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NAME OF SUPERVISOR/NOM DU DIRECTEUR DE THÈSE DR. E. SALEN

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SOME CONTRIBUTIONS TO THE TWO-SAMPLE
NONPARAMETRIC TESTS FOR LOCATION AND SCALE
UNDER MULTIPLE CENSORING

by

Jean-Paul Dionne, B.A.Sc., M.A., M.Ed.

A thesis submitted to the Faculty of
Graduate Studies and Research in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy

Department of Mathematics
Carleton University
Ottawa, Canada

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The undersigned recommend to the Faculty of
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"SOME CONTRIBUTIONS TO THE TWO-SAMPLE NONPARAMETRIC TESTS
FOR LOCATION AND SCALE UNDER MULTIPLE CENSORING"

submitted by Jean-Paul Dionne, M.Sc.
in partial fulfillment of the requirements
for the degree of Doctor of Philosophy

Professor P.K. Sen
External Examiner

Professor E. Saleh
Thesis Supervisor

Professor L.D. Nel
Chairman, Department of
Mathematics

ABSTRACT

The problem of linear estimation in the two-sample context with censored data has been studied extensively in the literature. Yet, until recently, the corresponding problem of hypothesis testing has received little attention. Based on the frequencies within the $k+1$ intervals specified by k arbitrary quantiles, optimum rank order tests are derived for the null hypothesis of identity of distributions against one and two-parameter families of alternatives.

In the case of one-parameter alternatives, the derived statistics yield locally most powerful rank order tests for the null hypothesis against Lehmann, location and scale alternatives, respectively. For Lehmann alternatives the proposed test generalizes the Savage test for the two-sample problem. The location and scale alternatives yield an extension of the work of Johnson and Mehrotra (1972), Mehrotra, Johnson and Battacharya (1977) and Gastwirth (1966). The optimal rank order statistics are derived from the likelihood ratio statistics. Under the null hypothesis they are asymptotically normal; their exact and asymptotic means and variances are readily computed. Moreover, with proper regularity conditions, the classes of alternatives are proved contiguous under the multiple censoring scheme. Hence, the distributions of the rank order statistics are also asymptotically normal under the alternatives and their asymptotic

means and variances can be computed. The Pitman efficiency of the censored sample tests relative to their complete sample counterparts is derived. The efficiencies coincide with those of the Best Linear Unbiased Estimators based on k quantiles studied by Ogawa (1962); therefore, the optimum spacings of the k quantiles are the same as those of the BLUE for a given density. Asymptotic efficiencies for small values of k are computed for the scoring functions corresponding to the normal, logistic and Cauchy distributions.

The two rank order statistics for the location and scale parameters, respectively, are combined in a quadratic form to produce a test for the location and scale parameters simultaneously. The resulting test, which is equivalent to the corresponding likelihood ratio test under the null hypotheses as well as the contiguous alternatives, is optimum in the sense of Wald (1943). The asymptotic means and variances are computed. The asymptotic efficiency relative to the complete sample counterpart is derived. With symmetric and absolutely continuous density functions, the bounds for the efficiencies correspond to the bounds of the ratio of two quadratic forms; a result due to Courant. Some examples of asymptotic efficiencies are also provided.

The technique can be applied to a variety of problems within the context of multiple censoring. For instance, with the one and two-sample problems most of the tests studied by Adichie (1967) and Hájek and Šidák (1967) have their counterparts within the multiple censored sample context. Likewise the technique may be extended to the c -sample problem.

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CHAPTER 1

INTRODUCTION

To test the null hypothesis of identity of distributions against specific families of alternatives, the experimenter may choose from a variety of methods. The observations may be assumed to come from a specific parametric distribution or else no assumptions on the form of the distribution are made and nonparametric methods are used. We take a position between these two extremes and assume the experimenter has some knowledge of the populations sampled but he cannot specify the distributions exactly. Typically, rank order tests match this position. Moreover, many of them have high asymptotic efficiency when compared with their parametric counterparts.

Although numerous rank order tests have appeared in the literature, the corresponding problem with censored data has received little attention until recently yet research with censored data has been almost exclusively centered on tail censored samples. Our objective is to extend some of those results to the multiple censored sample situation.

Within the two-sample problem context, locally most powerful rank tests (LMPRT) have been proposed for location, scale and

Lehmann alternatives, both for uncensored and tail censored samples. The asymptotic distributions were derived for alternatives of the fixed and contiguous type. Moreover, in most cases, their asymptotic power was compared with other tests for the same problem, using the Pitman notion of asymptotic relative efficiency. Most of those results were also extended to the two-parameter location and scale alternatives.

Our objective is to further extend the above results to the two-sample problem under multiple censoring of type II where censoring occurs within the $k + 1$ intervals specified by k fixed quantiles. The resulting tests are then based on frequencies within these intervals.

In some statistical applications it may be necessary and can be desirable to reach a decision on the basis of a part of the sample. For instance, the multiple censored sample procedures can be applied to some life testing and clinical trials problems.

1.1 Review of Literature

Consider two independent samples, $X_1, X_2, \dots, X_m$ and $Y_1, Y_2, \dots, Y_n$, from two populations with absolutely continuous cumulative distribution functions F_X and G_Y belonging to the same family of distributions. Let $W_1 \leq W_2 \leq \dots \leq W_N$, $N = m + n$, be the ordered observations in the combined sample of the m X -observations and n Y -observations. The order statistics are identified as coming from one population or the other by the rank orders

$Z = (Z_1, Z_2, \dots, Z_N)$ where $Z_i = 1$ if W_i is an X-observation or $Z_i = 0$ if W_i is a Y-observation.

We wish to test the null hypothesis

$$H_0 : F_X = G_Y = F(w; \theta_0) \quad (1.1)$$

against the alternative hypothesis

$$H_A : F_X = F(w; \theta_1) \text{ and } G_Y = F(w; \theta_2) \quad (1.2)$$

where θ_0, θ_1 and θ_2 belong to an interval of R or R^2 and $F(w; \theta)$ represents a family of cumulative distribution functions indexed by θ ; θ_1 and θ_2 are distinct, specified and very close to θ_0 .

To test H_0 against various alternatives we can use a variety of rank order statistics of the type

$$T_N = \sum_{i=1}^N c_i Z_i \quad (1.3)$$

where $c_1, c_2, \dots, c_N$ are constants. The scores c_i are usually obtained from a score generating function $J(u)$. Of particular interest to us are the scores defined by

$$c_i = E[J(U_{Ni})] \quad , i = 1, 2, \dots, N, \quad (1.4)$$

where U_{Ni} denotes the i th order statistic in a sample of size N from the uniform distribution on $(0, 1)$; the expectation is taken

under the null hypotheses. Under appropriate regularity conditions, usually satisfied by densities considered in this type of problem, the scores in (1.4) can be obtained as the expected value of the function $(\partial/\partial\theta)\log f(W_i;\theta)|_{\theta=\theta_0}$ of the i th smallest observation

W_i of a sample of size N under H_0 where $f(W;\theta)$ stands for the density corresponding to $F(W;\theta)$. Scores given by (1.4) occur in statistics yielding locally most powerful rank tests (LMPRT).

For location, scale and Lehmann alternatives, explicit forms of the statistic T_N , yielding LMPRT, are given by Hájek and Šidák (1967, section 4 of Chapter 2) and by Puri and Sen (1971, section 7 of Chapter 3). Moreover with specific densities we obtain more specialized tests; for instance, using the normal density with location alternatives we obtain the Fisher-Yates-Terry-Hoeffding test, later treated in detail by Terry (1952), and with scale alternatives, the Capon (1961) and Klotz (1962) tests; likewise, using the exponential distribution with scale alternatives we obtain the Savage (1956) test.

One central problem concerning rank order statistics is their asymptotic distribution under the null as well as the alternative hypotheses. Under some regularity conditions, these distributions are asymptotically normal. The first general attempt to treat the asymptotic distribution of rank order statistics in the two-sample case was made by Dwass (1956) using the method of U-statistics; the method was limited to polynomial score generating functions only. A major development occurred with the Chernoff and Savage (1958)

paper which became the basis for all further developments except those based on the contiguity approach; these alternatives are referred to as fixed alternatives. However, the regularity conditions on the score generating functions were rather stringent. The Chernoff and Savage conditions were relaxed by Govindarajulu, LeCam and Raghavachari (1966) and Pyke and Shorack (1968, 1969), to allow for discontinuities in the score generating functions.

Another approach consists in regarding the testing problem as a member of a sequence of similar testing problems. Along this line, Hájek (1962, 1968) and Hájek and Šidák (1967, Chapter 4) used methods based on the LeCam (1960) concept of contiguity for probability distributions in order to find the asymptotic distribution under local alternatives. In general, with appropriate regularity conditions, the LeCam Lemmas are used; see Hájek and Šidák (1967, 202-210). As a typical procedure, the contiguity of the sequences of hypotheses is obtained as an application of LeCam's first lemma to the asymptotic normality of the log-likelihood ratio function under H_0 . Then using LeCam's third lemma the asymptotic normality of the statistic under local alternatives is shown.

The asymptotic distribution of the statistic under alternatives is required whenever the asymptotic power of the test is considered. This power is often expressed as an asymptotic efficiency (ARE), in the Pitman sense, relative to a second test for the same problem; see Puri and Sen (1971), section 8, Chapter 3), Van Eeden (1963) or Hájek and Šidák (1967, Chapter 7).

Rank-order tests for single parameter alternatives, namely location and scale alternatives, have been combined in a quadratic form by Lepage (1975) and Rubison (1976) to produce a test for location and scale parameters simultaneously. It should be noted that the resulting quadratic form does not yield a LMPRT for the two parameter alternatives although the two component statistics did so for the respective one-parameter alternatives. An example of a test of this type is given by Lepage (1971) as a combination of the two sample Wilcoxon test for location and the Ansari-Bradley test for scale. When normal scores are used, the test is given by combination of the Van der Waerden test for location and the Klotz test for scale. Additional examples can be found in Rubison (1976). The asymptotic properties of the statistic proposed by Lepage (1975) were obtained for contiguous location and scale alternatives by an approach similar to Hájek and Šidák (1967). On the other hand, Rubison (1976) showed the asymptotic bivariate normality of the two-dimensional vector statistic for location and scale by extending the Chernoff and Savage Theorem. In either case, the quadratic form was shown to be asymptotically a chi-square random variable with two degrees of freedom; central under H_0 and with a non-centrality parameter under the alternatives.

When comparing the asymptotic power of two tests for the same problem we may use the notion of Pitman efficiency; with multiparameter families of alternatives it is obtained as the ratio of two non-centrality parameters; see Puri and Sen (1971, 121). It is our objective

to consider multiple censored samples and compare their asymptotic efficiency relative to their complete sample counterparts.

Over the last two decades, the two-sample problem with censored data has received increasing attention. Prior to Gastwirth (1965), rank order tests for the two-sample problem, in which censoring of the observations has occurred, appear to have been considered mainly by Savage (1956), Rao, Savage and Sobel (1960) and Halperin (1960) where the investigations concentrated on data censored either on the left or on the right. At this stage, we note that censoring may be classified in two general categories. When censoring takes place at certain fixed points it is called Type I censoring. When a fixed proportion of the sample size N is censored within an interval we have censoring of Type II. We restrict ourselves to Type II censoring.

Several results of Hájek and Šidák (1967) were extended to the two-sample situation where right or left censoring of Type II occurs. In particular, tests of the form (1.3) with the scores c_i averaged over the censored portion were shown to be LMPRT by Rao, Savage and Sobel (1960) for Lehman alternatives and by Johnson and Mehrotra (1972), for location and scale alternatives. In their paper, Rao, Savage and Sobel (1960) did not develop the asymptotic distribution and did not, consequently, compute the asymptotic efficiency of their statistic. However, Johnson and Mehrotra (1972) have shown the asymptotic normality of their statistics using the Pyke and Shorack (1968) version of the Chernoff and Savage (1958) Theorem but failed to include a discussion on the ARE.

Within the one-sample context Johnson (1974) has provided asymptotic results for the likelihood ratio statistic based on the r smallest observations, or equivalently, for right censored data. These results were extended by Mehrotra and Johnson (1976) to the two-sample case, under right censoring, showing the asymptotic normality of the log-likelihood ratio and the asymptotic equivalence of the latter to a LMPRT which in turn is asymptotically sufficient; as a corollary to their Theorem 3.2, the contiguity of the hypotheses follows. Here, these results are extended to multiple censoring occurring within the two-sample problem context, for a selection of one and two-parameter families of alternatives.

We shall say that multiple censoring occurs whenever censoring of Type II takes place within the $k + 1$ intervals specified by k nonconsecutive ordered observations determined by k fixed quantiles.

Although tests of hypothesis under multiple censoring have been given little attention until recently, the problem of linear estimation based on sample quantiles has been studied extensively. Ogawa (1951) proposed best linear unbiased estimators based on sample quantiles and discussed the optimum spacing of the sample quantiles. These estimators are linear functions of order statistics and have been considered by many authors, for instance, by Chan (1970), Hassanein (1969), Balmer, Boulton and Sack (1974), Cane (1974) for the parameters of the Cauchy distribution, by Chan (1969), Gupta and Gnanadesikan (1966) for the parameters of the logistic distribution, by Saleh and Ali (1966) for the parameters of the negative

exponential distribution and by Ogawa (1962) and Kulldorf (1961) for the parameters of the normal distribution. The optimum spacings of these estimators will be seen to correspond to the optimum spacings for our rank order tests in the respective cases.

Within the framework of the two-sample problem, under multiple censoring of Type II, a selection of one and two-parameter families of alternatives are considered. For the present study, the one-parameter location, scale and Lehmann alternatives and the two-parameter location and scale alternatives are selected. In the first instance, LMPRT tests are derived while in the second, an asymptotically optimum test in the Wald sense is obtained. In each case, the contiguity of the hypotheses hold under mild regularity conditions. Then, the asymptotic distribution of each test under the respective null and alternative hypotheses is studied. Finally, the optimum efficiency of the derived tests relative to their complete sample counterparts is discussed.

1.2 Outline of thesis

Within the context of the two-sample problem, Tests of hypothesis are considered where censoring of Type II occurs within the $k+1$ intervals specified by k arbitrary quantiles. The derived tests are then based on frequencies within these intervals. The presentation can be divided into three parts. The first part consists of Chapters 2 and 3, in which tests for one-parameter families of alternatives are studied. The second part consists of Chapter 4,

and deals with a two-parameter family of alternatives. Finally, the proofs for the contiguity of the hypotheses used in the previous chapters are collectively presented in the Chapter 5.

In Chapter 2, locally most powerful rank tests (LMPRT) for location and scale alternatives are derived. Following a discussion of some exact properties of the test statistics their limiting distributions are studied under the null as well as the location and scale alternatives, near to the null hypothesis. Next, the asymptotic efficiency of the derived tests relative to their complete sample counterparts is computed. Finally, the optimum choice of the k quantiles to maximize the above asymptotic relative efficiency is discussed.

The LMPRT for Lehmann alternatives are studied in Chapter 3. These alternatives are expressed either in terms of location or scale alternatives. Consequently, the appropriate regularity conditions under Lehmann alternatives can be translated from those assumed under location and scale alternatives. The technique parallels the one used in Chapter 2.

In Chapter 4, the test statistics, obtained earlier in Chapter 2, are combined into a quadratic form to produce a test for identity of distributions against the two-parameter location-scale alternatives. Although the component statistics yield LMPRT the resulting test is not so, rather it is shown to be asymptotically optimum in the Wald sense. Under contiguity, the asymptotic normality of the vector statistic is also obtained under the null as well as the local location-scale alternatives. Consequently, the asymptotic efficiency of the quadratic form relative to its complete sample

counterpart can be computed. Finally, a short discussion of the optimum spacings for specific density functions is presented.

In the fifth chapter, within the context of multiple censoring, the contiguity of the hypotheses used in the previous chapters is proved. The results are obtained successively for one and two-parameter families of alternatives. In each case, from the asymptotic normality of the log-likelihood ratio, the contiguity results by an application of LeCam's first Lemma.

In the final chapter, some concluding remarks together with specific propositions for further research within the multiple censored sample context are presented.

1.3 Preliminary Results

In this section, two basic notions are considered, namely: A locally most powerful test and contiguity.

As in Hájek and Šidák (1967, 63) let $\{q_{\Delta}; \Delta \neq 0\}$ be a sequence of densities and assume q_0 belongs to the null hypotheses H_0 . Then a locally most powerful test (LMP) is defined by:

Definition 1. A test will be called LMP for $\Delta=0$ against $\Delta>0$ at some level α , if it is uniformly most powerful at level α for H_0 against $K_{\epsilon} = \{q_{\Delta}, 0 < \Delta < \epsilon\}$ for some $\epsilon > 0$. Likewise, a test is LMP for $\Delta=0$ against $\Delta \neq 0$ if it is uniformly most powerful for H_0 against $K_{\epsilon} = \{q_{\Delta}, 0 < |\Delta| < \epsilon\}$ for some $\epsilon > 0$.

The notion of contiguity is basic in the study of limiting distributions of statistics under alternative hypotheses. The following definition appears in Matthes and Truax (1965).

Definition 2. Let $\{P_N; N \geq 1\}$ and $\{Q_N; N \geq 1\}$ be two sequences of probability measures defined on measurable spaces $\{X_N, \mathcal{A}_N; N \geq 1\}$. The sequence $\{Q_N\}$ is said to be contiguous to $\{P_N\}$ if for any sets $A_N \in \mathcal{A}_N$,

$$\lim_{N \rightarrow \infty} P_N(A_N) = 0 \text{ implies } \lim_{N \rightarrow \infty} Q_N(A_N) = 0.$$

As an important consequence, it was remarked by Hájek and Šidák (1967, 202) and Matthes and Truax (1965, 1214) that contiguity implies that any sequence of random variables converging to zero in P_N -probability converges to zero in Q_N -probability, $N \rightarrow \infty$.

The above conditions on the probability functions have been shown equivalent to some properties of the likelihood ratio L_N by LeCam (1960) in what is known as his First Lemma; it is reproduced here from Hájek and Šidák (1967, 204).

LeCam's First Lemma. If $\log L_N \sim N(-\frac{1}{2}\sigma^2, \sigma^2)$, then the sequences of alternative and null hypotheses are contiguous.

Limiting distributions of test statistics S_N under alternatives are important in view of the power properties of the respective tests. Under contiguity the proofs of limiting distributions under alternatives are simplified by the following lemma of LeCam, which is reproduced here from Hájek and Šidák (1967, 208).

LeCam's Third Lemma. Let S_N be a statistic and assume that under the null hypotheses $(S_N, \log L_N)$ is asymptotically normal $(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \sigma_{12})$

with $\mu_2 = -\frac{1}{2}\sigma_2^2$. Also assume the alternative and null hypotheses are contiguous. Then under the alternative hypotheses $S_N \sim N(\mu_1 + \sigma_{12}, \sigma_1^2)$.

The first lemma will be used to establish the contiguity of the sequences of hypotheses while the third lemma will be used to show the asymptotic normality of our statistics under the local alternatives.

CHAPTER 2

CLASS OF LOCALLY MOST POWERFUL RANK ORDER TESTS

FOR THE LOCATION AND SCALE PARAMETERS

Let $X_1, X_2, \dots, X_m$ and $Y_1, Y_2, \dots, Y_n$ be two independent samples of sizes m and n respectively, from two cumulative distribution functions F and G belonging to a one-parameter family of distributions. Let $W_1 \leq W_2 \leq \dots \leq W_N$, $N = m + n$, be the combined order statistics obtained from the two samples. Let $Z_1, Z_2, \dots, Z_N$ denote the rank order statistics, where $Z_i = 1$ if W_i is an X -observation or 0 if it is a Y -observation. Statistics based on linear combinations of the rank orders $Z_1, Z_2, \dots, Z_N$ for testing $H_0: F = G$ against location and scale alternatives have been proposed by many authors, for example Hájek and Šidák (1967) and Puri and Sen (1971, Chapter 3). In particular, in these references the authors discuss the locally most powerful tests.

Johnson and Mehrotra (1971) derived locally most powerful ranks tests for the single censored, right or left. The result was extended by Mehrotra, Johnson and Bhattacharya (1977) to the case where the data are left and right censored and censored once in the middle; however, they did not discuss the asymptotic distributions

of their statistics. The Pitman asymptotic efficiency of each statistic under censoring relative to its complete sample counterpart was not studied in either of the two papers.

In this chapter, we further extend the results to the case where censoring occurs in the $k+1$ intervals determined by k fixed quantiles. Moreover, this extension enables us to present a self-contained discussion of the problem based on the frequencies within the $k+1$ intervals.

We propose the statistics $T_{N1}^{(k)}$ and $T_{N2}^{(k)}$ for the location and scale parameters, respectively. We shall show that these statistics yield locally most powerful rank order tests (LMPRT). Moreover, they are asymptotically equivalent to the asymptotically most powerful grouped rank tests proposed by Gastwirth (1966).

After preliminary results we derive the locally most powerful rank tests, $T_{N1}^{(k)}$ and $T_{N2}^{(k)}$, for location and scale alternatives, respectively. Then we obtain the asymptotic distribution of each statistic under the null as well as the local alternatives. Finally, we study the asymptotic efficiency of $T_{N1}^{(k)}$ and $T_{N2}^{(k)}$ with respect to their complete sample counterparts, T_{N1} and T_{N2} .

2.1 Preliminary Results

Consider the combined order statistics $W_1 \leq W_2 \leq \dots \leq W_N$ from the samples of X and Y observations, of sizes m and n , respectively. Let k be a fixed natural number, and assume N is large,

$k \ll N$ and $m/N \rightarrow \lambda$ with $0 < \lambda < 1$, define the ranks $r_j = [N\lambda_j] + 1$
 $j = 1, 2, \dots, k$, with $[N\lambda_j]$ standing for the largest integer not
 exceeding $N\lambda_j$. These ranks specify the k quantiles $W_{r_1} < \dots < W_{r_k}$.

Letting M_{j+1} and N_{j+1} stand for the number of X and Y -observations,

strictly between W_{r_j} and $W_{r_{j+1}}$, we can write $M_{j+1} = \sum_{i=1} Z_i$ and

$N_{j+1} = \sum_{i=1} (1-Z_i)$ where i runs from r_j+1 to $r_{j+1}-1$, inclusive; we

also note that

$$M_{j+1} + N_{j+1} = r_{j+1} - r_j - 1 \quad (2.1)$$

$j = 0, 1, \dots, k$ with $r_0 = 0$ and $r_{k+1} = N + 1$. With this nota-
 tion, we define the random vector $Z^* = (M_1, Z_{r_1}, \dots, Z_{r_k}, M_{k+1})$.

Moreover, we select two families of cumulative distribution
 functions with absolutely continuous densities namely: the location
 parameter family denoted by $F(x-\theta)$, for θ real, and the scale family
 denoted by $F(e^{-\tau}x)$, for τ real. We note that σ is often used instead
 of $e^{-\tau}$.

Following Hájek and Šidák (1967, p. 65), we define the scores

$$a_N(i, f) = E \left\{ - \frac{f'(W_i)}{f(W_i)} \right\} \quad (2.2)$$

and

$$b_N(i, f) = E \left\{ -1 - W_i \frac{f'(W_i)}{f(W_i)} \right\} \quad (2.3)$$

for $i = 1, 2, \dots, N$, for the location and scale parameter, respectively; $f(w)$ stands for the density function corresponding to $F(w)$. Moreover, we define

$$\bar{a}_{j+1} = \sum_{i=r_j+1}^{r_{j+1}-1} a_N(i, f) / (r_{j+1} - r_j - 1) \quad (2.4)$$

and

$$\bar{b}_{j+1} = \sum_{i=r_j+1}^{r_{j+1}-1} b_N(i, f) / (r_{j+1} - r_j - 1). \quad (2.5)$$

for $j = 0, 1, \dots, k$ with $r_0 = 0$ and $r_{k+1} = N + 1$.

We first prove two lemmas which relate the expectation of the random variables $[f(W_s) - f(W_r)]/[F(W_s) - F(W_r)]$ and $[W_s f(W_s) - W_r f(W_r)]/[F(W_s) - F(W_r)]$ to the scores of the unobserved variables between W_r and W_s , for $s > r$. These results were shown by Johnson and Mehrotra (1972), for tail censored samples, in which case the random variables reduce to functions of only one ordered observation. We adapt their method to deal with functions of two ordered observations.

Lemma 2.1 If $f(x)$ is absolutely continuous and

$$\int_{-\infty}^{\infty} |f'(x)| dx < \infty, \quad (2.6)$$

then, for $r \leq s-2$,

$$E \left\{ - \frac{f(W_s) - f(W_r)}{F(W_s) - F(W_r)} \right\} = \sum_{i=r+1}^{s-1} a_N(i, f) / (s-r-1) \quad (2.7)$$

where the expectation is taken under H_0 .

Proof. Let the joint density of the i^{th} and the s^{th} ordered observa-

tions in a sample of size N be $g_{i,s}(w_i, w_s)$, where $i < s$, that is,

$$g_{i,s}(x, y) = c_{i,s} F(x)^{i-1} [F(y) - F(x)]^{s-i-1} [1 - F(y)]^{N-s} f(x) f(y) \quad (2.8)$$

for all $x < y$ and where $c_{i,s} = N! / \{(i-1)!(s-i-1)!(N-s)!\}$. Let

$$J(i, s, N) = E \left\{ - \frac{f(W_s) - f(W_i)}{F(W_s) - F(W_i)} \right\} \quad (2.9)$$

for $i = r+1, \dots, s-1$. Then

$$\begin{aligned} J(i, s, N) &= \int_{-\infty}^{\infty} \int_{-\infty}^y -[f(y) - f(x)] / [F(y) - F(x)] g_{i,s}(x, y) dx dy \\ &= c_{i,s} \int_{-\infty}^{\infty} [1 - F(y)]^{N-s} f(y) H(y) dy \end{aligned} \quad (2.10)$$

where

$$H(y) = \int_{-\infty}^y -[f(y) - f(x)] F(x)^{i-1} [F(y) - F(x)]^{s-i-2} f(x) dx.$$

Using the method of integration by parts with $H(y)$ taking $f(x)$ as one function, we can write $H(y)$ as

$$\begin{aligned}
 H(y) = & -F(x)^i [f(y)-f(x)] [F(y)-F(x)]^{s-i-2} \Big|_{-\infty}^y \\
 & - \int_{-\infty}^y F(x)^i [F(y)-F(x)]^{s-i-2} f'(x) dx \\
 & + (i-1) \int_{-\infty}^y [f(y)-f(x)] F(x)^{i-1} [F(y)-F(x)]^{s-i-2} f(x) dx \\
 & - (s-i-2) \int_{-\infty}^y [f(y)-f(x)] F(x)^i [F(y)-F(x)]^{s-i-3} f(x) dx. \quad (2.11)
 \end{aligned}$$

The first term in equation (2.11) is zero because $F(x) \rightarrow 0$ as $x \rightarrow -\infty$

Now, substituting $c_{i,s}/c_{i+1,s} = i/(s-i-1)$ and (2.11) in (2.10) we obtain

$$\begin{aligned}
 J(i,s,N) = & (c_{i,s}/c_{i+1,s}) \int_{-\infty}^{\infty} \int_{-\infty}^y [-f'(x)/f(x)] g_{i+1,s}(x,y) dx dy \\
 & - (i-1)J(i,s,N) + (s-i-1)^{-1} i(s-i-2)J(i+1,s,N) \quad (2.12)
 \end{aligned}$$

By straightforward integration, the first term of (2.12) yields

$i(s-i-1)^{-1} a_N(i+1,f)$; when substituted in (2.12), after some rearrangement, we have the recurrence formula

$$(s-i-1)J(i,s,N) = (s-i-2)J(i+1,s,N) + a_N(i+1,f) \quad (2.13)$$

Applying sequentially (2.13), when $i = r, r + 1, \dots, s - 1$, we obtain (2.7) upon using the definition of $J(i, s, N)$ in (2.9).

Lemma 2.2 If $f(x)$ is absolutely continuous and

$$\int_{-\infty}^{\infty} |xf'(x)| dx < \infty, \quad (2.14)$$

then

$$E \left\{ - \frac{W_s f(W_s) - W_r f(W_r)}{F(W_s) - F(W_r)} \right\} = \sum_{i=r+1}^{s-1} b_N(i, f) / (s-r-1) \quad (2.15)$$

where the expectation is taken under H_0 .

Proof: The proof is similar to that of Lemma 2.1 with

$$J_1(i, s, N) = E \left\{ - \frac{W_s f(W_s) - W_1 f(W_1)}{F(W_s) - F(W_1)} \right\}, \quad (2.16)$$

for $i = r + 1, \dots, s - 1$, instead of $J(i, s, N)$. Using the same procedure as in Lemma 2.1 we obtain

$$J_1(i, s, N) = c_{i, s} \int_{-\infty}^{\infty} [1 - F(y)]^{N-s} f(y) H_1(y) dy \quad (2.17)$$

where

$$H_1(y) = \int_{-\infty}^y [-yf(y) + xf(x)] F(x)^{i-1} [F(y) - F(x)]^{s-i-2} f(x) dx.$$

Then, using integration by parts with $H_1(y)$, taking $f(x)$ as one function and proceeding as in Lemma 2.1, we obtain the recurrence formula

$$(s-i-1)J_1(i,s,N) = (s-i-2)J_1(i+1,s,N) + b_N(i+1,f) \quad (2.18)$$

Applying sequentially (2.18), together with (2.16), the result (2.15) follows.

The next two theorems will be required in the proof of the asymptotic normality of the LMPRT statistics under location and scale alternatives. The first theorem relates the variance of

$[f(W_s)-f(W_r)]/[F(W_s)-F(W_r)]$ to the variances of $-f'(W_i)/f(W_i)$,

$i = r+1, \dots, s-1$ while the second relates the variance of

$[W_s f(W_s) - W_r f(W_r)]/[F(W_s)-F(W_r)]$ to the variances of $-1-W_i f'(W_i)/f(W_i)$,

$i = r+1, \dots, s-1$.

