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**DIRICHLET FORMS AND
INFINITE DIMENSIONAL ORNSTEIN-UHLENBECK PROCESSES**

by

Byron Schmuland, B.Sc., M.Sc.

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

Department of Mathematics and Statistics

Carleton University

Ottawa, Canada

August, 1987

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**Dirichlet Forms and
Infinite Dimensional Ornstein-Uhlenbeck Processes**

submitted by Byron Allan Schmuland, B.Sc., M.Sc., in partial fulfillment of the requirements for the degree of Doctor of Philosophy

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ABSTRACT

We apply the methods of probabilistic potential theory and Dirichlet forms to the study of a class of infinite dimensional stochastic processes, which includes the Ornstein-Uhlenbeck process. We obtain criteria for the sample path continuity of the process in a suitable Hilbert space, in terms of its invariant measure and diffusion coefficients. Applied to Ornstein-Uhlenbeck process these yield extensions of results given previously by other authors. By looking at the properties of its Dirichlet form, we deduce that the Ornstein-Uhlenbeck process is 'quasi-ergodic' in general, and genuinely ergodic when the drift is not too small. As well, we consider some local properties of the process such as the regularity of boundary points and the law of the iterated logarithm.

To conclude we present some open questions concerning these infinite dimensional Ornstein-Uhlenbeck processes.

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NOTATION

$X(t)$ and X_t ... will both be used for random variables with index t .

R ... The real line.

R^d ... d -dimensional Euclidean space.

Q ... The rational numbers.

$\tilde{f}$... A quasicontinuous version of f .

$C([0, \infty); R^d)$... The space of continuous functions from $[0, \infty)$ into R^d .

$C_0(X)$... The space of continuous functions from X into R with compact support.

$C^\infty(R^d)$... The space of infinitely differentiable functions from R^d into R .

$C_0^\infty(R^d)$... The space of infinitely differentiable functions from R^d into R , with compact support.

$C_b^1(R)$... The space of functions from R into R which are bounded, and have bounded continuous first derivative.

$L_{loc}^1(dt)$... The space of functions from R into R which are Lebesgue integrable over every compact set.

$N(\mu, \sigma^2)$... The Gaussian measure on R with mean μ and variance σ^2 .

H_λ ... The Hilbert space of sequences $x = (x_1, x_2, \dots)$ such that $\|x\|_\lambda^2 = \sum \lambda_i^2 x_i^2 < \infty$.

Let μ and ν be probability measures. Then

$\mu \ll \nu$ means that $\nu(B) = 0$ implies $\mu(B) = 0$.

$\mu \sim \nu$ means that $\nu(B) = 0$ if and only if $\mu(B) = 0$.

$\mu \perp \nu$ means that there is a measurable set B such that $\mu(B) = 0$ and $\nu(B) = 1$.

$\mu_n \Rightarrow \nu$ denotes weak convergence of measures.

To Giseon

INTRODUCTION

Recent years have seen the development of the theory of stochastic processes in infinite dimensional spaces. Such processes have proved important both in pure and applied mathematics. However, mathematically they are far from being completely understood, and they comprise a very active area of research. Part of the difficulty of working in infinite dimensions stems from the fact that the methods to construct and study stochastic processes in finite dimensions do not usually translate well to the infinite dimensional case. Thus, when one makes the jump to infinite dimensions, new and interesting mathematical problems arise. For example, the questions of existence and regularity of processes take on new difficulties as one may now ask in which infinite dimensional space the process lives and with respect to which topology does it have continuous sample paths? Before we can hope to answer such questions we must look for approaches to the subject which do not depend on the finite dimensionality of the state space. When our process is symmetrizable, a powerful approach is to be found in the theory of Dirichlet forms.

Let us take an intuitive look at what a Dirichlet form is and how it relates to stochastic processes. The object of study in this thesis is a stationary vector-valued Markov process $\{X(t) : t > 0\}$, where t is the time parameter and by stationary we mean that $X(t)$ has the same distribution, m , at each time t . Physically this process corresponds to a system which is in a state of macroscopic equilibrium but still subject to random changes at the microscopic level. For example, in a glass of water at room temperature, no change is seen with the naked eye although the molecules are moving about very rapidly. If we wish to study the behaviour of the individual particles which make up such a system, we clearly need to know more than the macroscopic steady state. For our process $\{X(t)\}$, which has

steady state m , this additional information is provided by the Dirichlet (energy) form, which describes the process' instantaneous rate of change.

Mathematically, a Dirichlet form is a type of bilinear form over a space of the form $L^2(X; m)$. In modern potential theory it plays the role which is played by the Dirichlet integral

$$\int \nabla f \nabla g \, dx \quad (0.1)$$

in classical potential theory on Euclidean space. Fukushima [7] and Silverstein [18], [19] have established a correspondence between Dirichlet forms on $L^2(X; m)$ and m -symmetric Markov processes on X , analogous to the well known relation between the classical Dirichlet integral and Brownian motion. That is, a Dirichlet form $\mathcal{E}$ can always be written as

$$\mathcal{E}(f, g) = \left(\sqrt{-A}f, \sqrt{-A}g \right)_{L^2} \quad (0.2)$$

where A is the infinitesimal generator of a Markov process on X . In the case of Brownian motion we let $X = \mathbb{R}^d$, $A = \frac{1}{2}\Delta$ and $m =$ Lebesgue measure and we see that (0.2) reduces to $1/2$ times (0.1), at least on $C_0^\infty(\mathbb{R}^d)$. The form may also be expressed directly in terms of the process by writing

$$\mathcal{E}(f, g) = \lim_{t \downarrow 0} \frac{1}{2t} E((f(X_t) - f(X_0))(g(X_t) - g(X_0))), \quad (0.3)$$

which shows precisely the relation between $\mathcal{E}$ and the instantaneous rate of change of the process.

In chapter 1 we present, without proof, the rudiments of the general theory of Dirichlet forms as it appears in [7]. In the second chapter we construct a class of Dirichlet forms over an infinite dimensional space X , each one depending on a probability measure m and a sequence of diffusion coefficients $\{\sigma_i^2\}$. The measure m will turn out to be the

invariant measure of the associated process $\{X(t)\} = \{(X_1(t), X_2(t), \dots, X_i(t), \dots)\}$, and σ_i may be regarded as the rate of activity or 'speed' of the coordinate process $\{X_i(t)\}$. This general construction allows us enough scope to build processes which exhibit very different types of behaviour. Hence by altering σ_i , while leaving m fixed, we may study qualitative differences of processes which have the same equilibrium distribution but have different dynamics. As expected, one gets more irregularity from a process with larger diffusion coefficients.

A number of fascinating relations have been established between the properties of the Dirichlet form and the behaviour of the associated Markov process. It is especially the connection between capacity and hitting times which makes the theory so effective in questions which cannot be answered by making observations at fixed times, a hitting time being a device which allows us to 'freeze' the process in a state that it may visit for only an instant. For example, if B is a set of m -measure zero, then $P(X(t) \in B) = m(B) = 0$ for every t ; however this is no guarantee that $P(X(t) \notin B \text{ for all } t) = 1$, i.e., that B is never hit by the process. Thus we cannot decide whether or not a set is negligible, in this sense, simply by looking at its m -measure. In order to make such fine distinctions, one needs to consider the random time defined by

$$\tau_B(\omega) = \inf\{t > 0 : X_t(\omega) \in B\}, \quad (0.4)$$

which is called the hitting time of the set B . Dirichlet space theory has revealed an intimate relationship between the hitting time of a set and a quantity known as its capacity. In particular, it arises that a set is hit by the process if and only if it has non-zero capacity. Now, to determine whether or not a set has positive capacity is usually a non-trivial problem. To help us in making this decision, we will introduce a new class of functions called the functions of finite energy. Coupled with Fukushima's powerful machinery these

functions will enable us, in section 2.2., to obtain capacity versions of the law of large numbers (Prop. 2.13.) and the law of the iterated logarithm (Prop. 2.14.). This section was inspired by Fukushima [8], where he did a similar analysis of Wiener space. Our main results come in the third section of the second chapter where we give criteria (Prop. 2.17. and 2.19.) for $\{X(t)\}$ to have continuous sample paths in various infinite dimensional spaces. It is also shown (Prop. 2.19.) that every process with diffusion coefficients $\{\sigma_i^2\}$ is ℓ^2 -continuous iff $\{\sigma_i^2\}$ is bounded.

In the final chapter of the thesis we turn to the important special case when m is a Gaussian measure, which is to say, that $\{X(t)\}$ is an Ornstein-Uhlenbeck process. The infinite dimensional Ornstein-Uhlenbeck process has been studied by various authors for the past fifteen years, with several different applications in mind. For instance, it has been used to describe physical phenomena subject to random forces [3], it has appeared in infinite dimensional filtering and quantum string theory [16] and has been suggested as a model for certain biological systems [3], [21] (see also the discussion in [2]).

We will be considering the Ornstein-Uhlenbeck process which solves the infinite dimensional stochastic differential equation

$$dX(t) = -AX(t)dt + \sqrt{\Sigma}dW(t), \quad (0.5)$$

where $A = \text{diag}(\alpha_i)$ and $\Sigma = \text{diag}(\sigma_i^2)$ and $\{W(t)\}$ is a cylindrical Brownian motion. In other words,

$$X(t) = (X_1(t), X_2(t), \dots, X_i(t), \dots) \quad (0.6)$$

where

$$dX_i(t) = -\alpha_i X_i(t)dt + \sigma_i dW_i(t), \quad i \in N \quad (0.7)$$

and $\{W_i(t) : t \geq 0\}, i \in N$ are independent Wiener processes. The numbers α_i and σ_i represent the mutually antagonistic tendencies of the process $\{X_i(t)\}$, the drift coefficient α_i ,

measures the strength of the drift toward the origin while the diffusion coefficient σ_i represents the amplitude of the noise, that is, the tendency to diffuse outward. In the Ornstein-Uhlenbeck process these will balance and give a process $\{X_i(t)\}$ which, in equilibrium, has distribution $N(0, \sigma_i^2/2\alpha_i)$ at each fixed time t . The infinite dimensional process (0.6) thus lives in the space $X = \prod_{i=1}^{\infty} R_i$ and has invariant measure $m = \prod_{i=1}^{\infty} N(0, \sigma_i^2/2\alpha_i)$.

In chapter 3 we analyze this Ornstein-Uhlenbeck process from various points of view in an attempt to better understand its sample path properties. In the first section, we use the material from the second chapter and results borrowed from the field of Gaussian processes to obtain fairly exact results (Prop. 3.4.) on sample path continuity in terms of the coefficients $\{\alpha_i\}$ and $\{\sigma_i^2\}$. This is a problem to which a complete solution has not yet appeared, and to our knowledge, we present the best results now available (see also [10]). In the subsequent sections we look at the process from other angles, obtaining results for example, on its ergodic behaviour (Prop. 3.9.) and on its law of the iterated logarithm at zero (Prop. 3.21.).

We conclude by discussing some unresolved questions raised by the material in the thesis. These indicate some promising avenues of research that are still largely unexplored.

CHAPTER 1

Preliminaries and definitions

The definitions and results found in the first four sections of this chapter are taken from the book [7] of Fukushima, to which the interested reader is referred for proofs and further details.

1.1. Dirichlet forms

Let X be a topological space, $\mathcal{B}$ its Borel σ -algebra and m a probability measure on $(X, \mathcal{B})$ which charges every non-empty open set, and let $L^2 = L^2(X; m)$. $\mathcal{E}$ is called a *symmetric form* on L^2 if the following hold:

- (i) $\mathcal{E}$ is defined on $\mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E})$, where $\mathcal{D}(\mathcal{E})$ is a dense linear subspace of L^2 .
- (ii) $\mathcal{E}$ is real valued and $\mathcal{E}(f, g) = \mathcal{E}(g, f)$, $\mathcal{E}(f + g, h) = \mathcal{E}(f, h) + \mathcal{E}(g, h)$,
 $t\mathcal{E}(f, g) = \mathcal{E}(tf, g)$, $\mathcal{E}(f, f) \geq 0$ for all $f, g, h \in \mathcal{D}(\mathcal{E})$, $t \in \mathbb{R}$.

Given a symmetric form $\mathcal{E}$ and $\alpha > 0$ we define $\mathcal{E}_\alpha$ by the following:

$$\mathcal{E}_\alpha(f, g) = \mathcal{E}(f, g) + \alpha(f, g), \quad \mathcal{D}(\mathcal{E}_\alpha) = \mathcal{D}(\mathcal{E}), \quad (1.1)$$

where (f, g) means the inner product in L^2 . The form $\mathcal{E}$ is called *closed* when $(\mathcal{D}(\mathcal{E}_1), \mathcal{E}_1)$ is a Hilbert space. When $\mathcal{E}$ is closed, $(\mathcal{D}(\mathcal{E}), \mathcal{E}_\alpha)$ is a Hilbert space for all $\alpha > 0$. $\mathcal{E}$ is

called *closable* when $(D(\mathcal{E}_1), \mathcal{E}_1)$ can be completed as a Hilbert space within L^2 . Thus $\mathcal{E}$ is closable when any $\mathcal{E}$ -Cauchy sequence $\{f_n\}$ in $D(\mathcal{E})$ which converges to zero in L^2 has the property that $\mathcal{E}(f_n, f_n) \rightarrow 0$.

If $\mathcal{E}$ is closable then we define $\bar{\mathcal{E}}$ as follows:

$$\begin{aligned} D(\bar{\mathcal{E}}) &= \{f \in L^2 : \exists \{f_n\} \in D(\mathcal{E}), f_n \rightarrow f \text{ in } L^2 \\ &\quad \text{and } \mathcal{E}(f_n - f_m, f_n - f_m) \rightarrow 0 \text{ as } n, m \rightarrow \infty\} \\ \bar{\mathcal{E}}(f, g) &= \lim_n \mathcal{E}(f_n, g_n). \end{aligned} \quad (1.2)$$

$\bar{\mathcal{E}}$ is well defined and is the smallest closed extension of $\mathcal{E}$.

A symmetric form $\mathcal{E}$ is called *Markovian* if the following hold: For every $\epsilon > 0$ there exists a function $\phi_\epsilon: R \rightarrow R$ such that $\phi_\epsilon(t) = t \quad \forall t \in [0, 1]$, $-\epsilon \leq \phi_\epsilon(t) \leq 1 + \epsilon \quad \forall t \in R$ and $0 \leq \phi_\epsilon(t') - \phi_\epsilon(t) \leq t' - t$ whenever $t < t'$. Also, $f \in D(\mathcal{E}) \Rightarrow \phi_\epsilon(f) \in D(\mathcal{E})$ and $\mathcal{E}(\phi_\epsilon(f), \phi_\epsilon(f)) \leq \mathcal{E}(f, f)$. A *Dirichlet form* is a symmetric form which is Markovian and closed.

There is a one to one correspondence between the family of closed symmetric forms $\mathcal{E}$ on L^2 and the family of non-positive definite self-adjoint operators A on L^2 . The correspondence is determined by:

$$\begin{aligned} D(\mathcal{E}) &= D(\sqrt{-A}) \\ \mathcal{E}(f, g) &= (\sqrt{-A}f, \sqrt{-A}g). \end{aligned} \quad (1.3)$$

A is the generator of a strongly continuous semigroup of symmetric operators on L^2 . That is, there exists a family $\{T_t, t > 0\}$ of symmetric operators with the following properties:

- (i) The domain of T_t is L^2 for each $t > 0$.
- (ii) $T_t T_s = T_{t+s}$ for all $t, s > 0$.
- (iii) $(T_t f, T_t f) \leq (f, f)$ for all $t > 0$ and $f \in L^2$.

(iv) $(T_t f - f, T_t f - f) \rightarrow 0$ as $t \rightarrow 0$ for every $f \in L^2$.

A is the generator of T_t in the sense that

$$Af = \lim_{t \rightarrow 0} (T_t f - f)/t \quad (1.4)$$

$$D(A) = \{f \in L^2; Af \text{ exists as a strong limit}\}.$$

A also generates a strongly continuous *resolvent*. That is, there exists a family $\{G_\alpha, \alpha > 0\}$ of symmetric operators which satisfy:

- (i) The domain of G_α is equal to L^2 for each $\alpha > 0$.
- (ii) $G_\alpha - G_\beta + (\alpha - \beta)G_\alpha G_\beta = 0$ for $\alpha, \beta > 0$.
- (iii) $(\alpha G_\alpha f, \alpha G_\alpha f) \leq (f, f)$ for every $\alpha > 0$ and $f \in L^2$.
- (iv) $(\alpha G_\alpha f - f, \alpha G_\alpha f - f) \rightarrow 0$ as $\alpha \rightarrow \infty$ for every $f \in L^2$.

The relation between A and its resolvent is given by:

$$Af = \alpha f - G_\alpha^{-1} f \quad (1.5)$$

$$D(A) = G_\alpha(L^2).$$

Each T_t is Markovian, i.e., $0 \leq f \leq 1$ m-a.s. implies $0 \leq T_t f \leq 1$ m-a.s. Each G_α is Markovian as well. Both T_t and G_α map L^2 into $D(\mathcal{E})$, and the convergence $T_t f \rightarrow f$ and $\alpha G_\alpha f \rightarrow f$ take place strongly in $D(\mathcal{E})$.

One also has the following approximations: Let $\mathcal{E}^t(f, g) = (f - T_t f, g)/t$ and $\mathcal{E}^\alpha(f, g) = \alpha(f - \alpha G_\alpha f, g)$ for $t > 0$, $\alpha > 0$, and $f, g \in L^2$. Then

$$D(\mathcal{E}) = \{f \in L^2 : \lim_{t \rightarrow 0} \mathcal{E}^t(f, f) < \infty\} \quad (1.6)$$

$$\mathcal{E}(f, g) = \lim_{t \rightarrow 0} \mathcal{E}^t(f, g), \quad f, g \in D(\mathcal{E}).$$

and also

$$D(\mathcal{E}) = \{f \in L^2 : \lim_{\alpha \rightarrow \infty} \mathcal{E}^\alpha(f, f) < \infty\} \quad (1.7)$$

$$\mathcal{E}(f, g) = \lim_{\alpha \rightarrow \infty} \mathcal{E}^\alpha(f, g), \quad f, g \in D(\mathcal{E}).$$

Since m charges each non-empty open set, a bounded continuous function on X may be regarded as a member of L^2 . $\mathcal{E}$ is called *regular* if $C_0(X) \cap D(\mathcal{E})$ is uniformly dense in $C_0(X)$ and $\mathcal{E}_1$ -dense in $D(\mathcal{E})$. Here $C_0(X)$ is the space of continuous functions on X with compact support. A form $\mathcal{E}$ is called *local* if it has the property that $\mathcal{E}(f, g) = 0$ whenever f and g have compact support and $\text{supp}[f] \cap \text{supp}[g] = \emptyset$.

1.2. Capacity and quasicontinuity

Suppose that $\mathcal{E}$ is a Dirichlet form on L^2 . We define the capacity of a non-empty open set G as follows:

$$\text{Cap}(G) = \begin{cases} \inf\{\mathcal{E}_1(f, f) : f \in \mathcal{L}_G\} & \mathcal{L}_G \neq \emptyset, \\ \infty & \mathcal{L}_G = \emptyset \end{cases} \quad (1.8)$$

where $\mathcal{L}_G = \{f : f \in D(\mathcal{E}) \text{ and } f \geq 1 \text{ m-a.s. on } G\}$. When $\mathcal{L}_G$ is non-empty, it is a closed convex subset of the Hilbert space $(D(\mathcal{E}_1), \mathcal{E}_1)$ and consequently the infimum above is achieved by a unique element of $D(\mathcal{E}_1)$. This element is denoted by e_G and is called the *1-equilibrium potential* of G . It is a consequence of the Markovian property of $\mathcal{E}$ that $0 \leq e_G \leq 1$ m-a.s.

For an arbitrary set B we let

$$\text{Cap}(B) = \inf\{\text{Cap}(G) : B \subseteq G, G \text{ open}\}. \quad (1.9)$$

The set function Cap defined in this way is a Choquet (outer) capacity and so has the following properties:

- (i) $m(B) \leq \text{Cap}(B)$ for Borel B .
- (ii) $B_1 \subseteq B_2 \Rightarrow \text{Cap}(B_1) \leq \text{Cap}(B_2)$.
- (iii) $B_n \uparrow B \Rightarrow \text{Cap}(B_n) \uparrow \text{Cap}(B)$.
- (iv) $\text{Cap}(\cup_{n=1}^{\infty} B_n) \leq \sum_{n=1}^{\infty} \text{Cap}(B_n)$.

These follow for open sets by the Markovian property of $\mathcal{E}$, and for arbitrary sets by the strong subadditivity of Cap over open sets. A property which holds for all $x \in X$, except on a set of capacity zero is said to hold *quasi-everywhere* (abbreviated q.e.). Let f be a function defined q.e. on X . We call f *quasicontinuous* if there exists for any $\epsilon > 0$ an open set $G \subset X$ such that $\text{Cap}(G) < \epsilon$ and $f|_{X \setminus G}$ is continuous. Here $f|_{X \setminus G}$ denotes the restriction of f to $X \setminus G$.

A sequence of closed sets such that $F_k \uparrow$ and $\text{Cap}(X - F_k) \downarrow 0$ as $k \uparrow \infty$, is called a *nest* on X . A nest $\{F_k\}$ is said to be *m-regular* if for each k , $\text{Supp}(I_{F_k} \cdot m) = F_k$, i.e., $m(U_x \cap F_k) \neq 0$ for any $x \in F_k$ and any open neighborhood U_x of x . We say that f belongs to $C(\{F_k\})$ when $f|_{F_k}$ is a continuous function for each k . Clearly a function is quasicontinuous if and only if $f \in C(\{F_k\})$ for some nest $\{F_k\}$. In fact, if $\{f_n\}$ is a countable family of quasicontinuous functions one can find a *m-regular* nest $\{F_k\}$ such that $f_n \in C(\{F_k\})$ for every n .

We have the following results concerning these concepts.

Proposition 1.1. *If $\mathcal{E}$ is regular, then each $f \in D(\mathcal{E})$ has a quasicontinuous version, denoted by $\tilde{f}$. If $f_n \rightarrow f$ in $\mathcal{E}_1$ -norm, then $\tilde{f}_n \rightarrow \tilde{f}$ q.e.*

Proposition 1.2. *If $\mathcal{E}$ is regular, then every Borel set B has a unique 1-equilibrium*

potential $e_B \in D(\mathcal{E})$, which minimizes the $\mathcal{E}_1$ -norm over the set

$$\{f : f \in D(\mathcal{E}) \text{ and } \bar{f} \geq 1 \text{ q.e. on } B\} \quad (1.10)$$

and such that $\text{Cap}(B) = \mathcal{E}_1(e_B, e_B)$.

