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It is clear that $a > 0$ if and only if $a \in M$. Similarly, $a < 0$ if and only if $-a \in M$. The five postulates in Definition 1.2.2 can then be restated as follows:

- (B1) If $a > 0$, then $a = a^*$
- (B2) $1 > 0$
- (B3) If $a > 0$ and $b > 0$, then $a+b > 0$
- (B4) If $a > 0$ and $\rho \in R$, then $\rho a \rho^* \geq 0$
- (B5) If $a = a^*$, then exactly one of the following holds:
 $a > 0$, $a = 0$, $a < 0$.

Definition 1.2.4. By a Baer ordering we mean any order relation (notation: $a \geq b$) such that postulates (B1)-(B5) hold.

We notice that the ordering $a \geq b$ is a Baer ordering if and only if the positive elements M of this ordering form a Baer set.

Definition 1.2.5. Let R be any Baer ordered $*$ -ring with ordering $\geq$ and let $a = a^*$. The absolute value of a , written $|a|$, is a if $a \geq 0$ and $-a$ otherwise.

Here are two examples of Baer ordered division rings.

Example 1.2.6. The real numbers R with the identity mapping as involution, the complex numbers C with complex conjugation as involution, and the real quaternions H with the conjugate transpose as involution can all be Baer ordered by the Baer ordering induced by the set $M = R^+$ of all positive real numbers.

BAER ORDERED $\star$ -RINGS

by

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A thesis submitted to the Faculty of
Graduate Studies and Research in Partial
fulfillment of the requirements for the degree of

Doctor of Philosophy

Department of Mathematics and Statistics

Carleton University

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July 1981

The undersigned recommend to the Faculty of Graduate Studies
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submitted by Long-Shung Shiao, B.Sc., M.Sc., in partial fulfillment
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ABSTRACT

Let R be a $\star$ -ring (i.e. an associative ring with unity element 1 equipped with an involution $\star$). An order relation $>$ on R is called a Baer ordering if $>$ satisfies the following conditions:

- (B1) $a > 0$ implies $a = a^\star$,
- (B2) $1 > 0$,
- (B3) $a > 0$ and $b > 0$ imply $a+b > 0$,
- (B4) If $a > 0$ and $p \in R$ then $pap^\star > 0$,
- (B5) For any $a = a^\star$, exactly one of the following holds:
 $a > 0$ or $a = 0$ or $a < 0$.

Any $\star$ -ring R together with a Baer ordering $>$ is called a Baer ordered $\star$ -ring. Let $V = V(R)$ denote the set of all elements $b \in R$ such that $bb^\star \leq r \cdot 1$ for some positive rational number r . R is said to be bounded if $R = V(R)$. Let $J = J(R)$ denote the set of elements $j \in R$ such that $jj^\star < r \cdot 1$ for each positive rational number r . R is said to be archimedean if $J \neq J(R) = \{0\}$.

We established the following involutory versions of Albert's Theorems:

Theorem 1: Let R be any Baer ordered $\star$ -division ring. Suppose that all skew elements of R are algebraic over an archimedean central subfield F . Then R is isomorphic to a $\star$ -subdivision ring of the real numbers R , the complex numbers C , or the real quaternions H .

Theorem 2. Let R be any Baer ordered $\star$ -domain with bounded archimedean $\star$ -center. If the symmetric elements of R satisfy a nontrivial polynomial identity then the division ring Q of quotients of R is at most 4-dimensional over its center $Z(Q)$.

Generalizing Holland's characterization theorem of archimedean Baer ordered $\star$ -division rings, we have the following

Theorem 3. Let R be a Baer ordered $\star$ -ring. Then R is archimedean and bounded if and only if R is isomorphic to a $\star$ -subring of R , C , or H .

We believe that a result concerning lifted Baer orderings due to Holland is probably true, but that there is a gap in the argument. We filled the gap and gave a complete proof of his result in the case where the residue $\star$ -division ring is a commutative field. We proved

Theorem 4. Let R be a $\star$ -division ring with smooth $\star$ -valuation w and residue $\star$ -field V_0 . Then every Baer ordering of V_0 can be lifted to R .

Finally, inspired by O.F.G. Schilling's construction of division rings we construct a new family of Baer ordered $\star$ -division rings with the prescribed Baer ordered residue $\star$ -field and value group under certain revised conditions on factor sets and certain conditions on the ground involution of the residue $\star$ -field.

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Lastly, but not least, I wish to thank my wife, Lee-Huey, and my son, Chang-Chi, whose love and understanding made it possible for me to finish my study.

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INTRODUCTION

The classical Hilbert ordering can be used to order the real numbers $\mathbb{R}$ but can not be used to order either the complex numbers $\mathbb{C}$ or the real quaternions $\mathbb{H}$. However, if we view $\mathbb{C}$ as a field with complex conjugation as involution (i.e. anti-automorphism of period 2) and view $\mathbb{H}$ as a division ring with transpose conjugate as involution, then it is clear that the symmetric elements of $\mathbb{C}$ or $\mathbb{H}$ are the real numbers $\mathbb{R}$ which can be totally ordered. The total ordering on the symmetric elements is of great interest to us because it seems more useful than Hilbert ordering in the area of rings endowed with an involution.

Any associative ring R with unity element 1 and equipped with an involution will be called a $\star$ -ring. Recall that a Baer ordering in a $\star$ -ring is an order relation $\geq$ on R such that

- (B1) $a > 0$ implies $a = a^\star$,
- (B2) $1 > 0$,
- (B3) $a > 0$ and $b > 0$ imply $a+b > 0$,
- (B4) If $a > 0$ and $p \in R$ then $pap^\star \geq 0$,
- (B5) For any $a = a^\star$, exactly one of the following holds:

$$a > 0 \text{ or } a = 0 \text{ or } a < 0.$$

The notion of Baer ordering was essentially first introduced by R. Baer in his book "Linear Algebra and Projective Geometry" [7, Appendix 1, Chapter IV]. But this subject did not develop further

until recent years, when two papers by Professor S.S. Holland (Jr.) ([20],[21]) appeared. Also, we would like to mention here that in a parallel direction Professor M. Chacron has been working on $\star$ -division rings whose symmetric elements are totally ordered by a so-called c -ordering (see [11], [12]).

Any $\star$ -ring R together with a Baer ordering is called a Baer ordered $\star$ -ring. The most immediate examples of Baer ordered $\star$ -rings are R , C , and H as mentioned above by setting $a > b$ if and only if $a-b$ is a positive real number. If a Baer ordered $\star$ -division ring R happens to be a commutative field with identity involution then R is a q -ordered field in the sense of A. Prestel [29]. The postulates of q -ordering differ from those of Hilbert ordering in requiring $\rho^2 a > 0$ for $a > 0$ and $0 \neq \rho \in R$ in place of requiring $ab > 0$ for $a > 0$ and $b > 0$. Thus every Hilbert ordered field is a q -ordered field and hence a Baer ordered field. This close relationship to Hilbert-ordering and q -ordering motivates our study of Baer ordered $\star$ -rings in this thesis.

Let R be a Baer ordered $\star$ -ring with Baer ordering $\geq$. An element $b \in R$ is said to be bounded if $bb^* \leq r \cdot 1$ for some $r \in \mathbb{Q}^+$ (the positive rational numbers). If $bb^* < r \cdot 1$ for each $r \in \mathbb{Q}^+$, we call b an infinitesimal. Let $V = V(R)$ denote the set of all bounded elements of R and let $J = J(R)$ denote the set of all infinitesimals of R . A Baer ordered $\star$ -ring R is said to be archimedean if $J = J(R) = \{0\}$.

In Chapter I of this dissertation we first give a systematic account of the structure of Baer ordered $\star$ -rings. We show that in any Baer ordered $\star$ -ring, V is a $\star$ -closed subring of R and J is a proper semiprime $\star$ -ideal in V . Also, we observe that $sk - ks \in J$ for any $s = s^* \in V$ and $k = -k^* \in V$. From this observation we derive that if R is bounded and archimedean then all symmetric elements of R are central. Concerning the quotient $\star$ -ring V/J , we prove that V/J can be Baer ordered by an induced Baer ordering of the ground ordering, and under such an ordering V/J is bounded and archimedean. When the $\star$ -ring R is a division ring, we have the following much stronger result:

Proposition 1.3.13. Let R be any Baer ordered $\star$ -division ring. Then the bounded subring V of R is a right valuation ring in the sense of I. Kaplansky [25] (i.e. for any a and b in V either $aR \subseteq bR$ or $bR \subseteq aR$) having $J = J(R)$ as its unique maximal $(\star)$ -ideal and hence $J = \text{Rad}(V)$, the Jacobson radical of V . Thus V/J is a Baer ordered $\star$ -division ring, which is archimedean.

An alternative form of Holland's characterization theorem of archimedean Baer ordered $\star$ -division rings [20, Theorem 2] is reviewed in

Theorem 1.3.16 (Holland). Any bounded (and hence archimedean) Baer ordered $\star$ -division ring R is $\star$ -isomorphic and order isomorphic to a $\star$ -division subring of the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, or the real quaternions $\mathbb{H}$.

We generalize Holland's result from division rings to simple rings in

Theorem 1.3.20. Let R be any bounded Baer ordered simple $\ast$ -ring. Then R is as in the conclusion of Theorem 1.3.16.

Any subring A of a ring R is said to be a right order subring if for any $x \in R$ there exist $a, b \in A$ such that $x = ab^{-1}$. S. Berberian found a very intimate relationship between any Baer ordered $\ast$ -ring R and its bounded subring $V(R)$ as follows.

Corollary 1.4.2 ([16]). Let R be any Baer ordered $\ast$ -ring. Then the bounded $\ast$ -subring $V = V(R)$ is a right and a left order subring in R provided $1 + xx^*$ is invertible for each $x \in R$.

We show that when R is restricted to a $\ast$ -division ring, and $S(R)$ is not contained in the center $Z(R)$ of R , the Corollary 1.4.2 can be strengthened to the following

Theorem 1.4.5. Let R be any Baer ordered $\ast$ -division ring such that $S(R) \not\subseteq Z(R)$. Then the subring $\overline{S(V)}$ of R generated by all bounded symmetric elements of R is a two-sided order subring in R .

In his paper [2], A. Albert has proved that any Hilbert ordered algebraic division ring is a field. We give the following skew version of Albert's result in

Theorem 1.5.1. Let R be any Baer ordered $\ast$ -division ring. Suppose that all skew elements of R are algebraic over an archimedean

central subfield F . Then R is as in the conclusion of Theorem 1.3.16.

It is known that every Hilbert ordering of a left Ore domain can be extended uniquely to a Hilbert ordering of the left quotient ring. It is quite natural to seek the analogous result for rings with involution. We have shown the following result, which was stated with a hint for a formal proof in Holland's paper [21].

Theorem 1.5.2. If R is any Baer ordered Ore $\ast$ -domain then the ordering can be extended uniquely to a Baer ordering on the $\ast$ -division ring Q of quotients of R .

In another paper [3], Albert has shown that if a Hilbert ordered ring R satisfies a nontrivial polynomial identity then R is commutative. We prove

Theorem 1.5.3. Let R be any Baer ordered $\ast$ -domain with bounded archimedean $\ast$ -center. If the symmetric elements of R satisfy a nontrivial polynomial identity then the division ring Q of quotients of R is at most 4-dimensional over its center $Z(Q)$ and hence $S(R) \subseteq Z(R)$.

We have already described the structure of bounded (hence archimedean) Baer ordered $\ast$ -division rings and bounded Baer ordered simple $\ast$ -rings. It is natural to try to generalize this result to bounded archimedean Baer ordered $\ast$ -rings. We devote our Chapter II to obtaining

the following characterization theorem.

Theorem 2.2.3. Let R be a Baer ordered $\ast$ -ring. Then R is archimedean and bounded if and only if R is isomorphic to a $\ast$ -subring of $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.

In Chapter III we turn our attention to $\ast$ -valuations and lifted Baer orderings on $\ast$ -division rings. Recall that in a $\ast$ -division ring R , a mapping w of $R - \{0\}$ into a totally ordered group G (written additively) is said to be a $\ast$ -valuation if w satisfies the following conditions:

- (V1) $w(xy) = w(x) + w(y)$ for $x, y \in R - \{0\}$,
- (V2) $w(x+y) \geq \min \{w(x), w(y)\}$ for $x, y, x+y \in R - \{0\}$,
- (V3) w is surjective,
- (V4) $w(x^\ast) = w(x)$ for $x \in R - \{0\}$.

The group G is called the value group of w . The set $V_w = \{x \in R \mid w(x) \geq 0\} \cup \{0\}$ is a $\ast$ -closed subring of R and $J_w = \{x \in R \mid w(x) > 0\} \cup \{0\}$ is the unique maximal $\ast$ -ideal of V_w . The $\ast$ -division ring V_w/J_w is called the residue $\ast$ -division ring of R with respect to w .

In a recent paper [21], Holland has shown that given a $\ast$ -division ring R , any Baer ordering on the residue $\ast$ -division ring V_w/J_w of R with respect to the $\ast$ -valuation w can be lifted to the division ring R itself under an extra assumption on the $\ast$ -valuation (smooth $\ast$ -valuation). The result as stated in [21, Lemma 3.1] is probably true, however, we believe that there is a gap in the argument. We

fill the gap and give a complete proof of Holland's lemma [21, Lemma 3.1], his theorem [21, Theorem 3.2], and his corollary [21, Corollary 3.3] in the case where the residue $\ast$ -division ring is a commutative field. Of course, we make the same assumption on the $\ast$ -valuation, that is, w is a smooth $\ast$ -valuation in the sense of Holland. Our proofs are essentially inspired from Holland's original work [21].

Further, when the Baer ordering in the residue $\ast$ -field is archimedean, we have

Theorem 3.5.11. Let R be a $\ast$ -division ring with smooth $\ast$ -valuation w . Let V be the $\ast$ -valuation ring of w and let J be the maximal ideal of V . If the residue $\ast$ -division ring $V_0 = V/J$ is a field and if $\geq_0$ is an archimedean Baer ordering in V_0 , then V is precisely the bounded subring of R and J is precisely the totality of infinitesimals with respect to the lifted ordering $\geq$ of $\geq_0$.

Finally, inspired from O.F.G. Schilling's construction of division ring [33] we construct a new family of Baer ordered $\ast$ -division rings with the prescribed Baer ordered residue $\ast$ -field and value group under certain revised conditions on factor sets and certain conditions on the ground involution of the residue $\ast$ -field.

