Uniformization Method for Two-Dimensional Queueing Systems

by

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Uniformization Method for Two-Dimensional Queueing Systems

Ge Wang

2009

Abstract

In the early 1970s, V. A. Malyshev proposed an approach of generating functions to asymptotical studies of stationary probabilities of a random walk. The generating function method is a classical method with the advantage of explicitly determining the constant besides the exact decay function. This method is also valid in principle for the doubly QBD process, or the random walks in the quarter plane. This thesis provides a detailed, systematic, step-by-step routine, associated with the famous 2-demand example which is originally solved by L. Flatto and S. Hahn, to help readers overcome the gaps that occur when solving queueing problems with the generating function method. There is also another example in this thesis to show the difficulties we may encounter when applying Flatto's method on general queueing models.

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Chapter 1

Introduction

1.1 Motivation

In single serial computing systems, to solve a problem, a processing method called algorithm is constructed and implemented, particularly, when we talk about single server queueing problems or uni-threaded computer software, as a serial stream of customers or instructions. These customers are served by a processing unit called server. One customer may be served at a time, and only when the server finished serving one customer, the next customer will be served.

A parallel queueing network, on the other hand, to increase the efficiency, uses multiple processing queues simultaneously to serve customers instead of single queue. In practice, this is accomplished by decomposing a problem into independent problem parts so that each processing queue can serve its part of the customers simultaneously with the others. In real worlds, processing queues can

be interpreted as resources such as multiple parallel processors, several networked computers, specialized hardwares, or any combination of the above, etc.

Optimally, the speed-up from parallelization would be linear if all elements of a composite problem are parallelizable. That is, doubling the number of processing queues should halve the total service time. However, realistically, the elements of the problem are usually not all parallelizable, and at the time that we reduce the total service time by increasing the number of processing queues, the complexity of the system is also increased enormously. The behavior of the system becomes beyond our control and several problems will occur, such as re-sequencing and synchronization, etc. Therefore, we have to find some mathematical methods to analyze these new kinds of parallel systems, and then re-design and optimize them according to the results developed by mathematicians.

The Intel Tera-scale Computing Research Program is a real case of such parallel computing integrations. The Intel corporation promotes this program as the computing technology for the next decade. By scaling multi-core architectures to 10s to 100s of cores and embracing a shift to parallel programming, it aims to improve performance, increase energy-efficiency, and make future applications more compelling and immersive. Recently, in Intel's experimental lab, an experimental multi-core chip called "POLARIS", achieves 1 TFLOPS (Trillion Floating point Operations Per Second) at 3.13 GHz. The 80-core chip can increase this result to 2 TFLOPS at 6.26 GHz. More details can be found in [17] by Richard Swinburne (2007).

Back to the world of mathematicians, as we all know, Markov processes on one-dimensional state spaces are well understood and various techniques have been developed to deal with such kind of problems. On the other hand, high-dimensional Markov chains are becoming the main interests of researchers in recent years since these kinds of models are more realistic and challenging. Moreover, as we mentioned before, engineers are also very enthusiastic on these models as multi-paralleled design is one of the fundamental foundations in modern industry. However, the methods of analysis on high dimensional Markov chains are somewhat limited. Developing such efficient techniques is currently a hot-spot of research.

In fact, as early as the 1960s and 1970s, Cox and Miller [6], Gaver [11] have discussed the diffusion approximation method to queuing systems. Adan et al. (1990,1991) [1] proposed a compensation method, which is effective for some special two-dimensional models. Another technique is called power-series algorithm, which is a numeric procedure based on power-series expansions of state probabilities and moments as a function of a parameter of the system, developed by Blanc (1987, 1990) [2], Hooghiemstra, Keane and Van de Ree (1988) [12]. B. Blaszczyszyn, A. Frey and V. Schmidt (1995) [3] applied factorial moment expansion to derive approximation formulae for stationary characteristics of multi-server queues with Markov-modulated arrival process and with the first-come-first-served queueing discipline. Besides, large deviation method is another popular method used to analyze multi-server queue. Foley and David R. McDonald (2005) [10] applied this method to modified Jackson Network; Amarjit Budhiraja, Paul Dupuis,

and Vasileios Maroulas (2008) [4] used large deviation method on infinite dimensional stochastic dynamical systems based on a variational representation for functionals of Brownian motion.

In the early 1970s, V. A. Malyshev [13], [14], [15] published several papers and proposed an approach of generating functions to asymptotical studies of stationary probabilities of a random walk on the lattice quarter-plane $\mathbb{Z}_+^2 = \{(m, n) : m, n = 0, 1, \dots\}$. Similar techniques can be found in [8] by L. Flatto, and H. P. McKean (1977). The generating function method is a classical method with the advantage of explicitly determining the constant besides the exact decay function. This method is also valid in principle for the doubly QBD process, or the random walks in the quarter plane, in which the generating function can be presented as an integral in terms of a conformal mapping, however, the exact expression of the generating function cannot be explicitly determined based on the theory of Riemann surfaces, algebraic ideas and reduction to a Riemann-Hilbert boundary value problem. As an example among all these works, J.W. Cohen (1994) [5] applied the generating method to two-dimensional algebraic nearest-neighbour random walk (queue with paired services). Random Walks in the Quarter-plane, authored by G. Fayolle, R. Iasnogorodski and V. Malyshev (1999) [7], systemically described the knowledge of generating function method and related analytical techniques, as well as the classification of certain models with their generating functions. The purpose of this thesis is to provide an example of how generating function technique can be used in two-dimensional Markov chain. The analytic technique to be demonstrated is based on complex analysis. Particularly, it uses the method

of solving Riemann-Hilbert boundary value problems to characterize a two-queue network. If the two queues in a network do not operate independently, the analysis of the distribution of their queue lengths becomes quite involved as standard one-dimensional queuing theory does not suffice. Complex boundary value problems can be formulated, however, the solutions to which fully characterize the queuing behavior.

1.2 Contribution

Although there are already plenty of work that had been done through the generating function method, most papers are lack of important details that cause difficulties for reader to understand and comprehend the entire procedure of solving queueing network with generating function method. This thesis provides a detailed, systematic, step-by-step routine, associated with the famous 2-demand example which is originally solved by L. Flatto and S. Hahn [9], to help readers overcome the gaps that occur when solving queueing problems with the generating function method.

Furthermore, I demonstrate another example in Chapter 5 which is a generalization of 2-demand model, to show the difficulties we may encounter when applying Flatto's method on general queueing models.

1.3 Organization

This thesis is organized as follows: In Chapter 2, I provide some background information and knowledge that are used in this thesis; In Chapter 3, I provide a detailed, systematic, step-by-step routine when solving queueing problems with Flatto's method. In Chapter 4, I demonstrate a detailed 2-demand example that was originally solved by L. Flatto and S. Hahn using the generating function technique. In Chapter 5, we will discuss some difficulties that we may encounter through another detailed example which is a generalization of 2-demand model.

Chapter 2

Background

This chapter provides the background information that is relevant to the research on solving queueing problems via uniformization method, and it is organized as follows.

Section 2.1 reviews the fundamental knowledge of the Markov Chains and mentions the concept of the stationary condition for an arbitrary Markov Chain that is always assumed satisfied in the later Chapters. Section 2.2.1 introduces the basic idea of constructing the generating functions. Section 2.3 is an overview of the so-called Riemann surface that is somehow the main ground we will work on. Section 2.4 introduces the way we parameterize the Riemann surface in this thesis.

2.1 Queueing Theory

In this section, we will review some basic conclusions in queueing theory, especially the results of the single-server exponential queueing system ($M/M/1$).

2.1.1 Poisson Process

Definition 2.1 *The counting process $\{N(t), t \geq 0\}$ is said to be a Poisson process having rate λ , $\lambda \geq 0$, if*

- $N(0) = 0$;
- *The process has independent increments;*
- *The number of events in any interval of length t is Poisson distributed with mean λt . That is, for all $s, t \geq 0$*

$$\mathbb{P}\{N(t+s) - N(s) = n\} = e^{-\lambda t} \frac{(\lambda t)^n}{n!} \quad n = 0, 1, \dots \quad (2.1)$$

A counting process is said to have *stationary increments* if the distribution of the number of events that occur in any interval of time just depends on the length of the time interval. From the definition above, it is easy to know that Poisson process has stationary increments.

An equivalent definition of Poisson process is stated as follows.

Definition 2.2 *The counting process $\{N(t), t \geq 0\}$ is said to be a Poisson process having rate λ , $\lambda \geq 0$, if*

- $N(0) = 0$;

- *The process has stationary and independent increments;*
- $\mathbb{P}\{N(h) = 1\} = \lambda h + o(h);$
- $\mathbb{P}\{N(h) \geq 2\} = o(h),$

where $o(h)$ is a function such that $\lim_{h \rightarrow 0} \frac{o(h)}{h} = 0$.

Consider a Poisson process $\{N(t), t \geq 0\}$ with rate λ , let T_n denote the elapsed time between the $(n-1)^{st}$ and the n^{th} events. The sequence $\{T_n, n = 1, 2, \dots\}$ is called the *sequence of inter-arrival times*.

Proposition 2.1 $T_n, n = 1, 2, \dots$, are independent identically distributed exponential random variables having rate λ .

2.1.2 Queueing Models

Suppose that customers arrive at a single-server service station with a Poisson process having rate λ , that is, the time between successive arrivals are independent exponential random variables with rate λ . Each customer, upon arrival, goes directly into the service if the server is idle; otherwise if there is another customer being served at this moment, the new-coming customer will join the queue. When the server finishes serving a customer, the customer leaves the system and the next customer (if there is any) in the queue will be served immediately. The successive service times are assumed to be independent exponential random variables having rate μ .

The preceding process is called the $M/M/1$ queue, where the two M s refer to

the fact that both the inter-arrival and the service distributions are exponential, and the 1 to the fact that there is a single server. Sometimes, we denote a queue as $M/M/1/N$ or $M/M/1/\infty$, where the last part is the capacity of this queue, that is, the total number of customers that could be in the system at the same time. Moreover, if there is more than one server in the system we can use $M/M/n$.

Furthermore, in $G/G/n/N$, the times between successive arrivals and services have arbitrary distribution G , and n, N are the same as we denoted before. Similarly, there are many different kinds of queues, such as $M/G/1$, $G/M/1$, $G/G/2$, $\dots$. In this thesis, we will only focus on the $M/M/1$ queue.

2.1.3 Stationarity of The Markov Chains

Proposition 2.2 (*Stability Condition for $M/M/1$ Queue*) *If $\rho(= \frac{\lambda}{\mu}) < 1$ then the queue is stable and has a unique stationary probability distribution. If $\rho \geq 1$ then the queue size grows to infinity.*

Let P_i be the proportion of time in state i and q_{ij} be the rate when in state i that the process goes to j .

Definition 2.3 *The continuous Markov chain is time reversible, in the sense that a process reversed in time has the same probabilistic structure as the original process, if $P_i q_{ij} = P_j q_{ji}$, for all i, j .*

2.2 Doubly QBD Random Walk

A doubly QBD random walk (DQBD-RW) is a discrete time Markov chain with the state space $\mathbb{Z}_+^2 = \{(i, j) : i, j = 0, 1, 2, \dots\}$ and transition probabilities $P_{\alpha, \beta}$, which satisfy the following *space homogeneity* and *boundedness* properties.

Property 2.1 (Space homogeneity) $\mathbb{Z}_+^2$ can be partitioned

$$\mathbb{Z}_+^2 = \bigcup_r S_r = S \cup S_x \cup S_y \cup S_0,$$

where

$$S = \{(i, j) : i, j = 1, 2, \dots\},$$

$$S_x = \{(i, 0) : i = 1, 2, \dots\},$$

$$S_y = \{(0, j) : j = 1, 2, \dots\},$$

$$S_0 = \{(0, 0)\}.$$

For each fixed S_r , if $\alpha \in S_r$ and $\alpha + (i, j) \in \mathbb{Z}_+^2$, then $P_{\alpha, \alpha + (i, j)}$ is independent of all α in S_r . In this case, $P_{\alpha, \alpha + (i, j)}$ will be denoted by $p_{i, j}$, $p_{i, j}^{(1)}$, $p_{i, j}^{(2)}$ and $p_{i, j}^{(0)}$ for $S_r = S, S_x, S_y$ and S_0 , respectively.

Property 2.2 (Boundedness of Jumps) For any $\alpha \in \mathbb{Z}_+^2$ and (i, j) such that $\alpha + (i, j) \in \mathbb{Z}_+^2$, $P_{\alpha, \alpha + (i, j)} = 0$ unless $|i| \leq 1$ and $|j| \leq 1$; equivalently, $p_{i, j} = 0$, $p_{i, j}^{(1)} = 0$, $p_{i, j}^{(2)} = 0$ and $p_{i, j}^{(0)} = 0$ unless $|i| \leq 1$ and $|j| \leq 1$.

The transition diagrams of a DQBD-RW is provided in Figure 2.2. Very often in applications, a continuous time Markov chain is a natural choice in modeling, which is then converted into a DQBD-RW in terms of uniformization.

Figure 2.1: Transition diagram of a DQBD-RW

2.2.1 Generating Functions

For a two-dimensional stationary Markov process $X(t) = (X_1(t), X_2(t))$ representing the length of the two queues, with state space $\{(m, n) : m, n = 0, 1, \dots\}$. The generating function is defined for $x, y \in \mathbb{C}$ with $|x|, |y| \leq 1$ as

$$P(x, y) = \mathbb{E}[x^{X_1} y^{X_2}] = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \pi_{m,n} x^m y^n \quad (2.2)$$

where

$$\pi_{m,n} = P[X_1(t) = m, X_2(t) = n] \quad (2.3)$$

is the stationary queue length distribution. A function equation is derived that relates $P(x, y)$ with $P(x, 0)$, $P(0, y)$ and $P(0, 0)$ as follows,

$$h_1(x, y)P(x, y) = h_2(x, y)P(x, 0) + h_3(x, y)P(0, y) + h_4(x, y)P(0, 0) \quad (2.4)$$

where

$$P(x, 0) = \sum_{m=0}^{\infty} \pi_{m,0} x^m, \quad P(0, y) = \sum_{n=0}^{\infty} \pi_{0,n} y^n, \quad P(0, 0) = \pi_{0,0} \quad (2.5)$$

2.3 Riemann Surfaces

In this section we give the definition and some basic properties of the Riemann surfaces.

Definition 2.4 *Defined on the complex plane $\mathbb{C}$, a Riemann surface is a one-complex-dimensional connected complex analytic manifold; that is, a two-real-dimensional connected manifold M with a maximal set of charts $\{U_\alpha, z_\alpha\}_{\alpha \in A}$ on M (that is, the $\{U_\alpha\}_{\alpha \in A}$ constitute an open cover of M and*

$$z_\alpha : U_\alpha \rightarrow \mathbb{C} \quad (2.6)$$

is a homeomorphism onto an open subset of the complex plane $\mathbb{C}$) such that the transition functions

$$f_{\alpha\beta} = z_\alpha \circ z_\beta^{-1} : z_\beta(U_\alpha \cap U_\beta) \rightarrow z_\alpha(U_\alpha \cap U_\beta) \quad (2.7)$$

are holomorphic whenever $U_\alpha \cap U_\beta \neq \emptyset$.

Any set of charts (not necessarily maximal) that cover M and satisfy condition (2.7) will be called a set of analytic coordinate charts.

Definition 2.5 *A continuous mapping*

$$f : M \rightarrow N \tag{2.8}$$

between Riemann surfaces is called holomorphic or analytic if for every local coordinate $\{U, z\}$ on M and every local coordinate $\{V, \zeta\}$ on N with $U \cap f^{-1}(V) \neq \emptyset$, the mapping

$$\zeta \circ f \circ z^{-1} : z(U \cap f^{-1}(V)) \rightarrow \zeta(V) \tag{2.9}$$

is holomorphic (as a mapping from $\mathbb{C}$ to $\mathbb{C}$).

The mapping f is called conformal if it is also one-to-one and onto. In this case (since holomorphic mappings are open or map onto a point)

$$f^{-1} : N \rightarrow M \tag{2.10}$$

is also conformal.

2.4 Parametrization of The Riemann Surface

In this section we will overview the way that we construct the Riemann Surface from the generating function equation and how to parameterize it.

2.4.1 Schwarz-Christoffel Theorem

First we will introduce the well-known Schwarz-Christoffel Theorem that is used to construct the Riemann surface from the generating function equation.

Theorem 2.1 *If a function W , continuous on the closed half-plane $\Im z \geq 0$ and holomorphic in the open half-plane $\Im z > 0$, is uniquely invertible on the real axis and transforms this axis into a polygonal line L (which is obvious closed and has no multiple points), then this function transforms conformally the open half-plane $\Im z > 0$ into the interior region of the polygon L , and the closed half-plane $\Im z \geq 0$ homeomorphically into the closure of this region.*

2.4.2 Schwarz Reflection Principle

Another technique used in this thesis to parameterize the Riemann surface is Schwarz Reflection Principle.

Suppose that f is an analytic function which is defined in the upper half-disk $|z|^2 < 1, \Re z > 0$. Further suppose that f extends to a continuous function on the real axis, and takes on real values on the real axis. Then f can be extended to an analytic function on the whole disk by the formula,

$$f(\bar{z}) = \overline{f(z)} \tag{2.11}$$

and the values for z reflected across the real axis are the reflections of $f(z)$ across the real axis. It is easy to check that the above function is complex differentiable in the interior of the lower half-disk. What is remarkable is that the resulting

function must be analytic along the real axis as well, despite no assumptions of differentiability.

This is called the Schwarz reflection principle, and is sometimes also known as Schwarz's symmetric principle.

For the reflected function to be continuous, it is necessary for the values at the boundary to be continuous and to fall on the line being reflected. The reflection principle also applies in the generality of reflecting along any line, not just the real axis, in which case the function f has to take values along a line in the range.

2.4.3 Riemann Surface: Genus 1 or Genus 0

The Riemann surface S generated from the generating function equation is divided into 2 cases: genus 1 or genus 0. The classification can be made by the following Lemma,

Lemma 2.1 (Fayolle, Iasnogorodski, Malyshev) *For all non-singular random walks, S has genus 0 if and only if one of the following relations takes place,*

$$M_x = M_y = 0, \tag{2.12}$$

$$p_{1,0} = p_{1,1} = p_{0,1} = 0, \tag{2.13}$$

$$p_{1,0} = p_{1,-1} = p_{0,-1} = 0, \tag{2.14}$$

$$p_{-1,0} = p_{-1,-1} = p_{0,-1} = 0, \tag{2.15}$$

$$p_{0,1} = p_{-1,0} = p_{-1,1} = 0 \tag{2.16}$$

where, $\mathbf{M}_x = \sum i p_{i,j}$, $\mathbf{M}_y = \sum j p_{i,j}$.

In fact, S has genus 0 if and only if the discriminant $D(x) = b^2(x) - 4a(x)c(x) = d_4x^4 + d_3x^3 + d_2x^2 + d_1x + d_0$ of the generating function equation $Q(x) = a(x)y^2 + b(x)y + c(x) = 0$ has a multiple zero.

Chapter 3

Uniformization

In this chapter, we will give a uniformized routine of solving 2-dimensional queueing problems via generating function method. Particularly, we will discuss some important issues that may cause difficulties when using this method. Summarily, I list the general steps of solving 2-dimensional queueing problems via generating function method as follows,

1. For the queueing model of interest, we need to set up the generating function equation, see section 3.1.
2. Obtain a stability condition of the model of interest, see section 3.2.
3. Analyze the kernel $h_1(x, y)$, see section 3.3.
4. Based on the study of kernel $h_1(x, y)$ in step 3, construct a Riemann surface through parametrization, see section 3.4.
5. Examine the relationship between $x(t)$ and $y(t)$, see section 3.5.
6. According to the properties of the elliptic function and the relationship of $x(t)$

and $y(t)$, continue the analytic curves of $P(x, 0)$ and $P(0, y)$ to the whole t -plane, see section 3.6.

7. Analyze the zeros and poles of $P(x, 0)$ and $P(0, y)$ in the t -plane, see section 3.7.

8. Finally, map the zeros and poles of $P(x, 0)$ and $P(0, y)$ in the t -plane back to the x -plane and y -plane, respectively, by which we can obtain the exact expression of $P(x, 0)$ and $P(0, y)$, see section 3.8.

In the following sections, I will state the detailed steps of solving 2-dimensional queueing problems via generating function method.

3.1 Setup of the Generating Function Equation

From [7], we can easily have the following general generating function equation,

Theorem 3.1 (Fayolle, Iasnogorodski and Malyshev)

$$h^*(x, y)\pi(x, y) = h_1^*(x, y)\pi_1(x) + h_2^*(x, y)\pi_2(y) + h_0^*(x, y)\pi_{0,0}, \quad (3.1)$$

where

$$\begin{aligned} h^*(x, y) &= xy \left(1 - \sum_{i=-1}^1 \sum_{j=-1}^1 p_{i,j} x^i y^j \right) \\ &= a(x)y^2 + b(x)y + c(x) = \tilde{a}(y)x^2 + \tilde{b}(y)x + \tilde{c}(y) \end{aligned} \quad (3.2)$$

$$h_1^*(x, y) = x \left(\sum_{i=-1}^1 \sum_{j=0}^1 p_{i,j}^{(1)} x^i y^j - 1 \right) \quad (3.3)$$

$$= \left(p_{-1,0}^{(1)} + p_{0,0}^{(1)} x + p_{1,0}^{(1)} x^2 \right) + \left(p_{-1,1}^{(1)} + p_{0,1}^{(1)} x + p_{1,1}^{(1)} x^2 \right) y - x,$$

$$h_2^*(x, y) = y \left(\sum_{i=0}^1 \sum_{j=-1}^1 p_{i,j}^{(2)} x^i y^j - 1 \right) \quad (3.4)$$

$$= \left(p_{0,-1}^{(2)} + p_{0,0}^{(2)} y + p_{0,1}^{(2)} y^2 \right) + \left(p_{1,-1}^{(2)} + p_{1,0}^{(2)} y + p_{1,1}^{(2)} y^2 \right) x - y,$$

$$h_0^*(x, y) = \left(\sum_{i=0}^1 \sum_{j=0}^1 p_{i,j}^{(0)} x^i y^j - 1 \right) \quad (3.5)$$

$$= \left(p_{0,0}^{(0)} + p_{1,0}^{(0)} x + p_{0,1}^{(0)} y + p_{1,1}^{(0)} xy \right) - 1,$$

with

$$a(x) = -(p_{-1,1} + p_{0,1}x + p_{1,1}x^2), \quad (3.6)$$

$$b(x) = -[p_{-1,0} + (p_{0,0} - 1)x + p_{1,0}x^2], \quad (3.7)$$

$$c(x) = -(p_{-1,-1} + p_{0,-1}x + p_{1,-1}x^2), \quad (3.8)$$

$$\tilde{a}(y) = -(p_{1,-1} + p_{1,0}y + p_{1,1}y^2), \quad (3.9)$$

$$\tilde{b}(y) = -[p_{0,-1} + (p_{0,0} - 1)y + p_{0,1}y^2], \quad (3.10)$$

$$\tilde{c}(y) = -(p_{-1,-1} + p_{-1,0}y + p_{-1,1}y^2). \quad (3.11)$$

And more generally, if the summation starts from 0 instead of 1, we can get the main equation used in this thesis based on Theorem 3.1 with some modifications,

Corollary 3.1

$$h_1(x, y)P(x, y) = h_2(x, y)P(x, 0) + h_3(x, y)P(0, y) + h_4(x, y)P(0, 0) \quad (3.12)$$

where

$$h_1(x, y) = h^*(x, y),$$

$$h_2(x, y) = h^*(x, y) + h_1^*(x, y)y,$$

$$h_3(x, y) = h^*(x, y) + h_2^*(x, y)x,$$

$$h_4(x, y) = h_0^*(x, y)xy - h^*(x, y) - h_1^*(x, y)y - h_2^*(x, y)x.$$

3.1.1 Queueing Examples

To better illustrate how the generating function method works on queueing problems, I will take several interesting queueing models as examples.

1. The Queueing System with Two Demands

The queueing system with two demands, originally analyzed by [9], detailed illustrated in Chapter 4. In this system, customers arrive according to a single Poisson arrival stream with rate λ . Upon arrival, each customer simultaneously creates two demands or jobs, which are served by two independent exponential servers, which are also independent of arrivals, with service rates μ_1 and μ_2 , respectively. In front of each server, there is a waiting line (or queue) of infinite waiting capacity. In each queue, jobs are served according to the First-Come-First-Served or FIFO discipline. Upon completion of its service, the job leaves the system immediately. For this system, the numbers of customers in queue 1 and queue 2, respectively, give a continuous time Markov chain. Without loss of generality, we assume that $\lambda + \mu_1 + \mu_2 = 1$. This Markov chain is ergodic if

and only if $\mu_1 > \lambda$ and $\mu_2 > \lambda$, which will be imposed throughout the monograph. Upon uniformization, this leads to the discrete version of this Markov chain, which is a DQBD-RW. Specifically, all non-zero transition probabilities are given by $p_{1,1} = p_{1,1}^{(1)} = p_{1,1}^{(2)} = p_{1,1}^{(0)} = \lambda$, $p_{-1,0} = p_{-1,0}^{(1)} = \mu_1$, $p_{0,-1} = p_{0,-1}^{(2)} = \mu_2$, $p_{0,0}^{(1)} = \mu_2$, $p_{0,0}^{(2)} = \mu_1$, and $p_{0,0}^{(0)} = \mu_1 + \mu_2$. The transition diagram is depicted in Figure 3.1.

