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ON SOME INFINITE DIMENSIONAL MARTINGALE PROBLEMS

AND RELATED

STOCHASTIC EVOLUTION EQUATIONS

by

Shoichi Mizuno

A thesis

submitted to the Faculty of Graduate Studies and Research

in partial fulfillment of the requirements for the degree

of

Doctor of Philosophy

Department of Mathematics

Carleton University

Ottawa, Canada

June, 1978

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The undersigned recommend to the Faculty of
Graduate Studies acceptance of the thesis

"ON SOME INFINITE DIMENSIONAL MARTINGALE PROBLEMS AND
RELATED STOCHASTIC EVOLUTION EQUATIONS"

submitted by Shoichi Mizuno, M.Sc.
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ABSTRACT

For a stochastic evolution equation, expressed in Ito form,

$$(S. E. E.) \quad dX_t = A(X)dt + F(X)dW_t,$$

we develop a method which consists of two stages. In the first stage the equations

$$(1) \quad dX_t = A(X)dt \quad (\text{deterministic})$$

and

$$(2) \quad dX_t = F(X)dW_t \quad (\text{a martingale problem})$$

are solved separately and in the second stage they are combined by using the product formula due to Trotter. It is important to note that the equation (2) should be considered as a martingale problem. The martingale problems specified by the following martingale equations

$$(i) \quad < (X_t - X_0, f)(X_t - X_0, g) > = \int_0^t (Q_s X_s, fg) ds.$$

$$(ii) \quad < (X_t - X_0, f)(X_t - X_0, g) > = \int_0^t (Q_s(x, y) X_s(x) X_s(y), f(x)f(y)) ds$$

$$(iii) \quad < (X_t - X_0, f)(X_t - X_0, g) > = \int_0^t \left[(X_s, fg) - (X_s, f)(X_s, g) \right] ds.$$

and related stochastic evolution equations are discussed.

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We owe the motivation of the problem (i) and related stochastic evolution equation to Dawson, the problem (ii) and related stochastic evolution equation to Shimizu and the problem (iii) and related stochastic evolution equation to Fleming.

The purpose of this thesis is to develop from a space-time homogeneous model a space-time non-homogeneous model for (i), from a spacially homogeneous time non-homogeneous model a space-time non-homogeneous model for (ii) and from a finite dimensional model of Sato an infinite dimensional model for (iii).

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INTRODUCTION

In statistical science we observe phenomena and try to discover the laws behind them. Many phenomena are random, such as the movement of pollen floating in water observed under a microscope, called Brownian motion after R. Brown who was the first to observe it. We are interested in constructing mathematical models to describe such phenomena, especially spatially distributed random phenomena, using probability theory. We owe the following philosophical point of view to Cramér [5]:

"The probability concept should be introduced by a purely mathematical definition, from which its fundamental properties and the classical theorems are deduced by purely mathematical operations. ... Against such a mathematical theory, no objections can be valid except on mathematical grounds. On the other hand, it should be emphasized that the mathematical theory does not prove anything about the real events that will occur. Probability formulas are just as unable to dictate the behavior of real events as are the formulas of classical mechanics to prescribe that the stars must attract one another according to the Newton law. It is only experience that can guide us here and show if our mathematical model yields an acceptable approximation of our observations."

Brownian motion has become a highly successful mathematical model, since Wiener [28] (see also commentary of Ito) constructed the mathematical foundation. The probability measure P on the continuous function space $C([0, \infty); \mathbb{R}^d)$, which describes Brownian motion, now carries the name of Wiener measure. Mathematical models derived from Brownian motion are called Brownian functionals (Hida [10]).

One important characterization of Brownian motion $B(t)$ comes from its martingale property. That is, setting $d=1$,

(i) $B(t)$ is a continuous square integrable martingale,

(ii) $(B(t))^2 - t$ is a martingale,

that is, the Doob-Meyer decomposition is

$$(B(t))^2 = ((B(t))^2 - t) + t,$$

t is the natural increasing process associated with $B(t)$ in this decomposition, and

$$(iii) \quad P \left\{ \lim_{n \rightarrow \infty} \sum_{k=1}^n \left[B\left(\frac{k}{n}t\right) - B\left(\frac{k-1}{n}t\right) \right]^2 = t \right\} = 1.$$

Wiener [28] called the increasing process t the, limit in (iii), "a nondifferentiability coefficient". For general martingales, we refer to Meyer [18], [19], and to Kunita-Watanabe [15] for properties (i), (ii) (Doob-Meyer decomposition) and (iii) the increasing process as the limiting quadratic variation.

In this dissertation, the increasing process is reviewed in a special case which covers the closure of those processes derived from Ito stochastic differential equations

$$dX_t = a(t, X)dB_t$$

with bounded coefficients $a(t, X)$, which are fundamental Brownian functionals.

If we specify the increasing process, that is,

$$\langle X_t^2 \rangle = A(X),$$

where $\langle X_t^2 \rangle$ denotes the increasing process in the Doob-Meyer decomposition, is given, then the "martingale problem" is to establish the existence of the corresponding martingale X_t . For example, for martingale problems on $C([0, \infty); \mathbb{R}^d)$ we refer to Stroock-Varadhan [23], on $C([0, \infty); H_{\text{weak}})$ to Viot [25] where H denotes a Hilbert space, and on $C([0, \infty); \mathbb{M})$ to Fleming-Viot [9] where $\mathbb{M}$ denotes a space of measures.

To construct a mathematical model by solving a martingale problem usually requires a compactness method. From this view point, weak convergence of probability measures on Polish spaces, especially on $C([0, \infty); \mathbb{M})$ is reviewed in this dissertation. For the Kolmogorov Theorem (continuity criteria in terms of a moment condition) we refer to Billingsley [2].

The following specific martingale problems are solved in this dissertation.

- (1) On $C([0, \infty); H)$, where H is a Hilbert space,

$$\langle (X_t - X_0, f)(X_t - X_0, g) \rangle = \int_0^t (Q_s(x, y), f(x)g(y)) ds.$$

- (2) On $C([0, \infty); \mathbb{M})$, where $\mathbb{M}$ is a space of measures,

$$\langle (X_t - X_0, f)(X_t - X_0, g) \rangle = \int_0^t (Q_s X_s, fg) ds.$$

(3) On $C([0, \infty); H)$

$$\langle (X_t - X_0, f), (X_t - X_0, g) \rangle = \int_0^t (b_s(x, y) X_s(x) X_s(y), f(x) f(y)) ds.$$

(4) On $C([0, \infty); \mathbb{M})$, for a, b constants,

$$\langle (X_t - X_0, f), (X_t - X_0, g) \rangle = \int_0^t [a^2 (X_s, fg) + b^2 (X_s, f)(X_s, g)] ds.$$

(5) On $C([0, \infty); \mathbb{M}_1)$ where $\mathbb{M}_1$ is the subspace of $\mathbb{M}$ with total measure 1,

$$\langle (X_t - X_0, f), (X_t - X_0, g) \rangle = \int_0^t [(X_s, fg) - (X_s, f)(X_s, g)] ds.$$

The mathematical model from the martingale problem (1) is a space-time correlated nonstationary Gaussian random field, from (2) a space-time nonhomogeneous completely random process, from (3) a space-time nonhomogeneous positive random field, from (4) a mixed type combining (2) and (3), and from (5) a gene frequency model in infinite dimensions.

The increasing processes in our martingale problems are all absolutely continuous with respect to Lebesgue measure. It suggests that the processes are Brownian functionals where Ito calculus is essential, especially Ito's formula for L_2 -martingales (refer to Kunita-Watanabe [15, Theorem 2.2 page 215]).

Once these martingale problems are solved, then we consider stochastic evolution equations expressed in Ito form

$$dX_t = A(X)dt + F(X)dW_t$$

where $dX_t = A(X)dt$ is a deterministic equation and $dX_t = F(X)dW_t$ is meant to denote the above corresponding martingale equation. To construct solutions we employ Trotter's product formula (cf. [24]).

The mathematical model derived from the related stochastic evolution equation for the martingale problem (2) is a space-time ^{non}homogeneous multiplicative process. Our model includes Dawson's model (see [7, page 36, Measure Diffusion Processes]) and Watanabe's model (see [26, page 157, The case of Diffusion Processes]). The model from the related stochastic evolution equation for the martingale problem (3) is motivated by Balakrishnan [1] and Dawson [7], for which Shimizu ^[22] obtained a rigorous explicit solution for spatially homogeneous time-nonhomogeneous white noise by using methods developed by Hida [10], [11]. Our model is for the more general case of space-time nonhomogeneous white noise. In a similar manner related stochastic evolution equations are considered for other martingale problems.

In the last chapter we describe some open problems which arose through our investigation.

CHAPTER I. PRELIMINARIES

§1.0- Introduction

The following known results are reviewed in this chapter.

Let $\mathcal{M}(S)$ (which we denote simply by $\mathcal{M}$ in case of no confusion,) denote the space of positive Radon measures, which we simply call Radon measures, equipped with the vague (weak*) topology on a locally compact separable metric space S .

(See Choquet [4] for the characterization: μ is a Radon measure iff μ is a positive linear form on $C_c(S)$, where $C_c(S)$ denotes the space of continuous functions with compact support on S .) Let $C([0, \infty); \mathcal{M})$, which we denote by V , denote the space of $\mathcal{M}(S)$ -valued continuous functions on $[0, \infty)$ equipped with the topology of uniform convergence on every bounded interval.

$\mathcal{M}(S)$, V are Polish spaces, i.e., metrizable to be complete and separable.

Let M denote a Polish space. Let $P(M)$ denote the space of probability measures on M equipped with the topology of weak convergence. $P(M)$ is metrizable. The family $\{P_n \in P(M); n = 1, 2, \dots\}$ is relatively compact iff it is uniformly tight.

One characterization of uniform tightness in $P(V)$ is reviewed which will be used in later chapters and which is based on Kolmogorov's theorem on uniform tightness in $P(C([0, \infty); \mathbb{R}))$ (see Billingsley [2]).

The characteristic functional for $P \in P(\mathbb{M})$ and Prohorov's theorem on uniform tightness in $P(\mathbb{M})$ are reviewed.

Let $P \in P(V)$ be given. Let $f \in C_c(S)$ be given. Let the process $(t, v) \rightarrow (v(t), f)$, where v represents the element of V , $v(t)$ the value in $\mathbb{M}$ at time t of v , and $(v(t), f)$ the integral of f by the measure $v(t)$, be called natural, denoted simply by $(v(t), f)$. Let P be called L_1 (L_2 , respectively) if the natural process $(v(t), f)$ is L_1 (L_2) for all f and t . Let P be called an L_1 (L_2 , respectively) martingale if the natural process $(v(t), f)$ is an L_1 (L_2)-martingale with respect to the increasing family of sub σ -fields $\mathcal{B}_t(V)$, $0 \leq t < \infty$, of $\mathcal{B}(V)$ where $\mathcal{B}_t(V)$ denotes the sub σ -field generated by $(v(s_1), f_1), (v(s_2), f_2), \dots, (v(s_n), f_n)$, $0 \leq s_1 < s_2 < \dots < s_n \leq t$, $f_j \in C_c(S)$, $1 \leq j \leq n$, $n = 1, 2, \dots$.

Let $P_n \in P(V)$ be L_1 (L_2 , respectively), $n = 1, 2, \dots$. Let the family $\{P_n\}$ be called uniformly L_1 (L_2) integrable if the following Radon measures on $[0, \infty)$

$$I_n(|(v(t), f)|; A): A \rightarrow \int_{\{|(v(t), f)| \in A\}} |(v(t), f)| P_n(dv)$$

$$\left(I_n((v(t), f)^2; A): A \rightarrow \int_{\{(v(t), f)^2 \in A\}} (v(t), f)^2 P_n(dv) \right).$$

are uniformly tight, i.e.,

$$\lim_{N \rightarrow \infty} \sup_n I_n(|(v(t), f)|; [N, \infty)) = 0$$

$$\left(\lim_{N \rightarrow \infty} \sup_n I_n((v(t), f)^2; (N, \infty)) = 0 \right) \text{ for every } t \geq 0.$$

Let $P_n \in P(V)$ $n = 1, 2, \dots$ be convergent weakly to $P \in P(V)$. Let P_n be uniformly L_1 (L_2 , respectively) integrable. Then the weak limit P is again an L_1 (L_2) martingale.

Let $P \in P(V)$ be an L_2 martingale. Let $M_2^C(\mathcal{B}_t)$ denote the space of $\mathcal{B}_t(V)$ adapted continuous square integrable martingales.

$M_2^C(\mathcal{B}_t)$ is a complete metric space.

The increasing process in the Doob-Meyer decomposition theorem for L_2 martingales is reviewed in the following case.

Let $X(t, v) \in M_2^C(\mathcal{B}_t)$, and satisfy the condition

$$E[(X(t) - X(s))^2 | \mathcal{B}_s] \leq N(T)(t-s) \text{ for all } s < t \in [0, T],$$

where $N(T)$ is a constant which depends only on T and the process X_t . Then the increasing process $\langle X(t, v)^2 \rangle$ exists uniquely,

$$\langle X(t, v)^2 \rangle = \lim_n \sum_{k=1}^n \left[X\left(\frac{k}{n}t\right) - X\left(\frac{k-1}{n}t\right) \right]^2 \text{ a.e.,}$$

and

$$(X(t, v))^2 - \langle X(t, v)^2 \rangle$$

is a martingale.

Let $M_{2,b}^C(\mathcal{B}_t)$ denote the subspace of $M_2^C(\mathcal{B}_t)$ consisting of above $X(t, v)$'s. Then the increasing process is uniquely extended to the closure of $M_{2,b}^C(\mathcal{B}_t)$.

(In general we refer to Meyer [18], [19], and Kunita-Watanabe [15]).

Let $P \in P(V)$ be an L_1 martingale. Let P_0 denote the probability distribution on M induced by $v(o)$. Let P_0 be concentrated on the atomic measure $\sum_j c_j \delta_{x_j}$. Then no further atoms are created from the martingale model. In addition the atoms do not change their locations.

§1.1 The spaces $M(S)$, V .

Notation 1.1.1

Let $S = (S, d)$ denote a locally compact metric space which is second countable; equivalently, a locally compact separable metric space, with the metric d .

Notation 1.1.2

Let $K_n : n = 1, 2, \dots$ denote a sequence of compact subsets of S such that (1) $K_n \subset K_{n+1}^o$ and $\bigcup_{n=1}^{\infty} K_n = S$, where K_{n+1}^o denotes the interior of K_{n+1} .

Example:

Let $\{y_n : n = 1, 2, \dots\}$ be dense in S . Let $r_n = (\sup\{r : B(y_n; r) \text{ is compact}\}) \wedge 1$, where $B(y_n; r)$ denotes the closed ball with the center y_n the radius r , $a \wedge b$ denotes the minimum of a, b . Put $K_n = \bigcup_{j=1}^n B(y_j; r_j (1-n^{-1}))$.

Notation 1.1.3

Let $\chi_n; n = 1, 2, \dots$ denote a sequence of continuous functions such that (1) $0 \leq \chi_n(x) \leq 1$, for all $x \in S$, (2) $\chi_n = 1$ on K_n and (3) $\chi_n = 0$ on K_{n+1}^c , where K_{n+1}^c denotes the complement of K_{n+1} .

Notation 1.1.4

Let $C_c(K_n) = (C_c(K_n), \|\cdot\|)$ denote the Banach space of continuous functions with support in K_n endowed with sup norm $\|\cdot\|$. (Remark; $C_c(K_n)$ is separable.)

Notation 1.1.5

Let $C_c(S)$ be the union of $C_c(K_n)$, $n = 1, 2, \dots$, endowed with the inductive limit topology.

Notation 1.1.6

Let $C_c^+(K_n)$ denote the cone of positive functions of $C_c(K_n)$.

Notation 1.1.7

Let $C_c^+(S)$ denote the cone of positive functions of $C_c(S)$.

Lemma 1.1.8

There exists a countable family F_{n+1} in $C_c^+(K_{n+1})$ such that F_{n+1} approximates $C_c^+(K_n)$ uniformly from below and from above.

Proof:

Let $\{\varphi_j^n: j = 1, 2, \dots\}$ be dense in $C_c^+(K_n)$. Put $F_{n+1} = \{(\varphi_j^n - q) \vee 0, (\varphi_j^n + q)\chi_n: j = 1, 2, \dots, q \geq 0 \text{ and rational}\}$, where $a \vee b$ denotes the maximum of a, b .

Notation 1.1.9

Let F denote the union of F_{n+1} , $n = 1, 2, \dots$

Remark 1.1.10

Let $C_0(S)$ denote the Banach space of continuous functions vanishing at ∞ : the point at infinity in the one point compactification of S , endowed with sup norm. Let $C_0^+(S)$ denote the cone of positive functions of $C_0(S)$. Then F approximates $C_0^+(S)$ uniformly from below. However there does not exist a countable family in $C_0^+(S)$ which approximates $C_0^+(S)$ uniformly from above.

Notation 1.1.11

Let $\mathcal{M} = \mathcal{M}(S, \text{vague})$ denote the space of positive Radon measures on S , equivalently, the space of positive linear functionals on $C_c(S)$, endowed with the vague (weak*) topology. (Remark: $\mathcal{M}$ is metrizable to be complete and separable.)

An example of a metric, d , on $\mathcal{M}(S)$ is given as follows.

Define:

$$d(\mu, \nu) = \sum_{n=1}^{\infty} 2^{-n} (|\langle \mu, f_n \rangle - \langle \nu, f_n \rangle| \wedge 1),$$

where

$$\mu, \nu \in \mathcal{M}, \{f_n : n = 1, 2, \dots\} = F \text{ and } \langle \mu, f_n \rangle = \int f_n(x) \mu(dx).$$

(completeness).

It suffices to show that if μ_n is Cauchy, then (μ_n, g) converges to a finite limit for each $g \in C_c^+(S)$. Let $g \in C_c^+(K_n)$. Let $p \in F_{n+2}$ be such that $\chi_{n+1} \leq p$. Let $\epsilon > 0$ be given. Let $f, g \in F_{n+1}$ be such that $0 \leq f \leq g \leq h$ and $0 \leq h - f \leq \epsilon p$. Then $\overline{\lim}(\mu_n, g) - \underline{\lim}(\mu_n, g) \leq \overline{\lim}(\mu_n, h) - \underline{\lim}(\mu_n, f) \leq \epsilon \overline{\lim}(\mu_n, p)$.

Since ϵ is arbitrary, $\lim(\mu_n, g)$ exists.

Remark 1.1.12

Let $K = \{\mu: (\mu, \chi_n) \leq k_n \text{ for given } k_n > 0, n = 1, 2, \dots\}$. Then K is compact in $\mathcal{M}(S)$.

Definition 1.1.13

A topological space is said to be Polish if its topology is metrizable to be complete and separable.

Using this terminology, $\mathcal{M}(S)$ is Polish.

Notation 1.1.14

Let V denote the space of $\mathcal{M}(S)$ -valued continuous functions on $[0, \infty)$, i.e., $V = C([0, \infty); \mathcal{M}(S))$, endowed with the uniform convergence topology on every bounded interval.

Lemma 1.1.15

V is Polish.

Proof:

$$\text{Define } d(\mu(\cdot), \nu(\cdot)) = \sum_{n=1}^{\infty} 2^{-n} \left[\left(\max_{0 \leq t \leq n} d(\mu(t), \nu(t)) \right) \wedge 1 \right].$$

Then this d gives a corresponding metric on V . Completeness is immediate.

(separability) Let $\mu, \nu \in \mathcal{M}(S)$. Then the segment which connects μ and ν lies in $\mathcal{M}(S)$. And on the segment, $d(\mu, a\mu + b\nu) \leq d(\mu, \nu)$, $a+b = 1$, $a \geq 0$, $b \geq 0$. Therefore $\mathcal{M}(S)$ -valued polygonal lines are dense in V , so that V is separable.

§1.2 The spaces $P(M)$, $P(V)$.

Notation 1.2.1

Let $P(M)$, $P(V)$, respectively, denote the space of probability measures on $(M, \mathcal{B}(M))$, $((V, \mathcal{B}(V)))$, where $\mathcal{B}(M)$, $(\mathcal{B}(V))$, denotes the topological σ -field of M , (V) .

Since M , V are Polish, we now review some important properties of probability measures on Polish spaces.

Notation 1.2.2

Let M denote a Polish space. Let $\mathcal{B}(M)$ denote the topological σ -field of M . Let $P(M)$ denote the space of probability measures on $(M, \mathcal{B}(M))$. Let $C_b(M)$ denote the space of bounded continuous functions on M .

Definition 1.2.3 (the weak topology on $P(M)$)

$P_n \in P(M)$ is said to converge to $P \in P(M)$ weakly if $\int f(x)P_n(dx)$ converges to $\int f(x)P(dx)$ for every $f \in C_b(M)$, denoted by $P_n \xrightarrow{w} P$.

Remark 1.2.4

$P \in P(M)$ is uniquely determined by the values $\int f(x)P(dx)$, $f \in C_b(M)$.

Remark 1.2.5 (regularity)

Let $A \in \mathcal{B}(M)$. Then $\sup\{P(F): A \supset F; \text{closed}\} = P(A) = \inf\{P(G): A \subset G; \text{open}\}$. (This property of P is called regularity).

Remark 1.2.6 (tightness)

In the above Remark 1.2.5 the condition " F ; closed" can be replaced by " K ; compact". (This is called tightness of P .)

The above Remark 1.2.6 is based on the fact: $\bigcap_{n=1}^{\infty} \bigcup_{j=1}^{N(n)} B(x_j^n; r_j^n)$ is relatively compact, if $\max_{1 \leq j \leq N(n)} r_j^n$ converges to 0 as n tends to infinity.

