\bullet})$	= the overbar indicates that it need not be the variation of a functional
$(\hat{\bullet})$	= The discrete boundary value of quantity $(\bullet)$
$(\tilde{\bullet})_{ij}$	= $-e_{ijk}(\bullet)_k$: cross product operator
$\langle \bullet \rangle$	= $\int_A (\bullet) dx_2 dx_3$
$\langle \langle \bullet \rangle \rangle$	$\langle (\bullet) \sqrt{g} \rangle = \int_A (\bullet) \sqrt{g} dx_2 dx_3$, $\sqrt{g} = 1 - x_2 k_3 - x_3 k_2$

Chapter 1: Introduction and Literature Review

1.1 Introduction

Helicopter, with its capability of vertical take-off and landing is a crucial means of aerial transportation. In fire-fighting rescue operations and missions for helping survivors of an earthquake or an avalanche, helicopters have played vital roles. The expansion of the domain of application of helicopters, however, faces with a few serious constraints. Among them is the relatively poor ride quality due to severe vibration and noise. Vibration can reduce the fatigue life of structural components, and hence, increase the operating costs. Furthermore, environmental consequences of noise and vibration have limited the range of application and the velocity of helicopters. That is why reducing noise and vibration is a major goal in the design of helicopters.

Smart materials are good candidates for providing a way to control noise and vibrations in helicopters. Embedded strain sensing and actuation in active structures can be used to reduce blade vibration, minimize blade vortex interaction, decrease noise and improve stability and response characteristics of the helicopter, Traugott et al. (2005). The efficient application of active materials in nonlinear composite rotating blades and generation of adaptive blades, however, requires the development of the necessary analytical tools.

Active beam models can, of course, be developed based on pure three dimensional (3-D) Finite Elements Method (FEM) models. However, modeling initially twisted and curved active rotor blades using 3-D FEM is computationally demanding and is not suitable for preliminary design or for control purposes. As an alternative, and since a helicopter rotor blade is a slender structural member, one may model it as a thin-walled composite beam. Such a beam model for a helicopter blade is an efficient alternative, Hopkins and Ormiston (2003). Beam modeling of these structures is simpler and is expected to yield sufficiently accurate results. This dimensional reduction transforms the 3-D rotating structure into a 1-D nonlinear rotating active composite beam which utilizes the cross-sectional properties obtained by a 2-D analysis.

The cross-sectional analysis can be classified as ad hoc or asymptotic. Asymptotic methods rely on the existence of a set of small parameters. Ad hoc analyses, however, invoke assumptions that do not hold in general. The most accurate and powerful of the ad hoc methods appears to be Jung et al. (2002). Conventional beam models rely on ad hoc assumptions on displacement or stress fields. An example is the Saint-Venant's torsion theory which assumes that the cross section of a beam remains rigid in its own plane as it twists (i.e warping is possible only along the longitudinal axis of the beam), Hodges (2006).

While simplifying ad hoc assumptions may be used for homogeneous and isotropic materials, for non-homogeneous and anisotropic beams, the in- and out-of-plane warping displacements are fully coupled and one should include both in- and out-of-plane

components of warping in the analysis. Ad hoc assumptions are neither necessary nor correct in any sense, Hodges (2006).

A beam theory suitable for composite rotor blade analysis should meet several requirements. The kinematics should include geometrical nonlinearities and initial twist and the strain should be assumed to be small (even though the displacements and rotations are not necessarily small). Based on the nonlinear-beam kinematics in Danielson and Hodges (1987), a theory which meets the mentioned requirements has been presented in Atilgan and Hodges (1991) which is valid for non-homogeneous, anisotropic beams undergoing large global rotation, small local rotation and small strain.

In the past two decades, research has been focused on the analysis of anisotropic composite beams using the Variational-Asymptotic Method (VAM). VAM is computationally more efficient than FEM and it starts from the elastic energy functional. It solves problems that can be formulated as minimization of a functional and have an inherently small dimension (e.g. beams, plates and shells). The solution has the common advantage of asymptotic methods of being mathematically well-grounded with no ad hoc assumptions about displacement or stress fields. Interestingly, there are no theoretical restrictions on the geometry of the cross section or on the materials of the problems for which VAM can be applied. It is especially proper for realistic modeling of initially curved and twisted anisotropic beams (like rotor blades), Hodges (2006).

VAM, as a powerful method for solving composite thin-walled beam problems was first introduced in Berdichevsky (1981). It splits the 3-D geometrically nonlinear elasticity analysis of active composite rotating blades into two parts. The first part is a

linear 2-D analysis to determine the cross-sectional stiffness and mass matrices as well as the warping functions. The 2-D problem is solved by minimizing strain energy with respect to warping. The second part is a nonlinear beam model (1-D analysis) which utilizes these matrices in order to solve the nonlinear intrinsic equations of motion of a beam. The outcomes are the 1-D displacement components of the reference line, as well as the generalized strains and generalized stresses. The combination of these two solutions provides the complete 3-D solution, Hodges (2006).

For certain simple cases like isotropic beams with relatively simple cross-sectional geometries or thin-walled beams made of laminated composite materials, the stiffness constants could be calculated in closed form. The stiffness matrices depend on the cross-sectional geometry, material properties, and the initial twist and bending curvature distributions along the beam. For complex cross-sections, a 2-D FEM discretization is used for minimizing the 2-D elastic energy functional.

The FEM code which implements the VAM linear cross sectional analysis is called the *Variational Asymptotic Beam Sectional Analysis* program (VABS). The research project that gave birth to VABS was initiated by Professor Hodges when he was first introduced to the Variational Asymptotic Method (VAM) by Professor Berdickevsky at Georgia Tech in 1989, Hodges (2006). Professor Cesnik is the author of the original version of VABS in Fortran 77 which was appeared in 1992. He continued his work on VABS for piezoelectric materials at MIT and later at University of Michigan. The original version of VABS was a research code but later research led to its transition to a production analysis tool for practicing engineers, Palacios (2005).

**Figure 1.1 Venn diagram for comparing the capabilities of the two VABS programs,
Ghorashi and Nitzsche (2007)**

At present, there are two versions of VABS: the Georgia Tech version (GT/VABS release 2.1, in Fortran 90/95), released and maintained by Professors Yu and Hodges, and the University of Michigan version (UM/VABS release 1.30), released and maintained by Professor Cesnik at the University of Michigan. A comparison of the capabilities and the outputs of these two codes can be found in Ghorashi and Nitzsche (2007). Figure 1.1 illustrates the capabilities of each of the VABS programs.

UM/VABS has the extra capability of obtaining 2-D asymptotically correct solutions for anisotropic active slender beams with embedded actuators. It has been implemented in the latest version of NASTRAN, Cesnik (2007). For analyzing the trapeze effect, however, one should use the GT/VABS. Trapeze effect is a nonlinear phenomenon and it deals with the variations of the torsional rigidity and the torsional natural frequency of rotor blades due to changes in the rotational speed and the axial force.

VABS codes can be used to calculate sectional properties of solid, open, closed and multi-cell thin-walled cross-sections. Using VABS, the classical, generalized Timoshenko and generalized Vlasov analyses are performed and sectional stiffness and mass matrices can be obtained.

The nonlinear 1-D theory helps applying the 2-D VAM to a real 3-D problem. The application of this theory results in 1-D displacements together with generalized strain and stress resultants. Finally, the stress and moment resultants and the recovery relations can be utilized to recover the 3-D stress and strain components.

1.2 Literature Review

A comprehensive monograph on the subject of nonlinear behaviour of composite beams is Hodges (2006). In this reference, the reasons why VAM is the optimal solution method and the advantages of using the 1-D intrinsic equations of beams have been discussed. It also includes a detailed up-to-date review of the literature. The major topics which are not covered in that reference are embedded actuators and the way one can analyze beams having complex boundary conditions. In what follows, a summary of the most significant contributions on the subject is listed.

A review of the various boundary conditions imposed on helicopter blades can be found in Rosen et al. (1991). In this reference, many combinations of boundary conditions at the root of the blade, including those with springs and dampers have been discussed. The formulation is applicable to both linear and nonlinear problems but it is valid only for isotropic materials. In Berdichevsky et al. (1992), VAM was applied to the

analysis of thin-walled closed anisotropic cross-section beams to obtain closed-form solution of the 4×4 stiffness matrix.

The nonlinear 1-D intrinsic equations of a beam were used in Shang and Hodges (1995) in order to present the stability analysis of a hingeless composite rotor blade in hovering flight. The stability equations were obtained by imposing small time-dependent perturbations to the steady state solutions and substituting the summation in the intrinsic equations of motion. Due to sparse coefficient matrices the method is computationally efficient and has low memory requirements. The obtained numerical results have been compared with available experimental data extracted from Sharpe (1986).

A revised and extended version of Shang and Hodges (1995) is Shang et al. (1999) where it has been shown that composite blades with appropriately chosen values of initial twist and curvature can exhibit significantly improved stability characteristics while simultaneously reducing steady state loads. It has also been shown that blades with positive pitch-flap coupling have an increased stability margin and a reduced structural load compared to those which do not.

In Hodges et al. (1996), VAM has been used in order to analyze the initially curved and twisted composite beams. The resulting nonlinear equations were solved numerically for both the nonlinear static deformation and the linearized free vibration about the static state of deformation. Results were compared with published exact solutions for isotropic beams and also with published experimental data for rotating isotropic and composite beams with swept tips, extracted from Epps and Chandra (1996). In both cases, good correlations have been observed.

The inclusion of active elements in the analysis was carried out by Cesnik and his co-workers. At the same time, VABS was developed in order to perform the cross-sectional analysis of composite beams. This new concept was first introduced in Cesnik and Hodges (1997) and applied to box and I-beams having initial twist and initial curvature.

In Cesnik and Shin (1998), an asymptotic formulation for analyzing multi-cell composite helicopter rotor blades with integral anisotropic active plies was presented. This paper discusses both the cross-sectional and the 1-D analyses. The theory has been applied to a two-cell thin-walled box beam as well as a single-cell airfoil-shape cross-section (NACA 0012). In Cesnik and Ortega-Morales (1999), the VAM 2-D analysis has been applied to include the effect of an embedded active element in the structure. An extended version of the same paper is Cesnik and Ortega-Morales (2001). In this paper, stiffness and actuation constants for an active box-beam, an active NACA 0012 blade, Mach-scaled CH47-D active blade section and the active twist rotor (ATR) prototype blade were calculated.

In Cesnik and Shin (2001-a and b) Anisotropic Piezo-composite Actuators (APA) were included and an asymptotic closed-form solution of the actuation force using the Berdichevsky et al. (1992) method was presented. However, Berdichevsky et al. (1992) had neglected the shell bending strain measures, leading to incorrect stiffness constants for certain cross-sections. Therefore, it consequently affected the active modeling which was derived from it. This flaw was later discussed in Volovoi and Hodges (2000), Volovoi et al. (2001) and Volovoi and Hodges (2002). In these references, the asymptotically correct theory for thin-walled beams was developed. It was demonstrated

how neglecting the shell bending strains can lead to an over-prediction of the torsional stiffness for certain cross-sections by almost 100%. It was also shown that, contrary to a widespread belief, *a cross-section of a composite beam is not rigid in its own plane*.

In Cesnik et al. (2001), the dynamic characteristics of the Active Twist Rotor (ATR) blades were investigated both analytically and experimentally. The ATR system is intended for vibration and potentially for noise reduction in helicopters through Individual Blade Control (IBC). The numerical results of the blade torsional loads showed an average error of 20% in magnitude and virtually no difference in phase for the blade frequency response.

The results of the VAM cross sectional analysis were validated in Yu et al. (2002). In this reference, VAM results for elliptical, channel and triangular prism bars, as well as box and I-beams were compared against other methods. It is especially shown how the classical method loses its validity in the nonlinear range of behaviour of composite members. Furthermore, it has been demonstrated that although VABS is restricted to beam applications, it provides a level of accuracy which is comparable to that of standard 3-D FEM codes, but with fewer computing and processing requirements.

In Yu and Hodges (2004), the results of VABS have been compared to those of 3-D elasticity. Identical results were reported for beams with elliptical and rectangular cross sections. It was then showed that VAM can be used to avoid difficulties in dealing with 3-D elastic problems; while obtaining results that are coincident with the exact solutions.

The asymptotically correct cross-sectional analysis of anisotropic active slender beams is presented in Cesnik and Palacios (2003) as well as in Palacios and Cesnik

(2005). In these references the concept of finite-section modes has been implemented in order to perform VAM calculations. The results were applied to the case of an active composite box beam. In Cesnik and Palacios (2003) it has been stated that the application of finite-section modes can approximate the stiffness constants and actuation within a 20% error. The theory in Palacios and Cesnik (2005) was also implemented in UM/VABS.

Traugott et al. (2005) applies VAM and the nonlinear 1-D analyses in order to obtain the response of an articulated active rotor blade. The cross-sectional model with embedded actuators used in Traugott et al. (2005), was based on the model developed in Patil and Johnson (2005). The effects of the rigid body modes of an articulated blade were analyzed and an MIMO controller based on full-state optimal control and optimal state estimation was presented with the aim of enhancing the damping characteristics of the weakly damped system. No details about the way the articulated blades were modeled and their boundary conditions have been given in Traugott et al. (2005).

In general, there are not many papers that discuss the method of solution of the 1-D intrinsic equations of a beam. In the solutions presented in Shang and Hodges (1995), Cesnik and Shin (1998) and Cesnik et al. (2001), the solution is performed in two steps. The first of them is calculating the steady state response. Then, the perturbed motion of the blade about the obtained steady state position is calculated by solving the perturbed steady state equations for small perturbations of the dependent variables.

This perturbed steady state solution is, of course, valid in the vicinity of the steady state response. If, however, obtaining the whole dynamics of the beam including its start

from static equilibrium and acceleration to full speed and even experiencing some perturbations afterwards is the aim, other solution methods should be sought. One of the purposes of this thesis is to present such an alternative solution.

1.3 Contributions of This Thesis

The objective of the current research is to analyze the geometrically nonlinear elasto-static and elasto-dynamic responses of hingeless or articulated composite beams with and without embedded actuators. The beam accelerates from its state of rest and reaches a constant speed of rotation. Both transient and steady state solutions are obtained. The analysis utilizes the results of the cross-sectional analysis and the solution of the nonlinear intrinsic equations of the beam is performed using finite differences, perturbations and the shooting method. To verify the solution, the results are compared against those of the perturbed steady state method.

The obtained simulation code is a powerful tool for analyzing the nonlinear response of composite rotor blades; and for the ultimate aim of efficient noise and vibration control of helicopters. The research was performed in the following steps:

1. Linear and nonlinear elasto-statics of passive isotropic or composite beams. These static solutions act as initial conditions for the dynamic analysis.
2. Nonlinear dynamics of passive clamped rotating composite beams (transient and steady state solutions) accelerating from rest.
3. Analyzing the effect of input perturbations on the response of rotating beams that are already at their steady state condition and verifying the results against those of the perturbed accelerating beam analysis.

4. Generalizing the formulation of the nonlinear dynamics of rigid articulated beams by considering the nonlinearly coupled flap and lag motions, hinge offset and aerodynamic loads.
5. Coupling the elastic model of the beam with the rigid body model in order to analyze the hinge boundary condition and analyzing the accelerating elastic articulated beam.
6. Linear and nonlinear statics and stress analysis of composite beams with embedded APA.
7. Nonlinear dynamic analysis of rotating composite beam with embedded APA and analyzing the sensitivity of the response of the beam to activating the actuators located at various angles.

In the above list, items numbered 2, 3, 6 and 7 are the contributions that have either been already published in Ghorashi and Nitzsche (2008) or accepted for publication in Ghorashi and Nitzsche (2009).

Chapter 2: Review of the Variational Asymptotic Method and the Intrinsic Equations of a Beam

The Variational-Asymptotic Method (VAM) was first introduced in Berdichevsky (1981). It is a powerful method for solving problems of composite thin-walled beams. VAM simplifies the procedure of finding stationary points of energy functional when this functional depends on one or more inherently small parameters. It is therefore the right tool for building accurate models for dimensionally reducible structures (e.g. beams, plates and shells), Hodges (2006).

The main assumptions of the theory are small strain, linearly elastic material, and the smallness of a relative to l and R , where a is the typical cross-sectional dimension, l is the wavelength of deformation along the beam axis, and R is the characteristic radius of initial curvature.

Figure 2.1 illustrates the way VAM splits the 3-D geometrically nonlinear elasticity problem into a 2-D analysis and a nonlinear 1-D analysis. The 2-D analysis is aimed at determining the cross-sectional stiffness and inertia matrices as well as the warping functions. The 1-D analysis is performed along the beam and it utilizes the results of the 2-D analysis in order to calculate the generalized stress and strain resultants as well as the 1-D displacements.

Figure 2.1 Beam analysis procedure by VAM, Hodges (2006)

Having performed the 1-D and 2-D analyses, one may complete the analysis by recovering the 3-D stress, strain and displacement fields.

2.1 Review of Cross-Sectional Modeling using VAM

Figure 2.2 illustrates the undeformed reference line of a straight beam as well as the displaced reference line of the deformed beam. A reference cross-section is also depicted in its initial and deformed positions. Two Cartesian coordinates are set up: the b -frame of the undeformed beam and the B -frame of the deformed beam.

The unit vector b_1 of the b -frame is tangential to the undeformed reference line; b_2 and b_3 are defining the plane of the reference cross-section. The origin of the B -frame is the origin of the b -frame translated by the displacement vector u . The unit vector B_1 is orthogonal to the non-warped but translated and rotated reference cross-section. Note that B_1 is not necessarily tangential to the deformed reference line, Traugott et al. (2005).

Figure 2.2 Frames and reference lines of the beam model, Traugott et al. (2005)

Since the behavior of an elastic body is completely determined by its energy, one may write the 3-D strain energy, minimize it subjected to the constraints and then solve for the warping displacements. In this way, the 3-D energy function is expressed in terms of 1-D quantities and a beam theory is derived.

The strain energy of the cross-section (per unit length) of the beam is,

$$U = \frac{1}{2} \langle \langle \Gamma^T D \Gamma \rangle \rangle \quad (2.1)$$

where Γ is the 3-D strain vector,

$$\Gamma = [\Gamma_{11} \quad 2\Gamma_{12} \quad 2\Gamma_{13} \quad \Gamma_{22} \quad 2\Gamma_{23} \quad \Gamma_{33}]^T \quad (2.2)$$

and,

$$\langle \langle \bullet \rangle \rangle = \langle \langle \bullet \rangle g \rangle = \int_A (\bullet) \sqrt{g} dx_2 dx_3, \quad \sqrt{g} = 1 - x_2 k_3 - x_3 k_2, \quad \langle \bullet \rangle = \int_A (\bullet) dx_2 dx_3 \quad (2.3)$$

A is the cross-sectional plane of the undeformed beam (the reference cross-section), D is the 6×6 symmetric material matrix in the local Cartesian system and g is the determinant of the metric tensor for the undeformed state.

Equation (2.1) for strain energy density implies a stress-strain law of the form,

$$\sigma = D\Gamma \quad (2.4)$$

where the 3-D stress components are elements of the following vector,

$$\sigma = [\sigma_{11} \quad \sigma_{12} \quad \sigma_{13} \quad \sigma_{22} \quad \sigma_{23} \quad \sigma_{33}]^T \quad (2.5)$$

The basic 2-D analysis problem is to minimize the strain energy functional U subject to four no rigid body motion conditions for the warping functions $w_i = w_i(x_1, x_2, x_3)$. The explicit form of these conditions is, Hodges (2006),

$$\langle w_i \rangle = 0 \quad ; \quad i = 1, 2, 3 \quad , \quad \langle x_2 w_3 - x_3 w_2 \rangle = 0 \quad (2.6)$$

They can alternatively be expressed as,

$$\langle w^T \psi \rangle = 0 \quad (2.7)$$

where,

$$w = \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} \quad \psi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -x_3 \\ 0 & 0 & 1 & x_2 \end{bmatrix} \quad (2.8)$$

To find the stationary value of U , the warping vector is assumed as,

$$w(x_1, x_2, x_3) = S(x_2, x_3) \cdot V(x_1) \quad (2.9)$$

where $S(x_2, x_3)$ represents the matrix of the FEM shape functions on the beam cross-section and V is a column matrix of the nodal values of the warping displacement over

the cross-section. Minimization of strain energy subjected to the orthogonality constraint (2.7) is obtained by using the method of Lagrange multipliers. Ignoring shear deformations the outcome is, Hodges (2006)

$$2U_0 = \begin{Bmatrix} \bar{\gamma}_{11} \\ \bar{\kappa}_1 \\ \bar{\kappa}_2 \\ \bar{\kappa}_3 \end{Bmatrix}^T \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{13} & \bar{S}_{14} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{23} & \bar{S}_{24} \\ \bar{S}_{13} & \bar{S}_{23} & \bar{S}_{33} & \bar{S}_{34} \\ \bar{S}_{14} & \bar{S}_{24} & \bar{S}_{34} & \bar{S}_{44} \end{bmatrix} \begin{Bmatrix} \bar{\gamma}_{11} \\ \bar{\kappa}_1 \\ \bar{\kappa}_2 \\ \bar{\kappa}_3 \end{Bmatrix} \quad (2.10)$$

In this quadratic form, the stiffness constants $\bar{S}_{ij}$ depend on the initial twist and curvature as well as on the geometry and material composition of the cross-section. This 4×4 model is sufficiently accurate for the analysis of long-wavelength static or low-frequency dynamic behavior of slender initially curved and twisted composite beams, Hodges et al. (1992). Using the classic stiffness matrix S ,

$$S = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{13} & \bar{S}_{14} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{23} & \bar{S}_{24} \\ \bar{S}_{13} & \bar{S}_{23} & \bar{S}_{33} & \bar{S}_{34} \\ \bar{S}_{14} & \bar{S}_{24} & \bar{S}_{34} & \bar{S}_{44} \end{bmatrix} \quad (2.11)$$

and for small strain, the 1-D constitutive law would be,

$$\begin{Bmatrix} F_1 \\ M_1 \\ M_2 \\ M_3 \end{Bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{13} & \bar{S}_{14} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{23} & \bar{S}_{24} \\ \bar{S}_{13} & \bar{S}_{23} & \bar{S}_{33} & \bar{S}_{34} \\ \bar{S}_{14} & \bar{S}_{24} & \bar{S}_{34} & \bar{S}_{44} \end{bmatrix} \begin{Bmatrix} \bar{\gamma}_{11} \\ \bar{\kappa}_1 \\ \bar{\kappa}_2 \\ \bar{\kappa}_3 \end{Bmatrix} \quad (2.12)$$

For homogeneous prismatic beams made of isotropic materials, the expression for classical strain energy per unit length is,

$$2U_0 = \begin{Bmatrix} \bar{\gamma}_{11} \\ \bar{\kappa}_1 \\ \bar{\kappa}_2 \\ \bar{\kappa}_3 \end{Bmatrix}^T \begin{bmatrix} EA & 0 & 0 & 0 \\ 0 & GJ & 0 & 0 \\ 0 & 0 & EI_2 & 0 \\ 0 & 0 & 0 & EI_3 \end{bmatrix} \begin{Bmatrix} \bar{\gamma}_{11} \\ \bar{\kappa}_1 \\ \bar{\kappa}_2 \\ \bar{\kappa}_3 \end{Bmatrix} \quad (2.13)$$

where the classical stiffness matrix is,

$$S = \begin{bmatrix} EA & 0 & 0 & 0 \\ 0 & GJ & 0 & 0 \\ 0 & 0 & EI_2 & 0 \\ 0 & 0 & 0 & EI_3 \end{bmatrix} \quad (2.14)$$

Here, EA is the extensional stiffness, GJ is the Saint-Venant torsional stiffness, EI_α is the bending stiffness about x_α ($\alpha=2, 3$), E is the Young's modulus, G is the shear modulus and the cross-sectional axes x_α are principal axes. Furthermore, $\bar{\gamma}_{11}$ is the extension of the reference line, $\bar{\kappa}_1$ is the elastic twist, and the elastic bending curvatures are denoted by $\bar{\kappa}_2$ and $\bar{\kappa}_3$.

