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**A Quantum Algebra
Associated with The Universal Algebra
of $\mathfrak{su}(2)$ and
Its Dual Space**

by

SULTAN SIAL, B.Sc.

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfilment of
the requirements for the degree of
Master of Science

Department of Mathematics and Statistics

Carleton University
Ottawa, Ontario
December, 1991

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and Its Dual Space**
submitted by
SULTAN SIAL, B.Sc.,
in partial fulfilment of the requirements
for the degree of Master of Science

Dr. John Dixon, Chairman,
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Dr. B.M. Puttaswamaiah, Thesis Supervisor

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1992

Abstract

The thesis contains an exposition of the theory of quantum algebras and quantum groups which arise from deformations of universal enveloping algebras of Lie algebras and Lie groups. In the first chapter, an introduction to the theory of universal enveloping algebras, deformations of such algebras, and thier relation to Hopf algebras is given.

The second chapter is concerned with the universal enveloping algebra $U = U(\mathfrak{su}(2))$ of the Lie algebra $\mathfrak{su}(2)$ or all 2×2 complex matrices of trace 0 and the deformed universal enveloping algebra $U_q = U(\mathfrak{su}(2))_q$. It is shown that irreducible linear representations of U_q are obtained by deforming the irreducible linear representations of U , equivalently of $\mathfrak{su}(2)$. Following Schwinger, the algebras U and U_q are described in terms of annihilation and creation operators. Furthermore, the algebras U and U_q are described in terms of multiplication and q -differentiation operators.

In the final chapter, it is shown that the algebra A_q may be viewed as the deformation of a function space on $\mathfrak{su}(2)$. It concludes with the recognition that the algebras A_q and U_q are in each others Hopf duals.

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Foreword

Quantum algebras and quantum groups are a fast developing field of mathematics and mathematical physics. Quantum algebras were developed by Jimbo [Ji] and Drinfeld [Dr] as an extension of the work of Kulish, Reshetikhin, Sklyanin [KuReSk] and Fadeev [FaReTa] on the quantum inverse scattering problem. The structures, here called quantum groups were developed independently by Woronowicz [Wo].

Why should these structures be studied? Since they arose out of the study of one physical problem, many applications have been found. They have been mentioned in various articles in connection with vertex models [DeV], conformal field theories (which are the continuum limits of vertex models), and knots and links [KiRe]. The chiral vertex operators of conformal field theory appear to be a quantum group comodule.

It has been suggested by Podles and Woronowicz [PoWo] that the symmetry properties of space-time on the micro scale can be described by a quantum group. They reason then that special relativity and quantum theory can be made compatible by describing a quantum deformation of the Lorentz group.

Sudbery [Su] shows how the consideration of noncommuting coordinates x and y describes local anyon statistics.

The spectra and vibrations of diatomic molecules is described using the Lie group $SU(1,1)$. The quantum algebra $U(su(1,1))_q$ is used by Bonatsos and Argyres [BoAr] to improve the agreement to empirical data.

As a final application one might mention the “ q -riosity” of Martin-Delgado [MD]. In [MD] it is shown that the quantum statistics of a system of free spins in a magnetic field can be described by the quantum algebra studied in this thesis. The deformation parameter q is given a physical meaning.

The aim of this thesis is to study the theory of the quantum algebra $U(su(2))_q$ as has been developed in the literature and tie the various loose ends together.

Proofs have been added where none were found in the literature or more intuitive proofs could be found.

There are two structures that go by the name of $SU(2)_q$ and both are called quantum groups. It became an aim of this thesis to spare future researchers the confusion that this causes. For this reason I have used different notations for these structures and have called one a quantum algebra and the other a quantum group.

The two structures are related but the proof that they are in each other's Hopf duals is due to several different papers, in one direction I have filled in the proofs for $[Su]$ and in the other I have done the same for $[MMNNSU]$. The approach using R-matrices is not discussed here.

I hope that this thesis gives a readable introduction from a very basic framework to a fascinating field.

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Chapter 1: Quantised Algebras

Introduction. This chapter deals with the notion of a quantised algebra, which is a one-parameter deformation of the universal enveloping algebra of a Lie algebra. A brief introduction to the enveloping algebra of a Lie algebra is given, and then the connection between these algebras and Hopf algebras is established. The enveloping algebras are then deformed in such a way as to retain the Hopf algebra structure.

§1. Universal Algebras. In the following, all vector spaces are vector spaces over the complex field C and a map means a linear map. The unadorned tensor product $U \otimes V$ is understood to mean $U \otimes_C V$. If U and V are vector spaces, a map f from U to V is written

$$f : U \rightarrow V \quad \text{or} \quad U \xrightarrow{f} V.$$

The identity map from U to U is denoted by I , I_U or $I : U \rightarrow U$. An algebra without further qualification shall mean an associative algebra with unity over the complex field C .

Any algebra A can be made into a Lie algebra by defining a Lie multiplication in A by

$$[ab] = [a, b] = ab - ba$$

for all $a, b \in A$. In fact, every Lie algebra can be obtained in this manner. The Lie algebra A_L so obtained is called the **Lie algebra associated with A** .

Given a (Lie) algebra L one may construct an associative algebra called the **universal enveloping algebra** associated with L . This is done by constructing an algebra that is generated as freely as possible by L .

Definition. Let L be a (Lie) algebra over the field C . A universal enveloping algebra of L is a pair (U, i) consisting of an algebra U over C and a homomorphism i from L into U (respectively U_L) such that:

If V is an algebra over C and j is a homomorphism from L into V (respectively V_L), then there is a unique homomorphism $\theta : U \rightarrow V$ such that $\theta i = j$.

$$\begin{array}{ccc} L & \xrightarrow{i} & U \\ & \searrow j & \downarrow \theta \\ & & V_L \end{array}$$

Uniqueness of the enveloping algebra. If the universal algebra of L exists, it is unique up to isomorphism.

Proof. If (U, i) and (V, j) are two universal enveloping algebras of

L , then there are homomorphisms $\theta : U \rightarrow V$ and $\phi : V \rightarrow U$ so that $\theta i = j$ and $\phi j = i$. Thus, $\theta \phi j = j$ and $\phi \theta i = i$. By uniqueness of ϕ and θ , $\theta \phi = 1_V$ and $\phi \theta = 1_U$. Thus two universal enveloping algebras of L are isomorphic.

The algebra (U, i) is generated by iL . Suppose that the algebra generated by iL is $V \subseteq U$. Then i is a homomorphism from L into V_L . Then there is a unique homomorphism $j : U \rightarrow V$ such that $ji = i$. Since $i = 1_U i$, uniqueness of the homomorphism implies that $j = 1_U$. Thus

$$U = 1_U U = jU \subseteq V \subseteq U$$

and so $U = V$ and iL generates U .

$$\begin{array}{ccc} L & \xrightarrow{i} & U \\ & \searrow i & \downarrow j \\ & & V_L \end{array}$$

Diagonal map. There is a unique algebra homomorphism (called the diagonal map) $\delta : U \rightarrow U \otimes U$ such that

$$\delta i(a) = i(a) \otimes 1 + 1 \otimes i(a)$$

for all $a \in L$.

Proof. Consider a map $\phi : L \rightarrow U \otimes U$ defined by

$$\phi(x) = i(x) \otimes 1 + 1 \otimes i(x)$$

for all $x \in L$. Now ϕ is a Lie algebra map since it is clear that ϕ is linear and it can be proven that $\phi[a, b] = [\phi(a), \phi(b)]$:

$$\begin{aligned}
 [\phi i(a), \phi i(b)] &= [i(a) \otimes 1 + 1 \otimes i(a), i(b) \otimes 1 + 1 \otimes i(b)] \\
 &= (i(a) \otimes 1 + 1 \otimes i(a))(i(b) \otimes 1 + 1 \otimes i(b)) \\
 &\quad - (i(b) \otimes 1 + 1 \otimes i(b))(i(a) \otimes 1 + 1 \otimes i(a)) \\
 &= i(a)i(b) \otimes 1 + i(a) \otimes i(b) + i(b) \otimes i(a) + 1 \otimes i(a)i(b) \\
 &\quad - i(b)i(a) \otimes 1 - i(b) \otimes i(a) - i(a) \otimes i(b) - 1 \otimes i(b)i(a) \\
 &= [i(a), i(b)] \otimes 1 + 1 \otimes [i(a), i(b)] \\
 &= \phi[i(a), i(b)].
 \end{aligned}$$

Thus, there is a unique algebra map $\delta : U \rightarrow U \otimes U$ such that

$$\begin{array}{ccc}
 U & \xrightarrow{\delta} & U \otimes U \\
 & \nwarrow i & \uparrow \phi \\
 & & L
 \end{array}$$

commutes, i.e. such that $\delta i(a) = \phi(a) = i(a) \otimes 1 + 1 \otimes i(a)$ as desired.

Tensor Algebra. Let V be a vector space over the complex field C . Consider the direct sum $T = T(V)$ of V defined

$$T = \sum_{n=0}^{\infty} T^n \quad \text{where} \quad T^n = V \otimes V \cdots \otimes V = \otimes^n V$$

for $n \geq 0$ and we take $T^0 = C$.

If $u = u_1 \otimes u_2 \otimes \cdots \otimes u_m \in T^m$ and $v = v_1 \otimes v_2 \otimes \cdots \otimes v_n \in T^n$, define multiplication in T by

$$uv = u_1 \otimes u_2 \otimes \cdots \otimes u_m \otimes v_1 \otimes v_2 \otimes \cdots \otimes v_n \in T^{m+n}.$$

The multiplication is associative and the distributive laws hold. Thus T is an infinite dimensional algebra over C with unity, called the **tensor algebra** of V .

The tensor algebra $T = T(V)$ is the free associative algebra on n generators ($n = \dim V$), in the sense that, given any linear map θ from V into an algebra A , there is a unique algebra mapping ϕ from T into A such that $\phi(1) = 1$ and $\phi i = \theta$, where i is the inclusion map.

$$\begin{array}{ccc} V & \xrightarrow{\theta} & A \\ i \downarrow & & \uparrow \phi \\ T & \xlongequal{\quad} & T \end{array}$$

In particular the map Δ from V into $T \otimes T$ given by $\Delta(a) = a \otimes 1 + 1 \otimes a$ is a linear map. Hence there is a unique algebra homomorphism from T to $T \otimes T$ that sends $x \in T$ to $1 \otimes x + x \otimes 1$.

If I is an ideal of T , then any algebra homomorphism θ from T to $T \otimes T$ such that $I \subseteq \ker(\theta)$ induces an algebra homomorphism

from T/I into $(T/I) \otimes (T/I)$. This observation will be used in the following repeatedly.

Theorem 1-1. Every Lie algebra has a universal enveloping algebra.

Proof: We sketch the proof. Let L be a given Lie algebra over a field k and T be the tensor algebra of L . The algebra T is an infinite dimensional algebra over k .

In the algebra T , let I be the ideal generated by

$$\{[ab] - a \otimes b + b \otimes a \in T \mid a, b \in L\},$$

$U = T/I$ and ν be the natural homomorphism from T to U . The restriction i of ν to $T^1 = L$ is an algebra mapping from L to U so that (U, i) is a universal enveloping algebra of L . Then we have

$$\begin{aligned} i[ab] - ia \otimes ib + ib \otimes ia &= i([ab] - a \otimes b + b \otimes a) \\ &= [ab] - a \otimes b + b \otimes a + I \\ &= I, \end{aligned}$$

so that $i[ab] = ia \otimes ib - ib \otimes ia$ in $U = T/I$, for all $a, b \in L$. The details of the proof may be found in [Se, LA3.2, Theorem 1.2].

Definition. Let $\{u_j \mid j \in J\}$ be a basis of L over k , where J is ordered and U is the universal enveloping algebra of U . The

monomials $u = u_{i_1} \otimes u_{i_2} \otimes \cdots \otimes u_{i_n}$ of degree n form a basis of T^n .

The monomial u is **standard** if $i_1 \leq i_2 \leq \cdots \leq i_n$.

Theorem 1-2. (Poincare-Birkoff-Witt). The elements $1 + I$ and $m + I$, with m ranging over all standard monomials, is a basis of U .

(See [Ja, p159]).

Corollary 1-3. L is imbedded in U .

Corollary 1-4. The diagonal map $\Delta : U \rightarrow U \otimes U$ defined by $\Delta(a) = 1 \otimes a + a \otimes 1$ for $a \in L$ is injective. See [Se, LA3.10, proposition 5.2].

Corollary 1-5. Let L be an algebra and let U be its universal enveloping algebra. If L_1 is a Lie subalgebra of L then the subalgebra U_1 of U generated by L_1 is a universal algebra of L_1 .

Furthermore if I is an ideal of L , then L/I can be identified with $(L + J)/J$ where J is the ideal of U generated by I and U/J is the universal enveloping algebra of $(L + I)/I$.

Notation. We will be working with deformations of U rather than L so we simplify our notation. We will write the product in U as xy instead of $x \otimes y$. The unity of U will be written as 1.

Basis of the enveloping algebra. We will identify iL with L , so L is contained in U_L since the identity mapping is a monomorphism of L into U_L . L generates U . Then by the above theorem if

$$\{u_j | j \in J\}$$

with J ordered, is a basis of L , then

$$\{1, \quad u = u_{i_1} u_{i_2} \dots u_{i_n} \quad | \quad i_1 \leq i_2 \dots\}$$

is a basis of U .

§2. Hopf Algebras. In this section we introduce a Hopf algebra and show that in order to check if an algebra is an Hopf algebra it is sufficient to check the Hopf axioms for a set of generators of the algebra. We give an important example of a Hopf algebra that will occur later.

Multiplication. Note that the multiplication on any algebra A can be linearly extended to a function $\mu : A \otimes A \rightarrow A$ in the form

$$\mu(a \otimes b) = ab$$

for all $a, b \in A$.

Definition. An Hopf algebra A is an associative unital algebra over C with the following properties (called Hopf axioms):

There are algebra homomorphisms $\Delta : A \rightarrow A \otimes A$, (called a **comultiplication**), $\epsilon : A \rightarrow C$ (called a **counit**), and $\text{id} : A \rightarrow A$ (called the identity) such that

$$(i) \quad (\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta$$

and

$$(ii) \quad (\epsilon \otimes \text{id})\Delta = (\text{id} \otimes \epsilon)\Delta$$

giving the following commutative diagrams:

$$\begin{array}{ccc}
 A & \xrightarrow{\Delta} & A \otimes A \\
 & \searrow^{\Delta \otimes \text{id}} & \\
 & & A \otimes A \otimes A \\
 & \searrow_{\text{id} \otimes \Delta} & \\
 & & A \otimes A \otimes A
 \end{array}
 \quad \parallel$$

$$\begin{array}{ccc}
 A & \xrightarrow{\Delta} & A \otimes A \\
 & \searrow^{\epsilon \otimes \text{id}} & \\
 & & A \cong C \otimes A \\
 & \searrow_{\text{id} \otimes \epsilon} & \\
 & & A \cong A \otimes C
 \end{array}
 \quad \parallel$$

There is an antihomomorphism $\gamma : A \rightarrow A$ (called an **antipode**) satisfying

$$(iii) \quad \mu(\gamma \otimes \text{id})\Delta = I\epsilon = \mu(\text{id} \otimes \gamma)\Delta$$

Thus we have the following diagram:

$$\begin{array}{ccc}
 & & A \otimes A \xrightarrow{\mu} A \\
 & \searrow^{\gamma \otimes \text{id}} & \\
 A & \xrightarrow{\Delta} & A \otimes A \\
 & \searrow_{\text{id} \otimes \gamma} & \\
 & & A \otimes A \xrightarrow{\mu} A
 \end{array}
 \quad \parallel$$

Coproduct. We refer to the action of the comultiplication on an element as the coproduct of that element.

Flip automorphism. We define σ by $\sigma(a \otimes b) = b \otimes a$. It can be shown that $\Delta(\gamma(a)) = (\gamma \otimes \gamma)(\sigma \Delta(a))$ for all a in the Hopf algebra.

A detailed study of Hopf algebras can be found in [Ab] and [Sw].

How can we check if a given algebra is a Hopf algebra? The next few lemmas will show that it is sufficient to check if the Hopf axioms hold on a set of generators of the algebra.

Lemma 2-1. Let A be an associative unital algebra over C with a comultiplication Δ and a counit ϵ which are homomorphisms, and an antipode γ which is an antihomomorphism such that Hopf axiom (i) holds on the generators. Then, axiom (i) holds on A .

Proof. Suppose that axiom (i) holds for elements a and b of A . Let the coproducts of a and b be $a^i \otimes a_i$ and $b^j \otimes b_j$, respectively. Then

$$\begin{aligned}
 (\Delta \otimes \text{id})\Delta(ab) &= (\Delta \otimes \text{id})\Delta(a)\Delta(b) \\
 &= (\Delta \otimes \text{id})(a^i b^j \otimes a_i b_j) \\
 &= \Delta(a^i)\Delta(b^j) \otimes a_i b_j \\
 &= (\Delta(a^i) \otimes a_i)(\Delta(b^j) \otimes b_j) \\
 &= (\Delta \otimes \text{id})\Delta(a)(\Delta \otimes \text{id})\Delta(b).
 \end{aligned}$$

Since axiom (i) holds for a and b we have that

$$(\Delta \otimes \text{id})\Delta(a)(\Delta \otimes \text{id})\Delta(b) = (\text{id} \otimes \Delta)\Delta(a)(\text{id} \otimes \Delta)\Delta(b).$$

Now this is the same as

$$\begin{aligned}
 & (a^i \otimes \Delta(a_i))(b^j \otimes \Delta(b_j)) \\
 &= (a^i b^j \otimes \Delta(a_i) \Delta(b_j)) \\
 &= (a^i b^j \otimes \Delta(a_i b_j)) \\
 &= (\text{id} \otimes \Delta)(a^i b^j \otimes a_i b_j) \\
 &= (\text{id} \otimes \Delta) \Delta(a) \Delta(b) \\
 &= (\text{id} \otimes \Delta) \Delta(ab).
 \end{aligned}$$

Since the axiom holds on the generators, induction gives that the axiom holds in general on A .

Lemma 2-2. Let A be as in the previous lemma and let axiom (ii) hold on the generators of A . Then axiom (ii) holds on A .

Proof. Suppose that axiom (ii) holds for elements a and b of A . Let the coproducts of a and b be $a^i \otimes a_i$ and $b^j \otimes b_j$, respectively. Then

$$\begin{aligned}
 (\epsilon \otimes \text{id}) \Delta(ab) &= (\epsilon \otimes \text{id}) \Delta(a) \Delta(b) \\
 &= (\epsilon \otimes \text{id})(a^i b^j \otimes a_i b_j) \\
 &= \epsilon(a^i) \epsilon(b^j) a_i b_j \\
 &= (\epsilon(a^i) a_i) (\epsilon(b^j) b_j) \\
 &= (\epsilon \otimes \text{id}) \Delta(a) (\epsilon \otimes \text{id}) \Delta(b).
 \end{aligned}$$

Since axiom (ii) holds for a and b we have that

$$(\epsilon \otimes \text{id}) \Delta(a) (\epsilon \otimes \text{id}) \Delta(b) = (\text{id} \otimes \epsilon) \Delta(a) (\text{id} \otimes \epsilon) \Delta(b).$$

Then the expression is

$$\begin{aligned}
 &= (a^i \epsilon(a_i))(b^j \epsilon(b_j)) \\
 &= a^i b^j \epsilon(a_i) \epsilon(b_j) \\
 &= (a^i b^j \epsilon(a_i b_j)) \\
 &= (\text{id} \otimes \epsilon)(a^i b^j \otimes a_i b_j) \\
 &= (\text{id} \otimes \epsilon) \Delta(a) \Delta(b) \\
 &= (\text{id} \otimes \epsilon) \Delta(ab).
 \end{aligned}$$

Since the axiom holds on the generators, induction gives that the axiom holds in general on A .

Lemma 2-3. Let A be as in the previous lemma and let axiom (iii) hold on the generators of A . Then axiom (iii) holds on A .

Proof. Suppose that axiom (iii) holds for elements a and b of A . Let the coproducts of a and b be $a^i \otimes a_i$ and $b^j \otimes b_j$, respectively. Then axiom (iii) gives

$$\gamma(b^j)b_j = \epsilon(b)I = b^j\gamma(b_j) \quad \text{and} \quad \gamma(a^i)a_i = \epsilon(a)I = a^i\gamma(a_i).$$

Then

$$\begin{aligned}
 \mu(\gamma \otimes \text{id})\Delta(ab) &= \mu(\gamma \otimes \text{id})\Delta(a)\Delta(b) \\
 &= \mu(\gamma \otimes \text{id})(a^i b^j \otimes a_i b_j) \\
 &= \mu(\gamma(b^j)\gamma(a^i) \otimes a_i b_j) \\
 &= \gamma(b^j)\gamma(a^i)a_i b_j \\
 &= \gamma(b^j)\epsilon(a)Ib_j \\
 &= \epsilon(a)I\epsilon(b)I \\
 &= \epsilon(ab)I.
 \end{aligned}$$

Also

$$\begin{aligned}
 \mu(\text{id} \otimes \gamma)\Delta(ab) &= \mu(\text{id} \otimes \gamma)\Delta(a)\Delta(b) \\
 &= \mu(\text{id} \otimes \gamma)(a^i b^j \otimes a_i b_j) \\
 &= \mu(a^i b^j \otimes \gamma(b_j)\gamma(a_i)) \\
 &= a^i b^j \gamma(b^j)\gamma(a^i) \\
 &= a^i \epsilon(b)I\gamma(a^i) \\
 &= \epsilon(a)I\epsilon(b)I \\
 &= \epsilon(ab)I \\
 &= \mu(\gamma \otimes \text{id})\Delta(ab).
 \end{aligned}$$

Since the axiom holds on the generators, induction gives that the axiom holds in general on A .

