Two-Dimensional Boundary Element Thermoelastic Fracture Mechanics Analysis of Cracked Anisotropic Bodies

by

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a thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfillment of
the requirements for the degree of
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Ottawa-Carleton Institute for
Mechanical and Aerospace Engineering

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Thesis Supervisor

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ABSTRACT

This thesis presents a two-dimensional parametric study of the behavior of uniformly cooled homogeneous linear elastic anisotropic bodies containing cracks using the boundary element method (BEM). The present work investigates the effects of varying material properties, and varying orientation of these material properties, on the magnitude of the stress intensity factors (SIFs) of the cracked bodies. Five cracked plate geometries are considered in this study, namely: (1) a plate with an edge-crack, (2) a plate with a central-crack, (3) a plate with a double edge-crack, (4) a plate with an edge-crack emanating from a circular notch, and (5) a plate with symmetric cracks emanating from a central hole. Where appropriate, finite element method (FEM) analyses are also performed in order to validate the results of the BEM analysis. The results of this study show that, for all crack geometries, the mode-I stress intensity factor, K_I^* , decreases as the anisotropy of the material properties is increased. Additionally, for all these cases, K_I^* decreases as the angle of orientation of the material properties, θ , increases with respect to the horizontal axis. The results also show that BEM is an accurate and efficient method for two-dimensional thermoelastic fracture mechanics analysis of cracked anisotropic bodies.

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NOMENCLATURE

a	- crack length
a_{ij}	- elastic compliance coefficient
BEM	- boundary element method
BIE	- boundary integral equation
c_{ijkl}	- anisotropic material stiffness coefficients
DBEM	- dual boundary element method
E	- Young's modulus of an isotropic material
E_{11}	- Young's modulus in the 1 st principal axis direction
E_{22}	- Young's modulus in the 2 nd principal axis direction
ETM	- exact transformation method
F	- Airy's stress function
f_i	- body force per unit volume in isotropy
f_i'	- effective body force in anisotropy
FEM	- finite element method
G	- shear modulus of an isotropic material
G_{ij}	- shear modulus of an anisotropic material
H	- plate height
J	- Jacobian of transformation
k_{ij}	- anisotropic heat conductivity coefficients
K_I	- mode-I stress intensity factor

K_{IC}	- material fracture toughness
K_{II}	- mode-II stress intensity factor
l	- crack-tip element length
LEFM	- linear elastic fracture mechanics
m	- element number
n	- node number
n_i	- unit outward normal
N^c	- shape function
p, P	- source point
p^a	- a-th node of the BEM mesh
$p^{d(b,c)}$	- c-th node of the b-th element in the BEM mesh
PIM	- particular integral method
q, Q	- field point
r	- distance from crack tip
r_{ij}	- distance between p and q (or P and Q)
S	- BEM boundary
t_i	- tractions
T_{ij}	- traction fundamental solution
u_i	- displacements
U_{ij}	- displacement fundamental solution
W	- plate width
x	- distance from crack-tip in direction parallel to the crack face
x_i	- Cartesian coordinate

X_i	- body force per unit volume in anisotropy
X_i'	- effective body force in anisotropy
y	- distance from crack-tip in direction perpendicular to the crack face
z_i	- characteristic directions
Θ	- temperature
Ω	- BEM domain
α	- coefficient of linear expansion of an isotropic material
α_{ij}	- coefficient of linear expansion of an anisotropic material
δ_{ij}	- Kronecker delta
ε_{ij}	- engineering strain tensor
ζ	- intrinsic local coordinates of each element
$\eta_{jk,m}$	- coefficient of mutual influence of the first kind
$\eta_{m,jk}$	- coefficient of mutual influence of the second kind
θ	- angular position of r with respect to x
μ_i	- characteristic roots of an anisotropic material
ν	- Poisson's ratio of an isotropic material
ν_{ij}	- Poisson's ratio of an anisotropic material
σ_{ij}	- stress tensor
φ	- angle between the global Cartesian coordinate system and the local Cartesian coordinate system with origin at the crack-tip

CHAPTER 1

INTRODUCTION

Over the past couple of decades, anisotropic materials have become more commonly used in the aerospace industry as a primary structural material in applications such as satellites and aircraft. In the aerospace industry, a major design consideration is the presence and behavior of cracks. As a result, much research has been conducted in the past two decades to study cracked anisotropic structures subjected to various types of mechanical loads, and much is now known about their behavior. However, in many cases, these structures are also subjected to thermal loads. There has, however, been very little research done to determine the effects anisotropic material properties have on the fracture behavior of a cracked body subjected to a temperature change. The goal of the present study is to examine some of these effects. This thesis presents results of two-dimensional thermoelastic fracture mechanics analysis of cracked anisotropic bodies using the boundary element method (BEM), also often known as the boundary integral equation (BIE) method.*

The boundary element method is a numerical method which has been shown to be very well suited to fracture mechanics analysis due to its ability to model problems with

* In this thesis, the terms 'BEM' and 'BIE method' will be used interchangeably.

high stress gradients. This method also has the desirable characteristic that only the boundary of the solution domain needs to be modeled, as opposed to alternative numerical schemes such as the finite element method (FEM), which require modeling of the entire solution domain.

The early groundwork for linear elastic fracture mechanics (LEFM) analysis in plane anisotropy was performed by Sih, Paris, and Irwin (1965); Bowie and Freese (1972); and Ghandi (1972). Sih, Paris, and Irwin (1965) used the modified mapping collocation (MMC) method to express the stresses and displacements in the crack-tip vicinity in terms of the mode-I stress intensity factor (SIF) for a two-dimensional anisotropic body. Bowie and Freese (1972) used the MMC method to determine the mode-I stress intensity factor, K_I , for a plane orthotropic body with a central-crack, while Ghandi (1972) also used the same technique to analyze a plane orthotropic body with a slant-crack. Although Rizzo and Shippy (1970) had presented, for the first time, a BEM formulation for plane anisotropy, no fracture mechanics analysis was carried out at this time. Snyder and Cruse (1975) followed this groundwork with an alternative formulation for two-dimensional anisotropic fracture mechanics based on modified Green's functions. This new formulation had the distinct advantage that the crack itself did not need to be modeled. However, it was only applicable to straight cracks and relatively simple geometric configurations.

Modeling of cracks remained somewhat problematic in BEM analysis until Blandford, Ingrassia, and Liggett (1981) suggested the use of sub-regioning as a convenient technique to model the two crack surfaces. The use of sub-regioning meant

that any crack of arbitrary shape and orientation could be modeled with relative ease. The accuracy of this technique is improved through the use of quarter-point traction singular elements at the crack-tip as proposed by Cruse and Wilson (1977). These elements are capable of capturing the $O(\sqrt{r})$ behavior of the displacements, and the $O(1/\sqrt{r})$ behavior of the tractions which occur near the crack-tip, thus greatly improving the accuracy of the obtained stress intensity factors. Martinez and Dominguez (1984) subsequently derived formulas, using these elements, with which the SIFs can be accurately obtained from the results of the BEM analyses.

Most major work in the area of two-dimensional LEFM analysis of anisotropic bodies in more recent years has been split between two main groups, the dual boundary element method (DBEM) along with its various forms, and the sub-regioning approach. DBEM [see e.g. Sollero and Aliabadi (1995); Pan (1997)] is a single domain approach, which instead of sub-regioning the solution domain, models the tractions on one crack face, and displacements on the other, to avoiding discontinuities which arise here. The formulation for DBEM is complicated, but the ability to model cracks without sub-regioning makes this method very well suited to the study of crack propagation. For the straightforward determination of SIFs only, the sub-regioning, or multiple domain, approach is still the simplest. With this method the sub-regioned body is analyzed, and the stresses and displacements near the crack-tip can be used to determine the SIFs with the now well established ‘traction formula’ and/or ‘displacement formula’ [Tan and Gao (1992)]. It should be noted that other methods have been proposed for calculating SIFs

which may also be applicable to two-dimensional anisotropic LEFM, such as the J_k -integral [Portella, Aliabadi, and Rooke (1992); Sollero and Aliabadi (1993)], weight functions [Aliabadi, Cartwright, and Rooke (1989)], and a singularity subtraction technique [Portela, Aliabadi, and Rooke (1991)]. They are mathematically more involved and require significant modifications and implementation into a BEM computer code.

There has been relatively little amount of work done in the area of two-dimensional thermoelastic fracture mechanics analysis for anisotropic bodies using BEM. This is due to the presence of extra volume integrals in the integral equations which arise when thermal effects are present, and the subsequent difficulty in transforming these volume integrals into surface integrals. In elasticity analysis, thermal loads are often treated as ‘pseudo-body forces’ because their terms appear together with those for body forces in the governing differential equation [see e.g. Solkonikoff (1956)]. The presence of volume integrals, however, destroys the main feature of the boundary element method as a boundary solution method, and therefore, transformation of these domain integrals is necessary to restore the method to a truly ‘boundary-only’ technique. This transformation was first achieved for body force loading by Zhang, Tan, and Afagh (1996, 1997) when they developed the exact transformation method (ETM) for body forces in two-dimensional anisotropy in BEM, and applied the formulation to analyze crack problems. Note that body forces have also been treated earlier using the particular integral method (PIM) [Pape and Banerjee (1987)], but for anisotropy, the approach is an approximate one. Shiah and Tan (1997) subsequently developed a direct domain

mapping technique to solve two-dimensional anisotropic field problems in BEM. This work facilitated the transformation of the thermoelastic domain integral into surface integrals by Shiah and Tan (1999) when they developed the ETM for two-dimensional thermoelastic anisotropy in BEM. This formulation was extended to allow for determination of interior point stresses [Shiah and Tan (2000a)], and employed for fracture mechanics analysis [Shiah and Tan (2000b)]. Other attempts using BEM to deal with anisotropic thermoelasticity have been made by Deb, Henry, and Wilson (1991); and Deb and Banerjee (1991) using PIM, but this method is not general and has not been applied to crack problems.

This research presented in this thesis is performed using the boundary element code BIE2DET developed by Dr. C. L. Tan at Carleton University. This is a code developed for two-dimensional elastic stress analysis of anisotropic bodies based on the exact transformation method (ETM). It may be used for problems involving cracks, body forces, as well as thermal loads. The program is based on the quadratic isoparametric element formulation, and stress intensity factors (SIFs) may be directly obtained and printed using the ‘traction formula’ and the ‘displacement formula’. Although it is capable of analyzing problems with non-uniform temperature distributions, only the case of uniform temperature distribution was considered in this study. The code was used to investigate the effect of varying material properties and varying material orientation of these material properties on the value of the SIFs for cracked rectangular plates under mode-I conditions. Before proceeding with the BEM parametric study, where possible, the analysis was also performed using the finite element method (FEM) program,

ABAQUS, for a limited range of cases treated. This provides a check and validation of the BEM results.

Chapter 2 of this thesis presents an overview of some basic equations in isotropy for two-dimensional elasticity and thermoelasticity, before the same is presented for anisotropy in Chapter 3. In the latter chapter, the basic equations for linear elastic fracture mechanics in two-dimensional anisotropy are also presented. Chapter 4 describes the boundary integral equation (BIE) method for two-dimensional isotropic and anisotropic thermoelasticity. The scope of the parametric study which is the focus of the present research is provided in Chapter 5, where the physical problems considered, the material parameters used, and their range are all defined. Numerical results are discussed in Chapter 6, and a summary and conclusions for the present investigation are presented in Chapter 7, together with some recommendations for future work.

CHAPTER 2

REVIEW OF SOME BASIC EQUATIONS IN ISOTROPY

This chapter details some basic equations in two-dimensional isotropic elasticity with mechanical loading, followed by those for thermoelasticity. These equations [see e.g. Solkonikoff (1956)] should help the reader already familiar with basic two-dimensional isotropy to better follow the developments in two-dimensional anisotropy presented in the next chapter.

2.1 Two-Dimensional Isotropic Elasticity

The governing differential equation for a two-dimensional isotropic linear elastic body in equilibrium is,

$$\sigma_{ij,j} + f_i = 0 \quad (2.1)$$

where σ_{ij} are the stress components in the Cartesian coordinate system, and f_i are the components of the body force per unit volume.

Assuming small strains, the corresponding strain-displacement relation can be written as,

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad (2.2)$$

where ε_{ij} are the engineering strains, and u_i are the displacements.

The elastic stresses can be directly related to strains via a constitutive relation, namely Hooke's Law. For plane stress, it can be written as,

$$\varepsilon_{ij} = \frac{1}{2G} \left[\sigma_{ij} - \frac{\nu}{1+\nu} \sigma_{kk} \delta_{ij} \right] \quad (2.3)$$

or, alternatively, in terms of displacements, u_i , as,

$$\sigma_{ij} = G \left[(u_{i,j} + u_{j,i}) + \delta_{ij} \frac{2\nu}{1-2\nu} u_{k,k} \right] \quad (2.4)$$

where G is the shear modulus, ν is the Poisson's ratio for the material, and δ_{ij} is the Kronecker delta, defined as follows,

$$\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} \quad (2.5)$$

In the case of plane strain, the quantity ν is substituted for ν^* where,

$$\nu^* = \frac{\nu}{1-\nu} \quad (2.6)$$

For isotropy, the shear modulus is also related to Young's Modulus E as,

$$G = \frac{E}{2(1+\nu)} \quad (2.7)$$

Substituting Hooke's law, equation (2.3), into the strain-displacement relation, equation (3.2), and using the equilibrium equation, equation (3.1), results in Navier's equations of equilibrium in terms of displacements,

$$G \left[u_{i,jj} + \frac{1}{1-2\nu'} u_{j,ji} \right] + f_i = 0 \quad (2.8)$$

where,

$$\nu' = \begin{cases} \left(\frac{\nu}{1+\nu} \right) & \text{for plane stress} \\ \nu & \text{for plane strain} \end{cases} \quad (2.9)$$

2.2 Two-Dimensional Isotropic Thermoelasticity

In the presence of a temperature field, equations (2.2), (2.4), and (2.8) become,

$$\varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) + \alpha \Theta \delta_{ij} \quad (2.10)$$

$$\sigma_{ij} = G \left[(u_{i,j} + u_{j,i}) + \delta_{ij} \frac{2\nu}{1-2\nu} u_{k,k} - \delta_{ij} \frac{2}{1-2\nu} \alpha \Theta \right] \quad (2.11)$$

and,

$$G \left[u_{i,jj} + \frac{1}{1-2\nu} u_{j,ji} \right] + f_i - G \frac{2}{1-2\nu} \alpha \Theta_{,i} = 0 \quad (2.12)$$

where Θ is the temperature. These equations are also valid for plane strain provided that α is replaced by α' where,

$$\alpha' = \alpha (1+\nu) \quad (2.13)$$

By comparing equations (2.8) and (2.12) it is evident that the effect of the temperature change is similar to that of a body force. Therefore, the term due to the

temperature change can be lumped together with the body force term, f_i , to create an ‘effective body force’ term, f'_i , where,

$$f'_i = f_i - \gamma \Theta_{,i} \quad (2.14)$$

and,

$$\gamma = \frac{2G \alpha}{1-2\nu} \quad (2.15)$$

It is important to note that before solving the thermoelastic problem, the temperature field, $\Theta(Q)$, must first be solved for. Steady state heat conduction for an isotropy body with constant conductivity in the absence of heat sources or sinks is governed by Laplace’s equation,

$$\Theta_{,ii} = 0 \quad (2.16)$$

where Θ is the temperature. The solution to a boundary value problem governed by equation (2.16) may be obtained analytically or numerically. This will not be discussed further as it is not within the scope of the present study.

CHAPTER 3

REVIEW OF SOME BASIC EQUATIONS IN ANISOTROPY

This chapter introduces some basic equations for two-dimensional anisotropy. It begins by discussing elasticity under mechanical loads, leading then into thermoelasticity in the subsequent section. This is then followed by some basic equations for anisotropic linear elastic fracture mechanics (LEFM), where the stress and displacement fields in the vicinity of a crack-tip in a plane anisotropic body are presented in terms of stress intensity factors.

3.1 Two-Dimensional Anisotropic Elasticity

The relationship between the strains and stresses for a homogeneous fully anisotropic elastic material in plane stress, under mechanical loading, is defined by the matrix equation,

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{16} \\ a_{12} & a_{22} & a_{26} \\ a_{16} & a_{26} & a_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} \quad (3.1)$$

where ϵ_{jk} and σ_{jk} are the engineering strains and stresses respectively, and a_{mn} are the compliance coefficients of the material. The compliance coefficients are related to the material properties as follows [Lekhnitskii (1963)],

$$\begin{aligned} a_{11} &= \frac{1}{E_1} & a_{16} &= \frac{\eta_{12,1}}{E_1} = \frac{\eta_{11,12}}{G_{12}} \\ a_{22} &= \frac{1}{E_2} & a_{26} &= \frac{\eta_{12,2}}{E_2} = \frac{\eta_{2,12}}{G_{12}} \\ a_{12} &= \frac{-\nu_{12}}{E_1} = \frac{-\nu_{21}}{E_2} & a_{66} &= \frac{1}{G_{12}} \end{aligned} \quad (3.2)$$

where E_k is the Young's modulus in the x_k -direction, G_{12} is the shear modulus in the principal x_1 - x_2 plane, ν_{jk} is the Poisson's ratio defined as the extensional strain in the x_k -direction due to a unit compressive strain in the x_j -direction, and $\eta_{jk,m}$ and $\eta_{m,jk}$ are the coefficients of mutual influence of the first and second kind respectively. Both mutual coefficients are zero in the case of special orthotropy.

For a two-dimensional anisotropic elastic body under external loads, Lekhnitskii (1963) has reformulated the boundary value problem in elasticity in terms of Airy's stress function, F , and has shown that for stable equilibrium, the following equation must be satisfied,

$$a_{22} \frac{\partial^4 F}{\partial \xi_1^4} - 2a_{26} \frac{\partial^4 F}{\partial \xi_1^3 \partial \xi_2} + (2a_{12} + a_{66}) \frac{\partial^4 F}{\partial \xi_1^2 \partial \xi_2^2} - 2a_{16} \frac{\partial^4 F}{\partial \xi_1 \partial \xi_2^3} + a_{11} \frac{\partial^4 F}{\partial \xi_2^4} = 0 \quad (3.3)$$

where, ξ_i are principal coordinates for the specified problem. The corresponding characteristic equation for equation (3.3) can be expressed as,

$$a_{22} - 2a_{26}\mu + (2a_{12} + a_{66})\mu^2 - 2a_{16}\mu^3 + a_{11}\mu^4 = 0 \quad (3.4)$$

For anisotropic materials, he showed that the four roots of this characteristic equation are distinct and always complex. Also, because the compliance coefficients are all real constants, the roots of the equation are complex conjugate pairs, denoted as,

$$\mu_1, \bar{\mu}_1; \mu_2, \bar{\mu}_2$$

These roots are used to define characteristic directions, expressed as generalized complex variables such that,

$$z_i = (x_1 - x_{p1}) + \mu_i(x_2 - x_{p2}) \quad i=1,2 \quad (3.5)$$

where x_1 and x_2 are the coordinates of the field point $q(x_1, x_2)$, and x_{p1} and x_{p2} are the coordinates of the load or source point $p(x_{p1}, x_{p2})$. The characteristic directions are then used in the fundamental solutions for two-dimensional anisotropic elasticity. The fundamental solutions correspond to the simplest conceivable solutions to the governing differential equation. They are an essential item for the derivation of the boundary integral equation, which will be discussed in Chapter 4.

3.2 Two-Dimensional Anisotropic Thermoelasticity

So far, we have dealt with the behavior of a two-dimensional homogeneous anisotropic elastic solid under mechanical loading. If thermal effects are present, the

stress-strain relationship for a two-dimensional homogeneous anisotropic elastic solid is now governed by the Duhamel-Neumann relation,

$$\sigma_{ij} = c_{ijkl} \varepsilon_{kl} - \gamma_{ij} \Theta \quad i, j, k, l = 1, 2 \quad (3.6)$$

where Θ is the temperature change, c_{ijkl} are the material stiffness coefficients, and,

$$\gamma_{ij} = c_{ijkl} \alpha_{kl} \quad (3.7)$$

and α_{kl} are the coefficients of linear expansion.

For static thermoelasticity, Navier's equations of equilibrium in terms of displacements is,

$$c_{ijkl} u_{k,lj} + X_i - \gamma_{ij} \Theta_{,j} = 0 \quad (3.8a)$$

or

$$c_{ijkl} u_{k,lj} + X_i' = 0 \quad (3.8b)$$

where X_i is the body force, and X_i' is the 'effective body force' including thermal effects.

Assuming a steady-state, uniformly distributed heat source over the body domain, the temperature distribution must satisfy the heat conduction equation,

$$k_{ij} \Theta_{,ij} = C_0 \quad i, j = 1, 2 \quad (3.9)$$

where C_0 is the heat source term and k_{ij} are the heat conductivity coefficients. From thermodynamics considerations and Onsager's reciprocity relation, these coefficients must satisfy the following conditions,

$$\begin{aligned} k_{11} &> 0 & k_{22} &> 0 \\ k_{12} &= k_{21} & k_{11}k_{22} - k_{12}^2 &= \Delta > 0 \end{aligned} \quad (3.10)$$

3.3 Stress and Displacement Fields Near a Crack-Tip

Linear elastic fracture mechanics (LEFM) involves the elastic stress analysis of a cracked body (see Figure 3.1). The primary goal of linear elastic fracture mechanics (LEFM) analysis is to obtain the stress intensity factors for the cracked body. These parameters characterize the stress and displacement distributions in the vicinity of the crack-tip. For two-dimensional fracture problems, the main concern is usually the opening mode, or mode-I, stress intensity factor, K_I , and the sliding mode, or mode-II, stress intensity factor, K_{II} (see Figure 3.2). Once these parameters are known, they can be compared with the fracture toughness of the material, K_{IC} or K_{IIC} , to determine, for example, if the body will fail catastrophically.