Theorem 2.1 If $\int_{-\infty}^{\infty} [f'(x)/f(x)]^2 f(x) dx < \infty$, then

$$(s-r-1) \text{Var} \left\{ \frac{f(W_s)-f(W_r)}{F(W_s)-F(W_r)} \right\} \leq \sum_{i=r+1}^{s-1} \text{Var} \left\{ -\frac{f'(W_i)}{f(W_i)} \right\} \quad (2.19)$$

for $r \leq s-2$

Proof. For the tail intervals the result has been shown by Mehrotra

(1971, 24). For convenience, we use

$$J(i) = J(i,s,N) = E \left\{ -\frac{f(W_s)-f(W_i)}{F(W_s)-F(W_i)} \right\} \quad (2.20)$$

and

$$I(i) = I(i,s,N) = E \left\{ \frac{f(W_s)-f(W_i)}{F(W_s)-F(W_i)} \right\}^2; \quad (2.21)$$

where $J(i,s,N)$ also appeared in (2.9). Then we can write

$$\text{Var} \left(\frac{f(W_s) - f(W_i)}{F(W_s) - F(W_i)} \right) = I(i) - J^2(i) \quad (2.22)$$

To show (2.19) it suffices to prove the recurrence inequality

$$(s-i-1)[I(i) - J^2(i)] \leq \text{Var}[f'(W_{i+1})/f(W_{i+1})] + (s-i-2)[I(i+1) - J^2(i+1)]; \quad (2.23)$$

the theorem follows by successive applications of (2.23) for

$i = r, \dots, s-1$.

First, consider the integral $I(i)$. From the definition (2.8),

$$\begin{aligned} I(i) &= \int_{-\infty}^{\infty} \int_{-\infty}^y \left(\frac{f(y) - f(x)}{F(y) - F(x)} \right)^2 g_{1,s}(x, y) dx dy \\ &= c_{1,s} \int_{-\infty}^{\infty} [1 - F(y)]^{N-s} f(y) H_2(y) dy \end{aligned}$$

where

$$H_2(y) = \int_{-\infty}^y [f(y) - f(x)]^2 F(x)^{i-1} [F(y) - F(x)]^{s-i-3} f(x) dx.$$

Using integration by parts with $F(x)^{i-1} f(x)$ as one function we obtain

$$\begin{aligned} H_2(y) &= i^{-1} F(x)^i [f(y) - f(x)]^2 [F(y) - F(x)]^{s-i-3} \Big|_{-\infty}^y \\ &+ 2i^{-1} \int_{-\infty}^y [f(y) - f(x)] F(x)^i [F(y) - F(x)]^{s-i-3} f'(x) dx \\ &+ (s-i-3)i^{-1} \int_{-\infty}^y [f(y) - f(x)]^2 F(x)^i [F(y) - F(x)]^{s-i-4} f(x) dx. \end{aligned}$$

It follows directly that the first term of $H_2(y)$ is zero, except for $i=s-1$ or $i=s-2$. For nontail intervals, when $i=s-1$ or $i=s-2$, the term also tends to zero by noting that $[f(y)-f(x)]/[F(y)-F(x)]$ tends to $f'(y)/f(y)$ when $x \rightarrow y$ and to $f(y)/F(y)$ when $x \rightarrow -\infty$. For tail intervals Mehrotra (1971) reached the same conclusion, though using a different argument.

Using $(s-i-1)c_{i,s} = ic_{i+1,s}$ in $H_2(y)$ and substituting in (2.24) we obtain

$$(s-i-1)I(i) = (s-i-3)I(i+1,s) + 2E \left\{ \frac{f(W_s)-f(W_i)}{F(W_s)-F(W_i)} \cdot \frac{f'(W_i)}{f(W_i)} \right\} \quad (2.25)$$

Combining the recurrence formulas (2.25) and (2.13) of Lemma 2.1, from the right side (2.22) we obtain

$$(s-i-1)[I(i)-J^2(i)] = (s-i-3)I(i+1)-2E \left\{ \frac{f(W_s)-f(W_i)}{F(W_s)-F(W_i)} \cdot \frac{f'(W_i)}{f(W_i)} \right\} \\ - (s-i-1)^{-1}[(s-i-2)J(i+1) + a_N(i+1,f)]^2, \quad (2.26)$$

We add and subtract $I(i+1)$ and $\text{Var}(f'(W_{i+1})/f(W_{i+1}))$ on the right side of (2.26). After some rearrangement we obtain

$$(s-i-1)[I(i)-J^2(i)] = \text{Var}[f'(W_{i+1})/f(W_{i+1})] + (s-i-2)[I(i+1)-J^2(i+1)] \\ - \text{Var}[h(W_{i+1}, W_s)] - (s-i-1)[a_N(i,f)-J(i+1)]^2, \quad (2.27)$$

which is an exact formula, where

$$h(W_{i+1}, W_s) = \frac{f(W_s) - f(W_{i+1})}{F(W_s) - F(W_{i+1})} + \frac{f'(W_{i+1})}{f(W_{i+1})}$$

Since the last two terms of (2.27) are negative the recurrence formula (2.23) holds and thus the theorem is proved.

Theorem 2.2 If $\int_{-\infty}^{\infty} [-1 - xf'(x)/f(x)]^2 f(x) dx < \infty$, then

$$(s-r-1) \text{Var} \left(\frac{W_s f(W_s) - W_r f(W_r)}{F(W_s) - F(W_r)} \right) \leq \sum_{i=r+1}^{s-1} \text{Var} \left(-1 - W_i \frac{f'(W_i)}{f(W_i)} \right) \quad (2.28)$$

for $r \leq s-2$.

Proof. Since the technique parallels the proof of the previous theorem we skip the details in the following. Again, for tail intervals, (2.28) was shown by Mehrotra (1971, 29). Let

$$J_1(i) = J_1(i, s, N) = E \left\{ - \frac{W_s f(W_s) - W_i f(W_i)}{F(W_s) - F(W_i)} \right\} \quad (2.29)$$

and

$$I_1(i) = I_1(i, s, N) = E \left\{ \left(\frac{W_s f(W_s) - W_i f(W_i)}{F(W_s) - F(W_i)} \right)^2 \right\}; \quad (2.30)$$

(2.29) corresponds to (2.16).

We first note that for nontail intervals, $[yf(y) - xf(x)]/[F(y) - F(x)]$ tends to $1 + yf'(y)/f(y)$ when $x \rightarrow y$; moreover, from Hájek and Šidák (1967, 20) it tends to $yf(y)/F(y)$ when $x \rightarrow -\infty$. Then, following the same procedure as in Theorem 2.1, with $I_1(i)$ instead of $I(i)$, $yf(y) - xf(x)$

instead of $f(y)-f(x)$ and (2.18), instead of (2.13), we obtain the equation

$$\begin{aligned} (s-i-1)[I_1(i)-J_1^2(i)] &= (s-i-3)I_1(i+1) \\ &+ 2E \left\{ - \frac{W_s f(W_s) - W_i f(W_i)}{F(W_s) - F(W_i)} \left(-1 - W_i \frac{f'(W_i)}{f(W_i)} \right) \right\} \\ &- (s-i-1)^{-1} [(s-i-2)J_1(i+1) + b_N(i+1, f)]^2 \end{aligned} \quad (2.31)$$

We add and subtract

$$\text{Var} \left(-1 - W_{i+1} \frac{f'(W_{i+1})}{f(W_{i+1})} \right) + E \left(- \frac{W_s f(W_s) - W_i f(W_i)}{F(W_s) - F(W_i)} \right)^2$$

to the right side of (2.31). After some rearrangement we obtain

$$\begin{aligned} (s-i-1)[I_1(i)-J_1^2(i)] &= \text{Var}[-1 - W_{i+1} f'(W_{i+1})/f(W_{i+1})] \\ &+ (s-i-2)[I_1(i+1)-J_1^2(i+1)] \\ &- \text{Var}[h_1(W_{i+1}, W_s)] \\ &- (s-i-1)^{-1} [b_N(i+1, f) - J_1(i+1)]^2 \end{aligned} \quad (2.32)$$

which is an exact formula, where

$$h_1(W_{i+1}, W_s) = -1 - W_{i+1} \frac{f'(W_{i+1})}{f(W_{i+1})} + \frac{W_s f(W_s) - W_{i+1} f(W_{i+1})}{F(W_s) - F(W_{i+1})}$$

Since the last two terms of (2.32) are negative we obtain the recurrence formula

$$(s-i-1)[I_1(i) - J_1^2(i)] = \text{Var}[-1 - W_{i+1} f'(W_{i+1}) / f(W_{i+1})] \\ + (s-i-2)[I_1(i+1) - J_1^2(i+1)]. \quad (2.33)$$

The theorem follows by successive applications of (2.33) for $i=r, \dots, s-1$.

Using a technique similar to that used in the proofs of Theorems 2.1 and 2.2 we can show exact results relating the covariances among $f'(W_i)/f(W_i)$ with the covariance between $f'(W_i)/f(W_i)$ and $[f(W_s) - f(W_r)]/[F(W_s) - F(W_r)]$ and likewise, covariances among $1 + W_i f'(W_i)/f(W_i)$ with the covariance between $1 + W_i f'(W_i)/f(W_i)$ and $[W_s f(W_s) - W_r f(W_r)]/[F(W_s) - F(W_r)]$ for $i=r+1, 1, \dots, s-1$. Since these results are not needed in the sequel they are omitted.

2.2 Locally Most Powerful Rank Order Tests

Within the context of multiple censoring of Type II, the results of Lemmas 2.1 and 2.2 are applied to obtain LMPRT statistics to test the null hypothesis against location and scale alternatives. These statistics are based on the random vector $Z^* = (M_1, Z_{r_1}, \dots, Z_{r_k}, M_{k+1})$.

The probability distribution of Z^* is obtained by the following theorem which is a generalization of Rao, Savage and Sobel (1960) and Johnson and Mehrotra (1972).

Theorem 2.3 The probability distribution of Z^* is given by

$$P(Z^*=z^*; F, G) = \left\{ \prod_{j=0}^k \binom{r_{j+1}-r_j-1}{m_{j+1}} / \binom{N}{m} \right\} E_W, \quad (2.34)$$

where

$$E_W = E \prod_{i=1}^k \left(\frac{g(W_{r_i})}{f(W_{r_i})} \right)^{1-z_{r_i}} \prod_{j=0}^k \left(\frac{G(W_{r_{j+1}})-G(W_{r_j})}{F(W_{r_{j+1}})-F(W_{r_j})} \right)^{m_{j+1}}. \quad (2.35)$$

Proof. The result is obtained by considering the product of two multinomial distributions over $2k+1$ disjoint intervals, namely: one random variable, $Z_{r_i} = 0$ or 1, occurs in the interval

$[W_{r_i}, W_{r_i}+dW_{r_i})$ for $i=1, 2, \dots, k$ and the $m_{j+1}+n_{j+1}$ random variables

$Z_i, i=r_j+1, \dots, r_{j+1}-1$, appear in the interval $[W_{r_j}+dW_{r_j}, W_{r_{j+1}})$ for

$j=0, 1, \dots, k$ with $W_{r_0} = -\infty$ and $W_{r_{k+1}} = \infty$. The integration gives (2.34).

Under the null hypothesis we obtain the following corollary to Theorem 2.3.

Corollary 2.1 Under the hypothesis $H_0: G=F$,

$$P_0(Z^*=z^*; F=G) = \left\{ \prod_{j=0}^k \binom{r_{j+1}-r_j-1}{m_{j+1}} / \binom{N}{m} \right\} \quad (2.36)$$

which is a multivariate hypergeometric distribution.

The likelihood ratio for Z^* , which is defined as the ratio of (2.34) and (2.36) is given by

$$L(z^*; \theta) = P(Z^*=z^*; F \neq G) / (Z^*=z^*; F=G) = E_W \quad (2.37)$$

where θ identifies the family of alternatives. We denote by $L'(z; \theta_0)$ the derivative of $L(z^*; \theta)$ with respect to θ evaluated at $\theta = \theta_0$. Assume θ is real valued and that $L'(z^*; \theta)$ exists in the neighborhood of $\theta = 0$. If the alternative is that $0 < \theta < \epsilon$, for some $\epsilon > 0$, then the locally most powerful test will reject the hypothesis $\theta = 0$ for the rank orders z^* which make $L'(z^*; 0)$ largest. When specializing to location and scale alternatives we obtain the following explicit results.

Theorem 2.4 If $\int_{-\infty}^{\infty} |f'(x)| dx < \infty$, then on the basis of Z^* , the test with critical region

$$T_{N1}^{(k)} = \sum_{i=1}^k (1 - Z_{r_i}) a_N(r_i, f) + \sum_{j=0}^k N_{j+1} \bar{a}_{j+1} \geq \text{constant}, \quad (2.38)$$

with $a_N(r_i, f)$ and $\bar{a}_{j+1}$ as defined in (2.2) and (2.4), is the locally most powerful rank order test for $H_0: G=F$ against $G(x) = F(x-\theta)$, $0 < \theta < \epsilon$, at the respective level.

Proof: We obtain $L(z^*; \theta)$ by substituting $G(x) = F(x-\theta)$ and $g(x) = f(x-\theta)$ into (2.37). Under the assumption of the theorem we can differentiate under the integral. Moreover with absolutely continuous densities the derivative of the integrand with respect to θ is a continuous function of θ and W_{r_j} , $j=1, \dots, k$. Then by direct computation of $L'(z^*; 0)$, using the expressions (2.2) and (2.4) and Lemma 2.1, we obtain $L'(z^*; 0) = T_{N1}^{(k)}$ and the theorem follows.

Theorem 2.5 If $\int_{-\infty}^{\infty} |xf'(x)| dx < \infty$, then on the basis of Z^* , the test with critical region

$$T_{N2}^{(k)} = \sum_{i=1}^k (1-Z_{r_i}) b_N(r_i, f) + \sum_{j=0}^k N_{j+1} \bar{b}_{j+1} \geq \text{constant}, \quad (2.39)$$

with $b_N(r_i, f)$ and b_{j+1} as defined by (2.3) and (2.5), is the locally most powerful rank order test for testing $H_0: G=F$ against $G(x) = F(e^{-\tau}x)$, $0 < \tau < \epsilon$, at the respective level.

Proof. The likelihood ratio $L(z^*; \tau)$ is obtained by substituting $G(x) = F(e^{-\tau}x)$ and $g(x) = e^{-\tau}f(e^{-\tau}x)$ into (2.37). As in Theorem 2.4 we can differentiate with respect to τ under the integral sign. Again, by direct computation of $L'(z^*; 0)$, using the expressions (2.3) and (2.5) and Lemma 2.2, we have $L'(z^*; 0) = T_{N2}^{(k)}$. The expression (2.39) follows.

When $F=G$, the probability distribution of Z^* is given by Corollary 2.1. Under specified location and scale alternatives we have the following approximations.

Corollary 2.2 If the assumptions of Lemma 2.1 hold, then under the alternatives $G(x) = F(x-\theta)$ for values of θ about 0,

$$P_{\theta}(Z^*=z^*) = P_0(Z^*=z^*; F=G) \{1 + \theta T_{N1}^{(k)} + O(\theta^2)\}. \quad (2.40)$$

Proof. Use the Taylor series expansion of $G(x) = F(x-\theta)$ and $g(x) = f(x-\theta)$ about $\theta=0$ in Theorem 2.3. Then from (2.38) and Corollary 2.1 we obtain (2.40).

Corollary 2.3 If the assumptions of Lemma 2.2 hold, then under the alternatives $G(x) = F(e^{-\tau}x)$ for values of τ about 0,

$$P_{\tau}(Z^*=z^*) = P_0(Z^*=z^*; F=G)\{1 + \tau T_{N2}^{(k)} + O(\tau^2)\}. \quad (2.41)$$

Proof. In Theorem 2.3 use the Taylor series expansion of

$G(x) = F(e^{-\tau}x)$ and $g(x) = e^{-\tau}f(e^{-\tau}x)$ about $\tau=0$. Then (2.41) follows from (2.39) and Corollary 2.1.

When restricted to the null hypothesis, with a technique similar to that used by Basu (1968), we obtain the exact mean and variance for $T_{N1}^{(k)}$ and $T_{N2}^{(k)}$; the asymptotic distributions under the null and alternative hypotheses will be studied in the next section.

The complete sample counterparts of $T_{N1}^{(k)}$ and $T_{N2}^{(k)}$ are given

by

$$T_{N1} = \sum_{i=1}^N (1-Z_i) a_N(i, f) \quad (2.42)$$

and

$$T_{N2} = \sum_{i=1}^N (1-Z_i) b_N(i, f) \quad (2.43)$$

where $a_N(i, f)$ and $b_N(i, f)$ are defined as (2.2) and (2.3). From

Hájek and Šidák (1967, 65f) the statistics T_{N1} and T_{N2} are locally

most powerful rank tests for testing H_0 against location and scale alternatives in the complete sample problem. Then, in addition to

the mean and variance for $T_{N1}^{(k)}$ and $T_{N2}^{(k)}$ we also obtain the covariance,

and consequently the correlation, between $T_{N1}^{(k)}$ and T_{N1} , $i=1, 2$. We summarize these null sample moments in the following theorems.

Theorem 2.6 Under $H_0: F=G$ and the assumptions of Lemma 2.1,

$$E_0(T_{N1}^{(k)}) = E_0(T_{N1}) = 0, \quad (2.44)$$

$$\text{Var}_0(T_{N1}^{(k)}) = \frac{mn}{N(N-1)} \left\{ \sum_{i=1}^k a_N^2(r_i, f) + \sum_{j=0}^k (r_{j+1} - r_j - 1) \bar{a}_{j+1}^2 \right\}, \quad (2.45)$$

$$\text{Var}_0(T_{N1}) = \frac{mn}{N(N-1)} \sum_{i=1}^N a_N^2(i, f), \quad (2.46)$$

$$\text{Cov}_0(T_{N1}^{(k)}, T_{N1}) = \text{Var}_0(T_{N1}^{(k)}), \quad (2.47)$$

and

$$r_0^2(T_{N1}^{(k)}, T_{N1}) = \text{Var}_0(T_{N1}^{(k)}) / \text{Var}_0(T_{N1}). \quad (2.48)$$

Proof. We note that $T_{N1}^{(k)}$ can be expressed as

$$T_{N1}^{(k)} = \sum_{s=1}^N (1 - Z_s) d_s \quad (2.49)$$

where

$$d_s = \begin{cases} a_N(r_i, f) & \text{for } s=r_i, i=1, 2, \dots, k \\ \bar{a}_{j+1} & \text{for } s=r_j+1, \dots, r_{j+1}-1, \text{ and } j=0, 1, \dots, k, \end{cases}$$

with $a_N(i, f)$ and $\bar{a}_{j+1}$ as defined in (2.2) and (2.4).

Since $E_o(1-Z_s) = n/N$ and $E_o(1-Z_s)(1-Z_r) = n(n-1)/N(N-1)$ we have

$$E_o(T_{N1}^{(k)}) = \frac{n}{N} \sum_{s=1}^N d_s \quad (2.50)$$

and

$$\text{Var}_o(T_{N1}^{(k)}) = \frac{mn}{N^2(N-1)} \left\{ N \sum_{s=1}^N d_s^2 - \left(\sum_{s=1}^N d_s \right)^2 \right\} \quad (2.51)$$

We note that for T_{N1} , defined by (2.42), the expressions (2.50) and (2.51) hold where $d_s = a_N(s, f)$ for $s=1, 2, \dots, N$. Moreover, Hájek

and Šidák (1967, p. 66) showed that $\sum_{i=1}^N a_N(i, f) = 0$, and hence,

using (2.50) and (2.51) the expressions (2.44), (2.45) and (2.46) follow.

Using the above notation, we also have

$$\text{Cov}_o(T_{N1}^{(k)}, T_{N1}) = \frac{mn}{N^2(N-1)} \left\{ N \sum_{s=1}^N d_s a_N(s, f) - \left(\sum_{s=1}^N d_s \right) \left(\sum_{s=1}^N a_N(s, f) \right) \right\} \quad (2.52)$$

where d_s represent the scores of $T_{N1}^{(k)}$ and $a_N(s, f)$, the scores of T_{N1} .

By direct computation we obtain (2.47). Finally, by definition

$r_o^2(x, y) = \text{Cov}_o^2(x, y) / [\text{Var}_o(x) \text{Var}_o(y)]$ which gives (2.48).

Theorem 2.7 Under $H_o: F=G$ and the assumptions of Lemma 2.2

$$E_o(T_{N2}^{(k)}) = E_o(T_{N2}) = 0, \quad (2.53)$$

$$\text{Var}_o(T_{N2}^{(k)}) = \frac{mn}{N(N-1)} \left\{ \sum_{i=1}^k b_N^2(r_i, f) + \sum_{j=0}^k (r_{j+1} - r_j - 1) \bar{b}_{j+1}^2 \right\}, \quad (2.54)$$

$$\text{Var}_o(T_{N2}) = \frac{mn}{N(N-1)} \sum_{i=1}^N b_N^2(i, f), \quad (2.55)$$

$$\text{Cov}(T_{N2}^{(k)}, T_{N2}) = \text{Var}_o(T_{N2}^{(k)}) \quad (2.56)$$

and

$$r_o^2(T_{N2}^{(k)}, T_{N2}) = \text{Var}_o(T_{N2}^{(k)}) / \text{Var}(T_{N2}) \quad (2.57)$$

Proof. Since the proof parallels that of Theorem 2.6, where $b_N(i, f)$ is used instead of $a_N(i, f)$, we omit it. We only note that by Hájek and Šidák (1967, p.66) we also have $\sum_{s=1}^N b_N(s, f) = 0$, under the assumptions of Lemma 2.2.

2.3 Asymptotic Distributions

For the asymptotic normality of $T_{N1}^{(k)}$ and $T_{N2}^{(k)}$ it suffices to consider the asymptotic distribution of the terms containing N_{j+1} , $j=0, \dots, k$ in (2.38) and (2.39); these N_{j+1} are jointly asymptotically multinormal under H_o as well as local alternatives. However, to study their asymptotic efficiency one requires their asymptotic mean and variance under the null as well as the alternative hypotheses. Under H_o , the asymptotic moments of $T_{Ni}^{(k)}$, $i=1, 2$ are obtained from the limiting weights. Using the concept of contiguity we obtain the

corresponding moments under the alternatives. The arguments concerning the contiguity of several families of alternatives are collected in Chapter 5.

In the sequel we shall need the next two lemmas.

Lemma 2.3 (Cramér, 1946 p. 299). The vector $(N_1, \dots, N_{k+1})$ is multivariate normal iff for every $\underline{a} = (a_1, \dots, a_{k+1})$, the linear combination $V_N = \sum_{j=0}^k a_{j+1} N_{j+1}$ is normal.

Lemma 2.4 (Chung, 1974 p. 93). Let $V_N^* = \sum_{j=0}^k d_{j+1} N_{j+1}$, where $d_1, \dots, d_{k+1}$ are constants. If $V_N = \sum_{j=0}^k a_{j+1} N_{j+1}$ is asymptotically normal and $d_{j+1}^{-1} a_{j+1} \rightarrow 0$, then V_N^* is also asymptotically normal.

To discuss the asymptotic normality of the statistics $T_{N1}^{(k)}$ and $T_{N2}^{(k)}$ we need their limiting weights which are obtained from the following result of Hoeffding (1953).

Lemma 2.5 Suppose $\int_{-\infty}^{\infty} |x| dF(x) < \infty$. Let $g(x)$ be a real valued, continuous function such that $|g(x)| \leq h(x)$ where the function $h(x)$ is convex with $\int_{-\infty}^{\infty} h(x) dF(x) < \infty$. Moreover, if $\frac{m}{N} \rightarrow F(y)$ as $N \rightarrow \infty$, where y is a continuity point of $F(x)$, then

$$\lim_{N \rightarrow \infty} N^{-1} \sum_{i=1}^m E\{g(W_i)\} = \int_{-\infty}^y g(x) dF(x). \quad (2.58)$$

By definition $r_j = [N\lambda_j] + 1$, and consequently $r_j/N \rightarrow \lambda_j$,

$j=1,2,\dots,k$. Then the limiting weights in $T_{N1}^{(k)}$ and $T_{N2}^{(k)}$ are given

by the two following lemmas.

Lemma 2.6 Under the same regularity conditions as in Theorem 2.4

and Lemma 2.5 where $g(x) = -f'(x)/f(x)$,

$$\lim_{N \rightarrow \infty} a_N(r_i, f) = - \frac{f'(F^{-1}(\lambda_i))}{f(F^{-1}(\lambda_i))}, \quad i=1, \dots, k \quad (2.59)$$

and

$$\alpha_{j+1} = \lim_{N \rightarrow \infty} \bar{a}_{j+1} = - \frac{f(F^{-1}(\lambda_{j+1})) - f(F^{-1}(\lambda_j))}{\lambda_{j+1} - \lambda_j} \quad (2.60)$$

for $j=0,1,\dots,k$ where $a_N(i, f)$ and $\bar{a}_{j+1}$ are defined as in (2.2)

and (2.4).

Proof. Expand $-f'(x)/f(x)$ in Taylor series about $F^{-1}(\lambda_j)$ where

$r_j/N \rightarrow \lambda_j$ and substitute into (2.2). Upon taking the expectation

(2.59) follows.

Whenever $g(x) = -f'(x)/f(x)$ satisfies the conditions of Lemma

2.5, we have

$$\lim_{N \rightarrow \infty} \sum_{i=1}^{r_j} a_N(i, f)/N = \int_{-\infty}^y -\{f'(x)/f(x)\} dF(x) = -f(F^{-1}(\lambda_j))$$

where $y = F^{-1}(\lambda_j)$. Then, from the definition of $\bar{a}_{j+1}$ in (2.4) we have

$$\bar{a}_{j+1} = (r_{j+1} - r_j - 1)^{-1} N \left\{ \sum_{i=1}^{r_{j+1}-1} a_N(i, f)/N - \sum_{i=1}^{r_j} a_N(i, f)/N \right\} \\ + (\lambda_{j+1} - \lambda_j)^{-1} [-f(F^{-1}(\lambda_{j+1})) + f(F^{-1}(\lambda_j))],$$

which is (2.60).

Lemma 2.7 Under the same regularity conditions as in Theorem 2.5

and Lemma 2.5, where $g(x) = -1 - xf'(x)/f(x)$,

$$\lim_{N \rightarrow \infty} b_N(r_i, f) = -1 - F^{-1}(\lambda_i) \frac{f'(F^{-1}(\lambda_i))}{f(F^{-1}(\lambda_i))}, \quad i=1, \dots, k, \quad (2.61)$$

and

$$\beta_{j+1} = \lim_{N \rightarrow \infty} \bar{b}_{j+1} = - \frac{F^{-1}(\lambda_{j+1})f(F^{-1}(\lambda_{j+1})) - F^{-1}(\lambda_j)f(F^{-1}(\lambda_j))}{\lambda_{j+1} - \lambda_j} \quad (2.62)$$

for $j=0, 1, \dots, k$, where $b_N(i, f)$ and $\bar{b}_{j+1}$ are defined as in (2.3)

and (2.5).

Proof. The same argument as that in Lemma 2.6 yields (2.61). Whenever

$g(x) = -1 - xf'(x)/f(x)$ satisfies the conditions of Lemma 2.5,

$$\lim_{N \rightarrow \infty} \sum_{i=1}^{r_j} b_N(i, f)/N = \int_{-\infty}^y [-1 - xf'(x)/f(x)] dF(x) \\ = -F^{-1}(\lambda_j)f(F^{-1}(\lambda_j))$$

where $y = F^{-1}(\lambda_j)$. Then, using the same technique as in Lemma 2.6, with $\bar{b}_{j+1}$ in (2.5) instead of $\bar{a}_{j+1}$, we obtain (2.62).

We note that the regularity conditions of Lemma 2.5, Theorem 2.4 and Theorem 2.5 are satisfied by most densities of interest, in particular this is so for the strongly unimodal densities.

We combine the above results with those of Chapter 5 to establish the asymptotic normality of the LMPRT statistics $T_{N1}^{(k)}$ and $T_{N2}^{(k)}$, under the null as well as the local alternative hypotheses.

For each statistic, we first show the asymptotic normality under the null hypothesis. Then, from Chapter 5, we observe that, under the null hypothesis, $\log L_N$ is asymptotically normal and that a result of the following type holds, that is, $\log L_N \rightarrow \theta S_{N\theta}^* - \frac{1}{2}\theta^2$ in probability, where $S_{N\theta}^*$ is asymptotically equivalent to $T_{Ni}^{(k)}$, $i=1,2$. Hence, applying the third LeCam's Lemma, we obtain the asymptotic distribution of $T_{Ni}^{(k)}$ under the local alternatives.

We first introduce some notation. Let

$$\mu_{i0} = \mu_0(T_{Ni}^{(k)}) = \lim_{N \rightarrow \infty} E_0(T_{Ni}^{(k)}), \quad (2.63)$$

and

$$\sigma_{i0}^2 = \sigma_0^2(T_{Ni}^{(k)}) = \lim_{N \rightarrow \infty} \text{Var}_0(T_{Ni}^{(k)}), \quad (2.64)$$

for $i=1,2$. Also, let $F^{-1}(\lambda_j) = u_j$ and $f_j = f(F^{-1}(\lambda_j)) = f(u_j)$

for $j=0,1,\dots,k+1$ with $f_0 = f_{k+1} = 0$.

Theorem 2.8 Assume the regularity conditions of Theorem 2.4 and Lemma 2.5 hold. Then, under the null hypothesis H_0 ,

$$T_{N1}^{(k)} \sim N(\mu_{10}, \sigma_{10}^2) \quad (2.65)$$

where $\mu_{10}=0$ and

$$N^{-1} \sigma_{10}^2 = \lambda(1-\lambda) \sum_{j=0}^k (f_{j+1} - f_j)^2 (\lambda_{j+1} - \lambda_j)^{-1}. \quad (2.66)$$

Proof. By virtue of Theorem 2.6, $\text{Var}_0(T_{N1}^{(k)}) = O(N)$, whereas

$\text{Var}_0\left(\sum_{i=1}^k Z_{r_i} a_{r_i}\right) = O(1)$, and hence, for the asymptotic normality of

$T_{N1}^{(k)}$, it suffices to consider only the statistic

$$T_{N1}^{*(k)} = \sum_{j=0}^k N_{j+1} \bar{a}_{j+1}; \quad (2.67)$$

both have the same limiting distribution.

The distribution of $\underline{N} = (N_1, N_2, \dots, N_{k+1})$, under H_0 , follows

from (2.36) of Corollary 2.1. From (2.36) the distribution of $\underline{N}$ is multivariate hypergeometric, and hence asymptotically multi-normal.