For thick beams or for beams in high frequency vibrations, the classic model should be replaced with the generalized Timoshenko model where,

$$2U = \begin{Bmatrix} \gamma_{11} \\ 2\gamma_{12} \\ 2\gamma_{13} \\ \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{Bmatrix}^T \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{12} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{13} & S_{23} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{14} & S_{24} & S_{34} & S_{44} & S_{45} & S_{46} \\ S_{15} & S_{25} & S_{35} & S_{45} & S_{55} & S_{56} \\ S_{16} & S_{26} & S_{36} & S_{46} & S_{56} & S_{66} \end{bmatrix} \begin{Bmatrix} \gamma_{11} \\ 2\gamma_{12} \\ 2\gamma_{13} \\ \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{Bmatrix} \quad (2.15)$$

or,

$$2U = \begin{Bmatrix} \gamma \\ \kappa \end{Bmatrix}^T \begin{bmatrix} A & B \\ B^T & D \end{bmatrix} \begin{Bmatrix} \gamma \\ \kappa \end{Bmatrix} \quad (2.16)$$

Therefore, the 1-D constitutive law in generalized Timoshenko model is,

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ M_1 \\ M_2 \\ M_3 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{12} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{13} & S_{23} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{14} & S_{24} & S_{34} & S_{44} & S_{45} & S_{46} \\ S_{15} & S_{25} & S_{35} & S_{45} & S_{55} & S_{56} \\ S_{16} & S_{26} & S_{36} & S_{46} & S_{56} & S_{66} \end{bmatrix} \begin{Bmatrix} \gamma_{11} \\ 2\gamma_{12} \\ 2\gamma_{13} \\ \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{Bmatrix} \quad (2.17)$$

or,

$$\begin{Bmatrix} F \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B^T & D \end{bmatrix} \begin{Bmatrix} \gamma \\ \kappa \end{Bmatrix} \quad (2.18)$$

Alternatively,

$$\begin{Bmatrix} \gamma \\ \kappa \end{Bmatrix} = \begin{bmatrix} R & Z \\ Z^T & T \end{bmatrix} \begin{Bmatrix} F \\ M \end{Bmatrix} \quad (2.19)$$

2.2 General Formulation of the 1-D Analysis

The nonlinear 1-D analysis along the beam utilizes the results of the cross-sectional analysis in order to calculate the generalized stress and strain resultants as well as the 1-D displacements.

2.2.1 Intrinsic Equations of Motion

The internal force and moment vectors F and M are partial derivatives of the strain energy U ,

$$F = \left(\frac{\partial U}{\partial \gamma} \right)^T, \quad M = \left(\frac{\partial U}{\partial \kappa} \right)^T \quad (2.20)$$

Similarly, the generalized sectional linear and angular momenta P and H are,

$$P = \left(\frac{\partial K.E.}{\partial V} \right)^T, \quad H = \left(\frac{\partial K.E.}{\partial \Omega} \right)^T \quad (2.21)$$

Now, recalling Hamilton's principle,

$$\int_1^2 \int_0^L \{ \delta(K.E. - U) + \delta \bar{W} \} dx_1 dt = \delta \bar{A} \quad (2.22)$$

for the case of generalized Timoshenko beam without embedded actuators, one obtains,

Hodges (2006)

$$\begin{aligned} \int_1^2 \int_0^L \{ \delta \bar{q}^T (F' + \tilde{K}F + f - \dot{P} - \tilde{\Omega}P) + \delta \bar{\psi}^T [M' + \tilde{K}M + (\tilde{e}_1 + \tilde{\gamma})F + m - \dot{H} - \tilde{\Omega}H - \tilde{V}P] \} dx_1 dt = \\ \int_0^L \left[\delta \bar{q}^T (\hat{P} - P) + \delta \bar{\psi}^T (\hat{H} - H) \right] \Big|_{t_1}^{t_2} dx_1 - \int_1^2 \left[\delta \bar{q}^T (\hat{F} - F) + \delta \bar{\psi}^T (\hat{M} - M) \right] \Big|_0^L dt \end{aligned} \quad (2.23)$$

The corresponding Euler-Lagrange equations are,

$$F' + \tilde{K}F + f = \dot{P} + \tilde{\Omega}P \quad (2.24)$$

$$M' + \tilde{K}M + (\tilde{e}_1 + \tilde{\gamma})F + m = \dot{H} + \tilde{\Omega}H + \tilde{V}P \quad (2.25)$$

Equations (2.24) and (2.25) are called the nonlinear intrinsic equations of motion of a beam. Here, F and M are column vectors of internal forces and moments, respectively. The first element of F is the axial force and the second and third elements are the shear forces, expressed in the deformed beam basis B . Similarly, the first element of M is the twisting moment and its second and third elements are the bending moments, again in the basis of the deformed beam frame B .

The scalar form of the intrinsic form of equations of motion is,

$$F_1' + K_2 F_3 - K_3 F_2 + f_1 = \dot{P}_1 + \Omega_2 P_3 - \Omega_3 P_2 \quad (2.26)$$

$$F_2' + K_3 F_1 - K_1 F_3 + f_2 = \dot{P}_2 + \Omega_3 P_1 - \Omega_1 P_3 \quad (2.27)$$

$$F_3' + K_1 F_2 - K_2 F_1 + f_3 = \dot{P}_3 + \Omega_1 P_2 - \Omega_2 P_1 \quad (2.28)$$

and,

$$M_1' + K_2 M_3 - K_3 M_2 + 2\gamma_{12} F_3 - 2\gamma_{13} F_2 + m_1 = \dot{H}_1 + \Omega_2 H_3 - \Omega_3 H_2 + V_2 P_3 - V_3 P_2 \quad (2.29)$$

$$M_2' + K_3 M_1 - K_1 M_3 + 2\gamma_{13} F_1 - (1 + \gamma_{11}) F_3 + m_2 = \dot{H}_2 + \Omega_3 H_1 - \Omega_1 H_3 + V_3 P_1 - V_1 P_3 \quad (2.30)$$

$$M_3' + K_1 M_2 - K_2 M_1 + (1 + \gamma_{11}) F_2 - 2\gamma_{12} F_1 + m_3 = \dot{H}_3 + \Omega_1 H_2 - \Omega_2 H_1 + V_1 P_2 - V_2 P_1 \quad (2.31)$$

For the special case of static behavior, equations (2.24) and (2.25) reduce to those of Reissner (1973). Also, by ignoring the shear deformation components in equations (2.24) and (2.25), and renaming $\kappa = \bar{\kappa}$ and $\gamma = \bar{\gamma}_{11} e_1$, the equations of motion for the classical theory of beams are obtained as, Hodges (2006),

$$F' + \tilde{K}F + f = \dot{P} + \tilde{\Omega}P \quad (2.32)$$

$$M' + \tilde{K}M + (1 + \bar{\gamma}_{11})\tilde{e}_1 F + m = \dot{H} + \tilde{\Omega}H + \tilde{V}P \quad (2.33)$$

where,

$$K = k + \kappa \quad (2.34)$$

The boundary conditions are another output of the application of the Hamilton's principle in which either force or moment can be specified or calculated at the ends of the beam.

2.2.2 Intrinsic Kinematical Equations

The nonlinear intrinsic kinematical equations of a beam that should be solved together with the equations of motion are, Hodges (2006),

$$V' + \tilde{K}V + (\tilde{e}_1 + \tilde{\gamma})\Omega = \dot{\gamma} \quad (2.35)$$

$$\Omega' + \tilde{K}\Omega = \dot{\kappa} \quad (2.36)$$

2.2.3 Momentum-Velocity Equations

The momentum-velocity equations are a set of linear algebraic equations, Hodges (2006),

$$\begin{Bmatrix} P \\ H \end{Bmatrix} = \begin{bmatrix} \mu\Delta & -\mu\tilde{\xi} \\ \mu\tilde{\xi} & i \end{bmatrix} \begin{Bmatrix} V \\ \Omega \end{Bmatrix} \quad (2.37)$$

where,

$$\xi = \begin{Bmatrix} 0 \\ x_2 \\ x_3 \end{Bmatrix}, \quad \bar{\xi} = \begin{Bmatrix} 0 \\ \bar{x}_2 \\ \bar{x}_3 \end{Bmatrix} \quad (2.38)$$

and the bars refer to the location of the centroid. Also,

$$i = \begin{bmatrix} i_2 + i_3 & 0 & 0 \\ 0 & i_2 & i_{23} \\ 0 & i_{23} & i_3 \end{bmatrix} \quad (2.39)$$

where,

$$i_2 = \rho \int_A x_3^2 dx_2 dx_3, \quad i_3 = \rho \int_A x_2^2 dx_2 dx_3, \quad i_{23} = -\rho \int_A x_2 x_3 dx_2 dx_3 \quad (2.40)$$

Furthermore,

$$\mu = \int_A (\rho) dx_2 dx_3 \quad (2.41)$$

2.2.4 Constitutive Equations

The constitutive equations are,

$$\begin{Bmatrix} \gamma \\ \kappa \end{Bmatrix} = \begin{bmatrix} R & Z \\ Z^T & T \end{bmatrix} \begin{Bmatrix} F \\ M \end{Bmatrix}, \quad \begin{Bmatrix} F \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B^T & D \end{bmatrix} \begin{Bmatrix} \gamma \\ \kappa \end{Bmatrix} \quad (2.42)$$

Such a linear material law is valid only for small local strains.

Equations (2.24), (2.25), (2.34), (2.35), (2.36), (2.37) and (2.42) form a system of nonlinear vector partial differential equations and linear algebraic vector equations. This system has a total of nine unknown vectors: F , M , V , Ω , P , H , γ , κ , and K , at every point along the beam and at every instant of time. All quantities refer to the B-frame of the deformed cross-section. Solution can be performed using initial and boundary conditions. Having solved this system, other variables of interest can be easily calculated.

2.2.5 Rodrigues Parameters

For direction cosines e_i of the unit vector in the direction of the axis of rotation, and α which is the angle of rotation, the Rodrigues parameters $\theta = [\theta_1 \ \theta_2 \ \theta_3]^T$ are defined as, Hodges (2006),

$$\theta_i = 2e_i \tan(\alpha/2) \quad (2.43)$$

Thus, the rotation matrix is,

$$C = \frac{[1 - (1/4)\theta^T \theta] \Lambda - \tilde{\theta} + (1/2)\theta \theta^T}{1 + (1/4)\theta^T \theta} \quad (2.44)$$

and also,

$$\kappa = \left(\frac{\Delta - \frac{1}{2}\tilde{\theta}}{1 + \frac{1}{4}\theta^T\theta} \right) \theta' + Ck - k \quad (2.45)$$

2.2.6 Strain-Displacement Equations

The generalized strain-displacement equations are, Hodges (2006),

$$\gamma = C(e_1 + u' + \tilde{k}u) - e_1 \quad (2.46)$$

$$\tilde{\kappa} = -C'C^T + C\tilde{k}C^T - \tilde{k} \quad (2.47)$$

2.2.7 Velocity-Displacement Equations

The generalized velocity-displacement equations are, Hodges (2006),

$$V = C(v + \dot{u} + \tilde{\omega}u) \quad (2.48)$$

$$\tilde{\Omega} = -\dot{C}C^T + C\tilde{\omega}C^T \quad (2.49)$$

Having calculated γ and κ before, equations (2.44) and (2.45) can now be solved for θ and C . Then, Eq. (2.46) can be solved for u which is the displacement vector on the beam reference line.

2.3 Recovery Relations and Their Application in Stress Analysis

2.3.1 Calculation of the 3-D Strain and 3-D Stress using the 2-D and 1-D Analyses

The 3-D strain can be expressed as, Hodges (2006)

$$\Gamma = \Gamma(w, w', \bar{\epsilon}) \quad (2.50)$$

where,

$$\Gamma = \Gamma_a w + \Gamma_\epsilon \bar{\epsilon} + \Gamma_R w + \Gamma_I w' \quad (2.51)$$

and,

$$\Gamma_a = \begin{bmatrix} 0 & 0 & 0 \\ \frac{\partial}{\partial x_2} & 0 & 0 \\ \frac{\partial}{\partial x_3} & 0 & 0 \\ 0 & \frac{\partial}{\partial x_2} & 0 \\ 0 & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ 0 & 0 & \frac{\partial}{\partial x_3} \end{bmatrix}, \quad \Gamma_\varepsilon = \frac{1}{\sqrt{g}} \begin{bmatrix} 1 & 0 & x_3 & -x_2 \\ 0 & -x_3 & 0 & 0 \\ 0 & x_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \bar{\varepsilon} = \begin{Bmatrix} \bar{\gamma}_{11} \\ \bar{\kappa}_1 \\ \bar{\kappa}_2 \\ \bar{\kappa}_3 \end{Bmatrix} \quad (2.52)$$

$$\Gamma_R = \frac{1}{\sqrt{g}} \left[\tilde{k} + \Delta k_1 \left(x_3 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3} \right) \right], \quad \Gamma_l = \frac{1}{\sqrt{g}} \begin{bmatrix} \Delta \\ O_3 \end{bmatrix} \quad (2.53)$$

The stresses are obtained using Hooke's law,

$$\sigma = D\Gamma \quad (2.54)$$

In a typical problem, first using the VAM logic and VABS the cross-sectional properties are obtained and then used in the 1-D analysis performed along the span of the beam. The 1-D analysis utilizes the intrinsic equations of motion, the intrinsic kinematical equations, the constitutive equations of the material of the beam, and the momentum-velocity equations.

2.4 Finite Difference Formulation in Time and Space

The Finite Difference Method (FDM) may be used for numerically approximating the solution of nonlinear partial differential equations (PDEs). This method, together with the shooting method, will be used in order to solve the system of nonlinear PDEs (2.32), (2.33), (2.35) and (2.36), or to calculate the steady state solution of this system.

Figure 2.3 The time-space grid for the numerical solution of a partial differential equation

Consider the space-time grid shown in Figure 2.3. An arbitrary variable ϕ at location x and time t is shown as,

$$\phi_i = \phi(x, t) \quad (2.55)$$

where i is the beam node number corresponding to the coordinate x . At the space-time grid points neighboring (x, t) the same variable can be expressed as,

$$\phi_{i+1} = \phi(x + \Delta x, t) \quad \phi_i^+ = \phi(x, t + \Delta t) \quad \phi_{i+1}^+ = \phi(x + \Delta x, t + \Delta t) \quad (2.56)$$

where the superscript “+” refers to the calculation at the next time step. Using Taylor series expansion in time and in space, Esmailzadeh and Ghorashi (1997),

$$\phi(x + \frac{\Delta x}{2}, t + \frac{\Delta t}{2}) = \frac{1}{4}(\phi_{i+1}^+ + \phi_i^+ + \phi_{i+1} + \phi_i) + O(\Delta x^2, \Delta t^2) \quad (2.57)$$

Also, for the partial derivatives at the center point,

$$\phi'(x + \frac{\Delta x}{2}, t + \frac{\Delta t}{2}) = \frac{1}{2\Delta x}(\phi_{i+1}^+ - \phi_i^+ + \phi_{i+1} - \phi_i) + O(\Delta x^2, \Delta t^2) \quad (2.58)$$

$$\dot{\phi}\left(x + \frac{\Delta x}{2}, t + \frac{\Delta t}{2}\right) = \frac{1}{2\Delta t}(\phi_{i+1}^+ - \phi_{i+1} + \phi_i^+ - \phi_i) + O(\Delta x^2, \Delta t^2) \quad (2.59)$$

Equations (2.57), (2.58) and (2.59) provide the second-order approximate finite difference expressions for a variable and its derivatives with respect to time and space. They were used in Ghorashi (1994) and in Esmailzadeh and Ghorashi (1997) to solve a moving load problem. In this research, by properly defining the initial and boundary conditions, these equations will be used to convert the discussed system of nonlinear partial differential equations into a set of linear algebraic difference equations.

Chapter 3: Linear Static Analysis of Composite Beams

In Chapter 2, the set of governing equations for the analysis of the elasto-dynamic response of a beam were reviewed. In this chapter, the finite difference method is used for calculating the linear static response of a beam and in this way, the performance of this solution method for solving the intrinsic equations of a beam is illustrated.

3.1 Analysis using the Finite Difference Method

In the linear static case, equations (2.24) and (2.25) reduce to,

$$F' + \tilde{k}F + f = 0 \quad (3.1)$$

$$M' + \tilde{k}M + \tilde{e}_1 F + m = 0 \quad (3.2)$$

Also, the static forms of equations (2.57) and (2.58) are,

$$\phi(x + \frac{\Delta x}{2}) = \frac{1}{2}(\phi_{i+1} + \phi_i) + O(\Delta x^2) \quad (3.3)$$

$$\phi'(x + \frac{\Delta x}{2}) = \frac{1}{\Delta x}(\phi_{i+1} - \phi_i) + O(\Delta x^2) \quad (3.4)$$

To complete the analysis, the linear forms of equations (2.45) and (2.46) are also required. These equations, for a beam with zero initial curvature and twist are, respectively,

$$\kappa = \theta' \quad (3.5)$$

$$\gamma = u' + \tilde{e}_1 \theta \quad (3.6)$$

Now, using the constitutive equation (2.42) one obtains,

$$u' = -\tilde{e}_1 \theta + RF + ZM \quad (3.7)$$

$$\theta' = Z^T F + TM \quad (3.8)$$

Once the boundary conditions at the two ends of the beam are specified, equations (3.1), (3.2), (3.7) and (3.8) will form a boundary value problem having a set of four first-order vector differential equations for four vector variables. The finite difference equations (3.3) and (3.4) can now be used to convert the set of first order differential equations (3.1), (3.2), (3.7) and (3.8) into the following set of difference equations,

$$\begin{bmatrix} I_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} \\ 0.5 \Delta x \tilde{e}_1 & I_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} \\ -0.5 \Delta x R & -0.5 \Delta x Z & I_{3 \times 3} & 0.5 \Delta x \tilde{e}_1 \\ -0.5 \Delta x Z^T & -0.5 \Delta x T & 0_{3 \times 3} & I_{3 \times 3} \end{bmatrix} \begin{bmatrix} F_{i+1} \\ M_{i+1} \\ u_{i+1} \\ \theta_{i+1} \end{bmatrix} = \begin{bmatrix} I_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} \\ -0.5 \Delta x \tilde{e}_1 & I_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} \\ 0.5 \Delta x R & 0.5 \Delta x Z & I_{3 \times 3} & -0.5 \Delta x \tilde{e}_1 \\ 0.5 \Delta x Z^T & 0.5 \Delta x T & 0_{3 \times 3} & I_{3 \times 3} \end{bmatrix} \begin{bmatrix} F_i \\ M_i \\ u_i \\ \theta_i \end{bmatrix} \quad (3.9)$$

or,

$$B_i q_{i+1} = A_i q_i \quad (3.10)$$

where,

$$B_i = \begin{bmatrix} I_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} \\ 0.5 \Delta x \tilde{e}_1 & I_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} \\ -0.5 \Delta x R & -0.5 \Delta x Z & I_{3 \times 3} & 0.5 \Delta x \tilde{e}_1 \\ -0.5 \Delta x Z^T & -0.5 \Delta x T & 0_{3 \times 3} & I_{3 \times 3} \end{bmatrix}, \quad q_{i+1} = \begin{bmatrix} F_{i+1} \\ M_{i+1} \\ u_{i+1} \\ \theta_{i+1} \end{bmatrix} \quad (3.11)$$

$$A_i = \begin{bmatrix} I_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} \\ -0.5 \Delta x \tilde{e}_1 & I_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} \\ 0.5 \Delta x R & 0.5 \Delta x Z & I_{3 \times 3} & -0.5 \Delta x \tilde{e}_1 \\ 0.5 \Delta x Z^T & 0.5 \Delta x T & 0_{3 \times 3} & I_{3 \times 3} \end{bmatrix}, \quad q_i = \begin{bmatrix} F_i \\ M_i \\ u_i \\ \theta_i \end{bmatrix} \quad (3.12)$$

Figure 3.1 Nodes along the beam and the coordinate system of the undeformed beam

In equation (3.9), there are eight unknown scalars. To obtain the unique solution of the problem, one needs to implement the boundary conditions by relating q_1 to q_N (N is the number of spatial nodes). To this end, using equation (3.10) one obtains,

$$q_i = \underbrace{A_i^{-1} B_i}_{a_i} q_{i+1} \quad (3.13)$$

So that,

$$q_1 = (a_1 a_2 \dots a_{N-1}) q_N, \quad H_{N-1} = a_1 a_2 \dots a_{N-1} \quad (3.14)$$

In order to complete the solution, the boundary conditions of the problem should be implemented. Consider the beam shown in Figure 3.1 where node 1 is clamped and node N is free. In this case, equation (3.14) relating q_1 and q_N gives,

$$\begin{Bmatrix} F_1 \\ M_1 \\ 0 \\ 0 \end{Bmatrix} = \underbrace{\begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ h_{41} & h_{42} & h_{43} & h_{44} \end{bmatrix}}_{H_{N-1}} \begin{Bmatrix} F_N \\ M_N \\ u_N \\ \theta_N \end{Bmatrix} \quad (3.15)$$

Equation (3.15) has four unknown vectors and four vector equations. Its solution is,

$$\begin{Bmatrix} u_N \\ \theta_N \end{Bmatrix} = - \begin{bmatrix} h_{33} & h_{34} \\ h_{43} & h_{44} \end{bmatrix}^{-1} \begin{bmatrix} h_{31} & h_{32} \\ h_{41} & h_{42} \end{bmatrix} \begin{Bmatrix} F_N \\ M_N \end{Bmatrix} \quad (3.16)$$

$$\begin{Bmatrix} F_1 \\ M_1 \end{Bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \end{bmatrix} \begin{Bmatrix} F_N \\ M_N \\ u_N \\ \theta_N \end{Bmatrix} \quad (3.17)$$

Having obtained the unknowns at the boundary nodes, one may use equation (3.13) to get the solution at all other nodes.

3.2 Case Study: Isotropic Rectangular Solid Model

Figure 3.2 illustrates a prismatic member having a solid rectangular section made of a homogeneous isotropic material for which,

$$\begin{aligned} E &= 1.792 \times 10^{13} \text{ N/m}^2, \quad \nu = 0.3 \\ A &= 0.02 \text{ m}^2, \quad \rho = 1770 \text{ kg/m}^3 \end{aligned} \quad (3.18)$$

Load is applied at the tip of the beam and its magnitude is,

$$F_1 = 10,000 \text{ N} \quad M_2 = 1,000 \text{ N.m} \quad (3.19)$$

Figure 3.2 The geometry of the beam and the coordinate system

Where the subscripts refer to the direction at which the forces and moments act, not the node numbers. Also, using equation (2.39),

$$i = \begin{bmatrix} 8.333 & 0 & 0 \\ 0 & 1.6667 & 0 \\ 0 & 0 & 6.6667 \end{bmatrix} \times 10^{-5} \times 1770 \quad (3.20)$$

The sectional stiffness matrix can be calculated by GT/VABS,

$$S = \begin{bmatrix} 0.358 \times 10^{12} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.1373 \times 10^{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.1074 \times 10^{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.354 \times 10^9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.298 \times 10^9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.119 \times 10^{10} \end{bmatrix} \quad (3.21)$$

So, using the nomenclature given in equation (2.42) one obtains,

$$R = \begin{bmatrix} 2.793 & 0 & 0 \\ 0 & 7.283 & 0 \\ 0 & 0 & 9.307 \end{bmatrix} \times 10^{-12}, T = \begin{bmatrix} 28.21 & 0 & 0 \\ 0 & 33.52 & 0 \\ 0 & 0 & 8.382 \end{bmatrix} \times 10^{-10}, Z = 0_{3 \times 3} \quad (3.22)$$

For the loading given in equation (3.19), the foregoing analysis produces the results which are illustrated in Figures 3.3 and 3.4. The corresponding analytic solution is,

$$u = x_1 \times 10^{-8} \left(\begin{Bmatrix} 2.7892 \\ 0 \\ 0 \end{Bmatrix} - 100x_1 \begin{Bmatrix} 0 \\ 0 \\ 1.673 \end{Bmatrix} \right) \quad (3.23)$$

$$\theta = x_1 \times \begin{Bmatrix} 0 \\ 3.352 \\ 0 \end{Bmatrix} \times 10^{-6} \quad (3.24)$$

Equations (3.23) and (3.24) are also plotted in Figures 3.3 and 3.4. It can be observed that there is an excellent agreement between these results and the numerical solution.