CHAPTER I

BAER ORDERED $\star$ -DOMAINS

§1.1 Introduction

In his paper [2] A. Albert showed that any ordered algebraic division ring is a field and in [3] he proved that an ordered ring satisfying a nontrivial polynomial identity must be commutative. Our main objective in this chapter will be to establish involutory versions of these theorems. In §1.2 we introduce the basic concepts and some examples. In §1.3 we investigate the structure of the bounded subring associated with a given Baer ordering and in §1.4 we discuss the ring of quotients of a Baer ordered domain. Finally, in §1.5 we offer the promised versions of Albert's theorems.

§1.2 Definitions, examples and basic properties

Definition 1.2.1. By a $\star$ -ring R we mean an associative ring with unity element 1, which is equipped with an involution $\star: x \rightarrow x^*$ (i.e. an anti-automorphism whose square is the identity mapping).

recall that an element a in R is called symmetric if $a = a^*$, and the set of all symmetric elements is denoted by $S = S(R)$. An element b in R is called skew if $b^* = -b$ and the set of all skew elements is denoted by $K = K(R)$. Let Q , Q^+ , and N denote, respectively, the rational numbers, the positive rational numbers, and the positive integers.

Convention. Throughout this work we will assume the prime subring of any $\star$ -ring R to be a field.

We now introduce the key notion of a Baer set according to the following definition:

Definition 1.2.2. Let R be any $\star$ -ring. By Baer set of R we mean a subset M of R having the following properties:

- (B1) $M \subseteq S(R)$
- (B2) $1 \in M$ but $0 \notin M$.
- (B3) If $a, b \in M$, then $a+b \in M$ (closed under addition)
- (B4) If $a \in M$ and $\rho \in R$, then $\rho a \rho^* \in M \cup \{0\} = M^\#$
- (B5) For each element a of $S(R)$ exactly one of the following holds:
 $a \in M$, $a = 0$, $-a \in M$ ($\star$ -trichotomy law).

The elements of M are called the positive elements of $S = S(R)$. The nonzero elements of $S(R)$ that are not in M are called the negative elements of $S(R)$.

The difference between our definition of Baer set and the one introduced first by S.S. Holland (Jr.) in the context of division rings is slight (see [20, p.207]). For any domain R , these two definitions will coincide.

Definition 1.2.3. Let R be a $\star$ -ring with a Baer set M . Given $a, b \in R$, we define:

$$a \geq b \iff a-b \in M \cup \{0\} = M^\#.$$

It is clear that $a > 0$ if and only if $a \in M$. Similarly, $a < 0$ if and only if $-a \in M$. The five postulates in Definition 1.2.2 can then be restated as follows:

- (B1) If $a > 0$, then $a = a^*$
- (B2) $1 > 0$
- (B3) If $a > 0$ and $b > 0$, then $a+b > 0$
- (B4) If $a > 0$ and $\rho \in R$, then $\rho a \rho^* \geq 0$
- (B5) If $a = a^*$, then exactly one of the following holds:
 $a > 0$, $a = 0$, $a < 0$.

Definition 1.2.4. By a Baer ordering we mean any order relation (notation: $a \geq b$) such that postulates (B1)-(B5) hold.

We notice that the ordering $a \geq b$ is a Baer ordering if and only if the positive elements M of this ordering form a Baer set.

Definition 1.2.5. Let R be any Baer ordered $*$ -ring with ordering $\geq$ and let $a = a^*$. The absolute value of a , written $|a|$, is a if $a \geq 0$ and $-a$ otherwise.

Here are two examples of Baer ordered division rings.

Example 1.2.6. The real numbers R with the identity mapping as involution, the complex numbers C with complex conjugation as involution, and the real quaternions H with the conjugate transpose as involution can all be Baer ordered by the Baer ordering induced by the set $M = R^+$ of all positive real numbers.

Each example given above is a Baer ordered $\star$ -division ring of finite dimension over its center. We now present a family of Baer ordered $\star$ -division rings due to Cohn and Holland, which are transcendental extensions of their centers (see [20, p.215-218]).

Example 1.2.7 (Cohn-Holland). Let Z_0 be an arbitrary formally real pythagorean field (i.e. non-zero squares add up to non-zero squares). Let $Z = Z_0[i]$ be a quadratic extension of Z_0 obtained by adjoining a square root i of -1 . Denote by $(\cdot)^*$ the Galois automorphism of Z , $(a + bi)^* = a - bi$, where $a, b \in Z_0$. Let $Z((x))$ be the field of formal Laurent series in the indeterminate x with coefficients in Z . A typical non-zero element in $Z((x))$ will be

$$f(x) = \sum_{n=0}^{\infty} a_n x^{n+k}, \quad a_n \in Z,$$

where k is an integer (positive, negative, or zero) and $a_0 \neq 0$. Define the involution on $Z((x))$ by setting

$$f(x) = \sum_{n=0}^{\infty} a_n^* x^{n+k}.$$

Let K be the ring of formal Laurent series in the indeterminate y with coefficients in $Z((x))$, where these coefficients are written on the left. A typical non-zero element in K will be

$$g(y) = \sum_{n=0}^{\infty} f_n(x) y^{n+q}, \quad f_n(x) \in Z((x)),$$

where q is an integer (positive, negative, or zero) and $f_0(x) \neq 0$.

We add elements in K componentwise, and the multiplication is subjected

to the commutation rule $xy^{-1} - y^{-1}x = i$ and its consequences. Define the involution on K by setting $g(y)^* = \sum_{n=0}^{\infty} y^{n+q} f_n(x)^*$. As shown by Holland [20, p.215-218], K is a Baer ordered $*$ -division ring.

We now proceed to some basic facts.

Remark 1.2.8. For a Baer ordered $*$ -ring R with ordering $\geq$, R is a domain (i.e. has no nonzero divisors of zero) if and only if $a > 0$, $0 \neq p \in R$ implies $pap^* > 0$.

Justification. If R is a domain, given $s > 0$ and $x \neq 0$ we have $xsx^* \neq 0$, which gives $xsx^* > 0$.

Conversely, if $xy = 0$ then we may assume $y \neq 0$ and hence $yy^* > 0$ by hypothesis. The equality $xyy^*x^* = 0$ will force $x = 0$ and so R must be a domain.

The following Remarks (1.2.9)-(1.2.13) were essentially given by Holland [20].

Remark 1.2.9. Let R be a Baer ordered $*$ -ring. Then R must be of characteristic 0.

Justification. By postulate (B2), $\underbrace{1+1+\dots+1}_{n \text{ times}} > 0$ for any positive integer n , so R is of characteristic 0.

Remark 1.2.10. Let R be a Baer ordered $*$ -ring. If a is an invertible element in R , then $a > 0 \Rightarrow a^{-1} > 0$.

Justification. $a^{-1} = a^{-1}a(a^{-1})^* \geq 0$ and $a^{-1} \neq 0$ give us $a^{-1} > 0$.

Remark 1.2.11. If a is invertible in the Baer ordered $\ast$ -ring R , then $a > 1 \Rightarrow a^{-1} < 1$.

Justification. $1 - a^{-1} = (1 - a^{-1})^2 + a^{-1}(a - 1)a^{-1} \geq 0$ and so $a^{-1} \leq 1$. But $a^{-1} \neq 1$, we have $a^{-1} < 1$.

Remark 1.2.12. Let R be a Baer ordered $\ast$ -ring and $a \in R$. If $0 < a < 1$ then $0 < \dots \leq a^{m+1} \leq a^m \leq \dots \leq a^2 \leq a < 1$.

Justification. $a > 0$ and $1 - a > 0$ imply $a(1 - a) = a(1 - a)[a + (1 - a)] = a(1 - a)a + (1 - a)a(1 - a) \geq 0$; or $0 \leq a^2 \leq a < 1$. From the last inequality, one can easily derive $0 < \dots \leq a^{m+1} \leq a^m \leq \dots \leq a^2 \leq a < 1$.

Remark 1.2.13. Let $\frac{m}{n}$ be a positive rational number and let R be a Baer ordered $\ast$ -ring. Then a is positive in R if and only if $\frac{m}{n}a$ is positive in R .

Justification. If $a > 0$, then $\frac{m}{n}a = \frac{1}{n}(mna) \frac{1}{n} > 0$. Conversely, if $\frac{m}{n}a > 0$ then $a = \frac{n}{m}(\frac{m}{n}a) > 0$.

Remark 1.2.14. Let R be any Baer ordered $\ast$ -ring and let $a \in S(R)$, $q \in Q^+$. Then $a^2 \leq q^2 \Leftrightarrow -q \leq a \leq q$.

Justification. Suppose $a^2 \leq q^2$. Then by Vidav's equality [33, equation (7)]

$$q - a = \frac{1}{2q}(q - a)^2 + \frac{1}{2q}(q^2 - a^2),$$

hence $q - a$ must be positive or zero. Replacing a by $-a$ gives $q + a \geq 0$, so $-q \leq a \leq q$.

Conversely, $q+a \geq 0$ and $q-a \geq 0$ give $(q-a)(q+a)[(q-a)+(q+a)] = (q-a)(q+a)(q-a) + (q+a)(q-a)(q+a) \geq 0$ or $(q^2 - a^2)2q \geq 0$ and so $q^2 - a^2 \geq 0$ by Remark 1.2.13.

§1.3 The bounded elements

The notion of bounded element was first studied by Vidav in the context of $\star$ -regular rings [34, p.75] and later by Berberian in the context of finite Baer $\star$ -rings [8, p.243]. In this section we will study the bounded elements in Baer ordered $\star$ -rings.

Definition 1.3.1. In any Baer ordered $\star$ -ring R with Baer set M an element $b \in R$ is said to be bounded if $bb^* \leq r \cdot 1$ for some $r \in Q^+$. If, further, $bb^* < r \cdot 1$ for each $r \in Q^+$, then call b an infinitesimal.

Definition 1.3.2. Let $V = V(R)$ be the set of all bounded elements of R . Let $J = J(R)$ be the set of all infinitesimals of R .

Definition 1.3.3. A Baer ordered $\star$ -ring R is said to be archimedean if $J = \{0\}$.

Now we begin with

Theorem 1.3.4. Let R be any Baer ordered $\star$ -ring. Then the set V of all bounded elements of R is a $\star$ -closed subring (briefly, $\star$ -subring) of R and the set J of all infinitesimals is a proper $\star$ -ideal of V (i.e. a two-sided ideal closed under involution). In fact, J is a semiprime ideal in V .

Proof. Given any two bounded elements, $a, b \in R$, using the equality $(a+b)(a+b)^* + (a-b)(a-b)^* = 2(aa^* + bb^*)$, we get that $(a+b)(a+b)^* \leq (a+b)(a+b)^* + (a-b)(a-b)^* = 2(aa^* + bb^*) \leq 2(q_1 + q_2) \cdot 1$ whenever $aa^* \leq q_1 \cdot 1$ and $bb^* \leq q_2 \cdot 1$ with $q_1, q_2 \in Q^+$. Also, $(ab)(ab)^* = abb^*a^* \leq q_2 aa^* \leq q_1 q_2 \cdot 1$. This shows that V is a subring of R . If $c \in V$ then $cc^* \leq q \cdot 1$ for some $q \in Q^+$. Since $c^*(q - cc^*)c \geq 0$ by (B4) and $(c^*c - q \cdot 1)^2 \geq 0$, using Berberian's identity [8, p.243]

$$q(q \cdot 1 - c^*c) = c^*(q - cc^*)c + (c^*c - q \cdot 1)^2,$$

we obtain $q(q \cdot 1 - c^*c) \geq 0$. By Remark 1.2.13, $q - c^*c \geq 0$. Thus $c^* \in V$. Therefore V is a $\star$ -subring of R . A similar argument shows that J is a $\star$ -ideal of V , and J is a proper ideal of V simply because $1 \in V$ but $1 \notin J$.

It remains to show that J is a semiprime ideal. Suppose $aVa \subseteq J$ with $a \in V$. Then $aa^*a \in J$ and so $aa^*aa^* \in J$. This gives us $aa^* \in J$, which implies $a \in J$. The proof is now complete.

Actually, we can show the following

Theorem 1.3.5. If R is a Baer ordered $\star$ -ring, then the $\star$ -ideal J is a completely prime ideal of V .

Proof. Given any $a \in V - J$ and $b \in V - J$, then $aa^* \not\leq \frac{1}{n} \cdot 1$ for some $n \in \mathbb{N}$ and $bb^* \not\leq \frac{1}{m} \cdot 1$ for some $m \in \mathbb{N}$. By the trichotomy law, we must have $aa^* \geq \frac{1}{n} \cdot 1$ and $bb^* \geq \frac{1}{m} \cdot 1$. From this we get that $(ab)(ab)^* = abb^*a^* \geq \frac{1}{m} aa^* \geq \frac{1}{mn}$ (from now on we will simply write $\frac{1}{n}$

for $\frac{1}{n} \cdot 1$), which shows $ab \in J$. Therefore, J is a completely prime ideal of V .

Remark 1.3.6. In the course of proving Theorem 1.3.4 we have shown something more, that is, $aa^* \leq q$ if and only if $a^*a \leq q$ where $q \in Q^+$.

In [20], Holland used the inequalities $(s+k)(s+k)^* \geq 0$ and $(s-k)(s-k)^* \geq 0$ to derive that

$$(*) \quad |[s,k]| \leq s^2 - k^2 \text{ for } s \in S(R) \text{ and } k \in K(R),$$

where $[s,k]$ denotes the Lie product $sk - ks$.

In [11], M. Chacron has shown that $[s,k]$ is an infinitesimal for any bounded symmetric s and skew k in the context of c -ordered division rings. Let us now show the analogous result in the case of Baer ordered $\star$ -rings.

Theorem 1.3.7. Let R be a Baer ordered $\star$ -ring. Let s and k be, respectively, a bounded symmetric and a bounded skew element of R . Then $[s,k] \in J = \tilde{J}(R)$.

Proof. Let $x = s+k$. Then x is bounded since s and k are bounded. To show $[s,k] \in J$ is equivalent to showing $[x^*,x] \in J$, since $[x^*,x] = 2[s,k]$. Suppose $[x^*,x] \notin J$. Then, by the trichotomy law, $[x^*,x]^2 \geq q^2$ for some $q \in Q^+$. From Remark 1.2.14 it follows that $|[x^*,x]| \geq q$; that is, $|x^*x - xx^*| \geq q$. Without loss of generality we may assume $x^*x - xx^* \geq q$. The latter inequality gives us

$x^*x \geq q + xx^* \geq q$. By Remark 1.3.6 we have $xx^* \geq q$ as well and thus $x^*x \geq q + xx^* \geq 2q$. Using an inductive argument, we have $xx^* \geq nq$ for any $n \in \mathbb{N}$, contrary to the boundedness of x . Therefore $[x^*, x] \in J$ and hence $[s, k] \in J$, completing the proof.