Therefore, by Lemma 2.3, the linear combination $V_{N1} = \sum_{j=0}^k \alpha_{j+1} N_{j+1}$ of

N_{j+1} , $j=0, 1, \dots, k$, is also asymptotically normal. Then, by Lemmas 2.4 and 2.6 it follows that $T_{N1}^{*(k)}$, and hence $T_{N1}^{(k)}$, are also asymptotically normal.

The asymptotic mean and variance of $T_{N1}^{(k)}$ are obtained as

limiting values of the sample moments in Theorem 2.6. The expression (2.44)

yields $\mu_{10}=0$ and (2.45) together with Lemma 2.6 yield (2.66) since

$$N^{-1} \sum_{j=0}^k (f'_j / f_j) \rightarrow 0 \text{ as } N \rightarrow \infty.$$

Theorem 2.9 Assume the regularity conditions of Theorem 2.5 and Lemma 2.5 hold. Then, under the null hypothesis H_0 ,

$$T_{N2}^{(k)} \sim N(\mu_{20}, \sigma_{20}^2) \quad (2.68)$$

where $\mu_{20}=0$ and

$$N^{-1} \sigma_{20}^2 = \lambda(1-\lambda) \sum_{j=0}^k (u_{j+1} f_{j+1} - u_j f_j)^2 (\lambda_{j+1} - \lambda_j)^{-1}. \quad (2.69)$$

Proof. Based on Theorem 2.7, Lemma 2.7 and the statistic

$$T_{N2}^{*(k)} = \sum_{j=0}^k N_{j+1} \bar{b}_{j+1}, \quad (2.70)$$

the argument parallels that of Theorem 2.8 and is omitted here.

To establish the asymptotic distribution of the LMPRT statistics $T_{N1}^{(k)}$ under local alternatives, we use the idea of contiguity.

First, consider the hypotheses specified by

$$H_0 = F_X(w) = G_Y(w) = F(w; \bar{\theta}) \quad (2.71)$$

and

$$H_A : F_X(w) = F(w) \text{ and } G_Y(w) = F(w; \theta) \quad (2.72)$$

with $\bar{\theta} = n\theta/N$. As in Hájek and Šidák (1967), the hypotheses (2.71) have been chosen as to give least favourable densities for H_0 against H_A in an asymptotic sense. In Chapter 5, we show the contiguity of sequences of hypotheses of this type for location and scale alternatives.

Corresponding to the hypotheses (2.71) and (2.72) we obtain the likelihood ratio $L_N(f)$ which is the ratio of the joint densities of

$W=(W_{r_1}, \dots, W_{r_k})$ and $Z^*=(M_1, Z_{r_1}, \dots, Z_{r_k}, M_{k+1})$ under H_A and H_0 ,

respectively; these densities can be derived from (2.34). As in

Theorem 2.4, for location alternatives, we write $F(w;\theta)=F(w-\theta)$ in

(2.71) and (2.72); also $f(w;\theta)=f(w-\theta)$. Likewise, for scale alter-

natives, as in Theorem 2.5, we use $F(w;\tau)=F(e^{-\tau}w)$ and $f(w;\tau)=e^{-\tau}f(e^{-\tau}w)$

Using the index $i=1$ or 2 to identify the location or scale case, respectively, we state two results which are proved in a slightly more general or in Chapter 5.

Theorem 2.10 If f is absolutely continuous, $\int_{-\infty}^{\infty} |f'(x)| dx < \infty$ and

$I(f) < \infty$, then

$$\log L_{N1} \xrightarrow{P_{\bar{\theta}}} \theta T_{N1}^{*(k)} - \frac{1}{2} \sigma_{N1}^2$$

with $T_{N1}^{*(k)}$ as defined by (2.67) and $\sigma_{N1}^2 = \theta^2 \sigma_{10}^2$.

Theorem 2.11 If f is absolutely continuous, $\int_{-\infty}^{\infty} |xf'(x)| dx < \infty$ and

$I_1(f) < \infty$, then

$$\log L_{N2} \xrightarrow{P_{\bar{\tau}}} \tau T_{N2}^{*(k)} - \frac{1}{2} \sigma_{N2}^2$$

with $T_{N2}^{*(k)}$ as defined by (2.70) and $\sigma_{N2}^2 = \tau^2 \sigma_{20}^2$.

We are now in a position to state the results corresponding to the limiting distributions of $T_{N1}^{(k)}$, $i=1,2$, under local alternatives.

To stress the dependence of L_{N1} , $T_{N1}^{(k)}$ and σ_{N1}^2 , $i=1,2$, on the density f we can write $L_{N1}(f)$, $T_{N1}^{(k)}(f)$ and $\sigma_{N1}^2(f)$.

Theorem 2.12 Under the assumptions of Theorem 2.10 and under the location alternatives $H_A: F_X(w)=F(w)$ and $G_Y(w)=F(w-\theta)$, where $\theta=h_N N^{-1/2}$ with $h_N \rightarrow h$ constant,

$$T_{N1}^{(k)} \sim N(\theta \sigma_{10}^2, \sigma_{10}^2) \quad (2.73)$$

where σ_{10}^2 is defined in Theorem 2.10.

Proof. Select a density f_0 and use the scores corresponding to it.

Then from Theorem 2.10

$$\log L_{N1}(f_0) \xrightarrow{P_0} \theta T_{N1}^{*(k)}(f_0) - \frac{1}{2} \sigma_{N1}^2(f_0). \quad (2.74)$$

For simplicity of presentation we shall abuse the notation and drop f_0 ; there is no ambiguity since the following expressions are all based on the density f_0 . In view of LeCam's third Lemma it suffices to obtain the asymptotic joint distribution of $(T_{N1}^{*(k)}, \log L_{N1})$ under the null hypothesis. By (2.74) the joint asymptotic distribution of $(T_{N1}^{*(k)}, \log L_{N1})$ is equivalent to that of $(T_{N1}^{*(k)}, \theta T_{N1}^{*(k)} - \frac{1}{2} \sigma_{N1}^2)$ which is bivariate normal. Under the null hypothesis, using Theorem 2.8 and its proof, we can write

$$(T_{N1}^{*(k)}, \theta T_{N1}^{*(k)} - \frac{1}{2} \sigma_{N1}^2) \sim N(0, -\frac{1}{2} \theta^2 \sigma_{10}^2, \sigma_{10}^2, \theta^2 \sigma_{10}^2, \theta \sigma_{10}^2).$$

Then, applying LeCam's third Lemma we obtain

$$T_{N1}^{*(k)} \sim N(\theta \sigma_{10}^2, \sigma_{10}^2) \quad (2.74)$$

under the local location alternatives. From the proof of Theorem 2.8, the statistics $T_{N1}^{*(k)}$ and $T_{N1}^{(k)}$ are asymptotically equivalent; hence, the theorem follows. A corresponding result holds for the scale alternatives.

Theorem 2.13 Under the assumptions of Theorem 2.11 and under the scale alternatives $H_A: F_X(w) = F(w)$ and $G_Y(w) = F(e^{-\tau} w)$ where $\tau = h_N N^{-1/2}$ with $h_N \rightarrow h$ constant,

$$T_{N2}^{(k)} \sim N(\tau \sigma_{20}^2, \sigma_{20}^2) \quad (2.75)$$

where σ_{20}^2 is defined in Theorem 2.11.

Proof. By Theorem 2.10 we have

$$\log L_{N2}(f_0) \xrightarrow{P_{\tau}^-} \tau T_{N2}^{*(k)}(f_0) - \frac{1}{2} \sigma_{N2}^2(f_0). \quad (2.76)$$

Using Theorem 2.9 instead of Theorem 2.8 and changing the index 1 to 2 in the proof of Theorem 2.12 we obtain that of Theorem 2.13.

The results of Theorems 2.8, 2.9, 2.12 and 2.13 were obtained by Hájek and Šidák (1967) for the complete sample case and by Mehrotra and Johnson (1976) for the right or left censored sample case. The results of Theorems 2.12 and 2.13 are important in view of the power properties of the respective tests.

2.4 Asymptotic Relative Efficiency

Whenever all statistics under consideration have asymptotically normal distributions under the null as well as the alternative hypothesis their asymptotic power can be compared using the concept of Asymptotic Relative Efficiency (ARE), referred to as the Pitman-Noether efficiency by Puri and Sen (1971, 113). In this section, our objective is to study the ARE of the tests based on $T_{N1}^{(k)}$, $i=1,2$, relative to their complete sample counterparts.

We first state the regularity conditions required for the limiting power of a test; these correspond to the assumptions of Theorem 3.8.1 of Puri and Sen (1971, 113-6) or to those of Van Eaden (1963). For the statistic T_N let $\mu_N(\theta) = E_\theta(T_N)$ and $\sigma_N^2(\theta) = \text{Var}_\theta(T_N)$ where the null and alternatives hypotheses are indexed by θ ; $H_0: \theta = \theta_0$ and $H_{AN}: \theta = \theta_N, \theta_N \neq \theta_0$, where $\theta_N = \theta_0 + hN^{-1/2}$ with h a positive constant independent of N . Moreover let $\mu'_N(\theta_0)$ stand for the derivative of $\mu_N(\theta)$, with respect to θ , evaluated at θ_0 . The regularity conditions are:

(a) $\lim_{N \rightarrow \infty} P_{\theta_0}(T_N \geq \lambda_{N,\alpha}) = \alpha$, with $0 < \alpha \leq 1$ a fixed value;

$T_N \geq \lambda_{N,\alpha}$ is the critical region of the corresponding test

A_N for testing $\theta = \theta_0$ against the alternative $\theta > \theta_0$;

(b) $\mu'_N(\theta_0) > 0$;

(c) $\lim_{N \rightarrow \infty} \mu'_N(\theta_N) / \mu'_N(\theta_0) = 1$;

$$(d) \lim_{N \rightarrow \infty} \sigma_N(\theta_N) / \sigma_N(\theta_0) = 1 ;$$

$$(e) \lim_{N \rightarrow \infty} N^{-1/2} \mu'_N(\theta_0) / \sigma_N(\theta_0) = c > 0 ;$$

(f) the distribution of $\{T_N - \mu_N(\theta_N)\} / \sigma_N(\theta_N)$ tends to a standardized normal distribution both under the null and the alternative hypotheses.

The quantity c in (e) is called the efficacy of the test corresponding to T_N ; we shall denote it by $\text{eff}(T_N)$. The limiting power of the test A_N corresponding to T_N is given by the following lemma of Puri and Sen (1971, 115).

Lemma 2.8 Under the assumptions (a)-(f) the limiting power of the test A_N is

$$1 - \Phi(\lambda_{N,\alpha} - hc) \quad (2.77)$$

where $\Phi(x)$ stands for the unit normal cumulative distribution. Note that by (a) $1 - \Phi(\lambda_{N,\alpha}) = \alpha$.

Whenever two tests, T_{1N} and T_{2N} , satisfy the assumptions (a)-(f) the asymptotic relative efficiency of T_{1N} relative to T_{2N} is given by:

$$\text{ARE}(T_{1N}, T_{2N}) = \lim_{N \rightarrow \infty} \left\{ \frac{\mu'_{1N}(\theta_0)}{\sigma_{1N}(\theta_0)} / \frac{\mu'_{2N}(\theta_0)}{\sigma_{2N}(\theta_0)} \right\}^2 \quad (2.78)$$

Using the notion of efficacy we can also write

$$ARE(T_{1N}, T_{2N}) = [\text{eff}(T_{1N})/\text{eff}(T_{2N})]^2. \quad (2.79)$$

Consider the statistics $T_{N1}^{(k)}$ and T_{N1} and the location alternatives $\theta_N = hN^{-1/2}$. Then, by Theorems 2.4, 2.8 and 2.12, the regularity conditions (a)-(f) are readily satisfied for $T_{N1}^{(k)}$. For T_{N1} , these conditions follow from Hájek and Šidák (1967, 215). Hence, we obtain the following results from Lemma 2.8 and (2.78); the first result being immediate.

Theorem 2.14 Under the location alternatives $\theta_N = hN^{-1/2}$, the asymptotic power of $T_{N1}^{(k)}$ is

$$1 - \phi(\lambda_{N1, \alpha} - hc)$$

where c is given by (e) with $T_{N1}^{(k)}$ and $1 - \phi(\lambda_{N1, \alpha}) = \alpha$.

Theorem 2.15 The asymptotic efficiency of $T_{N1}^{(k)}$ relative to T_{N1} is given by

$$ARE(T_{N1}^{(k)}, T_{N1}) = I^{-1}(f) \sum_{j=0}^k (f_{j+1} - f_j)^2 / (\lambda_{j+1} - \lambda_j) \quad (2.80)$$

where $f_j = f(F^{-1}(\lambda_j))$ with $f_0 = f_{k+1} = 0$ and $I(f) = \int_0^1 J^2(u) du$ where

$J(u) = -f'(F^{-1}(u))/f(F^{-1}(u))$ is the Fisher's information of f in the location parameter family of densities.

Proof. By Theorems 2.8 and 2.12, $T_{N1}^{(k)} \sim N(0, \sigma_{10}^2)$ and $T_{N1}^{(k)} \sim N(\theta_N \sigma_{10}, \sigma_{10}^2)$

under the null and the location alternative hypothesis, respectively,

where σ_{10}^2 is defined by (2.66). Then, from condition (e) above, we have

$$\begin{aligned} [\text{eff}(T_{N1}^{(k)})]^2 &= [\lim_{N \rightarrow \infty} N^{-1/2} \sigma_{10}]^2 \\ &= \lambda(1-\lambda) \sum_{j=0}^k (f_{j+1} - f_j)^2 / (\lambda_{j+1} - \lambda_j). \end{aligned} \quad (2.81)$$

Likewise, from Hájek and Šidák (1967, 215), $T_{N1} \sim N(0, \sigma_o^2)$ and

$T_{N1} \sim N(\theta_N \sigma_o^2, \sigma_o^2)$ under the null and the alternative hypothesis,

respectively, where $N^{-1} \sigma_o^2 = \lambda(1-\lambda) I(f)$. These asymptotic results

are also obtained by the Chernoff-Savage Theorem as stated in Chernoff

and Savage (1958) or in Puri and Sen (1971, 116) where their statistic

T_N correspond to T_{N1}/m . Then we obtain

$$\begin{aligned} [\text{eff}(T_{N1})]^2 &= [\lim_{N \rightarrow \infty} N^{-1/2} (mnN^{-1} I(f))^{1/2}]^2 \\ &= \lambda(1-\lambda) I(f). \end{aligned} \quad (2.82)$$

Applying the definition (2.79),

$$\text{ARE}(T_{N1}^{(k)}, T_{N1}) = [\text{eff}(T_{N1}^{(k)})]^2 / [\text{eff}(T_{N1})]^2$$

and (2.80) follows.

Now, consider the scale alternatives $\tau_N = hN^{-1/2}$. By Theorems

2.5, 2.9 and 2.13 the regularity conditions (a)-(f) are satisfied

for $T_{N2}^{(k)}$; for T_{N2} these follow from Hájek and Šidák (1967, 215).

Then, results similar to those of Theorems 2.14 and 2.15 for location alternatives are obtained for the scale alternatives.

Theorem 2.16 Under the scale alternatives $\tau_N = hN^{-1/2}$, the asymptotic power of $T_{N2}^{(k)}$ is

$$1 - \Phi\left(\frac{c}{\sigma_{20}} - hc\right) \quad (2.83)$$

where c is given by (a) with $T_{N2}^{(k)}$ and $1 - \Phi(\lambda_{N2, \alpha}) = \alpha$.

Theorem 2.17 The asymptotic efficiency of $T_{N2}^{(k)}$ relative to T_{N2} is given by

$$ARE(T_{N2}^{(k)}, T_{N2}) = I_1^{-1}(f) \sum_{j=0}^k (u_{j+1} f_{j+1} - u_j f_j)^2 / (\lambda_{j+1} - \lambda_j) \quad (2.84)$$

where $f_j = f(F^{-1}(\lambda_j))$ and $F^{-1}(\lambda_j) = u_j$ with $f_0 = f_{k+1} = 0$ and $I_1(f) = \int_0^1 J_1^2(u) du$

where $I_1(f) = -1/F^{-1}(u) f'(F^{-1}(u)) / f(F^{-1}(u))$ is the Fisher's information matrix of f in the scale parameter family of densities.

Proof. The proof parallels that of Theorem 2.15; we provide only the highlights of it. By Theorems 2.9 and 2.13 with σ_{20}^2 as defined by

(2.69) we obtain

$$[eff(T_{N2}^{(k)})]^2 = \lambda(1-\lambda) \sum_{j=0}^k (u_{j+1} f_{j+1} - u_j f_j)^2 / (\lambda_{j+1} - \lambda_j). \quad (2.85)$$

In the same manner as for Theorem 2.15 from Hájek and Šidák (1967, 215) we obtain

$$[\text{eff}(T_{N2})]^2 = \lambda(1-\lambda)I_1(f). \quad (2.86)$$

The ratio of (2.85) and (2.86) yields (2.84) which proves the theorem.

It may be noted that $\text{ARE}(T_{N1}^{(k)}, T_{N1})$ and $\text{ARE}(T_{N2}^{(k)}, T_{N2})$ in (2.80) and (2.84) can also be obtained as the limiting values of the expressions (2.48) and (2.57) for the correlation coefficients $r_{0N1}^2(T_{N1}^{(k)}, T_{N1})$ and $r_{0N2}^2(T_{N2}^{(k)}, T_{N2})$, respectively. This fact is in agreement with Van Eeden's (1963) Theorem provided some additional regularity conditions are satisfied by the joint distribution of $T_{N1}^{(k)}$ and T_{N1} , $i=1,2$; these hold in the present case.

We note that $\text{ARE}(T_{N1}^{(k)}, T_{N1})$ and $\text{ARE}(T_{N2}^{(k)}, T_{N2})$ coincide with the ARE of the corresponding BLUE/estimators for the location and scale parameter of a distribution based on k arbitrary but fixed quantiles. Therefore the optimum spacings of the quantiles, for the estimation as well as the testing problem, are the same.

2.5 Optimum Asymptotic Relative Efficiency

So far we have considered the statistics $T_{N1}^{(k)}$ and $T_{N2}^{(k)}$ based on k fixed quantiles. We now want to examine the selection of a fixed number of quantiles which would maximize $\text{ARE}(T_{N1}^{(k)}, T_{N1})$ and $\text{ARE}(T_{N2}^{(k)}, T_{N2})$. These correspond to the optimum quantiles for the BLUE estimators based on k sample quantiles. We then extract from the literature of BLUE estimators, some specific examples of optimum spacings, with the corresponding ARE for the location and scale parameter problems.

2.5.1 The Location Alternatives Case

For the location parameter problem the required optimum spacings are those which make $ARE(T_{N1}^{(k)}, T_{N1})$, in (2.80), maximum. In view of (2.80), it suffices to maximize.

$$Q_1 = \sum_{i=1}^{k+1} (f_i - f_{i-1})^2 / (\lambda_i - \lambda_{i-1}),$$

where the index j has been changed to index $i=j+1$.

1. Normal distribution. In the case of the normal distribution,

$f_i = (2\pi)^{-1/2} \exp(-u_i^2/2)$, $i=0, 1, \dots, k+1$. Ogawa (1962) showed that the required optimum spacings should be searched among the solutions of the simultaneous equations

$$2u_i + \frac{f_{i+1} - f_i}{\lambda_{i+1} - \lambda_i} + \frac{f_i - f_{i-1}}{\lambda_i - \lambda_{i-1}} = 0, \quad i=1, 2, \dots, k,$$

which satisfy $-\infty = u_0 < u_1 < u_2 < \dots < u_{k+1} = +\infty$. For any k , the solution is unique. In table 1, we list these solutions for $k=1(1)10$ showing the optimum spacings $\lambda_1, \dots, \lambda_k$ with the corresponding $ARE(T_{N1}^{(k)}, T_{N1})$. We note the spacings are symmetric about $\lambda = \lambda_k$. The ARE ranges from 0.6366 to 0.9808 as k goes from 1 to 10.

2. Logistic distribution. In the case of the logistic distribution,

$f(x) = e^{-x} / (1 + e^{-x})^2$, $-\infty < x < \infty$, $f_i = \lambda_i (1 - \lambda_i)$ and Q_1 reduces to

$$Q_1 = \sum_{i=1}^{k+1} (\lambda_i - \lambda_{i-1}) (1 - \lambda_i - \lambda_{i-1})^2$$

which is maximized by the equidistant spacings $\lambda_i = i/(k+1)$, as shown

by Gupta and Gnanadesikan (1966). Then, $ARE(T_{N1}^{(k)}, T_{N1}) = \frac{k(k+2)}{(k+1)^2}$.

Here, T_{N1} is the Wilcoxon test statistic. As examples, for $k=1$ the ARE equals 0.75 and for $k=10$ it is 0.9917.

3. Cauchy distribution. In the case of the Cauchy distribution

$$f(x) = \pi^{-1} (1+x^2)^{-1}, \quad -\infty < x < \infty, \quad f_i = \pi^{-1} \sin^2(\pi \lambda_i) \text{ and }$$

$$Q_1 = \sum_{i=1}^{k+1} \frac{\{\sin^2(\pi \lambda_i) - \sin^2(\pi \lambda_{i-1})\}^2}{\pi(\lambda_i - \lambda_{i-1})}$$

which was studied by Balmer, Boulton and Sack (1974). Assuming the optimal k -spacings are always symmetric about $\lambda = \frac{1}{2}$, Chan (1970) computed the ARE for $k=1(1)10$. These results are reproduced in Table 2 except for $k=3$ and 7. However, Balmer, Boulton and Sack (1974) showed the optimal spacings are asymmetric when $k=4m-1$ but otherwise are in agreement with Chan (1970). Their results for $k=3$ and 7 appear in Table 2, replacing those obtained earlier by Chan (1970).

2.5.2 The Scale Alternatives Case

Here, the optimum spacings are those which maximize $ARE(T_{N2}^{(k)}, T_{N2})$ in (2.84). In view of (2.84) it suffices to maximize

$$Q_2 = \sum_{i=1}^{k+1} \frac{(u_i f_i - u_{i-1} f_{i-1})^2}{\lambda_i - \lambda_{i-1}}$$

with index $i=j+1$ replacing j ; $f_0 = f_{k+1} = 0$, $\lambda_0 = 0$, $\lambda_{k+1} = 1$.

1. Normal distribution. In the case of the normal distribution, Ogawa (1962) demonstrated that Q_2 has, in general, more than one maximum. The spacings given in Table 3, for $k=1(1)9$, were obtained from Ogawa (1962) and Kulldorff (1961); they provide relative maxima and Kulldorff conjectured that they also provide the absolute maxima.

2. Logistic distribution. For the logistic distribution, $u_i = \ln(\lambda_i / (1 - \lambda_i))$ and $f_i = \lambda_i (1 - \lambda_i)$, $f_0 = f_{k+1} = 0$ and

$$Q_2 = \sum_{i=1}^{k+1} \frac{[\lambda_i (1 - \lambda_i) \ln(\lambda_i / (1 - \lambda_i)) - \lambda_{i-1} (1 - \lambda_{i-1}) \ln(\lambda_{i-1} / (1 - \lambda_{i-1}))]^2}{\lambda_i - \lambda_{i-1}}$$

The optimum spacings for $k=1(1)10$ have been calculated by Chan (1969); they are reproduced in Table 4.

3. Cauchy distribution. With the Cauchy distribution, $u_i = \tan(\lambda_i - \frac{1}{2})$, $f_i = \pi^{-1} \sin^2(\pi \lambda_i)$ and

$$Q_2 = \sum_{i=1}^{k+1} \frac{[\sin(2\pi \lambda_i) - \sin(2\pi \lambda_{i-1})]^2}{4\pi^2 (\lambda_i - \lambda_{i-1})}$$

The maximization of Q_2 has been studied by Chan (1970) and his results appear in Table 5, for $k=1(1)10$.

Using the projection of the score generating functions $J(u)$ and $J_1(u)$ on the space of step functions which increase at the points $\lambda_1, \dots, \lambda_k$, Gastwirth (1966) obtained asymptotically most powerful grouped rank tests which are asymptotically equivalent to $T_{N1}^{(k)}$ and $T_{N2}^{(k)}$. Gastwirth (1966) approach makes it possible to use a version of the Chernoff and Savage (1958) Theorem, which allows for discontinuities in the score generating functions, to study the asymptotic properties of the statistics.

Using a regression model, Sen (1967) proposed asymptotically most powerful rank order tests within the multiply censored sample context. A basic difference with our approach is that the boundaries of the intervals in which censoring occurs are random while ours are fixed. In spite of this basic difference the power properties of Sen (1967) statistics are given by the same formulae as ours.

Addendum: As suggested by the optimum asymptotic relative efficiencies in Tables 1-5, the expressions (2.80) and (2.84) tend to one as k increases to ∞ . We can show that the characteristic function of the multivariate hypergeometric distribution, in Corollary 2.1, tends to the characteristic function of the normal distribution as $k=k(N) \rightarrow \infty$. Hence, by the same argument as in Section 2.3, the asymptotic normality of $T_{N1}^{(k)}$ is obtained under H_0 as well as local alternatives. Also, $ARE(T_{N1}^{(k)}, T_{N1})$, $i=1, 2$, are well defined. Therefore, as $\max_j (r_{j+1} - r_j) N^{-1} \rightarrow 0$ the following corollary to Theorems 2.15 and 2.17 obtains.

Corollary 2.4 If $k=k(N) \rightarrow \infty$ such that $\max_j (r_{j+1} - r_j) N^{-1} \rightarrow 0$ then

$ARE(T_{N1}^{(k)}, T_{N1}) \rightarrow 1$, $i=1, 2$.

Table 1. Optimum spacings and ARE of $T_{N1}^{(k)}$ with respect to T_{N1} for the two-sample location problem for the normal distribution; $k=1(1)10$.^a

k	1	2	3	4	5	6	7	8	9	10
λ_1	.500	.270	.163	.107	.074	.055	.040	.031	.024	.020
λ_2		.730	.500	.351	.255	.195	.147	.115	.092	.076
λ_3			.837	.649	.500	.395	.308	.247	.202	.167
λ_4				.893	.745	.605	.500	.412	.343	.288
λ_5					.926	.805	.692	.588	.500	.427
λ_6						.945	.852	.753	.657	.573
λ_7							.960	.885	.798	.712
λ_8								.969	.908	.833
λ_9									.976	.924
λ_{10}										.980
ARE ^b	.6366	.8097	.8800	.9342	.9420	.9559	.9654	.9722	.9771	.9808

a From Sarhan and Greenberg (1962, Table 11D.1 p. 278).

b ARE stands for ARE ($T_{N1}^{(k)}, T_{N1}$).

Table 2. Optimum spacings and ARE of $T_{N1}^{(k)}$ with respect to T_{N1} for the two-sample location problem with the Cauchy distribution; $k=1(1)10^a$.

k	1	2	3	4	5	6	7	8	9	10
λ_1	.5000	.4435	.4504	.0928	.0976	.0999	.0997	.0616	.0626	.0632
λ_2		.5565	.5747	.4330	.4024	.3848	.3860	.1308	.1332	.1347
λ_3			.9059	.5670	.5000	.4635	.4660	.3796	.3668	.3577
λ_4				.9072	.5976	.5365	.5404	.4621	.4374	.4198
λ_5					.9024	.6152	.6216	.5379	.5000	.4739
λ_6						.9001	.8690	.6204	.5626	.5261
λ_7							.9383	.8692	.6332	.5802
λ_8								.9384	.8668	.6423
λ_9									.9374	.8653
λ_{10}										.9368
ARE ^b	.8106	.8577	.8991	.9393	.9582	.9658	.9719	.9779	.9821	.9845

a From Balmer, Boulton and Sack (1974, p. 241) for values of $k = 3$ and 7 and from Chan (1970), Table 1), otherwise.

b ARE stands for $ARE(T_{N1}^{(k)}, T_{N1})$.

Table 3. Optimum spacings and ARE of $T_{N2}^{(k)}$ with respect to T_{N2} for the two-sample scale problem for the normal distribution; $k=1(1)9^a$

k	1	2	3	4	5	6	7	8	9
λ_1	.942	.905	.021	.023	.010	.010	.005	.005	.003
λ_2		.982	.118	.127	.053	.056	.028	.029	.017
λ_3			.928	.873	.163	.171	.083	.085	.048
λ_4				.977	.867	.829	.196	.201	.110
λ_5					.975	.944	.827	.799	.222
λ_6						.990	.944	.915	.795
λ_7							.990	.971	.914
λ_8								.995	.977
λ_9									.996
ARE ^b	.304	.653	.729	.824	.858	.893	.912	.929	.939

a From Sarhan and Greenberg (1962, Table 10E.4 p 281) and Kulldorf (1961, Table 9.1 p 99).

b ARE stands for, $ARE(T_{N2}^{(k)}, T_{N2})$.

Table 4. Optimum spacings and ARE of $T_{N2}^{(k)}$ with respect to T_{N2} for the two-sample scale problem with logistic distribution: $k=1(1)10^a$.

k	1	2	3	4	5	6	7	8	9	10
λ_1	.0832	.1029	.0376	.0404	.0190	.0197	.0108	.0110	.0066	.0068
λ_2		.8971	.1576	.1690	.0821	.0850	.0480	.0490	.0303	.0308
λ_3			.8914	.8310	.2048	.2124	.1197	.1223	.0767	.0778
λ_4				.9596	.8259	.7876	.2379	.2434	.1504	.1526
λ_5					.9584	.9150	.7834	.7566	.2627	.2669
λ_6						.9803	.9134	.8777	.7532	.7331
λ_7							.9799	.9510	.8762	.8474
λ_8								.9890	.9503	.9222
λ_9									.9888	.9692
λ_{10}										.9932
ARE ^b	.3071	.6838	.7612	.8442	.8751	.9071	.9225	.9383	.9471	.9561

^a From Chan (1969, Table 2).

^b ARE stands for $ARE(T_{N2}^{(k)}, T_{N2})$.