Remark 1.2.7 (Equivalent Conditions)

The following five statements are equivalent.

- (1) $P_n \xrightarrow{w} P$.
- (2) $\int f(x) P_n(dx) \rightarrow \int f(x) P(dx)$ for each $f \in C_{b,u}(M)$, where $C_{b,u}(M)$ denotes the space of bounded uniformly continuous functions on M .
- (3) $\lim P_n(F) \leq P(F)$, for every F : closed.
- (4) $\lim P_n(G) \geq P(G)$, for every G : open.
- (5) $\lim P_n(A) = P(A)$, if $P(\partial A) = 0$, where ∂A denotes the boundary of A .

Proof:

(1) $\Rightarrow$ (2), (1) $\Rightarrow$ (3) $\Leftrightarrow$ (4) $\Rightarrow$ (5) are immediate, (2) $\Rightarrow$ (3) follows, since for every pair of F : closed $\subset G$: open, there exists a uniformly continuous function $f(x)$ which is 1 on F , 0 outside of G and $0 \leq f(x) \leq 1$ everywhere.

(Example. Let $f(x) = (1 - n \cdot d(F, x)) \vee 0$ for n large.) ((5) $\Rightarrow$ (1)). Let $\epsilon > 0$ be given. We note that $\#\{c: P\{x: f(x) = c\} > 0\}$ is countable. Avoiding such c 's, let

c_j : $j = 1, 2, \dots$ be of $c_{j+1} - c_j \leq \epsilon$ and $\bigcup E_j = M$ where $E_j = \{c_j \leq f(x) < c_{j+1}\}$.

(Remark: $P(\partial E_j) = 0$.) Consider $\sum c_j I_{E_j}(x) \leq f(x) \leq \sum c_{j+1} I_{E_j}(x)$. Integrating

the equation with respect to P_n , letting $n \rightarrow \infty$, considering the arbitrariness of ϵ , then (5) $\Rightarrow$ (1) follows. $\square$

Remark 1.2.8

$P(M)$ with the weak topology is metrizable, i.e., there exists a metric, d , on $P(M)$ such that $P_n \xrightarrow{w} P \Leftrightarrow d(P_n, P) \rightarrow 0$.

Example:

Let f_j be such that (1) continuous on M , (2) $0 \leq f_j \leq 1$, and (3) f_j generates topology. The following continuous map from M into a compact metric space $[0,1]^\infty$:

$$\begin{aligned}\pi : M &\rightarrow [0,1]^\infty \\ x &\rightarrow (f_j(x))\end{aligned}$$

gives a homeomorphism between M and the image $\pi(M)$. The Banach space $C_{b,u}(\pi(M))$ endowed with sup norm is separable. Let $\{g_n : n = 1, 2, \dots\}$ be countably dense in $C_{b,u}(\pi(M))$. Define a metric, d , on $P(M)$ as follows.

$$d(P, Q) = \sum_{n=1}^{\infty} 2^{-n} (|\int g_n(\pi(x))P(dx) - \int g_n(\pi(x))Q(dx)| \wedge 1).$$

Definition 1.2.9 (Uniform Tightness)

Let a family $\{P_n : n = 1, 2, \dots\}$ be called uniformly tight if for each $\epsilon > 0$, there exists a compact set $K = K(\epsilon)$ such that $P_n(K) > 1 - \epsilon$ holds for all $n = 1, 2, \dots$.

Remark 1.2.10 (Important)

A family $\{P_n \in P(M) : n = 1, 2, \dots\}$ is relatively compact iff it is uniformly tight.

Proof:

For necessity, we refer to Billingsley [2: page 37, THEOREM 1.2].

Sufficiency: Let $K_N: N = 1, 2, \dots$ be compact such that $K_N \subset K_{N+1}$ and

$P_n(K_N) > 1 - N^{-1}$ for all $n = 1, 2, \dots$. Let $P_n|_{K_N}$ be the restriction of P_n on K_N .
relatively

Then $\{P_n|_{K_N}: n = 1, 2, \dots\}$ is compact in the dual topology of $C(K_N)$. From the diagonal argument, there exists a subsequence $P_{n_k}: k = 1, 2, \dots$ such that P_{n_k} is

convergent on every K_N . Let $P_N = \lim_k P_{n_k}|_{K_N}$ on K_N . Let P be the projective limit of the sub-stochastic measures P_N . Then P is a limit point,

Now we return to the spaces $\mathbb{M}$, C , and characterize compact sets.

Remark 1.2.11

A countable family $\{\mu_n\}$ in $\mathbb{M}$ is relatively compact iff (μ_n, f) is relatively compact in $\mathbb{R}^1$ for each $f \in C_c(S)$. (The topology of $\mathbb{M}$ is point-f-wise convergence.) (Remark: The condition, " $C_c(S)$ " can be replaced by " $\mathcal{F}$ ".)

Notation 1.2.12

Let $W(= C([0, \infty); \mathbb{R}))$ denote the space of continuous functions on $[0, \infty)$ equipped with the uniform convergence topology on every bounded interval.

(Remark: W is a Polish space.)

Lemma 1.2.13

A countable family $\{\mu_n(t)\}$ in V is relatively compact iff for each $f \in C_c(S)$, $\{(\mu_n(t), f)\}$ is relatively compact in W . (Remark: the condition " $C_c(S)$ " can be replaced by " $\mathcal{F}$ ".)

Proof:

This follows immediately from the metric, d , on V .

Combining two concepts, we have the following result.

Lemma 1.2.14

$P_n \in P(M)$ ($P_n \in P(V)$, respectively) is uniformly tight iff $(\pi_f(P_n))$ is uniformly tight in $P(R^1)(P(W))$ for every $f \in C_c(S)$. (The condition " $f \in C_c(S)$ " can be replaced by " $f \in F$ ".)

THEOREM 1.2.15

Suppose the countable family $\{P_n\} \subset P(V)$ satisfies the condition:

for each $f \in F$,

(1) there exists positive constants $R = R(f)$ and $r = r(f)$ such that for all n ,

$$E_{P_n} [(\mu(0), f)^r] \leq R,$$

and

(2) there exist constants $a = a(f) > 0$, $b = b(f) > 0$, $R_T(f) > 0$; $T = 1, 2, \dots$ such that for all n and T ,

$$E_{P_n} |(\mu(t), f) - (\mu(s), f)|^a \leq R_T |t-s|^{1+b} \text{ if } s, t \in [0, T].$$

Then the family $\{P_n\}$ is uniformly tight.

Proof:

This follows from the fact that $(\pi_f(P_n))$ is uniformly tight in $P(W)$ for each $f \in F$ from Kolmogorov's theorem (see Billingsley [2: page 95, THEOREM 12.3]).

In Lemma 1.2.15 and Theorem 1.2.16, $\pi_f(P)$ denotes the probability measure induced by $(v(t), f)$.

§1.3 Characteristic functionals and random measures.

Definition 1.3.1

Let $P \in \mathcal{P}(\mathbb{M})$ be given. The functional on $C_c(S)$:

$$f \rightarrow \text{Ch}(f) = \int [\exp i(\mu, f)] P(d\mu), \quad f \in C_c(S),$$

is called the characteristic functional of P .

Remark 1.3.2

The characteristic functional is continuous.

Proof:

Let f_n be convergent to f in $C_c(S)$, i.e., for some $c_n \downarrow 0$, and $g \in F$, the following is true for all x and n :

$$0 \leq |f_n(x) - f(x)| \leq c_n g(x), \quad n = 1, 2, \dots$$

Let $\epsilon > 0$ be given. Let $K \subset \mathbb{M}$ be compact such that $P(K^c) < \epsilon$. We remark:

$|\exp ia| = 1$ and $|(\exp ia) - (\exp ib)| \leq |a - b|$, for a, b real. Then,

$$\begin{aligned} |\text{Ch}(f_n) - \text{Ch}(f)| &\leq \int_K |(\mu, f_n) - (\mu, f)| P(d\mu) + 2P(K^c) \\ &\leq c_n P(K) \left[\sup_{\mu \in K} (\mu, g) \right] + 2P(K^c). \end{aligned}$$

Let $n \rightarrow \infty$ and then $\epsilon \rightarrow 0$. Then the assertion is proved.

Remark 1.3.3

P is uniquely characterized by its characteristic functional. This is based on the following facts: (1) $\mathcal{B}(M)$ is generated by the algebra of finite dimensional cylinder sets and (2) the probability distribution Q on a finite dimensional Euclidean space R^d is uniquely determined by the values $Q(H)$ for all half spaces H .

Before proving the converse assertion, we introduce the following concept.

Definition 1.3.4

X is said to be a random measure if X is a measurable mapping from some probability space $(\Omega, \mathcal{F}, P)$ to $(M, \mathcal{B}(M))$.

Remark 1.3.5

A mapping X from a probability space $(\Omega, \mathcal{F}, P)$ to $(M, \mathcal{B}(M))$ is measurable iff (X, f) is measurable for all $f \in F$.

Lemma 1.3.6 (Existence)

Suppose the system of random variables $\{X_f : f \in C_c(S)\}$ on Ω satisfies the conditions:

- (1) $X_f \geq 0$, a.e., if $f \in C_c^+(S)$,
- (2) X_f is finite, a.e.,
- (3) $X_{cf} = cX_f$, a.e., for c real,
- (4) $X_{f+g} = X_f + X_g$, a.e..

Then X is a random measure and is uniquely determined by the above.

Proof:

Let Ω_0 denote the set of elements $w \in \Omega$ which satisfy:

- (i) $0 \leq X_{cf}(w) = cX_f(w) < \infty$, whenever $f \in F$ and c is positive rational,
- (ii) $X_{f+g}(w) = X_f(w) + X_g(w)$, whenever $f, g \in F$,
- (iii) $|X_f(w) - X_g(w)| \leq X_{|f-g|}(w)$, whenever $f, g \in F$, and
- (iv) $X_{|f-g|}(w) \leq X_{ch}(w)$, whenever $f, g, h \in F$, $|f-g| \leq ch$ and c is positive rational.

Define $X(w)$ on F for $w \in \Omega_0$ as follows:

$$X(w): f \rightarrow X_f(w), \quad f \in F.$$

Then $X(w)$ is uniquely extended to $\bar{F} (= C_c^+(S))$. To show this, let $f_n \in F$ be Cauchy, i.e., there exist a set of positive rationals $\{c_{m,n}\}$ and $h \in F$ such that $|f_m(x) - f_n(x)| \leq c_{m,n} h(x)$ for all m, n and $x \in S$, and $c_{m,n} \rightarrow 0$ as $m, n \rightarrow \infty$. From (iii), (iv) and (i), $X(w)(f_n)$ is convergent. Further, Ω_0 is a set of probability one, since it is a countable intersection of sets of probability one. $X(w)$ is extended to be continuous and linear on $C_c^+(S)$. Then $X(w)$ can be naturally and uniquely extended to be continuous and linear on $C_c(S)$. We note that $X(w)$ is positive and therefore $X(w)$ is a Radon measure for each $w \in \Omega_0$. $(X(w), f) = X_f(w)$, a.e., for $f \in F$, follows from the definition. The general case is easily verified if we notice that X_f converges to X_g , a.e., if f converges to g in $C_c(S)$. Measurability follows from the above Remark 1.3.5.

Now we consider the converse statement.

Definition 1.3.7

Let $\text{Ch}(\cdot)$ called a positive characteristic functional: (if the following conditions are satisfied)

- (1) $\text{Ch}(\cdot): f \in C_c(S) \rightarrow \text{Ch}(f) \in \mathbb{C}$ is continuous,
- (2) $\text{Ch}(0) = 1$,
- (3) $\text{Ch}(\cdot)$ is positive definite, i.e.,

$$\sum_{j,k} \text{Ch}(f_j - f_k) a_j \bar{a}_k \geq 0,$$

where $f_j \in C_c(S)$, $j = 1, 2, \dots, n$, and $a_j \in \mathbb{C}$, $j = 1, 2, \dots, n$, $n = 1, 2, \dots$,

- (4) For each $f \in C_c^+(S)$, the probability measure P_f defined by the characteristic function $\text{Ch}(\theta f)$ is concentrated on $[0, \infty)$,

$$\text{i.e., } \int_{-\infty}^{+\infty} [\exp i\theta x] P_f(dx) = \text{Ch}(\theta f) \quad \text{and} \quad P_f[0, \infty) = 1.$$

THEOREM 1.3.8

A positive characteristic functional $\text{Ch}(\cdot)$ is the characteristic functional of a unique $P \in \mathcal{P}(\mathbb{M})$.

Proof:

Put $C_c(S) = \{f_\lambda: \lambda \in \Lambda\}$. Let $\Lambda_0 = \{\lambda_1, \lambda_2, \dots, \lambda_n\} \subset \Lambda$ be finite.

The characteristic function $\text{Ch}(\theta_1 f_{\lambda_1} + \theta_2 f_{\lambda_2} + \dots + \theta_n f_{\lambda_n})$ defines a unique probability measure P_{Λ_0} on $\mathbb{R}^n$ (Bochner's theorem). The system of probability

measures $\{P_{\Lambda_0} : \Lambda_0 \subset \Lambda \text{ finite}\}$ is easily verified to be projective. Let P be the projective limit of P_{Λ_0} on $(R^\Lambda, \mathcal{B}(R^\Lambda))$ (Kolmogorov extension theorem). The system of random variables $\{X_{f_\lambda} : (x_\lambda) \rightarrow x_\lambda\}$ satisfies the conditions in Lemma 1.3.6. Therefore there exists a random measure X on R^Λ such that $(X, f_\lambda) = X_{f_\lambda}$, a.e., P . Let P_X be the probability measure induced on $\mathbb{M}$ by X . Then $\text{Ch}(\cdot)$ agrees with the characteristic functional of P_X .

Combining Lemma 1.2.15 and Theorem 1.3.8, we have the following result..

THEOREM 1.3.9 (Prohorov) (c.f. [20])

$P_n \in P(\mathbb{M})$ converges to $P \in P(\mathbb{M})$ weakly iff the characteristic functional of P_n converges to the characteristic functional of P .

§1.4 Conditional expectations and martingales.

Definition 1.4.1

A random measure X is said to be L_p if (X, f) is L_p for every $f \in C_c(S)$, $1 \leq p < \infty$.

Remark 1.4.2.

The L_1 -property is equivalent to: $E(X, f)$ is a Radon measure, denoted by $E(X)$ and called the intensity of X .

Definition 1.4.3

Let X be an L_1 random measure. Let $\mathcal{F}_0$ be a sub- σ -field of $\mathcal{F}$. The conditional expectation of X for the given $\mathcal{F}_0$ is the $\mathcal{F}_0$ -measurable random measure defined by the system of random variables $\{E[(X, f) | \mathcal{F}_0] : f \in C_c(S)\}$ (see Lemma 1.3.6), denoted by $E[X | \mathcal{F}_0]$. (We remark: $(E[X | \mathcal{F}_0], f) = E[(X, f) | \mathcal{F}_0]$, a.e.).

Notation 1.4.4

Let $\mathcal{B}_t = \mathcal{B}_t(V)$ denote the sub- σ -field of $\mathcal{B}(V)$ generated by $\{(v(s), f) : 0 \leq s \leq t, f \in C_c(S)\}$.

Definition 1.4.5

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{F}_t$ be an increasing family of sub- σ -fields of $\mathcal{F}$. Let X_t be an $\mathcal{F}_t$ -adapted $\mathcal{M}(S)$ -valued process and L_1 for every t . Then X_t is said to be a martingale if $E[X_t | \mathcal{F}_s] = X_s$, a.e., holds for $0 \leq s < t < \infty$.

Definition 1.4.6

Let $P \in P(V)$. P is said to be a martingale if the natural $\mathcal{B}_t$ -adapted $\mathcal{M}$ -valued process: $(t, v) \rightarrow v(t) \in \mathcal{M}$ is a martingale, i.e.,

$$E_P[v(t) | \mathcal{B}_s(V)] = v(s), \text{ a.e., } P, \text{ for } 0 \leq s < t < \infty.$$

Remark 1.4.7 (Criterion)

Let $P \in P(V)$ be L_1 . Then P is a martingale iff

$$\int F((v(s_1), f_1), (v(s_2), f_2), \dots, (v(s_d), f_d)) (v(t), f) P(dv) \\ = \int F((v(s_1), f_1), (v(s_2), f_2), \dots, (v(s_d), f_d)) (v(s), f) P(dv)$$

for every pair of

$$0 \leq s_1 \leq s_2 \leq \dots \leq s_d \leq s < t < \infty,$$

$$f_1, f_2, \dots, f_d, f \in C_c(S),$$

$$d = 1, 2, \dots, \text{ and}$$

$$F \in C_b(R^d).$$

The proof follows immediately.

Notation 1.4.8

Let $P \in \mathcal{P}(V)$ be L_p . Let $I_p(|(v(t), f)|^p; A)$ denote the finite measure on $[0, \infty)$ induced by $| (v(t), f) |^p$, i.e.,

$$I_p(|(v(t), f)|^p; A) = \int_{\{|(v(t), f)|^p \in A\}} |(v(t), f)|^p P(dv), \quad A \in \mathcal{B}[0, \infty).$$

We introduce the following concept which corresponds to uniform tightness.

Definition 1.4.9

$\{P_n\} \subset \mathcal{P}(V)$ is said to be L_p -uniformly integrable if for each t , $f \in C_c(S)$,

$$\lim_{N \rightarrow \infty} \left[\sup_n I_{P_n}(|(v(t), f)|^p; [N, \infty)) \right] = 0.$$

Lemma 1.4.10 (Analogue of Fatou's lemma)

Let P_n converge to P weakly in $P(V)$. Then

$$\lim_{n \rightarrow \infty} E_{P_n} [|(v(t), f)|^p] \geq E_P [|(v(t), f)|^p],$$

and the equality holds if $\{P_n\}$ is L_p -uniformly integrable.

Proof:

For each $N > 0$, $E_{P_n} [|(v(t), f)|^p] \geq E_{P_n} [|(v(t), f)|^p \wedge N]$. Let $n \rightarrow \infty$. Then,

$$\lim_{n \rightarrow \infty} E_{P_n} [|(v(t), f)|^p] \geq \lim_{n \rightarrow \infty} E_{P_n} [|(v(t), f)|^p \wedge N] = E_P [|(v(t), f)|^p \wedge N].$$

(The weak convergence is used in the last equality). Since N is arbitrary,

$$\lim_{n \rightarrow \infty} E_{P_n} [|(v(t), f)|^p] \geq E_P [|(v(t), f)|^p].$$

Now, let P_n be L_p -uniformly integrable. Then for every $N > 0$,

$$E_{P_n} [|(v(t), f)|^p] \leq E_{P_n} [|(v(t), f)|^p \wedge N] + \sup_n I_{P_n} (|(v(t), f)|^p; [N, \infty)).$$

Therefore,

$$\lim_{n \rightarrow \infty} E_{P_n} [|(v(t), f)|^p] \leq E_P [|(v(t), f)|^p \wedge N] + \sup_n I_{P_n} (|(v(t), f)|^p; [N, \infty)).$$

Let $N \rightarrow \infty$. Then,

$$\lim_{n \rightarrow \infty} E_{P_n} [|(v(t), f)|^p] \leq E_P [|(v(t), f)|^p].$$

From Remark 1.4.7 and Lemma 1.4.10, we obtain the following important theorem.

THEOREM 1.4.10

Let $\{P_n\} \subset P(V)$ be L_p -uniformly integrable martingales and be convergent to $P \in P(V)$ weakly. Then the weak limit P is again an L_p -martingale.

Also,

$$\begin{aligned} \lim_n \int F((v(s_1), f_1), (v(s_2), f_2), \dots, (v(s_d), f_d))(v(t), f) P_n(dv) \\ = \int F((v(s_1), f_1), (v(s_2), f_2), \dots, (v(s_d), f_d))(v(t), f) P(dv) \end{aligned}$$

holds under the condition in Remark 1.4.7.

§1.5 L_2 -martingales and the Doob-Meyer decomposition

Remark 1.5.1

If $P \in P(V)$ is L_p ($1 \leq p < \infty$), then P is automatically L_1 .

Remark 1.5.2

The natural increasing family of sub- σ -fields $\mathcal{B}_t(V)$: $0 \leq t$, of $\mathcal{B}(V)$ is left continuous, but not from the right.

Hereafter $P \in P(V)$ is chosen and fixed to be an L_2 -martingale throughout in this section. We introduce the following space $M_2^c(\mathcal{B}_t)$: $\mathcal{B}_t$ -adapted real valued continuous square integrable P -martingales (following Kunita-Watanabe [15]).

Notation 1.5.3

Let $M_2^c(\mathcal{B}_t)$ denote the space of processes $X_t(v)$ satisfying:

- (1) $X_t(v): [0, \infty) \times V \rightarrow \mathbb{R}$, (real valued),
- (2) $X_t(v)$ is $\mathcal{B}[0, \infty) \times \mathcal{B}(V)/\mathcal{B}(\mathbb{R})$ measurable, (measurability),
- (3) $P(\{v: [0, \infty) \ni t \rightarrow X_t(v) \text{ is continuous.}\}) = 1$, (continuity),
- (4) for every t , $X_t(v)$ is $\mathcal{B}_t(V)$ measurable, ($\mathcal{B}_t$ -adaptedness),
- (5) $E_P[X_t(v)^2] < \infty$, for every $t \in [0, \infty)$ (square integrability),
- (6) $E_P[X_t | \mathcal{F}_s] = X_s$, a.e., $0 \leq s \leq t$ (martingale property).