Figure 3.1: The Queueing Model with Two Demands

In this case,

$$h(x, y) = xy - (\mu_1 y + \mu_2 x + \lambda x^2 y^2),$$

$$h_1(x, y) = (\mu_1 + \mu_2 x) + \lambda x^2 y - x,$$

$$h_2(x, y) = (\mu_2 + \mu_1 y) + \lambda x y^2 - y,$$

$$h_0(x, y) = (\mu_1 + \mu_2) + \lambda x y - 1$$

2. The Classical Tandem Queues

Let the arrival rate be λ , and let the service rates in queue 1 and queue 2 be μ_1 and μ_2 , respectively. Without loss of generality, assume $\lambda + \mu_1 + \mu_2 = 1$. Let the number of customers in queue 1 and in queue 2, including the one in service, be X and Y , respectively. Upon uniformization, the discrete time version of the tandem queue model leads to $p_{1,0} = p_{1,0}^{(1)} = p_{1,0}^{(2)} = p_{1,0}^{(0)} = \lambda$, $p_{-1,1} = p_{-1,1}^{(1)} = \mu_1$, $p_{0,-1} = p_{0,-1}^{(2)} = \mu_2$, $p_{0,0}^{(1)} = \mu_2$, $p_{0,0}^{(2)} = \mu_1$ and $p_{0,0}^{(0)} = \mu_1 + \mu_2$. All other probabilities are zero. The transition diagram is given in Figure 3.2. Therefore,

Figure 3.2: The Tandem Queues with Two Stages

$$h(x, y) = xy - (\mu_1 y^2 + \mu_2 x + \lambda x^2 y),$$

$$h_1(x, y) = (\mu_2 x + \lambda x^2 + \mu_1 y) - x,$$

$$h_2(x, y) = (\mu_2 + \mu_1 y + \lambda xy) - y,$$

$$h_0(x, y) = (\mu_1 + \mu_2 + \lambda x) - 1.$$

3. The Tandem Queues with Coupled Processors

This model is exactly the same as the previous one except that in the tandem queue model with two coupled processors, $p_{-1,1}^{(1)} = \mu_1 + \mu_2^*$, $p_{0,0}^{(1)} = 1 - (\lambda + \mu_1 + \mu_2^*)$, $p_{0,-1}^{(2)} = \mu_1^* + \mu_2$, and $p_{0,0}^{(2)} = 1 - (\lambda + \mu_1^* + \mu_2)$, where $0 \leq \mu_1^* \leq \mu_1$ and $0 \leq \mu_2^* \leq \mu_2$.

The transition diagram is given in Figure 3.3. In this case,

Figure 3.3: Tandem Queues with Couple Processors

$$h(x, y) = xy - (\mu_1 y^2 + \mu_2 x + \lambda x^2 y),$$

$$h_1(x, y) = [(1 - (\lambda + \mu_1 + \mu_2^*))x + \lambda x^2 + (\mu_1 + \mu_2^*)y] - x,$$

$$h_2(x, y) = [(\mu_1^* + \mu_2) + (1 - (\lambda + \mu_1^* + \mu_2))y + \lambda xy] - y,$$

$$h_0(x, y) = (\mu_1 + \mu_2 + \lambda x) - 1$$

4. The Classical Open Jackson Network

This is a classical model for which we know the product-form solution for the joint stationary distribution of the queue length in the both queues. This network has two nodes denoted by node 1 and node 2. Node i , $i = 1, 2$, has exogenous arrivals according to the Poisson process with rate λ_i , and a single server whose service times are i.i.d. exponential with rate μ_i . Customers who complete their service at node i go to the other node j with probability $r_{i,j}$ or leave the network with probability $1 - r_{i,j}$. Without loss of generality, we assume that $\lambda_1 + \lambda_2 + \mu_1 + \mu_2 = 1$. See Figure 3.4 for the transition diagram.

Figure 3.4: Two-Node Jackson network

In this case,

$$\begin{aligned}
 h(x, y) &= xy - (\lambda_1 x^2 y + r_{2,1} \mu_2 x^2 + \lambda_2 x y^2 + r_{2,0} \mu_2 x + r_{1,2} \mu_1 y^2 + r_{1,0} \mu_1 y), \\
 h_1(x, y) &= (r_{1,0} \mu_1 + \mu_2 x + \lambda_1 x^2) + (r_{1,2} \mu_1 + \lambda_2 x) y - x, \\
 h_2(x, y) &= (r_{2,0} \mu_2 + \mu_1 y + \lambda_2 y^2) + (r_{2,1} \mu_2 + \lambda_1 y) x - y, \\
 h_0(x, y) &= (\mu_1 + \mu_2) + \lambda_1 x + \lambda_2 y - 1
 \end{aligned}$$

5. The Jackson Network with Coupled Processors

This system is a modification of the classical Jackson network. All assumptions about the model remain the same as for the classical one, except that when server i is idle, it will help serving customers in the other queue, if there is any, with service rate $\mu_i^* \leq \mu_i$, $i = 1, 2$. See Figure 3.5 for the transition diagram.

Figure 3.5: The Jackson Network with Coupled Processors

In this case,

$$h(x, y) = xy - (\lambda_1 x^2 y + r_{2,1} \mu_2 x^2 + \lambda_2 x y^2 + r_{2,0} \mu_2 x + r_{1,2} \mu_1 y^2 + r_{1,0} \mu_1 y),$$

$$h_1(x, y) = [r_{1,0}(\mu_1 + \mu_2^*) + (1 - \theta_1)x + \lambda_1 x^2] + [r_{1,2}(\mu_1 + \mu_2^*) + \lambda_2 x]y - x,$$

$$h_2(x, y) = [r_{2,0}(\mu_1^* + \mu_2) + (1 - \theta_2)y + \lambda_2 y^2] + [r_{2,1}(\mu_1^* + \mu_2) + \lambda_1 y]x - y,$$

$$h_0(x, y) = (\mu_1 + \mu_2) + \lambda_1 x + \lambda_2 y - 1$$

where

$$\theta_1 = \lambda_1 + \lambda_2 + \mu_1 + \mu_2^*, \quad \theta_2 = \lambda_1 + \lambda_2 + \mu_1^* + \mu_2.$$

Consider the symmetric join-the-shortest-queue (JSQ) model, in which customers arrive according to a Poisson process with rate λ to two identical exponential servers, each having its own queue, with a common service rate μ . Upon arrival, a customer joins the shortest queue. If the queue lengths are equal, the customer joins either queue with probability $1/2$. Let $Q_1(t)$ and $Q_2(t)$ be the lengths of the two queues at time t , then $(Q_1(t), Q_2(t))$ is a Markov chain, which is not immediately lead to a DQBD-RW. To convert it into a DQBD-RW, let $X(t) = \min(Q_1(t), Q_2(t))$ and $Y(t) = |Q_2(t) - Q_1(t)|$. Then, the discrete time version of $(X(t), Y(t))$ is a DQBD-RW with $p_{-1,1} = \mu$, $p_{0,-1} = p_{0,-1}^{(2)} = \mu$, $p_{1,-1} = p_{1,-1}^{(2)} = \lambda$, $p_{-1,1}^{(1)} = 2\mu$, $p_{0,1}^{(1)} = p_{0,1}^{(0)} = \lambda$, $p_{0,0}^{(2)} = \mu$, and $p_{0,0}^{(0)} = 2\mu$, whose transition diagram is given in Figure 3.6.

Figure 3.6: The Symmetric Join-The-Shortest-Queue Model

$$\begin{aligned} h(x, y) &= xy - (\mu y^2 + \mu x + \lambda x^2), \\ h_1(x, y) &= (2\mu + \lambda x)y - x, \\ h_2(x, y) &= (\mu + \mu y + \lambda x) - y, \\ h_0(x, y) &= (2\mu + \lambda y) - 1. \end{aligned}$$

This model is a modification of the classical join-the-shortest-queue model described in the previous subsection. The only difference for the modified model is the service rate when queue i is empty. In this case, the service rate for the other queue is modified from μ_j to $\mu_j + \mu_i^*$, where $\mu_i^* \leq \mu_i$. See Figure 3.7 for the transition diagram.

Figure 3.7: The Symmetric JSQ Model with Coupled Processors

For this model, we have

$$\begin{aligned}
 h(x, y) &= xy - (\mu y^2 + \mu x + \lambda x^2), \\
 h_1(x, y) &= (2\mu + \lambda x)y - x, \\
 h_2(x, y) &= (\mu + \mu y + \lambda x) - y, \\
 h_0(x, y) &= (2\mu + \lambda y) - 1.
 \end{aligned}$$

8. The Pre-Emptive Priority Queueing System

Consider the $M/M/1$ preemptive priority queueing system with two classes of customers, the higher and lower priority classes of customers arrive independently according to two Poisson processes with rates λ_1 and λ_2 , respectively. Higher priority customers have priority in service over lower priority customers. In each of these two classes, customers are served according to the FIFO queueing discipline. With the preemptive rule, the service of a lower priority customer is interrupted upon the arrival of a higher priority customer, that stays at the head of line waiting for restart of its service immediately after the last higher priority customer in the system completes its service. Both classes of customers require an exponential amount of service time and are served by a single server with possibly different service rates μ_1 and μ_2 for higher and lower priority customers, respectively. All service times are independent and also independent of the two arrival processes. If we use the number of higher and lower customers, respectively, in the system as two system variables, then we have a continuous time Markov chain. Without loss of generality, we assume that $\lambda_1 + \lambda_2 + \mu_1 + \mu_2 = 1$ and consider the discrete version of this Markov chain, which is a DQBD-RW. Then, we have $p_{-1,0} = p_{-1,0}^{(1)} = \mu_1$,

$p_{0,0} = p_{0,0}^{(1)} = \mu_2$, $p_{0,1} = p_{0,1}^{(1)} = p_{0,1}^{(2)} = p_{0,1}^{(0)} = \lambda_2$, $p_{1,0} = p_{1,0}^{(1)} = p_{1,0}^{(2)} = p_{1,0}^{(0)} = \lambda_1$, $p_{0,-1}^{(2)} = \lambda_2$, $p_{0,0}^{(2)} = \mu_1$, and $p_{0,0}^{(0)} = \mu_1 + \mu_2$. The transition diagram is given in Figure 3.8. Therefore,

Figure 3.8: The Priority Queueing System with Two Types of Customers

$$h(x, y) = xy - [\mu_1 + (\mu_2 + \lambda_2 y)x + \lambda_1 x^2] y,$$

$$h_1(x, y) = [\mu_1 + (\mu_2 + \lambda_2 y)x + \lambda_1 x^2] - x,$$

$$h_2(x, y) = [(\lambda_2 + \mu_1 y + \lambda_2 y^2) + \lambda_1 xy] - y,$$

$$h_0(x, y) = [(\mu_1 + \mu_2) + \lambda_2 y + \lambda_1 x] - 1.$$

Remark 3.1 From the expression of $h(x, y)$, we can easily identify that it can be re-written as,

$$h(x, y) = \{x - [\mu_1 + (\mu_2 + \lambda_2 y)x + \lambda_1 x^2]\} y \quad (3.13)$$

which implies that the system is singular, and the Riemann surface of y is of genus 0. However, the Riemann surface of x is still of genus 1, which is not meromorphic with the surface of y . In this case, we cannot apply the uniformization method to this system.

3.2 Stability Condition

After we constructed the generating function equation, we need to consider several issues based on the fundamental equation we have. First, we have to consider the genus of the Riemann surface and the stability condition of the queueing system since in this thesis we are only interested in the case that the Riemann surface is of genus 1 and the random walk itself is ergodic. It is easy to check that the Riemann surface is of genus 1 if and only if when $M \neq 0$, see Lemma 2.1. The stability check can be done by the following Theorem from [7].

Theorem 3.2 (Fayolle, Iasnogorodski and Malyshev) *When $M \neq 0$, the random walk is ergodic if, and only if, one of the three following conditions holds,*

$$\begin{aligned}
 (i) \quad & \begin{cases} M_x < 0, M_y < 0 \\ M_x M'_y - M_y M'_x < 0 \\ M_y M''_x - M_x M''_y < 0 \end{cases} \\
 (ii) \quad & M_x < 0, M_y \geq 0, M_y M''_x - M_x M''_y < 0 \\
 (iii) \quad & M_x \geq 0, M_y < 0, M_x M'_y - M_y M'_x < 0
 \end{aligned} \tag{3.14}$$

3.2.1 Queueing Examples

For those queueing examples we picked in the last section, we can have the following ergodic conditions.

1. The Queueing System with Two Demands

From the transition diagram, we can easily obtain the following result,

$$\mathbf{M}_x = \lambda - \mu_1 \quad \mathbf{M}_y = \lambda - \mu_2 \quad (3.15)$$

$$\mathbf{M}'_x = \lambda - \mu_1 \quad \mathbf{M}'_y = \lambda \quad (3.16)$$

$$\mathbf{M}''_x = \lambda \quad \mathbf{M}''_y = \lambda - \mu_2 \quad (3.17)$$

Then by the theorem above, we must have,

$$(i) \left\{ \begin{array}{l} (a) \lambda < \mu_1, \lambda < \mu_2 \\ (b) \lambda(\lambda - \mu_1) - (\lambda - \mu_1)(\lambda - \mu_2) < 0 \\ \Rightarrow \mu_2(\lambda - \mu_1) < 0 \\ (c) (\lambda - \mu_2)\lambda - (\lambda - \mu_1)(\lambda - \mu_2) < 0 \\ \Rightarrow \mu_1(\lambda - \mu_2) < 0 \end{array} \right. \quad (3.18)$$

$$(ii) \lambda < \mu_1, \quad \lambda \geq \mu_2$$

$$\mu_1(\lambda - \mu_2) < 0, (\text{not possible})$$

$$(iii) \lambda \geq \mu_1, \quad \lambda < \mu_2$$

$$\mu_1(\lambda - \mu_1) < 0, (\text{not possible})$$

Therefore, this system is stable if and only if $\lambda < \min\{\mu_1, \mu_2\}$.

2. The Tandem Queues

From the transition diagram, we can easily obtain the following result,

$$\mathbf{M}_x = \lambda - \mu_1 \quad \mathbf{M}_y = \mu_1 - \mu_2 \quad (3.19)$$

$$\mathbf{M}'_x = \lambda - \mu_1 \quad \mathbf{M}'_y = \mu_1 \quad (3.20)$$

$$\mathbf{M}''_x = \lambda \quad \mathbf{M}''_y = -\mu_2 \quad (3.21)$$

Then by the theorem above, we must have,

$$(i) \left\{ \begin{array}{l} (a) \lambda < \mu_1, \mu_1 < \mu_2 \\ (b) \mu_1(\lambda - \mu_1) - (\lambda - \mu_1)(\mu_1 - \mu_2) < 0 \\ \Rightarrow \mu_2(\lambda - \mu_1) < 0, i.e. \lambda < \mu_1 \\ (c) (\mu_1 - \mu_2)\lambda + (\lambda - \mu_1)\mu_2 < 0 \\ \Rightarrow \mu_1(\lambda - \mu_2) < 0, i.e. \lambda < \mu_1 < \mu_2 \end{array} \right. \quad (3.22)$$

$$(ii) \lambda < \mu_1, \quad \mu_1 \geq \mu_2$$

$$(\mu_1 - \mu_2)\lambda + (\lambda - \mu_1)\mu_2 < 0$$

$$\Rightarrow \mu_1(\lambda - \mu_2) < 0, i.e. \lambda < \mu_2 < \mu_1$$

$$(iii) \lambda \geq \mu_1, \quad \mu_1 < \mu_2$$

$$\mu_1(\lambda - \mu_1) - (\lambda - \mu_1)(\mu_1 - \mu_2) < 0$$

$$\Rightarrow \mu_2(\lambda - \mu_1) < 0, \text{ (not possible)}$$

Therefore, this system is stable if and only if $\lambda < \mu_i$ for $i = 1, 2$.

3. The Tandem Queues with Coupled Processors

From the transition diagram, we can easily obtain the following result,

$$\mathbf{M}_x = \lambda - \mu_1 \quad \mathbf{M}_y = \mu_1 - \mu_2 \quad (3.23)$$

$$\mathbf{M}'_x = \lambda - (\mu_1 + \mu_2^*) \quad \mathbf{M}'_y = \mu_1 + \mu_2^* \quad (3.24)$$

$$\mathbf{M}''_x = \lambda \quad \mathbf{M}''_y = -(\mu_1^* + \mu_2) \quad (3.25)$$

Then by the theorem above, we must have,

$$(i) \left\{ \begin{array}{l} (a) \lambda < \mu_1, \mu_1 < \mu_2 \\ (b) (\mu_1 + \mu_2^*)(\lambda - \mu_1) - (\lambda - \mu_1 - \mu_2^*)(\mu_1 - \mu_2) < 0 \\ \Rightarrow \mu_2^*(\lambda - \mu_2) + \mu_2(\lambda - \mu_1) < 0, i.e. \lambda < \mu_1 < \mu_2 \\ (c) (\mu_1 - \mu_2)\lambda + (\lambda - \mu_1)(\mu_1^* + \mu_2) < 0 \\ \Rightarrow \mu_1(\lambda - \mu_2) + \mu_1^*(\lambda - \mu_1) < 0, i.e. \lambda < \mu_1 < \mu_2 \end{array} \right. \quad (3.26)$$

$$(ii) \lambda < \mu_1, \quad \mu_1 \geq \mu_2$$

$$\mu_1(\lambda - \mu_2) + \mu_1^*(\lambda - \mu_1) < 0$$

$$(iii) \lambda \geq \mu_1, \quad \mu_1 < \mu_2$$

$$\mu_2^*(\lambda - \mu_2) + \mu_2(\lambda - \mu_1) < 0$$

Therefore, this system is stable if and only if one of the above conditions (i) (ii) (iii) holds.

4. The Classical Open Jackson Network

From the transition diagram, we can easily obtain the following result,

$$\begin{aligned} \mathbf{M}_x &= \lambda_1 + r_{21}\mu_2 - r_{12}\mu_1 - r_{10}\mu_1 & \mathbf{M}_y &= \lambda_2 + r_{12}\mu_1 - r_{21}\mu_2 - r_{20}\mu_2 \\ \mathbf{M}'_x &= \lambda_1 - (r_{12}\mu_1 + r_{10}\mu_1) & \mathbf{M}'_y &= \lambda_2 + r_{12}\mu_1 \\ \mathbf{M}''_x &= \lambda_1 + r_{21}\mu_2 & \mathbf{M}''_y &= \lambda_2 - r_{21}\mu_2 \end{aligned}$$

Then applying the theorem above, and after some simplifications, we obtain that this network is stable if and only if $\rho_1 < 1$ and $\rho_2 < 1$, where

$$\rho_i = \frac{\lambda_i + r_{j,i}\lambda_j}{(1 - r_{i,j}r_{j,i})\mu_i}, \quad i \neq j \text{ and } i, j = 1, 2.$$

5. The Jackson Network with Coupled Processors

The stationary condition for this system is quite complicated when applying the Theorem above. We can refer to the paper [10] by Robert D. Foley and David R. McDonald for the stationary condition on one type of modified Jackson Network.

6. The Symmetric JSQ Model

From the transition diagram, we can easily obtain the following result,

$$\mathbf{M}_x = \lambda - \mu \quad \mathbf{M}_y = -\mu \tag{3.27}$$

$$\mathbf{M}'_x = -2\mu \quad \mathbf{M}'_y = \lambda \tag{3.28}$$

$$\mathbf{M}''_x = \lambda \quad \mathbf{M}''_y = -\mu \tag{3.29}$$

Then by the theorem above, we must have,

$$(i) \left\{ \begin{array}{l} (a) \lambda < \mu, -\mu < 0 \\ (b) \lambda(\lambda - \mu) - \mu(2\mu) < 0 \\ \Rightarrow (\lambda + \mu)(\lambda - 2\mu) < 0, i.e. \lambda < 2\mu \\ (c) -\mu\lambda + (\lambda - \mu)\mu < 0 \\ \Rightarrow -\mu^2 < 0 \end{array} \right. \quad (3.30)$$

$$(ii) \lambda < \mu, \quad -\mu \geq 0 \quad (\text{not possible})$$

$$(iii) \lambda \geq \mu, \quad -\mu < 0$$

$$(\lambda + \mu)(\lambda - 2\mu) < 0, i.e. \lambda < 2\mu$$

The DQBD-RW is ergodic if and only if $\rho = \lambda/2\mu < 1$.

7. The JSQ Model with Couple Processors

From the transition diagram, we can easily obtain the following result,

$$M_x = \lambda - \mu \quad M_y = -\mu \quad (3.31)$$

$$M'_x = -2\mu \quad M'_y = \lambda \quad (3.32)$$

$$M''_x = \lambda \quad M''_y = -(\mu + \mu^*) \quad (3.33)$$

Then by the theorem above, we must have,

$$(i) \left\{ \begin{array}{l} (a) \lambda < \mu, -\mu < 0 \\ (b) \lambda(\lambda - \mu) - \mu(2\mu) < 0 \\ \Rightarrow (\lambda + \mu)(\lambda - 2\mu) < 0, i.e. \lambda < 2\mu \\ (c) -\mu\lambda + (\lambda - \mu)(\mu + \mu^*) < 0 \\ \Rightarrow -\mu^2 + (\lambda - \mu)^* < 0 \end{array} \right. \quad (3.34)$$

$$(ii) \lambda < \mu_1, \quad -\mu \geq 0 \quad (\text{not possible})$$

$$(iii) \lambda \geq \mu, \quad -\mu_1 < 0$$

$$(\lambda + \mu)(\lambda - 2\mu) < 0, i.e. \lambda < 2\mu$$

The DQBD-RW is ergodic if and only if $\rho = \lambda/2\mu < 1$.

8. The Pre-Emptive Priority Queueing System

From the transition diagram, we can easily obtain the following result,

$$\mathbf{M}_x = \lambda_1 - \mu_1 \quad \mathbf{M}_y = \lambda_2 \quad (3.35)$$

$$\mathbf{M}'_x = \lambda_1 - \mu_1 \quad \mathbf{M}'_y = \lambda_2 \quad (3.36)$$

$$\mathbf{M}''_x = \lambda_1 \quad \mathbf{M}''_y = \lambda_2 - \mu_2 \quad (3.37)$$

Then by the theorem above, we must have,

$$\begin{aligned}
(i) \quad & \lambda < \mu_1, \lambda_2 < 0 \quad (\text{not possible}) \\
(ii) \quad & \lambda < \mu_1, \quad \lambda_2 \geq 0 \\
& \lambda_2 \lambda_1 - (\lambda_1 - \mu_1)(\lambda_2 - \mu_2) < 0 \\
& \Rightarrow \lambda_1 \mu_2 + \mu_1 \lambda_2 - \mu_1 \mu_2 < 0, i.e. \frac{\lambda_1}{\mu_1} + \frac{\lambda_2}{\mu_2} < 1 \\
(iii) \quad & \lambda \geq \mu_1, \quad \lambda_2 < 0 \quad (\text{not possible})
\end{aligned} \tag{3.38}$$

Therefore, the DQBD-RW is ergodic if and only if

$$\rho = \rho_1 + \rho_2 < 1,$$

where

$$\rho_1 = \frac{\lambda_1}{\mu_1} \quad \text{and} \quad \rho_2 = \frac{\lambda_2}{\mu_2}.$$

3.3 Analysis of The Kernel $h_1(x, y)$

If we treat y as a function of x or x as a function of y , then let $h_1(x, y) = 0$, we can get the relationship between y and x , in which case one variable will be a quadratic function of the other variable. Next, focus on the delta function of the quadratic relation equation, if the Riemann surfaces of later analysis are genus 1, then the delta functions will have four different non-negative roots, usually 2 of which are between 0 and 1 and the other 2 are greater than 1. Particularly, if one of the four roots is equal to 0, then in later analysis, we should consider the other

variable to construct the Riemann surface. For the general case, we can refer to the following Lemma shown in [7].

Lemma 3.1 (Fayolle, Iasnogorodski and Malyshev) *For all non-singular random walks such that $M_y \neq 0$, $y(x)$ has two branch points x_1 and x_2 (resp. x_3 and x_4) inside (resp. outside) the unit circle. All these branch points lie on the real line.*

For the pair (x_3, x_4) , the following classification holds,

1. *If $p_{10} > 2\sqrt{p_{11}p_{1,-1}}$, then x_3 and x_4 are positive;*
2. *If $p_{10} = 2\sqrt{p_{11}p_{1,-1}}$, then one point (x_4) is infinite and the other (x_3) is positive;*
3. *If $p_{10} < 2\sqrt{p_{11}p_{1,-1}}$, one point is negative and the other is positive;*

Similarly, for the pair (x_1, x_2) ,

4. *If $p_{-1,0} > 2\sqrt{p_{-1,1}p_{-1,-1}}$, then x_1 and x_2 are positive;*
5. *If $p_{-1,0} = 2\sqrt{p_{-1,1}p_{-1,-1}}$, then one point is 0 and the other is positive;*
6. *If $p_{-1,0} < 2\sqrt{p_{-1,1}p_{-1,-1}}$, one point is negative and the other is positive;*

This lemma is also true for $x(y)$, up to a proper symmetric change of the parameters.

Lemma 3.2 (Fayolle, Iasnogorodski and Malyshev) *For all non-singular random walk such that $M_y = 0$, one of the branch points of $y(x)$ is equal to 1. In addition,*

1. *If $M_x < 0$, then two other branch points have a modulus bigger than 1 and the remaining one has a modulus less than 1;*

2. If $M_x > 0$, then two branch points are less than 1 and the remaining one has a modulus bigger than 1;

3.3.1 Queueing Examples

For those queueing examples we picked in section 3.1.1, we can easily determine the positions of the roots by the Lemma above. Besides, since among all these queueing examples $M_y \neq 0$, we immediately have $0 \leq x_1 < x_2 < 1 < x_3 < x_4$ or $0 \leq x_1 < x_2 < 1 < x_3 < +\infty$.

1. The Queueing System with Two Demands

From the transition diagram, we obtain that $p_{10} = p_{1,-1} = 0$, while $p_{11} = \lambda$ and $p_{-1,0} = p_{-1,1} = p_{-1,-1} = 0$. Therefore, according to the two Lemmas above, we have $0 < x_1 < x_2 < 1 < x_3 < x_4 = +\infty$.

2. The Tandem Queues & 3. Tandem Queues with Coupled Processors

From the transition diagram, we obtain that $p_{11} = p_{1,-1} = 0$, while $p_{10} = \lambda$ and $p_{-1,0} = p_{-1,-1} = 0$ while $p_{-1,1} = \mu_1$. Therefore, according to the two Lemmas above, we have $0 = x_1 < x_2 < 1 < x_3 < x_4$.