Table 5. Optimum spacings and ARE of $T_{N2}^{(k)}$ with respect to T_{N2} for the two-sample scale problem with the Cauchy distribution; $k=1(1)10^a$.

k	1	2	3	4	5	6	7	8	9	10
λ_1	.2055	.2500	.1796	.1855	.1504	.1524	.1306	.1315	.1164	.1169
λ_2		.7500	.3024	.3145	.2464	.2500	.2110	.2126	.1866	.1874
λ_3			.7435	.6855	.3415	.3476	.2849	.2874	.2488	.2500
λ_4				.8145	.6812	.6524	.3646	.3685	.3108	.3126
λ_5					.8124	.7500	.6494	.6315	.3804	.3831
λ_6						.8476	.7483	.7126	.6293	.6169
λ_7							.8467	.7874	.7112	.6874
λ_8								.8685	.7865	.7500
λ_9									.8680	.8126
λ_{10}										.8831
ARE ^b	.2887	.8106	.8630	.9226	.9399	.9582	.9657	.9739	.9780	.9821

a From Chan (1970, Table 2).

b ARE stands for ARE ($T_{N2}^{(k)}, T_{N2}$).

CHAPTER 3

A GENERALIZATION OF THE SAVAGE TEST

Consider the combined order statistics $W_1 \leq W_2 \leq \dots \leq W_N$ obtained from the two independent samples $X_1, X_2, \dots, X_m$ and $Y_1, Y_2, \dots, Y_n$ where $m+n=N$. Given a set of k quantiles we propose a statistic $S_N^{(k)}$, based on the frequencies in the $k+1$ classes specified by these quantiles. The statistic yields a locally most powerful rank order test for the null hypothesis against Lehmann alternatives. It generalizes the Savage (1956) test in that only k order statistics together with the respective frequencies of the X and Y -observations, in the $k+1$ intervals specified by the k quantiles, need be considered.

In the first section, we present a derivation of the locally most powerful rank order test together with some basic properties of $S_N^{(k)}$ under H_0 . This is followed, in section 2, by a discussion of the asymptotic normality of the statistic $S_N^{(k)}$ under H_0 as well as under local Lehmann alternatives. The content of the third section deals with the asymptotic efficiency of $S_N^{(k)}$ relative to its complete sample counterpart, the Savage statistic. In particular, a discussion of the optimum choice of the k quantiles to maximize the asymptotic relative efficiency is presented.

3.1 The Locally Most Powerful Rank Order Test

Based on the random vector $Z^* = (M_1, Z_{r_1}, \dots, Z_{r_k}, M_{k+1})$ we define the statistic

$$S_N^{(k)} = \sum_{i=1}^k Z_{r_i} (c_{r_i} - 1) + \sum_{j=0}^k M_{j+1} (\bar{c}_{j+1} - 1) \quad (3.1)$$

where

$$c_i = \sum_{j=1}^i (N-j+1)^{-1} \quad (3.2)$$

and

$$(r_{j+1} - r_j - 1) \bar{c}_{j+1} = \sum_{i=r_j+1}^{r_{j+1}-1} c_i \quad (3.3)$$

Since $\sum_{i=1}^N c_i = N$ and $r_{j+1} - r_j - 1 = M_{j+1} + N_{j+1}$ for $j=0, 1, \dots, k$ with $r_0=0$ and $r_{k+1}=N+1$, the expression (3.1) is equivalent to

$$S_N^{(k)} = \sum_{i=1}^k (1 - Z_{r_i}) (1 - c_{r_i}) + \sum_{j=0}^k N_{j+1} (1 - \bar{c}_{j+1}) \quad (3.4)$$

The statistic $S_N^{(k)}$ is an extended version of Basu (1968) statistic.

It is interesting to note that if $f(x) = \exp(x - e^x)$, $x \in \mathbb{R}$, then

$$J(u) = -f'(F^{-1}(u)) / f(F^{-1}(u)) = -1 - \ln(1-u), \quad (3.5)$$

and hence, $E[-f'(W_1)/f(W_1)] = c_1 - 1$. Likewise, if $f(x) = e^{-x}$, $x > 0$, then

$$J_1(u) = -1 - F^{-1}(u) f'(F^{-1}(u)) / f(F^{-1}(u)) = -1 - \ln(1-u) \quad (3.6)$$

and $E[-1-W_1 f'(W_1)/f(W_1)] = c_1 - 1$. This suggests that $S_N^{(k)}$ can be considered as a test statistic for the null hypothesis against either location or scale alternatives. Consequently, we could apply the results of Lemmas 2.1 and 2.2. We shall make use of this relationship for the asymptotic properties of $S_N^{(k)}$; to show the LMPRT property for $S_N^{(k)}$ we only need the following result which can be obtained directly.

Lemma 3.1 If $f(x)$ is absolutely continuous, then

$$E \{-1 - \ln(1 - F(W_1))\} = c_1 - 1 \quad (3.7)$$

and

$$E \left\{ \frac{(1 - F(W_s)) \ln(1 - F(W_s)) - (1 - F(W_r)) \ln(1 - F(W_r))}{F(W_s) - F(W_r)} \right\} = \sum_{r+1}^{s-1} (c_1 - 1) / (r - s - 1) \quad (3.8)$$

with $r \leq s - 2$, where the expectation is taken under H_0 . The latter expression reduces to $c_{j+1} - 1$ whenever $s = r_{j+1}$ and $r = r_j$.

Proof. Write each expectation in its integral form, using the density

$g_1(x) = N! [(i-1)!(N-i)!]^{-1} F(x)^{i-1} [1-F(x)]^{N-i} f(x)$ for the order statistic

W_1 and the joint density $g_{r,s}(x,y)$ given by (2.8) for the order sta-

tistics W_r and W_s , $r \leq s - 2$. The expression (3.8) is then obtained by

straightforward calculations using the properties of the beta density

function. The expression (3.7) is also obtained directly from an

application of the method of integration by parts followed by com-

putations like those used for (3.8).

From the likelihood ratio in (2.37), with $G(x)=1-(1-F(x))^{1+\Delta}$, we obtain the explicit result:

Theorem 3.1 If $f(x)=F'(x)$ exists and is absolutely continuous, then on the basis of Z^* ,

$$S_N^{(k)} \geq C, \quad (3.9)$$

with $S_N^{(k)}$ defined by (3.1) or (3.4), is the locally most powerful rank test for testing $H_0: G=F$ against Lehmann alternatives,

$G(x) = 1-(1-F(x))^{1+\Delta}$, $0 < \Delta < \epsilon$, for some $\epsilon > 0$, at the respective level.

Proof. From (2.37), the likelihood ratio $L(z^*; \Delta) = E_W$ is obtained

by substituting $G(x)=1-(1-F(x))^{1+\Delta}$ and $g(x)=(1+\Delta)(1-F(x))^\Delta f(x)$ into (2.35). We can differentiate under the integral sign since the derivative of the integrand is a continuous function of Δ and

W_{r_j} , $j=1, \dots, k$, over the respective regions. Then by direct com-

putation of $L'(z^*; 0)$, using Lemma 3.1, we obtain $L'(z^*; 0) = S_N^{(k)}$. The result follows.

The LMPRT for $\Delta < 0$ is obtained in a similar manner.

The Savage (1956) statistic, $S_N = \sum_{i=1}^N Z_i(c_i - 1)$ with c_i

given by (3.2), was extended by Gastwirth (1965) and Basu (1968) to the right censored sample situation; these two statistics were shown to be asymptotically equivalent and asymptotically most powerful.

Our statistic, $S_N^{(k)}$, further extends the Savage statistic to the

multiple censored sample case. Moreover, we have shown $S_N^{(k)}$ is a LMPRT for the two-sample problem under multiple censoring. By applying the same technique to the Basu Statistic we can also show it is a LMPRT statistic for the corresponding problem. The Basu statistic can be considered as a special case of $S_N^{(k)}$ where the censoring occurs in the rightmost interval only. However, unlike the LMPRT property above, not all the arguments used with $S_N^{(k)}$ equally apply to the Basu statistic. Here, with $S_N^{(k)}$, our arguments are based on frequencies within the $k+1$ intervals determined by k quantiles.

An approximation of the probability distribution of Z^* under local Lehmann alternatives is given by the following corollary to Theorem 2.3 which is analogous to Corollaries 2.2 and 2.3 for location and scale alternatives, respectively.

Corollary 3.1 Under the assumptions of Theorem 3.1, for the Lehmann alternatives $G(x) = 1 - (1 - F(x))^{1+\Delta}$ with values of Δ about 0,

$$P_{\Delta}(Z^* = z^*) = P(Z^* = z^*; F = G) \{1 + \Delta S_N^{(k)} + O(\Delta^2)\}, \quad (3.10)$$

with $S_N^{(k)}$, as defined by (3.1) or (3.4).

Proof. We expand $G(x) = 1 - (1 - F(x))^{1+\Delta}$ and $g(x) = (1 + \Delta)(1 - F(x))^{\Delta} f(x)$ in Taylor series about $\Delta = 0$ and substitute in (2.34). Then, using Lemma 3.1, Corollary 2.1, and the definition of $S_N^{(k)}$ by (3.1), the expression (2.34) can be written as (3.10).

The Savage (1956) statistic S_N , which can be written as

$$S_N = \sum_{i=1}^N (1-Z_i)(c_i-1) \quad (3.11)$$

with c_i defined by (3.2), represents the complete sample counterpart of $S_N^{(k)}$ given by (3.4). The statistic S_N yields a LMPRT for H_0 against Lehmann alternatives for the complete sample problem. We collect some small sample results for $S_N^{(k)}$ and S_N in the following theorem.

Theorem 3.2 Under $H_0: F=G$ and the assumptions of Lemma 3.1,

$$E_0(S_N^{(k)}) = E_0(S_N) = 0, \quad (3.12)$$

$$\text{Var}_0(S_N^{(k)}) = \frac{mn}{N(N-1)} \left\{ \sum_{i=1}^k (1-c_{r_i})^2 + \sum_{j=0}^k (r_{j+1}-r_j-1)(1-\bar{c}_{j+1})^2 \right\}, \quad (3.13)$$

$$\text{Var}_0(S_N) = \frac{mn}{N(N-1)} \sum_{i=1}^N (1-c_i)^2, \quad (3.14)$$

$$\text{Cov}_0(S_N^{(k)}, S_N) = \text{Var}_0(S_N^{(k)}) \quad (3.15)$$

and

$$r_0^2(S_N^{(k)}, S_N) = \text{Var}_0(S_N^{(k)}) / \text{Var}(S_N). \quad (3.16)$$

Proof. The proof parallels that of Theorem 2.6, where c_i-1 is used

instead of $a_N(i, f)$. We note that the relationship $\sum_{i=1}^N (c_i-1) = 0$ was

obtained by Savage (1956). Then proceeding as in Theorem 2.6 we obtain all the above results.

In the next section we show the asymptotic normality of $S_N^{(k)}$ under the null as well as local alternative hypotheses. It will be followed by a discussion of the asymptotic efficiency of $S_N^{(k)}$ relative to S_N .

3.2 Asymptotic Distributions.

As in Chapter 2, for the asymptotic normality of $S_N^{(k)}$ it suffices to consider the asymptotic distribution of the terms containing N_{j+1} , $j=0,1,\dots,k$ in (3.4); these N_{j+1} are jointly asymptotically normal under H_0 as well as local alternatives. Moreover, the asymptotic moments of $S_N^{(k)}$, under H_0 , are obtained as the limiting values of the small sample moments of $S_N^{(k)}$ which were computed in Theorem 3.2. Using basically the same technique as in Basu (1968), the limiting weights of $S_N^{(k)}$ are given by the following lemma.

Lemma 3.2. For c_{r_i} , $i=1,2,\dots,k$ and $\bar{c}_{j+1}$, $j=0,1,\dots,k$ defined in (3.2)

and (3.3),

$$\lim_{N \rightarrow \infty} (1 - c_{r_i}) = 1 + \ln(1 - \lambda_i) \quad (3.17)$$

and

$$\lim_{N \rightarrow \infty} (1 - \bar{c}_{j+1}) = \gamma_{j+1} = \frac{(1 - \lambda_{j+1}) \ln(1 - \lambda_{j+1}) - (1 - \lambda_j) \ln(1 - \lambda_j)}{\lambda_{j+1} - \lambda_j} \quad (3.18)$$

Proof. From the definition of c_i in (3.2) we obtain

$$\lim_{N \rightarrow \infty} c_{r_i} = \lim_{N \rightarrow \infty} \sum_{j=1}^{r_i} \{1 - (j-1)/N\}^{-1} N^{-1}$$

$$= \int_0^{\lambda_1} (1-x)^{-1} dx$$

$$= -\ln(1-\lambda_1)$$

since $r_i/N \rightarrow \lambda_1$; the expression (3.17) follows. Similarly, we can write

$$\lim_{N \rightarrow \infty} \sum_{i=1}^{r_j} c_i/N = \lim_{N \rightarrow \infty} \sum_{i=1}^{r_j} N^{-1} (r_j/N - (i-1)/N) / (1 - (i-1)/N)$$

$$= \int_0^{\lambda_j} (\lambda_j - x) / (1-x) dx$$

$$= \lambda_j + (1-\lambda_j) \ln(1-\lambda_j).$$

$$\text{since } \sum_{i=1}^{r_j} c_i = \sum_{i=1}^{r_j} \sum_{j=1}^i (N-j+1)^{-1} = \sum_{i=1}^{r_j} (r_j - i + 1)(N-i+1)^{-1} \text{ by direct}$$

computation. When substituted in the defining expression (3.3) for

$\bar{c}_{j+1}$ we obtain

$$\lim_{N \rightarrow \infty} \bar{c}_{j+1} = \lim_{N \rightarrow \infty} N(r_{j+1} - r_j)^{-1} \left\{ \sum_{i=1}^{r_{j+1}} c_i/N - \sum_{i=1}^{r_j} c_i/N \right\}$$

$$= (\lambda_{j+1} - \lambda_j)^{-1} \{ \lambda_{j+1} + (1-\lambda_{j+1}) \ln(1-\lambda_{j+1}) - \lambda_j - (1-\lambda_j) \ln(1-\lambda_j) \},$$

since $(r_{j+1} - r_j - 1)/N \rightarrow \lambda_{j+1} - \lambda_j$; the expression (3.18) follows.

We first show the asymptotic normality of $S_N^{(k)}$ under the null hypothesis. In Chapter 5, we show that the Lehmann alternatives for the family of exponential densities is a subclass of the location alternatives. Then, these Lehmann alternatives are shown to be contiguous to the null hypotheses. As an application of LeCam's third Lemma, we obtain the asymptotic distribution of $S_N^{(k)}$ under the local Lehmann alternatives.

$$\text{Let } \mu_o = \lim_{N \rightarrow \infty} E_o(S_N^{(k)}) \text{ and } \sigma_o^2 = \lim_{N \rightarrow \infty} \text{Var}_o(S_N^{(k)}).$$

Theorem 3.3. If f is absolutely continuous, then, under the null hypothesis H_o ,

$$S_N^{(k)} \sim N(\mu_o, \sigma_o^2) \quad (3.19)$$

where $\mu_o = 0$ and

$$N^{-1} \sigma_o^2 = \lambda(1-\lambda) \sum_{j=0}^k \frac{\{(1-\lambda_{j+1}) \ln(1-\lambda_{j+1}) - (1-\lambda_j) \ln(1-\lambda_j)\}^2}{\lambda_{j+1} - \lambda_j}. \quad (3.20)$$

Proof. Although the proof parallels that of Theorem 2.8, we reproduce it here. From Theorem 3.2, $\text{Var}_o(S_N^{(k)}) = O(N)$, whereas

$\text{Var}_o\left(\sum_{i=1}^k (1 - Z_{r_i})(c_{r_i} - 1)\right) = O(1)$, and hence, for the asymptotic normality of $S_N^{(k)}$, it suffices to consider only the statistic

$$S_N^{*(k)} = \sum_{j=0}^k N_{j+1} (1 - \bar{c}_{j+1}) ; \quad (3.21)$$

both have the same limiting distribution.

From (2.36) of Corollary 2.1, the distribution of $\underline{N} = (N_1, N_2, \dots, N_{k+1})$ is multivariate hypergeometric, and hence asymptotically multivariate normal, under H_0 . Therefore, $\sum_{j=0}^k \gamma_{j+1} N_{j+1}$,

which is a linear combination of N_{j+1} , $j=0, 1, \dots, k$, is also asymptotically normal, by Lemma 2.3. Then by Lemmas 2.4 and 3.2 it follows that $S_N^{*(k)}$, and hence $S_N^{(k)}$, is also asymptotically normal.

The asymptotic mean and variance of $S_N^{(k)}$ are obtained as the limiting values of (3.12) and (3.13), using Lemma 3.2, where it is observed that $(\lambda_{j+1} - \lambda_j - 1)/N \rightarrow \lambda_{j+1} - \lambda_j$.

To establish the asymptotic distribution of the LMPRT statistic $S_N^{(k)}$ under local Lehmann alternatives, for the family of exponential distributions, we use the idea of contiguity. First, the same argument as for (2.71) and (2.72), consider the hypotheses specified by

$$H_0 : F_X(w) = G_Y(w) = 1 - (1 - F(w))^{1+\bar{\Delta}} \quad (3.22)$$

and

$$H_A : F_X(w) = F(w) \text{ and } G_Y(w) = 1 - (1 - F(w))^{1+\Delta} \quad (3.23)$$

with $\bar{\Delta} = n\Delta/N$. For the family of exponential distributions

$F(w) = 1 - \exp(-e^{w-\delta})$, $-\infty < w < \infty$, the hypotheses H_{0N} and H_{AN} form

a subclass of the location hypotheses since

$$G(w) = 1 - (1 - F(w))^{1+\Delta} = F(w - \delta + \ln(1+\Delta)) .$$

The joint densities of $W = (W_{r_1}, \dots, W_{r_k})$ and $Z^* = (M_1, Z_{r_1}, \dots, Z_{r_k}, M_{k+1})$, under H_0 and H_A , are obtained by (2.34) using $F(w)$ and $F(w - \delta + \ln(1+\Delta))$ respectively; the likelihood ratio L_N is then obtained as the ratio of these two joint densities. We obtain the following theorem, for the distribution of L_N under the null hypothesis; it is proved in Chapter 5, and appears there as Theorem 5.5.

Theorem 3.4. For the exponential family of densities

$$f(w) = \exp(w - \delta - e^{w-\delta}), \quad \delta = O(N^{-1/2}),$$

$$\log L_N \xrightarrow{P_{\bar{\Delta}}} \Delta S_N^{*(k)} - \frac{1}{2} \sigma_N^2 \quad (3.24)$$

with $S_N^{*(k)}$ as defined by (3.21) and $\sigma_N^2 = \Delta^2 \sigma_0^2$ where σ_0^2 is defined by (3.20)

Under the alternatives hypothesis H_{ON} , for the exponential distributions, the asymptotic distribution of $S_N^{(k)}$ is given by the next theorem.

Theorem 3.5. Under the assumptions of Theorem 3.4 and under the Lehmann alternatives, $H_A: F_X(w) = F(w)$ and $G_Y(w) = 1 - (1 - F(w))^\Delta$ for the exponential distribution functions, where $\Delta = h_N N^{1/2}$ with $h_N \rightarrow h$ constant,

$$S_N^{(k)} \sim N(\Delta\sigma_o^2, \sigma_o^2) \quad (3.25)$$

where σ_o^2 is defined by (3.20).

Proof. In view of LeCam's third Lemma, it suffices to show the joint asymptotic normality of $(S_N^{(k)}, \log L_N)$ under the null hypothesis.

From Theorem 3.3, $S_N^{(k)}$ in (3.4) and $S_N^{*(k)}$ in (3.21) are asymptotically equivalent. Likewise, from Theorem 3.4, $\log L_N$ and $\Delta S_N^{*(k)} - \frac{1}{2}\sigma_N^2$ are also asymptotically equivalent. Therefore, the joint asymptotic distribution of $(S_N^{(k)}, \log L_N)$ is equivalent to that of

$(S_N^{*(k)}, \Delta S_N^{*(k)} - \frac{1}{2}\sigma_N^2)$ which is bivariate normal. Then, from Theorem 3.3 and its proof, we obtain, under the null hypothesis,

$$(S_N^{*(k)}, \Delta S_N^{*(k)} - \frac{1}{2}\sigma_N^2) \sim N(0, -\frac{1}{2}\Delta^2\sigma_o^2, \sigma_o^2, \Delta^2\sigma_o^2, \Delta\sigma_o^2).$$

Then, by the LeCam's third Lemma we obtain

$$S_N^{*(k)} \sim N(\Delta\sigma_o^2, \sigma_o^2)$$

under the local Lehmann alternatives. The result follows from the asymptotic equivalence of $S_N^{(k)}$ and $S_N^{*(k)}$.

The results of Theorems 3.4 and 3.5 are important to study the power properties of the test corresponding to $S_N^{(k)}$.

3.3 Asymptotic Relative Efficiency.

The statistic S_N proposed by Savage (1956), for the complete sample, is defined by

$$S_N = \sum_{i=1}^N Z_i (c_i - 1) \quad (3.26)$$

or

$$S_N = \sum_{i=1}^N (1 - Z_i) (1 - c_i) \quad (3.27)$$

where c_i is given by (3.2). In this section we present a discussion of the asymptotic relative efficiency (ARE) of $S_N^{(k)}$ in (3.4), to S_N , in the Pitman sense.

Let $\mu_N(\Delta) = E_{\Delta}(S_N^{(k)})$ and $\sigma_N^2(\Delta) = \text{Var}_{\Delta}(S_N^{(k)})$. By Theorems 3.1, 3.3 and 3.5, the regularity conditions (a)-(f), in section 2.4 of Chapter 2, are satisfied for the statistics S_N and $S_N^{(k)}$; the parameter Δ is used instead of θ . Then, we obtain the following results from Lemma 2.8 and the expression (2.78).

Theorem 3.6. Under the Lehmann alternatives $\Delta_N = hN^{-1/2}$, for the exponential family of distributions, the asymptotic power of $S_N^{(k)}$ is given by

$$1 - \Phi(\lambda_{N,\alpha} - hc)$$

where

$$c = \lim_{N \rightarrow \infty} N^{-1/2} \sigma_0 \quad (3.28)$$

where σ_0^2 is given by (3.20) and $1 - \Phi(\lambda_{N,\alpha}) = \alpha$.

Proof. By Theorems 3.3 and 3.5, $S_N^{(k)} \sim N(0, \sigma_0^2)$ and $S_N^{(k)} \sim N(\Delta\sigma_0^2, \sigma_0^2)$

under the null and the alternative hypothesis, respectively, where σ_0^2 is defined by (3.20). Then, from the condition (e) in section 2.4 of Chapter 2,

$$\begin{aligned} c &= \lim_{N \rightarrow \infty} N^{-1/2} \mu'(0) / \sigma_N(0) \\ &= \lim_{N \rightarrow \infty} N^{-1/2} \sigma_0 \end{aligned} \quad (3.29)$$

Then, the theorem follows directly from Lemma 2.8.

Theorem 3.7. The asymptotic efficiency of $S_N^{(k)}$ relative to S_N defined by (3.4) and (3.27), respectively, is given by

$$\text{ARE}(S_N^{(k)}, S_N) = \sum_{j=0}^k \frac{\{(1-\lambda_{j+1}) \ln(1-\lambda_{j+1}) - (1-\lambda_j) \ln(1-\lambda_j)\}^2}{\lambda_{j+1} - \lambda_j} \quad (3.30)$$

Proof. In view of (2.79), it suffices to compute the efficacy of $S_N^{(k)}$ and S_N . The efficacy of $S_N^{(k)}$ is obtained by (3.29) using σ_0^2 as defined by (3.20), that is,

$$\begin{aligned} [\text{eff}(S_N^{(k)})]^2 &= [\lim_{N \rightarrow \infty} N^{-1/2} \sigma_0]^2 \\ &= \lambda(1-\lambda) \sum_{j=0}^k \frac{\{(1-\lambda_{j+1}) \ln(1-\lambda_{j+1}) - (1-\lambda_j) \ln(1-\lambda_j)\}^2}{\lambda_{j+1} - \lambda_j} \end{aligned} \quad (3.31)$$

The asymptotic distribution of S_N , under the null and the alternative hypotheses, was derived by Chernoff and Savage (1958). The statistic S_N satisfies the regularity conditions (a)-(f), in section 2.4 of Chapter 2; in particular, the efficacy of S_N is given by

$$[\text{eff}(S_N)]^2 = \lambda(1-\lambda) \quad (3.32)$$

From (2.79), using (3.31) and (3.32),

$$\text{ARE}(S_N^{(k)}, S_N) = [\text{eff}(S_N^{(k)})/\text{eff}(S_N)]^2$$

yields (3.30).

It may be noted that the expression (3.30) is also given by $\rho^2(S_N^{(k)}, S_N)$, which is the limiting value of $r_o^2(S_N^{(k)}, S_N)$ in (3.16), under H_o . This fact is an agreement with Van Eeden (1963) under some further regularity conditions on the joint distribution of $S_N^{(k)}$ and S_N , which hold in the present case.

Letting $u_j = -\ln(1-\lambda_j)$, $j=1, \dots, k$, with $u_0 = 0$ and $u_{k+1} = \infty$, the expression (3.30) can be written as

$$Q_k(u_1, \dots, u_k) = \sum_{j=1}^{k+1} (u_j e^{-u_j} - u_{j-1} e^{-u_{j-1}})^2 / (e^{-u_{j-1}} - e^{-u_j}) \quad (3.33)$$

Thus the limiting Pitman's efficiency of $S_N^{(k)}$ relative to S_N is given by $Q_k(u_1, \dots, u_k)$ defined by (3.33).

To maximize the asymptotic efficiency of $S_N^{(k)}$ relative to S_N equates to maximizing Q_k for variations of $(u_1, \dots, u_k)$. The uniqueness of this maximization has been proved by Saleh and Ali (1966) which in turn is in agreement with the uniqueness of the optimum spacings of the quantiles given by Sarhan and Greenberg (1962); these are reproduced in Table 6, for $k=1(1)15$. From table 6, it is clear that to test H_0 , against Lehmann alternatives, using only one quantile, the quantile of order 0.7968 must be selected to achieve the maximum asymptotic efficiency 0.6476. The maximum efficiency increases to 0.9476 if one decides to use the five quantiles of order 0.3931, 0.6670, 0.8434, 0.9434, and 0.9885.

Table 6. Optimum spacings and relative efficiencies for the generalized Savage statistic $S_N^{(k)}$ for $k = 1(1)15$.^a

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
λ_1	.7968	.6386	.5296	.4514	.3931	.3478	.3121	.2827	.2583	.2381	.2207	.2057	.1924	.1808	.1710
λ_2	.9266	.8300	.7419	.6670	.6042	.5513	.5066	.4680	.4349	.4063	.3810	.3585	.3384	.3208	.3028
λ_3	.9655	.9067	.8434	.7828	.7277	.6782	.6340	.5946	.5596	.5284	.5001	.4745	.4515		
λ_4	.9810	.9434	.8978	.8506	.8047	.7613	.7212	.6841	.6502	.6191	.5905	.5643			
λ_5		.9885	.9631	.9297	.8928	.8551	.8181	.7827	.7491	.7175	.6880	.6605			
λ_6			.9746	.9496	.9205	.8896	.8583	.8274	.7974	.7686	.7413				
λ_7				.9948	.9626	.9394	.9140	.8874	.8606	.8340	.8081				
λ_8					.9963	.9865	.9715	.9528	.9317	.9091	.8858	.8624			
λ_9						.9973	.9897	.9778	.9625	.9448	.9255	.9053			
λ_{10}							.9979	.9920	.9824	.9697	.9548	.9383			
λ_{11}								.9984	.9936	.9858	.9752	.9625			
λ_{12}									.9987	.9949	.9883	.9794			
λ_{13}										.9990	.9958	.9903			
λ_{14}											.9991	.9965			
λ_{15}												.9993			
ARE ^b	.8076	.8203	.8910	.9269	.9476	.9606	.9693	.9754	.9798	.9832	.9857	.9874	.9894	.9907	.9918

^a From Sarhan and Greenberg (1962, Table 11D.1 p. 374-5)

^b ARE stands for $ARE(S_N^{(k)}; S_N)$

CHAPTER 4

CLASS OF RANK ORDER TESTS FOR THE LOCATION AND SCALE PARAMETERS SIMULTANEOUSLY

The test statistics $T_{N1}^{(k)}$ and $T_{N2}^{(k)}$ were derived in Chapter 2 for the location and scale parameter families of distributions, respectively. Our objective is now to combine these two statistics in a quadratic form, denoted by $LS_N^{(k)}$, to produce a test for the location and scale parameters simultaneously.

Although the statistics $T_{N1}^{(k)}$ and $T_{N2}^{(k)}$ yielded locally most powerful tests for location and scale alternatives the same property does not hold for the statistic $LS_N^{(k)}$. Rather, we show the corresponding test is optimal in the sense of Wald (1943). The limiting distribution of $LS_N^{(k)}$ is that of a chi-square random variable with two degrees of freedom, central under the null hypothesis and with a non-centrality parameter under the alternative hypothesis; in the second case, we use a contiguity argument. For each set of k prefixed quantiles we also compute the asymptotic efficiency of $LS_N^{(k)}$, arising from the multiply-censored sample situation, relative to its complete sample counterpart as found in Rubison (1976), or Duran, Tsai and

Lewis (1976), or Lepage (1977). Finally, we discuss the optimum choice of the k quantiles to maximize the above asymptotic relative efficiency.

4.1 The Test Statistic

We consider the family of absolutely continuous cumulative distribution functions $F(e^{-T}w-\theta)$ with absolutely continuous densities on R . Within the same multiply-censored sample context as in chapters 2 and 3, we require a statistic to test $H_0: (\theta, \tau) = (0, 0)$ against the local alternatives $(\theta, \tau) \neq (0, 0)$. In particular, if $\theta=0$ or $\tau=0$, the problem reduces to the scale or the location problem, respectively, which were discussed in Chapter 2. Here, we shall be concerned with the alternative hypothesis H_A which specify $G(w)=F(e^{-T}w-\theta)$ where $\theta=\theta_N=AN^{-1/2}$ and $\tau=\tau_N=BN^{-1/2}$ with A and B non-negative numbers independent of N , not both equal to zero.

We assume the joint density of $W=(W_{r_1}, \dots, W_{r_k})$ and $Z^*=(M_1, Z_{r_1}, M_2, \dots, Z_{r_k}, M_{k+1})$, under H_0 and K_N , can be expressed in terms of a fixed density f within the above family of densities and such that:

$$I(f) = \int_0^1 f^2(u) du < \infty \quad (4.1)$$

and

$$I_1(f) = \int_0^1 J_1^2(u) du < \infty \quad (4.2)$$

where $J(u) = -f'(F^{-1}(u))/f(F^{-1}(u))$ and $J_1(u) = -1 - F^{-1}(u)f'(F^{-1}(u))/f(F^{-1}(u))$.