Corollary 1.3.8. Let R be any Baer ordered $\ast$ -ring. Suppose R is bounded and archimedean. Then all symmetric elements are central.

Proof. Since R is bounded, $[s, k]$ is an infinitesimal for any $s \in S(R)$ and $k \in K(R)$ by Theorem 1.3.7. Since R is archimedean, $[s, k]$ must then be zero. For each x in R , write x as $s_1 + k_1$ with $s_1 \in S(R)$ and $k_1 \in K(R)$. Given any symmetric element s in R , $[s, [s, x]] = [s, [s, s_1]] + [s, [s, k_1]] = 0 + 0 = 0$. By Herstein's sublemma [18, p.5], s is central in R .

Now we propose to define on the quotient $\ast$ -ring V/J a Baer set, which is induced by the ground ordering.

Theorem 1.3.9. In any Baer ordered $\ast$ -ring R with Baer set M , the quotient $\ast$ -ring $\bar{V} = V/J$ carries a Baer ordering (called the quotient Baer ordering) whose Baer set is $\bar{M} = \{\bar{a} = a+J \mid a \in M \text{ and } a+J = b+J \text{ for some } b \in M\}$. Under this ordering $\bar{V}$ is bounded and archimedean.

Proof. We check postulates (B1)-(B5):

(B1) $\bar{M} \subseteq S(\bar{V})$ is obvious.

(B2) $\bar{1} \in \bar{M}$ and $\bar{0} \notin \bar{M}$ follow directly from the definition of $\bar{M}$.

(B3) Given $\bar{x}, \bar{y} \in \bar{M}$, $\bar{x} = \bar{a}$, $\bar{y} = \bar{b}$ for some $a, b \in M-J$. Thus $\bar{x} + \bar{y} = \overline{x+y} = \bar{a} + \bar{b} = \overline{a+b}$ and $a+b \in M$. We claim that $x+y \notin J$,

or equivalently, $a+b \notin J$. To see this note that $a, b \notin J$ implies $a^2 \geq q_a^2$ and $b^2 \geq q_b^2$ for some $q_a, q_b \in Q^+$, so $a \geq q_a$ and $b \geq q_b$. Therefore $a+b \geq q_a + q_b$ with $q_a + q_b \in Q^+$; that is, $a+b \notin J$.

This proves $\bar{x} + \bar{y} = \overline{x+y} \in \bar{M}$.

(B4) Let $\bar{\rho} \in \bar{V}$ and $\bar{x} \in \bar{M}$. Then $\bar{x} = \bar{a}$ for some $a \in M-J$ and we have $\bar{\rho} \bar{x} \bar{\rho}^* = \bar{\rho} \bar{a} \bar{\rho}^* = \overline{\rho a \rho^*}$. Since $\rho a \rho^* \in M \cup \{0\} = M^\#$, we get $\bar{\rho} \bar{x} \bar{\rho}^* \in \bar{M}^\# = \bar{M} \cup \{0\}$ for any $\bar{\rho} \in \bar{V}$.

(B5) For each nonzero symmetric element $\bar{x}$ in $\bar{V}$, $\bar{x} = \bar{x}^*$ implies $x - x^* \in J$. Writing $x = \frac{x+x^*}{2} + \frac{x-x^*}{2}$, we have $\bar{x} = \overline{\left(\frac{x+x^*}{2}\right)}$ with $\frac{x+x^*}{2} \notin J$. Therefore, $\frac{x+x^*}{2}$ is certainly nonzero. We have either $\frac{x+x^*}{2} \in M$ or $-(\frac{x+x^*}{2}) \in M$, which yields that either $\bar{x} = \overline{\left(\frac{x+x^*}{2}\right)} \in \bar{M}$ or $-\bar{x} = -\overline{\left(\frac{x+x^*}{2}\right)} \in \bar{M}$.

All conditions (B1)-(B5) are satisfied by $\bar{M}$ and so $\bar{M}$ qualifies as a Baer set for $\bar{V}$. Finally we wish to show that the quotient ring $\bar{V}$ with the quotient Baer ordering has no nonzero infinitesimals.

Suppose that $\bar{a} = a+J \in \bar{V}$ and $\bar{a}\bar{a}^* < \left(\frac{1}{n}\right)$ for all $n \in \mathbb{N}$; that is, for each $n \in \mathbb{N}$ there exists an $\bar{x}_n \in \bar{M}$ such that $\bar{a}\bar{a}^* + \bar{x}_n = \left(\frac{1}{n}\right)$ with $x_n \in M-J$. Then $aa^* + x_n = \frac{1}{n} + i$ for some $i \in J$. Suppose $aa^* \geq \frac{2}{n}$ for some $n \in \mathbb{N}$. Then $aa^* + x_n = \frac{1}{n} + i \geq \frac{2}{n} + x_n$ gives $i \geq \frac{1}{n} + x_n > \frac{1}{n}$, a contradiction. Therefore $a \in J$ or $\bar{a} = \bar{0}$, as wished. The theorem is now proved.

Definition 1.3.10. Let R be any Baer ordered $\ast$ -ring. A subset C of the bounded $\ast$ -subring V of R is $\ast$ -convex if whenever $0 \leq a \leq b$ with $a, b \in S(R)$ and $b \in C$ we have $a \in C$. A $\ast$ -ideal I of V is

said to be *-convex if it is *-convex as a subset.

Clearly, arbitrary unions and intersections of *-convex sets are *-convex. In the course of proving condition (B3) in Theorem 1.3.9 it can be seen that J is a *-convex ideal. Related to Theorem 1.3.9 is the following more general situation:

Corollary 1.3.11. Let R be any Baer ordered *-ring and let I be a proper convex *-ideal of the bounded subring V . Then the quotient *-ring V/I is also Baer orderable.

Proof. Use the same argument as in Theorem 1.3.9.

When the Baer ordered *-ring R happens to be a division ring, the bounded subring V of R has some stronger properties as will be seen in Proposition 1.3.13. Before proceeding, we introduce

Definition 1.3.12. Any associative ring R is said to be a right valuation ring (in the sense of Kaplansky [25]) if for any a and b in R either $aR \subseteq bR$ or $bR \subseteq aR$.

As is well-known, any right valuation ring R is such that all right ideals are linearly ordered under inclusion. Thus R has a unique maximal right ideal J , which coincides with the set of non-units.

Proposition 1.3.13. Let R be any Baer ordered *-division ring. Then the bounded subring V of R is a right valuation ring having $J = J(R)$ as its unique maximal (*)-ideal and hence $J = \text{Rad}(V)$, the Jacobson radical of V . Thus V/J is a Baer ordered *-division ring, which is archimedean.

Proof. Given any $x \in V-J$, we have $q_1 \leq xx^* \leq q_2$ for some $q_1, q_2 \in Q^+$. Therefore $\frac{1}{q_1} \geq x^{-1}x^{-1*} \geq \frac{1}{q_2}$, which means $x^{-1} \in V-J$. This says that V is a local ring with J as the unique maximal $(*)$ -ideal. Consequently, $J = \text{Rad}(V)$, the Jacobson radical of V .

Next, we wish to show that V is a right valuation ring. Given $x \in R$ we have x or $x^{-1} \in V$, for if $x \notin V$ then $xx^* > q$ for each $q \in Q^+$ and hence $x^{-1}x^{-1*} < \frac{1}{q}$ for each $q \in Q^+$, which implies $x^{-1} \in V$. Now for $a \neq 0, b \neq 0$ in V we have either $a^{-1}b \in V$ or $(a^{-1}b)^{-1} = b^{-1}a \in V$ and so either $bV \subseteq aV$ or $aV \subseteq bV$. Also, $a = 0$ or $b = 0$ implies that $aV \subseteq bV$ or $bV \subseteq aV$. In any case we have either $aV \subseteq bV$ or $bV \subseteq aV$ for any $a, b \in V$, proving the proposition.

Definition 1.3.14. Let R be a Baer ordered $*$ -division ring, let V be the bounded subring of R , and let J be the maximal ideal of V . The $*$ -division ring V/J is called the residue $*$ -division ring.

Remark 1.3.15. For R any Baer ordered $*$ -division ring, to say that R is archimedean is the same as saying that R is bounded.

Justification. From the property: $aa^* < q \iff a^{-1}a^{-1*} > \frac{1}{q}$ where $a \neq 0 \in R$ and $q \in Q^+$, it follows that R is archimedean if and only if R is bounded.

In [20], Holland has shown the following characterization of archimedean Baer ordered $*$ -division rings:

Theorem 1.3.16 (Holland). Any bounded (and hence archimedean) Baer ordered $\ast$ -division ring R is $\ast$ -isomorphic and order isomorphic to a $\ast$ -subdivision ring of the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, or the real quaternions $\mathbb{H}$.

As an immediate corollary we have

Corollary 1.3.17. In any Baer ordered $\ast$ -division ring R , the residue $\ast$ -division ring $\bar{V} = V/J$ of R is $\ast$ -and order isomorphic to a $\ast$ -subdivision ring of $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.

Theorem 1.3.18. Let R be a Baer ordered $\ast$ -division ring. Suppose that all symmetric elements are algebraic over an archimedean subfield F . Then R is at most 4-dimensional over its center.

Proof. Let $F_0 = F \cap S(R)$. We claim that R has no symmetric infinitesimals $\neq 0$. For if $s \in S(R)$ is an infinitesimal, then by hypothesis $F_0[s]$ is an ordered algebraic field extension of the ordered archimedean field F_0 . Thus $F_0[s]$ must also be archimedean and hence $s = 0$. Similarly, if $k = -k^\ast$ is an infinitesimal then k^2 is a symmetric infinitesimal. By the above $k^2 = 0$ follows and so $k = 0$. Consequently, all symmetric and skewes of R are bounded. From this we can derive that all elements of R are bounded. By Theorem 1.3.16, the conclusion follows.

One might ask if the conclusion in Theorem 1.3.18 is still true when F is replaced by an arbitrary central subfield of R . The answer to this question is negative, as is shown by the following example of Holland.

Example 1.3.19 [21, p.227]. Let $\omega = \exp(\frac{2\pi i}{3})$ and let x be an indeterminate. Set $Z_0 = Q(x^3)$, $Z = Q(\omega)(x^3)$, and $M = Q(\omega)(x)$. Take $\theta(x) = x\omega$ as a generator of the Galois group $\text{Gal}(M/Z)$, and define the involution $*$ on M by setting $\mu^* = \bar{\mu}$ for $\mu \in Q(\omega)$ and $x^* = x$. Consider the set $K = (M/Z, \theta, 2) = \{\mu_0 + \mu_1 y + \mu_2 y^2 \mid \mu_i \in M, y^3 = 2, y\mu = \theta(\mu)y\}$. As was checked by Holland, K is a nonabelian bicyclic $*$ -division ring of dimension 9 over the center Z . That this division ring K is Baer orderable was shown by using a smooth presection argument (see [21, p.227-228]). We will come back to this in Chapter III [Example 3.6.4].

Notice that in this example, y is symmetric and algebraic over Q and y is not central.

Our next theorem generalizes Theorem 1.3.16 from division rings to simple rings:

Theorem 1.3.20. Let R be any bounded Baer ordered simple $*$ -ring. Then R is as in the conclusion of Theorem 1.3.16.

Proof. Since R is bounded, $R = V(R)$. Since $J = J(R)$ is a proper ideal of $V = V(R)$, $J = \{0\}$ follows. By Corollary 1.3.8, $S(R) \subseteq Z(R)$. Since R is simple, by Kaplansky's theorem [17, p.157] R must either be a division ring or the 2×2 matrix ring F_2 over some field F . However, it can be shown that F_2 is not Baer orderable (cf. Chapter II, Section 3). Thus R must be a division ring and the theorem follows.

§1.4 Order subrings of R

Recall that a subring A of the $\star$ -ring R is said to be a right order subring if for any $x \in R$ there exist $a, b \in A$ such that $x = ab^{-1}$. The following Lemma 1.4.1 and Corollary 1.4.2 concerning order subrings are essentially due to Vidav [34, Lemma 6] and Berberian [16], respectively. For the convenience of the reader, we shall give the proofs.

Lemma 1.4.1. [34, Lemma 6]. Let R be a Baer ordered $\star$ -ring.

Suppose that $1 + xx^*$ is invertible for each $x \in R$. Then there exist t_1, t_2, t_3 in R such that $[(1 + xx^*)^{-1}]^2 + \sum_{i=1}^3 t_i t_i^* = 1$, and $(1 + xx^*)^{-1}, x^*(1 + xx^*)^{-1} \in V = V(R)$.

Proof. Let $a = xx^*$ and $y = (1 + xx^*)^{-1}$. Since $1 = (1+a)^2 - 2a - a^2$, $yy^* = y(1+a)^2 y^* - 2yay^* - ya^2 y^*$ or $yy^* + 2(yx)(yx)^* + (ya)(ya)^* = 1$.

Defining t_1 as follows:

$$t_i = \begin{cases} yx & \text{for } i = 1, 2 \\ ya & \text{for } i = 3, \end{cases}$$

we obtain $y = (1 + xx^*)^{-1} \in V$ and $(1 + xx^*)^{-1} x \in V$. Also, $x^*(1 + xx^*)^{-1} = [(1 + xx^*)^{-1} x]^* \in V$.

From Lemma 1.4.1 one can derive

Corollary 1.4.2 [16]. Let R be any Baer ordered $\star$ -ring. Then the bounded $\star$ -subring $V = V(R)$ is both a right and a left order subring in R provided $1 + xx^*$ is invertible for each $x \in R$.

Proof. Write $x = (1+xx^*)(1+xx^*)^{-1}x = [(1+xx^*)^{-1}]^{-1}[(1+xx^*)^{-1}x]$. Since $(1+xx^*)^{-1}$ and $(1+xx^*)^{-1}x$ are in V by Lemma 1.4.1, V is then a left order subring in R . Also, $(1+x^*x)^{-1}x^* \in V$ gives $x(1+x^*x)^{-1} = [(1+x^*x)^{-1}x^*]^* \in V$. Writing $x = x(1+x^*x)^{-1}(1+x^*x) = [x(1+x^*x)^{-1}][(1+x^*x)^{-1}]^{-1}$, V is then a right order subring in R .