**Figure 3.3 The distribution of displacement components along the isotropic beam
FDM (dashed) and exact solution (*)**

**Figure 3.4 The distribution of rotation components along the isotropic beam
FDM (dashed) and exact solution (*)**

Figure 3.5 The UM/VABS mesh for the box beam, Cesnik and Palacios (2003)

3.3 Case Study: Composite Box Model

Figure 3.5 illustrates a composite square box beam with constant properties along the beam axis and a cross section of 2.5cm between mid-lines. The upper and lower sides are made of four plies of AS4/3506-1 at 45° with the beam axis, and the lateral sides are made of four plies of a typical Anisotropic Piezo-composite Actuator (APA) at -45° . The thickness of each ply is 0.127mm and the length of the box is $L=1\text{m}$. This model has been discussed in Cesnik and Palacios (2003) and the UM/VABS input file for this case is among the examples provided with the software.

Figure 3.6 Cross-sectional view of a laminate and convention for material orientation within the element, Palacios (2005)

Table 3.1 Material properties of active box beam, Cesnik and Palacios (2003)

	E_{11} GPa	E_{22} GPa	G_{12} GPa	G_{23} GPa	ν_{12}	ν_{23}	d_{111} pm/V	d_{112} pm/V	t mm	Distance between Electrodes mm
AS4/3506-1	142	9.8	6.0	4.8	0.3	0.42	—	—	0.127	—
APA	42.2	17.5	5.5	4.4	0.354	0.42	381	-160	0.127	1.143

The mentioned ply angles are the angles of fibres with respect to the longitudinal x -axis as shown in Figure 3.6. Also, the material properties are listed in Table 3.1. It is furthermore assumed that $\rho=1,770\text{kg/m}^3$, $E_{33}=0.8E_{22}$ and $\nu_{13}=\nu_{23}=\nu_{12}$, Cesnik and Palacios (2003). Using the above-mentioned data, the stiffness matrix can be calculated by UM/VABS as,

$$S = \begin{bmatrix} 7.977 \times 10^5 & -0.9873 & -0.8575 & -1.5056 \times 10^3 & -7.3017 \times 10^{-3} & 1.348 \times 10^{-3} \\ -0.9873 & 2.5482 \times 10^5 & 4.6845 \times 10^{-3} & -3.897 \times 10^{-3} & 1.962 \times 10^3 & 5.9626 \times 10^{-5} \\ -0.8575 & 4.6845 \times 10^{-3} & 2.296 \times 10^5 & 1.0716 \times 10^{-2} & 9.912 \times 10^{-5} & -2.8055 \times 10^2 \\ -1.5056 \times 10^3 & -3.897 \times 10^{-3} & 1.0716 \times 10^{-2} & 86.95 & 2.1193 \times 10^{-4} & 1.6532 \times 10^{-4} \\ -7.3017 \times 10^{-3} & 1.962 \times 10^3 & 9.912 \times 10^{-5} & 2.1193 \times 10^{-4} & 90.397 & 3.6091 \times 10^{-6} \\ 1.348 \times 10^{-3} & 5.9626 \times 10^{-5} & -2.8055 \times 10^2 & 1.6532 \times 10^{-4} & 3.6091 \times 10^{-6} & 79.4434 \end{bmatrix} \quad (3.25)$$

Now, the box beam is subjected to the following tip loading,

$$F_1=100 \text{ N} \quad M_2=10 \text{ N.m} \quad (3.26)$$

Where the subscripts refer to the direction at which the forces and moments act, not the node numbers. Also,

$$i = \begin{bmatrix} 9.9555 & 0 & 0 \\ 0 & 4.9777 & 0 \\ 0 & 0 & 4.9777 \end{bmatrix} \times 10^{-9} \times 1770 \quad (3.27)$$

Figure 3.7 The distribution of displacement components along the composite box beam

Figure 3.8 The distribution of rotation components along the composite box beam

The response of the composite box beam to the loading in equation (3.26) has been obtained by the discussed finite difference method and the results are illustrated in Figures 3.7 and 3.8. It is observed that, because of the coupling terms in the stiffness matrix of the composite material the displacements and rotation terms are coupled and unlike the isotropic case, u_2 , as well as θ_1 and θ_3 are no longer zero.

Chapter 4: Nonlinear Static Analysis of Composite Beams

In Chapter 3, the FDM was used in order to convert a set of linear differential equations to the corresponding set of linear difference equations. These equations were then solved by implementing the boundary conditions. If, due to large deflections one needs to keep the nonlinear terms, then the governing differential equations cannot be linearized without compromising the accuracy of the solution. In this case, one may use the perturbation method in order to solve the problem.

4.1 The Governing Nonlinear Statics Equations

For the nonlinear static case, equations (2.24) and (2.25) would reduce to,

$$F' + \tilde{K}F + f = 0 \quad (4.1)$$

$$M' + \tilde{K}M + (\tilde{e}_1 + \tilde{\gamma})F + m = 0 \quad (4.2)$$

To convert the linear terms in equations (4.1) and (4.2) into equivalent finite difference expressions, one may use equations (3.3) and (3.4). In order to deal with the nonlinear terms in equations (4.1) and (4.2) the method of perturbations may be implemented. The idea is that for a fine mesh and a continuous variable ϕ with gradual changes, its value at the spatial node $i+1$ is close to its value at node i . So a small spatial perturbation $\check{\phi}_{i+1}$ can be defined as,

$$\phi_{i+1} = \phi_i + \check{\phi}_{i+1} \quad (4.3)$$

Substituting equation (4.3) into equations (3.3) and (3.4) gives,

$$\phi\left(x + \frac{\Delta x}{2}\right) = \phi_i + \frac{1}{2}\check{\phi}_{i+1} \quad (4.4)$$

$$\phi'\left(x + \frac{\Delta x}{2}\right) = \frac{1}{\Delta x}\check{\phi}_{i+1} \quad (4.5)$$

The nonlinear terms in equations (4.1) and (4.2) are products. Starting with equation (4.3) one obtains,

$$\phi\left(x + \frac{\Delta x}{2}\right) \times \gamma\left(x + \frac{\Delta x}{2}\right) = \phi_i \gamma_i + \frac{1}{2}\phi_i \check{\gamma}_{i+1} + \frac{1}{2}\check{\phi}_{i+1} \gamma_i \quad (4.6)$$

For $k=0$, using equations (4.4) to (4.6) in equations (4.1) and (4.2) results in,

$$\begin{aligned} & \frac{1}{\Delta x} \begin{Bmatrix} \check{F}_1 \\ \check{F}_2 \\ \check{F}_3 \end{Bmatrix}_{i+1} + \begin{bmatrix} 0 & -\kappa_3 & \kappa_2 \\ \kappa_3 & 0 & -\kappa_1 \\ -\kappa_2 & \kappa_1 & 0 \end{bmatrix}_i \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}_i + \frac{1}{2} \begin{bmatrix} 0 & -\kappa_3 & \kappa_2 \\ \kappa_3 & 0 & -\kappa_1 \\ -\kappa_2 & \kappa_1 & 0 \end{bmatrix}_i \begin{Bmatrix} \check{F}_1 \\ \check{F}_2 \\ \check{F}_3 \end{Bmatrix}_i + \\ & \frac{1}{2} \begin{bmatrix} 0 & -\check{\kappa}_3 & \check{\kappa}_2 \\ \check{\kappa}_3 & 0 & -\check{\kappa}_1 \\ -\check{\kappa}_2 & \check{\kappa}_1 & 0 \end{bmatrix}_{i+1} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}_i + \frac{1}{2} \begin{Bmatrix} f_{1,i+1} + f_{1,i} \\ f_{2,i+1} + f_{2,i} \\ f_{3,i+1} + f_{3,i} \end{Bmatrix} = 0 \end{aligned} \quad (4.7)$$

$$\begin{aligned} & \frac{1}{\Delta x} \check{M}_{i+1} + \left(\check{\kappa}_i M_i + \frac{1}{2} \check{\kappa}_i \check{M}_{i+1} + \frac{1}{2} \check{\kappa}_{i+1} M_i \right) + \check{e}_1 \left(F_i + \frac{1}{2} \check{F}_{i+1} \right) \\ & + \left(\check{\gamma}_i F_i + \frac{1}{2} \check{\gamma}_i \check{F}_{i+1} + \frac{1}{2} \check{\gamma}_{i+1} F_i \right) + \frac{1}{2} (m_{i+1} + m_i) = 0 \end{aligned} \quad (4.8)$$

Next, for an arbitrary node i , the constitutive equation (2.42) gives,

$$\check{\gamma}_{i+1} = R \check{F}_{i+1} + Z \check{M}_{i+1} \quad (4.9)$$

$$\check{\kappa}_{i+1} = Z^T \check{F}_{i+1} + T \check{M}_{i+1} \quad (4.10)$$

Equations (4.7) to (4.10) form a system of twelve linear algebraic equations for twelve unknown perturbation values at every node corresponding to variables F , M , γ and

κ . These equations can be rearranged in the following matrix form,

$$\begin{bmatrix} \frac{1}{dx} & \frac{-1}{2}\kappa_3 & \frac{1}{2}\kappa_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}F_3 & -\frac{1}{2}F_2 \\ \frac{1}{2}\kappa_3 & \frac{1}{dx} & \frac{-1}{2}\kappa_1 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}F_3 & 0 & \frac{1}{2}F_1 \\ \frac{-1}{2}\kappa_2 & \frac{1}{2}\kappa_1 & \frac{1}{dx} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}F_2 & -\frac{1}{2}F_1 & 0 \\ 0 & -\gamma_{13} & \gamma_{12} & \frac{1}{dx} & \frac{-1}{2}\kappa_3 & \frac{1}{2}\kappa_2 & 0 & \frac{1}{2}F_3 & -\frac{1}{2}F_2 & 0 & \frac{1}{2}M_3 & -\frac{1}{2}M_2 \\ \gamma_{13} & 0 & -\frac{1}{2}-\frac{1}{2}\gamma_{11} & \frac{1}{2}\kappa_3 & \frac{1}{dx} & \frac{-1}{2}\kappa_1 & -\frac{1}{2}F_3 & 0 & \frac{1}{2}F_1 & -\frac{1}{2}M_3 & 0 & \frac{1}{2}M_1 \\ -\gamma_{12} & \frac{1}{2}+\frac{1}{2}\gamma_{11} & 0 & \frac{-1}{2}\kappa_2 & \frac{1}{2}\kappa_1 & \frac{1}{dx} & \frac{1}{2}F_2 & -\frac{1}{2}F_1 & 0 & \frac{1}{2}M_2 & -\frac{1}{2}M_1 & 0 \\ R_{11} & R_{12} & R_{13} & Z_{11} & Z_{12} & Z_{13} & -1 & 0 & 0 & 0 & 0 & 0 \\ R_{21} & R_{22} & R_{23} & Z_{21} & Z_{22} & Z_{23} & 0 & -1 & 0 & 0 & 0 & 0 \\ R_{31} & R_{32} & R_{33} & Z_{31} & Z_{32} & Z_{33} & 0 & 0 & -1 & 0 & 0 & 0 \\ Z_{11} & Z_{21} & Z_{31} & T_{11} & T_{12} & T_{13} & 0 & 0 & 0 & -1 & 0 & 0 \\ Z_{12} & Z_{22} & Z_{32} & T_{21} & T_{22} & T_{23} & 0 & 0 & 0 & 0 & -1 & 0 \\ Z_{13} & Z_{23} & Z_{33} & T_{31} & T_{32} & T_{33} & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} \tilde{F}_1 \\ \tilde{F}_2 \\ \tilde{F}_3 \\ \tilde{M}_1 \\ \tilde{M}_2 \\ \tilde{M}_3 \\ \tilde{\gamma}_{11} \\ 2\tilde{\gamma}_{12} \\ 2\tilde{\gamma}_{13} \\ \tilde{\kappa}_1 \\ \tilde{\kappa}_2 \\ \tilde{\kappa}_3 \end{Bmatrix} = b_i \quad (4.11)$$

$$b_i = \begin{Bmatrix} \kappa_3 F_2 - \kappa_2 F_3 - \frac{1}{2}(f_{1,i} + f_{1,i+1}) \\ -\kappa_3 F_1 + \kappa_1 F_3 - \frac{1}{2}(f_{2,i} + f_{2,i+1}) \\ \kappa_2 F_1 - \kappa_1 F_2 - \frac{1}{2}(f_{3,i} + f_{3,i+1}) \\ \kappa_3 M_2 - \kappa_2 M_3 + 2\gamma_{13} F_2 - 2\gamma_{12} F_3 - \frac{1}{2}(m_{1,i} + m_{1,i+1}) \\ -\kappa_3 M_1 + \kappa_1 M_3 + F_3 - 2\gamma_{13} F_1 + \gamma_{11} F_3 - \frac{1}{2}(m_{2,i} + m_{2,i+1}) \\ \kappa_2 M_1 - \kappa_1 M_2 - F_2 + 2\gamma_{12} F_1 - \gamma_{11} F_2 - \frac{1}{2}(m_{3,i} + m_{3,i+1}) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (4.12)$$

Similarly for the displacements,

$$\begin{bmatrix} \frac{1}{\Delta x} - \frac{1}{4}\theta_1\kappa_1 & \frac{1}{2\Delta x}\theta_3 - \frac{1}{4}\theta_2\kappa_1 & -\frac{1}{2\Delta x}\theta_2 - \frac{1}{4}\theta_3\kappa_1 \\ -\frac{1}{2\Delta x}\theta_3 - \frac{1}{4}\theta_1\kappa_2 & \frac{1}{\Delta x} - \frac{1}{4}\theta_2\kappa_2 & \frac{1}{2\Delta x}\theta_1 - \frac{1}{4}\theta_3\kappa_2 \\ \frac{1}{2\Delta x}\theta_2 - \frac{1}{4}\theta_1\kappa_3 & -\frac{1}{2\Delta x}\theta_1 - \frac{1}{4}\theta_2\kappa_3 & \frac{1}{\Delta x} - \frac{1}{4}\theta_3\kappa_3 \end{bmatrix} \begin{Bmatrix} \check{\theta}_1 \\ \check{\theta}_2 \\ \check{\theta}_3 \end{Bmatrix}_{i+1} = \begin{Bmatrix} \kappa_1 + \frac{\check{\kappa}_1}{2} \\ \kappa_2 + \frac{\check{\kappa}_2}{2} \\ \kappa_3 + \frac{\check{\kappa}_3}{2} \end{Bmatrix}_i \left[1 + \frac{1}{4}(\theta_1^2 + \theta_2^2 + \theta_3^2) \right] \quad (4.13)$$

$$\tilde{u}_{i+1} = \Delta x \cdot \left\{ \frac{1}{2}(C_i + C_{i+1})^{-1} \left[\frac{1}{2}(\gamma_i + \gamma_{i+1}) + e_1 \right] - e_1 \right\} \quad (4.14)$$

The solution of equations (4.11) to (4.14) provides all of variables of interest along the beam.

Having calculated the displacement components one may also calculate the foreshortening. As shown in Figure 4.1, foreshortening is the axial contraction of the projected length of a beam due to a lateral load. This is a nonlinear phenomenon and cannot be predicted by linear modeling.

Referring to Figure 4.1, the exact foreshortening formula is,

$$\Delta_{x_1} = -\frac{1}{2} \int_0^{x_1} u_2'^2 \cdot d\xi_1 \quad (4.15)$$

Figure 4.1 Foreshortening in nonlinear bending of a cantilever beam

Using equation (4.5) for a beam laterally displaced in the x_2 and x_3 directions the approximate foreshortening formula would be,

$$\Delta_{x_{1,n}} = -\frac{1}{2\Delta x} \sum_{i=1}^n \left[(\tilde{u}_{2,i})^2 + (\tilde{u}_{3,i})^2 \right] \quad (4.16)$$

4.2 Case Study: Isotropic Rectangular Solid Model

The Isotropic Rectangular Solid Model introduced in section 3.2 is considered here again. The beam is subjected to the following tip loading,

$$\begin{aligned} F_1 &= 0\text{N}, F_2 = 100,000\text{N}, F_3 = 0\text{N} \\ M_1 &= 0\text{N.m}, M_2 = 0\text{N.m}, M_3 = 0\text{N.m} \end{aligned} \quad (4.17)$$

**Figure 4.2 The distribution of internal force components along the beam
linear static (*) and nonlinear static (dashed)**

Figure 4.3 The distribution of internal moment components along the beam linear static (*) and nonlinear static (dashed)

Figure 4.4 The distribution of displacement components along the beam linear static (*), nonlinear static (dashed) and foreshortening equation (circles)

**Figure 4.5 The distribution of rotation components along the beam
linear static (*) and nonlinear static (dashed)**

Figures 4.2 to 4.5 illustrate the outputs of the linear and nonlinear perturbation codes. It is observed that the linear and nonlinear solutions obtained for forces, moments, displacements and rotations are close; that is expected. Also, as seen in Figure 4.4, since foreshortening is a nonlinear phenomenon it is not predicted by the linear model and the linear model provides zero for the u_1 value. However, the nonlinear code and the exact foreshortening equation provide identical results.

4.3 Case Study: Composite Box Model

Consider the composite beam discussed in section 3.3 that is subjected to the tip loading,

$$\begin{aligned} F_1 &= 0\text{N}, F_2 = 0\text{N}, F_3 = 100\text{N} \\ M_1 &= 0\text{N.m}, M_2 = 0\text{N.m}, M_3 = 0\text{N.m} \end{aligned} \quad (4.18)$$

The results are shown in Figures 4.6 - 4.9 leading to similar conclusions as in section 4.2.

**Figure 4.6 The distribution of internal force components along the beam
linear static (*) and nonlinear static (dashed)**

**Figure 4.7 The distribution of internal moment components along the beam
linear static(*) and nonlinear static (dashed)**

**Figure 4.8 The distribution of displacement components along the beam
linear static (*), nonlinear static (dashed) and exact foreshortening equation (circles)**

**Figure 4.9 The distribution of rotation components along the beam
linear static(*) and nonlinear static (dashed)**

Chapter 5: Transient Nonlinear Dynamics of Accelerating Hingeless Rotating Beams

To obtain the transient nonlinear dynamics of an accelerating rotating beam, the dynamic form of the nonlinear intrinsic equations which were presented in Chapter 2 should be solved. One way to solve this problem is to use FDM and perturbations. To this end, the governing equations should be expressed in terms of equivalent finite differences or perturbations equations.

In this chapter, the complete sequence of the nonlinear motion of a rotating composite beam is analyzed. The beam is assumed to start its motion from rest and it accelerates until it reaches the steady state rotational speed. To perform the solution, the FDM and the perturbation method used in Chapter 4 for solving the nonlinear static problem will be expanded to include time-dependent variables. Most of the contents of this chapter have been accepted for publication in Ghorashi and Nitzsche (2009). In Chapter 6, the steady state solution obtained here will be compared with the result of the shooting method.

5.1 Derivation of the Generic Nonlinear Term

The system of governing equations to be solved includes equations (2.24), (2.25), (2.34), (2.35), (2.36), (2.37) and (2.42). To deal with the nonlinear terms in the

mentioned equations one may analyze the generic nonlinear vector term $\tilde{\phi}\lambda$ using its scalar components $\phi_m\lambda_n (m=1:3, n=1:3)$. Defining perturbations in time as,

$$\phi_{m,i}^+ = \phi_{m,i} + \hat{\phi}_{m,i} \quad (5.1)$$

one may use equation (4.3) in order to write,

$$\phi_{m,i+1}^+ = \phi_{m,i}^+ + \check{\phi}_{m,i+1} \quad (5.2)$$

as the new expression for the perturbations in space.

Substituting equations (5.1) and (5.2) into (2.57) results in the following equation for the generic nonlinear term at the center of the space-time grid shown in Figure 2.3,

$$\phi_m\lambda_n = \frac{1}{16} (3\phi_{m,i} + 2\hat{\phi}_{m,i} + \check{\phi}_{m,i+1} + \phi_{m,i+1}) (3\lambda_{n,i} + 2\hat{\lambda}_{n,i} + \check{\lambda}_{n,i+1} + \lambda_{n,i+1}) \quad (5.3)$$

For small perturbations equation (5.3) reduces to,

$$\begin{aligned} \phi_m\lambda_n = & \frac{1}{16} (9\phi_{m,i}\lambda_{n,i} + 3\phi_{m,i}\lambda_{n,i+1} + 3\phi_{m,i+1}\lambda_{n,i} + \phi_{m,i+1}\lambda_{n,i+1}) + \\ & \frac{1}{16} [2\hat{\phi}_{m,i}(3\lambda_{n,i} + \lambda_{n,i+1}) + \check{\phi}_{m,i+1}(3\lambda_{n,i} + \lambda_{n,i+1}) + 2\hat{\lambda}_{n,i}(3\phi_{m,i} + \phi_{m,i+1}) + \check{\lambda}_{n,i+1}(3\phi_{m,i} + \phi_{m,i+1})] \end{aligned} \quad (5.4)$$

or, using equations (5.1) and (5.2),

$$\begin{aligned} \phi_m\lambda_n = & \frac{1}{16} \times [(\phi_{m,i+1}^+ + \phi_{m,i}^+) (\lambda_{n,i+1} + 3\lambda_{n,i}) + (\lambda_{n,i+1}^+ + \lambda_{n,i}^+) (\phi_{m,i+1} + 3\phi_{m,i})] \\ & + \frac{1}{16} \times (\phi_{m,i+1}\lambda_{n,i+1} + \phi_{m,i+1}\lambda_{n,i} + \phi_{m,i}\lambda_{n,i+1} - 3\phi_{m,i}\lambda_{n,i}) \end{aligned} \quad (5.5)$$

5.2 The Finite Difference Formulation and Solution Algorithm

Using equation (5.5) the governing equations can be expressed in the following matrix form,

$$A_i q_i^+ + B_i q_{i+1}^+ = J_i \quad (5.6)$$

where the right hand side of equation (5.6) contains the currently known quantities and the column state vector q has 24 elements as follows,

$$q = [F_1 \ F_2 \ F_3 \ M_1 \ M_2 \ M_3 \ V_1 \ V_2 \ V_3 \ \Omega_1 \ \Omega_2 \ \Omega_3 \ P_1 \ P_2 \ P_3 \ H_1 \ H_2 \ H_3 \ \gamma_{11} \ 2\gamma_{12} \ 2\gamma_{13} \ \kappa_1 \ \kappa_2 \ \kappa_3]^T \quad (5.7)$$

Expressions for matrices A_i and B_i as well as the vector J_i are given in Appendix A.

Matrices A_i and B_i are 24×24 matrices and q_i and J_i are column vectors.

Each equation relating q_i to q_{i+1} is composed of 24 algebraic equations with 48 unknowns and as such is not solvable on its own. To solve the problem, one should utilize the initial and boundary conditions of the system and implement a solution method similar to that of Chapter 3. Using equation (5.6) one can relate the state vectors at the two ends of the beam at the next time step, i.e. q_1^+ and q_N^+ , directly to each other,

$$q_1^+ = M_{N-1}^{tot} q_N^+ + T_{N-1}^{tot} \quad (5.8)$$

where,

$$M_{N-1}^{tot} = a_1 a_2 a_3 a_4 \dots a_{N-1} \quad , \quad a_i = -A_i^{-1} B_i \quad (5.9)$$

$$T_{N-1}^{tot} = b_1 + a_1 b_2 + a_1 a_2 b_3 + \dots + a_1 a_2 a_3 \dots a_{N-2} b_{N-1} \quad , \quad b_i = A_i^{-1} J_i \quad (5.10)$$

Equation (5.8) can now be solved for the unknowns at the two ends of the beam. Then, equation (5.6) may be implemented in order to obtain the unknowns at all of the remaining nodes.