We summarize the regularity conditions as follows:

$$(B1) \int_{-\infty}^{\infty} |f'(w)| dw < \infty \quad \text{and} \quad \int_{-\infty}^{\infty} |wf'(w)| dw < \infty,$$

$$(B2) \quad I(f, K) = \int_0^1 [KJ(u) + J_1(u)]^2 du < \infty$$

where K is a constant; note that (B2) is equivalent to (4.1) and (4.2).

Then, to test H_0 against the alternatives H_A , we propose the statistic

$$LS_N^{(k)} = N^{-1} (T_{N1}^{(k)}, T_{N2}^{(k)}) \Sigma_N^{-1} (T_{N1}^{(k)}, T_{N2}^{(k)})', \quad (4.3)$$

where

$$T_{N1}^{(k)} = \sum_{i=1}^k (1 - Z_{r_i}) a_N(r_i, f) + \sum_{j=0}^k N_{j+1} \bar{a}_{j+1} \quad (4.4)$$

and

$$T_{N2}^{(k)} = \sum_{i=1}^k (1 - Z_{r_i}) b_N(r_i, f) + \sum_{j=0}^k N_{j+1} \bar{b}_{j+1} \quad (4.5)$$

as in Theorems 2.4 and 2.5 and Σ_N denotes the dispersion matrix of

$N^{-1/2} (T_{N1}^{(k)}, T_{N2}^{(k)})$; Σ_N is assumed non-singular.

The statistic $LS_N^{(k)}$ can be considered as the multiply-censored sample counterpart of the statistic $\mathcal{J}_N$ studied by Rubison (1976).

Whenever the complete sample statistics T_{N1} and T_{N2} , corresponding to

$T_{N1}^{(k)}$ and $T_{N2}^{(k)}$, respectively, are odd and even translation invariant

they are also asymptotically independent and Rubison's statistic reduces to the statistic studied by Duran, Tsai and Lewis (1976).

However, the statistics $T_{N1}^{(k)}$ and $T_{N2}^{(k)}$ do not, in general, remain

odd and even translation invariant. For instance, they would retain the invariance property for symmetric quantiles while they would not for most other selection of the quantiles; the symmetry property being only a sufficient condition for the invariance property. Hence, the case of interest lies in the study of $LS_N^{(k)}$ where $T_{N1}^{(k)}$ and $T_{N2}^{(k)}$ may be correlated even asymptotically.

Under the null hypothesis $H_0: G(w)=F(w)$, the probability distribution of Z^* which we denote by $P(Z^*=z^*; H_0)$, is provided by Corollary 2.1. Under the alternative H_A the following corollary to Theorem 2.3 holds.

Corollary 4.1 Under the alternatives $H_A : G(w) = F(e^{-\tau w - \theta})$ for values of θ and τ about 0,

$$P(Z^*=z^*; H_A) = P(Z^*=z^*; H_0) \{1 + \theta T_{N1}^{(k)} + \tau T_{N2}^{(k)} + O(\theta^2) + O(\tau^2)\} \quad (4.6)$$

where $T_{N1}^{(k)}$ and $T_{N2}^{(k)}$ are defined by (4.4) and (4.5), respectively.

Proof. From Theorem 2.3 and Corollary 2.1,

$$P(Z^*=z^*; H_A) = P(Z^*=z^*; H_0) E_W$$

where E_W is given by (2.35) with $G(w)=F(e^{-\tau w - \theta})$ and $g(w)=e^{-\tau w - \theta} f(e^{-\tau w - \theta})$.

We expand $G(w)$ and $g(w)$ in Taylor series about $(\theta, \tau) = (0, 0)$ and substitute in (2.35). Then using the expressions (4.4) and (4.5) with $a_N(i, f)$, $b_N(i, f)$, $\bar{a}_{j+1}$ and $\bar{b}_{j+1}$ as in (2.2)-(2.5), we obtain (4.6).

Under the null hypothesis, the small sample moments of the random vector $N^{-1/2}(T_{N1}^{(k)}, T_{N2}^{(k)})$ are provided by the following theorem.

Theorem 4.1 Under H_0 and the conditions (B1) and (B2); the random

vector $N^{-1/2}(T_{N1}^{(k)}, T_{N2}^{(k)})$ has mean vector

$$N^{1/2}(E_0(T_{N1}^{(k)}), E_0(T_{N2}^{(k)})) = (0, 0) \quad (4.7)$$

and dispersion matrix

$$\Sigma_{ON} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad (4.8)$$

where $A_{ii} = N^{-1} \text{Var}_0(T_{Ni}^{(k)})$, $i=1, 2$, are given by (2.45) and (2.54) in Theorems 2.6 and 2.7 and

$$\begin{aligned} A_{12} = A_{21} &= N^{-1} \text{Cov}_0(T_{N1}^{(k)}, T_{N2}^{(k)}) \\ &= \frac{mn}{N(N-1)} \left\{ \sum_{i=1}^k a_N(r_i, f) b_N(r_i, f) + \sum_{j=0}^k (r_{j+1} - r_j - 1) \bar{a}_{j+1} \bar{b}_{j+1} \right\} \quad (4.9) \end{aligned}$$

with $a_N(r_i, f)$, $b_N(r_i, f)$, $\bar{a}_{j+1}$, and $\bar{b}_{j+1}$ as defined by (2.2)-(2.5).

Proof. Expressions for A_{ii} , $i=1, 2$ and for (4.7) are obtained from

Theorems 2.6 and 2.7. Using the same argument as in these theorems

the expression (4.9) follows by direct computation.

The limiting distributions of the above random vector, and consequently of $LS_N^{(k)}$, under both H_0 and K_N , are studied in the next section.

4.2 Asymptotic Distributions

The asymptotic normality of the statistics $T_{N1}^{(k)}$ and $T_{N2}^{(k)}$ was established, under H_0 and the respective local alternatives, in Chapter 2. We now study the asymptotic distribution of $LS_N^{(k)}$ under H_0 as well as local location-scale hypotheses. By an argument similar to that of Theorem 2.8, we show that $N^{-1/2}(T_{N1}^{(k)}, T_{N2}^{(k)})$ is asymptotically normal under H_0 . Using the notion of contiguity we also show it is asymptotically normal under local location-scale alternatives. Consequently, we show the statistic $LS_N^{(k)}$ is distributed as a chi-square random variable with two degrees of freedom.

We first consider the asymptotic distributions of $N^{-1/2}(T_{N1}^{(k)}, T_{N2}^{(k)})$ and $LS_N^{(k)}$ under H_0 in the next two theorems.

Theorem 4.2. Under $H_0: F(w)=G(w)$ and the regularity conditions (B1)

and (B2), the random vector $N^{-1/2}(T_{N1}^{(k)}, T_{N2}^{(k)})$ is asymptotically bi-

variate normal with mean vector 0 and dispersion matrix Σ_0 with elements

$$\sigma_{11}^2 = \lambda(1-\lambda) \sum_{j=0}^k (f_{j+1} - f_j)^2 / (\lambda_{j+1} - \lambda_j) \quad (4.10)$$

$$\sigma_{22}^2 = \lambda(1-\lambda) \sum_{j=0}^k (u_{j+1} f_{j+1} - u_j f_j)^2 / (\lambda_{j+1} - \lambda_j) \quad (4.11)$$

and

$$\sigma_{12} = \sigma_{21} = \lambda(1-\lambda) \sum_{j=0}^k (f_{j+1} - f_j)(u_{j+1} f_{j+1} - u_j f_j) / (\lambda_{j+1} - \lambda_j) \quad (4.12)$$

where $f(F^{-1}(\lambda_j)) = f_j$ and $F^{-1}(\lambda_j) = u_j$ for $j=0,1,\dots,k+1$ with $f_0 = f_{k+1} = 0$.

Proof. For the asymptotic normality of $(T_{N1}^{(k)}, T_{N2}^{(k)})$ it suffices to show that every linear combination of $T_{N1}^{(k)}$ and $T_{N2}^{(k)}$ is asymptotically normal. From the definitions of $T_{N1}^{(k)}$ and $T_{N2}^{(k)}$ in (2.38) and (2.39), every linear combination of $T_{N1}^{(k)}$ and $T_{N2}^{(k)}$ can be written as

$$\begin{aligned} T_N^{(k)} &= c_1 T_{N1}^{(k)} + c_2 T_{N2}^{(k)} \\ &= \sum_{i=1}^k (1 - Z_{r_i}) [c_1 a_{1N}(r_i, f) + c_2 b_{1N}(r_i, f)] \\ &\quad + \sum_{j=0}^k N_{j+1} [c_1 \bar{a}_{j+1} + c_2 \bar{b}_{j+1}]. \end{aligned} \quad (4.13)$$

Using the same technique as in Theorems 2.6, 2.7 and 4.1 we obtain

$$E_0(T_N^{(k)}) = 0 \text{ and}$$

$$\begin{aligned} \text{Var}_0(T_N^{(k)}) &= \frac{1}{N(N-1)} \left\{ \sum_{i=1}^k [c_1 a_{1N}(r_i, f) + c_2 b_{1N}(r_i, f)]^2 \right. \\ &\quad \left. + \sum_{j=0}^k (r_{j+1} - r_j - 1) [c_1 \bar{a}_{j+1} + c_2 \bar{b}_{j+1}]^2 \right\}. \end{aligned}$$

We observe that $\text{Var}_0(T_N^{(k)}) = O(1)$ whereas $\text{Var}_0 \left\{ \sum_{i=1}^k (1-Z_{r_i}) [c_1 a_N(r_i, f) + c_2 b_N(r_i, f)] \right\} = O(1)$. Then, for the asymptotic normality of $T_N^{(k)}$ it suffices to consider only the statistic

$$T_N^{*(k)} = \sum_{j=0}^k N_{j+1} [c_1 \bar{a}_{j+1} + c_2 b_{j+1}]; \quad (4.14)$$

both have the same limiting distribution.

As in Theorem 2.8, the distribution of $\underline{N} = (N_1, \dots, N_{k+1})$, under H_0 , is asymptotically normal, and hence, the linear combination

$$V_N = \sum_{j=0}^k N_{j+1} [c_1 \alpha_{j+1} + c_2 \beta_{j+1}]$$

with α_{j+1} and β_{j+1} as in Lemmas 2.6 and 2.7, is also asymptotically normal. By Lemmas 2.4, 2.6 and 2.7, it follows that $T_N^{*(k)}$, and hence $T_N^{(k)}$, is also asymptotically normal. Moreover we have $N^{-\frac{1}{2}} T_N^{*(k)} \sim (0, (c_1, c_2) \Sigma_0 (c_1, c_2)')$. Consequently, $N^{-\frac{1}{2}} (T_{N1}^{(k)}, T_{N2}^{(k)})$ is asymptotically normal.

The asymptotic moments of $N^{-\frac{1}{2}} (T_{N1}^{(k)}, T_{N2}^{(k)})$ are derived from the small sample moments in Theorem 4.1 using the limiting weights in Lemmas 2.6 and 2.7.

As an immediate consequence of Theorem 4.2 we obtain the asymptotic distribution of $LS_N^{(k)}$ under H_0 .

Theorem 4.3. Under H_0 and the regularity conditions (B1) and (B2), the statistic $LS_N^{(k)}$ is distributed asymptotically as a central chi-square random variable with two degrees of freedom, which we write as

$$LS_N^{(k)} \sim \chi_2^2(0) \quad (4.15)$$

To discuss the distribution of $N^{-1/2}(T_{N1}^{(k)}, T_{N2}^{(k)})$, or $LS_N^{(k)}$,

under the location-scale alternatives we use the idea of contiguity.

To consider the least favourable densities in H_0 we use the hypotheses

$$H_0 : F_X(w) = G_Y(w) = F(e^{-\bar{\tau}w - \bar{\theta}}) \quad (4.16)$$

and

$$H_A : F_X(w) = F(w) \text{ and } G_Y(w) = F(e^{-\tau w - \theta}) \quad (4.17)$$

with $\bar{\tau} = n\tau/N$ and $\bar{\theta} = n\theta/N$; $\theta = h_{1N}N^{-1/2}$ and $\tau = h_{2N}N^{-1/2}$ with $h_{iN} \rightarrow h_i$

constant, $i=1, 2$. The joint densities of $W = (W_{r_1}, W_{r_2}, \dots, W_{r_k})$

and $Z^* = (M_1, Z_{r_1}, \dots, Z_{r_k}, M_{k+1})$, denoted by $P_N(z^*; H_A)$ and $P_N(z^*; H_0)$

under H_A and H_0 , respectively, can be derived from (2.34) by sub-

stitution of the distribution function in (4.16) and (4.17). The

likelihood ratio is defined by

$$L_N = P_N(z^*; H_A) / P_N(z^*; H_0).$$

We then have the following theorem which is proved in a slightly more general form in Chapter 5; it appears there as Theorem 5.7.

Theorem 4.4. If f is absolutely continuous and the regularity conditions (B1) and (B2) hold, then

$$\log L_N \xrightarrow{P_{\theta\tau}} \theta T_{N1}^{*(k)} + \tau T_{N2}^{*(k)} - \frac{1}{2} \sigma_N^2$$

with $T_{N1}^{*(k)}$ and $T_{N2}^{*(k)}$ as defined in (2.67) and (2.70) and

$$\sigma_N^2 = (\theta, \tau) \Sigma_0 (\theta, \tau)'$$

Although all the statistics used here depend on f we omit the reference to it in our notation. We first show the asymptotic normality of the random vector $N^{-1/2} (T_{N1}^{(k)}, T_{N2}^{(k)})$ under the alternatives.

Theorem 4.5. Under the regularity conditions (B1) and (B2) and under the location alternatives H_A , the asymptotic distribution of the vector $N^{-1/2} (T_{N1}^{(k)}, T_{N2}^{(k)})$ is given by

$$N^{-1/2} (T_{N1}^{(k)}, T_{N2}^{(k)}) \sim N((\theta, \tau) \Sigma_0, \Sigma_0) \quad (4.18)$$

where Σ_0 is defined as in Theorem 4.2.

Proof: For the asymptotic normality of $(T_{N1}^{(k)}, T_{N2}^{(k)})$ under H_A , it

suffices to show that every linear combination $T_N^{(k)} = c_1 T_{N1}^{(k)} + c_2 T_{N2}^{(k)}$,

as defined in (4.13), is asymptotically normal. In view of LeCam's

third Lemma it suffices to show the asymptotic normality of $(T_N^{(k)}, \log L_N)$

under the null hypothesis. We note that $T_N^{*(k)}$ in (4.14) can be written as:

$$T_N^{*(k)} = c_1 T_{N1}^{*(k)} + c_2 T_{N2}^{*(k)} \quad (4.19)$$

with $T_{N1}^{*(k)}$ and $T_{N2}^{*(k)}$ as in Theorem 4.4. Moreover, by Theorem 4.2, the statistics $T_N^{*(k)}$ and $T_N^{(k)}$ are asymptotically equivalent. Hence, $(T_N^{(k)}, \log L_N)$ and $(T_N^{*(k)}, \log L_N)$ have the same limiting distribution.

Now, to show the asymptotic bivariate normality of

$(T_N^{*(k)}, \log N)$, under H_0 , we need only show that every linear combination of $T_N^{*(k)}$ and $\log L_N$ is asymptotically normal. In view of (4.19) and Theorem 4.4, for any real numbers a and b , the distribution of $a T_N^{*(k)} + b \log L_N$ is asymptotically equivalent to that of

$$\begin{aligned} & a(c_1 T_{N1}^{*(k)} + c_2 T_{N2}^{*(k)}) + b(\theta T_{N1}^{*(k)} + \tau T_{N2}^{*(k)} - \frac{1}{2}\sigma_N^2) \\ &= (ac_1 + b\theta)T_{N1}^{*(k)} + (ac_2 + b\tau)T_{N2}^{*(k)} - \frac{1}{2}\sigma_N^2. \end{aligned} \quad (4.20)$$

From the proof of Theorem 4.2, the distribution of (4.20) is asymptotically normal $(-\frac{1}{2}\sigma_N^2, N(d_1, d_2)\Sigma_0(d_1, d_2)')$ with $d_1 = ac_1 + b\theta$ and $d_2 = ac_2 + b\tau$. Hence, $(T_N^{*(k)}, \log L_N)$, and consequently $(T_N^{(k)}, \log L_N)$, is asymptotically bivariate normal $(0, -\frac{1}{2}\sigma_N^2, N(c_1, c_2)\Sigma_0(c_1, c_2)')$, $\sigma_N^2, N^{\frac{1}{2}}(\theta, \tau)\Sigma_0(c_1, c_2)')$, under H_0 . Now, applying LeCam's third Lemma, as in Hájek and Šidák (1967, 208) we obtain, under H_A ,

$$N^{-\frac{1}{2}}T_N^{(k)} \sim N((\theta, \tau)\Sigma_0(c_1, c_2)', (c_1, c_2)\Sigma_0(c_1, c_2)')$$

for any real numbers c_1 and c_2 . Then, the result (4.18) follows.

As a consequence of Theorem 4.5, we obtain the following result which is basic for further discussion on asymptotic efficiency of tests based on $LS_N^{(k)}$.

Theorem 4.6. Under the regularity conditions (B1) and (B2) and under the alternatives H_A ,

$$LS_N^{(k)} \sim \chi_2^2(\Delta),$$

where $\chi_2^2(\Delta)$ stands for the non-central chi-square distribution with two degrees of freedom and non-centrality parameter

$$\Delta = \Delta(LS_N^{(k)}) = (\theta, \tau) \Sigma_0 (\theta, \tau)'. \quad (4.21)$$

Proof: This is an immediate consequence of Theorem 4.5 where we note that $\Delta = (\theta, \tau) \Sigma_0 \Sigma_0^{-1} \{(\theta, \tau) \Sigma_0\}'$ and that $\Sigma_0' = \Sigma_0$.

Since $(\theta, \tau) \Sigma_0 = (\theta \sigma_{11}^2 + \tau \sigma_{12}, \theta \sigma_{12} + \tau \sigma_{22}^2)$, it is interesting to note that Δ can also be expressed in terms of the correlation matrix Λ_0 under H_0 , namely:

$$\Delta = (\theta/\sigma_{11}, \tau/\sigma_{22}) \Sigma_0 \Lambda_0^{-1} \{(\theta/\sigma_{11}, \tau/\sigma_{22}) \Sigma_0\}'. \quad (4.22)$$

The results obtained in Theorems 4.2 and 4.6 are compatible with those obtained by Lepage (1977) and Rubison (1976) for the complete sample counterpart of $LS_N^{(k)}$; Lepage used the idea of contiguity while Rubison used Chernoff and Savage (1958) type techniques.

2 2

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The special case where $T_{N1}^{(k)}$ and $T_{N2}^{(k)}$ are asymptotically uncorrelated is of some interest, if only for comparison with Lepaige (1977) statistic U_V . In this case, $LS_N^{(k)}$ has asymptotically the same distribution as $N(T_{N1}^{(k)})^2 \text{Var}_O(T_{N1}^{(k)}) + N(T_{N2}^{(k)})^2 \text{Var}_O(T_{N2}^{(k)})$. Although the asymptotic independence of the complete sample statistics T_{N1} and T_{N2} is desirable and occurs frequently, the same property seldom carries over to the statistics $T_{N1}^{(k)}$ and $T_{N2}^{(k)}$. One instance where the asymptotic independence property for T_{N1} and T_{N2} still holds for $T_{N1}^{(k)}$ and $T_{N2}^{(k)}$ occurs when T_{N1} and T_{N2} are, respectively, odd and even translation invariant and the quantiles are symmetric.

The results of Theorem 4.6 are required for the study of power properties of the test corresponding to $LS_N^{(k)}$; some of these properties will be presented in section 4. First, we show the test is asymptotically optimum in the sense of Wald (1943).

4.3 Asymptotic Optimality

Although the statistic $LS_N^{(k)}$ is a natural extension of $T_{N1}^{(k)}$ and $T_{N2}^{(k)}$ to the two-sample location-scale problem, it does not yield a locally most powerful test for H_0 against H_A as defined in (4.16) and (4.17). In this section, we show $LS_N^{(k)}$ is asymptotically equivalent to the likelihood ratio test statistic, and hence, is asymptotically optimum in the Wald (1943) sense.

Let $L_N(W, Z^*; \theta, \tau)$ be the likelihood function of the vectors W and Z^* corresponding to the density $f(w; \theta, \tau)$. We define the likelihood ratio $\lambda_N^{(k)}$ as

$$\lambda_N^{(k)} = L(W, Z^*; 0, 0) / L(W^*, Z^*; \hat{\theta}, \hat{\tau}) \quad (4.23)$$

where $\hat{\theta}$ and $\hat{\tau}$ are the maximum likelihood estimators of θ and τ . Our objective is to show that $LS_N^{(k)}$ is asymptotically equivalent to $-2 \log \lambda_N^{(k)}$, under H_0 as well as H_A , and hence, yields an asymptotically optimum test in the Wald (1943) sense.

We make the following assumptions on the likelihood function $L(W, Z^*; \theta, \tau)$ as in Wald (1943); we use freely the notation $\gamma = (\gamma_1, \gamma_2) = (\theta, \tau)$. Moreover, let $D_i(f)$ stand for $(\partial/\partial \gamma_i)f$, $i=1, 2$ and likewise $D_{ij}^2(f)$ stand for $(\partial^2/\partial \gamma_i \partial \gamma_j)f$, $i, j=1, 2$.

Assumption 1. Denote by D_N the set of all sample points E_N for which the maximum likelihood estimator $\hat{\gamma}$ exists and the second order partial derivatives $D_{ij}^2(L(W, Z^*; \gamma))$, $i, j=1, 2$, are continuous functions of γ .

It is assumed that

$$\lim_{N \rightarrow \infty} P(D_N | \gamma) = 1, \quad (4.24)$$

uniformly in γ . If for a sample point E_N there exist several maximum likelihood-estimators we can select one of them by some given rule.

Hence we shall consider $\hat{\gamma}_N$ as a single-valued function of E_N , defined for all points of D_N . For brevity of notation we omit the index N , whenever possible, without ambiguity.

Assumption 2. For any positive ϵ ,

$$\lim_{N \rightarrow \infty} P[|\hat{\gamma} - \gamma| < \epsilon | \gamma] = 1 \quad (4.25)$$

uniformly in γ , where

$$|\hat{\gamma} - \gamma| = \left\{ \sum_{i=1}^2 (\hat{\gamma}_i - \gamma_i)^2 \right\}^{1/2} = \{(\hat{\theta} - \theta)^2 + (\hat{\tau} - \tau)^2\}^{1/2}.$$

For any x , any positive δ , and any γ_0 , let $\psi_{1j}(x, \gamma_0, \delta)$ and $\phi_{1j}(x, \gamma_0, \delta)$ denote the greatest lower bound and the least upper bound of $D_{1j}^2(\log f(x; \gamma))$, $i, j=1, 2$ with respect to γ where $|\gamma - \gamma_0| \leq \delta$.

Assumption 3

(a) For any sequences $\{\gamma_{1N}\}$, $\{\gamma_{2N}\}$ and $\{\delta_N\}$ for which $\lim_{N \rightarrow \infty} \gamma_{1N} = \lim_{N \rightarrow \infty} \gamma_{2N} = \gamma$

and $\lim_{N \rightarrow \infty} \delta_N = 0$, we have

$$\begin{aligned} \lim_{N \rightarrow \infty} E_{\gamma_{1N}} \psi_{1j}(x, \gamma_{2N}, \delta_N) &= \lim_{N \rightarrow \infty} E_{\gamma_{2N}} \phi_{1j}(x, \gamma_{1N}, \delta_N) \\ &= E_{\gamma} [D_{1j}^2(\log f(x; \gamma))] , \end{aligned} \quad (4.26)$$

uniformly in γ .

(b) There exists an $\epsilon > 0$ such that $E_{\gamma_1} [\psi_{1j}(x, \gamma_2, \delta)]^2$ and

$E_{\gamma_1} [\phi_{1j}(x, \gamma_2, \delta)]^2$ are bounded functions of (γ_1, γ_2) and δ in the domain D_{ϵ} defined by the inequalities $|\gamma_1 - \gamma_2| \leq \epsilon$ and $|\delta| \leq \epsilon$.

(c) The greatest lower bound with respect to γ of the absolute value of the determinant of the matrix $||E_Y[-D_{ij}^2(\log f(x;\gamma))]|$ is positive.

Assumption 4. We can differentiate under the integral sign, that is,

$$E_Y[D_i(f(x;\gamma))] = E_Y[D_{ij}^2(f(x;\gamma))] = 0 \quad (4.27)$$

for all $i, j=1, 2$.

Assumption 5. There exists a positive $\eta > 0$ such that

$$E_Y|D_i(\log f(x;\gamma))|^{2+\eta}, \quad (4.28)$$

$i=1, 2$, are bounded functions of γ .

Let the connected surface $S_c(\theta, \tau)$ be defined by

$$(\theta, \tau) ||\alpha_{ij}|| (\theta, \tau)' = c \quad (4.29)$$

where

$$\alpha_{ij} = E_Y[-D_{ij}^2(\log L(W, Z^*; \gamma))] , \quad i, j=1, 2.$$

For any point γ in $S_c(\theta, \tau)$ and any $\rho > 0$, let

$$\omega \equiv \omega(\gamma_0, \rho) = \{\gamma \in S_c(\theta, \tau) : |\gamma - \gamma_0| \leq \rho\} \quad (4.30)$$

Also let $\omega^* = T(\omega)$ be the image of ω by the non-singular transformation

T such that $T'T = ||\alpha_{ij}||$. Denoting the area of ω by $A(\omega)$ and that

of ω^* by $A(\omega^*)$, we define the weight function

$$\eta(\theta, \tau) = \lim_{\rho \rightarrow 0} A(\omega^*) / A(\omega) \quad (4.31)$$

We are now in a position to state the following theorem of Wald (1943, 478) concerning the optimality of the likelihood ratio test.

Theorem 4.7 (Wald, 1943) Let $S_c(\theta, \tau)$ and $\eta(\theta, \tau)$ be as in (4.29) and (4.31). Then for testing H_0 ; $F_X(w) = G_Y(w) = F(w)$, under the

Assumptions 1-5, the likelihood ratio test

- (a) has asymptotically the best average power with respect to the surfaces $S_c(\theta, \tau)$ and weight functions $\eta(\theta, \tau)$;
- (b) has asymptotically best constant power on the surfaces $S_c(\theta, \tau)$;
- (c) is an asymptotically most stringent test.

We recall the definition of the above asymptotic power properties of tests taken from Wald (1943). Let $\{C_n\}$ be a sequence of critical regions for testing the simple hypothesis $\gamma=0$.

Definition 4.1 The sequence $\{C_n\}$ of size α is said to have asymptotically best average power with respect to the family of surfaces $S_c(\gamma)$ and the weight function $\eta(\gamma)$ if for any sequence $\{D_n\}$ for which $P(D_n | 0) = \alpha$ we have

$$\limsup_{n \rightarrow \infty} \left\{ \text{u.b.} \left| \int_{S_c} P(D_n | \gamma) [\eta(\gamma) / A(S_c)] dA - \int_{S_c} P(C_n | \gamma) [\eta(\gamma) / A(S_c)] dA \right| \right\} \leq 0.$$

where

$$A(S_c) = \int_{S_c} \eta(\psi) dA.$$

Definition 4.2. The sequence $\{C_n\}$ is said to be of size α and to have asymptotically best constant power on the surfaces $S_c(\gamma)$ if the following three conditions are fulfilled:

(a) $P(C_n|0) = \alpha, n=1,2,\dots$

(b) $\lim_{n \rightarrow \infty} \left\{ 1.u.b. \left[1.u.b. P(C_n|\gamma) - g.l.b. P(C_n|\gamma) \right] \right\} = 0$

where the symbols 1.u.b. and g.l.b mean that the bounds are taken with respect to γ restricted to $S_c(\gamma)$.

(c) For any sequence $\{D_n\}$ which satisfies (a) and (b) we have

$$\lim_{n \rightarrow \infty} \{ 1.u.b. [P(D_n|\gamma) - P(C_n|\gamma)] \} = 0.$$

Definition 4.3. The sequence $\{C_n\}$ is said to be an asymptotically most stringent test on the level of significance α if $P(C_n|0) = \alpha$ and if for any $\{D_n\}$ for which $P(D_n|0) = \alpha$ we have

$$\lim_{n \rightarrow \infty} \{ 1.u.b. [P_n(\gamma, 0, \alpha) - P(C_n|\gamma)]$$

$$- 1.u.b. [P_n(\gamma, 0, \alpha) - P(D_n|\gamma)] \} \leq 0$$

where $P_n(\gamma, 0, \alpha)$ denotes the least upper bound of $P(C_n|\gamma)$ with respect to C_n , where C_n is restricted to regions for which $P(C_n|0) = \alpha$.

Whenever a test statistic satisfies the properties (a), (b) and (c) of Theorem 4.7 we say it is asymptotically optimum in the Wald sense. We are now in a position to prove the asymptotic equivalence of $LS_N^{(k)}$ and $-2\log\lambda_N^{(k)}$, and, consequently, that $LS_N^{(k)}$ yields an asymptotically optimum test in the Wald sense. For simplicity, we use the notation $L(\gamma)$ for $L(W, Z^*; \gamma) = L(W, Z^*; \theta, \tau)$.

Theorem 4.8 Under the regularity conditions of Theorem 4.7

$$LS_N^{(k)} + 2\log\lambda_N^{(k)} \xrightarrow{P_0} 0. \quad (4.32)$$

Proof. We first show the asymptotic equivalence between $-2\log\lambda_N^{(k)}$ and $N\hat{\gamma}_0' \hat{\gamma}$, then, the asymptotic equivalence between $LS_N^{(k)}$ and $N\hat{\gamma}_0' \hat{\gamma}$ and, consequently, we obtain the asymptotic equivalence between $LS_N^{(k)}$ and $-2\ln\lambda_N^{(k)}$ which proves the theorem.