For mode-I loading, the stresses, σ_{ij} , and displacements, u_i , near the crack-tip of a two-dimensional homogeneous anisotropic elastic body are [Sih, Paris, and Irwin (1965)],

$$\sigma_{11}^I = \frac{K_I}{\sqrt{2\pi r}} \Re \left[\frac{\mu_2 \mu_1}{\mu_1 - \mu_2} \left(\frac{\mu_2}{\sqrt{\cos\theta + \mu_2 \sin\theta}} - \frac{\mu_1}{\sqrt{\cos\theta + \mu_1 \sin\theta}} \right) \right] \quad (3.11)$$

$$\sigma_{22}^I = \frac{K_I}{\sqrt{2\pi r}} \Re \left[\frac{1}{\mu_1 - \mu_2} \left(\frac{\mu_1}{\sqrt{\cos\theta + \mu_2 \sin\theta}} - \frac{\mu_2}{\sqrt{\cos\theta + \mu_1 \sin\theta}} \right) \right] \quad (3.12)$$

$$\sigma_{12}^I = \frac{K_I}{\sqrt{2\pi r}} \Re \left[\frac{\mu_1 \mu_2}{\mu_1 - \mu_2} \left(\frac{1}{\sqrt{\cos\theta + \mu_1 \sin\theta}} - \frac{1}{\sqrt{\cos\theta + \mu_2 \sin\theta}} \right) \right] \quad (3.13)$$

and,

$$u_1^I = K_I \sqrt{\frac{2r}{\pi}} \Re \left[\frac{1}{\mu_1 - \mu_2} (\mu_1 r_{12} \sqrt{\cos\theta + \mu_2 \sin\theta} - \mu_2 r_{11} \sqrt{\cos\theta + \mu_1 \sin\theta}) \right] \quad (3.14)$$

$$u_2^I = K_I \sqrt{\frac{2r}{\pi}} \Re \left[\frac{1}{\mu_1 - \mu_2} (\mu_1 r_{22} \sqrt{\cos\theta + \mu_2 \sin\theta} - \mu_2 r_{21} \sqrt{\cos\theta + \mu_1 \sin\theta}) \right] \quad (3.15)$$

In equations (3.11) to (3.15) $\Re[\cdot]$ indicates the real part of the quantities in parentheses, and,

$$r_{j1} = a_{11} \mu_j^2 + a_{12} - a_{16} \mu_j \quad (3.16a)$$

$$r_{j2} = a_{12} \mu_j + \frac{a_{22}}{\mu_j} - a_{26} \quad (3.16b)$$

For mode-II conditions, the corresponding stress and displacement distributions are,

$$\sigma_{11}^{\text{II}} = \frac{K_{\text{II}}}{\sqrt{2\pi r}} \Re \left[\frac{1}{\mu_1 - \mu_2} \left(\frac{\mu_2^2}{\sqrt{\cos\theta + \mu_2 \sin\theta}} - \frac{\mu_1^2}{\sqrt{\cos\theta + \mu_1 \sin\theta}} \right) \right] \quad (3.17)$$

$$\sigma_{22}^{\text{II}} = \frac{K_{\text{II}}}{\sqrt{2\pi r}} \Re \left[\frac{1}{\mu_1 - \mu_2} \left(\frac{1}{\sqrt{\cos\theta + \mu_2 \sin\theta}} - \frac{1}{\sqrt{\cos\theta + \mu_1 \sin\theta}} \right) \right] \quad (3.18)$$

$$\sigma_{12}^{\text{II}} = \frac{K_{\text{II}}}{\sqrt{2\pi r}} \Re \left[\frac{1}{\mu_1 - \mu_2} \left(\frac{\mu_1}{\sqrt{\cos\theta + \mu_1 \sin\theta}} - \frac{\mu_2}{\sqrt{\cos\theta + \mu_2 \sin\theta}} \right) \right] \quad (3.19)$$

and,

$$u_1^{\text{II}} = K_{\text{II}} \sqrt{\frac{2r}{\pi}} \Re \left[\frac{1}{\mu_1 - \mu_2} (\mu_1 r_{12} \sqrt{\cos\theta + \mu_2 \sin\theta} - \mu_2 r_{11} \sqrt{\cos\theta + \mu_1 \sin\theta}) \right] \quad (3.20)$$

$$u_2^{\text{II}} = K_{\text{II}} \sqrt{\frac{2r}{\pi}} \Re \left[\frac{1}{\mu_1 - \mu_2} (r_{22} \sqrt{\cos\theta + \mu_2 \sin\theta} - r_{21} \sqrt{\cos\theta + \mu_1 \sin\theta}) \right] \quad (3.21)$$

Mixed mode stresses and displacements are obtained by superposition of the primary modes, i.e.,

$$\sigma_{ij} = \sigma_{ij}^{\text{I}} + \sigma_{ij}^{\text{II}} \quad i, j=1,2 \quad (3.22)$$

and,

$$u_j = u_j^{\text{I}} + u_j^{\text{II}} \quad (3.23)$$

The stresses ahead of the crack tip and in the plane of the crack ($\theta=0^\circ$) for an anisotropic body have been shown [Sih and Liebowitz (1968)] to be the same as those for an isotropic body, namely,

$$\sigma_{yy} = \frac{K_{\text{I}}}{\sqrt{2\pi r}} \quad \sigma_{xy} = \frac{K_{\text{II}}}{\sqrt{2\pi r}} \quad (3.24)$$

However, the displacements near the crack tip on the crack face ($\theta=180^\circ$) are relatively more complicated, and may be explicitly written as [Tan and Gao (1992)],

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \sqrt{\frac{2r}{\pi}} \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{Bmatrix} K_{\text{I}} \\ K_{\text{II}} \end{Bmatrix} \quad (3.25)$$

where,

$$\begin{aligned} D_{11} &= \Im \left[\frac{s_2 p_1 - s_1 p_2}{s_1 - s_2} \right] & D_{12} &= \Im \left[\frac{p_1 - p_2}{s_1 - s_2} \right] \\ D_{21} &= \Im \left[\frac{s_2 q_1 - s_1 q_2}{s_1 - s_2} \right] & D_{22} &= \Im \left[\frac{q_1 - q_2}{s_1 - s_2} \right] \end{aligned} \quad (3.26)$$

In equation (3.26), $\Im[\cdot]$ indicates the imaginary part of the quantity within parentheses.

The quantities p_j and q_j are defined as,

$$p_j = a'_{11}s_j^2 + a'_{12} - a'_{16}s_j \quad (3.27)$$

$$q_j = \frac{a'_{12}s_j^2 + a'_{22} - a'_{26}s_j}{s_j} \quad (3.28)$$

where s_j are material property parameters related to the roots of the characteristic equation as follows [Lekhnitskii (1963)],

$$s_j = \frac{\mu_j \cos \varphi + \sin \varphi}{\cos \varphi - \mu_j \sin \varphi} \quad (3.29)$$

and a'_{jk} are the compliance constants resulting from the transformation of a_{jk} from the global Cartesian coordinate system to the local Cartesian coordinate system with origin at the crack-tip (see Figure 3.1), φ is the angle between these two coordinate systems. The compliance constants, a'_{jk} , are related to a_{jk} and φ as follows,

$$\begin{aligned} a'_{11} = & a_{11} \cos^4 \varphi + (2a_{12} + a_{66}) \sin^2 \varphi \cos^2 \varphi \\ & + a_{22} \sin^4 \varphi + (a_{16} \cos^2 \varphi + a_{26} \sin^2 \varphi) \sin 2\varphi \end{aligned} \quad (3.30a)$$

$$\begin{aligned} a'_{22} = & a_{11} \sin^4 \varphi + (2a_{12} + a_{66}) \sin^2 \varphi \cos^2 \varphi \\ & + a_{22} \cos^4 \varphi - (a_{16} \cos^2 \varphi + a_{26} \sin^2 \varphi) \sin 2\varphi \end{aligned} \quad (3.30b)$$

$$\begin{aligned} a'_{12} = & a_{12} + (a_{11} + a_{22} - 2a_{12} - a_{66}) \sin^2 \varphi \cos^2 \varphi \\ & + \frac{1}{2}(a_{26} - a_{16}) \sin 2\varphi \cos 2\varphi \end{aligned} \quad (3.30c)$$

$$\begin{aligned} a'_{66} = & a_{66} + 2(a_{11} + a_{22} - 2a_{12} - a_{66}) \sin^2 \varphi \cos^2 \varphi \\ & + 2(a_{26} - a_{16}) \sin 2\varphi \cos 2\varphi \end{aligned} \quad (3.30d)$$

$$\begin{aligned}
a'_{16} = & [a_{22} \sin^2 \varphi - a_{11} \cos^2 \varphi \\
& + \frac{1}{2}(2a_{12} + a_{66}) \cos 2\varphi] \sin 2\varphi \\
& + (a_{16} \cos^2 \varphi)(\cos^2 \varphi + 3 \sin^2 \varphi) \\
& + (a_{26} \sin^2 \varphi)(3 \cos^2 \varphi - \sin^2 \varphi)
\end{aligned} \tag{3.30e}$$

$$\begin{aligned}
a'_{26} = & [a_{22} \sin^2 \varphi - a_{11} \cos^2 \varphi \\
& - \frac{1}{2}(2a_{12} + a_{66}) \cos 2\varphi] \sin 2\varphi \\
& + (a_{16} \sin^2 \varphi)(\cos^2 \varphi - \sin^2 \varphi) \\
& + (a_{26} \cos^2 \varphi)(\cos^2 \varphi - 3 \sin^2 \varphi)
\end{aligned} \tag{3.30f}$$

3.4 Closing Remarks

Numerical stress analysis is usually required to obtain the stress and displacement fields in a cracked body subjected to mechanical and/or thermal loads. From the relationships shown in the previous section relating the stress intensity factors to these fields, these stress intensity factors may be obtained. An efficient computational technique to this end is the boundary integral equation (BIE) method, also known as the boundary element method (BEM). The equations presented here, as well as the previous chapter, are used as a foundation for the development of the boundary integral equations, which are the basis of the BIE method. This will be discussed in the next chapter.

CHAPTER 4

REVIEW OF THE BOUNDARY INTEGRAL EQUATION METHOD

The boundary integral equation (BIE) method is an efficient alternative to the finite element method (FEM) for linear elastic stress analysis. The differences between the two methods arise from the fact that FEM finds its basis in partial differential equations, whereas the BIE method, as the name suggests, is based on integral equations. Although mathematically more complex, the BIE method offers the distinct advantage that only the boundary of the problem domain needs to be numerically modeled and analyzed. By requiring only boundary discretization, problems can be modeled with significantly less computing resources, and much simpler meshing and data preparation. These factors, along with the ability to resolve very high stress gradients with fewer elements, have made BEM popular for the study of fracture mechanics problems.

This chapter presents the direct formulation of the boundary integral equation method for both isotropic and anisotropic two-dimensional homogeneous elastic bodies. Analytical and numerical formulations will be developed for isotropic and anisotropic two-dimensional thermoelasticity. The chapter ends with the development of the BIE method for linear elastic fracture mechanics (LEFM) analysis.

4.1 Two-Dimensional Isotropic Thermoelasticity

Before dealing with the BIE method for anisotropy, it is perhaps expedient to start with the development of the technique for isotropy and then extend it after that to anisotropy. In what follows, lower case p and q are used to denote points within the domain, Ω , and upper case P and Q are used to denote points on the boundary, S (see Figure 4.1).

4.1.1 Analytical Formulation

One of the essential items in the formulation of the boundary integral equation method, are the fundamental solutions. A fundamental solution corresponds to the simplest conceivable non-trivial solution to the governing differential equation of a given problem. For the problem of a two-dimensional homogeneous isotropic elastic body subjected to external loads, the displacement fundamental solution for Navier's equations corresponds to Kelvin's problem in elasticity, and may be written as [see e.g. Tan (1987); Cruse (1988)],

$$U_{ji}(p,q) = \frac{1}{8\pi G(1-\nu)} \left[(3-4\nu)\delta_{ij} \log \frac{1}{r} + r_i r_j \right] \quad (4.1)$$

The corresponding traction fundamental solution may be written as,

$$T_{ji}(p, q) = \frac{-(1-2\nu)}{4\pi(1-\nu)} \frac{1}{r} \left\{ \frac{dr}{dn} \left[\delta_{ij} + \frac{2}{1-2\nu} r_i r_j \right] + r_i n_j - r_j n_i \right\} \quad (4.2)$$

In equations (4.1) and (4.2), $U_{ji}(p, q)$ and $T_{ji}(p, q)$ represent the displacements and tractions, respectively, in the x_i -direction at field point q , due to a unit load in the x_j -direction at source point p ; r is the distance between p and q ; and n_i is the unit outward normal at q .

The basis for the direct formulation of the boundary integral equation (BIE) for elasticity is Betti-Rayleigh's reciprocal work theorem. For an isotropic body subjected to tractions and body forces, the theorem can be expressed as follows [see e.g. Tan (1987)]

$$\oint_S t_i u_i' dS + \int_{\Omega} f_i u_i' d\Omega = \oint_S t_i' u_i dS + \int_{\Omega} f_i' u_i d\Omega \quad (4.3)$$

This equation states that if there is an elastic body subjected to two separate systems of forces, then the work that would be done by the first system of forces t_i, f_i in acting through the displacements, u_i' , due to the second system of forces, is the same as the work that would be done by the second system of forces t_i', f_i' in acting through the displacements, u_i , due to the first system of forces.

Now, if it is assumed that one of these systems corresponds to the unknown boundary tractions and displacements $t_i(P), u_i(P)$ and the other system corresponds to the known fundamental solutions $T_{ji}(P, Q), U_{ji}(P, Q)$ then,

$$\oint_S t_i(Q) U_{ji}(P, Q) dS + \int_{\Omega} f_i(Q) U_{ji}(P, q) d\Omega$$

$$= \oint_S u_i(Q) T_{ji}(P, Q) dS \quad (4.4)$$

However, due to the singular nature of the fundamental solutions, the kernels in this equation are discontinuous when $P=Q$, since $r=0$. A limiting process needs to be followed [see e.g. Tan (1987)], and the resulting integral equation for isotropic elastostatics becomes,

$$\begin{aligned} C_{ji}(P)u_i(P) + \oint_S u_i(Q)T_{ji}(P, Q)dS \\ = \oint_S t_i(Q)U_{ji}(P, Q)dS + \int_{\Omega} f_i(q)U_{ji}(p, q)d\Omega \end{aligned} \quad (4.5)$$

If thermal effects are present, it was shown in Chapter 2, equation (2.14), that thermal effects can be lumped into an ‘effective body force’ term, f_i' . Thus, simply by substituting f_i by f_i' in equation (4.5), the resulting integral equation for two-dimensional isotropic thermoelasticity becomes,

$$\begin{aligned} C_{ji}(P)u_i(P) + \oint_S u_i(Q)T_{ji}(P, Q)dS \\ = \oint_S t_i(Q)U_{ji}(P, Q)dS + \int_{\Omega} f_i'(q)U_{ji}(p, q)d\Omega \end{aligned} \quad (4.6)$$

The presence of a domain integral in equation (4.6) is undesirable since it prevents the solution process from being one of a true boundary solution technique. Fortunately, for isotropy, the process of converting the domain integral in equations (4.5) and (4.6) is feasible and well established [see e.g. Tan (1987)]. The resulting BIE for thermoelasticity in the absence of body forces is,

$$\begin{aligned}
& C_{ji}(P)u_i(P) + \oint_S u_i(Q)T_{ji}(P, Q)dS \\
& = \oint_S [t_i(Q) - n_i(Q)\gamma\Theta(Q)]U_{ji}(P, Q)dS \\
& \quad - \oint_S \frac{1-2\nu}{16\pi(1-\nu)G} [n_j(2\log r - 1) + 2r_{,j} \frac{dr}{dn}] \gamma\Theta(Q) dS \\
& \quad + \oint_S \frac{1-2\nu}{16\pi(1-\nu)G} [r r_{,j}(2\log r - 1)] \frac{\gamma d\Theta(Q)}{dn} dS
\end{aligned} \tag{4.7}$$

4.1.2 Numerical Implementation

In general, it is extremely difficult, or even impossible, to analytically solve the BIE, equation (4.7), shown in the previous section. Therefore, a numerical scheme must be employed for the problem solution. The first step of the numerical scheme involves discretization of the domain boundary, S , into a ‘mesh’ of M elements and N nodes (see Figure 4.2). The most commonly used and versatile treatment for two-dimensional elasticity problems involves the use of quadratic isoparametric line elements [see e.g. Tan and Gao (1992)]. With these elements (see Figure 4.3), both the geometry and primary solution variables are assumed to vary in a quadratic manner over each three-noded element, and both these variations are defined in terms of nodal values and the shape functions, $N^c(\zeta)$. These shape functions are defined as (see Figure 4.4),

$$N^1(\zeta) = -\frac{1}{2}\zeta(1-\zeta) \tag{4.8a}$$

$$N^2(\zeta) = (1 - \zeta^2) \quad (4.8b)$$

$$N^3(\zeta) = \frac{1}{2}\zeta(1 + \zeta) \quad (4.8c)$$

where ζ is the intrinsic or local coordinate of the element. The values of x_j , u_j , t_j , Θ , and $d\Theta/dn$ at any point on the element may be expressed in terms of the shape functions and the nodal values as follows,

$$x_j(\zeta) = N^c(\zeta)x_j^c \quad c=1,3 \quad (4.9)$$

$$u_j(\zeta) = N^c(\zeta)u_j^c \quad (4.10)$$

$$t_j(\zeta) = N^c(\zeta)t_j^c \quad (4.11)$$

$$\Theta(\zeta) = N^c(\zeta)\Theta^c \quad (4.12)$$

$$\frac{d\Theta}{dn}(\zeta) = N^c(\zeta)\frac{d\Theta^c}{dn} \quad (4.13)$$

Substitution of equations (4.8) to (4.13) into equation (4.7) results in the discretized BIE for thermoelasticity, as follows [see e.g. Tan (1987)],

$$\begin{aligned} & C_{jk}(P^a)u_k(P^a) \\ & + \sum_{b=1}^M \sum_{c=1}^3 u_k(P^{d(b,c)}) \oint_{S_b} T_{jk}(P^a, Q) N^c(\zeta) J(\zeta) dS \\ & = \sum_{b=1}^M \sum_{c=1}^3 \left[t_k(P^{d(b,c)}) - n_k \Theta(P^{d(b,c)}) \right] \oint_{S_b} U_{jk}(P^a, Q) N^c(\zeta) J(\zeta) dS \\ & - \sum_{b=1}^M \sum_{c=1}^3 \Theta(P^{d(b,c)}) \oint_{S_b} A_{jk}(P^a, Q) N^c(\zeta) J(\zeta) dS \end{aligned}$$

$$+ \sum_{b=1}^M \sum_{c=1}^3 \Psi(P^{d(b,c)}) \oint_{S_b} B_j(P^a, Q) N^c(\zeta) J(\zeta) dS \quad (4.14)$$

where $P^a=1, N$ is the a -th node of the mesh, $P^{d(b,c)}$ is the c -th node of the b -th element, and $J(\zeta)$ is the Jacobian of transformation, defined as,

$$|J(\zeta)| = \frac{dS}{d\zeta} \quad (4.15)$$

and,

$$\Psi(P^{d(b,c)}) = \frac{d\Theta}{dn}(P^{d(b,c)}) \quad (4.16)$$

$$A_j(P^a, Q) = \frac{1-2\nu}{16\pi(1-\nu)G} \left[n_j(2\log_e r - 1) + 2r_j \frac{dr}{dn} \right] \quad (4.17)$$

$$B_j(P^a, Q) = \frac{1-2\nu}{16\pi(1-\nu)G} [r r_j(2\log_e r - 1)] \quad (4.18)$$

The integrals in equation (4.14) can be evaluated using Gauss quadrature while taking special precautions for the case when $P^a=Q$ [see e.g. Tan (1987)]. Equation (4.14) represents a set of $2N$ linear algebraic equations in terms of the unknown displacements and tractions at the boundary nodes. This system of equations can be expressed in matrix form, with the unknown tractions and displacements, t_i and u_i , grouped together into a vector, $\{\underline{u}\}$, and known quantities multiplied by the appropriate coefficients and placed in a vector $\{\underline{b}\}$. Equation (4.14) then takes the form,

$$[A]\{\underline{u}\}=\{\underline{b}\} \quad (4.19)$$

Equation (4.19) can then be easily solved by standard matrix methods such as Gaussian elimination.

For some problems, such as those involving complicated or elongated geometries, multiple materials, or cracks, it may be desirable or even necessary to divide the domain into several sub-regions. Sub-regioning simply requires that a separate BIE be written for each sub-region, and that continuity of the displacements, and equilibrium of the tractions is satisfied at the interface boundaries between the sub-regions.

4.2 Two-Dimensional Anisotropic Thermoelasticity

The BIE for anisotropic thermoelasticity has the same form as equation (4.6) for isotropic thermoelasticity. However, the equivalent body force term for isotropy, f_i' , is replaced by the equivalent body force term for anisotropy, X_i' , namely,

$$\begin{aligned} C_{ji}(P)u_i(P) + \oint_S u_i(Q)T_{ji}(P, Q)dS \\ = \oint_S t_i(Q)U_{ji}(P, Q)dS + \int_{\Omega} X_i'(q)U_{ji}(p, q)d\Omega \end{aligned} \quad (4.20)$$

In equation (4.20), the displacement fundamental solution, $U_{ji}(p, q)$, is now,

$$U_{ij}(p, q) = 2\Re[r_{ji}A_{i1} \log z_1 + r_{j2}A_{i2} \log z_2] \quad i, j=1,2 \quad (4.21)$$

and the corresponding traction fundamental solution is,

$$T_{i1}(p, q) = 2n_1 \Re\left[\frac{\mu_1^2 A_{i1}}{z_1} + \frac{\mu_2^2 A_{i2}}{z_2}\right] - 2n_2 \Re\left[\frac{\mu_1 A_{i1}}{z_1} + \frac{\mu_2 A_{i2}}{z_2}\right] \quad (4.22a)$$

$$T_{i2}(p, q) = -2n_1 \Re\left[\frac{\mu_1 A_{i1}}{z_1} + \frac{\mu_2 A_{i2}}{z_2}\right] + 2n_2 \Re\left[\frac{A_{i1}}{z_1} + \frac{A_{i2}}{z_2}\right] \quad (4.22b)$$

As before, in equations (4.21) and (4.22), $U_{ji}(p, q)$ and $T_{ji}(p, q)$ represent the displacement and tractions, respectively, in the x_j -direction at point q , due to a unit point load at point p , in an infinite anisotropic elastic body. Also, r_{ji} are defined previously in equation (3.16); n_i are the components of the unit outward normal at the field point q , with respect to the x_1 - x_2 coordinate system; and A_{ij} are complex constants obtained from the following matrix equation,

$$[\Im\{B_1\}\Re\{B_1\}\Im\{B_2\}\Re\{B_2\}]\begin{bmatrix}\Re\{A_{i1}\} \\ \Im\{A_{i1}\} \\ \Re\{A_{i2}\} \\ \Im\{A_{i2}\}\end{bmatrix}=\begin{bmatrix}\frac{-1}{4\pi} & \delta_{i2} \\ \frac{1}{4\pi} & \delta_{i1} \\ 0 & 0 \\ 0 & 0\end{bmatrix} \quad (4.23)$$

where,

$$\{B_k\}=[i \quad \mu_k \quad r_{1k} \quad r_{2k}]^T \quad k=1,2 \quad (4.24)$$

and δ_{jk} is the Kronecker delta, defined in equation (2.5).

Substituting X_i' into equation (4.20) along with the additional thermal traction term, results in the integral equation for thermoelasticity, which, in the absence of body forces, can be written as,

$$\begin{aligned} C_{ji}(P)u_i(P)+\oint_S u_i(Q)T_{ji}(P,Q)dS=&\oint_S t_i(Q)U_{ji}(P,Q)dS \\ &+\oint_S \gamma_{ik}n_k \Theta U_{ji}(P,Q)dS-\int_{\Omega} \gamma_{ik} \Theta_{,k} U_{ji}(P,q)d\Omega \end{aligned} \quad (4.25)$$

where, n_k is the unit outward normal at field point Q .

In order to realize the full advantages of the boundary element method, it is important to transform the domain integral in equation (4.25) into surface integrals, thus rendering the equation a boundary only equation. This transformation has been achieved only quite recently through a direct domain mapping technique [Shiah and Tan (1997, 1999)]. In this method, the domain integral is mapped into a new coordinate system, $\underline{x}_i$, such that the governing heat conduction equation (3.9) takes the form of Poisson's equation,

$$\Theta_{\underline{ij}} = C_1 \quad (4.26)$$

where,

$$C_1 = \frac{C_0 k_{11}}{\Delta} \quad (4.27)$$

Essentially, this direct domain mapping technique converts the anisotropic domain integral into isotropic form in the new transformed domain, thus rendering it solvable by established means.