Remark 1.5.4

For a given $f \in C_c(S)$ we call the process $(t, v) \rightarrow (v(t), f)$ natural.

Then, the natural processes belong to $M_2^c(\mathcal{B}_t)$.

Definition 1.5.5

$X_t(v), X'_t(v) \in M_2^c(\mathcal{B}_t)$ are identified, denoted $X = X'$, if

$P(\{v: t \rightarrow X_t(v) = X'_t(v)\}) = 1$. The semi-norms $\|X\|_t$ are defined as follows..

$$\|X\|_t = \sqrt{E(X_t^2)} \quad t \geq 0.$$

We remark: $\|X\|_t$ is increasing, since X_t^2 is a sub-martingale, and the metric is defined as follows:

$$\|X - X'\| = \sum_{n=1}^{\infty} \frac{1}{2^n} (\|X - X'\|_n \wedge 1).$$

Lemma 1.5.6 (The key property of L_2 -martingale theory)

$M_2^c(\mathcal{B}_t)$ with the metric $\|X-X'\|$ is a complete metric space.

Proof:

First we prove that $\|X-X'\| = 0$ implies $X = X'$. For every $t = 1, 2, \dots$, $E|X_t - X'_t|^2 = 0$. Therefore $E|X_s - X'_s|^2 = 0$ for all $s \geq 0$. From this and sample path continuity, $X = X'$ follows. To prove completeness, let $X^{(n)}$ be Cauchy, i.e., $\lim_{n,m} E|X_t^{(n)} - X_t^{(m)}|^2 = 0$. From the Kolmogorov-Doob inequality, for all $t > 0$, and $c > 0$,

$$P\left(\max_{0 \leq s \leq t} |X_s^{(n)} - X_s^{(m)}| > c\right) \leq \frac{1}{c^2} \|X^{(n)} - X^{(m)}\|_t^2,$$

which implies that there exist a subsequence $X^{(n_j)}$ and a set $V_0 \in \mathcal{B}(V)$ of probability one such that for each $v \in V_0$, the continuous functions: $t \rightarrow X_t^{(n_j)}(v)$, $j = 1, 2, \dots$ converge uniformly on every bounded interval so that the limit: $X_t(v) = \lim_j X_t^{(n_j)}(v)$ is continuous. $X_t(v) \in M_2^c(\mathcal{B}_t)$ and $X^{(n)} \rightarrow X$ are immediate.

Remark 1.5.7

The above lemma holds in a more general setting, i.e., let $(\Omega, \mathcal{F}, P)$ be a probability space, $\mathcal{F}_t$ be an increasing family of sub- σ -fields of $\mathcal{F}$, and $M_2^c(\mathcal{F}_t)$ denote the space of $\mathcal{F}_t$ -adapted square integrable continuous martingales. Then $M_2^c(\mathcal{F}_t)$ is a complete metric space with the corresponding metric.

Remark 1.5.8

Put $\|X-X_0\|_t(v) = \max_{0 \leq s \leq t} |X_s(v) - X_0(v)|$. Let $X \in M_2^c$. Then $\|X-X_0\|_t(v)$ is a $\mathcal{B}_t$ -adapted continuous process. Continuity follows from sample path continuity. Measurability follows from:

$$\|X-X_0\|_t(v) = \lim_{n \rightarrow \infty} \max_{s \in I_n \cap [0, t]} |X_s(v) - X_0(v)|,$$

where

$$I_n = \left\{ \frac{k}{2^n} : k = 0, 1, \dots \right\}.$$

Definition 1.5.9

Let $X \in M_2^c$. The following $\sigma_1(v)$ is said to be the first hitting time, $\sigma_2(v)$ the first passage time of X :

$$\sigma_1(v) = \min\{t : \|X-X_0\|_t(v) \geq c\}$$

$$\sigma_2(v) = \inf\{t : \|X-X_0\|_t(v) > c\},$$

where $c > 0$ is given.

Remark 1.5.10

$\sigma_1(v)$, $\sigma_2(v)$ are both $\mathcal{B}(V)/\mathcal{B}(R)$ -measurable, since

$$\sigma_1(v) = \lim_N \lim_n \min \left\{ t : \|X-X_0\|_t(v) \geq c - \frac{1}{N}, t \in I_n \right\}$$

and

$$\sigma_2(v) = \lim_n \min \{ t : \|X-X_0\|_t(v) > c, t \in I_n \}.$$

The difference is the following: $\{\sigma_1 = t\}$ can be known by the information $\{X_s: s \leq t\}$, while $\{\sigma_2 = t\}$ cannot be known by the same information. $\{\sigma_1(v) \leq t\}$, $\{\sigma_2(v) \leq t\}$: the events which happened in the past are both $\mathcal{B}_t(v)$ -measurable.

In the Doob-Meyer decomposition theorem for martingales (see Meyer [18], [19] and Kunita-Watanabe [15]), the increasing family of sub- σ -fields is usually required to be right continuous.

Here we review this decomposition in the special case.

Definition 1.5.11

Let $M_{2,b}^c(\mathcal{B}_t)$ denote the subspace of $M_2^c(\mathcal{B}_t)$ consisting of X such that: there exists $N(T) > 0$ such that for every pair of $0 \leq s \leq t \leq T$,

$$E[(X_t - X_s)^2 | \mathcal{B}_s] \leq N(T)(t-s), \text{ a.e.}$$

This definition corresponds to those processes defined by Ito stochastic integral with bounded coefficients. The restriction still seems severe. However we sometimes obtain the condition by using the stopping time argument. (For stopping times, see Meyer [19]). Later we see that this class can be extended to its closure in $M_2^c(\mathcal{B}_t)$.

Remark 1.5.12. (Fundamental martingale inequalities)

Let $X_t \in M_2^c(\mathcal{B}_t)$. Then,

$$(1) \quad N P \left(\sup_{0 \leq s \leq t} |X(s)| > N \right) \leq \int \left\{ \sup_{0 \leq s \leq t} |X(s)| > N \right\} |X(t)| dP, \quad N > 0.$$

$$(c) \quad E \left[\sup_{s \leq t} |X(s)| \right] \leq 4 E |X(t)|.$$

Proof:

We refer to Meyer [14]. Let $X \in M_{2,b}^c(\mathcal{B}_t)$ be fixed.

Notation 1.5.1:

Let (Δ_n) denote a sequence of positive increasing real numbers:

$$(\Delta_n): c_0^n = 0 < c_1^n < c_2^n < c_3^n \dots \rightarrow \infty,$$

and

$$|\Delta|_n = \sup_k (c_{k+1}^n - c_k^n) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Notation 1.5.14.

$$Z_n(t) = \sum_{k=0}^{\infty} (X(c_{k+1}^n \wedge t) - X(c_k^n \wedge t))^2.$$

Notation 1.5.15

$$X_n(s) = \sum_{k=0}^{\infty} I_{[c_k^n, c_{k+1}^n)}(s) X(c_k^n).$$

Notation 1.5.16.

$$\int_0^t X_n(s) dX(s) = \sum_{k=0}^{\infty} X(c_k^n) (X(c_{k+1}^n \wedge t) - X(c_k^n \wedge t)).$$

Remark 1.5.17

$$(X_t - X_0)^2 - Z_n(t) = 2 \int_0^t X_n(s) dX(s) - 2X(0)(X(t) - X(0)).$$

Remark 1.5.18

$$Z_n(t) - Z_m(t) = 2 \int_0^t (X_m(s) - X_n(s)) dX(s),$$

here the right hand side is calculated for the mutual refinement of Δ_n, Δ_m .

Now, taking account of $X_t \in M_{2,b}^c(\mathcal{B}_t)$ and conditional expectation,

$$E(Z_n(t) - Z_m(t))^2 \leq 4N(T) \int_0^t E(X_m(s) - X_n(s))^2 ds, \quad 0 \leq t \leq T.$$

The integrand $(X_m(s) - X_n(s))^2$ is uniformly bounded by $4\|X\|_t^2$, which is integrable from Remark 1.5.1, and converges to 0 as $n, m \rightarrow \infty$. Therefore the martingales

$(X_t - X_0)^2 - Z_n(t)$ are convergent in $M_{2,b}^c(\mathcal{B}_t)$: the limit is denoted by $M(t)$. Now

let us assume (Δ_{n+1}) is a refinement of (Δ_n) , i.e., c_j^n appear in the sequence of c_j^{n+1} . It is easy to see that the limit $M(t)$ is unchanged. For some sub-

sequence, $Z_{n_k}(t)$ is convergent uniformly on every bounded time-interval. $Z_{n_k}(t)$ is increasing on $c_j^{n_k}$, $j = 1, 2, \dots$ so that the limit, denoted by $A(t)$ is increasing and continuous. Therefore we have the existence and it is uniquely obtained in the above manner, i.e., the limit $A(t)$ does not depend on the choice of (Δ_n) .

Remark 1.5.19

$M_{2,b}^c(\mathcal{B}_t)$ is a linear space, since $(X + Y)^2 \leq 2X^2 + 2Y^2$.

Remark 1.5.20 (Important)

$$\sqrt{\langle (X_t + Y_t - X_0 - Y_0)^2 \rangle} \leq \sqrt{\langle (X_t - X_0)^2 \rangle} + \sqrt{\langle (Y_t - Y_0)^2 \rangle}$$

where $\langle (X_t - X_0)^2 \rangle$ denotes the increasing process obtained in Remark 1.5.18.

The above relation is the direct consequence of approximations in Remark 1.5.18.

Remark 1.5.21

If $X_n(t) \in M_{2,b}^c(\mathcal{B}_t)$ converges in $M_2^c(\mathcal{B}_t)$, then $\sqrt{\langle (X_n(t) - X_n(0))^2 \rangle}$ is convergent.

Proof:

$$\begin{aligned} & \max_{0 \leq t \leq T} \left| \sqrt{\langle (X_n(t) - X_n(0))^2 \rangle} - \sqrt{\langle (X_m(t) - X_m(0))^2 \rangle} \right| \\ & \leq \max_{0 \leq t \leq T} \sqrt{\langle (X_n(t) - X_m(t) - X_n(0) + X_m(0))^2 \rangle} \\ & = \sqrt{\langle (X_n(T) - X_m(T) - X_n(0) + X_m(0))^2 \rangle}. \end{aligned}$$

$\rightarrow 0$ as $n, m \rightarrow \infty$ uniformly on every bounded time interval with probability one, so that the definition of the increasing process is uniquely extended to the closure of $M_{2,b}^c(\mathcal{B}_t)$ in $M_2^c(\mathcal{B}_t)$.

§1.6 L_2 -martingales which start from atomic measures.

Remark 1.6.1

Let $P \in \mathcal{P}(V)$ be an L_2 -martingale. Let K_0 be a fixed compact set. Let $\mathcal{B}(K_0)$ denote the topological σ -field of K_0 . Let $P \in \mathcal{P}(V)$ be an L_2 -martingale. Then, for each $B \in \mathcal{B}(K_0)$, the process $(v(t), I_B) \in M_2^c(\mathcal{B}_t)$. To prove this, we introduce the following definition.

Definition 1.6.2 (Dynkin π -system, d-system)

Let $\mathcal{G}$ be a subsystem of $\mathcal{B}(K_0)$. Then $\mathcal{G}$ is said to be a π -system if $A_k \in \mathcal{G}; k = 1, 2, \dots, n$ implies $\bigcap_k A_k \in \mathcal{G}$, i.e., closed under finite intersection. $\mathcal{G}$ is said to be a d-system if

$$(1) K_0 \in \mathcal{G},$$

$$(2) A, B \in \mathcal{G} \text{ and } A \subset B \text{ implies } B-A \in \mathcal{G}, \text{ and}$$

$$(3) A_n \in \mathcal{G}, A_n \subset A_{n+1}, n = 1, 2, \dots \text{ implies } \bigcup_n A_n \in \mathcal{G}.$$

Lemma 1.6.3 (Dynkin)

Let $\mathcal{G} = \{K \subset K_0 : \text{compact}\}$. Then, it is immediate that $\mathcal{G}$ is a π -system. Let $\mathcal{H}$ contain $\mathcal{G}$ and be a d-system. Then $\mathcal{H} = \mathcal{B}(K_0)$.

Now, we return to the proof of Remark 1.6.1. Put $\mathcal{H} = \{B \in \mathcal{B}(K_0) : (v(t), I_B) \in M_2^c\}$. Let $K \subset K_0$ be compact. Approximating I_K by continuous functions of compact support from above, then $K \in \mathcal{H}$ follows. It is easy to see that $\mathcal{H}$ is a d-system. Therefore Remark 1.6.1 is proved.

Remark 1.6.4

Let $P \in P(V)$ be an L_2 -martingale with $P\left(v(0) = \sum_{n=1}^{\infty} a_n \delta_{x_n}\right) = 1$. Let $X(t) = \sum_{n=1}^{\infty} (v(t), \{x_n\}) \delta_{x_n}$. Then $P(v(t) = X(t)) = 1$.

Proof:

From the above, $(v(t), \{x_n\}) \in M_2^c$. Hence $v(t) \geq X(t)$. Let $f \in C_0^+(S)$. Then $0 \leq E[(v(t) - X(t), f)] = E[(v(0) - X(0), f)] = 0$. Therefore $v(t) = X(t)$.

This remark says that if P starts from an atomic measure, then no further atoms are created. In addition the atoms do not change their locations.

Remark 1.6.5

Every Radon measure μ is decomposed uniquely into an atomic part μ_a and a nonatomic part μ_{na} . The mapping $\mu \rightarrow \mu_a$ is $\mathcal{B}(\mathbb{M})/\mathcal{B}(\mathbb{M})$ -measurable.

Proof:

Let $\{E_j^n: j = 1, 2, \dots\}$ denote a sequence of partitions of S , $n = 1, 2, \dots$ such that

- (1) $\bigcup_j E_j^n = S$, a disjoint union for every n ,
- (2) each E_j^n is included in some compact set,
- (3) for each n , every compact set K is covered by a finite number of E_j^n , i.e. $\#\{j; K \cap E_j^n \neq \emptyset \text{ for } n \text{ fixed}\} < \infty$.

(4) $\{E_j^{n+1}; j = 1, 2, \dots\}$ is a refinement of $\{E_j^n; j = 1, 2, \dots\}$ for $n = 1, 2, \dots$ and (5) $\sup_j \text{diam}(E_j^n) \rightarrow 0$ as $n \rightarrow \infty$.

Let x_j^n be a point of E_j^n , chosen and fixed. Let $\epsilon > 0$ be given.

For every n we construct a measurable mapping from $\mathbb{M}$ to $\mathbb{M}$ as follows:

$$\mu \rightarrow \sum_j I_{(\mu(E_j^n), \epsilon)} \mu(E_j^n) \delta_{x_j^n},$$

where

$$I_{(\mu(E_j^n), \epsilon)} = \begin{cases} 1 & \text{if } \mu(E_j^n) \geq \epsilon, \\ 0 & \text{if } \mu(E_j^n) < \epsilon. \end{cases}$$

Let $n \rightarrow \infty$ and then $\epsilon \rightarrow 0$. Then the limit is the atomic part of μ . Therefore

$\mu \rightarrow \mu_a$ is measurable.

CHAPTER II. MARTINGALE PROBLEMS (1) AND RELATED MEASURE VALUED MARTINGALE PROBLEMS (2)

§2.0 Introduction

Let S denote the space. Let $\{E_n\}$ be a partition of S . Let X_t denote random phenomena on S . Let (X_t, E_n) be given and be called observations. Then our motivation is to find the equation which describes the random phenomena X_t through the observations.

Along this line, one mathematical model is obtained in this chapter, which describes a space-time nonhomogeneous completely random process.

The method to obtain this model is the following. First we consider a martingale problem which describes the evolution of mass, based on space-time nonhomogeneous white noise. Second we consider the Poisson randomization, which describes the number and the location of particles coming from outside. The significance of Poisson randomization is that the particles pick up independent mass evolution governed by the above space-time nonhomogeneous martingale equations. Once this is done, then we observe the limit process, changing the intensity and the initial mass.

§2.1 The spaces, $U_p(\mu)$, $U_p^+(\mu)$ ($\mu \in \mathcal{M}(S)$).

Notation 2.1.1

Let $U_p(\mu)$ ($= C(\{0, \infty\}; L_p(\mu))$) denote the space of $L_p(\mu)$ -valued continuous functions on $[0, \infty)$, and $U_p^+(\mu)$ denote the cone in $U_p(\mu)$ consisting of positive functions endowed with the topology of uniform convergence on every bounded interval, $1 \leq p < \infty$.

Remark 2.1.2

$U_p(\mu)$ is Polish. This follows since $L_p(\mu)$ is Polish and the distance between two points on the segment which connects f and $g \in L_p(\mu)$ is smaller than the distance between the terminal points f, g .

Remark 2.1.3

Similarly, $U_p^+(\mu)$ is Polish.

Notation 2.1.4

Let $P(U_p(\mu))$, $(P(U_p^+(\mu)))$, respectively, denote the space of probability measures on $(U_p(\mu), \mathcal{B}(U_p(\mu)))$, $(U_p^+(\mu), \mathcal{B}(U_p^+(\mu)))$, where $\mathcal{B}(U_p(\mu))$, $(\mathcal{B}(U_p^+(\mu)))$ denotes the topological σ -field of $U_p(\mu)$, $(U_p^+(\mu))$.

Remark 2.1.5

Let $F \in U_p(\mu)$. Then for each $t \in [0, \infty)$, F is an $L_p(\mu)$ -function $F(t)$. Therefore $F(t)$ has a value $F(t)(x)$ for $x \in S$, uniquely defined almost everywhere μ . It is a fact that $F(t)(x)$ has a jointly measurable version in (t, x) . Therefore it is meaningful to consider a mapping $t \rightarrow F(t)(x)$ for each fixed point x . However it cannot be expected that $t \rightarrow F(t)(x)$ is continuous.

Remark 2.1.6

We would like to consider a mapping $(t, x): F \rightarrow F(t)(x)$ on $U_p(\mu)$ for given (t, x) . However this mapping may not be well-defined. To avoid this difficulty, we introduce the following definition.

Definition 2.1.7

Let $\{E_j^n: j \geq 1\}_n$ be as in the proof of Remark 1.6.5. Let the following $F_n(t)$ be called the n-th approximation of $F(t)$.

$$F(t) \rightarrow F_n(t) = \sum_{j \in \mu(E_j^n)} \frac{1}{\mu(E_j^n)} (F(t), I_{E_j^n}) I_{E_j^n}(x),$$

where $(F(t), I_{E_j^n})$ denotes $\int F(t)(x) I_{E_j^n}(x) \mu(dx)$ and if $\mu(E_j^n) = 0$, then $\frac{0}{0}$ is understood to be 0.

(Remark: The above mapping is a contraction in $U_p(\mu)$. Further $t \rightarrow F_n(t)(x)$ is continuous for each fixed $x \in S$, if $F \in U_p(\mu)$.)

Definition 2.1.8

Let $P \in P(U_p(\mu))$. Let the probability measure $P_n \in P(U_p(\mu))$ induced by F_n be called the n-th approximation of P .

§2.2 Poisson randomization

Remark 2.2.1

The mapping $(t, x): (F, x) \rightarrow F_n(t)(x)$, is $(U_p(\mu) \times S, \mathcal{B}(U_p(\mu)) \times \mathcal{B}(S))$ -measurable.

Remark 2.2.2

The mapping: $U_p^+(\mu) \times S \rightarrow V$

$$(F, x) \mapsto F_n(\cdot)(x) \delta_x,$$

is measurable.

$F_n(\cdot)(x) \delta_x$, for x fixed, describes the following: A particle falls onto the space S at time $t = 0$ at the location x and then its mass evolves as $F_n(t)(x)$ at the location x .

Definition 2.2.3 (Poisson randomization)

Let $P \in \mathcal{P}(U_p^+(\mu))$. Let P_n, F_n denote the n -th approximation of P, F , respectively. Let the following V -valued process $X_n^N(t)$ on $\sum_{m=0}^{\infty} (U_p(\mu) \times K)^m$ be called the Poisson randomization of P_n with intensity $N\mu$ on K , where $K \subset S$ is a fixed compact set:

$$X_n^N(t): (F_n^{(1)}, F_n^{(2)}, \dots, F_n^{(m)}, x_1, x_2, \dots, x_m) \in (U_p(\mu) \times K)^m \\ \rightarrow F_n^{(1)}(t, x_1) \delta_{x_1} + F_n^{(2)}(t, x_2) \delta_{x_2} + \dots + F_n^{(m)}(t, x_m) \delta_{x_m},$$

where

(1) $\sum_{m=0}^{\infty} (U_p(\mu) \times K)^m$ has the direct sum topology,

(2) each component $(U_p(\mu) \times K)^m$ has the product measure,

$$\frac{N^m}{m! e^{N\mu(K)}} P(dF^{(1)}) P(dF^{(2)}) \dots P(dF^{(m)}) \mu(dx_1) \mu(dx_2) \dots \mu(dx_m),$$

(3) for $m = 0$, the corresponding measurable space is $\{\emptyset\}$ and its total measure is $e^{-N\mu(K)}$,

(4) for $m = 0$, $X_n^N(\emptyset) = 0$.

The corresponding physical phenomena is as follows. Particles fall onto the space S at time 0, according to a Poisson random field with intensity $N\mu$. The particles stay at the same locations where they fell and the evolution of the mass of a particle at x is governed by $F_n(t, x)$. Since we introduced the product measure on each component (see (1) and (2) above), this method of constructing a measure valued process yields a completely random process, i.e., if $\text{supp}(f) \cap \text{supp}(g) = \emptyset$, then $(X_n^N(t), f)$ and $(X_n^N(t), g)$ are independent processes.