We have proved:

Theorem 2-4. If A is an algebra with unity, and Δ and ϵ are homomorphisms and γ is an antihomomorphism such that the Hopf axioms hold on a set of generators of A , then A is a Hopf algebra.

Proof. Follows from Lemmas 2-1, 2-2, and 2-3.

Examples 1. Let G be a group and CG the group algebra of G over C . Define $\Delta : CG \rightarrow CG \otimes CG$, $\epsilon : CG \rightarrow C$ and $\gamma : CG \rightarrow CG$ by

$$\Delta(x) = x \otimes x, \quad \gamma(x) = x^{-1}, \quad \epsilon(\alpha 1_{CG}) = \alpha 1_C$$

and $\epsilon(u) = 0$ for $u \neq \beta 1_{CG}$ for all $x \in G$. Then it is easily verified that CG is a Hopf algebra.

Definition. Let X be a set. A free algebra on X is an algebra F together with a mapping i from X to F such that if j is any function from X into an algebra A , there is a unique algebra homomorphism θ from F into A such that $\theta i = j$.

$$\begin{array}{ccc} & F & \\ i \uparrow & & \\ X & \xrightarrow{j} & A \end{array}$$

For example, let V be a vector space with X as a basis and T the tensor algebra of V . Then it is easily verified that T is a free algebra

on X . It is also easy to check that two free algebras on X are isomorphic. Hence we may assume that a free algebra on a set is unique.

2. Let F be a free algebra on $\{x, y, z\}$, I the ideal of F generated by $xy - 1$ and $yx - 1$, I_n the ideal generated by $xy - 1$, $yx - 1$ and $x^n z x^n - z$ and J_n the ideal generated by $xy - 1$, $yx - 1$ and $x^n - 1$, and

$$H_0 = F/I, \quad H_n = F/I_n \quad \text{and} \quad K_n = F/J_n.$$

Let $H = H_0, H_n$ or K_n . We define $\Delta : H \rightarrow H \otimes H$, $\epsilon : H \rightarrow C$ and $\gamma : H \rightarrow H$ by

$$\Delta(x) = x \otimes x, \Delta(y) = y \otimes y, \Delta(z) = z \otimes 1 + 1 \otimes z,$$

$$\epsilon(x) = 1, \quad \epsilon(y) = 1, \quad \epsilon(z) = 0,$$

$$\gamma(x) = x^{-1}, \quad \gamma(x^{-1}) = x, \quad \gamma(z) = -zx^{-1}.$$

Then H_0, H_n and K_n are Hopf algebras [Sw].

Theorem 2-5. Let F be a free algebra on $\{x, y, z, u\}$, $q (\neq 1)$ a positive real number and I the ideal of F generated by the elements

$$xy - qyx, \quad xz - qzx, \quad yu - quy,$$

$$zu - quz, \quad yz - zy, \quad xu - qyz - 1, \quad ux - q^{-1}yz - 1.$$

Then the algebra $A_q = F/I$ is an Hopf algebra.

Proof. The algebra A_q is generated by $\{x, y, z, u\}$ subjected to the relations

$$xy = qyx, \quad xz = qzx, \quad yu = quy, \quad zu = quz,$$

$$yz = zy, \quad xu - qyz = 1, \quad ux - q^{-1}yz = 1.$$

An element f of A_q is a finite linear combination of monomials in x, y, z and u . In view of the above defining relations, f can be expressed as a linear combination of the monomials of the form

$$x^k y^l z^m u^n.$$

To prove the result, we define

$$\epsilon : A_q \rightarrow C \text{ and } \gamma : A_q \rightarrow A_q$$

and verify the axioms of a Hopf algebra. Obviously, it suffices to define the functions Δ , ϵ and γ on a typical element of the form $f = x^k y^l z^m u^n$. Define the comultiplication by

$$\Delta(x^k y^l z^m u^n) = \Delta(x)^k \Delta(y)^l \Delta(z)^m \Delta(u)^n$$

where

$$\Delta(x) = x \otimes x + y \otimes z, \quad \Delta(y) = x \otimes y + y \otimes u,$$

$$\Delta(z) = z \otimes x + u \otimes x, \quad \Delta(u) = z \otimes y + u \otimes u.$$

We define the counit by

$$\epsilon(x^k y^l z^m u^n) = \delta_{l0} \delta_{m0}.$$

The antipode is defined by

$$\gamma(x^k y^l z^m u^n) = x^n (-qz)^m (-q^{-1}y)^l u^k.$$

We now extend these definitions by linearity.

Clearly, the functions μ , Δ , ϵ and γ are all linear transformations. We will show that Δ , and ϵ are homomorphisms, and that γ is an antihomomorphism. We will also show that the Hopf axioms hold on the generators. Then by theorem 2-4, A_q will be a Hopf algebra. We will do this in several steps:

Step 1: Hopf axiom (i) holds for the generators of A_q .

Proof. As

$$\begin{aligned} (\Delta \otimes \text{id})\Delta x &= (\Delta \otimes \text{id})(x \otimes x + y \otimes z) \\ &= \Delta x \otimes x + \Delta y \otimes z \\ &= x \otimes x \otimes x + y \otimes z \otimes x + x \otimes y \otimes z + y \otimes u \otimes z \\ &= x \otimes (x \otimes x + y \otimes z) + y \otimes (z \otimes x + u \otimes z) \\ &= x \otimes \Delta x + y \otimes \Delta z \\ &= (\text{id} \otimes \Delta)\Delta x, \end{aligned}$$

it follows that $(\Delta \otimes \text{id})\Delta x = (\text{id} \otimes \Delta)\Delta x$. Similarly, we have

$$(\Delta \otimes \text{id})\Delta t = (\text{id} \otimes \Delta)\Delta t \text{ for } t = y, z, u,$$

so that $(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta$ on the generators of A_q .

Step 2: Hopf axiom (ii) holds on the generators of A_q .

Proof.

$$(\epsilon \otimes \text{id})\Delta t = 0 = (\text{id} \otimes \epsilon)\Delta t$$

for $t = x, y, z, u$.

Step 3: Hopf axiom (iii) holds for all of the generators of A_q .

Proof. We have

$$\begin{aligned} \mu(\gamma \otimes \text{id})\Delta x &= \mu(\gamma \otimes \text{id})(x \otimes x + y \otimes z) \\ &= \gamma(x)x + \gamma(y)z \\ &= ux - q^{-1}yz = 1, \end{aligned}$$

$$\begin{aligned} \mu(\text{id} \otimes \gamma)\Delta x &= \mu(\text{id} \otimes \gamma)(x \otimes x + y \otimes z) \\ &= x\gamma(x) + u\gamma(z) \\ &= xu - qyz = 1, \end{aligned}$$

$$\begin{aligned}
\mu(\gamma \otimes \text{id})\Delta y &= \mu(\gamma \otimes \text{id})(x \otimes y + y \otimes u) \\
&= \gamma(x)y + \gamma(y)u \\
&= uy - q^{-1}yu \\
&= uy - uy = 0,
\end{aligned}$$

$$\begin{aligned}
\mu(\text{id} \otimes \gamma)\Delta y &= \mu(\text{id} \otimes \gamma)(x \otimes y + y \otimes u) \\
&= x\gamma(y) + y\gamma(u) \\
&= -q^{-1}xy + yx \\
&= -yx + yx = 0,
\end{aligned}$$

$$\mu(\gamma \otimes \text{id})\Delta z = 0 = \mu(\text{id} \otimes \gamma)\Delta z,$$

and

$$\mu(\gamma \otimes \text{id})\Delta u = 1 = \mu(\text{id} \otimes \gamma)\Delta u.$$

Hence we have

$$\mu(\gamma \otimes \text{id})\Delta = \mu(\text{id} \otimes \gamma)\Delta$$

on the generators of A_q .

Step 4: Δ is an algebra homomorphism.

Proof. We have to show that

$$\Delta(x^k y^l z^m u^n x^{k_1} y^{l_1} z^{m_1} u^{n_1}) = \Delta(x^m y^l z^m u^n) \Delta(x^{k_1} y^{l_1} z^{m_1} u^{n_1}),$$

for arbitrary exponents $k, l, m, n, k_1, l_1, m_1$ and n_1 .

As

$$\begin{aligned}
 u^n x &= u^{n-1}(ux) = u^{n-1}(1 + q^{-1}yz) = u^{n-1} + q^{-1}u^{n-1}yz \\
 &= u^{n-1} + q^{-1}q^{-n+1}yu^{n-1}z = u^{n-1} + q^{-n}y(u^{n-1}x) \\
 &= u^{n-1} + q^{-n}q^{-n+1}yzu^{n-1} = u^{n-1} + q^{-2n+1}yzu^{n-1},
 \end{aligned}$$

it follows that

$$x^k y^l z^m u^n x = x^k y^l z^m u^{n-1} + q^{-2n+1} x^k y^{l+1} z^{m+1} u^{n-1}$$

and

$$\begin{aligned}
 \Delta(x^k y^l z^m u^n x) &= (\Delta x)^k (\Delta y)^l (\Delta z)^m (\Delta u)^{n-1} \\
 &\quad + q^{-2n+1} (\Delta x)^k (\Delta y)^{l+1} (\Delta z)^{m+1} (\Delta u)^{n-1}.
 \end{aligned}$$

A similar argument shows that

$$(\Delta u)^n \Delta x = (\Delta u)^{n-1} + q^{-2n+1} \Delta z (\Delta u)^{n-1},$$

so that

$$\begin{aligned}
 \Delta(x^k y^l z^m u^n) \Delta x &= (\Delta x)^k (\Delta y)^l (\Delta z)^m (\Delta u)^n \Delta x \\
 &= (\Delta x)^k (\Delta y)^l (\Delta z)^m (\Delta u)^{n-1} \\
 &\quad + q^{-2n+1} (\Delta x)^k (\Delta y)^{l+1} (\Delta z)^{m+1} (\Delta u)^{n-1}.
 \end{aligned}$$

Hence it follows that

$$\Delta(x^k y^l z^m u^n x) = \Delta(x^k y^l z^m u^n) \Delta x.$$

By induction on k_1 , we get

$$\Delta(x^k y^l z^m u^n x^{k_1}) = \Delta(x^k y^l z^m u^n) \Delta(x^{k_1})$$

for all integer $k_1 \geq 0$. Next

$$\begin{aligned} \Delta(x^k y^l z^m u^n y) &= \Delta(x^k y^l z^m q^{-n} y u^n) \\ &= q^{-n} \Delta(x^k y^{l+1} z^m u^n) \\ &= q^{-n} (\Delta x)^k (\Delta y)^{l+1} (\Delta z)^m (\Delta u)^n \\ &= (\Delta x)^k (\Delta y)^l (\Delta z)^m q^{-n} \Delta y (\Delta u)^n \\ &= (\Delta x)^k (\Delta y)^l (\Delta z)^m (\Delta u)^n \Delta y \\ &= \Delta(x^k y^l z^m u^n) \Delta y. \end{aligned}$$

Induction on l_1 shows that

$$\Delta(x^k y^l z^m u^n x^{k_1}) = \Delta(x^k y^l z^m u^n) \Delta(x^{k_1}).$$

A similar argument shows that

$$\Delta(x^k y^l z^m u^n y^{l_1}) = \Delta(x^k y^l z^m u^n) \Delta(y^{l_1}).$$

Obviously, we have

$$\Delta(x^k y^l z^m u^n u^{n_1}) = \Delta(x^k y^l z^m u^n) \Delta(u^{n_1}).$$

Combining the above cases, it follows that Δ is an algebra homomorphism.

Step 5: ϵ is an algebra homomorphism.

Proof. We found that

$$x^k y^l z^m u^n x = x^k y^l z^m u^{n-1} + q^{-2n+1} x^k y^{l+1} z^{m+1} u^{n-1}.$$

Then

$$\begin{aligned} & \epsilon(x^k y^l z^m u^n x) \\ &= \epsilon(x^k y^l z^m u^{n-1}) + \epsilon(q^{-2n+1} x^k y^{l+1} z^{m+1} u^{n-1}) \\ &= \delta_{l0} \delta_{m0} \\ &= \epsilon(x^k y^l z^m u^n) \epsilon(x). \end{aligned}$$

We now argue by induction that

$$\epsilon(x^k y^l z^m u^n x^{k_1}) = \epsilon(x^k y^l z^m u^n) \epsilon(x^{k_1}).$$

$$\begin{aligned} \epsilon(x^k y^l z^m u^n y) &= q^{-n} \epsilon(x^k y^{l+1} z^m u^n) \\ &= 0 \\ &= \epsilon(x^k y^l z^m u^n) \epsilon(y). \end{aligned}$$

We now argue by induction that

$$\epsilon(x^k y^l z^m u^n y^{l_1}) = \epsilon(x^k y^l z^m u^n) \epsilon(y^{l_1}).$$

Similarly $\epsilon(x^k y^l z^m u^n z^{m_1}) = \epsilon(x^k y^l z^m u^n) \epsilon(z^{m_1})$.

Obviously, we have

$$\epsilon(x^k y^l z^m u^n u^{n_1}) = \epsilon(x^k y^l z^m u^n) \epsilon(u^{n_1}).$$

Combining the above cases, it follows that ϵ is an algebra homomorphism.

Step 6: γ is an antihomomorphism.

Proof. We defined $\gamma(x^k y^l z^m u^n) = x^n (-qz)^m (-q^{-1}y)^l u^k$ so

$$\begin{aligned} & \gamma(x^k y^l z^m u^n x) \\ &= \gamma(x^k y^l z^m u^{n-1} + q^{-2n+1} x^k y^{l+1} z^{m+1} u^{n-1}) \\ &= x^{n-1} (-qz)^m (-q^{-1}y)^l u^k + q^{-2n+1} x^{n-1} (-qz)^{m+1} (-q^{-1}y)^{l+1} u^k \\ &= (-q)^{m-l} x^{n-1} z^m y^l u^k + q^{-2n+1} (-q)^{m-l} x^{n-1} z^{m+1} y^{l+1} u^k. \end{aligned}$$

On the other hand

$$\begin{aligned} & \gamma(x) \gamma(x^k y^l z^m u^n) \\ &= u x^n (-qz)^m (-q^{-1}y)^l u^k \\ &= (-q)^{m-l} u x x^{n-1} z^m y^l u^k \\ &= (-q)^{m-l} (1 + q^{-1} y z) x^{n-1} z^m y^l u^k \\ &= (-q)^{m-l} x^{n-1} z^m y^l u^k + q^{-2n+1} (-q)^{l-m} x^{n-1} z^{m+1} y^{l+1} u^k \\ &= \gamma(x^k y^l z^m u^n x). \end{aligned}$$

By induction, $\gamma(x^k y^l z^m u^n x^{k_1}) = \gamma(x^{k_1}) \gamma(x^k y^l z^m u^n)$.

$$\begin{aligned} \gamma(x^k y^l z^m u^n y) &= q^{-n} \gamma(x^k y^{l+1} z^m u^n) \\ &= q^{-n} x^n (-qz)^m (-q^{-1}y)^{l+1} u^k \end{aligned}$$

while

$$\begin{aligned} \gamma(y) \gamma(x^k y^l z^m u^n) &= (-q^{-1}y) x^n (-qz)^m (-q^{-1}y)^l u^k \\ &= q^{-n} x^n (-qz)^m (-q^{-1}y)^{l+1} u^k. \end{aligned}$$

We now argue by induction that

$$\gamma(x^k y^l z^m u^n y^{l_1}) = \gamma(y^{l_1}) \gamma(x^k y^l z^m u^n).$$

Similarly $\gamma(x^k y^l z^m u^n z^{m_1}) = \gamma(z^{m_1}) \gamma(x^k y^l z^m u^n)$.

Similar to the proof for powers of x , we have

$$\gamma(x^k y^l z^m u^n u^{n_1}) = \gamma(u^{n_1}) \gamma(x^k y^l z^m u^n).$$

Combining the above cases, it follows that γ is an algebra antihomomorphism.

Δ and ϵ have been shown to be homomorphisms. γ is an antihomomorphism. The Hopf axioms hold on the generators of A_q . Then by theorem 2-4, A_q is a Hopf algebra.

§3. Universal enveloping algebras of a simple Lie algebras.

In this section we will show how to get the defining relations for a simple Lie algebra from a Cartan matrix. We then show that the universal enveloping algebra of a simple Lie algebra is a Hopf algebra.

Algebra U_L . The algebra $A = \langle X \rangle$ becomes a Lie algebra $\langle X \rangle_L$ with respect to the bracket operation $[xy] = xy - yx$ for all $x, y \in A$. The subalgebra F of the Lie algebra $\langle X \rangle_L$ generated by $\langle X \rangle$ is called the *free Lie algebra* on X . $\langle X \rangle$ is a universal enveloping algebra of $\langle X \rangle_L$. (See the result of Witt in [Ja]).

Cartan Matrix. A matrix (A_{ij}) of degree n with rational integer coefficients is called a Cartan matrix if

- (i) $A_{ii} = 2$ for all $i = 1, 2, \dots, n$.
- (ii) $A_{ij} \leq 0$ for all $i \neq j$ and $A_{ij} = 0$ iff $A_{ji} = 0$.
- (iii) The determinant of (A_{ij}) is nonzero.
- (iv) If $\{v_1, v_2, \dots, v_n\}$ is a basis of a n dimensional vector space over the field of rational numbers, the group W generated by the linear transformations S_j defined by

$$v_i S_j = v_i - A_{ij} v_j; \quad i, j = 1, 2, \dots, n$$

is a finite group.

We now use the Cartan matrix to construct a simple Lie algebra.

Simple Lie algebra for a Cartan matrix. Let (A_{ij}) be a Cartan matrix of degree n , F the free algebra generated by

$$X = \{ h_i, x_i, y_i \mid i = 1, 2, \dots, n \},$$

FL the subalgebra of F_L generated by X , I the ideal of FL generated by the elements

$$[h_i h_j], [x_i y_j] - \delta_{ij} h_i, [h_i x_j] - A_{ij} x_j, [h_i y_j] + A_{ij} y_j,$$

where $i, j = 1, 2, \dots, n$, $[xy] = xy - yx$ for any $x, y \in F$, $L = FL/I$, $\bar{I}$ the ideal of F generated by I and UL the quotient algebra $F/\bar{I}$. Then UL is the universal enveloping algebra of the Lie algebra L generated by $\{ h_i, x_i, y_i \mid i = 1, 2, \dots, n \}$ and subject to the relations:

$$[h_i h_j] = 0, [x_i y_j] = \delta_{ij} h_i, [h_i x_j] = A_{ij} x_j, [h_i y_j] = -A_{ij} y_j$$

for $i, j = 1, 2, \dots, n$.

Simple Lie algebras. All of the simple Lie algebras are given by the construction just described. The next theorem shows that the universal enveloping algebras of simple Lie algebras have a Hopf algebra structure when the comultiplication, counit, and antipode are defined appropriately.

Theorem 3-1. Let (A_{ij}) be a Cartan matrix of degree n . Then the enveloping algebra UL of the Lie algebra L generated by

$$\{h_i, x_i, y_i \mid i = 1, 2, \dots, n\}$$

with relations

$$[h_i h_j] = 0, [x_i y_j] = \delta_{ij} h_i, [h_i x_j] = A_{ij} x_j, [h_i y_j] = -A_{ij} y_j$$

for $i, j = 1, 2, \dots, n$ and where $[xy] = xy - yx$, is a Hopf algebra.

Proof. Write UL as U for simplicity. We now need to define maps $\Delta : U \rightarrow U \otimes U$, $\epsilon : U \rightarrow C$, $\gamma : U \rightarrow U$ and to check the Hopf axioms.

Define

$$\Delta h = 1_U \otimes h + h \otimes 1_U, \quad h \in L$$

$$\epsilon(\alpha 1_U) = \alpha 1_C \quad \epsilon(h) = 0, \forall h \neq \beta 1_U$$

and

$$\gamma(h) = -h \quad \forall h \in \{h_i, x_i, y_i | i = 1, 2, \dots, n\}.$$

The maps Δ and ϵ and γ are linear. Δ and ϵ can be extended to unique algebra homomorphisms. (ϵ is easy and Δ in this case is the diagonal map and is known to be a homomorphism (see [Se, proposition 5.2])). γ can be extended to an antihomomorphism.

For any $h \in \{h_i, x_i, y_i | i = 1, 2, \dots, n\}$ we have

$$\begin{aligned}
 (\Delta \otimes \text{id})\Delta h &= (\Delta \otimes \text{id})(1 \otimes h + h \otimes 1) \\
 &= \Delta 1 \otimes h + \Delta h \otimes 1 \\
 &= 1 \otimes 1 \otimes h + (1 \otimes h + h \otimes 1) \otimes 1 \\
 &= 1 \otimes 1 \otimes h + 1 \otimes h \otimes 1 + h \otimes 1 \otimes 1
 \end{aligned}$$

while

$$\begin{aligned}
 (\text{id} \otimes \Delta)\Delta h &= (\text{id} \otimes \Delta)(1 \otimes h + h \otimes 1) \\
 &= 1 \otimes \Delta h + h \otimes \Delta 1 \\
 &= 1 \otimes (1 \otimes h + h \otimes 1) + h \otimes 1 \otimes 1 \\
 &= 1 \otimes 1 \otimes h + 1 \otimes h \otimes 1 + h \otimes 1 \otimes 1
 \end{aligned}$$

and so Hopf axiom (i) holds on the generators.

Now we check Hopf axiom (ii):

$$(\epsilon \otimes \text{id})\Delta h = (\epsilon \otimes \text{id})(1 \otimes h + h \otimes 1) = h$$

while

$$(\text{id} \otimes \epsilon)\Delta h = (\text{id} \otimes \epsilon)(1 \otimes h + h \otimes 1) = h$$

and so axiom (ii) holds for the generators.

