***T*-Stress Solutions of Cracks Emanating from Circular Holes**

by

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A thesis submitted to
The Faculty of Graduate Studies and Research
in partial fulfilment of
the degree requirements of
Master of Applied Science

Ottawa-Carleton Institute for
Mechanical and Aerospace Engineering

Department of Mechanical and Aerospace Engineering
Carleton University
Ottawa, Ontario, Canada
April, 2006

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ABSTRACT

Geometries with circular holes under loads are commonly found in engineering practice, e.g. rivet and bolt holes in aircraft structures, and they are often the site of crack initiation. In this thesis, the elastic T -stress and stress intensity factor solutions for geometries with cracks emanating from the edge of circular hole(s) are obtained. The T -stress, in conjunction with the stress intensity factor, provides a more accurate characterization of the stress field in the vicinity of a crack. Such a two-parameter approach is increasingly being employed in fracture assessments of cracked components. The two-dimensional boundary element method (BEM) is employed for the determination of the parameters. The BEM procedure is first validated with available T -stress and stress intensity factor solutions in the literature. Some mesh design guidelines for using the BEM program are established from this analysis. In the present study, a wide range of cracked geometries involving plates with single and multiple holes are analyzed. The T -stress and stress intensity factor solutions for single hole geometries are first presented. The influence of adjacent holes on a central hole when cracks have formed is then discussed. It is shown that the size and distance between these adjacent holes have a significant effect on the T -stress and stress intensity factor, particularly for relatively small cracks. The determination of T -stress and stress intensity factors using the weight function method is also carried out for these geometries. It is demonstrated that this technique can be used to provide a quick and accurate means of obtaining T -stress and stress intensity factor solutions for cracked geometries with circular hole(s) when under complex loading conditions.

ACKNOWLEDGEMENTS

I would like to acknowledge my supervisors, Dr. Choon Lai Tan and Dr. Xin Wang, for their tremendous patience and guidance, and for their continuous injection of confidence. Thanks are also due to Jian Li and Pranav Dhoj Shah for their expert and admirable assistance. I would also like express my appreciation to Nancy Powell, Marlene Groves and Christie Egbert at the office of the Department of Mechanical and Aerospace Engineering for their wonderful help with cumbersome, yet unavoidable, administrative-related issues.

I would like to thank my parents for their nurturing and exceptional support, both emotionally as well as financially. I would also express my appreciation to Jen Tran for her tireless understanding and never-ending encouragement. Finally, I would like to give a shout out to my colleagues at Carleton University, including Travis “The Tizzle” Mikjaniec, Phillipe “Philbert” Gnreux, Andrew Furlong, Marc-Andr Mller, and Patrick Boisvert.

TABLE OF CONTENTS

ABSTRACT	iii
ACKNOWLEDGEMENTS	iv
TABLE OF CONTENTS	v
LIST OF TABLES	ix
LIST OF FIGURES	xiii
NOMENCLATURE	xx
CHAPTER 1 INTRODUCTION	1
CHAPTER 2 TWO-PARAMETER LINEAR ELASTIC FRACTURE MECHANICS APPROACH AND THEIR NUMERICAL EVALUATION	5
2.0 Introduction	5
2.1 Linear Elastic Fracture Mechanics (LEFM)	6
2.2 <i>T</i> -Stress and Two-Parameter LEFM	7
2.3 Weight Functions	9
2.3.1 Weight Functions for Stress Intensity Factors	10
2.3.2 Weight Functions for <i>T</i> -Stress	11
2.4 The Boundary Element Method for <i>T</i> -Stress Determination	13
2.4.1 Review of the BEM	14
2.4.2 Contour Integral for Calculating <i>T</i> -Stress Using BEM	17

2.4.3	Self-Regularized Boundary Element Method	22
2.5	Summary	25
CHAPTER 3 BEM VALIDATION AND TEST PROBLEMS.....		31
3.0	Introduction	31
3.1	Numerical Modelling and Mesh Design Guidelines	32
3.1.1	Relative Quarter-Point Crack-Tip Element Size (l/a) and Relative Contour Radii (r/a)	33
3.2	Test Cases for BEM Validation.....	34
3.2.1	Circular Disk with Internal Crack (Problem A).....	34
3.2.2	Circular Disk with Single Edge-Crack (Problem B).....	35
3.2.3	Single Edge-Cracked Plate (Problem C).....	35
3.2.4	U-Notch Cracked Plate (Problem D)	36
3.2.5	Infinite Plate with Double-Cracks Emanating from the Edge of a Circular Hole (Problem E)	36
3.3	Conclusions	37
CHAPTER 4 ELASTIC <i>T</i>-STRESS AND STRESS INTENSITY FACTOR SOLUTIONS FOR CRACKS EMANATING FROM CIRCULAR HOLES IN A RECTANGULAR PLATE.....		60
4.0	Introduction	60
4.1	<i>T</i> -Stress and Stress Intensity Factor from Crack(s) Emanating from a Circular Hole	61
4.1.1	Finite Plate with Double-Cracks Emanating from the Edge of a Circular Hole under Remote Tension (Problem F).....	61
4.1.2	Finite Plate with Double-Cracks Emanating from the Edge of a Circular Hole under Remote Bending (Problem G).....	63

4.1.3	Infinite Plate with a Single Crack Emanating from the Edge of a Circular Hole under Remote Tension (Problem H)	64
4.1.4	Finite Plate with a Single Crack Emanating from the Edge of a Circular Hole under Remote Tension (Problem I)	65
4.1.5	Finite Plate with a Single Crack Emanating from the Edge of a Circular Hole under Remote Bending (Problem J)	66
4.2	Influence of Adjacent Holes on T -Stress and Stress Intensity Factor	67
4.2.1	Infinite Plate with Double-Cracks Emanating from a Periodic Array of Holes under Remote Tension (Problem K)	68
4.2.2	Infinite Plate with a Single Crack Emanating from a Periodic Array of Holes under Remote Tension (Problem L)	69
4.2.3	Infinite Plate with Double-Cracks Emanating from the Edge of a Circular Hole Influence by Adjacent Holes under Remote Tension (Problem M)	70
4.3	Summary	72

CHAPTER 5 WEIGHT FUNCTION METHOD FOR DETERMINING FRACTURE PARAMETERS.....123

5.0	Introduction	123
5.1	Formulation of the Weight Function Method.....	124
5.1.1	Weight Function for Stress Intensity Factor	124
5.1.2	Weight Function for T -Stress.....	126
5.2	Verification Problems.....	127
5.2.1	Infinite Plate with Double-Cracks Emanating from the Edge of a Circular Hole under Remote Tension (Problem E).....	128
5.2.2	Finite Plate with Double-Cracks Emanating from the Edge of a Circular Hole under Remote Tension (Problem F).....	130
5.2.3	Finite Plate with a Single Crack Emanating from the Edge of a Circular Hole under Remote Tension (Problem I).....	132

5.3	Summary	133
CHAPTER 6	CONCLUSIONS.....	148
REFERENCES		151

LIST OF TABLES

Table 3.1: List of problems analyzed in Chapter 3	39
Table 3.2: Comparison of normalized stress intensity factor and T -stress solutions obtained by BEM and by Fett (2002) for Problem A; $a/R = 0.2$	40
Table 3.3: Comparison of normalized stress intensity factor and T -stress solutions obtained by BEM and by Fett (2002) for Problem B; $a/R = 0.4$	40
Table 3.4: Comparison of normalized stress intensity factor and T -stress solutions obtained by BEM and by Fett (2002) for Problem C; $a/W = 0.6$	41
Table 3.5: Comparison of normalized stress intensity factor and T -stress solutions between BEM and Fett (2002) for different relative crack lengths (a/R) for Problem A	41
Table 3.6: Comparison of normalized stress intensity factor and T -stress solutions between BEM and Fett (2002) for different relative crack lengths (a/R) for Problem B	42
Table 3.7: Comparison of normalized stress intensity factor and T -stress solutions between BEM and Fett (2002) for different relative crack lengths (a/W) for Problem C	42
Table 3.8: Comparison of normalized T -stress between BEM and Lewis (2005) for different relative crack lengths (a/R) for Problem D	43
Table 3.9: Comparison of normalized stress intensity factor and T -stress between BEM and those from literature for different relative crack lengths (a/R) for Problem E	43
Table 4.1: List of problems analyzed in Chapter 4	74
Table 4.2: Variation of normalized stress intensity factors and T -stress with relative crack lengths for Problem F; $R/W = 0.25$	75
Table 4.3: Variation of normalized stress intensity factors and T -stress with relative crack lengths for Problem F; $R/W = 0.50$	75
Table 4.4: Effect of height-to-width ratio (H/W) on stress intensity factor and T -stress for Problem F; $R/W = 0.25$	76

Table 4.5: Effect of height-to-width ratio (H/W) on stress intensity factor and T -stress for Problem F; $R/W = 0.50$	76
Table 4.6: Variation of normalized stress intensity factors and T -stress with relative crack lengths for Problem G; $R/W = 0.25$	77
Table 4.7: Variation of normalized stress intensity factors and T -stress with relative crack lengths for Problem G; $R/W = 0.50$	77
Table 4.8: Variations of normalized stress intensity factors and T -stress with relative crack lengths for Problem H	78
Table 4.9: Variation of normalized stress intensity factors and T -stress with relative crack lengths for Problem I; $R/W = 0.25$	78
Table 4.10: Variation of normalized stress intensity factors and T -stress with relative crack lengths for Problem I; $R/W = 0.50$	79
Table 4.11: Comparison of normalized stress intensity factors and T -stress between single crack and double-cracks emanating from a circular hole in a finite plate under remote tension; $R/W = 0.25$	79
Table 4.12: Comparison of normalized stress intensity factors and T -stress between single crack and double-cracks emanating from a circular hole in a finite plate under remote tension; $R/W = 0.50$	80
Table 4.13: Comparison of normalized stress intensity factors and T -stress between single crack and double-cracks emanating from a circular hole in a finite plate under remote bending; $R/W = 0.25$	80
Table 4.14: Comparison of normalized stress intensity factors and T -stress between single crack and double-cracks emanating from a circular hole in a finite plate under remote bending; $R/W = 0.50$	81
Table 4.15: Variation of normalized stress intensity factors and T -stress with relative crack lengths for Problem K; $R/W = 0.25$	81
Table 4.16: Variation of normalized stress intensity factors and T -stress with relative crack lengths for Problem K; $R/W = 0.50$	82
Table 4.17: Variation of normalized stress intensity factors and T -stress with relative crack lengths for Problem L; $R/W = 0.25$	82
Table 4.18: Variation of normalized stress intensity factors and T -stress with relative crack lengths for Problem L; $R/W = 0.50$	83

Table 4.19: Variation of normalized stress intensity factors and T -stress with relative crack lengths and radius ratios $\alpha = R_2/R_1$ for Problem M; $d^* = d/R_1 = 2.5$	83
Table 4.20: Variation of normalized stress intensity factors and T -stress with relative crack lengths and radius ratios $\alpha = R_2/R_1$ for Problem M; $d^* = d/R_1 = 4.0$	84
Table 4.21: Variation of normalized stress intensity factors and T -stress with relative crack lengths and radius ratios $\alpha = R_2/R_1$ for Problem M; $d^* = d/R_1 = 5.0$	84
Table 4.22: Variation of normalized stress intensity factors and T -stress with relative crack lengths and radius ratios $\alpha = R_2/R_1$ for Problem M; $d^* = d/R_1 = 10.0$	85
Table 5.1: Variation of normalized stress intensity factor and T -stress with relative crack lengths for Problem E under uniform and linear crack face loading	134
Table 5.2: Coefficients for normalized stress intensity factor Y under uniform and linear crack face loading for Problem E	134
Table 5.3: Coefficients for normalized T -stress V under uniform and linear crack face loading for Problem E	134
Table 5.4: Comparison of stress intensity factor and T -stress solutions between solutions obtained using the weight function method and those available in literature for Problem E	135
Table 5.5: Variation of normalized stress intensity factor and T -stress with relative crack lengths for Problem F under uniform and linear crack face loading	135
Table 5.6: Coefficients for normalized stress intensity factor Y under uniform and linear crack face loading for Problem F	136
Table 5.7: Coefficients for normalized T -stress V under uniform and linear crack face loading for Problem F	136
Table 5.8: Normalized stress distribution obtained using BEM for Problem F	137
Table 5.9: Comparison of stress intensity factor and T -stress solutions between solutions obtained using the weight function method and those obtained by BEM for Problem F	138

Table 5.10: Variation of normalized stress intensity factor and T -stress with relative crack lengths for Problem I under uniform and linear crack face loading	138
Table 5.11: Coefficients for normalized stress intensity factor Y under uniform and linear crack face loading for Problem I	139
Table 5.12: Coefficients for normalized T -stress V under uniform and linear crack face loading for Problem I	139
Table 5.13: Comparison of stress intensity factor and T -stress solutions between solutions obtained using the weight function method and those obtained by BEM for Problem I	139

LIST OF FIGURES

Fig. 2.1: Modes of deformation for a cracked body	27
Fig. 2.2: Two-dimensional stresses in the vicinity of the crack tip	27
Fig. 2.3: Weight Function for one-dimensional cracks	28
Fig. 2.4: Linear elastic body R with boundary S in a two-dimensional co-ordinate system	28
Fig. 2.5: Discretized domain R , with boundary S	29
Fig. 2.6: A quadratic isoparametric line element	29
Fig. 2.7: Integration paths and co-ordinate definitions	30
Fig. 3.1: A general crack problem showing crack length a , quarter-point crack-tip element length l , and contour radius r	44
Fig. 3.2: Problem A - Circular disk with internal crack under uniform radial tension at the circumference	44
Fig. 3.3: Problem B - Circular disk with single edge crack under uniform radial tension at the circumference	45
Fig. 3.4: Problem C - Single edge crack plate under remote bending	45
Fig. 3.5: A quarter of the Problem A domain being modelled with displacement constraints on the planes of symmetry (dotted lines)	46
Fig. 3.6: A typical BEM mesh showing element size and contour for Problem A; $a/R = 0.2$, $r/a = 0.5$	46
Fig. 3.7: A typical BEM mesh showing element size and contour for Problem B; $a/R = 0.6$, $r/a = 0.5$	47
Fig. 3.8: A typical BEM mesh showing element size and contour for Problem C; $a/W = 0.3$, $r/a = 0.5$, $H/W = 1.5$	48
Fig. 3.9: Percent discrepancy of stress intensity factor for different relative crack-tip element length (l/a) between BEM and Fett (2002) for Problem A ($a/R = 0.2$), Problem B ($a/R = 0.4$) and Problem C ($a/W = 0.6$)	49
Fig. 3.10: Percent discrepancy of T -stress for different relative crack-tip element length (l/a) between BEM and Fett (2002) for Problem A ($a/R = 0.2$)	49

Fig. 3.11: Percent discrepancy of T -stress for different relative crack-tip element length (l/a) between BEM and Fett (2002) for Problem B ($a/R = 0.4$)	50
Fig. 3.12: Percent discrepancy of T -stress for different relative crack-tip element length (l/a) between BEM and Fett (2002) for Problem C ($a/W = 0.6$)	50
Fig. 3.13: Variation of normalized T -stress with relative crack lengths for Problem A	51
Fig. 3.14: Variation of normalized stress intensity factors with relative crack lengths for Problem A	51
Fig. 3.15: The numerical model of one half of the Problem B geometry with displacement constraints on the plane of symmetry (dotted line)	52
Fig. 3.16: Variation of normalized T -stress with relative crack lengths for Problem B	52
Fig. 3.17: Variation of normalized stress intensity factors with relative crack lengths for Problem B	53
Fig. 3.18: Variation of normalized T -stress with relative crack lengths for Problem C	53
Fig. 3.19: Variation of normalized stress intensity factors with relative crack lengths for Problem C	54
Fig. 3.20: Problem D - U-notch cracked plate under remote tension	54
Fig. 3.21: A typical BEM mesh showing element size and contour integral for Problem D; $L/W = 0.3$, $H/W = 3.0$, $R/W = 0.025$, $a/R = 1.0$, $r/a = 0.5$	55
Fig. 3.22: A typical FEM mesh used by Lewis (2005) for the U-notch geometry	56
Fig. 3.23: Variation of normalized T -stress with relative crack lengths for Problem D	57
Fig. 3.24: Problem E - Double-cracks emanating from a circular hole in an infinite plate under remote tension	57
Fig. 3.25: A typical BEM mesh showing element size and contour integral for Problem E; $a/R = 2.0$, $r/a = 0.5$	58
Fig. 3.26: Variation of normalized T -stress with relative crack lengths for Problem E	59
Fig. 3.27: Variation of normalized stress intensity factors with relative crack lengths for Problem E	59
Fig. 4.1: Problem F - Finite plate with double-cracks emanating from the edge of a circular hole under remote tension	86
Fig. 4.2: A quarter of the Problem F domain being modelled with displacement constraints on the planes of symmetry (dotted lines)	86

Fig. 4.3: A typical BEM mesh showing element size and contour integral for Problem F; $R/W = 0.25$, $a/R = 2.0$, $H/W = 2.0$, $r/(a-R) = 0.5$	87
Fig. 4.4: Variations of normalized T -stress with relative crack length a/R for Problem F	88
Fig. 4.5: Variations of normalized stress intensity factors with relative crack length a/R for Problem F	88
Fig. 4.6: Effect of height-to-width ratio (H/W) on T -stress for Problem F; $R/W = 0.25$	89
Fig. 4.7: Effect of height-to-width ratio (H/W) on stress intensity factor for Problem F; $R/W = 0.25$	89
Fig. 4.8: Effect of height-to-width ratio (H/W) on T -stress for Problem F; $R/W = 0.50$	90
Fig. 4.9: Effect of height-to-width ratio (H/W) on stress intensity factor for Problem F; $R/W = 0.50$	90
Fig. 4.10: Problem G - Finite plate with double-cracks emanating from the edge of a circular hole under remote bending	91
Fig. 4.11: A half of the Problem G domain being modelled with displacement constraints on the planes of symmetry (dotted lines) and a nodal constraint at the lower right corner	91
Fig. 4.12: A typical BEM mesh showing element size and contour integral for Problem G; $R/W = 0.25$, $a/R = 2.0$, $H/W = 2.0$, $r/(a-R) = 0.5$	92
Fig. 4.13: Comparison of normalized T -stress between remote bending and remote tension on double-cracks emanating from a circular hole in a finite plate	93
Fig. 4.14: Comparison of normalized stress intensity factor between remote bending and remote tension on double-cracks emanating from a circular hole in a finite plate	93
Fig. 4.15: Problem H - Infinite plate with a single crack emanating from the edge of a circular hole under remote tension	94
Fig. 4.16: A typical BEM mesh showing element size and contour integral for Problem H; $a/R = 2.0$, $r/(a-R) = 0.5$	95
Fig. 4.17: Comparison of normalized T -stress between a single and double-crack configuration in an infinite plate under remote tension	96
Fig. 4.18: Comparison of normalized stress intensity factor between a single and double-crack configuration in an infinite plate under remote tension	96

Fig. 4.19: Problem I - Finite plate with a single crack emanating from the edge of a circular hole under remote tension	97
Fig. 4.20: A typical BEM mesh showing element size and contour integral for Problem I; $R/W = 0.25$, $a/R = 2.0$, $H/W = 2.0$, $r/(a-R) = 0.5$	98
Fig. 4.21: Variations of normalized T -stress with relative crack lengths for Problem I	99
Fig. 4.22: Variations of normalized stress intensity factor with relative crack lengths for Problem I	99
Fig. 4.23: Comparison of normalized T -stress between single crack and double-cracks emanating from a circular hole in a finite plate under remote tension	100
Fig. 4.24: Comparison of normalized stress intensity factor between single crack and double-cracks emanating from a circular hole in a finite plate under remote tension	100
Fig. 4.25: Problem J - Finite plate with a single crack emanating from the edge of a circular hole under remote bending	101
Fig. 4.26: A typical BEM mesh showing element size and contour integral for Problem J; $R/W = 0.25$, $a/R = 2.0$, $H/W = 2.0$, $r/(a-R) = 0.5$	102
Fig. 4.27: Comparison of normalized T -stress between single crack and double-cracks emanating from a circular hole in a finite plate under remote bending	103
Fig. 4.28: Comparison of normalized stress intensity factor between single crack and double-cracks emanating from a circular hole in a finite plate under remote bending	103
Fig. 4.29: Problem K - Infinite plate with double-cracks emanating from a periodic array of holes under remote tension	104
Fig. 4.30: A section of the Problem K domain being modelled with displacement constraints on the planes of symmetry (dotted lines)	105
Fig. 4.31: A typical BEM mesh showing element size and contour integral for Problem K; $R/W = 0.25$, $a/R = 2.0$, $H/W = 2.0$, $r/(a-R) = 0.5$	106
Fig. 4.32: Comparison of normalized T -stress between double-cracks in a finite and an infinite plate under remote tension	107