4. The Jackson Network & 5. Jackson Network with Coupled Processors

From the transition diagram, we obtain that $p_{10} = \lambda_1, p_{1,-1} = r_{21}\mu_2$, while $p_{11} = 0$ and $p_{-1,0} = r_{1,0}\mu_1, p_{-1,-1} = 0$ while $p_{-1,1} = r_{12}\mu_1$. Therefore, according to the two Lemmas above, we have $0 < x_1 < x_2 < 1 < x_3 < x_4$.

6. The Symmetric JSQ Model & 7. JSQ Model with Coupled Processors

From the transition diagram, we obtain that $p_{10} = p_{11} = 0$, while $p_{1,-1} = \lambda$ and $p_{-1,0} = p_{-1,-1} = 0$ while $p_{-1,1} = \mu$. Therefore, according to the two Lemmas above, we have $0 = x_1 < x_2 < 1 < x_3 < x_4 = +\infty$.

8. The Pre-Emptive Priority Queueing System

Since the system is singular, i.e., the Riemann surface of the system is of genus 0, we do not discuss the roots of the system here.

3.4 Parametrization of The Riemann Surface

Now if we treat y as a quadratic function of x , we need to map the upper half of the x -plane onto the t -plane to construct the Riemann surface, on the other hand, if we treat x as a quadratic function of y , we need to map the upper half of the y -plane onto the t -plane to construct the Riemann surface. The method of mapping is usually described as follows,

By the Schwarz-Chistoffel transformation, see section 2.4, there exists a closed rectangle $\mathcal{R}_0$ with vertices $0, \tilde{\omega}, \tilde{\omega} + \omega, \omega$, where $\tilde{\omega}, \omega/i > 0$, and a $1 - 1$ mapping $t(x)$ from the extended closed upper half plane $\mathbb{H}$ to $\mathcal{R}_0$ which contains $t(x)$, continuous on $\mathbb{H}$ and analytic on interior of $\mathbb{H}$. $t(x)$ maps x_1, x_2, x_3, x_4 to the vertices $0, \tilde{\omega}, \tilde{\omega} + \omega, \omega$, respectively, see Figure 3.9.

 Figure 3.9: General Mapping Rule from Upper-Half of x -plane to t -plane

Particularly, in the 2-demand example stated in section 3.1, we only have three distinct positive roots of $D(x)$, x_1 , x_2 and x_3 with $0 < x_1 < x_2 < 1 < x_3 < +\infty$. In this case, we map the point ∞ instead of x_4 in the x -plane to point ω in the t -plane.

Then according to the Reflection Principle, we duplicate the rectangle we constructed to the whole t -plane, see Figure 3.10

Figure 3.10: Reflection Principle

Now, look at the original rectangle that we construct, we can easily obtain that the point ∞ and 0 are on the vertical axis between 0 and ω , while the point 1 is on the opposite side between $\tilde{\omega}$ and $\tilde{\omega} + \omega$. In the example described in section 3.1, we have to consider and assume the other point $t = q$ such that $x(q) = \rho_1^{-1}$, in which case $y(q) = 1$, which potentially is the candidate of the zero or the pole of $P(x, 0)$ and $P(0, y)$.

3.5 The Relationship between $x(t)$ and $y(t)$

One of the key issues that affect the determinations of the final generating functions of $P(x, 0)$ and $P(0, y)$ is that we have to determine the relationship between $x(t)$ and $y(t)$. The procedure of the classification of the relationship starts from the comparison between the zeros of $x(t)$ and $y(t)$ and the poles of $x(t)$ and $y(t)$. Generally,

$$\begin{aligned}
0 \equiv h_1(x, y) &= a(x)y^2 + b(x)y + c(x) \\
&= \tilde{a}(y)x^2 + \tilde{b}(y)x + \tilde{c}(y) \\
&= -(p_{-1,1} + p_{0,1}x + p_{1,1}x^2)y^2 - [p_{-1,0} + (p_{0,0} - 1)x + p_{1,0}x^2]y \\
&\quad - (p_{-1,-1} + p_{0,-1}x + p_{1,-1}x^2) \\
&= -(p_{1,-1} + p_{1,0}y + p_{1,1}y^2)x^2 - [p_{0,-1} + (p_{0,0} - 1)y + p_{0,1}y^2]x \\
&\quad - (p_{-1,-1} + p_{-1,0}y + p_{-1,1}y^2)
\end{aligned} \tag{3.39}$$

We have to solve the equation by either taking y as a quadratic function of x , or taking x as a quadratic function of y .

$$x(y) = \frac{-\tilde{b}(y) \pm \sqrt{\tilde{b}(y)^2 - 4\tilde{a}(y)\tilde{c}(y)}}{2\tilde{a}(y)} \quad (3.40)$$

$$y(x) = \frac{-b(x) \pm \sqrt{b(x)^2 - 4a(x)c(x)}}{2a(x)} \quad (3.41)$$

where,

$$a(x) = -(p_{-1,1} + p_{0,1}x + p_{1,1}x^2), \quad (3.42)$$

$$b(x) = -[p_{-1,0} + (p_{0,0} - 1)x + p_{1,0}x^2], \quad (3.43)$$

$$c(x) = -(p_{-1,-1} + p_{0,-1}x + p_{1,-1}x^2), \quad (3.44)$$

$$\tilde{a}(y) = -(p_{1,-1} + p_{1,0}y + p_{1,1}y^2), \quad (3.45)$$

$$\tilde{b}(y) = -[p_{0,-1} + (p_{0,0} - 1)y + p_{0,1}y^2], \quad (3.46)$$

$$\tilde{c}(y) = -(p_{-1,-1} + p_{-1,0}y + p_{-1,1}y^2). \quad (3.47)$$

Then we usually apply the Taylor expansion to the square root of the delta functions to find the relationship between x and y .

3.5.1 Queueing Examples

For those queueing examples we picked in section 3.1.1, we can express the y as a function of x as follows,

1. The Queueing System with Two Demands

$$\begin{aligned}
0 \equiv h_1(x, y) &= (\lambda + \mu_1 + \mu_2)xy - \mu_1y - \mu_2x - \lambda x^2y^2 \\
&= (1 + \rho_1^{-1} + \rho_2^{-1})xy - \rho_1^{-1}y - \rho_2^{-1}x - x^2y^2 \\
&= -x^2y^2 + ((1 + \rho_1^{-1} + \rho_2^{-1})x - \rho_1^{-1})y - \rho_2^{-1}x
\end{aligned} \tag{3.48}$$

Next, if we treat y as a quadratic function of x , we take a look at the expression of $y(x)$,

$$y(x) = -\frac{-(1 + \rho_1^{-1} + \rho_2^{-1})x + \rho_1^{-1}}{2x^2} \mp \frac{\sqrt{\mathbf{D}_1(x)}}{2x^2} \tag{3.49}$$

after taking the Taylor expansion of $\mathbf{D}_1(x)$, we have,

$$y_+(x) = \frac{[-(1 + \rho_1^{-1} + \rho_2^{-1})x + \rho_1^{-1}] + \sqrt{\mathbf{D}_1(x)}}{2(-x^2)} = -\frac{\rho_2^{-1}}{\rho_1^{-1}}x + o(x) \tag{3.50}$$

$$\begin{aligned}
y_-(x) &= \frac{[-(1 + \rho_1^{-1} + \rho_2^{-1})x + \rho_1^{-1}] - \sqrt{\mathbf{D}_1(x)}}{2(-x^2)} \\
&= -\frac{\rho_1^{-1}}{x^2} + \frac{1 + \rho_2^{-1} + \rho_2^{-1}}{x} + \frac{\rho_2^{-1}}{\rho_1^{-1}}x + o(x)
\end{aligned} \tag{3.51}$$

which further implies the constant in the relationship between $x(t)$ and $y(t)$, and after some argument around the zero point of $x(t)$, we can obtain the exact relationship of $x(t)$ and $y(t)$.

In the 2-demand example described in section 3.1, we eventually have $y(t) = \frac{\rho_2^{-1}}{\rho_1^{-1}}x(t - 2m)$.

2. The Tandem Queues & 3. Tandem Queues with Coupled Processors

$$0 \equiv h_1(x, y) = xy - (\mu_1y^2 + \mu_2x + \lambda x^2y) \tag{3.52}$$

We will have,

$$y = \frac{x - \lambda x^2 \pm \sqrt{\mathbf{D}_1(x)}}{2\mu_1} \quad (3.53)$$

where,

$$\mathbf{D}_1(x) = (\lambda x - 1)^2 x^2 - 4\mu_1 \mu_2 x \quad (3.54)$$

Further Taylor expansion of $\mathbf{D}_1(x)$ is omitted for future work.

4. The Jackson Network & 5. Jackson Network with Coupled Processors

$$0 \equiv h_1(x, y) = xy - (\lambda_1 x^2 y + r_{2,1} \mu_2 x^2 + \lambda_2 x y^2 + r_{2,0} \mu_2 x + r_{1,2} \mu_1 y^2 + r_{1,0} \mu_1 y) \quad (3.55)$$

We will have,

$$y = \frac{x - \lambda_1 x^2 + r_{10} \mu_1 \pm \sqrt{\mathbf{D}_1(x)}}{2(\lambda_2 x + r_{12} \mu_1)} \quad (3.56)$$

where,

$$\mathbf{D}_1(x) = (\lambda_1 x^2 + r_{10} \mu_1 - x)^2 - 4(\lambda_2 x + r_{12} \mu_1)(r_{21} \mu_2 x^2 + r_{20} \mu_2 x) \quad (3.57)$$

Further Taylor expansion of $\mathbf{D}_1(x)$ is omitted for future work.

6. The Symmetric JSQ Model & 7. JSQ Model with Coupled Processors

$$0 \equiv h_1(x, y) = xy - (\mu y^2 + \mu x + \lambda x^2) \quad (3.58)$$

We will have,

$$y = \frac{x \pm \sqrt{\mathbf{D}_1(x)}}{2\mu} \quad (3.59)$$

where,

$$\mathbf{D}_1(x) = x^2 - 4\mu(\lambda x^2 - \mu x) \quad (3.60)$$

Further Taylor expansion of $\mathbf{D}_1(x)$ is omitted for future work.

6. The Pre-Emptive Priority Queueing System

$$0 \equiv h_1(x, y) = xy - [\mu_1 + (\mu_2 + \lambda_2 y)x + \lambda_1 x^2]y \quad (3.61)$$

We will have,

$$y = 0 \quad \text{or} \quad y = \frac{x - \mu_1 - \mu_2 x - \lambda_1 x^2}{\lambda_2 x} \quad (3.62)$$

Remark 3.2 *Again, from the analysis of the relationship between x and y in the pre-emptive priority queueing system, we find that x and y in this system are not meromorphic functions, in which case we cannot apply the uniformization method.*

3.5.2 Properties of Elliptic Functions

For the elliptic function $\wp(t; \tilde{\omega}, \omega)$ with the periods $m\tilde{\omega} + n\omega$, $m, n \in \mathbb{Z}$, we can determine the two periods $\tilde{\omega}, \omega$ by the following Lemma from [7],

Lemma 3.3 (Fayolle, Iasnogorodski and Malyshev) *One can choose $\tilde{\omega}, \omega$ so that ω be purely imaginary and $\tilde{\omega}$ be real. Such a choice is unique, up to signs. Hence we shall take $\tilde{\omega} > 0, \text{Im}(\omega) > 0$. Moreover, ω are given by the following integrals, taken on the intervals of the real axis,*

$$\omega = 2i \int_{x_1}^{x_2} \frac{dx}{\sqrt{-D_1(x)}}, \quad \tilde{\omega} = 2 \int_{x_2}^{x_3} \frac{dx}{\sqrt{D_1(x)}} \quad (3.63)$$

Formally this lemma is sufficient to determine the periods of elliptic functions we obtained from the generating functions. However, since usually we do not need the information about the periods of elliptic functions directly, the determination of the periods of elliptic functions can be usually omitted.

3.6 Analytic Continuation of The Algebraic Curve

Define $D_x = \{t | |x(t)| < 1\}$, $D_y = \{t | |y(t)| < 1\}$. The closures of these sets are $\overline{D}_x = \{t | |x(t)| \leq 1\}$, $\overline{D}_y = \{t | |y(t)| \leq 1\}$. Then the functions $P(x(t), 0)$ and $P(0, y(t))$ are analytic on D_x and D_y and continuous on $\overline{D}_x$ and $\overline{D}_y$, respectively. Let D_x, D_y be the components of D_x, D_y that containing 0 and $2m$, respectively. Generally, the mapping rule of $x(t)$ and $y(t)$ implies that: D_x and D_y are invariant under $t \mapsto t + 2\omega$; D_x is symmetric with respect to the real and imaginary axis, D_y is symmetric with respect to the horizontal line through $2m$ and imaginary

axis.

Let $A(t) = P(x(t), 0), t \in \overline{D}_x$, $B(t) = P(0, y(t)), t \in \overline{D}_y$. Then $A(t), B(t)$ are analytic on D_x, D_y and continuous on $\overline{D}_x, \overline{D}_y$ respectively. Define $A_1(t) = \frac{A(t)}{f(x(t))}$ where generally $f(x(t)) = \frac{h_2(x(t), y(t))}{h_3(x(t), y(t))}$.

The definition of $f(x(t))$ is according to the relationship between $P(x, 0)h_2(x, y)$ and $P(0, y)h_3(x, y)$ when $h_1(x, y) = 0$. We have to consider those points that will potentially be the zeros or the poles of $P(x, 0)$ and $P(0, y)$. Particularly, if the equation $P(x, 0)h_2(x, y) = P(0, y)h_3(x, y)$ can be factorized, latter analysis will be simplified significantly. Here, in example of queueing system with two demands, we have that $h_2(x, y) = x(y - 1)$ and $h_3(x, y) = y(x - 1)$. Obviously, in this case, the equation $P(x, 0)h_2(x, y) = P(0, y)h_3(x, y)$ on kernel is,

$$P(x, 0)x(y - 1) = P(0, y)y(x - 1) \quad (3.64)$$

Which can be factorized into the form,

$$\frac{P(x, 0)}{\frac{x-1}{x}} = \frac{P(0, y)}{\frac{y-1}{y}} \quad (3.65)$$

It is clear that equation (3.65) implies that we need to find the points that $x = 0, 1$ and $y = 0, 1$ since these points may affect the zeros and the poles of $P(x, 0)$ and $P(0, y)$.

Formally, we have the following three theorems from [7] for general analytic con-

tinuation.

Theorem 3.3 (Fayolle, Iasnogorodski and Malyshev) *$A(t)$ and $B(t)$ can be continued as meromorphic functions to the whole universal covering, where they satisfy equations,*

$$A(t + \tilde{\omega}) = A(t), \quad B(t + \tilde{\omega}) = B(t) \quad (3.66)$$

$$h_2(t)A(t) + h_3(t)B(t) + h_4(t)P(0, 0) = 0 \quad (3.67)$$

together with the relations

$$A(t) = A(\xi^*(\omega)), \quad B(t) = B(\eta^*(\omega)), \forall \omega \in \mathbb{C} \quad (3.68)$$

where,

$$\xi(\omega) = -\omega + 2a_1, \quad \eta(\omega) = -\omega + 2b_1 \quad (3.69)$$

and a_1, b_1 are fixed points of ξ^* and η^* , such that,

$$\xi(a_1) = a_1, \quad \eta(b_1) = b_1 \quad (3.70)$$

Theorem 3.4 (Fayolle, Iasnogorodski and Malyshev) *The functions $A(t)$ and $B(t)$ can be continued to the whole Riemann surface $\mathbb{S}$.*

Theorem 3.5 (Fayolle, Iasnogorodski and Malyshev) *Under the ergodic condition of the theorem 3.2, $A(t)$ is a meromorphic function in the complex plane $\mathbb{C}_x$ cut along $[x_3, x_4]$. A similar statement holds for $B(t)$, with the corresponding*

cut $[y_3, y_4]$ in $\mathbb{C}_y$.

The algebraic continuation is the most important step among the procedure of solving queueing problems via generating function method. Theoretically, any algebraic curve of $A(t)$ and $B(t)$ can be continued with these three theorems. However, practically, we may encounter some technical obstacles which are usually very time-consuming. For those queueing examples we discussed in the last few sections, the algebraic continuation would be a huge work which we decide to left them as future work.

Nevertheless, here we pick the 2-demand example, the algebraic continuation of which has a quick path through the determination of the period of $A_1(t)$, with the help from the relationship of $x(t)$ and $y(t)$.

Next, in the fundamental rectangle that describes $A_1(t)$, in which we make an argument that $A(t)$ is analytic, but $f(x(t))$ has zeros and poles in this rectangle, we can easily find the zeros and poles of $A_1(t)$. Usually, in this procedure, we also need one of the properties of the elliptic function, that is, in one period, the number of zeros and the number of poles must be equal.

Then, we can find the period of $A(t)$ by determining least common multiplier of the periods of $f(x(t))$ and $A_1(t)$. In the 2-demand example described in section 3.1, we have, $A_1(t)$ is an even elliptic function with periods $2\tilde{\omega}, 4m = \frac{4\omega}{3}$. $q - 4m$, p are the poles of $A_1(t)$ in $\mathbf{R}_{A_1}$. Similarly, we can determine that $\pm m$ are the zeros of $A_1(t)$ in $\mathbf{R}_{A_1}$. By a simple argument of the least common multiplier of

the periods of $f(x(t))$ and $A_1(t)$, we have the periods of $A(t)$ are $2\tilde{\omega}, 4\omega$.

3.7 Determination of The Zeros and Poles of $A(t)$ and $B(t)$

In the fundamental rectangle of $A(t)$, we classify the zeros and poles of $A(t)$ by marking out the zeros and poles of $A_1(t)$ and $f(x(t))$. After the comparison and classification, we can obtain the complete set of zeros and poles of $A(t)$ in its fundamental rectangle, i.e.,

Functions	Zeros	Poles
$A_1(t)$	All zeros of $A_1(t)$ in one period of $A(t)$	All poles of $A_1(t)$ in one period of $A(t)$
$f(x(t))$	All zeros of $f(x(t))$ in one period of $A(t)$	All poles of $f(x(t))$ in one period of $A(t)$
$A(t)$	Compare the zeros and poles of $A_1(t)$ and $f(x(t))$ and obtain the zeros and poles of $A(t)$	

In the 2-demand example described in section 3.1, in the fundamental rectangle of $A(t)$, we obtain the complete sets of zeros and poles for $A(t)$, when $\rho_1^{-1} < \rho_2^{-1}$,

Functions	Periods	Zeros	Poles
$A(t)$	$2\tilde{\omega}, 4\omega$	$\{-\omega, \omega, q + 2m, p + 6m\}$	$\{p - 4m, q, p + 4m, q + 4m\}$

3.8 Construction of $P(x, 0)$ and $P(0, y)$

Since we have obtained the complete set of zeros and poles of $A(t)$ and $B(t)$, we can construct functions $F(t)$ and $G(t)$ by putting all zeros of $A(t)$ and $B(t)$ in the

numerator while left all the poles of $A(t)$ and $B(t)$ in the denominator. Generally, we construct a meromorphic function of $A(t)$ by

$$F(t) = \frac{\text{zeros of } A(t)}{\text{poles of } A(t)} \quad (3.71)$$

$$G(t) = \frac{\text{zeros of } B(t)}{\text{poles of } B(t)} \quad (3.72)$$

Then we argue that since $F(t)$ and $A(t)$, $G(t)$ and $B(t)$ have identical sets of zeros and poles, we must have that $A(t) = c_1 F(t)$ and $B(t) = c_2 G(t)$. Next, we determine c_1 , c_2 by comparing some simple points of $P(x, 0)$ and $P(0, y)$, such as $P(1, 0)$ and $P(0, 1)$. Finally, we convert $F(t)$ and $G(t)$ back to x -plane and y -plane, by mapping the zeros and poles from t -plane back to x -plane and y -plane. However, this is another key issue of the generating function method since sometimes we cannot obtain the exact positions of all zeros and poles in the x -plane and y -plane from the information we already have. For details, we can refer to the example in Chapter 5.

Chapter 4

Uniformization Application: 2-Demand Model

In this chapter, we will describe a particular queueing network, and apply the generating function method step by step, as one of the detailed example in this thesis.

4.1 Model Description

Consider a two-parallel-queue system, where the customers (or packets, jobs, fragments, etc.), arrive at the queueing system according to a Poisson process. The rate of the two-demand source is λ , upon each arrival of which, it creates one new customer/job in both queues simultaneously at a time. As Figure 4.1 shows, there are two parallel single-server queues in the queueing network. Each server serves one customer at a time with an exponential service time rate $\mu_i, i = 1, 2$ and the two servers are independent. In each queue, the job created by an arriving

customer immediately starts being processed if the server is idle; otherwise, the job waits in a line (that is, it joins the queue). We assume that there is infinite waiting space at each server. We assume $\lambda < \mu_i, i = 1, 2$, so that each queue is stable.

Figure 4.1: Queueing Network with Two Demands

4.1.1 The Balance Equations

Let $\pi = \{\pi_{m,n} : m, n = 0, 1, \dots\}$ be the unique stationary distribution of the two-demand queueing system, then, from Figure 4.2, we can easily obtain the following balance equations.

Figure 4.2: Transition Diagram of The Queueing Network with Two Demands

$$\lambda\pi_{0,0} = \mu_1\pi_{1,0} + \mu_2\pi_{0,1} \quad (4.1)$$

$$(\lambda + \mu_1)\pi_{1,0} = \mu_1\pi_{2,0} + \mu_2\pi_{1,1} \quad (4.2)$$

$$(\lambda + \mu_1)\pi_{m,0} = \mu_1\pi_{m+1,0} + \mu_2\pi_{m,1} \quad (4.3)$$

$$(\lambda + \mu_2)\pi_{0,1} = \mu_2\pi_{0,2} + \mu_1\pi_{1,1} \quad (4.4)$$

$$(\lambda + \mu_2)\pi_{0,n} = \mu_2\pi_{0,n+1} + \mu_1\pi_{0,n} \quad (4.5)$$

$$(\lambda + \mu_1 + \mu_2)\pi_{1,1} = \mu_1\pi_{2,1} + \mu_2\pi_{1,2} + \lambda\pi_{0,0} \quad (4.6)$$

$$(\lambda + \mu_1 + \mu_2)\pi_{m,n} = \mu_1\pi_{m+1,n} + \mu_2\pi_{m,n+1} + \lambda\pi_{m-1,n-1} \quad (4.7)$$

4.1.2 The Functional Equations

Multiply (4.7) by $x^{m+1}y^{n+1}$ and summed all valid states $(m, n = 1, 2, \dots)$,

$$\begin{aligned}
 & (\lambda + \mu_1 + \mu_2) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} x^{m+1} y^{n+1} \pi_{m,n} \\
 &= \mu_1 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} x^{m+1} y^{n+1} \pi_{m+1,n} + \mu_2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} x^{m+1} y^{n+1} \pi_{m,n+1} \\
 &+ \lambda \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} x^{m+1} y^{n+1} \pi_{m-1,n-1}
 \end{aligned} \tag{4.8}$$

or,

$$\begin{aligned}
 & (\lambda + \mu_1 + \mu_2) xy \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} x^m y^n \pi_{m,n} \\
 &= \mu_1 y \sum_{m=2}^{\infty} \sum_{n=1}^{\infty} x^m y^n \pi_{m,n} + \mu_2 x \sum_{m=1}^{\infty} \sum_{n=2}^{\infty} x^m y^n \pi_{m,n} \\
 &+ \lambda x^2 y^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} x^m y^n \pi_{m,n}
 \end{aligned} \tag{4.9}$$

Similarly, multiply (4.5) by xy^{n+1} and summed all valid states $(n = 1, 2, \dots)$,

$$(\lambda + \mu_2) xy \sum_{n=1}^{\infty} y^n \pi_{0,n} = \mu_1 y \sum_{n=1}^{\infty} x y^n \pi_{1,n} + \mu_2 x \sum_{n=2}^{\infty} y^n \pi_{0,n} \tag{4.10}$$

In the similar way, multiply (4.3) by $x^{m+1}y$ and summed all valid states $(m = 1, 2, \dots)$,

$$(\lambda + \mu_1) xy \sum_{m=1}^{\infty} x^m \pi_{m,0} = \mu_1 y \sum_{m=2}^{\infty} x^m \pi_{m,0} + \mu_2 x \sum_{m=1}^{\infty} x^m y \pi_{m,1} \quad (4.11)$$

Note that,

$$P(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} x^m y^n \pi_{m,n} \quad (4.12)$$

$$P(0, y) = \sum_{n=0}^{\infty} y^n \pi_{0,n} \quad (4.13)$$

$$P(x, 0) = \sum_{m=0}^{\infty} x^m \pi_{m,0} \quad (4.14)$$

$$P(0, 0) = \pi_{0,0}, \quad (4.15)$$

Expand (4.1) by substituting in the RHS's of (4.10) and (4.11), we get,

$$\begin{aligned} & (\lambda + \mu_1 + \mu_2) xy \left[\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} x^m y^n \pi_{m,n} - \sum_{m=0}^{\infty} x^m \pi_{m,0} - \sum_{n=0}^{\infty} y^n \pi_{0,n} + \pi_{0,0} \right] \\ &= \mu_1 y \left[\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} x^m y^n \pi_{m,n} - \sum_{n=0}^{\infty} y^n \pi_{0,n} - \sum_{n=0}^{\infty} x y^n \pi_{1,n} - \sum_{m=2}^{\infty} x^m \pi_{m,0} \right] \\ &+ \mu_2 x \left[\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} x^m y^n \pi_{m,n} - \sum_{m=0}^{\infty} x^m \pi_{m,0} - \sum_{m=0}^{\infty} x^m y \pi_{m,1} - \sum_{n=2}^{\infty} y^n \pi_{0,n} \right] \\ &+ \lambda x^2 y^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} x^m y^n \pi_{m,n} \end{aligned} \quad (4.16)$$

That is,

$$\begin{aligned}
 & (\lambda + \mu_1 + \mu_2) xy [P(x, y) - P(x, 0) - P(0, y) + P(0, 0)] \\
 &= \mu_1 y [P(x, y) - P(0, y)] + \mu_2 x [P(x, y) - P(x, 0)] + \lambda x^2 y^2 P(x, y) \\
 & \quad - \left[(\lambda + \mu_2) xy \sum_{n=1}^{\infty} y^n \pi_{0,n} + \mu_1 x \pi_{1,0} \right] - \left[(\lambda + \mu_1) xy \sum_{m=1}^{\infty} x^m \pi_{m,0} + \mu_1 y \pi_{0,1} \right]
 \end{aligned} \tag{4.17}$$

Put (4.10) and (4.11) in, we can get,

$$\begin{aligned}
 & (\lambda + \mu_1 + \mu_2) xy [P(x, y) - P(x, 0) - P(0, y) + P(0, 0)] \\
 &= \mu_1 y [P(x, y) - P(0, y)] + \mu_2 x [P(x, y) - P(x, 0)] + \lambda x^2 y^2 P(x, y) \\
 & \quad - [(\lambda + \mu_2) xy (P(0, y) - P(0, 0)) + \mu_1 xy \pi_{1,0}] \\
 & \quad - [(\lambda + \mu_1) xy (P(x, 0) - P(0, 0)) + \mu_1 xy \pi_{0,1}]
 \end{aligned} \tag{4.18}$$

Substitute (4.1) in,

$$\begin{aligned}
 & (\lambda + \mu_1 + \mu_2) xy [P(x, y) - P(x, 0) - P(0, y) + P(0, 0)] \\
 &= \mu_1 y [P(x, y) - P(0, y)] + \mu_2 x [P(x, y) - P(x, 0)] + \lambda x^2 y^2 P(x, y) \\
 & \quad - [(\lambda + \mu_2) xy (P(0, y) - P(0, 0))] - (\lambda + \mu_1) xy (P(x, 0) - P(0, 0)) - \lambda xy P(0, 0)
 \end{aligned} \tag{4.19}$$

After re-arrangement, we obtain,

$$\begin{aligned}
 & [(\lambda + \mu_1 + \mu_2)xy - \mu_1y - \mu_2x - \lambda x^2y^2] P(x, y) \\
 = & [(\lambda + \mu_1 + \mu_2)xy - \mu_2x - (\lambda + \mu_1)xy] P(x, 0) \\
 & + [(\lambda + \mu_1 + \mu_2)xy - \mu_1y - (\lambda + \mu_2)xy] P(0, y) \\
 & + [-(\lambda + \mu_1 + \mu_2)xy + (\lambda + \mu_2) + (\lambda + \mu_1) - \lambda xy] P(0, 0) \\
 = & \mu_2x(y - 1)P(x, 0) + \mu_1y(x - 1)P(0, y)
 \end{aligned} \tag{4.20}$$

Now we can re-express (4.20) in the form of (2.4) as follows,

$$h_1(x, y)P(x, y) = h_2(x, y)P(x, 0) + h_3(x, y)P(0, y) + h_4(x, y)P(0, 0) \tag{4.21}$$

where,

$$h_1(x, y) = (\lambda + \mu_1 + \mu_2)xy - \mu_1y - \mu_2x - \lambda x^2y^2 \tag{4.22}$$

$$h_2(x, y) = \mu_2x(y - 1) \tag{4.23}$$

$$h_3(x, y) = \mu_1y(x - 1) \tag{4.24}$$

$$h_4(x, y) = 0 \tag{4.25}$$

We can easily check that this fundamental equation is exactly the same if we use the general form of Theorem 3.1 directly without the uniformization assumption that $\lambda + \mu_1 + \mu_2 = 1$.