For axiom (iii) we have

$$\begin{aligned}
 \mu(\gamma \otimes \text{id})\Delta h &= \mu(\gamma \otimes \text{id})(1 \otimes h + h \otimes 1) \\
 &= \mu(1 \otimes h + \gamma(h) \otimes 1) \\
 &= h - (h)1 \\
 &= 0
 \end{aligned}$$

while

$$\begin{aligned}
 \mu(\text{id} \otimes \gamma)\Delta h &= \mu(\text{id} \otimes \gamma)(1 \otimes h + h \otimes 1) \\
 &= \mu(1 \otimes \gamma(h) + h \otimes 1) \\
 &= -h + h \\
 &= 0
 \end{aligned}$$

and $\epsilon(h)I = 0$, so axiom (iii) holds on the generators.

The Hopf axioms hold on the generators. ϵ and Δ are homomorphisms, and γ is an antihomomorphism. Thus, U is a Hopf algebra by theorem 2-4.

§4. Deformed Algebras. In this section we deform the universal enveloping algebra of a Lie algebra according to the approach of Drinfeld in [Dr] and Jimbo in [Ji] and show that the Hopf algebra structure can be retained.

Generators and Relations. Let X be an arbitrary set, V a vector space over C with X as a basis and T the tensor algebra of V . Then $T = \langle X \rangle$ is a free algebra on the set X . For any $x \in X$ let $C[[x]]$ be the ring of **formal power series** in x over C . Let q be a positive real number not equal to 1. We define $q^x \in C[[x]]$ by

$$q^x = \sum_{n=0}^{\infty} \frac{(\ln q)^n x^n}{n!}$$

and we will assume that $q^x q^{-x} = q^{-x} q^x = 1$.

Definition. If f is a homomorphism on T and $x \in T$ then we will define an extension of f by

$$f(q^x) = q^{f(x)}.$$

Deformed Algebra UL_q . Let $q (\neq 1)$ be a positive real number, (A_{ij}) a Cartan matrix of degree n and F the free algebra generated by

$$\{h_i, x_i, y_i, q^{h_i/2}, q^{-h_i/2} \mid i = 1, 2, \dots, n\}$$

and I the ideal of F generated by

$$q^{h_i/2}q^{-h_i/2} - 1, \quad q^{-h_i/2}q^{h_i/2} - 1, \quad [h_i h_j],$$

$$[x_i y_j] = \delta_{ij} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}, \quad [h_i x_j] = A_{ij} x_j \text{ and } [h_i y_j] = -A_{ij} y_j$$

where $i, j = 1, 2, \dots, n$ and $[x, y] = xy - yx$. The algebra $UL_q = F/I$ is called a **deformed algebra** of UL and it has a unity, since F has a unity. The algebra UL_q is generated by

$$\{ h_i, x_i, y_i, q^{h_i/2}, q^{-h_i/2} \mid i = 1, 2, \dots, n \},$$

subject to the relations

$$q^{h_i/2}q^{-h_i/2} = q^{-h_i/2}q^{h_i/2} = 1, [h_i h_j] = 0, [x_i y_j] = \delta_{ij} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}},$$

$$[h_i x_j] = A_{ij} x_j, \quad [h_i y_j] = -A_{ij} y_j$$

for $i, j = 1, 2, \dots, n$.

Notation. We define the notation

$$[x]_q \equiv \frac{q^x - q^{-x}}{q - q^{-1}}.$$

Thus we can write $[x_i y_j] = \delta_{ij} [h_i]_q$ above. Where no confusion will result we drop the subscript q .

Connection to UL . If we set q equal to 1 we see that we get the definition of the original undeformed algebra in the following sense:

$$\begin{aligned}
 & [x_i y_i] \\
 &= \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}} \\
 &= \frac{\ln q h_i + (\ln^3 q / 3!) h_i^3 + (\ln^5 q / 5!) h_i^5 \dots}{\ln q + \ln^3 q / 3! + \ln^5 q / 5! \dots} \\
 &= \frac{h_i + (\ln^2 q / 3!) h_i^3 + (\ln^4 q / 5!) h_i^5 \dots}{1 + \ln^2 q / 3! + \ln^4 q / 5! \dots}
 \end{aligned}$$

and when we set $q = 1$ we get h_i which is the result of $[x_i y_i]$ in the undeformed algebra. Thus, by abuse of notation I write

$$\lim_{q \rightarrow 1} UL_q = UL.$$

The algebra UL_q is said to be obtained by deforming UL or L and will be called a **quantum algebra**.

We do not hope to deform the commutator relations of UL to get UL_q and still retain its structure. But we can still retain a Hopf algebra structure.

Theorem 4-1. Let (A_{ij}) be a Cartan matrix of degree n and let UL_q be the deformed algebra generated by

$$\{h_i, x_i, y_i, q^{h_i/2}, q^{-h_i/2}\}$$

subject to the relations

$$q^{h_i/2} q^{-h_i/2} = q^{-h_i/2} q^{h_i/2} = 1,$$

$$[h_i h_j] = 0, [x_i y_j] = \delta_{ij} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}$$

$$[h_i x_j] = A_{ij} x_j, [h_i y_j] = -A_{ij} y_j$$

for $i, j = 1, 2, \dots, n$. (We have written $q^{h_i/2} q^{h_i/2} = q^{h_i}$, etc.) Then UL_q is a Hopf algebra.

Proof. Let $V = UL_q$, The elements of UL_q are linear combinations of monomials in x_i, y_i, h_i, q^{h_i} and q^{-h_i} . The defining relations allow us to express an element as a linear combination of elements of the form

$$\Pi_{i=1}^n (x_i^{k_i} y_i^{l_i} h_i^{m_i} q^{\nu_i h_i}),$$

where $k_i, l_i, m_i \geq 0$ and ν_i are integers. As before, we define the maps $\epsilon : V \rightarrow C$, $\gamma : V \rightarrow V$ and verify the Hopf axioms. For this purpose, define

$$\Delta \Pi_{i=1}^n (x_i^{k_i} y_i^{l_i} h_i^{m_i} q^{\nu_i h_i}) = \Pi_{i=1}^n (\alpha_i^{k_i} \beta_i^{l_i} \gamma_i^{m_i} \delta_i^{\nu_i}),$$

where $\alpha_i = q^{-h_i/2} \otimes x_i + x_i \otimes q^{h_i/2}$, $\beta_j = q^{-h_j/2} \otimes y_j + y_j \otimes q^{h_j/2}$, $\gamma_i = 1 \otimes h_i + h_i \otimes 1$, and $\delta_i = q^{h_i} \otimes q^{h_i}$,

$$\epsilon(\alpha 1_V) = \alpha 1_C \text{ and } \epsilon(u) = 0 \text{ for } u \neq \beta 1_V$$

and

$$\gamma \prod_{i=1}^n (x_i^{k_i} y_i^{l_i} h_i^{m_i} q^{\nu_i h_i}) = \prod_{i=1}^n q^{-\nu_i h_i} (-h_i)^{m_i} (-q^{-1} y_i)^{l_i} (-q x_i)^{k_i}$$

for all $k_i, l_i, h_i \geq 0$ and all integers ν_i . We set

$$\epsilon(q^{\alpha h_i}) = 1$$

for all $\alpha \in C$.

Clearly the maps Δ , ϵ and γ are linear. It is easily checked that ϵ is an algebra homomorphism. First we observe the consequences of the relations, namely

$$\Delta q^{\alpha h_i} = q^{\alpha h_i} \otimes q^{\alpha h_i}, \quad \epsilon(q^{\alpha h_i}) = 1 \text{ for } \alpha \in C, \quad i = 1, 2, \dots, n$$

and

$$\begin{aligned} q^{h_i/2} x_j &= q^{A_{ij}/2} x_j q^{h_i/2}, & q^{-h_i/2} x_j &= q^{-A_{ij}/2} x_j q^{-h_i/2}, \\ q^{h_i/2} y_j &= q^{A_{ij}/2} y_j q^{h_i/2}, & q^{-h_i/2} y_j &= q^{A_{ij}/2} y_j q^{-h_i/2} \end{aligned}$$

for all $i, j = 1, 2, \dots, n$.

The last four relations are consequences of the following observations.

Claim. $h_i^n x_j = x_j (h_i + A_{ij})^n, \quad h_i^n y_j = y_j (h_i - A_{ij})^n.$

Proof. First we prove $h_i^n x_j = x_j(h_i + A_{ij})^n$. In the case $n = 1$ this is just the commutator relation. Assume that

$$h_i^k x_j = x_j(h_i + A_{ij})^k.$$

Then

$$h_i^{k+1} x_j = h_i h_i^k x_j = h_i x_j(h_i + A_{ij})^k = x_j(h_i + A_{ij})(h_i + A_{ij})^k$$

and so by mathematical induction $h_i^n x_j = x_j(h_i + A_{ij})^n$. The proof that $h_i^n y_j = y_j(h_i - A_{ij})^n$ follows in the same way.

Claim. $q^{\alpha h_i} x_j = x_j q^{\alpha A_{ij}} q^{\alpha h_i}$ and $q^{\alpha h_i} y_j = y_j q^{-\alpha A_{ij}} q^{\alpha h_i}$.

Proof.

$$\begin{aligned} q^{\alpha h_i} x_j &= \sum_{t=0}^{\infty} \frac{\alpha^t (\ln q)^t}{t!} h_i^t x_j \\ &= \sum_{t=0}^{\infty} \frac{\alpha^t (\ln q)^t}{t!} x_j (h_i + A_{ij})^t \\ &= x_j e^{\alpha(h_i + A_{ij}) \ln q} \\ &= x_j q^{\alpha A_{ij}} q^{\alpha h_i} \end{aligned}$$

and the proof of $q^{\alpha h_i} y_j = y_j q^{-\alpha A_{ij}} q^{\alpha h_i}$ follows similarly.

As a consequence of the relations we have

$$\Delta q^{\alpha h_i} = q^{\alpha h_i} \otimes q^{\alpha h_i}, \quad \epsilon(q^{\alpha h_i}) = 1 \quad \forall \alpha \in C, \quad h_i$$

and

$$q^{h_i/2}x_j = q^{A_{ij}/2}x_jq^{h_i/2}, \quad q^{-h_i/2}x_j = q^{-A_{ij}/2}x_jq^{-h_i/2},$$

$$q^{h_i/2}y_j = q^{-A_{ij}/2}y_jq^{h_i/2}, \quad q^{-h_i/2}y_j = q^{A_{ij}/2}y_jq^{-h_i/2}.$$

The last four relations follow from the previous observations. We also have

Antipode. $\gamma(x_j) = -qx_j, \quad \gamma(y_j) = -q^{-1}y_j.$

Proof. $\gamma(x_j) = -q^{h_i/2}x_jq^{-h_i/2} = -q^{A_{jj}/2}x_j = -qx_j$ and similarly $\gamma(y_j) = -q^{-1}y_j.$

Now we check if the Hopf axioms hold on the generators.

Axiom (i). Hopf axiom (i) holds on the generators.

Proof. For any $h = h_i$, we have

$$\begin{aligned} (\Delta \otimes \text{id})\Delta h &= (\Delta \otimes \text{id})(1 \otimes h + h \otimes 1) \\ &= \Delta 1 \otimes h + \Delta h \otimes 1 \\ &= 1 \otimes 1 \otimes h + (1 \otimes h + h \otimes 1) \otimes 1 \\ &= 1 \otimes 1 \otimes h + 1 \otimes h \otimes 1 + h \otimes 1 \otimes 1 \end{aligned}$$

while

$$\begin{aligned}
(\Delta \otimes \text{id})\Delta h &= (\Delta \otimes \text{id})(1 \otimes h + h \otimes 1) \\
&= \Delta 1 \otimes h + \Delta h \otimes 1 \\
&= 1 \otimes 1 \otimes h + (1 \otimes h + h \otimes 1) \otimes 1 \\
&= 1 \otimes 1 \otimes h + 1 \otimes h \otimes 1 + h \otimes 1 \otimes 1
\end{aligned}$$

and so axiom (i) holds for h_i . For any x_i , we have

$$\begin{aligned}
&(\text{id} \otimes \Delta)\Delta x_i \\
&= (\text{id} \otimes \Delta)(q^{-h_i/2} \otimes x_i + x_i \otimes q^{h_i/2}) \\
&= q^{-h_i/2} \otimes \Delta x_i + x_i \otimes \Delta q^{h_i/2} \\
&= q^{-h_i/2} \otimes (q^{-h_i/2} \otimes x_i + x_i \otimes q^{h_i/2}) + x_i \otimes q^{h_i/2} \otimes q^{h_i/2} \\
&= q^{-h_i/2} \otimes q^{-h_i/2} \otimes x_i + q^{-h_i/2} \otimes x_i \otimes q^{h_i/2} + x_i \otimes q^{h_i/2} \otimes q^{h_i/2},
\end{aligned}$$

while

$$\begin{aligned}
&(\Delta \otimes \text{id})\Delta x_i \\
&= (\Delta \otimes \text{id})(q^{-h_i/2} \otimes x_i + x_i \otimes q^{h_i/2}) \\
&= q^{-h_i/2} \otimes q^{-h_i/2} \otimes \Delta x_i + \Delta x_i \otimes q^{h_i/2} \\
&= q^{-h_i/2} \otimes q^{-h_i/2} \otimes x_i + (q^{-h_i/2} \otimes x_i + x_i \otimes q^{h_i/2}) \otimes q^{h_i/2} \\
&= q^{-h_i/2} \otimes q^{-h_i/2} \otimes x_i + q^{-h_i/2} \otimes x_i \otimes q^{h_i/2} + x_i \otimes q^{h_i/2} \otimes q^{h_i/2}
\end{aligned}$$

and so axiom (i) holds for the x_i . A similar proof works for y_i .

Axiom (ii). Hopf axiom (ii) holds for the generators.

Proof. For axiom (ii), we have for $h = h_i$

$$(\epsilon \otimes \text{id})\Delta h = (\epsilon \otimes \text{id})(1 \otimes h + h \otimes 1) = h$$

while

$$(\text{id} \otimes \epsilon)\Delta h = (\text{id} \otimes \epsilon)(1 \otimes h + h \otimes 1) = h$$

and so axiom (ii) holds for all h_i . For x_i , we have

$$\begin{aligned} (\epsilon \otimes \text{id})\Delta x_i &= (\epsilon \otimes \text{id})(q^{-h_i/2} \otimes x_i + x_i \otimes q^{h_i/2}) \\ &= q^{-\epsilon(h_i)/2} x_i + \epsilon(x_i) q^{h_i/2} \\ &= x_i \end{aligned}$$

while

$$\begin{aligned} (\text{id} \otimes \epsilon)\Delta x_i &= (\text{id} \otimes \epsilon)(q^{-h_i/2} \otimes x_i + x_i \otimes q^{h_i/2}) \\ &= q^{-h_i/2} \epsilon(x_i) + x_i q^{\epsilon(h_i)/2} \\ &= x_i \end{aligned}$$

so axiom (ii) holds for x_i . A similar proof works for y_i . Thus axiom (ii) holds on all the generators.

Axiom (iii). Hopf axiom (iii) holds on the generators.

Proof. For axiom (iii), we have for any $h = h_i$

$$\begin{aligned} \mu(\gamma \otimes \text{id})\Delta h &= \mu(\gamma \otimes \text{id})(1 \otimes h + h \otimes 1) \\ &= \mu(1 \otimes h + \gamma(h) \otimes 1) \\ &= h - h \\ &= 0 \end{aligned}$$

while

$$\begin{aligned}
 \mu(\text{id} \otimes \gamma)\Delta h &= \mu(\text{id} \otimes \gamma)(1 \otimes h + h \otimes 1) \\
 &= \mu(1 \otimes \gamma(h) + h \otimes 1) \\
 &= -h + h \\
 &= 0
 \end{aligned}$$

and so axiom (iii) holds for all h_i . For x_i , we have

$$\begin{aligned}
 \mu(\gamma \otimes \text{id})\Delta x_i &= \mu(\gamma \otimes \text{id})(q^{-h_i/2} \otimes x_i + x_i \otimes q^{h_i/2}) \\
 &= \mu(q^{-\gamma(h_i)/2}) \otimes x_i + \gamma(x_i) \otimes q^{h_i/2} \\
 &= q^{h_i/2} x_i + \gamma(x_i) q^{h_i/2} \\
 &= q^{h_i/2} x_i - q^{h_i/2} x_i q^{-h_i/2} q^{h_i/2} \\
 &= 0
 \end{aligned}$$

while

$$\begin{aligned}
 \mu(\text{id} \otimes \gamma)\Delta x_i &= \mu(\text{id} \otimes \gamma)(q^{-h_i/2} \otimes x_i + x_i \otimes q^{h_i/2}) \\
 &= \mu(q^{-h_i/2}) \otimes \gamma(x_i) + x_i \otimes q^{\gamma(h_i)/2} \\
 &= q^{-h_i/2} \gamma(x_i) + x_i q^{-h_i/2} \\
 &= -q^{-h_i/2} q^{h_i/2} x_i q^{-h_i/2} + x_i q^{-h_i/2} \\
 &= 0
 \end{aligned}$$

and $\epsilon(x_i)I = 0$ and so axiom (iii) holds for all x_i . A similar proof works for y_i . Thus axiom (iii) holds for all the generators.

Claim. Δ is an algebra homomorphism.

Proof. The proof in the general case is messy. By means of a simple case we show how this can be verified. Let $n = 1$, $x_1 = x$, $y_1 = y$, and $h_1 = h$, so that

$$[xy] = \frac{q^h - q^{-h}}{q - q^{-1}}, \quad [hx] = 2x, \quad [hy] = -2y, \quad q^h q^{-h} = 1 = q^{-h} q^h$$

and

$$[\Delta x \Delta y] = \frac{\Delta q^h - \Delta q^{-h}}{q - q^{-1}}, \quad [\Delta h \Delta x] = 2\Delta x, \quad [\Delta h \Delta y] = -2\Delta y,$$

and

$$\Delta q^h \Delta q^{-h} = 1 = \Delta q^{-h} \Delta q^h.$$

From the above relations we must show that

$$\Delta(x^k y^l h^m q^{nh} x^{k'} y^{l'} h^{m'} q^{n'h}) = \Delta(x^k y^l h^m q^{nh}) \Delta(x^{k'} y^{l'} h^{m'} q^{n'h}).$$

For this purpose, we observe that

$$hx = x(h+2), \quad h^t x = x(h+2)^t, \quad hy = y(h-2), \quad h^t y = y(h-2)^t$$

from which we deduce that

$$q^{\nu h} x = x q^{2\nu} q^{\nu h} \quad \text{and} \quad q^{\nu h} y = y q^{-2\nu} q^{\nu h}.$$

Proceeding by induction,

$$yx = xy + w(q^h - q^{-h}), \quad w = \frac{1}{q - q^{-1}},$$

$$y^2x = xy^2 + wy(q^{h-2} - q^{-h+2}) + wy(q^h - q^{-h}),$$

and so on.

We can express $y^l x$ as a sum of terms of the form $x^k y^l h^m q^{nh}$. After some careful shifting, the above sets of relations give

$$\Delta(x^k y^l h^m q^{nh} x) = \Delta(x^k y^l h^m q^{nh}) \Delta(x),$$

$$\Delta(x^k y^l h^m q^{nh} y) = \Delta(x^k y^l h^m q^{nh}) \Delta(y),$$

$$\Delta(x^k y^l h^m q^{nh} h) = \Delta(x^k y^l h^m q^{nh}) \Delta(h),$$

from which we conclude that

$$\Delta(x^k y^l h^m q^{nh} x^{k'} y^{l'} h^{m'} q^{n'h}) = \Delta(x^k y^l h^m q^{nh}) \Delta(x^{k'} y^{l'} h^{m'} q^{n'h})$$

so that Δ is an algebra homomorphism.

Claim. γ is an antihomomorphism.

Proof. Again the proof in the general case is messy. One can show the result for $n = 1$ in the same way as the previous proof.

Since γ is an antihomomorphism and Δ and ϵ are homomorphisms and the Hopf axioms hold, then by theorem 2-4, we can conclude that UL_q is a Hopf algebra.

Chapter 2: Deformation of $U(\mathfrak{su}(2))$

Introduction. The previous chapter dealt in general with deformations of universal enveloping algebras of simple Lie algebras. This section considers the particular case of $U(\mathfrak{su}(2))$. The deformed universal algebra $U(\mathfrak{su}(2))_q$ will be denoted by U_q in this section. We will discuss irreducible representations using linear operators on an abstract Hilbert space in analogy with the familiar case of $U(\mathfrak{su}(2))$.

§1. Deformation of $U(\mathfrak{su}(2))$. In this section we deform the universal enveloping algebra of $\mathfrak{su}(2)$ and describe all irreducible representations for it.

The algebra $\mathfrak{su}(2)$. The algebra $\mathfrak{su}(2)$ is generated by elements $\{h, x, y\}$ and is subject to relations

$$[h, x] = 2x, \quad [h, y] = -2y, \quad [x, y] = h.$$

The algebra $\mathfrak{su}(2)$ is isomorphic to the algebra $\mathfrak{sl}(2)$ of all 2×2 matrices of trace 0. Its universal enveloping algebra $U(\mathfrak{su}(2))$ is a Hopf algebra by theorem 3-1 of chapter 1. The algebra $\mathfrak{su}(2)$ has Cartan matrix [2]. In keeping with the usual physicist's notation, we write the defining relations as

$$[j_z, j_+] = j_+, \quad [j_z, j_-] = -j_-, \quad [j_+, j_-] = 2j_z,$$

where we have set $x = j_+$, $y = j_-$, $h = 2j_z$.