1.3. The link with Markov processes

There is an intimate connection between the theory of Dirichlet forms on $L^2(X; m)$ and the theory of m -symmetric Hunt processes on X . In the previous two sections we have seen what a Dirichlet form is, let us now consider the Hunt process side of it.

Let X be a locally compact metrizable space and $X_\Delta (= X \cup \Delta)$ its one point compactification. If X is already compact we adjoin Δ as an isolated point. Let S be a Borel subset of X and regard $S_\Delta (= S \cup \Delta)$ as a topological subspace of X_Δ . We define $\mathcal{S}$ as the family of all Borel subsets of S and $S_\Delta = \mathcal{S} \cup \{B \cup \Delta : B \in \mathcal{S}\}$.

Let $T = [0, \infty]$ be the time parameter and let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in T})$ be a filtered measurable space. That is, $\mathcal{F}$ is a σ -algebra of subsets of Ω , and $\{\mathcal{F}_t\}_{t \in T}$ is a non-decreasing sequence of sub σ -algebras of $\mathcal{F}$. A function $\sigma : \Omega \rightarrow [0, \infty]$ is called an $\{\mathcal{F}_t\}$ -stopping time if $\{\omega \in \Omega : \sigma(\omega) \leq t\} \in \mathcal{F}_t$ for all $t \in T$. We define an associated sub σ -algebra by $\mathcal{F}_\sigma = \{F \in \mathcal{F} : F \cap \{\sigma \leq t\} \in \mathcal{F}_t \text{ } \forall t \in T\}$. The filtration $\{\mathcal{F}_t\}$ is said to be right continuous if $\mathcal{F}_t = \mathcal{F}_{t+} = (\bigcap_{t' > t} \mathcal{F}_{t'})$.

$$M = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in T}, \{X_t\}_{t \in T}, \{P_x\}_{x \in S_\Delta})$$

is called a *Hunt process* on S if it satisfies the following:

(i) For each $x \in S_\Delta$, $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in T}, \{X_t\}_{t \in T}, P_x)$ is an adapted, S -valued stochastic process. That is, P_x is a probability measure on $(\Omega, \mathcal{F})$ and for each $t \in T$, $X_t: (\Omega, \mathcal{F}_t) \rightarrow (S_\Delta, S_\Delta)$ is a measurable mapping.

(ii) $P_x(X_t \in E)$ is $\mathcal{B}$ -measurable in $x \in S$ for each $t \geq 0$ and $E \in \mathcal{B}$.

(iii) $\{\mathcal{F}_t\}$ is right continuous.

(iv) $P_\Delta(X_t = \Delta) = 1 \quad \forall t \in T$.

(v) $P_x(X_0 = x) = 1 \quad \forall x \in S$.

(vi) $X_\infty(\omega) = \Delta \quad \forall \omega \in \Omega$.

(vii) $X_t(\omega) = \Delta \quad \forall t \geq \zeta(\omega)$ where $\zeta(\omega) = \inf\{t \geq 0 : X_t(\omega) = \Delta\}$.

(viii) For each $t \in T$, there exists a map $\theta_t: \Omega \rightarrow \Omega$ such that $X_s \circ \theta_t = X_{s+t}$, $s \geq 0$.

(ix) For each $\omega \in \Omega$, the sample path $t \rightarrow X_t(\omega)$ is right continuous on $[0, \infty)$ and has left limits on $(0, \infty)$ inside S_Δ .

(x) $P_x(X_{\sigma+s} \in E \mid \mathcal{F}_\sigma) = P_{X_\sigma}(X_s \in E)$, for each $s \geq 0$, $E \in S_\Delta$ and each $\{\mathcal{F}_t\}$ -stopping time σ .

(xi) For any $\{\mathcal{F}_t\}$ -stopping times σ_n increasing to σ we have

$$P_x(\lim_{n \rightarrow \infty} X_{\sigma_n} = X_\sigma, \sigma < \infty) = P_x(\sigma < \infty) \text{ for } x \in S_\Delta.$$

The function ζ which appears above in (vii) is called the *lifetime* of the process and the functions θ_t from (viii) are called the *shift operators*. Property (x) is called the *strong Markov property* and (xi) is called *quasi-left continuity*. We define the measure P_m by letting $P_m(B) = \int P_x(B) m(dx)$. The symbols E_x and E_m refer to expectations taken with respect to P_x and P_m , respectively.

Given a Hunt process we may define its *transition function* by $p_t(x, E) = P_x(X_t \in E)$ for $x \in S$, $t \in T$, and $E \in \mathcal{S}$. For a non-negative Borel function $f \in L^2$, we let $p_t f(x) = \int f(y) p_t(x, dy) = E_x(f(X_t))$. The transition function is said to be *m-symmetric* if

$$\int p_t f(x) g(x) m(dx) = \int f(x) p_t g(x) m(dx) \quad (1.11)$$

for all non-negative Borel functions f and g . By symmetry, each operator $f \mapsto p_t f$ can be extended from $C_0(X)$ to an operator T_t on L^2 . If $p_t f \rightarrow f$ pointwise for every $f \in C_0(X)$, then $\{T_t\}$ is a strongly continuous contraction semigroup and gives rise to a Dirichlet form $\mathcal{E}$ related to it as in (1.3) and (1.4). In the other direction we have the result in Proposition 1.3. We say that a Hunt process is *properly associated* with a Dirichlet form $\mathcal{E}$ if, for each non-negative $f \in L^2$ and each $t > 0$, $p_t f$ is a quasicontinuous version of $T_t f$.

Proposition 1.3. *If X is a separable metrizable locally compact space and $\mathcal{E}$ is a regular Dirichlet form on L^2 , then there exists a unique X -valued Hunt process $\{X_t\}$ properly associated with $\mathcal{E}$.*

The uniqueness is meant in the sense that two m -symmetric Hunt processes having the same regular Dirichlet space over L^2 , have a common properly exceptional set (see 1.19) outside of which their transition functions coincide.

When there is no killing, i.e., $T_t 1 = 1$, we may recover $\mathcal{E}$ from the associated process as follows:

$$\mathcal{E}(f, g) = \lim_{t \rightarrow 0} 1/2t E_m ((f(X_t) - f(X_0))(g(X_t) - g(X_0))) \quad (1.12)$$

$$D(\mathcal{E}) = \{f \in L^2 : \mathcal{E}(f, f) \text{ exists and is finite}\}$$

From this expression we see that $\mathcal{E}$ measures the instantaneous rate of change of the process. That is, it shows how fast and in what directions $\{X_t\}$ travels.

The behaviour of the process $\{X_t\}$ and the properties of $\mathcal{E}$ are closely related. For example, it is a consequence of the symmetry of $\mathcal{E}$ that $\{X_t\}$ is time reversible in the following sense:

If $f_0, f_1, \dots, f_n$ are non-negative Borel functions on X and $0 < t_1 < t_2 < \dots < t_n$, then

$$\begin{aligned} & \int E_x(f_0(X_0)f_1(X_{t_1}) \dots f_{n-1}(X_{t_{n-1}})f_n(X_{t_n})) m(dx) \\ &= \int E_x(f_n(X_0)f_{n-1}(X_{t_n-t_{n-1}}) \dots f_1(X_{t_2-t_1})f_0(X_{t_1})) m(dx). \end{aligned} \quad (1.13)$$

From this equation we deduce that if there is no killing inside X , then m is an invariant measure for the process. In order to see this, let B be any Borel subset of X and let $t > 0$.

Taking $n = 1$, $t_1 = t$, $f_0 = 1$ and $f_1 = 1_B$ above we get

$$\begin{aligned} & \int E_x(X_t \in B) m(dx) \\ &= \int E_x(X_0 \in X, X_t \in B) m(dx) \\ &= \int E_x(X_0 \in B, X_t \in X) m(dx) \\ &= \int E_x(X_0 \in B) m(dx) \\ &= m(B). \end{aligned} \quad (1.14)$$

The second last equality holds if there is no killing in X , and the last equality is because of the property (v) of the Hunt process.

1.4. Capacity and hitting times

In this section we suppose that we are given a regular Dirichlet form $\mathcal{E}$, and an X -valued Hunt process $\{X_t\}$ properly associated with it, along with all the usual conditions on X and m . In the previous section, we have seen some of the ways in which $\mathcal{E}$ and $\{X_t\}$ are related. In this section we consider sets of zero capacity and show that they are negligible as far as the process is concerned. In a sense to be made precise below, sets of no capacity are exactly those sets on which we may alter $\{X_t\}$ without affecting its equilibrium behaviour.

The *hitting time* of a Borel set B is defined to be

$$\tau_B(\omega) = \inf\{t > 0 : X_t(\omega) \in B\}. \quad (1.15)$$

The α -order *hitting distribution* H_α^B of B is given by

$$H_\alpha^B(x, E) = E_x(e^{-\alpha\tau_B} 1_E(X_{\tau_B})), \quad \alpha \geq 0, x \in X, E \in \mathcal{B}. \quad (1.16)$$

When $\alpha = 0$ we simply write H^B . For a non-negative Borel function f we let

$$H_\alpha^B f(x) = E_x(e^{-\alpha\tau_B} f(X_{\tau_B})). \quad (1.17)$$

A set N is called *exceptional* if it is contained in a Borel set B and

$$\int P_x(\tau_B < \infty) m(dx) = 0. \quad (1.18)$$

This says that for almost every starting point x , the process will not hit B (nor N !). A set N is called *properly exceptional* if N is Borel, $m(N) = 0$, and

$$P_x(X_t(\omega) \in X \setminus N, X_{t-}(\omega) \in X \setminus N \forall t \geq 0) = 1 \quad \forall x \in X \setminus N. \quad (1.19)$$

Here X_{t-} refers to the left limit which exists by (ix) from page 12 in the definition of the Hunt process.

The results in Propositions 1.4. and 1.7. below, form the basis of the approach taken in this thesis to the study of stochastic processes. For the class of Hunt processes $\{X_t\}$ covered by Fukushima's theory, these results allow us to draw probabilistic conclusions from analytic facts. Once we have the form $\mathcal{E}$ associated with $\{X_t\}$, we may use the material in Section 1.2. to determine if a given set has capacity zero or whether a given function is quasicontinuous. We are then able to use these two propositions to get information about the nature of the process.

To be specific, if a set B has zero capacity then from 1.4. and (1.18) we see that $P_m(\tau_B < \infty) = 0$. This means that the set of starting points from which B may be hit has m measure zero. Thus, B is not hit by the process in equilibrium (i.e., under P_m) and so is negligible in this sense. In fact, from the last sentence in 1.7. and the quasicontinuity of $H_1^B 1$, we see that the set of starting points from which the process may hit an exceptional set is itself a set of capacity zero.

Proposition 1.4. *A set N is exceptional iff it is contained in a properly exceptional set. $H_1^B 1(x) = E_x(e^{-\tau_B})$ is a quasicontinuous version of the function e_B given in Proposition 1.2. A Borel set B is exceptional iff $\text{Cap}(B) = 0$.*

A set N is called *polar* if it is contained in a Borel B such that $P_x(\tau_B < \infty) = 0$ for all $x \in X$. Clearly a polar set is exceptional but the converse is not true in general. In fact, we have the following:

Proposition 1.5. *The following are equivalent:*

- (i) *Every exceptional set is polar.*
- (ii) *$p_t(x, \cdot) \ll m(\cdot)$ for all $t > 0$ and $x \in X^0$.*

δ

We conclude this section with some topics related to the fine topology. Blumenthal's 0-1 law tells us that for each Borel set B and each $x \in X$, the probability $P_x(r_B = 0)$ is equal to either zero or one. In the latter case, x is called *regular* for B , and the set of regular points of B is denoted B^r . These are the points from which $\{X_t\}$ will enter immediately into B , in that sense they are the points close to B . It turns out that 'most' points in B are of this type as we have:

Proposition 1.6. *For every Borel B , the set $B \setminus B^r$ is exceptional.*

A set A is called *finely open* if for each $x \in A$ there exists a Borel set B such that B contains $X \setminus A$ and $P_x(r_B > 0) = 1$. Because of the right continuity of the sample paths of $\{X_t\}$, we see that the fine topology is finer than the original topology on X .

Proposition 1.7. *If f is quasicontinuous, then f is finely continuous q.e. That is, there exists a properly exceptional set N such that for any $x \in X \setminus N$,*

$$P_x(f(X_t) \text{ is right continuous and } \lim_{s \uparrow t} f(X_s) = f(X_{t-}) \forall t \geq 0) = 1. \quad (1.20)$$

If f is quasicontinuous and $f \geq 0$ m-a.s., then $f \geq 0$ q.e.

Finally we state the result that links sample path continuity of $\{X_t\}$ with the local property of $\mathcal{E}$.

Proposition 1.8. *The form $\mathcal{E}$ has the local property iff there exists a properly exceptional set N such that*

$$P_x(X_t \text{ is continuous in } t \in [0, \zeta)) = 1, \quad (1.21)$$

for every $x \in X \setminus N$. There is no killing if the following stronger property holds: If f and $g \in \mathcal{D}(\mathcal{E})$ both have compact support and if f is constant on a neighborhood of $\text{supp}[g]$, then $\mathcal{E}(f, g) = 0$.

1.5. Dirichlet forms in finite dimensions

Before we tackle infinite dimensions, let us gather some results about Dirichlet forms on R^d . We begin with some basic results on approximations by smooth functions.

Definition 1.9. Let $j \in C_0^\infty(R^d)$ such that $j(x) \geq 0$ for all x , and $\int j(x) dx = 1$. For $\delta > 0$, let $j_\delta(x) = \delta^{-d} j(\delta^{-1}x)$. The family $\{j_\delta : \delta > 0\}$ is a smooth approximate identity. For f in $L_{loc}^1(dy)$, we define f_δ by the convolution of f with j_δ , i.e.,

$$f_\delta(x) = (f * j_\delta)(x) = \int j_\delta(x - y) f(y) dy = \int j_\delta(y) f(x - y) dy. \quad (1.22)$$

Proposition 1.10.

(i) $f_\delta \in C^\infty$ and has compact support if f does. If f is continuous, then $f_\delta \rightarrow f$ uniformly on compacts as $\delta \rightarrow 0$.

(ii) If m is locally integrable and f is bounded and has compact support, then

$$\int |f_\delta - f|^2 |m| \rightarrow 0 \text{ as } \delta \rightarrow 0.$$

(iii) If $d = 1$ and f is absolutely continuous with bounded derivative, then

$$(f_\delta)' = (f')_\delta. \text{ If } d \geq 1 \text{ and } f \in C_b^1(R^d), \text{ then } \nabla(f_\delta) = (\nabla f)_\delta.$$

Proof: (i) This follows from well known properties of the convolution operation.

(ii) By the change of variables $r = y/\delta$ we have

$$f_\delta(x) - f(x) = \int (f(x-y) - f(x)) j_\delta(y) dy = \int (f(x-\delta r) - f(x)) j(r) dr. \quad (1.23)$$

Therefore,

$$\begin{aligned} \int |f_\delta(x) - f(x)|^2 |m(x)| dx &= \int \left| \int (f(x-\delta r) - f(x)) j(r) dr \right|^2 |m(x)| dx \\ &\leq \int j^2(r) \int |f(x-\delta r) - f(x)|^2 |m(x)| dx dr, \end{aligned} \quad (1.24)$$

for any locally integrable m . By standard Lebesgue integration theory, the integrand in r goes to zero pointwise as $\delta \rightarrow 0$. By bounded convergence, so does the integral, which gives us (ii).

(iii) For $t \in R$ and $h > 0$ we have,

$$\begin{aligned} 1/h (f_\delta(t+h) - f_\delta(t)) &= \int j_\delta(s) 1/h (f(t+h-s) - f(t-s)) ds \\ &= \int j_\delta(s) 1/h \int_{t-s}^{t+h-s} f'(r) dr ds. \end{aligned} \quad (1.25)$$

Now $1/h \int_{t-s}^{t+h-s} f'(r) dr \rightarrow f'(t-s)$ for Lebesgue a.e. $s \in R$ as $h \rightarrow 0$, and boundedly by the assumption that f' is bounded. Therefore, the integral converges and we get $(d^+/dx)f_\delta(t) = (j_\delta * f')(t)$. A similar argument for $h < 0$ gives the required result. The second assertion in (iii) follows in an analogous manner. □

Definition 1.1. As a tool for checking that a given form is Markovian, we fix a collection $\{\phi_\epsilon : \epsilon > 0\}$ of $C^\infty(R)$ functions by

$$\phi_\epsilon = j_\epsilon * (((-\epsilon) \vee t) \wedge (1 + \epsilon)), \quad (1.26)$$

where δ is so small that the diameter of the support of j_δ is less than ϵ . It is now easy to check that $\phi_\epsilon(t) = t$, $\forall t \in [0, 1]$; $-\epsilon \leq \phi_\epsilon(t) \leq 1 + \epsilon$, $\forall t \in \mathbb{R}$ and $0 \leq \phi_\epsilon(t') - \phi_\epsilon(t) \leq t' - t$ whenever $t < t'$.

Proposition 1.12. *Let $\sigma^2 > 0$ and m a non-negative Borel function such that $m \in L^1_{loc}(dt)$ and $1/m \in L^1_{loc}(dt)$. The form*

$$\begin{aligned} \mathcal{D}(\mathcal{E}) &= C_0^\infty(\mathbb{R}) \\ \mathcal{E}(f, g) &= \sigma^2/2 \int f'(t)g'(t)m(t) dt. \end{aligned} \tag{1.27}$$

is a symmetric Markovian form on $L^2(m(t)dt)$. It is closable and its closure is given by

$$\begin{aligned} \mathcal{D}(\bar{\mathcal{E}}) &= \{f \in L^2 : f \text{ is absolutely continuous and } f' \in L^2\} \\ \bar{\mathcal{E}}(f, g) &= \sigma^2/2 \int f'(t)g'(t)m(t) dt. \end{aligned} \tag{1.28}$$

Comment: In the one dimensional case, the problem of the closability of forms of the type (1.27) is completely solved, even with the condition $1/m \in L^1_{loc}(dt)$ removed. In this case the description of $\mathcal{D}(\bar{\mathcal{E}})$ is more delicate. For further material on these and related topics see [7; p. 47], [17].

Proof: The symmetry is clear. To show the Markovian property we use the functions given by Definition 1.11. Take $f \in C_0^\infty(\mathbb{R})$ and consider the composition $\phi_\epsilon(f)$. Then $\phi_\epsilon(f) \in \mathcal{D}(\mathcal{E})$ and $|\phi_\epsilon(f)'| = |\phi_\epsilon'(f)f'| \leq |f'|$ everywhere, as $|\phi_\epsilon'| \leq 1$. Hence,

$$\begin{aligned} \mathcal{E}(\phi_\epsilon(f), \phi_\epsilon(f)) &= \sigma^2/2 \int (\phi_\epsilon(f)'(t))^2 m(t) dt \\ &\leq \sigma^2/2 \int (f'(t))^2 m(t) dt \\ &= \mathcal{E}(f, f), \end{aligned} \tag{1.29}$$

so that $\mathcal{E}$ is seen to be Markovian.

To prove the second part, we must show that $(\mathcal{D}(\bar{\mathcal{E}}), \bar{\mathcal{E}}_1)$ is the smallest closed extension of $(\mathcal{D}(\mathcal{E}), \mathcal{E}_1)$. We start by showing that $(\mathcal{D}(\bar{\mathcal{E}}), \bar{\mathcal{E}}_1)$ is a Dirichlet form. As in the first

part, it is easily seen to symmetric and Markovian. We show that it is closed. Consider the following inequality which holds for $f \in D(\bar{\mathcal{E}})$ and $x < y \in R$.

$$(f(x) - f(y))^2 = \left(\int_x^y f'(t) dt \right)^2 \leq \left(\int_x^y (f'(t))^2 m(t) dt \right) \left(\int_x^y \frac{dt}{m(t)} \right) \quad (1.30)$$

Now if $\{f_n\}$ is $\bar{\mathcal{E}}_1$ -Cauchy, then we take f to be its limit in L^2 , and we define $f' = \lim_n f'_n$, also an L^2 limit. By taking a subsequence we may assume that f_n converges to f at some point x . But then by (1.30), we see that $|(f_n - f_m)(x) - (f_n - f_m)(y)| \rightarrow 0$ as $n, m \rightarrow \infty$ so that $\{f_n\}$ converges pointwise everywhere. Take f to be the version which is the pointwise limit. Then for $t > 0$,

$$f(t) = \lim_n f_n(t) = \lim_n f_n(0) + \int_0^t f'_n(s) ds = f(0) + \int_0^t f'(s) ds, \quad (1.31)$$

and similarly when $t < 0$. The convergence of the integrals follows by putting $f_n - f$ into (1.30). Thus f is absolutely continuous and has derivative f' , which shows that $f \in D(\bar{\mathcal{E}})$, and clearly $\bar{\mathcal{E}}_1(f_n - f, f_n - f) \rightarrow 0$ as $n \rightarrow \infty$. This proves that $(D(\bar{\mathcal{E}}), \bar{\mathcal{E}}_1)$ is a closed form.