To show the asymptotic equivalence between $-2\log\lambda_N^{(k)}$ and $N\hat{\gamma}_0' \hat{\gamma}$ we first observe that, from (4.23)

$$-2\log\lambda_N^{(k)} = -2[\log L(0) - \log(\hat{\gamma})]. \quad (4.33)$$

Using the Taylor expansion of $\log L(0)$ about $\gamma = \hat{\gamma}$, we can write

$$\begin{aligned} \log L(0) &= \log L(\hat{\gamma}) - \sum_{i=1}^2 \hat{\gamma}_i [D_i(\log L(\gamma))]_{\gamma=\hat{\gamma}} \\ &\quad + \frac{1}{2} \sum_{i,j=1}^2 \hat{\gamma}_i \hat{\gamma}_j [D_{ij}(\log L(\gamma))]_{\gamma=\gamma^*} \end{aligned} \quad (4.34)$$

where γ^* lies in the 2-dimensional rectangle $(0, \hat{\gamma})$. Since $\hat{\gamma}_1$ is the

maximum likelihood estimator of γ , by definition

$$[D_i(\log L(\gamma))]_{\gamma=\hat{\gamma}} = 0 \quad (4.35)$$

Then, from (4.33), (4.34) and (4.35), we can write

$$-2\log \lambda_N^{(k)} = N^{\frac{1}{2}} \hat{\gamma} \left| -N^{-1} [D_{ij}(\log L(\gamma))]_{\gamma=\hat{\gamma}} \right| N^{\frac{1}{2}} \hat{\gamma}.$$

Moreover, since

$$N^{-1} [-D_{ij}(\log L(\gamma))]_{\gamma=\hat{\gamma}} - E_{\gamma=0} \{N^{-1} [-D_{ij}(\log L(\gamma))]\} \xrightarrow{p_0} 0$$

for $i, j=1, 2$ and $N^{\frac{1}{2}} \hat{\gamma}$ has a non-degenerate limit distribution,

$$-2\log \lambda_N^{(k)} - \hat{\gamma} \left| E_{\gamma=0} [-D_{ij}(\log L(\gamma))] \right| \hat{\gamma} \xrightarrow{p_0} 0. \quad (4.36)$$

By direct computation, we have $D_i(\log L(\gamma)) = T_{Ni}^{(k)}(g)$, $i=1, 2$, where

$T_{Ni}^{(k)}(g)$ is of the form $T_{Ni}^{(k)}$, in (4.4) and (4.5), where the weights are

obtained by $a_N(i, g)$ and $b_N(i, g)$, instead of $a_N(i, f)$ and $b_N(i, f)$, with

$g(w) = e^{-T} f(e^{-T} w - \theta)$. Then, together with the regularity assumptions of Wald,

$$\begin{aligned} E_{\gamma=0} [-D_{ij}(\log L(\gamma))] &= E_{\gamma=0} [D_i(\log L(\gamma)) D_j(\log L(\gamma))] \\ &= E_{\gamma=0} [T_{Ni}^{(k)}(g) T_{Nj}^{(k)}(g)]. \end{aligned} \quad (4.37)$$

Since f is absolutely continuous, we have

$$||E_{\gamma=0}[T_{N1}^{(k)}(g)T_{Nj}^{(k)}(g)]|| \xrightarrow{P_0} ||E_{\gamma=0}[T_{N1}^{(k)}(f)T_{Nj}^{(k)}(f)]|| \xrightarrow{P_0} N\Sigma_0. \quad (4.38)$$

Combining (4.36), (4.37) and (4.38), we obtain

$$-2\log\lambda_N^{(k)} \xrightarrow{P_0} N\hat{\Sigma}_0 \hat{\Sigma}_0' \quad (4.39)$$

We now show that $LS_N^{(k)}$ and $N\hat{\Sigma}_0 \hat{\Sigma}_0'$ are asymptotically equivalent.

With Taylor's expansion about $\gamma = \hat{\gamma}$, we have

$$\begin{aligned} 0 &= [D_1(\log L(\gamma))]_{\gamma=\hat{\gamma}} \\ &= [D_1(\log L(\gamma))]_{\gamma=0} + \sum_{i=1}^2 \hat{\gamma}_i [D_{1j}^2(\log L(\gamma))]_{\gamma=\gamma^{**}} \end{aligned} \quad (4.40)$$

where γ^{**} lies in the rectangle $(0, \hat{\gamma})$. By direct computation,

$$[D_1(\log L(\gamma))]_{\gamma=0} = T_{N1}^{(k)}(f) \text{ with } T_{N1}^{(k)}(f) \text{ as defined in (4.4) and (4.5).}$$

When substituted in (4.40), we can write

$$(T_{N1}^{(k)}(f), T_{N2}^{(k)}(f)) = \hat{\gamma} ||[-D_{1j}^2(\log L(\gamma))]_{\gamma=\gamma^{**}}|| \quad (4.41)$$

where

$$||[-D_{1j}^2(\log L(\gamma))]_{\gamma=\gamma^{**}}|| - N\Sigma_0 \xrightarrow{P_0} 0 \quad (4.42)$$

Therefore, using (4.41), (4.42) and applying Slutsky's Theorem to the definition of $LS_N^{(k)}$ in (4.3) we have

$$LS_N^{(k)} \xrightarrow{P_0} N^{-1}(N\hat{\Sigma}_0 \hat{\Sigma}_0')^{-1}(N\hat{\Sigma}_0 \hat{\Sigma}_0') = N\hat{\Sigma}_0 \hat{\Sigma}_0' \quad (4.43)$$

Hence, (4.39) and (4.43) yield (4.32) which proves the theorem.

As a consequence of the contiguity of the alternatives H_A to the null hypotheses H_0 , which is proved in Chapter 5, we also obtain that

$$LS_N^{(k)} + 2 \log \lambda_N^{(k)} \xrightarrow{P_N} 0$$

under the alternatives.. We can now state the following theorem.

Theorem 4.9. Under the regularity conditions of Theorem 4.7, the statistic $LS_N^{(k)}$ yields an asymptotically optimum rank order test in the Wald sense.

We shall consider some asymptotic power properties in the next section.

4.4 Asymptotic Relative Efficiency

As a large sample method of power comparison between two tests, in the non-normal cases, we use the Pitman Asymptotic Relative Efficiency (ARE), as discussed in Puri and Sen (1971, 119f). We are particularly interested in the ARE of the two-dimensional statistic $LS_N^{(k)}$ relative to its complete sample counterparts $\mathcal{L}_N$ and U_N , studied by Rubison (1976) and Lepage (1977), respectively. It is interesting to note that Rubison's statistic reduces to Lepage's statistic for asymptotically uncorrelated component statistics within the context of contiguous alternatives.. Although it may be desirable to have asymptotically uncorrelated statistics for the location and scale parameters in the complete sample context, the corresponding

statistics are rarely so within the multiple censoring context; the property carries over under special conditions on the density functions and the choice of the quantiles.

With our notation, the Rubison (1976) statistic $\mathcal{L}_N$, applied to the location and scale statistics, can be written

$$\mathcal{L}_N = N^{-1}(T_{N1}, T_{N2})\Sigma_{ON}^{*-1}(T_{N1}, T_{N2}) \quad (4.44)$$

where T_{N1} and T_{N2} are defined by (2.42) and (2.43) and Σ_{ON}^* denotes the dispersion matrix of $N^{-1/2}(T_{N1}, T_{N2})$ under H_0 ; Σ_{ON}^* is assumed non-singular. It should be noted that T_{N1} and T_{N2} used by Rubison (1976) correspond to T_{N1}/m and T_{N2}/m in our notation.

To discuss the asymptotic power properties of $LS_N^{(k)}$ relative to those of $\mathcal{L}_N$ the limiting distributions of these statistics are required, specially under the alternative hypothesis H_A in (4.17). The asymptotic distributions of $LS_N^{(k)}$, both under H_0 and H_A , have been derived in section 4.2; those of $\mathcal{L}_N$ were obtained by Rubison (1976) and are stated in the following two Lemmas.

Lemma 4.1. Under H_0 and the regularity conditions (B1) and (B2) $\mathcal{L}_N$ is distributed asymptotically as a central chi-square random variable with two degrees of freedom.

Lemma 4.2. Under the alternatives H_A and the regularity conditions (B1) and (B2), $\mathcal{L}_N$ is distributed asymptotically as a non-central chi-square

random variable with two degrees of freedom and non-centrality parameter

$$\Delta(\mathbf{Z}_N) = (\theta, \tau) \Sigma_0^*(\theta, \tau)' , \quad (4.45)$$

where Σ_0^* denotes the limiting dispersion matrix of $N^{-1/2}(T_{N1}, T_{N2})$ under H_0 .

The dispersion matrix Σ_0^* is the complete sample analogue to Σ_0 ; the matrix Σ_0^* has entries

$$\sigma_{11}^{*2} = \lambda(1-\lambda)I(f) , \quad (4.46)$$

$$\sigma_{22}^{*2} = \lambda(1-\lambda)I_1(f) , \quad (4.47)$$

$$\sigma_{21}^* = \sigma_{12}^* = \lambda(1-\lambda)I_{12}(f) , \quad (4.48)$$

where $I(f)$ and $I_1(f)$ represent the Fisher's information defined by (4.1) and (4.2), and $I_{12}(f)$ is defined by

$$I_{12}(f) = \int_0^1 J(u)J_1(u)du . \quad (4.49)$$

To compare tests based on statistics having asymptotically chi-square distributions, Puri and Sen (1971, 121) have proposed the following generalization of the Pitman efficiency which is stated here only with the degree of generality required for our specific application.

Lemma 4.3. Assume the regularity conditions of Puri and Sen (1971, section 3.8.3) hold. If under the alternative hypotheses, the statistics Q_1 and Q_2 are distributed as chi-square random variables with two de-

degrees of freedom and centrality parameters $\Delta(Q_1)$ and $\Delta(Q_2)$, respectively, then

$$\text{ARE}(Q_1, Q_2) = \Delta(Q_1)/\Delta(Q_2). \quad (4.50)$$

The regularity conditions are analogous to the regularity conditions (a)-(f) in Section 2.4. As an application of Lemma 4.3 to $LS_N^{(k)}$ and $\mathcal{L}_N$ the following theorem follows directly from Theorem 4.6 and Lemma 4.2.

Theorem 4.10. Under the regularity conditions (B1) and (B2)

$$\text{ARE}(LS_N^{(k)}, \mathcal{L}_N) = \frac{(\theta, \tau) \Sigma_0 (\theta, \tau)'}{(\theta, \tau) \Sigma_0^* (\theta, \tau)'} \quad (4.51)$$

where $LS_N^{(k)}$ and $\mathcal{L}_N$ are defined by (4.3) and (4.44).

Proof. From Lemma 4.3 the expression (4.51) is the ratio of (4.21) and (4.45).

From the definitions of Σ_0 and Σ_0^* we can also write (4.51)

as

$$\text{ARE}(LS_N^{(k)}, \mathcal{L}_N) = \frac{\theta^2 \sigma_{11}^2 + 2\theta\tau\sigma_{12} + \tau^2 \sigma_{22}^2}{\theta^2 \sigma_{11}^{*2} + 2\theta\tau\sigma_{12}^* + \tau^2 \sigma_{22}^{*2}} \quad (4.52)$$

where σ_{11}^2 , σ_{12} and σ_{22}^2 are defined by (4.10), (4.11) and (4.12) and σ_{11}^{*2} , σ_{12}^* and σ_{22}^{*2} are defined by (4.46), (4.47) and (4.48).

Whenever T_{N1} and T_{N2} are asymptotically independent, $\sigma_{12}^* = 0$ and the statistic $\mathcal{L}_N$ coincides with the Lepage (1977) statistic U_v ;

we recall that $T_{N1}^{(k)}$ and $T_{N2}^{(k)}$ are in general correlated in this case.

We obtain the following corollary to Theorem 4.10.

Corollary 4.2. Assume the regularity conditions (B1) and (B2). If T_{N1} and T_{N2} are asymptotically uncorrelated, then

$$ARE(LS_N^{(k)}, \mathcal{I}_N) = \frac{\theta^2 \sigma_{11}^2 + 2\theta\tau\sigma_{12} + \tau^2 \sigma_{22}^2}{\theta^2 \sigma_{11}^{*2} + \tau^2 \sigma_{22}^{*2}} \quad (4.53)$$

It is interesting to note that when $\tau=0$ in (4.17) the hypothesis H_A reduces to that of the location alternatives and (4.53) reduces to $ARE(T_{N1}^{(k)}, T_{N1}) = \sigma_{11}^{*-2} \sigma_{11}^2$ which is the same as (2.80).

Likewise, when $\theta=0$ in (4.17) the hypothesis H_A reduces to that of the scale alternatives and (4.53) reduces to $ARE(T_{N2}^{(k)}, T_{N2}) = \sigma_{22}^{*-2} \sigma_{22}^2$ which is the same as (2.84).

4.5 Optimum Asymptotic Relative Efficiency

By considering the bounds on the ratio of two quadratic forms we can determine the supremum and the infimum of $ARE(LS_N^{(k)}, \mathcal{I}_N)$. These bounds are provided by the following property due to Courant; re. Puri and Sen (1971, 122).

Lemma 4.4. (Courant) If XAX' and XBX' are two positive definite quadratic forms in two variables, then the supremum and the infimum of the ratio XAX'/XBX' with respect to $X \neq 0$ are given by the largest and the smallest characteristic roots of AB^{-1} , and

$$\inf_X \frac{XAX'}{XBX'} \leq \left\{ AB^{-1} \right\}^{\frac{1}{2}} \leq \sup_X \frac{XAX'}{XBX'}. \quad (4.54)$$

Applying Lemma 4.4 to the expression (4.51) for $ARE(LS_N^{(k)}, \mathcal{L}_N)$

we obtain the following results.

Theorem 4.11. Whenever $(\theta, \tau)\Sigma_0(\theta, \tau)'$ and $(\theta, \tau)\Sigma_0^*(\theta, \tau)'$ are positive definite, for any fixed k , we have

$$\inf_{\theta, \tau} ARE(LS_N^{(k)}, \mathcal{L}_N) \leq \left\{ |\Sigma_0| / |\Sigma_0^*| \right\}^{\frac{1}{2}} \leq \sup_{\theta, \tau} ARE(LS_N^{(k)}, \mathcal{L}_N) \quad (4.55)$$

where $|\Sigma_0|$ and $|\Sigma_0^*|$ stand for the determinants of Σ_0 and Σ_0^* respectively.

Proof. The expression (4.55) follows directly from (4.54) where we observe that $|\Sigma_0 \Sigma_0^{*-1}| = |\Sigma_0| |\Sigma_0^*|^{-1}$.

Theorem 4.12. Under the same conditions as in Theorem 4.11,

$$\frac{1}{2} D_1^{-1} \{D - (D^2 - 4D_1 D_2)^{\frac{1}{2}}\} \leq ARE(LS_N^{(k)}, \mathcal{L}_N) \leq \frac{1}{2} D_1^{-1} \{D + (D^2 - 4D_1 D_2)^{\frac{1}{2}}\} \quad (4.56)$$

where

$$D = \sigma_{11}^2 \sigma_{22}^{*2} + \sigma_{22}^2 \sigma_{11}^{*2} - 2\sigma_{12} \sigma_{12}^*, \quad (4.57)$$

$$D_1 = \sigma_{11}^{*2} \sigma_{22}^{*2} - \sigma_{12}^{*2}, \quad (4.58)$$

and

$$D_2 = \sigma_{11}^2 \sigma_{22}^2 - \sigma_{12}^2. \quad (4.59)$$

with σ_{ij} and σ_{ij}^* , $i, j=1, 2$ defined by (4.10)-(4.12) and (4.46)-(4.48).

Proof. By Lemma 4.4, it suffices to find the largest and smallest characteristic roots of $\Sigma_0 \Sigma_0^{*-1}$. Hence, it suffices to solve the quadratic equation $|\Sigma_0 \Sigma_0^{-1} - tI_2| = 0$ where I_2 is the identity matrix of rank 2; using (4.57), (4.58) and (4.59) the equation can be written as $t^2 D_1 - tD + D_2 = 0$. Moreover, based on positive definite quadratic forms, by direct computation, $D^2 - 4D_1 D_2 > 0$ and $D > (D^2 - 4D_1 D_2)^{1/2}$, yielding positive roots. The results (4.56) follows.

We recall that Lepage (1977) results are restricted to statistics T_{N1} and T_{N2} which are asymptotically uncorrelated; this situation occurs, in particular, with odd and even translation invariant statistics as studied by Duran, Tsai and Lewis (1976). In this case the expression (4.56) holds with $\sigma_{12}^* = 0$. Although the statistics T_{N1} and T_{N2} may be asymptotically uncorrelated, the property does not hold, in general, for the statistics $T_{N1}^{(k)}$ and $T_{N2}^{(k)}$. A particular instance where the property carries over to $T_{N1}^{(k)}$ and $T_{N2}^{(k)}$ occurs when symmetric unimodal densities and symmetric quantiles are used. In such a case we have the following corollary to Theorem 4.12.

Corollary 4.3. Assume the statistics T_{N1} and T_{N2} as well as $T_{N1}^{(k)}$ and $T_{N2}^{(k)}$ are asymptotically uncorrelated. Then,

$$\text{ARE}(T_{N2}^{(k)}, T_{N2}) \leq \text{ARE}(LS_N^{(k)}, T_N) \leq \text{ARE}(T_{N1}^{(k)}, T_{N2}) \quad (4.60)$$

or equivalently,

$$\sigma_{22}^2 / \sigma_{22}^{*2} \leq \text{ARE}(\text{LS}_{N1}^{(k)}, \mathcal{L}_N) \leq \sigma_{11}^2 / \sigma_{11}^{*2} \quad (4.61)$$

where $\text{ARE}(T_{N1}^{(k)}, T_{N1})$ are given by (2.80) and (2.84) for $i=1,2$.

Proof. From the assumptions $\sigma_{12} = \sigma_{12}^* = 0$. When substituted in (4.57), (4.58) and (4.59) we obtain:

$$D = \sigma_{11}^2 \sigma_{22}^{*2} + \sigma_{22}^2 \sigma_{11}^{*2}, \quad D_1 = \sigma_{11}^2 \sigma_{22}^{*2} \quad \text{and} \quad D_2 = \sigma_{11}^{*2} \sigma_{22}^2.$$

Moreover $D^2 - 4D_1 D_2 = (\sigma_{11}^2 \sigma_{22}^{*2} - \sigma_{22}^2 \sigma_{11}^{*2})^2$. With these quantities substituted in (4.56); after some algebraic computations we have (4.61). Referring to (2.80) and (2.84), we also have (4.60).

To find the optimum spacings which maximize $\text{ARE}(\text{LS}_N^{(k)}, \mathcal{L}_N)$ we should maximize $D - (D^2 - 4D_1 D_2)^{1/2}$ in (4.56). The problem of optimum spacings in the two-parameter situation, even under simplified assumptions of symmetry, is far more complicated than the corresponding problem within the one-parameter situation. In general, for most density functions, there is no definite answer. As a compromise solution, some specific sets of spacings have been proposed and used with success; two popular ones are

$$\{1/(k+1), 2/(k+1), \dots, k/(k+1)\} \quad (4.62)$$

and

$$\{(1-1/2)/k, (2-1/2)/k, \dots, (k-1/2)/k\}. \quad (4.63)$$

Based on (4.63) and for $k=2,3,5,7,9$, with the logistic distribution, Chan (1969) obtained the ARE's 0.4006, 0.5766, 0.7449, 0.8217 and

0.8644; with the Cauchy distribution, Chan (1970) obtained the ARE's 0.3285, 0.5717, 0.7982, 0.8865 and 0.9204. Based on (4.62) and for $k=2,3,5,7,9$, with the Cauchy distribution, Chan (1970) obtained the ARE's 0.4677, 0.6570, 0.8315, 0.9018 and 0.9361.

For practical purposes we may adopt the following compromising method. Since $ARE(T_{N1}^{(k)}, T_{N1})$ is larger than $ARE(T_{N2}^{(k)}, T_{N2})$ and if there is no special emphasis on the accuracy of the scale parameter we should use the spacings which are optimum for $ARE(T_{N1}^{(k)}, T_{N1})$.

CHAPTER 5

CONTIGUOUS ALTERNATIVES

Within the two-sample problem context, Hájek and Šidák (1967) have shown the contiguity of alternative hypotheses for one-parameter location and scale families of densities. Using the same technique, Lepage (1975) extended the results to the two-parameter location and scale family of densities. The proposed objective of this study is to further extend these results to multiply censored samples.

With type II censoring, the Hájek and Šidák (1967) procedure is not easily applied. Rather, we refer to the work of Johnson (1974) and Johnson and Mehrotra (1976) with right censored samples for a different approach. Applying a similar technique to the multiple censored samples we show the contiguity of location, scale and simultaneous location and scale alternatives. First, under specified regularity conditions, we define sequences of alternative and null hypotheses for each family of densities. Then, from the asymptotic normality of the corresponding log-likelihood ratio, under the null hypothesis, we show the contiguity of the alternatives as an application of LeCam's first lemma.

5.1 Assumptions and Preliminary Results

As in the preceding chapters, let $W_1 \leq W_2 \leq \dots \leq W_N$, $N = m + n$, be the combined ordered observations from two independent samples, of sizes m and n , with cumulative distribution functions $F_X(x)$ and $G_Y(y)$, respectively. From the ordered observations W_i , $i = 1, 2, \dots, N$, the sample quantiles $W_{r_1} < W_{r_2} < \dots < W_{r_k}$ are specified by the k real numbers $0 < \lambda_1 < \dots < \lambda_k < 1$ using the ranks $r_j = [N\lambda_j] + 1$, $j = 1, 2, \dots, k$; they are represented by the vector $W = (W_{r_1}, W_{r_2}, \dots, W_{r_k})$. With $Z_i = 1$ if W_i is an X -observation, and 0 otherwise, the rank order vector Z^* is defined as $Z^* = (M_1, Z_{r_1}, M_2, Z_{r_2}, \dots, Z_{r_k}, M_{k+1})$ where M_{j+1} , $j=0, \dots, k$, represents the number of X -observations between W_{r_j} and $W_{r_{j+1}}$ with $W_{r_0} = -\infty$ and $W_{r_{k+1}} = \infty$.

The likelihood of Z^* can then be expressed as

$$p_N(W=w, Z^*=z^*) = C \prod_{i=1}^k f_X^{z_{r_i}(w_{r_i})} g_Y^{1-z_{r_i}(w_{r_i})} \times \prod_{j=0}^k [F_X(w_{r_{j+1}}) - F_X(w_{r_j})]^{m_{j+1}} [G_Y(w_{r_{j+1}}) - G_Y(w_{r_j})]^{n_{j+1}} \quad (5.1)$$

for $W_{r_1} < W_{r_2} < \dots < W_{r_k}$, and 0 otherwise, where

$$C = m!n! \left(\prod_{j=0}^k m_{j+1}! n_{j+1}! \right)^{-1}. \quad (5.2)$$

and $f_X(x)$ and $g_Y(y)$ are the respective densities of $F_X(x)$ and $G_Y(y)$.

We first make some smoothness assumptions regarding the families of densities $f(x;\theta)$, where θ belongs to an open set of R or R^2 .

Assumptions.

- (A1) The densities $f(x;\theta)$ are absolutely continuous and the set where they are positive does not depend on θ .
- (A2) The function $f^{1/2}(x;\theta)$ has a quadratic mean derivative with respect to θ at $\bar{\theta}$. That is, by Loève (1963, 470), the Fisher information $I(f,\theta)$ satisfies

$$I(f,\bar{\theta}) = \int_{-\infty}^{\infty} \left\{ \frac{\dot{f}(x;\bar{\theta})}{f(x;\bar{\theta})} \right\}^2 f(x;\bar{\theta}) dx < \infty. \quad (5.3)$$

For each θ , let

$$\phi(W;\theta) = [f(W;\theta)/f(W;\bar{\theta})]^{1/2} \quad (5.4)$$

and denote by $\dot{\phi}(W;\bar{\theta})$ its quadratic mean derivative with respect to θ at $\bar{\theta}$.

The assumption (A1), not needed in the uncensored case, could be relaxed to absolute continuity at the censoring points. A further relaxation could be obtained using assumptions (A) of Johnson (1974, 1141). The additional smoothness required here is satisfied by the densities usually encountered in similar problems and allows for a simplification of the proofs.

Next, consider sequences of null hypotheses,

$$H_{ON} = F_X(w) = G_Y(w) = F(w; \bar{\theta}_N) \quad (5.5)$$

and sequences of alternative hypotheses,

$$H_{AN} \quad F_X(w) = F(w; \theta_{1N}), \quad G_Y(w) = F(w; \theta_{2N}) \quad (5.6)$$

where θ_{sN} , $s = 1, 2$, are one or two dimensional vector of parameters and $\bar{\theta}_N = (m\theta_{1N} + n\theta_{2N})/N$. Moreover, let the alternatives be of the form $\theta_N = \bar{\theta}_N + h_N N^{-1/2}$ with $h_N \rightarrow h$ and denote by P_{θ_N} the probability function under θ_N .

Under the assumptions (A1) and (A2) and the null hypothesis (5.5) we obtain the following results as used by Johnson (1974) and proved by Roussas (1965):

$$\max_{1 \leq i \leq N} | \phi(W_i; \theta_N) - 1 | \xrightarrow{P_{\bar{\theta}_N}} 0, \quad (5.7)$$

$$N^{1/2}(\phi(W_i; \theta_N) - 1) \xrightarrow{q.m.} h_N \dot{\phi}(W_i; \bar{\theta}_N), \quad (5.8)$$

$$N^{1/2}(\phi^2(W_i; \theta_N) - 1) \xrightarrow{1st \text{ mean}} 2h_N \dot{\phi}(W_i; \bar{\theta}_N), \quad (5.9)$$

$$E \dot{\phi}(W_i; \theta_N) = 0, \quad (5.10)$$

where q.m. stands for quadratic mean. Although θ_N , h_N and P_{θ_N} depend on N , we shall allow the abuse of notation θ , h and P_θ in the sequel.

With $\phi_s(w) = \phi(w; \theta_s)$, let $\dot{\phi}_s(w)$ denote the quadratic mean derivative of $\phi_s(w)$ at $\theta = \bar{\theta}$ and

$$\begin{aligned} \psi_{s,j+1} &= \psi_{s,j+1}(w) = \psi_{s,j+1}(w_{r_{j+1}}, w_{r_j}; \theta_s, \bar{\theta}) \\ &= \frac{F(w_{r_{j+1}}; \theta_s) - F(w_{r_j}; \theta_s)}{F(w_{r_{j+1}}; \bar{\theta}) - F(w_{r_j}; \bar{\theta})} \end{aligned} \quad (5.11)$$

for $s=1, 2$, and $j=0, 1, \dots, k$. Let the joint density (5.1) of W and Z^* be denoted by $p_N(w, z^*; \bar{\theta})$ under the null hypothesis and by $p_N(w, z^*; \theta_1, \theta_2)$ under the alternative hypothesis H_{AN} . Then $\log L_N$ can be expressed as

$$\begin{aligned} \log L_N &= \log[p_N(w, z^*; \theta_1, \theta_2) / p_N(w, z^*; \bar{\theta})] \\ &= 2 \sum_{i=1}^k [z_{r_i} \log \phi_1(w_{r_i}) + (1-z_{r_i}) \log \phi_2(w_{r_i})] \\ &\quad + \sum_{j=0}^k [m_{j+1} \log \psi_{1,j+1} + n_{j+1} \log \psi_{2,j+1}]. \end{aligned} \quad (5.12)$$

To obtain a large sample expansion of $\log L_N$, each term in (5.12) is written as a sum of three terms using

$$\log Z = (Z-1) - \frac{1}{2}(Z-1)^2 + c(Z-1)^3 \quad (5.13)$$

where $|c| \leq 3$ for $|Z-1| \leq \frac{1}{2}$.

Then from (5.12) we have

$$\log L_N = A_{1N} - A_{2N} + A_{3N} \quad (5.14)$$

where

$$A_{1N} = 2 \sum_{i=1}^k [Z_{r_i} (\phi_1(W_{r_i}) - 1) + (1-Z_{r_i}) (\phi_2(W_{r_i}) - 1)] \\ + \sum_{j=0}^k [M_{j+1} (\psi_{1,j+1} - 1) + N_{j+1} (\psi_{2,j+1} - 1)], \quad (5.15)$$

$$A_{2N} = \sum_{i=1}^k [Z_{r_i} (\phi_1(W_{r_i}) - 1)^2 + (1-Z_{r_i}) (\phi_2(W_{r_i}) - 1)^2] \\ + \frac{1}{2} \sum_{j=0}^k [M_{j+1} (\psi_{1,j+1} - 1)^2 + N_{j+1} (\psi_{2,j+1} - 1)^2], \quad (5.16)$$

$$A_{3N} = 2 \sum_{i=1}^k [c_{1i} Z_{r_i} (\phi_1(W_{r_i}) - 1)^3 + c_{2i} (1-Z_{r_i}) (\phi_2(W_{r_i}) - 1)^3] \\ + \sum_{j=0}^k [d_{1,j+1} M_{j+1} (\psi_{1,j+1} - 1)^3 + d_{2,j+1} N_{j+1} (\psi_{2,j+1} - 1)^3] \quad (5.17)$$

where

$$|c_{si}| \leq 3 \text{ for } |\phi_s(W_{r_i}) - 1| \leq \frac{1}{2} \quad (5.18)$$

and

$$|d_{s,j+1}| \leq 3 \text{ for } |\psi_{s,j+1} - 1| \leq \frac{1}{2} \quad (5.19)$$

for $s = 1, 2$, $i = 1, 2, \dots, k$, and $j = 0, 1, \dots, k$.

Under the assumptions (A1) and (A2) and the null hypothesis, we obtain asymptotic expressions for A_{1N} , A_{2N} and A_{3N} in (5.14) and hence for $\log L_N$. Next we show that $\log L_N$ is asymptotically normal under the null hypothesis and, by a result due to LeCam, that the alternatives are contiguous. We consider sequentially the one and the two-parameter alternatives; the techniques used with the two-parameter case parallel those with the one-parameter case.

5.2 One-Parameter Families of Alternatives.

The contiguity of the alternatives will be shown through a sequence of lemmas. Many of these results are similar to those obtained by Johnson (1974) and Mehrotra and Johnson (1976) when adapted to the multiple censored sample context.