4.2 Stability Condition

From the transition diagram, we can easily obtain the following result,

$$\mathbf{M}_x = \lambda - \mu_1 \quad \mathbf{M}_y = \lambda - \mu_2 \quad (4.26)$$

$$\mathbf{M}'_x = \lambda - \mu_1 \quad \mathbf{M}'_y = \lambda \quad (4.27)$$

$$\mathbf{M}''_x = \lambda \quad \mathbf{M}''_y = \lambda - \mu_2 \quad (4.28)$$

Then by the theorem 3.2, we must have,

$$(i) \left\{ \begin{array}{l} (a) \lambda < \mu_1, \lambda < \mu_2 \\ (b) \lambda(\lambda - \mu_1) - (\lambda - \mu_1)(\lambda - \mu_2) < 0 \\ \Rightarrow \mu_2(\lambda - \mu_1) < 0 \\ (c) (\lambda - \mu_2)\lambda - (\lambda - \mu_1)(\lambda - \mu_2) < 0 \\ \Rightarrow \mu_1(\lambda - \mu_2) < 0 \end{array} \right. \quad (4.29)$$

$$(ii) \lambda < \mu_1, \quad \lambda \geq \mu_2$$

$$\mu_1(\lambda - \mu_2) < 0, \text{ (not possible)}$$

$$(iii) \lambda \geq \mu_1, \quad \lambda < \mu_2$$

$$\mu_1(\lambda - \mu_1) < 0, \text{ (not possible)}$$

Therefore, this system is stable if and only if $\lambda < \min\{\mu_1, \mu_2\}$.

4.3 A Uniformization Solution for $P(x, 0)$ and $P(0, y)$

In this section, to describe $P(x, y)$ explicitly, both $P(x, 0)$ and $P(0, y)$ need to be determined firstly.

4.3.1 Analysis of The Kernel

We note that $h_1(x, y)$ can be viewed as a 2-degree polynomial in y . For every value of x , there are 2 possible values of y , denoted by $y_+(x)$ and $y_-(x)$, such that $h_1(x, y_+(x)) = h_1(x, y_-(x)) = 0$, i.e. For every value of x , solving the equation,

$$\begin{aligned} 0 \equiv h_1(x, y) &= (\lambda + \mu_1 + \mu_2)xy - \mu_1y - \mu_2x - \lambda x^2 y^2 \\ &= (1 + \rho_1^{-1} + \rho_2^{-1})xy - \rho_1^{-1}y - \rho_2^{-1}x - x^2 y^2 \\ &= -x^2 y^2 + ((1 + \rho_1^{-1} + \rho_2^{-1})x - \rho_1^{-1})y - \rho_2^{-1}x \end{aligned} \quad (4.30)$$

Then we get,

$$\begin{aligned} y_+(x) &= \frac{[-(1 + \rho_1^{-1} + \rho_2^{-1})x + \rho_1^{-1}] + \sqrt{[-(1 + \rho_1^{-1} + \rho_2^{-1})x + \rho_1^{-1}]^2 - 4\rho_2^{-1}x^3}}{2(-x^2)} \\ &= -\frac{-(1 + \rho_1^{-1} + \rho_2^{-1})x + \rho_1^{-1}}{2x^2} - \frac{\sqrt{\mathbf{D}_1(x)}}{2x^2} \end{aligned} \quad (4.31)$$

where

$$\mathbf{D}_1(x) = [-(1 + \rho_1^{-1} + \rho_2^{-1})x + \rho_1^{-1}]^2 - 4\rho_2^{-1}x^3 \quad (4.32)$$

and

$$\begin{aligned}
 y_-(x) &= \frac{[-(1 + \rho_1^{-1} + \rho_2^{-1})x + \rho_1^{-1}] - \sqrt{[-(1 + \rho_1^{-1} + \rho_2^{-1})x + \rho_1^{-1}]^2 - 4\rho_2^{-1}x^3}}{2(-x^2)} \\
 &= -\frac{-(1 + \rho_1^{-1} + \rho_2^{-1})x + \rho_1^{-1}}{2x^2} + \frac{\sqrt{\mathbf{D}_1(x)}}{2x^2}
 \end{aligned} \tag{4.33}$$

More generally, we write a two valued function

$$y(x) = -\frac{-(1 + \rho_1^{-1} + \rho_2^{-1})x + \rho_1^{-1}}{2x^2} \mp \frac{\sqrt{\mathbf{D}_1(x)}}{2x^2} \tag{4.34}$$

The zero-space of $h_1(x, y)$ can be expressed as $\{(x, y) : x \in \mathbb{C}, y = y(x)\}$.

Similarly, if we treat $h_1(x, y)$ as a 2-degree polynomial in x , and for every value of y , there are 2 possible values of x , denoted by $x_+(y)$ and $x_-(y)$, such that $h_1(x_+(y), y) = h_1(x_-(y), y) = 0$, i.e. For every value of y , solving the equation (4.30), we can get,

$$\begin{aligned}
 x_+(y) &= \frac{[-(1 + \rho_1^{-1} + \rho_2^{-1})y + \rho_2^{-1}] + \sqrt{[-(1 + \rho_1^{-1} + \rho_2^{-1})y + \rho_2^{-1}]^2 - 4\rho_1^{-1}y^3}}{2(-y^2)} \\
 &= -\frac{-(1 + \rho_1^{-1} + \rho_2^{-1})y + \rho_2^{-1}}{2y^2} - \frac{\sqrt{\mathbf{D}_2(y)}}{2y^2}
 \end{aligned} \tag{4.35}$$

where

$$\mathbf{D}_2(y) = [-(1 + \rho_1^{-1} + \rho_2^{-1})y + \rho_2^{-1}]^2 - 4\rho_1^{-1}y^3 \tag{4.36}$$

and

$$\begin{aligned}
 x_-(y) &= \frac{[-(1 + \rho_1^{-1} + \rho_2^{-1})y + \rho_2^{-1}] - \sqrt{[-(1 + \rho_1^{-1} + \rho_2^{-1})y + \rho_2^{-1}]^2 - 4\rho_1^{-1}y^3}}{2(-y^2)} \\
 &= -\frac{-(1 + \rho_1^{-1} + \rho_2^{-1})y + \rho_2^{-1}}{2y^2} + \frac{\sqrt{\mathbf{D}_2(y)}}{2y^2}
 \end{aligned} \tag{4.37}$$

More generally, we write a two valued function

$$x(y) = -\frac{-(1 + \rho_1^{-1} + \rho_2^{-1})y + \rho_2^{-1}}{2y^2} \mp \frac{\sqrt{\mathbf{D}_2(y)}}{2y^2} \tag{4.38}$$

The zero-space of $h_1(x, y)$ also can be expressed as $\{(x, y) : y \in \mathbb{C}, x = x(y)\}$.

Proposition 4.1 (i) $\mathbf{D}_1(x)$ has three real and distinct zeros x_1, x_2, x_3 , with $0 < x_1 < x_2 < 1 < x_3$. $\mathbf{D}_1(x) > 0$ on $(-\infty, x_1) \cup (x_2, x_3)$ and $\mathbf{D}_1(x) < 0$ on $(x_1, x_2) \cup (x_3, \infty)$. A similar statement holds for y_1, y_2, y_3 of $\mathbf{D}_2(y)$.

(ii) $y_i = \frac{\rho_2^{-1}}{\rho_1^{-1}}x_i, 1 \leq i \leq 3$

(iii) $\rho_1^{-1} \leq x_3 \leq \rho_2^{-1} \leq y_3$, the equality signs hold if and only if $\rho_1^{-1} = \rho_2^{-1}$

PROOF. (i) It is easy to check that,

$$\mathbf{D}_1(0) = (\rho_1^{-1})^2 > 0 \tag{4.39}$$

$$\mathbf{D}_1\left(\frac{\rho_1^{-1}}{1 + \rho_1^{-1} + \rho_2^{-1}}\right) = -4\rho_2^{-1}\left(\frac{\rho_1^{-1}}{1 + \rho_1^{-1} + \rho_2^{-1}}\right)^3 < 0 \tag{4.40}$$

$$\mathbf{D}_1(1) = (\rho_2^{-1} - 1)^2 > 0 \tag{4.41}$$

$$\mathbf{D}_1(+\infty) = -\infty \tag{4.42}$$

Which imply that $\mathbf{D}_1(x)$ has roots x_1, x_2 in the intervals $(0, \frac{\rho_1^{-1}}{1+\rho_1^{-1}+\rho_2^{-1}})$, $(\frac{\rho_1^{-1}}{1+\rho_1^{-1}+\rho_2^{-1}}, 1)$, respectively, and has root x_3 in the interval $(1, +\infty)$. Besides, from the above equations, we can determine that $\mathbf{D}_1(x) > 0$ on $(-\infty, x_1) \cup (x_2, x_3)$ and $\mathbf{D}_1(x) < 0$ on $(x_1, x_2) \cup (x_3, \infty)$.

Similarly, we can obtain that,

$$\mathbf{D}_2(0) = (\rho_2^{-1})^2 > 0 \quad (4.43)$$

$$\mathbf{D}_2\left(\frac{\rho_2^{-1}}{1+\rho_1^{-1}+\rho_2^{-1}}\right) = -4\rho_1^{-1}\left(\frac{\rho_2^{-1}}{1+\rho_1^{-1}+\rho_2^{-1}}\right)^3 < 0 \quad (4.44)$$

$$\mathbf{D}_2(1) = (\rho_1^{-1} - 1)^2 > 0 \quad (4.45)$$

$$\mathbf{D}_2(+\infty) = -\infty \quad (4.46)$$

such that a similar statement holds for the roots y_1, y_2, y_3 of $\mathbf{D}_2$.

(ii) Since beginning with (4.32))

$$\begin{aligned} \mathbf{D}_1\left(\frac{\rho_1^{-1}}{\rho_2^{-1}}y\right) &= \left[-(1+\rho_1^{-1}+\rho_2^{-1})\frac{\rho_1^{-1}}{\rho_2^{-1}}y + \rho_1^{-1}\right]^2 - 4\rho_2^{-1}\left(\frac{\rho_1^{-1}}{\rho_2^{-1}}y\right)^3 \\ &= \frac{(\rho_1^{-1})^2}{(\rho_2^{-1})^2} \left[-(1+\rho_1^{-1}+\rho_2^{-1})y + \rho_2^{-1}\right]^2 - 4\frac{(\rho_1^{-1})^2}{(\rho_2^{-1})^2}\rho_1^{-1}(y)^3 \\ &= \frac{(\rho_1^{-1})^2}{(\rho_2^{-1})^2}\mathbf{D}_2(y) \end{aligned} \quad (4.47)$$

We immediately get the conclusion of (ii).

(iii) Without the loss of generality, setting $\rho_1^{-1} \leq \rho_2^{-1}$, we check that,

$$\mathbf{D}_1(\rho_1^{-1}) = (\rho_1^{-1})^2(\rho_2^{-1} - \rho_1^{-1})^2 \quad (4.48)$$

$$\mathbf{D}_1(\rho_2^{-1}) = [3(\rho_2^{-1})^2 + \rho_2^{-1} + \rho_1^{-1}(\rho_2^{-1} - 1)](\rho_2^{-1} - 1)(\rho_1^{-1} - \rho_2^{-1}) \quad (4.49)$$

$$\mathbf{D}_2(\rho_2^{-1}) = (\rho_2^{-1})^2 (\rho_2^{-1} - \rho_1^{-1})^2 \quad (4.50)$$

Therefore, in the case of $\rho_1^{-1} \leq \rho_2^{-1}$, $\mathbf{D}_1(\rho_2^{-1}) < 0 < \mathbf{D}_1(\rho_1^{-1})$ which implies that $\rho_1^{-1} < x_3 < \rho_2^{-1}$. On the other hand, $\mathbf{D}_2(\rho_2^{-1}) > 0$ and (4.46) implies that $\rho_2^{-1} < y_3$. Moreover, if $\rho_1^{-1} = \rho_2^{-1}$, we have the equations above equal to 0 such that $\rho_1^{-1} = \rho_2^{-1} = x_3 = y_3$. $\square$

4.3.2 Parametrization of The Algebraic Curve

By the Schwarz-Chistoffel transformation, see section 2.4, there exists a closed rectangle $\mathcal{R}_0$ with vertices $0, \tilde{\omega}, \tilde{\omega} + \omega, \omega$, where $\tilde{\omega}, \omega/i > 0$, and a $1-1$ mapping $t(x)$ from the extended closed upper half plane $\mathbb{H}$ to $\mathcal{R}_0$ which contains $t(x)$, continuous on $\mathbb{H}$ and analytic on interior of $\mathbb{H}$. $t(x)$ maps x_1, x_2, x_3, ∞ to the vertices $0, \tilde{\omega}, \tilde{\omega} + \omega, \omega$, respectively, see Figure 4.3. Let $x = x(t)$ be the inverse mapping. By repeated use of the reflection principle [16], $x(t)$ can be extended as a meromorphic function on the entire t -plane. Furthermore, $x(t)$ is an even and doubly periodic (i.e. elliptic) function with period $2\tilde{\omega}, 2\omega$.

Figure 4.3: Mapping Rule from Upper-Half of x-plane to t-plane

The function $y(t)$ is then defined as follows. Let $\mathbb{P}$ be the x -plane minus the slits $[x_1, x_2], [x_3, +\infty]$. For $x \in \mathbb{P}$, let $\sqrt{\mathbf{D}_1(x)}$ denote the square root that is analytic on $\mathbb{P}$ and $\sqrt{\mathbf{D}_1(0)} = \rho_1^{-1}$, $x = x(t)$ maps each of the strips $\mathbb{S}_k = \{t | k\mathbf{Im}\omega < \mathbf{Im}t < (k+1)\mathbf{Im}\omega\}$, where k is an integer onto $\mathbb{P}$.

Now define,

$$s(t) = \begin{cases} +\sqrt{\mathbf{D}_1(x(t))}, & \text{if } t \in \mathbb{S}_k, k \text{ is even} \\ -\sqrt{\mathbf{D}_1(x(t))}, & \text{if } t \in \mathbb{S}_k, k \text{ is odd} \end{cases} \quad (4.51)$$

For $\mathbf{Im}t = k\mathbf{Im}\omega$, let $s(t) = \lim_{u \rightarrow t} s(u)$, $u \in \mathbb{S}_k$. Therefore $s(t)$ is an elliptic function with periods $2\tilde{\omega}, 2\omega$.

Next, we define,

$$y(t) = \frac{(1 + \rho_1^{-1} + \rho_2^{-1})x(t) - \rho_1^{-1} + s(t)}{2x^2(t)} \quad (4.52)$$

Proposition 4.2 *The mapping $t \rightarrow \phi(t) = (x(t), y(t))$ has the following properties.*

- (i) $\phi(t)$ maps the t -plane onto $\mathbb{S}$.
- (ii) $\phi(t)$ is locally $1-1$, i.e. $\forall t$, there exists neighborhoods $\mathbb{N}_1$ about t and $\mathbb{N}_2$ about $\phi(t)$ which contains t , such that $t \rightarrow \phi(t)$ is $1-1$ from $\mathbb{N}_1$ onto $\mathbb{N}_2$.
- (iii) $\phi(t_1) = \phi(t_2)$ if and only if $t_2 = t_1 + 2m\tilde{\omega} + 2n\omega$, where m, n are arbitrary integers. In other words, $\mathbb{S}$ can be identified with the quotient group $\mathbb{C}/\mathbb{D}$,

where $\mathbb{C}$ is the additive group of complex numbers and $\mathbb{D}$ is the discrete subgroup $\{2m\tilde{\omega} + 2n\omega\}$.

From the proposition above, we can identify that $\mathbb{S}$ is a compact Riemann surface of genus 1.

The following facts about elliptic functions can be found in [16].

Theorem 4.1 *Let $\omega \neq 0$, ω_1/ω_2 not real. Let $\mathbb{P}$ be the fundamental parallelogram where $\{t = r_1\omega_1 + r_2\omega_2 | 0 < r_1, r_2 \leq 1\}$ and $a_1, \dots, a_k, b_1, \dots, b_l \in \mathbb{P}$. There exists an elliptic function $f(t)$ with periods ω_1, ω_2 whose zeros and poles in $\mathbb{P}$ are given respectively by $\{a_1, \dots, a_k\}, \{b_1, \dots, b_l\}$ if and only if $k = l$ and $\sum a_i - \sum b_i = m_1\omega_1 + m_2\omega_2$, where m_1 and m_2 are integers, in which case $f(t)$ is determined up to a constant multiple. $f(t)$ assumes every value k times in $\mathbb{P}$, where k is called the order of $f(t)$.*

For any two complex numbers a, b , the symbols $[a, b]$, (a, b) , $[a, b)$, $(a, b]$ denote the closed, open, and half open line segments directed from a to b . $[a, b)$ containing a and $(a, b]$ containing b . Let $\mathbb{R}$ be the fundamental rectangle $\{t | -\tilde{\omega} < \operatorname{Re} t \leq \tilde{\omega}, -\operatorname{Im}\omega < \operatorname{Im} t \leq \operatorname{Im}\omega\}$. $x(t)$ maps $[\omega, 0]$ one-to-one onto $[-\infty, x_1]$, and $[\tilde{\omega}, \tilde{\omega} + \omega]$ one-to-one onto $[x_2, x_3]$. As $0 < x_1 < x_2 < 1 < \rho_1^{-1} \leq x_3$, there exists points $m \in (0, \omega), p, q \in (\tilde{\omega}, \tilde{\omega} + \omega]$ such that $x(m) = 0$, $x(p) = 1$, $x(q) = \rho_1^{-1}$. Moreover, we denote $p < q$ since p precedes q on $(\tilde{\omega}, \tilde{\omega} + \omega]$.

4.3.3 The Relationship between $x(t)$ and $y(t)$

Till now, we have done the most part of preparations. In the next stage, we will show an important theorem in this thesis leading to the solution of the problem.

Theorem 4.2 *Let $\mathbb{R}$ be the fundamental rectangle $\{t \mid -\tilde{\omega} < \operatorname{Re} t \leq \tilde{\omega}, -\operatorname{Im} \omega < \operatorname{Im} t \leq \operatorname{Im} \omega\}$, and m, p, q as defined above.*

(i) *The zeros and the poles of $x(t)$ in $\mathbb{R}$ are respectively $\{m, -m\}, \{\omega, \omega\}$ (i.e., ω is a double pole of $x(t)$).*

(ii) *The zeros and the poles of $y(t)$ in $\mathbb{R}$ are $\{m, \omega\}, \{-m, -m\}$ (i.e., $-m$ is a double pole of $y(t)$).*

(iii) $m = \frac{\omega}{3}$.

(iv) $y(t) = \frac{\rho_2^{-1}}{\rho_1^{-1}} x(t - 2m)$.

(v) $\frac{p+q}{2} = \tilde{\omega} + 2m$, then,

$$\tilde{\omega} + m \leq p < \tilde{\omega} + 2m < q \leq \tilde{\omega} + \omega \quad (4.53)$$

the equality signs hold if and only if $\rho_1^{-1} = \rho_2^{-1}$.

(vi) *At the points $p - 2km, q - 2km, k = 0, 1, 2$, $x(t)$ and $y(t)$ assume the values listed in Table 4.1.*

t	$p - 4m$	$q - 4m$	$p - 2m$	$q - 2m$	p	q
$x(t)$	ρ_1^{-1}	1	$\frac{\rho_1^{-1}}{\rho_2^{-1}}$	$\frac{\rho_1^{-1}}{\rho_2^{-1}}$	1	ρ_1^{-1}
$y(t)$	$\frac{\rho_2^{-1}}{\rho_1^{-1}}$	ρ_2^{-1}	ρ_2^{-1}	$\frac{\rho_2^{-1}}{\rho_1^{-1}}$	1	1

Table 4.1: Values of $x(t)$ and $y(t)$

PROOF. (i) According to the mapping rule, $t(x)$ maps $x_1, x_2, x_3, +\infty$ to the vertices $0, \tilde{\omega}, \tilde{\omega} + \omega, \omega$, respectively, and $x(t)$ is the inverse mapping of $t(x)$. We can easily get $x(\omega) = +\infty$. On the other hand, by the definition of m , we have $x(m) = 0$. Then, by the reflection principle that we used to extend the mapping to the whole t -plane,

$$x(-m) = \overline{x(\overline{-m})} = \overline{x(m)} = \overline{0} = 0 \quad (4.54)$$

Similarly, by the reflection principle, ω is also the pole of $x(t)$ in the negative half t -plane. Therefore, $\{m, -m\}$ are the zero points of $x(t)$ in $\mathbb{R}$ and ω is the double pole of $x(t)$ in $\mathbb{R}$.

(ii) To find the zeros and the poles of $y(t)$, first we want to simplify the expression (4.52), using Taylor Expansion.