Remark 1.4.3. Let R be a Baer ordered $\star$ -ring such that $1+xx^*$ is invertible for each $x \in R$. Then the center $Z(V)$ of the bounded subring V coincides with the bounded central elements Z_0 of R .

Proof. It is obvious that $Z_0 \subseteq Z(V)$. Now let $z \in Z(V)$ and x an arbitrary element in R . Then $[z, (1+xx^*)^{-1}] = 0$ and $[z, (1+xx^*)^{-1}x] = 0$, since $(1+xx^*)^{-1} \in V$ and $(1+xx^*)^{-1}x \in V$. From these two equalities we see that $[z, x] = 0$, and so $z \in Z_0$. Thus $Z_0 = Z(V)$.

Remark 1.4.4. Let R be a Baer ordered $\star$ -ring such that $xx^* = 0$ implies $x = 0$ whenever x is bounded. If R is semiprime, so is the bounded subring V .

Proof. Suppose I is an ideal of V such that $I^2 = 0$. If $I = I^*$ then, for any $x \in I$, $xx^* \in I^2 = 0$, so $x = 0$, and $I = 0$ follows. If $I \neq I^*$, then consider the $\star$ -ideal $II^* \subseteq I$. Then $(II^*)^2 \subseteq I^2 = 0$ gives us $II^* = 0$ as in the preceding case. For any $x \in I$, we have $xx^* \in II^* = 0$ and hence $x = 0$. Thus $I = 0$. Consequently, V is semiprime.

When R is restricted to a $\star$ -division ring, and $S(R) \not\subseteq Z(R)$, the result of Corollary 1.4.2 can be strengthened as follows:

Theorem 1.4.5. Let R be any Baer ordered $\ast$ -division ring such that $S(R) \not\subseteq Z(R)$. Then the subring $\overline{S(V)}$ of R generated by all bounded symmetric elements of R is a two-sided order subring in R .

Before proving Theorem 1.4.5 we recall the following:

Definition 1.4.6. Given any associative ring R , a subset A of R is said to be a Lie subring of R if A is an additive subgroup such that for a, b in A , $[a, b] = ab - ba$ must also be in A . The additive subgroup U of A is said to be a Lie ideal of A if whenever $u \in U$ and $a \in A$ then $[u, a] \in U$.

The following lemma is well known. For the convenience of the reader we prove it here.

Lemma 1.4.7 (Herstein). Under the same assumption as in Theorem 1.4.5, $\overline{S(V)}$ is a Lie ideal in V .

Proof. For each $r \in V$, $s \in S(V)$, $[s, r] = sr - rs = (sr + r^\ast s) - (r + r^\ast)s \in S(V)$. Hence $[r, s_1 s_2 \cdots s_n] = s_1 [r, s_2 s_3 \cdots s_n] + [r, s_1] s_2 \cdots s_n \in \overline{S(V)}$. By induction, $[r, s_1 s_2 \cdots s_m] \in \overline{S(V)}$ for any $s_1, s_2, \dots, s_m \in S(V)$, $r \in V$, and $m \in \mathbb{N}$. Therefore $\overline{S(V)}$ is a Lie ideal of V .

Proof of Theorem 1.4.5. First we claim that $S(V)$ is not contained in the center $Z(V)$ of V . For if $S(V) \subseteq Z(V)$ then $[(1 + xx^\ast)^{-1}, (1 + yy^\ast)^{-1}] = 0$ for $x, y \in R$. Thus $[xx^\ast, yy^\ast] = 0$ for any $x, y \in R$. In particular, $[s_1^2, s_2^2] = 0$ for $s_1, s_2 \in S(R)$.

Linearizing in s_1, s_2 we obtain $[s_1, s_2] = 0$ for $s_1, s_2 \in S(R)$. In other words, any two symmetric elements in R commute. By a result of Dieudonné [14], $\overline{S(R)} = R$ or $S(R) \subseteq Z(R)$. In any case we have $S(R) \subseteq Z(R)$, a contradiction. Next, $S(V) \not\subseteq Z(V)$ and $\overline{S(V)}$ is a subring and a Lie ideal of V ; hence $\overline{S(V)}$ must contain a nonzero ideal I of V by [18, Lemma 1.3]. Since every element x in R is of the form $a^{-1}b$ for some $a, b \in V$ and $I \neq 0$, we can choose some nonzero element α from I and we then represent x by $x = a^{-1}b = (\alpha a)^{-1}(\alpha b)$. Thus I is an order subring in R . Since $I \subseteq \overline{S(V)}$ therefore $\overline{S(V)}$ is also an order subring in R .

§1.5 Theorems of A. Albert

A theorem of Albert asserts that any (Hilbert) ordered algebraic division ring is a field [2]. We show the following skew version of this theorem:

Theorem 1.5.1. Let R be any Baer ordered $\ast$ -division ring.

Suppose that all skew elements of R are algebraic over an archimedean central subfield F . Then R is as in the conclusion of Theorem 1.3.18.

Proof. If k is a skew infinitesimal, then k^2 is a symmetric infinitesimal. By using the same argument as in the proof of Theorem 1.3.18, we can show that k^2 must be 0 and hence that k is 0. From this we know that R contains no nonzero skew infinitesimals. Let $\overline{S(V)}$ be the subring generated by the bounded symmetric elements and let I be the ideal of infinitesimals in $\overline{S(V)}$. Then $I \cap I^*$ is

a $\star$ -ideal of $\overline{S(V)}$ containing no skew elements and so $I \cap I^\star$ is commutative. If $I \cap I^\star \neq 0$ then the prime ring $\overline{S(V)}$ is commutative [19, p.7] and hence in view of Remark 1.4.3, $S(R) \subseteq Z(R)$. If $I \cap I^\star = 0$, then from Theorem 1.3.7 we know that $[s, k] \in I \cap I^\star = 0$ for each $s = s^\star$ and $k = -k^\star$ in $\overline{S(V)}$, and consequently that $[s, k] = 0$. As before, every symmetric element of $\overline{S(V)}$ is then central in $\overline{S(V)}$ and hence central in R . If now $s = s^\star \in R$, then since s or s^{-1} belongs to $\overline{S(V)}$ we conclude that $s \in Z(R)$.

We now turn to the study of the ring of quotients of the Baer ordered Öre $\star$ -domains R . It is known that every Hilbert ordering of a left Öre domain can be extended uniquely to a (Hilbert) ordering of the left quotient ring. Here we show the following analogous result which was stated with a hint for a formal proof in Holland's paper [21].

Theorem 1.5.2. *If R is any Baer ordered Öre $\star$ -domain then the ordering of R can be extended uniquely to a Baer ordering on the $\star$ -division ring Q of quotients of R .*

Proof. Existence of the ordering: Let M_0 be the Baer set of R . Define $M = \{b^{-1}a \in Q \mid ab^\star \in M_0\}$. We claim that M qualifies as a Baer set for Q . Let us check the conditions (B1)-(B5) as follows.

(B1) For each $b^{-1}a \in M$, $ab^\star \in M_0$ by definition of M . Thus we have $(ab^\star) = (ab^\star)^\star$ or $ab^\star = ba^\star$, from which $b^{-1}a = (b^{-1}a)^\star$ follows. Therefore $M \subseteq S(Q)$.

(B2) Evidently, $1 \in M$ and $0 \notin M$.

(B3) Given $b^{-1}a, d^{-1}c \in M$, then $(b^{-1}a + d^{-1}c) = (xb)^{-1}(xa + yc)$

with $xb = yd$. Thus $(xa + yc)(xb)^* = xab^*x^* + yc(yd)^* =$

$x(ab^*)x^* + y(cd^*)y^* \in M_0$. Therefore $b^{-1}a + d^{-1}c \in M$.

(B4) Given $b^{-1}a \in M$, we have $ab^* \in M_0$. For each $0 \neq x \in R$,

$x(b^{-1}a)x^* = xb^{-1}ax^* = \alpha^{-1}(\alpha x)(\beta b)^{-1}(\beta ax^*)$ with $\alpha x = \beta b$.

Consequently $x(b^{-1}a)x^* = \alpha^{-1}(\beta ax^*)$ and $\beta ax^*\alpha^* = \beta a(\alpha x)^* =$

$\beta a(\beta b)^* = \beta ab^*\beta^* \in M_0$. By the definition of M , $x(b^{-1}a)x^* \in M$.

Now for each $n^{-1}m \in M$, $0 \neq y \in R$, we have $y^{-1}(n^{-1}m)y^{-1*} =$

$(ny)^{-1}m(y^*)^{-1} = (uny)^{-1}(um)(vy^*)^{-1}v$ with $um = vy^*$. Thus

$y^{-1}(u^{-1}m)y^{-1*} = (uny)^{-1}v$, and $v(un y)^* = vy^*n^*u^* = (um)n^*u^* =$

$u(mn^*)u^* \in M_0$. This shows that $y^{-1}(\beta^{-1}\alpha)y^{-1*}$ is in M .

Combining these relations we get that $(y^{-1}x)(\beta^{-1}\alpha)(y^{-1}x)^*$

$y^{-1}[x(\beta^{-1}\alpha)x^*]y^{-1*}$ is in M for each $\beta^{-1}\alpha \in M$ and $0 \neq y^{-1}x \in Q$.

(B5) (*-trichotomy law). For each $0 \neq b^{-1}a = (b^{-1}a)^*$ in Q we have

$b^{-1}a = a^*b^{-1*}$ or $ab^* = ba^* = (ab^*)^*$. This shows that ab^* is

in $S(R)$. Thus either $ab^* \in M_0$ or $-ab^* \in M_0$, which implies

either $b^{-1}a \in M$ or $-b^{-1}a \in M$.

The above discussion shows that conditions (B1)-(B5) are all satisfied by M , and thus the quotient ring Q can be ordered by M .

Uniqueness of the ordering: Let M be an arbitrary Baer set of the ring Q . We have $b^{-1}a \in M$ if and only if $b(b^{-1}a)b^* \in M$ or $ab^* \in M \cap R$. Now it is clear that $M \cap R = M_0$. Thus $b^{-1}a \in M$ if and only if $ab^* \in M_0$, and M is completely determined by M_0 .

In [3], Albert has also shown that if a Hilbert ordered ring R satisfies a nontrivial polynomial identity then R is commutative.

We prove

Theorem 1.5.3. *Let R be any Baer ordered $\ast$ -domain with bounded archimedean $\ast$ -center. If the symmetric elements of R satisfy a nontrivial polynomial identity then the division ring Q of quotients of R is at most 4-dimensional over its center $Z(Q)$, and hence $S(R) \subseteq Z(R)$.*

Proof. Since $S(R)$ satisfies a nontrivial polynomial identity, by a result of Amitsur [5] R also satisfies a nontrivial polynomial identity. We may assume, further, that R satisfies a standard identity. As is well-known, this implies that R must be an Ore domain, and so R has a left quotient ring, say Q . By Theorem 1.5.2, Q is also Baer orderable. A result of Amitsur [6] states that the left quotient ring Q will satisfy the above-mentioned standard identity for R as well. Applying Kaplansky's theorem [17, Theorem 6.3.1], Q must be finite dimensional over its center $Z(Q)$. It is known that $Z(Q) = Z(R)^{-1}Z(R) = \{a^{-1}b \mid a \neq 0, a, b \in Z(R)\}$. Since the $\ast$ -center $Z_0 = Z(R) \cap S(R)$ of R is bounded and archimedean and $xx^* \in Z_0$ for any $x \in Z(R)$, $Z(R)$ must be bounded and archimedean. Consequently, $Z(Q)$ is also archimedean. By Theorem 1.3.18, Q is at most 4-dimensional over its center $Z(Q)$, which completes the proof.

CHAPTER II

THE STRUCTURE OF ARCHIMEDEAN BOUNDED BAER ORDERED $\star$ -RINGS

§2.1 Introduction

In this chapter we will study the structure of archimedean bounded Baer ordered $\star$ -rings R . We will show that for a given archimedean bounded Baer ordered $\star$ -ring R , $S(R)$ is isomorphic to an ordered subring of the real numbers $\mathbb{R}$. From this we will derive that R must be isomorphic to a $\star$ -subring of $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.

§2.2 The structure of archimedean, bounded Baer ordered $\star$ -rings

As shown by Holland, any archimedean Baer ordered $\star$ -division ring is either a field or a normal quaternion. In this section we generalize this result to archimedean bounded Baer ordered $\star$ -rings R . Recall that for any such ring R the symmetric elements are all central, and thus $S = S(R)$ is a subring of R [Corollary 1.3.8]. We now prove:

Theorem 2.2.1. *Given any archimedean bounded Baer ordered $\star$ -ring R , the commutative ring $S = S(R)$ consisting of all symmetric elements in R can be embedded as an ordered subring in the field of real numbers $\mathbb{R}$ with the natural ordering.*

Proof. We first note that the archimedean condition forces each nonzero symmetric element s to have $|s| \geq \frac{1}{n(s)}$ for some $n(s) \in \mathbb{N}$ depending on s . For any $s_1, s_2 \in S(R) - \{0\}$, assume $|s_1| \geq \frac{1}{m}$

and $|s_2| \geq \frac{1}{n}$ for some $m, n \in \mathbb{N}$. Then $|s_2||s_1||s_2| \geq |s_2| \frac{1}{m}|s_2| = \frac{1}{m}|s_2|^2 \geq \frac{1}{m}(\frac{1}{n})^2$. Hence we get $s_1 s_2 \neq 0$ and $s_2 s_1 \neq 0$. In other words, nonzero symmetric elements in R do not annihilate one another. Thus the commutative ring $S = S(R)$ is actually a domain. Next, we wish to show that the induced ordering on S is a Hilbert ring ordering. It can easily be seen that both the mapping $+$: $S \times S \rightarrow S$ defined by $+$: $(x, y) \rightarrow x + y$ and scalar multiplication by rational numbers are continuous with respect to the interval topology induced by the usual absolute value. For the continuity of the multiplication $\cdot$: $(x, y) \rightarrow xy$, we first verify that the functions $f : x \rightarrow x^2$ and $g_a : x \rightarrow ax$ are continuous at 0 . We have that f is continuous at 0 for $|x| < 1$, which implies $|x^2| = |x|^2 \leq |x|$ (Remark 1.2.12). As for g_a , since $|a| < n$ for some $n \in \mathbb{N}$ by hypothesis, we can take $\delta = \frac{\epsilon}{1+n+n^2}$, and thus

$$\begin{aligned} |ax| &= \frac{1}{2} |(1+a)^2 x - (1+a^2)x| \\ &\leq \frac{1}{2} [|(1+a)^2 x| + |(1+a^2)x|] \\ &\leq \frac{1}{2} [(1+a)^2 + (1+a^2)] |x| \\ &\leq \frac{1}{2} [(1+a)^2 + (1+a^2)] \frac{\epsilon}{1+n+n^2} \\ &\leq \frac{1}{2} [(1+|a|)^2 + 1+a^2] \frac{\epsilon}{1+n+n^2} \\ &< \frac{1}{2} [(1+n)^2 + (1+n^2)] \frac{\epsilon}{1+n+n^2} < \epsilon. \end{aligned}$$

Hence g_a is continuous at 0 . The continuity of $x+y$ at $(0,0)$ and of x^2 at 0 imply the continuity of the composite function $(x+y)^2$

at $(0,0)$. Consequently, $xy = \frac{1}{2}[(x+y)^2 - x^2 - y^2]$ is again continuous at $(0,0)$. For arbitrary $x = a$, $y = b$ in S , from the inequality

$$\begin{aligned} |xy - ab| &= |(x-a)(y-b) + a(y-b) + b(x-a)| \\ &\leq |(x-a)(y-b)| + |a(y-b)| + |b(x-a)| \end{aligned}$$

it follows that xy is continuous at (a,b) for any $a, b \in S$. Since the rational numbers Q are dense in S , and since multiplication is continuous, it follows that the induced Baer ordering in S must be a Hilbert ordering.