The new $\underline{x}_i$ -coordinate system is related to the original Cartesian coordinate system by the following transformation,

$$\underline{x}_1 = \frac{\sqrt{\Delta}}{k_{11}} x_1 \quad (4.28)$$

$$\underline{x}_2 = x_2 - \frac{k_{12}}{k_{11}} x_1 \quad (4.29)$$

Thus, the new transformed volume integral can now be expressed as,

$$(\text{V.I.})_j = - \int_{\underline{\Omega}} \gamma_{\underline{ik}} \Theta_{\underline{ik}} U_{\underline{ij}} d\underline{\Omega} \quad (4.30)$$

where the underline indicates that the variables are now defined in the new mapped coordinate system, and $\gamma_{\underline{ik}}$ is defined in matrix form as,

$$\gamma_{\underline{ik}} = \begin{bmatrix} \gamma_{11} & \frac{(-\gamma_{11} k_{12} + \gamma_{12} k_{11})}{\sqrt{\Delta}} \\ \gamma_{21} & \frac{(-\gamma_{21} k_{12} + \gamma_{22} k_{11})}{\sqrt{\Delta}} \end{bmatrix} \quad (4.31)$$

The volume integral in equation (4.30) is analytically transformed into the following surface integral,

$$(V.I.)_j = \oint_S \left[(\gamma_{ik} Q_{ijk,t} \ominus - \gamma_{ik} Q_{ijk} \ominus_{,t} + C_1 \gamma_{ik} R_{ijkt}) n_t - \gamma_{ik} U_{ij} \ominus n_k \right] dS \quad (4.32)$$

where,

$$Q_{ijk} = 2\Re \left\{ \frac{r_{i1} A_{j1} \mu_{k1} z_1 \log z_1}{(\mu_{11}^2 + \mu_{21}^2)} + \frac{r_{i2} A_{j2} \mu_{k2} z_2 \log z_2}{(\mu_{21}^2 + \mu_{22}^2)} \right\} \quad (4.33)$$

$$Q_{ijk,t} = 2\Re \left\{ \frac{r_{i1} A_{j1} \mu_{k1} \mu_{t1} z_1 \log z_1}{(\mu_{11}^2 + \mu_{21}^2)} + \frac{r_{i2} A_{j2} \mu_{k2} \mu_{t2} z_2 \log z_2}{(\mu_{21}^2 + \mu_{22}^2)} \right\} \quad (4.34)$$

$$R_{ijkt} = 2\Re \left\{ \frac{r_{i1} A_{j1} \mu_{k1} (z_1^2 \log z_1 - z_1^2 / 2)}{4 \mu_{t1} (\mu_{11}^2 + \mu_{21}^2)} + \frac{r_{i2} A_{j2} \mu_{k2} (z_2^2 \log z_2 - z_2^2 / 2)}{4 \mu_{t2} (\mu_{21}^2 + \mu_{22}^2)} \right\} \quad (4.35)$$

and,

$$\mu_{ji} = \begin{bmatrix} \frac{(k_{11} + \mu_1 k_{12})}{\sqrt{\Delta}} & \frac{(k_{11} + \mu_2 k_{12})}{\sqrt{\Delta}} \\ \mu_1 & \mu_2 \end{bmatrix} \quad (4.36)$$

Thus, the resulting BIE for two-dimensional anisotropic thermoelasticity by the exact transformation method (ETM) can be expressed as,

$$\begin{aligned} C_{ji}(P) u_i(P) &+ \oint_S u_i(Q) T_{ji}(P, Q) dS \\ &= \oint_S t_i(Q) U_{ji}(P, Q) dS + \oint_S \gamma_{ik} n_k \ominus U_{ji}(P, Q) dS \\ &+ \oint_S [(\gamma_{ik} Q_{ijk,t} \ominus - \gamma_{ik} Q_{ijk} \ominus_{,t} + C_1 \gamma_{ik} R_{ijkt}) n_t] \end{aligned}$$

$$-\gamma_{ik} U_{ji}(P, Q) \Theta n_k] dS \quad (4.37)$$

The integrals in equation (4.37) pose no numerical problems because they are weakly singular at most. However, care must still be taken to avoid discontinuities which arise due to the geometry of the domain. It has been shown [Zhang, Tan, and Afagh (1996a), (1996b)] that such discontinuities can be avoided provided that,

(a) for a simply connected convex domain (Figure 4.5 (a)), the range of functions involving z are limited such that,

$$0 < \arg(z) \leq 2\pi \quad (4.38)$$

(b) for a simply connected but non-convex domain (Figure 4.5 (b)), the range of functions involving z are limited such that,

$$(\Phi - 2\pi) < \arg(z) \leq \Phi \quad (4.39)$$

where, $\Phi [0, 2\pi]$ is the angle between the outward normal and the x_1 -axis.

(c) for multiply connected domains (Figure 4.5 (c)), the domain is cut into infinitesimal strips, resulting in an extra line integral in equation (4.37) such that it now becomes,

$$\begin{aligned} & C_{ji}(P)u_i(P) + \oint_S u_i(Q)T_{ji}(P, Q)dS \\ &= \oint_S t_i(Q)U_{ji}(P, Q)dS + \oint_S \gamma_{ik} n_k \Theta U_{ji}(P, Q)dS \\ &+ \oint_{\underline{S}} [(\gamma_{ik} Q_{ijk,t} \Theta - \gamma_{ik} Q_{ijk} \Theta_{,t} + C_1 \gamma_{ik} R_{ijkt})n_t] \end{aligned}$$

$$-\gamma_{ik} U_{ij}(P, Q) \Theta n_k] dS + \sum_{n=1}^m \int_{l_{2n-1}}^{l_{2n-2}} L_j(\zeta_1) d\zeta_1 \quad (4.40)$$

where, m is the number of times that the domain is intersected by the negative ζ_1 -axis, and $L_j(\zeta_1)$ is defined as,

$$\begin{aligned} L_j(\zeta_1) = & 4 \pi \Theta \gamma_{ik} \left\{ \frac{k_{12}}{k_{11}} \Im \left\{ \frac{r_{i1} A_{j1} \mu_{11} \mu_{k1}}{(\mu_{11}^2 + \mu_{21}^2)} + \frac{r_{i2} A_{j2} \mu_{12} \mu_{k2}}{(\mu_{12}^2 + \mu_{22}^2)} \right\} \right. \\ & + \frac{\sqrt{\Delta}}{k_{11}} \Im \left\{ \frac{r_{i1} A_{j1} \mu_{21} \mu_{k1}}{(\mu_{11}^2 + \mu_{21}^2)} + \frac{r_{i2} A_{j2} \mu_{22} \mu_{k2}}{(\mu_{12}^2 + \mu_{22}^2)} \right\} \Bigg\} \\ & - 4 \pi \gamma_{ik} \left(\frac{k_{12}}{k_{11}} \Theta_{,1} + \frac{\sqrt{\Delta}}{k_{11}} \Theta_{,2} \right) \Im \left\{ \frac{r_{i1} A_{j1} \mu_{k1}}{(\mu_{11}^2 + \mu_{21}^2)} + \frac{r_{i2} A_{j2} \mu_{k2}}{(\mu_{12}^2 + \mu_{22}^2)} \right\} \\ & + C_1 \pi \zeta_1^2 \gamma_{ik} \left\{ \frac{k_{12}}{k_{11}} \Im \left\{ \frac{r_{i1} A_{j1} \mu_{k1}}{\mu_{11}(\mu_{11}^2 + \mu_{21}^2)} + \frac{r_{i2} A_{j2} \mu_{k2}}{\mu_{12}(\mu_{12}^2 + \mu_{22}^2)} \right\} \right. \\ & + \frac{\sqrt{\Delta}}{k_{11}} \Im \left\{ \frac{r_{i1} A_{j1} \mu_{k1}}{\mu_{21}(\mu_{11}^2 + \mu_{21}^2)} + \frac{r_{i2} A_{j2} \mu_{k2}}{\mu_{22}(\mu_{12}^2 + \mu_{22}^2)} \right\} \Bigg\} \\ & - 4 \pi \Theta \left(\frac{k_{12}}{k_{11}} \gamma_{i1} + \frac{\sqrt{\Delta}}{k_{11}} \gamma_{i2} \right) \Im \{ r_{i1} A_{j1} + r_{i2} A_{j2} \} \end{aligned} \quad (4.41)$$

Equation (4.41) is analytically exact for any physical domain. However, for thermoelasticity problems, temperature data is required along the negative ζ_1 -axis for every source point along the boundary. This is highly undesirable since it greatly increases the amount of computation required for the problem solution. Fortunately, this line integral can be avoided altogether through the selective use of sub-regioning to transform the multiply connected domain into a finite, and generally small, number of

simply connected non-convex domains, so that the much more simple argument of equation (4.39) can be employed (see Figure 4.6). The form of the discretized BIE, and the numerical procedures for solving it are the same for anisotropy as they are for isotropy, hence they will not be elaborated on further.

4.3 Two-Dimensional Linear Elastic Fracture Mechanics Analysis

The BIE method for two-dimensional homogeneous anisotropic thermoelasticity discussed in the previous section can be applied to the analysis of crack problems without requiring any approximation or further modification. All that is required are special ‘crack-tip’ elements to model the crack, and appropriate formulae to calculate the stress intensity factors from the computed displacements and tractions. These two requirements are now discussed.

In order to model anisotropic problems with a general crack, sub-regioning of the solution domain may be employed. The sub-regions are chosen such that the crack, of arbitrary orientation, lies along the sub-region interface. The nodes at this interface boundary are bonded along the un-cracked portion, and are free along the crack face (see Figure 4.7).

From equations (3.11) to (3.21) it is evident that near the crack-tip, displacements exhibit an $O(\sqrt{r})$ variation, and stresses exhibit an $O(1/\sqrt{r})$ variation. These variations

cannot be resolved by regular quadratic isoparametric line elements. However, it has been well established [Martinez and Dominguez (1984)] that by shifting the mid-node of the quadratic isoparametric line elements to the quarter-point, towards the crack-tip (see Figure 4.8), the near-tip $\sqrt{r}$ variation of displacements can be captured. The $1/\sqrt{r}$ variation of tractions can also be resolved, provided the traction shape functions are multiplied by $\sqrt{l/r}$, where l is the length of the crack-tip element.

With the crack being modeled using these quarter-point elements at the crack-tip, the stress intensity factors can be obtained directly from the computed displacements and tractions at the nodes of these elements, using either the ‘traction formula’,

$$K_I = \bar{t}_2^A \sqrt{2 \pi l} \quad (4.42a)$$

$$K_{II} = \bar{t}_1^A \sqrt{2 \pi l} \quad (4.42b)$$

or the ‘displacement formula’,

$$\begin{bmatrix} K_I \\ K_{II} \end{bmatrix} = \frac{\sqrt{(\pi/2l)}}{\text{Det}[D]} \begin{bmatrix} D_{22} & -D_{12} \\ -D_{21} & D_{11} \end{bmatrix} \begin{bmatrix} (-u_1^C + 4u_1^B - 3u_1^A) \\ (-u_2^C + 4u_2^B - 3u_2^A) \end{bmatrix} \quad (4.43)$$

as derived by Tan and Gao (1992).

In equations (4.42) and (4.43) the superscripts indicate the nodes of the crack-tip elements as shown in Figure 4.9. The computed traction at the crack-tip, $\bar{t}_j^A$ is related to the physical traction, t_j^A as follows,

$$\bar{t}_j^A = \lim_{r \rightarrow 0} t_j^A \sqrt{\frac{r}{l}} \quad (4.44)$$

and,

$$\text{Det}[D] = (D_{11}D_{22} - D_{21}D_{12}) \quad (4.45)$$

The displacement formula requires significantly more computation than the traction formula, and is known to be more sensitive to the size of the crack-tip element, l [see e.g. Tan and Gao (1992)]. However, the use of both formulae provides a useful cross-check of the results, and good agreement can also be an indicator of good mesh design.

CHAPTER 5

THERMOELASTIC FRACTURE MECHANICS ANALYSIS IN PLANE ANISOTROPY

Anisotropic materials are becoming more commonly used in the aerospace industry as a primary structural material in applications such as satellites, and commercial as well as military aircraft. In such applications, cracks are a vital concern and are a major consideration in design and analysis. Much research over the past few decades has been devoted to studying the behavior of cracked anisotropic materials under mechanical loads. However, in many cases these structures are also subjected to thermal loads, yet very little work has been done to determine the effect of a temperature change on a cracked anisotropic body.

The goal of the present study is to determine the effect of varying material properties on the value of the stress intensity factors, K_I and K_{II} , for anisotropic plates containing cracks and subjected to a temperature change. This chapter outlines the scope of the parametric study carried out to examine these effects. The physical problems considered are first described. Following this, the range of material properties used for this study is discussed. Then the general guidelines used to model the physical problems by both the boundary element method (BEM), and the finite element method (FEM) are

presented. Sample meshes are shown, and a comparison is made between BEM and FEM results for isotropy and quasi-isotropy, to provide an assessment of the adequacy of the mesh designs. This is a precursor to the anisotropic analysis which will be discussed in Chapter 6. The chapter closes with a qualitative comparison between BEM and FEM as numerical tools for the problems treated.

5.1 Physical Problems Considered

For the two-dimensional anisotropic thermoelastic fracture mechanics study performed here, the basic physical problem is that of a rectangular plate with a crack, whereupon the crack opens as a result of a uniform temperature drop. The plate is constrained in the vertical direction along the top and bottom edges, and constrained in the horizontal direction at the middle of the bottom edge (see Figure 5.1). These boundary conditions allow the plate to expand or contract in the horizontal direction but not in the vertical direction, while also eliminating rigid body motion in the numerical analysis. In all cases, the plates are cooled by a temperature of 100 °C, and the height to width ratio, H/W , is 2 (see Figure 5.1).

The five basic types of cracked plates investigated in this study are (a) an edge-cracked plate (see Figure 5.2), (b) a plate with a central crack (see Figure 5.3), (c) a plate with a double edge-crack (see Figure 5.4), (d) a plate with a crack emanating from a circular notch (see Figure 5.5), and (e) a plate with symmetric cracks emanating from a

circular hole (see Figure 5.6). The BIE method is performed for all five of the geometric cases, and FEM analysis is performed for cases (a) and (b) to provide a check on the BEM results. Also, for the BEM analysis, relative crack lengths considered are $a/W=0.1, 0.2, 0.3, 0.4, 0.5$, and for the FEM analysis the cases $a/W=0.1, 0.3, 0.5$ are considered, where 'a' is the crack length. The number of physical problems modeled using FEM was limited because data preparation in FEM is quite time consuming, as will be discussed later in this chapter. Therefore, the number of cases performed was limited to those considered necessary to validate the BEM results.

5.2 Material Properties

For two-dimensional anisotropy in plane stress, four elastic constants are normally required to fully define the material, namely E_{11} , E_{22} , G_{12} , and ν_{12} . In order to reduce the number of runs required in the parametric study, an alternative method of representing the material properties was used [Bowie and Freese (1972)], whereby these four elastic constants can be related to a pair of new material parameters, η_1 and η_2 , as follows,

$$\eta_1 \eta_2 = \sqrt{\frac{E_{11}}{E_{22}}} \quad (5.1a)$$

and

$$\eta_1 + \eta_2 = \sqrt{2 \left(\sqrt{\frac{E_{11}}{E_{22}}} + \frac{E_{11}}{2G_{12}} - \nu_{12} \right)} \quad (5.1b)$$

The material properties of some common anisotropic materials are shown in Table 5.1, along with the corresponding values of η_1 and η_2 . From this table, it can be seen that for all the listed materials, the value of η_1 lies in between 0.5 to 1, and the range of η_2 is between 2 to 7. By plotting the values of the four elastic constants for various values of η_1 and η_2 (see Figures 5.7 and 5.8 respectively) it was determined that the range of elastic constants for typical anisotropic materials can be adequately represented using four values of η_1 , and three values of η_2 . This results in the twelve sets of material properties shown in Table 5.2. Also, because there are only two equations, (5.1a) and (5.1b), but three unknowns, E_{22} , G_{12} , and ν_{12} , it was necessary to fix ν_{12} to be 0.3 for all twelve sets of properties. Due to the limited range of ν_{12} , and the presence of only thermal loading in this study, this restriction is not expected to have a significant effect on the results obtained.

In anisotropy, the angle between the global Cartesian coordinate system, and the material principal Cartesian coordinate system, is very important in defining the properties of a material. The range of this material property angle, θ , was chosen to be between 0 and 90 degrees, and the following five values are considered, $\theta=0,30,45,60,90$.

So far, thermal properties have not been represented in the sets of material properties chosen for the parametric study. This requires the creation of a third material parameter representing the ratio of the material coefficients of linear expansion, α_{11} and

α_{22} , in the principal directions. This parameter, which will be called η_3 for convenience, is defined as,

$$\eta_3 = \frac{\alpha_{22}}{\alpha_{11}} \quad (5.2)$$

The values α_{11} , α_{22} and η_3 for typical anisotropic materials are shown in Table 5.1. Again it is the ratio of coefficients of linear expansion that is important and not necessarily the explicit values. Therefore, α_{11} is set to be $0.01 \cdot 10^{-6}$ m/m/°C throughout this study. In order to determine the effect of varying η_3 , the stress intensity factors were computed for numerous values of η_3 , for $\theta=0,45,90$ and for the material parameter extremes, $[\eta_1, \eta_2]=[2,0.5]$ and $[\eta_1, \eta_2]=[7,1]$. The stress intensity factors, K_I and K_{II} , are all normalized to produce the normalized stress intensity factors, K_I^* and K_{II}^* , defined as follows,

$$K_I^* = \frac{K_I}{E_{11} \eta_3 \alpha_{11} \Delta T \sqrt{\pi a}} \quad (5.3a)$$

and

$$K_{II}^* = \frac{K_{II}}{E_{11} \eta_3 \alpha_{11} \Delta T \sqrt{\pi a}} \quad (5.3b)$$

These results are shown for the edge-crack and central-crack cases in Figures 5.9 to 5.20 as well as in Tables 5.3 and 5.4. It can be seen from these graphs that, in all cases, for values of η_3 greater than +5, or less than -5, the normalized stress intensity factors change very little. Therefore, any results obtained should be directly applicable to all materials

with η_3 greater than +10, or less than -10. Therefore, for the purpose of efficiency, only the case of $\eta_3=30$ is considered for further analysis.

Thus, for the BEM parametric study there are twelve sets of material properties $[\eta_1, \eta_2]$ (shown in Figure 5.2), five values of θ , one value of η_3 , and five crack lengths, for five different types of cracked plates, resulting in 1500 distinct anisotropic cases. For the FEM analysis there are three crack lengths and two types of cracked plates, resulting in 360 distinct anisotropic cases. These results are discussed in detail in the next chapter. But first, the next section will outline the data preparation and numerical modeling process.

5.3 Numerical Modeling

This section outlines the procedures which were followed throughout the course of this research for numerical modeling in BEM and FEM.

5.3.1 BEM Modeling

All BEM analysis was performed using the program BIE2DET developed by Professor C. L. Tan at Carleton University. This program has already been successful in treating fracture mechanics problems for two-dimensional anisotropic thermoelasticity in

previously published research [see e.g. Shiah and Tan (1999a), (1999b), (2000)]. This program calculates the stress intensity factors using both the ‘traction’ and ‘displacement’ formulas discussed in Chapter 4. It has been shown that the traction formula is the more accurate and robust of the two formulas [Tan and Gao (1992)], and is the one used for all BEM results presented in this thesis. The present analysis employed 2-D quadratic isoparametric elements over the entire boundary, except for the crack-tip elements. For the crack-tip elements, the mid-nodes on both sides of the crack-tip node were shifted towards the crack-tip quarter-point. Because of the anisotropic nature of the problems being modeled, it was not possible to take advantage of symmetry for mesh creation. Therefore, the problem domain had to be divided into two sub-regions along the crack face (refer to Figure 4.7). This sub-regioning therefore results in four distinct crack-tip elements, all of which must have their respective mid-nodes shifted to the quarter-point. In all cases a crack-tip element size of 10% of the overall crack length was used. Previous work by Tan and Gao (1992); Shiah and Tan (1999, 2000b) has shown this to be adequate for anisotropic problems. Other meshing guidelines followed include ensuring that elements which meet at opposing corners are approximately the same size; and ensuring that the other elements away from the crack-tip are graded, at approximately one and a half times the size of the previous element, and never more than double the size. These general guidelines were double-checked for each mesh by trying various crack-tip element sizes of 12%, 14%, and 18%, and also by trying coarser and finer meshes. In every case, a 10% crack-tip element size proved most versatile, providing good accuracy over the entire range of material properties, from the quasi-isotropic cases to the ‘most

anisotropic' case ($[\eta_1, \eta_2] = [7, 1]$). Some sample BEM meshes are shown in Figures 5.21 to 5.25.

5.3.2 FEM Modeling

All FEM modeling was performed using ABAQUS/Standard version 6.2 [Hibbitt, Karlsson, and Sorenson Inc. (2001)]. A major drawback to the FEM analysis is that for two-dimensional anisotropic thermoelasticity, two separate analyses are required for each run, as part of what is termed a 'sequentially coupled' analysis. This required first solving the heat transfer problem to obtain the temperature distribution even though a uniform temperature was applied. Once the temperature data at all the nodes has been determined through this heat transfer analysis, the temperature field data is then used in stress-displacement analysis. In both cases, the exact same mesh is used but with different commands in the input file, as well as different element types. For the thermal analysis DC2D8 elements were used, and for the mechanical analysis CPS8 elements were used. DC2D8 is an 8-noded quadrilateral heat conduction element, and CPS8 is an 8-noded quadrilateral plane stress element. Once again, due to the anisotropic nature of the problem the entire domain needed to be modeled. The crack-tip was modeled using 'standard' crack-tip elements formed by collapsing the three nodes on one side of the quadrilateral element to form the crack-tip, and shifting the mid-nodes of the adjacent edges to the quarter-point (see Figure 5.26). As with the BEM analysis, crack-tip

elements which were 10% of the total crack length, a , were used. The crack-tip region was modeled using six rings of sixteen elements each (see Figure 5.27). The first three rings were sized at 10% of the crack length each, and the next three were sized as 15% of the crack length each. ABAQUS/Standard used the ‘contour-integral approach’ to calculate stress intensity factors, and therefore, these ‘rings’ allow for the accurate calculation of the stress intensity factors. Mesh grading beyond the crack-tip region follows the same guidelines as for BEM, with an attempt made to keep elements at approximately one and a half times the size of neighboring elements as they move away from the crack-tip, while ensuring that no element is ever more than double the size of its neighbor. Some sample FEM meshes are shown in Figures 5.28 and 5.29.

5.3.3 Comparison of Isotropic and Quasi-Isotropic Results

The lower portion of Table 5.5 shows isotropic and quasi-isotropic results for the mode-I stress intensity factor, K_I^* , obtained from the BEM and FEM analyses. For both the BEM and FEM programs, the isotropic and quasi-isotropic results are the same, which shows that both programs are capable of handling the thermoelastic anisotropic fracture mechanics problems which are the focus of this study. These two sets of results also indicate that the meshes are of good quality since good agreement implies that the meshes are sufficiently refined to handle both isotropic and anisotropic analysis.