Fig. 4.33: Comparison of normalized stress intensity factor between double-cracks in a finite and an infinite plate under remote tension	107
Fig. 4.34: Problem L - Infinite plate with a single crack emanating from a periodic array of holes under remote tension	108
Fig. 4.35: A section of the Problem L domain being modelled with displacement constraints on the planes of symmetry (dotted lines)	108
Fig. 4.36: A typical BEM mesh showing element size and contour integral for Problem L; $R/W = 0.25$, $a/R = 2.0$, $H/W = 2.0$, $r/(a-R) = 0.5$	109
Fig. 4.37: Comparison of normalized T -stress between a single crack in a finite and an infinite plate under remote tension	110
Fig. 4.38: Comparison of normalized stress intensity factor between a single crack in a finite and an infinite plate under remote tension	110
Fig. 4.39: Comparison of normalized T -stress between single and double-crack configuration in an infinite array of holes under remote tension	111
Fig. 4.40: Comparison of normalized stress intensity factor between a single and double-crack configuration in an infinite array of holes under remote tension	111
Fig. 4.41: Problem M: Infinite plate with double-cracks emanating from the edge of a circular hole influence by adjacent holes under remote tension	112
Fig. 4.42: A quarter of the Problem M domain being modelled with displacement constraints on the planes of symmetry (dotted lines)	113
Fig. 4.43: A typical BEM mesh showing element size and contour integral for Problem M; $a/R_1 = 2.0$, $\alpha = R_2/R_1 = 0.5$, $d^* = d/R_1 = 4$, $r/(a-R_1) = 0.5$	114
Fig. 4.44: Variations of normalized T -stress with relative crack lengths for different relative hole distance $d^* = d/R_1$ for Problem M; $\alpha = R_2/R_1 = 0.25$	115
Fig. 4.45: Variations of normalized stress intensity factor with relative crack lengths for different relative hole distance $d^* = d/R_1$ for Problem M; $\alpha = R_2/R_1 = 0.25$	115
Fig. 4.46: Variations of normalized T -stress with relative crack lengths for different relative hole distance $d^* = d/R_1$ for Problem M; $\alpha = R_2/R_1 = 0.50$	116

Fig. 4.47: Variations of normalized stress intensity factor with relative crack lengths for different relative hole distance $d^* = d/R_I$ for Problem M; $\alpha = R_2/R_I = 0.50$	116
Fig. 4.48: Variations of normalized T -stress with relative crack lengths for different relative hole distance $d^* = d/R_I$ for Problem M; $\alpha = R_2/R_I = 0.75$	117
Fig. 4.49: Variations of normalized stress intensity factor with relative crack lengths for different relative hole distance $d^* = d/R_I$ for Problem M; $\alpha = R_2/R_I = 0.75$	117
Fig. 4.50: Variations of normalized T -stress with relative crack lengths for different relative hole distance $d^* = d/R_I$ for Problem M; $\alpha = R_2/R_I = 1.0$	118
Fig. 4.51: Variations of normalized stress intensity factor with relative crack lengths for different relative hole distance $d^* = d/R_I$ for Problem M; $\alpha = R_2/R_I = 1.0$	118
Fig. 4.52: Variations of normalized T -stress with relative crack lengths for different radius ratios $\alpha = R_2/R_I$ for Problem M; $d^* = d/R_I = 2.5$	119
Fig. 4.53: Variations of normalized stress intensity factor with relative crack lengths for different radius ratios $\alpha = R_2/R_I$ for Problem M; $d^* = d/R_I = 2.5$	119
Fig. 4.54: Variations of normalized T -stress with relative crack lengths for different radius ratios $\alpha = R_2/R_I$ for Problem M; $d^* = d/R_I = 4.0$	120
Fig. 4.55: Variations of normalized stress intensity factor with relative crack lengths for different radius ratios $\alpha = R_2/R_I$ for Problem M; $d^* = d/R_I = 4.0$	120
Fig. 4.56: Variations of normalized T -stress with relative crack lengths for different radius ratios $\alpha = R_2/R_I$ for Problem M; $d^* = d/R_I = 5.0$	121
Fig. 4.57: Variations of normalized stress intensity factor with relative crack lengths for different radius ratios $\alpha = R_2/R_I$ for Problem M; $d^* = d/R_I = 5.0$	121
Fig. 4.58: Variations of normalized T -stress with relative crack lengths for different radius ratios $\alpha = R_2/R_I$ for Problem M; $d^* = d/R_I = 10.0$	122
Fig. 4.59: Variations of normalized stress intensity factor with relative crack lengths for different radius ratios $\alpha = R_2/R_I$ for Problem M; $d^* = d/R_I = 10.0$	122
Fig. 5.1: Crack face loading $\sigma(x')$ of n^{th} order on an arbitrary geometry involving a circular hole; $x' = x - R$, $a' = a - R$	140

Fig. 5.2: Variations of normalized stress intensity factor with relative crack lengths for Problem E under uniform and linear crack face loading	141
Fig. 5.3: Variations of normalized T -stress with relative crack lengths for Problem E under uniform and linear crack face loading	141
Fig. 5.4: Variation of normalized stress intensity factors with relative crack lengths for Problem E	142
Fig. 5.5: Variation of normalized T -stress with relative crack lengths for Problem E	142
Fig. 5.6: Variations of normalized stress intensity factor with relative crack lengths for Problem F under uniform and linear crack face loading	143
Fig. 5.7: Variations of normalized T -stress with relative crack lengths for Problem F under uniform and linear crack face loading	143
Fig. 5.8: Normalized uncracked stress distribution in the y -direction on the perspective crack face due to remote tension for Problem F	144
Fig. 5.9: Variation of normalized stress intensity factors with relative crack lengths for Problem F	145
Fig. 5.10: Variation of normalized T -stress with relative crack lengths for Problem F	145
Fig. 5.11: Variations of normalized stress intensity factor with relative crack lengths for Problem I under uniform and linear crack face loading	146
Fig. 5.12: Variations of normalized T -stress with relative crack lengths for Problem I under uniform and linear crack face loading	146
Fig. 5.13: Variation of normalized stress intensity factors with relative crack lengths for Problem I	147
Fig. 5.14: Variation of normalized T -stress with relative crack lengths for Problem I	147

NOMENCLATURE

a	crack length
A_i	coefficients of least-squares fit for normalized T -stress
BIE	boundary integral equation
BEM	the boundary element method
B_i	coefficients of least-squares fit for normalized stress intensity factor
d	hole distance between adjacent holes and the central hole
D_1, D_2	weight function parameters for T -stress
E	Young's modulus
H	plate height; general elastic modulus
$ J(\xi) $	Jacobian of transformation
K	stress intensity factor
K_I	mode I (opening) stress intensity factor
K_{IC}	mode I fracture toughness
K_r	reference stress intensity factor
l	crack-tip element size
L	notch depth
LEFM	linear elastic fracture mechanics
$m(x, a)$	weight function for stress intensity factor
M	number of boundary elements

M_1, M_2, M_3	weight function parameters for stress intensity factor
n	power index for stress distribution
n_j	unit outward normal
N	number of nodes
$N^{(i)}$	shape functions
P	load or source point in BEM analysis
$P^{d(b,c)}$	c^{th} node of the b^{th} element
Q	field point; second parameter for elastic-plastic LEFM analysis
r	contour radius; distance between load and field point; r co-ordinate
R	circular hole radius; an arbitrary isotropic linear body; notch radius
S	boundary of a domain; far field stress
t_i	tractions
$t(x,a)$	weight function for T -stress
T	T -stress
T_{ji}	traction fundamental solutions
u_i	displacements
U_{ji}	displacement fundamental solutions
V_0, V_I	normalized T -stress for uniform and linear crack face loading
W	plate width
WF	weight function
x_i	Cartesian coordinates
X_i	body force per unit volume

Y_0, Y_I	normalized stress intensity factor for uniform and linear crack face loading
α	radius ratio
β	biaxiality ratio
δ_{ij}	Kronecker delta
σ_0	nominal stress
σ_{ij}	stress tensor
$\sigma(x)$	crack face stress distribution
θ	angular coordinate
ν	Poisson's ratio
ξ	natural coordinate for isoparametric element
Γ	integral contour
Ω	BEM domain
μ	shear modulus

CHAPTER 1 INTRODUCTION

Failure of components of mechanical systems or structures which are under stress can result in catastrophic consequences. The evaluation and prediction of failure of such components has therefore been widely studied. The stress intensity factor (K), which stems from the first term in the Williams (1957) series expansion for the stress field in the vicinity of the crack-tip, has long been developed as the sole characterizing parameter for linear elastic fracture mechanics (LEFM) analysis. Failure was assumed to occur as the geometry and loading condition dependent stress intensity factor reaches the fracture toughness of the material. Much emphasis has been invested in the determination of K solutions for numerous geometries and loading conditions, and a voluminous amount of these solutions are available in the literature today, e.g. Murakami (2003). Over the years, however, questions have been raised concerning this so-called single parameter LEFM approach to fracture assessment particularly under “low constraint” conditions in the vicinity of the crack-tip. Experimentally-obtained fracture toughness of materials are usually obtained under “high constraint” crack-tip conditions, such as the standard compact tension or three-point bending tests. When using these material properties for design against fracture, it has been frequently observed over the years that the stress intensity factor approach often leads to overly-conservative estimates of the failure loads

and hence the design. Larsson and Carlsson (1973) proposed to include the second, non-singular, term in the Williams (1957) series expansion to provide a better characterization of the stress field in the vicinity of the crack. This term is often referred to as the elastic T -stress, or simply the T -stress in fracture mechanics analysis.

Significant amounts of research have been performed on the implications of the elastic T -stress in LEFM analysis, such as on fracture toughness, crack growth rate and crack instability. The importance of the sign and magnitude of the T -stress was investigated by, for example, Leever et al. (1976), Cotterall and Rice (1980), and Bilby et al. (1986). Positive T -stress was found to increase crack-tip constraint, thus increasing the possibility of brittle fracture, while negative T -stress reduces crack-tip constraint and promotes plasticity development. It has been discovered that positive T -stress leads to out-of-plane crack growth, while stable in-plane crack growth was demonstrated with negative T -stress levels at the crack-tip. These findings have led quite recently to the T -stress being included as a second parameter in fracture assessment in design codes (see e.g. Ainsworth et al. (2002)).

T -stress solutions for different loading conditions and geometries have been presented by a number of authors; see e.g. Fett (2002), Li (2004). Surprisingly, however, only a limited amount of solutions have been performed for geometries involving stress concentrations, even though they are commonly found in practical engineering situations, e.g. notches and rivet holes. Recent examples of work with this focus include those by Lewis (2005), who investigated the effects of T -stress on a U-notch cracked plate, and by Broberg (2003), who obtained T -stress solutions for an infinite plate with double-cracks

emanating from the edge of a circular hole. The primary aim of the present study is to extend the work by Broberg (2003) by investigating finite plates with a single hole and plates with multiple holes under a range of loading conditions. For completeness, the stress intensity factor solutions will also be obtained and compared with those from the literature where possible.

In practical applications, components undergo complex stress fields, e.g. residual stresses, thermal stresses, stress concentrations. The weight function (WF) method, which is essentially a Green's function method, was developed to provide an efficient technique to analyze for stress intensity factor solutions. Wang (2002) extended this method to obtain T -stress solutions for these complex stress distributions. Although there are ample demonstration of this WF approach for the determination of K and T -stress solutions for a variety of geometries, its applicability to plates with circular holes has yet to be proven. Thus, this analytical method will also be investigated in the present study.

The boundary element method (BEM), also known as the boundary integral equation (BIE) method, will be the numerical stress tool used throughout this thesis. It offers significant advantages over other numerical methods, such as the finite element method (FEM). As the name implies, only the boundary of the domain needs to be discretized, thus reducing mesh complexity and data preparation time.

The organization of this thesis will be as follows. In the next chapter, the fundamentals of linear elastic fracture mechanics (LEFM), the weight function (WF) method, and the boundary element method (BEM) will be briefly reviewed. The procedures for the contour integral method for obtaining T -stress solutions and the self-

regularizing feature for BEM analysis will be discussed. The confidence of the BEM as the numerical technique for analysis will be validated in Chapter 3, where stress intensity factor and T -stress solutions are compared with those available in the literature. Numerical modelling procedures and mesh design guidelines will be described. Chapter 4 presents the results of the investigation on T -stress for problems of plates with cracks emanating from a single or multiple circular holes. Several of these problems will be reanalyzed using the weight function method in Chapter 5, to verify its validity. Conclusions from the work of this thesis will be summarized in Chapter 6.

CHAPTER 2 TWO-PARAMETER LINEAR ELASTIC FRACTURE MECHANICS APPROACH AND THEIR NUMERICAL EVALUATION

2.0 Introduction

In this chapter, the fundamentals of linear elastic fracture mechanics will be reviewed. The significance of T -stress and its recent developments will then be discussed. Due to complex loading conditions in practical environments, such as the presence of thermal and residual stresses, the weight function method will be introduced to obtain solutions for non-uniform stress distributions. In the last part of the chapter, the boundary element method (BEM) is discussed; it is the computational tool employed for all numerical analysis. The mutual or M-contour integral for T -stress evaluation in conjunction with BEM is described. To this end, self-regularized BEM will also be examined, as it allows further refinement of the modelling strategy and accuracy of the computed solution.

2.1 Linear Elastic Fracture Mechanics (LEFM)

There are three possible modes of deformation in the vicinity of the crack-tip for a cracked body under stress, as shown in Fig. 2.1. Mode I defines crack opening, when the crack growth is induced by the stressed body pulling apart (tension). Mode II characterizes sliding, or in-plane shear, when the top and bottom faces slide with respect to each other on the crack plane. Tearing, or out-of-plane shear, is defined by Mode III, when the top and bottom faces displace out of plane to open the crack. These modes can occur individually or simultaneously in the vicinity of a crack-tip. In this thesis, however, only Mode I deformation is considered, as it is the most commonly encountered mode of deformation in practice.

The stresses near the crack-tip have been determined by Williams (1957) to be as follows:

$$\sigma_{ij} = Ar^{-\frac{1}{2}}f_{ij}(\theta) + Br^0g_{ij}(\theta) + Cr^{\frac{1}{2}}h_{ij}(\theta) + \dots \quad (2.1)$$

where r and θ are polar coordinates with the origin at the crack-tip, as shown in Fig. 2.2, and the functions $f_{ij}(\theta)$, $g_{ij}(\theta)$, and $h_{ij}(\theta)$ are trigonometric functions of the angular location of the point with respect to the crack plane at a distance r . Parameters A , B , and C are proportional to the loading conditions. Close to the crack-tip, i.e. as $r \rightarrow 0$, the first term in eq. (2.1) dominates, while the higher order terms remain finite or approach zero. Thus, in the near crack-tip regions, eq. (2.1) reduces to

$$\sigma_{ij} = \frac{K_I}{\sqrt{2\pi r}} f_{ij}(\theta) \quad (2.2)$$

where K_I represents the Mode I stress intensity factor.

When K_I is equal or greater than the critical stress intensity factor K_{IC} (also known as fracture toughness), fracture of the stressed body is predicted to occur. This single parameter approach for fracture assessment has been used and adopted for many years; thus, K_I solutions for many different geometries and loading conditions can be found in the literature (e.g. Rooke and Cartwright (1974), Fett (2002), and Murakami (2003)).

2.2 T -Stress and Two-Parameter LEFM

In recent years, increasing emphasis is being placed onto adding the higher order terms in the Williams series expansion in the study of fracture behaviour of cracked bodies. Larsson and Carlsson (1973) showed that the inclusion of the second term in eq. (2.1) provides a more complete characterization of the stress field near the crack-tip, and predicts more precisely the size and shape of the plastic zones, which are different than those based on K_I alone. This second term is non-singular and acts in the direction of the plane of the crack, i.e. in the x_I -direction. Rice (1974) named this second term “ T -stress”, and the stress components in the crack-tip vicinity can be written as:

$$\begin{bmatrix} \sigma_x & \sigma_{xy} \\ \sigma_{xy} & \sigma_y \end{bmatrix} = \frac{K_I}{\sqrt{2\pi r}} \begin{bmatrix} f_{11}(\theta) & f_{12}(\theta) \\ f_{12}(\theta) & f_{22}(\theta) \end{bmatrix} + \begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix} \quad (2.3)$$

It has been further shown that the sign of the T -stress has a significant role in characterizing the stress field. Leever et al. (1976) and Cotterall and Rice (1980) have

shown that positive T -stress increases constraint on the crack-tip, which discourages plastic deformation and yielding. As a result, the fracture trajectory would diverge from the direction of the initial crack from the increase in crack-tip triaxiality. In contrast, negative T -stress reduces crack-tip constraint by reducing the mean stress, thus promoting plasticity development. This results in a more stable crack growth along the plane of the initial crack. Leever and Radon (1982) developed a dimensionless parameter called the biaxiality ratio, β , to normalize the effect of T -stress with respect to the stress intensity factor, where

$$\beta = \frac{T\sqrt{\pi a}}{K_I} \quad (2.4)$$

It was found by Leever and Radon (1982) that negative biaxiality ratio influences stable crack growth in the direction of the crack plane, while high positive β will result in directional instability of the crack. For elastic materials, the K - T two-parameter approach provides an accurate representation of the stress field. Betegon and Hancock (1991) and Du and Hancock (1991) have adopted the T -stress as the second parameter to the J -integral to account for elastic-plastic behaviour in the material. The J -integral proposed by Rice (1968) is a function of the strain energy density and the vector components of displacement and traction, and it is path independent. Later study by O'Dowd and Shih (1991) and Wang (1993) found that in order to accommodate plasticity from small-scale yielding to fully yielded conditions, the J - Q two-parameter approach is more suitable, where Q is related to T -stress and the material yield stress σ_y by (O'Dowd and Shih (1991))

$$Q = \frac{T}{\sigma_Y} \quad (2.5)$$

Since the work of Larsson and Carlsson (1973), a relatively huge volume of papers have been published on T -stress for a variety of geometries. For example, Fett (2002) has obtained one-dimensional crack T -stress solutions for many different geometries. Wang (2002) has evaluated T -stress solutions for semi-elliptical surface cracks in finite plates. The effect of a circular hole in an infinite plate on T -stress was studied by Broberg (2004). Li et al. (2005) obtained T -stress solutions for cracks in a thick-walled cylinder due to internal pressure. Crack stability related to T -stress has also been studied. Melin (2002) has discovered that the directional stability of the crack depends on not only J and T -stress, but also the material properties as well as geometric parameters. Tong (2002) found that fatigue crack growth behaviour can be described more in detail using T -stress, since conventional LEFM failed to explain the irregularity in crack growth caused by temperature elevation.

2.3 Weight Functions

T -stress solutions can be obtained through numerical analysis for many different geometries and loading conditions. In many practical situations, however, components undergo loading conditions with high degree of complexity. A stress field due to stress concentrations, thermal stresses and residual stresses is often more complex than those for which solutions are readily available. The weight function method provides a means of obtaining stress intensity factors and T -stresses in a relatively simple procedure for the

same geometry without the need for more complex repeated analysis once the weight functions have been determined.

2.3.1 Weight Functions for Stress Intensity Factors

Using the method of superposition, Bueckner (1970) first established that for a cracked body, as shown in Fig. 2.3(a), under stress loading S , the stress intensity factor is equivalent to the summation of the same cracked body under crack face loading $\sigma(x)$ (Fig. 2.3(b)) and the same uncracked body under the applied load S (Fig. 2.3(c)). The stress distribution $\sigma(x)$ is that on the prospective crack plane in the uncracked body under the load S . Since the stress intensity factor for the uncracked body is zero, i.e. $K_{\text{uncracked}} = 0$, the stress intensity factor for the entire system is only a function of σ_x , i.e. $K = K_{\text{crack pressure}}$. The calculation for the stress intensity factor for a specific geometry under any applied load can then be transformed to the same geometry under the corresponding crack face pressure. By integrating the product of the weight function $m(x, a)$ and the crack face pressure $\sigma(x)$, as shown,

$$K = \int_0^a \sigma(x) \cdot m(x, a) dx \quad (2.6)$$

the stress intensity factor can be obtained. The advantage of this method is that the weight function $m(x, a)$ depends only on the geometry of the body. For a specific geometry, the stress intensity factor for any $\sigma(x)$ can be determined using eq. (2.6) once the weight function has been derived.