We use the notation $\dot{F}(w; \bar{\theta})$ and $\dot{f}(w; \bar{\theta})$ for the derivative of $F(w; \theta)$ and $f(w; \theta)$ with respect to θ at $\bar{\theta}$. Moreover, we define

$$\dot{\psi}_{j+1} = \dot{\psi}_{j+1}(w) = \frac{\dot{F}(w_{r_{j+1}}; \bar{\theta}) - \dot{F}(w_{r_j}; \bar{\theta})}{F(w_{r_{j+1}}; \bar{\theta}) - F(w_{r_j}; \bar{\theta})} \quad (5.20)$$

for $j=0, 1, \dots, k$. We note that (5.20) does not depend explicitly on θ_s , $s=1, 2$.

We first prove some results for general one-parameter families of densities which satisfy the regularity conditions (A1) and (A2). These results will then be used to study the asymptotic properties of $\log L_N$ for location and scale families of densities.

Lemma 5.1 Under assumptions (A1) and (A2)

$$N^{\frac{1}{2}}[F(w; \theta_s) - F(w; \bar{\theta}) - N^{-\frac{1}{2}} h_s \dot{F}(w; \bar{\theta})] \rightarrow 0 \quad (5.21)$$

on bounded sets of w where

$$h_s \dot{F}(w; \bar{\theta}) = \int_{-\infty}^w 2h_s \dot{\phi}_s f(x; \bar{\theta}) dx \quad (5.22)$$

for $s=1, 2$, and $N^{\frac{1}{2}} h_s = \theta_s - \bar{\theta}$.

Proof: From the definition of $\phi_s(w)$ we have

$$\int_{-\infty}^w (\phi_s^2(x) - 1) f(x; \bar{\theta}) dx = F(w; \theta_s) - F(w; \bar{\theta}).$$

Then, rewriting the left hand side of (5.21) using (5.22) we obtain

$$\int_{-\infty}^w [N^{\frac{1}{2}}(\phi_s^2(x) - 1) - 2h_s \dot{\phi}_s(x)] f(x; \bar{\theta}) dx$$

which tends to 0 by (5.9). Moreover, (5.22) also holds from the definition of $\phi_s(w)$ since

$$\dot{\phi}_s(w) = \frac{\partial}{\partial \theta} \phi_s(w) \Big|_{\theta_s = \bar{\theta}} = \frac{1}{2} \dot{f}(w; \bar{\theta}) / f(w; \bar{\theta}).$$

Lemma 5.2 Under assumptions (A1) and (A2)

$$N^{\frac{1}{2}}(\psi_{s,j+1} - 1) \xrightarrow{P_{\bar{\theta}}} h_s \dot{\psi}_{j+1}(w) \quad (5.23)$$

on bounded sets of w , where $h_s = N^{\frac{1}{2}}(\theta_s - \bar{\theta})$, $\psi_{s,j+1}$ is defined by (5.11)

and $\dot{\psi}_{j+1}$ is defined by (5.20) with $s=1, 2$, $j=0, 1, \dots, k$.

Proof. Using the definition of $\psi_{s,j+1}$ in (5.11), we have, by Lemma 5.1,

$$N^{\frac{1}{2}}(\psi_{s,j+1}-1) = N^{\frac{1}{2}} \left[\frac{F(W_{r_{j+1}}; \theta_s) - F(W_{r_{j+1}}; \bar{\theta})}{F(W_{r_{j+1}}; \bar{\theta}) - F(W_{r_j}; \bar{\theta})} - \frac{F(W_{r_j}; \theta_s) - F(W_{r_j}; \bar{\theta})}{F(W_{r_{j+1}}; \bar{\theta}) - F(W_{r_j}; \bar{\theta})} \right]$$

$$\xrightarrow{P_{\bar{\theta}}} h_s \frac{F(W_{r_{j+1}}; \bar{\theta}) - F(W_{r_j}; \bar{\theta})}{F(W_{r_{j+1}}; \bar{\theta}) - F(W_{r_j}; \bar{\theta})}.$$

The expression (5.23) follows using the definition of $\dot{\psi}_{j+1}$ in (5.20).

Lemma 5.3. Under the assumptions (A1) and (A2)

$$A_{2N} \xrightarrow{P_{\bar{\theta}}} \frac{1}{2}(h_2 - h_1)^2 \lambda(1-\lambda) \sum_{j=0}^k (\lambda_{j+1} - \lambda_j) \dot{\psi}_{j+1}^2 = \frac{1}{2} A_{\bar{\theta}} \quad (5.24)$$

where $h_s = N^{\frac{1}{2}}(\theta_s - \bar{\theta})$, $s=1,2$ and A_{2N} is defined by (5.16).

Proof. From (5.8)

$$(\phi_s(W_{r_i}) - 1)^2 \rightarrow N^{-1} [h_s \dot{\psi}(W_{r_i}; \bar{\theta})]^2 \xrightarrow{P_{\bar{\theta}}} 0 \quad (5.25)$$

since $h_2 \dot{\psi}(W_{r_i}; \bar{\theta})$ is bounded on bounded sets of W . The first sum of

A_{2N} in (5.16), being a finite sum of terms of the type (5.25), converges

to 0 in $P_{\bar{\theta}}$ probability. It then suffices to show the second sum

of A_{2N} converges to (5.24).

Using $E(M_{j+1}) = (\lambda_{j+1} - \lambda_j)m$, $E(r_{j+1} - r_j - 1) = (\lambda_{j+1} - \lambda_j)N$ and $m/N \rightarrow \lambda$, together with (5.8) and Lemma 5.2, we can write

$$M_{j+1}(\psi_{1,j+1}^{-1})^2 = \frac{M_{j+1}}{r_{j+1} - r_j - 1} \left(\frac{r_{j+1} - r_j - 1}{N} \right) N(\psi_{1,j+1}^{-1})^2$$

$$\xrightarrow{P_{\bar{\theta}}} \lambda h_1^2 (\lambda_{j+1} - \lambda_j) \dot{\psi}_{j+1}^2(w), \quad (5.26)$$

and similarly, using $E(N_{j+1}) = (\lambda_{j+1} - \lambda)n$,

$$N_{j+1}(\psi_{2,j+1}^{-1})^2 \xrightarrow{P_{\bar{\theta}}} (1-\lambda) h_2^2 (\lambda_{j+1} - \lambda_j) \dot{\psi}_{j+1}^2(w). \quad (5.27)$$

Moreover, from $N\bar{\theta} = m\theta_1 + n\theta_2$ we have $\bar{\theta} \rightarrow \lambda\theta_1 + (1-\lambda)\theta_2$, so that,

$$h_1 = N^{\frac{1}{2}}(\theta_1 - \bar{\theta}) \rightarrow -N^{\frac{1}{2}}(1-\lambda)(\theta_2 - \theta_1) = -(1-\lambda)(h_2 - h_1), \quad h_2 \rightarrow \lambda(h_2 - h_1),$$

and hence, $\lambda h_1^2 + (1-\lambda)h_2^2 \rightarrow \lambda(1-\lambda)(h_2 - h_1)^2$. Then, from (5.26) and (5.27)

we have

$$M_{j+1}(\psi_{1,j+1}^{-1})^2 + N_{j+1}(\psi_{2,j+1}^{-1})^2 \xrightarrow{P_{\bar{\theta}}} \lambda(1-\lambda)(h_2 - h_1)^2 (\lambda_{j+1} - \lambda_j) \dot{\psi}_{j+1}^2(w).$$

Upon substitution of this result in the second sum of A_{2N} in (5.16) and

using the definition of $\dot{\psi}_{j+1}$ in (5.20) the lemma follows.

Lemma 5.4. Under assumptions (A1) and (A2)

$$A_{3N} \xrightarrow{P_{\bar{\theta}}} 0 \quad (5.28)$$

where A_{3N} is defined by (5.17).

Proof. From (5.7), $\max_{i,s} |\phi_s(W_{r_i}) - 1| \xrightarrow{P_{\bar{\theta}}} 0$. Moreover,

since $\dot{F}(W_{r_{j+1}}; \bar{\theta}) \xrightarrow{P_{\bar{\theta}}} \dot{F}(F^{-1}(\lambda_{j+1}))$, we can find an integer N_0 and a

constant K such that

$$P_{\bar{\theta}} \{ |\dot{\psi}_{j+1}| < K \} > 1 - \epsilon \quad \text{for } N \geq N_0.$$

Since $\theta_s - \bar{\theta} \rightarrow 0$, from Lemma 5.2, it follows that

$$\psi_{s,j+1}(W) - 1 \xrightarrow{P_{\bar{\theta}}} (\theta_s - \bar{\theta}) \dot{\psi}_{j+1} \xrightarrow{P_{\bar{\theta}}} 0.$$

Now, using the expressions, (5.16) - (5.19), we obtain

$$|A_{3N}| \leq 6 \max_{i,j,s} \{ |\phi_s(W_{r_i}) - 1|, |\psi_{s,j+1}(W) - 1| \} A_{2N}$$

which converges to 0 in $p_{\bar{\theta}}$ probability since A_{2N} is bounded by Lemma 5.3. This proves the lemma.

Lemma 5.5. Under the assumptions (A1) and (A2)

$$A_{1N} \xrightarrow{P_{\bar{\theta}}} (\theta_2 - \theta_1) S_{N\bar{\theta}}^* \quad (5.29)$$

where

$$S_{N\bar{\theta}}^* = \sum_{j=0}^k [(1-\lambda)M_{j+1} - \lambda N_{j+1}] \frac{\dot{F}(W_{r_{j+1}}; \bar{\theta}) - \dot{F}(W_{r_j}; \bar{\theta})}{F(W_{r_{j+1}}; \bar{\theta}) - F(W_{r_j}; \bar{\theta})}, \quad (5.30)$$

$m/N \rightarrow \lambda$, $n/N \rightarrow 1-\lambda$ and A_{1N} is defined by (5.15)

Proof. It suffices to consider the asymptotic value of the second term of A_{1N} in (5.15) since, by (5.7), the first sum converges to 0

in $p_{\bar{\theta}}$ probability. From the proof of Lemma 5.3 we have

$N^{-1/2}h_1 \rightarrow -(1-\lambda)(\theta_2 - \theta_1)$ and $N^{-1/2}h_2 \rightarrow \lambda(\theta_2 - \theta_1)$. Then, together with

Lemma 5.2 we obtain

$$M_{j+1}(\psi_{1,j+1}-1) \xrightarrow{p_{\bar{\theta}}} -M_{j+1}(1-\lambda)(\theta_2 - \theta_1) \dot{\psi}_{j+1}(W)$$

and

$$N_{j+1}(\psi_{2,j+1}-1) \xrightarrow{p_{\bar{\theta}}} N_{j+1}\lambda(\theta_2 - \theta_1) \dot{\psi}_{j+1}(W).$$

When these expressions are substituted in A_{1N} we obtain (5.29),

through the definition of $\dot{\psi}_{j+1}(W)$ as in (5.20).

Now, combining Lemmas 5.3, 5.4 and 5.5 we obtain an asymptotic expression for $\log L_N$.

Lemma 5.6. Under the assumptions (A1) and (A2)

$$\log L_N - (\theta_2 - \theta_1) S_{N\bar{\theta}}^* \xrightarrow{p_{\bar{\theta}}} -A_{\bar{\theta}} \quad (5.31)$$

where $S_{N\bar{\theta}}^*$ and $A_{\bar{\theta}}$ are defined in (5.30) and (5.24), respectively.

We now specialize the above results to specific one-parameter classes of alternatives; in each case we provide a second approximation to Lemma 5.3, and hence to Lemma 5.6. Then, we show the contiguity of each sequence of alternatives to its corresponding sequence of null hypotheses.

5.2.1 Location Alternatives.

With the location alternatives, the hypotheses H_{ON} and H_{AN} defined by (5.5) and (5.6) take the form

$$H_{ON} : F_X(w) = G_Y(w) = F(w - \bar{\theta}_N) \quad (5.32)$$

and

$$H_{AN} : F_X(w) = F(w - \theta_{1N}), G_Y(w) = F(w - \theta_{2N}) \quad (5.33)$$

where $\bar{\theta}_N = (m\theta_{1N} + n\theta_{2N})N^{-1}$ and $\theta_{sN} = d_{sN}N^{-1/2}$ with $d_{sN} \rightarrow$ constant for $s=1,2$. Moreover, we assume $d_{2N} - d_{1N} \rightarrow d > 0$. We note that $\theta_{sN} - \bar{\theta}_N$ can also be expressed as $\theta_{sN} - \bar{\theta}_N = h_{sN}N^{-1/2}$ where $h_{1N} = -\frac{n}{N}(d_{2N} - d_{1N}) \rightarrow -(1-\lambda)d$ and $h_{2N} = \frac{m}{N}(d_{2N} - d_{1N}) \rightarrow \lambda d$.

For the application, of the results of section 5.2, to the location case we note that $F(w; \theta) = F(w - \theta)$, $f(w; \theta) = f(w - \theta)$, $\dot{F}(w; \theta) = -f(w - \theta)$ and $\dot{\phi}_s(w) = -\frac{1}{2}f'(w - \theta)/f(w - \theta)$, $s=1,2$. We first obtain a second approximation for A_{2N} ; more specifically for ψ_{j+1} in (5.24).

Lemma 5.7 If f' is bounded in the neighborhood of $F^{-1}(\lambda_j)$, $j=1, \dots, k$ and if the assumptions (A1) and (A2) hold, then for the location case,

$$A_{2N} \xrightarrow{P_{\bar{\theta}}} \frac{1}{2} \sigma_{\bar{\theta}}^2 \quad (5.34)$$

where

$$\sigma_{\bar{\theta}}^2 = d^2 \lambda (1-\lambda) \sum_{j=0}^k \frac{[f_{j+1} - f_j]^2}{\lambda_{j+1} - \lambda_j} \quad (5.35)$$

with $f_j = f(F^{-1}(\lambda_j))$, $j=1, \dots, k$, and $f_0 = f_{k+1} = 0$; $d = \lim_{N \rightarrow \infty} (d_{2N} - d_{1N})$.

Proof. From the above notation, $h_2 - h_1 \rightarrow d$. In view of (5.24), it suffices to show that

$$\psi_{j+1}^2 \xrightarrow{P_{\bar{\theta}}} \left(\frac{f_{j+1} - f_j}{\lambda_{j+1} - \lambda_j} \right)^2 \quad (5.36)$$

Letting $F^{-1}(\lambda_j) = \xi_j$, $j=1, \dots, k$ and using Taylor's expansion of

$f(W_{r_j} - \bar{\theta})$ about ξ_j , we obtain

$$\begin{aligned} f(W_{r_j} - \bar{\theta}) &= f[\xi_j + (W_{r_j} - \bar{\theta} - \xi_j)] \\ &= f(\xi_j) + (W_{r_j} - \bar{\theta} - \xi_j) f'(W_{r_j}^*) \end{aligned} \quad (5.37)$$

where $W_{r_j}^*$ lies between $W_{r_j} - \bar{\theta}$ and ξ_j . From the assumptions on the

parameters θ_{sN} , $s=1, 2$, in the hypotheses H_{ON} and H_{AN} , we can express

$\bar{\theta}$ as $\bar{\theta} = \{(md_1 + nd_2)/N^{-1}\} N^{-1/2} = \delta_N N^{-1/2}$ with $\delta_N \rightarrow \text{constant}$. Since

$W_{r_j} \rightarrow \xi_j$, $\bar{\theta} = O(N^{-1/2})$ and f' is bounded for $W_{r_j}^*$ in the neighborhood of

$F^{-1}(\lambda_j)$, from (5.37) we have

$$f(W_{r_j} - \bar{\theta}) \xrightarrow{P_{\bar{\theta}}} f(\xi_j).$$

Similarly, $F(W_{r_j} - \bar{\theta}) \rightarrow F(\xi_j)$. When substituted in the expression (5.20)

for ψ_{j+1} , (5.36) follows, and hence, the lemma.

For the location alternatives case, the expression (5.30) becomes

$$S_{N\bar{\theta}}^* = \sum_{j=0}^k [(1-\lambda)M_{j+1} - \lambda N_{j+1}] \left\{ - \frac{f(W_{r_{j+1}} - \bar{\theta}) - f(W_{r_j} - \bar{\theta})}{F(W_{r_{j+1}} - \bar{\theta}) - F(W_{r_j} - \bar{\theta})} \right\}. \quad (5.38)$$

We recall from Chapter 2, that the LMPRT statistic in the location case is given by $T_{N1}^{(k)}$ in Theorem 2.4. Moreover, in the proof of Theorem 2.8 it was shown that $T_{N1}^{(k)}$ is asymptotically equivalent

$$\text{to } T_{N1}^{*(k)} = \sum_{j=0}^k N_{j+1} \bar{a}_{j+1} \text{ where } \bar{a}_{j+1} = \sum_i a_N(i, f) / (r_{j+1} - r_j - 1).$$

for $i = r_j + 1, \dots, r_{j+1} - 1$, $j = 0, 1, \dots, k$, with $a_N(i, f) = E\{-f'(W_i)/f(W_i)\}$.

Since $\sum_{i=1}^N a_N(i, f) = 0$, from Hájek and Šidák (1967, 66), the statistic

$$T_{N1}^{(k)} \text{ can also be expressed as } T_{N1}^{(k)} = - \sum_{i=1}^k \sum_{r_i} a_N(r_i, f) - \sum_{j=0}^k M_{j+1} \bar{a}_{j+1};$$

by the same argument as in Theorem 2.8, $T_{N1}^{(k)}$ is also asymptotically

equivalent to $T_{N1}^{*(k)} = - \sum_{j=0}^k M_{j+1} \bar{a}_{j+1}$. Hence, $T_{N1}^{*(k)}$ is asymptotically

equivalent to $T_{N1}^{*(k)} = \lambda T_{N1}^{*(k)} + (1-\lambda) T_{N1}^{*(k)}$, that is,

$$T_{N1}^{*(k)} = - \sum_{j=0}^k [(1-\lambda) M_{j+1} - \lambda N_{j+1}] \bar{a}_{j+1}. \quad (5.39)$$

The statistic $T_{N1}^{*(k)}$ in (5.39) is the same as $S_{N\bar{\theta}}^*$ in (5.38) except

that $\bar{a}_{j+1}$ is replaced by

$$\psi_{j+1}(W) = - \frac{f(W_{r_{j+1}} - \bar{\theta}) - f(W_{r_j} - \bar{\theta})}{F(W_{r_{j+1}} - \bar{\theta}) - F(W_{r_j} - \bar{\theta})}. \quad (5.40)$$

It follows that, under H_{ON} ,

$$E(S_{N\bar{\theta}}^*) = E(T_{N1}^{*(k)}) = 0. \quad (5.41)$$

To show the asymptotic normality of $\log L_N$, we first show the asymptotic equivalence of $S_{N\bar{\theta}}^*$ and $T_{N1}^{*(k)}$ which is obtained by the following theorem, in view of (5.42). Although $\log L_N$ depends on θ , we omit this reference, whenever feasible, without ambiguity.

Lemma 5.8. Under the assumptions (A1) and (A2) for the location family of densities,

$$\frac{1}{N} \text{Var}(S_{N\bar{\theta}}^* - T_{N1}^{*(k)}) \xrightarrow{P_{\bar{\theta}}} 0. \quad (5.42)$$

Proof. From (5.41),

$$\begin{aligned} \text{Var}(S_{N\bar{\theta}}^* - T_{N1}^{*(k)}) &= E(S_{N\bar{\theta}}^* - T_{N1}^{*(k)})^2 \\ &= E_W[E_Z(S_{N\bar{\theta}}^* - T_{N1}^{*(k)})^2 | W=w]. \end{aligned} \quad (5.43)$$

From (5.38) and (5.39), using $M_{j+1} = (r_{j+1} - r_j - 1) - N_{j+1}$, we have

$$S_{N\bar{\theta}}^* - T_{N1}^{*(k)} = -(1-\lambda) \sum_{j=0}^k (r_{j+1} - r_j - 1) (\dot{\psi}_{j+1}(W) - \bar{a}_{j+1}) + \sum_{j=0}^k N_{j+1} (\dot{\psi}_{j+1}(W) - \bar{a}_{j+1}). \quad (5.44)$$

Then, for fixed W , we obtain

$$\begin{aligned} E_Z(S_{N\bar{\theta}}^* - T_{N1}^{*(k)})^2 &\leq \sum_{j=0}^k E_Z[N_{j+1} (\dot{\psi}_{j+1}(W) - \bar{a}_{j+1})]^2 \\ &= \frac{mn}{N(N-1)} \sum_{j=0}^k (r_{j+1} - r_j - 1) (\dot{\psi}_{j+1}(W) - \bar{a}_{j+1})^2, \end{aligned} \quad (5.45)$$

referring to Theorem 2.6 for the computation of the second moments with respect to Z under the null hypothesis. From Theorem 2.1, in our notation,

$$(r_{j+1} - r_j - 1) E_W(\dot{\psi}_{j+1}(W) - \bar{a}_{j+1})^2 \leq \sum_{i=r_j+1}^{r_{j+1}-1} E_W \left[-\frac{f'(W_i - \bar{\theta})}{f(W_i - \bar{\theta})} - a_N(i, f) \right]^2. \quad (5.46)$$

Moreover, by Theorem 4 of Hájek and Šidák (1967, 157)

$$\lim_{N \rightarrow \infty} E_W \left[-\frac{f'(W_1 - \bar{\theta})}{f(W_1 - \bar{\theta})} - a_N(1, f) \right]^2 = 0. \quad (5.47)$$

Using the expressions (5.55) to (5.47),

$$N^{-1} \text{Var}(S_{N\bar{\theta}}^* - T_{N1}^{*(k)}) \leq N^{-1} [mn/N(N-1)] \sum_{i=1}^N E_W \left[- \frac{f'(W_i - \bar{\theta})}{f(W_i - \bar{\theta})} - a_N(i, f) \right]^2 p_{\bar{\theta}} \rightarrow 0,$$

where we observe that the summation has N terms while (5.45), after substitution of (5.46), has $N-k$ of them. The lemma follows.

Summarizing Lemmas 5.1-5.8, we obtain the following theorem.

Theorem 5.1. If f' is bounded in the neighborhood of $F^{-1}(\lambda_j)$,

$j=1, \dots, k$, and if the assumptions (A1) and (A2) hold for the location parameter family of densities, then

$$\log L_N - (\theta_2 - \theta_1) T_{N1}^{*(k)} \xrightarrow{p_{\bar{\theta}}} -\frac{1}{2} \sigma_{\bar{\theta}}^2 \quad (5.48)$$

where $T_{N1}^{*(k)} = \sum_{j=0}^k N_{j+1} \bar{a}_{j+1}$, $E(T_{N1}^{*(k)})=0$ and $\sigma_{\bar{\theta}}^2$ is defined by (5.35).

In the next theorem, we show the asymptotic normality of $\log L_N$ and, consequently, the contiguity of the sequences of hypotheses H_{AN} and H_{ON} defined by (5.32) and (5.33), respectively.

Theorem 5.2. If $N^{1/2}(\theta_2 - \theta_1) = d_2 - d_1 \rightarrow d$ and the assumptions (A1)

and (A2) hold, then under H_{ON}

$$\log L_N \sim N\left(-\frac{1}{2} \sigma_{\bar{\theta}}^2, \sigma_{\bar{\theta}}^2\right), \quad (5.49)$$

and hence the alternatives H_{AN} are contiguous to H_{ON} .

Proof. From Theorem 2.8, under the null hypothesis,

$$N^{-1} \text{Var}(T_{N1}^{*(k)}) \xrightarrow{P_{\bar{\theta}}} \lambda(1-\lambda) \sum_{j=0}^k \frac{[f_{j+1} - f_j]^2}{\lambda_{j+1} - \lambda_j} = d^{-2} \sigma_{\bar{\theta}}^2. \quad (5.50)$$

Again from Theorem 2.8, $T_{N1}^{*(k)}$, and hence $\log L_N$ is asymptotically

normal under H_{ON} . Since $E(T_{N1}^{*(k)}) = 0$, from (5.48), $E(\log L_N) = -\frac{1}{2} \sigma_{\bar{\theta}}^2$.

From (5.50) and the assumption $N^{\frac{1}{2}}(\theta_2 - \theta_1) \rightarrow d$,

$$\text{Var}(\log L_N) \rightarrow N(\theta_2 - \theta_1)^2 N^{-1} \text{Var}(T_{N1}^{*(k)}) \rightarrow \sigma_{\bar{\theta}}^2,$$

which shows (5.49).

The contiguity of the alternatives follows by a direct application of LeCam's first Lemma as stated in Chapter 1, or in Hájek and Šidák (1967, 204).

5.2.2 Scale Alternatives

For the scale alternatives, (5.5) and (5.6) read

$$H_{ON} : F_X(w) = G_Y(w) = F(e^{-\tau_N w}) \quad (5.51)$$

and

$$H_{AN} : F_X(w) = F(e^{-\tau_1 N w}), \quad G_Y(w) = F(e^{-\tau_2 N w}) \quad (5.52)$$

where $\bar{\tau}_N = (m\tau_{1N} + n\tau_{2N})N^{-1}$ and $\tau_{sN} = d_{sN}N^{-\frac{1}{2}}$ with $d_{sN} \rightarrow \text{constant}$ for $s=1,2$.

We also assume that $d_{2N} - d_{1N} \rightarrow d > 0$. We can write $\tau_{sN} - \bar{\tau}_N = h_{sN}N^{-\frac{1}{2}}$ where

$h_{1N} \rightarrow -(1-\lambda)d$ and $h_{2N} \rightarrow \lambda d$. To translate the results of Section 5.2 to the scale case we use $F(w;\tau) = F(e^{-\tau}w)$, $f(w;\tau) = e^{-\tau}f(e^{-\tau}w)$, $\dot{F}(w;\tau) = -e^{-\tau}wf(e^{-\tau}w)$ and $\dot{\phi}_s(w) = \frac{1}{2}\{-1 - e^{-\tau}w f'(e^{-\tau}w)/f(e^{-\tau}w)\}$, $s=1,2$.

For an approximation of A_{2N} we could substitute the above expressions in (5.24). Rather, we obtain a more practical approximation of A_{2N} in the next lemma.

Lemma 5.9. If f' is bounded in the neighborhood of $F^{-1}(\lambda_j)$, $j=1,2,\dots,k$, and if the assumptions (A1) and (A2) hold, then for the scale case

$$A_{2N} \xrightarrow{P_{\tau}} \frac{1}{2} \sigma_{\tau}^2 \quad (5.53)$$

where

$$\sigma_{\tau}^2 = d^2 \lambda(1-\lambda) \sum_{j=0}^k \frac{[u_{j+1}f_{j+1} - u_j f_j]^2}{\lambda_{j+1} - \lambda_j} \quad (5.54)$$

with $f_j = f(F^{-1}(\lambda_j))$ and $F^{-1}(\lambda_j) = u_j$, $j=1,\dots,k$, $f_0 = f_{k+1} = 0$, $u_0 = -\infty$,

$u_{k+1} = +\infty$; $d = \lim_{N \rightarrow \infty} (d_{2N} - d_{1N})$.

Proof. With the scale family of distributions, the expression (5.20) reads

$$\dot{\psi}_{j+1}(w) = - \frac{e^{-\tau}w_{r_{j+1}} f(e^{-\tau}w_{r_{j+1}}) - e^{-\tau}w_{r_j} f(e^{-\tau}w_{r_j})}{F(e^{-\tau}w_{r_{j+1}}) - F(e^{-\tau}w_{r_j})} \quad (5.55)$$

Since $h_2 - h_1 \rightarrow d$, in view of Lemma 5.3, it suffices to show that

$$\psi_{j+1}^2 \xrightarrow{p_{\tau}^-} \left(\frac{u_{j+1} f_{j+1} - u_j f_j}{\lambda_{j+1} - \lambda_j} \right)^2. \quad (5.56)$$

As in Lemma 5.7, from the assumptions on the parameters τ_{sN} , $s=1,2$ in H_{ON} and H_{AN} we have $\bar{\tau} = \delta_N N^{-1/2}$ with $\delta_N \rightarrow \text{constant}$. Therefore, we can write

$$\exp(-\bar{\tau}) = \exp(-\delta_N N^{-1/2}) = 1 - \delta_N N^{-1/2} + O\left(\frac{1}{N}\right). \quad (5.57)$$

Letting $F^{-1}(\lambda_j) = \xi_j$, $j=1, \dots, k$ and using Taylor's expansion of

$f(e^{-\bar{\tau}} W_{r_j})$ about ξ_j , we obtain

$$\begin{aligned} f(e^{-\bar{\tau}} W_{r_j}) &= f[\xi_j + (e^{-\bar{\tau}} W_{r_j} - \xi_j)] \\ &= f(\xi_j) + (e^{-\bar{\tau}} W_{r_j} - \xi_j) f'(W_{r_j}^{**}) \end{aligned} \quad (5.58)$$

where $W_{r_j}^{**}$ lies between $e^{-\bar{\tau}} W_{r_j}$ and ξ_j . Now, from (5.57),

$$\begin{aligned} e^{-\bar{\tau}} W_{r_j} - \xi_j &= e^{-\bar{\tau}} W_{r_j} - W_{r_j} + (W_{r_j} - \xi_j) \\ &= W_{r_j} \{-\delta_N N^{-1/2} + O\left(\frac{1}{N}\right)\} + (W_{r_j} - \xi_j) \xrightarrow{p_{\tau}^-} 0 \end{aligned} \quad (5.59)$$

since $W_{r_j} \rightarrow \xi_j = F^{-1}(\lambda_j)$ for $0 < \lambda_j < 1$, and hence, is bounded. Since f'

is bounded for $W_{r_j}^{**}$ in the neighborhood of $F^{-1}(\lambda_j)$, from (5.58) and (5.59)

$f(e^{-\bar{\tau}W_{r_j}}) \xrightarrow{p_{\bar{\tau}}} f(\xi_j)$; similarly, $F(e^{-\bar{\tau}W_{r_j}}) \rightarrow F(\xi_j)$; moreover,

$e^{-\bar{\tau}W_{r_j}} = e^{-\bar{\tau}W_{r_j} - \xi_j + \xi_j} \xrightarrow{p_{\bar{\tau}}} \xi_j$. We substitute these expressions in

(5.55) and the resulting (5.55) into (5.24) of Lemma 5.3 to complete the proof of our lemma.