See Jagers [13] for completely random measures.

§2.3 The characteristic functional of X_n^N .

The characteristic functional of $X_n^N(t)$:

$$\begin{aligned} & E[\exp\{i(X_n^N, f)\}] \\ &= \sum_{m=0}^{\infty} \frac{N^m}{m!} e^{-N\lambda(K)} \left[\iint_K \exp\{i F_n(t, x) f(x)\} P(dF) \mu(dx) \right]^m \\ &= \exp \iint_K [-1 + \exp\{i F_n(t, x) f(x)\}] N \cdot P(dF) \mu(dx). \end{aligned}$$

Now, let us assume that $P_N \in P(U_p(\mu))$ is given for $N = 1, 2, \dots$, and replace $N \cdot P(dF)$ above by $N \cdot P_N(dF)$. Then we are concerned with the convergence of the probability measures on V induced by X_n^N as N tends to infinity. Bose [3; page 90, Theorem 3.2.2] has demonstrated the convergence, assuming that P_N is determined by "the transition density function $p(\frac{1}{N}, t, z)$ with respect to the speed measure m of a diffusion with the initial condition $\frac{1}{N}$ and an absorbing boundary at the origin", where $p(\frac{1}{N}, t, z)$ is spatially homogeneous. We introduce the following method which enables us to consider spatially nonhomogeneous cases.

First, we introduce Hilbert space-valued space-time nonhomogeneous white noise $W(t, x)$.

Second, we consider stochastic differential equations:

$$dX_t = a(t, X) dW(t, x), \quad X_0 = \frac{1}{N} f(x).$$

Third, we consider the Poisson randomization of X with intensity

$N\mu$.

Fourth, we discuss the convergence as N tends to infinity.

Hereafter $\mu \in \mathcal{M}(S)$ is restricted to be in the case that $\text{supp}(\mu) = K$ is compact and $\mu(K) = 1$, and $U_2(\mu)$ is denoted by U_2 .

§2.4 Hilbert space-valued white noise

The section is devoted to the following theorem.

THEOREM 2.4.1

There exists a martingale $P \in P(U_2)$ with the increasing process:

$$\langle F(t), f \rangle \langle F(t), g \rangle = \int_0^t (Q(s, x, y), f(x)g(y)) ds,$$

where

- (1) $(Q(s, x, y), f(x)g(y))$ denotes $\iint Q(s, x, y) f(x)g(y) \mu(dx) \mu(dy)$,
- (2) $Q(s, x, y)$ is real valued and continuous on $[0, \infty) \times K \times K$,
- (3) $Q(s, x, y) = Q(s, y, x)$, i.e., symmetric,
- (4) positive definite, i.e.,

$$\sum_{i,j}^n Q(s, x_i, x_j) y_i y_j \geq 0, \quad (x_1, x_2, \dots, x_n) \in K^n$$

$$(y_1, y_2, \dots, y_n) \in \mathbb{R}^n$$

$$n = 1, 2, \dots$$

Remark 2.4.2

At each time s , $u(s, x, y)$ is decomposed in $L_2(\mu \times \nu)$ as follows.

$$u(s, x, y) = \sum_{j=1}^{\infty} \lambda_j(s) \varphi_j(s, x) \varphi_j(s, y),$$

where

(1) $\lambda_j(s)$ are the eigenvalues of $Q(s): f \rightarrow \int f(y) u(s, x, y) \mu(dy)$,

(2) $\{\varphi_j(s, x)\}$ are the eigenfunctions corresponding to $\{|\lambda_j(s)|\}$ and form a complete orthonormal system in the image of $Q(s)$,

(3) $\sum_{j=1}^{\infty} \lambda_j^2(s) = \|Q^2(s, x, y) \mu(dx) \mu(dy)\| < \infty$ (Hilbert Schmidt),

(4) $\sum_{j=1}^{\infty} \lambda_j(s) = \int Q(s, x, x) \mu(dx) < \infty$ (nuclear),

(5) $\sum_{j=1}^{\infty} \lambda_j(s) \varphi_j(s, x)^2 = Q(s, x, x)$ in $L_1(\mu)$.

We approximate the problem as follows. Put $t_k^n = \frac{k}{2^n}$ for $k = 0, 1, 2, \dots$, and $n = 1, 2, \dots$. Put $\xi_n(t) = \sum_{k=0}^{\infty} t_k^n I_{[t_k^n, t_{k+1}^n)}(t)$. The n -th approximating

problem is as follows: $P_n \in P(U_2)$ to be a martingale with the increasing process;

$$\langle F(t), f \rangle \langle F(t), g \rangle = \int_0^t (Q(\xi_n(s), x, y), f(x)g(y)) ds.$$

The solution to this martingale problem is immediately obtained by the probability measure induced by the following:

$$F_{(n)}(t, x) = \sum_{k=0}^{\infty} \sum_{j=1}^{\infty} \sqrt{\lambda_j(t_k^n)} (B_j(t_{k+1}^n \wedge t) - B_j(t_k^n \wedge t)) \varphi_j(t_k^n, x),$$

where $\{B_j: j \geq 1\}$ are independent Brownian motions.

Definition 2.4.3 (c. f. [1])

Let $P \in P(U_2)$ be called an L_2 -martingale if $(F(t), f)$ is an L_2 -martingale, $f \in L_2(\mu)$, with respect to the increasing family of sub- σ -fields $\mathcal{B}_t(U_2)$ or $\mathcal{B}(U_2)$ generated by $F(s)$, $s \leq t$ and $E\|F(t)\|_2^2 < \infty$, where $(F(t), f) = \int F(t)(x)f(x)\mu(dx)$ and $\|F(t)\|_2^2 = \int F(t)^2\mu(dx)$.

Remark 2.4.4

$F_{(n)}$ is U_2 -valued. To prove this, it suffices to show that for each k ,

$$Y_k(t, x) = \sum_{j=1}^{\infty} \sqrt{\lambda_j(t_k^n)} (B_j(t_{k+1}^n \wedge t) - B_j(t_k^n \wedge t)) \varphi_j(t_k^n, x),$$

belongs to U_2 . Let Y_k^N denote the N -th partial sum of Y_k . Obviously, Y_k^N belongs to U_2 , since Brownian paths are continuous. Y_k^N converges uniformly to Y_k in the sense that:

$$E \left(\sup_t \|Y_k^N(t) - Y_k^{N'}(t)\|_2^2 \right), \quad (Y_k^N \text{ is a martingale}),$$

$$\leq 4 E \|Y_k^N(t_{k+1}^n) - Y_k^{N'}(t_{k+1}^n)\|_2^2$$

$$= 4 \sum_{j=N+1}^{N'} \lambda_j(t_k^n)(t_{k+1}^n - t_k^n), \text{ assuming } N < N',$$

tends to 0 as N, N' tend to infinity. Therefore Y_k belongs to U_2 .

Remark 2.4.5.

The probability measure $P_n \in P(U_2)$ induced by $F^{(n)}$ is an L_2 -martingale. This follows from Remark 2.4.4 and the fact that Y_k^N is an L_2 -martingale.

Remark 2.4.6

$F(t) = \sum_{j=1}^{\infty} (F(t), \psi_j) \psi_j(x)$ holds in $U_2(\mu)$, where $\{\psi_j; j \geq 1\}$ is a fixed c.o.n.s. of $L_2(\mu)$.

Notation 2.4.7

Let $(U_2 \times S, \mathcal{B}(U_2) \times \mathcal{B}(S), P_n \times \mu; \mathcal{B}_t(U_2) \times \mathcal{B}(S))$ denote the product probability space and $M_2^c(\mathcal{B}_t(U_2) \times \mathcal{B}(S))$ denote the space of $\mathcal{B}_t(U_2) \times \mathcal{B}(S)$ -adapted continuous square integrable $P_n \times \mu$ martingales.

Remark 2.4.8

$F(t, x) = \sum_j (F(t), \psi_j) \psi_j(x)$ is convergent in $M_2^c(\mathcal{B}_t(U_2) \times \mathcal{B}(S))$, since

$$E \left[\int F(t, x)^2 \mu(dx) \right] = \int_0^t \int Q(\xi_n(s), x, x) \mu(dx) ds < \infty.$$

Remark 2.4.9

Let $\mathcal{Q}_c(\mathcal{B}_t(U_2) \times \mathcal{B}(S))$ denote the space of absolutely continuous $(\mathcal{B}_t(U_2) \times \mathcal{B}(S))$ -adapted processes. Then $\langle F(t,x)^2 \rangle$ in $\mathcal{Q}_c(\mathcal{B}_t(U_2) \times \mathcal{B}(S))$ is as follows:

$$\begin{aligned} \langle F(t,x)^2 \rangle &= \left\langle \left(\sum_{k=0}^{\infty} Y_k(t,x) \right)^2 \right\rangle \\ &= \sum_{k=0}^{\infty} \sum_{j=1}^{\infty} \lambda_j(t_k^n) \varphi_j(t_k^n, x)^2 (t_{k+1}^n \wedge t - t_k^n \wedge t) \\ &= \int_0^t Q(\xi_n(s), x, x) ds. \end{aligned}$$

Remark 2.4.10

Let $(U_2 \times S_x \times S_y, \mathcal{B}(U_2) \times \mathcal{B}(S_x) \times \mathcal{B}(S_y), P_n \times \mu \times \mu; \mathcal{B}_t(U_2) \times \mathcal{B}(S_x) \times \mathcal{B}(S_y))$ denote the product probability space, and $M_2^c(\mathcal{B}_t(U_2) \times \mathcal{B}(S_x) \times \mathcal{B}(S_y))$ denote the space of $(\mathcal{B}_t(U_2) \times \mathcal{B}(S_x) \times \mathcal{B}(S_y))$ -adapted continuous square integrable $P_n \times \mu \times \mu$ martingales. An example is: $F(t,x)I_S(y), (F(t,y)I_S(x))$. Now, we are interested in the absolutely continuous part of the product $F(t,x)F(t,y)$.

Remark 2.4.11

In $\mathcal{Q}_c(\mathcal{B}_t(U_2) \times \mathcal{B}(S_x) \times \mathcal{B}(S_y))$ with $P_n \times \mu \times \mu$,

$$\begin{aligned} \langle F(t)(x)F(t)(y) \rangle &= \sum_{k=0}^{\infty} \sum_{j=1}^{\infty} \lambda_j(t_k^n) \varphi_j(t_k^n, x) \varphi_j(t_k^n, y) (t_{k+1}^n \wedge t - t_k^n \wedge t) \\ &= \int_0^t Q(\xi_n(s), x, y) ds. \end{aligned}$$

Lemma 2.4.12

$$\begin{aligned} \langle (F(t), f)(F(t), g) \rangle &= \langle (F(t)(x)F(t)(y), f(x)g(y)) \rangle \\ &= \int \int \langle (F(t, x)F(t, y)) \rangle f(x)g(y) \mu(dx) \mu(dy) \end{aligned}$$

holds, in $\mathcal{A}_c(\mathcal{B}_t(U_2))$. (Direct calculation)

Combining Lemma 2.4.12 and Remark 2.4.11, we have the statement verified for the approximating problem.

Secondly, we consider the weak convergence of P_n in $P(U_2)$ as n tends to infinity. We show that $\{P_n\}$ is uniformly tight. To prove this, we introduce the following lemma.

Lemma 2.4.13 (A special case of Dini's theorem)

Let $a_j^n \geq 0$, $a_j \geq 0$. Let $X^{(n)} = \sum_{j=1}^{\infty} a_j^n < \infty$ and $X = \sum_{j=1}^{\infty} a_j < \infty$.

Suppose $X^{(n)}$ is convergent to X and that each partial sum $\sum_{j=1}^N a_j^{(n)}$ is

convergent to the partial sum $X_N = \sum_{j=1}^N a_j$. Then for every given $\epsilon > 0$, there exist N and N' such that $X^{(n)} - X_N^{(n)} = \sum_{j=N+1}^{\infty} a_j^n < \epsilon$ for all $n \geq N'$.

Proof:

$$X^{(n)} - X_N^{(n)} = (X^{(n)} - X) + (X - X_N) + (X_N - X_N^{(n)}).$$

Choose N_0 be such that $|X^{(n)} - X| < \frac{\epsilon}{3}$ for all $n \geq N_0$. Choose N_1 be such that

$X - X_{N_1} < \frac{\epsilon}{3}$. For above N_1 , choose N_2 be such that $|X_{N_1} - X_{N_1}^{(n)}| < \frac{\epsilon}{3}$ for all

$n \geq N_2$. Put $N = N_1$ and $N' = N_0 \vee N_2$. Then the assertion is proved.

Lemma 2.4.14

$$\lim_{N \rightarrow \infty} \sup_n P_n \left(\sup_{0 \leq s \leq t} \sum_{j=N+1}^{\infty} (F(t), \psi_j)^2 > \delta \right) = 0, \quad \delta > 0.$$

Proof:

Put,

$$X^{(n)} \equiv E_{P_n} \left[\sum_{j=1}^{\infty} (F(t), \psi_j)^2 \right] = \int_0^t \int Q(\xi_n(s), x, x) \mu(dx) ds,$$

$$X \equiv \int_0^t \int Q(s, x, x) \mu(dx) ds,$$

$$X_N^{(n)} \equiv E_{P_n} \left[\sum_{j=1}^N (F(t), \psi_j)^2 \right] = \sum_{j=1}^N \int_0^t (Q(\xi_n(s), x, y), \psi_j(x) \psi_j(y)) ds$$

and

$$X_N \equiv \sum_{j=1}^N \int_0^t (Q(s, x, y), \psi_j(x) \psi_j(y)) ds.$$

Then, $X_N^{(n)} \uparrow X^{(n)}$ as $N \rightarrow \infty$, $X_N \uparrow X$ as $N \rightarrow \infty$, $X_N^{(n)} \rightarrow X_N$ as $n \rightarrow \infty$ and $X^{(n)} \rightarrow X$ as $n \rightarrow \infty$. Applying Lemma 2.4.13, we have $X^{(n)} - X_N^{(n)}$ is uniformly small for all n when N is large. From this and the martingale inequality

$$\begin{aligned} & P_n \left(\sup_{0 \leq s \leq t} \sum_{j=N+1}^{\infty} (F(s), \psi_j)^2 > \delta \right) \\ & \leq \frac{4}{\delta} E_{P_n} \left[\sum_{j=N+1}^{\infty} (F(t), \psi_j)^2 \right] \end{aligned}$$

the Lemma follows.

Lemma 2.4.15

$$E_{P_n} [(F(t), f) - (F(s), f)]^4 = \left[\int_s^t Q(\xi_n(u), x, y; f(x)f(y)) du \right]^2.$$

Proof:

This is obtained by the direct calculation.

Remark 2.4.16

From Lemma 2.4.14 and Lemma 2.4.15, the uniform tightness of $\{P_n\}$ in $P(U_2)$ follows.

THEOREM 2.4.17

Let P be a limit point of $\{P_n\}$. Then $P \times \mu \times \mu$ is a solution to the martingale problem:

$$\langle F(t)(x)F(t)(y) \rangle = \int_0^t Q(s, x, y) ds.$$

Proof:

(1) The martingale property follows from $P_{n_j} \rightarrow P$ weakly and

$\max_{0 \leq t \leq T} |(F(t), f)|$ is uniformly integrable, i.e., putting $\max_{0 \leq t \leq T} |(F(t), f)| = \|(F, f)\|_T$,

$$\lim_{N \rightarrow \infty} \sup_{n \in \{N\}} \int_{\|(F, f)\|_T \geq N} \|(F, f)\|_T^2 P_n(dF) = 0.$$

(2) Square integrability follows from

$$\lim_{N \rightarrow \infty} \sup_{n \in \{N\}} \int_{\|(F, f)\|_T^2 \geq N} \|(F, f)\|_T^2 P_n(dF) = 0.$$

(3) $F(t)(x)F(t)(y) - \int_0^t Q(\xi_n(s), x, y) ds$ is a $P_n \times \mu \times \mu$ martingale.

Let $n \rightarrow \infty$. Then $F(t)(x)F(t)(y) - \int_0^t Q(s, x, y) ds$ is a $P \times \mu \times \mu$ martingale.

To prove the convergence we use the fact that $\|(F, f)\|_T^4$ is uniformly P_n -bounded in view of the martingale inequality.

$$\begin{aligned} E_{P_n} \|(F, f)\|_T^4 &\leq \left(\frac{4}{3}\right)^4 E_{P_n} [(F(T), f)]^4 \\ &= \left(\frac{4}{3}\right)^4 \left[\int_0^T (Q(\xi_n(s), x, y), f(x)f(y)) ds \right]^2. \end{aligned}$$

§2.5 Stochastic differential equations.

Given the solution $P \times \mu \times \mu$ to the martingale problem:

$$\langle F(t)(x)F(t)(y) \rangle = \int_0^t Q(s, x, y) ds,$$

obtained in the previous section, with $F(t)(x)$ denoted by $W(t, x)$, we would like to consider a stochastic differential equation:

$$dX_t = \sqrt{X_t} dW(t, x) \quad (2.5.1)$$

$$X_0 = f(x) \geq 0, \quad f \in L_2(\mu).$$

Since $W(t, x)$ is spatially correlated, the solution X_t will be spatially correlated.

Since the square root function is not Lipschitz continuous, the method will require the compactness argument. Although it is possible to calculate the point-wise

characteristic functional (see the next section), it is left for the future to calculate the spatial correlation. In view of the Poisson randomization only the point-wise characteristic functional will be necessary.

We approximate the problem in the space variable. Let $W_n(t, x)$ denote the n -th approximation of $W(t, x)$ (see Definition 2.1.7). That is, we consider

$$dX_t = \sqrt{X_t} dW_n(t, x), \quad (2.5.2)$$

with

$$X_0 = f_n(x),$$

where $f_n(x)$ is the n -th approximation of f .

Lemma 2.5.1

$W_n(t, x)$ is convergent to $W(t, x)$ in $M_2^C(B_t(U_2) \times B(S))$.

Proof:

It suffices to show

$$\lim_{n \rightarrow \infty} E_P \int (W(t, x) - W_n(t, x))^2 \mu(dx) = 0.$$

In fact,

$$\begin{aligned} & E_P \int (W(t, x) - W_n(t, x))^2 \mu(dx) \\ &= E_P \int W(t, x)^2 \mu(dx) - 2E_P \int W(t, x) W_n(t, x) \mu(dx) + E_P \int W_n(t, x)^2 \mu(dx). \end{aligned}$$

The first term = $\int_0^t \int Q(s, x, x) \mu(dx) ds$, the second term = $-2 \times \sum_j (\mu(E_j^n))^{-1} \int_0^t (Q(s, x, y), I_{E_j^n}(x) I_{E_j^n}(y)) ds$ and the third term = $-\frac{1}{2} \times$ (the second term).

Hence,

$$\begin{aligned} & E_P \int (W(t,x) - W_n(t,x))^2 \mu(dx) \\ &= \sum_j (\mu(E_j^n))^{-1} \int_0^t ((Q(s,x,x) - Q(s,x,y)); I_{E_j^n}(x) I_{E_j^n}(y)) ds. \end{aligned}$$

Let $\epsilon > 0$ be given. Let n be sufficiently large so that $|Q(s,x,x) - Q(s,x,y)| < \epsilon$ for all $(x,y) \in E_j^n \times E_j^n$, $j = 1, 2, \dots$. This is possible, since Q is uniformly continuous and $\text{diam}(E_j^n)$ is uniformly small for large n . Therefore,

$$\begin{aligned} & E_P \int (W(t,x) - W_n(t,x))^2 \mu(dx) \\ &\leq \sum_j (\mu(E_j^n))^{-1} \epsilon (\mu(E_j^n))^2 t \\ &= \epsilon \mu(K) t, \end{aligned}$$

where $K = \text{supp}(\mu)$, which is assumed to be compact. Therefore the lemma is proved.

We next approximate the problem in the time variable. Introducing

$$\xi_m(t) = \sum_k t_k^m I_{[t_k^m, t_{k+1}^m)}(t),$$

we consider

$$dX_t = \sqrt{X_{\xi_m(t)}} dW_n(t,x), \quad (2.5.3)$$

$$X_0 = f_n(x).$$

The solution is obtained, inductively, in each time interval $[t_k^m, t_{k+1}^m)$ as follows.

Let x be fixed. Then,

$$X_t = X(t_k^m) + \sqrt{X(t_k^m)} (W_n(t, x) - W_n(t_k^m, x))$$

$$X_0 = f_n(x),$$

till X_t hits 0. Here we introduce the stopping time σ which is a function of the sample path $W \in U_2$ and the location $x \in S$, namely, $\sigma(W, x)$ is the first time when X_t hits 0. We have:

$$(1) \quad X_{t \wedge \sigma} \text{ belongs to } M_2^c(\mathcal{B}_t(U_2) \times \mathcal{B}(S)), P \times \mu.$$

$$(2) \quad \text{In } \mathcal{C}(\mathcal{B}_t(U_2) \times \mathcal{B}(S)),$$

$$\langle (X_{t \wedge \sigma} - X_0)^2 \rangle = \int_0^{t \wedge \sigma} X_{\xi_m}(s) d \langle W_n(s) \rangle^2$$

Assuming $x \in E_j^n$, then

$$\langle (X_{t \wedge \sigma} - X_0)^2 \rangle = [\mu(E_j^n)]^{-2} \int_0^t X_{\xi_m}(s) ((Q(s, y, z), I_{E_j^n}(y) I_{E_j^n}(z))) ds.$$

Lemma 2.5.2

Let $X_n^m(t)$ denote the solution X_t of (2.5.3). Let P_n^m denote the probability measure on U_2 induced by $X_n^m(t)$. Let $t_k^m = \frac{k}{2^m}$. Then $\{P_n^m: m = 1, 2, \dots\}$ is uniformly tight in $P(U_2)$.