The weight function $m(x, a)$ is a generalization of Green's function, where the system responds to an impulse, as shown in Fig. 2.3(d). Rice (1972) simplified the expression for the weight function to:

$$m(x, a) = \frac{H}{K_r} \cdot \frac{\partial u_r(x, a)}{\partial a} \quad (2.7)$$

where
$$H = \begin{cases} E & \text{for plane stress} \\ E/(1-\nu)^2 & \text{for plane strain} \end{cases} \quad (2.8)$$

In eq. (2.8), E is the Young's modulus and ν is the Poisson's ratio. The corresponding crack opening displacement field is denoted as $u_r(x, a)$. Since it is difficult to obtain, numerical approximations for $u_r(x, a)$ have been developed. Shen and Glinka (1991) found that the following four-term weight function approximation could be applied to several one-dimensional edge- and through-cracks:

$$m(x, a) = \frac{2}{\sqrt{2\pi(a-x)}} \left[1 + M_1 \left(1 - \frac{x}{a} \right)^{\frac{1}{2}} + M_2 \left(1 - \frac{x}{a} \right) + M_3 \left(1 - \frac{x}{a} \right)^{\frac{3}{2}} \right] \quad (2.9)$$

where M_1 , M_2 , and M_3 are geometric constants for a specific cracked body.

2.3.2 Weight Functions for T -Stress

The weight function for T -stress can be developed in a similar manner as for stress intensity factors described above. Consider the cracked body under applied load system S , as shown in Fig. 2.3(a). The stress field of this cracked body can be divided into two parts: the uncracked body of the same geometry under the original applied load S (Fig. 2.3(c)), resulting in the stress field $\sigma(x)$ on the prospective crack plane, and the cracked

body subjected to crack face loading corresponding to $\sigma(x)$, as shown in Fig. 2.3(b), similar to that seen before for the stress intensity factor. Hence, the problem as described by Fig. 2.3(a) can now be represented by superposition as follows:

$$T = T_{crackpressure} + T_{uncracked} \quad (2.10)$$

Recall that the regular stress field (Fig. 2.3(c)) has no singularity at the crack-tip, thus, the corresponding stress intensity factor is zero. The same cannot be said for the T -stress, as Wang (2002) has shown that:

$$T_{uncracked} = (\sigma_x - \sigma_y)_{cracktip} \quad (2.11)$$

The T -stress under the crack face pressure $\sigma(x)$ can be determined by integrating the product of the T -stress weight function, $t(x,a)$, and $\sigma(x)$, as shown:

$$T_{crackpressure} = \int_0^a \sigma(x) \cdot t(x,a) dx \quad (2.12)$$

The T -stress weight function depends only on the crack geometry, and is independent of loading conditions, similar to the K_I weight function. The weight function is essentially Green's function, a pair of unit loads on the crack face, for T -stress (Fig. 2.3(d)). Therefore, substituting eq. (2.11) and eq. (2.12) into eq. (2.10), the T -stress for the stressed crack body under load S (Fig. 2.3(a)) is found to be:

$$T = \int_0^a \sigma(x) \cdot t(x,a) dx + (\sigma_x - \sigma_y)_{cracktip} \quad (2.13)$$

For any arbitrary loading condition, eq. (2.13) provides a very efficient way to obtain the elastic T -stress. For a specific geometry, once $t(x,a)$ is found, and

$(\sigma_x - \sigma_y)_{cracktip}$ is determined by stress analysis of the uncracked body, the corresponding T -stress can then be calculated for any arbitrary loading condition. Many numerical approximations have been developed for the determination of the T -stress weight function. Wang (2002) proposed the following two-term approximation for T -stress weight function for edge cracks:

$$t(x, a) = \frac{2}{\pi a} \left[D_1 \left(1 - \frac{x}{a} \right)^{1/2} + D_2 \left(1 - \frac{x}{a} \right)^{3/2} \right] \quad (2.14)$$

where D_1 and D_2 are geometric constants of the cracked body. Wang (2002) has demonstrated that this approximation is applicable to a wide range of both edge and internal cracks, while Li (2004) has shown it is also valid for cracked thick-walled cylinders.

2.4 The Boundary Element Method for T -Stress Determination

Over the years, several numerical methods have been developed for the determination of stress intensity factors. Due to its late recognition as a fracture assessment parameter, developments for obtaining T -stress have yet to reach the same level as for the stress intensity factor. One established computational method is the boundary element method (BEM). This method is an alternative to the finite element method (FEM) frequently used for numerical analysis of engineering problems. The BEM, also known as the boundary integral equation (BIE) method, involves the transformation of the governing differential equations into an integral equation over the boundary. Thus, only the boundary of the test domain needs to be modelled. This feature offers significant

advantages over the FEM in efforts taken in mesh designs and data preparation, as well as analysis time. Due to its efficiency and high accuracy for problems with high gradients in the solution variables, such as crack problems, the BEM has become a popular tool for elastic fracture mechanics analysis. Recent work performed by, e.g. Ortiz and Cisilino (2005), demonstrated that BEM can be employed to analyze bimaterial interface cracks in three-dimensions for mixed modes stress intensity factors as well.

T-stress determination using BEM has also been studied in recent years. Sladek et al. (1997) has developed contour integral formulae, in conjunction with BEM, to evaluate the *T*-stress for two-dimensional geometries. This work was subsequently extended to analyze *T*-stress in dynamic loading conditions (Sladek et al. (1999)) and three-dimensional cracked cases (Sladek and Sladek (2000)). Also, Tan and Wang (2003) had derived a simple formula employing the use of quarter-point quadratic isoparametric elements for obtaining *T*-stress solutions of cracked body using BEM. The contour integral method described by Sladek et al. (1997) will be further discussed later in this section, as it will serve as the analysis tool for this thesis.

2.4.1 Review of the BEM

The analytical formulation of BEM starts from the differential equations for equilibrium. By the use of the fundamental solution to the governing differential equations and a reciprocal theorem, they are transformed into integral equations. For brevity, the indicial notation will be used hereafter, in which subscript indices (1,2,3) will replace the Cartesian co-ordinate direction of (x,y,z).

In the present work, only two-dimensional isotropic, linear elastostatic problems are considered. Consider an isotropic linear body, R , with boundary S as shown in Fig. 2.4. Beginning with the Navier's equation of equilibrium, and applying Betti-Rayleigh's reciprocal work theorem and fundamental solutions, they are transformed into integral equations over the surface of the solution domain. Following the usual limiting process, and in the absence of body forces, the boundary integral equation for elastostatics is (Tan (1987)):

$$C_{ji}(P)u_i(P) + \int_S u_i(Q)T_{ji}(P,Q)dS(Q) = \int_S t_i(Q)U_{ji}(P,Q)dS(Q) \quad (2.15)$$

where $U_{ji}(P,Q)$ and $T_{ji}(P,Q)$ are the Kelvin's fundamental solutions for displacements and tractions to Navier's equation, respectively. More specifically, $U_{ji}(P,Q)$ and $T_{ji}(P,Q)$ are the displacement and traction, respectively, in the x_i -direction at $Q(x)$ due to the application of a unit concentrated load in the x_j -direction at $P(x)$. In eq. (2.15), $u_i(P,Q)$ and $t_i(P,Q)$ denote the unknown displacements and tractions, respectively, and

$$C_{ji}(P) = \lim_{\varepsilon \rightarrow 0} \int_{S_\varepsilon} T_{ji}(P,Q)dS(Q) \quad (2.16)$$

where $C_{ji}(P)$ depends on the local geometry of the surface at point P , and S_ε is the boundary of the small region of exclusion around P .

Equation (2.15) is an integral equation that relates the boundary tractions to the boundary displacements. It is generally too difficult to solve this equation analytically; hence, a numerical technique must be used. To this end, the boundary S is represented by a sequence of M line elements, as shown in Fig. 2.5. As presented by Tan (1987), the

quadratic isoparametric element (Fig. 2.6) is used here to describe the variation of element geometry and functions. It is defined by three equally spaced nodes with intrinsic co-ordinates $\xi = -1$, $\xi = 0$, and $\xi = 1$, respectively. The corresponding shape functions are described as follows:

$$\begin{aligned} N^{(1)}(\xi) &= -\frac{1}{2}\xi(1-\xi) \\ N^{(2)}(\xi) &= (1-\xi)^2 \\ N^{(3)}(\xi) &= \frac{1}{2}\xi(1+\xi) \end{aligned} \quad -1 \leq \xi \leq 1 \quad (2.17)$$

where the superscripts on the shape function $N^{(i)}(\xi)$ represents the local nodes of the element. Expressing in terms of shape functions and nodal values, the geometric parameters, displacements and tractions can be written as:

$$\begin{aligned} x_j(\xi) &= N^c(\xi)x_j^c \\ u_j(\xi) &= N^c(\xi)u_j^c \\ t_j(\xi) &= N^c(\xi)t_j^c \end{aligned} \quad c = 1, 3 \quad (2.18)$$

Substituting eq. (2.18) into eq. (2.15), the discretized form of the boundary integral equation is:

$$\begin{aligned} &C_{ji}(P^a)u_i(P^a) + \sum_{b=1}^M \sum_{c=1}^3 u_i(P^{d(b,c)}) \int_S T_{ji}(P^a, Q) N^c(\xi) |J(\xi)| dS \\ &= \sum_{b=1}^M \sum_{c=1}^3 t_i(P^{d(b,c)}) \int_S U_{ji}(P^a, Q) N^c(\xi) |J(\xi)| dS \\ &P^a = 1, N \end{aligned} \quad (2.19)$$

where M = number of elements in the domain

N = number of nodes in the domain = $2M$

$P^{d(b,c)}$ = c^{th} node of the b^{th} element

$$b = 1, M$$

$$c = 1, 3$$

$$|J(\xi)| = \text{Jacobian of transformation}$$

Equation (2.19) represents a total of $4M$ linear algebraic equations for the unknown displacements and tractions at the boundary nodes of the discretized domain. The unknown displacements and tractions on the boundary S can now be solved using standard matrix manipulations.

2.4.2 Contour Integral for Calculating T -Stress Using BEM

The contour integral described by Rice (1968) was introduced to overcome the difficulties in determining strain concentrations when analyzing notched and cracked geometries. Since then, this contour integral technique has been widely used to solve a variety of problems. The evaluation of elastic T -stress using the contour integral was introduced by Sladek et al. (1997). It is based on Betti-Rayleigh's reciprocal work theorem and an auxiliary field, and it is particularly suitable for implementation with the BEM. These authors derived a path independent contour integral around the crack-tip, termed the mutual or M-integral, which is directly related to the T -stress. The contour integral is along a closed path sufficiently far from the crack-tip. This in turn obviates the need to compute the stress field near the vicinity of the crack, where the crack-tip stress singularity can cause large numerical errors. The required field variables along the integration path are obtained from the BEM analysis.

The following briefly describes the key steps in the derivation of the M-contour integral and its relationship with the T -stress. Consider a cracked isotropic, elastic domain R with boundary S , as shown in Fig. 2.7. Inside this domain, consider a closed integration path comprised of Γ_o , Γ_C^+ , and Γ_C^- . Using Gauss' divergence theorem, Hooke's law and strain-displacement relations, the Betti-Rayleigh's reciprocal theorem for two sets of equilibrium states of the sub-domain can be expressed as:

$$\int_{\Gamma} (\sigma_{ij}' u_i n_j - \sigma_{ij} u_i' n_j) \cdot d\Gamma = \int_{\Omega} (X_i' u_i - X_i u_i') \cdot d\Omega \quad (2.20)$$

where X_i and X_i' are body forces in two load states, respectively; and n_j is the unit outward normal of the contour Γ of integration in sub-domain Ω . A small circular region bounded by Γ_ε near the crack-tip has to be excluded due to the existence of the singularity. This region is small and will reduce to zero in the limiting process. The contour of integration $\Gamma = \Gamma_o + \Gamma_C^+ + \Gamma_C^- - \Gamma_\varepsilon$ is a closed path in the counterclockwise direction.

Without the loss of generality, assume that in eq. (2.20), the primed and unprimed states correspond to an auxiliary and the unknown fields, respectively. Assume an auxiliary field where $\sigma_{ij}' = 0$ on crack faces and body forces $X_i' = 0$. Equation (2.20) can be rewritten as:

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\Gamma_\varepsilon} (\sigma_{ij}' u_i n_j - \sigma_{ij} u_i' n_j) \cdot d\Gamma &= \int_{\Gamma_o} (\sigma_{ij}' u_i n_j - \sigma_{ij} u_i' n_j) \cdot d\Gamma \\ &\quad - \int_{\Gamma_C^+} (\sigma_{ij}^+ u_i' n_j^+ - \sigma_{ij}^- u_i' n_j^-) \cdot d\Gamma \\ &\quad - \lim_{\varepsilon \rightarrow 0} \int_{\Omega - \Omega_\varepsilon} X_i u_i' \cdot d\Omega \end{aligned} \quad (2.21)$$

where n_j^+ and n_j^- are unit outward normal on the upper and lower crack face, respectively, and $n_j^+ = -n_j^-$. Also, assume that $\sigma_{ij}^+ = \sigma_{ij}^-$ for small equilibrium stress loads. Using the relationship between traction and stress, i.e. $t_i = \sigma_{ij}n_j$, eq. (2.21) can be transformed into:

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon} (t_i' u_i - t_i u_i') \cdot d\Gamma &= \int_0 (t_i' u_i - t_i u_i') \cdot d\Gamma - \int_{c^+} 2t_i^+ u_i' d\Gamma \\ &\quad - \lim_{\varepsilon \rightarrow 0} \int_{\Omega - \Omega_\varepsilon} X_i u_i' d\Omega \end{aligned} \quad (2.22)$$

This equation can be used to derive integral formulas for evaluating various fracture parameters by choosing different auxiliary fields. Bueckner (1989) employed singular auxiliary fields to obtain integral equations for stress intensity factor K_I . However, to obtain the non-vanishing contribution to elastic T -stress, a special auxiliary field has to be determined. It must also eliminate the singular integrand contribution in the process. It was proposed by Sladek et al. (1997) to use an auxiliary field that is one order higher than the one utilized by Kfoury (1986). The auxiliary displacements and tractions are obtained by differentiating Kfoury's field, with respect to x_I , and they are proportional to $1/r$ and $1/r^2$, respectively. This auxiliary field is represented by:

$$\begin{aligned} u_1'(r, \theta) &= -\frac{f}{\pi E r} (1 - \nu^2) \cdot \left(\cos \theta - \frac{1}{1 - \nu} \sin^2 \theta \cos \theta \right) \\ u_2'(r, \theta) &= -\frac{f}{2\pi E r} (1 + \nu) \cdot (1 - 2\nu - \cos 2\theta) \cdot (-\sin \theta) \\ \sigma_{11}'(r, \theta) &= \frac{f}{\pi r^2} \cos^2 \theta \cdot (\cos^2 \theta - 3\sin^2 \theta) \\ \sigma_{12}'(r, \theta) &= \frac{f}{\pi r^2} \sin 2\theta \cdot \cos 2\theta \end{aligned}$$

$$\sigma_{22}'(r, \theta) = \frac{f}{\pi r^2} \sin^2 \theta \cdot (3 \cos^2 \theta - \sin^2 \theta) \quad (2.23)$$

where f is a static point force applied at the crack-tip in the crack propagating direction (Fig. 2.7).

The unknown asymptotic displacements and tractions can be separated into two parts:

$$u_i = u_i^S + u_i^T \quad (2.24)$$

$$\sigma_{ij} = \sigma_{ij}^S + \sigma_{ij}^T \quad (2.25)$$

The terms with the superscript “S” represent those of the singular stress field. σ_{ij}^S is equivalent to eq. (2.2), while u_i^S is given as:

$$u_i^S = \frac{K_I}{4\mu} \sqrt{\frac{r}{2\pi}} f_i(\theta) \quad (2.26)$$

where μ is the shear modulus, and ν is Poisson’s ratio.

The terms with superscript “T” are given as follows:

$$\sigma_{ij}^T = T \cdot \delta_{i1} \delta_{j1} \quad (2.27)$$

$$u_i^T = \frac{Tr}{E} [\delta_{i1}(1 - \nu^2) \cos \theta - \delta_{i2} \nu(1 + \nu) \sin \theta] \quad (2.28)$$

Substituting the above auxiliary field solution and the non-singular term of the Williams series expansion into the left-hand-side of eq.(2.22), gives

$$\lim_{\epsilon \rightarrow 0} \int_{\Gamma_\epsilon} (t_i' u_i^T - t_i^T u_i') \cdot d\Gamma = \frac{1 - \nu^2}{E} T \cdot f \quad (2.29)$$

Performing the same task again, but this time with u_i^S and σ_{ij}^S , the left-hand-side of eq.(2.22) disappears, i.e.

$$\lim_{\varepsilon \rightarrow 0} \int_{\Gamma_s} (t_i' u_i^S - t_i^S u_i') \cdot d\Gamma = 0 \quad (2.30)$$

By substituting eq. (2.29) and eq. (2.30) into eq. (2.22), the T -stress term, without any body forces, can finally be determined:

$$T = \frac{E}{f(1-\nu^2)} \int_{\Gamma_0} (t_i' u_i - t_i u_i') \cdot d\Gamma - \frac{2E}{f(1-\nu^2)} \int_{\Gamma_c^+} t_i^+ u_i' d\Gamma \quad (2.31)$$

Equation (2.31) can be further simplified as the second term in the equation, for Mode I loading, is always zero. Hence, the T -stress solution now becomes

$$T = \frac{E}{f(1-\nu^2)} \int_{\Gamma_0} (t_i' u_i - t_i u_i') \cdot d\Gamma \quad (2.32)$$

By substituting eq. (2.22) into eq. (2.32), a more explicit T -stress solution can be obtained:

$$T = \frac{E}{f(1-\nu^2)} \int_{\Gamma_0} \frac{f}{\pi r^2} \left(F_{ij}(\theta) u_i - t_i \frac{F_{ij}(\theta)}{E} \right) \cdot d\Gamma \quad (2.33)$$

where $F_{ij}(\theta)$ contains trigonometric functions of the angular location. This explicit T -stress expression can now be solved numerically using, for example, a Gaussian quadrature scheme. The nodal displacements and tractions along the integration path, u_i and t_i , can be computed from the BEM analysis.

2.4.3 Self-Regularized Boundary Element Method

As aforementioned, $U_{ji}(p, Q)$ and $T_{ji}(p, Q)$ are the displacement and traction fundamental solutions to Navier's equation, respectively. They are explicitly expressed as:

$$U_{ji}(p, Q) = \frac{1}{16\pi(1-\nu)\mu r} [(3-4\nu)(\delta_{ij} + r_{,i} r_{,j})] \quad (2.34)$$

$$T_{ji}(p, Q) = \frac{-1}{8\pi(1-\nu)r^2} \left\{ \frac{\partial r}{\partial n} [(1-2\nu)\delta_{ij} + 3r_{,i} r_{,j}] - (1-2\nu)(n_j r_{,i} - n_i r_{,j}) \right\} \quad (2.35)$$

where r here is the distance between the source and the field point. Differentiating eq. (2.15) with respect to the co-ordinates at p , and using Hooke's Law, the Somigliana's identity for stresses can be obtained:

$$\sigma_{ij}(p) = \int_S t_k(Q) D_{kij}(p, Q) ds(Q) = \int_S u_k(Q) S_{kij}(p, Q) ds(Q) \quad (2.36)$$

where the kernels $D_{kij}(p, Q)$ and $S_{kij}(p, Q)$, containing derivatives of $U_{ji}(p, Q)$ and $T_{ji}(p, Q)$, can be written in the following forms:

$$D_{kij}(p, Q) = \frac{1}{8\pi(1-\nu)r^2} [(1-2\nu)(\delta_{ki} r_{,j} + \delta_{kj} r_{,i} + \delta_{ij} r_{,k}) + 3r_{,i} r_{,j} r_{,k}] \quad (2.37)$$

$$S_{kij}(p, Q) = \frac{\mu}{4\pi(1-\nu)r^3} \left\{ 3 \frac{\partial r}{\partial n} [(1-2\nu)\delta_{ij} r_{,k} + \nu(\delta_{ki} r_{,j} + \delta_{kj} r_{,i}) - 5r_{,i} r_{,j} r_{,k}] + 3\nu(n_i r_{,j} r_{,k} - n_j r_{,i} r_{,k}) + (1-2\nu)(3n_k r_{,i} r_{,j} + n_j \delta_{ik} \delta_{jk}) - (1-4\nu)n_k \delta_{ij} \right\} \quad (2.38)$$

As evident in the order of r that is present in the denominator of these equations, $U_{ji}(p, Q)$ is weakly singular, $T_{ji}(p, Q)$ and $D_{kij}(p, Q)$ are strongly singular, and

$S_{kij}(p, Q)$ is hypersingular. This may contribute to numerical errors when evaluating displacements and stresses at interior points near the boundaries since the integrands in the BIE become nearly singular. These integrals are difficult to evaluate by standard quadrature procedures as errors will increase when the interior point approaches the domain boundary, i.e. as $r \rightarrow 0$. This is sometimes referred to as the boundary layer effect in the BEM community. To overcome this problem, an obvious strategy is to increase the number of boundary elements, by reducing their lengths, in the vicinity of the interior points. This approach is, however, cumbersome as the mesh near the interior points needs to be extremely refined.