Start with,

$$\sqrt{\mathbf{D}_1(x)} = \left\{ \left[-(1 + \rho_1^{-1} + \rho_2^{-1})x + \rho_1^{-1} \right]^2 - 4\rho_2^{-1}x^3 \right\}^{\frac{1}{2}} \quad (4.55)$$

Taking the 1st derivative,

$$\begin{aligned} \left(\sqrt{\mathbf{D}_1(x)} \right)' &= \frac{1}{2} \frac{1}{\sqrt{\mathbf{D}_1(x)}} \left\{ 2[(1 + \rho_1^{-1} + \rho_2^{-1})x - \rho_1^{-1}](1 + \rho_1^{-1} + \rho_2^{-1}) - 12\rho_2^{-1}x^2 \right\} \\ &= \frac{1}{\sqrt{\mathbf{D}_1(x)}} \left\{ [(1 + \rho_1^{-1} + \rho_2^{-1})x - \rho_1^{-1}](1 + \rho_1^{-1} + \rho_2^{-1}) - 6\rho_2^{-1}x^2 \right\} \end{aligned} \quad (4.56)$$

Then the 2nd derivative,

$$\begin{aligned}
 \left(\sqrt{\mathbf{D}_1(x)}\right)'' &= \frac{1}{\left(\sqrt{\mathbf{D}_1(x)}\right)^2} \left\{ [(1 + \rho_1^{-1} + \rho_2^{-1})^2 - 12\rho_2^{-1}x] \sqrt{\mathbf{D}_1(x)} \right\} \\
 &\quad - \frac{\left(\sqrt{\mathbf{D}_1(x)}\right)'}{\left(\sqrt{\mathbf{D}_1(x)}\right)^2} \left\{ [(1 + \rho_1^{-1} + \rho_2^{-1})x - \rho_1^{-1}](1 + \rho_1^{-1} + \rho_2^{-1}) - 6\rho_2^{-1}x^2 \right\} \\
 &= \frac{1}{\sqrt{\mathbf{D}_1(x)}} [(1 + \rho_1^{-1} + \rho_2^{-1})^2 - 12\rho_2^{-1}x] \\
 &\quad - \frac{\left\{ [(1 + \rho_1^{-1} + \rho_2^{-1})x - \rho_1^{-1}](1 + \rho_1^{-1} + \rho_2^{-1}) - 6\rho_2^{-1}x^2 \right\}^2}{\left(\sqrt{\mathbf{D}_1(x)}\right)^3}
 \end{aligned} \tag{4.57}$$

And the 3rd derivative,

$$\begin{aligned}
 &\left(\sqrt{\mathbf{D}_1(x)}\right)^{(3)} \\
 &= \frac{-12\rho_2^{-1}\sqrt{\mathbf{D}_1(x)} - [(1 + \rho_1^{-1} + \rho_2^{-1})^2 - 12\rho_2^{-1}x] \left(\sqrt{\mathbf{D}_1(x)}\right)'}{\left(\sqrt{\mathbf{D}_1(x)}\right)^2} \\
 &\quad - \frac{2 \left\{ [(1 + \rho_1^{-1} + \rho_2^{-1})x - \rho_1^{-1}](1 + \rho_1^{-1} + \rho_2^{-1}) - 6\rho_2^{-1}x^2 \right\}}{\left(\sqrt{\mathbf{D}_1(x)}\right)^3} \\
 &\quad \times [(1 + \rho_1^{-1} + \rho_2^{-1})^2 - 12\rho_2^{-1}x] \\
 &\quad + \frac{3 \left(\sqrt{\mathbf{D}_1(x)}\right)'}{\left(\sqrt{\mathbf{D}_1(x)}\right)^4} \left\{ [(1 + \rho_1^{-1} + \rho_2^{-1})x - \rho_1^{-1}](1 + \rho_1^{-1} + \rho_2^{-1}) - 6\rho_2^{-1}x^2 \right\}^2
 \end{aligned}$$

After rearrangement and simplification, we get,

$$\begin{aligned}
 & \left(\sqrt{\mathbf{D}_1(x)} \right)^{(3)} \\
 = & - \frac{12\rho_2^{-1}}{\sqrt{\mathbf{D}_1(x)}} \\
 & - \frac{[(1 + \rho_1^{-1} + \rho_2^{-1})^2 - 12\rho_2^{-1}x] \{[(1 + \rho_1^{-1} + \rho_2^{-1})x - \rho_1^{-1}](1 + \rho_1^{-1} + \rho_2^{-1}) - 6\rho_2^{-1}x^2\}}{\left(\sqrt{\mathbf{D}_1(x)} \right)^3} \\
 & - \frac{2 \{[(1 + \rho_1^{-1} + \rho_2^{-1})x - \rho_1^{-1}](1 + \rho_1^{-1} + \rho_2^{-1}) - 6\rho_2^{-1}x^2\} [(1 + \rho_1^{-1} + \rho_2^{-1})^2 - 12\rho_2^{-1}x]}{\left(\sqrt{\mathbf{D}_1(x)} \right)^3} \\
 & + \frac{3}{\left(\sqrt{\mathbf{D}_1(x)} \right)^5} \{[(1 + \rho_1^{-1} + \rho_2^{-1})x - \rho_1^{-1}](1 + \rho_1^{-1} + \rho_2^{-1}) - 6\rho_2^{-1}x^2\}^3
 \end{aligned} \tag{4.58}$$

and so on.

Then for x near 0, by (4.55), (4.56), (4.57), (4.58), we obtain that,

$$\sqrt{\mathbf{D}_1(0)} = \rho_1^{-1} \tag{4.59}$$

$$\begin{aligned}
 \left(\sqrt{\mathbf{D}_1(0)} \right)' &= \frac{1}{\rho_1^{-1}} [-\rho_1^{-1}(1 + \rho_1^{-1} + \rho_2^{-1})] \\
 &= -(1 + \rho_1^{-1} + \rho_2^{-1})
 \end{aligned} \tag{4.60}$$

$$\begin{aligned}
 \left(\sqrt{\mathbf{D}_1(0)}\right)'' &= \frac{(1 + \rho_1^{-1} + \rho_2^{-1})^2}{\rho_1^{-1}} - \frac{1}{(\rho_1^{-1})^3} [-\rho_1^{-1}(1 + \rho_1^{-1} + \rho_2^{-1})]^2 \\
 &= 0
 \end{aligned} \tag{4.61}$$

$$\begin{aligned}
 \left(\sqrt{\mathbf{D}_1(0)}\right)^{(3)} &= -\frac{12\rho_2^{-1}}{\rho_1^{-1}} - \frac{(1 + \rho_1^{-1} + \rho_2^{-1})^2 [-\rho_1^{-1}(1 + \rho_1^{-1} + \rho_2^{-1})]}{(\rho_1^{-1})^3} \\
 &\quad - \frac{-2\rho_1^{-1}(1 + \rho_1^{-1} + \rho_2^{-1})^3}{(\rho_1^{-1})^3} + \frac{3[-\rho_1^{-1}(1 + \rho_1^{-1} + \rho_2^{-1})^3]}{(\rho_1^{-1})^5} \\
 &= -12\frac{\rho_2^{-1}}{\rho_1^{-1}}
 \end{aligned} \tag{4.62}$$

Therefore, the Taylor Expansion of $\sqrt{\mathbf{D}_1(0)}$ near $x = 0$ is,

$$\begin{aligned}
 \sqrt{\mathbf{D}_1(x)} &= \frac{\sqrt{\mathbf{D}_1(0)}}{0!} + \frac{\left(\sqrt{\mathbf{D}_1(0)}\right)'}{1!}x + \frac{\left(\sqrt{\mathbf{D}_1(0)}\right)''}{2!}x^2 + \frac{\left(\sqrt{\mathbf{D}_1(0)}\right)^{(3)}}{3!}x^3 + \dots \\
 &= \rho_1^{-1} - (1 + \rho_1^{-1} + \rho_2^{-1})x - 2\frac{\rho_2^{-1}}{\rho_1^{-1}}x^3 + o(x^3)
 \end{aligned} \tag{4.63}$$

Now the $y_+(x)$ and $y_-(x)$ becomes,

$$y_+(x) = \frac{[-(1 + \rho_1^{-1} + \rho_2^{-1})x + \rho_1^{-1}] + \sqrt{\mathbf{D}_1(x)}}{2(-x^2)} = -\frac{\rho_2^{-1}}{\rho_1^{-1}}x + o(x) \tag{4.64}$$

$$\begin{aligned}
 y_-(x) &= \frac{[-(1 + \rho_1^{-1} + \rho_2^{-1})x + \rho_1^{-1}] - \sqrt{\mathbf{D}_1(x)}}{2(-x^2)} \\
 &= -\frac{\rho_1^{-1}}{x^2} + \frac{1 + \rho_2^{-1} + \rho_2^{-1}}{x} + \frac{\rho_2^{-1}}{\rho_1^{-1}}x + o(x)
 \end{aligned} \tag{4.65}$$

Therefore, according to the definition of $s(t)$, $y(t) = y_1(x(t))$ for t near m and $y(t) = y_2(x(t))$ for t near $-m$. Then from (4.64) and (4.65), $y(t)$ has a simple zero point at $t = m$ and a double pole at $t = -m$. Besides, since $x(\omega) = +\infty$ and $s(t) = \lim_{u \rightarrow t} s(u)$, we have $y(t) = 0$ at $t = \omega$.

Thus, as a conclusion, the zeros and the poles of $y(t)$ in $\mathbb{R}$ are $\{m, \omega\}$ and $\{-m, -m\}$, respectively.

(iii) In (ii), we have found the zeros and the poles of $y(t)$ in $\mathbb{R}$. Now applying Theorem 4.1 to $y(t)$, we have

$$\begin{aligned}
 a_1 &= m, \quad a_2 = \omega, \quad b_1 = b_2 = -m \\
 \Rightarrow \quad m + \omega - (-m) - (-m) &= m_1 \cdot 2\tilde{\omega} + m_2 \cdot 2\omega
 \end{aligned} \tag{4.66}$$

Solve m in $(0, \omega)$, we get

$$m_1 = 0, \quad m_2 = 1, \quad m = \frac{\omega}{3} \tag{4.67}$$

(iv) From (ii) and (iii), we obtain that the zeros and the poles of $y(t)$ in $\mathbb{R}$ are $\{\frac{\omega}{3}, \omega\}$ and $\{-\frac{\omega}{3}, -\frac{\omega}{3}\}$, respectively. On the other hand, from (i) and (iii), $\{\frac{\omega}{3}, -\frac{\omega}{3}\}$

are the zero points of $x(t)$ in $\mathbb{R}$ and ω is the double pole of $x(t)$ in $\mathbb{R}$. Through both results we have acquired, we can easily deduce that $x(t-2m) = x(t-\frac{2\omega}{3})$ and $y(t)$ have identical sets of zeros and poles. Therefore, we must have $y(t) = c \cdot x(t-2m)$, where c is a constant. Next, we determine the value of c by the following argument: as $t = m$ is a simple zero point of $x(t)$, let $t = m + \epsilon$ where $\epsilon \rightarrow 0$, we notice that,

$$\begin{aligned} y(m + \epsilon) &= c \cdot x(m + \epsilon - 2m) = c \cdot x(-m + \epsilon) \\ &= c \cdot x(m - \epsilon) \quad (\text{by reflection principle}) \\ &\sim c \cdot x(m + \epsilon) \quad (t = m \text{ is a simple zero point}) \end{aligned} \tag{4.68}$$

Since for t near m , $y(t) = y_1(x(t))$. Compare (4.68) with (4.64) which gives,

$$\begin{aligned} y(m + \epsilon) &= -\frac{\rho_2^{-1}}{\rho_1^{-1}} x(m + \epsilon) + o(x(m + \epsilon)) \\ &\sim -\frac{\rho_2^{-1}}{\rho_1^{-1}} x(m + \epsilon) \quad (\text{as } \epsilon \rightarrow 0) \end{aligned} \tag{4.69}$$

We get $c = \frac{\rho_2^{-1}}{\rho_1^{-1}}$.

(v) Since $x(p) = 1$ and $x(q) = \rho_1^{-1}$, and $p, q \in (\tilde{\omega}, \tilde{\omega} + \omega]$, from (4.52), we get,

$$\begin{aligned} y(p) &= \frac{[(1 + \rho_1^{-1} + \rho_2^{-1})x(p) - \rho_1^{-1}] + \sqrt{[-(1 + \rho_1^{-1} + \rho_2^{-1})x(p) + \rho_1^{-1}]^2 - 4\rho_2^{-1}x^3(p)}}{2(x^2(p))} \\ &= \frac{1}{2}\{(1 + \rho_2^{-1}) + (1 - \rho_2^{-1})\} \\ &= 1 \end{aligned}$$

$$\begin{aligned}
 y(q) &= \frac{[(1 + \rho_1^{-1} + \rho_2^{-1})x(q) - \rho_1^{-1}] + \sqrt{[-(1 + \rho_1^{-1} + \rho_2^{-1})x(q) + \rho_1^{-1}]^2 - 4\rho_2^{-1}x^3(q)}}{2(x^2(q))} \\
 &= \frac{1}{2(\rho_1^{-1})^2} \left\{ (\rho_1^{-1} + \rho_2^{-1})\rho_1^{-1} + \sqrt{(\rho_1^{-1} + \rho_2^{-1})^2 (\rho_1^{-1})^2 - 4\rho_2^{-1}(\rho_1^{-1})^3} \right\} \\
 &= 1
 \end{aligned}$$

Applying Theorem 4.1 to $y(t) - 1$, we get,

$$\begin{aligned}
 a_1 &= p, \quad a_2 = q, \quad b_1 = b_2 = -m \\
 \Rightarrow \quad p + q - (-m) - (-m) &= m_1 \cdot 2\tilde{\omega} + m_2 \cdot 2\omega
 \end{aligned} \tag{4.70}$$

Solve p, q in $(\tilde{\omega}, \tilde{\omega} + \omega]$, we get

$$m_1 = 1, \quad m_2 = 1, \quad \frac{p+q}{2} = \tilde{\omega} + \frac{2\omega}{3} = \tilde{\omega} + 2m \tag{4.71}$$

From the mapping rule of $x(t)$, we know that $q = \tilde{\omega} + \omega$ if and only if $x(q) = \rho_1^{-1} = x_3$, which will happen if and only $\rho_1^{-1} = \rho_2^{-1}$, by Proposition 4.1.

(vi) By the reflection principle, $x(t) = \overline{x(\bar{t})}$, so we get $x(\bar{p}) = 1$, $x(\bar{q}) = \rho_1^{-1}$, and $\bar{p}, \bar{q} \in (\tilde{\omega}, \tilde{\omega} - \omega]$. Then,

$$\begin{aligned}
 p + \bar{p} &= 2\tilde{\omega} = 2 \left(\frac{p+q}{2} - 2m \right) \quad (\text{by(v)}) \\
 \Rightarrow \quad \bar{p} &= q - 4m \\
 q + \bar{q} &= 2\tilde{\omega} = 2 \left(\frac{p+q}{2} - 2m \right) \quad (\text{by(v)}) \\
 \Rightarrow \quad \bar{q} &= p - 4m
 \end{aligned} \tag{4.72}$$

Substitute $x(\bar{p}) = 1$, $x(\bar{q}) = \rho_1^{-1}$ into $y(t)$, and notice that $p, q \in (\tilde{\omega}, \tilde{\omega} - \omega]$, from (4.52), we get,

$$\begin{aligned}
 y(\bar{p}) &= \frac{[(1 + \rho_1^{-1} + \rho_2^{-1})x(\bar{p}) - \rho_1^{-1}] - \sqrt{[-(1 + \rho_1^{-1} + \rho_2^{-1})x(\bar{p}) + \rho_1^{-1}]^2 - 4\rho_2^{-1}x^3(\bar{p})}}{2(x^2(\bar{p}))} \\
 &= \frac{1}{2}\{(1 + \rho_2^{-1}) - (1 - \rho_2^{-1})\} \\
 &= \rho_2^{-1} \\
 y(\bar{q}) &= \frac{[(1 + \rho_1^{-1} + \rho_2^{-1})x(\bar{q}) - \rho_1^{-1}] - \sqrt{[-(1 + \rho_1^{-1} + \rho_2^{-1})x(\bar{q}) + \rho_1^{-1}]^2 - 4\rho_2^{-1}x^3(\bar{q})}}{2(x^2(\bar{q}))} \\
 &= \frac{1}{2(\rho_1^{-1})^2} \left\{ (\rho_1^{-1} + \rho_2^{-1})\rho_1^{-1} - \sqrt{(\rho_1^{-1} + \rho_2^{-1})^2 (\rho_1^{-1})^2 - 4\rho_2^{-1} (\rho_1^{-1})^3} \right\} \\
 &= \frac{\rho_2^{-1}}{\rho_1^{-1}}
 \end{aligned} \tag{4.73}$$

Besides, from (vi),

$$x(p - 2m) = \frac{\rho_1^{-1}}{\rho_2^{-1}}y(p) = \frac{\rho_1^{-1}}{\rho_2^{-1}} \tag{4.74}$$

$$x(q - 2m) = \frac{\rho_1^{-1}}{\rho_2^{-1}}y(q) = \frac{\rho_1^{-1}}{\rho_2^{-1}} \tag{4.75}$$

$$y(p - 2m) = \frac{\rho_2^{-1}}{\rho_1^{-1}}x(p - 4m) = \frac{\rho_2^{-1}}{\rho_1^{-1}}x(\bar{p}) = \frac{\rho_2^{-1}}{\rho_1^{-1}} \tag{4.76}$$

$$y(q - 2m) = \frac{\rho_2^{-1}}{\rho_1^{-1}}x(q - 4m) = \frac{\rho_2^{-1}}{\rho_1^{-1}}x(\bar{q}) = \rho_2^{-1} \tag{4.77}$$

which finish the proof. $\square$

4.3.4 Analytic Continuation of $P(x(t), 0), P(0, y(t))$

Let $\mathbf{D}_x = \{t | |x(t)| < 1\}$, $\mathbf{D}_y = \{t | |y(t)| < 1\}$. The closures of these sets are $\overline{\mathbf{D}}_x = \{t | |x(t)| \leq 1\}$, $\overline{\mathbf{D}}_y = \{t | |y(t)| \leq 1\}$. Then the functions $P(x(t), 0)$ and $P(0, y(t))$ are analytic on $\mathbf{D}_x$ and $\mathbf{D}_y$ and continuous on $\overline{\mathbf{D}}_x$ and $\overline{\mathbf{D}}_y$, respectively. Let D_x, D_y be the components of $\mathbf{D}_x, \mathbf{D}_y$ that containing 0 and $2m$, respectively. The mapping rule of $x(t)$ and $y(t)$ implies that: D_x and D_y are invariant under $t \rightarrow t + 2\omega$; D_x is symmetric with respect to the real and imaginary axis, D_y is symmetric with respect to the horizontal line through $2m$ and imaginary axis.

Let $A(t) = P(x(t), 0), t \in \overline{\mathbf{D}}_x$, $B(t) = P(0, y(t)), t \in \overline{\mathbf{D}}_y$. Then $A(t), B(t)$ are analytic on $\mathbf{D}_x, \mathbf{D}_y$ and continuous on $\overline{\mathbf{D}}_x, \overline{\mathbf{D}}_y$ respectively.

Theorem 4.3 *$A(t)$ and $B(t)$ can be continued as meromorphic functions to the entire t -plane. The extended functions satisfy,*

$$A(t) = A(-t) = A(t + 2\tilde{\omega}) \quad (4.78)$$

$$A(t - 4m) = g(t)A(t) \quad (4.79)$$

$$B(t) = u(t)A(t) \quad (4.80)$$

where,

$$f(x) = \frac{x-1}{x}, \quad g(t) = \frac{f(x(t-4m))}{f(x(t))}, \quad u(t) = -\frac{\rho_2^{-1} f(y(t))}{\rho_1^{-1} f(x(t))} \quad (4.81)$$

PROOF. By the mapping rule and the reflection principle, $\overline{\mathbf{D}}_{\mathbf{x}}$ is invariant under $t \rightarrow -t$ and $t \rightarrow t + 2\tilde{\omega}$, so we have,

$$x(t) = x(-t) = x(t + 2\tilde{\omega}) \quad (4.82)$$

Then the following equation follows,

$$A(t) = A(-t) = A(t + 2\tilde{\omega}), \quad \forall t \in \overline{\mathbf{D}}_{\mathbf{x}} \quad (4.83)$$

Similarly, by 4.2(iv) and (4.82), we know that,

$$\begin{aligned} y(4m - t) &= \frac{\rho_2^{-1}}{\rho_1^{-1}} x(2m - t) \\ &= \frac{\rho_2^{-1}}{\rho_1^{-1}} x(t - 2m) \\ &= y(t) \end{aligned} \quad (4.84)$$

which implies that,

$$y(t) = y(4m - t) = y(t + 2\tilde{\omega}) \quad (\text{by reflection principle}) \quad (4.85)$$

Then the following equation follows,

$$B(t) = B(4m - t) = B(t + 2\tilde{\omega}), \quad \forall t \in \overline{\mathbf{D}}_{\mathbf{y}} \quad (4.86)$$

On the other hand, from the fundamental equation (4.21), we know that if we set $h_1(x, y) = 0$, we will get,

$$\begin{aligned}
 0 &= h_2(x, y)P(x, 0) + h_3(x, y)P(0, y) + h_4(x, y)P(0, 0) \\
 P(0, y) &= -\frac{h_2(x, y)}{h_3(x, y)}P(x, 0) \quad (h_4(x, y) = 0) \\
 &= -\frac{\mu_2 x (y - 1)}{\mu_1 y (x - 1)}P(x, 0) \\
 &= -\frac{\rho_2^{-1} y - 1}{\rho_1^{-1} y} \frac{x}{x - 1} P(x, 0) \\
 &= -\frac{\rho_2^{-1} f(y)}{\rho_1^{-1} f(x)} P(x, 0)
 \end{aligned} \tag{4.87}$$

The if we treat y, x as functions of t , the equation above becomes,

$$\begin{aligned}
 B(t) &= -\frac{\rho_2^{-1} f(y(t))}{\rho_1^{-1} f(x(t))} A(t) \\
 &= u(t)A(t) \quad t \in \overline{D}_x \cap \overline{D}_y
 \end{aligned} \tag{4.88}$$

Now according to the relationship of $A(t)$ and $B(t)$ above, we can extend $A(t)$ to $\overline{D}_{x \cup y} = \overline{D}_x \cup \overline{D}_y$ by defining $A(t) = \frac{B(t)}{u(t)}$, $\forall t \in \overline{D}_y$. Then if we shift $\overline{D}_{x \cup y}$ by $4m$ denoted by $\overline{D}_{x \cup y}^{4m}$, and let $\overline{D}^*$ be the image of $\overline{D}_x \cap \overline{D}_y$ from $t \rightarrow 4m - t$, we can observe that $\overline{D}^* = \overline{D}_{x \cup y} \cap \overline{D}_{x \cup y}^{4m}$. That is,

$\forall t \in \overline{D}_y$ and $4m - t \in \overline{D}_x \cap \overline{D}_y$, the following equation holds,

$$\begin{aligned}
 u(t)A(t) &= B(t) = B(4m - t) = u(4m - t)A(4m - t) = u(4m - t)A(t - 4m) \\
 \Rightarrow A(t - 4m) &= \frac{u(t)}{u(4m - t)} A(t) = g(t)A(t)
 \end{aligned} \tag{4.89}$$

Now according to the equation above, we can extend $A(t)$ to $\overline{\mathbf{D}}_{\mathbf{x} \cup \mathbf{y}}^{4\mathbf{m}}$ by defining $A(t) = \frac{A(t-4m)}{g(t)}$. Continuing using this expansion technique repeatedly, we can finally extend $A(t)$ to the whole t -plane. Then we can easily extend $B(t)$ by defining $B(t) = u(t)A(t)$. Eventually, we will have the theorem valid for all t by analytic continuation. $\square$

4.3.5 Formulae for $P(\mathbf{x}, 0)$, $P(0, \mathbf{y})$ and $P(\mathbf{x}, \mathbf{y})$

By the following theorem, we determine the formulae for $P(x, 0)$, $P(0, y)$, and thus by (4.20), we can obtain the formula for $P(x, y)$.

Theorem 4.4

$$P(x, 0) = \frac{\rho_2^{-1} - 1}{\rho_2^{-1}} \frac{\phi(x)}{\phi(1)}, \quad |x| \leq 1 \quad (4.90)$$

$$P(0, y) = \frac{\rho_1^{-1} - 1}{\rho_1^{-1}} \frac{\psi(y)}{\psi(1)}, \quad |y| \leq 1 \quad (4.91)$$

where,

$$\phi(x) = \frac{\sqrt{x_3 - x} + \sqrt{x_3 - 1}}{\left(\sqrt{x_3 - x} + \sqrt{x_3 - \frac{\rho_1^{-1}}{\rho_2^{-1}}}\right) \left(\sqrt{x_3 - x} + \sqrt{x_3 - \rho_1^{-1}}\right)} \quad (4.92)$$

$$\psi(y) = \frac{\sqrt{y_3 - y} + \sqrt{y_3 - 1}}{\left(\sqrt{y_3 - y} + \sqrt{y_3 - \frac{\rho_2^{-1}}{\rho_1^{-1}}}\right) \left(\sqrt{y_3 - y} + \sqrt{y_3 - \rho_2^{-1}}\right)} \quad (4.93)$$

Particularly, if $\rho_1^{-1} = \rho_2^{-1}$, we have

$$P(x, 0) = P(0, x) = \frac{(\rho_1^{-1} - 1)^{\frac{3}{2}}}{\rho_1^{-1}(\rho_1^{-1} - x)^{\frac{1}{2}}} \quad (4.94)$$

Remark 4.1 *The analytic continuations of $P(x, 0)$, $P(0, y)$ are the x -plane minus the slits $[x_3, \infty)$ and the y -plane minus the slits $[y_3, \infty)$. If $\rho_1^{-1} < \rho_2^{-1}$, $P(x, 0)$ is analytic in the slit x -plane except for a simple pole at ρ_1^{-1} , while $P(0, y)$ is analytic in the slit y -plane. If $\rho_1^{-1} = \rho_2^{-1}$, both $P(x, 0)$, $P(0, y)$ are analytic in the slit x -plane and y -plane.*

PROOF. Let $A_1(t) = \frac{A(t)}{f(x(t))}$, where $f(x) = \frac{x-1}{x}$, as defined in Theorem 4.3. Then from the property of $A(t)$ shown in Theorem 4.3, we have,

$$A_1(-t) = \frac{A(-t)}{f(x(-t))} = \frac{A(t)}{f(x(t))} = A_1(t) \quad (\text{reflection principle}) \quad (4.95)$$

$$A_1(t + 2\tilde{\omega}) = \frac{A(t + 2\tilde{\omega})}{f(x(t + 2\tilde{\omega}))} = A_1(t) \quad (4.96)$$

$$A_1(t + 4m) = \frac{A(t + 4m)}{f(x(t + 4m))} = \frac{A(t)}{u(t + 4m)f(x(t + 4m))} = A_1(t) \quad (4.97)$$

The equations above indicate that $A_1(t)$ is an even elliptic function with periods $2\tilde{\omega}, 4m = \frac{4\omega}{3}$. As stated before (see sec. 4.3.2), $x(t)$ is an even and doubly periodic function with periods $2\tilde{\omega}, 2\omega$, we can conclude that $A(t)$ is an even elliptic function with periods $2\tilde{\omega}, 4\omega$. Now we want to determine the zeros and poles of $A_1(t)$ in its fundamental rectangle $\mathbb{R}_{A_1} = \{t \mid -\tilde{\omega} < \text{Ret} \leq \tilde{\omega}, -2\text{Imm} < \text{Imt} \leq 2\text{Imm}\}$.