For the scale case, the expression (5.30) reads

$$S_{N\tau}^* = - \sum_{j=0}^k [(1-\lambda)M_{j+1} - \lambda N_{j+1}] \left\{ - \frac{e^{-\bar{\tau}W_{r_{j+1}}} f(e^{-\bar{\tau}W_{r_{j+1}}}) - e^{-\bar{\tau}W_{r_j}} f(e^{-\bar{\tau}W_{r_j}})}{F(e^{-\bar{\tau}W_{r_{j+1}}}) - F(e^{-\bar{\tau}W_{r_j}})} \right\}. \quad (5.60)$$

We recall from Chapter 2, that the LMPRT statistic in the scale alternatives case is given by $T_{N2}^{(k)}$ in Theorem 2.5. By the same argument as that used for the location case, in section 5.2.1, we obtain that $T_{N2}^{(k)}$ and $T_{N2}^{*(k)}$, defined by:

$$T_{N2}^{*(k)} = - \sum_{j=0}^k [(1-\lambda)M_{j+1} - \lambda N_{j+1}] \bar{b}_{j+1}, \quad (5.61)$$

are asymptotically equivalent where the weight $b_N(i, f)$ replaces

$a_N(i, f)$ used in the location case; $b_N(i, f) = E\{-1-W_i f'(W_i)/f(W_i)\}$.

Since $T_{N2}^{*(k)}$ in (5.61) is of the same form as $S_{N\tau}^*$ in (5.60) we

obtain, under H_{ON} ,

$$E(S_{N\tau}^*) = E(T_{N2}^{*(k)}) = 0. \quad (5.62)$$

The asymptotic equivalence of $S_{N\tau}^*$ and $T_{N2}^{*(k)}$ is established by the following lemma which parallels Lemma 5.8.

Lemma 5.10. Under the assumptions (A1) and (A2) for the scale family of densities,

$$\frac{1}{N} \text{Var}(S_{N\tau}^* - T_{N2}^{*(k)}) \xrightarrow{P_{\tau}^-} 0. \quad (5.63)$$

Proof. The argument is the same as for Lemma 5.8 where ψ_{j+1} is defined by (5.55), $b_N(i, f)$ and $\bar{b}_{j+1}$ are used instead of $a_N(i, f)$ and $\bar{a}_{j+1}$ and where Theorem 2.2 is used instead of Theorem 2.1.

Combining Lemmas 5.1-5.6, 5.9 and 5.10 we obtain the following result.

Theorem 5.3. If f' is bounded in the neighborhood of $F^{-1}(\lambda_j)$, $j=1, \dots, k$, and if the assumptions (A1) and (A2) hold for the scale parameter family of densities, then

$$\log L_N - (\tau_2 - \tau_1) T_{N2}^{*(k)} \xrightarrow{P_{\tau}^-} -\frac{1}{2} \sigma_{\tau}^2 \quad (5.64)$$

where $T_{N2}^{*(k)} = \sum_{j=0}^k N_{j+1} \bar{b}_{j+1}$, $E(T_{N2}^{*(k)}) = 0$, and σ_{τ}^2 is defined by (5.54).

The asymptotic normality of $\log L_N$ and the contiguity of the sequences of hypotheses H_{AN} and H_{ON} defined by (5.51) and (5.52) are shown in the following theorem.

Theorem 5.4: If $N^{1/2}(\tau_2 - \tau_1) = d_2 - d_1 \rightarrow d$ and the assumptions (A1) and (A2) hold, then under H_{ON} ,

$$\log L_N \sim N\left(-\frac{1}{2}\sigma_{\tau}^2, \sigma_{\tau}^2\right), \quad (5.65)$$

and hence, the alternatives H_{AN} are contiguous to H_{ON} .

Proof. From Theorem 2.9, under the null hypothesis,

$$N^{-1} \text{Var}(T_{N2}^{*(k)}) \xrightarrow{P_{\tau}} \lambda(1-\lambda) \sum_{j=0}^k \frac{[u_{j+1}^{f_{j+1}} - u_j^{f_j}]^2}{\lambda_{j+1} - \lambda_j} = d^{-2} \sigma_{\tau}^2. \quad (5.66)$$

For the remaining part of the proof, the argument is the same as for Theorem 5.2 where Theorem 2.9 is used instead of Theorem 2.8; the proof is thus omitted.

5.2.3 Lehmann Alternatives

In the case of Lehmann alternatives the hypotheses (5.5) and (5.6) take the form

$$H_{ON} : F_X(w) = G_Y(w) = 1 - (1 - F(w))^{1+\bar{\Delta}_N} \quad (5.66)$$

$$H_{AN} : F_X(w) = 1 - (1 - F(w))^{1+\Delta_{1N}}, \quad G_Y(w) = 1 - (1 - F(w))^{1+\Delta_{2N}} \quad (5.67)$$

where $\bar{\Delta}_N = (m\Delta_{1N} + n\Delta_{2N})N^{-1}$ and $\Delta_{sN} = d_{sN}N^{-1/2}$ with $d_{sN} \rightarrow$ constant for $s=1,2$.

We shall limit our considerations to the family of exponential distributions given by $F(w) = 1 - \exp(-e^{w-\delta})$ with densities $f(w) = \exp(w-\delta)e^{-e^{w-\delta}}$, $-\infty < w < \infty$. By direct algebraic computation, we have $G(w) = 1 - (1 - F(w))^{1+\Delta} = F(w-\delta + \ln(1+\Delta))$ and $g(w) = (1+\Delta)(1-F(w))^\Delta f(w) = f(w-\delta + \ln(1+\Delta))$. Then, the hypotheses H_{ON} and H_{AN} of Lehmann type can be considered as a subclass of the location hypotheses for the family of exponential distributions.

Like in the location case, assume that, $\delta_{sN} - \ln(1+\Delta_{sN}) = d_{sN} N^{-1/2}$, $s=1,2$, with $d_{sN} \rightarrow$ non-zero constant and also that $d_{2N} - d_{1N} \rightarrow d > 0$. The above assumption is compatible with those made for the location and the scale case since, for instance, $\delta_{sN} = a_{sN} N^{-1/2}$ and $\Delta_{sN} = b_{sN} N^{-1/2}$ would give $\delta_{sN} - \ln(1+\Delta_{sN}) = a_{sN} N^{-1/2} - b_{sN} N^{-1/2} + O(\frac{1}{N})$ where we would require that $a_{sN} - b_{sN} \rightarrow$ non-zero constant.

Recall from Chapter 3 that the LMPRT statistic in the Lehmann alternatives case is given by $S_N^{(k)}$ in (3.4). In the proof of Theorem 3.3 we also show that $S_N^{(k)}$ is asymptotically equivalent to

$$S_N^{*(k)} = \sum_{j=0}^k N_{j+1} (1 - \bar{c}_{j+1}) \quad (5.68)$$

where $c_i = \sum_{j=1}^i (N-j-1)^{-1}$ and $\bar{c}_{j+1} = \sum_{i=1}^{r_{j+1}-r_j-1} c_i / (r_{j+1}-r_j-1)$, $i=r_j+1, \dots, r_{j+1}-1$, $j=0, 1, \dots, k$.

We can restate Theorems 5.1 and 5.2 in the present special case of the location alternatives.

Theorem 5.5. For the exponential family of densities

$f(w) = \exp(w\delta - e^{w\delta})$, $\delta = O(N^{-1/2})$, the assumptions (A1) and (A2) hold and

$$\log L_N - (\Delta_2 - \Delta_1) S_N^{*(k)} \xrightarrow{P_{\Delta}} -\frac{1}{2} \sigma_{\Delta}^2 \quad (5.69)$$

where $S_N^{*(k)}$ is defined by (5.68) and

$$\sigma_{\Delta}^2 = d^2 \lambda(1-\lambda) \sum_{j=0}^k \frac{[(1-\lambda_{j+1}) \ln(1-\lambda_{j+1}) - (1-\lambda_j) \ln(1-\lambda_j)]^2}{\lambda_{j+1} - \lambda_j} \quad (5.70)$$

with $N^{1/2}(\Delta_2 - \Delta_1) = d_2 - d_1 \rightarrow d$.

Proof. The proof is the same as that of Theorem 5.1 where we note

that $\sum_{i=1}^N (1-c_i) = 0$. Since the weights of $S_N^{(k)}$ are density function independent, the limiting weights in (5.70), obtained directly from Lemma 3.2, are also density function independent.

Theorem 5.6. If $N^{1/2}(\Delta_2 - \Delta_1) = d_2 - d_1 \rightarrow d$ and the same assumptions as in Theorem 5.5 hold, then under the null hypothesis,

$$\log L_N \sim N(-\frac{1}{2} \sigma_{\Delta}^2, \sigma_{\Delta}^2) \quad (5.71)$$

and hence, the alternatives H_{AN} in (5.67) are contiguous to H_{ON} in (5.66) for the exponential family of distributions.

Proof. The proof, being the same as in Theorem 5.2, is omitted.

So far, under the relatively mild assumptions (A1) and (A2), we have shown the contiguity of one-parameter families of densities specializing to the location, scale and Lehmann alternatives. In the next section we shall extend the result to the two-parameter location and scale family of densities.

5.3 Location-Scale Alternatives.

Our objective is to extend the contiguity property, shown in section 5.2 for location or scale alternatives, to the simultaneous location and scale alternatives. With the appropriate identification of the key results, the arguments will parallel those used in section 5.2. Consequently, it will suffice to only highlight the arguments and interpret the results in the present context.

Here, the observations are drawn from independent samples with cumulative distribution functions belonging to the family $F(x; \theta, \tau) = F(e^{-\tau}x - \theta)$, with densities $f(x; \theta, \tau) = e^{-\tau}f(e^{-\tau}x - \theta)$ where θ and τ are defined in open intervals. For the densities $f(x; \theta, \tau)$, we retain the smoothness assumption (A1), but we use a modified version of (A2) which will be defined after the introduction of the following symbolism.

For each pair (θ, τ) , let

$$\phi = \phi(W; \theta, \tau, \bar{\theta}, \bar{\tau}) = e^{-\tau}f(e^{-\tau}W - \theta) / [e^{-\bar{\tau}}f(e^{-\bar{\tau}}W - \bar{\theta})] \quad (5.72)$$

and the quadratic mean derivatives with respect to θ and τ at $(\bar{\theta}, \bar{\tau})$ be denoted by $\dot{\phi}_L$ and $\dot{\phi}_S$ respectively, that is,

$$\dot{\phi}_L = \dot{\phi}_L(W) = -\frac{1}{2} f'(e^{-\bar{\tau}W-\bar{\theta}}) / f(e^{-\bar{\tau}W-\bar{\theta}}), \quad (5.73)$$

and

$$\dot{\phi}_S = \dot{\phi}_S(W) = -\frac{1}{2} [1 - e^{-\bar{\tau}W} f'(e^{-\bar{\tau}W-\bar{\theta}}) / f(e^{-\bar{\tau}W-\bar{\theta}})]. \quad (5.74)$$

The assumption (A2) is modified to read:

Assumption (A2)'. The Fisher's information $I(f, \bar{\theta}, \bar{\tau})$ satisfies

$$I(f, \bar{\theta}, \bar{\tau}) = \int_{-\infty}^{\infty} \dot{\phi}_{LS}^2 e^{-\bar{\tau}x} f(e^{-\bar{\tau}x-\bar{\theta}}) dx < \infty \quad (5.75)$$

where

$$\dot{\phi}_{LS} = K_1 \dot{\phi}_L + K_2 \dot{\phi}_S \quad (5.74)$$

for constants K_1 and K_2 . The condition (5.75) is similar to the definition (2.5) used by Lepage (1975, 673).

Now, consider the sequences of hypotheses

$$H_{ON} : F_X(w) = G_Y(w) = F(e^{-\bar{\tau}_N w - \bar{\theta}_N}) \quad (5.77)$$

and

$$H_{AN} : F_X(w) = F(e^{-\tau_{1N} w - \theta_{1N}}), \quad G_Y(w) = F(e^{-\tau_{2N} w - \theta_{2N}}) \quad (5.78)$$

where $\bar{\tau}_N = (m\tau_{1N} + n\tau_{2N})N^{-1}$, $\bar{\theta}_N = (m\theta_{1N} + n\theta_{2N})N^{-1}$, $\tau_{sN} = d_{\tau sN} N^{-1/2}$

and $\theta_{sN} = d_{\theta sN} N^{-1/2}$ with $d_{\tau sN} \rightarrow \text{constant}$ and $d_{\theta sN} \rightarrow \text{constant}$, $s=1,2$.

We also assume that $d_{\theta 2N} - d_{\theta 1N} \rightarrow d_{\theta} > 0$ and $d_{\tau 2N} - d_{\tau 1N} \rightarrow d_{\tau} > 0$.

As in sections 5.2.1 and 5.2.2, we write $\theta_{sN} - \bar{\theta}_N = h_{\theta sN} N^{-1/2}$ where

$h_{01N} \rightarrow -(1-\lambda)d_{\theta}$ and $h_{02N} \rightarrow \lambda d_{\theta}$, and likewise, $\tau_{sN}^{-\tau_N} = h_{\tau s N}^{N-1/2}$ where $h_{\tau 1N} \rightarrow -(1-\lambda)d_{\tau}$ and $h_{\tau 2N} \rightarrow \lambda d_{\tau}$. By considering the hypotheses H_{ON} and H_{AN} , defined by (5.32) and (5.33) for the location alternatives, and by (5.51) and (5.52) for the scale alternatives, we can observe how they are combined to give (5.77) and (5.78).

Let $p_{\bar{\theta}\bar{\tau}}$ denote the probability under H_{ON} . We note the properties (5.7) to (5.10) hold with ϕ defined by (5.72) and with $h_N \phi(W_i; \bar{\theta})$ replaced by $\dot{\phi}_{LS} = h_{\theta s} \dot{\phi}_L + h_{\tau s} \dot{\phi}_S$ defined through (5.73) and (5.74).

Upon substitution of (5.77) and (5.78) into (5.1), we obtain the likelihoods of Z^* under H_{ON} and H_{AN} which we denote by $p_N(w, z^*; \bar{\theta}, \bar{\tau})$ and $p_N(w, z^*; \theta_1, \theta_2, \tau_1, \tau_2)$, respectively. Using (5.72) we write

$$\phi_s(W) = \phi(W; \theta_s, \tau_s, \bar{\theta}, \bar{\tau}), \quad s=1,2; \quad (5.79)$$

we also define

$$\psi_{s,j+1} = \frac{F(e^{-\tau_s W_{r_{j+1}}} - \theta_s) - F(e^{-\tau_s W_{r_j}} - \theta_s)}{F(e^{-\bar{\tau} W_{r_{j+1}}} - \bar{\theta}) - F(e^{-\bar{\tau} W_{r_j}} - \bar{\theta})}, \quad s=1,2. \quad (5.80)$$

By the same procedure as in section 5.1, we obtain

$$\log L_N = A_{1N} - A_{2N} + A_{3N} \quad (5.81)$$

as in (5.14), with A_{1N} , A_{2N} and A_{3N} as defined by (5.15), (5.16)

and (5.17), respectively, but with ϕ_s and $\psi_{s,j+1}$ given by (5.79) and (5.80).

To simplify the expression of the following results we introduce the notation

$$L_{j+1}(W) = - \frac{f(e^{-\bar{\tau}W_{r_{j+1}}} - \bar{\theta}) - f(e^{-\bar{\tau}W_{r_j}} - \bar{\theta})}{F(e^{-\bar{\tau}W_{r_{j+1}}} - \bar{\theta}) - F(e^{-\bar{\tau}W_{r_j}} - \bar{\theta})} \quad (5.82)$$

and

$$S_{j+1}(W) = - \frac{e^{-\bar{\tau}W_{r_{j+1}}} f(e^{-\bar{\tau}W_{r_{j+1}}} - \bar{\theta}) - e^{-\bar{\tau}W_{r_j}} f(e^{-\bar{\tau}W_{r_j}} - \bar{\theta})}{F(e^{-\bar{\tau}W_{r_{j+1}}} - \bar{\theta}) - F(e^{-\bar{\tau}W_{r_j}} - \bar{\theta})} \quad (5.83)$$

The following results parallel the main results of section 5.2.

However, we should note they are stated here under the assumptions

(A1) and (A2)', the latter being defined by (5.75). The expression

(5.21) of Lemma 5.1 now reads

$$N^{\frac{1}{2}} [F(e^{-\bar{\tau}S_{W-\bar{\theta}}}) - F(e^{-\bar{\tau}W_{-\bar{\theta}}})] \xrightarrow{P_{\bar{\theta}\bar{\tau}}} -h_{\theta S} f(e^{-\bar{\tau}W_{-\bar{\theta}}}) - h_{\tau S} e^{-\bar{\tau}W} f(e^{-\bar{\tau}W_{-\bar{\theta}}}) \quad (5.84)$$

where

$$-h_{\theta S} f(e^{-\bar{\tau}W_{-\bar{\theta}}}) = \int_{-\infty}^W 2h_{\theta S} \phi_L e^{-\bar{\tau}x} f(e^{-\bar{\tau}x - \bar{\theta}}) dx$$

and

$$-h_{\tau S} e^{-\bar{\tau}W} f(e^{-\bar{\tau}W_{-\bar{\theta}}}) = \int_{-\infty}^W 2h_{\tau S} \phi_S e^{-\bar{\tau}x} f(e^{-\bar{\tau}x - \bar{\theta}}) dx$$

with $h_{\theta s} = N^{\frac{1}{2}}(\theta_s - \bar{\theta})$ and $h_{\tau s} = N^{\frac{1}{2}}(\tau_s - \bar{\tau})$, $s=1,2$. The expression (5.23)

of Lemma 5.2 becomes

$$N^{\frac{1}{2}}(\psi_{s,j+1} - 1) \xrightarrow{P_{\bar{\theta}\bar{\tau}}} -h_{\theta s} L_{j+1}(w) - h_{\tau s} S_{j+1}(w) \quad (5.85)$$

with $\psi_{s,j+1}$, $s=1,2$, defined by (5.80) and $L_{j+1}(w)$ and $S_{j+1}(w)$ by (5.82)

and (5.83). Likewise, analogous to (5.25), we have

$$N(\psi_{s,j+1} - 1)^2 \xrightarrow{P_{\bar{\theta}\bar{\tau}}} [h_{\theta s} L_{j+1}(w) + h_{\tau s} S_{j+1}(w)]^2 \quad (5.86)$$

Then using (5.84), (5.85), (5.86) and proceeding as in

Lemmas 5.3, 5.4 and 5.5 we obtain

$$A_{2N} \xrightarrow{P_{\bar{\theta}\bar{\tau}}} \frac{1}{2} A_{\bar{\theta}\bar{\tau}} \quad (5.87)$$

where

$$A_{\bar{\theta}\bar{\tau}} = \lambda(1-\lambda) \sum_{j=0}^k (\lambda_{j+1} - \lambda_j) [(h_{\theta 2} - h_{\theta 1}) L_{j+1}(w) + (h_{\tau 2} - h_{\tau 1}) S_{j+1}(w)]^2, \quad (5.88)$$

$$A_{3N} \xrightarrow{P_{\bar{\theta}\bar{\tau}}} 0 \quad (5.89)$$

and

$$A_{1N} \xrightarrow{P_{\bar{\theta}\bar{\tau}}} (\theta_2 - \theta_1) S_{N\bar{\theta}}^* + (\tau_2 - \tau_1) S_{N\bar{\tau}}^* = S_{N\bar{\theta}\bar{\tau}}^* \quad (5.90)$$

where L_{j+1} and S_{j+1} are defined by (5.82) and (5.83),

$$S_{N\theta}^* = - \sum_{j=0}^k [(1-\lambda)M_{j+1} - \lambda N_{j+1}] L_{j+1}(W), \quad (5.91)$$

$$S_{N\tau}^* = - \sum_{j=0}^k [(1-\lambda)M_{j+1} - \lambda N_{j+1}] S_{j+1}(W). \quad (5.92)$$

The expressions (5.91) and (5.92) correspond to (5.30) and (5.60) for location and scale alternatives, respectively. As in Lemma 5.6, from the above discussion we have

$$\log L_N - (\theta_2 - \theta_1) S_{N\theta}^* - (\tau_2 - \tau_1) S_{N\tau}^* \xrightarrow{P_{\theta\tau}} -\frac{1}{2} A_{\theta\tau} \quad (5.93)$$

Analogously to Lemmas 5.7 and 5.9, we obtain a second asymptotic approximation for A_{2N} by the following lemma.

Lemma 5.11. If f' is bounded in the neighborhood of $F^{-1}(\lambda_j)$, $j=1, \dots, k$ and if the assumptions (A1) and (A2)' hold, then for the location-scale case,

$$A_{2N} \xrightarrow{P_{\theta\tau}} \frac{1}{2} \sigma_{\theta\tau}^2 \quad (5.94)$$

where

$$\sigma_{\theta\tau}^2 = \lambda(1-\lambda) \sum_{j=0}^k \frac{[d_{\theta}(f_{j+1} - f_j) + d_{\tau}(u_{j+1} f_{j+1} - u_j f_j)]^2}{\lambda_{j+1} - \lambda_j} \quad (5.95)$$

with $f_j = f(F^{-1}(\lambda_j))$ and $F^{-1}(\lambda_j) = u_j$, $j=1, \dots, k$, $f_0 = f_{k+1} = 0$, $u_0 = -\infty$, $u_{k+1} = +\infty$;

$$d_{\theta} = \lim_{N \rightarrow \infty} (d_{\theta 2N} - d_{\theta 1N}) \text{ and } d_{\tau} = \lim_{N \rightarrow \infty} (d_{\tau 2N} - d_{\tau 1N}).$$

Proof. The proof is obtained by the same argument as that of Lemmas 5.7 and 5.9, using the results shown in the proof of these lemmas.

Now, consider the weights defined by

$$c_i = (\theta_2 - \theta_1)E(2\dot{\phi}_L) + (\tau_2 - \tau_1)E(2\dot{\phi}_S) \quad (5.96)$$

for $\dot{\phi}_L$ and $\dot{\phi}_S$ as in (5.73) and (5.74). Under multiple censoring, we obtain a rank order statistic of the same form as $T_{N1}^{(k)}$ or $T_{N2}^{(k)}$, obtained in Theorems 2.4 or 2.5, namely:

$$T_{N12}^{(k)} = \sum_{i=1}^k (1 - Z_{r_i}) c_{r_i} + \sum_{j=0}^k N_{j+1} \bar{c}_{j+1} \quad (5.97)$$

where $\bar{c}_{j+1} = \sum_i c_i / (r_{j+1} - r_j - 1)$, $i = r_j + 1, \dots, r_{j+1} - 1$, $j = 0, \dots, k$. Using

Lemmas 2.1 and 2.2 we can write

$$\bar{c}_{j+1} = (\theta_2 - \theta_1)E(L_{j+1}(W)) + (\tau_2 - \tau_1)E(S_{j+1}(W)). \quad (5.98)$$

Proceeding as in sections 5.2.1 and 5.2.2, re: (5.39) and (5.64), we obtain that $T_{N12}^{(k)}$ is asymptotically equivalent to

$$T_{N12}^{*(k)} = - \sum_{j=0}^k [(1 - \lambda)M_{j+1} - \lambda N_{j+1}] \bar{c}_{j+1} \quad (5.99)$$

with $\bar{c}_{j+1}$ given by (5.98).

We observe that $S_{N\theta\tau}^*$, defined in (5.90), can be written as

$$S_{N\bar{\theta}\tau}^* = - \sum_{j=0}^k \{ (1-\lambda)M_{j+1} - \lambda N_{j+1} \} \{ (\theta_2 - \theta_1)L_{j+1}(W) + (\tau_2 - \tau_1)S_{j+1}(W) \}, \quad (5.100)$$

using the expressions (5.91) and (5.92). The statistic $T_{N12}^{*(k)}$ is

then the same as $S_{N\bar{\theta}\tau}^*$, except that $\bar{c}_{j+1}$ is replaced by

$(\theta_2 - \theta_1)L_{j+1}(W) + (\tau_2 - \tau_1)S_{j+1}(W)$. Using (5.41) and (5.62), we obtain

$$E(T_{N12}^{*(k)}) = E(S_{N\bar{\theta}\tau}^*) = 0. \quad (5.101)$$

In view of (5.101), the following lemma establishes the asymptotic equivalence of $T_{N12}^{*(k)}$ and $S_{N\bar{\theta}\tau}^*$.

Lemma 5.12. Under the assumptions (A1) and (A2)', for the location-scale family of densities,

$$\frac{1}{N} \text{Var}(S_{N\bar{\theta}\tau}^* - T_{N12}^{*(k)}) \xrightarrow{P_{\bar{\theta}\tau}} 0. \quad (5.102)$$

Proof. By the same argument as in the proofs of Lemmas 5.8 and 5.10, from expressions analogous to (5.43) and (5.45), we can write

$$\begin{aligned} \text{Var}(S_{N\bar{\theta}\tau}^* - T_{N12}^{*(k)}) &\leq mn[N(N-1)]^{-1} \sum_{j=0}^k (r_{j+1} - r_j - 1) \\ &\quad \times \text{Var}_N \{ (\theta_2 - \theta_1)L_{j+1}(W) + (\tau_2 - \tau_1)S_{j+1}(W) \}. \end{aligned} \quad (5.103)$$

Now, since $\text{cov}_W(L_{j+1}, S_{j+1}) \leq \max(\text{Var}_W(L_{j+1}), \text{Var}_W(S_{j+1}))$,

we have

$$\begin{aligned} & \text{Var}_W \{ (\theta_2 - \theta_1) L_{j+1}(W) + (\tau_2 - \tau_1) S_{j+1}(W) \} \\ & \leq 3 \{ (\theta_2 - \theta_1)^2 \text{Var}_W(L_{j+1}(W)) + (\tau_2 - \tau_1)^2 \text{Var}_W(S_{j+1}(W)) \}. \end{aligned} \quad (5.104)$$

After the substitution of (5.104) in (5.103) we apply Lemmas 5.8 and 5.10, successively, thus proving the lemma.

Summarizing the above results we obtain the following theorem

Theorem 5.7 If f' is bounded in the neighborhood of $F^{-1}(\lambda_j)$, $j=1, \dots, k$, and if the assumptions (A1) and (A2)' hold for the location-scale family of densities, then

$$\log L_N - T_{N12}^{*(k)} \xrightarrow{P_{\theta\tau}} -\frac{1}{2} \sigma_{\theta\tau}^2 \quad (5.105)$$

where $T_{N12}^{*(k)}$ and $\sigma_{\theta\tau}^2$ are defined by (5.99) and (5.95), respectively;

$$E(T_{N12}^{*(k)}) = 0.$$

The asymptotic normality of $\log L_N$ and the contiguity of the hypotheses is established in the following theorem.

Theorem 5.8. If $N^{1/2}(\theta_2 - \theta_1) = h_{\theta 2} - h_{\theta 1} \rightarrow \text{constant}$, $N^{1/2}(\tau_2 - \tau_1) = h_{\tau 2} - h_{\tau 1} \rightarrow \text{constant}$ and the assumptions (A1) and (A2)' hold, then under H_{ON}

$$\log L_N \sim N\left(-\frac{1}{2} \sigma_{\theta\tau}^2, \sigma_{\theta\tau}^2\right), \quad (5.106)$$

and hence, the alternatives H_{AN} in (5.78) are contiguous to the null hypotheses H_{ON} in (5.77).

Proof. By the same argument as in Theorem 4.2 $T_{N12}^{*(k)}$ is asymptotically

normal. Following the same procedure as in the proof of Theorems 5.2 and 5.4, our theorem follows.

For the two-sample problem under multiple censoring of type II we have shown the contiguity of several sequences of alternative and null hypotheses. For the one-parameter families we have focussed our attention on location, scale and Lehmann alternatives while for the two-parameter families we have considered the simultaneous location and scale alternatives.

In each case the argument followed the same pattern. From an expansion of the log-likelihood ratio as a sum of three terms, an asymptotic expression of the former was obtained from the asymptotic expression of the three latter components. Under the null hypothesis, the asymptotic normality of the statistics used in Chapters 2, 3 and 4 yielded the asymptotic normality of the log-likelihood ratio. Then, the contiguity of the sequences of alternatives to the null hypotheses followed by an application of LeCam's first lemma.

CHAPTER 6

CONCLUDING REMARKS

Within the two-sample problem context and under multiple censoring, the optimality of some rank order tests are investigated. These optimum rank order tests for the null hypothesis of identity of distributions against one and two parameter families of alternatives are derived on the basis of the frequencies within the $k+1$ intervals specified by k arbitrary quantiles.

For one-parameter alternatives, the statistics yield locally most powerful rank order tests for the null hypothesis against location, scale and Lehmann alternatives. The two locally most powerful tests for the location and scale parameters are combined in a quadratic form to produce a test for the two-parameter location and scale alternatives which is asymptotically equivalent to the likelihood ratio test and hence, optimal in the Wald sense.

Under the null hypothesis, the joint density of the frequencies within the $k+1$ intervals specified by the k quantiles is multivariate hypergeometric and hence, asymptotically multivariate normal. Moreover the asymptotic distributions of the derived statistics are obtained; they are normal for the one-parameter case and chi-square with two degrees of freedom for the two-parameter case.

With relatively mild regularity conditions, which are satisfied by most of the density functions usually encountered with this type of problem, the sequences of alternatives are contiguous to the corresponding sequences of null hypotheses. Using LeCam's Lemma, the asymptotic distributions of the above statistics are also obtained under the alternatives; these are required for the study of asymptotic power properties. The tests resulting from the derived statistics are then compared with their complete sample counterparts using the Pitman efficiency.

The method could also be applied to the one-sample problem. For instance, the results of Groeneveld (1972) who proposed asymptotically optimal group rank tests for the location parameter of a symmetric distribution, could be extended to rank order tests with location and scale parameters. In much the same way, as it is done in the thesis within the two-sample problem context, the statistics proposed by Adichie (1967 a,b) for regression parameters, within the complete one-sample context, can be studied under multiple censoring. Likewise, tests proposed by Hájek and Šidák (1967) for the null hypothesis against regression in location and regression in scale can be extended in the multiple censored sample context. The method can also be applied to the general alternatives considered by Hájek and Šidák (1967, p. 70-71).

The rank order statistics were obtained on the basis of score-generating functions derived from specific density functions. The ARE was computed on the basis that the populations from which the samples were drawn have this same specific density as probability law.

The robustness of these statistics, that is, the performance of the tests when the samples are drawn from populations with various densities, is then of practical interest.

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