Also contained in Table 5.5 are the BEM and FEM K_I^* results for cases with quasi-isotropic mechanical properties, but anisotropic thermal properties ($\eta_3=30$). These results help give an indication of how well the two programs will agree once the material properties are rotated from the principal axis. The two programs are in good agreement for all cases except for $\theta=90$ degrees. This may be because the FEM program would not execute for $\theta=90$ degrees, so $\theta=89.9$ degrees was used for these cases instead. However, this discrepancy is of little concern since the K_I^* values in these cases are typically an order of magnitude smaller than those of the rest of the study.

5.3.4 Concluding Remarks

For analyzing two-dimensional anisotropic thermoelasticity problems, BEM seems to have numerous advantages over FEM. Even for the relatively simple geometries which were modeled in this study, mesh creation was much faster in BEM since only the boundary of the domain is discretized, as opposed to discretizing the entire domain for FEM. For more complicated geometries this advantage would be even more pronounced. In discretizing only the boundary, BEM allows not only the use of fewer elements, but also much simpler elements. This becomes very noticeable when modeling the crack-tip region. Whereas BEM only required shifting four nodes to model the crack in the single edge-crack case, FEM requires the much more complicated treatment

discussed in the previous section, where sixteen elements must have an edge ‘collapsed’ and nodes shifted. With more complicated geometries the modeling to the crack-tip region would become very cumbersome. Even for simple geometries, mesh refinement and modification for FEM was also very time consuming. The number of FEM cases which could be executed and compared with BEM results was very limited for this reason. Solution time was much quicker with BEM than FEM, largely due to the significantly smaller file sizes, and partly because the ABAQUS/Standard software created many extra files for use with the post-processing portion of the program which was not needed for the type of problem solved here. A major disadvantage of the FEM analysis was the need to execute a ‘sequentially-coupled’ analysis in order to solve the thermoelastic problem. This not only increased the time required for data preparation, and created more files, but also effectively doubled run-time.

All in all, data preparation was much easier and faster for BEM, and run times were shorter. The boundary element method is evidently very well suited to the analysis of two-dimensional anisotropic thermoelastic fracture mechanics problems.

CHAPTER 6

NUMERICAL RESULTS AND DISCUSSION

This chapter presents the stress intensity factor (SIF) results obtained from the parametric study outlined in the previous chapter. The general trends resulting from variation of the material parameters η_1 and η_2 , which are used to define the anisotropic properties of the material, are first presented. These parameters are directly related to the material elastic constants, and thus will give an indication of the overall effect that variation of the mechanical properties has on the SIFs for anisotropic plates under thermal loads. The results of varying the orientation of the material properties are presented next, and the effect on the value of the SIFs is discussed. This is followed by a look at the effect of crack length on the SIFs for each plate geometry. The chapter ends with a comparison of the separate crack cases and some concluding remarks.

It is important to note that, in general, only the results for the mode-I normalized stress intensity factor, K_I^* , are plotted and discussed. Values for the mode-II normalized stress intensity factor, K_{II}^* , were typically an order of magnitude smaller than K_I^* and are therefore only of secondary importance. However, the results for K_{II}^* are contained in tables along with the K_I^* results. Also note that throughout this thesis, θ is in units of degrees.

The maximum value of K_I^* in this study is 1.880 which occurs in the case of a plate containing a double edge-crack with $\theta=0$ degrees, $a/W=0.1$, $\eta_1=2$, $\eta_2=0.5$, and $\eta_3=30$. Likewise, the maximum value of K_I^* for the other cases also occurs for the same set of properties. The maximum value for these other four cases are; 1.408 for the plate with symmetric cracks emanating from a central hole, 1.388 for the plate with an edge-crack emanating from a circular notch, 1.043 for the plate with an edge-crack, and 0.680 for the plate with a central-crack. Minimum values of K_I^* occur in all cases at $\theta=90$ degrees, $a/W=0.5$, $\eta_1=7$, $\eta_2=1$, and $\eta_3=30$ and is typically a value between 0.01 and 0.03.

6.1 Effects of Material Properties, η_1 and η_2

The variations of the normalized stress intensity factor, K_I^* , with varying values of η_1 and η_2 for a uniformly cooled plate with an edge-crack can be seen in Figures 6.1 to 6.25. Only figures for the edge-crack case are presented here because the general trends are virtually identical for the other four cases. The data for these figures is also listed in Tables 6.1 to 6.12. Similarly, the data for the other four cases is also contained in the corresponding tables, from Table 6.13 to 6.36.

From Figures 6.1 to 6.25 one can clearly see that regardless of material property angle, θ , or relative crack length, a/W , the normalized SIF, K_I^* , decreases as either of the material parameters, η_1 or η_2 , is increased. The only exception is the case of $\theta=90$, where

K_I^* does not vary significantly with either parameter, nor with crack length. However, the case where $\theta=90$ is a special case which will be described later, in the next section dealing with the variation of material angle θ . Note that independently increasing either material parameter η_1 or material parameter η_2 corresponds to an increase in the ‘degree of anisotropy’ of the material properties. Therefore, it is fair to say that, in general, the normalized stress intensity factor, K_I^* , decreases as the material becomes ‘more anisotropic’ and this is true of all five crack cases in this study.

6.2 Effects of Material Property Angle, θ

The variation of normalized SIF, K_I^* , with varying material property angle, θ , for various material properties and crack lengths is shown in Figures 6.26 (b) to 6.85 (b), and is listed in Tables 6.1 to 6.36. There are twelve plots for each of the five crack cases in this study, corresponding to the twelve sets of material properties, $[\eta_1, \eta_2]$, for a total of sixty figures. The individual cases will not be discussed independently here since the trends for all five are the same for varying material angle, θ . However, for the benefit of the reader, note that the first twelve figures (6.26 to 6.37) correspond to the edge-crack case, and the next twelve correspond to the central-crack case, and so on through all five cases.

Before discussing the results, it is important to recall the physical boundary conditions which are applied to the plates (see Figure 5.1). As described in the previous

chapter the plates are constrained at the top and bottom edges, and it is this constraint which induces the crack to open as the plate is cooled. When $\theta=0$, E_{11} is oriented in the horizontal direction, and α_{22} is oriented in the vertical direction. This means that when $\theta=0$ the plates are most stiff in the horizontal direction, and maximum shrinkage of the plates is occurring in the vertical direction which is the least stiff. As θ increases, both E_{11} and α_{22} are rotated counterclockwise until $\theta=90$, where now E_{11} is vertical, and α_{22} is horizontal. Therefore, $\theta=90$ corresponds to a condition of maximum stiffness in the vertical direction and maximum shrinkage in the horizontal direction.

For all five crack cases, the general trend observed is that the normalized SIF, K_I^* , is maximum at $\theta=0$ and decreases in a sinusoidal fashion to a minimum value at $\theta=90$. The maximum value at $\theta=0$ is expected in this situation since it corresponds both to maximum vertical shrinkage and minimum vertical stiffness. Thus the crack opens by the maximum amount, yielding the maximum value of K_I^* . At $\theta=90$, the opposite situation arises. When $\theta=90$, shrinkage occurs mainly in the horizontal direction where the plate is free to expand and/or contract. There is the minimum amount of shrinkage in the vertical direction, and the maximum amount of stiffness in that direction, resulting in relatively low stresses at the crack-tip, and the minimum value of K_I^* . It is expected that if the material properties were rotated further, from $\theta=90$ to $\theta=180$, the observed trends would be mirrored, and the normalized SIF, K_I^* , would increase back to a maximum at $\theta=180$, and this cyclic behavior should continue through to $\theta=360$.

6.3 Effects of Crack Length, a/W

Although the five crack cases analyzed in this study all exhibit the same behavior for varying material parameters, η_1 and η_2 , and varying material property angle, θ , there are some differences in observed trends as it relates to crack length. Certain cases exhibit a greater dependence of the normalized stress intensity factors on crack length.

The variations of K_I^* with relative crack length, a/W , for the five crack cases are shown in Figures 6.1(a) to 6.85(a). It can be seen in these figures that there is a trend of decreasing K_I^* with increasing relative crack length, and this trend is generally present for any combination of material properties, η_1 and η_2 , or material property angle, θ . The only exception is the central-crack case. For this case, K_I^* is, for all intents and purposes, constant for any combination of material properties, η_1 and η_2 , or material property angle, θ . It is also evident from the results that, for a given set of material properties, the decrease of K_I^* with crack length is relatively more dramatic for the geometric cases with a hole or a semi-circular notch. This is particularly so at the smaller relative crack lengths where the effects of the stress concentrations of these geometric discontinuities are still apparent, and is therefore to be expected. It is interesting to note also that the rate of decrease of K_I^* , with increasing values of a/W , for the double edge-crack geometry is also significantly greater than that for the edge-crack case.

It is worth noting that although the normalized stress intensity factor, K_I^* , may decrease with increasing relative crack length, a/W , the absolute value of K_I actually increases with crack size in all cases.

6.4 Concluding Remarks

Wherever possible FEM analysis has been performed and compared with the BEM results discussed above. In all cases, the FEM results follow the exact same trends as those obtained by the BIE method, generally to within 2-5%. This provides much confidence in the numerical results obtained. There are exceptions to this range of percent deviations between the BEM and FEM results, but they occur in those cases where the SIFs are negligibly small in magnitude.

CHAPTER 7

SUMMARY AND CONCLUSIONS

A two-dimensional thermoelastic fracture mechanics analysis of anisotropic bodies has been performed using the boundary element method (BEM). The physical problems considered involved a rectangular plate with (a) an edge-crack, (b) a central-crack, (c) a double edge-crack, (d) an edge-crack emanating from a circular notch, and (e) symmetric cracks emanating from a central hole. A range of relative crack sizes was treated for each of these geometries. Before presenting the results of the numerical analysis in this thesis, the basic relations in the theory of elasticity, for both isotropy and anisotropy, have been reviewed. An outline of the formulation of the boundary integral equation (BIE) method, which BEM is also commonly known as, has been given for thermoelastic stress analysis, first for isotropy, then for anisotropy. The key equations for the elastic displacement and stress fields near the crack-tip of a plane anisotropic body have been presented, and the procedures for obtaining the stress intensity factors (SIFs) for such problems using BEM have also been described.

In this study the BEM analysis has also been complemented by some additional analysis performed using the finite element method (FEM), for the purpose of comparison. The goal of this research was primarily to determine the effects of varying

anisotropic material properties on the value of the normalized stress intensity factors (SIF) for cracked anisotropic plates subjected to a uniform temperature change. To this end, two parameters, η_1 and η_2 , related to the mechanical properties of a material, have been introduced to characterize the ‘degree of anisotropy’, and to minimize the number of analyses required in the parametric study. For an isotropic material, $\eta_1=\eta_2=1$, while for most practical structural composites, η_1 lies between 2 and 7, and η_2 lies between 0.5 and 1. In the cases analyzed, both the BEM and FEM results showed very good agreement with one another, with deviations generally being less than 5 percent.

There are a few general conclusions which can be made from the results obtained. First, as the material becomes ‘more anisotropic’, or in other words as η_1 and η_2 deviate further from unity, the value of the normalized stress intensity factor (SIF), K_I^* , decreases for a given crack size. Second, for the boundary conditions used in this study, as the material property angle, θ , is increased, K_I^* also decreases in a sinusoidal manner from $\theta=0$ to $\theta=90$ degrees. It is expected that K_I^* will continue to vary in a sinusoidal fashion from $\theta=90$ degrees to $\theta=360=0$ degrees. Third, the normalized stress intensity factor, K_I^* , in general, decreases with increasing relative crack length, for a given set of anisotropic material properties for the geometric cases considered. There is an exception however for the plate with the central-crack, where K_I^* was found, for all practical purposes, not to change with relative crack length. As expected, the presence of geometric discontinuities such as a hole or a notch was found to have a more pronounced influence on the magnitude, and the rate of change, of K_I^* , particularly at smaller relative crack lengths.

This study has shown that BEM is a quick and efficient numerical tool for the analysis of two-dimensional cracked anisotropic bodies under thermal loads. It has been found to be a much more efficient alternative to FEM for such problems.

7.1 Recommendations for Future Work

In this study material parameters were used to cover a very wide range of anisotropic material properties. Some common materials which fall into this range include composite laminates, but it should be noted here that laminates were not analyzed in this study. This study did not take into account many factors important in the analysis of cracked composite laminates, such as fibre-bridging and inter-laminar stress, thus care should be taken when trying to apply the presented results to such materials. It would have also been desirable to widen the scope of this study to include a range of the height to width ratio of the plate, H/W , as well as to investigate a range of hole sizes for the two cases with circular cut-outs. The latter would provide a better insight into relationship between the stresses around the cut-out and the crack-tip, and how they affect the value of the normalized stress intensity factors. Future work could also be carried out to study angled cracks. This would allow the investigation of mixed-mode loading regardless of material property orientation, which would possibly allow insight into problems that might not be subject to exactly the same physical boundary conditions as presented here.

Additional work is also required to further study the effects of thermal anisotropy, or material parameter η_3 as defined in this study, on the SIFs for the cracked plate.

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CORRIGENDUM

Please note that, in the tables and figures that follow, subscripts have been inadvertently left out of the expressions for the normalized stress intensity factors, K_I^* and K_{II}^* . The expressions should read as follows,

$$K_I^* = K_I / E_{11} \eta_3 \alpha_{11} \Delta T (\pi a)^{1/2}$$

and,

$$K_{II}^* = K_{II} / E_{11} \eta_3 \alpha_{11} \Delta T (\pi a)^{1/2}$$

Table 5.1: Typical material properties for various orthotropic materials, and corresponding values for material parameters, η_1 , η_2 , and η_3

Material	E ₁₁ [Gpa]	E ₂₂ [Gpa]	G ₁₂ [Gpa]	ν ₁₂	α ₁₁	α ₂₂	η ₁	η ₂	η ₃
					[10 ⁻⁶ m/m/°C]				
AS/3501 ¹ Graphite/Epoxy	138	9	6.9	0.3	0.88	31	4.31	0.91	35.23
Scotchply 1002 ¹ E-glass/Epoxy	38.6	8.27	4.14	0.26	6.3	20	2.87	0.75	3.17
E-glass/470-36 ¹ E-glass/Vinylester	24.4	6.87	2.89	0.32	6.3	20	2.71	0.70	3.17
High Modulus ^{2,3} Graphite/Epoxy	221	6.9	4.8	0.3	-0.9	27	6.69	0.85	-30.00
Ultra High Modulus ^{2,3} Graphite/Epoxy	303	6.9	6.6	0.31	-1.44	30.6	6.66	1.00	-21.25
Kevlar 49/Epoxy ^{2,3}	86	5.5	2.1	0.34	-3.6	54	6.32	0.63	-15.00
S-glass/Epoxy ^{2,3}	55	14	3.4	0.28	6.3	19.8	3.92	0.51	3.14
Glass/Epoxy ⁴	38.6	8.27	4.14	0.26	8.6	22.1	2.96	0.73	2.57
Boron/Epoxy ⁴	204	18.5	5.59	0.23	6.1	30.3	5.98	0.55	4.97
Graphite/Epoxy ⁴	181	10.3	7.17	0.28	0.02	22.5	4.89	0.86	1125
γ-Alumina/Epoxy ⁵	137	18	7.25	0.3	0.04	0.37	4.23	0.65	9.25
E-glass/Epoxy ⁵	45	12	5.5	0.19	3.7	30	2.70	0.72	8.11
S-glass/Epoxy ⁵	55	16	7.6	0.28	3.5	32	2.47	0.75	9.14

¹ Gibson (1994), ² Vinson (1987), ³ Mallick (1993), ⁴ Kaw (1997), ⁵ Schwartz (1997)

Table 5.2: Material properties used in study

η_1	η_2	η_3	ν_{12}
2	0.5	30	0.3
3	0.5	30	0.3
5	0.5	30	0.3
7	0.5	30	0.3
2	0.75	30	0.3
3	0.75	30	0.3
5	0.75	30	0.3
7	0.75	30	0.3
2	1	30	0.3
3	1	30	0.3
5	1	30	0.3
7	1	30	0.3

Table 5.3: Normalized stress intensity factor, $K_I^* = K_I / E \eta_3 \alpha T (\pi a)^{1/2}$, with varying material parameter, η_3 , normalized crack length, a/W , and material angle, θ , for a uniformly cooled plate with an edge-crack

Edge-Crack		$\eta_1=2, \eta_2=0.5$					$\eta_1=7, \eta_2=1$				
Angle, θ [Deg]	η_3	a/W					a/W				
		0.1	0.2	0.3	0.4	0.5	0.1	0.2	0.3	0.4	0.5
0	-30	1.044	0.979	0.885	0.792	0.711	0.022	0.023	0.024	0.024	0.023
	-10	1.045	0.980	0.886	0.793	0.712	0.022	0.023	0.024	0.024	0.023
	-5	1.046	0.981	0.887	0.794	0.713	0.022	0.023	0.024	0.024	0.023
	-3	1.048	0.982	0.882	0.795	0.714	0.023	0.023	0.024	0.024	0.023
	-1	1.067	0.999	0.903	0.809	0.726	0.024	0.024	0.024	0.024	0.023
	1	1.041	0.978	0.884	0.791	0.710	0.021	0.022	0.024	0.024	0.023
	3	1.040	0.975	0.882	0.789	0.709	0.022	0.023	0.024	0.024	0.023
	5	1.041	0.976	0.883	0.790	0.710	0.022	0.023	0.024	0.024	0.023
	10	1.042	0.978	0.884	0.791	0.710	0.022	0.023	0.024	0.024	0.023
	30	1.043	0.978	0.884	0.792	0.711	0.022	0.023	0.024	0.024	0.023
45	-30	0.356	0.352	0.339	0.320	0.297	0.031	0.031	0.031	0.030	0.028
	-10	0.335	0.331	0.319	0.301	0.279	0.033	0.033	0.032	0.031	0.029
	-5	0.302	0.299	0.288	0.272	0.253	0.035	0.035	0.034	0.033	0.031
	-3	0.259	0.256	0.247	0.234	0.218	0.038	0.037	0.037	0.035	0.034
	-1	0.046	0.046	0.046	0.046	0.046	0.054	0.051	0.049	0.047	0.046
	1	0.695	0.687	0.660	0.620	0.571	0.007	0.011	0.013	0.012	0.009
	3	0.475	0.470	0.452	0.426	0.393	0.022	0.024	0.024	0.024	0.021
	5	0.432	0.427	0.411	0.387	0.358	0.026	0.027	0.027	0.026	0.024
	10	0.399	0.395	0.380	0.359	0.332	0.028	0.029	0.029	0.028	0.026
	30	0.378	0.374	0.360	0.340	0.314	0.029	0.030	0.030	0.029	0.027
90	-30	-0.048	-0.043	-0.039	-0.035	-0.032	-0.028	-0.020	-0.016	-0.014	-0.013
	-10	-0.117	-0.108	-0.098	-0.088	-0.079	-0.083	-0.060	-0.049	-0.043	-0.038
	-5	-0.221	-0.206	-0.186	-0.167	-0.150	-0.166	-0.121	-0.098	-0.085	-0.076
	-3	-0.360	-0.337	-0.304	-0.272	-0.245	-0.276	-0.201	-0.163	-0.141	-0.127
	-1	-1.056	-0.989	-0.894	-0.800	-0.719	-0.827	-0.603	-0.489	-0.424	-0.381
	1	1.030	0.967	0.875	0.783	0.703	0.827	0.602	0.488	0.423	0.381
	3	0.335	0.315	0.285	0.255	0.229	0.275	0.201	0.163	0.141	0.127
	5	0.196	0.185	0.167	0.150	0.134	0.165	0.120	0.098	0.085	0.076
	10	0.091	0.087	0.079	0.070	0.063	0.082	0.060	0.049	0.042	0.038
	30	0.022	0.022	0.020	0.018	0.016	0.027	0.020	0.016	0.014	0.013

Table 5.4: Normalized stress intensity factor, $K_I^* = K_I / E\eta_3 \alpha T(\pi a)^{1/2}$, with varying material parameter, η_3 , normalized crack length, a/W , and material angle, θ , for a uniformly cooled plate with a central-crack

Central-Crack		$\eta_1=2, \eta_2=0.5$					$\eta_1=7, \eta_2=1$				
Angle, θ [Deg]	η_3	a/W					a/W				
		0.1	0.2	0.3	0.4	0.5	0.1	0.2	0.3	0.4	0.5
0	-30	0.680	0.675	0.661	0.644	0.625	0.014	0.014	0.014	0.014	0.014
	-10	0.681	0.676	0.661	0.645	0.625	0.014	0.014	0.014	0.014	0.014
	-5	0.682	0.676	0.662	0.645	0.626	0.014	0.014	0.014	0.014	0.015
	-3	0.683	0.677	0.663	0.646	0.627	0.014	0.014	0.014	0.014	0.015
	-1	0.696	0.690	0.675	0.657	0.637	0.014	0.014	0.014	0.015	0.015
	1	0.678	0.673	0.660	0.643	0.624	0.014	0.014	0.014	0.014	0.015
	3	0.677	0.672	0.658	0.642	0.622	0.014	0.014	0.014	0.014	0.014
	5	0.678	0.673	0.659	0.643	0.623	0.014	0.014	0.014	0.014	0.014
	10	0.679	0.674	0.660	0.643	0.624	0.014	0.014	0.014	0.014	0.014
	30	0.680	0.674	0.660	0.644	0.624	0.014	0.014	0.014	0.014	0.014
45	-30	0.219	0.221	0.223	0.225	0.227	0.016	0.016	0.017	0.018	0.019
	-10	0.207	0.208	0.210	0.212	0.214	0.017	0.018	0.019	0.019	0.020
	-5	0.187	0.188	0.192	0.192	0.193	0.020	0.020	0.021	0.021	0.022
	-3	0.162	0.162	0.164	0.165	0.166	0.023	0.022	0.023	0.023	0.024
	-1	0.036	0.033	0.034	0.033	0.033	0.041	0.036	0.037	0.035	0.034
	1	0.421	0.428	0.431	0.436	0.441	0.000	0.000	0.000	0.000	0.003
	3	0.290	0.294	0.296	0.299	0.302	0.006	0.009	0.010	0.012	0.014
	5	0.264	0.267	0.270	0.272	0.275	0.010	0.011	0.013	0.014	0.016
	10	0.245	0.248	0.250	0.252	0.255	0.012	0.013	0.015	0.016	0.017
	30	0.232	0.234	0.237	0.239	0.241	0.014	0.015	0.016	0.017	0.018
90	-30	-0.032	-0.031	-0.030	-0.029	-0.028	-0.022	-0.019	-0.017	-0.015	-0.013
	-10	-0.077	-0.075	-0.074	-0.071	-0.069	-0.064	-0.057	-0.049	-0.043	-0.039
	-5	-0.145	-0.143	-0.140	-0.136	-0.132	-0.129	-0.113	-0.099	-0.086	-0.077
	-3	-0.236	-0.233	-0.228	-0.222	-0.215	-0.215	-0.189	-0.164	-0.144	-0.129
	-1	-0.689	-0.682	-0.668	-0.651	-0.631	-0.643	-0.567	-0.492	-0.431	-0.386
	1	0.670	0.666	0.652	0.636	0.617	0.643	0.566	0.492	0.431	0.386
	3	0.217	0.217	0.212	0.207	0.201	0.214	0.189	0.164	0.144	0.128
	5	0.127	0.127	0.124	0.122	0.118	0.128	0.113	0.098	0.086	0.077
	10	0.059	0.059	0.058	0.057	0.056	0.064	0.056	0.049	0.043	0.038
	30	0.013	0.014	0.014	0.014	0.014	0.021	0.019	0.016	0.014	0.013