An alternate scheme to overcome the near-singularity problem is to “regularize” the integrals, either locally on certain segments on the boundary (e.g. Huang and Cruse (1994)) or globally (e.g. Cruse and Richardson (1996, 1999)), so that the integration can be carried out normally using standard quadrature techniques. In this thesis, the self-regularized forms from Cruse and Richardson (1996, 1999), will be adopted along with contour integral formulas to derive accurate T -stress solutions.

Cruse and Richardson (1996) proposed a self-regularized displacement BIE (DBIE). This can be derived from the Somigliana displacement identity by subtracting a simple solution that corresponds to rigid body motion, as shown below:

$$\begin{aligned} u_i(p) - u_j(P) = & - \int_S [u_i(Q) - u_i(P)] T_{ji}(p, Q) dS(Q) \\ & + \int_S t_i(Q) U_{ji}(p, Q) dS(Q) \end{aligned} \quad (2.39)$$

where p is the source point (internal point) and P is the regularizing point at the boundary usually taken to be close to p . Both of the integrals in this equation are regular at every internal point. This suggests that the boundary layer effect is eliminated and no special attention is required for internal points close to boundary nodes.

For stress, Cruse and Richardson (1996) also derived a self-regularized BIE (SBIE) from the Somigliana stress identity. This can be obtained by subtracting and adding back the first and second terms of the Taylor series expansion to the original form of the SBIE. The procedure is equivalent to subtracting and adding back a simple solution corresponding to a state of constant stress in the body that is equal to the boundary stress at a surface point P , and can be expressed as

$$\begin{aligned} \sigma_{ij}(p) - \sigma_{ij}(P) = & - \int_S [u_k(Q) - u_k^L(Q)] S_{kij}(p, Q) dS(Q) \\ & + \int_S [t_k(Q) - t_k^L(Q)] D_{kij}(p, Q) dS(Q) \end{aligned} \quad (2.40)$$

where $u_k^L(Q)$ and $t_k^L(Q)$ represent the linear state of displacements and tractions associated with the boundary stress at P , and are given by

$$u_k^L(Q) = u_k(P) + u_{k,m}(P)[x_m(Q) - x_m(P)] \quad (2.41)$$

$$t_k^L(Q) = \sigma_{km}(P)n_m(Q) \quad (2.42)$$

The coefficient $u_{k,m}(P)$ is the displacement gradients at the regularizing point P . $x_m(Q)$ and $x_m(P)$ are the Cartesian co-ordinates of the field point and regularizing point, respectively; and $\sigma_{km}(P)$ represents the stress components at P . Following the procedures described by Cruse and Richardson (1999), a system of equations relating

local traction and displacement tangential derivatives with the displacement gradients can be obtained:

$$\begin{Bmatrix} t_1 \\ t_2 \\ u_{1,\xi} \\ u_{2,\xi} \end{Bmatrix} = \begin{bmatrix} \frac{2\mu(1-\nu)}{1-2\nu}n_1 & \mu n_2 & \mu n_2 & \frac{2\mu\nu}{1-2\nu}n_1 \\ \frac{2\mu\nu}{1-2\nu}n_2 & \mu n_1 & \mu n_1 & \frac{2\mu(1-\nu)}{1-2\nu}n_2 \\ -J(\xi)n_2 & 0 & J(\xi)n_1 & 0 \\ 0 & -J(\xi)n_2 & 0 & J(\xi)n_1 \end{bmatrix} \begin{Bmatrix} u_{1,1} \\ u_{2,1} \\ u_{1,2} \\ u_{2,2} \end{Bmatrix} \quad (2.43)$$

where $J(\xi)$ is the Jacobian of transformation. Finally, by inverting the above matrix, the displacement gradients at the regularizing point can be obtained in terms of local tractions and displacements

$$u_{k,l}(P) \approx u_{k,l}(\xi^P) = A_{klr}(\xi^P)t_r(\xi^P) + B_{klr}(\xi^P)\sum_{i=1}^3 N_i'(\xi^P)u_r^i \quad (2.44)$$

where $A_{klr}(\xi^P)$ and $B_{klr}(\xi^P)$ are the mapping functions. Using this regularized technique, the solutions from BEM analysis have been shown to be highly accurate (Cruse and Richardson (1999)) even for interior points very close to the domain boundary.

2.5 Summary

The theory of linear elastic fracture mechanics (LEFM) has been reviewed. The T -stress (non-singular) term in the Williams (1957) series expansion has been shown to be a significant factor in describing more accurately the stress field in the vicinity of a crack. Due to the complexity of loading conditions in practical applications, the weight function method was also reviewed to provide a convenient method for solving these types of

problems. The review of the boundary element method (BEM) was carried out as it would serve as the numerical tool for all analysis. Finally, T -stress determination using the mutual contour integral method, in conjunction with self-regularizing BEM to minimize errors in evaluating the contours, was discussed.

Fig. 2.1: Modes of deformation for a cracked body

Fig. 2.2: Two-dimensional stresses in the vicinity of the crack tip

Fig. 2.3: Weight Function for one-dimensional cracks

Fig. 2.4: Linear elastic body R with boundary S in a two-dimensional co-ordinate system

Fig. 2.5: Discretized domain R , with boundary S

Fig. 2.6: A quadratic isoparametric line element

Fig. 2.7: Integration paths and co-ordinate definitions

CHAPTER 3 BEM VALIDATION AND TEST PROBLEMS

3.0 Introduction

In this chapter, the effectiveness and accuracy of the boundary element method (BEM) for determining stress intensity factor, K_I , and T -stress is verified. The BEM computer program used for all analysis was a variation of the one for two-dimensional stress analysis using the quadratic isoparametric numerical formulation; it was developed by Prof. C.L. Tan at Carleton University. Numerical modelling procedure will first be discussed, and proper crack-tip element size and contour radii for testing crack problems are established to ascertain accurate results, when using the computer program. Different test geometries, as listed in Table 3.1, will be analyzed using the BEM and comparisons are then made with corresponding solutions in the literature.

3.1 Numerical Modelling and Mesh Design Guidelines

BEM fracture mechanics analysis entails the use of the well-known quarter-point crack tip element and the contour integral method, as shown in Fig. 3.1. Li (2004) has shown that relative crack-tip element (l/a) and relative contour radius sizes (r/a) can have some effect on the accuracy of computed BEM results. For self-regularizing BEM, it is necessary to again establish these parameters that provide comparable results to those available in literature before carrying out any analysis. Three different geometries were tested: a circular disk with internal crack under uniform radial tension at the circumference (Problem A), as shown in Fig. 3.2; a circular disk with single edge-crack under uniform radial tension at the circumference (Problem B), as shown in Fig. 3.3, and a single edge-crack plate under remote bending (Problem C), as shown in Fig. 3.4. The relative crack sizes considered in these problems was arbitrarily chosen to be $a/R = 0.2$ and 0.4 for Problems A and B, respectively, and $a/W = 0.6$ for Problem C. Numerical results for these problems are available in the literature, such as the ones obtained from Fett (2002) using the method of boundary collocation.

The numerical modelling of each of these test cases can take advantage of symmetry, hence only a fraction of the geometry needs to be analyzed. For example, only a quarter of the Problem A domain needs to be modelled (Fig. 3.5). The vertical plane of symmetry is constrained in the x_1 -direction, while free to displace in the x_2 -direction. The horizontal plane (uncracked portion) is constrained in the x_2 -direction, while being able to displace freely in the x_1 -direction. Also due to the symmetry of the

problem, only a semi-circular contour is needed for the T -stress determination, as shown in Fig. 3.6. Two-dimensional standard quadratic isoparametric elements were used except for the crack-tip elements, with the sizes gradually being enlarged away from the crack-tip. Figure 3.6 shows the BEM mesh employed for Problem A, with the contour integral path used in the interior of the numerical domain. Plane strain condition was assumed. The integration of the contour can be performed numerically using Gaussian quadrature scheme. The contour was represented by 12 discretized circular elements, and 10th order Gaussian quadrature was used for numerical integration. The corresponding BEM meshes used for Problems B and C are shown in Figures 3.7 and 3.8, respectively.

3.1.1 Relative Quarter-Point Crack-Tip Element Size (l/a) and Relative Contour Radii (r/a)

The relative quarter-point crack-tip element lengths $l/a = 0.05, 0.10, 0.15$, and 0.20 and relative contour radii $r/a = 0.4, 0.5$, and 0.6 were tested. The results for the normalized stress intensity factor ($K_I / \sigma \sqrt{\pi a}$) and T -stress (T/σ) are shown in Tables 3.2 – 3.4. Generally, excellent agreement with less than 2% discrepancy between the BEM results and that obtained by Fett (2002) was achieved for all three test problems considered, suggesting that the solutions obtained by the self-regularizing BEM were fairly insensitive to the crack-tip element size or contour radii. As shown in Fig. 3.9, the $l/a = 0.10$ case exhibited the least amount of discrepancy, for all geometries. Thus, this relative crack-tip element length will be adopted for all of the BEM analysis in this thesis. The T -stress results for the three different geometries are shown in Fig. 3.10 – 3.12. All

of the r/a ratios tested provided equally accurate results, and although it is not necessary to use all three ratios to verify the results, they are nevertheless considered for all the analysis carried out in this thesis as a check on the numerical solutions of the T -stress obtained.

3.2 Test Cases for BEM Validation

Having established appropriate relative crack-tip element lengths and contour paths for three arbitrarily chosen test problems above, a more comprehensive test involving a wider range of crack sizes for those three geometries was carried out. In addition, two other problems with a crack or cracks emanating from stress concentrations were also treated. They were a U-notch cracked plate (Problem D) and an infinite plate with two cracks emanating from the edge of a circular hole (Problem E).

3.2.1 Circular Disk with Internal Crack (Problem A)

Two-dimensional BEM analysis was performed for the circular disk with internal crack under uniform radial tension at the circumference, as shown in Fig. 3.2. As previously mentioned, the numerical modelling of this geometry can be reduced to one quarter of its original geometry, by virtue of symmetry. Relative crack size of $a/R = 0.1 - 0.6$ were analyzed and the results are presented in Table 3.5 and Fig. 3.13 and 3.14. The BEM results show very good agreement indeed with those by Fett (2002), using the method of boundary collocation, for both the normalized stress intensity factor and the T -stress.

3.2.2 Circular Disk with Single Edge-Crack (Problem B)

The circular disk with single edge-crack under uniform normal traction at the circumference case as shown in Fig. 3.3, was analyzed with two-dimensional BEM. Again, symmetry can be employed to reduce the size of the solution domain. However, since this domain is only symmetric about the x_1 -axis, one half of the physical problem was modelled (Fig. 3.15), with the plane of symmetry constrained in the x_2 -direction. It is also shown in Fig. 3.15 that due to the lack of constraint in the x_1 -direction, a boundary nodal constraint is applied at the lower right corner to preclude rigid body motion. Relative crack size of $a/R = 0.1 - 0.6$ were analyzed, and the results are shown in Table 3.6 and Fig. 3.16 and 3.17. The obtained results from BEM are again compared to those by Fett (2002), with discrepancies of generally less than 2%.

3.2.3 Single Edge-Cracked Plate (Problem C)

Two-dimensional BEM analysis was performed on the single edge-cracked plate under remote bending (Fig. 3.4). Again, similar to Problem B, only half of the domain was numerically modelled by virtue of symmetry. Relative crack size of $a/W = 0.1 - 0.6$ were treated for a height-to-width ratio (H/W) of 1.5. The results are listed in Table 3.7, and shown graphically in Fig. 3.18 and 3.19. Generally very good agreement was again achieved compared to the values from Fett (2002), for both the normalized stress intensity factor, $K_I / \sigma \sqrt{\pi a}$, and T -stress, T/σ .

3.2.4 U-Notch Cracked Plate (Problem D)

The U-notch cracked plate under remote tension problem (Fig. 3.20) was solved numerically in BEM for the normalized T -stress. Lewis (2005) has recently obtained T -stress solutions for this geometry using the finite element method (FEM). It therefore provides another comparison check of the effectiveness of BEM. The plate geometry is defined by the height $2H$, width W , notch depth L , notch radius R , and crack length a , as shown in Fig. 3.20. A typical BEM mesh for this problem is shown in Fig. 3.21, while a typical FEM mesh used by Lewis (2005) is shown in Fig. 3.22. It is evident that the FEM modelling scheme involves the use of substantially more elements than the BEM. The modelling complexity for FEM, especially around the crack-tip, can also be seen when comparing Fig. 3.21 and 3.22. Relative crack lengths of $a/R = 0.1, 0.2, 0.3, 0.5$, and 1.0 were analyzed for the case of $L/W = 0.3$, $H/W = 3.0$, $R/W = 0.025$. The normalized T -stresses evaluated using BEM are shown in Table 3.8 and in Fig. 3.23. It can be seen that the two sets of numerical results are in excellent agreement, which further provides confidence in the self-regularizing BEM for its efficiency and accuracy.

3.2.5 Infinite Plate with Double-Cracks Emanating from the Edge of a Circular Hole (Problem E)

T -stress solutions for double-cracks emanating from a circular hole in an infinite plate, as shown in Fig. 3.24, are available from the recent paper by Broberg (2004). Unlike the previous four cases, this test case involves an infinite plate. Numerical representation of such problems can be achieved by treating a relatively large plate. It is usually sufficient

to model the size of the domain boundary to be 20 times the length of the crack, i.e. $(a-R)/W = 0.05$ and $(a-R)/H = 0.05$, where W and H represent the width and height of the numerical domain, respectively. Symmetry can again be taken advantage of for this infinite plate case, thus only one quarter of the domain needs to be analyzed with the appropriate boundary conditions as in Problem A. The BEM mesh used for analysis for this problem is shown in Fig. 3.25.

Relative crack sizes of $a/R = 1.1 - 10.0$ were considered. The results from the two-dimensional BEM analysis are shown in Table 3.9, and graphically in Fig. 3.26 and 3.27. The normalized T -stresses (T/σ) obtained from the BEM analysis show very good agreement with those from Broberg (2004); the discrepancies were less than 2%. Comparison was also drawn for the normalized stress intensity factor, $K_I / \sigma \sqrt{\pi(a-R)}$, between the calculated BEM results and those from Bowie (1965); the discrepancy was less than 1.4%. Thus, it is established here that the BEM is an excellent tool for solving this type of geometry for K_I and T -stress.

3.3 Conclusions

The two-dimensional self-regularized boundary element method (BEM) was employed to analyze the stress intensity factor and T -stress solutions for five different test problems. Relative crack-tip element size (l/a) was found to have a small effect on K_I , but not on T . It was shown that the contour integral method generates accurate solutions for T -stress for different contour radii (r/a). It was established that relative crack-tip element size l/a of 0.1 and contour radius r/a of 0.5 would generate accurate results using the BEM. It

has been established from this part of the study that the BEM gave numerical results of K_I and T -stress which were in excellent agreement with those established in the literature, with discrepancies which were generally less than 2% for the problems considered.

Table 3.1: List of problems analyzed in Chapter 3

Problem	Definition	Figure #
A	Circular Disk with Internal Crack Under Uniform Radial Tension	3.2
B	Circular Disk with Single Edge-Crack Under Uniform Radial Tension	3.3
C	Single Edge-Cracked Plate Under Remote Bending	3.4
D	U-Notch Cracked Plate Under Remote Tension	3.20
E	Infinite Plate with Double-Cracks Emanating from the Edge of a Circular Hole Under Remote Tension	3.24

Table 3.2: Comparison of normalized stress intensity factor and T -stress solutions obtained by BEM and by Fett (2002) for Problem A; $a/R = 0.2$

l/a	$K_I / \sigma \sqrt{\pi a}$			T/σ						
	FETT (2002)	BEM	% DIFF	FETT (2002)	BEM ($r/a=0.4$)	% DIFF	BEM ($r/a=0.5$)	% DIFF	BEM ($r/a=0.6$)	% DIFF
0.05	1.0629	1.0684	0.53	-0.0800	-0.0786	-1.75	-0.0788	-1.56	-0.0786	-1.75
0.10	1.0629	1.0639	0.09	-0.0800	-0.0788	-1.50	-0.0789	-1.38	-0.0788	-1.50
0.15	1.0629	1.0686	0.54	-0.0800	-0.0797	-0.38	-0.0790	-1.25	-0.0786	-1.75
0.20	1.0629	1.0734	1.00	-0.0800	-0.0783	-2.13	-0.0783	-2.13	-0.0782	-2.25

Table 3.3: Comparison of normalized stress intensity factor and T -stress solutions obtained by BEM and by Fett (2002) for Problem B; $a/R = 0.4$

l/a	$K_I / \sigma \sqrt{\pi a}$			T/σ						
	FETT (2002)	BEM	% DIFF	FETT (2002)	BEM ($r/a=0.4$)	% DIFF	BEM ($r/a=0.5$)	% DIFF	BEM ($r/a=0.6$)	% DIFF
0.05	1.5673	1.5822	-0.95	0.7406	0.7384	-0.30	0.7389	-0.23	0.7388	-0.25
0.10	1.5673	1.5806	-0.84	0.7406	0.7382	-0.33	0.7387	-0.26	0.7382	-0.33
0.15	1.5673	1.5878	-1.30	0.7406	0.7381	-0.34	0.7386	-0.27	0.7385	-0.29
0.20	1.5673	1.5999	-2.08	0.7406	0.7378	-0.38	0.7383	-0.31	0.7382	-0.33

Table 3.4: Comparison of normalized stress intensity factor and T -stress solutions obtained by BEM and by Fett (2002) for Problem C; $a/W = 0.6$

l/a	$K_I / \sigma \sqrt{\pi a}$			T/σ						
	FETT (2002)	BEM	% DIFF	FETT (2002)	BEM ($r/a=0.4$)	% DIFF	BEM ($r/a=0.5$)	% DIFF	BEM ($r/a=0.6$)	% DIFF
0.05	1.9140	1.9134	0.03	0.8313	0.8321	0.10	0.8324	0.14	0.8324	0.14
0.10	1.9140	1.9138	0.01	0.8313	0.8319	0.08	0.8323	0.13	0.8319	0.08
0.15	1.9140	1.9174	-0.18	0.8313	0.8317	0.05	0.8320	0.09	0.8321	0.10
0.20	1.9140	1.8856	1.48	0.8313	0.8322	0.11	0.8323	0.13	0.8325	0.15

Table 3.5: Comparison of normalized stress intensity factor and T -stress solutions between BEM and Fett (2002) for different relative crack lengths (a/R) for Problem A

a/R	l/a	$K_I / \sigma \sqrt{\pi a}$			T/σ		
		FETT (2002)	BEM	% DIFF	FETT (2002)	BEM	% DIFF
0.100	0.100	1.0167	1.0326	1.56	-0.0216	-0.0201	-7.01
0.200	0.100	1.0629	1.0639	-0.09	-0.0806	-0.0789	-2.06
0.300	0.100	1.1368	1.1502	-1.19	-0.1712	-0.1714	0.10
0.400	0.100	1.2415	1.2590	-1.42	-0.2937	-0.2974	1.27
0.500	0.100	1.3844	1.4080	-1.70	-0.4569	-0.4610	0.91
0.600	0.100	1.5785	1.6005	-1.39	-0.6868	-0.6656	-3.09