It is well known that any archimedean bounded Hilbert ordered ring can be embedded as an ordered subring in the real numbers R with the natural ordering. The proof is now complete.

From the structure of the ring of symmetric $S(R)$ let us obtain some information on the structure of the ring R itself. This is the

Lemma 2.2.2. *Let R be a bounded archimedean Baer ordered $\ast$ -ring. Then every nonzero symmetric element in R is regular in R .*

Proof. We know that $S(R) \subseteq Z(R)$. By Theorem 2.3.1, nonzero symmetric elements do not annihilate each other. Now suppose that $0 \neq s = s^*$ is not regular, we have $xs = sx = 0$ for some $0 \neq x \in R$. From this it follows that $x^*xs = 0$ and $(x + x^*)s = 0$, which yields $x^*x = 0$ and $x + x^* = 0$, since $s \neq 0$. From the last two equalities it follows that x must be skew with square 0. Since $xRs = xsR = 0$,

* $xR \subseteq K = K(R)$ and $(xR)^2 = 0$ by the preceding discussion. By Theorem 1.3.4, R is semiprime. Thus $xR = 0$ and so $x = 0$, a contradiction. Therefore every nonzero symmetric element is regular.

We come now to the main theorem of this chapter.

Theorem 2.2.3. Let R be a Baer ordered $\star$ -ring. Then R is archimedean and bounded if and only if R is isomorphic to a $\star$ -subring of R , C , or H .

Proof. It is evident that if R is isomorphic to a $\star$ -subring of R , C , or H then R is archimedean and bounded.

Conversely, if R is archimedean and bounded then R is semiprime by Theorem 1.3.4. Also, we know from Theorem 2.2.1 that nonzero symmetric elements in R do not annihilate each other. Thus by a result due to Lanski [26, Theorem 6], we have the following possibilities:

(a) R is a domain;

(b) $R_S = \{s^{-1}r \mid r \in R, 0 \neq s \in S\}$, the localization of R at $S^\times = S - \{0\}$, is the complete 2×2 matrix ring

$$F_2 = \begin{bmatrix} F & F \\ F & F \end{bmatrix} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in F \right\} \text{ over a field } F;$$

(c) R is a subdirect sum of A and A^{op} , where A is a domain, A^{op} is the opposite ring of A , and the involution in R is induced by exchange involution in $A \oplus A^{\text{op}}$.

In case (a), since $S(R) \subseteq Z(R)$, R satisfies a polynomial identity. Further, by [17, Corollary to Theorem 6.3.2], R satisfies

a standard identity. As is known, this implies that R is an Ore domain. The $\star$ -division ring Q of quotients of R is also Baer orderable by Theorem 1.5.9. Clearly, Q will inherit the archimedean property and boundedness from R . By Holland's theorem (Theorem 1.3.16), Q is isomorphic to a $\star$ -subdivision ring of R , C , or H and hence R is isomorphic to a $\star$ -subring of R , C or H .

In case (b), as shown in [26], R must be a prime ring. Let $S^+ = \{s \in S(R) \mid s > 0 \text{ in } R\}$. Consider the central localization $R_{S^+} = \{s^{-1}r \mid r \in R, s \in S^+\}$ (cf. [32]). We claim that $R_S = R_{S^+}$. It is obvious that $R_{S^+} \subseteq R_S$. For any $s \in S^X = S - \{0\}$, we have $s^2 > 0$ by Theorem 2.2.1. Thus given $s^{-1}r \in R_S$ with $s \in S^X$ and $r \in R$, $s^{-1}r = (s^2)^{-1}(sr) \in R_{S^+}$. So $R_S = R_{S^+}$. Now we want to show that R_{S^+} is Baer orderable. Let M be the Baer set associated with the Baer ordering in R . Consider the set $M_1 = \{s^{-1}r \in S(R_{S^+}) \mid r \in M\}$. We claim that M_1 defines a Baer ordering on R_{S^+} . We check properties

(B1) - (B5):

(B1) - (B2) are immediate from the definition of M_1

(B3) Given $s_1^{-1}r_1, s_2^{-1}r_2 \in M_1$, we have $s_1^{-1}r_1 + s_2^{-1}r_2 = (s_1s_2)^{-1}(s_2r_1 + s_1r_2) \in M$ since $s_1s_2 \in S^+$ and $s_2r_1 + s_1r_2 \in S^+$ by Theorem 2.2.1.

(B4) For any $s^{-1}r \in M_1$ and $\alpha^{-1}\beta \in R_{S^+}$, we have $(\alpha^{-1}\beta)(s^{-1}r)(\alpha^{-1}\beta)^* = (\alpha s \alpha)^{-1}(\beta r \beta^*) \in M_1 \cup \{0\} = M_1^\#$.

(B5) For any $0 \neq s^{-1}r \in S(R_{S^+})$, $r = r^*$. We have either $r \in M$ or $-r \in M$. From this it follows that either $s^{-1}r \in M_1$ or $-s^{-1}r \in M_1$.

Hence $R_S = R_{S^+}$ is Baer orderable. Now we claim that the involution $*$ on the matrix ring R_S must be of the symplectic type. If not, then by a result of Jacobson [22, p.311, Case A], $*$ must be of the

following type $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^* = \begin{bmatrix} r_1^{-1} \bar{a} r_1 & r_1^{-1} \bar{c} r_2 \\ r_2^{-1} \bar{b} r_1 & r_2^{-1} \bar{d} r_2 \end{bmatrix}$ where F has an

involution $\lambda \rightarrow \bar{\lambda}$ and $\bar{r}_i = r_i$ ($i=1,2$) are fixed invertible symmetric elements of F . If we choose $x = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, then $xx^* = \begin{bmatrix} 0 & 0 \\ 0 & r_1^{-1} r_2 \end{bmatrix} \neq 0$

and $x^*x = \begin{bmatrix} r_1^{-1} r_2 & 0 \\ 0 & 0 \end{bmatrix} \neq 0$, while $(x^*x)(xx^*) = 0$, contrary to the

fact that nonzero symmetric elements do not annihilate each other in R_S .

With the symplectic involution we get that F is isomorphic to

$S(F_2) = \left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \mid a \in F \right\} = Z(F_2)$, and so F can then be identified with a subfield Z_1 of $\mathbb{R}$. Thus the Baer ordering in R_S is uniquely defined. However, if we choose $\alpha = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\alpha\alpha^* = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} < \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, contradicting property (B4) of a Baer ordering.

Therefore case (b) is ruled out.

In case (c), since R has a unity element, A must also have a unity element, say 1 . Identifying R with a subring of $A \oplus A^{\text{op}}$, we may write each element of R in the form (a,b) where $a,b \in A$ and the unity element e of R is $(1,1)$. Now $S(R) = \{(a,a) \mid a \in A\} \approx A$. By Theorem 2.3.1 we may identify A with a subring of $\mathbb{R}$. Hence we may view R as a subring of $\mathbb{R} \oplus \mathbb{R}^{\text{op}}$ with the exchange involution. We claim that all elements of R must be of the form (p,p) , $p \in \mathbb{R}$. Assume the contrary; then there exists $(a,b) \in R$ with $a \neq b$.

There is no loss of generality in assuming $a < b$. Choose a rational number q such that $a < q < b$ and let $x = (q, q) - (a, b) = (q-a, q-b)$.

$$\begin{aligned} \text{Then we have } xx^* &= (q-a, q-b)(q-b, q-a) \\ &= ((q-a)(q-b), (q-a)(q-b)) \\ &< (0, 0), \end{aligned}$$

which contradicts property (B4) of a Baer ordering. Hence every element of R is of the form (p, p) with $p \in \mathbb{R}$. Therefore R is a subring of $\mathbb{R}$. With this our theorem is established.

CHAPTER III

*-VALUATIONS AND BAER ORDERED *-DIVISION RINGS

§3.1 Introduction

In this chapter we shall be concerned with lifting Baer orderings, and will apply this to get new examples of Baer ordered *-division rings.

In §3.2 we first investigate some basic properties of valuations and in §3.3 we review some facts about *-valuations. In §3.4 we make a comparison between valuations and *-valuations. In §3.5 we give a complete proof of Holland's lemma [21, Lemma 3.1], his theorem [21, Theorem 3.2], and his corollary [21, Corollary 3.3] in the case where the residue *-division ring is a (commutative) field. In §3.6 we construct a new family of Baer ordered *-division rings with prescribed value groups and residue *-fields.

§3.2 Valuations

Definition 3.2.1 (Schilling. [33]). Let R be any *-division ring.

A mapping w of $R^* = R - \{0\}$ into a totally ordered group G (written additively, though not necessarily abelian) is said to be a valuation if w satisfies the following conditions (V1) - (V3):

- (V1) $w(xy) = w(x) + w(y) \quad (x, y \neq 0),$
- (V2) $w(x+y) \geq \min \{w(x), w(y)\} \quad (x+y \neq 0),$
- (V3) w is surjective.

The group G is called the value group.

Convention. We set $w(0) = \infty$, where ∞ is a symbol such that $\infty > a$ and $\infty + \infty = a + \infty = \infty + a = \infty$ for each $a \in G$.

Definition 3.2.2. A subring V of R is said to be total if for every $a \in R^\times$, $a \in V$ or $a^{-1} \in V$. V is said to be invariant if $aVa^{-1} = V$ for all $a \in R^\times$.

By a valuation ring we mean a total invariant subring of R .

Remark 3.2.3. It can easily be verified that for any valuation w , the set $V = \{x \in R \mid w(x) \geq 0\}$ is a valuation ring in R . Conversely, every valuation ring in R determines a valuation on R (cf. [33]). Also, $J = \{x \in R \mid w(x) > 0\}$ is the unique maximal ideal in V .

Remark 3.2.4 ([33]). For any n th root of unity α , $w(\alpha) = 0$ since $0 = w(\alpha^n) = nw(\alpha)$ implies $w(\alpha) = 0$. In particular, $w(1) = w(-1) = 0$.

The following remark holds for any valuation.

Remark 3.2.5. If $w(\alpha) \neq w(\beta)$, then $w(\alpha + \beta) = \min\{w(\alpha), w(\beta)\}$.

Here is one criterion for a total subring to be a valuation ring:

Proposition 3.2.6. Let R be a $\ast$ -division ring and let V be a total subring of R . Then the following two statements are equivalent:

- (i) V is invariant.
- (ii) Every one-sided principal ideal in V is two-sided.

Proof. (i) $\Rightarrow$ (ii): Suppose that V is invariant. Then $aVa^{-1} = V$ for any $a \in V^\times = V - \{0\}$ and hence $aV = Va$. This shows that every one-sided principal ideal in V is actually two-sided.

(ii) $\Rightarrow$ (i): Suppose that every one-sided principal ideal in V is two-sided. Given any element $a \in R^\times$, if $a \in V$ then both the left principal ideal Va generated by a and the right principal ideal aV generated by a must be two-sided and hence $aV = Va$, which gives $aVa^{-1} = V$. If $a \notin V$ then $a^{-1} \in V$ by assumption. Similarly, we have $a^{-1}V = Va^{-1}$ and so $aVa^{-1} = V$. Therefore V is invariant.

Recall that the element u of R is said to be unitary if u is invertible and such that $uu^* = u^*u = 1$.

Proposition 3.2.7. Let R be any $\ast$ -division ring with a valuation w such that $w(2) = 0$. Then the valuation ring V_w of w contains all unitary elements.

Proof. For any unitary element u in R , $uu^* = u^*u = 1$ and so $w(u^*) = -w(u)$. We know that $-1 \in V_w$ and any unitary $u \neq -1$ can be written in the form $u = \frac{1-k}{1+k}$ for some skew k . We distinguish two cases.

Case 1: $k \in V_w$. Then $1+k \in V_w$. We claim that $1+k \notin J_w$, the maximal ideal of V_w . For otherwise $(1+k)^* \in J_w$ or $1-k \in J_w$, which implies $2 = (1+k) + (1-k) \in J_w$, a contradiction. Therefore we have $(1+k)^{-1} \in V_w$ and so $u = \frac{1-k}{1+k} \in V_w$.

Case 2: $k \notin V_w$. Then $k^{-1} \in J_w$ and so $\frac{1}{1 + \frac{1}{k}} \in V_w$. Therefore,

$$u = \frac{1-k}{1+k} = \frac{\frac{1}{k} - 1}{\frac{1}{k} + 1} \in V_w.$$

In any case, $u \in V_w$ and the proof is complete.

§3.3 *-valuations

Definition 3.3.1 ([21, p.221]). A valuation w on a *-division ring R is said to be a *-valuation on R if it satisfies the following additional condition (V4). $w(x^*) = w(x)$ for $x \in R^\times$.

Some immediate consequences of the definition are:

Remark 3.3.2 ([21]). A *-division ring R admits only *-valuations with abelian value group G .