Table 5.5: Normalized stress intensity factor, $K_I^* = K_I / E \eta_3 \alpha T (\pi a)^{1/2}$, for varying normalized crack length, a/W , with isotropic and quasi-isotropic material properties

<i>Edge-Crack</i>		<i>Quasi-isotropic</i>		<i>Central-Crack</i>		<i>Quasi-isotropic</i>	
Angle, θ [Deg]	a/W	$\eta_3=30$		Angle, θ [Deg]	a/W	$\eta_3=30$	
		BEM	FEM			BEM	FEM
0	0.1	1.082	1.110	0	0.1	0.684	0.706
	0.3	0.972	0.993		0.3	0.679	0.696
	0.5	0.808	0.826		0.5	0.666	0.681
30	0.1	0.847	0.842	30	0.1	0.540	0.535
	0.3	0.765	0.753		0.3	0.535	0.528
	0.5	0.641	0.626		0.5	0.525	0.516
45	0.1	0.587	0.573	45	0.1	0.376	0.365
	0.3	0.533	0.513		0.3	0.373	0.359
	0.5	0.449	0.427		0.5	0.366	0.352
60	0.1	0.318	0.305	60	0.1	0.205	0.194
	0.3	0.291	0.273		0.3	0.204	0.191
	0.5	0.247	0.227		0.5	0.200	0.187
90	0.1	0.023	0.037	90	0.1	0.014	0.024
	0.3	0.023	0.033		0.3	0.015	0.023
	0.5	0.019	0.028		0.5	0.016	0.023
<i>Quasi-isotropic</i>		<i>Isotropic</i>		<i>Quasi-isotropic</i>		<i>Isotropic</i>	
BEM	FEM	BEM	FEM	BEM	FEM	BEM	FEM
0.036	0.037	0.036	0.037	0.023	0.024	0.023	0.024
0.032	0.033	0.032	0.033	0.023	0.023	0.023	0.023
0.027	0.028	0.027	0.028	0.022	0.023	0.022	0.023

Table 6.1: Normalized stress intensity factors, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$, and $K_{II}^* = K_{II} / E\eta_3\alpha T(\pi a)^{1/2}$ with varying material angle, θ , and normalized crack length, a/W , for a uniformly cooled plate with an edge-crack; $\eta_1=2$, $\eta_2=0.5$

Angle, θ [Deg]	a/W	K_I^*			K_{II}^*		
		BEM	FEM	% Error	BEM	FEM	% Error
0	0.5	0.711	0.730	2.67	0.000	0.000	-
	0.4	0.792			0.000		
	0.3	0.884	0.908	2.71	0.000	0.000	-
	0.2	0.978			0.000		
	0.1	1.043	1.073	2.88	0.000	0.000	-
30	0.5	0.477	0.472	1.05	0.083	0.000	-
	0.4	0.519			0.084		
	0.3	0.557	0.556	0.18	0.085	0.000	-
	0.2	0.587			0.085		
	0.1	0.601	0.605	0.67	0.087	0.000	-
45	0.5	0.314	0.309	1.59	0.077	0.000	-
	0.4	0.340			0.077		
	0.3	0.360	0.357	0.83	0.079	0.000	-
	0.2	0.374			0.079		
	0.1	0.378	0.381	0.79	0.081	0.000	-
60	0.5	0.182	0.171	6.04	0.083	0.000	-
	0.4	0.197			0.083		
	0.3	0.211	0.201	4.74	0.084	0.000	-
	0.2	0.220			0.085		
	0.1	0.224	0.219	2.23	0.086	0.000	-
90	0.5	0.016	0.024	50.00	0.000	0.000	-
	0.4	0.018			0.000		
	0.3	0.020	0.030	50.00	0.000	0.000	-
	0.2	0.022			0.000		
	0.1	0.022	0.036	63.64	0.000	0.000	-

Table 6.2: Normalized stress intensity factors, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$, and $K_{II}^* = K_{II} / E\eta_3\alpha T(\pi a)^{1/2}$ with varying material angle, θ , and normalized crack length, a/W , for a uniformly cooled plate with an edge-crack; $\eta_1=3$, $\eta_2=0.5$

Angle, θ [Deg]	a/W	K_I^*			K_{II}^*		
		BEM	FEM	% Error	BEM	FEM	% Error
0	0.5	0.330	0.338	2.42	0.000	0.000	-
	0.4	0.366			0.000		
	0.3	0.405	0.415	2.47	0.000	0.000	-
	0.2	0.442			0.000		
	0.1	0.463	0.475	2.59	0.000	0.000	-
30	0.5	0.239	0.235	1.67	0.046	0.017	63.04
	0.4	0.258			0.048		
	0.3	0.274	0.271	1.09	0.050	0.020	60.00
	0.2	0.284			0.051		
	0.1	0.286	0.286	0.00	0.052	0.022	57.69
45	0.5	0.170	0.168	1.18	0.035	0.013	62.86
	0.4	0.182			0.036		
	0.3	0.191	0.189	1.05	0.037	0.015	59.46
	0.2	0.196			0.038		
	0.1	0.196	0.197	0.51	0.039	0.016	58.97
60	0.5	0.112	0.104	7.14	0.033	0.006	81.82
	0.4	0.121			0.034		
	0.3	0.128	0.122	4.69	0.035	0.008	77.14
	0.2	0.134			0.036		
	0.1	0.137	0.133	2.92	0.036	0.009	75.00
90	0.5	0.016	0.020	25.00	0.000	0.000	-
	0.4	0.018			0.000		
	0.3	0.021	0.026	23.81	0.000	0.000	-
	0.2	0.025			0.000		
	0.1	0.028	0.034	21.43	0.000	0.000	-

Table 6.3: Normalized stress intensity factors, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$, and $K_{II}^* = K_{II} / E\eta_3\alpha T(\pi a)^{1/2}$ with varying material angle, θ , and normalized crack length, a/W , for a uniformly cooled plate with an edge-crack; $\eta_1=5$, $\eta_2=0.5$

Angle, θ [Deg]	a/W	K_I^*			K_{II}^*		
		BEM	FEM	% Error	BEM	FEM	% Error
0	0.5	0.123	0.126	2.44	0.000	0.000	-
	0.4	0.136			0.000		
	0.3	0.150	0.153	2.00	0.000	0.000	-
	0.2	0.161			0.000		
	0.1	0.166	0.170	2.41	0.000	0.000	-
30	0.5	0.096	0.096	0.00	0.023	0.017	26.09
	0.4	0.102			0.025		
	0.3	0.107	0.108	0.93	0.027	0.020	25.93
	0.2	0.110			0.027		
	0.1	0.109	0.110	0.92	0.027	0.021	22.22
45	0.5	0.074	0.074	0.00	0.017	0.014	17.65
	0.4	0.078			0.019		
	0.3	0.080	0.081	1.25	0.019	0.015	21.05
	0.2	0.081			0.019		
	0.1	0.080	0.082	2.50	0.019	0.015	21.05
60	0.5	0.054	0.051	5.56	0.014	0.007	50.00
	0.4	0.057			0.014		
	0.3	0.061	0.058	4.92	0.015	0.008	46.67
	0.2	0.063			0.016		
	0.1	0.064	0.062	3.13	0.016	0.009	43.75
90	0.5	0.014	0.016	14.29	0.000	0.000	-
	0.4	0.016			0.000		
	0.3	0.018	0.021	16.67	0.000	0.000	-
	0.2	0.023			0.000		
	0.1	0.028	0.032	14.29	0.000	0.000	-

Table 6.4: Normalized stress intensity factors, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$, and $K_{II}^* = K_{II} / E\eta_3\alpha T(\pi a)^{1/2}$ with varying material angle, θ , and normalized crack length, a/W , for a uniformly cooled plate with an edge-crack; $\eta_1=7$, $\eta_2=0.5$

Angle, θ [Deg]	a/W	K_I^*			K_{II}^*		
		BEM	FEM	% Error	BEM	FEM	% Error
0	0.5	0.064	0.065	1.56	0.000	0.000	-
	0.4	0.071			0.000		
	0.3	0.077	0.079	2.60	0.000	0.000	-
	0.2	0.082			0.000		
	0.1	0.084	0.087	3.57	0.000	0.000	-
30	0.5	0.052	0.053	1.92	0.015	0.014	6.67
	0.4	0.055			0.016		
	0.3	0.058	0.059	1.72	0.017	0.016	5.88
	0.2	0.058			0.017		
	0.1	0.058	0.059	1.72	0.017	0.016	5.88
45	0.5	0.042	0.043	2.38	0.012	0.011	8.33
	0.4	0.044			0.013		
	0.3	0.045	0.046	2.22	0.013	0.012	7.69
	0.2	0.045			0.013		
	0.1	0.044	0.046	4.55	0.012	0.012	0.00
60	0.5	0.032	0.031	3.13	0.008	0.005	37.50
	0.4	0.033			0.008		
	0.3	0.035	0.035	0.00	0.009	0.006	33.33
	0.2	0.036			0.009		
	0.1	0.036	0.036	0.00	0.009	0.007	22.22
90	0.5	0.012	0.014	16.67	0.000	0.000	-
	0.4	0.014			0.000		
	0.3	0.016	0.018	12.50	0.000	0.000	-
	0.2	0.020			0.000		
	0.1	0.027	0.029	7.41	0.000	0.000	-

Table 6.5: Normalized stress intensity factors, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$, and $K_{II}^* = K_{II} / E\eta_3\alpha T(\pi a)^{1/2}$ with varying material angle, θ , and normalized crack length, a/W , for a uniformly cooled plate with an edge-crack; $\eta_1=2$, $\eta_2=0.75$

Angle, θ [Deg]	a/W	K_I^*			K_{II}^*		
		BEM	FEM	% Error	BEM	FEM	% Error
0	0.5	0.379	0.386	1.85	0.000	0.000	-
	0.4	0.415			0.000		
	0.3	0.447	0.455	1.79	0.000	0.000	-
	0.2	0.470			0.000		
	0.1	0.480	0.491	2.29	0.000	0.000	-
30	0.5	0.323	0.314	2.79	0.057	0.022	61.40
	0.4	0.351			0.060		
	0.3	0.375	0.367	2.13	0.062	0.027	56.45
	0.2	0.392			0.064		
	0.1	0.399	0.394	1.25	0.065	0.029	55.38
45	0.5	0.247	0.237	4.05	0.047	0.017	63.83
	0.4	0.268			0.050		
	0.3	0.288	0.279	3.12	0.052	0.022	57.69
	0.2	0.302			0.054		
	0.1	0.309	0.303	1.94	0.055	0.024	56.36
60	0.5	0.156	0.147	5.77	0.033	0.008	75.76
	0.4	0.170			0.034		
	0.3	0.185	0.177	4.32	0.036	0.010	72.22
	0.2	0.198			0.037		
	0.1	0.206	0.200	2.91	0.039	0.012	69.23
90	0.5	0.019	0.023	21.05	0.000	0.000	-
	0.4	0.021			0.000		
	0.3	0.024	0.029	20.83	0.000	0.000	-
	0.2	0.027			0.000		
	0.1	0.029	0.036	24.14	0.000	0.000	-

Table 6.6: Normalized stress intensity factors, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$, and $K_{II}^* = K_{II} / E\eta_3\alpha T(\pi a)^{1/2}$ with varying material angle, θ , and normalized crack length, a/W , for a uniformly cooled plate with an edge-crack; $\eta_1=3$, $\eta_2=0.75$

Angle, θ [Deg]	a/W	K_I^*			K_{II}^*		
		BEM	FEM	% Error	BEM	FEM	% Error
0	0.5	0.178	0.181	1.69	0.000	0.000	-
	0.4	0.193			0.000		
	0.3	0.205	0.209	1.95	0.000	0.000	-
	0.2	0.212			0.000		
	0.1	0.213	0.217	1.88	0.000	0.000	-
30	0.5	0.163	0.160	1.84	0.037	0.025	32.43
	0.4	0.175			0.040		
	0.3	0.185	0.182	1.62	0.042	0.030	28.57
	0.2	0.190			0.043		
	0.1	0.191	0.189	1.05	0.043	0.031	27.91
45	0.5	0.134	0.131	2.24	0.028	0.020	28.57
	0.4	0.145			0.031		
	0.3	0.153	0.151	1.31	0.032	0.024	25.00
	0.2	0.159			0.034		
	0.1	0.160	0.159	0.63	0.034	0.025	26.47
60	0.5	0.094	0.091	3.19	0.016	0.009	43.75
	0.4	0.102			0.018		
	0.3	0.111	0.109	1.80	0.020	0.012	40.00
	0.2	0.119			0.021		
	0.1	0.124	0.122	1.61	0.022	0.014	36.36
90	0.5	0.017	0.019	11.76	0.000	0.000	-
	0.4	0.020			0.000		
	0.3	0.023	0.025	8.70	0.000	0.000	-
	0.2	0.027			0.000		
	0.1	0.031	0.034	9.68	0.000	0.000	-

Table 6.7: Normalized stress intensity factors, $K_I^* = K_I / E \eta_3 \alpha T (\pi a)^{1/2}$, and $K_{II}^* = K_{II} / E \eta_3 \alpha T (\pi a)^{1/2}$ with varying material angle, θ , and normalized crack length, a/W , for a uniformly cooled plate with an edge-crack; $\eta_1=5$, $\eta_2=0.75$

Angle, θ [Deg]	a/W	K_I^*			K_{II}^*		
		BEM	FEM	% Error	BEM	FEM	% Error
0	0.5	0.067	0.068	1.49	0.000	0.000	-
	0.4	0.072			0.000		
	0.3	0.076	0.077	1.32	0.000	0.000	-
	0.2	0.077			0.000		
	0.1	0.076	0.078	2.63	0.000	0.000	-
30	0.5	0.066	0.066	0.00	0.020	0.019	5.00
	0.4	0.070			0.022		
	0.3	0.073	0.074	1.37	0.023	0.022	4.35
	0.2	0.074			0.023		
	0.1	0.073	0.074	1.37	0.023	0.021	8.70
45	0.5	0.059	0.059	0.00	0.016	0.015	6.25
	0.4	0.063			0.017		
	0.3	0.065	0.066	1.54	0.018	0.018	0.00
	0.2	0.066			0.019		
	0.1	0.066	0.067	1.52	0.018	0.018	0.00
60	0.5	0.044	0.045	2.27	0.008	0.007	12.50
	0.4	0.048			0.009		
	0.3	0.052	0.053	1.92	0.010	0.009	10.00
	0.2	0.056			0.011		
	0.1	0.057	0.058	1.75	0.012	0.011	8.33
90	0.5	0.015	0.016	6.67	0.000	0.000	-
	0.4	0.016			0.000		
	0.3	0.019	0.020	5.26	0.000	0.000	-
	0.2	0.023			0.000		
	0.1	0.029	0.031	6.90	0.000	0.000	-

Table 6.8: Normalized stress intensity factors, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$, and $K_{II}^* = K_{II} / E\eta_3\alpha T(\pi a)^{1/2}$ with varying material angle, θ , and normalized crack length, a/W , for a uniformly cooled plate with an edge-crack; $\eta_1=7$, $\eta_2=0.75$

Angle, θ [Deg]	a/W	K_I^*			K_{II}^*		
		BEM	FEM	% Error	BEM	FEM	% Error
0	0.5	0.035	0.036	2.86	0.000	0.000	-
	0.4	0.038			0.000		
	0.3	0.039	0.040	2.56	0.000	0.000	-
	0.2	0.039			0.000		
	0.1	0.039	0.039	0.00	0.000	0.000	-
30	0.5	0.036	0.037	2.78	0.013	0.014	7.69
	0.4	0.038			0.014		
	0.3	0.040	0.041	2.50	0.015	0.016	6.67
	0.2	0.040			0.015		
	0.1	0.039	0.040	2.56	0.015	0.015	0.00
45	0.5	0.033	0.035	6.06	0.011	0.011	0.00
	0.4	0.036			0.012		
	0.3	0.037	0.038	2.70	0.012	0.013	8.33
	0.2	0.037			0.012		
	0.1	0.036	0.038	5.56	0.012	0.013	8.33
60	0.5	0.026	0.027	3.85	0.005	0.005	0.00
	0.4	0.028			0.006		
	0.3	0.030	0.032	6.67	0.006	0.007	16.67
	0.2	0.032			0.007		
	0.1	0.032	0.034	6.25	0.007	0.008	14.29
90	0.5	0.013	0.013	0.00	0.000	0.000	-
	0.4	0.014			0.000		
	0.3	0.016	0.017	6.25	0.000	0.000	-
	0.2	0.020			0.000		
	0.1	0.027	0.028	3.70	0.000	0.000	-

Table 6.9: Normalized stress intensity factors, $K_I^* = K_I / E\eta_3 \alpha T(\pi a)^{1/2}$, and $K_{II}^* = K_{II} / E\eta_3 \alpha T(\pi a)^{1/2}$ with varying material angle, θ , and normalized crack length, a/W , for a uniformly cooled plate with an edge-crack; $\eta_1=2$, $\eta_2=1$

Angle, θ [Deg]	a/W	K_I^*			K_{II}^*		
		BEM	FEM	% Error	BEM	FEM	% Error
0	0.5	0.239	0.243	1.67	0.000	0.000	-
	0.4	0.257			0.000		
	0.3	0.270	0.274	1.48	0.000	0.000	-
	0.2	0.276			0.000		
	0.1	0.275	0.280	1.82	0.000	0.000	-
30	0.5	0.230	0.223	3.04	0.049	0.029	40.82
	0.4	0.249			0.052		
	0.3	0.264	0.257	2.65	0.055	0.035	36.36
	0.2	0.273			0.056		
	0.1	0.275	0.269	2.18	0.057	0.037	35.09
45	0.5	0.194	0.187	3.61	0.038	0.023	39.47
	0.4	0.211			0.042		
	0.3	0.227	0.220	3.08	0.045	0.029	35.56
	0.2	0.240			0.047		
	0.1	0.246	0.240	2.44	0.048	0.032	33.33
60	0.5	0.131	0.127	3.05	0.020	0.011	45.00
	0.4	0.144			0.022		
	0.3	0.159	0.155	2.52	0.024	0.015	37.50
	0.2	0.173			0.026		
	0.1	0.182	0.179	1.65	0.027	0.018	33.33
90	0.5	0.020	0.022	10.00	0.000	0.000	-
	0.4	0.022			0.000		
	0.3	0.025	0.028	12.00	0.000	0.000	-
	0.2	0.029			0.000		
	0.1	0.031	0.036	16.13	0.000	0.000	-

Table 6.10: Normalized stress intensity factors, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$, and $K_{II}^* = K_{II} / E\eta_3\alpha T(\pi a)^{1/2}$ with varying material angle, θ , and normalized crack length, a/W , for a uniformly cooled plate with an edge-crack; $\eta_1=3$, $\eta_2=1$

Angle, θ [Deg]	a/W	K_I^*			K_{II}^*		
		BEM	FEM	% Error	BEM	FEM	% Error
0	0.5	0.114	0.115	0.88	0.000	0.000	-
	0.4	0.121			0.000		
	0.3	0.124	0.126	1.61	0.000	0.000	-
	0.2	0.125			0.000		
	0.1	0.122	0.124	1.64	0.000	0.000	-
30	0.5	0.117	0.115	1.71	0.032	0.027	15.63
	0.4	0.125			0.035		
	0.3	0.131	0.129	1.53	0.037	0.031	16.22
	0.2	0.133			0.037		
	0.1	0.132	0.131	0.76	0.037	0.031	16.22
45	0.5	0.106	0.105	0.94	0.025	0.021	16.00
	0.4	0.115			0.027		
	0.3	0.122	0.121	0.82	0.030	0.026	13.33
	0.2	0.127			0.031		
	0.1	0.129	0.128	0.78	0.031	0.027	12.90
60	0.5	0.079	0.079	0.00	0.011	0.010	9.09
	0.4	0.087			0.013		
	0.3	0.096	0.096	0.00	0.015	0.014	6.67
	0.2	0.105			0.017		
	0.1	0.110	0.111	0.91	0.018	0.017	5.56
90	0.5	0.017	0.019	11.76	0.000	0.000	-
	0.4	0.020			0.000		
	0.3	0.023	0.024	4.35	0.000	0.000	-
	0.2	0.027			0.000		
	0.1	0.032	0.034	6.25	0.000	0.000	-

Table 6.11: Normalized stress intensity factors, $K_I^* = K_I / E \eta_3 \alpha T (\pi a)^{1/2}$, and $K_{II}^* = K_{II} / E \eta_3 \alpha T (\pi a)^{1/2}$ with varying material angle, θ , and normalized crack length, a/W , for a uniformly cooled plate with an edge-crack; $\eta_1=5$, $\eta_2=1$

Angle, θ [Deg]	a/W	K_I^*			K_{II}^*		
		BEM	FEM	% Error	BEM	FEM	% Error
0	0.5	0.043	0.044	2.33	0.000	0.000	-
	0.4	0.045			0.000		
	0.3	0.046	0.047	2.17	0.000	0.000	-
	0.2	0.045			0.000		
	0.1	0.044	0.044	0.00	0.000	0.000	-
30	0.5	0.048	0.049	2.08	0.017	0.018	5.88
	0.4	0.051			0.019		
	0.3	0.052	0.053	1.92	0.019	0.020	5.26
	0.2	0.052			0.019		
	0.1	0.051	0.052	1.96	0.019	0.019	0.00
45	0.5	0.047	0.048	2.13	0.014	0.015	7.14
	0.4	0.051			0.016		
	0.3	0.053	0.054	1.89	0.017	0.017	0.00
	0.2	0.054			0.017		
	0.1	0.053	0.055	3.77	0.017	0.018	5.88
60	0.5	0.038	0.039	2.63	0.007	0.007	0.00
	0.4	0.042			0.008		
	0.3	0.046	0.048	4.35	0.009	0.010	11.11
	0.2	0.050			0.010		
	0.1	0.052	0.053	1.92	0.011	0.011	0.00
90	0.5	0.014	0.015	7.14	0.000	0.000	-
	0.4	0.016			0.000		
	0.3	0.019	0.020	5.26	0.000	0.000	-
	0.2	0.023			0.000		
	0.1	0.030	0.031	3.33	0.000	0.000	-

Table 6.12: Normalized stress intensity factors, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$, and $K_{II}^* = K_{II} / E\eta_3\alpha T(\pi a)^{1/2}$ with varying material angle, θ , and normalized crack length, a/W , for a uniformly cooled plate with an edge-crack; $\eta_1=7$, $\eta_2=1$