5.3 The Case of a Rotating Hingeless Beam

The above-mentioned method is now used to perform the nonlinear dynamic analysis of a rotating hingeless (cantilevered) composite beam. To obtain the beam response at transient and steady state conditions it is gently accelerated and the time variation of Ω_3 at the root of the beam is assumed to be,

$$\Omega_3(t) = \Omega_{3,ss} \times \frac{t^2}{k_\Omega^2 + t^2} \quad (5.11)$$

where k_Ω is a constant.

For a hingeless beam with its root (node 1) on the axis of rotation, the root boundary conditions are,

$$V = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}, \quad \Omega = \begin{Bmatrix} 0 \\ 0 \\ \Omega_3 \end{Bmatrix} \quad (5.12)$$

and at the tip (node N),

$$F = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}, \quad M = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad (5.13)$$

Now, enough information for solving the problem is available.

As one can expect, the most significant force that is generated in the rotating beam is the axial force F_1 . It is advantageous to derive an alternative expression for this force in order to be used for verification. Using the free body diagram of an element of the beam and Newton's second law of motion one obtains,

$$F_1 = \frac{1}{2} \rho A \Omega_3^2 (L^2 - x_1^2) \quad (5.14)$$

Equation (5-14) provides a benchmark for evaluating the accuracy of the numerical results.

Finally, to make the model more realistic, the effect of the weight of the beam in its dynamic response is taken into account. To this end, the effect of weight is modeled by the following gradually increasing force per unit length,

$$f_3(t) = \rho A g \times \frac{t^2}{k_g^2 + t^2} \quad (5.15)$$

where k_g is a constant.

5.4 Case Study: Isotropic Rectangular Solid Model

The Isotropic Rectangular Solid Model introduced in section 3.2 is considered here again. The beam rotates about x_3 at the angular velocity shown in Figure 5.1. The distribution of the induced internal forces along the span at $t=2s$ is illustrated in Figure 5.2. In Figure 5.3, the time history diagrams of these forces at the mid-span of the beam are plotted. Figure 5.4 is the same as Figure 5.3 except for the inclusion of the effect of the weight of the beam. The weight is observed to affect F_3 the most, as expected. Figures 5.3 and 5.4 also reveal that the obtained results for F_1 are very close to those of equation (5.14).

Figure 5.5 depicts the variation of the moment components along the beam at $t=0.5s$ when the weight effects are included. In Figures 5.6 and 5.7 the moment components at the mid-span of the beam have been plotted.

Figure 5.1 The time history diagram of the angular velocity Ω_3 at the root and the corresponding bending moment at the clamped root

Figure 5.2 The variation of the internal force components along the beam at $t=2$ s

Figure 5.3 Time history diagram of the internal forces at the mid-span (solid line), equation (5.14) (dashed)

Figure 5.4 Time history diagram of the internal forces at the mid-span (solid line) equation (5.14) (dashed); the weight effect included

Figure 5.5 The variation of the internal moment components along the beam at $t=0.5s$; the weight effect included

Figure 5.6 The time history diagram of the internal moment components at the mid-span

**Figure 5.7 The time history diagram of the internal moment components at the mid-span;
the weight effect included**

Figure 5.8 The variation of the velocity components along the beam at $t=2s$ for the nonlinear case (solid and dashed lines) and according to $V_2=x_1$. Ω_3 (*)

Figure 5.9 The time-space variation of the induced bending moment M_3 at the root; the weight effect included

The only difference between these two figures is in the gravity effect which mainly affects M_2 , as expected. Figure 5.8 illustrates the distributions of velocity components along the beam at $t=2s$. The linear variation of V_2 has, of course, been expected. Finally, the variation of M_3 at the root of the beam in time and space is given in Figure 5.9.

5.5 Verification using the Nonlinear Static Model

The results of the nonlinear dynamic method can be compared with those of the nonlinear static model of Chapter 4 for a non-rotating hingeless beam that is subjected to a gradually applied lateral load. If the load application in the dynamic model is gradual enough, the outcome of this quasi-static loading should be close to and actually converge to that of the nonlinear statics.

As an example, consider the Isotropic Rectangular Solid model introduced in section 3.2. If the 1m beam is divided into 19 elements, a tip load $\hat{F}_2 = 25\text{N}$ would be almost equivalent to the triangularly distributed load $f_2 = 950\text{ N/m}$ applied only at the tip element. The moment of the gradually applied dynamic load about the mid-span of the beam is, therefore, almost 12.06 N.m. The corresponding static moment, however, is 12.5 N.m.

The internal moment components at the mid-span of the beam are depicted in Figure 5.10. In Figure 5.11, the moment output of the nonlinear statics model along the length of the beam is presented. It can be seen that the value which Figure 5.10 is converging to is very close to the static moment at the middle of the beam shown in Figure 5.11.

Figure 5.10 Time history diagrams of internal moment components at the mid-span of the clamped beam using the nonlinear dynamic model

Figure 5.11 Internal moments along the beam under the application of an $F_2=25N$ tip load; the linear static (*) the nonlinear static (dashed)

5.6 Case Study: Composite Box Model

Consider the composite beam discussed in section 3.3. The cross-sectional area is $5.08 \times 10^{-5} m^2$ and the model has 50 nodes along its span. The beam accelerates from rest to 100 rad/s. The response of the beam has been illustrated in Figures 5.12 to 5.15.

Figure 5.12 Time history diagram of Ω_3 and the bending moment at the clamped root

Figure 5.13 The time-space variation of M_3 at the root

Figure 5.14 The variation of the internal force components along the beam at $t=3s$

Figure 5.15 The variation of the internal moment components along the beam at $t=3s$

Chapter 6: Steady State and Perturbed Steady State Nonlinear Dynamics of Hingeless Rotating Beams

The finite difference solution presented in Chapter 5 can provide the transient and the steady state responses of a rotating beam. However, there are cases where one may only be interested in the steady state response. So, finding an alternative solution which can also be used for verifying the results of Chapter 5 is in order.

In this Chapter, a solution based on the shooting method used for the numerical solution of nonlinear boundary value problems is introduced. The outcome would be the steady state response of the nonlinear hingeless (cantilevered) rotating beam. The mathematical algorithm has been discussed in detail in Esmailzadeh, Ghorashi and Mehri (1995), for example. Also, having obtained the steady state solution, the response to small input perturbations is calculated. This outcome is then compared with the dynamic solution obtained in Chapter 5. Most of the contents of this chapter have already been published in Ghorashi and Nitzsche (2008) or accepted for publication in Ghorashi and Nitzsche (2009).

6.1 Formulation of the Boundary Value Problem

The steady state form of the governing differential equations (2.24), (2.25), (2.35) and (2.36) are equations (6.1) to (6.12) listed below,

$$F_1' = -K_2 F_3 + K_3 F_2 + \Omega_2 P_3 - \Omega_3 P_2 - f_1 \quad (6.1)$$

$$F_2' = -K_3 F_1 + K_1 F_3 + \Omega_3 P_1 - \Omega_1 P_3 - f_2 \quad (6.2)$$

$$F_3' = -K_1 F_2 + K_2 F_1 + \Omega_1 P_2 - \Omega_2 P_1 - f_3 \quad (6.3)$$

$$M_1' = -K_2 M_3 + K_3 M_2 - 2\gamma_{12} F_3 + 2\gamma_{13} F_2 + \Omega_2 H_3 - \Omega_3 H_2 + V_2 P_3 - V_3 P_2 - m_1 \quad (6.4)$$

$$M_2' = -K_3 M_1 + K_1 M_3 - 2\gamma_{13} F_1 + (1 + \gamma_{11}) F_3 + \Omega_3 H_1 - \Omega_1 H_3 + V_3 P_1 - V_1 P_3 - m_2 \quad (6.5)$$

$$M_3' = -K_1 M_2 + K_2 M_1 - (1 + \gamma_{11}) F_2 + 2\gamma_{12} F_1 + \Omega_1 H_2 - \Omega_2 H_1 + V_1 P_2 - V_2 P_1 - m_3 \quad (6.6)$$

$$V_1' = -K_2 V_3 + K_3 V_2 - 2\gamma_{12} \Omega_3 + 2\gamma_{13} \Omega_2 \quad (6.7)$$

$$V_2' = -K_3 V_1 + K_1 V_3 + (1 + \gamma_{11}) \Omega_3 - 2\gamma_{13} \Omega_1 \quad (6.8)$$

$$V_3' = -K_1 V_2 + K_2 V_1 - (1 + \gamma_{11}) \Omega_2 + 2\gamma_{12} \Omega_1 \quad (6.9)$$

$$\Omega_1' = -K_2 \Omega_3 + K_3 \Omega_2 \quad (6.10)$$

$$\Omega_2' = -K_3 \Omega_1 + K_1 \Omega_3 \quad (6.11)$$

$$\Omega_3' = -K_1 \Omega_2 + K_2 \Omega_1 \quad (6.12)$$

These equations form a system of twelve scalar nonlinear ordinary differential equations in terms of the components of F , M , V and Ω . They should be solved together with the hingeless boundary conditions (5.12) and (5.13) at the root and at the tip of the beam.

In the coming sections, it will be shown how the solution to this problem can be obtained by converting it into an equivalent initial value problem and solving iteratively.

6.2 The Solution Algorithm and Formulation

The objective is to calculate the nonlinear steady state response of a composite rotating beam by solving the nonlinear boundary value problem formulated in section 6.1

and using the shooting method. To this end, by guessing the unknown initial conditions at the root, the boundary value problem is converted into an initial value problem. Next, the Runge-Kutta method is used in order to solve this initial value problem. The checkpoint is whether this solution would satisfy the known boundary conditions at the tip of the beam. If these conditions are satisfied, the correct solution to the boundary value problem has been obtained. Otherwise, the assumed initial conditions at the root are in error and should be modified.

The modification is performed iteratively by the use of the Newton-Raphson algorithm. The solution of the initial value problem and updating the initial conditions are repeated until the correct solution to the problem is obtained.

Denote the known (target) values of the boundary conditions at tip of the beam by β_j ($j=1$ to 6). These conditions are related to physical quantities with current (actual) values g_j ($j=1$ to 6). These quantities are used for verifying the accuracy of the implemented initial conditions at the root of the beam.

The unknown initial conditions at the root are shown by α_i ($i=1$ to 6). The guessed values of these variables at the root are denoted by α_{i0} ($i=1$ to 6). Each g_j quantity at the free end is a function of the adopted values for the initial conditions. Using those guessed values, a corresponding estimation for β_j at the free end is obtained and denoted by β_{j0} ,

$$g_j(\alpha_{i0}, L) = \beta_{j0} \quad (6.13)$$

The proper initial conditions at the root α_i , are those for which g_j becomes equal to the known boundary value β_j . The desired g_j can be related to $g_j(\alpha_{i0}, L)$ using the following Taylor series expansion,

$$g_j(\alpha_i, L) \approx g_j(\alpha_{i0}, L) + \sum_{i=1}^6 \left(\frac{\partial g_j}{\partial \alpha_i}(\alpha_{i0}, L) \times \Delta \alpha_i \right) \quad (6.14)$$

For ideal initial conditions the left hand side of equation (6.18) is zero. So,

$$0 = \begin{Bmatrix} g_1(\alpha_{i0}, L) \\ g_2(\alpha_{i0}, L) \\ g_3(\alpha_{i0}, L) \\ g_4(\alpha_{i0}, L) \\ g_5(\alpha_{i0}, L) \\ g_6(\alpha_{i0}, L) \end{Bmatrix} + \begin{bmatrix} \frac{\partial g_1}{\partial \alpha_1} & \frac{\partial g_1}{\partial \alpha_2} & \frac{\partial g_1}{\partial \alpha_3} & \frac{\partial g_1}{\partial \alpha_4} & \frac{\partial g_1}{\partial \alpha_5} & \frac{\partial g_1}{\partial \alpha_6} \\ \frac{\partial g_2}{\partial \alpha_1} & \frac{\partial g_2}{\partial \alpha_2} & \frac{\partial g_2}{\partial \alpha_3} & \frac{\partial g_2}{\partial \alpha_4} & \frac{\partial g_2}{\partial \alpha_5} & \frac{\partial g_2}{\partial \alpha_6} \\ \frac{\partial g_3}{\partial \alpha_1} & \frac{\partial g_3}{\partial \alpha_2} & \frac{\partial g_3}{\partial \alpha_3} & \frac{\partial g_3}{\partial \alpha_4} & \frac{\partial g_3}{\partial \alpha_5} & \frac{\partial g_3}{\partial \alpha_6} \\ \frac{\partial g_4}{\partial \alpha_1} & \frac{\partial g_4}{\partial \alpha_2} & \frac{\partial g_4}{\partial \alpha_3} & \frac{\partial g_4}{\partial \alpha_4} & \frac{\partial g_4}{\partial \alpha_5} & \frac{\partial g_4}{\partial \alpha_6} \\ \frac{\partial g_5}{\partial \alpha_1} & \frac{\partial g_5}{\partial \alpha_2} & \frac{\partial g_5}{\partial \alpha_3} & \frac{\partial g_5}{\partial \alpha_4} & \frac{\partial g_5}{\partial \alpha_5} & \frac{\partial g_5}{\partial \alpha_6} \\ \frac{\partial g_6}{\partial \alpha_1} & \frac{\partial g_6}{\partial \alpha_2} & \frac{\partial g_6}{\partial \alpha_3} & \frac{\partial g_6}{\partial \alpha_4} & \frac{\partial g_6}{\partial \alpha_5} & \frac{\partial g_6}{\partial \alpha_6} \end{bmatrix} \begin{Bmatrix} \Delta \alpha_1 \\ \Delta \alpha_2 \\ \Delta \alpha_3 \\ \Delta \alpha_4 \\ \Delta \alpha_5 \\ \Delta \alpha_6 \end{Bmatrix} \quad (6.15)$$

The Jacobian matrix in equation (6.15) includes the sensitivities of the calculated boundary values at the tip with respect to the assumed initial conditions at the root of the beam. These derivatives are calculated using, for example,

$$\frac{\partial g_j}{\partial \alpha_2}(\alpha_{i0}, L) = \frac{g_j(\alpha_{i0}, \alpha_{20} + \varepsilon, \alpha_{30}, \alpha_{40}, \alpha_{50}, \alpha_{60}, L) - g_j(\alpha_{i0}, \alpha_{20} - \varepsilon, \alpha_{30}, \alpha_{40}, \alpha_{50}, \alpha_{60}, L)}{2\varepsilon} \quad (6.16)$$

So, the best modifications of the initial conditions would be,

$$\begin{Bmatrix} \Delta \alpha_1 \\ \Delta \alpha_2 \\ \Delta \alpha_3 \\ \Delta \alpha_4 \\ \Delta \alpha_5 \\ \Delta \alpha_6 \end{Bmatrix} = - \begin{bmatrix} \frac{\partial g_1}{\partial \alpha_1} & \frac{\partial g_1}{\partial \alpha_2} & \frac{\partial g_1}{\partial \alpha_3} & \frac{\partial g_1}{\partial \alpha_4} & \frac{\partial g_1}{\partial \alpha_5} & \frac{\partial g_1}{\partial \alpha_6} \\ \frac{\partial g_2}{\partial \alpha_1} & \frac{\partial g_2}{\partial \alpha_2} & \frac{\partial g_2}{\partial \alpha_3} & \frac{\partial g_2}{\partial \alpha_4} & \frac{\partial g_2}{\partial \alpha_5} & \frac{\partial g_2}{\partial \alpha_6} \\ \frac{\partial g_3}{\partial \alpha_1} & \frac{\partial g_3}{\partial \alpha_2} & \frac{\partial g_3}{\partial \alpha_3} & \frac{\partial g_3}{\partial \alpha_4} & \frac{\partial g_3}{\partial \alpha_5} & \frac{\partial g_3}{\partial \alpha_6} \\ \frac{\partial g_4}{\partial \alpha_1} & \frac{\partial g_4}{\partial \alpha_2} & \frac{\partial g_4}{\partial \alpha_3} & \frac{\partial g_4}{\partial \alpha_4} & \frac{\partial g_4}{\partial \alpha_5} & \frac{\partial g_4}{\partial \alpha_6} \\ \frac{\partial g_5}{\partial \alpha_1} & \frac{\partial g_5}{\partial \alpha_2} & \frac{\partial g_5}{\partial \alpha_3} & \frac{\partial g_5}{\partial \alpha_4} & \frac{\partial g_5}{\partial \alpha_5} & \frac{\partial g_5}{\partial \alpha_6} \\ \frac{\partial g_6}{\partial \alpha_1} & \frac{\partial g_6}{\partial \alpha_2} & \frac{\partial g_6}{\partial \alpha_3} & \frac{\partial g_6}{\partial \alpha_4} & \frac{\partial g_6}{\partial \alpha_5} & \frac{\partial g_6}{\partial \alpha_6} \end{bmatrix}^{-1} \times \begin{Bmatrix} g_1(\alpha_{i0}, L) \\ g_2(\alpha_{i0}, L) \\ g_3(\alpha_{i0}, L) \\ g_4(\alpha_{i0}, L) \\ g_5(\alpha_{i0}, L) \\ g_6(\alpha_{i0}, L) \end{Bmatrix} \quad (6.17)$$

Table 6.1 The initial conditions at the root and their improvements

iteration	$\alpha_1=F_1(0)$	$\alpha_2=F_2(0)$	$\alpha_3=F_3(0)$	$\alpha_4=M_1(0)$	$\alpha_5=M_2(0)$	$\alpha_6=M_3(0)$
1	100	100	1000	1000	1000	1000
2	1.77e+5	1.437e-1	-8.897e-1	-3.561e-3	4.973e-1	7.132e-2
3	1.77e+5	8.935e-12	5.735e-13	-2.697e-11	-1.541e-12	4.315e-12

The calculated increments are then used to improve the initial guess values,

$$\alpha_i = \alpha_{i0} + \Delta\alpha_i \quad (6.18)$$

Now the whole procedure can be repeated using the new set of assumed initial conditions (6.18). By repeating this algorithm the unknown initial conditions would gradually improve. The procedure can be terminated when a properly defined convergence criterion like,

$$\sum_{j=1}^6 |g_j(\alpha_i, L)| < \varepsilon \quad (6.19)$$

is satisfied. At this instant, the proper initial conditions, and consequently, the correct steady state response of the beam have been obtained with enough accuracy.

6.3 Case Study: Isotropic Rectangular Solid Model

The Isotropic Rectangular Solid Model introduced in section 3.2 is considered here again. A root angular velocity of $\Omega_3=100$ rad /s is applied and the steady state response of the beam is sought.

Table 6.1 presents the sequence of improvements of the calculated initial conditions at the root. The corresponding sequence of error values has been listed in Table 6.2. It can be figured out that even though the initial guesses have been far from the correct initial conditions; the method has been able to obtain the correct solution in just a few iterations.

Table 6.2 The errors at the tip corresponding to the initial conditions in Table 6.1

iteration	$g_1=F_1(L)$	$g_2=F_2(L)$	$g_3=F_3(L)$	$g_4=M_1(L)$	$g_5=M_2(L)$	$g_6=M_3(L)$
1	-1.769e+5	1.001e+2	9.99e+2	1.00e+3	1.9998e+3	8.999e+2
2	-4.948e-3	1.437e-1	-8.897e-1	-3.561e-3	-3.924e-1	-7.236e-2
3	1.4188e-1	8.935e-12	5.735e-13	-2.697e-11	-9.678e-13	-4.619e-12

Figures 6.1 to 6.4 illustrate the corresponding steady state response of the beam. As expected, the axial force F_1 is the dominant force component. Also, the linear variation of V_2 along the beam is compatible with the constant Ω_3 along the beam.

To verify the results, the obtained steady state solution has been compared with the transient finite difference solution discussed in Chapter 5. The transient solution is expected to converge to the steady state response obtained by the shooting method. The two steady state solutions are illustrated in Fig.6.5 and are seen to be almost identical. While the shooting method provides the steady state solution in a few seconds, the transient finite difference solution has to run for a few minutes to obtain the same steady state solution.

Figure 6.1 The steady state internal force components along the beam

Figure 6.2 The steady state internal moment components along the beam

Figure 6.3 The steady state velocity components along the beam

Figure 6.4 The steady state angular velocity components along the beam

Figure 6.5 The steady state values of the internal force F_1 along the beam using the shooting method (solid line) the FDM (circles)

6.4 Case Study: Verification Example, Hodges (2008)

The case of a rotating beam with the following non-dimensional data has been given in Hodges (2008),

$$\begin{aligned} \Omega_3 &= 1.4426 & L &= 10 & k &= 0 & R(1,1) &= 1 \times 10^{-7} \\ Z &= 0 & T(1,1) &= 1 \times 10^{-6} & T(2,2) &= 1 \times 10^{-6} & T(3,3) &= 1 \times 10^{-8} \\ \mu &= 4 & i(1,1) &= 0.1 \end{aligned} \quad (6.20)$$

Other elements of R and i are assumed to be zero. Figures 6.6 to 6.9 illustrate the results obtained by the present method and those of Hodges (2008). The two steady state solutions are dominated by the axial force and the results are very close.

Figure 6.6 Steady state force solution along the beam; present code (*) Hodges (2008) (circles)

Figure 6.7 Steady state moment solution along the beam; present code (*) Hodges (2008) (circles)

Figure 6.8 Steady state velocity solution along the beam; present code (*) Hodges (2008) (circles)

Figure 6.9 Steady state angular velocity solution along the beam; present code (*) Hodges (2008) (circles)

6.5 Case Study: Passive Airfoil Model

The case of a passive composite airfoil has been discussed in Hodges et al. (2007). The beam comprises of a spanwise uniform cantilever beam with the cross-sectional configuration shown in Figure 6.10.

Figure 6.10 Sketch of the rotor blade cross-section, Hodges et al. (2007)

The blade section has a chord length 20.2 inches and a web located 8.4025 inches from the leading edge. The layup orientations for the D-spar are $[45^\circ/-45^\circ/0^\circ/0^\circ/0^\circ]$ from outside to inside. The layup angle is 45° for the trailing edge skin and 0° for the web. For the current analysis a 120 inches long blade is considered. Using VABS, Hodges (2007):

$$\begin{aligned}\mu &= 1.54601 \times 10^{-3} \text{ lb. sec}^2/\text{inch}^2 & I_2 &= 1.28544 \times 10^{-3} \text{ lb. sec}^2 \\ I_3 &= 2.68960 \times 10^{-2} \text{ lb. sec}^2\end{aligned}\quad (6.21)$$

The generalized Timoshenko model obtained from VABS for this beam section is represented by the following stiffness matrix, Hodges (2007)

$$S = \begin{bmatrix} 128.550 & -0.651479 & 0.026700 & -1.17823 & 25.0459 & -168.948 \\ -0.651479 & 9.68119 & -0.414611 & 0.021110 & 0.706635 & 1.03049 \\ 0.026700 & -0.414611 & 1.44328 & -0.001493 & -0.054592 & 0.116498 \\ -1.17823 & 0.021110 & -0.001493 & 34.5784 & 1.30984 & 1.36493 \\ 25.0459 & 0.706635 & -0.054592 & 1.30984 & 91.8417 & 25.3056 \\ -168.948 & 1.03049 & 0.116498 & 1.36493 & 25.3056 & 1492.29 \end{bmatrix} \times 10^6 \quad (6.22)$$

Figure 6.11 illustrates the steady state distribution of F_1 at 100rad/s using the shooting method.