We conclude by showing that $C_0^\infty(R)$ is $\bar{\mathcal{E}}_1$ -dense in $D(\bar{\mathcal{E}})$. It suffices to show that $C_0^\infty(R)$ is dense in the bounded functions of $D(\bar{\mathcal{E}})$ since these are already dense ([7; Theorem 1.4.2 (iii)]). Let $f \in D(\bar{\mathcal{E}})$ be bounded, and take K a compact set so large that $\int_{R \setminus K} f^2(t) + (f')^2 m(t) dt < \epsilon/2$. Let $h \in C_0^\infty$ such that $0 \leq h(t) \leq 1$ and $|h'(t)| \leq 1$ for all $t \in R$ and $h(t) = 1$ for $t \in K$. By Proposition 1.10. (ii) we see that $(fh)_\delta \rightarrow fh$ and $(fh)_\delta' \rightarrow (fh)'$ in L^2 as $\delta \rightarrow 0$. Let $\|\cdot\|$ denote the norm in $L^2(m)$. Since $\|fh - f\| < \epsilon$ and $\|(fh)' - f'\| \leq \|f'h - f'\| + \|h'f\| \leq \epsilon/2 + \epsilon/2 = \epsilon$, by taking δ small enough we can guarantee that $\|(fh)_\delta - f\| < \epsilon$ and $\|(fh)_\delta' - f'\| < \epsilon$. Therefore $\bar{\mathcal{E}}_1(f - (fh)_\delta, f - (fh)_\delta) \leq \sigma^2/2\epsilon + \epsilon$ and since ϵ is arbitrary we see that $C_0^\infty(R)$ is dense in $D(\bar{\mathcal{E}})$. $\square$

Let us now consider the case $d > 1$ and a form $\mathcal{E}$ given by:

$$\begin{aligned} \mathcal{D}(\mathcal{E}) &= C_0^\infty(R^d) \\ \mathcal{E}(f, g) &= 1/2 \int \sum_{i=1}^d \sigma_i^2 \left(\frac{\partial}{\partial x_i} f \right)(x) \left(\frac{\partial}{\partial x_i} g \right)(x) m(dx). \end{aligned} \quad (1.32)$$

Using the functions ϕ_e defined in 1.11., one may easily confirm that $\mathcal{E}$ is indeed a symmetric Markovian form. In order to get closability we use a multidimensional analogue of the condition that $1/m$ is locally integrable.

Let y be a vector in R^d , and let $\eta: R^d \rightarrow R^d$ be the projection onto the orthogonal complement of y , which we write as $\langle y \rangle^\perp$. We let $\eta \oplus ty$ denote the orthogonal decomposition of a generic element $x \in R^d$. Now let $\{\rho(\cdot|\eta)\}$ be a regular conditional probability distribution of m given $\sigma\{\langle y \rangle^\perp\}$. By regularity, $\rho(\cdot|\eta)$ only charges $\{\eta \oplus ty : t \in R\}$ which we identify with the real line. Letting $\nu = m \circ \eta^{-1}$, we get the decomposition, for integrable f ,

$$\int f(x) m(dx) = \int_{\eta(R^d)} \int_R f(\eta \oplus ty) \rho(dt|\eta) \nu(d\eta). \quad (1.33)$$

Condition 1.13. Suppose that for ν -a.e. η , $\rho(\cdot|\eta)$ is absolutely continuous with respect to Lebesgue measure and its density $\rho(\cdot|\eta)$ has the property that $1/\rho(\cdot|\eta) \in L_{loc}^1(dt)$.

It has been shown that if 1.13. holds for enough vectors, then the form (1.32) is closable. An exact description of its closure is not as simple as in the one dimensional case. Naturally we no longer have the concept of absolute continuity, however the following obtains:

Proposition 1.14. *If Condition 1.13. holds for m and each e_i in the standard basis of R^d , then the form (1.32) is closable. If it holds for m and y , then every $f \in \mathcal{D}(\mathcal{E})$ has a quasicontinuous version f_y such that $f_y(x + ty)$ is an absolutely continuous function of t , for m almost every $x \in R^d$.*

Proof: The first assertion follows from the paper of Kusuoka [14], where it is proved in infinite dimensions. As for the second part, consider the following inequality which is analogous to (1.30) from the 1-dimensional case.

$$\begin{aligned} (f(\eta \oplus sy) - f(\eta \oplus ry))^2 &\leq \left(\int_r^s \left(\frac{\partial}{\partial y} f \right) (\eta \oplus ty) dt \right)^2 \\ &\leq \left(\int_r^s \left(\frac{\partial}{\partial y} f \right)^2 \rho(t|\eta) dt \right) \left(\int_r^s \frac{dt}{\rho(t|\eta)} \right). \end{aligned} \quad (1.34)$$

Now let $f \in \mathcal{D}(\tilde{\mathcal{E}})$ and let f_n be smooth functions converging to f in $\tilde{\mathcal{E}}_1$ -norm. By taking a subsequence, if necessary, we may assume that f_n converges q.e. to $\tilde{f}$. We claim that the function $f_\nu = \limsup f_n$ has the required properties. Since $\frac{\partial}{\partial y} f_n = \sum_{i=1}^d y_i \frac{\partial}{\partial x_i} f_n$ converges in L^2 , we have that for ν -a.e. η ,

$$\int \left(\frac{\partial}{\partial y} f_n - \frac{\partial}{\partial y} f_m \right)^2 (\eta \oplus ty) \rho(dt|\eta) \rightarrow 0. \quad (1.35)$$

Let A be a set of full ν measure such that $\eta \in A$ implies that $\rho(dt|\eta)$ is absolutely continuous with respect to Lebesgue measure, $1/\rho(\cdot|\eta) \in L^1_{loc}(dt)$, $f_n(\eta \oplus sy)$ converges for some s , and (1.35) holds. From (1.34) we see that if $\eta \in A$, then $f_n(\eta \oplus sy)$ is uniformly convergent on compacts as a function of s . Define $\frac{\partial}{\partial y} f_\nu(\eta \oplus sy)$ as the $L^2(\rho(dt|\eta))$ limit of $\frac{\partial}{\partial y} f_n(\eta \oplus sy)$, then using (1.34) and the absolute continuity of $f_n(\eta \oplus sy)$ we get for $t > s$,

$$\begin{aligned} f_\nu(\eta \oplus ty) &= \lim_{n \rightarrow \infty} f_n(\eta \oplus ty) \\ &= \lim_{n \rightarrow \infty} f_n(\eta \oplus sy) + \int_s^t \left(\frac{\partial}{\partial y} f_n \right) (\eta \oplus ry) dr \\ &= f_\nu(\eta \oplus sy) + \int_s^t \left(\frac{\partial}{\partial y} f_\nu \right) (\eta \oplus ry) dr, \end{aligned} \quad (1.36)$$

and similarly when $t < s$. □

1.6. The Ornstein-Uhlenbeck process

Fukushima's existence result (Proposition 1.3.) tells us roughly that there is a Hunt process (X_t, P_x) properly associated with every regular Dirichlet form $\mathcal{E}$. The corresponding uniqueness result [7; Theorem 4.3.6] says that any two m -symmetric Hunt processes, with the same regular Dirichlet space over L^2 , have a common properly exceptional set outside of which their transition functions coincide. This is as unique as you can get within Dirichlet space theory but it allows for a certain amount of freedom. If singletons are polar, for example, then for a fixed x one can replace P_x by the measure $\tilde{P}_x$ defined by $\tilde{P}_x(X_t(\omega) = x \quad \forall t < \infty) = 1$, without changing the Dirichlet form. As a result, the question 'How does the process behave when started at x ?' is ill-posed.

In some cases, however, we may begin with the transition function of a Hunt process and prove that it is properly associated with some regular Dirichlet form $\mathcal{E}$. When that happens, we may be able to choose a 'nice' or 'canonical' family $\{P_x\}_{x \in X}$, and using this version of the Hunt process we can legitimately ask what happens when we begin at a particular starting point.

The above comments serve as an explanation of why we are about to go through the trouble of constructing a Dirichlet form from a given transition function rather than relying on Proposition 1.3. which gives the transition function from the Dirichlet form.

Let $p(t, x, \cdot)$ be the transition function for a 1-dimensional Brownian motion, i.e.,

$$p(t, x, B) = 1/(\sqrt{2\pi t}) \int_B \exp(-|y - x|^2/2t) dy, \quad (1.38)$$

for $t > 0$, $x \in \mathbb{R}$ and $B \in \mathcal{B}$.

Definition 1.15. Fix constants $\alpha, \sigma > 0$. We define the Ornstein-Uhlenbeck transition function with drift α and diffusion σ as

$$o(t, x, B) = p\left(\frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t}), e^{-\alpha t}x, B\right), \quad (1.39)$$

for $t > 0, x \in R$ and $B \in \mathcal{B}$. In other words, for fixed x and t , $o(t, x, \cdot) = N(xe^{-\alpha t}, \sigma^2/2\alpha(1 - e^{-2\alpha t}))$.

We begin by showing that $\{o(t, x, \cdot)\}$ has the semigroup property, using the semigroup property of $\{p(t, x, \cdot)\}$ and the scaling property $p(t, cx, cB) = p(t/c^2, x, B)$. If $x \in R$, $s, t > 0$ and $B \in \mathcal{B}$, then

$$\begin{aligned} & \int o(s, x, dy) o(t, y, B) \\ &= \int p\left(\frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha s}), e^{-\alpha s}x, dy\right) o(t, y, B) \\ &= \int p\left(\frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha s}), e^{\alpha t}\left(e^{-\alpha(t+s)}x\right), e^{\alpha t}d(e^{-\alpha t}y)\right) o(t, y, B) \\ &= \int p\left(\frac{\sigma^2}{2\alpha}(e^{-2\alpha t} - e^{-2\alpha(t+s)}), e^{-\alpha(t+s)}x, d(e^{-\alpha t}y)\right) p\left(\frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t}), e^{-\alpha t}y, B\right) \\ &= p\left(\frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha(s+t)}), e^{-\alpha(s+t)}x, B\right) \\ &= o(s+t, x, B). \end{aligned} \quad (1.40)$$

This transition function is m -symmetric where $m = N(0, \sigma^2/2\alpha)$. To prove this it suffices to show that the measure $o(t, x, dy)m(dx)$ on R^2 has a density with respect to Lebesgue measure which is symmetric in x and y . Except for a constant this density is given by

$$\begin{aligned} & \exp\left(-\frac{\alpha}{\sigma^2} \frac{(y - e^{-\alpha t}x)^2}{(1 - e^{-2\alpha t})}\right) \exp\left(-\frac{\alpha}{\sigma^2} x^2\right) \\ &= \exp\left(-\frac{\alpha}{\sigma^2} \left(\frac{(y - e^{-\alpha t}x)^2}{(1 - e^{-2\alpha t})} + x^2\right)\right) \\ &= \exp\left(-\frac{\alpha}{\sigma^2} \left(\frac{y^2 - 2xye^{-\alpha t} + e^{-2\alpha t}x^2 + x^2 - e^{-2\alpha t}x^2}{(1 - e^{-2\alpha t})}\right)\right) \\ &= \exp\left(-\frac{\alpha}{\sigma^2} \left(\frac{y^2 - 2xye^{-\alpha t} + x^2}{(1 - e^{-2\alpha t})}\right)\right), \end{aligned} \quad (1.41)$$

which proves the result.

Let us now look at the regularity of this kernel. If f is a non-negative (or bounded) Borel function, then we define

$$o_t f(x) = k_t \int f(y) \exp \left(-\frac{\alpha (y - e^{-\alpha t} x)^2}{\sigma^2 (1 - e^{-2\alpha t})} \right) dy, \quad (1.42)$$

where $k_t = 1/\sqrt{2\pi\sigma^2/2\alpha(1 - e^{-2\alpha t})}$. Since we can write $o_t f(x) = k_t (f * g)(e^{-\alpha t} x)$ where g is the smooth function $g(w) = \exp(-2\alpha w^2/2\sigma^2(1 - e^{-2\alpha t}))$, we conclude that $o_t f \in C^\infty$.

Furthermore,

$$(d/dx)o_t f(x) = k_t \frac{2\alpha}{\sigma^2(1 - e^{-2\alpha t})} e^{-\alpha t} \int f(y) 2(y - e^{-\alpha t} x) \exp \left(-\frac{\alpha (y - e^{-\alpha t} x)^2}{\sigma^2 (1 - e^{-2\alpha t})} \right) dy, \quad (1.43)$$

which is bounded if f is. Thus $f \mapsto o_t f$ maps bounded functions with compact support into $C_b^1(\mathbb{R})$.

Now let us consider regularity of $o_t f$ in t . For $f \in C_0^\infty$ we get

$$o_t f(x) - f(x) = k_t \int (f(y) - f(x)) \exp \left(-\frac{\alpha (y - e^{-\alpha t} x)^2}{\sigma^2 (1 - e^{-2\alpha t})} \right) dy. \quad (1.44)$$

By performing the change of variables, $w = (y - e^{-\alpha t} x)/\sqrt{\sigma^2/2\alpha(1 - e^{-2\alpha t})}$, and expanding f in a partial Taylor's series we get the following, where $\mu = N(0, 1)$.

$$\begin{aligned} o_t f(x) - f(x) &= 1/\sqrt{2\pi} \int \left\{ f \left(e^{-\alpha t} x + \sqrt{\frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t})} w \right) - f(x) \right\} \mu(dw) \\ &= 1/\sqrt{2\pi} \int f'(x) \left\{ \sqrt{\frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t})} w + (e^{-\alpha t} - 1)x \right\} \mu(dw) \\ &\quad + \frac{1}{2\sqrt{2\pi}} \int f''(x) \left\{ \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t}) w^2 + 2wx(e^{-\alpha t} - 1)\sqrt{\frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t})} + (e^{-\alpha t} - 1)^2 x^2 \right\} \mu(dw) \\ &\quad + o(t). \end{aligned} \quad (1.45)$$

The last term is $o(t)$ because it is $O\left(\left(\sqrt{\sigma^2/2\alpha}(1 - e^{-2\alpha t})w + (e^{-\alpha t} - 1)x\right)^3\right)$.

We see, first of all, that $o_t f \rightarrow f$ pointwise, and using the m -symmetry one can conclude that $f \mapsto o_t f$ can be extended to a bounded operator T_t on $L^2(m)$, and that $\{T_t\}$ is a strongly continuous semigroup. In particular, it gives rise to a Dirichlet form $\mathcal{E}$. Now, because μ has mean zero, the integrals with a factor of w in them vanish so

$$\begin{aligned} o_t f(x) - f(x) &= 1/\sqrt{2\pi} \int f'(x)(e^{-\alpha t} - 1)x\mu(dw) \\ &\quad + 1/2\sqrt{2\pi} \int f''(x) \left\{ \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t})w^2 + (e^{-\alpha t} - 1)^2 x^2 \right\} \mu(dw) \\ &\quad + o(t). \end{aligned} \tag{1.46}$$

Noting that $(e^{-\alpha t} - 1)^2$ is $o(t)$, if we divide the above expression by t and let it go to zero, we find

$$Af(x) = \lim_{t \downarrow 0} (o_t f(x) - f(x))/t = (\sigma^2/2)f''(x) - \alpha x f'(x). \tag{1.47}$$

For $f \in C_0^\infty(R)$, this convergence takes place in $L^2(m)$ and so by (1.4) we find that $C_0^\infty(R) \subset \mathcal{D}(A) \subset \mathcal{D}(\mathcal{E})$.

We will now confirm that on $C_0^\infty(R)$, our form $\mathcal{E}$ is equal to $\sigma^2/2 \int \tilde{f}'g'dm$. We let $\phi(x) = 1/\sqrt{2\pi\sigma^2/2\alpha} \exp(-2\alpha x^2/2\sigma^2)$ and note that $\phi'(x) = -2\alpha x/\sigma^2 \phi(x)$. Using

integration by parts we get,

$$\begin{aligned}
 & \sigma^2/2 \int f'(x)g'(x)\phi(x) dx \\
 &= -\sigma^2/2 \int g(x) (f'(x)\phi(x))' dx \\
 &= -\sigma^2/2 \int g(x) \{f''(x)\phi(x) + f'(x)\phi'(x)\} dx \\
 &= -\sigma^2/2 \int g(x)\phi(x) \{f''(x) - 2\alpha x/\sigma^2 f'(x)\} dx \quad (1.48) \\
 &= \int g(x)\phi(x) \{-\sigma^2/2 f''(x) + \alpha x f'(x)\} dx \\
 &= (-Af, g) \\
 &= \mathcal{E}(f, g).
 \end{aligned}$$

Thus our form $\mathcal{E}$ and the form $\sigma^2/2 \int f'g'dm$ agree on $C_0^\infty(R)$. They must also agree on the $\mathcal{E}_1$ closure of $C_0^\infty(R)$, which by Proposition 1.12, includes $C_b^1(R)$. But we've seen that T_t maps $C_0^\infty(R)$ into $C_b^1(R)$ and since $T_t f \rightarrow f$ strongly in $\mathcal{D}(\mathcal{E})$, we find that $C_b^1(R)$ is dense in $\mathcal{D}(\mathcal{E})$. In other words, $C_b^1(R)$ is dense in the domains of both forms, and since they agree there, they must be the same form. We therefore have the following:

Proposition 1.16. *The Dirichlet form $\mathcal{E}$ associated with the Ornstein-Uhlenbeck transition function with drift α and diffusion σ is given by (1.28), where m is the density of $N(0, \sigma^2/2\alpha)$.*

We now outline how the process is constructed. Let $\Omega = C([0, \infty); R)$ and $X_t(\omega) = \omega(t)$ and $X_\infty(\omega) = \Delta$ for all $\omega \in \Omega$. We let $\mathcal{F}_t^\circ = \sigma\{X_s : s \leq t\}$ for $0 \leq t \leq \infty$, $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s^\circ$ for $0 \leq t < \infty$ and $\mathcal{F}_\infty = \mathcal{F}_\infty^\circ$. For each $x \in R$, we define P_x by:

$$\begin{aligned}
 & P_x(X_{t_1} \in A_1, \dots, X_{t_n} \in A_n) \\
 &= \int_{A_1} \dots \int_{A_n} o(t_1, x, dx_1) o(t_2 - t_1, x_1, dx_2), \dots, o(t_n - t_{n-1}, x_{n-1}, dx_n), \quad (1.49)
 \end{aligned}$$

where $A_i \in \mathcal{B}(R)$ and $0 \leq t_1 < t_2 < \dots < t_n < \infty$. For $x = \Delta$ we let $P_\Delta(X_t = \Delta \forall t) =$

1. One uses the semigroup property of $\{o(t, x, \cdot)\}$ to show that P_x is well-defined. By

mimicking the proof for Brownian motion, one can show that each P_x can be extended to be countably additive on $(\Omega, \mathcal{F}_\infty)$. Using the fact proved above that $\{o(t, x, \cdot)\}$ is Feller, or by again mimicking the proof for Brownian motion, one can show that

$$M = (\Omega, \mathcal{F}_\infty, \{\mathcal{F}_t\}_{t \in [0, \infty]}, \{X_t\}_{t \in [0, \infty]}, \{P_x\}_{x \in R_A})$$

is a Hunt process. The only thing to check is the strong Markov property.

Since $o(t, x, A) = o(t, 0, A - e^{-\alpha t}x)$ we find that $P_x = P_0 \circ \Psi_x^{-1}$ where $(\Psi_x \omega)(t) = xe^{-\alpha t} + \omega(t)$. Thus the Ornstein-Uhlenbeck process started at x is simply the sum of $xe^{-\alpha t}$ and the process begun at 0.

Suppose that we are now interested in a multidimensional process

$$\{X(t)\} = \{(X_1(t), \dots, X_d(t))\}, \quad (1.50)$$

whose coordinates are independent Ornstein-Uhlenbeck processes with drift coefficients $\{\alpha_i\}_{i=1}^d$ and diffusion coefficients $\{\sigma_i\}_{i=1}^d$. We begin with the transition function, which is formed by taking the product of the transition functions from (1.39), i.e.,

$$o_d(t, (x_1, \dots, x_d), \cdot) = \prod_{i=1}^d o(t, x_i, \cdot), \quad (1.51)$$

where the right hand side is the product measure on R^d . Of course $o(t, x_i, \cdot)$ is chosen with drift α_i and diffusion σ_i . It is now routine to confirm that this defines an m -symmetric transition function where $m = \prod_{i=1}^d N(0, \sigma_i^2/2\alpha_i)$ and that the associated Dirichlet form is the closure of $\mathcal{E}$ given in (1.32). We have therefore,

Proposition 1.17. *The Dirichlet form $\mathcal{E}$ associated with the transition function $\{o_d(t, x, \cdot)\}$ is the closure of the form given in (1.32), with $m = \prod_{i=1}^d N(0, \sigma_i^2/2\alpha_i)$.*

Note that we still have $P_x = P_0 \circ \Psi_x^{-1}$ on $\Omega = C([0, \infty); R^d)$, where now $(\Psi_x \omega)(t) = (x_1 e^{-\alpha_1 t}, \dots, x_d e^{-\alpha_d t}) + \omega(t)$.

Finally, we anticipate the contents of the next section by noting that there is nothing to stop us considering the infinite dimensional Ornstein-Uhlenbeck process,

$$\{X(t)\} = \{(X_1(t), \dots, X_i(t), \dots)\}. \quad (1.52)$$

In this case we get

$$o_\infty(t, x, \cdot) = \prod_{i=1}^{\infty} o(t, x_i, \cdot) = \prod_{i=1}^{\infty} N\left(x_i e^{-\alpha_i t}, \frac{\sigma_i^2}{2\alpha_i}(1 - e^{-2\alpha_i t})\right), \quad (1.53)$$

and

$$m = \prod_{i=1}^{\infty} m_i = \prod_{i=1}^{\infty} N\left(0, \frac{\sigma_i^2}{2\alpha_i}\right). \quad (1.54)$$

Now $\Omega = C([0, \infty); R^\infty)$ where R^∞ has the product topology, and we still get $P_x = P_0 \circ \Psi_x^{-1}$ where $(\Psi_x \omega)(t) = (x_1 e^{-\alpha_1 t}, \dots, x_i e^{-\alpha_i t}, \dots) + \omega(t)$. The associated form is the closure of (2.1) defined in the next chapter. We note that the closability of $\mathcal{E}$ can be obtained from either the infinite dimensional version of 1.14. or from Proposition 2.1.

CHAPTER 2

Dirichlet forms in infinite dimensions

2.1. Dirichlet forms on $\prod_{i=1}^{\infty} R_i$.

In order to apply the theory of Chapter 1 to the study of infinite dimensional processes, we define a class of Dirichlet forms over an infinite dimensional space X . These forms are given formally by setting $d = \infty$ in (1.32). The main difficulty in justifying this extension is to get around Fukushima's requirement that X be locally compact. This is accomplished by a trivial compactification $I: X \rightarrow \hat{X}$, where by trivial we mean that $\text{Cap}(\hat{X} \setminus X) = 0$.

Let $X = \prod_{i=1}^{\infty} R_i$ equipped with the product topology, $\mathcal{B}$ its Borel σ -algebra and m a probability measure on $(X, \mathcal{B})$ which charges every non-empty open set. Let $\pi_k: X \rightarrow R^k$ be the projection onto the first k coordinates, i.e., $\pi_k(x_1, x_2, \dots) = (x_1, x_2, \dots, x_k)$. We let $\mathcal{B}_k$ denote the σ -algebra generated by π_k . It consists of the sets which only depend on the first k coordinates. Any set in the algebra $\cup_k \mathcal{B}_k$ is called a *cylinder set*, and any function measurable with respect to this algebra is called a *cylinder function*. We also define e_i to be the vector with a one in the i^{th} position and zero otherwise.