Angle, θ [Deg]	a/W	K_I^*			K_{II}^*		
		BEM	FEM	% Error	BEM	FEM	% Error
0	0.5	0.023	0.023	0.00	0.000	0.000	-
	0.4	0.024			0.000		
	0.3	0.024	0.024	0.00	0.000	0.000	-
	0.2	0.023			0.000		
	0.1	0.022	0.023	4.55	0.000	0.000	-
30	0.5	0.026	0.028	7.69	0.011	0.013	18.18
	0.4	0.027			0.011		
	0.3	0.028	0.030	7.14	0.012	0.014	16.67
	0.2	0.027			0.011		
	0.1	0.027	0.028	3.70	0.011	0.013	18.18
45	0.5	0.027	0.028	3.70	0.009	0.011	22.22
	0.4	0.029			0.011		
	0.3	0.030	0.031	3.33	0.011	0.012	9.09
	0.2	0.030			0.011		
	0.1	0.029	0.031	6.90	0.011	0.012	9.09
60	0.5	0.023	0.024	4.35	0.005	0.005	0.00
	0.4	0.025			0.006		
	0.3	0.027	0.029	7.41	0.006	0.007	16.67
	0.2	0.030			0.007		
	0.1	0.030	0.032	6.67	0.007	0.008	14.29
90	0.5	0.013	0.013	0.00	0.000	0.000	-
	0.4	0.014			0.000		
	0.3	0.016	0.017	6.25	0.000	0.000	-
	0.2	0.020			0.000		
	0.1	0.027	0.028	3.70	0.000	0.000	-

Table 6.13: Normalized stress intensity factors, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$, and $K_{II}^* = K_{II} / E\eta_3\alpha T(\pi a)^{1/2}$ with varying material angle, θ , and normalized crack length, a/W , for a uniformly cooled plate with a central-crack; $\eta_1=2$, $\eta_2=0.5$

Angle, θ [Deg]	a/W	K_I^*			K_{II}^*		
		BEM	FEM	% Error	BEM	FEM	% Error
0	0.5	0.624	0.641	2.72	0.000	0.000	-
	0.4	0.644			0.000		
	0.3	0.660	0.679	2.88	0.000	0.000	-
	0.2	0.674			0.000		
	0.1	0.680	0.704	3.53	0.000	0.000	-
30	0.5	0.378	0.377	0.26	0.060	0.000	-
	0.4	0.377			0.060		
	0.3	0.376	0.377	0.27	0.062	0.000	-
	0.2	0.375			0.061		
	0.1	0.373	0.377	1.07	0.068	0.000	-
45	0.5	0.241	0.239	0.83	0.056	0.000	-
	0.4	0.239			0.056		
	0.3	0.237	0.236	0.42	0.058	0.000	-
	0.2	0.234			0.057		
	0.1	0.232	0.234	0.86	0.063	0.000	-
60	0.5	0.143	0.137	4.20	0.060	0.000	-
	0.4	0.143			0.060		
	0.3	0.142	0.137	3.52	0.062	0.000	-
	0.2	0.141			0.061		
	0.1	0.141	0.137	2.84	0.068	0.000	-
90	0.5	0.014	0.021	50.00	0.000	0.000	-
	0.4	0.014			0.000		
	0.3	0.014	0.023	64.29	0.000	0.000	-
	0.2	0.014			0.000		
	0.1	0.013	0.023	76.92	0.000	0.000	-

Table 6.14: Normalized stress intensity factors, $K_I^* = K_I / E\eta_3 \alpha T (\pi a)^{1/2}$, and $K_{II}^* = K_{II} / E\eta_3 \alpha T (\pi a)^{1/2}$ with varying material angle, θ , and normalized crack length, a/W , for a uniformly cooled plate with a central-crack; $\eta_1=3$, $\eta_2=0.5$

Angle, θ [Deg]	a/W	K_I^*			K_{II}^*		
		BEM	FEM	% Error	BEM	FEM	% Error
0	0.5	0.280	0.287	2.50	0.000	0.000	-
	0.4	0.288			0.000		
	0.3	0.295	0.303	2.71	0.000	0.000	-
	0.2	0.301			0.000		
	0.1	0.303	0.313	3.30	0.000	0.000	-
30	0.5	0.182	0.180	1.10	0.026	0.005	80.77
	0.4	0.180			0.025		
	0.3	0.178	0.177	0.56	0.024	0.002	91.67
	0.2	0.176			0.023		
	0.1	0.175	0.175	0.00	0.024	0.000	-
45	0.5	0.126	0.125	0.79	0.020	0.004	80.00
	0.4	0.123			0.019		
	0.3	0.121	0.120	0.83	0.018	0.002	88.89
	0.2	0.119			0.017		
	0.1	0.117	0.117	0.00	0.018	0.000	-
60	0.5	0.088	0.083	5.68	0.022	0.002	90.91
	0.4	0.087			0.021		
	0.3	0.086	0.081	5.81	0.021	0.001	95.24
	0.2	0.084			0.020		
	0.1	0.084	0.080	4.76	0.022	0.000	-
90	0.5	0.016	0.019	18.75	0.000	0.000	-
	0.4	0.017			0.000		
	0.3	0.018	0.022	22.22	0.000	0.000	-
	0.2	0.019			0.000		
	0.1	0.019	0.023	21.05	0.000	0.000	-

Table 6.15: Normalized stress intensity factors, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$, and $K_{II}^* = K_{II} / E\eta_3\alpha T(\pi a)^{1/2}$ with varying material angle, θ , and normalized crack length, a/W , for a uniformly cooled plate with a central-crack; $\eta_1=5$, $\eta_2=0.5$

Angle, θ [Deg]	a/W	K_I^*			K_{II}^*		
		BEM	FEM	% Error	BEM	FEM	% Error
0	0.5	0.102	0.104	1.96	0.000	0.000	-
	0.4	0.104			0.000		
	0.3	0.106	0.109	2.83	0.000	0.000	-
	0.2	0.108			0.000		
	0.1	0.109	0.113	3.67	0.000	0.000	-
30	0.5	0.069	0.070	1.45	0.010	0.005	50.00
	0.4	0.068			0.008		
	0.3	0.066	0.067	1.52	0.007	0.002	71.43
	0.2	0.065			0.006		
	0.1	0.064	0.065	1.56	0.005	0.000	-
45	0.5	0.052	0.052	0.00	0.007	0.005	28.57
	0.4	0.050			0.006		
	0.3	0.047	0.048	2.13	0.005	0.002	60.00
	0.2	0.046			0.004		
	0.1	0.044	0.045	2.27	0.004	0.000	-
60	0.5	0.041	0.039	4.88	0.008	0.002	75.00
	0.4	0.040			0.007		
	0.3	0.038	0.037	2.63	0.006	0.001	83.33
	0.2	0.037			0.006		
	0.1	0.036	0.035	2.78	0.006	0.000	-
90	0.5	0.014	0.016	14.29	0.000	0.000	-
	0.4	0.016			0.000		
	0.3	0.018	0.019	5.56	0.000	0.000	-
	0.2	0.019			0.000		
	0.1	0.021	0.023	9.52	0.000	0.000	-

Table 6.16: Normalized stress intensity factors, $K_I^* = K_I / E\eta_3 \alpha T (\pi a)^{1/2}$, and $K_{II}^* = K_{II} / E\eta_3 \alpha T (\pi a)^{1/2}$ with varying material angle, θ , and normalized crack length, a/W , for a uniformly cooled plate with a central-crack; $\eta_1=7$, $\eta_2=0.5$

Angle, θ [Deg]	a/W	K_I^*			K_{II}^*		
		BEM	FEM	% Error	BEM	FEM	% Error
0	0.5	0.052	0.053	1.92	0.000	0.000	-
	0.4	0.053			0.000		
	0.3	0.054	0.056	3.70	0.000	0.000	-
	0.2	0.055			0.000		
	0.1	0.056	0.058	3.57	0.000	0.000	-
30	0.5	0.036	0.037	2.78	0.006	0.004	33.33
	0.4	0.035			0.004		
	0.3	0.034	0.035	2.94	0.003	0.002	33.33
	0.2	0.033			0.002		
	0.1	0.032	0.033	3.13	0.002	0.000	-
45	0.5	0.028	0.029	3.57	0.005	0.004	20.00
	0.4	0.027			0.004		
	0.3	0.025	0.026	4.00	0.003	0.002	33.33
	0.2	0.024			0.002		
	0.1	0.022	0.024	9.09	0.001	0.000	-
60	0.5	0.023	0.023	0.00	0.004	0.002	50.00
	0.4	0.022			0.003		
	0.3	0.021	0.021	0.00	0.003	0.002	33.33
	0.2	0.020			0.002		
	0.1	0.019	0.019	0.00	0.002	0.000	-
90	0.5	0.013	0.014	7.69	0.000	0.000	-
	0.4	0.014			0.000		
	0.3	0.016	0.017	6.25	0.000	0.000	-
	0.2	0.018			0.000		
	0.1	0.021	0.022	4.76	0.000	0.000	-

Table 6.17: Normalized stress intensity factors, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$, and $K_{II}^* = K_{II} / E\eta_3\alpha T(\pi a)^{1/2}$ with varying material angle, θ , and normalized crack length, a/W , for a uniformly cooled plate with a central-crack; $\eta_1=2$, $\eta_2=0.75$

Angle, θ [Deg]	a/W	K_I^*			K_{II}^*		
		BEM	FEM	% Error	BEM	FEM	% Error
0	0.5	0.299	0.305	2.01	0.000	0.000	-
	0.4	0.301			0.000		
	0.3	0.303	0.310	2.31	0.000	0.000	-
	0.2	0.305			0.000		
	0.1	0.304	0.314	3.29	0.000	0.000	-
30	0.5	0.251	0.245	2.39	0.031	0.006	80.65
	0.4	0.251			0.029		
	0.3	0.250	0.245	2.00	0.028	0.002	92.86
	0.2	0.250			0.027		
	0.1	0.250	0.245	2.00	0.029	0.000	-
45	0.5	0.194	0.188	3.09	0.027	0.005	81.48
	0.4	0.194			0.025		
	0.3	0.193	0.187	3.11	0.025	0.002	92.00
	0.2	0.193			0.023		
	0.1	0.192	0.187	2.60	0.025	0.000	-
60	0.5	0.128	0.122	4.69	0.021	0.003	85.71
	0.4	0.129			0.020		
	0.3	0.130	0.124	4.62	0.020	0.001	95.00
	0.2	0.130			0.019		
	0.1	0.130	0.125	3.85	0.021	0.000	-
90	0.5	0.018	0.021	16.67	0.000	0.000	-
	0.4	0.018			0.000		
	0.3	0.019	0.023	21.05	0.000	0.000	-
	0.2	0.019			0.000		
	0.1	0.019	0.023	21.05	0.000	0.000	-

Table 6.18: Normalized stress intensity factors, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$, and $K_{II}^* = K_{II} / E\eta_3\alpha T(\pi a)^{1/2}$ with varying material angle, θ , and normalized crack length, a/W , for a uniformly cooled plate with a central-crack; $\eta_1=3$, $\eta_2=0.75$

Angle, θ [Deg]	a/W	K_I^*			K_{II}^*		
		BEM	FEM	% Error	BEM	FEM	% Error
0	0.5	0.134	0.137	2.24	0.000	0.000	-
	0.4	0.135			0.000		
	0.3	0.135	0.138	2.22	0.000	0.000	-
	0.2	0.136			0.000		
	0.1	0.135	0.139	2.96	0.000	0.000	-
30	0.5	0.120	0.118	1.67	0.016	0.007	56.25
	0.4	0.119			0.013		
	0.3	0.118	0.116	1.69	0.012	0.003	75.00
	0.2	0.116			0.010		
	0.1	0.116	0.114	1.72	0.010	0.000	-
45	0.5	0.101	0.099	1.98	0.012	0.006	50.00
	0.4	0.099			0.010		
	0.3	0.098	0.096	2.04	0.009	0.003	66.67
	0.2	0.096			0.008		
	0.1	0.095	0.094	1.05	0.007	0.000	-
60	0.5	0.077	0.075	2.60	0.009	0.003	66.67
	0.4	0.076			0.008		
	0.3	0.076	0.074	2.63	0.007	0.001	85.71
	0.2	0.075			0.006		
	0.1	0.075	0.073	2.67	0.006	0.000	-
90	0.5	0.017	0.019	11.76	0.000	0.000	-
	0.4	0.018			0.000		
	0.3	0.020	0.022	10.00	0.000	0.000	-
	0.2	0.020			0.000		
	0.1	0.021	0.023	9.52	0.000	0.000	-

Table 6.19: Normalized stress intensity factors, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$, and $K_{II}^* = K_{II} / E\eta_3\alpha T(\pi a)^{1/2}$ with varying material angle, θ , and normalized crack length, a/W , for a uniformly cooled plate with a central-crack; $\eta_1=5$, $\eta_2=0.75$

Angle, θ [Deg]	a/W	K_I^*			K_{II}^*		
		BEM	FEM	% Error	BEM	FEM	% Error
0	0.5	0.048	0.049	2.08	0.000	0.000	-
	0.4	0.049			0.000		
	0.3	0.049	0.050	2.04	0.000	0.000	-
	0.2	0.049			0.000		
	0.1	0.049	0.050	2.04	0.000	0.000	-
30	0.5	0.046	0.046	0.00	0.006	0.005	16.67
	0.4	0.045			0.005		
	0.3	0.043	0.044	2.33	0.004	0.002	50.00
	0.2	0.042			0.002		
	0.1	0.041	0.042	2.44	0.002	0.000	-
45	0.5	0.042	0.042	0.00	0.006	0.005	16.67
	0.4	0.040			0.004		
	0.3	0.038	0.039	2.63	0.003	0.002	33.33
	0.2	0.037			0.002		
	0.1	0.035	0.036	2.86	0.001	0.000	-
60	0.5	0.035	0.036	2.86	0.004	0.003	25.00
	0.4	0.034			0.003		
	0.3	0.033	0.033	0.00	0.002	0.001	50.00
	0.2	0.032			0.001		
	0.1	0.031	0.032	3.23	0.001	0.000	-
90	0.5	0.015	0.016	6.67	0.000	0.000	-
	0.4	0.016			0.000		
	0.3	0.018	0.019	5.56	0.000	0.000	-
	0.2	0.020			0.000		
	0.1	0.021	0.023	9.52	0.000	0.000	-

Table 6.20: Normalized stress intensity factors, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$, and $K_{II}^* = K_{II} / E\eta_3\alpha T(\pi a)^{1/2}$ with varying material angle, θ , and normalized crack length, a/W , for a uniformly cooled plate with a central-crack; $\eta_1=7$, $\eta_2=0.75$

Angle, θ [Deg]	a/W	K_I^*			K_{II}^*		
		BEM	FEM	% Error	BEM	FEM	% Error
0	0.5	0.025	0.025	0.00	0.000	0.000	-
	0.4	0.025			0.000		
	0.3	0.025	0.025	0.00	0.000	0.000	-
	0.2	0.025			0.000		
	0.1	0.025	0.026	4.00	0.000	0.000	-
30	0.5	0.024	0.025	4.17	0.004	0.004	0.00
	0.4	0.023			0.002		
	0.3	0.022	0.023	4.55	0.002	0.002	0.00
	0.2	0.021			0.000		
	0.1	0.021	0.022	4.76	0.000	0.000	-
45	0.5	0.023	0.024	4.35	0.004	0.004	0.00
	0.4	0.022			0.003		
	0.3	0.020	0.021	5.00	0.002	0.002	0.00
	0.2	0.019			0.001		
	0.1	0.018	0.019	5.56	0.000	0.000	-
60	0.5	0.020	0.021	5.00	0.002	0.002	0.00
	0.4	0.019			0.001		
	0.3	0.018	0.019	5.56	0.001	0.001	0.00
	0.2	0.017			0.000		
	0.1	0.016	0.017	6.25	0.000	0.000	-
90	0.5	0.013	0.013	0.00	0.000	0.000	-
	0.4	0.014			0.000		
	0.3	0.016	0.017	6.25	0.000	0.000	-
	0.2	0.019			0.000		
	0.1	0.021	0.022	4.76	0.000	0.000	-

Table 6.21: Normalized stress intensity factors, $K_I^* = K_I / E \eta_3 \alpha T (\pi a)^{1/2}$, and $K_{II}^* = K_{II} / E \eta_3 \alpha T (\pi a)^{1/2}$ with varying material angle, θ , and normalized crack length, a/W , for a uniformly cooled plate with a central-crack; $\eta_1=2$, $\eta_2=1$

Angle, θ [Deg]	a/W	K_I^*			K_{II}^*		
		BEM	FEM	% Error	BEM	FEM	% Error
0	0.5	0.175	0.178	1.71	0.000	0.000	-
	0.4	0.174			0.000		
	0.3	0.173	0.177	2.31	0.000	0.000	-
	0.2	0.173			0.000		
	0.1	0.172	0.177	2.91	0.000	0.000	-
30	0.5	0.174	0.168	3.45	0.022	0.008	63.64
	0.4	0.172			0.019		
	0.3	0.171	0.166	2.92	0.018	0.003	83.33
	0.2	0.170			0.016		
	0.1	0.170	0.165	2.94	0.016	0.000	-
45	0.5	0.153	0.148	3.27	0.018	0.007	61.11
	0.4	0.153			0.016		
	0.3	0.152	0.147	3.29	0.014	0.003	78.57
	0.2	0.151			0.013		
	0.1	0.151	0.146	3.31	0.013	0.000	-
60	0.5	0.111	0.108	2.70	0.010	0.004	60.00
	0.4	0.112			0.009		
	0.3	0.113	0.110	2.65	0.009	0.002	77.78
	0.2	0.113			0.008		
	0.1	0.114	0.111	2.63	0.008	0.000	-
90	0.5	0.019	0.021	10.53	0.000	0.000	-
	0.4	0.020			0.000		
	0.3	0.020	0.022	10.00	0.000	0.000	-
	0.2	0.021			0.000		
	0.1	0.021	0.023	9.52	0.000	0.000	-

Table 6.22: Normalized stress intensity factors, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$, and $K_{II}^* = K_{II} / E\eta_3\alpha T(\pi a)^{1/2}$ with varying material angle, θ , and normalized crack length, a/W , for a uniformly cooled plate with a central-crack; $\eta_1=3$, $\eta_2=1$

Angle, θ [Deg]	a/W	K_I^*			K_{II}^*		
		BEM	FEM	% Error	BEM	FEM	% Error
0	0.5	0.079	0.080	1.27	0.000	0.000	-
	0.4	0.078			0.000		
	0.3	0.077	0.079	2.60	0.000	0.000	-
	0.2	0.077			0.000		
	0.1	0.076	0.079	3.95	0.000	0.000	-
30	0.5	0.083	0.082	1.20	0.011	0.007	36.36
	0.4	0.082			0.009		
	0.3	0.080	0.079	1.25	0.007	0.003	57.14
	0.2	0.079			0.006		
	0.1	0.078	0.077	1.28	0.005	0.000	-
45	0.5	0.080	0.079	1.25	0.009	0.006	33.33
	0.4	0.078			0.007		
	0.3	0.077	0.076	1.30	0.006	0.003	50.00
	0.2	0.075			0.004		
	0.1	0.074	0.073	1.35	0.004	0.000	-
60	0.5	0.067	0.067	0.00	0.005	0.004	20.00
	0.4	0.067			0.004		
	0.3	0.066	0.066	0.00	0.003	0.002	33.33
	0.2	0.065			0.002		
	0.1	0.065	0.065	0.00	0.002	0.000	-
90	0.5	0.018	0.019	5.56	0.000	0.000	-
	0.4	0.019			0.000		
	0.3	0.020	0.021	5.00	0.000	0.000	-
	0.2	0.021			0.000		
	0.1	0.022	0.023	4.55	0.000	0.000	-

Table 6.23: Normalized stress intensity factors, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$, and $K_{II}^* = K_{II} / E\eta_3\alpha T(\pi a)^{1/2}$ with varying material angle, θ , and normalized crack length, a/W , for a uniformly cooled plate with a central-crack; $\eta_1=5$, $\eta_2=1$

Angle, θ [Deg]	a/W	K_I^*			K_{II}^*		
		BEM	FEM	% Error	BEM	FEM	% Error
0	0.5	0.028	0.029	3.57	0.000	0.000	-
	0.4	0.028			0.000		
	0.3	0.028	0.028	0.00	0.000	0.000	-
	0.2	0.028			0.000		
	0.1	0.028	0.028	0.00	0.000	0.000	-
30	0.5	0.032	0.032	0.00	0.004	0.005	25.00
	0.4	0.030			0.003		
	0.3	0.029	0.030	3.45	0.002	0.002	0.00
	0.2	0.028			0.000		
	0.1	0.027	0.028	3.70	0.000	0.000	-
45	0.5	0.033	0.034	3.03	0.004	0.005	25.00
	0.4	0.032			0.003		
	0.3	0.030	0.031	3.33	0.002	0.002	0.00
	0.2	0.029			0.001		
	0.1	0.027	0.028	3.70	0.000	0.000	-
60	0.5	0.031	0.032	3.23	0.003	0.003	0.00
	0.4	0.030			0.002		
	0.3	0.029	0.030	3.45	0.001	0.001	0.00
	0.2	0.028			0.001		
	0.1	0.027	0.028	3.70	0.000	0.000	-
90	0.5	0.015	0.016	6.67	0.000	0.000	-
	0.4	0.016			0.000		
	0.3	0.018	0.019	5.56	0.000	0.000	-
	0.2	0.020			0.000		
	0.1	0.022	0.023	4.55	0.000	0.000	-

Table 6.24: Normalized stress intensity factors, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$, and $K_{II}^* = K_{II} / E\eta_3\alpha T(\pi a)^{1/2}$ with varying material angle, θ , and normalized crack length, a/W , for a uniformly cooled plate with a central-crack; $\eta_1=7$, $\eta_2=1$

Angle, θ [Deg]	a/W	K_I^*			K_{II}^*		
		BEM	FEM	% Error	BEM	FEM	% Error
0	0.5	0.014	0.015	7.14	0.000	0.000	-
	0.4	0.014			0.000		
	0.3	0.014	0.015	7.14	0.000	0.000	-
	0.2	0.014			0.000		
	0.1	0.014	0.014	0.00	0.000	0.000	-
30	0.5	0.016	0.017	6.25	0.002	0.003	50.00
	0.4	0.015			0.001		
	0.3	0.015	0.016	6.67	0.000	0.001	-
	0.2	0.014			0.000		
	0.1	0.013	0.015	15.38	0.000	0.000	-
45	0.5	0.018	0.019	5.56	0.003	0.004	33.33
	0.4	0.017			0.002		
	0.3	0.016	0.017	6.25	0.001	0.002	100.00
	0.2	0.015			0.000		
	0.1	0.014	0.015	7.14	0.000	0.000	-
60	0.5	0.018	0.019	5.56	0.002	0.002	0.00
	0.4	0.017			0.001		
	0.3	0.016	0.017	6.25	0.001	0.001	0.00
	0.2	0.015			0.000		
	0.1	0.014	0.015	7.14	0.000	0.000	-
90	0.5	0.013	0.013	0.00	0.000	0.000	-
	0.4	0.014			0.000		
	0.3	0.016	0.017	6.25	0.000	0.000	-
	0.2	0.019			0.000		
	0.1	0.021	0.022	4.76	0.000	0.000	-

Table 6.25: Normalized stress intensity factors, $K_I^* = K_I / E\eta_3 \alpha T (\pi a)^{1/2}$, and $K_{II}^* = K_{II} / E\eta_3 \alpha T (\pi a)^{1/2}$ with varying material angle, θ , and normalized crack length, a/W , for a uniformly cooled plate with a double edge-crack; $\eta_1 = 2$