Figure 6.11 The steady state variation of F_1 along the beam

6.6 Perturbed Steady State Analysis

Consider a rotating beam already in its steady state condition. Now the response of this beam to a small input perturbation is of interest. Referring to Figure 2.3, for every dependent variable one may write,

$$\phi\left(x + \frac{\Delta x}{2}, t + \frac{\Delta t}{2}\right) = \phi_{ss}\left(x + \frac{\Delta x}{2}\right) + \phi_p\left(x + \frac{\Delta x}{2}, t + \frac{\Delta t}{2}\right) \quad (6.23)$$

That is, the whole solution for the variable of interest is the summation of its steady state value and perturbations about the steady state. Using equation (6.23), for the two variables ϕ_m and λ_n , the perturbation part of the generic nonlinear term $\phi_m \lambda_n$ can be written as,

$$\left\{ \phi_m\left(x + \frac{\Delta x}{2}, t + \frac{\Delta t}{2}\right) \lambda_n\left(x + \frac{\Delta x}{2}, t + \frac{\Delta t}{2}\right) \right\}_p = \phi_{m,ss}\left(x + \frac{\Delta x}{2}\right) \lambda_{n,p}\left(x + \frac{\Delta x}{2}, t + \frac{\Delta t}{2}\right) + \phi_{m,p}\left(x + \frac{\Delta x}{2}, t + \frac{\Delta t}{2}\right) \lambda_{n,ss}\left(x + \frac{\Delta x}{2}\right) \quad (6.24)$$

Or using equation (2.57),

$$\left\{ \phi_m\left(x + \frac{\Delta x}{2}, t + \frac{\Delta t}{2}\right) \lambda_n\left(x + \frac{\Delta x}{2}, t + \frac{\Delta t}{2}\right) \right\}_p = \frac{1}{8} [(\phi_{m,ss,i+1} + \phi_{m,ss,i})(\lambda_{n,p,i+1}^+ + \lambda_{n,p,i}^+ + \lambda_{n,p,i+1} + \lambda_{n,p,i})] + \frac{1}{8} [(\lambda_{n,ss,i+1} + \lambda_{n,ss,i})(\phi_{m,p,i+1}^+ + \phi_{m,p,i}^+ + \phi_{m,p,i+1} + \phi_{m,p,i})] \quad (6.25)$$

Implementation of equation (6.25) converts the dynamic governing equations (2.24), (2.25), (2.34), (2.35), (2.36), (2.37) and (2.42) to the following matrix form,

$$A_{ss,i} q_{p,i}^+ + B_{ss,i} q_{p,i+1}^+ = J_i \quad (6.26)$$

where q_p contains the perturbations of the variables given in equation (5.7). The rest of the solution is similar to that of Chapter 5.

6.7 Case Study: Isotropic Rectangular Solid Model

The Isotropic Rectangular Solid Model introduced in section 3.2 is considered. The steady state angular velocity is $\Omega_3=100$ rad/s and a root angular velocity perturbation of,

$$\Omega_{3,p,1} = \sin(100t) \text{ rad/s} \quad (6.27)$$

is applied.

Figure 6.12 The perturbations of angular velocity and bending moment at the root

Figure 6.13 The perturbations of force components at the root

Figures 6.12 and 6.13 illustrate the input angular velocity perturbations as well as the resulting perturbations in the bending moment and the force components at the root.

Having calculated the perturbed dependent variables, one may now use equation (6.23) to get the complete dynamic response. In Figures 6.14 to 6.17 the steady state results are plotted in solid lines until $t=2.667\text{s}$. At this instant the perturbation,

$$\Omega_{3,p,1} = \sin(93.5t) \text{ rad/s} \quad (6.28)$$

is applied at the root of the beam. In Figures 6.14 to 6.17, the effect of this perturbation on other variables, using the foregoing algorithm has been illustrated in solid lines.

Figure 6.14 The steady state, accelerating and perturbed steady state angular velocities at the root

Figure 6.15 The steady state, accelerating and perturbed axial force at the root

Figure 6.16 The steady state, accelerating and perturbed shear force at the root

Figure 6.17 The steady state, accelerating and perturbed bending moment at the root

Alternatively, one may use the algorithm discussed in Chapter 5 for an accelerating beam to do the same job. In this case, the beam starts to rotate from rest and at $t=2.667$ s when the beam has an angular velocity of 93.5 rad/s, the perturbation shown in equation (6.28) is applied. In Figures 6.14 to 6.17, the results corresponding to this algorithm are plotted in dash-dotted lines.

It can be observed that the predictions of the perturbed steady state method discussed in this chapter are close to those of the accelerating beam discussed in Chapter 5. The results of such an analysis can therefore be used in order to estimate the degree of sensitivity of each of the output variables to input perturbations.

Chapter 7: Rigid and Elastic Articulated Rotating Composite Beams

Chapters 5 and 6 discussed the nonlinear elasto-dynamics of rotating hingeless (cantilever) beams. The blades in many helicopters, however, are articulated and have hinges in order to reduce the induced root moments. Analyzing the dynamics of these beams requires the inclusion of rigid body motions as a new set of unknown variables.

One may start the analysis by ignoring the elastic deformations. In fact, for a number of important helicopter problems with hinged blades, the blade rigidity assumption is adequate. “It is fortunate that, in spite of the considerable flexibility of rotor blades, much of helicopter theory can be affected by regarding the blade as rigid, with obvious simplifications in the analysis”, Done and Balmford (2001). That is why the analysis of rigid body motions is so essential for understanding the behaviour of articulated blades.

However, still there are cases for which the flexibility of the beam should be taken into consideration. As an example, one may consider the elastic deformations of a beam that can affect aerodynamic loading and the natural frequencies of vibration of the beam.

In this chapter, first a review of the analysis of rigid articulated rotating beams having flap, lead-lag and feathering hinges is presented. In this analysis, the angular velocity of the rotor is considered to be constant and the helicopter undergoes rectilinear motion at

constant speed. Then the elastic articulated rotating beams are discussed. The root boundary condition is formulated by the use of the solution of the rigid articulated beam. Having done that, the root boundary condition will be used together with the elastic rotating beam formulation (derived in Chapter 5) to solve the elastic articulated rotating beam problem.

7.1 Introduction to Articulated Blades

A typical hinge arrangement is shown in Figure 7.1. The use of these hinges in order to reduce the induced moments has been a main development in the manufacture of helicopters. The most important of these hinges is the flapping hinge which allows the blade to move in a plane containing the beam and the shaft. Now, a blade which is free to flap experiences large Coriolis moments in the plane of rotation that is why a further hinge—called the drag or lead-lag hinge—is provided to relieve these moments. Lastly, the blade can be feathered about a third axis, usually parallel to the blade span, to enable the blade pitch angle to be changed, Done and Balmford (2001).

Figure 7.1 Hinge arrangement, Tomas-Rodriguez and Sharp (2007)

In hovering flight, using the collective pitch, the flow field is azimuthally axisymmetric and so each blade encounters the same aerodynamic environment. The blades flap up and lag back with respect to the hub and reach a steady equilibrium position under the action of these steady (non-azimuthally varying) aerodynamic and centrifugal forces.

In forward flight, using the cyclic pitch, the asymmetry of the onset flow and dynamic pressure over the disk produces aerodynamic forces that are functions of blade azimuth position (i.e. cyclically varying airloads are now produced). The flapping hinge allows each blade to freely flap up and down in a periodic manner with respect to the azimuth angle under the action of the varying aerodynamic loads, Leishman (2006).

7.2 Euler and Extended Euler Equations in Rigid Body Dynamics

Figure 7.2 illustrates a rigid rotor beam that rotates about the fixed point O on the shaft and has a hinge system which is effectively concentrated at a single point P .

Figure 7.2 Rotating beam with O on the shaft axis and P at the hinge location

If the reference axes are coincident with the principal axes of inertia (at mass center G or point O fixed in the beam and fixed in space about which the beam rotates) then the moment equations about G or O would be, Meriam and Kraige (2003)

$$\sum M_x = I_{xx}\dot{\omega}_x - (I_{yy} - I_{zz})\omega_y\omega_z \quad (7.1)$$

$$\sum M_y = I_{yy}\dot{\omega}_y - (I_{zz} - I_{xx})\omega_z\omega_x \quad (7.2)$$

$$\sum M_z = I_{zz}\dot{\omega}_z - (I_{xx} - I_{yy})\omega_x\omega_y \quad (7.3)$$

Equations (7.1) to (7.3) are the Euler's equations of motion for a rotating rigid body.

If points O and P are not identical, the hinge would have non-zero velocity and the beam would rotate about point P fixed in the beam but not fixed in space. In such a case, the modified Euler equations listed below should be used,

$$\sum M_{Px} = I_{xx}\dot{\omega}_x - (I_{yy} - I_{zz})\omega_y\omega_z + m(y_G a_{Pz} - z_G a_{Py}) \quad (7.4)$$

$$\sum M_{Py} = I_{yy}\dot{\omega}_y - (I_{zz} - I_{xx})\omega_z\omega_x + m(z_G a_{Px} - x_G a_{Pz}) \quad (7.5)$$

$$\sum M_{Pz} = I_{zz}\dot{\omega}_z - (I_{xx} - I_{yy})\omega_x\omega_y + m(x_G a_{Py} - y_G a_{Px}) \quad (7.6)$$

In helicopter blades, the center of gravity is practically on the x -axis. Therefore,

$$\vec{\rho} = x_G \hat{i} \quad , \quad y_G = 0 \quad , \quad z_G = 0 \quad (7.7)$$

Consequently equations (7.4) to (7.6) reduce to,

$$\sum M_{Px} = I_{xx}\dot{\omega}_x - (I_{yy} - I_{zz})\omega_y\omega_z \quad (7.8)$$

$$\sum M_{Py} = I_{yy}\dot{\omega}_y - (I_{zz} - I_{xx})\omega_z\omega_x - mx_G a_{Pz} \quad (7.9)$$

$$\sum M_{Pz} = I_{zz}\dot{\omega}_z - (I_{xx} - I_{yy})\omega_x\omega_y + mx_G a_{Py} \quad (7.10)$$

It should be noted that equations (7.8) to (7.10) are different from (A.1.14-16) and (A.1.18-19) in Done and Balmford, (2001) and there are sign errors in those equations.

Having obtained the general form of the ‘extended’ Euler equations, one can use them to derive the equations of motion of a rigid articulated blade in flapping, lagging, and feathering.

7.3 Coupled Equations of Motion for Rigid Articulated Blade

Figure 7.3 illustrates a rotating beam with flap and lead-lag hinges mounted at distance eR from the axis of rotation (the hinge offset is eR , where R is the rotor radius shown in the figure). The flap angle β is measured relative to a plane perpendicular to the shaft axis and its positive sense is in the negative y direction (flapping up). The positive direction of the lead-lag angle is defined as the one for a backward moving blade. In what follows, the nonlinear and coupled flap and lead-lag equations of motion are derived for a rotating rigid blade with coincident flap and lead-lag hinges.

Figure 7.3 Blade flapping, Done and Balmford (2001)

Figure 7.4 Deflected rotating rigid blade with hub and blade coordinate systems, Done and Balmford (2001)

Figure 7.4 illustrates a blade coordinate system which is attached to the blade and its origin is located at the hinge. Also the axes are aligned with the principal directions of inertia of the blade at the root.

Considering the shaft angular velocity Ω , as well as the flap and lead-lag motions, and transforming them to the blade coordinate system, the total angular velocity of the blade can be written as,

$$\vec{\omega} = (\Omega \sin \beta \cos \xi) \hat{i} + (\Omega \sin \beta \sin \xi - \dot{\beta}) \hat{j} + (\Omega \cos \beta - \dot{\xi}) \hat{k} \quad (7.11)$$

Also, the acceleration of the hinge point P in the beam coordinate system can be written as,

$$\vec{a}_p = eR\Omega^2 (-\cos \beta \cos \xi \hat{i} - \cos \beta \sin \xi \hat{j} + \sin \beta \hat{k}) \quad (7.12)$$

For small β and ξ , substitution of equations (7.11) and (7.12) into equation (7.10) gives,

$$-\frac{1}{I_{zz}} \sum M_{pz} = \ddot{\xi} + v_{\xi}^2 \Omega^2 \xi + 2\Omega\beta\dot{\beta} \quad (7.13)$$

where,

$$v_{\xi}^2 = \frac{mx_{Gn}eR^2}{I_{zz}}, \quad x_G = Rx_{Gn} \quad (7.14)$$

Equation (7.13) is one of the two generalized coupled flap-lead-lag equations of motion.

Considering Figure 7.5, if a mechanical lag damper is also included in the model, the equation of motion would be modified to,

$$\ddot{\xi} + C_{\xi}\dot{\xi} + v_{\xi}^2\Omega^2\xi + 2\Omega\beta\dot{\beta} = -\frac{1}{I_{zz}}\sum M_{pz} \quad (7.15)$$

Similarly, equations (7.9), (7.11) and (7.12) result in,

$$\ddot{\beta} + \Omega^2\left(1 + \frac{mx_{Gn}eR^2}{I_{yy}}\right)\beta - \Omega(2\dot{\xi}\beta + \dot{\beta}\xi) = -\frac{1}{I_{yy}}\sum M_{py} \quad (7.16)$$

But, on page 394 of Johnson (1994) and on page 197 of Leishman (2006) one reads,

$$\ddot{\beta} + \Omega^2v_{\beta}^2\beta - 2\Omega\dot{\xi}\beta = -\frac{1}{I_{yy}}\sum M_{py}, \quad v_{\beta}^2 = 1 + \frac{mx_{Gn}eR^2}{I_{yy}} \quad (7.17)$$

Figure 7.5 Articulated blade model showing the lag damper, Friedman (2003)

While, time has so far been used as the independent variable, it is advantageous to change the independent variable to the azimuth angle since the aerodynamic loads are functions of this angle. The azimuth angle of a blade, shown in Figure 7.4 and depicted by ψ , is the angle between the spanwise direction of the blade and the rear centerline of the helicopter. Using azimuth angle as independent variable, the coupled flap-lag equations (7.15) and (7.16) become,

$$\frac{d^2 \beta}{d\psi^2} + \left(1 + \frac{mx_{Gn}eR^2}{I_{yy}}\right)\beta - \left(2\beta \frac{d\xi}{d\psi} + \xi \frac{d\beta}{d\psi}\right) = -\frac{1}{\Omega^2 I_{yy}} \sum M_{py} \quad (7.18)$$

$$\frac{d^2 \xi}{d\psi^2} + \frac{C_\xi}{\Omega} \frac{d\xi}{d\psi} + \left(\frac{mx_{Gn}eR^2}{I_{zz}}\right)\xi + 2\beta \frac{d\beta}{d\psi} = -\frac{1}{\Omega^2 I_{zz}} \sum M_{pz} \quad (7.19)$$

7.4 Case Study: Articulated Isotropic Rectangular Solid Model

The Isotropic Rectangular Solid Model introduced in section 3.2 is considered here again. In this case, the moments of inertia of the beam about its root are,

$$I_{yy} = I_{x_2x_2} = 11.8295 \text{ kg.m}^2, \quad I_{zz} = I_{x_3x_3} = 11.9180 \text{ kg.m}^2 \quad (7.20)$$

Also,

$$x_{Gn} = 0.48 \quad e = 0.04 \quad R = 1.04 \text{ m} \quad (7.21)$$

In hover, for a rotor angular velocity of $\Omega=100$ rad/s and moments, $M_y = -1000 \text{ N.m}$ and $M_z = -10 \text{ N.m}$, the coupled flap-lag equations would be,

$$\ddot{\beta} + (100)^2 (1 + 0.0621)\beta - 100(2\dot{\xi}\beta + \dot{\beta}\xi) = \frac{1000}{11.8295} \quad (7.22)$$

$$\ddot{\xi} + 0.0617 \times (100)^2 \xi + 200\beta\dot{\beta} = \frac{10}{11.9180} \quad (7.23)$$

Figure 7.6 Time history diagrams for lead-lag and flap motions

Figure 7.7 Time history diagrams for lead-lag and flap motions (zoomed)

Figures 7.6 and 7.7 illustrate the solution of equations (7.22) and (7.23). The effects of motion coupling and the contribution of the term missing in equation (7.17) are illustrated in these figures.

7.5 Aerodynamic Damping in Articulated Blades with Hinge Offset

Considering Figure 7.8 and including the effects of blade pitch, flap motion and the induced velocity field, the lift force per unit length would be, Leishman (2006),

$$L = \frac{1}{2} \rho U_T^2 c C_{l,\alpha} \left(\theta - \frac{\dot{\beta}(r - eR)}{U_T} - \frac{v_i}{U_T} \right) \quad (7.24)$$

where the induced velocity v_i is the velocity imparted to the mass of air contained in the control volume at the rotor disk. Also, the in-plane (tangential) component of the velocity of the blade element in hover or axial climb is,

$$U_T = r\Omega \quad (7.25)$$

and c is the local blade chord. The moment of aerodynamic forces about the hinge location at $r = eR$ can be written as, Leishman (2006),

Figure 7.8 Blade section aerodynamics, Johnson (1994)

$$-\sum M_{py} = \int_{eR}^R L(r - eR) dr \quad (7.26)$$

Using equation (7.18) and substituting equation (7.24) into (7.26) and integration, result in the general form of the coupled nonlinear flap-lag equation of motion including aerodynamic damping and hinge offset.

$$\begin{aligned} \frac{d^2 \beta}{d\psi^2} + \left(\frac{\gamma}{8} (1-e)^3 \left(1 + \frac{1}{3} e \right) \right) \frac{d\beta}{d\psi} + \left(1 + \frac{mx_{Gn} e R^2}{I_{yy}} \right) \beta - \left(2\beta \frac{d\xi}{d\psi} + \xi \frac{d\beta}{d\psi} \right) = \\ \frac{\gamma}{2} \left[\left(\frac{1}{4} + \frac{1}{12} e^4 - \frac{1}{3} e \right) \theta - \left(\frac{1}{3} (1-e^3) - \frac{1}{2} e (1-e^2) \right) \lambda_i \right] \end{aligned} \quad (7.27)$$

where the Lock number is defined as,

$$\gamma = \frac{\rho c C_{L\alpha} R^4}{I_{yy}} \quad (7.28)$$

and the rotor induced inflow ratio is,

$$\lambda_i = \frac{v_i}{R\Omega} \quad (7.29)$$

For the lead-lag motion, if P denotes the power required to drive one blade, the necessary moment would be, Leishman (2006)

$$\sum M_{pz} = - \left(\frac{P}{\Omega} \right) \left(1 - \frac{2\dot{\xi}}{\Omega} \right) \quad (7.30)$$

Substituting equation (7.30) into (7.19) results in,

$$\frac{d^2 \xi}{d\psi^2} + \left(\frac{C_\xi}{\Omega} + \frac{2P}{\Omega^3 I_{zz}} \right) \frac{d\xi}{d\psi} + \left(\frac{mx_{Gn} e R^2}{I_{zz}} \right) \xi + 2\beta \frac{d\beta}{d\psi} = \frac{1}{\Omega^2 I_{zz}} \left(\frac{P}{\Omega} \right) \quad (7.31)$$

Equations (7.27) and (7.31) are the general set of flap and lead-lag equations of motion where the effects of hinge offset, aerodynamic damping and nonlinear coupling have all been taken into account.

7.6 Case Study: Articulated Beam with Aerodynamic Damping

The Isotropic Rectangular Solid Model introduced in section 3.2 is considered here again. Assuming,

$$x_{Gn} = 0.48 \quad e = 0.04 \quad R = 1\text{m} \quad (7.32)$$

and using equations (7.27) and (7.31) with the drag moment $M_z = -10\text{N.m}$, $\Omega = 100 \text{ rad/s}$, $\gamma = 8$ and a collective pitch angle of 2° , the set of nonlinear flap and lead-lag equations of motion become,

$$\ddot{\beta} + 89.6\dot{\beta} + (100)^2(1 + 0.0621)\beta - 100(2\dot{\xi}\beta + \dot{\beta}\xi) = 330.45 \quad (7.33)$$

$$\ddot{\xi} + 0.0617 \times (100)^2 \xi + 200\beta\dot{\beta} = \frac{10}{11.9180} \quad (7.34)$$

Figure 7.9 illustrates the solution where the effect of damping on the flap motion and the formation of the coning angle are evident. It is seen, however, that the effect of aerodynamic damping, being directly imposed on the flap motion is not significant on the lead-lag motion. Figure 7.10 illustrates a zoomed view of Figure 7.9.

It is observed that with aerodynamic damping in effect, and due to the application of the step input which represents the collective pitch, the flap motion converges to a new steady state condition, which corresponds to the coning angle.

Figure 7.9 Time history diagrams of flap and lead-lag motions when aerodynamic damping is included

Figure 7.10 Time history diagrams of flap and lead-lag motions when aerodynamic damping is included (zoomed)

Now, to control the lead-lag motion, that is otherwise naturally undamped, mechanical dampers are implemented. Thus the equations of motion would be,

$$\ddot{\beta} + 89.6\dot{\beta} + (100)^2(1 + 0.0621)\beta - 100(2\dot{\xi}\beta + \dot{\beta}\xi) = 330.45 \quad (7.35)$$

$$\ddot{\xi} + 10\dot{\xi} + 0.0617 \times (100)^2 \xi + 200\beta\dot{\beta} = \frac{10}{11.9180} \quad (7.36)$$

Figures 7.11 and 7.12 illustrate the corresponding solutions where the effect of damping on both motions is evident. As a result, due to the application of the step input (collective pitch) both flap and lead-lag motions converge to new steady states. These steady states represent a coning angle and a steady lag angle for the beam, respectively. Figures 7.11 and 7.12 reveal that while there are some differences in the dynamic responses, all of the three methods predict the same steady state response.

Figure 7.11 Time history diagrams of flap and lead-lag when aerodynamic damping and lead-lag dampers are included

Figure 7.12 Time history diagrams of flap and lead-lag when aerodynamic damping and lead-lag dampers are included (zoomed)

7.7 Elastic Articulated Composite Rotating Beam

In the previous sections of this chapter the behaviour of rigid articulated beams was discussed. In Chapter 5, however, the elastodynamic response of hingeless rotor blades was presented. So, now one may combine the two solutions and perform the analysis of elastic articulated beams.

While for a rigid beam one can calculate the flap and lead-lag motions by solving the set of two coupled nonlinear differential equations, the same is not easily achievable for elastic beams. The main problem is that instead of having a purely initial value problem, one should solve a combined initial and boundary value problem whose root boundary conditions depend on the unknown solution of the differential equations of motion.

In this chapter, in order to circumvent the mentioned problem, the angular velocity boundary conditions of the elastic beam at its root are taken from the corresponding solution for its rigid articulated beam counterpart. Therefore, first using rigid body dynamics of the beam, the rigid-body motions of lead-lag and flap are calculated. Then these solutions are utilized for calculating the angular velocity boundary conditions of the elastic beam at its root. Having done that, one can proceed for solving the nonlinear elastic problem of the rotating beam using the method discussed in Chapter 5. In this way, combining the algorithms for analyzing the rigid articulated beam and the elastic hingeless beam problems generates an algorithm which is suitable for analyzing elastic articulated beams.

The solution for the motion of the beam is then performed in the following steps:

Step 1: The elastic beam, initially modelled as hingeless, is accelerated from rest to its full speed.

Step 2: The beam continues to rotate at full speed and experiences a steady state condition while its root is still clamped. The purpose of this part is to obtain an idea of the stability of the response of the beam after the acceleration phase and before the aerodynamic loads are applied and the hinges are released.

Step 3: While the beam is still rotating at full speed, the flap and lead-lag hinges are released and at the same time the beam is subjected to aerodynamic loadings. To simulate this phase of motion, first the rigid body code is run and the flap and lead-lag solutions are calculated by solving equations (7.27) and (7.31). Then the angular velocity

components at the root of the beam are calculated using equation (7.11). Having obtained this unknown, one may use equation (5.8) to get,

$$\left\{ \begin{array}{c} F^+ \\ M^+ \\ \left\{ \begin{array}{c} 0 \\ eR\Omega_{ss} \\ 0 \end{array} \right\} \\ \left\{ \begin{array}{c} \Omega_{ss} \sin \beta \cos \xi \\ \Omega_{ss} \sin \beta \sin \xi - \dot{\beta} \\ \Omega_{ss} \cos \beta - \dot{\xi} \end{array} \right\} \\ \left\{ \begin{array}{c} 0 \\ \mu eR\Omega_{ss} \\ 0 \end{array} \right\} \\ \left\{ \begin{array}{c} (i_2 + i_3)\Omega_{ss} \sin \beta \cos \xi \\ i_2(\Omega_{ss} \sin \beta \sin \xi - \dot{\beta}) \\ i_3(\Omega_{ss} \cos \beta - \dot{\xi}) \end{array} \right\} \\ \gamma^+ \\ \kappa^+ \end{array} \right\}_1 = M_{N-1}^{tot} \left\{ \begin{array}{c} 0 \\ 0 \\ V^+ \\ \Omega^+ \\ P^+ \\ H^+ \\ 0 \\ 0 \end{array} \right\}_N + T_{N-1}^{tot} \quad (7.37)$$

Equation (7.37) should now be solved for the remaining unknowns at the two nodes. After calculating the remaining unknowns at nodes 1 and N , one may use equation (5.6) in order to calculate the unknown variables at all interior nodes, by starting either from node 1 or node N . It should be pointed out that when dealing with ground resonance of helicopter blades, the use of rigid body assumption for modeling the boundary condition of articulated blades results in less accurate solutions.