Remark 3.3.3. Given any *-valuation w , we have $w(x) = 0$ for each unitary x in R .

Justification. From $xx^* = 1$ it follows $0 = w(1) = w(x) + w(x^*) = 2w(x)$ and consequently $w(x) = 0$.

Definition 3.3.4 ([21]). A subring V of the *-division ring R is said to be a *-valuation ring if V is total and V contains x^*x^{-1} for each $x \in R^\times$.

Remark 3.3.5. It can be shown that for any *-valuation w , the set $V = \{x \in R \mid w(x) \geq 0\}$ is a *-valuation ring in R . Conversely, every *-valuation ring in R determines a *-valuation on R . (cf. [21, p.221]).

Remark 3.3.6. Any $\star$ -valuation ring V and its unique maximal ideal J are both $\star$ -closed (i.e. $V^\star = V$ and $J = J^\star$).

Definition 3.3.7. Let w be a $\star$ -valuation on the $\star$ -division ring R . Let V_w be the $\star$ -valuation ring associated with w and let J_w be the maximal ideal of V_w . Then the $\star$ -division ring V_w/J_w is called the residue $\star$ -division ring of R with respect to w .

3.4 A comparison between valuations and $\star$ -valuations

In this section we present a natural condition that converts a valuation ring into a $\star$ -valuation ring.

Theorem 3.4.1. Let R be a $\star$ -division ring and V a valuation ring in R . If V is $\star$ -closed, then V is a $\star$ -valuation ring of R .

Proof. In order to show that V is a $\star$ -valuation it suffices to show that $x^\star x^{-1} \in V$ for any $0 \neq x \in R$. Assume the contrary, $x^\star x^{-1} \notin V$ for some $x \neq 0$ in R , then $(x^\star x^{-1})^{-1} = xx^\star \in V$ since V is a total subring of R . Also, the $\star$ -closure of V implies $x^{-1} x^\star = (xx^\star)^\star \in V$. Since V is invariant, we have $x(x^{-1} x^\star)x^{-1} \in V$. Consequently, $x^\star x^{-1} \in V$ which is a contradiction. This proves the theorem.

3.5 Lifting orderings

Let R be a division ring with involution having the $\star$ -valuation w . In a recent paper [21], Holland has shown that any Baer ordering on the residue $\star$ -division ring V_w of R can be lifted to the division ring R under an extra assumption on the valuation (smooth $\star$ -valuation)

The result as stated in [21, Lemma 3.1] is probably true however we believe that there is a gap in the argumentation. The main difficulty comes from the fact that the equation

$$xx^* = a_0 \in R$$

has in general infinitely many (or no) solutions. For if x_0 is a solution then $x_0 u$ is also a solution for each unitary u in R . In what will follow, we shall be able to circumvent this indeterminacy of x in the case where the residue $\star$ -division ring V_0 is a field. Of course, we make the same assumption on the $\star$ -valuation, that is, w is a smooth $\star$ -valuation in the sense due to Holland.

First let us recall some facts:

Remark 3.5.1 ([21]). Since both the $\star$ -valuation ring V and its maximal ideal J are invariant, any inner automorphisms A_a of R induced by the element $a \in R$ ($A_a : x \mapsto axa^{-1}$) induces an automorphism, $\bar{A}_a$, on the residue $\star$ -division ring V_0 .

Remark 3.5.2 ([21]). Given any symmetric or skew element a of R , the mapping $\bar{A}_a \circ \star$ is an involution on V_0 .

Definition 3.5.3 ([21]). Any symmetric or skew element a is said to be smooth if the induced involution, $\bar{A}_a \circ \star$ on V_0 is a conjugate of $\star$. In symbols: $\bar{A}_a \circ \star = \Gamma_a \circ \star \circ \Gamma_a^{-1}$ for some automorphism Γ_a of V_0 .

Definition 3.5.4. ([21]). A $\star$ -valuation $w : R \rightarrow G$ is said to be smooth if:

- (1) $w(2) = 0$,
- (2) each equivalence class $Cl(g) = \{x \in R \mid w(x) = g\}$ contains a

smooth element a_g , which is symmetric if $Cl(g) \cap S(R) \neq \emptyset$
 and skew otherwise.

Remark 3.5.5 ([21]). For any $x \in R$, xx^* is smooth.

Justification. For any $a \in R$, $\bar{A}_{xx^*}(a^*) = xx^*(a^*)(xx^*)^{-1} =$
 $x(x^*a^*x^{-1})x^{-1} = \bar{A}_x(\bar{A}_{x^{-1}}(a))^* = \bar{A}_x \circ * \circ \bar{A}_{x^{-1}}(a) = \bar{A}_x \circ * \circ \bar{A}_x^{-1}(a)$.
 Hence $\bar{A}_{xx^*} \circ * = \bar{A}_x \circ * \circ \bar{A}_x^{-1}$. This shows that xx^* is smooth.

Definition 3.5.6. Given a $*$ -valuation $w : R \rightarrow G$, a function
 $w^{-1} : G \rightarrow R$ is said to be a presection if

- (0) $w^{-1}(g) \in Cl(g)$ for each $g \in G$.
- (1) $w^{-1}(0) = 1$
- (2) $w^{-1}(g) = \begin{cases} \text{a symmetric, if } Cl(g) \cap S(R) \neq \emptyset \\ \text{a skew, otherwise} \end{cases}$
- (3) $w^{-1}(2g) = \beta\beta^*$ for some $\beta \in R$.
- (4) $w^{-1}(g+2g_0) = \gamma w^{-1}(g)\gamma^*$ for some $\gamma \in R$.

Definition 3.5.7. A presection is said to be smooth if each
 $w^{-1}(g)$ is smooth and the selected family of automorphisms of V_0 ,
 $\{\Gamma_{w^{-1}(g)} \mid g \in G\}$, has the following properties:

- (1)' $\Gamma_1 = I$, the identity mapping of V_0 .
- (3)' $\Gamma_{w^{-1}(2g)} = \bar{A}_{\beta u}$ for some unitary element u , given that $w^{-1}(2g) = \beta\beta^*$.
- (4)' $\Gamma_{w^{-1}(g+2g_0)} = \bar{A}_{\delta} \circ \Gamma_{w^{-1}(g)}$ for some $\delta \in R$ with $w(\delta) = g_0$.

The proofs of the next two theorems were inspired from Holland's
 original work ([21]).

Theorem 3.5.8. Any smooth $\ast$ -valuation has a smooth presection.

Proof. Let E be a system of representatives from the family of all cosets of $2G$ in G with the convention that 0 is chosen as the representative for the coset $2G$ (i.e. $0 \in E$). We write G as a direct sum $E \oplus 2G$. Then each element a in G can be written in a unique way in the form $a = b + c$, where $b \in E$ and $c \in 2G$. First we define w^{-1} on E as follows: consider a choice function. We choose $w^{-1}(0) = 1$. We let the corresponding automorphism Γ_1 be the identity mapping of V_0 . For each $g \neq 0$ in E we set $w^{-1}(g) = a_g$, where a_g is the smooth element appearing in Definition 3.5.4. Then let Γ_{a_g} be the automorphism of V_0 corresponding to a_g .

We now proceed to the definition of w^{-1} on $2G$. Consider the set N of all nonzero norms xx^* in R . It is easy to see that $w(N) = \{w(xx^*) \mid x \neq 0\} = 2G$. Let $w|N$ denote the restriction of w to N . Consider the set F of all equivalence classes $[aa^*] = \{xx^* \mid w(xx^*) = w(aa^*)\}$. By the axiom of choice, there is a choice function γ on the set of sets F . Given $g \in G$, let tt^* be the selected element from the equivalence class of $2g$. We set $w^{-1}(2g) = tt^*$. For the corresponding automorphism of the elements tt^* we claim that by choosing the inner automorphism $\Gamma_{tt^*} = \bar{A}_t$ there is an ambiguity in the choice of Γ_{tt^*} . For if $tt^* = t_1 t_1^*$ then $t_1^{-1} tt^* (t_1^*)^{-1} = 1$ and consequently $t_1^{-1} t = u = (u^*)^{-1}$ so that $t = t_1 u$ and $\bar{A}_t = \bar{A}_{t_1} \circ \bar{A}_u$ follow. Now since u is a unitary element, u is a unit of V . Because $V_0 = V/J$ is commutative, for each $x \in V$ we have $uxu^{-1} - x = (ux - xu)u^{-1} \in Ju^{-1} = J$. This means that the inner automorphism induced by u on V_0 is the identity mapping and so $\bar{A}_t = \bar{A}_{t_1}$.

We now complete the definition of w^{-1} as follows. We define first $w^{-1}(g)$. We have $w^{-1}(2g_0) = bb^*$ for some $b \in R$. Consider the set S of sets $H_{e,g} = \{tw^{-1}(e)t^* \mid w(tt^*) = 2g\}$, where e ranges over E , and g ranges over $G - \{0\}$. There is a choice function ψ for the set S . We define $w^{-1}(e \oplus 2g) = \psi(H_{e,g}) = tw^{-1}(e)t^*$ for some t with $w(tt^*) = 2g$. For the corresponding automorphism we set $\Gamma_{w^{-1}(e \oplus 2g)} = \Gamma_{w^{-1}(2g)} \circ \Gamma_{w^{-1}(e)}$.

We claim that for any element $d = b + 2c \in G$, $w^{-1}(d) = tw^{-1}(b)t^*$ for some t with $w(tt^*) = 2c$ and that $\Gamma_{w^{-1}(d)} = \bar{A}_\gamma \circ \Gamma_{w^{-1}(b)}$ for some γ with $w(\gamma) = c$. First, we decompose b as $b = e_1 \oplus 2g_1 \in E \oplus 2G$, then $w^{-1}(b) = t_1 w^{-1}(e_1) t_1^*$ for some t_1 . Since $d = b + 2c = e_1 \oplus 2(g_1 + c)$, we have $w^{-1}(d) = s w^{-1}(e) s^*$ for some s with $w(s) = g_1 + c$. Then $w^{-1}(d) = s t_1^{-1} t_1 w^{-1}(e_1) t_1^* t_1^{-1} s^* = tw^{-1}(b)t^*$, where $t = s t_1^{-1}$.

Next, $\Gamma_{w^{-1}(d)} = \Gamma_{w^{-1}(e_1 \oplus 2(g_1 + c))} = \bar{A}_{\beta u} \circ \Gamma_{w^{-1}(e_1)}$ where $\beta\beta^* = w^{-1}(2(g_1 + c))$ and u is a unitary element. Since $\Gamma_{w^{-1}(b)} = \Gamma_{w^{-1}(e_1 \oplus 2g_1)} = \bar{A}_{kv} \circ \Gamma_{w^{-1}(e_1)}$ for some $kk^* = w^{-1}(2g_1)$ and v is unitary, we obtain

$$\begin{aligned} \Gamma_{w^{-1}(d)} &= \bar{A}_{\beta u} \circ \Gamma_{w^{-1}(e_1)} \\ &= \bar{A}_{\beta u} \circ \bar{A}_{(kv)^{-1}} \circ (\bar{A}_{kv} \circ \Gamma_{w^{-1}(e_1)}) \\ &= \bar{A}_{\beta u v^{-1} k^{-1}} \circ \Gamma_{w^{-1}(b)} \\ &= \bar{A}_\delta \circ \Gamma_{w^{-1}(b)} \quad \text{where } \delta = \beta u v^{-1} k^{-1} \text{ and } w(\delta) = c. \end{aligned}$$

This proves our theorem.

Theorem 3.5.9. Let R be a $\star$ -division ring with a smooth $\star$ -valuation w such that its residue $\star$ -division ring V_0 is a field. Then there exists a mapping N of R onto V_0 such that

- (1) $N(a) = 0 \iff a = 0$,
- (2) for each x and each symmetric a in R , there exists an X in V_0 such that $N(xax^*) = X N(a) X^*$,
- (3)
$$N(a) = \begin{cases} N(a)^* & \text{when } w^{-1}(w(a)) \text{ is symmetric} \\ -N(a)^* & \text{otherwise, and} \end{cases}$$
- (4) if $N(a) + N(b) \neq 0$, then $N(a+b)$ is either $N(a)$, $N(b)$, or $N(a) + N(b)$ according as $w(a) < w(b)$, $w(a) > w(b)$, or $w(a) = w(b)$, respectively.

Proof. Define the mapping $N : R \rightarrow V_0$ by setting $N(a) = \Gamma_{w^{-1}(w(a))}^{-1} (a(w^{-1}(w(a))))^{-1}$ where $(\cdot)$ denotes the canonical $\star$ -homomorphism of the $\star$ -valuation ring V onto $V_0 = V/J$, and $N(0) = \bar{0}$.

Then

- (1) is clear.
- (2) We write $w(xax^*) = e + 2g_0$ where $e = w(a)$ and $g_0 = w(x)$. Then we have $w^{-1}(w(xax^*)) = tw^{-1}(a)t^*$ for some $t \in R$ and $\Gamma_{w^{-1}(w(xax^*))}^{-1} = \bar{A}_\delta \circ \Gamma_{w^{-1}(w(a))}^{-1}$ for some δ with $w(\delta) = w(x)$.