First, we show that $A(t)$ is analytic in the strip $\mathbb{T}_1 = \{t \mid |\text{Imt}| \leq 2\text{Imm}\}$. We have already shown that $A(t)$ is analytic in the interior of $\overline{\mathbf{D}}$. Noting that $\{t \mid 0 \leq \text{Imt} \leq 2\text{Imm}\} \subset \overline{\mathbf{D}}$ and $A(-t) = A(t)$, we can conclude that $A(t)$ is analytic in $\mathbb{T}_1$.

On the other hand, by Proposition 4.1, the zeros of $g(t)$ in $\mathbb{R}$ are computed

to be $q, -m$, both of which are outside $\mathbf{D}_y$. Again by this Proposition, the zeros and poles of $f(x(t))$ in $\mathbf{R}$ are $q - 4m, p$ and $\pm m$, respectively. Since $A(q - 4m) = A(p) = P(1, 0) = \frac{\rho_2^{-1} - 1}{\rho_2^{-1}} > 0$, we can conclude that $q - 4m, p$ are the poles of $A_1(t)$ in $\mathbf{R}_{A_1}$. Similarly, we can determine that $\pm$ are the zeros of $A_1(t)$ in $\mathbf{R}_{A_1}$.

Comparing with the zeros and poles of $x(t)$ and $A_1(t)$, as well as the points that $x(t) = 1$, in the fundamental rectangle of $A(t)$, $\mathbf{R}_A = \{t \mid -\tilde{\omega} < \mathbf{Re} t \leq \tilde{\omega}, -2\mathbf{Im}\omega < \mathbf{Im} t \leq 2\mathbf{Im}\omega\}$,

Functions	Periods	Zeros	Poles
$x(t)$	$2\tilde{\omega}, 2\omega$	$\{-5m, -m, m, 5m\}$	$\{-\omega, \omega\}$
$A_1(t)$	$2\tilde{\omega}, 4m$	$\left\{ \begin{array}{c} -5m, -3m, -m, \\ m, 3m, 5m \end{array} \right\}$	$\left\{ \begin{array}{c} p - 4m, p, p + 4m, \\ q - 4m, q, q + 4m \end{array} \right\}$
$x(t) = 1$	$\{q - 4m, p, p + 6m, q + 2m\}$		

We obtain the complete sets of zeros and poles for $A(t)$, when $\rho_1^{-1} < \rho_2^{-1}$,

Functions	Periods	Zeros	Poles
$A(t)$	$2\tilde{\omega}, 4\omega$	$\{-\omega, \omega, q + 2m, p + 6m\}$	$\{p - 4m, q, p + 4m, q + 4m\}$

When $\rho_1^{-1} = \rho_2^{-1}$, since $q + 2m = p + 4m, p + 6m = q + 4m$, the complete sets of zeros and poles for $A(t)$ become $\{\pm\omega\}$ and $\{p - 4m, q\}$, respectively.

Let $\lambda(t)$ be an elliptic function with periods $2\tilde{\omega}, 4\omega$ and with the complete sets of zeros and poles $\{\tilde{\omega} \pm \omega\}$ and $\{\pm\omega\}$. By Theorem 4.1, as in this case $\sum a_i - \sum b_i = 2\tilde{\omega}$, we can conclude that such $\lambda(t)$ exists. From the mapping rule and Theorem

4.2(i), we know that $\lambda^2(t)$ and $x_3 - x(t)$ have identical zeros and poles. We may assume that $\lambda(t) = \sqrt{x_3 - x(t)}$, where $\sqrt{x_3 - x(t)} \geq 0$ on $[\tilde{\omega} - \omega, \tilde{\omega} + \omega]$, and $\sqrt{x_3 - x(t)} \leq 0$ on $[\tilde{\omega} + \omega, \tilde{\omega} + 3\omega]$. Note that $p - 4m, q \in [\tilde{\omega} - \omega, \tilde{\omega} + \omega]$ and $q + 2m, q + 4m, p + 4m, p + 6m \in [\tilde{\omega} + \omega, \tilde{\omega} + 3\omega]$, by Theorem 4.2, we conclude that,

$$\lambda(p - 4m) = \lambda(q) = \sqrt{x_3 - \rho_1^{-1}} \quad (4.98)$$

$$\lambda(q + 2m) = \lambda(q + 6m) = -\sqrt{x_3 - 1} \quad (4.99)$$

$$\lambda(p + 4m) = \lambda(q + 4m) = -\sqrt{x_3 - \frac{\rho_1^{-1}}{\rho_2^{-1}}} \quad (4.100)$$

Therefore, $A(t)$ and $F(t)$ have identical zeros and poles, such that $A(t) = cF(t)$ for some constant c , where,

$$F(t) = \frac{\lambda(t) + \sqrt{x_3 - 1}}{\left(\lambda(t) + \sqrt{x_3 - \frac{\rho_1^{-1}}{\rho_2^{-1}}}\right) \left(\lambda(t) - \sqrt{x_3 - \rho_1^{-1}}\right)} \quad (4.101)$$

Let t vary over $\overline{\mathbf{D}}_{\mathbf{x}}$, then $x(t)$ varies over $|x| \leq 1$. Meantime, $A(t) = P(x(t), 0)$ and $\lambda(t) = \sqrt{x_3 - x(t)}$. We conclude that,

$$P(x, 0) = cF(t) = c\phi(x), \quad |x| \leq 1 \quad (4.102)$$

To determine c , let $x(t) = 1$, we get $c = \frac{\rho_2^{-1}-1}{\rho_2^{-1}\phi(1)}$ from $c\phi(1) = P(1, 0) = \frac{\rho_2^{-1}-1}{\rho_2^{-1}}$, thus, (4.90) holds.

Similarly, comparing with the zeros and poles of $x(t)$, $y(t)$ and $A(t)$, as well

as the points that $x(t) = 1$ and $y(t) = 1$, in the fundamental rectangle of $B(t)$,

$$\mathbf{R}_B = \{t \mid -\tilde{\omega} < \mathbf{Re} t \leq \tilde{\omega}, \mathbf{Im}(-2\omega + 2m) < \mathbf{Im} t \leq \mathbf{Im}(2\omega + 2m)\},$$

Functions	Periods	Zeros	Poles
$x(t)$	$2\tilde{\omega}, 2\omega$	$\{-m, m, 5m, 7m\}$	$\{-\omega, \omega\}$
$y(t)$	$2\tilde{\omega}, 2\omega$	$\{-\omega, m, \omega, 7m\}$	$\{-m, 5m\}$
$A(t)$	$2\tilde{\omega}, 4\omega$	$\left\{ \begin{array}{l} -\omega, q + 2m, \\ \omega, p + 6m \end{array} \right\}$	$\left\{ \begin{array}{l} p - 4m, q, p + 4m, \\ q + 4m, p + 8m \end{array} \right\}$
$x(t) = 1$	$\{q - 4m, p, p + 6m, q + 2m, q + 8m\}$		
$y(t) = 1$	$\{p, p + 6m, q, q + 6m\}$		

We obtain the complete sets of zeros and poles for $B(t)$, when $\rho_1^{-1} < \rho_2^{-1}$,

Functions	Periods	Zeros	Poles
$B(t)$	$2\tilde{\omega}, 4\omega$	$\left\{ \begin{array}{l} -m, p + 6m, \\ 5m, q + 6m \end{array} \right\}$	$\left\{ \begin{array}{l} p + 4m, q + 4m, \\ p + 8m, q + 8m \end{array} \right\}$

When $\rho_1^{-1} = \rho_2^{-1}$, since $q + 6m = p + 8m, p + 6m = q + 4m$, the complete sets of zeros and poles for $B(t)$ become $\{-m, 5m\}$ and $\{p + 4m, q + 8m\}$, respectively.

Let $\mu(t)$ be an elliptic function with periods $2\tilde{\omega}, 4\omega$ and with the complete sets of zeros and poles $\{\tilde{\omega} - m, \tilde{\omega} + 5m\}$ and $\{-m, 5m\}$. By Theorem 4.1, as in this case $\sum a_i - \sum b_i = 2\tilde{\omega}$, we can conclude that such $\mu(t)$ exists. From the mapping rule and Theorem 4.2(i), we know that $\mu^2(t)$ and $y_3 - y(t)$ have identical zeros and poles. We may assume that $\mu(t) = \sqrt{y_3 - y(t)}$, where $\sqrt{y_3 - y(t)} \geq 0$ on $[\tilde{\omega} - m, \tilde{\omega} + 5m]$, and $\sqrt{y_3 - y(t)} \leq 0$ on $[\tilde{\omega} + 5m, \tilde{\omega} + 11m]$. Note that $p - 4m, q \in [\tilde{\omega} - m, \tilde{\omega} + 5m]$ and $p + 6m, q + 6m, p + 8m, q + 8m \in [\tilde{\omega} + 5m, \tilde{\omega} + 11m]$, by Theorem 4.2, we

conclude that,

$$\mu(p + 6m) = \mu(q + 6m) = -\sqrt{x_3 - \rho_1^{-1}} \quad (4.103)$$

$$\mu(p + 8m) = -\sqrt{x_3 - \frac{\rho_2^{-1}}{\rho_1^{-1}}} \quad (4.104)$$

$$\mu(q + 8m) = -\sqrt{x_3 - \rho_2^{-1}} \quad (4.105)$$

Therefore, $B(t)$ and $G(t)$ have identical zeros and poles, such that $B(t) = c'G(t)$ for some constant c' , where,

$$G(t) = \frac{\mu(t) + \sqrt{y_3 - 1}}{\left(\mu(t) + \sqrt{y_3 - \frac{\rho_2^{-1}}{\rho_1^{-1}}}\right) \left(\mu(t) + \sqrt{y_3 - \rho_2^{-1}}\right)} \quad (4.106)$$

Let t vary over $\overline{\mathbf{D}}_y$, then $y(t)$ varies over $|y| \leq 1$. Meantime, $B(t) = P(0, y(t))$ and $\mu(t) = \sqrt{y_3 - y(t)}$. We conclude that,

$$P(0, y) = c'G(t) = c'\psi(y), \quad |y| \leq 1 \quad (4.107)$$

To determine c' , let $y(t) = 1$, we get $c' = \frac{\rho_1^{-1}-1}{\rho_1^{-1}\psi(1)}$ from $c'\psi(1) = P(0, 1) = \frac{\rho_1^{-1}-1}{\rho_1^{-1}}$, thus, (4.91) holds. $\square$

Chapter 5

Generalization of The 2-Demand Model

In this chapter, we will describe a generalized 2-demand queueing network, and apply the generating function method step by step, and here we summarize the difficulties we encounter in the analysis at the beginning of this chapter.

1. We cannot determine the exact relationship between some important points on the t -plane after the parametrization, i.e., the relationship between m , r and ω .
2. We cannot determine the relationship between $x(t)$ and $y(t)$ because of insufficient information about m , r and ω , see section 5.2.3.
3. We cannot determine the periods of $A(t)$ and $B(t)$ on the t -plane, see section 5.2.4.
4. We cannot determine some zeros and poles of $A(t)$ and $B(t)$ on the t -plane, see section 5.2.5.

5.1 Model Description

Consider a generalized two-parallel-queue system, where there are three sources of customers (or packets, jobs, fragments, etc.), each of which arrives at the queueing system according to a Poisson process. Among these three arriving streams, the rate of the two-demand source is λ , upon each arrival of which, it creates one new customer/job in both queues simultaneously at a time. The other two types of arrivals are dedicated to queue 1 and queue 2, with arrival rates λ_1 and λ_2 , respectively. As Figure 5.1 shows, there are two parallel single-server queues in the queueing network. Each server serves one customer at a time with an exponential service time rate $\mu_i, i = 1, 2$ and the two servers are independent. Upon arrival, if the arriving customer belongs to the two-demand type, it creates one new job in both queues simultaneously at a time; On the other hand, if the arrival is dedicated to queue $i, i = 1, 2$, the arriving customer will create a single job to queue $i, i = 1, 2$. In each queue, the job created by an arriving customer immediately starts being processed if the server is idle; otherwise, the job waits in a line (that is, it joins the queue). We assume that there is infinite waiting space at each server. We assume $\lambda_i + \lambda < \mu_i, i = 1, 2$, so that each queue is stable.

5.1.1 The Balance Equations

Let $\pi = \{\pi_{m,n} : m, n = 0, 1, \dots\}$ be the unique stationary distribution of the two-demand queueing system, and denote λ^* be the summation of the arrival rates, i.e. $\lambda^* = \lambda + \lambda_1 + \lambda_2$, then, according to the transition rates relationship shown

Figure 5.1: Generalized Queueing Network Model with Two Demands

in Figure 5.2, we have the following balance equations,

Figure 5.2: State Transition Rates of The Generalized Two-Demand Queueing Model

$$\lambda^* \pi_{0,0} = \mu_1 \pi_{1,0} + \mu_2 \pi_{0,1}$$

$$(\lambda^* + \mu_1) \pi_{1,0} = \lambda_1 \pi_{0,0} + \mu_1 \pi_{2,0} + \mu_2 \pi_{1,1}$$

$$\begin{aligned}
(\lambda^* + \mu_1) \pi_{m,0} &= \lambda_1 \pi_{m-1,0} + \mu_1 \pi_{m+1,0} + \mu_2 \pi_{m,1} \\
(\lambda^* + \mu_2) \pi_{0,1} &= \lambda_2 \pi_{0,0} + \mu_2 \pi_{0,2} + \mu_1 \pi_{1,1} \\
(\lambda^* + \mu_2) \pi_{0,n} &= \lambda_2 \pi_{0,n-1} + \mu_2 \pi_{0,n+1} + \mu_1 \pi_{0,n} \\
(\lambda^* + \mu_1 + \mu_2) \pi_{1,1} &= \mu_1 \pi_{2,1} + \mu_2 \pi_{1,2} + \lambda \pi_{0,0} + \lambda_2 \pi_{1,0} + \lambda_1 \pi_{0,1} \\
(\lambda^* + \mu_1 + \mu_2) \pi_{m,n} &= \mu_1 \pi_{m+1,n} + \mu_2 \pi_{m,n+1} + \lambda \pi_{m-1,n-1} + \lambda_2 \pi_{m,n-1} + \lambda_1 \pi_{m-1,n}
\end{aligned}$$

5.1.2 The Functional Equations

By [7], we obtain,

$$\begin{aligned}
& [(\lambda^* + \mu_1 + \mu_2) xy - \mu_1 y - \mu_2 x - \lambda x^2 y^2 - \lambda_1 x^2 y - \lambda_2 x y^2] P(x, y) \\
& = \mu_2 x (y - 1) P(x, 0) + \mu_1 y (x - 1) P(0, y)
\end{aligned} \tag{5.1}$$

Now we can re-express (5.1) in the form of (2.4) as follows,

$$h_1(x, y) P(x, y) = h_2(x, y) P(x, 0) + h_3(x, y) P(0, y) + h_4(x, y) P(0, 0) \tag{5.2}$$

where,

$$h_1(x, y) = (\lambda^* + \mu_1 + \mu_2) xy - \mu_1 y - \mu_2 x - \lambda x^2 y^2 - \lambda_1 x^2 y - \lambda_2 x y^2 \tag{5.3}$$

$$h_2(x, y) = \mu_2 x (y - 1) \tag{5.4}$$

$$h_3(x, y) = \mu_1 y (x - 1) \tag{5.5}$$

$$h_4(x, y) = 0 \tag{5.6}$$

Remark 5.1 *Expressions (5.3)-(5.6) can be first found in Wright (1992) [18] 1.3-1.6. In this thesis I represented all these expressions with the details omitted.*

5.2 Uniformization Method on $P(x, 0)$ and $P(0, y)$

In this section, we will try to apply the uniformization method to the generalized 2-demand model and state the difficulties we encounter.

5.2.1 Analysis of The Kernel

We note that $h_1(x, y)$ can be viewed as a 2-degree polynomial in y . For every value of x , there are two possible values of y , denoted by $y_+(x)$ and $y_-(x)$, such that $h_1(x, y_+(x)) = h_1(x, y_-(x)) = 0$, i.e. For every value of x , dividing both sides by λ^* , solving the equation,

$$\begin{aligned} 0 &\equiv h_1(x, y) = (\lambda^* + \mu_1 + \mu_2)xy - \mu_1y - \mu_2x - \lambda x^2y^2 - \lambda_1x^2y - \lambda_2xy^2 \\ 0 &= (1 + \rho_1^{-1} + \rho_2^{-1})xy - \rho_1^{-1}y - \rho_2^{-1}x - \gamma x^2y^2 - \gamma_1x^2y - \gamma_2xy^2 \\ 0 &= -(\gamma x^2 + \gamma_2x)y^2 + \{[(1 + \rho_1^{-1} + \rho_2^{-1})x - \rho_1^{-1}] - \gamma_1x^2\}y - \rho_2^{-1}x \end{aligned} \quad (5.7)$$

where,

$$\begin{aligned} \gamma &= \frac{\lambda}{\lambda^*} \\ \gamma_1 &= \frac{\lambda_1}{\lambda^*} \\ \gamma_2 &= \frac{\lambda_2}{\lambda^*} \\ \rho_1^{-1} &= \frac{\mu_1}{\lambda^*} \end{aligned}$$

$$\rho_2^{-1} = \frac{\mu_2}{\lambda^*} \quad (5.8)$$

Then we get,

$$\begin{aligned} y_+(x) &= - \frac{[\gamma_1 x^2 + \rho_1^{-1} - (1 + \rho_1^{-1} + \rho_2^{-1})x]}{2(\gamma x^2 + \gamma_2 x)} \\ &\quad + \frac{\sqrt{\{[(1 + \rho_1^{-1} + \rho_2^{-1})x - \rho_1^{-1}] - \gamma_1 x^2\}^2 - 4\rho_2^{-1}x(\gamma x^2 + \gamma_2 x)}}{2(\gamma x^2 + \gamma_2 x)} \quad (5.9) \\ &= \frac{(1 + \rho_1^{-1} + \rho_2^{-1})x - \gamma_1 x^2 - \rho_1^{-1} + \sqrt{\mathbf{D}_1(x)}}{2(\gamma x^2 + \gamma_2 x)} \end{aligned}$$

and

$$\begin{aligned} y_-(x) &= - \frac{[\gamma_1 x^2 + \rho_1^{-1} - (1 + \rho_1^{-1} + \rho_2^{-1})x]}{2(\gamma x^2 + \gamma_2 x)} \\ &\quad - \frac{\sqrt{\{[(1 + \rho_1^{-1} + \rho_2^{-1})x - \rho_1^{-1}] - \gamma_1 x^2\}^2 - 4\rho_2^{-1}x(\gamma x^2 + \gamma_2 x)}}{2(\gamma x^2 + \gamma_2 x)} \quad (5.10) \\ &= \frac{(1 + \rho_1^{-1} + \rho_2^{-1})x - \gamma_1 x^2 - \rho_1^{-1} - \sqrt{\mathbf{D}_1(x)}}{2(\gamma x^2 + \gamma_2 x)} \end{aligned}$$

where

$$\mathbf{D}_1(x) = \{[(1 + \rho_1^{-1} + \rho_2^{-1})x - \rho_1^{-1}] - \gamma_1 x^2\}^2 - 4\rho_2^{-1}x(\gamma x^2 + \gamma_2 x) \quad (5.11)$$

More generally, we write a two valued function

$$y(x) = \frac{(1 + \rho_1^{-1} + \rho_2^{-1})x - \gamma_1 x^2 - \rho_1^{-1} \pm \sqrt{\mathbf{D}_1(x)}}{2(\gamma x^2 + \gamma_2 x)} \quad (5.12)$$

Hence, the zero-space of $h_1(x, y)$ can be expressed as $\{(x, y) : x \in \mathbb{C}, y = y(x)\}$.

Similarly, if we treat $h_1(x, y)$ as a 2-degree polynomial in x , and for every value of y , there are 2 possible values of x , denoted by $x_+(y)$ and $x_-(y)$, such that $h_1(x_+(y), y) = h_1(x_-(y), y) = 0$. That is, for every value of y , solving the equation (5.7), we can get,

$$\begin{aligned} x_+(y) &= -\frac{[\gamma_2 y^2 + \rho_2^{-1} - (1 + \rho_1^{-1} + \rho_2^{-1})y]}{2(\gamma y^2 + \gamma_1 y)} \\ &\quad + \frac{\sqrt{\{[(1 + \rho_1^{-1} + \rho_2^{-1})y - \rho_2^{-1}] - \gamma_2 y^2\}^2 - 4\rho_2^{-1}y(\gamma y^2 + \gamma_1 y)}}{2(\gamma y^2 + \gamma_1 y)} \quad (5.13) \\ &= \frac{(1 + \rho_1^{-1} + \rho_2^{-1})y - \gamma_2 y^2 - \rho_2^{-1} + \sqrt{\mathbf{D}_2(y)}}{2(\gamma y^2 + \gamma_1 y)} \end{aligned}$$

and

$$\begin{aligned} x_-(y) &= -\frac{[\gamma_2 y^2 + \rho_2^{-1} - (1 + \rho_1^{-1} + \rho_2^{-1})y]}{2(\gamma y^2 + \gamma_1 y)} \\ &\quad - \frac{\sqrt{\{[(1 + \rho_1^{-1} + \rho_2^{-1})y - \rho_2^{-1}] - \gamma_2 y^2\}^2 - 4\rho_2^{-1}y(\gamma y^2 + \gamma_1 y)}}{2(\gamma y^2 + \gamma_1 y)} \quad (5.14) \\ &= \frac{(1 + \rho_1^{-1} + \rho_2^{-1})y - \gamma_2 y^2 - \rho_2^{-1} - \sqrt{\mathbf{D}_2(y)}}{2(\gamma y^2 + \gamma_1 y)} \end{aligned}$$

where

$$\mathbf{D}_2(y) = \{[(1 + \rho_1^{-1} + \rho_2^{-1})y - \rho_2^{-1}] - \gamma_2 y^2\}^2 - 4\rho_2^{-1}y(\gamma y^2 + \gamma_1 y) \quad (5.15)$$

More generally, we write a two valued function

$$x(y) = \frac{(1 + \rho_1^{-1} + \rho_2^{-1})y - \gamma_2 y^2 - \rho_2^{-1} \pm \sqrt{\mathbf{D}_2(y)}}{2(\gamma y^2 + \gamma_1 y)} \quad (5.16)$$

Hence, zero-space of $h_1(x, y)$ also can be expressed as $\{(x, y) : y \in \mathbb{C}, x = x(y)\}$.

Proposition 5.1 (i) $\mathbf{D}_1(x)$ has four real and distinct zeros x_1, x_2, x_3 and x_4 with $0 < x_1 < x_2 < 1 < x_3 < x_4$. $\mathbf{D}_1(x) > 0$ on $(-\infty, x_1) \cup (x_2, x_3) \cup (x_4, +\infty)$ and $\mathbf{D}_1(x) < 0$ on $(x_1, x_2) \cup (x_3, x_4)$. A similar statement holds for y_1, y_2, y_3 and y_4 of $\mathbf{D}_2(y)$. i.e. $\mathbf{D}_2(y)$ has four real and distinct zeros y_1, y_2, y_3 and y_4 with $0 < y_1 < y_2 < 1 < y_3 < y_4$. $\mathbf{D}_2(y) > 0$ on $(-\infty, y_1) \cup (y_2, y_3) \cup (y_4, +\infty)$ and $\mathbf{D}_2(y) < 0$ on $(y_1, y_2) \cup (y_3, y_4)$.

(ii) If $\frac{\rho_1^{-1}}{1-\gamma_2} \leq \frac{\rho_2^{-1}}{1-\gamma_1}$, $\frac{\rho_1^{-1}}{1-\gamma_2} \leq x_3$; if $\frac{\rho_1^{-1}}{1-\gamma_2} > \frac{\rho_2^{-1}}{1-\gamma_1}$, $\frac{\rho_2^{-1}}{1-\gamma_1} \leq y_3$.

PROOF. (i) In $\mathbf{D}_1(x)$, let $x = 0, x = 1$ and $x = \infty$, it is easy to check that,

$$\mathbf{D}_1(0) = (\rho_1^{-1})^2 > 0 \quad (5.17)$$

$$\begin{aligned} \mathbf{D}_1(1) &= (1 + \rho_2^{-1} - \gamma_1)^2 - 4\rho_2^{-1}(\gamma + \gamma_2) \\ &= (\gamma + \gamma_2 + \rho_2^{-1})^2 - 4\rho_2^{-1}(\gamma + \gamma_2) \\ &= (\gamma + \gamma_2 - \rho_2^{-1})^2 \geq 0 \end{aligned} \quad (5.18)$$

$$\mathbf{D}_1(+\infty) = +\infty \quad (5.19)$$

Then from the quadratic part of $\mathbf{D}_1(x)$, denoted by $u(x) = \gamma_1 x^2 - (1 + \rho_1^{-1} + \rho_2^{-1})x + \rho_1^{-1}$, which has two real roots x_1^u, x_2^u , we observe that

$$u(0) = \rho_1^{-1} > 0 \quad u(1) = -(\gamma + \gamma_2 + \rho_2^{-1}) < 0 \quad u(+\infty) = +\infty \quad (5.20)$$

Which implies that $0 < x_1^u < 1 < x_2^u$.