Deformed Algebra. According to the scheme that was discussed in chapter 1, the deformed algebra that we are concerned with in this section is generated by $\{j_+, j_-, j_z, q^{j_z}, q^{-j_z}\}$ subject to the relations

$$[j_z, j_+] = j_+, \quad [j_z, j_-] = -j_-, \quad [j_+, j_-] = [2j_z].$$

This algebra is given the structure of a Hopf algebra by defining a comultiplication Δ , a counit ϵ , and an antipode γ according to

$$\Delta(j_+) = q^{-j_z} \otimes j_+ + j_+ \otimes q^{j_z},$$

$$\Delta(j_-) = q^{-j_z} \otimes j_- + j_- \otimes q^{j_z},$$

$$\Delta(j_z) = 1 \otimes j_z + j_z \otimes 1,$$

$$\epsilon(1) = 1, \quad \epsilon(j_z) = \epsilon(j_{\pm}) = 0,$$

and

$$\gamma(j_+) = -q^{j_z} j_+ q^{-j_z} = -q j_+,$$

$$\gamma(j_-) = -q^{j_z} j_- q^{-j_z} = -q^{-1} j_+,$$

$$\gamma(j_z) = -j_z.$$

Representations. If A is an algebra over C , then a representation T of A of degree n is an algebra homomorphism from A into the algebra $END(V)$ of endomorphisms of an n -dimensional vector space

V over C . Thus, if T is a representation we have

$$T(a + b) = T(a) + T(b), \quad T(\beta a) = \beta T(a), \quad T(ab) = T(a)T(b)$$

for all $a, b \in A$ and $\beta \in C$. The representation T of an algebra A can be extended to a representation of $U(A)$, the universal enveloping algebra of A . To see if T is a representation it is sufficient to define T on the generators and check that it preserves the defining relations.

We say that a representation T of an algebra A on a vector space V is **irreducible** if there are no invariant (invariant under the elements of $T(A)$) subspaces $V' \subset V$, given that $V' \neq (0)$.

Irreducible representations. We want to describe finite dimensional representations, particularly irreducible representations, of U_q . We have some assistance in this in that there is a correspondence between the finite dimensional irreducible representations of $U(su(2))$ and finite dimensional irreducible representations of its deformation U_q . An irreducible representation of U_q depends on q and in the limit as $q \rightarrow 1$ it gives an irreducible representation of $U(su(2))$. Corresponding representations will be of the same degree.

Irreducible Representations of $U(su(2))$. All of the finite dimensional irreducible representations T^j of $su(2)$ are determined by integers or semi-integers $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$ and the representation

T^j is given by

$$T^j(j_+) = J_+, \quad T^j(j_-) = J_-, \quad T^j(j_z) = J_z,$$

where one thinks of the operators J_+, J_-, J_z as acting on a Hilbert space V_j with basis vectors $|j, m\rangle$ according to

$$\begin{aligned} J_+|j, m\rangle &= \sqrt{(j-m)(j+m+1)}|j, m+1\rangle \\ J_-|j, m\rangle &= \sqrt{(j+m)(j-m+1)}|j, m-1\rangle \\ J_z|j, m\rangle &= m|j, m\rangle \end{aligned}$$

with $m = -j, -j+1, \dots, j$. These representations then are extendable to $U(su(2))$.

Representations of $U_q = U(su(2))_q$. By U_q we mean the algebra obtained by deforming $U(su(2))$ according to the scheme of section 4 of chapter 1. U_q according to this deformation is the algebra F/I , where F is the free algebra generated by

$$\{j_+, j_-, j_z, q^{j_+}, q^{-j_+}\}$$

and I is the ideal of F generated by the elements

$$\begin{aligned} q^{j_+}q^{-j_+} - 1, \quad q^{-j_+}q^{j_+} - 1, \quad [j_z j_z], \\ [j_+ j_-] - \frac{q^{2j_z} - q^{-2j_z}}{q - q^{-1}}, \quad [j_z j_+] - j_+, [j_z j_-] + j_-, \end{aligned}$$

where $[xy] = xy - yx$.

In view of our remarks on representations, one way of obtaining representations of U_q would be to give representations T_q^j for integer or semi-integer j that reduce to the irreducible representations T^j in the limit $q \rightarrow 1$. If we then have 'deformed' J_+, J_-, J_z , they must obey

$$\begin{aligned} [J_z, J_+] &= J_+, \\ [J_z, J_-] &= -J_-, \\ [J_+, J_-] &= [2J_z] \equiv \frac{q^{2J_z} - q^{-2J_z}}{q - q^{-1}}. \end{aligned}$$

A realisation of U_q . In the limit $q \rightarrow 1$ we are back to $U(su(2))$. If we are to define new operators acting on some Hilbert space they must reduce to the classical case in the limit $q \rightarrow 1$. One approach would be to deform the coefficients in the above relations continuously with q so that they reduce to the usual coefficients when $q \rightarrow 1$. This suggests that we might try replacing some of the coefficients x in the $su(2)$ equations that define the operators with $[x]$.

Deformed operators. The standard deformation of the operators J_+, J_- , and J_z is

$$\begin{aligned} J_+ |j, m\rangle &= \sqrt{[j-m][j+m+1]} |j, m+1\rangle, \\ J_- |j, m\rangle &= \sqrt{[j+m][j-m+1]} |j, m-1\rangle, \\ J_z |j, m\rangle &= m |j, m\rangle. \end{aligned}$$

Given these definitions, we check the commutator relations by finding the action of a commutator on a basis vector.

Lemma 1-1. $[J_+, J_-]|j, m\rangle = [2m]|j, m\rangle$.

Proof.

$$\begin{aligned}
 & [J_+, J_-]|j, m\rangle \\
 &= J_+ J_- |j, m\rangle - J_- J_+ |j, m\rangle \\
 &= \sqrt{[j+m][j-m+1]} J_+ |j, m-1\rangle \\
 &\quad - \sqrt{[j-m][j+m+1]} J_- |j, m+1\rangle \\
 &= \sqrt{[j+m][j-m+1]} \sqrt{[j-m+1][j+m]} |j, m\rangle \\
 &\quad - \sqrt{[j-m][j+m+1]} \sqrt{[j+m+1][j-m]} |j, m\rangle \\
 &= [j+m][j-m+1] |j, m\rangle - [j-m][j+m+1] |j, m\rangle.
 \end{aligned}$$

The definition of $[x]$ gives that

$$\begin{aligned}
 & [J_+, J_-]|j, m\rangle \\
 &= \frac{(q^{j+m} - q^{-j-m})(q^{j-m+1} - q^{-j+m-1})}{(q - q^{-1})(q - q^{-1})} |j, m\rangle \\
 &\quad - \frac{(q^{j-m} - q^{-j+m})(q^{j+m+1} - q^{-j-m-1})}{(q - q^{-1})(q - q^{-1})} |j, m\rangle \\
 &= \frac{q(q^{2m} - q^{-2m}) - q^{-1}(q^{2m} - q^{-2m})}{(q - q^{-1})(q - q^{-1})} |j, m\rangle \\
 &= [2m]|j, m\rangle.
 \end{aligned}$$

We want this to be equal to $[2J_z]|j, m\rangle$. We see that this is true from

$$\begin{aligned}
& [2J_z]|j, m\rangle \\
&= \frac{q^{2J_z} - q^{-2J_z}}{q - q^{-1}}|j, m\rangle \\
&= \frac{(1 + \ln q 2J_z + \dots) - (1 - \ln q 2J_z + \dots)}{q - q^{-1}}|j, m\rangle \\
&= \frac{(1 + \ln q 2m + \dots)|j, m\rangle - (1 - \ln q 2m + \dots)|j, m\rangle}{q - q^{-1}} \\
&= \frac{q^{2m} - q^{-2m}}{q - q^{-1}}|j, m\rangle \\
&= [2m]|j, m\rangle.
\end{aligned}$$

Lemma 1-2. $[J_z, J_+] = J_+$.

Proof.

$$\begin{aligned}
& [J_z, J_+]|j, m\rangle \\
&= J_z J_+|j, m\rangle - J_+ J_z|j, m\rangle \\
&= \sqrt{[j - m][j + m + 1]} J_z|j, m + 1\rangle - m J_+|j, m\rangle \\
&= (m + 1) \sqrt{[j - m][j + m + 1]}|j, m + 1\rangle \\
&\quad - m \sqrt{[j - m][j + m + 1]}|j, m + 1\rangle \\
&= \sqrt{[j - m][j + m + 1]}|j, m + 1\rangle \\
&= J_+|j, m\rangle
\end{aligned}$$

and so $[J_z, J_+] = J_+$ as desired.

Lemma 1-3. $[J_z, J_-] = -J_-$.

Proof.

$$\begin{aligned}
 & [J_z, J_-]|j, m\rangle \\
 &= J_z J_-|j, m\rangle - J_- J_z|j, m\rangle \\
 &= \sqrt{[j+m][j-m+1]} J_z|j, m-1\rangle - M J_-|j, m\rangle \\
 &= (m-1)\sqrt{[j+m][j-m+1]}|j, m-1\rangle \\
 &\quad - m\sqrt{[j+m][j-m+1]}|j, m-1\rangle \\
 &= -\sqrt{[j+m][j-m+1]}|j, m-1\rangle \\
 &= -J_-|j, m\rangle
 \end{aligned}$$

which is the desired result.

Thus we have

Theorem 1-4. An irreducible representation of U_q is defined by

$$\begin{aligned}
 J_+|j, m\rangle &= \sqrt{[j-m][j+m+1]}|j, m+1\rangle, \\
 J_-|j, m\rangle &= \sqrt{[j+m][j-m+1]}|j, m-1\rangle, \\
 J_z|j, m\rangle &= m|j, m\rangle.
 \end{aligned}$$

We know from our previous remarks that such representations are irreducible. An important question is "are these *all* of the ir-

reducible representations?". A result of Rosso says yes for q not a root of unity. See [Ro] for further details.

§2. Harmonic oscillator description. We give a brief summary of Schwinger's description of the $U(su(2))$ algebra in terms of the simple harmonic oscillator of quantum mechanics. Having done this we will be in a position to see if this approach can be extended to U_q .

Oscillator algebra. To describe the quantum harmonic oscillator we consider operators a and a^+ , called **annihilation** and **creation** operators that act on a Hilbert space (called a Fock space) with basis $\{|n\rangle \mid n = 0, 1, 2, \dots\}$ according to

$$\begin{aligned} a|n\rangle &= \sqrt{n}|n-1\rangle, & (a|0\rangle &= 0) \\ a^+|n\rangle &= \sqrt{n+1}|n+1\rangle, \end{aligned}$$

where $n = 0, 1, \dots$ label the energy levels of the harmonic oscillator.

The commutator $[a, a^+]$ acting on $|n\rangle$ gives

$$\begin{aligned} [a, a^+]|n\rangle &= aa^+|n\rangle - a^+a|n\rangle \\ &= a\sqrt{n+1}|n+1\rangle - a^+\sqrt{n}|n-1\rangle \\ &= (n+1)|n\rangle - n|n\rangle \\ &= |n\rangle \end{aligned}$$

and so $[a, a^+] = 1$. Thus we have the commutator relations

$$[a, a] = 0, \quad [a^+, a^+] = 0, \quad [a, a^+] = 1.$$

We define **number operators** N and $N + 1$ by

$$N = a^+ a, \quad N + 1 = a a^+$$

and it can be directly verified that

$$N|n\rangle = n|n\rangle \quad \text{and} \quad (N + 1)|n\rangle = (n + 1)|n\rangle.$$

The commutator relations for N are given by

$$\begin{aligned} [N, a]|n\rangle &= Na|n\rangle - aN|n\rangle \\ &= N\sqrt{n}|n-1\rangle - an|n\rangle \\ &= (n-1)\sqrt{n}|n-1\rangle - n\sqrt{n}|n-1\rangle \\ &= -\sqrt{n}|n-1\rangle \\ &= -a|n\rangle \end{aligned}$$

$$\begin{aligned} [N, a^+]|n\rangle &= Na^+|n\rangle - a^+N|n\rangle \\ &= N\sqrt{n+1}|n+1\rangle - a^+n|n\rangle \\ &= (n+1)\sqrt{n+1}|n+1\rangle - n\sqrt{n+1}|n+1\rangle \\ &= \sqrt{n+1}|n+1\rangle \\ &= a^+|n\rangle \end{aligned}$$

and so

$$[N, a] = -a \quad [N, a^+] = a^+$$

are the commutator relations for N .

Schwinger's description of $su(2)$. In Schwinger's description of $su(2)$ we have two sets of annihilation and creation operators, a_i and $a_i^\dagger$, acting on a Hilbert space with basis $|n_1; n_2\rangle$; $n_i = 0, 1, 2, \dots$, where

$$a_1|n_1; n_2\rangle = \sqrt{n_1}|n_1 - 1; n_2\rangle,$$

$$a_1^\dagger|n_1; n_2\rangle = \sqrt{n_1 + 1}|n_1 + 1; n_2\rangle,$$

$$a_2|n_1; n_2\rangle = \sqrt{n_2}|n_1; n_2 - 1\rangle,$$

$$a_2^\dagger|n_1; n_2\rangle = \sqrt{n_2 + 1}|n_1; n_2 + 1\rangle,$$

$$[a_i, a_i^\dagger] = 1, \quad N_i = a_i^\dagger a_i,$$

$$[a_1, a_2] = 0, \quad [a_1, a_2^\dagger] = 0, \quad [a_2, a_1^\dagger] = 0.$$

In the construction of Schwinger, we define

$$J_+ \equiv a_1^\dagger a_2 \quad J_- \equiv a_2^\dagger a_1$$

and we write $|j, m\rangle$ as $|j + m; j - m\rangle$.

Then the commutator $[J_+, J_-]$ is given by

$$\begin{aligned} [J_+, J_-] &= a_1^\dagger a_2 a_2^\dagger a_1 - a_2^\dagger a_1 a_1^\dagger a_2 \\ &= a_1^\dagger (N_2 + 1) a_1 - a_2^\dagger (N_1 + 1) a_2 \\ &= N_1 (N_2 + 1) - N_2 (N_1 + 1) \\ &= N_1 - N_2 \end{aligned}$$

and so

$$2J_z \equiv N_1 - N_2$$

is the obvious definition.

Now check that the operators J_+ , J_- , and J_z act on $|j, m\rangle$ properly.

$$\begin{aligned} J_+|j, m\rangle &= a_1^\dagger a_2|j + m; j - m\rangle \\ &= \sqrt{j + m + 1}\sqrt{j - m}|j + m + 1; j - m - 1\rangle \end{aligned}$$

which is the vector identified with $\sqrt{j + m + 1}\sqrt{j - m}|j, m + 1\rangle$ in our old notation.

$$\begin{aligned} J_-|j, m\rangle &= a_2^\dagger a_1|j + m; j - m\rangle \\ &= \sqrt{j - m + 1}\sqrt{j + m}|j + m - 1; j - m + 1\rangle \end{aligned}$$

which is the vector identified with $\sqrt{j + m}\sqrt{j - m + 1}|j, m - 1\rangle$ in our old notation.

$$\begin{aligned} J_z|j, m\rangle &= \frac{N_1 - N_2}{2}|j + m; j - m\rangle \\ &= \frac{2m}{2}|j + m; j - m\rangle \end{aligned}$$

which is the vector identified with $m|j, m\rangle$ in our old notation.

Thus, the operators behave properly and the usual commutation relations will hold. (Checking the commutator relations would be

the same as checking them for the usual J_+ , J_- , J_z since they behave the same way on the basis $|j, m \rangle$.) A description of the theory of the harmonic oscillator and Schwinger's approach to the angular momentum operators may be found in [Sa].

§3. Harmonic oscillator description of U_q . Can we extend Schwinger's method to $U_q = U(su(2))_q$? Yes, if we deform the oscillator algebra in the right way. This approach has been followed in [Bi] and [Mac]. Assume that our goal can be achieved and try to find the necessary conditions. We define our “deformed” operators J_+, J_- by

$$J_+ \equiv a_1^\dagger a_2 \quad J_- \equiv a_2^\dagger a_1.$$

For any quantity x , let $[x]$ be the result of q -deformation. We need

$$\begin{aligned} J_+ |j, m\rangle &= a_1^\dagger a_2 |j + m; j - m\rangle \\ &= \sqrt{[j + m + 1]} \sqrt{[j - m]} |j + m + 1; j - m - 1\rangle \end{aligned}$$

$$\begin{aligned} J_- |j, m\rangle &= a_2^\dagger a_1 |j + m, j - m\rangle \\ &= \sqrt{[j + m]} \sqrt{[j - m + 1]} |j + m - 1, j - m + 1\rangle \end{aligned}$$

and so it seems natural to define

$$\begin{aligned} a |n\rangle &= \sqrt{[n]} |n - 1\rangle \quad (a |0\rangle = 0) \\ a^\dagger |n\rangle &= \sqrt{[n + 1]} |n + 1\rangle \end{aligned}$$

where $n = 0, 1, \dots$ label the energy levels of the deformed q - oscillator.

With this definition one can check that

$$|n\rangle = \frac{(a^\dagger)^n |0\rangle}{\sqrt{[n]!}}.$$

where $[n]! \equiv [n][n-1]\dots[1]$.

Lemma 3-1. $[a, a^+] = [N+1] - [N]$ where we define *number operators* N and $N+1$ by

$$N|n\rangle = n|n\rangle \quad \text{and} \quad N+1|n\rangle = (n+1)|n\rangle.$$

Proof.

$$\begin{aligned} [a, a^+]|n\rangle &= aa^+|n\rangle - a^+a|n\rangle \\ &= a\sqrt{[n+1]}|n+1\rangle - a^+\sqrt{[n]}|n-1\rangle \\ &= [n+1]|n\rangle - [n]|n\rangle \\ &= ([N+1] - [N])|n\rangle \end{aligned}$$

One directly verifies

Lemma 3-2. $aa^+ = [N+1]$ and $a^+a = [N]$.

Lemma 3-3. $[N, a] = -a$ and $[N, a^+] = a^+$.

Proof. The commutator relations for N are given by

$$\begin{aligned} [N, a]|n\rangle &= Na|n\rangle - aN|n\rangle \\ &= N\sqrt{[n]}|n-1\rangle - a n|n\rangle \\ &= (n-1)\sqrt{[n]}|n-1\rangle - n\sqrt{[n-1]}|n\rangle \\ &= -\sqrt{[n]}|n-1\rangle \\ &= -a|n\rangle \end{aligned}$$

$$\begin{aligned}
[N, a^+]|n\rangle &= Na^+|n\rangle - a^+N|n\rangle \\
&= N\sqrt{[n+1]}|n+1\rangle - a^+n|n\rangle \\
&= (n+1)\sqrt{[n+1]}|n+1\rangle - n\sqrt{[n+1]}|n+1\rangle \\
&= \sqrt{[n+1]}|n+1\rangle \\
&= a^+|n\rangle
\end{aligned}$$

and so

$$[N, a] = -a, \quad [N, a^+] = a^+$$

are the commutator relations for N .

To define J_z , we need to find the commutator $[J_+, J_-]$. This is given by the following lemma.

Lemma 3-4. $[J_+, J_-] = [N_1 - N_2]$.

Proof.

$$\begin{aligned}
[J_+, J_-] &= a_1^+ a_2 a_2^+ a_1 - a_2^+ a_1 a_1^+ a_2 \\
&= a_1^+ [N_2 + 1] a_1 - a_2^+ [N_1 + 1] a_2 \\
&= [N_1][N_2 + 1] - [N_2][N_1 + 1].
\end{aligned}$$

By the definition of $[x]$, this is the same as

$$\frac{q^{N_1} - q^{-N_1}}{q - q^{-1}} \frac{q^{N_2+1} - q^{-N_2-1}}{q - q^{-1}} - \frac{q^{N_2} - q^{-N_2}}{q - q^{-1}} \frac{q^{N_1+1} - q^{-N_1-1}}{q - q^{-1}}.$$

Multiplying out, we get

$$\begin{aligned}
 & \frac{q^{N_2-N_1-1} - q^{N_1-N_2-1} + q^{-N_2+N_1+1} - q^{-N_1+N_2+1}}{(q - q^{-1})^2} \\
 &= \frac{(q - q^{-1})(q^{N_1-N_2} - q^{-N_1+N_2})}{(q - q^{-1})^2} \\
 &= [N_1 - N_2].
 \end{aligned}$$

Thus

$$2J_z \equiv N_1 - N_2$$

is the obvious definition . We have proved the following theorem:

Theorem 3-5. An irreducible representation of $\hat{U}_q$ is define by

$$J_+ = a_1^+ a_2, \quad J_- = a_1 a_2^+, \quad 2J_z = N_1 - N_2.$$

§4. q -analogue of the Bargmann space. In this section we will find a deformed Bargmann space and deformed differential operators. These will be used in the next section to give a representation of U_q .

We can represent the undeformed harmonic oscillator in terms of a polynomial space in a complex variable called a Bargmann space. We write

$$a = \partial/\partial x, \quad a^+ = x, \quad |n\rangle = \frac{x^n}{\sqrt{n!}}.$$

It is easily seen that a and a^+ satisfy the desired commutation relations. We would like to find a deformed Bargmann space that would give us a representation of $U(su(2))_q$. This has been done in [BrMcZhGo].

We need a basis that will become the same as the usual basis in the limit $q \rightarrow 1$. Let us tentatively set

$$|n\rangle = \frac{x^n}{\sqrt{[n]!}}.$$

Then we need

$$a^+|n\rangle = \sqrt{[n+1]}|n+1\rangle = \sqrt{[n+1]} \frac{x^{n+1}}{\sqrt{[n+1]!}} = \frac{x^{n+1}}{\sqrt{[n]!}}.$$

By inspection, setting $a^+ = x$ (meaning multiplication by x) works.