Table 3.6: Comparison of normalized stress intensity factor and T -stress solutions between BEM and Fett (2002) for different relative crack lengths (a/R) for Problem B

a/R	l/a	$K_I / \sigma \sqrt{\pi a}$			T/σ		
		FETT (2002)	BEM	% DIFF	FETT (2002)	BEM	% DIFF
0.100	0.100	1.2112	1.2152	0.33	0.5252	0.5112	-2.67
0.200	0.100	1.3135	1.3272	-1.04	0.5852	0.5778	-1.26
0.300	0.100	1.4311	1.4501	-1.33	0.6561	0.6535	-0.39
0.400	0.100	1.5673	1.5806	-0.84	0.7406	0.7387	-0.26
0.500	0.100	1.7267	1.7545	-1.61	0.8427	0.8409	-0.21
0.600	0.100	1.9149	1.9466	-1.66	0.9673	0.9655	-0.19

Table 3.7: Comparison of normalized stress intensity factor and T -stress solutions between BEM and Fett (2002) for different relative crack lengths (a/W) for Problem C

a/W	l/a	$K_I / \sigma \sqrt{\pi a}$			T/σ		
		FETT (2002)	BEM	% DIFF	FETT (2002)	BEM	% DIFF
0.100	0.100	1.0929	1.0628	-2.75	-0.3838	-0.3794	-1.15
0.200	0.100	1.0567	1.0697	-1.23	-0.2380	-0.2394	0.60
0.300	0.100	1.1240	1.1331	-0.81	-0.0770	-0.0799	3.72
0.400	0.100	1.2611	1.2682	-0.57	0.1195	0.1203	0.64
0.500	0.100	1.4971	1.5016	-0.30	0.3923	0.3967	1.12
0.600	0.100	1.9140	1.8856	1.48	0.8325	0.8323	-0.02

Table 3.8: Comparison of normalized T -stress between BEM and Lewis (2005) for different relative crack lengths (a/R) for Problem D

a/R	T/σ		
	LEWIS (2005)	BEM	% DIFF
0.1	-4.7238	-4.7939	1.48
0.2	-3.3638	-3.3781	0.43
0.3	-2.5128	-2.5289	0.64
0.5	-1.5632	-1.5605	-0.17
1.0	-0.7931	-0.8034	1.30

Table 3.9: Comparison of normalized stress intensity factor and T -stress between BEM and those from literature for different relative crack lengths (a/R) for Problem E

a/R	$K_I / \sigma \sqrt{\pi(a-R)}$			T/σ		
	BOWIE (1965)	BEM	% DIFF	BROBERG (2004)	BEM	% DIFF
1.1	2.8017	2.8396	1.35	-1.0584	-1.0471	-1.07
1.2	2.4135	2.4463	1.36	-0.7925	-0.7951	0.33
1.5	1.8369	1.8622	1.38	-0.5616	-0.5557	-1.05
1.6	1.7500	1.7555	0.31	N/A	-0.5319	N/A
2.0	1.4902	1.5011	0.73	-0.609	-0.6037	-0.87
3.0	1.2800	1.2803	0.02	N/A	-0.7776	N/A
4.0	1.2000	1.2103	0.86	N/A	-0.8809	N/A
5.0	1.1510	1.1541	0.27	-0.8984	-0.9161	1.97
10.0	1.0794	1.0827	0.31	-0.9714	-0.9901	1.93

Fig. 3.1: A general crack problem showing crack length a , quarter-point crack-tip element length l , and contour radius r

Fig. 3.2: Problem A - Circular disk with internal crack under uniform radial tension at the circumference

Fig. 3.3: Problem B - Circular disk with single edge crack under uniform radial tension at the circumference

Fig. 3.4: Problem C - Single edge crack plate under remote bending

Fig. 3.5: A quarter of the Problem A domain being modelled with displacement constraints on the planes of symmetry (dotted lines)

Fig. 3.6: A typical BEM mesh showing element size and contour for Problem A; $a/R = 0.2, r/a = 0.5$

Fig. 3.7: A typical BEM mesh showing element size and contour for Problem B; $a/R = 0.6$, $r/a = 0.5$

Fig. 3.8: A typical BEM mesh showing element size and contour for Problem C; $a/W = 0.3$, $r/a = 0.5$, $H/W = 1.5$

Fig. 3.9: Percent discrepancy of stress intensity factor for different relative crack-tip element length (l/a) between BEM and Fett (2002) for Problem A ($a/R = 0.2$), Problem B ($a/R = 0.4$) and Problem C ($a/W = 0.6$)

Fig. 3.10: Percent discrepancy of T -stress for different relative crack-tip element length (l/a) between BEM and Fett (2002) for Problem A ($a/R = 0.2$)

Fig. 3.11: Percent discrepancy of T -stress for different relative crack-tip element length (l/a) between BEM and Fett (2002) for Problem B ($a/R = 0.4$)

Fig. 3.12: Percent discrepancy of T -stress for different relative crack-tip element length (l/a) between BEM and Fett (2002) for Problem C ($a/W = 0.6$)

Fig. 3.13: Variation of normalized T -stress with relative crack lengths for Problem A

Fig. 3.14: Variation of normalized stress intensity factors with relative crack lengths for Problem A

Fig. 3.15: The numerical model of one half of the Problem B geometry with displacement constraints on the plane of symmetry (dotted line)

Fig. 3.16: Variation of normalized T -stress with relative crack lengths for Problem B

Fig. 3.17: Variation of normalized stress intensity factors with relative crack lengths for Problem B

Fig. 3.18: Variation of normalized T -stress with relative crack lengths for Problem C

Fig. 3.19: Variation of normalized stress intensity factors with relative crack lengths for Problem C

Fig. 3.20: Problem D - U-notch cracked plate under remote tension

Fig. 3.21: A typical BEM mesh showing element size and contour integral for Problem D; $L/W = 0.3$, $H/W = 3.0$, $R/W = 0.025$, $a/R = 1.0$, $r/a = 0.5$

Fig. 3.22: A typical FEM mesh used by Lewis (2005) for the U-notch geometry

Fig. 3.23: Variation of normalized T -stress with relative crack lengths for Problem D

Fig. 3.24: Problem E - Double-cracks emanating from a circular hole in an infinite plate under remote tension

Fig. 3.25: A typical BEM mesh showing element size and contour integral for Problem E;

$$a/R = 2.0, r/a = 0.5$$

Fig. 3.26: Variation of normalized T -stress with relative crack lengths for Problem E

Fig. 3.27: Variation of normalized stress intensity factors with relative crack lengths for Problem E

CHAPTER 4 ELASTIC T -STRESS AND STRESS INTENSITY FACTOR SOLUTIONS FOR CRACKS EMANATING FROM CIRCULAR HOLES IN A RECTANGULAR PLATE

4.0 Introduction

This chapter examines the effect of geometry and loading conditions on elastic T -stress and stress intensity factors for cracks emanating from the edge of a circular hole in a rectangular plate. In Chapter 3, the effects of such cracks in an infinite plate were studied. The evaluation of similar types of geometry will be further investigated. First, the influence of a crack or cracks emanating from a single circular hole will be analyzed. The T -stress and stress intensity factor solutions will then be compared to those obtained for geometries involving multiple circular holes. Table 4.1 lists the problems analyzed in this chapter.

4.1 ***T*-Stress and Stress Intensity Factor from Crack(s) Emanating from a Circular Hole**

T-stress and stress intensity factor of a crack or cracks emanating from a single circular hole will be evaluated in this section. First, double-cracks in a finite plate under remote tension and bending will be discussed. Single-crack configuration for an infinite plate under remote tension will be studied. The analysis of this crack configuration is then extended to finite plate under remote tension and bending.

4.1.1 **Finite Plate with Double-Cracks Emanating from the Edge of a Circular Hole under Remote Tension (Problem F)**

Two-dimensional BEM was employed to analyze the effects on *T*-stress and stress intensity factor of double-cracks emanating from a circular hole in a finite plate under remote tension. The geometry of the problem is defined by the plate width $2W$, plate height $2H$, and hole radius R , as shown in Fig. 4.1. Similar to Problem E in the previous chapter, the domain needed to be analyzed can be reduced to a quarter of its physical domain, due to symmetry, as shown in Fig. 4.2. Figure 4.3 shows the BEM mesh used for analysis. The stress intensity factor and *T*-stress solutions for the relative hole radius R/W of 0.25 and 0.50 and relative height-to-width ratio (H/W) of 2 was analyzed. Relative crack lengths a/R of 1.1, 1.2, 1.6, 2, 2.4, 2.8, 3.2 were analyzed for the $R/W = 0.25$ case, and a/R of 1.1-1.6 for the $R/W = 0.50$ case.

Tables 4.2 and 4.3 show the results for the normalized T -stress T/σ and stress intensity factor $K_I/\sigma\sqrt{\pi(a-R)}$ for $R/W = 0.25$ and $R/W = 0.50$, respectively. The normalized stress intensity factor solutions obtained are compared to those by Newman (1971); generally very good agreement of less than 2% discrepancy was achieved between the two sets of results for both R/W cases. The results were also shown graphically in Fig. 4.4 and 4.5 for T/σ and $K_I/\sigma\sqrt{\pi(a-R)}$, respectively. It can be seen in Fig. 4.4 that the normalized T -stress results for all cases are negative, thus indicating low constraint conditions. Higher T -stress gradient was noticed as R/W increases, illustrating the significant influence of the finite boundary. The presence of the circular hole was also noticed from the dramatic incline for small cracks, and decline for larger cracks. With small cracks, the stress distribution in the vicinity of the hole is significantly altered due to the presence of the hole. As the crack becomes longer, there is less influence from the circular hole, which explains the declining trend in the larger cracks region. Furthermore, it is evident that a finite boundary causes the T -stress to continually decrease, while the infinite boundary case shows an asymptotic trend that levels off for increasing crack lengths.

The effect of changing the height-to-width ratio H/W was also studied. The height-to-width ratio of 2.5 and 3 were tested as well. The normalized stress intensity factor and T -stress solutions are listed in Tables 4.4 and 4.5, and shown graphically in Fig. 4.6 – 4.9. It is evident that all stress intensity factor and T -stress values remained almost constant for the three height-to-width ratios. This suggests that any testing of this type of geometry need not exceed a height-to-width ratio of 2.

4.1.2 Finite Plate with Double-Cracks Emanating from the Edge of a Circular Hole under Remote Bending (Problem G)

The geometry of double-cracks emanating from a circular hole under remote bending problem is identical to that of Problem F, with only a change in the loading conditions, as shown in Fig. 4.10. The numerical analysis can be reduced to half of its physical domain, as shown in Fig. 4.11. A nodal constraint in the x_I -direction is again required to preclude rigid body motion. Figure 4.12 shows typical BEM mesh used for analysis. Note that two different contours are required to evaluate the two crack-tips, although the primary focus will be the results from the tension side of the plate under the bending load. For comparison with the tension problem, relative crack lengths a/R of 1.1, 1.2, 1.6, 2, 2.4, 2.8, 3.2 for the relative hole radius $R/W = 0.25$ case, and a/R of 1.1-1.6 for the $R/W = 0.50$ case were again analyzed.

The normalized stress intensity factor and T -stress solutions are presented in Tables 4.6 and 4.7 for $R/W = 0.25$ and 0.50, respectively. Figures 4.13 and 4.14 compare the remote bending and remote tension solutions for T -stress and K_I , respectively. It is evident that the bending load has resulted in smaller magnitude than the tension case for both the normalized K_I and T -stress, due to the compression portion of the applied load. The variation of the bending solution with a/R is also noticed to be more gradual, especially for small cracks where more rapid changes in K_I and T -stress are present for the remote tension problem. As before, all normalized T -stress solutions obtained are negative in magnitude.

4.1.3 Infinite Plate with a Single Crack Emanating from the Edge of a Circular Hole under Remote Tension (Problem H)

The problem with a single crack emanating from a circular hole in an infinite plate (Fig. 4.15) was analyzed using two-dimensional BEM. The numerical domain of this problem can be reduced to half of its physical domain, due to symmetry. Figure 4.16 shows a typical BEM mesh for this problem; as before in Problem E, the boundaries of the numerical domain is set at 20 times the crack length. The boundary conditions of this problem are applied in the same manner as for Problem C, with the plane of symmetry constrained in the x_2 -direction, and a nodal constraint in the x_1 -direction to preclude rigid body motion. Relative crack lengths a/R of 1.1 – 4.0 were analyzed.

The normalized stress intensity factor, $K_I / \sigma \sqrt{\pi(a-R)}$, and T -stress, T/σ , solutions are listed in Table 4.8 and illustrated in Fig. 4.17 and 4.18. Very good agreement of generally less than 3% discrepancy was achieved for K_I between the results obtained from BEM and those by Newman (1971). It is shown in Fig. 4.17 that the normalized T -stress has the same trends as the double-crack case (Problem E) and is negative in magnitude. Lower constraint at the crack-tip is therefore implied from the T -stress obtained. It is observed that the T -stress for the single crack case peaks at a higher value and has a steeper decline than the corresponding double-crack case. Since the T -stress solutions for both the single and double crack problems are similar, it is evident that, for rough approximation, evaluation of T -stress is only necessary to be performed once.

4.1.4 Finite Plate with a Single Crack Emanating from the Edge of a Circular Hole under Remote Tension (Problem I)

For the finite plate with a single crack emanating from a circular hole problem, as shown in Fig. 4.19, symmetry can again be used for BEM analysis, and only half of the domain needs to be modelled. Figure 4.20 shows the typical BEM mesh used for this geometry. The boundary conditions used for this problem are identical to those applied for Problem H. Relative crack lengths a/R of 1.1, 1.2, 1.6, 2, 2.4, 2.8, 3.2 were analyzed for the relative hole radius $R/W = 0.25$ case, and a/R of 1.1-1.6 for the $R/W = 0.50$ case (these ratios are identical to Problem F for comparison purposes).

Tables 4.9 and 4.10 show the normalized stress intensity factor $K_I / \sigma \sqrt{\pi(a-R)}$ and T -stress T/σ solutions for the two different hole radii cases. Figures 4.21 and 4.22 compare the normalized T -stress and stress intensity factor results, respectively, with the infinite plate case (Problem H). It can be seen that the normalized T -stress solutions exhibit a peak value, then declines as a/R is increased for all R/W cases. The case with the largest circular hole, i.e. $R/W = 0.50$, exhibits a significantly more severe variation than the other two. As for K_I , it is evident that the infinite plate problem has the lowest magnitude, while the presence of the finite boundary increased the crack-tip stress intensity factor for the other two cases.

From the results obtained from BEM analysis, it was found that the normalized stress intensity factor and T -stress trends were similar for both finite plate problems with single and double-cracks. Tables 4.11 and 4.12 compare the two sets of data, and Fig. 4.23 and 4.24 illustrate them graphically. It is evident that the difference between the

single and double-crack cases increases significantly as a/R increases for the normalized stress intensity factor for both R/W ratios. This was expected as the asymmetry of the single crack problem reduces the nominal net-section stress at the plane of the crack, resulting in a much lower stress intensity factor. The cause for the increase in difference between the two crack configurations as a/R increases stems from the fact that the geometry becomes increasingly asymmetric, which enlarges the reduction in the magnitude of the nominal net-section stress. The asymmetry aspect also holds true for the normalized T -stress, as the difference between single and double-crack increases with a/R . It should be noted that the percentage difference for K_I is higher than that for T/σ , especially for large a/R ratios, which implies that the stress intensity factor is much more sensitive to geometric asymmetry than the T -stress. Figure 4.23 more clearly illustrates this fact as the single and double crack configuration values remain close to each other, and diverges slightly as a/R is increased. The implication here is that for small crack lengths, analysis for T -stress is only necessary on either the single or double-crack geometry, with the results being fairly accurate for both configurations. Again, similar to previous geometries tested, the normalized T -stress solutions are negative in magnitude.

4.1.5 Finite Plate with a Single Crack Emanating from the Edge of a Circular Hole under Remote Bending (Problem J)

This problem has the exact geometry of Problem G, only with a single crack instead of two, as shown in Fig. 4.25. BEM analysis was performed by modelling half of the physical domain, as shown in Fig. 4.26, with the boundary conditions similar to that of

Problem H. The same relative crack lengths a/R of 1.1, 1.2, 1.6, 2, 2.4, 2.8, 3.2 were analyzed for the relative hole radius $R/W = 0.25$ case, and a/R of 1.1-1.6 for the $R/W = 0.50$ case.

The solutions for normalized stress intensity factor and T -stress are presented in Tables 4.13 and 4.14, and are compared to those of the double-crack case. Figures 4.27 and 4.28 graphically illustrate the T -stress and stress intensity factor results, respectively. It is evident that both the K_I and T -stress solutions for the single crack case followed the same trend as the double-crack case. The stress intensity factor solutions can be observed to diverge between the two crack configurations for longer crack lengths (up to 14.5% difference). On the other hand, the T -stress solutions showed that it is less sensitive to the crack configuration, even for longer cracks. As shown in Table 4.13, a difference of only 4.7% exists between the single and double-crack problem for the longest crack analyzed, i.e. $a/R = 3.2$.

4.2 Influence of Adjacent Holes on T -Stress and Stress Intensity Factor

T -stress and stress intensity factor solutions will be presented for geometries that have multiple holes in this section. The solutions obtained will also be compared to those of the single-hole geometry. The geometries considered include: double and single crack emanating from a periodic array of holes in an infinite plate under remote tension, and the influence of adjacent holes not in the plane of the crack on double-cracks emanating from a circular hole in an infinite plate under remote tension.

4.2.1 Infinite Plate with Double-Cracks Emanating from a Periodic Array of Holes under Remote Tension (Problem K)

The problem of periodic array of holes with double-cracks (Fig. 4.29) is essentially an infinite array of the Problem F geometry. A section of the physical domain was modelled with the same boundary conditions as Problem F, with the addition of an x_I -direction constraint on the extra plane of symmetry, as shown in Fig. 4.30. Figure 4.31 shows a typical BEM mesh for the problem at hand. Relative height-to-width ratio H/W of 2 was used as the domain geometry, since it was established in Section 4.1.1 that greater H/W values yielded the same results. The relative crack lengths a/R and hole radius R/W considered are identical to those of Problem F, namely, a/R of 1.1, 1.2, 1.6, 2, 2.4, 2.8, 3.2 for the $R/W = 0.25$ case, and a/R of 1.1-1.6 for the $R/W = 0.50$ case.

The numerical results for the stress intensity factor and T -stress are listed in Tables 4.15 and 4.16. Compared to the corresponding solutions for the finite plate problem (Problem F) in Fig. 4.32 and 4.33, it can be seen that K_I is significantly reduced for the infinite array case, especially for the $R/W = 0.50$ case. The normalized T -stress solutions decreased very gradually for increasing a/R values (when the uncracked portion of the plate become small), in contrast to the declining behaviour in T -stress for the finite plate case. As explained in Section 4.1.1, the T -stress declines for the finite plate as the effect from the presence of the hole is reduced when the crack grows longer. This effect is negated as an infinite array creates extra constraint between the holes, which explains the large deviation between the two sets of data.

4.2.2 Infinite Plate with a Single Crack Emanating from a Periodic Array of Holes under Remote Tension (Problem L)

The geometry of Problem L is composed of an infinite array of the Problem I geometry, as shown in Fig. 4.34. The numerical modelling of this geometry is only necessary on a small section of physical domain, as shown in Fig. 4.35. The height-to-width ratio H/W is again taken to be 2 for convenience. Figure 4.35 also shows the boundary conditions of the BEM mesh, where the two vertical boundaries are constrained in the x_I -direction to represent the infinite array. A typical BEM mesh for this problem is shown in Fig. 4.36.

The normalized solutions for stress intensity factor and T -stress are given in Tables 4.17 and 4.18. Figures 4.37 and 4.38 compare these results to those of the finite plate problem (Problem H). It is evident that the trends follow those described in Section 4.2.1, where the stress intensity factor is substantially reduced and the T -stress declines at a gradual pace for increasing crack lengths. The numerically determined results are also compared against the infinite array with double-crack case (Problem K), as shown in Fig. 4.39 and 4.40. It can be seen that the K_I solutions of the two crack configurations diverged substantially as a/R increased, while the T -stress solutions exhibited only small changes. For both R/W cases the T -stress solutions begin to diverge only when $a/R > 1.5$, suggesting that for small relative crack lengths, the T -stress for one crack configuration is also accurate for the other.

4.2.3 Infinite Plate with Double-Cracks Emanating from the Edge of a Circular Hole Influence by Adjacent Holes under Remote Tension (Problem M)

The effects of adjacent holes not in the plane of the crack on double-cracks emanating from a circular hole are studied in this section. The geometry of the problem is shown in Fig. 4.41, with hole radius R_1 , adjacent hole radius R_2 , hole-distance d , and crack length a . Only a quarter of the physical domain needs to be modelled, by virtue of symmetry. Figure 4.42 shows the boundary conditions applied and Fig. 4.43 shows a typical BEM mesh used for analysis. Relative crack lengths a/R_1 of 1.1, 1.2, 1.4, 1.6, 2, 3, 4, and 10 was analyzed for relative hole-distance d^* of 2.5, 4, 5, 10 and radius ratio α of 0 – 1.0, where $d^* = d/R_1$ and $\alpha = R_2/R_1$.