Fix a sequence $\{\sigma_i\}_{i \in \mathbb{N}}$ of positive constants and define:

$$\begin{aligned} \mathcal{D}(\mathcal{E}) &= \{f \circ \pi_k : \exists k \in \mathbb{N}, f \in C_0^\infty(R^k)\} \\ \mathcal{E}(f \circ \pi_{k_1}, g \circ \pi_{k_2}) &= 1/2 \int \sum_{i=1}^{k_1 \wedge k_2} \sigma_i^2 \left(\frac{\partial}{\partial x_i} f \circ \pi_{k_1} \right)(x) \left(\frac{\partial}{\partial x_i} g \circ \pi_{k_2} \right)(x) m(dx). \end{aligned} \quad (2.1)$$

This definition is independent of the way that $f \circ \pi_k$ is represented as a cylinder function.

For simplicity we will usually write

$$\mathcal{E}(f, g) = 1/2 \int \sum \sigma_i^2 \left(\frac{\partial}{\partial x_i} f \right) \left(\frac{\partial}{\partial x_i} g \right) dm. \quad (2.2)$$

Because the cylinder sets generate $\mathcal{B}$, $\mathcal{D}(\mathcal{E})$ is dense in L^2 so that condition (i) of the definition of a symmetric form is fulfilled. Condition (ii) follows easily so that $\mathcal{E}$ is a symmetric form.

Let $\{\phi_\epsilon\}$ be as in Definition 1.11. If $f \in \mathcal{D}(\mathcal{E})$, then $\phi_\epsilon(f) \in \mathcal{D}(\mathcal{E})$ and

$$\begin{aligned} \mathcal{E}(\phi_\epsilon(f), \phi_\epsilon(f)) &= 1/2 \int \sum \sigma_i^2 (\phi'_\epsilon(f))^2 \left(\frac{\partial}{\partial x_i} f \right)^2 dm \\ &\leq 1/2 \int \sum \sigma_i^2 \left(\frac{\partial}{\partial x_i} f \right)^2 dm \\ &= \mathcal{E}(f, f) \end{aligned} \quad (2.3)$$

since $|\phi'_\epsilon| \leq 1$. Thus $\mathcal{E}$ is a Markovian form. We would like to know when this form is closable. We note that since we have taken m to be a probability measure it follows from 1.10. (i) and (iii), that if $\mathcal{E}$ is closable, then $\mathcal{D}(\mathcal{E})$ will contain all C_b^1 cylinder functions. Now the closability of (2.1) depends on the measure m and even for the analogous problem in finite dimensions, it is not easy to characterize such measures m . Of course, several sufficient conditions are known (cf. [7], [17]). In the following proposition we give sufficient conditions for the infinite dimensional case.

Proposition 2.1. *Suppose that m is a product measure and that the form $\mathcal{E}$ on $\mathcal{D}(\mathcal{E}) \cap L^2(X, \mathcal{B}_k, m)$ is closable in $L^2(X, \mathcal{B}_k, m)$ for each k . Then $\mathcal{E}$ on $\mathcal{D}(\mathcal{E})$ is closable in $L^2(X, \mathcal{B}, m)$.*

Proof : Let P_k be the projection in L^2 onto $L^2(X, \mathcal{B}_k, m)$. As m is a product measure we may write down an explicit formula for $P_k f$ when $f (= f \circ \pi_l) \in \mathcal{D}(\mathcal{E})$. For $l \leq k$ we

have $P_k f = f$ and for $k < l$ we get

$$(P_k f)(x_1, x_2, \dots, x_k) = \int f(x_1, x_2, \dots, x_k, x_{k+1}, \dots, x_l) \pi_1(dx_{k+1} \dots dx_l). \quad (2.4)$$

When $f \in \mathcal{D}(\mathcal{E})$ we may differentiate under the integral above and conclude that $P_k f \in \mathcal{D}(\mathcal{E})$ and that

$$\frac{\partial}{\partial x_i} P_k f = \begin{cases} P_k \frac{\partial}{\partial x_i} f & \text{when } i \leq k \\ 0 & \text{otherwise.} \end{cases} \quad (2.5)$$

Hence,

$$\begin{aligned} 2\mathcal{E}(P_k f, P_k f) &= \sum \sigma_i^2 \left\| \frac{\partial}{\partial x_i} P_k f \right\|_{L^2}^2 = \sum \sigma_i^2 \left\| P_k \frac{\partial}{\partial x_i} f \right\|_{L^2}^2 \\ &\leq \sum \sigma_i^2 \left\| \frac{\partial}{\partial x_i} f \right\|_{L^2}^2 = 2\mathcal{E}(f, f). \end{aligned} \quad (2.6)$$

Furthermore,

$$\begin{aligned} 2\mathcal{E}(P_k f, g) &= \sum \sigma_i^2 \int \left(\frac{\partial}{\partial x_i} (P_k f) \right) \left(\frac{\partial}{\partial x_i} g \right) dm \\ &= \sum \sigma_i^2 \left(P_k \left(\frac{\partial}{\partial x_i} f \right), \left(\frac{\partial}{\partial x_i} g \right) \right)_{L^2} \\ &= \sum \sigma_i^2 \left(\frac{\partial}{\partial x_i} f, P_k \left(\frac{\partial}{\partial x_i} g \right) \right)_{L^2} \\ &= \sum \sigma_i^2 \left(\frac{\partial}{\partial x_i} f, \frac{\partial}{\partial x_i} (P_k g) \right)_{L^2} \\ &= 2\mathcal{E}(f, P_k g). \end{aligned} \quad (2.7)$$

We can now show that $\mathcal{E}$ is closable. Let $\{f_n\} \in \mathcal{D}(\mathcal{E})$ be $\mathcal{E}$ -Cauchy and convergent to zero in L^2 and let $g (= g \circ \pi_k) \in \mathcal{D}(\mathcal{E})$. Then $\mathcal{E}(f_n, g) = \mathcal{E}(f_n, P_k g) = \mathcal{E}(P_k f_n, P_k g)$. The inequality (2.6) shows that $\{P_k f_n\}_{n \in \mathbb{N}}$ is $\mathcal{E}$ -Cauchy and since $f_n \rightarrow 0$ in L^2 we have $P_k f_n \rightarrow 0$ in $L^2(X, \mathcal{B}_k, m)$. By supposition, $\mathcal{E}$ restricted to $\mathcal{D}(\mathcal{E}) \cap L^2(X, \mathcal{B}_k, m)$ is closable on $L^2(X, \mathcal{B}_k, m)$ and so $\mathcal{E}(P_k f_n, P_k f_n)$ converges to zero as $n \rightarrow \infty$. The inequality

$$\mathcal{E}(f_n, g) = \mathcal{E}(P_k f_n, P_k g) \leq \sqrt{\mathcal{E}(P_k f_n, P_k f_n)} \sqrt{\mathcal{E}(P_k g, P_k g)} \quad (2.8)$$

shows that $\mathcal{E}(f_n, g) \rightarrow 0$ as $n \rightarrow \infty$. For $n, m \in \mathbb{N}$ we have

$$\mathcal{E}(f_n - f_m, f_n - f_m) = \mathcal{E}(f_n, f_n) + \mathcal{E}(f_m, f_m) - 2\mathcal{E}(f_n, f_m). \quad (2.9)$$

Since $\{f_n\}$ is $\mathcal{E}$ -Cauchy we can find N so large that $n, m \geq N$ implies that the left hand side is less than some given ϵ . Fixing $m = N$ and letting $n \rightarrow \infty$ makes the third term on the right hand side go to zero, by (2.8). Thus we find that $\limsup_n \mathcal{E}(f_n, f_n) < \epsilon$ and since ϵ is arbitrary we have the result. $\square$

For example, if m is the product of Gaussian measures, then by Proposition 1.17., it is known that $\mathcal{E}$ is closable on $\mathcal{D}(\mathcal{E}) \cap L^2(X, \mathcal{B}_k, m)$ in $L^2(X, \mathcal{B}_k, m)$ for each k . By Proposition 2.1. above, $\mathcal{E}$ is closable in $L^2(X; m)$.

Corollary 2.2. *Under the conditions of Proposition 2.1., $P_k f \rightarrow f$ in $\mathcal{E}_1$ -norm for each $f \in \mathcal{D}(\mathcal{E})$.*

Proof: In the proof of the previous proposition, we have seen that P_k is a projection in $(\mathcal{D}(\mathcal{E}), \mathcal{E}_1)$ so that $\mathcal{E}_1(P_k f, P_k f) \leq \mathcal{E}_1(f, f)$ for all k . Also, $\mathcal{E}_1(P_k f, G_1 u) = (P_k f, u) \rightarrow (f, u) = \mathcal{E}_1(f, G_1 u)$ for all $u \in L^2$. Since $G_1(L^2)$ is dense in $(\mathcal{D}(\mathcal{E}), \mathcal{E}_1)$ we see that $P_k f \rightarrow f$ weakly in $(\mathcal{D}(\mathcal{E}), \mathcal{E}_1)$. The strong convergence follows since P_k is a projection. That is, $\mathcal{E}_1(P_k f - f, P_k f - f) \leq 2\mathcal{E}_1(f, f) - 2\mathcal{E}_1(P_k f, f) \rightarrow 0$ as $k \rightarrow \infty$. $\square$

As mentioned earlier, the infinite dimensional version of Proposition 1.14. would also guarantee the closability of (2.1). That is, if condition 1.13. holds for m and every e_i , then (2.1) is closable (see [14]). We also get the following:

Proposition 2.3. *Suppose that (2.1) is closable. If $E(X_i^2) < \infty$ for all $i \in N$ and if $\sum_{i=1}^{\infty} y_i^2 / \sigma_i^2 < \infty$, then every $f \in \mathcal{D}(\mathcal{E})$ has a quasicontinuous version f_y such that $f_y(x + ty)$ is an absolutely continuous function of t for all $x \in X$. Consequently, if $\mathcal{E}(f, f) = 0$ then f_y is constant along every line parallel to y .*

Proof: Since $E(X_i^2) < \infty$ for all $i \in N$, there exist coefficients λ_i so that $m(H_\lambda) = m(\{x : \sum \lambda_i^2 x_i^2 < \infty\}) = 1$. Using this Hilbert space one makes sense of the regular conditional probability distribution $\{\rho(\cdot|\eta)\}$ of m with respect to $\sigma\{\langle y \rangle^\perp\}$. The proof follows as in Proposition 1.14. once we show that the $\mathcal{E}_1$ convergence of smooth functions f_n forces the L^2 convergence of $\frac{\partial}{\partial y} f_n$. But

$$\int \left(\frac{\partial}{\partial y} f_n \right)^2 = \int \left(\sum y_i \frac{\partial}{\partial x_i} f_n \right)^2 \leq \int \left(\sum y_i^2 / \sigma_i^2 \right) \sum \sigma_i^2 \left(\frac{\partial}{\partial x_i} f_n \right)^2, \quad (2.10)$$

and the right hand side is just $2 \sum y_i^2 / \sigma_i^2$ times $\mathcal{E}(f_n, f_n)$ and so the result follows. $\square$

So far we have a Markovian form $\mathcal{E}$ which in some cases is closable. In order to speak about the associated Markov process we must realize $\mathcal{E}$ as a regular, symmetric form on $L^2(\hat{X})$ where $\hat{X}$ is a compact separable metrizable space, (see Proposition 1.3.). Let $\hat{X} = \prod_{i=1}^\infty \bar{R}_i$ where $\bar{R}$ is the one-point compactification of R , and consider the embedding

$$I: X \rightarrow \hat{X}, \quad I(x_1, x_2, \dots) = (x_1, x_2, \dots). \quad (2.11)$$

$\hat{X}$ is a compact separable metrizable space, $I(X)$ is a dense Borel subset of $\hat{X}$ and $I: X \rightarrow \hat{X}$ is a homeomorphism. Now $\hat{m} = m \circ I^{-1}$ is a probability measure on $\hat{X}$ which charges each non-empty open set and $\hat{m}(\hat{X} \setminus I(X)) = m(\phi) = 0$. There is an obvious isomorphism between $L^2(\hat{X}; \hat{m})$ and $L^2(X; m)$ by

$$f \mapsto \hat{f}, \quad \hat{f} = f \circ I \quad m\text{-a.s.} \quad (2.12)$$

This isomorphism also preserves order so that a symmetric Markovian form $\mathcal{E}$ on $L^2(X; m)$ may be regarded as a symmetric Markovian form $\hat{\mathcal{E}}$ on $L^2(\hat{X}; \hat{m})$. We let

$$D(\hat{\mathcal{E}}) = \{\hat{f} : f \in D(\mathcal{E})\} \quad (2.13)$$

$$\hat{\mathcal{E}}(\hat{f}, \hat{g}) = \mathcal{E}(f, g).$$

Clearly $\mathcal{E}$ is closable (closed) if and only if $\hat{\mathcal{E}}$ is closable (closed).

We see that $\mathcal{D}(\hat{\mathcal{E}})$ is a point separating algebra of continuous functions which does not vanish at any point in $\hat{X}$. By the Stone-Weierstrass theorem its uniform closure is $C(\hat{X})$, so that $\hat{\mathcal{E}}$ is regular. Therefore, if m is such that $\mathcal{E}$ is closable, then by Fukushima's construction there is an $\hat{X}$ -valued Hunt process $\{\hat{X}(t), t \geq 0\}$ associated with the closure of $\hat{\mathcal{E}}$.

We finally remark that $\widehat{\text{Cap}}(\hat{X} \setminus I(X)) = 0$ so that the process may be regarded as X -valued. This may be seen as follows: for every $n \geq 3$ let $f_n \in C_b^1(R)$ with $|f_n(x)| \leq 1$ and $|f'_n(x)| \leq 1$ for all $x \in R$, and $f_n(x) = 0$ when $|x| \leq n-2$ and $f_n(x) = 1$ when $|x| \geq n$. Let x_k denote the k^{th} coordinate of $x \in \hat{X}$. Then $f_{nk} (= f_n(x_k))$ is in $\mathcal{D}(\hat{\mathcal{E}})$ and

$$\frac{\partial}{\partial x_i} f_{nk}(x) = \begin{cases} 0, & i \neq k \\ f'_n(x_k), & i = k. \end{cases} \quad (2.14)$$

The function f_{nk} is equal to 1 on the open set $A_{nk} = \{x \in \hat{X} : |x_k| > n\}$ and vanishes outside of $A_{(n-2)k}$. Therefore, by the definition of capacity (see (1.8)),

$$\begin{aligned} \widehat{\text{Cap}}(A_{nk}) &\leq \hat{\mathcal{E}}_1(f_{nk}, f_{nk}) \\ &= \sigma_k^2/2 \int (f'_{nk})^2 d\hat{m} + \int f_{nk}^2 d\hat{m} \\ &\leq (\sigma_k^2/2 + 1)\hat{m}(A_{(n-2)k}). \end{aligned} \quad (2.15)$$

Now $\bar{R}_1 \times \bar{R}_2 \times \dots \times \bar{R}_{k-1} \times \{\infty\} \times \bar{R}_{k+1} \times \dots = \bigcap_{n=3}^{\infty} A_{nk}$ and

$$\begin{aligned} 0 &\leq \widehat{\text{Cap}}(\bigcap_n A_{nk}) \leq \inf_n \widehat{\text{Cap}}(A_{nk}) \leq (\sigma_k^2/2 + 1) \inf_n \hat{m}(A_{nk}) \\ &= (\sigma_k^2/2 + 1)\hat{m}(\bigcap_n A_{nk}) \leq (\sigma_k^2/2 + 1)\hat{m}(\hat{X} \setminus I(X)) = 0. \end{aligned} \quad (2.16)$$

Hence, by (iv) on page 10,

$$\begin{aligned} \widehat{\text{Cap}}(\hat{X} \setminus I(X)) &= \widehat{\text{Cap}}\left(\bigcup_{k=1}^{\infty} \bigcap_{n=3}^{\infty} A_{nk}\right) \\ &\leq \sum_{k=1}^{\infty} \widehat{\text{Cap}}\left(\bigcap_{n=3}^{\infty} A_{nk}\right) \\ &= 0. \end{aligned} \quad (2.17)$$

We will generally suppress any reference to the compactification and simply regard everything as happening on X .

Let us consider again the case when m is a product measure which makes $\mathcal{E}$ closable. The proof of Proposition 2.1. shows us that $P_k(\mathcal{D}(\mathcal{E})) \subset \mathcal{D}(\mathcal{E})$ and $\mathcal{E}(P_k f, P_k f) \leq \mathcal{E}(f, f)$ from which it follows that the closure of the form $\mathcal{E}$ on $\mathcal{D}(\mathcal{E}) \cap L^2(X, \mathcal{B}_k, m)$ is the restriction of the closure of $\mathcal{E}$ to $L^2(X, \mathcal{B}_k, m)$. This means that the k -dimensional process associated with the closure of $\mathcal{E}$ on $\mathcal{D}(\mathcal{E}) \cap L^2(X, \mathcal{B}_k, m)$ is simply the projection onto the first k coordinates of the process associated with the closure of $\mathcal{E}$ on $\mathcal{D}(\mathcal{E})$.

For example, when $m = \prod_{i=1}^{\infty} N(0, \sigma_i^2/2\alpha_i)$ is Gaussian product measure the coordinates of the k -dimensional process are known, by Proposition 1.17., to be independent Ornstein-Uhlenbeck processes where the i^{th} coordinate process has drift coefficient α_i and diffusion coefficient σ_i . Our infinite dimensional process is thus given by

$$\{X(t)\} = \{(X_1(t), X_2(t), \dots, X_i(t), \dots)\} \quad (2.18)$$

which is the concatenation of independent Ornstein-Uhlenbeck processes with appropriate coefficients..

Let us conclude this section with a discussion of the local property of $\mathcal{E}$. To do so, we give another useful description of $\mathcal{E}$, due to Albeverio and Høegh-Krohn (1977). If $\{\sigma_i\}_{i \in \mathbb{N}}$ is an arbitrary sequence of constants, we define the Hilbert space H_σ as $\{x \in X : \sum_{i=1}^{\infty} \sigma_i^2 x_i^2 < \infty\}$, with the inner product $\langle x, y \rangle_\sigma = \sum_{i=1}^{\infty} \sigma_i^2 x_i y_i$. We define an operator

$$\nabla: \mathcal{D}(\nabla) \rightarrow H_\sigma \hat{\otimes}_2 L^2(X; m) \quad (2.19)$$

where $\mathcal{D}(\nabla) = \mathcal{D}(\mathcal{E})$ and $\hat{\otimes}_2$ refers to the completed Hilbert-Schmidt tensor product of

the two Hilbert spaces H_σ and L^2 . The operator ∇ is simply the gradient, that is

$$\nabla f = \left(\frac{\partial}{\partial x_1} f, \frac{\partial}{\partial x_2} f, \dots, \frac{\partial}{\partial x_k} f, 0, 0, \dots \right) \quad (2.20)$$

where $f (= f \circ \pi_k)$ only depends on k variables.

Now

$$\mathcal{E}(f, g) = 1/2 \langle \nabla f, \nabla g \rangle = 1/2 \int \langle \nabla f, \nabla g \rangle_\sigma dm \quad (2.21)$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in the Hilbert space $H_\sigma \hat{\otimes} L^2(X; m)$. Closability of $\mathcal{E}$ is equivalent to the closability of the operator ∇ and we have in this case

$$\begin{aligned} \mathcal{D}(\bar{\mathcal{E}}) &= \mathcal{D}(\bar{\nabla}) \\ \bar{\mathcal{E}}(f, g) &= 1/2 \langle \bar{\nabla} f, \bar{\nabla} g \rangle. \end{aligned} \quad (2.22)$$

Using the product rule we derive the following identity for $f, g \in \mathcal{D}(\mathcal{E})$,

$$\nabla(fg) = f\nabla g + g\nabla f. \quad (2.23)$$

When g belongs to $\mathcal{D}(\mathcal{E})$ it is bounded with (finitely many) bounded partials. It is easy to see that the relation above extends to $f \in \mathcal{D}(\bar{\mathcal{E}})$ and $g \in \mathcal{D}(\mathcal{E})$ giving

$$\bar{\nabla}(fg) = f\nabla g + g\bar{\nabla}f. \quad (2.24)$$

Let us consider the local property of $\bar{\mathcal{E}}$. Let $f \in \mathcal{D}(\bar{\mathcal{E}})$ and let $x \in X$ such that $f(x) = 1$ m-a.s. on a neighborhood U of x . As we are using the product topology we may take U to be an open cylinder set $U = U_k \times \prod_{i=k+1}^{\infty} R_i$ where U_k is open in R^k . Let g_k be a $C_0^\infty(R^k)$ function with $\nabla g_k \neq 0$ and $g_k \neq 0$ on a neighborhood $V_k \subseteq U_k$ of $\pi_k(x)$ and such that g_k vanishes outside of U_k . Let $g (= g_k \circ \pi_k) \in \mathcal{D}(\mathcal{E})$. Since $fg = g$, (2.24) tells us that

$$\nabla g = \bar{\nabla}(fg) = f\nabla g + g\bar{\nabla}f. \quad (2.25)$$

Since $f = 1$ m -a.s. on U and $\nabla g \neq 0$ on $V = V_k \times \prod_{i=k+1}^{\infty} R_i$, we see that ∇f vanishes m -a.s. on a neighborhood of x . In other words we have proved that for $f \in D(\mathcal{E})$, ∇f vanishes on any open set where f is almost surely constant. Therefore, if f and $g \in D(\mathcal{E})$ and f is constant on a neighborhood of $\text{supp}[g]$, then

$$\mathcal{E}(f, g) = 1/2 \int \langle \nabla f, \nabla g \rangle_{\sigma} dm = 0 \quad (2.26)$$

and so $\mathcal{E}$ has the stronger property described in Proposition 1.8. Using this proposition we see that, in the case when $\mathcal{E}$ is closable, the associated Markov process has continuous sample paths in the product topology, and has m as an invariant measure.

2.2. Functions of finite energy and applications

This section is devoted to verifying the quasicontinuity of some simple extended-real valued functions on X . The propositions which follow (2.11-2.14) demonstrate the power and flexibility of the Dirichlet space techniques. The reader is invited to apply Proposition 1.4. and discover what significance these have in terms of the sample paths of $\{X(t)\}$.