We also need that

$$a|n\rangle = \sqrt{[n]}|n-1\rangle = \sqrt{[n]} \frac{x^{n-1}}{\sqrt{[n-1]!}} = \frac{[n]x^{n-1}}{\sqrt{[n]!}}.$$

Keeping this in mind, and also that a must reduce to the usual derivative in the limit of $q \rightarrow 1$, we define a to be the q -derivative $\tilde{\partial}$ where

$$\tilde{\partial}f(x) \equiv \frac{f(qx) - f(q^{-1}x)}{q - q^{-1}}.$$

The last thing left to do is to define the number operators N and $N + 1$ by

$$N|n\rangle = n|n\rangle \quad (N + 1)|n\rangle = (n + 1)|n\rangle.$$

Lemma 4-1. $[a, a^+] = [N + 1] - [N]$.

Proof.

$$\begin{aligned} (aa^+ - a^+a)|n\rangle &= (\tilde{\partial}x - x\tilde{\partial})\frac{x^n}{\sqrt{[n]!}} \\ &= \tilde{\partial}x^{n+1}\sqrt{[n]!} - x[n]\frac{x^{n-1}}{\sqrt{[n]!}} \\ &= [n + 1]x^n\sqrt{[n]!} - [n]\frac{x^n}{\sqrt{[n]!}} \\ &= ([n + 1] - [n])|n\rangle. \end{aligned}$$

Lemma 4-2. $[N, a] = -a$ and $[N, a^+] = a^+$.

Proof. We have constructed our operators so that the required relations hold.

§5. Realization of U_q in deformed differential operators. We have seen that the deformed harmonic oscillator may be represented in terms of multiplication and q -differentiation. We used the following identifications

$$a^+ \rightarrow x, \quad a \rightarrow \tilde{\partial}$$

where the $\tilde{\partial}$ stands for q -differentiation wrt x . Then N corresponds to $x\tilde{\partial}$ and $|n\rangle$ corresponds to $\frac{x^n}{\sqrt{[n]!}}$.

To get a realization of U_q , we used a representation

$$J_+ = a_1^+ a_2 \quad J_- = a_1 a_2^+ \quad 2J_z = N_1 - N_2,$$

which in the scheme that we are presently considering becomes

$$J_+ = x_1 \tilde{\partial}_2, \quad J_- = x_2 \tilde{\partial}_1, \quad J_z = \frac{1}{2}(x_1 \tilde{\partial}_1 - x_2 \tilde{\partial}_2).$$

A basis of the representation space V_j of the irreducible representations corresponding to j is

$$\Gamma_j = \{|j, m\rangle = \frac{x_1^{j+m} x_2^{j-m}}{\sqrt{[j+m]![j-m]!}}, m = -j, -j+1, \dots, j\}.$$

These describe the irreducible representations. In the next section, the extension to $U(su(n))_q$ will be described. For more details see [DaGuYa].

§6. Representations of $U(su(n))_q$. In this section we will extend our ideas to get irreducible representations of $U(g)_q$, where $g = su(n)$.

Definition of $su(n)$. By $su(n)$ one means the Lie algebra with Chevalley basis

$$H_i, X_i^+, X_i^-$$

satisfying

$$\begin{aligned} [H_i, H_j] &= 0, & [X_i^+, X_j^-] &= \delta_{ij} H_i, \\ [H_i, X_j^+] &= A_{ij} X_j^+, & [H_i, X_j^-] &= -A_{ij} X_j^-. \end{aligned}$$

We intend to deform the universal enveloping algebras of these Lie algebras to algebras $U(su(n))_q$ that satisfy

$$\begin{aligned} [H_i, H_j] &= 0, & [X_i^+, X_j^-] &= \delta_{ij} [H_i], \\ [H_i, X_j^+] &= A_{ij} X_j^+, & [H_i, X_j^-] &= -A_{ij} X_j^-. \end{aligned}$$

as was discussed in the first chapter. We will do this for $U(su(3))_q$ and then extend the result.

Boson realization of $U(su(3))_q$. In constructing the q -oscillator, we came up with an algebra that would be generated by operators a, a^+ and N that satisfy

$$[N, a] = -a, \quad [N, a^+] = a^+, \quad [a, a^+] = [N + 1] - [N]$$

and the last relation may be written $aa^+ - q^{-1}a^+a = q^N$. The generators acted on the basis kets according to

$$\begin{aligned} a|n\rangle &= \sqrt{[n]}|n-1\rangle, \\ a^+|n\rangle &= \sqrt{[n+1]}|n+1\rangle, \\ N|n\rangle &= n|n\rangle. \end{aligned}$$

We then used this algebra to obtain a realization of $U(su(2))_q$. Can we extend this technique to other $U(su(n))$? Yes, if we extend our bose algebra.

In the case of $U(su(3))_q$ we have operators $H_1, H_2, X_1^\pm, X_2^\pm$ that satisfy

$$[H_1, H_2] = 0$$

$$\begin{aligned} [H_1, X_1^+] &= 2X_1^+ & [H_1, X_1^-] &= -2X_1^- \\ [H_1, X_2^+] &= -X_2^+ & [H_1, X_2^-] &= X_2^- \\ [H_2, X_1^+] &= -X_1^+ & [H_2, X_1^-] &= X_1^- \\ [H_2, X_2^+] &= 2X_2^+ & [H_2, X_2^-] &= -2X_2^- \end{aligned}$$

$$\begin{aligned} [X_1^+, X_1^-] &= [H_1] & [X_1^+, X_2^-] &= 0 \\ [X_2^+, X_1^-] &= 0 & [X_2^+, X_2^-] &= [H_2]. \end{aligned}$$

If we were to ignore for the moment elements with suffix 2, we would have the relations for $U(su(2))_q$ and we would set

$$X_1^+ = a_1^+ a_2 \quad X_1^- = a_1 a_2^+ \quad H_1 = N_1 - N_2.$$

Similarly we would write

$$X_2^+ = a_2^+ a_3 \quad X_2^- = a_2 a_3^+ \quad H_2 = N_2 - N_3$$

and consider that we had operators acting on basis kets $|n_1, n_2, n_3\rangle$.

Lemma 6-1. The relations involving operators of the same index are satisfied.

Proof. These are the relations for $U(su(2))_q$ and we checked that they are satisfied when we constructed the oscillator representation of $U(su(2))_q$.

Lemma 6-2. The remaining relations are satisfied.

Proof.

$$\begin{aligned} [H_1, X_2^+] &= [N_1 - N_2, a_2^+ a_3] \\ &= [N_1, a_2^+ a_3] - [N_2, a_2^+ a_3] \\ &= -[N_2, a_2^+] a_3 \\ &= -a_2^+ a_3 \\ &= -X_2^+. \end{aligned}$$

$$\begin{aligned}
[H_1, X_2^-] &= [N_1 - N_2, a_2 a_3^+] \\
&= [N_1, a_2 a_3^+] - [N_2, a_2 a_3^+] \\
&= -[N_2, a_2] a_3^+ \\
&= a_2 a_3^+ \\
&= X_2^-.
\end{aligned}$$

$$\begin{aligned}
[H_2, X_1^+] &= [N_2 - N_3, a_1^+ a_2] \\
&= [N_2, a_1^+ a_2] - [N_3, a_1^+ a_2] \\
&= [N_2, a_2] a_1^+ \\
&= -a_2 a_1^+ \\
&= -X_1^+.
\end{aligned}$$

$$\begin{aligned}
[H_2, X_1^-] &= [N_2 - N_3, a_1 a_2^+] \\
&= [N_2, a_1 a_2^+] - [N_3, a_1 a_2^+] \\
&= [N_2, a_2^+] a_1 \\
&= a_2^+ a_1 \\
&= X_1^-.
\end{aligned}$$

$$[X_1^+, X_2^-] = [a_1^+ a_2, a_2 a_3^+] = 0.$$

$$[X_2^+, X_1^-] = [a_2^+ a_3, a_1 a_2^+] = 0.$$

We have proven

Theorem 6-3. A representation of $U(su(3))_q$ is given by operators

$$X_1^+ = a_1^+ a_2 \quad X_1^- = a_1 a_2^+ \quad H_1 = N_1 - N_2.$$

$$X_2^+ = a_2^+ a_3 \quad X_2^- = a_2 a_3^+ \quad H_2 = N_2 - N_3$$

acting on an extended Fock space with basis $|n_1, n_2, n_3\rangle$.

Representation on extended Bargmann space. Exactly as in the case of $U(su(2))_q$, in terms of differential operators we would have

$$X_1^+ = x_1 \tilde{\partial}_2 \quad X_2^+ = x_2 \tilde{\partial}_3$$

$$X_1^- = \tilde{\partial}_1 x_2 \quad X_2^- = \tilde{\partial}_2 x_3$$

$$H_1 = x_1 \partial_1 - x_2 \partial_2 \quad H_2 = x_2 \partial_2 - x_3 \partial_3$$

The space that the oscillators work on is

$$|n_1, n_2, n_3\rangle \equiv \frac{x_1^{n_1}}{\sqrt{[n_1]!}} \frac{x_2^{n_2}}{\sqrt{[n_2]!}} \frac{x_3^{n_3}}{\sqrt{[n_3]!}}$$

Irreducible representations are given by setting $n_1 + n_2 + n_3$ to some constant.

§7. Oscillator description of $U(su(n))_q$. We now see how to deal with the more general case of $U(su(n))_q$.

Defining relations of $U(su(n))_q$. According to the scheme of chapter 1 the defining relations of $U(su(n))_q$ are:

$$\begin{aligned} [H_i, H_j] &= 0 & [X_i^+, X_j^-] &= \delta_{ij}[H_i] \\ [H_i, X_j^+] &= A_{ij}X_j^+ & [H_i, X_j^-] &= -A_{ij}X_j^-. \end{aligned}$$

where the cartan matrix A has 2's on the main diagonal, -1's on the diagonals above and below the main diagonal, and zeros everywhere else.

Oscillator operators. Define operators

$$X_i^+ = a_i^+ a_{i+1} \quad X_i^- = a_i a_{i+1}^+ \quad H_i = N_i - N_{i+1}$$

which act on a deformed Fock space with basis $|n_1; \dots; n_n\rangle$. We can now check the relations.

Theorem 7-1. The defining relations for $U(su(n))_q$ hold for the operators defined above.

Proof. Since the number operators commute, $[H_i, H_j] = 0$.

In the case that $i \neq j$, $[X_i^+, X_j^-] = 0$ just as in the case of $U(su(3))_q$. In the case where $i = j$, we are back to the $U(su(2))_q$ case and the desired result holds.

Let us check if $[H_i, X_j^+] = A_{ij}X_j^+$. We again have to check a number of cases. If $i = j$, we have that $[H_i, X_i^+] = 2X_i^+$ just as we checked for $U(su(2))_q$. If $i = j + 1$, then

$$\begin{aligned} [H_{j+1}, X_j^+] &= [N_{j+1} - N_{j+2}, a_j^+ a_{j+1}] \\ &= a_j^+ [N_{j+1}, a_{j+1}] \\ &= -a_j^+ a_{j+1} \\ &= A_{ij}X_i^+ \end{aligned}$$

as desired. If $i = j - 1$, then

$$\begin{aligned} [H_{j-1}, X_j^+] &= [N_{j-1} - N_j, a_j^+ a_{j+1}] \\ &= -[N_j, a_j^+] a_{j+1} \\ &= -a_j^+ a_{j+1} \\ &= A_{ij}X_i^+ \end{aligned}$$

as desired. If $i \neq j + 1, j - 1$, then the operators will commute and so $[H_i, X_j^+] = 0 = A_{ij}X_j^+$. Thus in all cases $[H_i, X_j^+] = A_{ij}X_j^+$.

Is $[H_i, X_j^-] = -A_{ij}X_j^-$? There are a number of cases. If $i = j$, we have $[H_i, X_i^-] = 2X_i^-$ just as we checked for $U(su(2))_q$. In the

case $i = j + 1$, then

$$\begin{aligned}
 [H_{j+1}, X_j^-] &= [N_{j+1} - N_{j+2}, a_j a_{j+1}^+] \\
 &= a_j [N_{j+1}, a_{j+1}^+] \\
 &= a_j a_{j+1}^+ \\
 &= -A_{ij} X_i^-
 \end{aligned}$$

as desired. If $i = j - 1$, then

$$\begin{aligned}
 [H_{j-1}, X_j^-] &= [N_{j-1} - N_j, a_j a_{j+1}^+] \\
 &= -[N_j, a_j] a_{j+1}^+ \\
 &= a_j a_{j+1}^+ \\
 &= -A_{ij} X_i^-
 \end{aligned}$$

as desired. If $i \neq j + 1, j - 1$, then the operators will commute and then $[H_i, X_j^-] = 0 = -A_{ij} X_j^-$. Thus in all cases $[H_i, X_j^-] = -A_{ij} X_j^-$.

Thus all of the relations are satisfied for $U(su(n))_q$.

§8. Matrix Representations of $U(su(n))_q$. Consider first the deformed oscillator realization of $U(su(2))_q$. We have creation and annihilation operators acting on a deformed Fock space according to

$$J_+ = a_1^\dagger a_2, \quad J_- = a_1 a_2^\dagger, \quad J_z = \frac{1}{2}(N_1 - N_2).$$

Notice that in acting on a basis ket $|n_1; n_2\rangle$, the total $n_1 + n_2$ does not change. Thus we may think of the deformed Fock space used in the representation of $U(su(2))_q$ as being the direct sum of spaces with the same sum $n_1 + n_2$.

As an example we consider the subspace spanned by basis kets that have $n_1 + n_2 = 2$.

$$\{|2, 0\rangle, |1, 1\rangle, |0, 2\rangle\}$$

and now look at the action of the $U(su(2))_q$ operators on this basis.

$$J_+ |2, 0\rangle = a_1^\dagger a_2 |2, 0\rangle = 0$$

$$J_+ |1, 1\rangle = a_1^\dagger a_2 |1, 1\rangle = \sqrt{[2]} |2, 0\rangle$$

$$J_+ |0, 2\rangle = a_1^\dagger a_2 |0, 2\rangle = \sqrt{[2]} |1, 1\rangle$$

and so

$$\begin{pmatrix} 0 & \sqrt{[2]} & 0 \\ 0 & 0 & \sqrt{[2]} \\ 0 & 0 & 0 \end{pmatrix}$$

is a matrix representation of J_+ .

$$J_-|2,0\rangle = a_1 + a_2^+|2,0\rangle = \sqrt{[2]}|1,1\rangle$$

$$J_-|1,1\rangle = a_1 + a_2^+|1,1\rangle = \sqrt{[2]}|0,2\rangle$$

$$J_-|0,2\rangle = a_1 + a_2^+|0,2\rangle = 0$$

and so

$$\begin{pmatrix} 0 & 0 & 0 \\ \sqrt{[2]} & 0 & 0 \\ 0 & \sqrt{[2]} & 0 \end{pmatrix}$$

is a matrix representation of J_- .

$$J_z|2,0\rangle = \frac{1}{2}(N_1 - N_2)|2,0\rangle = |2,0\rangle$$

$$J_z|1,1\rangle = \frac{1}{2}(N_1 - N_2)|1,1\rangle = 0$$

$$J_z|0,2\rangle = \frac{1}{2}(N_1 - N_2)|0,2\rangle = -|0,2\rangle$$

and so

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

is a matrix representation of J_z . The same approach could be used to construct higher dimensional representations of $U(su(2))_q$.

This approach extends immediately to the $U(su(n))_q$ deformed universal algebras. An interesting line of future inquiry might be to consider if one can get representations of the quantum group by means of the exponential or q -exponential map acting on our matrix representations. The same results have been found in [SuFu].

Chapter 3: Deformed Function Spaces

Introduction. In this chapter we will reintroduce our Hopf algebra A_q and show that it may be viewed as a deformation of a function space on $SU(2)$. Then A_q and $U(su(2))_q$ are shown to be in each others Hopf duals.

§1. Polynomial functions. In this section we will show that the algebra of polynomial functions on certain matrix groups is a Hopf algebra. We also consider the notion of a Hopf corepresentation.

Algebra of polynomial functions of $SL(2, \mathbb{C})$. Consider the Lie group

$$G = SL(2, \mathbb{C}) = \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \mid xw - yz = 1 \right\}$$

which has coordinate functions a, b, c , and d defined on it according to

$$\begin{aligned} a \begin{pmatrix} x & y \\ z & w \end{pmatrix} &= x \\ b \begin{pmatrix} x & y \\ z & w \end{pmatrix} &= y \\ c \begin{pmatrix} x & y \\ z & w \end{pmatrix} &= z \\ d \begin{pmatrix} x & y \\ z & w \end{pmatrix} &= w \end{aligned}$$

which map G to $\mathbb{C}$.

The functions a, b, c, d generate, in the algebra of functions $G \rightarrow \mathbb{C}$, the subalgebra $POL(G)$ of polynomial functions on G . The algebra is associative and has the identity 1 given by $1(g) = 1$ for all $g \in G$. Now $POL(G)$ as a unital algebra is isomorphic to a unital algebra A with a priori non-commuting generators $\{a, b, c, d\}$ and relations

$$ab-ba, \quad ac-ca, \quad ad-da, \quad bc-cb, \quad bd-db, \quad cd-dc, \quad ad-bc-I.$$

Thus we can write $A = POL(G)$.

Hopf corepresentation. Let A be a Hopf algebra. A *matrix corepresentation* of A is a matrix of generators $\{u_{ij} | i \leq i, j \leq n\}$ of A

$$u = \begin{pmatrix} u_{11} & \dots & u_{1n} \\ \vdots & & \vdots \\ u_{n1} & \dots & u_{nn} \end{pmatrix}$$

such that

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$$

and

$$\epsilon(u_{ij}) = \delta_{ij}.$$

From Hopf axiom (iii), we have

$$\sum_k \gamma(u_{ik}) u_{kj} = \delta_{ij} = \sum_k u_{ik} \gamma(u_{kj}).$$

Definition. Two matrix corepresentations u and v of dimension n of a Hopf algebra are *equivalent* if u and v are $n \times n$ matrices and there is an invertible $n \times n$ matrix s such that

$$su = vs.$$

Definition. A corepresentation u of A is *irreducible* if it is not equivalent to a representation of the form

$$v = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

where $v_{ij} = 0$ for all (i, j) such that $m + 1 \leq i \leq n, 1 \leq j \leq m$ for some

$$m, \quad 1 \leq m \leq n - 1.$$

Hopf Algebra Structure of A . Let us make $A = POL(G)$ a Hopf algebra such that

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a Hopf corepresentation.

A is a unital associative algebra. From the definition of matrix corepresentation we have

$$\Delta a = a \otimes a + b \otimes c,$$

$$\Delta b = a \otimes b + b \otimes d,$$

$$\Delta c = c \otimes a + d \otimes c,$$

$$\Delta d = c \otimes b + d \otimes d,$$

and

$$\epsilon(a) = \epsilon(d) = 1, \quad \epsilon(b) = \epsilon(c) = 0.$$

We define the antipode by

$$\gamma(a) = d, \quad \gamma(b) = -b, \quad \gamma(c) = -c, \quad \gamma(d) = a.$$

We now have the Hopf algebra A_q discussed previously with $q = 1$. Thus we can conclude that A is a Hopf algebra.

There are certain features that will reoccur later. Note that the conditions for a Hopf corepresentation may also be written as

$$\Delta T = T \hat{\otimes} T$$

where the multiplication $\hat{\otimes}$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \hat{\otimes} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} a \otimes a + b \otimes c & a \otimes b + b \otimes d \\ c \otimes a + d \otimes c & c \otimes b + d \otimes d \end{pmatrix}.$$

We also have

$$\epsilon(T) = I_2, \quad \text{the identity matrix}$$

and

$$\gamma(T) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

so that

$$\gamma(T)T = T\gamma(T) = I_2.$$

A discussion of Hopf algebras and Hopf corepresentations from the point of view of deformed function spaces may be found in [Ko]. Further discussions of these concepts may be found in [Ta].

§2. Deformed Polynomial Algebra. In this section we will get a deformed Hopf algebra $SU(2)_q$ as a deformation of the algebra of polynomial functions on $SU(2)$.

Quantum Planes. The notion of quantum plane was introduced by Manin and is discussed in [Man].

Let R be an algebra with unity and let V be the collection of all column 2-tuples of V . We define the *quantum plane* to be

$$R_q[2, 0] = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in V \mid xy = q^{-1}yx \right\}$$

and the set

$$R_q[0, 2] = \left\{ \begin{pmatrix} \zeta \\ \eta \end{pmatrix} \in V \mid \zeta^2 = 0, \quad \eta^2 = 0, \quad \zeta\eta + q\eta\zeta = 0 \right\}$$

is called the **dual plane**, where $q(\neq 0, 1)$ is a real number.

The algebra $GL(2)_q$. We consider matrices

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

that take the quantum plane to itself and the dual plane to itself under the assumption that the components of the matrix T commute with the elements of the quantum plane and the dual plane. We get a number of constraints on the elements of T . From

$$T \begin{pmatrix} x \\ y \end{pmatrix} \in R_q[2, 0]$$

and

$$T \begin{pmatrix} \zeta \\ \eta \end{pmatrix} \in R_q[0, 2]$$

we get

$$(i) \quad (ax + by)(cx + dy) = q^{-1}(cx + dy)(ax + by),$$

$$(ii) \quad (a\zeta + b\eta)^2 = 0,$$

$$(iii) \quad (c\zeta + d\eta)^2 = 0,$$

$$(iv) \quad (a\zeta + b\eta)(c\zeta + d\eta) + q(c\zeta + d\eta)(a\zeta + b\eta) = 0.$$

From (i) we have

$$\begin{aligned} LHS &= acxx + adxy + bcyx + bdy y \\ &= acxx + (ad + qbc)xy + bdy y \end{aligned}$$

while

$$\begin{aligned} RHS &= q^{-1}(caxx + cbxy + dayx + dbyy) \\ &= q^{-1}(caxx + (cb + qda)xy + dbyy) \\ &= q^{-1}caxx + (q^{-1}cb + da)xy + q^{-1}dbyy \end{aligned}$$

and so we conclude that $ac = q^{-1}ca$, $ad - da = q^{-1}cb - qbc$, and $bd = q^{-1}db$.