Thus we have

$$N(xax^*) = \Gamma_{w^{-1}(w(xax^*))}^{-1} ((xax^*)(w^{-1}(w(xax^*))))^{-1}$$

$$\begin{aligned}
 &= \Gamma^{-1}_{w^{-1}(w(xax^*))} \overline{((xax^*)t^{*-1}(w^{-1}(w(a)))^{-1}t^{-1})} \\
 &= \Gamma^{-1}_{w^{-1}(w(xax^*))} \circ \overline{\overline{A}_t(t^{-1}xax^*t^{*-1}(w^{-1}(w(a)))^{-1})} \\
 &= [\Gamma^{-1}_{w^{-1}(w(xax^*))} \circ \overline{\overline{A}_t(t^{-1}x)}] [\Gamma^{-1}_{w^{-1}(w(xax^*))} \circ \overline{\overline{A}_t(ax^*t^{*-1}(w^{-1}(w(a)))^{-1})}]
 \end{aligned}$$

Since $\Gamma^{-1}_{w^{-1}(w(xax^*))} = \Gamma^{-1}_{w^{-1}(w(a))} \circ \overline{\overline{A}_\delta^{-1}}$, composing this with $\overline{\overline{A}_t}$ on the right we get

$$\begin{aligned}
 \Gamma^{-1}_{w^{-1}(w(xax^*))} \circ \overline{\overline{A}_t} &= \Gamma^{-1}_{w^{-1}(w(a))} \circ \overline{\overline{A}_\delta^{-1}} \circ \overline{\overline{A}_t} \\
 &= \Gamma^{-1}_{w^{-1}(w(a))} \circ \overline{\overline{A}_{\delta^{-1}t}}
 \end{aligned}$$

It follows that

$$\begin{aligned}
 N(xax^*) &= [\Gamma^{-1}_{w^{-1}(w(xax^*))} \circ \overline{\overline{A}_t(t^{-1}x)}] [\Gamma^{-1}_{w^{-1}(w(xax^*))} \circ \overline{\overline{A}_t(ax^*t^{*-1}(w^{-1}(w(a)))^{-1})}] \\
 &= [\Gamma^{-1}_{w^{-1}(w(a))} \circ \overline{\overline{A}_{\delta^{-1}t}(t^{-1}x)}] [\Gamma^{-1}_{w^{-1}(w(a))} \circ \overline{\overline{A}_{\delta^{-1}t}(ax^*t^{*-1}(w^{-1}(w(a)))^{-1})}] \\
 &= [\Gamma^{-1}_{w^{-1}(w(a))} \circ \overline{(t^{-1}x)}] [\Gamma^{-1}_{w^{-1}(w(a))} \circ \overline{(ax^*t^{*-1}(w^{-1}(w(a)))^{-1})}] ,
 \end{aligned}$$

since $w(\delta^{-1}t) = 0$ implies that $\delta^{-1}t \in V_0$ and V_0 is commutative.

Let $X = \Gamma^{-1}_{w^{-1}(w(a))} \circ \overline{(t^{-1}x)}$ and $Y = \Gamma^{-1}_{w^{-1}(w(a))} \circ \overline{(ax^*t^{*-1}(w^{-1}(w(a)))^{-1})}$.

Then $Y = \Gamma^{-1}_{w^{-1}(w(a))} \circ \overline{\overline{A}_{w^{-1}(w(a))}[(w^{-1}(w(a)))^{-1}ax^*t^{*-1}]}$

$$\begin{aligned}
 &= \{ \Gamma_{w^{-1}(w(a))}^{-1} \circ \bar{A}_{w^{-1}(w(a))}^{-1} \overline{[(w^{-1}(w(a)))^{-1}a]} \} \\
 &\quad \{ \Gamma_{w^{-1}(w(a))}^{-1} \circ \bar{A}_{w^{-1}(w(a))}^{-1} \overline{(x^* t^{*-1})} \} \\
 &= \{ \Gamma_{w^{-1}(w(a))}^{-1} \overline{(a(w^{-1}(w(a))))^{-1}} \} \{ * \circ \Gamma_{w^{-1}(w(a))}^{-1} \overline{(t^{-1}x)} \}, \\
 &\quad \text{since } \Gamma_{w^{-1}(w(a))}^{-1} \circ \bar{A}_{w^{-1}(w(a))}^{-1} \circ * = * \circ \Gamma_{w^{-1}(w(a))}^{-1} \\
 &= N(a)X^*.
 \end{aligned}$$

Consequently, $N(xax^*) = XN(a)X^*$ where $X = \Gamma_{w^{-1}(w(a))}^{-1} \overline{(t^{-1}x)}$.

$$\begin{aligned}
 (3) \quad N(a^*) &= \Gamma_{w^{-1}(w(a^*))}^{-1} \overline{(a^*(w^{-1}(w(a^*))))^{-1}} \\
 &= \Gamma_{w^{-1}(w(a^*))}^{-1} \circ \bar{A}_{w^{-1}(w(a^*))}^{-1} \overline{((w^{-1}(w(a^*)))^{-1}a^*)} \\
 &= \begin{cases} \Gamma_{w^{-1}(w(a))}^{-1} \circ \bar{A}_{w^{-1}(w(a))}^{-1} \circ * \circ \overline{(a(w^{-1}(w(a))))^{-1}} & \text{if } w^{-1}(w(a)) \in S(R) \\ -\Gamma_{w^{-1}(w(a))}^{-1} \circ \bar{A}_{w^{-1}(w(a))}^{-1} \circ * \circ \overline{(a(w^{-1}(w(a))))^{-1}} & \text{if } w^{-1}(w(a)) \in K(R) \end{cases} \\
 &= \pm \Gamma_{w^{-1}(w(a))}^{-1} \circ \Gamma_{w^{-1}(w(a))}^{-1} \circ * \circ \Gamma_{w^{-1}(w(a))}^{-1} \overline{(a(w^{-1}(w(a))))^{-1}} \\
 &= \pm N(a)^*.
 \end{aligned}$$

(4) To show that $N(a+b) = N(a)$ or $N(b)$ or $N(a) + N(b)$, we consider three cases.

Case 1. If $w(a) > w(b)$, then

$$\begin{aligned}
 N(a+b) &= \Gamma_{w^{-1}(w(a+b))}^{-1} \left(((a+b)(w^{-1}(w(a+b))))^{-1} \right) \\
 &= \Gamma_{w^{-1}(w(b))}^{-1} \left(((a+b)(w^{-1}(w(b))))^{-1} \right) \\
 &= \Gamma_{w^{-1}(w(b))}^{-1} \left((a(w^{-1}(w(b))))^{-1} \right) + \Gamma_{w^{-1}(w(b))}^{-1} \left((b(w^{-1}(w(b))))^{-1} \right) \\
 &= \Gamma_{w^{-1}(w(b))}^{-1} \left((b(w^{-1}(w(b))))^{-1} \right), \text{ since } w(a(w^{-1}(w(b))))^{-1} \\
 &= w(a) - w(b) > 0 \text{ gives } a(w^{-1}(w(b))))^{-1} = \bar{0} \\
 &= N(b) .
 \end{aligned}$$

Case 2. If $w(a) < w(b)$, then by symmetry we have $N(a+b) = N(a)$.

Case 3. If $w(a) = w(b)$, then $b = ua$ for some unit u . Thus, $a+b = (1+u)a$. We claim that $1+u \notin J$. For if $1+u \in J$ then $\bar{u} = -\bar{1}$, and so

$$\begin{aligned}
 N(b) &= N(ua) \\
 &= \Gamma_{w^{-1}(w(ua))}^{-1} \left(((ua)(w^{-1}(w(ua))))^{-1} \right) \\
 &= \Gamma_{w^{-1}(w(a))}^{-1} \left((\bar{u} a(w^{-1}(w(a))))^{-1} \right) \\
 &= [\Gamma_{w^{-1}(w(a))}^{-1}(\bar{u})] [\Gamma_{w^{-1}(w(a))}^{-1} \left((a(w^{-1}(w(a))))^{-1} \right)] \\
 &= -\bar{1} \cdot N(a) \\
 &= -N(a), \text{ a contradiction.}
 \end{aligned}$$

Hence $1+u \notin J$ and so $w(1+u) = 0$. This gives that $w(a+b) = w((1+u)a) = w(a) = w(b)$. Therefore

$$\begin{aligned} N(a+b) &= \Gamma_{w^{-1}(w(a+b))}^{-1} \left(((a+b)(w^{-1}(w(a+b))))^{-1} \right) \\ &= \Gamma_{w^{-1}(w(a))}^{-1} (a(w^{-1}(w(a))))^{-1} + \Gamma_{w^{-1}(w(b))}^{-1} (b(w^{-1}(w(b))))^{-1} \\ &= N(a) + N(b) . \end{aligned}$$

The proof is now complete.

Theorem 3.5.10. Let R be a $\star$ -division ring with smooth $\star$ -valuation w and residue $\star$ -field $V_0 = V/J$. Then every Baer ordering of V_0 can be lifted to R .

Proof. Let N be the mapping of R onto V_0 as defined in Theorem 3.5.9 and let $M = \{a \in S(R) \mid N(a) > 0\}$. We claim that M is a Baer set of R .

(B1) $M \subseteq S(R)$ is obvious from the definition of M .

(B2) Since $N(1) = \bar{1} > 0$ and $N(0) = \bar{0}$, $1 \in M$ and $0 \notin M$.

(B3) Given $a, b \in M$, $N(a) > 0$ and $N(b) > 0$ imply that $N(a+b) = N(a)$ or $N(b)$ or $N(a) + N(b)$ by Theorem 3.5.9. In any case $N(a+b) > 0$ and so $a+b \in M$.

(B4) For any $x \in R$, $a \in M$, we have $N(xax^*) = X N(a) X^*$ for some X in V_0 by Theorem 3.5.9. Hence $N(xax^*) \geq 0$, which implies $xax^* \in M^\# = M \cup \{0\}$.

(B5) For each $0 \neq s \in S(R)$, $N(s)^* = N(s^*)$ by (3) in Theorem 3.5.9.

Therefore $N(s)^* = N(s) \neq 0$, so that either $N(s) > 0$ or $N(s) < 0$. This shows that $s \in M$ or $-s \in M$, and the proof is complete.

Theorem 3.5.11. Let R be a $\star$ -division ring with smooth $\star$ -valuation w . Let V be the $\star$ -valuation ring of w and let J be the maximal ideal of V . If the residue $\star$ -division ring $V_0 = V/J$ is a field and if $\geq_0$ is an archimedean Baer ordering in V_0 then V is precisely the bounded subring of R and J is precisely the totality of infinitesimals with respect to the lifted ordering $\geq$ of $\geq_0$.

Proof. First note that if a is a symmetric unit, i.e. $a = a^* \in V-J$, then $w(a) = 0$ and so $w^{-1}(w(a)) = 1$ and $\Gamma_{w^{-1}(w(a))}^{-1} = I$, the identity mapping of V_0 . By definition, we have
$$N(a) = \Gamma_{w^{-1}(w(a))}^{-1} (a(w^{-1}(w(a))))^{-1} = \bar{a}. \text{ Thus } a > 0 \text{ if and only if } \bar{a} >_0 0.$$

Denote the bounded subring and its maximal ideal with respect to the lifted ordering $\geq$ by $V(\geq)$ and $J(\geq)$ respectively. We wish to show that $V \subseteq V(\geq)$ and $J \subseteq J(\geq)$. Let a be a symmetric element in $V-J$ with $\bar{a} >_0 0$. Then $\bar{a} \leq_0 q$ for some $q \in Q^+$ since the ordering $\geq_0$ is archimedean. Thus $\overline{q-a} \geq_0 \bar{0}$ and so $q-a \geq 0$ and $q \geq a$. This shows that $a \in V(\geq)$. Now for any element $x \in V-J$, $xx^* \in V-J$ and hence $xx^* \in V(\geq)$, which implies $x \in V(\geq)$. Consequently,

$V-J \subseteq V(>)$. We claim that $J \subseteq J(\geq)$. For if $j = j^* \in J$, then $q-j \in V-J$ for any $q \in Q^+$, since J is the Jacobson radical of V . Then $\overline{q-j} = \bar{q} > \bar{0}$ gives us $q-j > 0$ and $q > j$. Since q is an arbitrary positive rational number, we must have $j \in J(\geq)$. Now for any $x \in J$, $xx^* \in J$ and thus $xx^* \in J(\geq)$, which implies $x \in J(\geq)$. From the above discussion we get $V \subseteq V(\geq)$ and $J \subseteq J(\geq)$. Further, we claim that $V = V(\geq)$. For otherwise, there exists an $x \in V(\geq)$ with $x \notin V$. So $x^{-1} \in J$ yields that $x^{-1} \in J(\geq)$, which forces $x \in V(\geq)$, a contradiction. Consequently, $V = V(\geq)$ and hence $J = J(\geq)$, which completes the proof.

As an immediate consequence, we have

Corollary 3.5.12. Under the same hypothesis as in Theorem 3.5.11, the $\star$ -valuation ring V is the bounded subring of the lifted ordering of each Baer ordering on V_0 .

As a digression to our discussion of the order relation on the division ring R under consideration, let us briefly recall the notion of natural ordering.

Definition 3.5.13. Let R be a $\star$ -ring and let $M \subseteq R$. Then M is said to be a $\star$ -set if:

- (B1) $M \subseteq S(R)$
- (B2) $1 \in M$ but $0 \notin M$
- (B3) If $a, b \in M$, then $a+b \in M$

(B4) If $a \in M$ and $p \in R$; then $pap^* \in M \cup \{0\} = M^\#$.

The order relation $a \geq b$ defined by $a-b \in M^\# = M \cup \{0\}$ is called a

*-ordering. The unique minimal *-ordering defined by $M_{\text{nat}} = \left\{ \sum_{i=1}^n x_i x_i^* \mid \right.$

$Q \neq x_i \in R \}$ is called the natural ordering of R .

D. Handelman has shown that if R is a field with involution or 4-dimensional with the symmetric S forming a formally real field then the natural ordering is an intersection of Baer orderings [16]. Equivalently, $x \in M_{\text{nat}}$ if and only if $x \in M$, for each Baer set M . This is an involutory version of the Artin-Schreier theorem and it makes sense to ask if it holds generally for division rings R possessing some Baer ordering. In this connection we derive the following fact.

Proposition 3.5.14. Let R be a *-division ring with a smooth *-valuation w such that its residue *-division ring $V_0 = V/J$ is a field. Each *-ordering on V_0 can be lifted to a *-ordering on R .

Proof. See the proof of Theorem 3.5.10.

Consider the lifting of the natural ordering on the field V_0 . This is a *-ordering $\geq_0$ on the division ring R . We claim that $\geq_0$ is an intersection of Baer orderings on R . For since by Handelman's result the natural ordering on V_0 is an intersection of Baer orderings on V_0 and since the lifted ordering of an intersection of orderings is the intersection of the liftings of the components we get the desired result. We have shown the

Theorem 3.5.15. Let R be a $\star$ -division ring with a smooth $\star$ -valuation w such that its residue $\star$ -division ring is a field. The lifted ordering of the natural ordering on V_0 is an intersection of Baer orderings.

Theorem 3.5.15 offers a quick dividend in the hypothetical case where the lifted ordering of the natural ordering turns out to be the natural ordering of R but this seems to be generally not true. The question we addressed to ourselves remains of unknown status. For related results see [13].