For the remainder of the work, we consider the form $\mathcal{E}$ defined by (2.1). It is assumed that $\mathcal{E}$, and hence the operator ∇ given in (2.19), is closable. For convenience we will denote their closures again by $\mathcal{E}$ and ∇ .

We recall that for every $f \in D(\mathcal{E})$, there is an H_{σ} -valued gradient $\nabla f \in L^2(X \rightarrow H_{\sigma}; m)$ and that

$$\mathcal{E}(f, f) = 1/2 \int \|\nabla f\|_{\sigma}^2 dm. \quad (2.27)$$

where $\|\cdot\|_\sigma$ is the norm in H_σ . It will be useful to consider a class of functions f for which $\nabla f \in L^2(X \rightarrow H_\sigma; m)$ but which do not necessarily belong to $L^2(X; m)$.

Definition 2.4. An extended real valued Borel function f is said to have *finite energy* if there exists a sequence $\{f_n\} \in \mathcal{D}(\mathcal{E})$ such that $\tilde{f}_n \rightarrow f$ q.e. and ∇f_n converges in $L^2(X \rightarrow H_\sigma; m)$. Recall that $\tilde{f}_n$ refers to the quasicontinuous version of f_n which exists by Proposition 1.1. We denote the limit of ∇f_n by ∇f , although it depends on the choice of f_n (see the comment following Proposition 2.5.). The space of all functions of finite energy will be denoted by $\mathcal{F}$.

Comment: It is easy to see that if $\{f_n\} \in \mathcal{F}$ and $\tilde{f}_n \rightarrow f$ q.e. and ∇f_n converges in $L^2(X \rightarrow H_\sigma; m)$, then $f \in \mathcal{F}$ and $\nabla f = \lim_n \nabla f_n$.

Proposition 2.5. Suppose that $g : \mathbb{R} \rightarrow \mathbb{R}$ is continuously differentiable with bounded derivative and that g has (possibly infinite) limits at $-\infty$ and ∞ . If $f \in \mathcal{F}$, then $g \circ f \in \mathcal{F}$ and $\nabla(g \circ f) = g'(f)\nabla f$. Furthermore, if g is bounded, then $g \circ f \in \mathcal{D}(\mathcal{E})$ and is quasicontinuous.

Proof: If f is a smooth cylinder function, then $g \circ f$ is a C_b^1 cylinder function, which belongs to $\mathcal{D}(\mathcal{E})$ and whose gradient is given by $g'(f)\nabla f$. For general $f \in \mathcal{D}(\mathcal{E})$, let f_n be smooth cylinder functions converging to f in $\mathcal{E}_1$ -norm. By taking a subsequence we may assume that $\tilde{f}_n \rightarrow f$ q.e. and $\nabla f_n \rightarrow \nabla f$ m -a.s. By continuity of g , $g \circ \tilde{f}_n \rightarrow g \circ f$ q.e. Now,

$$\begin{aligned} & \|\nabla(g \circ f_n - g \circ f_m)\|_\sigma^2 \\ &= \|\nabla(g \circ f_n)\|_\sigma^2 + \|\nabla(g \circ f_m)\|_\sigma^2 - 2\langle \nabla(g \circ f_n), \nabla(g \circ f_m) \rangle_\sigma \\ &= (g'(f_n))^2 \|\nabla f_n\|_\sigma^2 + (g'(f_m))^2 \|\nabla f_m\|_\sigma^2 - 2g'(f_n)g'(f_m) \langle \nabla f_n, \nabla f_m \rangle_\sigma. \end{aligned} \quad (2.28)$$

By continuity of g' , we see that $\|\nabla(g \circ f_n - g \circ f_m)\|_\sigma^2$ converges to zero m -a.s. as $n, m \rightarrow \infty$.

Furthermore, since $\|\nabla f_n\|_0^2$ converges in $L^1(X; m)$, it is uniformly integrable. Since g' is bounded, (2.28) tells us that $\|\nabla(g \circ f_n - g \circ f_m)\|_0^2$ is also uniformly integrable. The a.s. convergence together with the uniform integrability give the convergence in L^1 as $n, m \rightarrow \infty$. In other words, $\nabla(g \circ f_n)$ converges in $L^2(X \rightarrow H_0; m)$. Thus $g \circ f \in \mathcal{F}$ and

$$\nabla(g \circ f) = \lim_n \nabla(g \circ f_n) = \lim_n g'(f_n) \nabla f_n = g'(f) \nabla f. \quad (2.29)$$

If $f \in \mathcal{F}$, then we repeat the above argument, using the approximating sequence $\{f_n\}$ which appears in the definition of $\mathcal{F}$. We need the limits of g at $-\infty$ and ∞ to ensure that $g \circ f_n$ converges to $g \circ f$ q.e. If g is bounded, then the convergence also takes place in $L^2(X; m)$ and so $g \circ f \in \mathcal{D}(\mathcal{E})$, the quasicontinuity follows from Proposition 1.1. $\square$

Comment: For each k , let $g_k \in C_b^1(R)$ such that $g_k(x) = x$ whenever $|x| \leq k$. If $f \in \mathcal{F}$, then $g_k \circ f \in \mathcal{D}(\mathcal{E})$ and $\nabla(g_k \circ f) = g'_k(f) \nabla f$ for each k . We therefore see that ∇f is determined m -a.s. on the set where $|f|$ is finite.

Also, since

$$(f \wedge k) \vee (-k) = ((g_k \circ f) \wedge k) \vee (-k) \quad (2.30)$$

is quasicontinuous for each k , we conclude that f is quasicontinuous into $[-\infty; \infty]$. Applying Proposition 1.7. then gives,

$$\int P_x(t \rightarrow f(X(t)) \text{ is } \tilde{R} \text{ continuous}) m(dx) = 1. \quad (2.31)$$

Hence, if $P_m(f(X(t)) < \infty \forall t) = 1$, then $f(X(t))$ has a.s. continuous sample paths under P_m . It is this observation which leads, via Corollary 2.8., to the main results (2.17 & 2.19) of this chapter.

Example 2.6 : Functions in $\mathcal{F}$

(i) The most obvious example of a function in $\mathcal{F}$ is the coordinate function $f = x_k$.

In this case, $\nabla f = (0, 0, \dots, 0, 1, 0, \dots)$.

(ii) The functions $f_n = \sum_{i=1}^n \lambda_i^2 x_i^2$ are in $\mathcal{F}$, and

$$\nabla f_n = 2(\lambda_1^2 x_1, \lambda_2^2 x_2, \dots, \lambda_n^2 x_n, 0, 0, \dots). \quad (2.32)$$

If $\sum_{i=1}^{\infty} \lambda_i^4 \sigma_i^2 x_i^2 \in L^1(X; m)$, then ∇f_n converges in $L^2(X \rightarrow H_\sigma; m)$ and so

$$f = \sum_{i=1}^{\infty} \lambda_i^2 x_i^2 \in \mathcal{F}. \quad (2.33)$$

(iii) Suppose that $\{\lambda_i, \sigma_i\}$ is bounded by M . Let g be continuously differentiable with bounded derivative and (possibly infinite) limits at $-\infty$ and ∞ . Suppose as well that $g'(x)x$ is bounded. If $f_n = \sum_{i=1}^n \lambda_i^2 x_i^2$, then $g \circ f_n \in \mathcal{F}$ and $\nabla(g \circ f_n) = g'(f_n)\nabla f_n$, thus

$$\begin{aligned} \|\nabla(g \circ f_n)\|_\sigma^2 &= (g'(f_n))^2 4 \sum_{i=1}^n \sigma_i^2 \lambda_i^4 x_i^2 \\ &\leq (g'(f_n))^2 4M^2 \sum_{i=1}^n \lambda_i^2 x_i^2 \\ &= (g'(f_n))^2 4M^2 f_n. \end{aligned} \quad (2.34)$$

By our choice of g , this converges boundedly to $g(\sum_{i=1}^{\infty} \lambda_i^2 x_i^2)$ at every $x \in X$. Hence,

if $m < n$, then

$$\|\nabla(g \circ f_n - g \circ f_m)\|_\sigma^2 = \|\nabla(g \circ f_n)\|_\sigma^2 + \|\nabla(g \circ f_m)\|_\sigma^2 - 8 \left(\sum_{i=1}^m \sigma_i^2 \lambda_i^4 x_i^2 \right) g'(f_n)g'(f_m) \quad (2.35)$$

converges boundedly to zero at each $x \in X$, and so also in $L^1(X; m)$.

Therefore $g(\sum_{i=1}^{\infty} \lambda_i^2 x_i^2) \in \mathcal{F}$.

(iv) Let g be as above and suppose that $\{\sigma_i\}$ is bounded and $p \geq 2$. Then $f_n =$

$\sum_{i=1}^n |x_i|^p \in \mathcal{F}$ and

$$\nabla f_n = p(\operatorname{sgn}(x_1)|x_1|^{p-1}, \operatorname{sgn}(x_2)|x_2|^{p-1}, \dots, \operatorname{sgn}(x_n)|x_n|^{p-1}, 0, 0, \dots), \quad (2.36)$$

where $\operatorname{sgn} x = x/|x|$. From $p \geq 2$ follows $2(p-1) \geq p$, and so $\|\cdot\|_{C^{(p-1)}} \leq \|\cdot\|_C$.

Therefore,

$$\begin{aligned} \|\nabla f_n\|_0^2 &= p \sum \sigma_i^2 |x_i|^{2(p-1)} \leq M \sum |x_i|^{2(p-1)} \\ &\leq M \left(\sum |x_i|^p \right)^{2(p-1)/p} = M f_n^2 \left(\sum |x_i|^p \right)^{-2/p} \end{aligned} \quad (2.37)$$

where M bounds $p\sigma_i^2$. Since $\|\nabla f_n\|_0^2 \leq M f_n^2$ when $f_n \geq 1$, we may repeat the argument in (iii) to conclude that $g(\sum_{i=1}^\infty |x_i|^p) \in \mathcal{F}$.

(v) Consider the maximum function $\max: R^k \rightarrow R$,

$$\max(x_1, x_2, \dots, x_k) = x_1 \vee x_2 \vee \dots \vee x_k. \quad (2.38)$$

Let $j: R^k \rightarrow R$ be a non-negative $C_0^\infty(R^k)$ function which integrates to 1. As in Definition 1.9., we let $m_\delta = \max * j_\delta$ where $*$ denotes convolution. Then $m_\delta \in C^\infty(R^k)$, $|\max - m_\delta| \leq \delta$ everywhere,

$$0 \leq \frac{\partial}{\partial x_j} m_\delta \leq 1 \quad \text{for } 1 \leq j \leq k \quad (2.39)$$

and

$$\sum_{j=1}^k \frac{\partial}{\partial x_j} m_\delta \equiv 1. \quad (2.40)$$

We define $\frac{\partial}{\partial x_j} \max$ as the pointwise limit of $\frac{\partial}{\partial x_j} m_\delta$ as $\delta \rightarrow 0$. If A_j denotes the set of points in R^k whose j^{th} coordinate is a strict maximum, and ∂A_j its boundary, then

$$\left(\frac{\partial}{\partial x_j} \max \right) (x) = 1_{A_j}(x) + 1_{\partial A_j}(x) |\{i : x \in \partial A_i\}|^{-1}. \quad (2.41)$$

Here $|A|$ means the cardinality of the set A . It is now easy to see that if $f_1, f_2, \dots, f_k$ are m -a.s. finite and belong to $\mathcal{F}$, then $f = \max(f_1, f_2, \dots, f_k) \in \mathcal{F}$ and $\nabla f = (\frac{\partial}{\partial x_1} f, \frac{\partial}{\partial x_2} f, \dots)$ where

$$\frac{\partial}{\partial x_i} f = \sum_{j=1}^k \left(\frac{\partial}{\partial x_j} \max \right) (f_1, f_2, \dots, f_k) \left(\frac{\partial}{\partial x_i} f_j \right) \quad (2.42)$$

Definition 2.7. A function $f \in \mathcal{F}$ is said to be *harmonic* when $\|\nabla f\|_0^2 = 0$ m -a.s. or $\mathcal{E}(f, f) = 0$. This means that the rate of change of the process $\{X(t)\}$ in the ' f direction' is zero. We make this more precise in the following proposition.

Proposition 2.8. The function f is in $\mathcal{F}$ and $\mathcal{E}(f, f) = 0$ iff

$$\int P_x(f(X(t)) = f(X(0)) \forall t) m(dx) = 1. \quad (2.43)$$

If m is a product measure which satisfies Condition 1.13. for each e_i , then f is a constant q.e.

Proof: If $g \in C_b^1(R)$, then by Proposition 2.5., $g \circ f \in \mathcal{D}(\mathcal{E})$ and $\mathcal{E}(g \circ f, g \circ f) = 0$. If $r \in R$, we may take a sequence $\{g_n\}$ of such functions such that $g_n \downarrow 1_{(-\infty, r]}$ pointwise. Since $\mathcal{E}(g_n \circ f, g_n \circ f) = 0$ for each n , the sequence $g_n \circ f$ is trivially $\mathcal{E}$ -Cauchy. Also $g_n \circ f \rightarrow 1_{\{f \leq r\}}$ pointwise and hence in $L^2(X; m)$. Thus $g_n \circ f \rightarrow 1_{\{f \leq r\}}$ in $\mathcal{E}_1$ -norm and $\mathcal{E}(1_{\{f \leq r\}}, 1_{\{f \leq r\}}) = 0$. Since f is quasicontinuous, so is $g_n \circ f$ for each n , and hence also the $\mathcal{E}_1$ limit. From Proposition 1.7. we get

$$\int P_x(1_{\{f \leq r\}}(X(t)) \text{ is continuous}) m(dx) = 1. \quad (2.44)$$

Hence $P_x(1_{\{f \leq r\}}(X(t)) \text{ is continuous}) = P_x(1_{\{f \leq r\}}(X(t)) \text{ is constant}) = 1$ holds simultaneously for $r \in Q$, for m almost every $x \in X$. Taking the intersection over $r \in Q$ for such x gives (2.43).

Conversely, suppose that f is Borel and (2.43) holds. Then it also holds for the function $f_n = ((-n) \vee f) \wedge n$ which lies in L^2 . Now $T_t f_n = f_n$ for all $t \geq 0$, and using equation (1.6) we see that $f_n \in \mathcal{D}(\mathcal{E})$ and $\mathcal{E}(f_n, f_n) = 0$. Obviously f_n is finely continuous so we may apply [7; Theorem 4.3.2. (ii)] and conclude that it is quasicontinuous. By the comment following Definition 2.4. we see that $f \in \mathcal{F}$ and $\mathcal{E}(f, f) = 0$.

As for the second assertion, recall from Proposition 1.14. that for each k there is a version of f_k of f , which is absolutely continuous in the e_k direction. Since $\mathcal{E}(f, f) = 0$ we have $\frac{\partial}{\partial x_k} f_k = 0$ m -a.s. and so $f_k(\eta \oplus te_k)$ is a constant in t for ν -a.s. η . Thus $f_k(\eta)$ is a version of f and since $f_k(\eta) \in \sigma(x_1, \dots, x_{k-1}, x_{k+1}, \dots)$ we have $f \in \sigma(x_1, \dots, x_{k-1}, x_{k+1}, \dots) \vee \mathcal{N}$ where $\mathcal{N}$ are the m -null sets. Taking the intersection over k gives $f \in \mathcal{T} \vee \mathcal{N}$ where $\mathcal{T}$ is the tail σ -algebra. Since m is a product measure, $\mathcal{T}$ is trivial and so f is a constant m -a.s. By the quasicontinuity of f , this can be strengthened to a q.e. statement. $\square$

Corollary 2.9. *If $f \in \mathcal{F}$, then for m -a.e. starting point $x \in X$*

$$\begin{aligned} f(x) < \infty &\Rightarrow P_x(i \rightarrow f(X(t)) \text{ is continuous}) = 1 \\ f(x) = \infty &\Rightarrow P_x(f(X(t)) = \infty \forall t) = 1. \end{aligned} \quad (2.45)$$

Proof: Let $g_n \in C_b^1(R)$ such that $g_n(x) = 0$ when $x \leq n$, and $g_n(x) = 1$ for x sufficiently large, and

$$|g'_n(x)|^2 \leq 1/n \mathcal{E}(f, f) \quad \forall x \in R. \quad (2.46)$$

By Proposition 2.5., $g_n \circ f \in \mathcal{D}(\mathcal{E})$ and $\mathcal{E}(g_n \circ f, g_n \circ f) \leq 1/n$. The sequence is therefore $\mathcal{E}$ -Cauchy, it converges pointwise and in $L^2(X; m)$ to $1_{\{f=\infty\}}$, and since each $g_n \circ f$ is quasicontinuous, so is the limit. Since $\mathcal{E}(1_{\{f=\infty\}}, 1_{\{f=\infty\}}) = 0$, we may apply Proposition 2.8. and conclude that

$$\int P_x(1_{\{f=\infty\}}(X(t)) = 1_{\{f=\infty\}}(X(0)) \forall t) m(dx) = 1. \quad (2.47)$$

The result now follows from (2.31). $\square$

Example 2.10 : Harmonic functions

(i) If $f \in \mathcal{F}$, then $1_{\{f=-\infty\}}$ and $1_{\{f=\infty\}}$ belong to $\mathcal{F}$ and are harmonic. See the previous corollary.

(ii) From example 2.6 (v) we know that $f_n = (\lambda_1 x_1 \vee \lambda_2 x_2 \vee \dots \vee \lambda_n x_n) \in \mathcal{F}$, where x_i refers to the coordinate function as in 2.6 (i). For a fixed $x \in X$, from (2.42) we get $\nabla f_n(x) = 1/k \sum_{j=1}^k \nabla \lambda_{i_j, x_{i_j}}(x)$ where $0 \leq i_1 < i_2 < \dots < i_k \leq n$ and $\lambda_{i_j, x_{i_j}}(x)$ are in a k way tie for $(\lambda_1 x_1 \vee \lambda_2 x_2 \vee \dots \vee \lambda_n x_n)(x)$. Let us consider what happens at this x as $n \rightarrow \infty$.

Case 1 If $\lambda_{i_1, x_{i_1}} = \lambda_{i_2, x_{i_2}} = \dots = \lambda_{i_k, x_{i_k}}$ are in a k way tie for $\lambda_1 x_1 \vee \lambda_2 x_2 \vee \dots$, then $\nabla f_n(x) = 1/k \sum_{j=1}^k \nabla \lambda_{i_j, x_{i_j}}$ for $n \geq i_k$.

Case 2 If $\lambda_{i_1, x_{i_1}} = \lambda_{i_2, x_{i_2}} = \dots$ are in an infinite tie for $\lambda_1 x_1 \vee \lambda_2 x_2 \vee \dots$, then $\|\nabla f_n\|_\sigma^2(x) = 1/k^2 \sum_{j=1}^k \lambda_{i_j}^2 \sigma_{i_j}^2$ will converge to zero if $\{\lambda_{i_j} \sigma_{i_j}\}$ is bounded.

Case 3 If $\lambda_1 x_1 \vee \lambda_2 x_2 \vee \dots$ does not occur in the sequence $\{\lambda_{i_j} \sigma_{i_j}\}$, then $\|\nabla f_n\|_\sigma^2(x) = 1/k^2 \sum_{j=1}^k \lambda_{i_j}^2 \sigma_{i_j}^2$, where $i_1 \rightarrow \infty$. This will converge to zero if $\{\lambda_{i_j} \sigma_{i_j}\}$ does.

Thus, if $\{\lambda_{i_j} \sigma_{i_j}\}$ converges to zero, then ∇f_n converges pointwise and, since $\|\nabla f_n\|_\sigma^2 \leq \sup_i \{\lambda_{i_j} \sigma_{i_j}\}$, boundedly. Therefore ∇f_n converges in $L^2(X \rightarrow H_\sigma; m)$ and so $f = \lim_n f_n = \lambda_1 x_1 \vee \lambda_2 x_2 \vee \dots \in \mathcal{F}$ and $\|\nabla f\|_\sigma^2 \leq \sup_i \{\lambda_{i_j} \sigma_{i_j}\}$. If we now consider the sequence $f_n = \lambda_n x_n \vee \lambda_{n+1} x_{n+1} \vee \dots$ we find that $f_n \in \mathcal{F}$, $\|\nabla f_n\|_\sigma^2 \leq \sup_{n \leq i} \lambda_{i_j} \sigma_{i_j} \rightarrow 0$ and so $f = \lim_n f_n = \limsup_n (\lambda_n x_n) \in \mathcal{F}$ and is harmonic.

(iii) Let $s_n = x_1 + x_2 + \dots + x_n$ for each $n \in N$. Then $\lambda_n s_n \in \mathcal{F}$ and

$$\nabla \lambda_n s_n = \lambda_n (1, 1, \dots, 1, 0, 0, \dots). \quad (2.48)$$

If $f_n = \lambda_1 s_1 \vee \lambda_2 s_2 \vee \dots \vee \lambda_n s_n$, then $\nabla f_n = 1/k \sum_{j=1}^k \nabla \lambda_{i_j} s_{i_j}$, where $0 \leq i_1 < i_2 < \dots < i_k \leq n$.

... $i_k \leq n$ and λ_i, s_i are in a k way tie for $\lambda_1 s_1 \vee \lambda_2 s_2 \vee \dots \vee \lambda_n s_n$,

$$\|\nabla f_n\|_\sigma^2 \leq 1/k \sum_{j=1}^k \|\nabla(\lambda_i, s_i)\|_\sigma^2 = 1/k \sum_{j=1}^k \left(\lambda_i^2 \sum_{l=1}^{v_j} \sigma_l^2 \right) \quad (2.49)$$

The same arguments as in (ii) show us that if $\lambda_j^2(\sum_{l=1}^{v_j} \sigma_l^2) \rightarrow 0$ as $j \rightarrow \infty$, then $f = \limsup_n(\lambda_n s_n) \in \mathcal{F}$ and is harmonic.

We will now use these functions to prove capacity versions of classical results. Since m is a probability measure on $\prod_{i=1}^\infty R_i$, the one dimensional projections $X_n(x) = x_n$ yield a sequence $\{X_n\}_{n \geq 1}$ of real random variables.