Proof:

It suffices to show that $\{X_n^m(t)(x): m = 1, 2, \dots\}$ is uniformly tight for each point x . Let x belong to E_j^n . Put

$$\eta(s) = (\mu(E_j^n))^{-2} (Q(s, x, y), I_{E_j^n}(x) I_{E_j^n}(y)) .$$

Then

$$\begin{aligned} < (X_n^m(t) - X_n^m(s))^2 > = \int_{s \wedge \sigma}^{t \wedge \sigma} X_n^m(\xi_m(u)) \eta(u) du \\ &\leq \int_s^t X_n^m(\xi_m(u)) \eta(u) du . \end{aligned}$$

Therefore

$$E(X_n^m(t) - X_n^m(s))^2 \leq f_n(x) \int_s^t \eta(u) du .$$

$$\begin{aligned} E(X_n^m(t))^2 &= E(X_n^m(t) - f_n(x))^2 + f_n(x)^2 \\ &\leq f_n(x) \int_0^t \eta(u) du + f_n(x)^2 . \end{aligned}$$

Applying Ito's formula,

$$\begin{aligned} E|X_n^m(t) - X_n^m(s)|^3 &= \frac{3\lambda^2}{2} \int_s^t E \left[|X_n^m(u) - X_n^m(s)| d < (X_n^m(u) - X_n^m(s))^2 > \right] \\ &\leq 3 \int_s^t \sqrt{E(X_n^m(u) - X_n^m(s))^2} \sqrt{E(X_n^m(\xi_m(u)))^2} \eta(u) du . \\ &\leq \text{const. } (t-s)^{1.5} , \end{aligned}$$

whenever $t, s \in [0, T]$ and const. does not depend on m . Therefore, $\{P_n^m; m = 1, 2, \dots\}$ is uniformly tight.

THEOREM 2.5.3

Let P_n be a limit point of $\{P_n^m : m=1, 2, \dots\}$. Then $P_n \times \mu$ is a solution to the martingale problem :

assuming $x \in E_j^n$,

$$\langle (F(t)(x) - f_n(x))^2 \rangle = \int_0^t F(s)(x) Q_n(s, x) ds$$

where $Q_n(s, x) = (\mu(E_j^n))^{-2} (Q(s, y, z), I_{E_j^n}(y) I_{E_j^n}(z))$.

Proof:

$P_n^m \times \mu$ is a solution to the martingale problem :

$$\langle (F(t) - f_n(x))^2 \rangle = \int_0^{t \wedge \sigma} F(\xi_m(s))(x) Q_n(s, x) ds ..$$

Noting that both sides are uniformly integrable, which can be shown by direct calculation, we let m tend to infinity, so that

$$\langle (F(t) - f_n(x))^2 \rangle = \int_0^t F(s)(x) Q_n(s, x) ds.$$

We remark P_n is supported on the cone $\{ \sum_j a_j(t) I_{E_j^n}(x) : a_j(t) \text{ is a nonnegative continuous function of } t \text{ which stays at } 0 \text{ after it hits } 0 \text{ for the first time and}$

$a_j(0) = f_n(x) \text{ at } x \in E_j^n \}$.

§2.6 The point-wise characteristic functional of P_n .

Since P_n is supported on the cone mentioned above, we can consider the point-wise characteristic functional:

$$E_{P_n} [\exp(i\theta F(t)(x))] .$$

This section is devoted to the following theorem.

THEOREM 2.6.1

$$E_{P_n} [\exp(i\theta F(t)(x))] = \exp \left\{ i \frac{\theta f_n(x)}{1 - \frac{i}{2} \theta \int_0^t Q_n(s, x) ds} \right\}$$

Proof:

Let x be fixed. Note that $F(0)(x) = f_n(x)$. Putting $f_n(x) = \zeta$, we consider the following function $b(t, \zeta)$.

$$b(t, \zeta) = \exp \left\{ i \frac{\theta \zeta}{1 - \frac{i}{2} \theta \int_0^t Q_n(s, x) ds} \right\} .$$

It is easy to check that:

$$(1) \quad b(0, \zeta) = \exp(i\theta \zeta) .$$

$$(2) \quad \frac{\partial b}{\partial t} - \frac{1}{2} \zeta Q_n(t, x) \frac{\partial^2 b}{\partial \zeta^2} = 0 .$$

For fixed $t_0 > 0$ and $t \in [0, t_0]$, we set

$$c(t, \zeta) = b(t_0 - t, \zeta) .$$

Putting $\zeta = F(t)(x)$, we consider the process $c(t, F(t)(x))$.

Applying Ito's formula, we have

$c(t, F(t)) - c(0, F(0))$ is a martingale

$$+ \int_0^t \frac{\partial c}{\partial t}(s, F(s)) ds + \frac{1}{2} \int_0^t \frac{\partial^2 c}{\partial \zeta^2}(s, F(s)) d\langle F(s) \rangle_s$$

$$= \int_0^t \left[-\frac{\partial b}{\partial t}(t_0 - s, F(s)) + \frac{1}{2} \frac{\partial^2 b}{\partial \zeta^2}(t_0 - s, F(s)) F(s) \chi_n(s, x) \right] ds$$

$= 0$, from (2) above.

Therefore, $c(t, F(t)) - c(0, F(0))$ is a martingale so that $E(c(t, F(t))) = E(c(0, F(0)))$ i.e., $E(b(t_0 - t, F(t))) = E(b(t_0, F(0)))$.

Letting $t \uparrow t_0$, we have $E(b(0, F(t_0))) = E(b(t_0, f_n(x)))$, that is

$$E[\exp i\theta F(t_0)(x)] = \exp \left\{ \frac{i\theta f_n(x)}{1 - \frac{1}{2} \theta \int_0^{t_0} Q_n(s, x) ds} \right\}.$$

§2.7 The Poisson randomization of P_n with intensity N_μ .

From the property of P_n (see the support of P_n in the proof of Theorem 2.5.3), we can directly consider the Poisson randomization. Let $X_n^N(t)$ denote the corresponding V -valued process. We remark that after the Poisson randomization, the particles pick up spatially nonhomogeneous independent evolutions of mass, thus only the point-wise characteristic functional of P_n is needed (see §2.2).

From (1.1), using g instead of f , the characteristic functional of $X_n^H(t)$ is as follows.

$$\begin{aligned} & E[\exp\{i(X_n^H(t), g)\}] \\ &= \exp \left[\iint [-1 + \exp\{iF(t, x)g(x)\}] P_n(dF)(N_n)(dx) \right] \\ &= \exp \left[\int \left[-1 + \exp \left\{ \frac{ig(x)f_n(x)}{1 - \frac{1}{2}g(x) \int_0^t Q_n(s, x)ds} \right\} \right] N_n(dx) \right]. \end{aligned}$$

Now, let us relate N and P_n in the following manner. Let $r(x) = \frac{1}{N}$. Therefore $f_n(x) = \frac{1}{N}$ everywhere. Hence,

$$\lim_{n \rightarrow \infty} E[\exp i(X_n^H(t), g)]$$

$$= \exp \int \left[-1 + \exp \left\{ \frac{ig(x) \frac{1}{N}}{1 - \frac{1}{2}g(x) \int_0^t Q(s, x, x)ds} \right\} \right] N_n(dx),$$

since

$$Q_n(s, x) = (\mu(E_j^n))^{-2} (Q(s, y, z), I_{E_j^n}(y) I_{E_j^n}(z)),$$

assuming $x \in E_j^n$, is convergent to $Q(s, x, x)$ and $g \in C_c(S)$, where the convergence is the bounded convergence.

Now, let $N \rightarrow \infty$. Then

$$\lim_{N \rightarrow \infty} \lim_{n \rightarrow \infty} E[\exp i(X_n^H(t), g)]$$

$$= \exp \left\{ 1 \int \frac{g(x)}{1 - \frac{1}{2}g(x) \int_0^t Q(s, x, x)ds} \mu(dx) \right\},$$

Since it is equal to the derivative of the function,

$$\exp \frac{ig(x)y}{1 - \frac{1}{2} g(x) \int_0^t Q(s, x, x) ds}$$

of y evaluated at $y = 0$, and other terms of s and x are bounded so that we can apply the Lebesgue bounded convergence theorem to obtain the conclusion.

§2.8 The uniform tightness of X_n^N .

We would like to show that the limit of X_n^N is realized in V . Let P_n^N denote the probability measure induced on V by X_n^N .

THEOREM 2.8.1

$\{P_n^N; n, N\}$ is uniformly tight in $P(V)$.

Proof:

Assume that m -particles at $x_1, x_2, \dots, x_m$ are observed. Then, putting $X_n^N = X$ for simplicity,

$$(X(t) - X(s), g) = \sum_{k=1}^m F_k(t, x_k) g(x_k).$$

Under this condition, applying Ito's formula,

$$\begin{aligned} & E |(X(t) - X(s), g)|^3 \\ &= E \int_s^t |(X(u) - X(s), g)| d < (X_n^N(u) - X_n^N(s), g)^2 >. \end{aligned}$$

(since x_k 's have independent mass evolutions),

$$E \int_s^t |(X(u) - X(s), g)| \sum_{k=1}^m F_k(u, x_k) g^2(x_k) \eta(u, x_k) du$$

(introducing $T: t, s \in [0, T], c$: the bound of g and

$$\begin{aligned} D(T) &= \sup \{ |Q(s, x, y)| : 0 \leq s \leq T, (x, y) \in K \times K \} , \\ &\leq E \int_s^t |(X(u) - X(s), g)| (X(u) - X(0), 1) du \, c^2 D(T) \\ &\quad + E \int_s^t |(X(u) - X(s), g)| (X(0), 1) du \, c^2 D(T) \\ &\leq \int_s^t \sqrt{E(X(u) - X(s), g)^2} \sqrt{E(X(u) - X(0), 1)^2} du \, c^2 D(T) \\ &\quad + \int_s^t \sqrt{E(X(u) - X(s), g)^2} \frac{m}{N} du \, c^2 D(T). \end{aligned}$$

Noting that the $\{F_k\}$ are independent, this equals to

$$\begin{aligned} &\int_s^t \sqrt{\sum_{k=1}^m E(F_k(u) - F_k(s))^2 g(x_k)^2} \sqrt{\sum_{k=1}^m E(F_k(u) - F_k(0))^2} du \, c^2 D(T) \\ &+ \int_s^t \sqrt{\sum_{k=1}^m E(F_k(u) - F_k(s))^2 g(x_k)^2} du \, \frac{m}{N} c^2 D(T) \end{aligned}$$

(from the martingale equation,)

$$\begin{aligned} &\leq \int_s^t \sqrt{\sum_k \frac{1}{N} \int_s^u Q_n(v, x_k) dv} \sqrt{\sum_k \frac{1}{N} \int_0^u Q_n(v, x_k) dv} du \, c^3 D(T) \\ &+ \int_s^t \sqrt{\sum_k \frac{1}{N} \int_s^u Q_n(v, x_k) dv} du \, \frac{m}{N} c^3 D(T) \\ &\leq \frac{m}{N} \int_s^t \sqrt{u-s} du \, c^3 (D(T))^2 \sqrt{T} \\ &+ \frac{m}{N} \int_s^t \sqrt{u-s} du \, c^3 D(T) \sqrt{D(T)} \end{aligned}$$

(introducing another constant $C(T)$),

$$\leq \left(\frac{m}{N} + \frac{m}{N} \sqrt{\frac{m}{N}} \right) C(T) (t-s)^{1.5}.$$

Therefore the total expectation is less than

$$\sum_m \frac{(N_\mu(K))^m}{m! \exp(N_\mu(K))} \left(\frac{m}{N} + \frac{m}{N} \sqrt{\frac{m}{N}} \right) C(T) (t-s)^{1.5}.$$

Since $\sum_m \frac{(N_\mu(K))^m}{m! \exp(N_\mu(K))} \left(\frac{m}{N} + \frac{m}{N} \sqrt{\frac{m}{N}} \right)$ is uniformly bounded, say by D' , we have for all N and n ,

$$E |X_n^N(t) - X_n^N(s, g)|^3 \leq D' C(T) (t-s)^{1.5}.$$

Therefore, together with the uniform tightness of $P_n^N(0)$, (Theorem 1.2.15), the uniform tightness of $\{P_n^N\}$ follows.

§2.9 Martingale property of P .

THEOREM 2.9.1

Let P be a limit point of $\{P_n^N: n, N \geq 1\}$ on V . Then P is a solution to the martingale problem:

$$\langle v(t) - v(0), f \rangle \langle v(t) - v(0), g \rangle = \int_0^t (Q(s, x, x) v(s), fg) ds,$$

with the initial condition $v(0) = \mu$.

Proof:

P_n^N is a solution to the martingale problem:

$$\langle (v(t)-v(0); f) (v(t)-v(0), g) \rangle = \int_0^t (Q_n(s, x) v(s), fg) ds$$

with the initial condition $P_n^N(0)$. Here $P_n^N(0)$ is the following compound Poisson random measure: the mass of each particle is $\frac{1}{N}$ and the location and the number of particles are distributed as a Poisson random field with the intensity $N\mu$.

Since $\lim_n \sup_{s \in [0, T]} |Q(s, x, x) - Q_n(s, x)| = 0$, then $\lim_n Q_n(s, x) v(s) =$

$Q(s, x, x) v(s)$ in V .

From higher moment conditions, $(v(t)-v(0), f) (v(t)-v(0), g)$ and $\int_0^t (Q_n(s, x) v(s), fg)$ are uniformly P_n^N -integrable. Therefore P is a martingale and satisfies the martingale equation.

Remark 2.9.2

Since P is a limit point of (P_n^N) , P has the characteristic functional,

$$E_P[\exp i(v(t), g)] = \exp \left[i \int \frac{g(x)}{1 - \frac{1}{2} g(x) \int_0^t Q(s, x, x) ds} \mu(dx) \right].$$

CHAPTER III. STOCHASTIC EVOLUTION EQUATIONS RELATED TO THE MARTINGALE PROBLEM (3).

§3.0 Introduction

For the solution of the martingale problem:

$$\langle (W(t), f)(W(t), g) \rangle = \int_0^t (Q(s, x, y), f(x)g(y)) ds, \quad (3.0.1)$$

obtained in Theorem 2.4.1, here $F(t)$ is denoted by $W(t)$ and called space-time white noise, we would like to consider the following stochastic evolution equation:

$$\begin{aligned} dX_t &= A X_t dt + X_t dW_t \\ X_0 &= f. \end{aligned}$$

Let us assume that the operator A generates a semi-group T_t . There are two approaches to this problem. One is to view the equation as a random linear perturbation, from which we are led to the formal solution:

$$X_t(f) = \sum_{n=0}^{\infty} X_n(t)(f), \quad (3.0.2)$$

where

$$X_n(t)(f) = \int_0^t T(t-s) \left[X_{n-1}(s)(f) dW_s \right] \quad (3.0.3)$$

$$X_0(t)(f) = T(t)(f).$$

We note that $X_{n-1}(s)(f)dW(s)$ denotes the multiplication of $X_{n-1}(s)(f)$ and $dW(s)$, while the operator $T(t-s)$ acts on $X_{n-1}(s)(f)dW_s$.

The other is to view the equation as a combination of:

$$(1) \quad dX_t = A X_t dt$$

and

$$(2) \quad dX_t = X_t dW_t.$$

From (1), we have the semi-group $T(t)$ which is not random. From (2), we have (see §3.3):

$$S(t) = \exp \left[W(t, x) - \frac{1}{2} \int_0^t Q(s, x, x) ds \right], \quad (3.0.4)$$

which is a random multiplication operator. $S(t)$ is not a semi-group, but a random evolution. However we can combine the two evolutions $T(t)$ and $S(t)$ in the following manner via Trotter's product formula. We put:

$$S(t_k^n, t_{k+1}^n) = \exp \left[W(t_{k+1}^n) - W(t_k^n) - \frac{1}{2} \int_{t_k^n}^{t_{k+1}^n} Q(s) ds \right] \quad (3.0.5)$$

for

$$\left\{ t_k^n = \frac{k}{2^n} \right\}, \quad n = 1, 2, \dots$$

The $\{S(t_k^n, t_{k+1}^n)\}$ are also multiplication operators. We define:

$$X^{(n)}(t) = T(t - t_{k_0}^n) S(t_{k_0}^n, t) \prod_k T\left(\frac{1}{2^n}\right) S(t_k^n, t_{k+1}^n)$$

where the product is in the sense of operators, k_0 is the maximum integer for which $t_{k_0}^n$ does not exceed t .

The question arises as to whether

$$\lim_{n \rightarrow \infty} X^{(n)}(t)(f) = X(t)(f) \quad (3.0.6)$$

§3.1 The Basic Setting.

To avoid technical difficulties and to show the convergence smoothly, we introduce the following setting:

$$S = R^d,$$

$$A = \frac{1}{2} \Delta,$$

$$Q(t, x, y);$$

(1) $\in C([0, \infty)) \cap C_c^2(R_x^d \times R_y^d)$, i.e., continuous and twice differentiable with respect to x and y with compact support,

(2) the above support of Q in $(R^d \times R^d)$ is uniformly contained in some fixed compact set $K \times K$ for all t ,

(3) $dx(K) = 1$, (the Lebesgue measure of K),

(4) symmetric in x and y ,

(5) positive definite.

Remark 3.1.1

$$T(t)(f) = \left(\frac{1}{2\pi t}\right)^{\frac{d}{2}} \int f(y) \exp\left[-\frac{|x-y|^2}{2t}\right] dy,$$

where $|x-y|^2$ denotes $(x_1 - y_1)^2 + (x_2 - y_2)^2 + \dots + (x_d - y_d)^2$.

If $f \in C_c^2(R^d)$,

$$T(t)(f) - f = \int_0^t T(s) \left(\frac{1}{2} \Delta f\right) ds.$$

Therefore

$$\|T(t)(f) - f\| \leq \frac{d}{2} \|f\|_2 t,$$

where $\| \cdot \|$ denotes the sup norm and $\|f\|_2 = \sup(\|f\|, \|D_x f\|, \|D_{x_i} D_{x_j} f\|)$, $D_x f$ the first derivatives, $D_{x_i} D_{x_j} f$ the second derivatives.

The condition (3) can be relaxed. If $dx(K) \neq 1$, then we consider $(dx(K))^{-1} dx$ instead of the Lebesgue measure dx .

§3.2 Stochastic integrals. (The first approach)

Since the operator $T(t-s)$ formally acts on $X_{n-1}(s)(f)dW(s)$, we must define it explicitly.

Remark 3.2.1 For example.

$$X_1(t)(f) = \int_0^t T(t-s)[T(s)(f)dW(s)], f \in C_c^\infty(\mathbb{R}^d) \quad (3.2.1)$$

is defined in the following manner.

Let t_0 be fixed. Let t vary in the interval $[0, t_0]$. Let $t_k^n = \frac{k}{2^n}$ as before. Put

$$Y_n(t, W, x) = \sum_{k; t_k^n < t} T(t - t_k^n)[T(t_k^n)(f)(W(t_{k+1}^n \wedge t) - W(t_k^n \wedge t))]$$

where T is acting as an integral operator on $L_2(\mathbb{R}^d)$.

Y_n is a measurable function of (t, W, x) with respect to the product measure $dt dP dx$. If Y_n is convergent in $L_2(dt dP dx)$, then we define the above integral $X_1(t)$ by the t -section of $\lim_n Y_n$. Before giving the proof we make several remarks.

Remark 3.2.2

The space-time white noise $W(t, x)$ associated with $q(t, x, y) dx dy$, here $\mu \in \mathbb{M}(\mathbb{R}^d)$ is assumed to be the Lebesgue measure dx , disturbs the bounded region. For the continuity of $W(t, x)$ with respect to x , we refer to Dudley [8] and Jain-Marcus [14].

Remark 3.2.3

The expectation E and the operator T are interchangeable.

Remark 3.2.4

If $k \neq j$, then $T(s_1)[T(s_2)(f)(W(t_{k+1}^n \wedge t) - W(t_k^n \wedge t))]$ and $T(s_3)[T(s_4)(f)(W(t_{j+1}^n \wedge t) - W(t_j^n \wedge t))]$ are P -orthogonal.

Notation 3.2.5

Let $(T_x T_y)(s)$ denote the product semi-group on $L_2(\mathbb{R}^d \times \mathbb{R}^d)$. Let $(\bar{T}_x \bar{T}_y)(s)(f) = (T_x T_y)(s)(f)(x, y)|_{x=y}$.

Lemma 3.2.6

Y_n of (3.2.1) is convergent in $L_2(dt dP dx)$.