The BEM results for normalized stress intensity factor $K_I / \sigma \sqrt{\pi(a - R_1)}$ and T -stress T/σ are shown in Tables 4.19 – 4.22. The results are also shown graphically for the different radius ratios α in Fig. 4.44 – 4.51. It can be seen that for $\alpha = 0.25$, only very small deviations for both T -stress and K_I solutions exist from the case with no adjacent holes ($R_2 = 0$), for all d^* ratios. This is especially true for $a/R_1 < 3$, as the numerical values of both T -stress and K_I for all d^* ratios are, for all practical purposes, identical. For larger α values, it is evident that the relative hole-distance d^* contributes significantly for both T -stress and K_I . The effects can be easily noticed in Fig. 4.50 for T -stress and Fig. 4.51 for K_I , where the solutions for $\alpha = 1.0$ are significantly affected. The normalized stress intensity factor decreased significantly as d^* decreased, illustrating the effects as the adjacent holes move closer to the central-hole. The normalized T -stress

solutions showed that as d^* was reduced, the peak of the curve also decreased, while exhibiting a more gradual trend.

To understand the effects of the radius ratio α , the T -stress and K_I were again plotted for a specific d^* value, as shown in Fig. 4.52 – 4.59. It is clear that the effect for increasing the size of the adjacent holes is similar to that of decreasing the distance between the holes. It is evident that for very short cracks, i.e. $a/R_I < 1.4$, enlarging the adjacent holes results in corresponding decreases in magnitude in K_I from the $\alpha = 0$ case. Figures 4.52 and 4.53 best illustrate this behaviour due to the close proximity of the adjacent holes. It is indicated that the magnitude of the T -stress is also reduced, but remains negative. For cracks in the range of $1.4 < a/R_I < 4$, it is evident that the introduction for adjacent holes decreased both the T -stress and K_I . The implication of this is that by having adjacent holes near a circular hole with small double-cracks, the crack-tip constraint can be substantially reduced, if this is permitted.

It can be observed that for longer crack lengths, such as when $a/R_I > 4$, the effects of the adjacent holes are minimal as the solution for all α values converged. This implies that for large cracks, the presence of adjacent holes is not very significant for T -stress and K_I analysis. Furthermore, it can be seen that for adjacent holes located far from the central-hole (Fig. 4.58 and 4.59), the deviations between the different α values are small. This suggests that for relative hole-distance $d^* > 10$, the problem can be simplified to that of the infinite plate problem with only one circular hole (Problem E).

4.3 Summary

Elastic T -stress and stress intensity factor solutions for cracked plates involving circular holes were determined in this chapter. For all geometries and loading conditions, it was observed that for cracks emanating from circular holes, the T -stress first becomes less negative over relatively small increases in crack length. Beyond a certain relative crack length, the decrease in constraint (i.e. T -stress become more negative in magnitude) begins to occur. Of significance to note is that for all the geometries and loading conditions analyzed, the T -stress obtained is negative in value. Low crack-tip constraint is therefore indicated, implying that the stress field characterized solely by the stress intensity factor may be overly conservative. From the present work, the following observations can also be made:

- The presence of a finite boundary further accentuates the rate of change of the above described behaviour, as expected.
- The stress intensity factors obtained for remote bending loads were determined to be significantly less than those for remote tension loads.
- Comparisons between single and double crack configurations indicated that the T -stress solutions only diverged beyond a relatively large crack size for remote tension loaded problems.
- The problem with an infinite array of holes indicated that the T -stress declines more gradually for increasing crack lengths, as compared to the steeper declining behaviour for the single-hole problem.

- The influence of adjacent holes not in the plane of the crack in an infinite plate was also studied. Both the adjacent hole-radius and the distance between holes were shown to have an effect on both the T -stress and stress intensity factor. Increasing the adjacent hole-radius significantly reduces the stress intensity factor, while generating gentler gradients in the T -stress behaviour. By locating the adjacent holes closer to the central-hole, the maximum value of the T -stress decreases and the stress intensity factor is also reduced. Moreover, the introduction of adjacent holes far from the centrally-cracked hole has little effect on both the T -stress and the stress intensity factor.

Table 4.1: List of problems analyzed in Chapter 4

Problem	Definition	Figure #
F	Finite Plate with Double-Cracks Emanating from the Edge of a Circular Hole Under Remote Tension	4.1
G	Finite Plate with Double-Cracks Emanating from the Edge of a Circular Hole Under Remote Bending	4.10
H	Infinite Plate with a Single Crack Emanating from the Edge of a Circular Hole Under Remote Tension	4.15
I	Finite Plate with a Single Crack Emanating from the Edge of a Circular Hole Under Remote Tension	4.19
J	Finite Plate with a Single Crack Emanating from the Edge of a Circular Hole Under Remote Bending	4.25
K	Infinite Plate with Double-Cracks Emanating from a Periodic Array of Holes Under Remote Tension	4.29
L	Infinite Plate with a Single Crack Emanating from a Periodic Array of Holes Under Remote Tension	4.34
M	Infinite Plate with Double-Cracks Emanating from the Edge of a Circular Hole Influence by Adjacent Holes Under Remote Tension	4.41

Table 4.2: Variation of normalized stress intensity factors and T -stress with relative crack lengths for Problem F; $R/W = 0.25$

a/R	$\frac{K}{\sigma \sqrt{\pi (a - R)}}$			T/σ
	NEWMAN (1971)	BEM	% DIFF	
1.1	3.0579	3.1069	1.60	-1.1331
1.2	2.6837	2.6690	-0.55	-0.8295
1.6	2.0207	2.0092	-0.57	-0.6332
2	1.8587	1.8379	-1.12	-0.8196
2.4	1.8569	1.8504	-0.35	-1.1155
2.8	2.0030	2.0038	0.04	-1.5561
3.2	2.3415	2.3877	1.97	-2.2696

Table 4.3: Variation of normalized stress intensity factors and T -stress with relative crack lengths for Problem F; $R/W = 0.50$

a/R	$\frac{K}{\sigma \sqrt{\pi (a - R)}}$			T/σ
	NEWMAN (1971)	BEM	% DIFF	
1.1	4.0957	4.1420	1.13	-1.4563
1.2	3.7193	3.7197	0.01	-1.1085
1.3	3.5297	3.5182	-0.32	-1.0265
1.4	3.4543	3.4444	-0.29	-1.1404
1.5	3.4918	3.4794	-0.36	-1.4282
1.6	3.6182	3.6199	0.05	-1.9362

Table 4.4: Effect of height-to-width ratio (H/W) on stress intensity factor and T -stress for Problem F; $R/W = 0.25$

a/R	$\frac{K}{\sigma \sqrt{\pi(a-R)}}$			T/σ		
	$H/W = 2.0$	$H/W = 2.5$	$H/W = 3.0$	$H/W = 2.0$	$H/W = 2.5$	$H/W = 3.0$
1.1	3.1069	3.1025	3.1002	-1.1331	-1.1329	-1.1326
1.2	2.6690	2.6648	2.6643	-0.8359	-0.8351	-0.8351
1.6	2.0092	2.0053	2.0052	-0.6359	-0.6357	-0.6359
2.0	1.8379	1.8342	1.8342	-0.8345	-0.8347	-0.8352
2.4	1.8504	1.8472	1.8472	-1.1397	-1.1406	-1.1417
2.8	2.0038	1.9888	1.9893	-1.5832	-1.5923	-1.5958
3.2	2.3877	2.3167	2.3165	-2.3658	-2.4279	-2.4273

Table 4.5: Effect of height-to-width ratio (H/W) on stress intensity factor and T -stress for Problem F; $R/W = 0.50$

a/R	$\frac{K}{\sigma \sqrt{\pi(a-R)}}$			T/σ		
	$H/W = 2.0$	$H/W = 2.5$	$H/W = 3.0$	$H/W = 2.0$	$H/W = 2.5$	$H/W = 3.0$
1.1	4.1420	4.1205	4.1194	-1.4563	-1.4532	-1.4529
1.2	3.7197	3.7006	3.6995	-1.1085	-1.1069	-1.1068
1.3	3.5182	3.5009	3.5000	-1.0265	-1.0295	-1.0296
1.4	3.4444	3.4287	3.4278	-1.1404	-1.1483	-1.1486
1.5	3.4794	3.4659	3.4652	-1.4282	-1.4443	-1.4446
1.6	3.6199	3.6065	3.6058	-1.9362	-1.9484	-1.9488

Table 4.6: Variation of normalized stress intensity factors and T -stress with relative crack lengths for Problem G; $R/W = 0.25$

a/R	$\frac{K}{\sigma\sqrt{\pi(a-R)}}$	T/σ
1.1	0.5026	-0.2133
1.2	0.4307	-0.1835
1.6	0.3678	-0.2895
2.0	0.3867	-0.4384
2.4	0.4366	-0.5958
2.8	0.5143	-0.7892
3.2	0.6432	-1.0964

Table 4.7: Variation of normalized stress intensity factors and T -stress with relative crack lengths for Problem G; $R/W = 0.50$

a/R	$\frac{K}{\sigma\sqrt{\pi(a-R)}}$	T/σ
1.1	1.0622	-0.4291
1.2	0.9644	-0.4045
1.3	0.9187	-0.4602
1.4	0.9074	-0.5688
1.5	0.9278	-0.7247
1.6	0.9816	-0.9414

Table 4.8: Variations of normalized stress intensity factors and T -stress with relative crack lengths for Problem H

a/R	$\frac{K}{\sigma \sqrt{\pi(a-R)}}$			T/σ
	NEWMAN (1971)	BEM	% DIFF	
1.1	2.7900	2.8707	2.89	-1.0573
1.2	2.3528	2.4160	2.69	-0.7587
1.4	1.8779	1.9201	2.25	-0.5502
1.6	1.6357	1.6372	0.09	-0.5303
1.8	1.4758	1.4584	-1.18	-0.5664
2.0	1.3667	1.3362	-2.23	-0.6164
2.2	1.2811	1.2495	-2.47	-0.6664
2.4	1.2175	1.1837	-2.77	-0.7117
2.6	1.1611	1.1339	-2.34	-0.7514
2.8	1.1141	1.0942	-1.79	-0.7857
4.0	0.9430	0.9666	2.50	-0.9152

Table 4.9: Variation of normalized stress intensity factors and T -stress with relative crack lengths for Problem I; $R/W = 0.25$

a/R	$\frac{K}{\sigma \sqrt{\pi(a-R)}}$	T/σ
1.1	3.0719	-1.1298
1.2	2.6082	-0.8126
1.6	1.8077	-0.6126
2.0	1.5300	-0.7950
2.4	1.4350	-1.0432
2.8	1.4505	-1.3776
3.2	1.5928	-1.9479

Table 4.10: Variation of normalized stress intensity factors and T -stress with relative crack lengths for Problem I; $R/W = 0.50$

a/R	$\frac{K}{\sigma \sqrt{\pi(a-R)}}$	T/σ
1.1	4.0741	-1.4524
1.2	3.5691	-1.0676
1.3	3.2647	-0.9798
1.4	3.0813	-1.0872
1.5	2.9989	-1.3553
1.6	3.0106	-1.8006

Table 4.11: Comparison of normalized stress intensity factors and T -stress between single crack and double-cracks emanating from a circular hole in a finite plate under remote tension; $R/W = 0.25$

a/R	$\frac{K}{\sigma \sqrt{\pi(a-R)}}$			T/σ		
	1 crack	2 cracks	% DIFF*	1 crack	2 cracks	% DIFF*
1.1	3.0719	3.1069	1.13	-1.1298	-1.1331	0.29
1.2	2.6082	2.6690	2.28	-0.8126	-0.8295	2.03
1.6	1.8077	2.0092	10.03	-0.6126	-0.6332	3.26
2.0	1.5300	1.8379	16.75	-0.7950	-0.8196	3.01
2.4	1.4350	1.8504	22.45	-1.0432	-1.1155	6.48
2.8	1.4505	2.0038	27.62	-1.3776	-1.5561	11.47
3.2	1.5928	2.3877	33.29	-1.9479	-2.2696	14.17

$$* \quad \%DIFF = \left(\frac{2cracks - 1crack}{2cracks} \right) \cdot 100\%$$

Table 4.12: Comparison of normalized stress intensity factors and T -stress between single crack and double-cracks emanating from a circular hole in a finite plate under remote tension; $R/W = 0.50$

a/R	$\frac{K}{\sigma \sqrt{\pi(a-R)}}$			T/σ		
	1 crack	2 cracks	% DIFF*	1 crack	2 cracks	% DIFF*
1.1	4.0741	4.1420	1.64	-1.4524	-1.4563	0.27
1.2	3.5691	3.7197	4.05	-1.0676	-1.1085	3.69
1.3	3.2647	3.5182	7.21	-0.9798	-1.0265	4.55
1.4	3.0813	3.4444	10.54	-1.0872	-1.1404	4.67
1.5	2.9989	3.4794	13.81	-1.3553	-1.4282	5.10
1.6	3.0106	3.6199	16.83	-1.8006	-1.9362	7.00

$$* \quad \%DIFF = \left(\frac{2cracks - 1crack}{2cracks} \right) \cdot 100\%$$

Table 4.13: Comparison of normalized stress intensity factors and T -stress between single crack and double-cracks emanating from a circular hole in a finite plate under remote bending; $R/W = 0.25$

a/R	$\frac{K}{\sigma \sqrt{\pi(a-R)}}$			T/σ		
	1 crack	2 cracks	% DIFF*	1 crack	2 cracks	% DIFF*
1.1	0.5055	0.5026	-0.57	-0.2141	-0.2133	-0.35
1.2	0.4398	0.4307	-2.12	-0.1860	-0.1835	-1.36
1.6	0.3980	0.3678	-8.24	-0.2930	-0.2895	-1.21
2.0	0.4325	0.3867	-11.82	-0.4443	-0.4384	-1.35
2.4	0.4964	0.4366	-13.71	-0.6093	-0.5958	-2.27
2.8	0.5889	0.5143	-14.52	-0.8167	-0.7892	-3.48
3.2	0.7346	0.6432	-14.22	-1.1474	-1.0964	-4.65

$$* \quad \%DIFF = \left(\frac{2cracks - 1crack}{2cracks} \right) \cdot 100\%$$

Table 4.14: Comparison of normalized stress intensity factors and T -stress between single crack and double-cracks emanating from a circular hole in a finite plate under remote bending; $R/W = 0.50$

a/R	$\frac{K}{\sigma \sqrt{\pi(a-R)}}$			T/σ		
	1 crack	2 cracks	% DIFF*	1 crack	2 cracks	% DIFF*
1.1	1.0756	1.0622	-1.26	-0.4336	-0.4291	-1.05
1.2	1.0010	0.9644	-3.80	-0.4130	-0.4045	-2.10
1.3	0.9796	0.9187	-6.63	-0.4705	-0.4602	-2.24
1.4	0.9914	0.9074	-9.26	-0.5812	-0.5688	-2.18
1.5	1.0341	0.9278	-11.45	-0.7421	-0.7247	-2.40
1.6	1.1099	0.9816	-13.07	-0.9699	-0.9414	-3.03

Table 4.15: Variation of normalized stress intensity factors and T -stress with relative crack lengths for Problem K; $R/W = 0.25$

a/R	$\frac{K}{\sigma \sqrt{\pi(a-R)}}$	T/σ
1.1	2.9017	-1.0708
1.2	2.2813	-0.7891
1.6	1.7701	-0.5391
2.0	1.6202	-0.5704
2.4	1.6079	-0.6191
2.8	1.6911	-0.6550
3.2	1.9118	-0.6817

Table 4.16: Variation of normalized stress intensity factors and T -stress with relative crack lengths for Problem K; $R/W = 0.50$

a/R	$\frac{K}{\sigma \sqrt{\pi (a - R)}}$	T/σ
1.1	2.9989	-1.1670
1.2	2.6980	-0.9253
1.3	2.5386	-0.7875
1.4	2.4696	-0.7084
1.5	2.4822	-0.6621
1.6	2.5760	-0.6331

Table 4.17: Variation of normalized stress intensity factors and T -stress with relative crack lengths for Problem L; $R/W = 0.25$

a/R	$\frac{K}{\sigma \sqrt{\pi (a - R)}}$	T/σ
1.1	2.8724	-1.0706
1.2	2.2414	-0.7703
1.6	1.6261	-0.5320
2.0	1.3926	-0.5918
2.4	1.2918	-0.6685
2.8	1.2633	-0.7331
3.2	1.3042	-0.7909

Table 4.18: Variation of normalized stress intensity factors and T -stress with relative crack lengths for Problem L; $R/W = 0.50$

a/R	$\frac{K}{\sigma \sqrt{\pi(a-R)}}$	T/σ
1.1	2.9680	-1.1647
1.2	2.6286	-0.8957
1.3	2.4138	-0.7496
1.4	2.2767	-0.6667
1.5	2.2044	-0.6180
1.6	2.1897	-0.5900

Table 4.19: Variation of normalized stress intensity factors and T -stress with relative crack lengths and radius ratios $\alpha = R_2/R_1$ for Problem M; $d^* = d/R_1 = 2.5$

a/R_1	$K_I / \sigma \sqrt{\pi(a-R_1)}$					T/σ				
	$\alpha=0$	$\alpha=0.25$	$\alpha=0.50$	$\alpha=0.75$	$\alpha=1.0$	$\alpha=0$	$\alpha=0.25$	$\alpha=0.50$	$\alpha=0.75$	$\alpha=1.0$
1.1	2.8396	2.8362	2.7081	2.4700	2.0694	-1.0471	-1.0512	-1.0051	-0.9204	-0.7788
1.2	2.4463	2.4229	2.3168	2.1258	1.8258	-0.7951	-0.7635	-0.7389	-0.6945	-0.6204
1.4	N/A	1.9840	1.9160	1.7882	1.5749	N/A	-0.5678	-0.5759	-0.5866	-0.5878
1.6	1.7555	1.7352	1.6811	1.5963	1.4634	-0.5319	-0.5369	-0.5562	-0.5851	-0.6177
2.0	1.5011	1.4892	1.4718	1.4249	1.3605	-0.6037	-0.6018	-0.6251	-0.6621	-0.7066
3.0	1.2803	1.2955	1.2941	1.2905	1.2844	-0.7776	-0.7859	-0.7942	-0.8078	-0.8241
4.0	1.2103	1.2357	1.2372	1.2400	1.2453	-0.8809	-0.8905	-0.8902	-0.8890	-0.8835
10.0	1.0827	1.1129	1.1137	1.1157	1.1181	-0.9901	-1.0067	-1.0010	-0.9904	-0.9722

Table 4.20: Variation of normalized stress intensity factors and T -stress with relative crack lengths and radius ratios $\alpha = R_2/R_1$ for Problem M; $d^* = d/R_1 = 4.0$

a/R_1	$K_I / \sigma \sqrt{\pi(a - R_1)}$					T/σ				
	$\alpha=0$	$\alpha=0.25$	$\alpha=0.50$	$\alpha=0.75$	$\alpha=1.0$	$\alpha=0$	$\alpha=0.25$	$\alpha=0.50$	$\alpha=0.75$	$\alpha=1.0$
1.1	2.8396	2.8396	2.7324	2.5552	2.3149	-1.0471	-1.0541	-1.0159	-0.9509	-0.8636
1.2	2.4463	2.4280	2.3430	2.2026	2.0125	-0.7951	-0.7649	-0.7421	-0.7043	-0.6526
1.4	N/A	1.9828	1.9188	1.8122	1.6655	N/A	-0.5647	-0.5605	-0.5528	-0.5391
1.6	1.7555	1.7505	1.7023	1.6227	1.5118	-0.5319	-0.5383	-0.5415	-0.5455	-0.5486
2.0	1.5011	1.5055	1.4760	1.4267	1.3568	-0.6037	-0.6109	-0.6223	-0.6393	-0.6595
3.0	1.2803	1.2924	1.2838	1.2679	1.2443	-0.7776	-0.7881	-0.8001	-0.8188	-0.8422
4.0	1.2103	1.2340	1.2316	1.2262	1.2186	-0.8809	-0.8935	-0.9007	-0.9126	-0.9287
10.0	1.0827	1.1126	1.1129	1.1137	1.1147	-0.9901	-1.0214	-1.0076	-1.0049	-1.0022