From (ii) we have

$$ab\zeta\eta + ba\eta\zeta = (ab - q^{-1}ba)\zeta\eta = 0$$

and so $ab = q^{-1}ba$.

From (iii) we conclude in the same way that $cd = q^{-1}dc$. Expanding (iv) gives

$$ad\zeta\eta + bc\eta\zeta + q(cb\zeta\eta + da\eta\zeta) = (ad - da)\zeta\eta + (qcb - q^{-1}bc)\zeta\eta = 0$$

and so we conclude that $ad - da = q^{-1}bc - qcb$ which when compared to the previous result $ad - da = q^{-1}cb - qbc$ lead us to also conclude that $bc = cb$.

Summing up we have the relations

$$ab = q^{-1}ba, \quad ac = q^{-1}ca, \quad bd = q^{-1}db,$$

$$(*) \quad cd = q^{-1}dc, \quad ad - da = (q^{-1} - q)bc, \quad bc = cb.$$

The associative unital algebra generated by a, b, c and d with the relations $(*)$ is denoted $GL(2)_q$ in the literature and the matrix T is said to be associated to A_q .

Quantum Determinant. An interesting feature of this algebra is the *quantum determinant*. In terms of the coordinate functions on

$GL(2, C)$ the determinant function is given by $DET = ad - bc = da - cb$ where a, b, c , and d are the obvious coordinate functions.

We have here a deformation of $A = POL(GL(2, C))$ and the closest thing to the determinant is $DET_q = ad - q^{-1}bc = da - qcb$.

Lemma 2-1. The quantum determinant commutes with the generators of $GL(2)_q$ and thus with all of $GL(2)_q$.

Proof.

$$\begin{aligned}
 aDET_q &= a(da - qbc) \\
 &= ada - qabc \\
 &= ada - q^{-1}bca \\
 &= (ad - q^{-1}bc)a \\
 &= DET_q a
 \end{aligned}$$

$$\begin{aligned}
 bDET_q &= b(ad - q^{-1}bc) \\
 &= bad - q^{-1}bcb \\
 &= adb - q^{-1}bcb \\
 &= (ad - q^{-1}bc)b \\
 &= DET_q b
 \end{aligned}$$

$$\begin{aligned}
cDET_q &= c(ad - q^{-1}bc) \\
&= cad - q^{-1}bcc \\
&= adc - q^{-1}bcc \\
&= (ad - q^{-1}bc)c \\
&= DET_q c
\end{aligned}$$

$$\begin{aligned}
dDET_q &= d(ad - q^{-1}bc) \\
&= dad - q^{-1}dbc \\
&= dad - bdc \\
&= dad - qbcd \\
&= (da - qbc)d \\
&= DET_q d
\end{aligned}$$

Quantum group $SL(2)_q$. If we add the constraint $DET_q = 1$ we have an algebra that is called the quantum group $SL(2)_q$.

To get to the quantum group $SU(2)_q$, we need to consider the notion of a Hopf $*$ -algebra.

Hopf $*$ -algebras. A $*$ -algebra A is an associative algebra with an anti-linear mapping $*$: $A \rightarrow A$ such that for $a \in A$ $(a^*)^* = a$ and $(ab)^* = b^*a^*$. If A has a unit I , then $I^* = I$.

A Hopf $*$ -algebra is a Hopf algebra A that is also a $*$ -algebra such that the counit and comultiplication are $*$ -homomorphisms and the antipode γ satisfies

$$\gamma \circ * \circ \gamma \circ * = \text{id}$$

with id being the identity on A .

Quantum group $SU(2)_q$. We will make the deformed function space $SL(2)_q$ into a Hopf $*$ -algebra by giving a $*$ -operation defined by

$$a^* = d, \quad d^* = a, \quad b^* = -qc, \quad c^* = -q^{-1}b.$$

Note that this means that the determinant condition can be written as $aa^* + cc^* = 1$. When we add the $*$ -structure we call our algebra $SU(2)_q$.

We can check that the axioms for a $*$ -algebra hold on the generators and then we use induction to see that they hold on the entire algebra.

Theorem 2-2. The algebra A_q is a Hopf algebra.

Proof. This was theorem 2-5 of chapter 1, with q instead of $1/q$.

§3. Hopf duals of Hopf algebras. In this section we will consider a way of defining a dual space of a Hopf algebra that will also be a Hopf algebra. We fill in some details of [MMNNSU].

Suppose that we have a set A of linear functions on a Hopf algebra H . Then we make the following definitions for all $a \in H$ and for all $\phi, \psi \in A$:

(i) The multiplication on A is defined by

$$\mu_A(\phi \otimes \psi)(a) \equiv (\phi \otimes \psi)\Delta a.$$

(ii) The comultiplication Δ_A is defined by

$$\Delta_A \phi(a \otimes b) = \phi(ab)$$

for all $\phi \in A$ and $a, b \in H$.

(iii) The antipode γ_A is defined by

$$\gamma_A(\phi)(a) = \phi(\gamma(a))$$

where γ is the antipode of H .

(iv) The counit ϵ_A is defined by

$$\epsilon_A(\phi) = \phi(1_H).$$

Remark. Note that not all dual spaces A are suitable. Either we will have to restrict ourselves to some subset of A or we will have to allow infinite sums of tensor products. We will assume in the following that the set A is sufficiently restricted that finite sums of tensor products suffice.

We now need to check that with definitions i) - iv) we have indeed defined a Hopf dual. This will be done in several steps.

Step 1. Δ_A is a homomorphism.

Proof. Suppose that $\Delta_A(\phi) = \sum \phi^k \otimes \phi_k$ and $\Delta_A(\psi) = \sum \phi^l \otimes \phi_l$. Then

$$\begin{aligned}\Delta_A(\phi\psi)(a \otimes b) &= \phi\psi(ab) \\ &= \mu_A(\phi \otimes \psi)(ab) \\ &= (\phi \otimes \psi)\Delta(ab) \\ &= (\phi \otimes \psi)[\Delta(a)\Delta(b)]\end{aligned}$$

Suppose that $\Delta(a) = \sum a^i \otimes a_i$ and $\Delta(b) = \sum b^j \otimes b_j$. Then

$$\begin{aligned}\Delta_A(\phi\psi)(a \otimes b) &= \sum_{i,j} \phi(a^i b^j) \psi(a_i b_j) \\ &= \sum_{i,j} \Delta_A(\phi)(a^i \otimes b^j) \Delta_A(\psi)(a_i \otimes b_j) \\ &= \sum_{i,j,k,l} \phi^k(a^i) \phi_k(b^j) \psi^l(a_i) \psi_l(b_j)\end{aligned}$$

On the other hand

$$\begin{aligned}
 \Delta(\phi)\Delta(\psi)(a \otimes b) &= \sum_{k,l} \phi^k \phi^l \otimes \phi_k \psi_l (a \otimes b) \\
 &= \sum_{k,l} \phi^k \phi^l(a) \phi_k \psi_l(b) \\
 &= \sum_{k,l} [\phi^k \otimes \phi^l] \Delta(a) \cdot [\phi_k \otimes \psi_l] \Delta(b) \\
 &= \sum_{k,l,i,j} \phi^k(a^i) \phi^l(a_i) \phi_k(b^j) \psi_l(b_j)
 \end{aligned}$$

and by comparing to the former result, we see that $\Delta(\phi)\Delta(\psi) = \Delta(\phi\psi)$. Also

$$\begin{aligned}
 \Delta(\phi + \psi)(a \otimes b) &= (\phi + \psi)(ab) \\
 &= \phi(ab) + \psi(ab) \\
 &= \Delta(\phi)(a \otimes b) + \Delta(\psi)(a \otimes b)
 \end{aligned}$$

so we conclude that Δ is a homomorphism.

Step 2. ϵ_A is a homomorphism.

Proof.

$$\epsilon_A(\phi\psi) = \phi\psi(1_{U_q}) = (\phi \otimes \psi)(1 \otimes 1) = \phi(1)\psi(1) = \epsilon_A(\phi)\epsilon_A(\psi).$$

$$\epsilon_A(\phi + \psi) = (\phi + \psi)(1_{U_q}) = \phi(1) + \psi(1) = \epsilon_A(\phi) + \epsilon_A(\psi).$$

and so ϵ_A is a homomorphism.

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Step 3. γ_A is an antihomomorphism.

Proof. Suppose that ϕ and ψ are in A and u is in U_q . Then we have that

$$\begin{aligned}
 \gamma_A(\phi\psi)(u) &= \phi\psi\gamma(u) \\
 &= (\phi \otimes \psi)\Delta(\gamma(u)) \\
 &= (\phi \otimes \psi)(\gamma \otimes \gamma)\sigma(\Delta(u)) \\
 &= (\psi \otimes \phi)(\gamma \otimes \gamma)(\Delta(u)) \\
 &= (\psi \circ \gamma)(\phi \circ \gamma)(\Delta(u)) \\
 &= \gamma_A(\psi)\gamma_A(\phi)(u)
 \end{aligned}$$

and

$$\begin{aligned}
 \gamma_A(\phi + \psi)(u) &= (\phi + \psi)\gamma(u) \\
 &= \phi(\gamma(u)) + \psi(\gamma(u)) \\
 &= [\gamma_A(\phi) + \gamma_A(\psi)](u)
 \end{aligned}$$

so γ_A is an antihomomorphism.

Step 4. The first Hopf axiom holds.

Proof. This is just a statement of associativity. Since $\phi[(ab)c] = \phi[a(bc)]$ we have from the definition of Δ_A that

$$\Delta_A \phi[(ab) \otimes c] = \Delta_A \phi[a \otimes (bc)].$$

Suppose that $\Delta_A \phi = \sum \phi^i \otimes \phi_i$. Then the above equation becomes

$$\sum \phi^i(ab)\phi_i(c) = \sum \phi^i(a)\phi_i(bc).$$

Again by the definition of Δ_A we have

$$\sum \Delta_A \phi^i(a \otimes b)\phi_i(c) = \sum \phi^i(a)\Delta_A \phi_i(b \otimes c)$$

which can also be written as

$$(\Delta_A \otimes \text{id})\Delta_A \phi(a \otimes b \otimes c) = (\text{id} \otimes \Delta_A)\Delta_A \phi(a \otimes b \otimes c) = .$$

Thus the first Hopf axiom is satisfied.

Step 5. The second Hopf axiom holds.

Proof. This follows from the existence of an identity in H . Since $1a = a1$ for $a \in H$, we have that

$$\Delta_A \phi(1 \otimes a) = \Delta_A \phi(a \otimes 1).$$

Then

$$\sum \phi^i \otimes \phi_i(1 \otimes a) = \sum \phi^i \otimes \phi_i(a \otimes 1)$$

which may be written as

$$\sum \phi^i(1)\phi_i(a) = \sum \phi^i(a)\phi_i(1).$$

By the definition of ϵ_A the above is

$$\sum \epsilon_A(\phi^i)\phi_i(a) = \sum \phi^i(a)\epsilon_A(\phi_i).$$

We rewrite this as

$$(\epsilon_A \otimes \text{id})(\sum \phi^i \otimes \phi_i)(a) = (\text{id} \otimes \epsilon_A)(\sum \phi^i \otimes \phi_i)(a)$$

to get the desired result.

Step 6. If the counit of H is the unit of A , then the third Hopf axiom holds.

Proof. We need to check that

$$\mu_A(\text{id}_A \otimes \gamma_A)\Delta_A\phi = \epsilon_A(\phi)1_A = \mu_A(\text{id}_A \otimes \gamma_A)\Delta_A\phi.$$

for all $\phi \in U_q$. The *LHS* is

$$\begin{aligned} & \mu_A(\text{id}_A \otimes \gamma_A)\Delta_A\phi \\ &= \mu_A(\text{id}_A \otimes \gamma_A) \sum \phi^i \otimes \phi_i \\ &= \sum \phi^i \phi_i \circ \gamma, \end{aligned}$$

so

$$\begin{aligned} LHS(a) &= \sum (\phi^i \phi_i \circ \gamma)(a) \\ &= \sum (\phi^i \otimes \phi_i \circ \gamma)\Delta(a) \\ &= \sum (\phi^i \otimes \phi_i)(\text{id} \otimes \gamma)\Delta(a) \\ &= \Delta_A\phi(\text{id} \otimes \gamma)\Delta(a) \\ &= \phi\mu(\text{id} \otimes \gamma)\Delta(a). \end{aligned}$$

Similarly $RHS(a) = \phi\mu(\gamma \otimes \text{id})\Delta(a)$. Then $RHS = LHS$, because U_q is a Hopf algebra.

What about the middle? We need

$$\phi\mu(\text{id} \otimes \gamma)\Delta a = \phi\epsilon(a)1_{U_q} = \epsilon(a)\phi(1) = \epsilon_A(\phi)\epsilon(a) = \epsilon_A(\phi)1_A(\phi)$$

so we need that the unit of A is the counit of U_q .

Theorem 3-1. If the unit of A is the counit of H , then A is a Hopf algebra.

Proof. Steps 1-6 and then theorem 2-4 of chapter 1.

A result that will be useful later on is

Lemma 3-2. The counit of H commutes with all of the elements of A .

Proof. We have that $(\text{id} \otimes \epsilon)\Delta(u) = (\epsilon \otimes \text{id})\Delta(u)$ for $u \in H$. Now if $\Delta(u) = \sum u^i \otimes u_i$, then $\sum u^i \epsilon(u_i) = \epsilon(u^i)(u_i)$. Thus

$$\begin{aligned}\phi(LHS) &= \sum \epsilon(u_i)\phi(u^i) \\ &= (\epsilon \otimes \phi)\Delta(u) \\ &= \epsilon\phi(u),\end{aligned}$$

while

$$\begin{aligned}
 \phi(RHS) &= \sum \phi(u_i)\epsilon(u^i) \\
 &= (\phi \otimes \epsilon)\Delta(u) \\
 &= \phi\epsilon(u).
 \end{aligned}$$

Since $LHS = RHS$, we have that $\phi\epsilon = \epsilon\phi$.

§4. Dual space of $SU(2)_q$.

In this section we will discuss the Hopf dual of our space $SU(2)_q$. We will show that the Hopf dual of the deformed polynomial functions on $SU(2)$ contains the algebra $U(su(2))_q$.

Suppose that we start with the algebra $GL(2)_q$ which consists of the deformed coordinate functions on $GL(2)$. We wish to define certain real valued functions on the algebra $GL(2)_q$. We do this by taking formal derivatives of polynomials in $GL(2)_q$ and then evaluating them at

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For instance,

$$\frac{\partial}{\partial b} a^k d^l b^m c^n = m a^k d^l b^{m-1} c^n$$

and when we set

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

we get $\delta_{m1}\delta_{n0}$. We use this to define a linear functional E by

$$\langle E | a^k d^l b^m c^n \rangle = \delta_{m1}\delta_{n0}.$$

Differentiating wrt c gives

$$\frac{\partial}{\partial c} a^k d^l b^m c^n = n a^k d^l b^m c^{n-1}$$

and when we set

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

we get $\delta_{m0}\delta_{n1}$. We use this to define a linear functional F by

$$\langle F | a^k d^l b^m c^n \rangle = \delta_{m0}\delta_{n1}.$$

Also

$$\frac{\partial}{\partial a} a^k d^l b^m c^n = k a^{k-1} d^l b^m c^n$$

and when we set

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

we get $k\delta_{m0}\delta_{n0}$.

$$\frac{\partial}{\partial d} a^k d^l b^m c^n = l a^k d^{l-1} b^m c^n$$

and when we set

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

we get $l\delta_{m0}\delta_{n0}$.

We use these results to define linear functionals

$$H = \frac{\partial}{\partial a}|_1 - \frac{\partial}{\partial d}|_1$$

and

$$I = \frac{\partial}{\partial a}|_1 + \frac{\partial}{\partial d}|_1$$

so that

$$\langle H | a^k d^l b^m c^n \rangle = (k - l) \delta_{m0} \delta_{n0}$$

and

$$\langle I | a^k d^l b^m c^n \rangle = (k + l) \delta_{m0} \delta_{n0}.$$

These are the definitions given in [Su] by Sudbery. Sudbery then claims that the relations of Woronowicz hold, namely:

$$[H, E] = 2E, \quad [H, F] = -2F, \quad qEF - q^{-1}FE = \frac{q^{2H} - 1}{q - q^{-1}}.$$

We will show that these relations hold by using the theory developed in the previous section for Hopf dual spaces.

We will prove the third relation first. We do this by finding the effect of the operators EF and FE on a typical element $a^k d^l b^m c^n$ of $SU(2)_q$ and then calculating $qEF - q^{-1}FE$.

Claim.

$$EF = q^{-mn} \delta_{m1} \delta_{n1} + \delta_{m0} \delta_{n0} (q^{-2l} + q^{-2(l-1)} + \dots q^{-2(l-k+1)}).$$

Proof. From the theory of Hopf algebras and their duals described in the previous section EF is defined in the following way:

$$\langle EF | a^k d^l b^m c^n \rangle = \langle E \otimes F | \Delta(a^k d^l b^m c^n) \rangle$$

which equals

$$< E \otimes F | (a \otimes a + b \otimes c)^k (c \otimes b + d \otimes d)^l (a \otimes b + b \otimes d)^m (c \otimes a + d \otimes c)^n > .$$

From the definitions of E and F we know that there must be one b in the first factor of a term and no c 's, or that term does not contribute. Similarly, the second factor should have one c and no b 's. Thus

$$< E \otimes F | \Delta(a^k d^l b^m c^n) >$$

is the same as

$$< E \otimes F | (a \otimes a + b \otimes c)^k (d \otimes d)^l (b \otimes d)^m (d \otimes c)^n > .$$

We can throw out some more terms using the same considerations.

For $(a \otimes a + b \otimes c)^k$ the terms that will contribute are:

$$\begin{aligned} & a^k \otimes a^k \\ & + (a^{k-1} \otimes a^{k-1})(b \otimes c) + (a^{k-2} \otimes a^{k-2})(b \otimes c)(a \otimes a) \\ & + \cdots + (b \otimes c)(a^{k-1} \otimes a^{k-1}). \end{aligned}$$

Then $< EF | a^k d^l b^m c^n > =$

$$\begin{aligned} q^{-mn} < E \otimes F | \{ & (a^k \otimes a^k) \\ & + (a^{k-1} \otimes a^{k-1})(b \otimes c) \\ & \vdots \\ & + (b \otimes c)(a^{k-1} \otimes a^{k-1}) \} (d^{l+n} b^m \otimes d^{l+n} c^n) > . \end{aligned}$$

Expanding out we have

$$\begin{aligned}
 & q^{-mn} \langle E \otimes F | a^k d^{l+n} b^m \otimes a^k d^{l+n} c^n \rangle \\
 & + q^{-mn} \langle E \otimes F | a^{k-1} b d^{l+n} b^m \otimes a^{k-1} c d^{l+n} c^n \rangle \\
 & + q^{-mn} \langle E \otimes F | a^{k-2} b a d^{l+n} b^m \otimes a^{k-2} c a d^{l+n} c^n \rangle \\
 & \vdots \\
 & + q^{-mn} \langle E \otimes F | b a^{k-1} d^{l+n} b^m \otimes c a^{k-1} d^{l+n} c^n \rangle .
 \end{aligned}$$

By the definition of E and F the first term above is $q^{-mn} \delta_{m1} \delta_{n1}$ and by using the relations on A_q we get that the other terms are

$$\alpha \langle E \otimes F | a^{k-1} d^{l+n} b^{m+1} \otimes a^{k-1} d^{l+n} c^{n+1} \rangle .$$

where $\alpha = (q^{-2(l+n)} + q^{-2(l+n-1)} + \dots + q^{-2(l+n-k+1)})$. Adding up and using the definitions of E and F gives that $\langle E \otimes F | a^k d^l b^m c^n \rangle$ equals

$$q^{-mn} \delta_{m1} \delta_{n1} + \delta_{m0} \delta_{n0} (q^{-2l} + q^{-2(l-1)} + \dots + q^{-2(l-k+1)}) .$$

Claim.

$$\langle FE | a^k d^l b^m c^n \rangle = (1 + q^{-2} + \dots + q^{-2(l-1)}) \delta_{m0} \delta_{n0} + q \delta_{m1} \delta_{n1} .$$

Proof. Using similar reasoning as in the proof of the previous claim gives that $\langle FE | a^k d^l b^m c^n \rangle$, which is defined as

$$\langle F \otimes E | (a \otimes a + b \otimes c)^k (c \otimes b + d \otimes d)^l (a \otimes b + b \otimes d)^m (c \otimes a + d \otimes c)^n \rangle$$

equals

$$< F \otimes E | (a \otimes a)^k (c \otimes b + d \otimes d)^l (a \otimes b)^m (c \otimes a)^n > .$$

For $(c \otimes b + d \otimes d)^l$ the only terms that can contribute are

$$\begin{aligned} & d^l \otimes d^l \\ & + (d^{l-1} \otimes d^{l-1})(c \otimes b) + (d^{l-2} \otimes d^{l-2})(c \otimes b)(d \otimes d) \\ & + \cdots + (c \otimes b)(d^{l-1} \otimes d^{l-1}). \end{aligned}$$

Then we see that

$$\begin{aligned} & < FE | a^k d^l b^m c^n > \\ & = < F \otimes E | a^k d^l a^m c^n \otimes a^k d^l b^m a^n > \\ & + < F \otimes E | a^k d^{l-1} c a^m c^n \otimes a^k d^{l-1} b b^m a^n > \\ & + < F \otimes E | a^k d^{l-2} c d a^m c^n \otimes a^k d^{l-2} b d b^m a^n > \\ & \vdots \\ & + < F \otimes E | a^k c d^{l-1} a^m c^n \otimes a^k b d^{l-1} b^m a^n > . \end{aligned}$$

Now consider the first term. We can have only one c in the first factor and one b in the second factor otherwise we have zero. Thus, the first term is

$$\begin{aligned} & \delta_{m1} \delta_{n1} < F \otimes E | a^k d^l a c \otimes a^k d^l b a > \\ & = q \delta_{m1} \delta_{n1} < F \otimes E | a^k d^l a c \otimes a^k d^l a b > . \end{aligned}$$

Recall that $da = ad - (q^{-1} - q)bc$ and so

$$d^l a = d^{l-1} da = d^{l-1}(ad - (q^{-1} - q)bc) = d^{l-1}ad - (q^{-1} - q)d^{l-1}bc.$$

Substituting this in and taking into account the definition of $F \otimes E$ we get that the first term is

$$\begin{aligned} & q\delta_{m1}\delta_{n1} \langle F \otimes E | a^k d^l ac \otimes a^k d^l ab \rangle \\ &= q\delta_{m1}\delta_{n1} \langle F \otimes E | a^k d^{l-1} c \otimes a^k d^{l-1} b \rangle \\ &= q\delta_{m1}\delta_{n1}. \end{aligned}$$

We now see how to approach the other terms and we get

$$\begin{aligned} & \delta_{m0}\delta_{n0} \langle F \otimes E | a^k d^{l-1} c \otimes a^k d^{l-1} b \rangle \\ &+ \delta_{m0}\delta_{n0} \langle F \otimes E | a^k d^{l-2} cd \otimes a^k d^{l-2} bd \rangle \\ & \vdots \\ &+ \delta_{m0}\delta_{n0} \langle F \otimes E | a^k cd^{l-1} \otimes a^k bd^{l-1} \rangle \end{aligned}$$

and applying the relations on A_q gives

$$\beta\delta_{m0}\delta_{n0} \langle F \otimes E | a^k d^{l-1} c \otimes a^k d^{l-1} b \rangle$$

where $\beta = 1 + q^{-2} + \dots + q^{-2(l-1)}$. Summing up then,

$$\langle FE | a^k d^l b^m c^n \rangle = (1 + q^{-2} + \dots + q^{-2(l-1)})\delta_{m0}\delta_{n0} + q\delta_{m1}\delta_{n1}.$$

We are now in a position to prove the third Woronowicz relation by making use of our results for EF and FE .