Proposition 2.11. *If $\{\lambda_n \sigma_n\}$ is a bounded sequence, and $\lambda_n X_n \rightarrow 0$ in probability, then $\lambda_n X_n \rightarrow 0$ in capacity. That is, for each $\epsilon > 0$, $\text{Cap}(|\lambda_n X_n| > \epsilon) \rightarrow 0$ as $n \rightarrow \infty$.*

Proof: Let $0 < \delta < \epsilon$ and $f \in C_b^1(R)$ such that $|f(x)| \leq 1$ for all x , $f(x) = 0$ when $|x| \leq \delta$ and $f(x) = 1$ when $|x| \geq \epsilon$. Let $g_n = f(\lambda_n x_n)$, then $g_n \in \mathcal{D}(\mathcal{E})$ and $\|\nabla g_n\|_\sigma^2 = (f'(\lambda_n x_n))^2 \sigma_n^2 \lambda_n^2$, which can be bounded uniformly in n . Let $A_\epsilon = \{x \in X : |\lambda_n X_n(x)| > \epsilon\}$. The function g_n is equal to 1 on the open set A_ϵ , and both g_n and $\|\nabla g_n\|_\sigma^2$ vanish outside of A_δ . Thus

$$\begin{aligned} \text{Cap}(|\lambda_n X_n| > \epsilon) &\leq \mathcal{E}_1(g_n, g_n) \\ &= 1/2 \int \|\nabla g_n\|_\sigma^2 dm + \int g_n^2 dm \\ &\leq (M/2 + 1)m(|\lambda_n X_n| > \delta) \end{aligned} \quad (2.50)$$

for some constant M . Letting $n \rightarrow \infty$ gives the result. $\square$

Proposition 2.12. *If $\{\lambda_n \sigma_n\}$ is a bounded sequence and $\lambda_n X_n \rightarrow 0$ m-a.s., then $\lambda_n X_n \rightarrow 0$ q.e.*

Proof: Let $0 < \delta < \epsilon$, and f as in the previous proposition. Let $g_{mn} = f(\lambda_m x_m \vee \lambda_{m+1} x_{m+1} \vee \dots \vee \lambda_n x_n)$ for $1 \leq m \leq n$. Then $g_{mn} \in \mathcal{D}(\mathcal{E})$ and

$$\|\nabla g_{mn}\|_{\sigma}^2 = (f'(\lambda_m x_m \vee \dots \vee \lambda_n x_n))^2 \|\nabla(\lambda_m x_m \vee \dots \vee \lambda_n x_n)\|_{\sigma}^2 \quad (2.51)$$

which can be bounded uniformly in n and m . Let $A_{\epsilon} = \{\cup_{i=m}^n (\lambda_i X_i > \epsilon)\}$. Then $g_{mn} = 1$ on the open set A_{ϵ} , and both g_{mn} and $\|\nabla g_{mn}\|_{\sigma}^2$ vanish outside of A_{δ} . Thus

$$\begin{aligned} \text{Cap}(A_{\epsilon}) &\leq \mathcal{E}_1(g_{mn}, g_{mn}) \\ &\leq (M/2 + 1)m(A_{\delta}) \\ &\leq (M/2 + 1)m(\cup_{i=m}^{\infty} (\lambda_i X_i > \delta)) \end{aligned} \quad (2.52)$$

for some constant M . Letting $n \rightarrow \infty$ gives

$$\text{Cap}(\cup_{i=m}^{\infty} (\lambda_i X_i > \epsilon)) \leq (M/2 + 1)m(\cup_{i=m}^{\infty} (\lambda_i X_i > \delta)), \quad (2.53)$$

and so

$$\begin{aligned} \text{Cap}(\limsup_n \lambda_n X_n > \epsilon) &\leq \lim_m \text{Cap}(\cup_{i=m}^{\infty} (\lambda_i X_i > \epsilon)) \\ &\leq (M/2 + 1) \lim_m m(\cup_{i=m}^{\infty} (\lambda_i X_i > \delta)) \\ &\leq (M/2 + 1)m\left(\limsup_n \lambda_n X_n > \delta\right) \\ &= 0. \end{aligned} \quad (2.54)$$

Hence

$$\begin{aligned} \text{Cap}(\limsup_n \lambda_n X_n > 0) &= \text{Cap}\left(\cup_{j=1}^{\infty} (\limsup_n \lambda_n X_n > 1/j)\right) \\ &\leq \sum_{j=1}^{\infty} \text{Cap}(\limsup_n \lambda_n X_n > 1/j) \\ &= 0. \end{aligned} \quad (2.55)$$

Replacing λ_n by $-\lambda_n$ we get

$$\text{Cap}(\liminf_n \lambda_n X_n < 0) = 0$$

and so by subadditivity,

$$\text{Cap}(\lambda_n X_n \neq 0) = 0. \quad \square \quad (2.56)$$

Proposition 2.13. The law of large numbers

If $\{X_i\}$ are i.i.d. with mean zero, and $\sum_{i=1}^n \sigma_i^2/n^2 \rightarrow 0$ as $n \rightarrow \infty$, then

$$(X_1 + X_2 + \cdots + X_n)/n \rightarrow 0 \quad \text{q.e.} \quad (2.57)$$

Proof: Let $f = \limsup_n S_n/n$, then by the law of large numbers, $m(f = 0) = 1$. Since $\sum_{i=1}^n \sigma_i^2/n^2 \rightarrow 0$, we see by letting $\lambda_n = 1/n$ in example 2.10 (iii) that $f \in \mathcal{F}$ and is harmonic. By Proposition 2.8.,

$$\int P_x(f(X(t)) = 0 \quad \forall t) m(dx) = 1 \quad (2.58)$$

and so $\text{Cap}(f \neq 0) = 0$. Replacing $1/n$ by $-1/n$ we get by a similar argument

$$\text{Cap}(\liminf_n S_n/n \neq 0) = 0, \quad (2.59)$$

and so by subadditivity,

$$\text{Cap}(S_n/n \neq 0) = 0. \quad \square \quad (2.60)$$

Proposition 2.14. The law of the iterated logarithm

If $\{X_i\}$ are i.i.d. with mean zero and variance one, and $\sum_{i=1}^n \sigma_i^2/(2n \log \log n) \rightarrow 0$, then

$$\limsup_n \frac{S_n}{\sqrt{2n \log \log n}} = 1 \quad \text{q.e.} \quad (2.61)$$

Proof: Because of the condition on $\{\sigma_i^2\}$, we may use $\lambda_n = 1/\sqrt{2n \log \log n}$ in example 2.10 (iii) and conclude that $f = \limsup_n (S_n/\sqrt{2n \log \log n})$ belongs to $\mathcal{F}$ and is harmonic. The same arguments used in Proposition 2.13. yield the result. $\square$

2.3. Support of $\{X(t)\}$ and sample path continuity

This section is the core of the thesis. Here we see that the Dirichlet space techniques may be profitably employed in establishing space-time regularity of the associated stochastic processes. The major results come in 2.17. and 2.19. Much of the work has been done in the previous section but here we are more explicit about the conclusions one can draw about the sample paths of $\{X(t)\}$.

By a *support* for $\{X(t)\}$ we will mean a set whose complement has zero capacity. As pointed out earlier, these sets support the process in the sense that a set of capacity zero has an infinite hitting time P_m -a.s. In Propositions 2.17. and 2.19. we will find a support for $\{X(t)\}$ by showing that the capacity of a certain set is zero under some conditions on the diffusion coefficients σ_i . It is perhaps worth pointing out that some conditions on the diffusion coefficients are necessary. This is the motivation for Proposition 2.16. We remind the reader that all symmetric forms encountered are assumed to be closable and that we continue to abuse notation in allowing $\mathcal{E}$ to denote both the unclosed form and also its closure.

Lemma 2.15. *Suppose that $\mathcal{E}$ is the closure in $L^2(R; m(t)dt)$ of*

$$\mathcal{E}(f, g) = \frac{\sigma^2}{2} \int f'(t)g'(t)m(t) dt, \quad D(\mathcal{E}) = C_0^\infty(R). \quad (2.62)$$

If $1/m \in L^1_{loc}(dt)$, then for each $x \in R$, $Cap(\{x\}) \rightarrow 1$ as $\sigma \rightarrow \infty$.

Comment: In terms of the process $\{X(t)\}$, the condition $1/m \in L^1_{loc}(dt)$ is equivalent to $P_m(\tau_{\{x\}} < \infty) = 1$ for every $x \in R$. In letting $\sigma \rightarrow \infty$ we are simply shrinking the time scale so that $\{x\}$ is hit arbitrarily soon.

Proof: In the proof of 1.12. we showed that $\mathcal{E}_1$ convergence implies pointwise convergence when $1/m \in L^1_{loc}(dt)$. Therefore, for any $x \in R$, the map $x: \mathcal{D}(\mathcal{E}) \rightarrow R$ defined by $x(f) = f(x)$ is continuous on $(\mathcal{D}(\mathcal{E}), \mathcal{E}_1)$ and so the measure δ_x is of finite energy integral (see [7; p.69]). Consequently, $\text{Cap}(\{x\}) > 0$ and so there exists a 1-equilibrium potential e_x for $\{x\}$ such that $e_x \in \mathcal{D}(\mathcal{E})$ and

$$0 \leq e_x(t) \leq 1, \quad e_x(x) = 1$$

$$\text{Cap}(\{x\}) = \mathcal{E}_1(e_x, e_x) = \frac{\sigma^2}{2} \int (e'_x(t))^2 m(t) dt + \int (e_x(t))^2 m(t) dt. \quad (2.63)$$

Since m is a probability measure and $1 \in \mathcal{D}(\mathcal{E})$, the set function Cap is bounded above by 1. Therefore $\int (e'_x(t))^2 m(t) dt \leq 2/\sigma^2$, and so for any $y \in R$,

$$\begin{aligned} (1 - e_x(y))^2 &= (e_x(x) - e_x(y))^2 \\ &\leq \left(\int_x^y (e'_x(t))^2 m(t) dt \right) \left(\int_x^y \frac{dt}{m(t)} \right) \\ &\leq 2/\sigma^2 \left(\int_x^y \frac{dt}{m(t)} \right). \end{aligned} \quad (2.64)$$

Let $y_1 < x < y_2$ so that $m(t : y_1 \leq t \leq y_2) > 1 - \epsilon$ and take σ so large that $e_x(y_1) \wedge e_x(y_2) > \sqrt{1 - \epsilon}$. Since $e_x(t)$ is a non-increasing function of $|x - t|$, we see that

$$(1 - \epsilon)^2 \leq \int_{y_1}^{y_2} (e_x(t))^2 m(t) dt \leq \text{Cap}(\{x\}). \quad (2.65)$$

Since ϵ is arbitrary we are finished. $\square$

Proposition 2.16. *Let m_i be probability measures on R satisfying the conditions of Lemma 2.15. For each $i \in N$ let $a_i < b_i$ and form the set $B = \prod_{i=1}^{\infty} [a_i, b_i]$. We can find $\{\sigma_i\}$ so large that $\text{Cap}(B)$ is arbitrarily close to 1 under the Dirichlet form (2.1).*

Proof: The process $\{X(t)\} = \{(X_1(t), X_2(t), \dots)\}$ associated with the form (2.1) is formed from independent copies of the processes $\{X_i(t)\}$ associated with the closure of

$$\mathcal{E}(f, g) = \frac{\sigma_i^2}{2} \int f'(t)g'(t)m_i(t) dt, \quad \mathcal{D}(\mathcal{E}) = C_0^\infty(R). \quad (2.66)$$

Let $\epsilon > 0$ and choose $\epsilon_i > 0$ so that $\prod_{i=1}^{\infty} (1 - \epsilon_i) > 1 - \epsilon$, also let $A_n = \prod_{i=1}^n (a_i, b_i) \times \prod_{i=n+1}^{\infty} R_i$ and $B_n = \prod_{i=1}^n [a_i, b_i] \times \prod_{i=n+1}^{\infty} R_i$. We will give an easy probabilistic proof that one can, in principle, find coefficients σ_i so large that the set B is hit arbitrarily soon by the process $\{X(t)\}$. We proceed by induction on the coordinates. By the lemma above, we can choose σ_1 so that

$$P_m(\tau_{A_1} < \epsilon) > 1 - \epsilon_1. \quad (2.67)$$

Suppose that $\sigma_1, \sigma_2, \dots, \sigma_n$ have been chosen so that

$$P_m(\tau_{A_n} < \epsilon) > \prod_{i=1}^n (1 - \epsilon_i). \quad (2.68)$$

Since Cap is a Choquet capacity there is a compact set $K \subset A_n$ such that

$$P_m(\tau_K < \epsilon) > \prod_{i=1}^n (1 - \epsilon_i). \quad (2.69)$$

Now A_n is open and $\{(X_1(t), X_2(t), \dots, X_n(t))\}$ has continuous sample paths so that the random variable defined by

$$S_n(\omega) = \inf\{t - \tau_K(\omega) : t > \tau_K(\omega), X(t, \omega) \notin A_n\} \quad (2.70)$$

is almost surely strictly positive. Choose δ_n so small that

$$P_m(\tau_K < \epsilon, S_n > \delta_n) > \prod_{i=1}^n (1 - \epsilon_i). \quad (2.71)$$

We divide $[0, \epsilon]$ into N equal segments of length $< \delta_n/2$. By Lemma 2.15. we can choose σ_{n+1} so large that

$$P_m(\tau_{(a_{n+1}, b_{n+1})} < 1/N) > 1 - \frac{\epsilon_{n+1}}{N+1}, \quad (2.72)$$

where $\tau_{(a_{n+1}, b_{n+1})}$ refers to the hitting time with respect to $\{X_{n+1}(t)\}$. Since $\{X_{n+1}(t)\}$ is in equilibrium, $\tau_{(a_{n+1}, b_{n+1})} \circ \theta_t$ are identically distributed for all fixed t and so,

$$P_m(\tau_{(a_{n+1}, b_{n+1})} \circ \theta_{k/N} < 1/N \quad \forall 0 \leq k \leq N) > 1 - \epsilon_{n+1}. \quad (2.73)$$

By independence of the coordinates

$$P_m(\tau_K < \epsilon, S_n > \delta_n, \tau_{(a_{n+1}, b_{n+1})} \circ \theta_{k/N} < 1/N \quad \forall 0 \leq k \leq N) > \prod_{i=1}^{n+1} (1 - \epsilon). \quad (2.74)$$

For each ω in the above event, $\tau_{A_{n+1}}(\omega) < \epsilon$ because $X(\cdot, \omega)$ spends an entire segment of time of length δ_n in A_n after hitting it, and $X_{n+1}(\cdot, \omega)$ hits (a_{n+1}, b_{n+1}) during any interval of length $1/N < \delta_n/2$. Hence, for each n , $P_m(\tau_{B_n} < \epsilon) > 1 - \epsilon$ and since $\{B_n\}$ is a decreasing sequence of closed sets, $P_m(\tau_{\cap B_n} < \epsilon) > 1 - \epsilon$. Now $\cap_n B_n = \prod_{i=1}^{\infty} [a_i, b_i] = B$ and since ϵ is arbitrary we find that we can make $\text{Cap}(B)$ arbitrarily close to 1. $\square$

Now we would like to discuss the question of sample path continuity. Recall that $\mathcal{E}$ has the local property (2.26) so that $\{X(t)\}$ has continuous sample paths in $\prod_{i=1}^{\infty} R_i$ equipped with the product topology (Prop. 1.8.). In other words, the coordinates $\{X_i(t)\}$ are continuous processes. If $\{\lambda_i\}_{i \in \mathbb{N}}$ is a sequence of constants, we let $\|x\|_{\lambda}^2: X \rightarrow [0, \infty]$ by $\|x\|_{\lambda}^2 = \sum_{i=1}^{\infty} \lambda_i^2 x_i^2$. We define $H_{\lambda} = \{x \in X : \|x\|_{\lambda}^2 < \infty\}$, so that $\|\cdot\|_{\lambda}$ is a Hilbertian seminorm on H_{λ} . We are interested in determining when H_{λ} supports $\{X(t)\}$ and when $\{X(t)\}$ has H_{λ} -continuous sample paths, as well. We point out that since the coordinate processes are continuous, in order to assert that $\{X(t)\}$ is H_{λ} -continuous it suffices to show the continuity of the seminorm process $\{\|X(t)\|_{\lambda}\}$. This follows from standard properties of Hilbert space.

Proposition 2.17. *If $\sum_{i=1}^{\infty} \sigma_i^2 \lambda_i^4 x_i^2 \in L^1(X; m)$, then for m -a.e. starting point $x \in X$,*

$$\begin{aligned} \|x\|_{\lambda}^2 < \infty &\Rightarrow P_x(t \rightarrow X(t) \text{ is } H_{\lambda} \text{ - continuous}) = 1 \\ \|x\|_{\lambda}^2 = \infty &\Rightarrow P_x(\|X(t)\|_{\lambda}^2 = \infty \quad \forall t) = 1. \end{aligned} \quad (2.75)$$

In particular, if $m(H_{\lambda}) = 1$, then the process is supported by H_{λ} and has continuous sample paths, i.e., $\{X(t)\}$ is an H_{λ} -valued diffusion.

Proof: By example 2.6 (ii) we see that $\|x\|_\lambda^2$ is a function of finite energy. The result follows from Corollary 2.8. $\square$

Example 2.18. When $m = \prod_{i=1}^\infty N(0, \sigma_i^2/2\alpha_i)$ our process $\{X(t)\}$ is the Ornstein-Uhlenbeck process. In order to apply the above result and get an H_λ -valued diffusion, we require $m(H_\lambda) = 1$ and $\sum_{i=1}^\infty \sigma_i^2 \lambda_i^4 x_i^2 \in L^1(X; m)$. Because we are using Gaussian measure this is equivalent to $\sum_{i=1}^\infty \lambda_i^2 (\sigma_i^2/2\alpha_i) < \infty$ and $\sum_{i=1}^\infty \sigma_i^2 \lambda_i^4 (\sigma_i^2/2\alpha_i) < \infty$. In case of unit noise ($\sigma_i \equiv 1$) and $H_\lambda = \ell^2$ ($\lambda_i \equiv 1$), we see that the sums above are the same so that $m(\ell^2) = 1 \Rightarrow \{X(t)\}$ is an ℓ^2 -valued diffusion. Thus, in this case, $m(\ell^2) = 1 \iff \{X(t)\}$ is an ℓ^2 -valued diffusion.

Proposition 2.19. If $\{\sigma_i, \lambda_i\}$ is bounded, then for m -a.e. starting point $x \in X$,

$$\begin{aligned} \|x\|_\lambda^2 < \infty &\Rightarrow P_x(t \rightarrow X(t) \text{ is } H_\lambda\text{-continuous}) = 1 \\ \|x\|_\lambda^2 = \infty &\Rightarrow P_x(\|X(t)\|_\lambda^2 = \infty \forall t) = 1. \end{aligned} \quad (2.76)$$

If $\{\sigma_i, \lambda_i\}$ is unbounded, then there exists a measure m which makes $\mathcal{E}$ closable, and such that $m(H_\lambda) = 1$ but $\{X(t)\}$ exits H_λ .

Proof: We use example 2.6 (iii), taking the function $g \in C_b^1(R)$ such that $g'(x)x$ is bounded. We require, as well, that $g|_{[0, \infty)} \rightarrow [0, \infty)$ is a homeomorphism and $g(\infty) = \infty$. For example, we could take a nice function such that $g(x) = \log(x)$ for sufficiently large x . Since $g(\|x\|_\lambda^2)$ then belongs to $\mathcal{F}$, we get the conclusion of Corollary 2.8. for this function. Noting that $g(\|x\|_\lambda^2) = \infty$ iff $\|x\|_\lambda^2 = \infty$, and applying the continuous function g^{-1} to $g(\|X(t)\|_\lambda^2)$ we get the desired conclusion in terms of $\|x\|_\lambda^2$.

Suppose, on the other hand, that $\{\sigma_i, \lambda_i\}$ is unbounded. We wish to construct a measure m so that $m(H_\lambda) = 1$ but $\text{Cap}(X \setminus H_\lambda) \neq 0$. The measure m will be a product measure, $m = \prod_{i=1}^\infty m_i$, where m_i is a probability measure on R . We require that m_i has a density

with respect to Lebesgue measure which is bounded away from zero on compacts. By Proposition 2.1. this measure makes $\mathcal{E}$ closable. Since $\{\sigma, \lambda_i\}$ is unbounded, we may choose a subsequence such that $|\sigma_{i_k}^2 \lambda_{i_k}^2| \geq k^3$. We will define m_i carefully when i runs over this subsequence. On the other indices we require only that $\sum \lambda_i^2 \text{Var}(X_i) < \infty$.

On the subsequence we want $E(X_{i_k}) = 0$, and that m_{i_k} has density greater than or equal to $|\lambda_{i_k}|/k^2$ on $[-|\lambda_{i_k}|^{-1}, |\lambda_{i_k}|^{-1}]$. This uses up $2/k^2$ of the probability and gives $(1/3) 1/(\lambda_{i_k}^2 k^2)$ as a lower bound on the variance. We use the rest of the probability to give a density which is bounded away from zero on compacts, and such that $\text{Var}(X_{i_k}) \leq 1/(\lambda_{i_k}^2 k^2)$. Thus

$$\sum \text{Var}(\lambda_{i_k} X_{i_k}) \leq \sum \lambda_{i_k}^4 \text{Var}(X_{i_k}) \leq \sum 1/k^2 < \infty \quad (2.77)$$

and so $\sum_{i=1}^{\infty} \lambda_i^2 \text{Var}(X_i) < \infty$ which means that $m(H_\lambda) = 1$.