Next, we substitute x_1^u, x_2^u into $\mathbf{D}_1(x)$,

$$\mathbf{D}_1(x_1^u) = -4\rho_2^{-1}x_1^u (\gamma(x_1^u)^2 + \gamma_2 x_1^u) < 0 \quad (5.21)$$

$$\mathbf{D}_1(x_2^u) = -4\rho_2^{-1}x_2^u (\gamma(x_2^u)^2 + \gamma_2 x_2^u) < 0 \quad (5.22)$$

All these inequalities above imply that $\mathbf{D}_1(x)$ has two roots x_1, x_2 in the intervals $(0, x_1^u), (x_1^u, 1)$, respectively, and has the other two roots x_3, x_4 in the interval $(1, x_2^u), (x_2^u, +\infty)$. Besides, from the above equations, we can determine that $\mathbf{D}_1(x) > 0$ on $(-\infty, x_1) \cup (x_2, x_3) \cup (x_4, +\infty)$ and $\mathbf{D}_1(x) < 0$ on $(x_1, x_2) \cup (x_3, x_4)$. Similarly, we can obtain that,

$$\mathbf{D}_2(0) = (\rho_2^{-1})^2 > 0 \quad (5.23)$$

$$\begin{aligned} \mathbf{D}_2(1) &= (1 + \rho_1^{-1} - \gamma_2)^2 - 4\rho_1^{-1}(\gamma + \gamma_1) \\ &= (\gamma + \gamma_1 - \rho_1^{-1})^2 \geq 0 \end{aligned} \quad (5.24)$$

$$\mathbf{D}_2(+\infty) = +\infty \quad (5.25)$$

And from the quadratic part of $\mathbf{D}_2(y)$, denoted by $v(y) = \gamma_2 y^2 - (1 + \rho_1^{-1} + \rho_2^{-1})y + \rho_2^{-1}$, which has two real roots y_1^v, y_2^v , we observe that

$$v(0) = \rho_2^{-1} > 0 \quad v(1) = -(\gamma + \gamma_1 + \rho_1^{-1}) < 0 \quad v(+\infty) = +\infty \quad (5.26)$$

Which implies that $0 < y_1^v < 1 < y_2^v$. Next, we substitute y_1^v, y_2^v into $\mathbf{D}_2(y)$,

$$\mathbf{D}_2(y_1^v) = -4\rho_1^{-1}y_1^v (\gamma(y_1^v)^2 + \gamma_1 y_1^v) < 0 \quad (5.27)$$

$$\mathbf{D}_2(y_2^v) = -4\rho_1^{-1}y_2^v (\gamma(y_2^v)^2 + \gamma_1 y_2^v) < 0 \quad (5.28)$$

Thus, a similar statement holds for the roots y_1, y_2, y_3 and y_4 of $\mathbf{D}_2(y)$.

(ii) Setting $\frac{\rho_1^{-1}}{1-\gamma_2} \leq \frac{\rho_2^{-1}}{1-\gamma_1}$, we check that,

$$\begin{aligned} & \mathbf{D}_1 \left(\frac{\rho_1^{-1}}{1-\gamma_2} \right) \\ &= \left\{ \left[(1 + \rho_1^{-1} + \rho_2^{-1}) \frac{\rho_1^{-1}}{1-\gamma_2} - \rho_1^{-1} \right] - \gamma_1 \left(\frac{\rho_1^{-1}}{1-\gamma_2} \right)^2 \right\}^2 \\ & \quad - 4\rho_2^{-1} \frac{\rho_1^{-1}}{1-\gamma_2} \left[\gamma \left(\frac{\rho_1^{-1}}{1-\gamma_2} \right)^2 + \gamma_2 \frac{\rho_1^{-1}}{1-\gamma_2} \right] \\ &= \left(\frac{\rho_1^{-1}}{1-\gamma_2} \right)^2 \left[\rho_1^{-1} \frac{\gamma}{1-\gamma_2} + \gamma_2 - \rho_2^{-1} \right]^2 \\ &\geq 0 \end{aligned} \quad (5.29)$$

Similarly,

$$\begin{aligned}
& \mathbf{D}_2 \left(\frac{\rho_2^{-1}}{1-\gamma_1} \right) \\
&= \left\{ \left[(1 + \rho_1^{-1} + \rho_2^{-1}) \frac{\rho_2^{-1}}{1-\gamma_1} - \rho_2^{-1} \right] - \gamma_2 \left(\frac{\rho_2^{-1}}{1-\gamma_1} \right)^2 \right\} \\
&\quad - 4\rho_1^{-1} \frac{\rho_2^{-1}}{1-\gamma_1} \left[\gamma \left(\frac{\rho_2^{-1}}{1-\gamma_1} \right)^2 + \gamma_1 \frac{\rho_2^{-1}}{1-\gamma_1} \right] \\
&= \left(\frac{\rho_2^{-1}}{1-\gamma_1} \right)^2 \left[\rho_2^{-1} \frac{\gamma}{1-\gamma_1} + \gamma_1 - \rho_1^{-1} \right]^2 \\
&\geq 0
\end{aligned} \tag{5.30}$$

On the other hand, we can easily verify that,

$$\begin{aligned}
x_2^u &= \frac{1 + \rho_1^{-1} + \rho_2^{-1} + \sqrt{(1 + \rho_1^{-1} + \rho_2^{-1})^2 - 4\gamma_1\rho_1^{-1}}}{2\gamma_1} \\
&\geq \frac{1 + \rho_1^{-1} + \rho_2^{-1} + \sqrt{(\gamma_1 + \rho_1^{-1} + \rho_2^{-1})^2 - 4\gamma_1\rho_1^{-1}}}{2\gamma_1} \\
&\geq \frac{1 + \rho_1^{-1} + \rho_2^{-1} + \sqrt{(\rho_1^{-1} + \rho_2^{-1} - \gamma_1)^2}}{2\gamma_1} \\
&\geq \frac{2\rho_1^{-1} + 2\rho_1^{-1} + 1 - \gamma_1}{2\gamma_1} \\
&\geq \frac{2\rho_1^{-1} + 2\rho_1^{-1}}{2\gamma_1} \\
&\geq \frac{\left(\frac{1-\gamma_1}{1-\gamma_2} + 1 \right) \rho_1^{-1}}{\gamma_1} \\
&\geq \frac{\rho_1^{-1}}{1-\gamma_2}
\end{aligned} \tag{5.31}$$

In the above inequalities, the second last step we used the assumption $\frac{\rho_1^{-1}}{1-\gamma_2} \leq \frac{\rho_2^{-1}}{1-\gamma_1}$, and the last step we applied the relation that $\gamma_1 + \gamma_2 + \gamma = 1$ such that

$$2 - \gamma_1 - \gamma_2 = 1 + \gamma > \gamma_1$$

Similarly, when $\frac{\rho_1^{-1}}{1-\gamma_2} \geq \frac{\rho_2^{-1}}{1-\gamma_1}$

$$\begin{aligned}
 y_2^v &= \frac{1 + \rho_1^{-1} + \rho_2^{-1} + \sqrt{(1 + \rho_1^{-1} + \rho_2^{-1})^2 - 4\gamma_2\rho_2^{-1}}}{2\gamma_2} \\
 &\geq \frac{1 + \rho_1^{-1} + \rho_2^{-1} + \sqrt{(\rho_1^{-1} + \rho_2^{-1} - \gamma_2)^2}}{2\gamma_2} \\
 &\geq \frac{\left(\frac{1-\gamma_2}{1-\gamma_1} + 1\right) \rho_2^{-1}}{\gamma_2} \\
 &\geq \frac{\rho_2^{-1}}{1-\gamma_1}
 \end{aligned} \tag{5.32}$$

Therefore, in the case of $\frac{\rho_1^{-1}}{1-\gamma_2} \leq \frac{\rho_2^{-1}}{1-\gamma_1}$, $\mathbf{D}_1(x_2^u) < 0 \leq \mathbf{D}_1(\frac{\rho_1^{-1}}{1-\gamma_2})$ which implies that $x_3 \geq \frac{\rho_1^{-1}}{1-\gamma_2}$. On the other hand, when $\frac{\rho_1^{-1}}{1-\gamma_2} \geq \frac{\rho_2^{-1}}{1-\gamma_1}$, $\mathbf{D}_2(y_2^v) < 0 \leq \mathbf{D}_2(\frac{\rho_2^{-1}}{1-\gamma_1})$ which implies that $y_3 \geq \frac{\rho_2^{-1}}{1-\gamma_1}$.

□

Remark 5.2 *Proposition 5.1 is first shown in Wright (1992) [18] Proposition 4.1(a) without a proof. I represented this proposition with a detailed proof here for uses later in this thesis. However, I cannot prove the proposition without the condition I assumed, future work is needed here.*

5.2.2 Parametrization of the Algebraic Curve

By the Schwarz-Chistoffel transformation, see section 2.4, there exists a closed rectangle $\mathcal{R}_0$ with vertices $0, \tilde{\omega}, \tilde{\omega} + \omega, \omega$, where $\tilde{\omega}, \omega/i > 0$, and a $1 - 1$ mapping $t(x)$ from the extended closed upper half plane $\mathbb{H}$ to $\mathcal{R}_0$ which contains $t(x)$,

continuous on $\mathbb{H}$ and analytic on interior of $\mathbb{H}$. $t(x)$ maps x_1, x_2, x_3, x_4 to the vertices $0, \tilde{\omega}, \tilde{\omega} + \omega, \omega$, respectively, see figure 3.9. Let $x = x(t)$ be the inverse mapping. By repeated use of the reflection principle [16], $x(t)$ can be extended as a meromorphic function on the entire t -plane. Furthermore, $x(t)$ is an even and doubly periodic (i.e. elliptic) function with period $2\tilde{\omega}, 2\omega$.

The function $y(t)$ is then defined as follows. Let $\mathbb{P}$ be the x -plane minus the slits $[x_1, x_2], [x_3, x_4]$. For $x \in \mathbb{P}$, let $\sqrt{\mathbf{D}_1(x)}$ denote the square root that is analytic on $\mathbb{P}$ and $\sqrt{\mathbf{D}_1(0)} = \rho_1^{-1}$, $x = x(t)$ maps each of the strips, defined by $\mathbb{S}_k = \{t | k\mathbf{Im}\omega < \mathbf{Im}t \leq (k+1)\mathbf{Im}\omega\}$, where k is an integer onto $\mathbb{P}$.

Now define,

$$s(t) = \begin{cases} +\sqrt{\mathbf{D}_1(x(t))}, & \text{if } t \in \mathbb{S}_k, k \text{ is even} \\ -\sqrt{\mathbf{D}_1(x(t))}, & \text{if } t \in \mathbb{S}_k, k \text{ is odd} \end{cases} \quad (5.33)$$

For $\mathbf{Im}t = \mathbf{Im}(2k\omega \pm n)$, let $s(t) = \lim_{u \rightarrow t} s(u)$, $u \in \mathbb{S}_k$, where $x(\pm n) = \infty$. Therefore $s(t)$ is an elliptic function with periods $2\tilde{\omega}, 2\omega$.

Next, we define,

$$y(t) = \frac{(1 + \rho_1^{-1} + \rho_2^{-1})x(t) - \rho_1^{-1} - \gamma_1 x(t)^2 + s(t)}{2(\gamma x(t)^2 + \gamma_2 x(t))} \quad (5.34)$$

Similar as the statement in Chapter 4, for any two complex numbers a, b , the symbols $[a, b]$, (a, b) , $[a, b)$, $(a, b]$ denote the closed, open, and half open line seg-

ments directed from a to b . $[a, b)$ containing a and $(a, b]$ containing b . Let $\mathcal{R}$ be the fundamental rectangle $\{t \mid -\tilde{\omega} < \mathbf{Re} t \leq \tilde{\omega}, -\mathbf{Im} \omega < \mathbf{Im} t \leq \mathbf{Im} \omega\}$. $x(t)$ maps $[\omega, 0]$ one-to-one onto $[x_4, x_1]$, and $[\tilde{\omega}, \tilde{\omega} + \omega]$ one-to-one onto $[x_2, x_3]$. As $0 < x_1 < x_2 < 1 < \frac{\rho_1^{-1}}{1-\gamma_2} \leq x_3 < x_4$, there exists points $m, n, r \in (0, \omega), p, q \in (\tilde{\omega}, \tilde{\omega} + \omega]$ such that $x(m) = 0, x(n) = \infty, x(r) = \frac{\gamma_2}{\gamma}, x(p) = 1, x(q) = \frac{\rho_1^{-1}}{1-\gamma_2}$. Moreover, we denote $p < q$ since p precedes q on $(\tilde{\omega}, \tilde{\omega} + \omega]$.

5.2.3 The Relationship between $x(t)$ and $y(t)$

Till now, we have done the most part of preparations. In the next stage, we will show an important theorem in this thesis leading to the solution of the problem.

Assumption 5.1 *Let $\mathcal{R}$ be the fundamental rectangle $\{t \mid -\tilde{\omega} < \mathbf{Re} t \leq \tilde{\omega}, -\mathbf{Im} \omega < \mathbf{Im} t \leq \mathbf{Im} \omega\}$, and m, n, r, p, q as defined above.*

(i) *The zeros and the poles of $x(t)$ in $\mathcal{R}$ are respectively $\{m, -m\}, \{n, -n\}$.*

(ii) *The zeros and the poles of $y(t)$ in $\mathcal{R}$ are $\{m, n\}, \{-m, -r\}$.*

(iii) *$m = \frac{n}{3}$.*

(iv) *$y(t) = \frac{\rho_2^{-1}}{\rho_1} x(t - 2m)$.*

(v) *$\frac{p+q}{2} = \tilde{\omega} + 2m$, then, $\tilde{\omega} + m \leq p < \tilde{\omega} + 2m < q \leq \tilde{\omega} + \omega$.*

(vi) *At the points $p - 2km, q - 2km, k = 0, 1, 2$, $x(t)$ and $y(t)$ assume the values listed in Table 5.1.*

PROOF. (i) According to the mapping rule, $t(x)$ maps x_1, x_2, x_3, x_4 to the vertices $0, \tilde{\omega}, \tilde{\omega} + \omega, \omega$, respectively, and $x(t)$ is the inverse mapping of $t(x)$. As

t	$p - 4m$	$q - 4m$	$p - 2m$	$q - 2m$	p	q
$x(t)$	$\frac{\rho_1^{-1}}{1-\gamma_2}$	1	$\frac{\rho_1^{-1}}{\rho_2^{-1}}$	$\frac{\rho_1^{-1}}{\rho_2^{-1}}$	1	$\frac{\rho_1^{-1}}{1-\gamma_2}$
$y(t)$	$\frac{\rho_2^{-1}}{\gamma \frac{\rho_2^{-1}}{1-\gamma_2} + \gamma_2}$	$\frac{\rho_2^{-1}}{1-\gamma_1}$	$\frac{\rho_2^{-1}}{1-\gamma_2}$	$\frac{\rho_2^{-1}}{\rho_1^{-1}}$	1	1

Table 5.1: Values of $x(t)$ and $y(t)$

defined above, by the definition of m, n , we have $x(m) = 0$, $x(n) = \infty$. Then, by the reflection principle that we used to extend the mapping to the whole t -plane,

$$x(-m) = \overline{x(-\overline{m})} = \overline{x(m)} = \overline{0} = 0 \quad (5.35)$$

$$x(-n) = \overline{x(-\overline{n})} = \overline{x(n)} = \overline{\infty} = \infty \quad (5.36)$$

Therefore, $\{m, -m\}$ are the zero points of $x(t)$ in $\mathcal{R}$ and $\{n, -n\}$ are the poles of $x(t)$ in $\mathcal{R}$.

(ii) To find the zeros and the poles of $y(t)$, first we want to simplify the expression (5.34), using Taylor Expansion.

Start with,

$$\sqrt{\mathbf{D}_1(x)} = \{[(1 + \rho_1^{-1} + \rho_2^{-1})x - \rho_1^{-1}] - \gamma_1 x^2\}^2 - 4\rho_2^{-1}x(\gamma x^2 + \gamma_2 x)^{\frac{1}{2}} \quad (5.37)$$

Taking the 1st derivative,

$$\begin{aligned} \left(\sqrt{\mathbf{D}_1(x)}\right)' &= \frac{1}{\sqrt{\mathbf{D}_1(x)}} \{[(1 + \rho_1^{-1} + \rho_2^{-1})x - \rho_1^{-1}] - \gamma_1 x^2\}(1 + \rho_1^{-1} + \rho_2^{-1}) \\ &\quad - \frac{1}{\sqrt{\mathbf{D}_1(x)}} \{6\rho_2^{-1}\gamma x^2 + 4\rho_2^{-1}\gamma_2 x\} \end{aligned} \quad (5.38)$$

Then the 2nd derivative,

$$\begin{aligned}
& \left(\sqrt{\mathbf{D}_1(x)} \right)'' \\
&= \frac{1}{\sqrt{\mathbf{D}_1(x)}} \left\{ [(1 + \rho_1^{-1} + \rho_2^{-1}) - 2\gamma_1 x]^2 - 2\gamma_1 [(1 + \rho_1^{-1} + \rho_2^{-1})x - \rho_1^{-1} - \gamma_1 x^2] \right\} \\
&\quad - \frac{1}{\sqrt{\mathbf{D}_1(x)}} \{ 12\rho_2^{-1}\gamma x + 4\rho_2^{-1}\gamma_2 \} \\
&\quad - \frac{\{ [(1 + \rho_1^{-1} + \rho_2^{-1})x - \rho_1^{-1} - \gamma_1 x^2](1 + \rho_1^{-1} + \rho_2^{-1}) - 6\rho_2^{-1}\gamma x^2 + 4\rho_2^{-1}\gamma_2 x \}^2}{\left(\sqrt{\mathbf{D}_1(x)} \right)^3}
\end{aligned} \tag{5.39}$$

And after some rearrangements and simplifications, we get the 3rd derivative,

$$\begin{aligned}
& \left(\sqrt{\mathbf{D}_1(x)} \right)^{(3)} \\
&= \frac{\{ 2(1 + \rho_1^{-1} + \rho_2^{-1} - 2\gamma_1 x)(-2\gamma_1) - 2\gamma_1(1 + \rho_1^{-1} + \rho_2^{-1} - 2\gamma_1 x) - 12\rho_2^{-1}\gamma \}}{\sqrt{\mathbf{D}_1(x)}} \\
&\quad - \frac{\{ [(1 + \rho_1^{-1} + \rho_2^{-1})x - \rho_1^{-1} - \gamma_1 x^2](1 + \rho_1^{-1} + \rho_2^{-1} - 2\gamma_1 x) - 6\rho_2^{-1}\gamma x^2 - 4\rho_2^{-1}\gamma_2 x \}}{\left\{ \sqrt{\mathbf{D}_1(x)} \right\}^3} \\
&\quad + \frac{\{ (1 + \rho_1^{-1} + \rho_2^{-1} - 2\gamma_1 x)^2 - 2\gamma_1[(1 + \rho_1^{-1} + \rho_2^{-1})x - \rho_1^{-1} - \gamma_1 x^2] - 12\rho_2^{-1}\gamma x - 4\rho_2^{-1}\gamma_2 \}}{\left(\sqrt{\mathbf{D}_1(x)} \right)^5} \\
&\quad + \frac{\{ [(1 + \rho_1^{-1} + \rho_2^{-1})x - \rho_1^{-1} - \gamma_1 x^2](1 + \rho_1^{-1} + \rho_2^{-1}) - 6\rho_2^{-1}\gamma x^2 + 4\rho_2^{-1}\gamma_2 x \}^3}{\left(\sqrt{\mathbf{D}_1(x)} \right)^5}
\end{aligned} \tag{5.40}$$

and so on.

Then for x near 0, by (5.37), (5.38), (5.39), (5.40), we obtain that,

$$\begin{aligned}
\sqrt{\mathbf{D}_1(0)} &= \rho_1^{-1} \\
\left(\sqrt{\mathbf{D}_1(0)}\right)' &= -(1 + \rho_1^{-1} + \rho_2^{-1}) \\
\left(\sqrt{\mathbf{D}_1(0)}\right)'' &= 2\gamma_1 - 4\frac{\rho_2^{-1}}{\rho_1^{-1}}\gamma_2 \\
\left(\sqrt{\mathbf{D}_1(0)}\right)^{(3)} &= -12\frac{\rho_2^{-1}}{\rho_1^{-1}}\gamma - 12\frac{\rho_2^{-1}}{(\rho_1^{-1})^2}\gamma_2(1 + \rho_1^{-1} + \rho_2^{-1})
\end{aligned} \tag{5.41}$$

Therefore, the Taylor Expansion of $\sqrt{\mathbf{D}_1(0)}$ near $x = 0$ is,

$$\begin{aligned}
\sqrt{\mathbf{D}_1(x)} &= \frac{\sqrt{\mathbf{D}_1(0)}}{0!} + \frac{\left(\sqrt{\mathbf{D}_1(0)}\right)'}{1!}x + \frac{\left(\sqrt{\mathbf{D}_1(0)}\right)''}{2!}x^2 + \frac{\left(\sqrt{\mathbf{D}_1(0)}\right)^{(3)}}{3!}x^3 + \dots \\
&= \rho_1^{-1} - (1 + \rho_1^{-1} + \rho_2^{-1})x + \left(\gamma_1 - 2\frac{\rho_2^{-1}}{\rho_1^{-1}}\gamma_2\right)x^2 \\
&\quad - 2\left(\frac{\rho_2^{-1}}{\rho_1^{-1}}\gamma - \frac{\rho_2^{-1}}{(\rho_1^{-1})^2}\gamma_2(1 + \rho_1^{-1} + \rho_2^{-1})\right)x^3 + o(x^3)
\end{aligned} \tag{5.42}$$

Now the $y_+(x)$ and $y_-(x)$ becomes,

$$\begin{aligned}
y_+(x) &= \frac{(1 + \rho_1^{-1} + \rho_2^{-1})x - \gamma_1x^2 - \rho_1^{-1} + \sqrt{\mathbf{D}_1(x)}}{2(\gamma x^2 + \gamma_2x)} \\
&= -\frac{\rho_2^{-1}}{\rho_1^{-1}}x + o(x)
\end{aligned} \tag{5.43}$$

$$\begin{aligned}
y_-(x) &= \frac{(1 + \rho_1^{-1} + \rho_2^{-1})x - \gamma_1 x^2 - \rho_1^{-1} - \sqrt{\mathbf{D}_1(x)}}{2(\gamma x^2 + \gamma_2 x)} \\
&= -\frac{\rho_1^{-1}}{\gamma x^2 + \gamma_2 x} + \frac{1 + \rho_2^{-1} + \rho_2^{-1}}{\gamma x + \gamma_2} - \frac{\gamma_1 - \frac{\rho_2^{-1}}{\rho_1^{-1}}\gamma_2}{\gamma x + \gamma_2}x \\
&\quad + \frac{\frac{\rho_2^{-1}}{\rho_1^{-1}}\gamma - \frac{\rho_2^{-1}}{(\rho_1^{-1})^2}\gamma_2(1 + \rho_1^{-1} + \rho_2^{-1})}{\gamma x + \gamma_2}x^2 + o(x)
\end{aligned} \tag{5.44}$$

Therefore, according to the definition of $s(t)$, $y(t) = y_+(x(t))$ for t near m and $y(t) = y_-(x(t))$ for t near $-m$. Then from (5.43) and (5.44), $y(t)$ has a simple zero point at $t = m$ and simple poles at $t = -m$ and $t = -r$. Besides, since $x(n) = +\infty$ and $s(t) = \lim_{u \rightarrow t} s(u)$, we have $y(t) = 0$ at $t = n$.

Thus, as a conclusion, the zeros and the poles of $y(t)$ in $\mathcal{R}$ are $\{m, n\}$ and $\{-m, -r\}$, respectively.

(iii) In (ii), we have found the zeros and the poles of $y(t)$ in $\mathcal{R}$. Now applying Theorem 4.1 to $y(t)$, we have

$$\begin{aligned}
a_1 &= m, \quad a_2 = n, \quad b_1 = -m, \quad b_2 = -r \\
\Rightarrow \quad m + n - (-m) - (-r) &= m_1 \cdot 2\tilde{\omega} + m_2 \cdot 2\omega
\end{aligned} \tag{5.45}$$

Solve m in $(0, \omega)$, we get

$$m_1 = 0, \quad m_2 = 1, \quad 2m + n + r = 2\omega \tag{5.46}$$

where $m \leq r \leq n$.

Now delighted by L. Flatto and S. Hahn [9], we notice that, from (ii), we obtain that the zeros and the poles of $y(t)$ in $\mathcal{R}$ are $\{m, n\}$ and $\{-m, -r\}$, respectively, while on the other hand, from (i), $\{m, -m\}$ are the zero points of $x(t)$ in $\mathcal{R}$ and $\{n, -n\}$ are the poles of $x(t)$ in $\mathcal{R}$. Through both results we have acquired, with the potential period offset denoted by ξ , we can claim that,

$$\begin{aligned} n - \xi &= m & m - \xi &= -m \\ -m - \xi &= -n & -r - \xi &= n - 2\omega \end{aligned}$$

Associated with (5.46), solving equations above, we get,

$$\xi = 2m \tag{5.47}$$

$$m = \frac{n}{3} \tag{5.48}$$

$$r = 2\omega - 5m \tag{5.49}$$

Remark 5.3 *Everything seems to be fine so far. However, we cannot find formal proof for the above statements. The main reason is that we do not have the exact relationship between m , r , and ω .*

Under these assumptions, we also obtain that,

$$\frac{\omega}{4} \leq m \leq \frac{\omega}{3} \leq r \leq n \leq \omega \tag{5.50}$$

where the equality holds only when $\gamma_1 = 0$, which case is already discussed in [9].

(iv) Consequently, we must have $y(t) = c \cdot x(t - 2m)$, where c is a constant and $\xi = 2m$ is the period offset. Next, we determine the value of c by the following argument:

as $t = m$ is a simple zero point of $x(t)$, let $t = m + \epsilon$ where $\epsilon \rightarrow 0$, we notice that,

$$\begin{aligned}
 y(m + \epsilon) &= c \cdot x(m + \epsilon - 2m) \\
 &= c \cdot x(-m + \epsilon) \\
 &= c \cdot x(m - \epsilon) \quad (\text{by reflection principle}) \\
 &\sim c \cdot -x(m + \epsilon)
 \end{aligned} \tag{5.51}$$

The last step is true, because $t = m$ is a simple zero point.

Since for t near m , $y(t) = y_1(x(t))$. Compare (5.51) with (5.43) which gives,

$$\begin{aligned}
 y(m + \epsilon) &= -\frac{\rho_2^{-1}}{\rho_1^{-1}} x(m + \epsilon) + o(x(m + \epsilon)) \\
 &\sim -\frac{\rho_2^{-1}}{\rho_1^{-1}} x(m + \epsilon) \quad (\text{as } \epsilon \rightarrow 0)
 \end{aligned} \tag{5.52}$$

We get $c = \frac{\rho_2^{-1}}{\rho_1^{-1}}$.