7.8 Case Study: Damped Elastic Articulated Blade in Hover

The Isotropic Rectangular Solid Model introduced in section 3.2 is considered here again but it is modeled as an articulated beam. The beam is assumed to have a hinge offset ratio

of $e=0.04$ and a steady state shaft angular velocity of 100 rad/s. The accelerating part of the motion lasts for 0.5s, during which the beam is still clamped at its root and accelerates to full speed.

The second part of the motion lasts for 0.1s in which the beam continues to rotate at the steady state speed while the root is still clamped. Finally, in the last 0.5s of the motion, with the beam still rotating at its full speed, the flap and lead-lag hinges are released and at the same time the beam is subjected to two moment pulses in flap and lead-lag directions. One moment corresponds to the application of a collective pitch and the second one originates from the drag force in the lead-lag motion.

The corresponding equations of motion for the rigid body motion of the beam in flap and lead-lag would be,

$$\ddot{\beta} + 89.6\dot{\beta} + (100)^2(1 + 0.0621)\beta - 100(2\dot{\xi}\beta + \dot{\beta}\xi) = 0.0845 \quad (7.38)$$

$$\ddot{\xi} + 0.0617 \times (100)^2 \xi + 200\beta\dot{\beta} = 0.0839 \quad (7.39)$$

The lead-lag motion here has no mechanical damper. Using the solution of equations (7.38) and (7.39) in the algorithm discussed in section 7.7, the complete elasto-dynamic response of the elastic articulated beam has been obtained and is illustrated in Figures 7.13 to 7.21.

The flap and lead-lag motions have been plotted in Figure 7.14. It can be observed that even though the flap and lead-lag motions are coupled, imposing damping on the flap motion only has negligible effect on the lead-lag motion. The gentle variations of the root angular velocity and the resulting root moment can be observed in Figure 7.15. As expected, the maximum dynamic moment happens when the angular acceleration of the

beam is the greatest. The moment diagrams in Figure 7.16 illustrate the induced moments at the root of the beam after releasing the hinges. The values are seen to be small and always oscillating around zero. This is an indication of the accuracy of the solution and the correct hinge modelling. This check point provides a very important verification tool and is an effective benchmark.

The time history diagrams of the induced internal forces at the root of the beam are plotted in Figure 7.17. It can be observed that the obtained result for F_1 (solid line) is very close to that of the approximate solution given in equation (5.14), which is plotted as a dashed line. In Figure 7.18 the time history diagrams of various components of angular velocity are illustrated. The convergence of Ω_2 to zero corresponds to the convergence of the flap angle to the constant coning angle in hover.

Figure 7.13 Time history diagrams of beam internal moment components at the root

Figure 7.14 Time history diagrams of lead-lag and flap motions

Figure 7.15 Time history diagrams of shaft angular velocity and the M_3 bending moment induced in the beam at the root

Figure 7.16 Time history diagrams of beam internal moment components at the root

Figure 7.17 Time history diagrams of beam internal force components at the root; present solution (solid line) equation (5.14) (dashed line)

Figure 7.18 Time history diagrams of beam angular velocity components at the root

Figure 7.19 Variation of beam internal force components along its span at $t=0.85s$

Figure 7.20 Variation of beam internal moment components along its span at $t=0.85s$

Figure 7.21 Variation of beam angular velocity components along its span at $t=0.85s$

The variation of the internal forces, induced moments and angular velocities along the span of the beam at $t=0.85s$ are illustrated in Figures 7.19 to 7.21, respectively. In Figures 7.19 and 7.20 the free boundary conditions at the tip of the beam are observed to have been satisfied, as expected.

7.9 Case Study: Damped Elastic Composite Airfoil

Consider the case of a composite airfoil similar to what is discussed in Cesnik et al. (2003). The UM/VABS input file for this case is among the examples provided with the software. The airfoil is a NACA 4415 airfoil with double-cells and has a spar located at 38.6% chord from the leading edge, as is shown in Figure 7.22.

Figure 7.22 Cross-section of the airfoil

Figure 7.23 Ply layups and orientation angles of the airfoil cross-section, Cesnik et al. (2003)

Figure 7.23 illustrates the ply lay-up definitions and orientation angles on the section. A passive 0° ply is used to enclose the cross-section. The inner layers consist of 90° , $+45^\circ$, -45° and 0° active plies (i.e. $[0, +90, +45, -45, 0]$). The angles are measured with respect to the axis along the wing span. Here, no actuation is imposed. The case of active blade will be discussed in Chapter 8.

The properties of the applied passive and active materials are shown in Table 8.3. Using those data and UM/VABS, the stiffness matrix of the cross-section was calculated as,

$$S = \begin{bmatrix} 1.12577 \times 10^9 & 3.615437 \times 10^3 & -1.28217 \times 10^4 & -1.64732 \times 10^5 & -2.168324 \times 10^5 & -5.681057 \times 10^6 \\ 3.615437 \times 10^3 & 3.15555 \times 10^8 & -4.04582 \times 10^5 & -1.0509 \times 10^7 & 7.05125 \times 10^4 & 4.33983 \times 10^3 \\ -1.28217 \times 10^4 & -4.04582 \times 10^5 & 2.79485 \times 10^7 & -1.06215 \times 10^7 & 5.81197 \times 10^2 & 1.08681 \times 10^4 \\ -1.64732 \times 10^5 & -1.0509 \times 10^7 & -1.06215 \times 10^7 & 1.75149 \times 10^7 & -2.4470 \times 10^3 & 2.00316 \times 10^3 \\ -2.168324 \times 10^5 & 7.05125 \times 10^4 & 5.81197 \times 10^2 & -2.4470 \times 10^3 & 1.39200 \times 10^7 & 1.16753 \times 10^5 \\ -5.681057 \times 10^6 & 4.33983 \times 10^3 & 1.08681 \times 10^4 & 2.00316 \times 10^3 & 1.16753 \times 10^5 & 3.3672 \times 10^8 \end{bmatrix} \quad (7.40)$$

Now with the full speed angular velocity of 30 rad/s, a 4% hinge offset ratio, unit applied moments in the 2 and 3 directions and using,

Figure 7.24 Time history diagrams of beam internal moment components at the root

Figure 7.25 Time history diagrams of shaft angular velocity and the M_3 bending moment induced in the beam at the root

Figure 7.26 Time history diagrams of beam internal force components at the root; present solution (solid line) equation (5.14) (dashed line)

Figure 7.27 Variation of beam internal force components along its span at $t=0.85$ s

Figure 7.28 Time history diagrams of beam internal moment components at the root

Figure 7.29 Time history diagrams of beam angular velocity components at the root

$$I = \begin{bmatrix} 77.255 & 0 & 0 \\ 0 & 3.1362 & -0.20052 \\ 0 & -0.20052 & 74.119 \end{bmatrix} \times 10^{-2} \quad (7.41)$$

Figures 7.24 to 7.29 illustrate the solution for this case and result in conclusions that are similar to those of the previous case.

Chapter 8: Static and Dynamic Analysis of Beams with Embedded Anisotropic Piezocomposite Actuators

In this chapter, the impact of inclusion of embedded Anisotropic Piezocomposite Actuators (APA) in a beam structure is analyzed. This is a step towards modeling adaptive rotor blades. Most of the contents of this chapter have been accepted for publication in Ghorashi and Nitzsche (2009).

Intelligent structures are characterized by sensors to monitor the structure and actuators which are capable of changing the state of the structure and can be networked through feedback control logic to produce robust control which can withstand or adapt to variations in controlled plant responses, environmental disturbances, and control system failures, Inman and Cudney (2000).

Figure 8.1 Basic induced-strain responses of piezoelectric materials; (a) direct strains (b) shear strain (c) shear strain, Giurgiutiu (2008)

Piezoelectricity, as shown in Figure 8.1, is a coupling between electrical and mechanical systems. The direct piezoelectric effect is the generator or sensor mode of application and the converse piezoelectric effect is the motor or actuator mode. All applied electrical fields cause deformations in the PZT. However, the resulting deformation depends on the relation of the poling direction and the electric field direction. If these two directions are parallel, the PZT will expand or contract but does not distort. If these two directions are not parallel, the PZT would expand and distort at the same time. When these directions are perpendicular, the PZT will only distort with no change in volume.

8.1 The 1-D Beam Formulation with Embedded Piezoelectric Element

For linear piezoelectric materials, the interaction between the electrical and mechanical variables can be described by linear relations. The constitutive equations can be written in the following matrix format,

$$\{S\} = [s]\{T\} + [d]^T\{E\} + \{\alpha\}\theta \quad (8.1)$$

$$\{D\} = [d]\{T\} + [\epsilon]\{E\} + \{\tilde{D}\}\theta \quad (8.2)$$

Where, S is the strain; D is the electric displacement (charge per unit area), s is the compliance (strain per unit stress), d is the piezoelectric moduli (m/V), ϵ is the piezodielectric matrix (F/m), α_i is the coefficient of thermal expansion and $\tilde{D}_i$ is the electric displacement temperature coefficient. Also, T is the stress vector, E is the electric field intensity (V/m) and θ is the temperature.

With actuators in the structure, the applied force and moment vector per unit length on the structure, at every location and every time, can be written as the summation of a mechanically applied one and another term which is due to the actuators.

$$\begin{Bmatrix} f \\ m \end{Bmatrix} = \begin{Bmatrix} f_M \\ m_M \end{Bmatrix} + \begin{Bmatrix} f_A \\ m_A \end{Bmatrix} \quad (8.3)$$

For a certain actuation scenario, UM/VABS can provide the vector of actuator forces for each active material. Therefore, the whole actuator force is obtained by adding up all of these forces. Substitution of equation (8.3) into equations (2.24) and (2.25) gives,

$$F' + \tilde{K}F + (f_M + f_A) = \dot{P} + \tilde{\Omega}P \quad (8.4)$$

$$M' + \tilde{K}M + (\tilde{e}_1 + \tilde{\gamma})F + (m_M + m_A) = \dot{H} + \tilde{\Omega}H + \tilde{V}P \quad (8.5)$$

In a linear static problem with no initial curvature and no shear forces equations (8.4) and (8.5) reduce to,

$$F' + (f_M + f_A) = 0 \quad (8.6)$$

$$M' + (m_M + m_A) = 0 \quad (8.7)$$

8.2 Case Study: Static Active Composite Box Model

Consider the composite beam discussed in section 3.3 but having length $L=0.5\text{m}$. This time active plies of the box are activated and two cases are considered. The first one is bending deformation obtained by imposing the left hand side of the box to -2000V and its right hand side to +2000V. The second one is twist deformation that is obtained when both active sides are subjected to +2000V.

Table 8.1 Actuation force and moment in bending actuation

	UM/VABS	Cesnik and Palacios (2003)
f_3 (N/m)	227.98	232.67
m_3 (N.m/m)	-1.231	-1.283

Table 8.2 Actuation force and moment in twist actuation

	UM/VABS	Cesnik and Palacios (2003)
f_1 (N/m)	16.860	17.219
m_1 (N.m/m)	3.518	3.666

The actuation forces for the bending and twist actuations were calculated by UM/VABS and are listed in Tables 8.1 and 8.2, respectively. The results are also compared with the ones in Cesnik and Palacios (2003) which also used UM/VABS for the analysis. As expected, the two sets of results are close.

Now, for the twist actuation, using the calculated actuation force and moment in the 1-D equations of motion the following clamped end reaction forces are obtained,

$$\begin{aligned} F_{1A} &= 8.43\text{N} \\ M_{1A} &= 1.759\text{N.m} \end{aligned} \tag{8.8}$$

Using the constitutive equation at the clamped end, one obtains, $\kappa_1=0.042$ rad/m and finally,

Figure 8.2 The stretched image of the airfoil for enhancing clarity

$$\theta_1 = 0.042x \quad (8.9)$$

Therefore, the twist angle at the free end is 1.209° . This value is close to 1.195° given in Cesnik and Palacios (2003).

8.3 Case Study: Static Active Composite Airfoil

Consider the composite airfoil discussed in Cesnik et al. (2003) and in Chapter 7; but this time with the actuation of piezocomposite actuators embedded in it. In order to show the details more clearly, the stretched image given in Figure 8.2 will be used in the illustrations.

Table 8.3 Material properties of active composite airfoil, Cesnik et al. (2003)

	E_{11}	E_{22}	E_{33}	G_{12}	G_{13}	G_{23}	ν_{12}	ν_{13}	ν_{23}	ρ	d_{11}	d_{12}	d_{13}
	GPa	GPa	GPa	GPa	GPa	GPa				kg/m ³	pm/V	pm/V	pm/V
Passive	19.3	9.8	9.8	5.5	5.5	4.4	0.35	0.35	0.496	1716	—	—	—
Active										4060	310	-130	-130

Table 8.4 Actuation forces and moments generated by active plies at various directions

	0° ply actuation	90° ply actuation	±45° plies actuation	All together
Extension (N/m)	71,042	-21,653	49,705	99,095
Shear F_2 (N/m)	0.603	5.089	2.658	8.349
Shear F_3 (N/m)	1.095	19.705	-16.799	4.002
Twist (N.m/m)	-0.288	15.67	-238.13	-222.746
Bending M_2 (N.m/m)	-41.44	15.19	-33.03	-59.28
Bending M_3 (N.m/m)	454.29	-409.01	575.276	620.557

The material properties of the applied passive and active materials are given in Table 8.3. Each layer has a thickness of 3429 μ m and a constant electric potential of +1000V between the two electrodes at a distance of 1100 μ m has been applied to the actuators. The spar has no active layers.

Using UM/VABS, forces, moments and stress components generated as a result of activating plies at various directions were calculated and some of the results are listed in Table 8.4. As expected, the active spanwise ply actuation generates the maximum axial force F_1 . Also, the $\pm 45^\circ$ plies are mainly responsible for twist generation.

Figures 8.3 to 8.8 were plotted by a code written in MatLab. These figures illustrate the distribution of various stress components across the cross section when only the 90° plies are activated by the 1000V actuation.

Figure 8.3 Distribution of the T_{11} stress component due to 1000V actuation of the 90° plies

Figure 8.4 Distribution of the T_{22} stress component due to 1000V actuation of the 90° plies

Figure 8.5 Distribution of the T_{33} stress component due to 1000V actuation of the 90° plies

Figure 8.6 Distribution of the T_{12} stress component due to 1000V actuation of the 90° plies

Figure 8.7 Distribution of the T_{13} stress component due to 1000V actuation of the 90° plies

Figure 8.8 Distribution of the T_{23} stress component due to 1000V actuation of the 90° plies

8.4 Case Study: Steady State Response of Rotating Active Airfoil

Consider the case of actuation of piezocomposite actuators embedded in the composite airfoil discussed in section 8.3. But this time the 3m beam is rotating at an angular velocity of 100rad/s and its steady state response under different actuation scenarios is of interest. To this end, and using $\mu=248.35$ kg/m, the method discussed in Chapter 6 was utilized.

First, considering no activation, the steady state response of the beam was obtained. To analyze the effect of coupling, two cases were considered. First, all of the terms in the stiffness and mass matrices were included in the analysis. Then, off-diagonal terms were ignored. The difference between these two solutions provides an overall estimation of the impact of coupling. The corresponding results are plotted in Figures 8.9 to 8.12.

Figure 8.9 The variation of internal force components along the beam; coupled solution (solid line) uncoupled solution (dashed or circles)

Figure 8.10 The variation of internal moment components along the beam; coupled solution (solid line) uncoupled solution (dashed)

Figure 8.11 The variation of velocity components along the beam; coupled solution (solid line) uncoupled solution (dashed or circles)

Figure 8.12 The variation of angular velocity components along the beam; coupled solution (solid line) uncoupled solution (dashed or circles)

Figure 8.13 The steady state variation of internal force components along the beam due to various modes of activation of Anisotropic Piezocomposite Actuators

Figure 8.14 The steady state variation of internal force components along the beam due to various modes of activation of Anisotropic Piezocomposite Actuators (zoomed)

Figure 8.15 The steady state variation of internal moment components along the beam due to various modes of activation of Anisotropic Piezocomposite Actuators

Figure 8.16 The steady state variation of internal moment components along the beam due to various modes of activation of Anisotropic Piezocomposite Actuators (zoomed)

Next, the effect of activation of each active ply was investigated. To this end, these plies were activated one by one by applying a +1000V potential to them and the corresponding steady state solutions were obtained. Finally, all of the active plies were activated simultaneously. The corresponding results are plotted in Figures 8.13 to 8.16. These diagrams can be used for controlling the response of the beam and the load distribution along it. It is observed that the actuators have significant controllability on M_3 , but little control on F_1 .

8.5 Case Study: Rotating Articulated Active Composite Airfoil

Consider the case of actuation of the piezocomposite actuators embedded in the composite airfoil discussed in Section 7.9. Using 1% of the full actuation forces and moments shown in Table 8.4, and a full speed angular velocity of 30rad/s, distributions

Figure 8.17 Time history diagrams of shaft angular velocity and the M_3 bending moment induced in the beam at its root

Figure 8.18 Variation of beam internal force components along its span at $t=0.85s$

Figure 8.19 Time history diagrams of beam internal force components at the root

of forces with and without actuation were calculated and plotted in Figures 8.17 to 8.19. As expected, and is observed in Figures 8.18 and 8.19, for this low inertia case with zero M_3 at the root, actuation which happens at $t=0.6$ s has maximum influence on the axial force F_1 .

Chapter 9: Discussion and Conclusions

The structural analysis of geometrically nonlinear passive and active composite rotating beams was presented. The analysis included linear and nonlinear statics, nonlinear dynamics, clamped and articulated boundary conditions, accelerating, steady state and perturbed steady state solutions, the effects of aerodynamic loading and damping as well as those of the embedded actuators.

The developed solutions use VABS which is based on a linear Variational-Asymptotic Method (VAM) for cross-sectional analysis. This method splits the 3-D geometrically nonlinear elasticity problem of the rotating active composite beams into a nonlinear 1-D analysis along the beam and a 2-D analysis across its cross-section. The solution of the 2-D problem results in the cross-sectional stiffness and mass matrices as well as the warping functions. These results are then used in all further 3-D simulations of the rotating beam without the need to repeat the 2-D analysis. The solution of the 1-D problem is performed by solving the nonlinear intrinsic differential equations of a beam. Combining these two solutions provides the complete 3-D structural solution.

The study started with the linear and nonlinear static analysis of passive isotropic or composite beams. Such a solution could act as the initial condition for the corresponding dynamic analysis.

Next, using the finite difference method and perturbations, the transient solution of the nonlinear intrinsic differential equations was presented. This solution provided the elasto-dynamic response of an accelerating hingeless composite beam. The specific problem considered involved an accelerating rotor blade that started its motion from rest and converged to a steady state condition. In this way, transient and the steady state solutions of internal forces, moments, velocities and angular velocities were calculated along the beam.

The steady state behavior of a rotating beam was then obtained by solving the time-independent form of the governing nonlinear intrinsic differential equations of a beam. The resulting boundary value problem was converted into a series of initial value problems. The solution was then performed by the application of the shooting method. After each iteration, the unknown initial conditions were improved by the use of the Newton-Raphson algorithm. These initial conditions correspond to the boundary conditions of the original boundary value problem. The solution was repeated and when a convergence criterion was satisfied, the solution of the boundary value problem and the steady state response of the beam were obtained. This solution compared very well with the other steady state solution obtained using the finite difference method and perturbations.

Having obtained the steady state response, the effect of imposing input perturbations on a beam which is already in its steady state condition was analyzed. Such a solution is, of course, valid only in the vicinity of the steady state response. The results were verified against those of the perturbed accelerating beam.

In the next step, using the extended Euler equations, the rigid body dynamics of a rotating articulated beam subjected to aerodynamic loading was analyzed. Formulations for the nonlinearly coupled flap-lag dynamics of a rigid beam including the effects of aerodynamics and hinge offset were developed. It was shown that the derived formulas are generalizations of those existing in the literature.

The obtained rigid body solution at the root of the beam was coupled with the elastic rotating articulated beam formulation in order to serve as its hinge boundary condition. Therefore, the rigid body solution was fed into the already developed finite difference code for the solution of the accelerating hingeless beam problem. The specific problem considered involved an accelerating rotating beam that started its motion from rest and converged to a steady state angular velocity. The solution was shown to be in good agreement with approximate formulas for the axial force and with the imposed boundary conditions.

Finally, the effect of inclusion of embedded Anisotropic Piezocomposite Actuators in the composite beam structure was analyzed. Both static and dynamic cases were considered and the sensitivity of the response of the beam to activating the actuators located at various angles was illustrated. The transient form of this analysis can be used for the purpose of noise and vibration reduction by controlling the response of an active rotating beam.

The developed structural codes can be integrated with SMARTROTOR developed by the Rotorcraft Research Group of Carleton University and its international collaborators. To this end, the deformation output of the structural codes can be used by

SMARTROTOR to generate the corresponding aerodynamic forces and moments. These forces and moments would then be fed back to the structural codes to provide an update of the deformation field. In this way, the structural and aerodynamic codes communicate back and forth to generate an aerodynamically and structurally correct solution for the nonlinear aero-elasto-dynamic response of active composite helicopter blades. Such a simulation code would be a powerful tool for the ultimate aim of efficient noise and vibration control of helicopters. At present, Michael G. Martin, a graduate student in Carleton University, is working on the integration of one of the structural codes developed in this thesis with SMARTROTOR.

Furthermore, research may be performed in order to generalize the presented articulated blade model so that it can be more suitable in situations like run up and ground resonance.