Proof:

Let $n < m$. Then,

$$E \int_0^t \int (Y_n - Y_m)^2 dt dx$$

$$= E \int_0^t \int \sum_{j,k; t_j \leq t_k^m < t_{j+1}^n} \left\{ T(t - t_k^m) (W(t_{k+1}^m \wedge t) - W(t_k^m \wedge t)) [T(t_k^m) f] \right. \\ \left. - T(t - t_j^n) (W(t_{k+1}^m \wedge t) - W(t_k^m \wedge t)) [T(t_j^n) f] \right\}^2 dt dx$$

$$(\text{put } \Delta W = [W(t_{k+1}^m \wedge t) - W(t_k^m \wedge t)],)$$

$$\leq 2 \sum \sum E \int_0^t \int \left\{ T(t - t_k^m) (\Delta W) [T(t_k^m) - T(t_j^n)] f \right\}^2$$

$$+ \left\{ [T(t - t_k^m) - T(t - t_j^n)] (\Delta W) [T(t_j^n) f] \right\}^2 dt dx$$

$$= 2 \sum \sum \int \int \bar{T}_x \bar{T}_y (t - t_k^m) \int_{t_k^m \wedge t}^{t_{k+1}^m \wedge t} Q(s, x, y) ds [T_x(t_k^m) - T_x(t_j^n)] [T_y(t_k^m) - T_y(t_j^n)] (f(x) f(y)) \\ + \frac{[T_x(t - t_k^m) - T_x(t - t_j^n)] [T_y(t - t_k^m) - T_y(t - t_j^n)]}{[T_x(t - t_k^m) - T_x(t - t_j^n)] [T_y(t - t_k^m) - T_y(t - t_j^n)]} \int_{t_k^m \wedge t}^{t_{k+1}^m \wedge t} Q(s, x, y) ds T_x T_y(t_j^n) (f(x) f(y)) dt dx$$

$$\leq \text{const. } \frac{1}{2^n}$$

Therefore Y_n is convergent in $L_2(dt, dP dx)$.

Remark 3.2.7

$X_n(t)(f)$ is defined inductively as

$$X_n = \int_0^t T(t-s) \left[X_{n-1}(s)(f) dW(s) \right]. \quad (3.2.2)$$

Remark 3.2.8

$(X(t)(f), g) = \sum_n (X_n(t)(f), g)$ is a sum of orthogonal processes.

Remark 3.2.9

$$E(X_n(t)(f), g)^2 = (Z_n(t)(f(x)f(y)), g(x)g(y)), \quad (3.2.3)$$

where

$$Z_n(t)(f(x)f(y)) = \int_0^t \int_0^{t_1} \dots \int_0^{t_{n-1}} T_x T_y(t-t_1) Q(t_1, x, y)$$

$$T_x T_y(t_1-t_2) Q(t_2, x, y) \dots T_x T_y(t_{n-1}-t_n) Q(t_n, x, y) T_x T_y(t_n) [f(x)f(y)] dt_1 \dots dt_n.$$

$\sum_n Z_n(t)$ is the time non-homogeneous semi-group with the infinitesimal operator:

$$\left(\frac{1}{2} \Delta_x \right) + \left(\frac{1}{2} \Delta_y \right) + Q(t, x, y)$$

where Q is the multiplication operator.

Remark 3.2.10

$Z_n(t)(f(x)f(y))$ can be described as

$$Z_n = E(X_n(t)(f)(x) X_n(t)(f)(y)). \quad (3.2.4)$$

§3.3 The Second Approach.

Consider

$$dX_t = X_t dw_t$$

$$X_0 = f.$$

By successive approximation

$$X_t = \sum_n X_{(n)}(t)(f),$$

where

$$X_{(0)}(t)(f) = f, \text{ and } X_{(n)}(t) = \int_0^t X_{(n-1)}(s)(f) dw(s).$$

Remark 3.3.1 (Hermite polynomial)

$$H_n(x, y) = \frac{(-y)^n}{n!} \exp\left(\frac{x^2}{2y}\right) \frac{d^n}{dx^n} \exp\left(-\frac{x^2}{2y}\right), \quad y > 0, \quad n \geq 0.$$

Then n -th Hermite polynomial $H_n(x, y)$ with the parameter y satisfies

$$D_x H_n = H_{n-1} \quad (3.3.1)$$

$$D_{xx} H_n = \frac{x}{y} H_{n-1} - \frac{n}{y} H_n \quad (3.3.2)$$

$$H_n - \frac{x}{n} H_{n-1} + \frac{y}{n} H_{n-2} = 0 \quad (3.3.3)$$

$$D_y H_n = \frac{n}{y} H_n - \frac{x}{y} H_{n-1} + \frac{1}{2} H_{n-2}. \quad (3.3.4)$$

Therefore, from (3.3.4) and (3.3.2),

$$D_y H_n + \frac{1}{2} D_{xx} H_n = \frac{n}{2y} H_n - \frac{x}{2y} H_{n-1} + \frac{1}{2} H_{n-2},$$

from (3.3.3), the above = 0, i.e.,

$$D_y H_n + \frac{1}{2} D_{xx} H_n = 0, \quad (3.3.5)$$

where D_x , (D_{xx}, D_y) , respectively) denotes "the derivative of" with respect to x .

Remark 3.3.2 (Hermite polynomial expansion).

$$\exp\left(x - \frac{1}{2}y\right) = \sum_{n=0}^{\infty} H_n(x, y), \quad y > 0. \quad (3.3.6)$$

See Hida [10] and Ito [12] for Hermite polynomials.

Remark 3.3.3

$$\text{Remember that } S(t) = \exp\left[W(t, x) - \frac{1}{2} \int_0^t Q(s, x, x) ds\right].$$

From (3.3.6),

$$S(t) = \sum_{n=0}^{\infty} H_n(W(t, x), \int_0^t Q(s, x, x) ds). \quad (3.3.7)$$

Applying the generalized Ito's formula (see Kunita-Watanabe [15]) to $H_n(W(t, x), \int_0^t Q(s, x, x) ds)$, we have

$$H_n = \int_0^t D_x H_n dW + \int_0^t D_y H_n Q(s, x, x) ds + \frac{1}{2} \int_0^t D_{xx} H_n d\langle W^2 \rangle,$$

from (3.3.1), the first term = $\int_0^t H_{n-1} dW$, since $d\langle W^2 \rangle = Q(s, x, x) ds$, the

third term = $\frac{1}{2} \int_0^t D_{xx} H_n Q(s, x, x) ds$. From (3.3.5), (the second term) + (the third

term) = 0. Therefore, $H_n = \int_0^t H_{n-1} dW(s)$. Inductively,

$$H_n = \int_0^t \int_0^{t_1} \dots \int_0^{t_{n-1}} dW(t_n) dW(t_{n-1}) \dots dW(t_1). \quad (3.3.8)$$

Now, it is easy to see that

$$X_{(n)}(t, f) = f(x) H_n(W(t, x), \int_0^t q(s, x, x) ds). \quad (3.1.1)$$

Lemma 3.1.4

$S(t) = \exp \left[W(t, x) - \frac{1}{2} \int_0^t q(s, x, x) ds \right]$ is a solution to the stochastic differential equation:

$$dX_t = X_t dW_t$$

$$X_0 = 1.$$

Further

$$S(t) = \sum_n H_n(W(t, x), \int_0^t q(s, x, x) ds)$$

is convergent in M_2^c .

Proof:

From the Ito's formula, $S(t)$ satisfies the stochastic differential equation.

For convergence it suffices to show $E(S(t))^2 < \infty$. In fact,

$$\begin{aligned} E(S(t))^2 &= \sum_{n=0}^{\infty} \int_0^t \int_0^{t_1} \dots \int_0^{t_{n-1}} q(t_1) q(t_2) \dots q(t_n) dt_1 dt_2 \dots dt_n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\int_0^t q(s) ds \right)^n \\ &= \exp \left[\int_0^t q(s, x, x) ds \right] < \infty. \end{aligned}$$

Lemma 3.3.5

$S(t)$ is a solution to the martingale problem:

$$\langle S(t), f \rangle \langle S(t), g \rangle = \int_0^t (q(s, x, y) S(s)(x) S(s)(y), f(x) g(y)) ds.$$

Proof:

It is obtained by direct calculation.

§3.4 The Convergence

We would like to show the convergence of (3.0.5) in $L_2(dP)$. For simplicity, let $X^{(n)}(t)(f)$ be denoted by Y_n .

Remark 3.4.1

$$E(S_t(x) S_t(y)) = \exp \int_0^t q(s, x, y) ds. \quad (3.4.1)$$

Remark 3.4.2

$$E(Y_n(t)(f)(x) Y_n(t)(f)(y)) \quad (3.4.2)$$

$$= T_x T_y (t - t_k^n) \exp \int_{t_k^n}^t q(s, x, y) ds \prod_k T_x T_y \left(\frac{1}{2^n} \right) \exp \int_{t_k^n}^{t_{k+1}^n} q(s, x, y) ds \dots f(x) f(y),$$

which converges to $T_x T_y * Q(t, x, y) (f(x) f(y))$, where $T_x T_y * Q$ denotes the combined semi-group.

Remark 3.4.3

From Remark 3.2.8, Remark 3.2.9,

$$E(X(t)(f)(x)X(t)(f)(y)) = T_x^* T_y^* \varphi(t, x, y)(f(x)f(y)). \quad (3.4.3)$$

To calculate $E(Y_n(t)(f)(x)X(t)(f)(y))$, denoted by I_n , we consider several remarks.

Remark 3.4.4

$$I_n = \sum_m E(Y_n(t)(f)(x)X_m(t)(f)(y)). \quad (3.4.4)$$

Remark 3.4.5

$$X_m(t)(f)(y) = \int_0^t \int_0^{t_1} \dots \int_0^{t_{m-1}} T(t_1 - t_2) \left[\dots \left[T(t_{m-1} - t_m) \left[T(t_m) f \, dw(t_m) \right] \dots dw(t_2) \right] dw(t_1) \right] (y). \quad (3.4.5)$$

$$\int_0^t \int_0^{t_1} \dots \int_0^{t_{m-1}} = \sum \int_{t_{k(1)}^{t_{k(1)+1}^n} \wedge t}^{t_{k(2)+1}^n} \int_{t_{k(2)}^{t_{k(2)+1}^n} \wedge t_1}^{t_{k(m)+1}^n} \dots \int_{t_{k(m)}^{t_{k(m)+1}^n} \wedge t_{m-1}}^{t_{m-1}^n}$$

$$\sum_{k(1), \dots, k(m); k(1) \geq k(2) \geq \dots \geq k(m)}$$

For $k(1) \geq \dots \geq k(m)$, let $1 \leq i(1) < i(2) < \dots < i(N) = m$ denote where the jumps occur, i.e.,

$$k(1) = \dots = k(i(1)) > k(i(1)+1) = \dots = k(i(2)) > \dots$$

Remark 3.4.6

$$S(t_k^n, t_{k+1}^n) = \sum_j H_j(t_k^n) \quad (3.4.6)$$

where H_j is the j -th Hermite polynomial and

$$H_j(t_k^n) = \int_{t_k^n}^{t_{k-1}^n} \int_{t_k^n}^{t_1^n} \dots \int_{t_k^n}^{t_{j-1}^n} dw(t_j) \dots dw(t_2) dw(t_1) \quad (3.4.7)$$

Remark 3.4.7

$$Y_n(t)(f) = \sum_m \sum_{j_0+j_1+\dots+j_{k_0}=m} T(t-t_{k_0}^n) H_{j_{k_0}}(t_{k_0}^n) \dots T\left(\frac{1}{2^n}\right) H_{j_{k_0-1}}(t_{k_0-1}^n) \dots T\left(\frac{1}{2^n}\right) H_{j_0}(t_0^n) f \quad (3.4.8)$$

Remark 3.4.8

In Remark 3.4.7, the corresponding term for $k(1), \dots, k(m)$ is

$$H_{i(1)}(t_{k(i(1))}^n), H_{i(2)-i(1)}(t_{k(i(2))}^n), \dots, H_{i(N)-i(N-1)}(t_m^n) \quad \text{Let } j_0, j_1, \dots, j_{k_0} \text{ be}$$

as above. The corresponding operator is

$$T\left(t-t_{k_0}^n + \frac{1}{2^n} (k_0 - k(i(1)))\right) H_{i(1)}(t_{k(i(1))}^n) T\left(\frac{1}{2^n} (k(i(1)) - k(i(2)))\right) \dots H_{i(2)-i(1)}(t_{k(i(2))}^n) \dots T\left(\frac{1}{2^n} (k(i(N-1)) - k(i(N)))\right) H_{i(N)-i(N-1)}(t_m^n) T\left(\frac{1}{2^n} k(m)\right) f \quad (3.4.9)$$

Now, the expectation of $Y_n(t)(f)(x)X(t)(f)(y)$ is the sum of

$$T_x(t - t_{k(i(1))}^n) \left[\int_{t_{k(i(1))}^n}^{t_{k(i(1))+1}^n} \dots \int_{t_{k(i(1))}^n}^{t_{i(1)-1}^n} T_y(t - t_1) Q(t_1, x, y) \right.$$

$$T_y(t_1 - t_2) Q(t_2) \dots T_y(t_{i(1)-1} - t_{i(1)}) Q(t_{i(1)}) T_y(t_{i(1)} - t_{k(i(1))}^n) dt_{i(1)} \dots dt_1 \Big]$$

$$T_x(t_{k(i(1))}^n - t_{k(i(2))}^n) \left[\int_{t_{k(i(2))}^n}^{t_{k(i(2))+1}^n} \dots \int_{t_{k(i(2))}^n}^{t_{i(2)-1}^n} T_y(t_{k(i(1))}^n - t_1) \right.$$

$$Q(t_1) T_y(t_1 - t_2) Q(t_2) \dots T_y(t_{i(2)-1} - t_{i(2)}) Q(t_{i(2)}) dt_{i(2)-1} \dots dt_1 \Big] \quad (3.4.10)$$

$$Q(t_{i(2)-1} - t_{i(1)}) T_y(t_{i(2)-1} - t_{k(i(2))}^n) dt_{i(2)-1} \dots dt_1 \Big] \dots T_x(t_{k(m)}^n) f(x) f(y) .$$

Summing, we obtain

$$E(Y_n(t)(f)(x)X(t)(f)(y))$$

$$T_x(t - t_{k_0}^n) (T_y * Q)(t_{k_0}^n, t) T_x\left(\frac{1}{2^n}\right) (T_y * Q)(t_{k_0-1}^n, t_{k_0}^n) \\ \dots T_x\left(\frac{1}{2^n}\right) (T_y * Q)(0, t_1^n) f(x) f(y) \quad (3.4.11)$$

where

$$(T_y * Q)(t_k^n, t_{k+1}^n)$$

$$= \sum_m \int_{t_k^n}^{t_{k+1}^n} \int_{t_k^n}^{t_1} \dots \int_{t_k^n}^{t_{m-1}^n} T_y(t_{k+1}^n - t_1) Q(t_1, x, y) T_y(t_1 - t_2) Q(t_2) \dots$$

$$\dots T_y(t_{m-1} - t_m) Q(t_m) T(t_m - t_k^n) dt_m \dots dt_2 dt_1 ,$$

which is the (t_k^n, t_{k+1}^n) -section of the combined time-nonhomogeneous semi-group

$(T_y * Q)(t)$ of $T_y(t)$ and $Q(t, x, y)$.

If we let $n \rightarrow \infty$ in (3.4.11), then it converges to

$$(T_x * T_y * Q)(f(x)f(y))(t).$$

Therefore, we have proved the following theorem.

THEOREM 3.4.9

The equation (3.0.6) holds in the sense that

$$\lim_n E(X^{(n)}(t)(f) - X(t)(f), g)^2 = 0, \quad \text{for } f \in C_c^\infty(\mathbb{R}^d), \quad g \in L_2(\mathbb{R}^d).$$

Corollary 3.4.10

$X_t(f)$ of (3.0.2) is non-negative almost everywhere.

CHAPTER IV. STOCHASTIC EVOLUTION EQUATIONS RELATED
TO THE MARTINGALE PROBLEM (2).

§4.0 Introduction

Given the solution of the martingale problem on $P(V)$,

$$\langle (v(t)-v(0), f) (v(t)-v(0), g) \rangle = \int_0^t (Q(s, x, x) v(s), fg) ds, \quad v(0) = \mu, \quad (4.0.1)$$

(see Theorem 2.9.1), we would like to consider the following related stochastic evolution equation, expressed in Dawson's form (see [6]),

$$dX_t = AX_t dt + dW_{\sqrt{X_t}}. \quad (4.0.2)$$

where $dW_{\sqrt{X_t}}$ is equivalent to the martingale problem (4.0.1). Since the equation is non-linear, the second approach of Chapter III is the appropriate one. That is, we consider separately,

$$dX_t = AX_t dt, \quad (4.0.3),$$

and

$$dX_t = dW_{\sqrt{X_t}}, \quad (4.0.4)$$

and combine them via Trotter's product formula.

We first let X_t evolve by equation (4.0.3) on the time interval $[0, t_1]$.

Then we let X_t evolve by equation (4.0.4) on the next time interval $[t_1, t_2]$ and so on.

We assume that the equation (4.0.3) is solved by the semi-group $T(t)$. We let $T^*(t)$ denote the dual semi-group of $T(t)$.

Then, putting $t_k^n = \frac{k}{2^n}$, the n -th approximating solution $X_n(t)$, has the characteristic functional,

$$E(\exp(i(X_n(t), f))) = \exp \left[i \int (T^* \left(\frac{1}{2^n} \right) S(t_0^n, t_1^n) \dots T^*(t - t_{k_0}^n) S(t_{k_0}^n, t) f)(x) \mu(dx) \right], \quad (4.0.5)$$

where

$$S(t_k^n, t_{k+1}^n): f \rightarrow \frac{f(x)}{1 - \frac{i}{2} f(x) \int_{t_k^n}^{t_{k+1}^n} Q(s, x, x) ds}$$

In equation (4.0.5), T^* is linear whereas S is nonlinear. Trotter's technique applied to a non-linear perturbation is investigated in Marsden [17]. In the present problem the global existence of the combined semi-group will be established below.

§4.1 The Basic Setting

To make the problem precise, consider the following setting. (cf.

§3.1)

Notation 4.1.1

Let S denote the one point compactification of R^d , and A denote $\frac{1}{2} \Delta$.

Then for $\mu \in \mathcal{M}(S)$, $T(t)(\mu)(dx)$ is given by

$$T(t)(\mu)(dx) = \left[\left(\frac{1}{2\pi t} \right)^{\frac{d}{2}} \int \exp \left(- \frac{|x-y|^2}{2t} \right) \mu(dy) \right] dx, \quad (4.1.1)$$

where dx denotes the Lebesgue measure and $|x-y|^2$ denotes $(x_1-y_1)^2 + (x_2-y_2)^2 + \dots + (x_n-y_n)^2$. (We note that since S is compact, the total measure, $\mu(S)$ is finite.)

Since in (4.0.1) the only values of $Q(s,x,y)$ which appear are those on the diagonal $x = y$, we can set $Q(s,x,x) = Q(s,x)$.

Assumption 4.1.2

Let $Q(s,x)$ be non-negative, twice continuously differentiable in x and assume that the derivatives are bounded and constant at infinity, that is,

$$\|Q\|_2(t) = \sup_{\substack{x,j,k \\ 0 \leq s \leq t}} \{ |Q(s,x)|, |D_{x_j} Q(s,x)|, |D_{x_j} D_{x_k} Q(s,x)| \} < \infty. \quad (4.1.2)$$

Remark 4.1.3

Q is not assumed to have compact support on R^d . $P \in P(V)$ is obtained in the following manner. First, we divide R^d into disjoint sets $\{K_j\}$ such that each K_j is relatively compact and the boundary ∂K_j is of μ -measure 0. Second, we construct P_j on $\bar{K}_j$ which is a solution to the martingale problem:

$$\langle (v(t)-v(0), f)(v(t)-v(0), g) \rangle = \int_0^t (Q(s,x)|_{\bar{K}_j} v(s), fg) ds.$$

Finally, P is defined to be the convolution of $\{P_j\}$.

Notation 4.1.3

Let C^{c+} denote the Banach space of complex-valued continuous functions with non-negative imaginary part on S endowed with the sup norm $\|\cdot\|$.

Notation 4.1.4

Let C_1^{c+} (C_2^{c+} , respectively) denote the Banach sub-space of C^{c+} of functions with continuous first derivatives (of twice continuously differentiable functions) endowed with the norm $\|\cdot\|_1$, ($\|\cdot\|_2$), where $\|\cdot\|_1$, ($\|\cdot\|_2$), denotes

$$\|f\|_1 = \sup\{|f(x)|, |D_{x_k} f(x)|; x \in S, 1 \leq k \leq d\}$$

$$(\|f\|_2 = \sup\{|f(x)|, |D_{x_k} f(x)|, |D_{x_k x_j} f(x)|; x \in S, 1 \leq k, j \leq d\}).$$

Notation 4.1.5

Let $S(s, t)$ denote the non-linear operator from C^{c+} to C^{c+} (and at the same time from C_1^{c+} to C_1^{c+} and further from C_2^{c+} to C_2^{c+}), given by

$$S(s, t): f(x) \rightarrow \frac{f(x)}{1 - \frac{1}{2} f(x) \int_s^t Q(u, x) du}. \quad (4.1.3)$$

Remark 4.1.6

$S(s, t)$ is a time nonhomogeneous evolution, i.e.,

$$S(s, t)S(t, u)f = S(s, u)f, \quad \text{here } 0 \leq s \leq t \leq u. \quad (4.1.4)$$

Proof:

$$S(s, t) (S(t, u) f) = \frac{(S(t, u) f)}{1 - \frac{1}{2} (S(t, u) f) \int_s^t Q(u) du}$$

(multiplying both numerator and denominator by $1 - \frac{1}{2} f \int_t^u Q(v) dv$),

$$= \frac{f}{1 - \frac{1}{2} f \int_t^u Q(v) dv - \frac{1}{2} f \int_s^t Q(v) dv}$$

$$= S(s, u) f.$$

Remark 4.1.6

Let $T(t)$ denote the semi-group on $C^{c+}(S)$ defined as follows:

$$T(t)(f)(x) = \left(\frac{1}{2\pi t} \right)^{\frac{d}{2}} \int f(y) \exp \left(- \frac{|x-y|^2}{2t} \right) dy \quad (4.1.5)$$

Note that we use the same notation $T(t)$ in (4.1.1) and (4.1.5). Then

$(T(t)(f), \mu(dx)) = (f, T(t)(\mu)(dx))$ holds for $f \in C^{c+}(S)$, $\mu \in \mathcal{M}(S)$.