Consider the capacity of the cylinder set $A_i = \{x \in X : |\lambda_i x_i| > 1\}$ when $\lambda_i \neq 0$. Because m is a product measure, the capacity of such a set is just the capacity of $(-\infty, -|\lambda_i|^{-1}) \cup (|\lambda_i|^{-1}, \infty)$ under the Dirichlet form $\mathcal{E}_i$ on R which is the closure of

$$\begin{aligned} \mathcal{E}_i(f, g) &= 1/2 \int \sigma_i^2 f'(t) g'(t) m_i(t) dt \\ \mathcal{D}(\mathcal{E}_i) &= C_b^1(R) \end{aligned} \quad (2.78)$$

on $L^2(R; m_i)$. Let us take $i = i_k$ from among the subsequence. Since $m_i(t)$ is bounded away from zero on compacts, each function in the domain of the closure of $(\mathcal{E}_i, \mathcal{D}(\mathcal{E}_i))$ has an absolutely continuous version. In particular, the 1-equilibrium potential of the set $(-\infty, -|\lambda_i|^{-1}) \cup (|\lambda_i|^{-1}, \infty)$ has an absolutely continuous version which we will call e_i . Thus,

$$\text{Cap}(A_i) = \sigma_i^2/2 \int (e_i'(t))^2 m_i(t) dt + \int e_i^2(t) m_i(t) dt. \quad (2.79)$$

Now for any $t \in [-|\lambda_s|^{-1}, |\lambda_s|^{-1}]$ we have,

$$\begin{aligned} (e_s(t) - 1)^2 &= \left(\int_{-|\lambda_s|^{-1}}^t e'_s(s) ds \right)^2 \\ &\leq \left(\int_{-|\lambda_s|^{-1}}^t (e'_s(s))^2 m_s(s) ds \right) \left(\int_{-|\lambda_s|^{-1}}^t \frac{ds}{m_s(s)} \right) \\ &\leq \left(\int (e'_s(s))^2 m_s(s) ds \right) (2k^2/\lambda_s^2) \end{aligned} \quad (2.80)$$

We also note that

$$\int (e'_s(s))^2 m_s(s) dt \leq 2/\sigma_s^2 \text{Cap}(A_s) \leq 2/\sigma_s^2. \quad (2.81)$$

Therefore

$$(e_s(t) - 1)^2 \leq 4k^2/\sigma_s^2 \lambda_s^2 \leq 4/k. \quad (2.82)$$

or

$$e_s^2(t) \geq (1 - 2/\sqrt{k})^2. \quad (2.83)$$

Integrating with respect to $m_s(dt)$ gives

$$\text{Cap}(A_s) \geq (1 - 2/\sqrt{k})^2. \quad (2.84)$$

Therefore, for any n ,

$$1 = \sup_{s \geq n} \text{Cap}(A_s) \leq \text{Cap}(\cup_{i=n}^{\infty} A_i) \leq 1. \quad (2.85)$$

This means that $\{X(t)\}$ will hit $(\cup_{i=n}^{\infty} A_i)$ immediately after any time t , with probability

1. Therefore the set

$$\bigcap_{n \in \mathbb{N}} \bigcap_{r, q \in \mathbb{Q}} \{\omega \in \Omega : \exists t \in [r, q] \ni X(t, \omega) \in \cup_{i=n}^{\infty} A_i\} \quad (2.86)$$

has probability 1. By continuity of the sample paths, for each ω in the set above,

$$O_n(\omega) = \{t \in [0, \infty) : X(t, \omega) \in \cup_{i=n}^{\infty} A_i\} \quad (2.87)$$

is open and dense. By the Baire category theorem, taking the intersection over n yields a dense set. Therefore, with probability 1, the process hits $\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i = \limsup A_i$ on a dense set of time points. If $x \in \limsup A_i$, then $|\lambda_i x_i| > 1$ infinitely often and so $\|x\|_{\lambda}^2 = \sum \lambda_i^2 x_i^2 = \infty$. Thus the process leaves H_{λ} , quite often in fact. This proves the proposition. $\square$

In summary, Proposition 2.19. tells us that any of the processes under consideration will be supported by some H_{λ} , but that no H_{λ} supports them all. Whether or not H_{λ} supports a given process depends on the relationship between the coefficients $\{\lambda_i\}$, the diffusion coefficients $\{\sigma_i\}$ and the measure m , and must be determined from case to case.

2.4. Further properties of harmonic functions

In this section, we explore some further properties of harmonic functions. We assume that m is a product measure which satisfies Condition 1.13. for each basis vector e_i . In the following two propositions, we identify harmonic functions via their averaging property and we show that harmonicity is a local property.

Definition 2.20. For a Borel set B , we let

$$\mathcal{F}_B = \{g \in \mathcal{D}(\mathcal{E}) : \bar{g} = 0 \text{ q.e. on } X \setminus B\} \quad (2.88)$$

$\mathcal{F}_B$ is a closed subspace of $(\mathcal{D}(\mathcal{E}), \mathcal{E}_1)$. A function $f \in \mathcal{D}(\mathcal{E})$ is said to be *harmonic on the set B* when $\mathcal{E}(f, g) = 0$ for all $g \in \mathcal{F}_B$.

In Theorem 4.4.1. of [7], Fukushima showed that for a non-negative, quasicontinuous $f \in \mathcal{D}(\mathcal{E})$, $H_{\alpha}^B f$ is a quasicontinuous version of the projection of f onto the orthogonal complement of $\mathcal{F}_{X \setminus B}$ in $(\mathcal{D}(\mathcal{E}), \mathcal{E}_{\alpha})$ for $\alpha > 0$. We use this fact to prove the following:

Proposition 2.21. *Suppose that B is a set whose complement is not exceptional.*

If f is quasicontinuous and bounded on $\bar{B}$, then f is harmonic on B iff $f = H^{X \setminus B} f$ q.e.

Proof: For notational convenience we drop the superscript from $H^{X \setminus B}$.

($\Rightarrow$) If f is harmonic on B , then $\mathcal{E}(f, g) = 0$ for every $g \in \mathcal{F}_B$. Now by Fukushima's result above we have

$$\mathcal{E}_\alpha(H_\alpha f, g) = \mathcal{E}(H_\alpha f, g) + \alpha(H_\alpha f, g) = 0 \quad (2.89)$$

for $\alpha > 0$ and $g \in \mathcal{F}_B$. Therefore we have $\mathcal{E}(f - H_\alpha f, g) = \alpha(H_\alpha f, g)$. Letting $g = f - H_\alpha f$ we see that $\mathcal{E}(f - H_\alpha f, f - H_\alpha f) = \alpha(H_\alpha f, f - H_\alpha f)$. Since $(H_\alpha f, f - H_\alpha f)$ is bounded, letting $\alpha \rightarrow 0$ gives $\mathcal{E}(f - H_\alpha f, f - H_\alpha f) \rightarrow 0$, which means that $\{H_\alpha f : \alpha > 0\}$ is $\mathcal{E}$ -Cauchy. We already know that $H_\alpha f \rightarrow Hf$ boundedly pointwise and so in L^2 as $\alpha \rightarrow 0$. Since $\mathcal{E}$ is closed, we have $Hf \in \mathcal{D}(\mathcal{E})$ and $\mathcal{E}_1(Hf - H_\alpha f, Hf - H_\alpha f) \rightarrow 0$. Now f is quasicontinuous and so are all the $H_\alpha f$ by Fukushima's result above. Their $\mathcal{E}_1$ limit, Hf , is also quasicontinuous by Proposition 1.1. Therefore $f - Hf$ belongs to $\mathcal{D}(\mathcal{E})$, is quasicontinuous, and $\mathcal{E}(f - Hf, f - Hf) = 0$, and so by Proposition 2.8., $f - Hf$ is equal to a constant q.e. Since they agree q.e. on the non-exceptional set $X \setminus B$, $f = Hf$ q.e.

($\Leftarrow$) Suppose that $f = Hf$ q.e. Since $f \mapsto H_\alpha f$ is a projection in $(\mathcal{D}(\mathcal{E}), \mathcal{E}_\alpha)$, we see that $\mathcal{E}_\alpha(H_\alpha f, H_\alpha f) \leq \mathcal{E}_\alpha(f, f)$. It follows therefore that $\mathcal{E}(H_\alpha f, H_\alpha f)$ are bounded for α near 0 and consequently the Cesàro means $H_n f = 1/n \sum_{i=1}^n H_{\alpha_i} f$ of some subsequence converge in $\mathcal{E}$. But $\lim_{n \rightarrow \infty} H_n f = Hf = f$ in L^2 and since $\mathcal{E}$ is closed $\mathcal{E}(H_n f - f, H_n f - f) \rightarrow 0$. In particular $\mathcal{E}(f, g) = \lim_{n \rightarrow \infty} \mathcal{E}(H_n f, g) = 0$ for $g \in \mathcal{F}_B$ so we see that f is harmonic on B . $\square$

Lemma 2.22. *Suppose that f is a function of finitely many variables, so that $f = P_k f$ for some k , and f is bounded on $\bar{G}$, where G is a bounded open set. Then f is*

harmonic on G iff it is locally harmonic on G , i.e., for each $x \in G$ there is an open neighborhood U_x of x such that f is harmonic on U_x .

Proof: Let O_n be open sets so that $P_m(\tau_{O_n} \upharpoonright \tau_{(X \setminus O)}) = 1$. The sets $K_n = X \setminus O_n$ are bounded and closed, so their projections on R^k are compact. By projections we mean the image under π_k defined at the beginning of the chapter. Let us fix n and note that $\{\pi_k U_x\}$ is an open cover for $\pi_k K_n$ and so we may find a partition of unity subordinate to this cover. Let h_j be a finite family of $C_0^\infty(R^k)$ functions with $\text{supp}(h_j) \subset U_x$, for some x , and such that $\sum h_j \equiv 1$ on K_n . If $g \in \mathcal{F}_{K_n}$, then $\mathcal{E}(f, g) = \mathcal{E}(P_k f, g) = \mathcal{E}(f, P_k g)$ and this is equal to $\mathcal{E}(f, \sum h_j P_k g) = \sum \mathcal{E}(f, h_j P_k g) = 0$ since each $h_j P_k g \in \mathcal{F}_{U_x}$. Now $g \in \mathcal{F}_G$ implies $g^{(i)} \in \mathcal{F}_{K_n}$ for some n , and so $\mathcal{E}(f, g) = \lim_{i \rightarrow \infty} \mathcal{E}(f, g^{(i)}) = 0$ (see [7; (1.4.2) (iv)]). Thus f is harmonic on G . $\square$

Proposition 2.23. If G is a bounded open set in X and f is bounded on $\bar{G}$, then f is harmonic on G iff it is locally harmonic on G .

Proof: Projecting onto the first k coordinates we find that $P_k f$ is locally harmonic on G and so is harmonic there, by the above result. Using Corollary 2.2., we get for every $g \in \mathcal{F}_G$, $\mathcal{E}(f, g) = \lim_{k \rightarrow \infty} \mathcal{E}(P_k f, g) = 0$, so that f is harmonic on G . $\square$

It is easily checked by the Markov property and the fact that $p_t 1 \equiv 1$ that if B is any Borel set then $H^B 1(x) = P_x(\tau_B < \infty)$ is harmonic. In fact, by the symmetry (1.13) we get

$$\begin{aligned} \int (I - T_t) H^B 1 \cdot 1 \, dm \\ = \int H^B 1 \cdot (I - T_t) 1 \, dm = 0. \end{aligned} \quad (2.90)$$

Now $T_t H^B 1(x) = P_x(\tau_B \circ \theta_t < \infty)$ which is less than or equal to $H^B 1(x)$. Therefore the

function $(I - T_t)H^B 1$ is non-negative and since its integral with respect to $m(dx)$ vanishes, it is equal to zero m -a.s. From (1.6) we may conclude that $\mathcal{E}(H^B 1, H^B 1) = 0$. Because of the assumptions we made on m , our process is ergodic in the following sense:

Proposition 2.24. *For each Borel set B we have $H^B 1 = 0$ q.e. if B is exceptional, and $H^B 1 = 1$ q.e. otherwise.*

Proof: By 2.8., we know that $H^B 1$ is equal to some constant q.e. If B is exceptional, then $H^B 1$ vanishes m -a.s. so that the constant must be zero. On the other hand, $H^B 1$ equals 1 on the set B^r of regular points of B . If B is not exceptional, then neither is $B \cap B^r$, because $B \setminus B^r$ is exceptional. In this case the constant must be one. $\square$

Comment: If the exceptional sets in the equations $H^B 1 = 0$ q.e. or $H^B 1 = 1$ q.e. are polar, then they are empty. This is because both sides of the equations are finely continuous functions and the complement of a polar set is finely dense.

Example 2.25. We show that the exceptional sets above need not be empty. We consider the Ornstein-Uhlenbeck process and suppose that $\sum \sigma_i^2 / 2\alpha_i < \infty$ and $\sum \sigma_i^4 / 2\alpha_i < \infty$ so that $\{X(t)\}$ is an ℓ^2 -valued diffusion. Now the Ornstein-Uhlenbeck process beginning at a point x is the Ornstein-Uhlenbeck process begun at 0 plus the vector

$$(x_1 e^{-\alpha_1 t}, \dots, x_n e^{-\alpha_n t}, \dots). \quad (2.91)$$

Thus, if we assume as well that the set $\{\alpha_i\}$ is bounded then the process begun in ℓ^2 will remain there for all time and the process begun outside ℓ^2 will remain outside for all time. This is because the vector (2.91) behaves in that way. Thus

$$H^{\ell^2} 1(x) = P_x(\tau_{\ell^2} < \infty) = \begin{cases} 1 & x \in \ell^2 \\ 0 & x \in X \setminus \ell^2 \end{cases} \quad (2.92)$$

and so here $X \setminus \ell^2$ is the non-empty exceptional set.

CHAPTER 3

The infinite dimensional Ornstein-Uhlenbeck process

In the final chapter of this thesis we turn our attention to infinite dimensional Ornstein-Uhlenbeck processes which are of great interest in different parts of pure and applied mathematics. The Ornstein-Uhlenbeck process is of course covered by the theory in the second chapter, but being a Gaussian process it is particularly amenable to further analysis. In each section of this chapter we look at the process from a different angle with a view to getting an overall picture of the sample path behaviour of $\{X(t)\}$. To conclude, we consider some of the questions raised which would make interesting problems for further research.

We will be looking at the infinite dimensional Ornstein-Uhlenbeck process whose transition function is given by (1.53) and invariant measure by (1.54). We know that this process is associated with the Dirichlet form (2.1), and we will always use the canonical version of P_x given in (1.49).

3.1. Sample path continuity

Let us begin with the question of sample path continuity for the Ornstein-Uhlenbeck process $\{X(t)\}$ in the Hilbert space ℓ^2 . The space ℓ^2 is chosen for ease of exposition, for results on the space H_λ one simply replaces σ , by σ_λ , below: We would like to know

for which sets of coefficients $\{\sigma_i^2\}$ and $\{\alpha_i\}$ the process given by (1.49) has ℓ^2 -continuous sample paths. A minimum requirement, of course, is that $m(\ell^2) = 1$ which is to say $E_m \|X(t)\|^2 = \sum \sigma_i^2 / 2\alpha_i < \infty$. By applying Proposition 2.17. we get the following.

Proposition 3.1. *If $\sum \sigma_i^2 / 2\alpha_i < \infty$ and $\sum \sigma_i^4 / 2\alpha_i < \infty$, then $\{X(t)\}$ has ℓ^2 -continuous sample paths P_m -a.s.*

In the Gaussian case, 2.19. gives us no more than this since if $\sum \sigma_i^2 / 2\alpha_i < \infty$ and $\{\sigma_i^2\}$ is bounded, then the conditions of 3.1. already hold. Therefore Proposition 3.1. is the best result we can get using the results from Chapter 2.

The following proposition gives a necessary condition for the ℓ^2 continuity of the process. The condition is on the growth of the coefficients $\{\alpha_i\}$ as $i \rightarrow \infty$.

Proposition 3.2. *If $\{X(t)\}$ has ℓ^2 -continuous sample paths P_m -a.s., then for every $\epsilon > 0$,*

$$\limsup_i \frac{\alpha_i}{\exp(\epsilon \alpha_i / \sigma_i^2)} < \infty. \quad (3.1)$$

Proof: It is a proof by contradiction. Let us assume that (3.1) fails to hold for some $\epsilon > 0$. We will now show that $\{X(t)\}$ must leave the space ℓ^2 . We show, in fact, that the set

$$A = \{x : |x_i| > \delta \text{ i.o.}\} \quad (3.2)$$

has capacity 1 for any $0 < \delta < \sqrt{\epsilon}$. This means that the process is in A on a dense set of time points, P_m -a.s. Since A clearly lies outside of ℓ^2 , we have the result.

To show this, we adapt the proof of 2.19., letting $m_i = N(0, \sigma_i^2 / 2\alpha_i)$ and letting e_i

be the 1-equilibrium potential of the set $(-\infty, -\delta] \cup [\delta, \infty)$. A bound for

$$\int_{-\delta}^{\delta} \frac{ds}{m_1(s)} \quad (3.3)$$

is given by 2δ times the supremum of $1/m_1(s)$ over the range of integration. The supremum is attained at the endpoints and is equal to

$$\sqrt{2\pi} \sqrt{\frac{\sigma_1^2}{2\alpha_1}} \exp\left(\frac{\delta^2 \alpha_1}{\sigma_1^2}\right). \quad (3.4)$$

As in the proof of 2.19., we get the following bound:

$$\begin{aligned} (e_1(t) - 1)^2 &\leq \frac{2}{\sigma_1^2} 2\delta \sqrt{2\pi} \sqrt{\frac{\sigma_1^2}{2\alpha_1}} \exp\left(\frac{\delta^2 \alpha_1}{\sigma_1^2}\right) \\ &\leq K \frac{1}{\alpha_1} \sqrt{\frac{\alpha_1}{\sigma_1^2}} \exp\left(\frac{\delta^2 \alpha_1}{\sigma_1^2}\right), \end{aligned} \quad (3.5)$$

for all $t \in R$, where K is some constant. Since $\delta^2 < \epsilon$, for large i we have

$$(e_1(t) - 1)^2 \leq K \frac{1}{\alpha_i} \exp\left(\frac{\epsilon \alpha_i}{\sigma_i^2}\right). \quad (3.6)$$

Since (3.1) fails to hold, there is a subsequence $\{i_k\}$ such that

$$\lim_k \frac{1}{\alpha_{i_k}} \exp\left(\frac{\epsilon \alpha_{i_k}}{\sigma_{i_k}^2}\right) = 0. \quad (3.7)$$

Repeating the argument in the proof of 2.19. shows that $\text{Cap}(A) = 1$, which yields the result. □

We have used the Dirichlet form to find a necessary condition and also a (different) sufficient condition for the ℓ^2 continuity of the Ornstein-Uhlenbeck process. Ideally we would like a necessary and sufficient condition, but barring that, we may try to improve either of Propositions 3.1. or 3.2. in order to get sharper results. Now we notice that the proof of 2.17., and hence of 3.1., does not even use the fact that m is a Gaussian measure, only

that $\sum \sigma_i^2 E(X_i^2) < \infty$. It therefore applies in cases where m has much fatter tails than a Gaussian measure. So it seems reasonable to attempt to improve our results by working on this end. This is what now follows.

For the purpose of comparison, we present a result on the continuity of $\{X(t)\}$ obtained using techniques from the field of Gaussian processes. It is proved by using a minor modification of a result due to Iscoe, Marcus, and McDonald [10].

Proposition 3.3. *Suppose that f is strictly increasing and convex in a neighborhood of zero, that $f(0) = 0$ and*

$$\int_0^\epsilon \frac{\sqrt{f^{-1}(u)}}{u\sqrt{\log(1/u)}} du < \infty \quad (3.8)$$

for some $\epsilon > 0$. If $\sum \sigma_i^2/2\alpha_i < \infty$ and $\limsup f(\sigma_i^2/\alpha_i)\alpha_i < \infty$, then $\{X(t)\}$ is ℓ^2 continuous under P_m .

Proof: Let $x = (x_1, x_2, \dots)$ such that $\sum_{i=1}^\infty x_i^2 = 1$, and let M be a bound for $f(\sigma_i^2/\alpha_i)\alpha_i$. Since we are considering the behavior in a neighborhood of zero, without loss of generality we take f to be strictly increasing and convex. Note that since $f(0) = 0$, convexity implies that $f(yz) \leq yf(z)$ for all z and all $0 \leq y \leq 1$. We obtain therefore,

$$\begin{aligned} & f\left(\sum x_i^2 E(X_i(t) - X_i(s))^2\right) \\ &= f\left(\sum x_i^2 \sigma_i^2/\alpha_i (1 - e^{-\alpha_i|s-t|})\right) \\ &\leq \sum x_i^2 f(\sigma_i^2/\alpha_i) (1 - e^{-\alpha_i|s-t|}) \\ &\leq \sum x_i^2 f(\sigma_i^2/\alpha_i) (\alpha_i|s-t| \wedge 1) \\ &= \sum_{\alpha_i|s-t| > 1} x_i^2 f(\sigma_i^2/\alpha_i) + \sum_{\alpha_i|s-t| \leq 1} x_i^2 f(\sigma_i^2/\alpha_i) \alpha_i|s-t| \\ &\leq \sum_{\alpha_i|s-t| > 1} x_i^2 M|s-t| + \sum_{\alpha_i|s-t| \leq 1} x_i^2 M|s-t| \\ &= M|s-t|, \end{aligned} \quad (3.9)$$

and so

$$\sum x_i^2 E(X_i(t) - X_i(s))^2 \leq f^{-1}(M|s - t|). \quad (3.10)$$

Now $1 - f^{-1}(M|u|)$ is an even convex function of u and so is a characteristic function ([6; p. 509]). Thus there is a stationary Gaussian process $\{G(t)\}$ with $E(G(t)G(s)) = 1 - f^{-1}(M|s - t|)$ so that $E(G(t) - G(s))^2 = 2f^{-1}(M|s - t|)$. By (3.8), we see that Fernique's condition holds so that $\{G(t)\}$ has continuous sample paths (see [11]). We now consider the Gaussian process $\{\langle x, X(t) \rangle\}$ indexed by $(x, t) \in B \times [0, N]$ where $N > 0$ and B is the unit ball in ℓ^2 . We put a metric d on the index space by

$$d((x, t), (y, s)) = \left\{ 2f^{-1}(M|t - s|) + 2 \sum (x_i - y_i)^2 \sigma_i^2 / 2\alpha_i \right\}^{1/2},$$

and note that $B \times [0, N]$ is compact with respect to this metric. We wish to show that the process $\{\langle x, X(t) \rangle : (x, t) \in B \times [0, N]\}$ is continuous when the index set has this metric. We do this by comparing it with a process known to be continuous. Let us first find a bound on the increment variance of $\{\langle x, X(t) \rangle\}$.

$$\begin{aligned} E|\langle x, X(t) \rangle - \langle y, X(s) \rangle|^2 &= \sum_{i=1}^{\infty} E|\langle x_i, X_i(t) \rangle - \langle y_i, X_i(s) \rangle|^2 \\ &\leq 2 \sum_{i=1}^{\infty} x_i^2 E|X_i(t) - X_i(s)|^2 + 2 \sum_{i=1}^{\infty} (x_i - y_i)^2 E(X_i(s))^2 \\ &\leq 2f^{-1}(M|t - s|) + 2 \sum_{i=1}^{\infty} (x_i - y_i)^2 \sigma_i^2 / 2\alpha_i \\ &= d^2((x, t), (y, s)). \end{aligned} \quad (3.11)$$

We now define a Gaussian process $\{H(x) : x \in B\}$, by

$$H(x) = \sum_{i=1}^{\infty} \sqrt{\sigma_i^2 / 2\alpha_i} x_i Z_i, \quad (3.12)$$

where Z_i are i.i.d. $N(0, 1)$ random variables. We want to show that H has continuous paths, and by linearity it suffices to show continuity at zero. Let $B(\delta) = \{x :$

$\sum x_i^2 \sigma_i^2 / 2\alpha_i \leq \delta$ and consider the following inequality.

$$\begin{aligned}
 E \sup_{x \in B(\delta)} |H(x)| &= E \sup_{x \in B(\delta)} \left| \sum_{i=1}^{\infty} \sqrt{\sigma_i^2 / 2\alpha_i} x_i Z_i \right| \\
 &\leq E \sup_{x \in B(\delta)} \left| \sum_{i=1}^N \sqrt{\sigma_i^2 / 2\alpha_i} x_i Z_i \right| + E \sup_{x \in B(\delta)} \left| \sum_{i=N+1}^{\infty} \sqrt{\sigma_i^2 / 2\alpha_i} x_i Z_i \right| \\
 &\leq E \sup_{x \in B(\delta)} \left(\sum_{i=1}^N Z_i^2 \right)^{1/2} \left(\sum_{i=1}^N x_i^2 \sigma_i^2 / 2\alpha_i \right)^{1/2} \\
 &\quad + E \sup_{x \in B(\delta)} \left(\sum_{i=N+1}^{\infty} x_i^2 \right)^{1/2} \left(\sum_{i=N+1}^{\infty} Z_i^2 \sigma_i^2 / 2\alpha_i \right)^{1/2} \\
 &\leq \sqrt{\delta} E \left(\sum_{i=1}^N Z_i^2 \right)^{1/2} + \sum_{i=N+1}^{\infty} \sigma_i^2 / 2\alpha_i.
 \end{aligned} \tag{3.13}$$

This can be made arbitrarily small by choosing N large and then δ small. The random variables $\sup \{|H(x)| : x \in B(\delta)\}$ are non-negative, non-decreasing in δ , and go to zero in L^1 as $\delta \rightarrow 0$. Therefore they go to zero a.s. as $\delta \rightarrow 0$ which is what we wanted to prove.