Table 4.21: Variation of normalized stress intensity factors and T -stress with relative crack lengths and radius ratios $\alpha = R_2/R_1$ for Problem M; $d^* = d/R_1 = 5.0$

a/R_1	$K_I / \sigma \sqrt{\pi(a - R_1)}$					T/σ				
	$\alpha=0$	$\alpha=0.25$	$\alpha=0.50$	$\alpha=0.75$	$\alpha=1.0$	$\alpha=0$	$\alpha=0.25$	$\alpha=0.50$	$\alpha=0.75$	$\alpha=1.0$
1.1	2.8396	2.8458	2.7640	2.6291	2.4452	-1.0471	-1.0575	-1.0274	-0.9782	-0.9109
1.2	2.4463	2.4323	2.3687	2.2642	2.1200	-0.7951	-0.7667	-0.7488	-0.7191	-0.6784
1.4	N/A	1.9859	1.9346	1.8503	1.7343	N/A	-0.5647	-0.5581	-0.5468	-0.5304
1.6	1.7555	1.7528	1.7142	1.6501	1.5611	-0.5319	-0.5375	-0.5355	-0.5320	-0.5263
2.0	1.5011	1.5059	1.4797	1.4361	1.3750	-0.6037	-0.6096	-0.6132	-0.6187	-0.6251
3.0	1.2803	1.2922	1.2810	1.2624	1.2365	-0.7776	-0.7879	-0.7960	-0.8086	-0.8249
4.0	1.2103	1.2331	1.2288	1.2213	1.2106	-0.8809	-0.8944	-0.9015	-0.9131	-0.9284
10.0	1.0827	1.1118	1.1134	1.1142	1.1142	-0.9901	-1.0079	-1.0080	-1.0087	-1.0086

Table 4.22: Variation of normalized stress intensity factors and T -stress with relative crack lengths and radius ratios $\alpha = R_2/R_1$ for Problem M; $d^* = d/R_1 = 10.0$

a/R_1	$K_I / \sigma \sqrt{\pi(a - R_1)}$					T/σ				
	$\alpha=0$	$\alpha=0.25$	$\alpha=0.50$	$\alpha=0.75$	$\alpha=1.0$	$\alpha=0$	$\alpha=0.25$	$\alpha=0.50$	$\alpha=0.75$	$\alpha=1.0$
1.1	2.8396	2.8508	2.8229	2.7655	2.7104	-1.0471	-1.0577	-1.0479	-1.0421	-1.0117
1.2	2.4463	2.4352	2.4147	2.3690	2.3284	-0.7951	-0.7655	-0.7597	-0.7582	-0.7379
1.4	N/A	1.9827	1.9644	1.9285	1.8898	N/A	-0.5615	-0.5572	-0.5561	-0.5422
1.6	1.7555	1.7466	1.7325	1.7025	1.6747	-0.5319	-0.5307	-0.5274	-0.5289	-0.5171
2.0	1.5011	1.4918	1.4807	1.4572	1.4324	-0.6037	-0.5959	-0.5928	-0.5963	-0.5796
3.0	1.2803	1.2686	1.2611	1.2464	1.2308	-0.7776	-0.7700	-0.7678	-0.7735	-0.7628
4.0	1.2103	1.1942	1.1889	1.1791	1.1677	-0.8809	-0.8672	-0.8665	-0.8727	-0.8667
10.0	1.0827	1.0805	1.0801	1.0809	1.0790	-0.9901	-0.9822	-0.9834	-0.9867	-0.9882

Fig. 4.1: Problem F - Finite plate with double-cracks emanating from the edge of a circular hole under remote tension

Fig. 4.2: A quarter of the Problem F domain being modelled with displacement constraints on the planes of symmetry (dotted lines)

Fig. 4.3: A typical BEM mesh showing element size and contour integral for Problem F;
 $R/W = 0.25$, $a/R = 2.0$, $H/W = 2.0$, $r/(a-R) = 0.5$

Fig. 4.4: Variations of normalized T -stress with relative crack length a/R for Problem F

Fig. 4.5: Variations of normalized stress intensity factors with relative crack length a/R for Problem F

Fig. 4.6: Effect of height-to-width ratio (H/W) on T -stress for Problem F; $R/W = 0.25$

Fig. 4.7: Effect of height-to-width ratio (H/W) on stress intensity factor for Problem F;
 $R/W = 0.25$

Fig. 4.8: Effect of height-to-width ratio (H/W) on T -stress for Problem F; $R/W = 0.50$

Fig. 4.9: Effect of height-to-width ratio (H/W) on stress intensity factor for Problem F;

$$R/W = 0.50$$

Fig. 4.10: Problem G - Finite plate with double-cracks emanating from the edge of a circular hole under remote bending

Fig. 4.11: A half of the Problem G domain being modelled with displacement constraints on the planes of symmetry (dotted lines) and a nodal constraint at the lower right corner

Fig. 4.12: A typical BEM mesh showing element size and contour integral for Problem G; $R/W = 0.25$, $a/R = 2.0$, $H/W = 2.0$, $r/(a-R) = 0.5$

Fig. 4.13: Comparison of normalized T -stress between remote bending and remote tension on double-cracks emanating from a circular hole in a finite plate

Fig. 4.14: Comparison of normalized stress intensity factor between remote bending and remote tension on double-cracks emanating from a circular hole in a finite plate

Fig. 4.15: Problem H - Infinite plate with a single crack emanating from the edge of a circular hole under remote tension

Fig. 4.16: A typical BEM mesh showing element size and contour integral for Problem H; $a/R = 2.0$, $r/(a-R) = 0.5$

Fig. 4.17: Comparison of normalized T -stress between a single and double-crack configuration in an infinite plate under remote tension

Fig. 4.18: Comparison of normalized stress intensity factor between a single and double-crack configuration in an infinite plate under remote tension

Fig. 4.19: Problem I - Finite plate with a single crack emanating from the edge of a circular hole under remote tension

Fig. 4.20: A typical BEM mesh showing element size and contour integral for Problem I;

$$R/W = 0.25, a/R = 2.0, H/W = 2.0, r/(a-R) = 0.5$$

Fig. 4.21: Variations of normalized T -stress with relative crack lengths for Problem I

Fig. 4.22: Variations of normalized stress intensity factor with relative crack lengths for Problem I

Fig. 4.23: Comparison of normalized T -stress between single crack and double-cracks emanating from a circular hole in a finite plate under remote tension

Fig. 4.24: Comparison of normalized stress intensity factor between single crack and double-cracks emanating from a circular hole in a finite plate under remote tension

Fig. 4.25: Problem J - Finite plate with a single crack emanating from the edge of a circular hole under remote bending

Fig. 4.26: A typical BEM mesh showing element size and contour integral for Problem J;

$$R/W = 0.25, a/R = 2.0, H/W = 2.0, r/(a-R) = 0.5$$

Fig. 4.27: Comparison of normalized T -stress between single crack and double-cracks emanating from a circular hole in a finite plate under remote bending

Fig. 4.28: Comparison of normalized stress intensity factor between single crack and double-cracks emanating from a circular hole in a finite plate under remote bending

Fig. 4.29: Problem K - Infinite plate with double-cracks emanating from a periodic array of holes under remote tension

Fig. 4.30: A section of the Problem K domain being modelled with displacement constraints on the planes of symmetry (dotted lines)

Fig. 4.31: A typical BEM mesh showing element size and contour integral for Problem K; $R/W = 0.25$, $a/R = 2.0$, $H/W = 2.0$, $r/(a-R) = 0.5$

Fig. 4.32: Comparison of normalized T -stress between double-cracks in a finite and an infinite plate under remote tension

Fig. 4.33: Comparison of normalized stress intensity factor between double-cracks in a finite and an infinite plate under remote tension

Fig. 4.34: Problem L - Infinite plate with a single crack emanating from a periodic array of holes under remote tension

Fig. 4.35: A section of the Problem L domain being modelled with displacement constraints on the planes of symmetry (dotted lines)

Fig. 4.36: A typical BEM mesh showing element size and contour integral for Problem L;
 $R/W = 0.25$, $a/R = 2.0$, $H/W = 2.0$, $r/(a-R) = 0.5$

Fig. 4.37: Comparison of normalized T -stress between a single crack in a finite and an infinite plate under remote tension

Fig. 4.38: Comparison of normalized stress intensity factor between a single crack in a finite and an infinite plate under remote tension

Fig. 4.39: Comparison of normalized T -stress between single and double-crack configuration in an infinite array of holes under remote tension

Fig. 4.40: Comparison of normalized stress intensity factor between a single and double-crack configuration in an infinite array of holes under remote tension

Fig. 4.41: Problem M: Infinite plate with double-cracks emanating from the edge of a circular hole influence by adjacent holes under remote tension

Fig. 4.42: A quarter of the Problem M domain being modelled with displacement constraints on the planes of symmetry (dotted lines)

Fig. 4.43: A typical BEM mesh showing element size and contour integral for Problem M; $a/R_1 = 2.0$, $\alpha = R_2/R_1 = 0.5$, $d^* = d/R_1 = 4$, $r/(a-R_1) = 0.5$

Fig. 4.44: Variations of normalized T -stress with relative crack lengths for different relative hole distance $d^* = d/R_1$ for Problem M; $\alpha = R_2/R_1 = 0.25$

Fig. 4.45: Variations of normalized stress intensity factor with relative crack lengths for different relative hole distance $d^* = d/R_1$ for Problem M; $\alpha = R_2/R_1 = 0.25$

Fig. 4.46: Variations of normalized T -stress with relative crack lengths for different relative hole distance $d^* = d/R_1$ for Problem M; $\alpha = R_2/R_1 = 0.50$

Fig. 4.47: Variations of normalized stress intensity factor with relative crack lengths for different relative hole distance $d^* = d/R_1$ for Problem M; $\alpha = R_2/R_1 = 0.50$

Fig. 4.48: Variations of normalized T -stress with relative crack lengths for different relative hole distance $d^* = d/R_1$ for Problem M; $\alpha = R_2/R_1 = 0.75$

Fig. 4.49: Variations of normalized stress intensity factor with relative crack lengths for different relative hole distance $d^* = d/R_1$ for Problem M; $\alpha = R_2/R_1 = 0.75$

Fig. 4.50: Variations of normalized T -stress with relative crack lengths for different relative hole distance $d^* = d/R_1$ for Problem M; $\alpha = R_2/R_1 = 1.0$

Fig. 4.51: Variations of normalized stress intensity factor with relative crack lengths for different relative hole distance $d^* = d/R_1$ for Problem M; $\alpha = R_2/R_1 = 1.0$

Fig. 4.52: Variations of normalized T -stress with relative crack lengths for different radius ratios $\alpha = R_2/R_1$ for Problem M; $d^* = d/R_1 = 2.5$

Fig. 4.53: Variations of normalized stress intensity factor with relative crack lengths for different radius ratios $\alpha = R_2/R_1$ for Problem M; $d^* = d/R_1 = 2.5$

Fig. 4.54: Variations of normalized T -stress with relative crack lengths for different radius ratios $\alpha = R_2/R_1$ for Problem M; $d^* = d/R_1 = 4.0$

Fig. 4.55: Variations of normalized stress intensity factor with relative crack lengths for different radius ratios $\alpha = R_2/R_1$ for Problem M; $d^* = d/R_1 = 4.0$

Fig. 4.56: Variations of normalized T -stress with relative crack lengths for different radius ratios $\alpha = R_2/R_1$ for Problem M; $d^* = d/R_1 = 5.0$

Fig. 4.57: Variations of normalized stress intensity factor with relative crack lengths for different radius ratios $\alpha = R_2/R_1$ for Problem M; $d^* = d/R_1 = 5.0$

Fig. 4.58: Variations of normalized T -stress with relative crack lengths for different radius ratios $\alpha = R_2/R_1$ for Problem M; $d^* = d/R_1 = 10.0$

Fig. 4.59: Variations of normalized stress intensity factor with relative crack lengths for different radius ratios $\alpha = R_2/R_1$ for Problem M; $d^* = d/R_1 = 10.0$

CHAPTER 5 WEIGHT FUNCTION METHOD FOR DETERMINING FRACTURE PARAMETERS

5.0 Introduction

In this chapter, the accuracy and effectiveness of the weight function method for obtaining T -stress and stress intensity factor solutions for cracked plates with circular holes are established. The weight function (WF) method provides a means to obtain these fracture parameters under complex stress fields. The formulation and determination of the weight functions for stress intensity factor and T -stress will first be discussed. Three problems will be analyzed using WF and the solutions will be compared to those available in literature or to those obtained directly from BEM, as presented in the previous chapters.

5.1 Formulation of the Weight Function Method

This section studies the weight function (WF) method as it relates to problems involving plates with circular holes. As previously mentioned in Chapter 2, the weight functions for stress intensity factor and T -stress involves the use of geometric constants. It is therefore necessary to develop the procedure in which they can be derived.

5.1.1 Weight Function for Stress Intensity Factor

The formulation of the WF method involves the absolute length of the crack, which does not include the radius of the hole. Thus, for convenience, the position ahead of the circular hole in the x -direction and the crack length are denoted with prime superscript (Fig. 5.1), i.e.

$$\begin{aligned} x' &= x - R \\ a' &= a - R \end{aligned} \quad (5.1)$$

Equation (2.9) now becomes

$$m(x', a') = \frac{2}{\sqrt{2\pi(a'-x')}} \left[1 + M_1 \left(1 - \frac{x'}{a'} \right)^{\frac{1}{2}} + M_2 \left(1 - \frac{x'}{a'} \right) + M_3 \left(1 - \frac{x'}{a'} \right)^{\frac{3}{2}} \right] \quad (5.2)$$

where M_1 , M_2 , and M_3 are geometric constants for a specific cracked body. Two reference stress states, as shown in Fig. 5.1, in the form of

$$\sigma(x') = \sigma_0 \left(1 - \frac{x'}{a'} \right)^n, \quad (5.3)$$

where n denotes the order of the crack face loading condition and σ_0 is the nominal stress, are used to solve for these geometric constants. A third condition proposed by Shen and Glinka (1991), which states that the derivative of the WF at $x' = 0$ is zero, provides the necessary means to solve for all three constants. The two reference stress states used are uniform and linear crack face loading. For uniform crack face loading ($n = 0$), i.e. $\sigma(x') = \sigma_0$,

$$K_{1r} = \sigma_0 \sqrt{\pi a'} \cdot Y_0 \quad (5.4)$$

and for linear crack face loading ($n = 1$), i.e. $\sigma(x') = \sigma_0 \left(1 - \frac{x'}{a'}\right)$,

$$K_{2r} = \sigma_0 \sqrt{\pi a'} \cdot Y_1 \quad (5.5)$$

where Y_0 and Y_1 are the boundary correction factors for uniform and linear crack face loading, respectively. Figure 5.1 shows the application of the crack face loading. The boundary correction factors are determined using BEM analysis for the each of the crack length tested, and approximated by simple polynomials using least-squares fit, in the following form:

$$Y = B_0 + B_1 \left(\frac{a'}{R}\right) + B_2 \left(\frac{a'}{R}\right)^2 + B_3 \left(\frac{a'}{R}\right)^3 + B_4 \left(\frac{a'}{R}\right)^4 + B_5 \left(\frac{a'}{R}\right)^5 \quad (5.6)$$

Using the null derivative of WF at $x' = 0$ and the two reference stress states, the geometric parameters can be shown to be:

$$\begin{aligned}
M_1 &= \pi\sqrt{2}\left[\frac{14}{3}Y_0 - \frac{25}{3}Y_1\right] - \frac{64}{9} \\
M_2 &= \frac{5}{3}\left[\frac{\pi}{\sqrt{2}}(24Y_1 - 12Y_0) + 7\right] \\
M_3 &= -5\pi\sqrt{2}[2Y_1 - Y_0] - \frac{16}{3}
\end{aligned} \tag{5.7}$$

The weight function for stress intensity factor $m(x', a')$ can now be obtained using eq. (5.2).

5.1.2 Weight Function for T -Stress

Similar to the stress intensity factor, the T -stress weight function also requires modification to account for the absolute crack length. Equation (2.14) now becomes

$$t(x', a') = \frac{2}{\pi a'} \left[D_1 \left(1 - \frac{x'}{a'}\right)^{1/2} + D_2 \left(1 - \frac{x'}{a'}\right)^{3/2} \right] \tag{5.8}$$

where D_1 and D_2 are geometric constants of the cracked body. Two reference stress states are necessary to determine these parameters. Uniform and linear crack face loading are applied once again. For uniform crack face loading ($n = 0$), i.e. $\sigma(x') = \sigma_0$,

$$T_{1r} = \sigma_0 \cdot V_0 \tag{5.9}$$

and for linear crack face loading ($n = 1$), i.e. $\sigma(x') = \sigma_0 \left(1 - \frac{x'}{a'}\right)$,

$$T_{2r} = \sigma_0 \cdot V_1 \tag{5.10}$$

where V_0 and V_1 are the normalized T -stress for uniform and linear crack face loading, respectively. The normalized T -stress solution is determined using BEM analysis for the

each of the crack length tested, and approximated by simple polynomials using least-squares fit, in the following form:

$$V = A_0 + A_1 \left(\frac{a'}{R} \right) + A_2 \left(\frac{a'}{R} \right)^2 + A_3 \left(\frac{a'}{R} \right)^3 + A_4 \left(\frac{a'}{R} \right)^4 + A_5 \left(\frac{a'}{R} \right)^5 \quad (5.11)$$

Using these reference stress states, the geometric constants can now be derived, as follows:

$$\begin{aligned} D_1 &= \frac{15\pi}{16} (5V_0 - 7V_1) \\ D_2 &= \frac{5\pi}{16} (35V_1 - 21V_0) \end{aligned} \quad (5.12)$$

The T -stress weight function $t(x', a')$ can therefore be obtained with eq. (5.8).

5.2 Verification Problems

Verification of the weight function (WF) method is performed on three different problems. The results obtained from WF for stress intensity factor and T -stress will be compared to those obtained from direct BEM analysis or those from the literature. The three verification problems are (a) an infinite plate with double-cracks at the edge of a hole (Problem E), (b) a finite plate with double-cracks at the edge of a hole (Problem F), and (c) a finite plate with a single crack at the edge of a hole (Problem I).

5.2.1 Infinite Plate with Double-Cracks Emanating from the Edge of a Circular Hole under Remote Tension (Problem E)

The infinite plate with double-cracks emanating from a circular hole problem was analyzed using the WF method. The BEM was again employed to determine stress intensity factor and T -stress solutions due to crack face loading. The geometry was described previously in Chapter 3, and graphically shown in Fig. 3.24. The numerical modelling of this problem is identical to that described in Chapter 3, except for the loading condition. The remote boundary is now a free boundary while the crack face is subjected to either uniform or linear variation of loading. Since the interest here is to validate the accuracy of the WF method, only some sample relative crack lengths from those considered in Chapter 3 were analyzed.

The normalized stress intensity factor and T -stress solutions for different relative crack lengths are listed in Table 5.1, and graphically illustrated in Fig. 5.2 and 5.3. Note that due to the formulation of the WF method, both K_I and T -stress solutions are plotted against $(a-R)/R$, instead of a/R as previously performed. The polynomial fitted approximations of less than 1% error are also shown in Fig 5.2 and 5.3, with the values of the coefficients B and A given in Tables 5.2 and 5.3, respectively. By substituting the polynomial approximations of V and Y into eq. (5.12) and (5.7), respectively, the WF parameters for T -stress and stress intensity factor can be obtained.

The stress intensity factor solution is determined by integrating the product of the uncracked stress distribution $\sigma(x')$ and the weight function $m(x',a')$ over the crack length, according to eq. (2.6). This stress distribution $\sigma(x')$ is the uncracked stress

distribution at the prospective crack face in the y -direction for an infinite plate with a circular hole under remote tension and can be expressed as follows (Timoshenko and Goodier (1970)):

$$\frac{\sigma(x')}{\sigma_n} = \frac{\sigma_y}{\sigma_n} = \left[1 + \frac{1}{2} \left(1 + \frac{x'}{R} \right)^{-2} - \frac{3}{2} \left(1 + \frac{x'}{R} \right)^{-4} \right] \quad (5.13)$$

where σ_n is the remote tension load. The results for stress intensity factor are presented in Table 5.4 and Fig. 5.4 and are compared to those obtained by Bowie (1965). Excellent agreement, with generally less than 2% discrepancy, is achieved between the two sets of results.