§4.2 Non-linear Perturbation

This section is devoted to the following theorem.

THEOREM 4.2.1

$$\text{Let } t_k^n = \frac{k}{2^n}, \quad k = 0, 1, 2, \dots, n = 1, 2, \dots$$

Let $K_n(t)$ denote the following continuous operator from C^{c+} to C^{c+} .

$$K_n(t)f = T\left(\frac{1}{2^n}\right) S(0, t_1^n) T\left(\frac{1}{2^n}\right) S(t_1^n, t_2^n) \dots T(t - t_{k_0}^n) S(t_{k_0}^n, t)f \quad (4.2.1)$$

where k_0 is the maximum integer for which $t_{k_0}^n$ does not exceed t . Then $\lim_n K_n(t)$ converges. Let $K(t)$ denote $\lim_n K_n(t)$. (4.2.2)

Then $K(t)$ satisfies

$$K(t): C^{c+} \rightarrow C^{c+} \text{ is continuous,} \quad (4.2.3)$$

$$\text{for } f \in C^{c+}, \text{ the mapping } t \in [0, \infty) \rightarrow K(t)f \in C^{c+} \text{ is continuous.} \quad (4.2.4)$$

We prove Theorem 4.2.1 step by step. An outline of the proof is given in the following remark.

Remark 4.2.2

First we show that the family of operators $\{K_n(t); n \geq 1, 0 \leq t \leq t_0\}$ is equi-continuous on C^{c+} . Second we show that $K_n(t)$ converges uniformly in t on a dense subset of C^{c+} , namely on C_2^{c+} . Then this implies that $K(t)$ satisfies (4.2.3) and (4.2.4).

Lemma 4.2.3

$$\|S(s, t)f\| \leq \|f\| \quad (4.2.5)$$

$$\|S(s, t)f - S(s, t)g\| \leq \|f - g\| \quad (4.2.6)$$

Proof:

(4.2.5) follows since $Q(s, x) \geq 0$ and $\operatorname{Im}(f) \geq 0$ imply that the real part of the denominator ≥ 1 . (4.2.6) follows since

$$S(s,t)f - S(s,t)g = \frac{f-g}{(1 - \frac{i}{2} f \int_s^t Q(u)du)(1 - \frac{i}{2} g \int_s^t Q(u)du)}$$

and the absolute value of the denominator ≥ 1 .

Lemma 4.2.4.

$$\|T(t)f\| \leq \|f\| \quad (4.2.7)$$

$$\|T(t)f - T(t)g\| \leq \|f-g\| \quad (4.2.8)$$

Proof:

(4.2.7) and (4.2.8) are the direct consequence of the fact that $T(t)$ is a linear contraction semi-group.

Lemma 4.2.5.

$$\|K_n(t)(f) - K_n(t)(g)\| \leq \|f-g\| \quad (4.2.9)$$

Proof:

Use Lemma 4.2.3 and Lemma 4.2.4 consecutively.

Remark 4.2.6

Lemma 4.2.5 proves that $\{K_n(t): n \geq 1, t\}$ is equi-continuous.

Remark 4.2.7

On C_{j+}^{c+} , T_t is a contraction semi-group. $j = 1, 2$.

$$\|T_t f\|_j \leq \|f\|_j \quad (4.2.10)$$

$$\|T_t f - T_t g\|_j \leq \|f-g\|_j \quad (4.2.11)$$

Remark 4.2.8

$(C_2^{c+}, \|\cdot\|_2) \hookrightarrow (C_1^{c+}, \|\cdot\|_1) \hookrightarrow (C^{c+}, \|\cdot\|)$ and the inclusion is dense.

Lemma 4.2.9

$$\|(S(s,t)f)\|_1 \leq \|f\|_1 + \frac{1}{2} \|f\|^2 \|q\|_1(t-s). \quad (4.2.12)$$

Proof:

Let D_x denote the differential operator with respect to the one coordinate among $x_1, x_2, \dots, x_d$. Since $D_x(S(s,t)f) = (D_x f + \frac{1}{2} f^2 \int_s^t D_x q(u,x) du) (1 - \frac{1}{2} f \int_s^t q(u) du)^{-2}$, the lemma follows. Let

$\|q\|_1(t)$ denote $\sup\{|q(u,x)|, |D_{x_k} q(u,x)| : 0 \leq u \leq t, x \in S, 1 \leq k \leq d\}$.

Lemma 4.2.10

$$\|S(s,t)f\|_2 \leq \|f\|_2 + 4\|f\|_1^2 \|q\|_2(t)(t-s) + \|f\|^3 \|q\|_1^2(t)(t-s)^2. \quad (4.2.13)$$

Proof:

This follows from the following Lemma 4.2.11.

Lemma 4.2.11

$$\begin{aligned} \|D_x D_y(S(s,t)f)\| &\leq \|f\|_2 + 4\|f\|_1^2 \|q\|_2(t)(t-s) \\ &\quad + \|f\|^3 \|q\|_1^2(t)(t-s)^2. \end{aligned} \quad (4.2.14)$$

Proof:

This follows immediately from

$$D_x D_y (S(s, t) f) = \frac{D_x D_y f + \frac{1}{2} f^2 \int_s^t D_x D_y Q(u) du}{(1 - \frac{1}{2} f \int_s^t Q(u) du)^2}$$

$$+ \frac{1 (f D_x f \int_s^t D_y Q du + f D_y f \int_s^t D_x Q du + D_x f D_y f \int_s^t Q du) - \frac{f^3}{2} \int_s^t D_x Q du \int_s^t D_y Q du}{(1 - \frac{1}{2} f \int_s^t Q(u) du)^3}$$

Lemma 4.2.12 (Crucial)

If $B \subset C_2^{c+}$ is bounded, then $\{K_n(t)f: n \geq 1, 0 \leq t \leq t_0, f \in B\}$ is bounded in C_2^{c+} . Here t_0 is a fixed number.

Proof:

$$\|K_n(t)f\|_2 \leq \|S(0, t_1^n) K_n(t_1^n, t)f\|_2 \quad (\text{from (4.2.10)})$$

$$\leq \|K_n(t_1^n, t)f\|_2 + 4\|K(t_1^n, t)f\|_1^2 \|Q\|_2(t_0) \left(\frac{1}{2^n}\right) + \|K(t_1^n, t)f\|_1^3 \|Q\|_1^2(t_0) \left(\frac{1}{2^n}\right)^2,$$

(from (4.2.13)), where $K_n(t_1^n, t)$ denotes the (t_1^n, t) -section of $K_n(t)$.

$$\|K_n(t_1^n, t)f\|_1 \leq \|S(t_1^n, t_2^n) K_n(t_2^n, t)f\|_1 \quad (\text{since } T \text{ is contraction}),$$

$$\leq \|K_n(t_2^n, t)f\|_1 + \frac{1}{2} \|K_n(t_2^n, t)f\|^2 \|Q\|_1(t_0) \left(\frac{1}{2^n}\right) \quad (\text{from (4.2.12)}),$$

$$\leq \|K_n(t_2^n, t)f\|_1 + \frac{1}{2} \|f\|^2 \|Q\|_1(t_0) \left(\frac{1}{2^n}\right) \quad (K_n \text{ is contraction}),$$

(inductively)

$$\leq \|f\|_1 + \frac{1}{2} \|f\|^2 \|Q\|_1(t_0) (t - t_1^n).$$

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Therefore

$$\begin{aligned} & \|K_n(t)f\|_2 \\ & \leq \|K_n(t_1^n, t)f\|_2 + 4(\|f\|_1 + \frac{1}{2}\|f\|^2\|Q\|_1(t_0)(t-t_1^n)^2)\|Q\|_2(t_0)(\frac{1}{2^n}) \\ & \quad + \|f\|^3\|Q\|_1^2(t_0)(\frac{1}{2^n})^2, \end{aligned}$$

(replacing $(t-t_1^n)$ by t and calculating inductively)

$$\leq \|f\|_2 + 4(\|f\|_1 + \frac{1}{2}\|f\|^2\|Q\|_1(t_0)t^2)\|Q\|_2(t_0)t + \|f\|^3\|Q\|_1^2(t_0)(\frac{1}{2^n})^2 2^n t.$$

Therefore the lemma follows.

Lemma 4.2.13 (The Estimate of the Commutator)

For each $M > 0$ and $t_0 > 0$, there exists a constant $c_0 = c_0(M, t_0)$ such that for all f and all s, t, u with $\|f\|_2 \leq M$ and $0 \leq s \leq t \leq u \leq t_0$,

$$\|T(u-s)S(s;u)f - T(t-s)S(s,t)T(u-t)S(t,u)f\| \leq c_0(t-s)(u-t). \quad (4.2.15)$$

Proof:

$$\begin{aligned} & \|T(u-s)S(s,u)f - T(t-s)S(s,t)T(u-t)S(t,u)f\| \\ & = \|T(t-s)[T(u-t)S(s,u)f - S(s,t)T(u-t)S(t,u)f]\| \quad (T \text{ is linear}), \\ & \leq \|T(u-t)S(s,u)f - S(s,t)T(u-t)S(t,u)f\| \quad (T \text{ is contraction}), \\ & \leq \|T(u-t)(S(s,u)f - S(t,u)f) - [S(s,u)f - S(t,u)f]\| \\ & \quad + \|S(s,u)f - S(t,u)f + T(u-t)S(t,u)f - S(s,t)T(u-t)S(t,u)f\|. \end{aligned}$$

The first term

$$\begin{aligned}
 & \| (T(u-t) - I) (S(s, u) f - \frac{1}{2} S(t, u) f \int_s^t Q(u) du) \| \\
 & \leq \frac{d}{2} (u-t) \| S(s, u) f - \frac{1}{2} S(t, u) f \int_s^t Q(u) du \|_2 \\
 & \leq \frac{d}{2} \frac{9}{2} \| S(s, u) f \|_2 \| S(t, u) f \|_2 \| \int_s^t Q(u) du \|_2 (u-t) \\
 & \leq 3d \| S(s, u) f \|_2 \| S(t, u) f \|_2 \| Q \|_2 (t_0) (t-s) (u-t)
 \end{aligned}$$

$\| S(s, u) f \|_2 \| S(t, u) f \|_2$ is bounded under the same conditions (see Lemma 4.2.10).

The second term

$$\begin{aligned}
 & S(s, u) f - S(t, u) f + T(u-t) S(t, u) f - S(s, t) T(u-t) S(t, u) f \\
 & = (S(s, u) f - \frac{1}{2} S(t, u) f \int_s^t Q(u) du - \frac{1}{2} S(s, t) T(u-t) S(t, u) f \\
 & \quad \times T(u-t) S(t, u) f \int_s^t Q(u) du \\
 & = \frac{1}{2} \int_s^t Q(u) du [(S(t, u) f - T(u-t) S(t, u) f) (S(s, u) f) \\
 & \quad + (T(u-t) S(t, u) f) (S(s, t) S(t, u) f - S(s, t) T(u-t) S(t, u) f)]
 \end{aligned}$$

Therefore

$$\begin{aligned}
 & \| \text{second term} \| \\
 & \leq \frac{1}{2} \| Q \| (t_0) (t-s) \left[\frac{d}{2} (u-t) \| S(t, u) f \|_2 \| f \| \right. \\
 & \quad \left. + \| f \| \| S(t, u) f - T(u-t) S(t, u) f \| \right] \quad (\text{see Lemma 4.2.3}) , \\
 & \leq \frac{1}{2} \| Q \| (t_0) (t-s) \frac{d}{2} (u-t) \cdot 2 \cdot \| f \| \| S(t, u) f \|_2 ,
 \end{aligned}$$

Here $\| S(t, u) f \|_2$ is bounded. Therefore the lemma is proved.

Lemma 4.2.14

Let $B \subset C_2^{c+}$ be bounded. Let t_0 be fixed and let t_k^n, t be such that $0 \leq t_k^n \leq t \leq t_0$. Then there exists a constant $c_1 = c_1(B, t_0)$ such that

$$\|T(t-t_k^n)S(t_k^n, t)f - K_n(t_k^n, t)f\| \leq c_1(t-t_k^n)^2; \quad (4.2.16)$$

for all $f \in B$ and t_k^n, t .

Proof:

From the triangle inequality,

$$\begin{aligned} & \|T(t-t_k^n)S(t_k^n, t)f - K_n(t_k^n, t)f\| \\ & \leq \sum_j \|K_n(t_k^n, t_j^n)T(\frac{1}{2^n})T(t-t_{j+1}^n)S(t_j^n, t_{j+1}^n)S(t_{j+1}^n, t)f \\ & \quad - K_n(t_k^n, t_j^n)T(\frac{1}{2^n})S(t_j^n, t_{j+1}^n)T(t-t_{j+1}^n)S(t_{j+1}^n, t)f\| \end{aligned}$$

(using the contraction property of K),

$$\leq \sum_j \|T(t-t_j^n)S(t_j^n, t)f - T(\frac{1}{2^n})S(t_j^n, t_{j+1}^n)T(t-t_{j+1}^n)S(t_{j+1}^n, t)f\|$$

(using Lemma 4.2.13)

$$\begin{aligned} & \leq \sum_j c_0 \frac{1}{2^n} (t-t_{j+1}^n) \\ & \leq c_1(t-t_k^n)^2. \end{aligned}$$

Lemma 4.2.15

Let $f \in C_2^{c+}$. Then there exists a constant c_2 such that for all n, m with $m > n$,

$$\|K_n(t)f - K_m(t)f\| \leq c_2 \frac{t}{2^n}, \quad (4.2.17)$$

for all t with $0 \leq t \leq t_0$. The constant c_2 depends only on f and t_0 .

Proof:

First we remark that $\{K_m(t_k^n, t)f : m > n, k, 0 \leq t \leq t_0\}$ is bounded in C_2^{c+} . (This can be proved in the same manner as in Lemma 4.2.12).

By the triangle inequality,

$$\begin{aligned} & \|K_n(t)f - K_m(t)f\| \\ & \leq \sum_k \|K_n(0, t_k^n)K_n(t_k^n, t_{k+1}^n)K_m(t_{k+1}^n, t)f - K_n(0, t_k^n)K_m(t_k^n, t_{k+1}^n)K_m(t_{k+1}^n, t)f\| \end{aligned}$$

(using the contraction property),

$$\leq \sum_k \|K_n(t_k^n, t_{k+1}^n)K_m(t_{k+1}^n, t)f - K_m(t_k^n, t_{k+1}^n)K_m(t_{k+1}^n, t)f\|$$

(since $K_n(t_k^n, t_{k+1}^n) = T(t_{k+1}^n - t_k^n)S(t_k^n, t_{k+1}^n)$, and from the above remark, applying Lemma 4.2.14),

$$\leq \sum_k c_1 \left(\frac{1}{2^n}\right)^2$$

(since the number of terms $\leq 2^n t$),

$$\leq c_2 \frac{t}{2^n}.$$

Therefore the lemma is proved.

Remark 4.2.16

Lemma 4.2.15 together with Lemma 4.2.5 proves Theorem 4.2.1.

§4.3 The limit process X_t

In this section, we summarize the properties of the limit process X_t of the approximating processes $\{X_n(t)\}$ with characteristic functionals $\{\exp i(\mu, K_n(t)f)\}$, without giving full details.

THEOREM 4.3.1 (Conditional characteristic functional)

Let $0 \leq s \leq t$. Let X_s be given the conditional characteristic functional $E_{X_s}(\exp i(X_t, f))$ for given X_s is as follows.

$$E_{X_s}(\exp i(X_t, f)) = \exp i(X_s, K(s, t)(f)). \quad (4.3.1)$$

THEOREM 4.3.2

$K(s, t)(f)$ satisfies the following functional equation.

$$K(s, t)(f) = T_{t-s}(f) + \frac{1}{2} \int_s^t T(u-s)Q(u)(K(u, t)(f))^2 du. \quad (4.3.2)$$

THEOREM 4.3.3

In case that $Q(t)$ is time homogeneous, the equation (4.3.2) becomes

$$K(t-s)(f) = T(t-s)(f) + \frac{1}{2} \int_0^{t-s} T(t-s-u)Q(K(u)(f))^2 du, \quad (4.3.3)$$

(cf. Watanabe [26]).

Replacing f by θf in (4.3.2), and denoting $(\frac{d}{d\theta})^n K(s, t)(\theta f)$ by $F_n(\theta, s, t)$, then $F_n(0, s, t)$ can be calculated in the following manner recursively.

$$F_0(0, s, t) = 0. \quad (4.3.4)$$

$$F_1(\theta, s, t) = T_{t-s}(f) + \frac{1}{2} \int_s^t T(u-s)Q(u)2F_0F_1(\theta, u, t)du. \quad (4.3.5)$$

From this,

$$F_1(0, s, t) = T_{t-s}(f). \quad (4.3.6)$$

(from (4.3.5))

$$F_2(\theta, s, t) = i \int_s^t T(u-s)Q(u)(F_0F_2 + F_1^2)(\theta, u, t)du. \quad (4.3.7)$$

Therefore,

$$F_2(0, s, t) = i \int_s^t T(u-s)Q(u)((T(t-u)(f))^2)du. \quad (4.3.8)$$

From (4.3.7),

$$F_3(\theta, s, t) = i \int_s^t T(u-s)Q(u)(F_0F_3 + 3F_1F_2)(\theta, u, t)du \quad (4.3.9)$$

Therefore,

$$F_3(0, s, t) = 3i \int_s^t T(u-s)Q(u)(T(t-u)(f)F_2(0, u, t))du. \quad (4.3.10)$$

From (4.3.9),

$$F_4(\theta, s, t) = i \int_s^t T(u-s)Q(u)(F_0F_4 + 4F_1F_3 + 3F_2^2)(\theta, u, t)du \quad (4.3.11)$$

Therefore,

$$F_4(0, s, t) = i \int_s^t T(u-s)Q(u)(4F_1F_3 + 3F_2^2)(0, u, t)du. \quad (4.3.12)$$

and so on.

THEOREM 4.3.4

The approximating processes $X_n(t)$ are uniformly tight in $P(V)$.
Therefore the limit process $X(t)$ is realized in V .

Proof:

Instead of showing the full calculation, we simply point out that from the equations $F_j(0, s, t)$ above, it is easy to show that there exists a constant $c = c(t_0)$ for each given $t_0 > 0$ such that

$$E|(X_t, f) - (X_s, f)|^4 \leq c|t-s|^2 \quad (4.3.12)$$

whenever $f \in C_2(S)$ and $0 \leq s \leq t \leq t_0$. Therefore the probability measure P_X induced by X is supported on V . (c.f. Theorem 1.2.15).

CHAPTER V. A CONSERVATIVE MARTINGALE PROBLEM (5).

§5.0. Introduction

The following measure valued martingale problem

$$\langle (v(t)-v(o), f)(v(t)-v(o), g) \rangle = \int_0^t [(v(s), fg) - (v(s), f)(v(s), g)] ds \quad (5.0.1)$$

$v(o) = \mu$, where $\mu(S) = 1$, is considered in this chapter.

We show that there exists a solution $P \in P(V)$ which is supported on the simplex $V_1 = \{v(t): v(t)(S) = 1\}$ at each time t . This implies that the total mass is conserved in each sample evolution. Then we show that the initially scattered unit mass will be asymptotically accumulated in a single point at time $t = \infty$ whose location is distributed according to the initial distribution μ .

§5.1 The construction of a solution $P \in P(V)$.

Assume that $P \in P(V)$ is a solution to the martingale problem (5.0.1).

Dividing the space S into a disjoint union of $E_1, E_2, \dots, E_n$, then we have the system $\{X_j(t) = (v(t), E_j): 1 \leq j \leq n\}$ which induces the probability measure, denoted by P_n , on $W^n = C([0, \infty); R^n)$. Using the same notation $X_j(t)$, the process $\{X_j(t)\}$ associated with P_n is a solution to the following martingale problem.

$$\langle (X_j(t)-X_j(0))(X_k(t)-X_k(0)) \rangle = \int_0^t (\delta_{jk} - X_j(s)X_k(s)) ds, \quad (5.1.1)$$

$X_j(0) = \mu(E_j)$. (Note that $\sum_j X_j(0) = 1$.)

This finite dimensional martingale problem has been investigated and has been shown to have a unique solution (see Sato [21]), P_n , which is supported on the unit simplex, that is, $P_n \{ \sum_j X_j(t) = 1 \} = 1$.

Since the initial condition satisfies $\mu(S) = 1$, we assume that S is compact (otherwise, consider the one point compactification).

Notation 5.1.1

Let $\{E_j^n: 1 \leq j \leq N(n)\}$, $n \geq 1$ denote a sequence of partitions of S such that

$$\bigcup_j E_j^n = S \text{ (a disjoint union),} \quad (5.1.2)$$

$$\{E_j^{n+1}: j\} \text{ is a refinement of } \{E_j^n: j\}, \quad (5.1.3)$$

$$\{E_j^{n+1}, 1 \leq j \leq N(n+1)\} \quad (5.1.4)$$

is arranged in order such that for each $k: 1 \leq k \leq N(n)$,

$$E_k^n = \bigcup_{j=M(n+1,k-1)+1}^{M(n+1,k)} E_j^{n+1}$$

where

$$0 = M(n+1,0) < M(n+1,1) < M(n+1,2) < \dots < M(n+1,N(n)) = N(n+1),$$

$$\sup_j \text{diam}(E_j^n) \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (5.1.5)$$

Notation 5.1.2

Let $W^\infty = C([0, \infty); \mathbb{R}^\infty)$ with coordinate representation given by

$$W^\infty = W^{N(1)} \times W^{N(2)} \times W^{N(3)} \times \dots$$

The $W^{N(n)}$ -section is given by

$$(x_1^n(t), x_2^n(t), \dots, x_{N(n)}^n(t)) . \quad (5.1.6)$$

Notation 5.1.3

Let π_{n+1} denote the following projection from $W^{N(n+1)}$ to $W^{N(n)}$.