(v) Since $x(p) = 1$ and $x(q) = \frac{\rho_1^{-1}}{1-\gamma_2}$, and $p, q \in (\tilde{\omega}, \tilde{\omega} + \omega]$, from (5.34), we

get,

$$\begin{aligned}
 y(p) &= \frac{(1 + \rho_1^{-1} + \rho_2^{-1}) - \rho_1^{-1} - \gamma_1 + \sqrt{(1 + \rho_2^{-1} - \gamma_1)^2 - 4\rho_2^{-1}(\gamma + \gamma_2)}}{2(\gamma + \gamma_2)} \\
 &= \frac{(1 + \rho_1^{-1} + \rho_2^{-1}) - \rho_1^{-1} - \gamma_1 + \sqrt{(\gamma + \gamma_2 + \rho_2^{-1})^2 - 4\rho_2^{-1}(\gamma + \gamma_2)}}{2(\gamma + \gamma_2)} \quad (5.53) \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 y(q) &= \frac{(1 + \rho_1^{-1} + \rho_2^{-1}) \frac{\rho_1^{-1}}{1-\gamma_2} - \rho_1^{-1} - \gamma_1 \left(\frac{\rho_1^{-1}}{1-\gamma_2} \right)^2}{\gamma \left(\frac{\rho_1^{-1}}{1-\gamma_2} \right)^2 + \gamma_2 \frac{\rho_1^{-1}}{1-\gamma_2}} \\
 &\quad + \frac{\sqrt{\left(\frac{\rho_1^{-1}}{1-\gamma_2} \right)^2 \left\{ \rho_2^{-1} + \gamma_2 + \rho_1^{-1} \left(1 - \frac{\gamma_1}{1-\gamma_2} \right) \right\}^2 - 4 \left(\frac{\rho_1^{-1}}{1-\gamma_2} \right)^2 \rho_2^{-1} \left[\left(\frac{\rho_1^{-1}\gamma}{1-\gamma_2} \right) + \gamma_2 \right]}}{2 \left(\gamma \left(\frac{\rho_1^{-1}}{1-\gamma_2} \right)^2 + \gamma_2 \frac{\rho_1^{-1}}{1-\gamma_2} \right)} \\
 &= \frac{(1 + \rho_1^{-1} + \rho_2^{-1}) \frac{\rho_1^{-1}}{1-\gamma_2} - \rho_1^{-1} - \gamma_1 \left(\frac{\rho_1^{-1}}{1-\gamma_2} \right)^2 + \frac{\rho_1^{-1}}{1-\gamma_2} \left[\rho_1^{-1} \frac{\gamma}{1-\gamma_2} + \gamma_2 - \rho_2^{-1} \right]}{2 \left(\gamma \left(\frac{\rho_1^{-1}}{1-\gamma_2} \right)^2 + \gamma_2 \frac{\rho_1^{-1}}{1-\gamma_2} \right)} \\
 &= 1
 \end{aligned} \tag{5.54}$$

Applying Theorem 4.1 to $y(t) - 1$, we get,

$$\begin{aligned}
 a_1 &= p, \quad a_2 = q, \quad b_1 = -m, \quad b_2 = -r \\
 \Rightarrow \quad p + q - (-m) - (-r) &= m_1 \cdot 2\tilde{\omega} + m_2 \cdot 2\omega
 \end{aligned} \tag{5.55}$$

By (5.49), solve p, q in $(\tilde{\omega}, \tilde{\omega} + \omega]$, we get

$$m_1 = 1, \quad m_2 = 1, \quad \frac{p+q}{2} = \tilde{\omega} + \omega - \frac{m+r}{2} = \tilde{\omega} + 2m \quad (5.56)$$

(vi) By the reflection principle, $x(t) = \overline{x(\bar{t})}$, so we get $x(\bar{p}) = 1$, $x(\bar{q}) = \frac{\rho_1^{-1}}{1-\gamma_2}$, and $\bar{p}, \bar{q} \in (\tilde{\omega}, \tilde{\omega} - \omega]$. Then,

$$\begin{aligned} p + \bar{p} &= 2\tilde{\omega} = 2 \left(\frac{p+q}{2} - 2m \right) && \text{(by(v))} \\ \Rightarrow \quad \bar{p} &= q - 4m \\ q + \bar{q} &= 2\tilde{\omega} = 2 \left(\frac{p+q}{2} - 2m \right) && \text{(by(v))} \\ \Rightarrow \quad \bar{q} &= p - 4m \end{aligned} \quad (5.57)$$

Substitute $x(\bar{p}) = 1$, $x(\bar{q}) = \frac{\rho_1^{-1}}{1-\gamma_2}$ into $y(t)$, and notice that $p, q \in (\tilde{\omega}, \tilde{\omega} - \omega]$, from (5.34), we get,

$$\begin{aligned} y(\bar{p}) &= \frac{(1 + \rho_1^{-1} + \rho_2^{-1}) - \rho_1^{-1} - \gamma_1 - \sqrt{(1 + \rho_2^{-1} - \gamma_1)^2 - 4\rho_2^{-1}(\gamma + \gamma_2)}}{2(\gamma + \gamma_2)} \\ &= \frac{\rho_2^{-1}}{\gamma + \gamma_2} \\ y(\bar{q}) &= \frac{(1 + \rho_1^{-1} + \rho_2^{-1})\frac{\rho_1^{-1}}{1-\gamma_2} - \rho_1^{-1} - \gamma_1 \left(\frac{\rho_1^{-1}}{1-\gamma_2} \right)^2}{\gamma \left(\frac{\rho_1^{-1}}{1-\gamma_2} \right)^2 + \gamma_2 \frac{\rho_1^{-1}}{1-\gamma_2}} \\ &\quad - \frac{\sqrt{\left(\frac{\rho_1^{-1}}{1-\gamma_2} \right)^2 \left\{ \rho_2^{-1} + \gamma_2 + \rho_1^{-1} \left(1 - \frac{\gamma_1}{1-\gamma_2} \right) \right\}^2 - 4 \left(\frac{\rho_1^{-1}}{1-\gamma_2} \right)^2 \rho_2^{-1} \left[\rho_1^{-1} \left(\frac{\gamma}{1-\gamma_2} \right) + \gamma_2 \right]}}{2 \left(\gamma \left(\frac{\rho_1^{-1}}{1-\gamma_2} \right)^2 + \gamma_2 \frac{\rho_1^{-1}}{1-\gamma_2} \right)} \\ &= \frac{\rho_2^{-1}}{\gamma \left(\frac{\rho_1^{-1}}{1-\gamma_2} \right) + \gamma_2} \end{aligned}$$

Besides, from (vi),

$$x(p - 2m) = \frac{\rho_1^{-1}}{\rho_2^{-1}} y(p) = \frac{\rho_1^{-1}}{\rho_2^{-1}} \quad (5.58)$$

$$x(q - 2m) = \frac{\rho_1^{-1}}{\rho_2^{-1}} y(q) = \frac{\rho_1^{-1}}{\rho_2^{-1}} \quad (5.59)$$

$$y(p - 2m) = \frac{\rho_2^{-1}}{\rho_1^{-1}} x(p - 4m) = \frac{\rho_2^{-1}}{\rho_1^{-1}} x(\bar{q}) = \frac{\rho_2^{-1}}{1 - \gamma_2} \quad (5.60)$$

$$y(q - 2m) = \frac{\rho_2^{-1}}{\rho_1^{-1}} x(q - 4m) = \frac{\rho_2^{-1}}{\rho_1^{-1}} x(\bar{p}) = \frac{\rho_2^{-1}}{\rho_1^{-1}} \quad (5.61)$$

which finish the proof. $\square$

5.2.4 Analytic Continuation of $P(x(t), 0), P(0, y(t))$

Let $\mathbf{D}_x = \{t | |x(t)| < 1\}$, $\mathbf{D}_y = \{t | |y(t)| < 1\}$. The closures of these sets are $\overline{\mathbf{D}}_x = \{t | |x(t)| \leq 1\}$, $\overline{\mathbf{D}}_y = \{t | |y(t)| \leq 1\}$. Then the functions $P(x(t), 0)$ and $P(0, y(t))$ are analytic on $\mathbf{D}_x$ and $\mathbf{D}_y$ and continuous on $\overline{\mathbf{D}}_x$ and $\overline{\mathbf{D}}_y$, respectively. Let D_x, D_y be the components of $\mathbf{D}_x, \mathbf{D}_y$ that containing 0 and $2m$, respectively. The mapping rule of $x(t)$ and $y(t)$ implies that: D_x and D_y are invariant under $t \rightarrow t + 2\omega$; D_x is symmetric with respect to the real and imaginary axis, D_y is symmetric with respect to the horizontal line through $2m$ and imaginary axis.

Let $A(t) = P(x(t), 0), t \in \overline{\mathbf{D}}_x$, $B(t) = P(0, y(t)), t \in \overline{\mathbf{D}}_y$. Then $A(t), B(t)$ are analytic on $\mathbf{D}_x, \mathbf{D}_y$ and continuous on $\overline{\mathbf{D}}_x, \overline{\mathbf{D}}_y$ respectively.

Theorem 5.1 *$A(t)$ and $B(t)$ can be continued as meromorphic functions to the*

entire t -plane. The extended functions satisfy,

$$A(t) = A(-t) = A(t + 2\tilde{\omega}) \quad (5.62)$$

$$A(t - 4m) = g(t)A(t) \quad (5.63)$$

$$B(t) = u(t)A(t) \quad (5.64)$$

where,

$$f(x) = \frac{x-1}{x}, \quad g(t) = \frac{f(x(t-4m))}{f(x(t))}, \quad u(t) = -\frac{\rho_2^{-1} f(y(t))}{\rho_1^{-1} f(x(t))} \quad (5.65)$$

PROOF. By the mapping rule and the reflection principle, $\overline{\mathbf{D}}_{\mathbf{x}}$ is invariant under $t \rightarrow -t$ and $t \rightarrow t + 2\tilde{\omega}$, so we have,

$$x(t) = x(-t) = x(t + 2\tilde{\omega}) \quad (5.66)$$

Then the following equation follows,

$$A(t) = A(-t) = A(t + 2\tilde{\omega}), \quad \forall t \in \overline{\mathbf{D}}_{\mathbf{x}} \quad (5.67)$$

Similarly, by Theorem 5.1(iv) and (5.66), we know that,

$$y(4m - t) = \frac{\rho_2^{-1}}{\rho_1^{-1}} x(2m - t) = \frac{\rho_2^{-1}}{\rho_1^{-1}} x(t - 2m) = y(t) \quad (5.68)$$

which implies that,

$$y(t) = y(4m - t) = y(t + 2\tilde{\omega}) \quad (\text{by reflection principle}) \quad (5.69)$$

Then the following equation follows,

$$B(t) = B(4m - t) = B(t + 2\tilde{\omega}), \quad \forall t \in \overline{\mathbf{D}}_y \quad (5.70)$$

On the other hand, from the fundamental equation (5.2), we know that if we set $h_1(x, y) = 0$, we will get,

$$0 = h_2(x, y)P(x, 0) + h_3(x, y)P(0, y) + h_4(x, y)P(0, 0)$$

$$\begin{aligned} P(0, y) &= -\frac{h_2(x, y)}{h_3(x, y)}P(x, 0) \quad (h_4(x, y) = 0) \\ &= -\frac{\mu_2 x (y - 1)}{\mu_1 y (x - 1)}P(x, 0) \\ &= -\frac{\rho_2^{-1} y - 1}{\rho_1^{-1} y} \frac{x}{x - 1} P(x, 0) \\ &= -\frac{\rho_2^{-1} f(y)}{\rho_1^{-1} f(x)} P(x, 0) \end{aligned} \quad (5.71)$$

Then if we treat y, x as functions of t , the equation above becomes,

$$\begin{aligned} B(t) &= -\frac{\rho_2^{-1} f(y(t))}{\rho_1^{-1} f(x(t))} A(t) \\ &= u(t)A(t) \quad t \in \overline{\mathbf{D}}_x \cap \overline{\mathbf{D}}_y \end{aligned} \quad (5.72)$$

Now according to the relationship of $A(t)$ and $B(t)$ above, we can extend $A(t)$ to $\overline{D}_{x \cup y} = \overline{D}_x \cup \overline{D}_y$ by defining $A(t) = \frac{B(t)}{u(t)}$, $\forall t \in \overline{D}_y$. Then if we shift $\overline{D}_{x \cup y}$ by $4m$ denoted by $\overline{D}_{x \cup y}^{4m}$, and let $\overline{D}^*$ be the image of $\overline{D}_x \cap \overline{D}_y$ from $t \rightarrow 4m - t$, we can observe that $\overline{D}^* = \overline{D}_{x \cup y} \cap \overline{D}_{x \cup y}^{4m}$. That is,

$\forall t \in \overline{D}_y$ and $4m - t \in \overline{D}_x \cap \overline{D}_y$, the following equation holds,

$$\begin{aligned} u(t)A(t) &= B(t) = B(4m - t) = u(4m - t)A(4m - t) = u(4m - t)A(t - 4m) \\ \Rightarrow A(t - 4m) &= \frac{u(t)}{u(4m - t)}A(t) = g(t)A(t) \end{aligned} \tag{5.73}$$

Now according to the equation above, we can extend $A(t)$ to $\overline{D}_{x \cup y}^{4m}$ by defining $A(t) = \frac{A(t-4m)}{g(t)}$. Continuing using this expansion technique repeatedly, we can finally extend $A(t)$ to the whole t -plane. Then we can easily extend $B(t)$ by defining $B(t) = u(t)A(t)$. Eventually, we will have the theorem valid for all t by analytic continuation. $\square$

5.2.5 Difficulties in Finding Formulae for $P(x, 0)$, $P(0, y)$

In this section we will show the difficulties of determining the formulae for $P(x, 0)$, $P(0, y)$.

Similar as we did in the last chapter, let $A_1(t) = \frac{A(t)}{f(x(t))}$, where $f(x) = \frac{x-1}{x}$, as defined in Theorem 5.1. Then from the property of $A(t)$ shown in Theorem

5.1, we have,

$$A_1(-t) = \frac{A(-t)}{f(x(-t))} = \frac{A(t)}{f(x(t))} = A_1(t) \quad (\text{reflection principle}) \quad (5.74)$$

$$A_1(t + 2\tilde{\omega}) = \frac{A(t + 2\tilde{\omega})}{f(x(t + 2\tilde{\omega}))} = A_1(t) \quad (5.75)$$

$$A_1(t + 4m) = \frac{A(t + 4m)}{f(x(t + 4m))} = \frac{A(t)}{u(t + 4m)f(x(t + 4m))} = A_1(t) \quad (5.76)$$

Remark 5.4 *The equations above indicate that $A_1(t)$ is an even elliptic function with periods $2\tilde{\omega}, 4m$. As stated before (see section 5.2.2), $x(t)$ is an even and doubly periodic function with periods $2\tilde{\omega}, 2\omega$. Here we encounter another difficulty since there is not enough evidence to show that $A(t)$ is an even elliptic function with known periods since there is no direct relationship between m and t . This makes further analysis quite difficult and directly leads to the fact that we cannot map the points back to the x -plane and the y -plane from the t -plane.*

Here if we assume the periods of $A(t)$ are $2\tilde{\omega}, 4\omega$ as in the classical 2-demand model, we can further our study by determining the zeros and poles of $A_1(t)$ in its fundamental rectangle $\mathcal{R}_{A_1} = \{t \mid -\tilde{\omega} < \text{Ret} \leq \tilde{\omega}, -2\text{Imm} < \text{Imt} \leq 2\text{Imm}\}$.

First, we show that $A(t)$ is analytic in the strip $\mathcal{T}_1 = \{t \mid |\text{Imt}| \leq 2\text{Imm}\}$. We have already shown that $A(t)$ is analytic in the interior of $\overline{\mathbf{D}}$. Noting that $\{t \mid 0 \leq \text{Imt} \leq 2\text{Imm}\} \subset \overline{\mathbf{D}}$ and $A(-t) = A(t)$, we can conclude that $A(t)$ is analytic in $\mathcal{T}_1$.

On the other hand, by Proposition 5.1, the zeros of $g(t)$ in $\mathcal{R}$ are computed

to be $q, -m, -r$, all of which are outside $\mathbf{D}_y$. Again by this Proposition, the zeros and poles of $f(x(t))$ in $\mathbf{R}$ are $q - 4m, p$ and $\pm m$, respectively. Since $A(q - 4m) = A(p) = P(1, 0) = \frac{\rho_2^{-1}-1}{\rho_2^{-1}} > 0$, we can conclude that $q - 4m, p$ are the poles of $A_1(t)$ in $\mathbf{R}_{A_1}$. Similarly, we can determine that $\pm m$ are the zeros of $A_1(t)$ in $\mathbf{R}_{A_1}$.

Comparing with the zeros and poles of $x(t)$ and $A_1(t)$, as well as the points that $x(t) = 1$, in the fundamental rectangle of $A(t)$, $\mathbf{R}_A = \{t \mid -\tilde{\omega} < \mathbf{Re}t \leq \tilde{\omega}, -2\mathbf{Im}\omega < \mathbf{Im}t \leq 2\mathbf{Im}\omega\}$,

Functions	Periods	Zeros	Poles
$x(t)$	$2\tilde{\omega}, 2\omega$	$\{m - 2\omega, -m, m, -m + 2\omega\}$	$\{n - 2\omega, -n, n, -n + 2\omega\}$
$A_1(t)$	$2\tilde{\omega}, 4m$	$\left\{ \begin{array}{l} -5m, -3m, -m, \\ m, 3m, 5m \end{array} \right\}$	$\left\{ \begin{array}{l} p - 4m, p, p + 4m, \\ q - 4m, q, q + 4m \end{array} \right\}$
$x(t) = 1$		$\{q - 4m, p, p + 2\omega, q - 4m + 2\omega\}$	

Table 5.2: Zeros and poles of $x(t)$ and $A_1(t)$

We obtain the complete sets of zeros and poles for $A(t)$, when $\rho_1^{-1} < \rho_2^{-1}$,

When $\frac{\rho_1^{-1}}{1-\gamma_2} = \frac{\rho_2^{-1}}{1-\gamma_1}$, since $q + 2m = p + 4m, p + 6m = q + 4m$, the complete sets

Functions	Periods	Zeros	Poles
$A(t)$	$2\tilde{\omega}, 4\omega$	$\left\{ \begin{array}{l} -5m, -3m, 3m, 5m, \\ q - 4m + 2\omega, p + 2\omega \end{array} \right\}$	$\left\{ \begin{array}{l} m - 2\omega, p - 4m, q, \\ 2\omega - m, p + 4m, q + 4m \end{array} \right\}$

Table 5.3: Zeros and poles $A(t)$

of zeros and poles for $A(t)$ become $\{\pm\omega\}$ and $\{p - 4m, q\}$, respectively.

Let $\lambda(t)$ be an elliptic function with periods $2\tilde{\omega}, 4\omega$ and with the complete sets of zeros and poles $\{\tilde{\omega} \pm \omega\}$ and $\{\pm n\}$. By Theorem 4.1, as in this case $\sum a_i - \sum b_i = 2\tilde{\omega}$, we can conclude that such $\lambda(t)$ exists. From the mapping rule and Theorem 5.1(i), we know that $\lambda^2(t)$ and $x_3 - x(t)$ have identical zeros and poles. We may assume that $\lambda(t) = \sqrt{x_3 - x(t)}$, where $\sqrt{x_3 - x(t)} \geq 0$ on $[\tilde{\omega} - \omega, \tilde{\omega} + \omega]$, and $\sqrt{x_3 - x(t)} \leq 0$ on $[\tilde{\omega} + \omega, \tilde{\omega} + 3\omega]$. Note that $-5m, 5m, m - 2\omega, 2\omega - m, p - 4m, q \in [\tilde{\omega} - \omega, \tilde{\omega} + \omega]$ and $q - 4m + 2\omega, q + 4m, p + 4m, p + 2\omega \in [\tilde{\omega} + \omega, \tilde{\omega} + 3\omega]$, by Theorem 5.1, we conclude that,

$$\lambda(-5m) = \lambda(5m) = \sqrt{x_3 + \frac{\gamma_2}{\gamma}} \quad (5.77)$$

$$\lambda(-m) = \lambda(m) = \sqrt{x_3} \quad (5.78)$$

$$\lambda(p - 4m) = \lambda(q) = \sqrt{x_3 - \frac{\rho_1^{-1}}{1 - \gamma_2}} \quad (5.79)$$

$$\lambda(q - 4m + 2\omega) = \lambda(p + 2\omega) = -\sqrt{x_3 - 1} \quad (5.80)$$

$$\lambda(p + 4m) = -\sqrt{x_3 - \kappa_1} \quad (5.81)$$

$$\lambda(q + 4m) = -\sqrt{x_3 - \kappa_2} \quad (5.82)$$

where $\kappa_1 \in [\frac{\rho_1^{-1}}{\rho_2^{-1}}, \frac{\rho_1^{-1}}{1 - \gamma_2}]$, $\kappa_2 \in [\frac{\rho_1^{-1}}{\rho_2^{-1}}, 1]$.

Remark 5.5 Here we have another problem of mapping the zeros and poles back to the x -plane and y -plane from the t -plane. In fact, as we cannot determine the exact periods of $A(t)$ and do not have enough information about the relationship between r , m and ω , we can hardly solve the queueing problem via generating function method successfully.

Similarly, comparing with the zeros and poles of $x(t)$, $y(t)$ and $A(t)$, as well as the points that $x(t) = 1$ and $y(t) = 1$, in the fundamental rectangle of $B(t)$,

$$\mathbf{R}_B = \{t \mid -\tilde{\omega} < \mathbf{Re}t \leq \tilde{\omega}, \mathbf{Im}(-2\omega + 2m) < \mathbf{Im}t \leq \mathbf{Im}(2\omega + 2m)\},$$

Functions	Periods	Zeros	Poles
$x(t)$	$2\tilde{\omega}, 2\omega$	$\{-m, m, -m + 2\omega, m + 2\omega\}$	$\{-n, n\}$
$y(t)$	$2\tilde{\omega}, 2\omega$	$\{n - 2\omega, m, n, m + \omega\}$	$\{-m, -r, -m + 2\omega, -r + 2\omega\}$
$A(t)$	$2\tilde{\omega}, 4\omega$	$\left\{ \begin{array}{c} -n, n, 5m \\ q - 4m + 2\omega, p + 2\omega \end{array} \right\}$	$\left\{ \begin{array}{c} 2\omega - m, q, p + 4m, \\ p - 4m + 4\omega, q + 4m, p + 8m \end{array} \right\}$
$x(t) = 1$ $y(t) = 1$		$\{q - 4m, p, p + 2\omega, q - 4m + 2\omega, q - 4m + 4\omega\}$ $\{p, p + 2\omega, q, q + 2\omega\}$	

Table 5.4: Zeros and poles of $x(t)$, $y(t)$ and $A(t)$

We obtain the complete sets of zeros and poles for $B(t)$, when $\rho_1^{-1} < \rho_2^{-1}$,

Functions	Periods	Zeros	Poles
$B(t)$	$2\tilde{\omega}, 4\omega$	$\left\{ \begin{array}{c} -m, -r, p + 2\omega, \\ 5m, q + 2\omega \end{array} \right\}$	$\left\{ \begin{array}{c} p + 4m, p - 4m + 4\omega \\ q + 4m, q - 4m + 4\omega, n - 2\omega \end{array} \right\}$

Table 5.5: Zeros and poles of $B(t)$

When $\rho_1^{-1} = \rho_2^{-1}$, since $q + 6m = p + 8m, p + 6m = q + 4m$, the complete sets of zeros and poles for $B(t)$ become $\{-m, 5m\}$ and $\{p + 4m, q + 8m\}$, respectively.

Let $\mu(t)$ be an elliptic function with periods $2\tilde{\omega}, 4\omega$ and with the complete sets of zeros and poles $\{\tilde{\omega} - m, \tilde{\omega} + 5m\}$ and $\{-m, -r\}$. By Theorem 4.1, as in this case $\sum a_i - \sum b_i = 2\tilde{\omega}$, we can conclude that such $\mu(t)$ exists. From the mapping rule and Theorem 5.1(i), we know that $\mu^2(t)$ and $y_3 - y(t)$ have identical zeros and poles. We may assume that $\mu(t) = \sqrt{y_3 - y(t)}$, where $\sqrt{y_3 - y(t)} \geq 0$ on $[\tilde{\omega} - m, \tilde{\omega} + 5m]$, and $\sqrt{y_3 - y(t)} \leq 0$ on $[\tilde{\omega} + 5m, \tilde{\omega} + 11m]$. Note that $5m \in [\tilde{\omega} - m, \tilde{\omega} + 5m]$ and

$p + 4m, q + 4m, p + 2\omega, q + 2\omega, p - 4m + 4\omega, q - 4m + 4\omega \in [\tilde{\omega} + 5m, \tilde{\omega} + 11m]$, by Theorem 5.1, we conclude that,

$$\mu(p + 2\omega) = \mu(q + 2\omega) = -\sqrt{y_3 - 1} \quad (5.83)$$

$$\mu(p - 4m + 4\omega) = -\sqrt{y_3 - \frac{\rho_2^{-1}}{\gamma \frac{\rho_1^{-1}}{1-\gamma_2} + \gamma_2}} \quad (5.84)$$

$$\mu(q - 4m + 4\omega) = -\sqrt{y_3 - \frac{\rho_2^{-1}}{1 - \gamma_1}} \quad (5.85)$$

$$\mu(p + 4m) = -\sqrt{y_3 - \kappa'_1} \quad (5.86)$$

$$\mu(q + 4m) = -\sqrt{y_3 - \kappa'_2} \quad (5.87)$$

Remark 5.6 *The study of the generalized 2-demand model ends here because of the uncertainty and insufficient information. However, in future research, we can apply other methods and techniques to overcome these difficulties we listed in the thesis, such as the techniques by Wright in [18].*

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