Lemma 4-1.

$$qEF - q^{-1}FE = \frac{q^{2H} - 1}{q - q^{-1}}.$$

Proof. First note that

$$\begin{aligned} & q^{-2l} + q^{-2l+2(1)} + q^{-2l+2(2)} + \dots + q^{-2l+2(k-1)} \\ &= q^{-2l}(1 + q^2 + q^{2(2)} + \dots + q^{2(k-1)}) \\ &= q^{-2l} \frac{q^{2k} - 1}{q^2 - 1} \end{aligned}$$

and

$$(1 + q^{-2} + q^{-2(2)} + \dots + q^{-2(l-1)}) = \frac{q^{-2l} - 1}{q^{-2} - 1}.$$

Substituting these two formulae into our formulae for EF and FE gives that $qEF - q^{-1}FE$ is

$$qq^{-2l} \frac{q^{2k} - 1}{q^2 - 1} \delta_{m0} \delta_{n0} - q^{-1} \frac{q^{-2l} - 1}{q^{-2} - 1} \delta_{m0} \delta_{n0}.$$

We then calculate directly that this is

$$\begin{aligned}
&= \frac{q^{2k-2l+1} - q^{-2l+1}}{q^2 - 1} \delta_{m0} \delta_{n0} - \frac{q^{-2l-1} - q^{-1}}{q^{-2} - 1} \delta_{m0} \delta_{n0} \\
&= \frac{q^{2k-2l-1} - q^{2k-2l+1} + q - q^{-1}}{(q^2 - 1)(q^{-2} - 1)} \delta_{m0} \delta_{n0} \\
&= \frac{q^{-1}(q^{2k-2l} - 1) + q(1 - q^{2k-2l})}{(q^2 - 1)(q^{-2} - 1)} \delta_{m0} \delta_{n0} \\
&= \frac{(q - q^{-1})(1 - q^{2k-2l})}{(q^2 - 1)(q^{-2} - 1)} \delta_{m0} \delta_{n0} \\
&= \frac{(q^2 - 1)(1 - q^{2k-2l})}{(q^2 - 1)(q^{-1} - q)} \delta_{m0} \delta_{n0} \\
&= \frac{q^{2k-2l} - 1}{q - q^{-1}} \delta_{m0} \delta_{n0} \\
&= \frac{q^{2H} - 1}{q - q^{-1}}.
\end{aligned}$$

Thus the third Woronowicz relation is proven. Now we prove the first relation in the same way. First we find the result of the operators HE and EH on a typical element of $SU(2)_q$ and then we find the commutator.

Claim. $HE = \delta_{n0} \delta_{m1} (k + 1 - l).$

Proof. $\langle HE | a^k d^l b^m c^n \rangle$ is defined to be

$$\langle H \otimes E | (a \otimes a + b \otimes c)^k (c \otimes b + d \otimes d)^l (a \otimes b + b \otimes d)^m (c \otimes a + d \otimes c)^n \rangle.$$

By the definition of $H \otimes E$, this is the same as

$$\begin{aligned} & \delta_{n0} < H \otimes E | (a \otimes a)^k (d \otimes d)^l (a \otimes b)^m > \\ & = \delta_{n0} < H \otimes E | a^k d^l a^m \otimes a^k d^l b^m > \\ & = \delta_{n0} \delta_{m1} < H \otimes E | a^k d^l a \otimes a^k d^l b >, \end{aligned}$$

and from the definition of H

$$\begin{aligned} & = \delta_{n0} \delta_{m1} < H \otimes E | a^{k+1} d^l \otimes a^k d^l b > \\ & = \delta_{n0} \delta_{m1} (k + 1 - l). \end{aligned}$$

Claim. $EH = \delta_{n0} \delta_{m0} (k - l - 1)$.

Proof. $< EH | a^k d^l b^m c^n >$ is defined to be

$$< E \otimes H | (a \otimes a + b \otimes c)^k (c \otimes b + d \otimes d)^l (a \otimes b + b \otimes d)^m (c \otimes a + d \otimes c)^n >,$$

which by the definition of $E \otimes H$ is the same as

$$\begin{aligned} & \delta_{n0} < E \otimes H | (a \otimes a)^k (d \otimes d)^l (b \otimes d)^m > \\ & = \delta_{n0} < E \otimes H | a^k d^l b^m \otimes a^k d^l d^m > \end{aligned}$$

and by the definition of E , this is

$$\begin{aligned} & = \delta_{n0} \delta_{m1} < E \otimes H | a^k d^l b \otimes a^k d^{l+1} > \\ & = \delta_{n0} \delta_{m1} (k - l - 1). \end{aligned}$$

We now can prove the first Woronowicz relation:

Lemma 4-2. $[H, E] = 2E$.

Proof. $[H, E] = \delta_{n0}\delta_{m1}(k - l + 1) - \delta_{n0}\delta_{m1}(k - l - 1) = 2\delta_{n0}\delta_{m1}$.

But $E = \delta_{n0}\delta_{m1}$. Thus, $[H, E] = 2E$.

We next prove the second Woronowicz relation by finding the operators HF and FH and then taking the commutator.

Claim. $HF = \delta_{m0}\delta_{n1}(k - l - 1)$.

Proof. $\langle HF | a^k d^l b^m c^n \rangle$ is defined to be

$$\langle H \otimes F | (a \otimes a + b \otimes c)^k (c \otimes b + d \otimes d)^l (a \otimes b + b \otimes d)^m (c \otimes a + d \otimes c)^n \rangle$$

which by the definition of $H \otimes F$ is the same as

$$\begin{aligned} & \delta_{m0} \langle H \otimes F | (a \otimes a)^k (d \otimes d)^l (d \otimes c)^n \rangle \\ &= \delta_{m0} \langle H \otimes F | a^k d^{l+n} \otimes a^k d^l c^n \rangle \end{aligned}$$

and by the definition of F , this is

$$\begin{aligned} &= \delta_{m0}\delta_{n1} \langle H \otimes F | a^k d^{l+1} \otimes a^k d^l c \rangle \\ &= \delta_{m0}\delta_{n1}(k - l - 1) \end{aligned}$$

Claim. $FH = \delta_{n0}\delta_{m1}(k - l + 1)$.

Proof. $\langle FH|a^k d^l b^m c^n \rangle$ is defined to be

$$\langle F \otimes H|(a \otimes a + b \otimes c)^k (c \otimes b + d \otimes d)^l (a \otimes b + b \otimes d)^m (c \otimes a + d \otimes c)^n \rangle$$

which by the definition of $F \otimes H$ is

$$\begin{aligned} & \delta_{m0} \langle F \otimes H|(a \otimes a)^k (d \otimes d)^l (c \otimes a)^n \rangle \\ &= \delta_{m0} \langle F \otimes H|a^k d^l c^n \otimes a^k d^l a^n \rangle \end{aligned}$$

by definition of F

$$\begin{aligned} &= \delta_{m0} \delta_{n1} \langle F \otimes H|a^k d^l c \otimes a^k d^l a \rangle \\ &= \delta_{m0} \delta_{n1} \langle F \otimes H|a^k d^l c \otimes a^{k+1} d^l \rangle \\ &= \delta_{n0} \delta_{m1} (k - l + 1). \end{aligned}$$

We calculate the commutator to prove the second Woronowicz relation:

Lemma 4-3. $[H, F] = -2F$

Proof. First, recall that $F = \delta_{m0} \delta_{n1}$. Then from the results for HF and FH , we have that

$$[H, F] = (k - l - 1)F - (k - l + 1)F = -2F.$$

Next, we find the commutator $[I, E]$.

Claim. $IE = \delta_{n0}\delta_{m1}(k + l + 1).$

Proof. $\langle IE|a^k d^l b^m c^n \rangle$ equals

$$\langle I \otimes E|(a \otimes a + b \otimes c)^k (c \otimes b + d \otimes d)^l (a \otimes b + b \otimes d)^m (c \otimes a + d \otimes c)^n \rangle$$

which by the definition of $I \otimes E$ is equal to

$$\begin{aligned} & \delta_{n0} \langle I \otimes E|(a \otimes a)^k (d \otimes d)^l (a \otimes b)^m \rangle \\ &= \delta_{n0} \langle I \otimes E|a^k d^l a^m \otimes a^k d^l b^m \rangle \\ &= \delta_{n0}\delta_{m1} \langle I \otimes E|a^k d^l a \otimes a^k d^l b \rangle \\ &= \delta_{n0}\delta_{m1} \langle I \otimes E|a^{k+1} d^l \otimes a^k d^l b \rangle \\ &= \delta_{n0}\delta_{m1}(k + l + 1). \end{aligned}$$

Claim. $EI = \delta_{n0}\delta_{m1}(k + l + 1) = IE.$

Proof. $\langle EI|a^k d^l b^m c^n \rangle$ equals

$$\langle E \otimes I|(a \otimes a + b \otimes c)^k (c \otimes b + d \otimes d)^l (a \otimes b + b \otimes d)^m (c \otimes a + d \otimes c)^n \rangle$$

which by the definition of $E \otimes I$ is

$$\begin{aligned} & \delta_{n0} \langle E \otimes I|(a \otimes a)^k (d \otimes d)^l (b \otimes d)^m \rangle \\ &= \delta_{n0} \langle E \otimes I|a^k d^l b^m \otimes a^k d^{l+m} \rangle \\ &= \delta_{n0}\delta_{m1} \langle E \otimes I|a^k d^l b \otimes a^k d^{l+1} \rangle \\ &= \delta_{n0}\delta_{m1}(k + l + 1). \end{aligned}$$

Lemma 4-4. $[I, E] = 0$.

We find the commutator $[I, F]$.

Claim. $IF = \delta_{m0}\delta_{n1}(k + l + 1)$.

Proof. $\langle IF|a^k d^l b^m c^n \rangle$ equals

$$\langle I \otimes F|(a \otimes a + b \otimes c)^k (c \otimes b + d \otimes d)^l (a \otimes b + b \otimes d)^m (c \otimes a + d \otimes c)^n \rangle$$

which is

$$\begin{aligned} & \langle I \otimes F|(a \otimes a)^k (d \otimes d)^l (d \otimes c)^n \rangle \\ &= \delta_{m0} \langle I \otimes F|a^k d^{l+n} \otimes a^k d^l c^n \rangle \\ &= \delta_{m0}\delta_{n1} \langle I \otimes F|a^k d^{l+1} \otimes a^k d^l c \rangle \\ &= \delta_{m0}\delta_{n1}(k + l + 1) \end{aligned}$$

Claim. $FI = \delta_{m0}\delta_{n1}(k + l + 1) = IF$.

Proof. $\langle FI|a^k d^l b^m c^n \rangle$ equals

$$\langle F \otimes I|(a \otimes a + b \otimes c)^k (c \otimes b + d \otimes d)^l (a \otimes b + b \otimes d)^m (c \otimes a + d \otimes c)^n \rangle$$

which is

$$\begin{aligned}
& \delta_{m0} < F \otimes I | (a \otimes a)^k (d \otimes d)^l (c \otimes a)^n > \\
& = \delta_{m0} < F \otimes I | a^k d^l c^n \otimes a^k d^l a^n > \\
& = \delta_{m0} \delta_{n1} < F \otimes I | a^k d^l c \otimes a^k d^l a > \\
& = \delta_{m0} \delta_{n1} < F \otimes I | a^k d^l c \otimes a^{k+1} d^l > \\
& = \delta_{m0} \delta_{n1} (k + l + 1).
\end{aligned}$$

Lemma 4-5. $[I, F] = 0$.

Next, we find the commutator $[I, H]$.

Claim. $IH = \delta_{m0} \delta_{n0} (k + l)(k - l)$.

Proof. $< IH | a^k d^l b^m c^n >$ equals

$$< I \otimes H | (a \otimes a + b \otimes c)^k (c \otimes b + d \otimes d)^l (a \otimes b + b \otimes d)^m (c \otimes a + d \otimes c)^n >$$

which is

$$\begin{aligned}
& \delta_{m0} \delta_{n0} < I \otimes H | (a \otimes a)^k (d \otimes d)^l > \\
& = \delta_{m0} \delta_{n0} < I \otimes H | a^k d^l \otimes a^k d^l > \\
& = \delta_{m0} \delta_{n0} (k + l)(k - l).
\end{aligned}$$

Claim. $HI = \delta_{m0} \delta_{n0} (k - l)(k + l) = IH$.

Proof. $\langle HI|a^k d^l b^m c^n \rangle$ equals

$$\langle H \otimes I | (a \otimes a + b \otimes c)^k (c \otimes b + d \otimes d)^l (a \otimes b + b \otimes d)^m (c \otimes a + d \otimes c)^n \rangle$$

which is

$$\begin{aligned} & \delta_{m0} \delta_{n0} \langle H \otimes I | (a \otimes a)^k (d \otimes d)^l \rangle \\ &= \delta_{m0} \delta_{n0} \langle H \otimes I | a^k d^l \otimes a^k d^l \rangle \\ &= \delta_{m0} \delta_{n0} (k-l)(k+l). \end{aligned}$$

Lemma 4-6. $[I, H] = 0$.

Summing up the results of lemmas 4-1 to 4-6 we have

$$\begin{aligned} [H, E] &= 2E, & [H, F] &= -2F \\ [I, E] &= [I, F] = [I, H] = 0, \end{aligned}$$

and

$$qEF - q^{-1}FE = \frac{q^{2H} - 1}{q - q^{-1}}.$$

Ignoring the operator I , we now have the relations of Woronowicz.

Jimbo-Drinfeld relations. We would like to get the relations of the quantised algebra that we started with in chapter 2.

Definition. Define

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix} = Q$$

and let

$$\langle J_+ | f \rangle = \frac{\partial f}{\partial b} |_Q, \quad \langle J_- | f \rangle = \frac{\partial f}{\partial c} |_Q, \quad \text{and} \quad J_z = H/2.$$

Remark. Note the correction to the definitions of the J_μ given in [Su].

Theorem 4-7. The linear functionals J_+ , J_- , and J_z defined above satisfy the Jimbo/Drinfeld relations

$$[J_z, J_+] = J_+, \quad [J_z, J_-] = -J_-, \quad [J_+, J_-] = [2J_z].$$

Proof. The proof will be done in several steps. Let $f = a^k d^l b^m c^n$. Since the elements of A_q are sums of such elements it is sufficient to show that the desired relations hold on f . We have

$$\langle J_+ | f \rangle = \frac{\partial f}{\partial b} |_Q = q^{\frac{l-k}{2}} \delta_{m1} \delta_{n0},$$

$$\langle J_- | f \rangle = \frac{\partial f}{\partial c} |_Q = q^{\frac{l-k}{2}} \delta_{m0} \delta_{n1},$$

and

$$\langle J_z | f \rangle = \langle H/2 | f \rangle = \frac{k-l}{2} \delta_{m0} \delta_{n0}.$$

We now check the commutator relations:

Claim. $J_+ J_- = q^{l+1-k} q^{-1} \delta_{m1} \delta_{n1} + \frac{q^{2k}-1}{q^2-1} q^{l-k+1} q^{-2l} \delta_{m0} \delta_{n0}$.

Proof. $\langle J_+ J_- | a^k d^l b^m c^n \rangle$ is defined to be

$$\langle J_+ \otimes J_- | (a \otimes a + b \otimes c)^k (c \otimes b + d \otimes d)^l (a \otimes b + b \otimes d)^m (c \otimes a + d \otimes c)^n \rangle.$$

From the definitions of the operators J_+ and J_- , this is

$$\begin{aligned} & \langle J_+ \otimes J_- | \{ (a^k \otimes a^k) + \frac{q^{2k}-1}{q^2-1} (a^{k-1} b \otimes a^{k-1} c) \} \\ & \quad (d^l b^m d^n \otimes d^l d^m c^n) \rangle \\ &= \langle J_+ \otimes J_- | a^k d^l b^m d^n \otimes a^k d^l d^m c^n \rangle \\ & \quad + \frac{q^{2k}-1}{q^2-1} \langle J_+ \otimes J_- | a^{k-1} b d^l b^m d^n \otimes a^{k-1} c d^l d^m c^n \rangle \\ &= \langle J_+ \otimes J_- | a^k d^l b d \otimes a^k d^l d c \rangle \delta_{m1} \delta_{n1} \\ & \quad + \frac{q^{2k}-1}{q^2-1} \langle J_+ \otimes J_- | a^{k-1} b d^l \otimes a^{k-1} c d^l \rangle \delta_{m0} \delta_{n0} \\ &= \langle J_+ \otimes J_- | a^k d^{l+1} b \otimes a^k d^{l+1} c \rangle q^{-1} \delta_{m1} \delta_{n1} \\ & \quad + \frac{q^{2k}-1}{q^2-1} \langle J_+ \otimes J_- | a^{k-1} d^l b \otimes a^{k-1} d^l c \rangle q^{-2l} \delta_{m0} \delta_{n0} \\ &= q^{l+1-k} q^{-1} \delta_{m1} \delta_{n1} + \frac{q^{2k}-1}{q^2-1} q^{l-k+1} q^{-2l} \delta_{m0} \delta_{n0}. \end{aligned}$$

Claim. $J_- J_+ = q^{l-k-1} q \delta_{m1} \delta_{n1} + q^{l-1-k} \delta_{m0} \delta_{n0}$.