First, let $x = (x_1^{n+1}, \dots, x_{N(n+1)}^{n+1}) \in W^{N(n+1)}$ be indexed as

$$x = (x_1^{n+1}, x_2^{n+1}, \dots, x_{M(n+1,1)}^{n+1}, x_{M(n+1,1)+1}^{n+1}, \dots, x_{M(n+1,2)}^{n+1}, \dots, x_{M(n+1,N(n))}^{n+1}) .$$

Define

$$\pi_{n+1}(x) = \left(\sum_{j=1}^{M(n+1,1)} x_j^{n+1}, \sum_{j=M(n+1,1)+1}^{M(n+1,2)} x_j^{n+1}, \dots, \sum_{j=M(n+1,N(n)-1)+1}^{M(n+1,N(n))} x_j^{n+1} \right) .$$

Remark 5.1.4

Consider the martingale problem on $W^{N(n+1)}$:

$$\langle (x_j^{n+1}(t) - x_j^{n+1}(0)), x_k^{n+1}(t) - x_k^{n+1}(0) \rangle = \int_0^t (\delta_{jk} - x_j^{n+1}(s)) x_k^{n+1}(s) ds , \quad (5.1.7)$$

$x_j^{n+1}(0) = \mu(E_j^{n+1})$, $1 \leq j, k \leq N(n+1)$. (Remark: $\mu(S) = 1$ is assumed.) The

existence of a solution to this martingale problem is essentially shown by Stroock-

Varadhan [23: page 375, Theorem 4.2]. The uniqueness is shown by Sato [21: page

653-page 656]. (Sato showed that the equivalent stochastic differential equation

has a path-wise unique solution, and then applied the theorem of Yamada-Watanabe [29: page 158] which states that pathwise uniqueness implies uniqueness in the sense of the probability law.)

Applying this result, we obtain the following lemma.

Lemma 5.1.5

Let $P \in P(W^{N(n+1)})$ be the unique solution to the martingale problem (5.1.7). Then $\pi_{n+1}(P) \in P(W^{N(n)})$ is the solution to the corresponding martingale problem.

Proof:

Direct calculation tells us that $\pi_{n+1}(P)$ satisfies the martingale equation. From the uniqueness, $\pi_{n+1}(P)$ agrees with the one which is obtained from the corresponding martingale problem.

Remark 5.1.6

In Lemma 5.1.5, P is supported on a simplex such that

$$P\left(\sum_j x_j^{n+1}(t) = 1, x_j^{n+1} \geq 0\right) = 1. \quad (5.1.8)$$

Lemma 5.1.7

For $n+1$ fixed, let $P \in P(W^{N(n+1)})$ be the solution to the martingale problem (5.1.7). Let $P_{n+1} \in P\left(\prod_{j=1}^{n+1} W^{N(j)}\right)$ denote the probability measure

induced by the following mapping:

$$x \in W^{N(n+1)} \rightarrow (\pi_2(\pi_3(\dots(\pi_{n+1}(x))))), \dots, \pi_{n+2}(\pi_{n+1}(x)), \pi_{n+1}(x), x).$$

Now, varying $n+1$, let $\{P_{n+1}\}$ denote the system of probability measures obtained from the above. Then $\{P_{n+1}\}$ is projective.

Proof:

This follows immediately from Lemma 5.1.5.

Notation 5.1.8

Let P_∞ denote the projective limit of P_{n+1} on W^∞ . $(W^\infty, \mathcal{B}(W^\infty), P_\infty; \mathcal{B}_t(W^\infty))$ is the basic probability space. $M_2^c(\mathcal{B}_t(W^\infty))$ is the basic martingale space.

Remark 5.1.9

A function $f(x)$ on S is said to be simple if for some $\{E_j^n: j \geq 1\}$ (c.f. Notation 5.1.1),

$$f(x) = \sum_j a_j I_{E_j^n}(x).$$

Definition 5.1.10

Let $f(x)$ be simple. Define (X_t, f) ($M_2^c(\mathcal{B}_t(W^\infty))$ -valued) to be

$$(X_t, f) = \sum_j a_j x_j^n(t). \quad (5.1.9)$$

Remark 5.1.11

Simple calculation shows that for f simple,

$$E(X_t, f)^2 = (\mu, f^2)(1-e^{-t}) + e^{-t}(\mu, f)^2, \quad (5.1.10)$$

and

$$E(X_t - \mu, f)^2 = ((\mu, f^2) - (\mu, f)^2)(1-e^{-t}). \quad (5.1.11)$$

In view of Remark 5.1.11, the definition (X_t, f) does not depend on the expression for f . Since $\{f: f \text{ is simple}\}$ is dense in $L_2(\mu)$ (see (5.1.5)), we have the following theorem.

THEOREM 5.1.12

Let $g \in L_2(\mu)$. Let f_n be simple and converge to g . Then (X_t, f_n) converges in M_2^c . Define $(X_t, g) = \lim_n (X_t, f_n)$. Then X_t is a V -valued martingale and the probability measure P_X on V induced by X_t is a solution to the martingale problem (5.0.1).

Proof:

Convergence follows from (5.1.10); further (X_t, g) does not depend on the choice of f_n . X_t is a positive generalized random field so that X_t is a measure valued process. Since (X_t, g) , $g \in L_2(\mu)$, is continuous, X_t is V -valued. X_t satisfies the martingale equation for f, g simple, and thus in general.

Corollary 5.1.13

P_X is supported on the set of probability distributions on S , that is,

$$P_X(v(t)(S) = 1) = 1.$$

§5.2 The Asymptotic Behaviour of X_t as $t \rightarrow \infty$.

This section is devoted to the following theorem. Roughly speaking, this theorem states that the spatially distributed unit mass at time $t = 0$ is all accumulated in a single point at time $t = \infty$ whose location is distributed according to the initial distribution.

THEOREM 5.2.1

The measure-valued martingale X_t of Theorem 5.1.12 is convergent in L_2 as t tends to infinity, i.e.,

$$\lim_{t \rightarrow \infty} E((X_t, f) - (X_s, f))^2 = 0 \quad (5.2.1)$$

Put $X_\infty = \lim_{t \rightarrow \infty} X_t$. ($f \rightarrow \lim_{t \rightarrow \infty} (X_t, f)$ is a random measure.) Then the limit distribution P_{X_∞} on $\mathcal{M}(S)$ coincides with that induced by the mapping:

$$(S, \mathcal{B}(S), \mu) \rightarrow \mathcal{M}(S)$$

$$x \rightarrow \delta_x$$

Proof:

$$E((X_t, f) - (X_s, f))^2 = ((\mu, f^2) - (\mu, f)^2)(e^{-s} - e^{-t}), \quad s < t.$$

Therefore (5.2.1) follows

The limit distribution P_{X_∞} can be identified from the expressions for the moments. That is, it suffices to show that

$$\lim_{t \rightarrow \infty} E((X_t, f_1)(X_t, f_2) \dots (X_t, f_n)) = (\mu, f_1 f_2 \dots f_n).$$

where f_j is bounded continuous and non-negative on S . $1 \leq j \leq n$.

The proof is by induction. When $n = 1$, it is immediate since X_t is a martingale. When $n = 2$, $E((X_t, f_1)(X_t, f_2)) = (1 - e^{-t})((\mu, f_1 f_2) - (\mu, f_1)(\mu, f_2)) + (\mu, f_1)(\mu, f_2) \rightarrow (\mu, f_1 f_2)$.

To complete the proof by induction, we assume that it is true for n . From Ito's formula, for $n+1$,

$$E((X_t, f_1)(X_t, f_2) \dots (X_t, f_{n+1})) = (\mu, f_1)(\mu, f_2) \dots (\mu, f_{n+1}) + \frac{1}{2} \int_0^t \sum_{j,k} E((X_s, f_1) \dots (X_s, f_j) \dots (X_s, f_k) \dots (X_s, f_{n+1})(X_s, f_j f_k)) ds - \binom{n+1}{2} \int_0^t E((X_s, f_1)(X_s, f_2) \dots (X_s, f_{n+1})) ds.$$

Put

$$E((X_t, f_1)(X_t, f_2) \dots (X_t, f_{n+1})) = F(t),$$

$$\frac{1}{2} \sum_{j,k} E((X_s, f_1) \dots (X_s, f_j) \dots (X_s, f_k) \dots (X_s, f_{n+1})(X_s, f_j f_k)) = G(s),$$

$$\binom{n+1}{2} = c$$

and

$$(\mu, f_1)(\mu, f_2) \dots (\mu, f_{n+1}) = c_0.$$

Then

$$F(t) = c_0 + \int_0^t G(s) ds - c \int_0^t F(s) ds.$$

We note that $(X_t, f_1)(X_t, f_2) \dots (X_t, f_{n+1})$ is a positive submartingale. Therefore $F(t)$ is increasing. For $t > s$,

$$F(t) - F(s) + c \int_s^t F(u) du = \int_s^t G(u) du.$$

(or equivalently $F'(t) + cF(t) = G(t)$). From this, we have

$$0 \leq cF(t) \leq G(t), \quad (5.2.2)$$

and

$$0 \leq F'(t) \leq G(t). \quad (5.2.3)$$

Since $\lim_{t \rightarrow \infty} G(t) = c(\mu, f_1 f_2 \dots f_{n+1})$ by the induction hypothesis, it follows that

$$\lim_{t \rightarrow \infty} F'(t) = 0, \quad (F(t) \text{ is increasing and bounded}) \quad (5.2.4)$$

$$\lim_{t \rightarrow \infty} cF(t) = \lim_{t \rightarrow \infty} G(t) = c(\mu, f_1 f_2 \dots f_{n+1}). \quad (5.2.5)$$

Therefore,

$$\lim_t E((X_t, f_1)(X_t, f_2) \dots (X_t, f_{n+1})) = (\mu, f_1 f_2 \dots f_{n+1}), \quad n \geq 1,$$

thus completing the induction step. This proves that

$$\lim_t E(X_t, f)^n = (\mu, f^n), \quad n \geq 1.$$

Therefore, computing the characteristic functional of X_t ,

$$\lim_t E(\exp(i(X_t, f)))$$

$$= \lim_t \sum_{n=0}^{\infty} \frac{i^n}{n!} E(X_t, f)^n \quad (|(X_t, f)| \leq \|f\|)$$

$$= \sum_{n=0}^{\infty} \frac{i^n}{n!} (\mu, f^n) \quad (\text{the tail term is uniformly small with respect to } t),$$

$$= (\mu, \sum_n \frac{i^n}{n!} f^n) \\ = \int e^{if(x)} \mu(dx),$$

which is the characteristic functional of

$$(S, \mathcal{B}(S), \mu) \rightarrow \mathcal{M}(S)$$

$$x \rightarrow \delta_x$$

Since (X_t, f) converges to (X_∞, f) in L_2 , the characteristic functional of X_∞ is given by $E(e^{i(X_\infty, f)}) = \int e^{if(x)} \mu(dx)$. Therefore the theorem is established.

CHAPTER VI. A MIXED-TYPE MARTINGALE PROBLEM (4)

§6.0 Introduction

In this chapter, we consider the martingale problem on V associated with the martingale equation

$$\langle v(t) - v(0), f \rangle \langle v(t) - v(0), g \rangle = \int_0^t [a^2(v(s), fg) + b^2(v(s), f) \langle v(s), g \rangle] ds, \quad (6.0.1)$$

$v(0) = \mu$ with $\text{supp}(\mu) = K$ compact (note that $\mu(K) < \infty$) where a, b are non-negative constants with $b > 0$. This martingale problem is a mixture of martingale problems (2) and (3) described in the introduction with $Q = 1$.

We employ the same approach as in Chapter V and use the same notation as that of Chapter V.

§6.1 The construction of a solution P on V .

For each n , let $X_j^n: 1 \leq j \leq N(n)$ be the solution of the stochastic differential equation:

$$\begin{pmatrix} dx_1^n \\ dx_2^n \\ \vdots \\ dx_{N(n)}^n \end{pmatrix} = \begin{pmatrix} bx_1^n & a\sqrt{x_1^n} & 0 & \dots & 0 \\ bx_2^n & 0 & a\sqrt{x_2^n} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ bx_{N(n)}^n & 0 & 0 & \dots & a\sqrt{x_{N(n)}^n} \end{pmatrix} \begin{pmatrix} dB \\ dB_1^n \\ \vdots \\ dB_{N(n)-1}^n \\ dB_{N(n)}^n \end{pmatrix} \quad (6.1.1)$$

with the initial condition $X_j^n(0) = \mu(E_j^n)$, $1 \leq j \leq N(n)$, where $B, B_1^n, B_2^n, \dots, B_{N(n)}^n$ are independent $N(n)+1$ dimensional Brownian motions. The existence of the solution to the above system is established in Watanabe [27]. $\{X_j^n\}$ satisfies the martingale equation:

$$\langle (X_j^n(t) - X_j^n(0))(X_k^n(t) - X_k^n(0)) \rangle = \int_0^t [a^2(\delta_{jk} X_j^n(s)) + b^2 X_j^n(s) X_k^n(s)] ds.$$

In the same manner, we construct a system of probability measures $\{P_n\}$ where P_n is on $W^{N(1)} \times W^{N(2)} \times \dots \times W^{N(n)}$. The consistency condition depends on the uniqueness of the martingale problem on $W^{N(n)}$. The uniqueness of the solution to the martingale problem follows from the pathwise uniqueness of the stochastic differential equation. The pathwise uniqueness is given by the following lemma.

Lemma 6.1.1

The system of the stochastic differential equations for X_j^n : $1 \leq j \leq N(n)$ has a pathwise unique solution.

Proof:

Let X_j^n and Y_j^n : $1 \leq j \leq N(n)$ be solutions with respect to the same $N(n) + 1$ dimensional Brownian motion $B, B_1^n, B_2^n, \dots, B_{N(n)}^n$. Then

$$X_j^n(t) - Y_j^n(t) = \int_0^t b(X_j^n(s) - Y_j^n(s)) dB(s) + \int_0^t a(\sqrt{X_j^n(s)} - \sqrt{Y_j^n(s)}) dB_j^n(s), \quad 1 \leq j \leq N(n).$$

The following proof is essentially the same as part of Yamada-Watanabe [29: page 164 Theorem 1]. The only difference is that their system is diagonal while the above system is off diagonal. Their method applies to this case as follows.

Note that

$$\int_0^1 \frac{du}{a^2 u + b^2 u^2} = \infty.$$

Therefore the positive decreasing sequence c_n which is uniquely determined by

$$\int_{c_1}^1 \frac{du}{a^2 u + b^2 u^2} = 1, \quad \int_{c_2}^{c_1} \frac{du}{a^2 u + b^2 u^2} = 2, \dots, \quad \int_{c_n}^{c_{n-1}} \frac{du}{a^2 u + b^2 u^2} = n, \dots$$

is convergent to 0. For each n , let $\psi_n(u)$ be a continuous function with compact support such that

$$(1) \quad \text{supp}(\psi_n) \subset (c_n, c_{n-1}),$$

$$(2) \quad 0 \leq \psi_n(u) \leq \frac{2}{n(a^2 u + b^2 u^2)},$$

$$(3) \quad \int_{c_n}^{c_{n-1}} \psi_n(u) du = 1.$$

We note that (3) is possible since from (2), $0 \leq \int_{c_n}^{c_{n-1}} \frac{2}{n(a^2 u + b^2 u^2)} du = 2$. Put

$$\varphi_n(w) = \int_0^{|w|} dv \int_0^v \psi_n(u) du$$

(here v is a parameter on $[0, \infty)$). Then

$$\varphi_n(w) \in C^2(\mathbb{R})$$

$$|\varphi_n'(w)| \leq 1$$

and

$$\varphi_n(w) \uparrow |w| \text{ as } n \rightarrow \infty.$$

Applying Ito's formula to $\varphi_n(X_j^n(t) - Y_j^n(t))$, we have

$$\begin{aligned} \varphi_n(X_j^n(t) - Y_j^n(t)) &= \text{a martingale} + \frac{1}{2} \int_0^t \varphi''_n(X_j^n(s) - Y_j^n(s)) \times [b^2((X_j^n(s) - Y_j^n(s))^2 \\ &\quad + a^2(\sqrt{X_j^n(s)} - \sqrt{Y_j^n(s)})^2] ds. \end{aligned}$$

Therefore, noting that $|\sqrt{x} - \sqrt{y}| \leq \sqrt{|x-y|}$ for x, y positive,

$$\begin{aligned} E \varphi_n(X_j^n(t) - Y_j^n(t)) &\leq \frac{1}{2} \int_0^t E(\varphi''_n(X_j^n(s) - Y_j^n(s)) \\ &\quad \times [a^2|X_j^n(s) - Y_j^n(s)| + b^2((X_j^n(s) - Y_j^n(s))^2] ds \\ &\leq \frac{1}{2} \int_0^t E \frac{2}{n} \frac{I_{\{c_n \leq |X_j^n(s) - Y_j^n(s)| \leq c_{n-1}\}}}{a^2|X_j^n(s) - Y_j^n(s)| + b^2((X_j^n(s) - Y_j^n(s))^2} \\ &\quad \times [a^2|X_j^n(s) - Y_j^n(s)| + b^2((X_j^n(s) - Y_j^n(s))^2] ds \\ &\leq \frac{1}{n} t \rightarrow 0 \quad (\text{as } n \rightarrow \infty). \end{aligned}$$

Therefore, from the monotone convergence theorem,

$$E(|X_j^n(t) - Y_j^n(t)|) = 0.$$

From the sample path continuity of X_j^n and Y_j^n , $X_j^n(t) = Y_j^n(t)$ with probability one. Therefore the proof of pathwise uniqueness is complete.

Applying the Yamada-Watanabe Theorem, pathwise uniqueness implies uniqueness in law.

Therefore, the uniqueness of the solution to the martingale problem also follows. Therefore the system of probability measures $\{P_n\}$ is consistent,

i.e., projective. Let P be the projective limit of P_n on W^∞ . Define a generalized positive martingale process X_t on W^∞ first for simple functions f :

$$(X_t, f) = \sum_j c_j x_j^n(t)$$

where

$$f(x) = \sum_j c_j I_{E_j^n}(x), \quad x \in K.$$

Applying Ito's formula, we have a functional equation for $E(X_t, f)^2$:

$$E(X_t, f)^2 = (\mu, f)^2 + \int_0^t [a^2(\mu, f^2) + b^2 E(X_s, f)^2] ds.$$

Solving this,

$$E(X_t, f)^2 = (\mu, f)^2 e^{b^2 t} + \frac{a^2(\mu, f^2)}{b^2} (e^{b^2 t} - 1),$$

which implies that the definition of X_t does not depend on the expression for f .

Since $\{f: f \text{ is simple}\}$ is dense in $L_2(\mu)$, X_t can be uniquely extended to all $g \in L_2(\mu)$, namely,

$$(X_t, g) = \lim_n (X_t, f_n)$$

where f_n is simple and convergent to g in $L_2(\mu)$. Note that (X_t, f_n) is convergent in M_2^c .

THEOREM 6.1.2

Let P be the probability measure induced by X_t on V . Then P is a solution to the martingale problem.

$$\langle (v(t)-v(o), f) (v(t)-v(o), g) \rangle = \int_0^t \left[a^2 (v(s), fg) + b^2 (v(s), f) (v(s), g) \right] ds$$

$$v(o) = \mu .$$

•Proof:

Since (X_t, f) is continuous for every f , X_t is supported on V . The martingale equation holds when f, g are simple, therefore it holds in general.

CHAPTER VII. FINAL REMARKS

We summarize some problems which are left unsolved in this dissertation and further developments for the future.

To obtain solutions of (2.5.1), (2.5.2), we used a "weak" method.

(7.1) Does the approximating solution $X_n^m(t)$ of (2.5.3) converge strongly in M_2^C as m tends to infinity?

(7.2) Is (2.5.1) solvable in U_2 ?

In the Poisson randomization (Definition 2.2.3), particles are found at fixed locations and pick up independent time evolutions.

(7.3) What is the appropriate model which enables particles to move throughout the space and permits interaction among them?

(7.4) How differently is the mass being carried to the point at infinity between the two different stochastic evolution equations discussed in Chapter III and Chapter IV?

The model X_t obtained in Chapter V possesses a very particular martingale equation, namely,

$$\langle (X_t - X_0, f)^2 \rangle = \int_0^t [(X_s, f^2) - (X_s, f)^2] ds.$$

The initial condition X_0 is restricted to a probability distribution on the space S . Then X_s is also a probability distribution on the space S . Therefore the form $(X_s, f^2) - (X_s, f)^2$ is the variance of f with respect to X_s .

(7.5) What is the appropriate extension of this model?

In Chapter VI, we constructed a mixed-type process. For this model, in the case $a = 0$, $b > 0$, then X_t ($t > 0$) is supported on the absolutely continuous measures with respect to the initial measure X_0 , while if $a > 0$ and $b = 0$ then X_t is supported on the atomic measures. Therefore the question arises

(7.6) to describe the support of X_t for the case that $a > 0$ and $b > 0$.

The system of equations (6.1.1) is rather simple.

(7.7) What is the natural extension of this system?

(7.8) For the martingale problems of Chapter V and Chapter VI, the related approximating stochastic evolution equations can be easily solved. Do they converge in the limit?

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