Angle, θ [Deg]	a/W	K_I^*			K_{II}^*		
		$\eta_2 = 0.5$	$\eta_2 = 0.75$	$\eta_2 = 1$	$\eta_2 = 0.5$	$\eta_2 = 0.75$	$\eta_2 = 1$
0	0.5	0.623	0.298	0.175	0.000	0.000	0.000
	0.4	0.743	0.365	0.217	0.000	0.000	0.000
	0.3	0.907	0.459	0.277	0.000	0.000	0.000
	0.2	1.187	0.619	0.381	0.000	0.000	0.000
	0.1	1.880	1.014	0.640	0.000	0.000	0.000
30	0.5	0.377	0.251	0.173	0.060	0.031	0.022
	0.4	0.463	0.307	0.214	0.060	0.035	0.027
	0.3	0.581	0.385	0.270	0.061	0.041	0.034
	0.2	0.780	0.518	0.365	0.061	0.050	0.046
	0.1	1.260	0.839	0.596	0.062	0.070	0.072
45	0.5	0.241	0.194	0.153	0.056	0.027	0.018
	0.4	0.298	0.238	0.187	0.056	0.030	0.022
	0.3	0.376	0.297	0.234	0.057	0.035	0.028
	0.2	0.509	0.397	0.313	0.057	0.042	0.038
	0.1	0.827	0.638	0.501	0.058	0.058	0.059
60	0.5	0.143	0.128	0.111	0.060	0.021	0.010
	0.4	0.174	0.155	0.134	0.060	0.023	0.013
	0.3	0.216	0.191	0.165	0.061	0.025	0.016
	0.2	0.288	0.251	0.216	0.061	0.029	0.021
	0.1	0.461	0.396	0.339	0.062	0.036	0.031
90	0.5	0.014	0.018	0.019	0.000	0.000	0.000
	0.4	0.018	0.021	0.022	0.000	0.000	0.000
	0.3	0.023	0.026	0.027	0.000	0.000	0.000
	0.2	0.031	0.034	0.034	0.000	0.000	0.000
	0.1	0.053	0.055	0.053	0.000	0.000	0.000

Table 6.26: Normalized stress intensity factors, $K_I^* = K_I / E\eta_3 \alpha T (\pi a)^{1/2}$, and $K_{II}^* = K_{II} / E\eta_3 \alpha T (\pi a)^{1/2}$ with varying material angle, θ , and normalized crack length, a/W , for a uniformly cooled plate with a double edge-crack; $\eta_1 = 3$

Angle, θ [Deg]	a/W	K_I^*			K_{II}^*		
		$\eta_2 = 0.5$	$\eta_2 = 0.75$	$\eta_2 = 1$	$\eta_2 = 0.5$	$\eta_2 = 0.75$	$\eta_2 = 1$
0	0.5	0.280	0.134	0.078	0.000	0.000	0.000
	0.4	0.335	0.165	0.098	0.000	0.000	0.000
	0.3	0.412	0.209	0.126	0.000	0.000	0.000
	0.2	0.543	0.284	0.176	0.000	0.000	0.000
	0.1	0.870	0.473	0.300	0.000	0.000	0.000
30	0.5	0.182	0.120	0.083	0.026	0.016	0.011
	0.4	0.225	0.149	0.104	0.029	0.020	0.015
	0.3	0.285	0.190	0.134	0.034	0.026	0.022
	0.2	0.386	0.258	0.184	0.041	0.036	0.032
	0.1	0.629	0.423	0.304	0.056	0.058	0.055
45	0.5	0.126	0.101	0.080	0.020	0.012	0.009
	0.4	0.158	0.126	0.100	0.022	0.016	0.013
	0.3	0.202	0.160	0.127	0.026	0.021	0.018
	0.2	0.275	0.216	0.172	0.031	0.029	0.027
	0.1	0.452	0.351	0.278	0.043	0.047	0.046
60	0.5	0.088	0.077	0.067	0.022	0.009	0.005
	0.4	0.107	0.094	0.082	0.023	0.011	0.007
	0.3	0.135	0.118	0.102	0.025	0.013	0.010
	0.2	0.180	0.156	0.135	0.028	0.018	0.015
	0.1	0.287	0.246	0.211	0.034	0.027	0.024
90	0.5	0.016	0.017	0.018	0.000	0.000	0.000
	0.4	0.019	0.020	0.020	0.000	0.000	0.000
	0.3	0.022	0.024	0.024	0.000	0.000	0.000
	0.2	0.029	0.030	0.030	0.000	0.000	0.000
	0.1	0.046	0.046	0.045	0.000	0.000	0.000

Table 6.27: Normalized stress intensity factors, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$, and $K_{II}^* = K_{II} / E\eta_3\alpha T(\pi a)^{1/2}$ with varying material angle, θ , and normalized crack length, a/W , for a uniformly cooled plate with a double edge-crack; $\eta_1=5$

Angle, θ [Deg]	a/W	K_I^*			K_{II}^*		
		$\eta_2 = 0.5$	$\eta_2 = 0.75$	$\eta_2 = 1$	$\eta_2 = 0.5$	$\eta_2 = 0.75$	$\eta_2 = 1$
0	0.5	0.101	0.048	0.028	0.000	0.000	0.000
	0.4	0.122	0.060	0.036	0.000	0.000	0.000
	0.3	0.150	0.076	0.046	0.000	0.000	0.000
	0.2	0.199	0.104	0.065	0.000	0.000	0.000
	0.1	0.322	0.176	0.113	0.000	0.000	0.000
30	0.5	0.069	0.046	0.032	0.010	0.006	0.004
	0.4	0.087	0.058	0.041	0.013	0.009	0.007
	0.3	0.111	0.075	0.053	0.017	0.013	0.011
	0.2	0.153	0.103	0.074	0.024	0.021	0.018
	0.1	0.252	0.172	0.125	0.039	0.036	0.032
45	0.5	0.052	0.042	0.033	0.008	0.006	0.004
	0.4	0.066	0.053	0.042	0.010	0.008	0.007
	0.3	0.086	0.069	0.055	0.013	0.012	0.011
	0.2	0.119	0.094	0.075	0.019	0.018	0.016
	0.1	0.198	0.155	0.124	0.031	0.031	0.029
60	0.5	0.041	0.035	0.031	0.008	0.004	0.003
	0.4	0.051	0.045	0.039	0.009	0.005	0.004
	0.3	0.065	0.057	0.050	0.011	0.007	0.006
	0.2	0.089	0.076	0.067	0.014	0.011	0.010
	0.1	0.142	0.122	0.105	0.020	0.018	0.016
90	0.5	0.014	0.015	0.015	0.000	0.000	0.000
	0.4	0.016	0.017	0.017	0.000	0.000	0.000
	0.3	0.019	0.019	0.019	0.000	0.000	0.000
	0.2	0.024	0.024	0.024	0.000	0.000	0.000
	0.1	0.036	0.036	0.035	0.000	0.000	0.000

Table 6.28: Normalized stress intensity factors, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$, and $K_{II}^* = K_{II} / E\eta_3\alpha T(\pi a)^{1/2}$ with varying material angle, θ , and normalized crack length, a/W , for a uniformly cooled plate with a double edge-crack; $\eta_1=7$

Angle, θ [Deg]	a/W	K_I^*			K_{II}^*		
		$\eta_2 = 0.5$	$\eta_2 = 0.75$	$\eta_2 = 1$	$\eta_2 = 0.5$	$\eta_2 = 0.75$	$\eta_2 = 1$
0	0.5	0.052	0.025	0.014	0.000	0.000	0.000
	0.4	0.062	0.031	0.018	0.000	0.000	0.000
	0.3	0.077	0.039	0.024	0.000	0.000	0.000
	0.2	0.102	0.053	0.033	0.000	0.000	0.000
	0.1	0.166	0.091	0.058	0.000	0.000	0.000
30	0.5	0.036	0.024	0.016	0.005	0.003	0.002
	0.4	0.046	0.031	0.021	0.008	0.006	0.004
	0.3	0.059	0.040	0.028	0.011	0.009	0.006
	0.2	0.083	0.056	0.040	0.017	0.014	0.012
	0.1	0.136	0.094	0.068	0.028	0.025	0.022
45	0.5	0.028	0.023	0.018	0.005	0.004	0.003
	0.4	0.037	0.029	0.024	0.007	0.006	0.005
	0.3	0.048	0.039	0.031	0.009	0.008	0.007
	0.2	0.067	0.054	0.043	0.013	0.013	0.011
	0.1	0.113	0.089	0.072	0.024	0.023	0.021
60	0.5	0.023	0.020	0.018	0.004	0.002	0.002
	0.4	0.030	0.026	0.023	0.005	0.003	0.003
	0.3	0.038	0.033	0.030	0.006	0.005	0.005
	0.2	0.053	0.045	0.040	0.009	0.007	0.007
	0.1	0.085	0.073	0.064	0.014	0.012	0.012
90	0.5	0.013	0.013	0.013	0.000	0.000	0.000
	0.4	0.014	0.014	0.014	0.000	0.000	0.000
	0.3	0.017	0.017	0.017	0.000	0.000	0.000
	0.2	0.021	0.021	0.020	0.000	0.000	0.000
	0.1	0.030	0.030	0.029	0.000	0.000	0.000

Table 6.29: Normalized stress intensity factors, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$, and $K_{II}^* = K_{II} / E\eta_3\alpha T(\pi a)^{1/2}$ with varying material angle, θ , and normalized crack length, a/W , for a uniformly cooled plate with an edge-crack emanating from a circular notch; $\eta_1=2$

Angle, θ [Deg]	a/W	K_I^*			K_{II}^*		
		$\eta_2 = 0.5$	$\eta_2 = 0.75$	$\eta_2 = 1$	$\eta_2 = 0.5$	$\eta_2 = 0.75$	$\eta_2 = 1$
0	0.5	0.706	0.377	0.239	0.000	0.000	0.000
	0.4	0.795	0.424	0.267	0.000	0.000	0.000
	0.3	0.916	0.480	0.298	0.000	0.000	0.000
	0.2	1.086	0.548	0.331	0.000	0.000	0.000
	0.1	1.388	0.667	0.391	0.000	0.000	0.000
30	0.5	0.476	0.322	0.230	0.083	0.055	0.047
	0.4	0.531	0.360	0.256	0.084	0.059	0.052
	0.3	0.597	0.403	0.286	0.084	0.064	0.057
	0.2	0.680	0.457	0.321	0.085	0.069	0.063
	0.1	0.826	0.550	0.382	0.088	0.080	0.076
45	0.5	0.314	0.246	0.193	0.076	0.046	0.037
	0.4	0.350	0.274	0.216	0.077	0.049	0.041
	0.3	0.390	0.308	0.243	0.078	0.053	0.046
	0.2	0.438	0.349	0.276	0.079	0.057	0.051
	0.1	0.524	0.422	0.335	0.081	0.066	0.062
60	0.5	0.182	0.156	0.131	0.082	0.032	0.019
	0.4	0.201	0.173	0.146	0.083	0.034	0.021
	0.3	0.225	0.195	0.166	0.083	0.036	0.024
	0.2	0.254	0.224	0.193	0.084	0.039	0.027
	0.1	0.305	0.276	0.242	0.086	0.043	0.034
90	0.5	0.016	0.019	0.020	0.000	0.000	0.000
	0.4	0.018	0.022	0.022	0.000	0.000	0.000
	0.3	0.021	0.025	0.025	0.000	0.000	0.000
	0.2	0.026	0.030	0.031	0.000	0.000	0.000
	0.1	0.034	0.040	0.041	0.000	0.000	0.000

Table 6.30: Normalized stress intensity factors, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$, and $K_{II}^* = K_{II} / E\eta_3\alpha T(\pi a)^{1/2}$ with varying material angle, θ , and normalized crack length, a/W , for a uniformly cooled plate with an edge-crack emanating from a circular notch; $\eta_1=3$

Angle, θ [Deg]	a/W	K_I^*			K_{II}^*		
		$\eta_2 = 0.5$	$\eta_2 = 0.75$	$\eta_2 = 1$	$\eta_2 = 0.5$	$\eta_2 = 0.75$	$\eta_2 = 1$
0	0.5	0.328	0.178	0.114	0.000	0.000	0.000
	0.4	0.369	0.199	0.127	0.000	0.000	0.000
	0.3	0.424	0.224	0.140	0.000	0.000	0.000
	0.2	0.498	0.252	0.153	0.000	0.000	0.000
	0.1	0.627	0.301	0.176	0.000	0.000	0.000
30	0.5	0.239	0.163	0.117	0.045	0.036	0.031
	0.4	0.266	0.182	0.131	0.048	0.040	0.036
	0.3	0.297	0.202	0.145	0.051	0.044	0.040
	0.2	0.334	0.226	0.160	0.055	0.049	0.044
	0.1	0.399	0.268	0.187	0.064	0.060	0.054
45	0.5	0.171	0.134	0.106	0.034	0.027	0.024
	0.4	0.190	0.149	0.118	0.036	0.031	0.027
	0.3	0.210	0.166	0.132	0.038	0.034	0.031
	0.2	0.233	0.187	0.149	0.041	0.038	0.035
	0.1	0.275	0.223	0.178	0.046	0.045	0.043
60	0.5	0.112	0.094	0.079	0.032	0.016	0.011
	0.4	0.124	0.104	0.088	0.034	0.017	0.013
	0.3	0.138	0.117	0.100	0.035	0.020	0.015
	0.2	0.155	0.135	0.118	0.037	0.022	0.018
	0.1	0.186	0.167	0.148	0.040	0.027	0.024
90	0.5	0.016	0.017	0.018	0.000	0.000	0.000
	0.4	0.018	0.020	0.020	0.000	0.000	0.000
	0.3	0.021	0.023	0.023	0.000	0.000	0.000
	0.2	0.026	0.028	0.028	0.000	0.000	0.000
	0.1	0.036	0.038	0.039	0.000	0.000	0.000

Table 6.31: Normalized stress intensity factors, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$, and $K_{II}^* = K_{II} / E\eta_3\alpha T(\pi a)^{1/2}$ with varying material angle, θ , and normalized crack length, a/W , for a uniformly cooled plate with an edge-crack emanating from a circular notch; $\eta_1=5$

Angle, θ [Deg]	a/W	K_I^*			K_{II}^*		
		$\eta_2 = 0.5$	$\eta_2 = 0.75$	$\eta_2 = 1$	$\eta_2 = 0.5$	$\eta_2 = 0.75$	$\eta_2 = 1$
0	0.5	0.122	0.067	0.044	0.000	0.000	0.000
	0.4	0.138	0.075	0.048	0.000	0.000	0.000
	0.3	0.158	0.084	0.052	0.000	0.000	0.000
	0.2	0.184	0.093	0.056	0.000	0.000	0.000
	0.1	0.228	0.109	0.064	0.000	0.000	0.000
30	0.5	0.096	0.066	0.048	0.023	0.020	0.017
	0.4	0.107	0.074	0.054	0.025	0.022	0.019
	0.3	0.118	0.082	0.059	0.028	0.025	0.022
	0.2	0.131	0.090	0.064	0.031	0.028	0.024
	0.1	0.156	0.106	0.074	0.039	0.036	0.031
45	0.5	0.074	0.059	0.047	0.017	0.015	0.014
	0.4	0.082	0.066	0.053	0.019	0.018	0.016
	0.3	0.090	0.072	0.058	0.021	0.020	0.018
	0.2	0.098	0.080	0.065	0.023	0.022	0.021
	0.1	0.114	0.094	0.076	0.026	0.027	0.025
60	0.5	0.054	0.045	0.038	0.013	0.008	0.006
	0.4	0.060	0.050	0.043	0.014	0.009	0.008
	0.3	0.066	0.056	0.049	0.015	0.011	0.009
	0.2	0.073	0.064	0.057	0.017	0.012	0.011
	0.1	0.087	0.078	0.071	0.019	0.016	0.015
90	0.5	0.014	0.015	0.015	0.000	0.000	0.000
	0.4	0.016	0.016	0.016	0.000	0.000	0.000
	0.3	0.018	0.019	0.019	0.000	0.000	0.000
	0.2	0.023	0.023	0.023	0.000	0.000	0.000
	0.1	0.033	0.033	0.033	0.000	0.000	0.000

Table 6.32: Normalized stress intensity factors, $K_I^* = K/E\eta_3\alpha T(\pi a)^{1/2}$, and $K_{II}^* = K_{II}/E\eta_3\alpha T(\pi a)^{1/2}$ with varying material angle, θ , and normalized crack length, a/W , for a uniformly cooled plate with an edge-crack emanating from a circular notch; $\eta_1=7$

Angle, θ [Deg]	a/W	K_I^*			K_{II}^*		
		$\eta_2 = 0.5$	$\eta_2 = 0.75$	$\eta_2 = 1$	$\eta_2 = 0.5$	$\eta_2 = 0.75$	$\eta_2 = 1$
0	0.5	0.063	0.035	0.023	0.000	0.000	0.000
	0.4	0.071	0.039	0.025	0.000	0.000	0.000
	0.3	0.082	0.043	0.027	0.000	0.000	0.000
	0.2	0.095	0.048	0.029	0.000	0.000	0.000
	0.1	0.116	0.055	0.032	0.000	0.000	0.000
30	0.5	0.052	0.036	0.026	0.015	0.013	0.010
	0.4	0.058	0.040	0.029	0.016	0.015	0.012
	0.3	0.064	0.044	0.032	0.019	0.017	0.014
	0.2	0.071	0.049	0.034	0.021	0.019	0.015
	0.1	0.083	0.057	0.040	0.026	0.023	0.019
45	0.5	0.042	0.034	0.027	0.012	0.011	0.009
	0.4	0.047	0.038	0.030	0.013	0.012	0.011
	0.3	0.051	0.041	0.033	0.014	0.014	0.012
	0.2	0.055	0.045	0.037	0.016	0.015	0.014
	0.1	0.063	0.053	0.043	0.018	0.018	0.016
60	0.5	0.032	0.026	0.023	0.007	0.005	0.004
	0.4	0.035	0.029	0.026	0.008	0.006	0.005
	0.3	0.038	0.032	0.029	0.009	0.007	0.007
	0.2	0.042	0.037	0.034	0.010	0.008	0.008
	0.1	0.050	0.045	0.042	0.012	0.010	0.010
90	0.5	0.012	0.013	0.013	0.000	0.000	0.000
	0.4	0.014	0.014	0.014	0.000	0.000	0.000
	0.3	0.016	0.016	0.016	0.000	0.000	0.000
	0.2	0.020	0.020	0.020	0.000	0.000	0.000
	0.1	0.029	0.029	0.028	0.000	0.000	0.000

Table 6.33: Normalized stress intensity factors, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$, and $K_{II}^* = K_{II} / E\eta_3\alpha T(\pi a)^{1/2}$ with varying material angle, θ , and normalized crack length, a/W , for a uniformly cooled plate with symmetric edge-cracks emanating from a central hole; $\eta_1=2$

Angle, θ [Deg]	a/W	K_I^*			K_{II}^*		
		$\eta_2 = 0.5$	$\eta_2 = 0.75$	$\eta_2 = 1$	$\eta_2 = 0.5$	$\eta_2 = 0.75$	$\eta_2 = 1$
0	0.5	0.946	0.468	0.277	0.000	0.000	0.000
	0.4	0.997	0.481	0.280	0.000	0.000	0.000
	0.3	1.065	0.502	0.290	0.000	0.000	0.000
	0.2	1.171	0.544	0.312	0.000	0.000	0.000
	0.1	1.408	0.647	0.370	0.000	0.000	0.000
30	0.5	0.592	0.362	0.273	0.063	0.042	0.038
	0.4	0.603	0.372	0.276	0.066	0.042	0.036
	0.3	0.624	0.399	0.284	0.074	0.044	0.034
	0.2	0.667	0.422	0.302	0.074	0.046	0.036
	0.1	0.789	0.509	0.358	0.085	0.063	0.056
45	0.5	0.385	0.304	0.241	0.069	0.047	0.033
	0.4	0.388	0.309	0.244	0.071	0.045	0.030
	0.3	0.397	0.319	0.252	0.074	0.043	0.027
	0.2	0.421	0.340	0.267	0.074	0.043	0.028
	0.1	0.495	0.400	0.314	0.077	0.052	0.040
60	0.5	0.224	0.199	0.172	0.107	0.034	0.019
	0.4	0.228	0.204	0.177	0.105	0.033	0.017
	0.3	0.236	0.212	0.185	0.099	0.032	0.016
	0.2	0.251	0.227	0.198	0.099	0.031	0.015
	0.1	0.295	0.266	0.233	0.096	0.035	0.021
90	0.5	0.024	0.028	0.028	0.000	0.000	0.000
	0.4	0.025	0.029	0.030	0.000	0.000	0.000
	0.3	0.027	0.031	0.033	0.000	0.000	0.000
	0.2	0.030	0.035	0.036	0.000	0.000	0.000
	0.1	0.037	0.042	0.044	0.000	0.000	0.000

Table 6.34: Normalized stress intensity factors, $K_I^* = K_I / E\eta_3 \alpha T (\pi a)^{1/2}$, and $K_{II}^* = K_{II} / E\eta_3 \alpha T (\pi a)^{1/2}$ with varying material angle, θ , and normalized crack length, a/W , for a uniformly cooled plate with symmetric edge-cracks emanating from a central hole; $\eta_1=3$

Angle, θ [Deg]	a/W	K_I^*			K_{II}^*		
		$\eta_2 = 0.5$	$\eta_2 = 0.75$	$\eta_2 = 1$	$\eta_2 = 0.5$	$\eta_2 = 0.75$	$\eta_2 = 1$
0	0.5	0.428	0.209	0.127	0.000	0.000	0.000
	0.4	0.450	0.215	0.128	0.000	0.000	0.000
	0.3	0.481	0.226	0.133	0.000	0.000	0.000
	0.2	0.532	0.247	0.145	0.000	0.000	0.000
	0.1	0.647	0.300	0.176	0.000	0.000	0.000
30	0.5	0.287	0.190	0.134	0.036	0.026	0.022
	0.4	0.290	0.191	0.134	0.036	0.024	0.019
	0.3	0.298	0.196	0.136	0.037	0.023	0.018
	0.2	0.316	0.208	0.139	0.040	0.026	0.020
	0.1	0.376	0.249	0.171	0.058	0.046	0.039
45	0.5	0.203	0.161	0.128	0.035	0.023	0.018
	0.4	0.202	0.162	0.129	0.034	0.021	0.016
	0.3	0.205	0.164	0.130	0.031	0.018	0.013
	0.2	0.214	0.172	0.135	0.031	0.017	0.012
	0.1	0.251	0.201	0.157	0.036	0.025	0.020
60	0.5	0.139	0.121	0.105	0.033	0.015	0.010
	0.4	0.140	0.123	0.107	0.032	0.014	0.009
	0.3	0.143	0.126	0.110	0.031	0.012	0.007
	0.2	0.150	0.133	0.116	0.030	0.012	0.006
	0.1	0.174	0.154	0.135	0.031	0.014	0.009
90	0.5	0.024	0.026	0.026	0.000	0.000	0.000
	0.4	0.026	0.028	0.028	0.000	0.000	0.000
	0.3	0.029	0.031	0.031	0.000	0.000	0.000
	0.2	0.033	0.035	0.035	0.000	0.000	0.000
	0.1	0.041	0.043	0.044	0.000	0.000	0.000