Similar to the stress intensity factor, the T -stress due to crack face pressure is also determined by integrating the product of $\sigma(x')$ and the weight function $t(x', a')$ over the crack length, as described by eq. (2.12). The uncracked T -stress is not zero in this case and can be solved using eq. (2.11). The uncracked stress distribution at the prospective crack face in the x -direction for an infinite plate with a circular hole can be expressed as (Timoshenko and Goodier (1970)):

$$\frac{\sigma_x}{\sigma_n} = \left[\frac{3}{2} \left(1 + \frac{x'}{R} \right)^{-2} - \frac{3}{2} \left(1 + \frac{x'}{R} \right)^{-4} \right] \quad (5.14)$$

Hence, an analytical solution for uncracked T -stress can be obtained, from eq. (2.11):

$$\begin{aligned} T_{uncracked} &= (\sigma_x - \sigma_y)_{cracktip} \\ \frac{T_{uncracked}}{\sigma_n} &= \left(\frac{\sigma_x}{\sigma_n} - \frac{\sigma_y}{\sigma_n} \right)_{cracktip} = \left[\left(1 + \frac{x'}{R} \right)^{-2} - 3 \left(1 + \frac{x'}{R} \right)^{-4} - 1 \right] \end{aligned} \quad (5.15)$$

The T -stress solutions can now be obtained by the addition of $T_{uncracked}$ and $T_{crackpressure}$, using eq. (2.10). The normalized T -stress solutions are compared to those obtained by Broberg (2004), as shown in Table 5.4 and Fig. 5.5, and very good agreement can be observed.

5.2.2 Finite Plate with Double-Cracks Emanating from the Edge of a Circular Hole under Remote Tension (Problem F)

The finite plate with two symmetric cracks at the edge of a circular hole, as shown in Fig. 4.1, was analyzed using the WF method. Two-dimensional BEM was used to obtain the reference normalized stress intensity factor Y and T -stress V . The same BEM mesh as described in Chapter 4 was used, but with crack face loading conditions. Relative hole radius $R/W = 0.25$ and a height-to-width ratio $H/W = 2$ with the same corresponding crack lengths were tested.

The obtained results for uniform and linear crack face loading are shown in Table 5.5, and in Fig. 5.6 and 5.7. Both the boundary correction factor Y and normalized T -stress V were approximated using polynomial fits with less than 2% error, with the values of the coefficients of B and A given in Tables 5.6 and 5.7. The WF geometric constants for stress intensity factor and T -stress can then be determined, as previously described. No analytical expressions are available in literature for the uncracked stress distribution on the prospective crack face $\sigma(x')$. Thus, the BEM was used to obtain this stress distribution for the uncracked finite plate. The results for the stresses normalized with respect to σ_n , in the x - and y -direction are given in Table 5.8. The stress σ_y is also

shown graphically in Fig. 5.8, and is approximated using 6th order polynomial least-squares fit of less than 2% error from the computed values. The approximated polynomial of this stress is:

$$\begin{aligned} \frac{\sigma(x')}{\sigma_n} = \frac{\sigma_y}{\sigma_n} = & 3.245 - 7.2875 \cdot \frac{x'}{R} + 13.486 \cdot \left(\frac{x'}{R}\right)^2 - 14.729 \cdot \left(\frac{x'}{R}\right)^3 \\ & + 9.1404 \cdot \left(\frac{x'}{R}\right)^4 - 2.9511 \cdot \left(\frac{x'}{R}\right)^5 + 0.3828 \cdot \left(\frac{x'}{R}\right)^6 \end{aligned} \quad (5.16)$$

The stress intensity factor solutions at the various crack lengths are determined by again integrating the product of $\sigma(x')$ and the weight function $m(x', a')$ over the crack length. Table 5.9 and Fig. 5.9 compares the WF and BEM obtained solutions. It can be seen that good agreement is attained, with the discrepancies all within 3 percent.

Using the results for σ_x in Table 5.8, the “uncracked” T -stress can be calculated at the corresponding crack-tip locations, as listed. The crack face pressure induced T -stress is determined by integrating the product of the BEM obtained uncracked stress distribution and the weight function $t(x', a')$. The normalized T -stress for the entire system, at various crack lengths, can then be determined by the addition of $T_{uncracked}$ and $T_{crackpressure}$. Very good agreement is again observed when comparing the results obtained by the WF method and those directly by the BEM, as shown in Table 5.9 and Fig. 5.10.

5.2.3 Finite Plate with a Single Crack Emanating from the Edge of a Circular Hole under Remote Tension (Problem I)

Verification of the weight function method was also performed on the finite plate with a single crack at the edge of a circular hole problem (Fig. 4.19). The normalized stress intensity factor and T -stress for the two reference stress states were once again determined using two-dimensional BEM.

The results for normalized stress intensity factor and T -stress for relative hole radius R/W of 0.25 for the two crack face loading conditions are shown in Table 5.10. Polynomial fits of less than 2% error of the boundary correction factor Y and normalized T -stress V are shown in Fig. 5.11 and 5.12, with the values of the coefficients of B and A given in Tables 5.11 and 5.12. The WF parameters for stress intensity factor and T -stress can therefore be obtained using these expressions. The uncracked stress distribution on the prospective crack face in both x - and y -directions is identical to that of Problem F (since identical uncracked geometry).

The computed results of stress intensity factor using the WF method are tabulated in Table 5.13 (and graphically shown in Fig. 5.13) and compared to those obtained directly using BEM. Discrepancy of less than 3% can be seen between the solutions obtained by the two methods. Also shown in the table are the T -stress values calculated using the WF method and those obtained directly from BEM. Figure 5.14 shows the T -stress results graphically. The discrepancies between these T -stress results are less than 4 percent.

5.3 Summary

Weight functions for determining stress intensity factor and T -stress have been obtained for three test problems here. They demonstrate the applicability of the WF method for the plate problems with cracks at a circular hole. Very good agreement was achieved between the present solutions and those previously established.

Table 5.1: Variation of normalized stress intensity factor and T -stress with relative crack lengths for Problem E under uniform and linear crack face loading

a/R	$\frac{a-R}{R}$	Uniform Loading		Linear Loading	
		$Y = \frac{K_I}{\sigma_0 \sqrt{\pi(a-R)}}$	$V = T / \sigma_0$	$Y = \frac{K_I}{\sigma_0 \sqrt{\pi(a-R)}}$	$V = T / \sigma_0$
1.1	0.1	1.0674	0.3898	0.4031	0.2897
1.2	0.2	1.0299	0.3453	0.3802	0.2579
1.5	0.5	0.9877	0.2403	0.3533	0.1845
1.6	0.6	0.9820	0.2154	0.3496	0.1673
2.0	1.0	0.9750	0.1464	0.3448	0.1194
3.0	2.0	0.9786	0.0699	0.3467	0.0664

Table 5.2: Coefficients for normalized stress intensity factor Y under uniform and linear crack face loading for Problem E

Crack Face Loading	B_0	B_1	B_2	B_3	B_4	B_5
Uniform ($n = 0$)	1.096	-0.3606	0.3308	-0.0899	0	0
Linear ($n = 1$)	0.421	-0.224	0.2035	-0.0551	0	0

Table 5.3: Coefficients for normalized T -stress V under uniform and linear crack face loading for Problem E

Crack Face Loading	A_0	A_1	A_2	A_3	A_4	A_5
Uniform ($n = 0$)	0.4405	-0.5347	0.2988	-0.0543	-0.0039	0
Linear ($n = 1$)	0.3264	-0.3898	0.2466	-0.0692	0.0054	0

Table 5.4: Comparison of stress intensity factor and T -stress solutions between solutions obtained using the weight function method and those available in literature for Problem E

a/R	$K_I / \sigma \sqrt{\pi(a-R)}$			T/σ		
	BOWIE (1965)	WEIGHT FUNCTIONS	% DIFF	BROBERG (2004)	WEIGHT FUNCTIONS	% DIFF
1.1	2.8017	2.7600	-1.49	-1.0584	-1.0659	0.71
1.2	2.4135	2.3885	-1.04	-0.7925	-0.8205	3.53
1.5	1.8369	1.8763	2.15	-0.5616	-0.5646	0.53
1.6	1.7500	1.7753	1.44	N/A	-0.5594	N/A
2.0	1.4902	1.5047	0.98	-0.6090	-0.6168	1.28
3.0	1.2800	1.2679	-0.95	N/A	-0.7833	N/A

Table 5.5: Variation of normalized stress intensity factor and T -stress with relative crack lengths for Problem F under uniform and linear crack face loading

a/R	$\frac{a-R}{R}$	Uniform Loading		Linear Loading	
		$Y = \frac{K_I}{\sigma_0 \sqrt{\pi(a-R)}}$	$V = T / \sigma_0$	$Y = \frac{K_I}{\sigma_0 \sqrt{\pi(a-R)}}$	$V = T / \sigma_0$
1.1	0.100	1.0676	0.4066	0.4018	0.2964
1.2	0.200	1.0445	0.3555	0.3892	0.2548
1.6	0.600	1.0443	0.2129	0.3868	0.1533
2.0	1.000	1.1018	0.0881	0.4207	0.0727
2.4	1.400	1.2026	-0.0747	0.4794	-0.0277
2.8	1.800	1.3653	-0.3453	0.5733	-0.1871
3.2	2.200	1.6758	-0.8553	0.7493	-0.4754

Table 5.6: Coefficients for normalized stress intensity factor Y under uniform and linear crack face loading for Problem F

Crack Face Loading	B_0	B_1	B_2	B_3	B_4	B_5
Uniform ($n = 0$)	1.0966	-0.3630	0.6359	-0.3549	0.0888	0
Linear ($n = 1$)	0.4193	-0.2128	0.3649	-0.1992	0.0492	0

Table 5.7: Coefficients for normalized T -stress V under uniform and linear crack face loading for Problem F

Crack Face Loading	A_0	A_1	A_2	A_3	A_4	A_5
Uniform ($n = 0$)	0.4516	-0.4942	0.1703	0.0061	-0.0473	0
Linear ($n = 1$)	0.3336	-0.4216	0.2489	-0.0785	-0.0107	0

Table 5.8: Normalized stress distribution obtained using BEM for Problem F

x/R	$(x-R)/R$	$\frac{\sigma_x}{\sigma_n}$	$\frac{\sigma_y}{\sigma_n}$	$\frac{T_{uncracked}}{\sigma_n} = \left(\frac{\sigma_x}{\sigma_n} - \frac{\sigma_y}{\sigma_n} \right)_{cracktip}$
1.00	0.00	0.004	3.280	-3.276
1.02	0.02	0.057	3.097	-3.040
1.04	0.04	0.125	2.967	-2.842
1.06	0.06	0.154	2.842	-2.688
1.08	0.08	0.200	2.734	-2.534
1.10	0.10	0.228	2.626	-2.399
1.12	0.12	0.262	2.536	-2.273
1.14	0.14	0.281	2.444	-2.162
1.16	0.16	0.307	2.368	-2.061
1.18	0.18	0.320	2.289	-1.969
1.20	0.20	0.339	2.225	-1.886
1.22	0.22	0.348	2.157	-1.810
1.24	0.24	0.362	2.103	-1.741
1.26	0.26	0.366	2.044	-1.678
1.28	0.28	0.376	1.997	-1.622
1.30	0.30	0.377	1.946	-1.569
1.32	0.32	0.384	1.906	-1.522
1.34	0.34	0.383	1.861	-1.478
1.36	0.36	0.387	1.827	-1.439
1.38	0.38	0.385	1.787	-1.402
1.40	0.40	0.387	1.757	-1.370
1.42	0.42	0.384	1.722	-1.338
1.44	0.44	0.385	1.697	-1.312
1.46	0.46	0.380	1.665	-1.285
1.48	0.48	0.379	1.643	-1.263
1.50	0.50	0.374	1.615	-1.240
1.52	0.52	0.373	1.595	-1.222
1.54	0.54	0.367	1.569	-1.202
1.56	0.56	0.365	1.552	-1.188
1.58	0.58	0.358	1.527	-1.169
1.60	0.60	0.359	1.518	-1.158
1.70	0.70	0.329	1.433	-1.104
1.80	0.80	0.309	1.378	-1.069
1.90	0.90	0.278	1.324	-1.046
2.00	1.00	0.257	1.291	-1.034
2.10	1.10	0.230	1.253	-1.024
2.20	1.20	0.210	1.232	-1.023
2.30	1.30	0.186	1.204	-1.018
2.40	1.40	0.168	1.190	-1.022
2.50	1.50	0.148	1.168	-1.020
2.60	1.60	0.132	1.157	-1.025
2.70	1.70	0.115	1.138	-1.023
2.80	1.80	0.101	1.129	-1.028
2.90	1.90	0.086	1.111	-1.026
3.00	2.00	0.074	1.103	-1.029
3.10	2.10	0.060	1.085	-1.025
3.20	2.20	0.050	1.078	-1.027

Table 5.9: Comparison of stress intensity factor and T -stress solutions between solutions obtained using the weight function method and those obtained by BEM for Problem F

a/R	$K_I / \sigma \sqrt{\pi(a-R)}$			T/σ		
	BEM	WEIGHT FUNCTIONS	% DIFF	BEM	WEIGHT FUNCTIONS	% DIFF
1.1	3.1069	3.0141	-2.99	-1.1331	-1.1622	2.57
1.2	2.6690	2.6894	0.76	-0.8295	-0.8446	1.83
1.6	2.0092	2.0257	0.82	-0.6332	-0.6415	1.30
2.0	1.8379	1.8202	-0.96	-0.8196	-0.8249	0.64
2.4	1.8504	1.8327	-0.96	-1.1155	-1.1072	-0.74
2.8	2.0038	2.0227	0.94	-1.5561	-1.5807	1.58
3.2	2.3877	2.3682	-0.82	-2.2696	-2.3135	1.93

Table 5.10: Variation of normalized stress intensity factor and T -stress with relative crack lengths for Problem I under uniform and linear crack face loading

a/R	$\frac{a-R}{R}$	Uniform Loading		Linear Loading	
		$Y = \frac{K_I}{\sigma_0 \sqrt{\pi(a-R)}}$	$V = T / \sigma_0$	$Y = \frac{K_I}{\sigma_0 \sqrt{\pi(a-R)}}$	$V = T / \sigma_0$
1.1	0.100	1.0593	0.4088	0.3945	0.2902
1.2	0.200	1.0161	0.3667	0.3659	0.2558
1.6	0.600	0.9538	0.2234	0.3267	0.1597
2.0	1.000	0.9467	0.1068	0.3204	0.0843
2.4	1.400	0.9726	-0.0247	0.3333	0.0037
2.8	1.800	1.0407	-0.2246	0.3680	-0.1102
3.2	2.200	1.1889	-0.6243	0.4423	-0.3206

Table 5.11: Coefficients for normalized stress intensity factor Y under uniform and linear crack face loading for Problem I

Crack Face Loading	B_0	B_1	B_2	B_3	B_4	B_5
Uniform ($n = 0$)	1.1015	-0.5092	0.5853	-0.2959	0.0651	0
Linear ($n = 1$)	0.4208	-0.3220	0.3651	-0.1814	0.0382	0

Table 5.12: Coefficients for normalized T -stress V under uniform and linear crack face loading for Problem I

Crack Face Loading	A_0	A_1	A_2	A_3	A_4	A_5
Uniform ($n = 0$)	0.4478	-0.4016	-0.0097	0.1353	-0.0675	0
Linear ($n = 1$)	0.3211	-0.3373	0.1177	0.0032	-0.0215	0

Table 5.13: Comparison of stress intensity factor and T -stress solutions between solutions obtained using the weight function method and those obtained by BEM for Problem I

a/R	$K_I / \sigma \sqrt{\pi(a - R)}$			T/σ		
	BEM	WEIGHT FUNCTIONS	% DIFF	BEM	WEIGHT FUNCTIONS	% DIFF
1.1	3.0719	2.9810	-2.96	-1.1298	-1.1524	2.00
1.2	2.6082	2.5742	-1.31	-0.8126	-0.8477	4.32
1.6	1.8077	1.7967	-0.60	-0.6126	-0.6267	2.31
2.0	1.5300	1.5192	-0.70	-0.7950	-0.7790	-2.01
2.4	1.4350	1.4246	-0.73	-1.0432	-1.0188	-2.34
2.8	1.4505	1.4641	0.94	-1.3776	-1.3769	-0.05
3.2	1.5928	1.5976	0.30	-1.9479	-1.9205	-1.41

Fig. 5.1: Crack face loading $\sigma(x')$ of n^{th} order on an arbitrary geometry involving a circular hole; $x' = x - R$, $a' = a - R$

Fig. 5.2: Variations of normalized stress intensity factor with relative crack lengths for Problem E under uniform and linear crack face loading

Fig. 5.3: Variations of normalized T -stress with relative crack lengths for Problem E under uniform and linear crack face loading

Fig. 5.4: Variation of normalized stress intensity factors with relative crack lengths for Problem E

Fig. 5.5: Variation of normalized T -stress with relative crack lengths for Problem E

Fig. 5.6: Variations of normalized stress intensity factor with relative crack lengths for Problem F under uniform and linear crack face loading

Fig. 5.7: Variations of normalized T -stress with relative crack lengths for Problem F under uniform and linear crack face loading

Fig. 5.8: Normalized uncracked stress distribution in the y -direction on the perspective crack face due to remote tension for Problem F

Fig. 5.9: Variation of normalized stress intensity factors with relative crack lengths for Problem F

Fig. 5.10: Variation of normalized T -stress with relative crack lengths for Problem F

Fig. 5.11: Variations of normalized stress intensity factor with relative crack lengths for Problem I under uniform and linear crack face loading

Fig. 5.12: Variations of normalized T -stress with relative crack lengths for Problem I under uniform and linear crack face loading

Fig. 5.13: Variation of normalized stress intensity factors with relative crack lengths for Problem I

Fig. 5.14: Variation of normalized T -stress with relative crack lengths for Problem I

CHAPTER 6 CONCLUSIONS

The scope of this thesis was to determine a measure of crack-tip constraint of plates with circular holes. To this end, the stress intensity factor (K) and T -stress solutions for these geometries were obtained. The geometries analyzed include plates with a single circular hole under remote tension and bending, and plates with multiple holes under remote tension. The boundary element method (BEM), in conjunction with the mutual or M-contour integral, was used to determine the K and T -stress solutions numerically.

The principles of linear elastic fracture mechanics (LEFM) and the significance of T -stress as a second parameter for LEFM analysis have been briefly described in Chapter 2. The formulation of the weight function (WF) method and the BEM for obtaining K and T -stress solutions were discussed. To overcome cumbersome meshing procedures around the crack-tip, self-regularized BEM, as described in Chapter 2, was employed.

The validation of the BEM as the numerical tool for this thesis was discussed in Chapter 3. Several problems with published values of stress intensity factor and T -stress, including those with stress concentrations, were evaluated using the BEM. Comparisons of the solutions with those from the literature demonstrated the accuracy of the BEM, as well as efficiency against other numerical techniques. From the BEM-obtained solutions, some general modelling guidelines were established.

The stress intensity factor and T -stress solutions obtained from BEM for plates with circular hole(s) were presented in Chapter 4. Some general trends were observed for all of the analyzed geometries. First, the T -stresses obtained from BEM were all negative in value, which indicate low constraint conditions. Second, the negative T -stresses become increasingly less negative for increasing crack length, then declines rapidly beyond a certain crack length. Plates with a single circular hole were first evaluated. The presence of a finite boundary around a circular hole was found to increase the rate-of-change of the “uptrend-downtrend” behaviour of the T -stress as the crack size becomes larger, when compared to the corresponding crack lengths of the infinite plate case. Also, the difference in value for T -stress between single and double-crack configuration was found to be significant only for relatively large crack lengths. Bending loads on the finite plates were also studied. A significant reduction from the values obtained for the tension case for the stress intensity factors was observed.

The effects of adding adjacent holes were also examined. For the infinite array of holes problems, it was found that the declining behaviour of the T -stress, as described above, was much more gradual. The addition of adjacent holes not in the plane of the crack was also investigated for the infinite plate problem. Increasing the size of the adjacent holes reduced the magnitude of the stress intensity factors and generated gentler gradients in the T -stress behaviour. Also, by reducing the distance between the holes, both stress intensity factors and T -stress decreased in value, resulting in even lower crack-tip constraint.

The weight function method has been shown to be an efficient technique in determining K and T -stress solutions for complex stress distributions for cracked-geometries. Several test problems were reanalyzed using this method, as described in Chapter 5. Very good agreement was achieved between the solutions obtained directly from BEM and those obtained using the WF method, hence validating its effectiveness for plates with crack(s) emanating from circular holes.

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