§3.6 Schilling's division rings

In [33], Schilling constructed division rings with valuation having a prescribed value group and residue division ring. We shall generalize his construction to the context of division rings with involution with the idea of getting a new family of examples of Baer ordered $\star$ -division rings. We shall first recall his construction. We are given any division ring D and any totally ordered group G (written additively, though not necessarily abelian). We consider a homomorphism $\psi : G \rightarrow \text{Aut}(D)$, and we associate to each $g \in G$ a symbol t_g . We let ϕ be a factor set from $G \times G$ into $D - \{0\}$. This means that for each $r, s, t \in G$ we have

$$\phi(r, s) \phi(r+s, t) = \phi(s, t)^r \phi(r, s+t),$$

where $\phi(s, t)^r = \psi(r)(\phi(s, t))$. We now consider the set R of all formal power series

$$\sum_{i=0}^{\infty} a_i t_{g_i}, \text{ where } a_i \in D \text{ and } (g_i)_{i=0,1,\dots,n,\dots}$$

is a well ordered sequence of elements in G (without infinite repetitions).

We set

- (1) $t_g t_h = \psi(g, h) t_{g+h}$,
- (2) $t_g a t_g^{-1} = a^g$ where $a^g = \psi(g)(a)$,
- (3) $t_o = 1$,
- (4) $t_{-g} = (t_g)^{-1}$.

We add formal power series componentwise and multiply them according to rules (1)-(4). It was pointed out by Schilling that R is a division ring and the complete proof was given by B.H. Neumann in [27].

Now suppose that D has an involution $*$ and D is a field (this is a technical condition, needed for lifting a possible Baer ordering in D). Also assume that G is an abelian group (this is clearly necessary if we are to construct a $*$ -valuation with value group G). We shall revise the definition of a factor set as follows. We require, further, that

$$(i) \quad \phi(h, g) = (\phi(g, h))^*{}^{g+h}$$

Also, for each automorphism $\psi(g)$ in $\text{Aut}(D)$, we set:

$$(ii) \quad * \circ \psi(g) \circ * = (\psi(g))^{-1} \text{ for any } g \in G.$$

Theorem 3.6.1. Consider the mapping $*$ on R defined by $(\sum_{i=0}^{\infty} a_{g_i} t_{g_i})^* = \sum_{i=0}^{\infty} t_{g_i} a_{g_i}^*$. Then necessary and sufficient conditions for $*$ to be an involution on R are that

$$(*1) \quad \phi(h, g) = (\phi(g, h))^*{}^{g+h},$$

$$(*2) \quad * \circ \psi(g) \circ * = \psi(g)^{-1} \text{ for any } g \in G.$$

Proof. First we show that $*$ is of period 2 if and only if (*2) holds. This means that $((a t_g)^*)^* = a t_g \iff (t_g a^*)^* = a t_g \iff$

$$((a^*)^g t_g)^* = a t_g \iff t_g ((a^*)^g)^* = a t_g \iff (((a^*)^g)^*)^g t_g = a t_g.$$

Since a_g is an arbitrary element in D , from the last equality we get $* \circ \psi(g) \circ * = \psi(g)^{-1}$ for any $g \in G$. Thus $*$ is of period 2 if and only if $* \circ \psi(g) \circ * = \psi(g)^{-1}$ for any $g \in G$.

Next, suppose that $*$ is an involution on R . Then $*$ is an anti-automorphism of R . That is, $[(a_g t_g)(b_h t_h)]^* = (b_h t_h)^*(a_g t_g)^*$ for any $a_g, b_h \in D$, $g, h \in G$. Now

$$\begin{aligned} [(a_g t_g)(b_h t_h)]^* &= (b_h t_h)^*(a_g t_g)^* \\ \Leftrightarrow (a_g b_h^g t_{gh})^* &= (t_h b_h^*)(t_g a_g^*) \\ \Leftrightarrow [(a_g b_h^g) \phi(g, h) t_{gh}]^* &= (b_h^*)^h t_h t_g a_g^* \\ \Leftrightarrow t_{gh} [a_g b_h^g \phi(g, h)]^* &= (b_h^*)^h \phi(h, g) t_{gh} a_g^* \quad (\text{since } t_{gh} = t_{h+g}) \\ \Leftrightarrow [(a_g b_h^g)^* \phi(g, h)^*]^{g+h} &= (b_h^*)^h \phi(h, g) (a_g^*)^{g+h} t_{g+h} \\ \Leftrightarrow [\phi(g, h)^*]^{g+h} &= \phi(h, g) \quad (\text{since } * \circ \psi(g) \circ * = \psi(g)^{-1} \text{ and} \end{aligned}$$

D is commutative). The proof is complete.

In what will follow we shall require both postulates (*1) and (*2) so that the division ring R will have an involution extending that of D .

Theorem 3.6.2. Let $w : R \rightarrow G$ be the mapping $w(\sum_{i=0}^{\infty} a_i t_{g_i}) = g_0$.

Then w is a $*$ -valuation with value group G and the residue $*$ -division ring is precisely D .

Proof. One can easily check that w is a $\star$ -valuation. It is clear that the $\star$ -valuation ring of w is $V = \left\{ \sum_{i=0}^{\infty} a_i t_{g_i} \mid g_0 \geq 0 \right\}$ and its maximal ideal is $J = \left\{ \sum_{i=0}^{\infty} a_i t_{g_i} \mid g_0 > 0 \right\}$. Therefore the residue $\star$ -division ring is $V/J \approx D$ as desired, proving the theorem.

What is the center $Z(R)$ of R ? Let I be the set of all those $g \in G$ such that $a^g = a$ for all $a \in D$ and $\phi(g, h) = \phi(h, g)$ for any $h \in G$. Let $Z_1 = \{a \in D \mid a^g = a \text{ for all } g \in G\}$. It is easy to see that $Z(R) = \left\{ \sum_{i=0}^{\infty} a_i t_{g_i} \mid g_i \in I \text{ and } a_{g_i} \in Z_1 \right\}$. If $\text{card}(I) = n < \infty$, say $I = \{g_1, \dots, g_n\}$, then $Z(R)$ is a rational function field $Z_1(t_{g_1}, t_{g_2}, \dots, t_{g_n})$. If $\text{card}(I) = \infty$ then $Z(R)$ is the field consisting of all formal power series $\sum_{i=0}^{\infty} a_i t_{g_i}$ with $g_i \in I$ and $a_{g_i} \in Z_1$.

We now give sufficient conditions for the $\star$ -valuation as defined in Theorem 3.6.2 to be a smooth $\star$ -valuation.

Theorem 3.6.3. Suppose that the group G maps onto a subgroup of automorphisms $\bar{G}$ of $\text{Aut}(D)$ such that

- (i) $\bar{G}$ is a finite cyclic group with cyclic generator σ , and
- (ii) $\star \circ \sigma \circ \star = \sigma^{-1}$ (condition $(\star 2)$).

If, further, the order of $\bar{G}$ is odd then the $\star$ -valuation w is a smooth $\star$ -valuation.

Proof. Clearly $w(2) = 0$. We need only check that each equivalence

class $Cl(g) = \{x \in R \mid w(x) = g\}$ contains a smooth element. For each $g \in G$, we claim that t_g is a smooth element in $Cl(g) = \{x \in R \mid w(x) = g\}$. Since $\sigma \circ \sigma \circ \sigma = \sigma^{-1}$, we have $\sigma \circ \sigma^n \circ \sigma = \sigma^{-n}$ for any integer n . Thus $\sigma \circ \psi(g) \circ \sigma = \psi(g)^{-1}$ and hence the subgroup G_1 generated by $\psi(g)$ and σ is a dihedral group of order $2m$ for some odd m (since m divides $|G|$ and $|G|$ is odd). Let $\langle \psi(g) \circ \sigma \rangle$ and $\langle \sigma \rangle$ be the subgroups of G_1 generated by $\psi(g) \circ \sigma$ and σ , respectively. Both $\langle \psi(g) \circ \sigma \rangle$ and $\langle \sigma \rangle$ are Sylow 2-subgroups of G , so $\langle \psi(g) \circ \sigma \rangle$ and $\langle \sigma \rangle$ are conjugate and hence $\psi(g) \circ \sigma$ and σ are conjugate. That is, $\psi(g) \circ \sigma = \Gamma_g \circ \sigma \circ \Gamma_g^{-1}$ for some $\Gamma_g \in G_1 \subseteq \text{Aut}(D)$. Let $\bar{A}_{t_g}$ be the automorphism of D induced by t_g . Then $\bar{A}_{t_g} = \psi(g)$ and so $\bar{A}_{t_g} \circ \sigma = \psi(g) \circ \sigma = \Gamma_g \circ \sigma \circ \Gamma_g^{-1}$. This shows that t_g is smooth. The proof is now complete.

Example 3.6.4. Let p be a prime integer $\equiv 3 \pmod{4}$, let $\omega = \exp\left(\frac{2\pi i}{p}\right)$, and let $\Lambda = \mathbb{Q}(\omega + \bar{\omega})$ be the maximal real subfield of the cyclotomic field $\mathbb{Q}(\omega)$. Let x be an indeterminate, set $F_0 = \Lambda(x^p)$, $F_1 = \mathbb{Q}(\omega)(x^p)$, and $F = \mathbb{Q}(\omega)(x)$. Then F is a normal extension of F_0 and the Galois group $\text{Gal}(F/F_0)$ is generated by σ and τ , where $\sigma(x) = x\omega$ is a generator of $\text{Gal}(F/F_1)$ and τ is an automorphism of F such that $\tau^* = \bar{\tau}$ (complex conjugate of τ) for $\xi \in \mathbb{Q}(\omega)$ and $\tau^* = x$. It is known that $\mathbb{Q}(\omega)$ has degree 2 over $\Lambda = \mathbb{Q}(\omega + \bar{\omega})$ [28, Art. 183]. From this it follows that $F_1 = \mathbb{Q}(\omega)(x^p)$ has degree 2 over $F_0 = \Lambda(x^p)$. It can be easily checked that $\sigma \circ \sigma \circ \sigma = \sigma^{-1}$ and hence $\text{Gal}(F/F_0)$ is precisely the dihedral group D_{2p} . Also, we claim that the τ -field $F = \mathbb{Q}(\omega)(x)$ is Baer orderable.

First, we show that the polynomial ring $Q(\omega)[x]$ in the indeterminate x over $Q(\omega)$ is Baer orderable. Since every symmetric element of $Q(\omega)[x]$ must be of the form $\sum_{\text{finite}} a_i x^i$ with $a_i \in R$ for each i , we may order $Q(\omega)[x]$ lexicographically by taking all symmetric elements with positive leading coefficients as a Baer set (one can check routinely conditions (B1)-(B5)). Thus, by Theorem 15.2, we can extend the Baer ordering of $Q(\omega)[x]$ to a Baer ordering of $Q(\omega)(x)$. This shows that $F = Q(\omega)(x)$ is Baer orderable.

We are given $F = Q(\omega)(x)$ as the prescribed residue $\ast$ -field and the integers $\mathbb{Z}$ as the prescribed value group. Consider the set

$R = \left\{ \sum_{n=n_0}^{\infty} a_n t_n \mid a_n \in F, n \in \mathbb{Z} \right\}$ of formal power series. We define the addition and left scalar multiplication on R in the obvious way. We choose the factor set in the following particular manner:

$$\phi(r,s) = 1 \text{ for all } r,s \in \mathbb{Z} \text{ and}$$

$$\psi(n) = \sigma^n = \underbrace{\sigma \circ \sigma \circ \sigma \dots \sigma}_{n \text{ times}}, \text{ the composite function of } \sigma \text{ with itself } n \text{ times.}$$

As before we define the multiplication in R according to the following rules (1)-(4):

- (1) $t_n t_m = t_{n+m}$,
- (2) $t_n a t_n^{-1} = \psi(n)(a) = \sigma^n(a)$,
- (3) $t_0 = 1$,
- (4) $t_{-n} = (t_n)^{-1}$,

Also, we see that

$$(*1) \quad \phi(m, n) = (\phi(n, m))^*^{m+n} \quad \text{and}$$

$$(*2) \quad * \circ \psi(n) \circ * = \psi(n)^{-1}$$

hold trivially in our present case. Therefore, by Theorem 3.6.1, we can extend the involution $*$ from F to R and so R becomes a $*$ -division ring.

It is easy to see that the mapping $w : R \rightarrow \mathbb{Z}$ defined by $w(\sum_{n=n_0}^{\infty} a_n t_n) = n_0$ is a $*$ -valuation on R . The $*$ -valuation ring of w is $V = \{ \sum_{n=n_0}^{\infty} a_n t_n \mid w(\sum_{n=n_0}^{\infty} a_n t_n) = n_0 \geq 0 \}$, the maximal ideal of V is $J = \{ \sum_{n=n_0}^{\infty} a_n t_n \mid w(\sum_{n=n_0}^{\infty} a_n t_n) = n_0 > 0 \}$, and so we have residue $*$ -division ring $V/J \approx F$ and the value group is precisely $\mathbb{Z}$. We shall show, further that w is a smooth $*$ -valuation. First, it is evident that $w(2) = 0$. Next, for each $n \in \mathbb{Z}$, we may choose $w^{-1}(n) = t_n \in S(R)$. Then composing the automorphism $\bar{A}_{t_n}$ of $V/J \approx F$ (here we identify V/J with F) induced by t_n with $*$ gives $\bar{A}_{t_n} \circ * = \sigma^n \circ *$ which is of order 2 in $\text{Gal}(F/F_0) = D_{2p}$. Since both $\sigma^n \circ *$ and $*$ are of order 2 in the dihedral group D_{2p} with p odd prime, it is known that $\sigma^n \circ *$ must be a conjugate of $*$. Thus $\sigma^n \circ * = \bar{\Gamma}_{t_n} \circ * = \bar{\Gamma}_{t_n} \circ * \circ \bar{\Gamma}_{t_n}^{-1}$ for some $\bar{\Gamma}_{t_n} \in \text{Gal}(F/F_0) \subseteq \text{Aut}(F)^*$ and hence t_n is smooth. Consequently, w is a smooth $*$ -valuation. By Theorem 3.5.10, the Baer ordering of F can be lifted to R . This shows that R is a Baer, ordered $*$ -division ring.

We conclude this work by a few questions which we have not been able to answer.

Question 1. Is any bounded Baer ordered $\ast$ -ring necessarily an Ore domain? If not, does some construction permit us to embed a Baer ordered $\ast$ -ring into a Baer ordered $\ast$ -division ring?

Question 2. Does there exist a Baer ordered $\ast$ -ring which is not a domain?

Question 3. For any Baer ordered $\ast$ -division ring R , is the bounded subring $V = V(R)$ invariant (i.e. V is preserved under conjugation)?

Question 4. In any Baer ordered $\ast$ -division ring R , is the natural ordering equal to the intersection of all Baer orderings in R ?

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