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Appendix: Matrices A , B and Vector J in Chapter 5

$$J_i(1) = \begin{pmatrix} \frac{1}{2\Delta x}(F_{1,i} - F_{1,i+1}) + \frac{1}{16}(\kappa_{3,i+1}F_{2,i+1} + \kappa_{3,i+1}F_{2,i} + \kappa_{3,i}F_{2,i+1} - 3\kappa_{3,i}F_{2,i}) \\ -\frac{1}{16}(\kappa_{2,i+1}F_{3,i+1} + \kappa_{2,i+1}F_{3,i} + \kappa_{2,i}F_{3,i+1} - 3\kappa_{2,i}F_{3,i}) - \frac{1}{4}(f_{1,i+1} + f_{1,i} + f_{1,i+1}^+ + f_{1,i}^+) \\ -\frac{1}{2\Delta t}(P_{1,i} + P_{1,i+1}) - \frac{1}{16}(\Omega_{3,i+1}P_{2,i+1} + \Omega_{3,i+1}P_{2,i} + \Omega_{3,i}P_{2,i+1} - 3\Omega_{3,i}P_{2,i}) \\ + \frac{1}{16}(\Omega_{2,i+1}P_{3,i+1} + \Omega_{2,i+1}P_{3,i} + \Omega_{2,i}P_{3,i+1} - 3\Omega_{2,i}P_{3,i}) \end{pmatrix} \quad (\text{A.1})$$

$$J_i(2) = \begin{pmatrix} \frac{1}{2\Delta x}(F_{2,i} - F_{2,i+1}) + \frac{1}{16}(\kappa_{1,i+1}F_{3,i+1} + \kappa_{1,i+1}F_{3,i} + \kappa_{1,i}F_{3,i+1} - 3\kappa_{1,i}F_{3,i}) \\ -\frac{1}{16}(\kappa_{3,i+1}F_{1,i+1} + \kappa_{3,i+1}F_{1,i} + \kappa_{3,i}F_{1,i+1} - 3\kappa_{3,i}F_{1,i}) - \frac{1}{4}(f_{2,i+1} + f_{2,i} + f_{2,i+1}^+ + f_{2,i}^+) \\ -\frac{1}{2\Delta t}(P_{2,i} + P_{2,i+1}) - \frac{1}{16}(\Omega_{1,i+1}P_{3,i+1} + \Omega_{1,i+1}P_{3,i} + \Omega_{1,i}P_{3,i+1} - 3\Omega_{1,i}P_{3,i}) \\ + \frac{1}{16}(\Omega_{3,i+1}P_{1,i+1} + \Omega_{3,i+1}P_{1,i} + \Omega_{3,i}P_{1,i+1} - 3\Omega_{3,i}P_{1,i}) \end{pmatrix} \quad (\text{A.2})$$

$$J_i(3) = \begin{pmatrix} \frac{1}{2\Delta x}(F_{3,i} - F_{3,i+1}) + \frac{1}{16}(\kappa_{2,i+1}F_{1,i+1} + \kappa_{2,i+1}F_{1,i} + \kappa_{2,i}F_{1,i+1} - 3\kappa_{2,i}F_{1,i}) \\ -\frac{1}{16}(\kappa_{1,i+1}F_{2,i+1} + \kappa_{1,i+1}F_{2,i} + \kappa_{1,i}F_{2,i+1} - 3\kappa_{1,i}F_{2,i}) - \frac{1}{4}(f_{3,i+1} + f_{3,i} + f_{3,i+1}^+ + f_{3,i}^+) \\ -\frac{1}{2\Delta t}(P_{3,i} + P_{3,i+1}) - \frac{1}{16}(\Omega_{2,i+1}P_{1,i+1} + \Omega_{2,i+1}P_{1,i} + \Omega_{2,i}P_{1,i+1} - 3\Omega_{2,i}P_{1,i}) \\ + \frac{1}{16}(\Omega_{1,i+1}P_{2,i+1} + \Omega_{1,i+1}P_{2,i} + \Omega_{1,i}P_{2,i+1} - 3\Omega_{1,i}P_{2,i}) \end{pmatrix} \quad (\text{A.3})$$

$$J_i(4) = \left(\begin{aligned} & \frac{1}{2\Delta x} (M_{1,i} - M_{1,i+1}) + \frac{1}{16} (\kappa_{3,i+1} M_{2,i+1} + \kappa_{3,i+1} M_{2,i} + \kappa_{3,i} M_{2,i+1} - 3\kappa_{3,i} M_{2,i}) \\ & - \frac{1}{16} (\kappa_{2,i+1} M_{3,i+1} + \kappa_{2,i+1} M_{3,i} + \kappa_{2,i} M_{3,i+1} - 3\kappa_{2,i} M_{3,i}) - \frac{1}{4} (m_{1,i+1} + m_{1,i} + m_{1,i+1}^+ + m_{1,i}^+) \\ & - \frac{1}{2\Delta t} (H_{1,i} + H_{1,i+1}) - \frac{1}{16} (\Omega_{3,i+1} H_{2,i+1} + \Omega_{3,i+1} H_{2,i} + \Omega_{3,i} H_{2,i+1} - 3\Omega_{3,i} H_{2,i}) \\ & + \frac{1}{16} (\Omega_{2,i+1} H_{3,i+1} + \Omega_{2,i+1} H_{3,i} + \Omega_{2,i} H_{3,i+1} - 3\Omega_{2,i} H_{3,i}) \\ & + \frac{1}{8} (\gamma_{13,i+1} F_{2,i+1} + \gamma_{13,i+1} F_{2,i} + \gamma_{13,i} F_{2,i+1} - 3\gamma_{13,i} F_{2,i}) \\ & - \frac{1}{8} (\gamma_{12,i+1} F_{3,i+1} + \gamma_{12,i+1} F_{3,i} + \gamma_{12,i} F_{3,i+1} - 3\gamma_{12,i} F_{3,i}) \\ & - \frac{1}{16} (V_{3,i+1} P_{2,i+1} + V_{3,i+1} P_{2,i} + V_{3,i} P_{2,i+1} - 3V_{3,i} P_{2,i}) \\ & + \frac{1}{16} (V_{2,i+1} P_{3,i+1} + V_{2,i+1} P_{3,i} + V_{2,i} P_{3,i+1} - 3V_{2,i} P_{3,i}) \end{aligned} \right) \quad (\text{A.4})$$

$$J_i(5) = \left(\begin{aligned} & \frac{1}{2\Delta x} (M_{2,i} - M_{2,i+1}) + \frac{1}{16} (\kappa_{1,i+1} M_{3,i+1} + \kappa_{1,i+1} M_{3,i} + \kappa_{1,i} M_{3,i+1} - 3\kappa_{1,i} M_{3,i}) \\ & - \frac{1}{16} (\kappa_{3,i+1} M_{1,i+1} + \kappa_{3,i+1} M_{1,i} + \kappa_{3,i} M_{1,i+1} - 3\kappa_{3,i} M_{1,i}) - \frac{1}{4} (m_{2,i+1} + m_{2,i} + m_{2,i+1}^+ + m_{2,i}^+) \\ & - \frac{1}{2\Delta t} (H_{2,i} + H_{2,i+1}) - \frac{1}{16} (\Omega_{1,i+1} H_{3,i+1} + \Omega_{1,i+1} H_{3,i} + \Omega_{1,i} H_{3,i+1} - 3\Omega_{1,i} H_{3,i}) \\ & + \frac{1}{16} (\Omega_{3,i+1} H_{1,i+1} + \Omega_{3,i+1} H_{1,i} + \Omega_{3,i} H_{1,i+1} - 3\Omega_{3,i} H_{1,i}) + \frac{1}{4} (F_{3,i} + F_{3,i+1}) \\ & + \frac{1}{16} (\gamma_{11,i+1} F_{3,i+1} + \gamma_{11,i+1} F_{3,i} + \gamma_{11,i} F_{3,i+1} - 3\gamma_{11,i} F_{3,i}) \\ & - \frac{1}{8} (\gamma_{13,i+1} F_{1,i+1} + \gamma_{13,i+1} F_{1,i} + \gamma_{13,i} F_{1,i+1} - 3\gamma_{13,i} F_{1,i}) \\ & - \frac{1}{16} (V_{1,i+1} P_{3,i+1} + V_{1,i+1} P_{3,i} + V_{1,i} P_{3,i+1} - 3V_{1,i} P_{3,i}) \\ & + \frac{1}{16} (V_{3,i+1} P_{1,i+1} + V_{3,i+1} P_{1,i} + V_{3,i} P_{1,i+1} - 3V_{3,i} P_{1,i}) \end{aligned} \right) \quad (\text{A.5})$$

$$J_i(6) = \left(\begin{aligned} & \frac{1}{24x} (M_{3,i} - M_{3,i+1}) + \frac{1}{16} (\kappa_{2,i+1} M_{1,i+1} + \kappa_{2,i+1} M_{1,i} + \kappa_{2,i} M_{1,i+1} - 3\kappa_{2,i} M_{1,i}) \\ & - \frac{1}{16} (\kappa_{1,i+1} M_{2,i+1} + \kappa_{1,i+1} M_{2,i} + \kappa_{1,i} M_{2,i+1} - 3\kappa_{1,i} M_{2,i}) - \frac{1}{4} (m_{3,i+1} + m_{3,i} + m_{3,i+1}^+ + m_{3,i}^+) \\ & - \frac{1}{24} (H_{3,i} + H_{3,i+1}) - \frac{1}{16} (\Omega_{2,i+1} H_{1,i+1} + \Omega_{2,i+1} H_{1,i} + \Omega_{2,i} H_{1,i+1} - 3\Omega_{2,i} H_{1,i}) \\ & + \frac{1}{16} (\Omega_{1,i+1} H_{2,i+1} + \Omega_{1,i+1} H_{2,i} + \Omega_{1,i} H_{2,i+1} - 3\Omega_{1,i} H_{2,i}) - \frac{1}{4} (F_{2,i} + F_{2,i+1}) \\ & + \frac{1}{16} (\gamma_{11,i+1} F_{2,i+1} + \gamma_{11,i+1} F_{2,i} + \gamma_{11,i} F_{2,i+1} - 3\gamma_{11,i} F_{2,i}) \\ & + \frac{1}{8} (\gamma_{12,i+1} F_{1,i+1} + \gamma_{12,i+1} F_{1,i} + \gamma_{12,i} F_{1,i+1} - 3\gamma_{12,i} F_{1,i}) \\ & - \frac{1}{16} (V_{2,i+1} P_{1,i+1} + V_{2,i+1} P_{1,i} + V_{2,i} P_{1,i+1} - 3V_{2,i} P_{1,i}) \\ & + \frac{1}{16} (V_{1,i+1} P_{2,i+1} + V_{1,i+1} P_{2,i} + V_{1,i} P_{2,i+1} - 3V_{1,i} P_{2,i}) \end{aligned} \right) \quad (A.6)$$

$$J_i(7) = \left(\begin{aligned} & \frac{1}{24x} (V_{1,i} - V_{1,i+1}) + \frac{1}{16} (\kappa_{3,i+1} V_{2,i+1} + \kappa_{3,i+1} V_{2,i} + \kappa_{3,i} V_{2,i+1} - 3\kappa_{3,i} V_{2,i}) \\ & - \frac{1}{16} (\kappa_{2,i+1} V_{3,i+1} + \kappa_{2,i+1} V_{3,i} + \kappa_{2,i} V_{3,i+1} - 3\kappa_{2,i} V_{3,i}) \\ & + \frac{1}{8} (\gamma_{13,i+1} \Omega_{2,i+1} + \gamma_{13,i+1} \Omega_{2,i} + \gamma_{13,i} \Omega_{2,i+1} - 3\gamma_{13,i} \Omega_{2,i}) \\ & - \frac{1}{8} (\gamma_{12,i+1} \Omega_{3,i+1} + \gamma_{12,i+1} \Omega_{3,i} + \gamma_{12,i} \Omega_{3,i+1} - 3\gamma_{12,i} \Omega_{3,i}) - \frac{1}{24} (\gamma_{11,i} + \gamma_{11,i+1}) \end{aligned} \right) \quad (A.7)$$

$$J_i(8) = \left(\begin{aligned} & \frac{1}{24x} (V_{2,i} - V_{2,i+1}) + \frac{1}{16} (\kappa_{1,i+1} V_{3,i+1} + \kappa_{1,i+1} V_{3,i} + \kappa_{1,i} V_{3,i+1} - 3\kappa_{1,i} V_{3,i}) \\ & - \frac{1}{16} (\kappa_{3,i+1} V_{1,i+1} + \kappa_{3,i+1} V_{1,i} + \kappa_{3,i} V_{1,i+1} - 3\kappa_{3,i} V_{1,i}) + \frac{1}{4} (\Omega_{3,i} + \Omega_{3,i+1}) \\ & + \frac{1}{16} (\gamma_{11,i+1} \Omega_{3,i+1} + \gamma_{11,i+1} \Omega_{3,i} + \gamma_{11,i} \Omega_{3,i+1} - 3\gamma_{11,i} \Omega_{3,i}) \\ & - \frac{1}{8} (\gamma_{13,i+1} \Omega_{1,i+1} + \gamma_{13,i+1} \Omega_{1,i} + \gamma_{13,i} \Omega_{1,i+1} - 3\gamma_{13,i} \Omega_{1,i}) - \frac{1}{24} (\gamma_{12,i} + \gamma_{12,i+1}) \end{aligned} \right) \quad (A.8)$$

$$J_i(9) = \left(\begin{aligned} & \frac{1}{24x} (V_{3,i} - V_{3,i+1}) + \frac{1}{16} (\kappa_{2,i+1} V_{1,i+1} + \kappa_{2,i+1} V_{1,i} + \kappa_{2,i} V_{1,i+1} - 3\kappa_{2,i} V_{1,i}) \\ & - \frac{1}{16} (\kappa_{1,i+1} V_{2,i+1} + \kappa_{1,i+1} V_{2,i} + \kappa_{1,i} V_{2,i+1} - 3\kappa_{1,i} V_{2,i}) - \frac{1}{4} (\Omega_{2,i} + \Omega_{2,i+1}) \\ & - \frac{1}{16} (\gamma_{11,i+1} \Omega_{2,i+1} + \gamma_{11,i+1} \Omega_{2,i} + \gamma_{11,i} \Omega_{2,i+1} - 3\gamma_{11,i} \Omega_{2,i}) \\ & + \frac{1}{8} (\gamma_{12,i+1} \Omega_{1,i+1} + \gamma_{12,i+1} \Omega_{1,i} + \gamma_{12,i} \Omega_{1,i+1} - 3\gamma_{12,i} \Omega_{1,i}) - \frac{1}{24} (\gamma_{13,i} + \gamma_{13,i+1}) \end{aligned} \right) \quad (A.9)$$

$$J_i(10) = \left(\frac{1}{24x} (\mathcal{Q}_{1,i} - \mathcal{Q}_{1,i+1}) + \frac{1}{16} (\kappa_{3,i+1} \mathcal{Q}_{2,i+1} + \kappa_{3,i+1} \mathcal{Q}_{2,i} + \kappa_{3,i} \mathcal{Q}_{2,i+1} - 3\kappa_{3,i} \mathcal{Q}_{2,i}) \right) - \frac{1}{16} (\kappa_{2,i+1} \mathcal{Q}_{3,i+1} + \kappa_{2,i+1} \mathcal{Q}_{3,i} + \kappa_{2,i} \mathcal{Q}_{3,i+1} - 3\kappa_{2,i} \mathcal{Q}_{3,i}) - \frac{1}{24y} (\kappa_{1,i} + \kappa_{1,i+1}) \quad (\text{A.10})$$

$$J_i(11) = \left(\frac{1}{24x} (\mathcal{Q}_{2,i} - \mathcal{Q}_{2,i+1}) + \frac{1}{16} (\kappa_{1,i+1} \mathcal{Q}_{3,i+1} + \kappa_{1,i+1} \mathcal{Q}_{3,i} + \kappa_{1,i} \mathcal{Q}_{3,i+1} - 3\kappa_{1,i} \mathcal{Q}_{3,i}) \right) - \frac{1}{16} (\kappa_{3,i+1} \mathcal{Q}_{1,i+1} + \kappa_{3,i+1} \mathcal{Q}_{1,i} + \kappa_{3,i} \mathcal{Q}_{1,i+1} - 3\kappa_{3,i} \mathcal{Q}_{1,i}) - \frac{1}{24y} (\kappa_{2,i} + \kappa_{2,i+1}) \quad (\text{A.11})$$

$$J_i(12) = \left(\frac{1}{24x} (\mathcal{Q}_{3,i} - \mathcal{Q}_{3,i+1}) + \frac{1}{16} (\kappa_{2,i+1} \mathcal{Q}_{1,i+1} + \kappa_{2,i+1} \mathcal{Q}_{1,i} + \kappa_{2,i} \mathcal{Q}_{1,i+1} - 3\kappa_{2,i} \mathcal{Q}_{1,i}) \right) - \frac{1}{16} (\kappa_{1,i+1} \mathcal{Q}_{2,i+1} + \kappa_{1,i+1} \mathcal{Q}_{2,i} + \kappa_{1,i} \mathcal{Q}_{2,i+1} - 3\kappa_{1,i} \mathcal{Q}_{2,i}) - \frac{1}{24y} (\kappa_{3,i} + \kappa_{3,i+1}) \quad (\text{A.12})$$

$$J_i(13) = -\frac{1}{4} (P_{1,i} + P_{1,i+1}) + \frac{\mu}{4} (V_{1,i} + V_{1,i+1}) + \frac{\mu}{4} \bar{x}_3 (\Omega_{2,i} + \Omega_{2,i+1}) - \frac{\mu}{4} \bar{x}_2 (\Omega_{3,i} + \Omega_{3,i+1}) \quad (\text{A.13})$$

$$J_i(14) = -\frac{1}{4} (P_{2,i} + P_{2,i+1}) + \frac{\mu}{4} (V_{2,i} + V_{2,i+1}) - \frac{\mu}{4} \bar{x}_3 (\Omega_{1,i} + \Omega_{1,i+1}) \quad (\text{A.14})$$

$$J_i(15) = -\frac{1}{4} (P_{3,i} + P_{3,i+1}) + \frac{\mu}{4} (V_{3,i} + V_{3,i+1}) + \frac{\mu}{4} \bar{x}_2 (\Omega_{1,i} + \Omega_{1,i+1}) \quad (\text{A.15})$$

$$J_i(16) = -\frac{1}{4} (H_{1,i} + H_{1,i+1}) - \frac{\mu}{4} \bar{x}_3 (V_{2,i} + V_{2,i+1}) + \frac{\mu}{4} \bar{x}_2 (V_{3,i} + V_{3,i+1}) + \frac{1}{4} i(1,1) (\Omega_{1,i} + \Omega_{1,i+1}) + \frac{1}{4} i(1,2) (\Omega_{2,i} + \Omega_{2,i+1}) + \frac{1}{4} i(1,3) (\Omega_{3,i} + \Omega_{3,i+1}) \quad (\text{A.16})$$

$$J_i(17) = -\frac{1}{4} (H_{2,i} + H_{2,i+1}) + \frac{\mu}{4} \bar{x}_3 (V_{1,i} + V_{1,i+1}) + \frac{1}{4} i(2,1) (\Omega_{1,i} + \Omega_{1,i+1}) + \frac{1}{4} i(2,2) (\Omega_{2,i} + \Omega_{2,i+1}) + \frac{1}{4} i(2,3) (\Omega_{3,i} + \Omega_{3,i+1}) \quad (\text{A.17})$$

$$J_i(18) = -\frac{1}{4}(H_{3,i} + H_{3,i+1}) - \frac{\mu}{4}\bar{x}_2(V_{1,i} + V_{1,i+1}) \\ + \frac{1}{4}i(3,1)(\Omega_{1,i} + \Omega_{1,i+1}) + \frac{1}{4}i(3,2)(\Omega_{2,i} + \Omega_{2,i+1}) + \frac{1}{4}i(3,3)(\Omega_{3,i} + \Omega_{3,i+1}) \quad (\text{A.18})$$

$$J_i(19) = -\frac{1}{4}(\gamma_{11,i} + \gamma_{11,i+1}) + \frac{1}{4}R(1,1)(F_{1,i} + F_{1,i+1}) + \frac{1}{4}R(1,2)(F_{2,i} + F_{2,i+1}) \\ + \frac{1}{4}R(1,3)(F_{3,i} + F_{3,i+1}) + \frac{1}{4}Z(1,1)(M_{1,i} + M_{1,i+1}) + \frac{1}{4}Z(1,2)(M_{2,i} + M_{2,i+1}) \\ + \frac{1}{4}Z(1,3)(M_{3,i} + M_{3,i+1}) \quad (\text{A.19})$$

$$J_i(20) = -\frac{1}{2}(\gamma_{12,i} + \gamma_{12,i+1}) + \frac{1}{4}R(2,1)(F_{1,i} + F_{1,i+1}) + \frac{1}{4}R(2,2)(F_{2,i} + F_{2,i+1}) \\ + \frac{1}{4}R(2,3)(F_{3,i} + F_{3,i+1}) + \frac{1}{4}Z(2,1)(M_{1,i} + M_{1,i+1}) + \frac{1}{4}Z(2,2)(M_{2,i} + M_{2,i+1}) \\ + \frac{1}{4}Z(2,3)(M_{3,i} + M_{3,i+1}) \quad (\text{A.20})$$

$$J_i(21) = -\frac{1}{2}(\gamma_{13,i} + \gamma_{13,i+1}) + \frac{1}{4}R(3,1)(F_{1,i} + F_{1,i+1}) + \frac{1}{4}R(3,2)(F_{2,i} + F_{2,i+1}) \\ + \frac{1}{4}R(3,3)(F_{3,i} + F_{3,i+1}) + \frac{1}{4}Z(3,1)(M_{1,i} + M_{1,i+1}) + \frac{1}{4}Z(3,2)(M_{2,i} + M_{2,i+1}) \\ + \frac{1}{4}Z(3,3)(M_{3,i} + M_{3,i+1}) \quad (\text{A.21})$$

$$J_i(22) = -\frac{1}{4}(\kappa_{1,i} + \kappa_{1,i+1}) + \frac{1}{4}Z(1,1)(F_{1,i} + F_{1,i+1}) + \frac{1}{4}Z(2,1)(F_{2,i} + F_{2,i+1}) \\ + \frac{1}{4}Z(3,1)(F_{3,i} + F_{3,i+1}) + \frac{1}{4}T(1,1)(M_{1,i} + M_{1,i+1}) + \frac{1}{4}T(1,2)(M_{2,i} + M_{2,i+1}) \\ + \frac{1}{4}T(1,3)(M_{3,i} + M_{3,i+1}) \quad (\text{A.22})$$

$$J_i(23) = -\frac{1}{4}(\kappa_{2,i} + \kappa_{2,i+1}) + \frac{1}{4}Z(1,2)(F_{1,i} + F_{1,i+1}) + \frac{1}{4}Z(2,2)(F_{2,i} + F_{2,i+1}) \\ + \frac{1}{4}Z(3,2)(F_{3,i} + F_{3,i+1}) + \frac{1}{4}T(2,1)(M_{1,i} + M_{1,i+1}) + \frac{1}{4}T(2,2)(M_{2,i} + M_{2,i+1}) \\ + \frac{1}{4}T(2,3)(M_{3,i} + M_{3,i+1}) \quad (\text{A.23})$$

$$J_i(24) = -\frac{1}{4}(\kappa_{3,i} + \kappa_{3,i+1}) + \frac{1}{4}Z(1,3)(F_{1,i} + F_{1,i+1}) + \frac{1}{4}Z(2,3)(F_{2,i} + F_{2,i+1}) \\ + \frac{1}{4}Z(3,3)(F_{3,i} + F_{3,i+1}) + \frac{1}{4}T(3,1)(M_{1,i} + M_{1,i+1}) + \frac{1}{4}T(3,2)(M_{2,i} + M_{2,i+1}) \\ + \frac{1}{4}T(3,3)(M_{3,i} + M_{3,i+1}) \quad (\text{A.24})$$

$$A_i(1:24,1:3) = \begin{bmatrix}
\frac{-1}{2\Delta x} & \frac{-1}{16}(\kappa_{3,i+1} + 3\kappa_{3,i}) & \frac{1}{16}(\kappa_{2,i+1} + 3\kappa_{2,i}) \\
\frac{1}{16}(\kappa_{3,i+1} + 3\kappa_{3,i}) & \frac{-1}{2\Delta x} & \frac{-1}{16}(\kappa_{1,i+1} + 3\kappa_{1,i}) \\
\frac{-1}{16}(\kappa_{2,i+1} + 3\kappa_{2,i}) & \frac{1}{16}(\kappa_{1,i+1} + 3\kappa_{1,i}) & \frac{-1}{2\Delta x} \\
0 & \frac{-1}{8}(\gamma_{13,i+1} + 3\gamma_{13,i}) & \frac{1}{8}(\gamma_{12,i+1} + 3\gamma_{12,i}) \\
\frac{1}{8}(\gamma_{13,i+1} + 3\gamma_{13,i}) & 0 & \frac{-1}{4} - \frac{1}{16}(\gamma_{11,i+1} + 3\gamma_{11,i}) \\
\frac{-1}{8}(\gamma_{12,i+1} + 3\gamma_{12,i}) & \frac{1}{4} + \frac{1}{16}(\gamma_{11,i+1} + 3\gamma_{11,i}) & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\frac{-1}{4}R(1,1) & \frac{-1}{4}R(1,2) & \frac{-1}{4}R(1,3) \\
\frac{-1}{4}R(2,1) & \frac{-1}{4}R(2,2) & \frac{-1}{4}R(2,3) \\
\frac{-1}{4}R(3,1) & \frac{-1}{4}R(3,2) & \frac{-1}{4}R(3,3) \\
\frac{-1}{4}Z(1,1) & \frac{-1}{4}Z(2,1) & \frac{-1}{4}Z(3,1) \\
\frac{-1}{4}Z(1,2) & \frac{-1}{4}Z(2,2) & \frac{-1}{4}Z(3,2) \\
\frac{-1}{4}Z(1,3) & \frac{-1}{4}Z(2,3) & \frac{-1}{4}Z(3,3)
\end{bmatrix} \quad (A.25)$$