Table 6.35: Normalized stress intensity factors, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$, and $K_{II}^* = K_{II} / E\eta_3\alpha T(\pi a)^{1/2}$ with varying material angle, θ , and normalized crack length, a/W , for a uniformly cooled plate with symmetric edge-cracks emanating from a central hole; $\eta_1=5$

Angle, θ [Deg]	a/W	K_I^*			K_{II}^*		
		$\eta_2 = 0.5$	$\eta_2 = 0.75$	$\eta_2 = 1$	$\eta_2 = 0.5$	$\eta_2 = 0.75$	$\eta_2 = 1$
0	0.5	0.157	0.078	0.047	0.000	0.000	0.000
	0.4	0.165	0.080	0.048	0.000	0.000	0.000
	0.3	0.178	0.085	0.051	0.000	0.000	0.000
	0.2	0.198	0.094	0.056	0.000	0.000	0.000
	0.1	0.242	0.114	0.068	0.000	0.000	0.000
30	0.5	0.111	0.074	0.052	0.015	0.012	0.010
	0.4	0.111	0.074	0.051	0.014	0.012	0.008
	0.3	0.113	0.074	0.051	0.013	0.010	0.007
	0.2	0.119	0.078	0.053	0.015	0.012	0.007
	0.1	0.146	0.097	0.066	0.036	0.030	0.024
45	0.5	0.085	0.068	0.055	0.015	0.012	0.010
	0.4	0.085	0.068	0.054	0.015	0.010	0.008
	0.3	0.083	0.067	0.053	0.012	0.008	0.006
	0.2	0.085	0.068	0.053	0.011	0.007	0.005
	0.1	0.098	0.078	0.061	0.013	0.010	0.008
60	0.5	0.066	0.057	0.051	0.012	0.007	0.006
	0.4	0.065	0.057	0.051	0.011	0.006	0.005
	0.3	0.065	0.057	0.051	0.010	0.005	0.004
	0.2	0.067	0.058	0.052	0.010	0.004	0.003
	0.1	0.075	0.066	0.058	0.010	0.005	0.003
90	0.5	0.021	0.021	0.021	0.000	0.000	0.000
	0.4	0.023	0.024	0.024	0.000	0.000	0.000
	0.3	0.027	0.027	0.027	0.000	0.000	0.000
	0.2	0.031	0.032	0.032	0.000	0.000	0.000
	0.1	0.041	0.041	0.041	0.000	0.000	0.000

Table 6.36: Normalized stress intensity factors, $K_I^* = K_I / E\eta_3 \alpha T (\pi a)^{1/2}$, and $K_{II}^* = K_{II} / E\eta_3 \alpha T (\pi a)^{1/2}$ with varying material angle, θ , and normalized crack length, a/W , for a uniformly cooled plate with symmetric edge-cracks emanating from a central hole; $\eta_1 = 7$

Angle, θ [Deg]	a/W	K_I^*			K_{II}^*		
		$\eta_2 = 0.5$	$\eta_2 = 0.75$	$\eta_2 = 1$	$\eta_2 = 0.5$	$\eta_2 = 0.75$	$\eta_2 = 1$
0	0.5	0.081	0.041	0.025	0.000	0.000	0.000
	0.4	0.086	0.042	0.025	0.000	0.000	0.000
	0.3	0.092	0.044	0.027	0.000	0.000	0.000
	0.2	0.103	0.049	0.030	0.000	0.000	0.000
	0.1	0.126	0.060	0.036	0.000	0.000	0.000
30	0.5	0.059	0.039	0.027	0.008	0.007	0.005
	0.4	0.058	0.039	0.026	0.007	0.006	0.004
	0.3	0.059	0.039	0.026	0.006	0.005	0.003
	0.2	0.062	0.040	0.027	0.007	0.005	0.003
	0.1	0.077	0.051	0.035	0.022	0.018	0.014
45	0.5	0.048	0.038	0.031	0.011	0.008	0.007
	0.4	0.049	0.038	0.030	0.013	0.008	0.006
	0.3	0.046	0.037	0.029	0.008	0.005	0.004
	0.2	0.046	0.036	0.028	0.007	0.004	0.003
	0.1	0.051	0.041	0.031	0.007	0.005	0.004
60	0.5	0.038	0.033	0.030	0.007	0.004	0.004
	0.4	0.037	0.032	0.030	0.006	0.003	0.003
	0.3	0.037	0.032	0.029	0.005	0.003	0.003
	0.2	0.037	0.032	0.029	0.004	0.002	0.002
	0.1	0.041	0.035	0.033	0.004	0.002	0.002
90	0.5	0.018	0.018	0.018	0.000	0.000	0.000
	0.4	0.020	0.021	0.020	0.000	0.000	0.000
	0.3	0.024	0.024	0.023	0.000	0.000	0.000
	0.2	0.028	0.029	0.028	0.000	0.000	0.000
	0.1	0.038	0.038	0.038	0.000	0.000	0.000

Figure 3.1: An arbitrary cracked body

Figure 3.2: Common fracture modes; (a) opening mode, Mode I, and (b) sliding mode, Mode II

Figure 4.1: An arbitrary linear elastic body

Figure 4.2: An arbitrary discretized body

Figure 4.3: A quadratic isoparametric line element

Figure 4.4: Shape functions; (a) $N^1(\zeta)$, (b) $N^2(\zeta)$, and (c) $N^3(\zeta)$

Figure 4.5: (a) a simply connected convex domain, (b) a simply connected non-convex domain, and (c) a multiply-connected domain

Figure 4.6: A multiply-connected domain divided into two simply connected non-convex sub-regions

Figure 4.7: An arbitrary cracked body after sub-regioning

Figure 4.8: Quadratic isoparametric crack-tip elements\

Figure 4.9: Node symbols for crack-tip elements

Figure 5.1: Boundary conditions for problem analysis

Figure 5.2: Plate with an edge-crack

Figure 5.3: Plate with a central-crack

Figure 5.4: Plate with a double edge-crack

Figure 5.5: Plate with an edge-crack emanating from a circular notch

Figure 5.6: Plate with symmetric cracks emanating from a central hole

Figure 5.7: Material elastic constants, E_{22} and G_{12} with varying material parameter η_1

Figure 5.8: Material elastic constants, E_{22} and G_{12} with varying material parameter η_2

Figure 5.9: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material parameter, η_3 , for a uniformly cooled plate with an edge-crack; $\theta=0$, $\eta_1=2$, $\eta_2=0.5$

Figure 5.10: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material parameter, η_3 , for a uniformly cooled plate with an edge-crack; $\theta=0$, $\eta_1=7$, $\eta_2=1$

Figure 5.11: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material parameter, η_3 , for a uniformly cooled plate with an edge-crack; $\theta=45$, $\eta_1=2$, $\eta_2=0.5$

Figure 5.12: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material parameter, η_3 , for a uniformly cooled plate with an edge-crack; $\theta=45$, $\eta_1=7$, $\eta_2=1$

Figure 5.13: Variation of the normalized stress intensity factor, $K_I^* = K_I / E \eta_3 \alpha T (\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material parameter, η_3 , for a uniformly cooled plate with an edge-crack; $\theta=90$, $\eta_1=2$, $\eta_2=0.5$

Figure 5.14: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material parameter, η_3 , for a uniformly cooled plate with an edge-crack; $\theta=90$, $\eta_1=7$, $\eta_2=1$

Figure 5.15: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material parameter, η_3 , for a uniformly cooled plate with a central-crack; $\theta=0$, $\eta_1=2$, $\eta_2=0.5$

Figure 5.16: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material parameter, η_3 , for a uniformly cooled plate with a central-crack; $\theta=0$, $\eta_1=7$, $\eta_2=1$

Figure 5.17: Variation of the normalized stress intensity factor, $K_I^* = K_I / E \eta_3 \alpha T (\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material parameter, η_3 , for a uniformly cooled plate with a central-crack; $\theta=45$, $\eta_1=2$, $\eta_2=0.5$

Figure 5.18: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material parameter, η_3 , for a uniformly cooled plate with a central-crack; $\theta=45$, $\eta_1=7$, $\eta_2=1$

Figure 5.19: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material parameter, η_3 , for a uniformly cooled plate with a central-crack; $\theta=90$, $\eta_1=2$, $\eta_2=0.5$

Figure 5.20: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material parameter, η_3 , for a uniformly cooled plate with a central-crack; $\theta=90$, $\eta_1=7$, $\eta_2=1$

Figure 5.21: Sample BEM mesh for the edge-crack case; $a/W=0.3$

Figure 5.22: Sample BEM mesh for the central-crack case; $a/W=0.4$

Figure 5.23: Sample BEM mesh for the double edge-crack case; $a/W=0.5$

Figure 5.24: Sample BEM mesh for the case of an edge-crack emanating from a circular notch; $a/W=0.3$

Figure 5.25: Sample BEM mesh for the case of symmetric cracks emanating from a central hole; $a/W=0.5$

Figure 5.26: Two-dimensional 8-noded crack-tip element for FEM analysis

Figure 5.27: Crack-tip region in FEM analysis

(180 elements, 579 nodes)

Figure 5.28: Sample FEM mesh for the edge-crack case; $a/W=0.3$

(432 elements, 1356 nodes)

Figure 5.29: Sample FEM mesh for the central-crack case; $a/W=0.3$

Figure 6.1: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$ with (a) material parameter η_1 , and (b) material parameter η_2 , for a uniformly cooled plate with an edge-crack; $\theta=0$, $a/W=0.1$

Figure 6.2: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$ with (a) material parameter η_1 , and (b) material parameter η_2 , for a uniformly cooled plate with an edge-crack; $\theta=0$, $a/W=0.2$

Figure 6.3: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$ with (a) material parameter η_1 , and (b) material parameter η_2 , for a uniformly cooled plate with an edge-crack; $\theta=0$, $a/W=0.3$

Figure 6.4: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3 \alpha T (\pi a)^{1/2}$ with (a) material parameter η_1 , and (b) material parameter η_2 , for a uniformly cooled plate with an edge-crack; $\theta=0$, $a/W=0.4$

Figure 6.5: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$ with (a) material parameter η_1 , and (b) material parameter η_2 , for a uniformly cooled plate with an edge-crack; $\theta=0$, $a/W=0.5$

Figure 6.6: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$ with (a) material parameter η_1 , and (b) material parameter η_2 , for a uniformly cooled plate with an edge-crack; $\theta=30$, $a/W=0.1$

Figure 6.7: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$ with (a) material parameter η_1 , and (b) material parameter η_2 , for a uniformly cooled plate with an edge-crack; $\theta=30$, $a/W=0.2$

Figure 6.8: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3 \alpha T (\pi a)^{1/2}$ with (a) material parameter η_1 , and (b) material parameter η_2 , for a uniformly cooled plate with an edge-crack; $\theta=30$, $a/W=0.3$

Figure 6.9: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$ with (a) material parameter η_1 , and (b) material parameter η_2 , for a uniformly cooled plate with an edge-crack; $\theta=30$, $a/W=0.4$

Figure 6.10: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3 \alpha T (\pi a)^{1/2}$ with (a) material parameter η_1 , and (b) material parameter η_2 , for a uniformly cooled plate with an edge-crack; $\theta=30$, $a/W=0.5$

Figure 6.11: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$ with (a) material parameter η_1 , and (b) material parameter η_2 , for a uniformly cooled plate with an edge-crack; $\theta=45$, $a/W=0.1$

Figure 6.12: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$ with (a) material parameter η_1 , and (b) material parameter η_2 , for a uniformly cooled plate with an edge-crack; $\theta=45$, $a/W=0.2$

Figure 6.13: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3 \alpha T(\pi a)^{1/2}$ with (a) material parameter η_1 , and (b) material parameter η_2 , for a uniformly cooled plate with an edge-crack; $\theta=45$, $a/W=0.3$

Figure 6.14: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$ with (a) material parameter η_1 , and (b) material parameter η_2 , for a uniformly cooled plate with an edge-crack; $\theta=45$, $a/W=0.4$

Figure 6.15: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$ with (a) material parameter η_1 , and (b) material parameter η_2 , for a uniformly cooled plate with an edge-crack; $\theta=45$, $a/W=0.5$

Figure 6.16: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$ with (a) material parameter η_1 , and (b) material parameter η_2 , for a uniformly cooled plate with an edge-crack; $\theta=60$, $a/W=0.1$

Figure 6.17: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$ with (a) material parameter η_1 , and (b) material parameter η_2 , for a uniformly cooled plate with an edge-crack; $\theta=60$, $a/W=0.2$

Figure 6.18: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$ with (a) material parameter η_1 , and (b) material parameter η_2 , for a uniformly cooled plate with an edge-crack; $\theta=60$, $a/W=0.3$

Figure 6.19: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$ with (a) material parameter η_1 , and (b) material parameter η_2 , for a uniformly cooled plate with an edge-crack; $\theta=60$, $a/W=0.4$

Figure 6.20: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$ with (a) material parameter η_1 , and (b) material parameter η_2 , for a uniformly cooled plate with an edge-crack; $\theta=60$, $a/W=0.5$

Figure 6.21: Variation of the normalized stress intensity factor, $K_I^* = K_I / E \eta_3 \alpha T (\pi a)^{1/2}$ with (a) material parameter η_1 , and (b) material parameter η_2 , for a uniformly cooled plate with an edge-crack; $\theta=90$, $a/W=0.1$

Figure 6.22: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$ with (a) material parameter η_1 , and (b) material parameter η_2 , for a uniformly cooled plate with an edge-crack; $\theta=90$, $a/W=0.2$

Figure 6.23: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3 \alpha T (\pi a)^{1/2}$ with (a) material parameter η_1 , and (b) material parameter η_2 , for a uniformly cooled plate with an edge-crack; $\theta=90$, $a/W=0.3$

Figure 6.24: Variation of the normalized stress intensity factor, $K_I^* = K_I / E \eta_3 \alpha T (\pi a)^{1/2}$ with (a) material parameter η_1 , and (b) material parameter η_2 , for a uniformly cooled plate with an edge-crack; $\theta=90$, $a/W=0.4$

Figure 6.25: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3 \alpha T (\pi a)^{1/2}$ with (a) material parameter η_1 , and (b) material parameter η_2 , for a uniformly cooled plate with an edge-crack; $\theta=90$, $a/W=0.5$

Figure 6.26: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with an edge-crack; $\eta_1=2$, $\eta_2=0.5$

Figure 6.27: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with an edge-crack; $\eta_1=3$, $\eta_2=0.5$

Figure 6.28: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3 \alpha T (\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with an edge-crack; $\eta_1=5$, $\eta_2=0.5$

Figure 6.29: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with an edge-crack; $\eta_1=7$, $\eta_2=0.5$

Figure 6.30: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with an edge-crack; $\eta_1=2$, $\eta_2=0.75$

Figure 6.31: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3 \alpha T (\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with an edge-crack; $\eta_1=3$, $\eta_2=0.75$

Figure 6.32: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3 \alpha T (\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with an edge-crack; $\eta_1=5$, $\eta_2=0.75$

Figure 6.33: Variation of the normalized stress intensity factor, $K_I^* = K_I / E \eta_3 \alpha T (\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with an edge-crack; $\eta_1=7$, $\eta_2=0.75$

Figure 6.34: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with an edge-crack; $\eta_1=2$, $\eta_2=1$

Figure 6.35: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with an edge-crack; $\eta_1=3$, $\eta_2=1$

Figure 6.36: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with an edge-crack; $\eta_1=5$, $\eta_2=1$

Figure 6.37: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3 \alpha T (\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with an edge-crack; $\eta_1=7$, $\eta_2=1$

Figure 6.38: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with a central-crack; $\eta_1=2$, $\eta_2=0.5$

Figure 6.39: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3 \alpha T (\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with a central-crack; $\eta_1=3$, $\eta_2=0.5$

Figure 6.40: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3 \alpha T (\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with a central-crack; $\eta_1=5$, $\eta_2=0.5$

Figure 6.41: Variation of the normalized stress intensity factor, $K_I^* = K_I / E \eta_3 \alpha T (\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with a central-crack; $\eta_1=7$, $\eta_2=0.5$

Figure 6.42: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with a central-crack; $\eta_1=2$, $\eta_2=0.75$

Figure 6.43: Variation of the normalized stress intensity factor, $K_I^* = K_I / E \eta_3 \alpha T (\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with a central-crack; $\eta_1=3$, $\eta_2=0.75$

Figure 6.44: Variation of the normalized stress intensity factor, $K_I^* = K_I / E \eta_3 \alpha T (\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with a central-crack; $\eta_1 = 5$, $\eta_2 = 0.75$

Figure 6.45: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with a central-crack; $\eta_1=7$, $\eta_2=0.75$

Figure 6.46: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with a central-crack; $\eta_1=2$, $\eta_2=1$

Figure 6.47: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with a central-crack; $\eta_1=3$, $\eta_2=1$

Figure 6.48: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with a central-crack; $\eta_1=5$, $\eta_2=1$

Figure 6.49: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with a central-crack; $\eta_1=7$, $\eta_2=1$

Figure 6.50: Variation of the normalized stress intensity factor, $K_I^* = K_I / E \eta_3 \alpha T (\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with a double edge-crack; $\eta_1=2$, $\eta_2=0.5$

Figure 6.51: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with a double edge-crack; $\eta_1=3$, $\eta_2=0.5$

Figure 6.52: Variation of the normalized stress intensity factor, $K_I^* = K_I / E \eta_3 \alpha T (\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with a double edge-crack; $\eta_1=5$, $\eta_2=0.5$

Figure 6.53: Variation of the normalized stress intensity factor, $K_I^* = K_I / E \eta_3 \alpha T (\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with a double edge-crack; $\eta_1=7$, $\eta_2=0.5$

Figure 6.54: Variation of the normalized stress intensity factor, $K_I^* = K_I / E \eta_3 \alpha T (\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with a double edge-crack; $\eta_1=2$, $\eta_2=0.75$

Figure 6.55: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with a double edge-crack; $\eta_1=3$, $\eta_2=0.75$

Figure 6.56: Variation of the normalized stress intensity factor, $K_I^* = K_I / E \eta_3 \alpha T (\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with a double edge-crack; $\eta_1=5$, $\eta_2=0.75$

Figure 6.57: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3 \alpha T (\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with a double edge-crack; $\eta_1=7$, $\eta_2=0.75$

Figure 6.58: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with a double edge-crack; $\eta_1=2$, $\eta_2=1$

Figure 6.59: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with a double edge-crack; $\eta_1=3$, $\eta_2=1$

Figure 6.60: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with a double edge-crack; $\eta_1=5$, $\eta_2=1$

Figure 6.61: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with a double edge-crack; $\eta_1=7$, $\eta_2=1$

Figure 6.62: Variation of the normalized stress intensity factor, $K_I^* = K_I / E \eta_3 \alpha T (\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with an edge-crack emanating from a circular notch; $\eta_1 = 2$, $\eta_2 = 0.5$

Figure 6.63: Variation of the normalized stress intensity factor, $K_I^* = K_I / E \eta_3 \alpha T (\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with an edge-crack emanating from a circular notch; $\eta_1 = 3$, $\eta_2 = 0.5$

Figure 6.64: Variation of the normalized stress intensity factor, $K_I^* = K_I / E \eta_3 \alpha T (\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with an edge-crack emanating from a circular notch; $\eta_1=5$, $\eta_2=0.5$

Figure 6.65: Variation of the normalized stress intensity factor, $K_I^* = K_I / E \eta_3 \alpha T (\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with an edge-crack emanating from a circular notch; $\eta_1=7$, $\eta_2=0.5$

Figure 6.66: Variation of the normalized stress intensity factor, $K_I^* = K_I / E \eta_3 \alpha T (\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with an edge-crack emanating from a circular notch; $\eta_1=2$, $\eta_2=0.75$

Figure 6.67: Variation of the normalized stress intensity factor, $K_I^* = K_I / E \eta_3 \alpha T (\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with an edge-crack emanating from a circular notch; $\eta_1=3$, $\eta_2=0.75$

Figure 6.68: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with an edge-crack emanating from a circular notch; $\eta_1=5$, $\eta_2=0.75$

Figure 6.69: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with an edge-crack emanating from a circular notch; $\eta_1=7$, $\eta_2=0.75$

Figure 6.70: Variation of the normalized stress intensity factor, $K_I^* = K_I / E \eta_3 \alpha T (\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with an edge-crack emanating from a circular notch; $\eta_1=2$, $\eta_2=1$

Figure 6.71: Variation of the normalized stress intensity factor, $K_I^* = K_I / E \eta_3 \alpha T (\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with an edge-crack emanating from a circular notch; $\eta_1=3$, $\eta_2=1$

Figure 6.72: Variation of the normalized stress intensity factor, $K_I^* = K_I / E \eta_3 \alpha T (\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with an edge-crack emanating from a circular notch; $\eta_1=5$, $\eta_2=1$

Figure 6.73: Variation of the normalized stress intensity factor, $K_I^* = K_I / E \eta_3 \alpha T (\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with an edge-crack emanating from a circular notch; $\eta_1=7$, $\eta_2=1$

Figure 6.74: Variation of the normalized stress intensity factor, $K_I^* = K_I / E \eta_3 \alpha T (\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with symmetric cracks emanating from a central hole; $\eta_1=2$, $\eta_2=0.5$

Figure 6.75: Variation of the normalized stress intensity factor, $K_I^* = K_I / E \eta_3 \alpha T (\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with symmetric cracks emanating from a central hole; $\eta_1=3$, $\eta_2=0.5$

Figure 6.76: Variation of the normalized stress intensity factor, $K_I^* = K_I / E \eta_3 \alpha T (\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with symmetric cracks emanating from a central hole; $\eta_1=5$, $\eta_2=0.5$

Figure 6.77: Variation of the normalized stress intensity factor, $K_I^* = K_I / E \eta_3 \alpha T (\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with symmetric cracks emanating from a central hole; $\eta_1=7$, $\eta_2=0.5$

Figure 6.78: Variation of the normalized stress intensity factor, $K_I^* = K_I / E \eta_3 \alpha T (\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with symmetric cracks emanating from a central hole; $\eta_1=2$, $\eta_2=0.75$

Figure 6.79: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3 \alpha T (\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with symmetric cracks emanating from a central hole; $\eta_1=3$, $\eta_2=0.75$

Figure 6.80: Variation of the normalized stress intensity factor, $K_I^* = K_I / E \eta_3 \alpha T (\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with symmetric cracks emanating from a central hole; $\eta_1=5$, $\eta_2=0.75$

Figure 6.81: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with symmetric cracks emanating from a central hole; $\eta_1=7$, $\eta_2=0.75$

Figure 6.82: Variation of the normalized stress intensity factor, $K_I^* = K_I / E \eta_3 \alpha T (\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with symmetric cracks emanating from a central hole; $\eta_1=2$, $\eta_2=1$

Figure 6.83: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3\alpha T(\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with symmetric cracks emanating from a central hole; $\eta_1=3$, $\eta_2=1$

Figure 6.84: Variation of the normalized stress intensity factor, $K_I^* = K_I / E \eta_3 \alpha T (\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with symmetric cracks emanating from a central hole; $\eta_1=5$, $\eta_2=1$

Figure 6.85: Variation of the normalized stress intensity factor, $K_I^* = K_I / E\eta_3 \alpha T (\pi a)^{1/2}$ with (a) normalized crack length, a/W , and (b) material angle, θ for a uniformly cooled plate with symmetric cracks emanating from a central hole; $\eta_1=7$, $\eta_2=1$