Proof. $\langle J_- J_+ | a^k d^l b^m c^n \rangle$ is defined to be

$$\langle J_- \otimes J_+ | (a \otimes a + b \otimes c)^k (c \otimes b + d \otimes d)^l (a \otimes b + b \otimes d)^m (c \otimes a + d \otimes c)^n \rangle$$

which from consideration of the number of b 's and c 's is the same as $J_- \otimes J_+$ acting on

$$(a^k \otimes a^k) \{d^l \otimes d^l + \frac{q^{-2l} - 1}{q^{-2} - 1} d^{l-1} b \otimes d^{l-1} c\} (a^m c^n \otimes b^m c^n).$$

Thus

$$\begin{aligned} & \langle J_- J_+ | a^k d^l b^m c^n \rangle \\ &= \langle J_- \otimes J_+ | a^k d^l a^m c^n \otimes a^k d^l b^m a^n \rangle \\ & \quad + \frac{q^{-2l} - 1}{q^{-2} - 1} \langle J_- \otimes J_+ | a^k d^{l-1} b a^m c^n \otimes a^k d^{l-1} c b^m a^n \rangle \\ &= \langle J_- \otimes J_+ | a^k d^l a c \otimes a^k d^l b a \rangle \delta_{m1} \delta_{n1} \\ & \quad + \frac{q^{-2l} - 1}{q^{-2} - 1} \langle J_- \otimes J_+ | a^k d^{l-1} b \otimes a^k d^{l-1} c \rangle \delta_{m0} \delta_{n0} \\ &= q \langle J_- \otimes J_+ | a^{k+1} d^l c \otimes a^{k+1} d^l b \rangle \delta_{m1} \delta_{n1} \\ & \quad + \frac{q^{-2l} - 1}{q^{-2} - 1} q^{l-1-k} \delta_{m0} \delta_{n0} \\ &= q^{l-k-1} q \delta_{m1} \delta_{n1} + q^{l-1-k} \delta_{m0} \delta_{n0}. \end{aligned}$$

Commutator. $[J_+, J_-] = [2J_z].$

Proof. The commutator $[J_+, J_-]$ is

$$\begin{aligned}
 & q^{l-k} \delta_{m1} \delta_{n1} + q^{-l-k+1} \frac{q^{2k-1} - q^{-1}}{q - q^{-1}} \delta_{m0} \delta_{n0} \\
 & - q^{l-k} \delta_{m1} \delta_{n1} - q^{l-k-1} \frac{q^{-2l+1} - q}{q^{-1} - q} \delta_{m0} \delta_{n0} \\
 & = \frac{(q^{2k-1} - q^{-1})q^{-l-k+1} + q^{l-k-1}(q^{-2l+1} - q)}{q - q^{-1}} \delta_{m0} \delta_{n0} \\
 & = \frac{q^{k-l} - q^{l-k}}{q - q^{-1}} \delta_{m0} \delta_{n0} \\
 & = \frac{q^{(k-l)\delta_{m0}\delta_{n0}} - q^{(l-k)\delta_{m0}\delta_{n0}}}{q - q^{-1}} \\
 & = \frac{q^{2J_z} - q^{-2J_z}}{q - q^{-1}}.
 \end{aligned}$$

Now we check the commutator $[J_z, J_+]$:

Claim. $J_z J_+ = \frac{k+1-l}{2} q^{(l-k)/2} \delta_{m1} \delta_{n0}.$

Proof. $\langle J_z J_+ | a^k d^l b^m c^n \rangle$ is defined to be

$$\langle J_z \otimes J_+ | (a \otimes a + b \otimes c)^k (c \otimes b + d \otimes d)^l (a \otimes b + b \otimes d)^m (c \otimes a + d \otimes c)^n \rangle$$

which from consideration of the number of b 's and c 's is the same as

$$\begin{aligned}
 & \langle J_z \otimes J_+ | a^k d^l a^m \otimes a^k d^l b^m \rangle \delta_{n0} \\
 & = \langle J_z \otimes J_+ | a^{k+m} d^l \otimes a^k d^l b^m \rangle \delta_{n0} \\
 & = \langle J_z \otimes J_+ | a^{k+m} d^l \otimes a^k d^l b \rangle \delta_{n0} \delta_{m1} \\
 & = \frac{k+1-l}{2} q^{(l-k)/2} \delta_{m1} \delta_{n0}.
 \end{aligned}$$

Claim. $J_+ J_z = q^{(l-k)/2} \frac{k-l-1}{2} \delta_{m1} \delta_{n0}$.

Proof. $\langle J_+ J_z | a^k d^l b^m c^n \rangle$ is defined to be

$$\langle J_+ \otimes J_z | (a \otimes a + b \otimes c)^k (c \otimes b + d \otimes d)^l (a \otimes b + b \otimes d)^m (c \otimes a + d \otimes c)^n \rangle$$

which from consideration of the number of b 's and c 's is the same as

$$\begin{aligned} & \langle J_+ \otimes J_z | a^k d^l b^m \otimes a^k d^l d^m \rangle \delta_{n0} \\ &= \langle J_+ \otimes J_z | a^k d^l b \otimes a^k d^{l+1} \rangle \delta_{n0} \delta_{m1} \\ &= q^{(l-k)/2} \frac{k-l-1}{2} \delta_{m1} \delta_{n0}. \end{aligned}$$

Commutator. $[J_z, J_+] = J_+$.

Proof. The result is immediate from the results for $J_z J_+$ and $J_+ J_z$.

Now we check the commutator $[J_z, J_-]$:

Claim. $J_z J_- = \frac{k-l-1}{2} q^{(l-k)/2} \delta_{m0} \delta_{n1}$.

Proof. $\langle J_z J_- | a^k d^l b^m c^n \rangle$ is defined to be

$$\langle J_z \otimes J_- | (a \otimes a + b \otimes c)^k (c \otimes b + d \otimes d)^l (a \otimes b + b \otimes d)^m (c \otimes a + d \otimes c)^n \rangle$$

which from consideration of the number of b 's and c 's is the same as

$$\begin{aligned} & \langle J_z \otimes J_- | a^k d^l d^n \otimes a^k d^l c^n \rangle \delta_{m0} \\ &= \langle J_z \otimes J_- | a^k d^{l+1} \otimes a^k d^l c \rangle \delta_{m0} \delta_{n1} \\ &= \frac{k-l-1}{2} q^{(l-k)/2} \delta_{m0} \delta_{n1}. \end{aligned}$$

Claim. $J_- J_z = q^{(l-k)/2} \frac{k+1-l}{2} \delta_{m0} \delta_{n1}$.

Proof. $\langle J_- J_z | a^k d^l b^m c^n \rangle$ is defined to be

$$\langle J_- \otimes J_z | (a \otimes a + b \otimes c)^k (c \otimes b + d \otimes d)^l (a \otimes b + b \otimes d)^m (c \otimes a + d \otimes c)^n \rangle$$

which from consideration of the number of b 's and c 's is the same as

$$\begin{aligned} & \langle J_- \otimes J_z | a^k d^l c^n \otimes a^k d^l a^n \rangle \delta_{m0} \\ &= \langle J_- \otimes J_z | a^k d^l c \otimes a^k d^l a \rangle \delta_{n1} \delta_{m0} \\ &= \langle J_- \otimes J_z | a^k d^l c \otimes a^{k+1} d^l \rangle \delta_{n1} \delta_{m0} \\ &= q^{(l-k)/2} \frac{k+1-l}{2} \delta_{m0} \delta_{n1}. \end{aligned}$$

Commutator. $[J_z, J_-] = -J_-$.

Proof. The result is immediate from the previous two claims.

We have now proved theorem 4-7.

Remark. It can be shown that the operators E, F , and H that were first defined by Woronowicz are related to the Jimbo/Drinfeld operators by

$$E = q^{-1/2} J_+ q^{J_z}, \quad F = q^{1/2} J_+ q^{J_z}, \quad H = 2J_z.$$

Proof. See [Su].

§5. Dual space of $U(su(2))_q$.

We have seen that our deformed universal algebra may be regarded as being in the dual space of the deformed function space A_q . In this section we would like to go the other way using the approach of Sudbery in [Su] and that of [MMNNSU]. The main idea is to consider a matrix representation of $U_q = U(su(2))_q$ and then consider the basis elements of $SU(2)_q$ as coordinate functions on the matrices in the representation of U_q .

Irreducible representation. One realisation of the classical angular momentum operators is

$$J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad J_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad 2J_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These matrices obey the usual commutator relations. In fact, these matrices even obey the deformed commutator relations.

We can now recall the definitions of a, b, c , and d as coordinate functions and apply these to our deformed universal algebra. We will use this definition to see if the desired commutation relations hold.

Star-Structure. We will give a star structure to U_q ($0 < q < 1$) by defining

$$q^{h/2*} = q^{h/2}, \quad x^* = y, \quad y^* = x.$$

This structure is readily seen to be compatible with the star structure of $SU(2)_q$. One simply checks that it is consistent with the generators of $SU(2)_q$ defined as coordinate functions.

A_q as the Dual Space of U_q . If we use the realization of U_q given at the beginning of the section, we can define functions a, b, c, d on U_q by

$$T(J_+) = \begin{pmatrix} a(J_+) & b(J_+) \\ c(J_+) & d(J_+) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and similarly we have

$$T(J_-) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and

$$T(J_z) = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$$

and

$$T(q^{J_z}) = \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}.$$

The relation $T(uv) = T(u)T(v)$ for all $u, v \in U_q$ implies that

$$ab(J_+) = (a \otimes b)(\Delta J_+)$$

and

$$a(J_+ J_-) = \Delta_A a(J_+ \otimes J_-) \text{ etc.}$$

We will show that the Hopf subalgebra generated by

$$\{a, \quad b, \quad c, \quad d\}$$

as defined and using the Hopf actions as given for A_q obeys the defining relations (*) of A_q .

Lemma 5-1. The counit of U_q commutes with the elements of U_q^* .

Proof. Lemma 3-2 of chapter 3.

Lemma 5-2 The counit acts as a unit for elements of U_q^* that act on the generators of U_q .

Proof. We have the desired result for each of the generators of U_q :

$$\epsilon\phi(J_z) = \epsilon \otimes \phi(1 \otimes J_z + J_z \otimes 1) = \phi(J_z)$$

$$\epsilon\phi(J_+) = \epsilon \otimes \phi(q^{-J_z} \otimes J_+ + J_+ \otimes q^{J_z}) = \phi(J_+)$$

$$\epsilon\phi(J_-) = \epsilon \otimes \phi(q^{-J_z} \otimes J_- + J_- \otimes q^{J_z}) = \phi(J_-)$$

so now we have to argue that the result can be extended.

Theorem 5-3. The counit is an identity in U_q^* .

Proof.

$$\epsilon\phi(J_+) = \phi(J_+).$$

Now $\epsilon\phi(J_+^n) = \epsilon \otimes \phi(\Delta J_+)^n$. But the coproduct of J_+ is $q^{-J_z} \otimes J_+ + J_+ \otimes q^{J_z}$. Only the first term is not sent to zero by ϵ so

$$\epsilon\phi(J_+^n) = \epsilon \otimes \phi(q^{-nJ_z} \otimes J_+^n) = \phi(J_+^n).$$

Taking this as a clue we find that

$$\begin{aligned}
 & \epsilon \phi(J_+^k J_-^l J_z^m q^{nJ_z}) \\
 &= (\epsilon \otimes \phi)(q^{-kJ_z} \otimes J_+^k)(q^{-lJ_z} \otimes J_-^l)(1 \otimes J_z^m)(q^{nJ_z} \otimes q^{nJ_z}) \\
 &= \epsilon(q^{(n-k-l)J_z}) \phi(J_+^k J_-^l J_z^m q^{nJ_z})
 \end{aligned}$$

which is $\phi(J_+^k J_-^l J_z^m q^{nJ_z})$.

Now we need to show that the counit ϵ is the same as the quantum determinant $ad - q^{-1}bc$. We first find that this is true on the generators.

Lemma 5-4. $ad - q^{-1}bc$ is the counit on the generators of U_q .

Proof.

$$(ad - q^{-1}bc)(J_z) = (a \otimes d)(1 \otimes J_z + J_z \otimes 1) = 0 = \epsilon(J_z),$$

$$(ad - q^{-1}bc)(J_+) = (a \otimes d)(q^{-J_z} \otimes J_+ + J_+ \otimes q^{J_z}) = 0 = \epsilon(J_+)$$

$$(ad - q^{-1}bc)(J_-) = (a \otimes d)(q^{-J_z} \otimes J_- + J_- \otimes q^{J_z}) = 0 = \epsilon(J_-)$$

$$(ad - q^{-1}bc)(q^{J_z}) = (a \otimes d)(q^{J_z} \otimes q^{J_z}) = 1 = \epsilon(q^{J_z})$$

$$(ad - q^{-1}bc)(q^{-J_z}) = (a \otimes d)(q^{-J_z} \otimes q^{-J_z}) = 1 = \epsilon(q^{-J_z})$$

so we conclude that $ad - q^{-1}bc = \epsilon$ for all of the generators of U_q .

Theorem 5-5. $ad - q^{-1}bc$ is the counit on U_q .

Proof. Suppose that DET_q is the counit on x and y in U_q . Then

$$\begin{aligned}
 (ad - q^{-1}bc)(xy) &= \Delta(ad - q^{-1}bc)(x \otimes y) \\
 &= (ad - q^{-1}bc) \otimes (ad - q^{-1}bc)(x \otimes y) \\
 &= (ad - q^{-1}bc)(x)(ad - q^{-1}bc)(y) \\
 &= \epsilon(x)\epsilon(y) \\
 &= \epsilon(xy).
 \end{aligned}$$

Thus we can argue by induction that the determinant is the counit on all of U_q . Thus the identity of the dual space is the same as the counit of the Hopf algebra.

We would like the defining relations (*) of A_q to hold. We already have that $ad - q^{-1}bc = 1$. We first see that the relations hold for the generators of U_q .

Claim. $ab = q^{-1}ba$ on the generators of U_q .

Proof.

$$ab(J_z) = (a \otimes b)(1 \otimes J_z + J_z \otimes 1) = 0,$$

$$ba(J_z) = (b \otimes a)(1 \otimes J_z + J_z \otimes 1) = 0,$$

so $ab(J_z) = q^{-1}ba(J_z)$.

$$ab(J_+) = (a \otimes b)(q^{-J_z} \otimes J_+ + J_+ \otimes q^{J_z}) = q^{-1/2},$$

$$ba(J_+) = (b \otimes a)(q^{-J_z} \otimes J_+ + J_+ \otimes q^{J_z}) = q^{1/2},$$

$$\text{so } ab(J_+) = q^{-1}ba(J_+).$$

$$ab(J_-) = (a \otimes b)(q^{-J_z} \otimes J_- + J_- \otimes q^{J_z}) = 0$$

$$ba(J_-) = (b \otimes a)(q^{-J_z} \otimes J_- + J_- \otimes q^{J_z}) = 0$$

$$\text{so } ab(J_-) = q^{-1}ba(J_-).$$

$$ab(q^{J_z}) = (a \otimes b)(q^{J_z} \otimes q^{J_z}) = 0$$

$$ba(q^{J_z}) = (b \otimes a)(q^{J_z} \otimes q^{J_z}) = 0$$

$$\text{so } ab(q^{J_z}) = q^{-1}ba(q^{J_z}).$$

$$ab(q^{-J_z}) = (a \otimes b)(q^{-J_z} \otimes q^{-J_z}) = 0$$

$$ba(q^{-J_z}) = (b \otimes a)(q^{-J_z} \otimes q^{-J_z}) = 0$$

$$\text{so } ab(q^{-J_z}) = q^{-1}ba(q^{-J_z}).$$

Thus we conclude that $ab = q^{-1}ba$ for the generators of U_q .

Claim. $ac = q^{-1}ca$ on the generators of U_q .

Proof.

$$ac(J_z) = (a \otimes c)(1 \otimes J_z + J_z \otimes 1) = 0,$$

$$ca(J_z) = (c \otimes a)(1 \otimes J_z + J_z \otimes 1) = 0,$$

$$\text{so } ac(J_z) = q^{-1}ca(J_z).$$

$$ac(J_+) = (a \otimes c)(q^{-J_z} \otimes J_+ + J_+ \otimes q^{J_z}) = 0,$$

$$ca(J_+) = (c \otimes a)(q^{-J_z} \otimes J_+ + J_+ \otimes q^{J_z}) = 0,$$

$$\text{so } ac(J_+) = q^{-1}ca(J_+).$$

$$ac(J_-) = (a \otimes c)(q^{-J_z} \otimes J_- + J_- \otimes q^{J_z}) = q^{-1/2}$$

$$ca(J_-) = (c \otimes a)(q^{-J_z} \otimes J_- + J_- \otimes q^{J_z}) = q^{1/2}$$

$$\text{so } ac(J_-) = q^{-1}ca(J_-).$$

$$ac(q^{J_z}) = (a \otimes c)(q^{J_z} \otimes q^{J_z}) = 0$$

$$ca(q^{J_z}) = (c \otimes a)(q^{J_z} \otimes q^{J_z}) = 0$$

$$\text{so } ac(q^{J_z}) = q^{-1}ca(q^{J_z}).$$

$$ac(q^{-J_z}) = (a \otimes c)(q^{-J_z} \otimes q^{-J_z}) = 0$$

$$ca(q^{-J_z}) = (c \otimes a)(q^{-J_z} \otimes q^{-J_z}) = 0$$

$$\text{so } ac(q^{-J_z}) = q^{-1}ca(q^{-J_z}).$$

Thus we conclude that $ac = q^{-1}ca$ for the generators of U_q .

Claim. $bc = cb$ on the generators of U_q .

Proof.

$$bc(J_z) = (b \otimes c)(1 \otimes J_z + J_z \otimes 1) = 0,$$

$$cb(J_z) = (c \otimes b)(1 \otimes J_z + J_z \otimes 1) = 0,$$

so $bc(J_z) = cb(J_z)$.

$$bc(J_+) = (b \otimes c)(q^{-J_z} \otimes J_+ + J_+ \otimes q^{J_z}) = 0,$$

$$cb(J_+) = (c \otimes b)(q^{-J_z} \otimes J_+ + J_+ \otimes q^{J_z}) = 0,$$

so $bc(J_+) = cb(J_+)$.

$$bc(J_-) = (b \otimes c)(q^{-J_z} \otimes J_- + J_- \otimes q^{J_z}) = 0$$

$$cb(J_-) = (c \otimes b)(q^{-J_z} \otimes J_- + J_- \otimes q^{J_z}) = 0$$

so $bc(J_-) = cb(J_-)$.

$$bc(q^{J_z}) = (b \otimes c)(q^{J_z} \otimes q^{J_z}) = 0$$

$$cb(q^{J_z}) = (c \otimes b)(q^{J_z} \otimes q^{J_z}) = 0$$

so $bc(q^{J_z}) = cb(q^{J_z})$.

$$bc(q^{-J_z}) = (b \otimes c)(q^{-J_z} \otimes q^{-J_z}) = 0$$

$$cb(q^{-J_z}) = (c \otimes b)(q^{-J_z} \otimes q^{-J_z}) = 0$$

so $bc(q^{-J_z}) = cb(q^{-J_z})$.

Thus we conclude that $bc = cb$ for the generators of U_q .

Claim. $bd = q^{-1}db$ on the generators of U_q .

Proof.

$$bd(J_z) = (b \otimes d)(1 \otimes J_z + J_z \otimes 1) = 0,$$

$$db(J_z) = (d \otimes b)(1 \otimes J_z + J_z \otimes 1) = 0,$$

so $bd(J_z) = q^{-1}db(J_z)$.

$$bd(J_+) = (b \otimes d)(q^{-J_z} \otimes J_+ + J_+ \otimes q^{J_z}) = q^{-1/2},$$

$$db(J_+) = (d \otimes b)(q^{-J_z} \otimes J_+ + J_+ \otimes q^{J_z}) = q^{1/2},$$

so $bd(J_+) = q^{-1}db(J_+)$.

$$bd(J_-) = (b \otimes d)(q^{-J_z} \otimes J_- + J_- \otimes q^{J_z}) = 0$$

$$db(J_-) = (d \otimes b)(q^{-J_z} \otimes J_- + J_- \otimes q^{J_z}) = 0$$

so $bd(J_-) = q^{-1}db(J_-)$.

$$bd(q^{J_z}) = (b \otimes d)(q^{J_z} \otimes q^{J_z}) = 0$$

$$db(q^{J_z}) = (d \otimes b)(q^{J_z} \otimes q^{J_z}) = 0$$

so $bd(q^{J_z}) = q^{-1}db(q^{J_z})$.

$$bd(q^{-J_z}) = (b \otimes d)(q^{-J_z} \otimes q^{-J_z}) = 0$$

$$db(q^{-J_z}) = d \otimes b(q^{-J_z} \otimes q^{-J_z}) = 0$$

so $bd(q^{-J_z}) = q^{-1}db(q^{-J_z})$.

Thus we conclude that $bd = q^{-1}db$ for the generators of U_q .

Claim. $cd = q^{-1}dc$ on the generators of U_q .

Proof.

$$cd(J_z) = (c \otimes d)(1 \otimes J_z + J_z \otimes 1) = 0,$$

$$dc(J_z) = (d \otimes c)(1 \otimes J_z + J_z \otimes 1) = 0,$$

so $cd(J_z) = q^{-1}dc(J_z)$.

$$cd(J_+) = (c \otimes d)(q^{-J_z} \otimes J_+ + J_+ \otimes q^{J_z}) = 0$$

$$dc(J_+) = (d \otimes c)(q^{-J_z} \otimes J_+ + J_+ \otimes q^{J_z}) = 0,$$

so $cd(J_+) = q^{-1}dc(J_+)$.

$$cd(J_-) = (c \otimes d)(q^{-J_z} \otimes J_- + J_- \otimes q^{J_z}) = q^{-1/2}$$

$$dc(J_-) = (d \otimes c)(q^{-J_z} \otimes J_- + J_- \otimes q^{J_z}) = q^{1/2}$$

so $cd(J_-) = q^{-1}dc(J_-)$.

$$cd(q^{J_z}) = (c \otimes d)(q^{J_z} \otimes q^{J_z}) = 0$$

$$dc(q^{J_z}) = (d \otimes c)(q^{J_z} \otimes q^{J_z}) = 0$$

so $cd(q^{J_z}) = q^{-1}dc(q^{J_z})$.

$$cd(q^{-J_z}) = (c \otimes d)(q^{-J_z} \otimes q^{-J_z}) = 0$$

$$db(q^{-J_z}) = (d \otimes b)(q^{-J_z} \otimes q^{-J_z}) = 0$$

so $cd(q^{-J_z}) = q^{-1}dc(q^{-J_z})$.

Thus we conclude that $cd = q^{-1}dc$ for the generators of U_q .
Now we extend the result to all of U_q .

Extending arguments. Now we have to extend our results to the entire algebra U_q from the results on the generators.

Theorem 5-6. The defining relations (*) of A_q hold.

Proof. First, suppose that all of the relations hold on x and y in U_q . Then, since Δ preserves the desired relations (it is a homomorphism):

$$ab(xy) = \Delta(ab)(x \otimes y) = (\Delta a)(\Delta b)(x \otimes y)$$

to get that

$$ab(xy) = (\Delta a)(\Delta b)(x \otimes y) = q^{-1}(\Delta b)(\Delta a)(x \otimes y) = q^{-1}ba(xy).$$

Then by induction the relation above holds. The same argument works for all of the other relations.

Remark. We have seen that $U_q = U(su(2))_q$ is in the Hopf dual of A_q and vice versa. Suppose that we had started with the algebra A ($q = 1$) and the algebra $U = U(su(2))$. Then A is the Hopf dual of U because all of the linear functions on U can be made from the component functions in A .

As U is deformed we saw that A_q is in the Hopf dual of U_q and that it satisfies the required commutation relations of Woronowicz. In this sense then A_q is a 'natural' Hopf dual for U_q .

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