$$A_7(1:24,4:6) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{-1}{2\Delta x} & \frac{-1}{16}(\kappa_{3,i+1} + 3\kappa_{3,i}) & \frac{1}{16}(\kappa_{2,i+1} + 3\kappa_{2,i}) \\ \frac{1}{16}(\kappa_{3,i+1} + 3\kappa_{3,i}) & \frac{-1}{2\Delta x} & \frac{-1}{16}(\kappa_{1,i+1} + 3\kappa_{1,i}) \\ \frac{-1}{16}(\kappa_{2,i+1} + 3\kappa_{2,i}) & \frac{1}{16}(\kappa_{1,i+1} + 3\kappa_{1,i}) & \frac{-1}{2\Delta x} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{-1}{4}Z(1,1) & \frac{-1}{4}Z(1,2) & \frac{-1}{4}Z(1,3) \\ \frac{-1}{4}Z(2,1) & \frac{-1}{4}Z(2,2) & \frac{-1}{4}Z(2,3) \\ \frac{-1}{4}Z(3,1) & \frac{-1}{4}Z(3,2) & \frac{-1}{4}Z(3,3) \\ \frac{-1}{4}T(1,1) & \frac{-1}{4}T(1,2) & \frac{-1}{4}T(1,3) \\ \frac{-1}{4}T(2,1) & \frac{-1}{4}T(2,2) & \frac{-1}{4}T(2,3) \\ \frac{-1}{4}T(3,1) & \frac{-1}{4}T(3,2) & \frac{-1}{4}T(3,3) \end{bmatrix} \quad (\text{A.26})$$

$$A_i(1:24,7:9) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{-1}{16}(P_{3,i+1} + 3P_{3,i}) & \frac{1}{16}(P_{2,i+1} + 3P_{2,i}) \\ \frac{1}{16}(P_{3,i+1} + 3P_{3,i}) & 0 & \frac{-1}{16}(P_{1,i+1} + 3P_{1,i}) \\ \frac{-1}{16}(P_{2,i+1} + 3P_{2,i}) & \frac{1}{16}(P_{1,i+1} + 3P_{1,i}) & 0 \\ \frac{-1}{2\Delta x} & \frac{-1}{16}(\kappa_{3,i+1} + 3\kappa_{3,i}) & \frac{1}{16}(\kappa_{2,i+1} + 3\kappa_{2,i}) \\ \frac{1}{16}(\kappa_{3,i+1} + 3\kappa_{3,i}) & \frac{-1}{2\Delta x} & \frac{-1}{16}(\kappa_{1,i+1} + 3\kappa_{1,i}) \\ \frac{-1}{16}(\kappa_{2,i+1} + 3\kappa_{2,i}) & \frac{1}{16}(\kappa_{1,i+1} + 3\kappa_{1,i}) & \frac{-1}{2\Delta x} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{-1}{4}\mu & 0 & 0 \\ 0 & \frac{-1}{4}\mu & 0 \\ 0 & 0 & \frac{-1}{4}\mu \\ 0 & \frac{1}{4}\mu \bar{x}_3 & \frac{-1}{4}\mu \bar{x}_2 \\ \frac{-1}{4}\mu \bar{x}_3 & 0 & 0 \\ \frac{1}{4}\mu \bar{x}_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (\text{A.27})$$

$$A_i(1:24,10:12) = \begin{bmatrix} 0 & \frac{-1}{16}(P_{3,i+1} + 3P_{3,i}) & \frac{1}{16}(P_{2,i+1} + 3P_{2,i}) \\ \frac{1}{16}(P_{3,i+1} + 3P_{3,i}) & 0 & \frac{-1}{16}(P_{1,i+1} + 3P_{1,i}) \\ \frac{-1}{16}(P_{2,i+1} + 3P_{2,i}) & \frac{1}{16}(P_{1,i+1} + 3P_{1,i}) & 0 \\ 0 & \frac{-1}{16}(H_{3,i+1} + 3H_{3,i}) & \frac{1}{16}(H_{2,i+1} + 3H_{2,i}) \\ \frac{1}{16}(H_{3,i+1} + 3H_{3,i}) & 0 & \frac{-1}{16}(H_{1,i+1} + 3H_{1,i}) \\ \frac{-1}{16}(H_{2,i+1} + 3H_{2,i}) & \frac{1}{16}(H_{1,i+1} + 3H_{1,i}) & 0 \\ 0 & \frac{-1}{8}(\gamma_{13,i+1} + 3\gamma_{13,i}) & \frac{1}{8}(\gamma_{12,i+1} + 3\gamma_{12,i}) \\ \frac{1}{8}(\gamma_{13,i+1} + 3\gamma_{13,i}) & 0 & \frac{-1}{4} - \frac{1}{16}(\gamma_{11,i+1} + 3\gamma_{11,i}) \\ \frac{-1}{8}(\gamma_{12,i+1} + 3\gamma_{12,i}) & \frac{1}{4} + \frac{1}{16}(\gamma_{11,i+1} + 3\gamma_{11,i}) & 0 \\ \frac{-1}{2\Delta x} & \frac{-1}{16}(\kappa_{3,i+1} + 3\kappa_{3,i}) & \frac{1}{16}(\kappa_{2,i+1} + 3\kappa_{2,i}) \\ \frac{1}{16}(\kappa_{3,i+1} + 3\kappa_{3,i}) & \frac{-1}{2\Delta x} & \frac{-1}{16}(\kappa_{1,i+1} + 3\kappa_{1,i}) \\ \frac{-1}{16}(\kappa_{2,i+1} + 3\kappa_{2,i}) & \frac{1}{16}(\kappa_{1,i+1} + 3\kappa_{1,i}) & \frac{-1}{2\Delta x} \\ 0 & \frac{-1}{4}\mu\bar{x}_3 & \frac{1}{4}\mu\bar{x}_2 \\ \frac{1}{4}\mu\bar{x}_3 & 0 & 0 \\ \frac{-1}{4}\mu\bar{x}_2 & 0 & 0 \\ \frac{-1}{4}i(1,1) & \frac{-1}{4}i(1,2) & \frac{-1}{4}i(1,3) \\ \frac{-1}{4}i(2,1) & \frac{-1}{4}i(2,2) & \frac{-1}{4}i(2,3) \\ \frac{-1}{4}i(3,1) & \frac{-1}{4}i(3,2) & \frac{-1}{4}i(3,3) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (\text{A.28})$$

$$A_7(1:24,13:15) = \begin{bmatrix} \frac{-1}{2\Delta t} & \frac{1}{16}(\Omega_{3,i+1} + 3\Omega_{3,i}) & \frac{-1}{16}(\Omega_{2,i+1} + 3\Omega_{2,i}) \\ \frac{-1}{16}(\Omega_{3,i+1} + 3\Omega_{3,i}) & \frac{-1}{2\Delta t} & \frac{1}{16}(\Omega_{1,i+1} + 3\Omega_{1,i}) \\ \frac{1}{16}(\Omega_{2,i+1} + 3\Omega_{2,i}) & \frac{-1}{16}(\Omega_{1,i+1} + 3\Omega_{1,i}) & \frac{-1}{2\Delta t} \\ 0 & \frac{1}{16}(V_{3,i+1} + 3V_{3,i}) & \frac{-1}{16}(V_{2,i+1} + 3V_{2,i}) \\ \frac{-1}{16}(V_{3,i+1} + 3V_{3,i}) & 0 & \frac{1}{16}(V_{1,i+1} + 3V_{1,i}) \\ \frac{1}{16}(V_{2,i+1} + 3V_{2,i}) & \frac{-1}{16}(V_{1,i+1} + 3V_{1,i}) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (A.29)$$

$$A_7(1:24,16:18) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{-1}{2\Delta t} & \frac{1}{16}(\Omega_{3,i+1} + 3\Omega_{3,i}) & \frac{-1}{16}(\Omega_{2,i+1} + 3\Omega_{2,i}) \\ \frac{-1}{16}(\Omega_{3,i+1} + 3\Omega_{3,i}) & \frac{-1}{2\Delta t} & \frac{1}{16}(\Omega_{1,i+1} + 3\Omega_{1,i}) \\ \frac{1}{16}(\Omega_{2,i+1} + 3\Omega_{2,i}) & \frac{-1}{16}(\Omega_{1,i+1} + 3\Omega_{1,i}) & \frac{-1}{2\Delta t} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (\text{A.30})$$

(A.31)

$$A_i(1:24,22:24) = \begin{bmatrix} 0 & \frac{1}{16}(F_{3,i+1} + 3F_{3,i}) & \frac{-1}{16}(F_{2,i+1} + 3F_{2,i}) \\ \frac{-1}{16}(F_{3,i+1} + 3F_{3,i}) & 0 & \frac{1}{16}(F_{1,i+1} + 3F_{1,i}) \\ \frac{1}{16}(F_{2,i+1} + 3F_{2,i}) & \frac{-1}{16}(F_{1,i+1} + 3F_{1,i}) & 0 \\ 0 & \frac{1}{16}(M_{3,i+1} + 3M_{3,i}) & \frac{-1}{16}(M_{2,i+1} + 3M_{2,i}) \\ \frac{-1}{16}(M_{3,i+1} + 3M_{3,i}) & 0 & \frac{1}{16}(M_{1,i+1} + 3M_{1,i}) \\ \frac{1}{16}(M_{2,i+1} + 3M_{2,i}) & \frac{-1}{16}(M_{1,i+1} + 3M_{1,i}) & 0 \\ 0 & \frac{1}{16}(V_{3,i+1} + 3V_{3,i}) & \frac{-1}{16}(V_{2,i+1} + 3V_{2,i}) \\ \frac{-1}{16}(V_{3,i+1} + 3V_{3,i}) & 0 & \frac{1}{16}(V_{1,i+1} + 3V_{1,i}) \\ \frac{1}{16}(V_{2,i+1} + 3V_{2,i}) & \frac{-1}{16}(V_{1,i+1} + 3V_{1,i}) & 0 \\ \frac{-1}{2\Delta t} & \frac{1}{16}(\Omega_{3,i+1} + 3\Omega_{3,i}) & \frac{-1}{16}(\Omega_{2,i+1} + 3\Omega_{2,i}) \\ \frac{-1}{16}(\Omega_{3,i+1} + 3\Omega_{3,i}) & \frac{-1}{2\Delta t} & \frac{1}{16}(\Omega_{1,i+1} + 3\Omega_{1,i}) \\ \frac{1}{16}(\Omega_{2,i+1} + 3\Omega_{2,i}) & \frac{-1}{16}(\Omega_{1,i+1} + 3\Omega_{1,i}) & \frac{-1}{2\Delta t} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} \quad (\text{A.32})$$

$$B_i(1:24,1:3) = \begin{bmatrix} \frac{1}{24x} & \frac{-1}{16}(\kappa_{3,i+1} + 3\kappa_{3,i}) & \frac{1}{16}(\kappa_{2,i+1} + 3\kappa_{2,i}) \\ \frac{1}{16}(\kappa_{3,i+1} + 3\kappa_{3,i}) & \frac{1}{24x} & \frac{-1}{16}(\kappa_{1,i+1} + 3\kappa_{1,i}) \\ \frac{-1}{16}(\kappa_{2,i+1} + 3\kappa_{2,i}) & \frac{1}{16}(\kappa_{1,i+1} + 3\kappa_{1,i}) & \frac{1}{24x} \\ 0 & \frac{-1}{8}(\gamma_{13,i+1} + 3\gamma_{13,i}) & \frac{1}{8}(\gamma_{12,i+1} + 3\gamma_{12,i}) \\ \frac{1}{8}(\gamma_{13,i+1} + 3\gamma_{13,i}) & 0 & \frac{-1}{4} - \frac{1}{16}(\gamma_{11,i+1} + 3\gamma_{11,i}) \\ \frac{-1}{8}(\gamma_{12,i+1} + 3\gamma_{12,i}) & \frac{1}{4} + \frac{1}{16}(\gamma_{11,i+1} + 3\gamma_{11,i}) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{-1}{4}R(1,1) & \frac{-1}{4}R(1,2) & \frac{-1}{4}R(1,3) \\ \frac{-1}{4}R(2,1) & \frac{-1}{4}R(2,2) & \frac{-1}{4}R(2,3) \\ \frac{-1}{4}R(3,1) & \frac{-1}{4}R(3,2) & \frac{-1}{4}R(3,3) \\ \frac{-1}{4}Z(1,1) & \frac{-1}{4}Z(2,1) & \frac{-1}{4}Z(3,1) \\ \frac{-1}{4}Z(1,2) & \frac{-1}{4}Z(2,2) & \frac{-1}{4}Z(3,2) \\ \frac{-1}{4}Z(1,3) & \frac{-1}{4}Z(2,3) & \frac{-1}{4}Z(3,3) \end{bmatrix} \quad (\text{A.33})$$

$$B_i(1:24,4:6) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{2\Delta x} & \frac{-1}{16}(\kappa_{3,i+1} + 3\kappa_{3,i}) & \frac{1}{16}(\kappa_{2,i+1} + 3\kappa_{2,i}) \\ \frac{1}{16}(\kappa_{3,i+1} + 3\kappa_{3,i}) & \frac{1}{2\Delta x} & \frac{-1}{16}(\kappa_{1,i+1} + 3\kappa_{1,i}) \\ \frac{-1}{16}(\kappa_{2,i+1} + 3\kappa_{2,i}) & \frac{1}{16}(\kappa_{1,i+1} + 3\kappa_{1,i}) & \frac{1}{2\Delta x} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{-1}{4}Z(1,1) & \frac{-1}{4}Z(1,2) & \frac{-1}{4}Z(1,3) \\ \frac{-1}{4}Z(2,1) & \frac{-1}{4}Z(2,2) & \frac{-1}{4}Z(2,3) \\ \frac{-1}{4}Z(3,1) & \frac{-1}{4}Z(3,2) & \frac{-1}{4}Z(3,3) \\ \frac{-1}{4}T(1,1) & \frac{-1}{4}T(1,2) & \frac{-1}{4}T(1,3) \\ \frac{-1}{4}T(2,1) & \frac{-1}{4}T(2,2) & \frac{-1}{4}T(2,3) \\ \frac{-1}{4}T(3,1) & \frac{-1}{4}T(3,2) & \frac{-1}{4}T(3,3) \end{bmatrix} \quad (\text{A.34})$$

$$B_i(1:24,7:9) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{-1}{16}(P_{3,i+1} + 3P_{3,i}) & \frac{1}{16}(P_{2,i+1} + 3P_{2,i}) \\ \frac{1}{16}(P_{3,i+1} + 3P_{3,i}) & 0 & \frac{-1}{16}(P_{1,i+1} + 3P_{1,i}) \\ \frac{-1}{16}(P_{2,i+1} + 3P_{2,i}) & \frac{1}{16}(P_{1,i+1} + 3P_{1,i}) & 0 \\ \frac{1}{2\Delta x} & \frac{-1}{16}(\kappa_{3,i+1} + 3\kappa_{3,i}) & \frac{1}{16}(\kappa_{2,i+1} + 3\kappa_{2,i}) \\ \frac{1}{16}(\kappa_{3,i+1} + 3\kappa_{3,i}) & \frac{1}{2\Delta x} & \frac{-1}{16}(\kappa_{1,i+1} + 3\kappa_{1,i}) \\ \frac{-1}{16}(\kappa_{2,i+1} + 3\kappa_{2,i}) & \frac{1}{16}(\kappa_{1,i+1} + 3\kappa_{1,i}) & \frac{1}{2\Delta x} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{-1}{4}\mu & 0 & 0 \\ 0 & \frac{-1}{4}\mu & 0 \\ 0 & 0 & \frac{-1}{4}\mu \\ 0 & \frac{1}{4}\mu \bar{x}_3 & \frac{-1}{4}\mu \bar{x}_2 \\ \frac{-1}{4}\mu \bar{x}_3 & 0 & 0 \\ \frac{1}{4}\mu \bar{x}_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (\text{A.35})$$

$$B_i(1:24,10:12) = \begin{bmatrix} 0 & \frac{-1}{16}(P_{3,i+1} + 3P_{3,i}) & \frac{1}{16}(P_{2,i+1} + 3P_{2,i}) \\ \frac{1}{16}(P_{3,i+1} + 3P_{3,i}) & 0 & \frac{-1}{16}(P_{1,i+1} + 3P_{1,i}) \\ \frac{-1}{16}(P_{2,i+1} + 3P_{2,i}) & \frac{1}{16}(P_{1,i+1} + 3P_{1,i}) & 0 \\ 0 & \frac{-1}{16}(H_{3,i+1} + 3H_{3,i}) & \frac{1}{16}(H_{2,i+1} + 3H_{2,i}) \\ \frac{1}{16}(H_{3,i+1} + 3H_{3,i}) & 0 & \frac{-1}{16}(H_{1,i+1} + 3H_{1,i}) \\ \frac{-1}{16}(H_{2,i+1} + 3H_{2,i}) & \frac{1}{16}(H_{1,i+1} + 3H_{1,i}) & 0 \\ 0 & \frac{-1}{8}(\gamma_{13,i+1} + 3\gamma_{13,i}) & \frac{1}{8}(\gamma_{12,i+1} + 3\gamma_{12,i}) \\ \frac{1}{8}(\gamma_{13,i+1} + 3\gamma_{13,i}) & 0 & \frac{-1}{4} - \frac{1}{16}(\gamma_{11,i+1} + 3\gamma_{11,i}) \\ \frac{-1}{8}(\gamma_{12,i+1} + 3\gamma_{12,i}) & \frac{1}{4} + \frac{1}{16}(\gamma_{11,i+1} + 3\gamma_{11,i}) & 0 \\ \frac{1}{2\Delta x} & \frac{-1}{16}(\kappa_{3,i+1} + 3\kappa_{3,i}) & \frac{1}{16}(\kappa_{2,i+1} + 3\kappa_{2,i}) \\ \frac{1}{16}(\kappa_{3,i+1} + 3\kappa_{3,i}) & \frac{1}{2\Delta x} & \frac{-1}{16}(\kappa_{1,i+1} + 3\kappa_{1,i}) \\ \frac{-1}{16}(\kappa_{2,i+1} + 3\kappa_{2,i}) & \frac{1}{16}(\kappa_{1,i+1} + 3\kappa_{1,i}) & \frac{1}{2\Delta x} \\ 0 & \frac{-1}{4}\mu\bar{x}_3 & \frac{1}{4}\mu\bar{x}_2 \\ \frac{1}{4}\mu\bar{x}_3 & 0 & 0 \\ \frac{-1}{4}\mu\bar{x}_2 & 0 & 0 \\ \frac{-1}{4}i(1,1) & \frac{-1}{4}i(1,2) & \frac{-1}{4}i(1,3) \\ \frac{-1}{4}i(2,1) & \frac{-1}{4}i(2,2) & \frac{-1}{4}i(2,3) \\ \frac{-1}{4}i(3,1) & \frac{-1}{4}i(3,2) & \frac{-1}{4}i(3,3) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (\text{A.36})$$

$$B_i(1:24,13:15) = \begin{bmatrix} \frac{-1}{2\Delta t} & \frac{1}{16}(\Omega_{3,i+1} + 3\Omega_{3,i}) & \frac{-1}{16}(\Omega_{2,i+1} + 3\Omega_{2,i}) \\ \frac{-1}{16}(\Omega_{3,i+1} + 3\Omega_{3,i}) & \frac{-1}{2\Delta t} & \frac{1}{16}(\Omega_{1,i+1} + 3\Omega_{1,i}) \\ \frac{1}{16}(\Omega_{2,i+1} + 3\Omega_{2,i}) & \frac{-1}{16}(\Omega_{1,i+1} + 3\Omega_{1,i}) & \frac{-1}{2\Delta t} \\ 0 & \frac{1}{16}(V_{3,i+1} + 3V_{3,i}) & \frac{-1}{16}(V_{2,i+1} + 3V_{2,i}) \\ \frac{-1}{16}(V_{3,i+1} + 3V_{3,i}) & 0 & \frac{1}{16}(V_{1,i+1} + 3V_{1,i}) \\ \frac{1}{16}(V_{2,i+1} + 3V_{2,i}) & \frac{-1}{16}(V_{1,i+1} + 3V_{1,i}) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (\text{A.37})$$

$$B_i(1:24,19:21) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{16}(F_{3,i+1} + 3F_{3,i}) & \frac{-1}{16}(F_{2,i+1} + 3F_{2,i}) \\ \frac{-1}{16}(F_{3,i+1} + 3F_{3,i}) & 0 & \frac{1}{16}(F_{1,i+1} + 3F_{1,i}) \\ \frac{1}{16}(F_{2,i+1} + 3F_{2,i}) & \frac{-1}{16}(F_{1,i+1} + 3F_{1,i}) & 0 \\ \frac{-1}{24t} & \frac{1}{16}(\Omega_{3,i+1} + 3\Omega_{3,i}) & \frac{-1}{16}(\Omega_{2,i+1} + 3\Omega_{2,i}) \\ \frac{-1}{16}(\Omega_{3,i+1} + 3\Omega_{3,i}) & \frac{-1}{24t} & \frac{1}{16}(\Omega_{1,i+1} + 3\Omega_{1,i}) \\ \frac{1}{16}(\Omega_{2,i+1} + 3\Omega_{2,i}) & \frac{-1}{16}(\Omega_{1,i+1} + 3\Omega_{1,i}) & \frac{-1}{24t} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (\text{A.39})$$

$$\begin{aligned}
B_i(1:24,22:24) = & \begin{bmatrix}
0 & \frac{1}{16}(F_{3,i+1} + 3F_{3,i}) & \frac{-1}{16}(F_{2,i+1} + 3F_{2,i}) \\
\frac{-1}{16}(F_{3,i+1} + 3F_{3,i}) & 0 & \frac{1}{16}(F_{1,i+1} + 3F_{1,i}) \\
\frac{1}{16}(F_{2,i+1} + 3F_{2,i}) & \frac{-1}{16}(F_{1,i+1} + 3F_{1,i}) & 0 \\
0 & \frac{1}{16}(M_{3,i+1} + 3M_{3,i}) & \frac{-1}{16}(M_{2,i+1} + 3M_{2,i}) \\
\frac{-1}{16}(M_{3,i+1} + 3M_{3,i}) & 0 & \frac{1}{16}(M_{1,i+1} + 3M_{1,i}) \\
\frac{1}{16}(M_{2,i+1} + 3M_{2,i}) & \frac{-1}{16}(M_{1,i+1} + 3M_{1,i}) & 0 \\
0 & \frac{1}{16}(V_{3,i+1} + 3V_{3,i}) & \frac{-1}{16}(V_{2,i+1} + 3V_{2,i}) \\
\frac{-1}{16}(V_{3,i+1} + 3V_{3,i}) & 0 & \frac{1}{16}(V_{1,i+1} + 3V_{1,i}) \\
\frac{1}{16}(V_{2,i+1} + 3V_{2,i}) & \frac{-1}{16}(V_{1,i+1} + 3V_{1,i}) & 0 \\
\frac{-1}{24r} & \frac{1}{16}(\Omega_{3,i+1} + 3\Omega_{3,i}) & \frac{-1}{16}(\Omega_{2,i+1} + 3\Omega_{2,i}) \\
\frac{-1}{16}(\Omega_{3,i+1} + 3\Omega_{3,i}) & \frac{-1}{24r} & \frac{1}{16}(\Omega_{1,i+1} + 3\Omega_{1,i}) \\
\frac{1}{16}(\Omega_{2,i+1} + 3\Omega_{2,i}) & \frac{-1}{16}(\Omega_{1,i+1} + 3\Omega_{1,i}) & \frac{-1}{24r} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\frac{1}{4} & 0 & 0 \\
0 & \frac{1}{4} & 0 \\
0 & 0 & \frac{1}{4}
\end{bmatrix}
\end{aligned} \tag{A.40}$$