We now note that the Gaussian process

$$W(x, t) = G(t) + \sqrt{2}H(x), \quad (x, t) \in B \times [0, N],$$

where G is chosen to be independent of H , has continuous sample paths and that

$$E |W(x, t) - W(y, s)|^2 = d^2((x, t), (y, s)). \tag{3.14}$$

By the comparison theorem ([11], Lemma 2.1), (3.11) and (3.14) we conclude that $\{\langle x, X(t) \rangle : (x, t) \in B \times [0, N]\}$ is continuous. Since $B \times [0, N]$ is compact the paths are uniformly continuous and so for $\epsilon > 0$ there exists $\delta > 0$ (depending on ω), so that $|s - t| < \delta \Rightarrow |\langle x, X(t) \rangle - \langle y, X(s) \rangle| < \epsilon$. Thus,

$$\sup_{|s-t| < \delta} \|X(t) - X(s)\| = \sup_{x \in B, |s-t| < \delta} |\langle x, X(t) - X(s) \rangle| < \epsilon. \tag{3.15}$$

This shows that $\{X(t)\}$ is an ℓ^2 continuous process. $\square$

In order to apply this result we try various test functions f . To get the sharpest results we would want f to go to zero as quickly as possible at zero, and yet still satisfy (3.8). Consider the function $f(x) = \exp(-ce^{\beta(x)}/x)$ where c is a positive constant and $\beta(x)$ is a negative, non-decreasing function. For $\beta(x) = -\delta \log(1/x)$ or $\beta(x) = -(2+\delta) \log \log(1/x)$ we find that f satisfies the conditions of Proposition 3.3., the integral condition can be checked by performing the change of variables $y = f^{-1}(u)$. But, for example, setting $\beta(x)$ to a constant is not good enough, in that case f goes to zero too fast and (3.8) fails. We combine the results of Proposition 3.2. and 3.3. with $\beta(x) = -(2+\delta) \log \log(1/x)$ to get the following:

Proposition 3.4. Suppose that $\sum \sigma_i^2/2\alpha_i < \infty$. If, for some $\epsilon > 0$,

$$\sup_i \frac{\alpha_i}{\exp(\epsilon \alpha_i / \sigma_i^2)} = \infty, \quad (3.16)$$

then $\{X(t)\}$ is not ℓ^2 continuous.

If, for some $c > 0$ and $\delta > 0$,

$$\sup_i \frac{\alpha_i}{\exp\left(c \{\log(\alpha_i / \sigma_i^2)\}^{-(2+\delta)} \alpha_i / \sigma_i^2\right)} < \infty, \quad (3.17)$$

then $\{X(t)\}$ is ℓ^2 continuous.

Before leaving this section, we suggest a method which perhaps uses Proposition 3.3. to better advantage. Roughly speaking, the conditions in this proposition require that $f(\sigma_i^2/\alpha_i)$ be comparable to $1/\alpha_i$, i.e., that σ_i^2/α_i be comparable to $f^{-1}(1/\alpha_i)$, and that f^{-1} satisfies (3.8). To get Proposition 3.4., we chose a function which satisfied (3.8), and

asked that σ_i^2/α_i and $f^{-1}(1/\alpha_i)$ were comparable. We could take the opposite tack and define a function f^{-1} by the relation

$$f^{-1}(1/\alpha_i) = \sigma_i^2/\alpha_i, \quad (3.18)$$

and see whether or not it satisfies (3.8). Let us suppose that $\{1/\alpha_i\}$ and $\{\sigma_i^2/\alpha_i\}$ decrease monotonically to zero. We can extend the function f^{-1} linearly between the points $1/\alpha_i$ to get an increasing function which is zero at the origin. The slope of the i^{th} segment is $\sigma_i^2\alpha_{i+1} - \sigma_{i+1}^2\alpha_i/\alpha_{i+1} - \alpha_i$, so that the concavity of f^{-1} requires that this sequence be non-decreasing for large i . Since $d/du \left(2\sqrt{\log(1/u)} \right) = 1/(u\sqrt{\log(1/u)})$ we get a bound on the integral in (3.8) by taking

$$\sum_{i=1}^{\infty} \sqrt{\frac{\sigma_i^2}{2\alpha_i}} \left(\sqrt{\log \alpha_{i+1}} - \sqrt{\log \alpha_i} \right). \quad (3.19)$$

In summary we have the following,

Proposition 3.5. *Suppose that $\sum \sigma_i^2/2\alpha_i < \infty$ and $\{1/\alpha_i\}$, $\{\sigma_i^2/\alpha_i\}$ are decreasing and $\{\sigma_i^2\alpha_{i+1} - \sigma_{i+1}^2\alpha_i/\alpha_{i+1} - \alpha_i\}$ is increasing for large i . If (3.19) is finite, then $\{X(t)\}$ is ℓ^2 continuous.*

3.2. Exceptional sets

In this section we give an analysis of the exceptional sets, that is, the sets of capacity zero, for the infinite dimensional Ornstein-Uhlenbeck process. We pose for ourselves the following question—"From which points $x \in X$ will the process hit the set B ?" We begin by recalling from Proposition 2.24., that for every Borel B , either $H^B 1 = 1$ m-a.e., or $H^B 1 = 0$ m-a.e. That is, either B is hit from almost every starting point or from almost

no starting point. Both equalities may be strengthened to q.e. by quasicontinuity, and both will be everywhere equalities when polar and exceptional are the same.

From Proposition 1.5. we know that if $o(t, x, \cdot) \ll m(\cdot)$ for all $t > 0$ and $x \in X$, then the notions of polar and exceptional coincide. In particular, this will happen for the finite dimensional Ornstein-Uhlenbeck process. In infinite dimensions we still have $o(t, x, \cdot) \Rightarrow \delta_x$ as $t \rightarrow 0$ and $o(t, x, \cdot) \Rightarrow m(\cdot)$ as $t \rightarrow \infty$, but now there is a critical time $t_0 \in [0, \infty]$, depending on x and the coefficients σ_i^2 and α_i , such that $o(t, x, \cdot) \ll m(\cdot)$ for $t < t_0$ but $o(t, x, \cdot) \not\ll m(\cdot)$ for $t > t_0$. This phenomenon motivates the proposition below, which refines 1.5. and is valid for m -symmetric Hunt processes with $p_t 1$ identically equal to one, for all $t > 0$.

Proposition 3.6.

(i) Suppose that there exists $t_0 \in [0, \infty]$ such that $p_t(x, \cdot) \ll m(\cdot)$ for $t > t_0$ and $p_t(x, \cdot) \not\ll m(\cdot)$ for $t < t_0$. Then there exists an exceptional set B such that $P_x(X(t) \in B \forall 0 \leq t < t_0) = 1$. Furthermore, for any exceptional B , $P_x(\tau_B \circ \theta_{t_0} < \infty) = 0$.

(ii) If $p_t(x, \cdot) \sim p_t(y, \cdot)$ for all $t > t_0$, then for any Borel B , $P_x(\tau_B \circ \theta_{t_0} < \infty)$ and $P_y(\tau_B \circ \theta_{t_0} < \infty)$ are zero or one simultaneously.

Proof: (i) Let $t < t_0$ and A_t be a Borel set such that $p_t(x, A_t) > 0$ but $m(A_t) = 0$. For any s , $p_s(\cdot, A_t) = 0$ m -a.s., as it is a version of $T_s 1_{A_t}$, and by quasicontinuity it vanishes q.e. Now if $s < t$, then the equation

$$p_t(x, A_t) = \int p_{t-s}(x, dy) p_s(y, A_t) > 0 \quad (3.20)$$

shows that under P_x , the process hits the set $A_{s,t} = \{y : p_s(y, A_t) > 0\}$ at time $t - s$ with positive probability. Since $\text{Cap}(A_{s,t}) = 0$, the set $\text{Hit } A_{s,t} = \{y : P_y(\tau_{A_{s,t}} < \infty) > 0\}$

has capacity zero. Now define $u(\cdot) = P(\tau_{A_{s,t}} < \infty)$, it then follows from the strong Markov property that $\{u(X(t)), P_x\}$ is a non-negative supermartingale for every $x \in X$. A well-known martingale result (Theorem 4.16. [5]) tells us that the set $\{y : u(y) = 0\}$ is absorbing. Since $\{X(t)\}$ begun at x must hit $A_{s,t}$ at time $t - s$, it must remain in $\text{Hit } A_{s,t}$ until that time. Take $t_n = t_0 - 1/n$ and $s_n = t_n - 1/n$ and let $B = \bigcup_{n=1}^{\infty} \text{Hit } A_{s_n, t_n}$. Then $\text{Cap}(B) = 0$ and $P_x(X(t) \in B \forall 0 \leq t < t_0) = 1$.

Now let $t > t_0$, B an exceptional set, and $u(\cdot) = P(\tau_B < \infty)$. The function u is finely continuous and vanishes m -a.s. Since $p_t(x, \cdot) \ll m(\cdot)$, it vanishes $p_t(x, \cdot)$ -a.s. and so $p_t u(x) = 0$. Hence

$$p_{t_0} u(x) = P_x(\tau_B \circ \theta_{t_0} < \infty) = \lim_{t \downarrow t_0} p_t u(x) = 0. \quad (3.21)$$

(ii) This follows similarly. Let B is a Borel set and u as above. Then

$$p_{t_0} u(\cdot) = P(\tau_B \circ \theta_{t_0} < \infty) = \uparrow \lim_{t \downarrow t_0} P(\tau_B \circ \theta_t < \infty) = \uparrow \lim_{t \downarrow t_0} p_t u(\cdot). \quad (3.22)$$

Now if $p_{t_0} u(x) = 0$ then $p_t u(x) = 0$ for all $t > t_0$. Since $p_t(x, \cdot)$ and $p_t(y, \cdot)$ are equivalent for such t , we see that $p_t u(y) = 0$ when $t > t_0$. Letting $t \downarrow t_0$ gives $p_{t_0} u(y) = 0$ which is the required result. For the value 1, consider the function $1 - u$ and argue as above. $\square$

In order to apply the above result to the Ornstein-Uhlenbeck process, we recall the following facts on product Gaussian measures (see [20]). All sums and products below have index i running from 1 to ∞ .

Lemma 3.7.

- (i) The measures $\mu_1 = \prod N(x_i, \sigma_i^2)$ and $\mu_2 = \prod N(y_i, \rho_i^2)$ are either equivalent or orthogonal. They are equivalent iff μ_1, μ_2, μ_3 , and μ_4 are all equivalent where $\mu_3 = \prod N(x_i, \rho_i^2)$ and $\mu_4 = \prod N(y_i, \sigma_i^2)$.

$$(ii) \prod N(0, \rho_i^2) \sim \prod N(0, \sigma_i^2) \text{ iff } \sum (\sigma_i^2/\rho_i^2 - 1)^2 < \infty.$$

$$(iii) \prod N(y_i, \sigma_i^2) \sim \prod N(x_i, \sigma_i^2) \text{ iff } \sum (x_i - y_i)^2/\sigma_i^2 < \infty.$$

Using this lemma we get the following results on the Ornstein-Uhlenbeck transition measures.

Lemma 3.8.

$$(i) o(t, x, \cdot) \sim o(t, y, \cdot) \text{ iff } \sum_{i=1}^{\infty} (x_i - y_i)^2 2\alpha_i e^{-2\alpha_i t} / (1 - e^{-2\alpha_i t}) \sigma_i^2 < \infty.$$

$$(ii) o(t, x, \cdot) \sim m(\cdot) \text{ iff } \sum_{i=1}^{\infty} x_i^2 e^{-2\alpha_i t} (2\alpha_i / \sigma_i^2) < \infty \text{ and } \sum_{i=1}^{\infty} e^{-4\alpha_i t} < \infty.$$

Proof: (i) is immediate from 3.7., and (ii) follows by an intermediate comparison with $\prod_{i=1}^{\infty} N(x_i e^{-\alpha_i t}, \sigma_i^2 / 2\alpha_i)$. □

Suppose that $x, y \in X$ are such that

$$\sum_{i=1}^{\infty} \frac{(x_i - y_i)^2}{\sigma_i^2} e^{-\alpha_i t} < \infty \quad \forall t > 0. \quad (3.23)$$

Since the function $xe^{-(x/2)/(1-e^{-x})}$ is bounded on $(0, \infty)$ we see that for fixed t , $2\alpha_i e^{-\alpha_i t} / (1 - e^{-2\alpha_i t})$ is bounded in i . Multiplying this into the sum above and using 3.8.(ii) we see that $o(t, x, \cdot) \sim o(t, y, \cdot)$ for all $t > 0$. From Proposition 3.6.(ii) we conclude that the process hits the same sets beginning from x and y . That is, $P_x(\tau_B < \infty)$ and $P_y(\tau_B < \infty)$ are zero or one simultaneously. Since $P(\tau_B < \infty) = 0$ or 1 , m -a.e., this is a stronger version of Proposition 2.3. which asserts that the function $P(\tau_B < \infty)$ is constant along almost every line parallel to $x - y$.

The next proposition shows that if the drift is relatively large, then there is a natural state space for $\{X(t)\}$.

Proposition 3.9. *If $\sum e^{-\alpha_i t} < \infty$ for all $t > 0$, then there is a state space $\tilde{X}$ for*

$\{X(t)\}$ in which exceptional sets are polar.

Proof: For $t > 0$, let $H_t = \{x : \sum x_i^2 e^{-2\alpha_i t} (2\alpha_i / \sigma_i^2) < \infty\}$. This is just H_λ as defined in the last chapter, with $\lambda_i^2 = e^{-2\alpha_i t} (2\alpha_i / \sigma_i^2)$. Now $E(\sum x_i^2 e^{-2\alpha_i t} (2\alpha_i / \sigma_i^2)) = \sum e^{-2\alpha_i t}$ so that $m(H_t) = 1$. Also $\lambda_i^2 \sigma_i^2 = 2\alpha_i e^{-2\alpha_i t}$ is bounded so by Proposition 2.19, $\text{Cap}(X \setminus H_t) = 0$. Let $\{t_n\}$ be a sequence converging to 0, and define $\tilde{X} = \cap_{n=1}^{\infty} H_{t_n}$. Then $\text{Cap}(X \setminus \tilde{X}) \leq \sum_{n=1}^{\infty} \text{Cap}(X \setminus H_{t_n}) = 0$ so that $\tilde{X}$ serves as a state space for the stationary process $\{X(t)\}$.

Now $\sum e^{-4\alpha_i t} < \infty$ so by Lemma 3.8.(ii) we see that $o(t, x, \cdot) \ll m(\cdot)$, for each $x \in \tilde{X}$.

The result now follows from Proposition 1.5. $\square$

Comment: The conditions of the proposition are met when α_i grows like any positive power of i . This is because $\sum e^{-i^\beta} < \infty$ for any $\beta > 0$.

In what sense is the state space described above 'natural'? As we mentioned at the beginning of this section, our process is quasi-ergodic in the sense that for every Borel B , $H^B 1 = 1$ q.e. or $H^B 1 = 0$ q.e. Under the conditions of the above proposition, the process restricted to $\tilde{X}$ is genuinely ergodic, so that $H^B 1 \equiv 1$ or $H^B 1 \equiv 0$ on $\tilde{X}$ for $B \subset \tilde{X}$. Another way of putting it is that when the coefficients α_i are large enough, every set is either entered or exited. That is, there exists no $B \subset \tilde{X}$ such that both B and $\tilde{X} \setminus B$ are invariant. To see this we argue as follows: By Proposition 2.24, every invariant set has m measure zero or one, so that if such a B exists we may assume without loss of generality that $m(B) = 0$. But in this case, $\text{Cap}(B) = 0$ because B can only be hit from within B . Then by Proposition 3.9, B must be polar and hence must be exited immediately. Therefore $\tilde{X} \setminus B$ is not invariant. This contradiction shows that no such B can exist.

If $\sum e^{-2\alpha_i t} = \infty$ for some $t > 0$, then $\sum e^{-4\alpha_i t/2} = \infty$ and so from Proposition 3.6.(i) we know that from each starting point $x \in X$, the process lives in an exceptional set up until time $t/2$. Whether or not it hits an exceptional set after this time depends on x and the coefficients $\{\alpha_i\}$ and $\{\sigma_i^2\}$. Note that we can arrange for any t_0 to be the critical time, by letting $\alpha_i = \log i / 4t_0$. Then

$$\sum e^{-4\alpha_i t} = \sum e^{-4(\log i / 4t_0)t} = \sum i^{-(t/t_0)} \quad (3.24)$$

which sums up iff $t > t_0$.

In the extreme case when $\sum e^{-4\alpha_i t} = \infty$ for all t , say when $\{\alpha_i\}$ is bounded, then from every starting point $x \in X$, the process lives in an exceptional set (depending on x) for all time. In very extreme cases, the picture which emerges of this process is that of many non-communicating processes living in disjoint layers of space. Collectively these processes make up the stationary process, although individually the layers have no measure and hence no capacity (by the argument two paragraphs ago). Hence, any layer is negligible and may be excluded from the state space without affecting the Dirichlet form. The layers are not polar sets and any collection of them with full m measure will serve as a state space for the stationary process.

In order to make quantitative statements about these layers we will compare the stationary process $\{X(t)\}$ with $\{X^0(t)\}$, which is the process begun at the origin. Intuitively, $\{X^0(t)\}$ begins as a point mass at the origin and, as time goes on, it evolves towards the stationary process. When $\{\alpha_i\}$ and $\{\sigma_i^2\}$ are small, the rate at which $\{X^0(t)\}$ expands can be so small that it may have significantly better continuity properties than the stationary process. This will result in the phenomenon described above.

Now taking a single coordinate we solve the stochastic differential equation to get

$$X_i^0(t) = e^{-\alpha_i t} \int_0^t \sigma_i e^{\alpha_i r} dW_i(r), \quad (3.25)$$

and so

$$\begin{aligned} E(X_i^0(t)X_i^0(s)) &= e^{-\alpha_i(s+t)} \int_0^{s \wedge t} \sigma_i^2 e^{2\alpha_i r} dr \\ &= e^{-\alpha_i(s+t)} \frac{\sigma_i^2}{2\alpha_i} (e^{2\alpha_i(s \wedge t)} - 1) \\ &= \frac{\sigma_i^2}{2\alpha_i} (e^{-\alpha_i|s-t|} - e^{-\alpha_i(s+t)}) \end{aligned} \quad (3.26)$$

From this formula we get

$$E(X_i^0(t))^2 = \sigma_i^2/2\alpha_i (1 - e^{-2\alpha_i t}) \quad (3.27)$$

and so

$$\begin{aligned} E(X_i^0(t) - X_i^0(s))^2 &= \sigma_i^2/2\alpha_i (1 - e^{-2\alpha_i t}) + (1 - e^{-2\alpha_i s}) - 2(e^{-\alpha_i|s-t|} - e^{-\alpha_i(s+t)}) \\ &= \sigma_i^2/2\alpha_i (2(1 - e^{-\alpha_i|s-t|}) - (e^{-\alpha_i t} - e^{-\alpha_i s})^2) \end{aligned} \quad (3.28)$$

Recall that for the stationary process,

$$E(X_i(t))^2 = \sigma_i^2/2\alpha_i \text{ and } E(X_i(t) - X_i(s))^2 = \sigma_i^2/2\alpha_i 2(1 - e^{-\alpha_i|s-t|}), \quad (3.29)$$

and so we see that the increment variance is of the same order for both processes while the variance of $\{X_i^0(t)\}$ is smaller than that of $\{X_i(t)\}$, significantly so when α_i are small.

Using an argument similar to that in Proposition 3.3. we get:

Proposition 3.10. *If $\sum \lambda_i^2 \sigma_i^2 < \infty$, then $\{X^0(t)\}$ has H_λ continuous sample paths.*

This is exactly what we would expect. In fact, $\{X^0(t)\}$ solves the infinite dimensional stochastic differential equation,

$$dX^0(t) = -AX^0(t)dt + \sqrt{\Sigma}dW(t), \quad X^0(0) = 0 \quad (3.30)$$

where $A = \text{diag}(\alpha_i)$, $\Sigma = \text{diag}(\sigma_i^2)$ and $\{W(t)\}$ is a cylindrical Brownian motion. In this setup A acts as a control on $\{X^0(t)\}$. Now the conditions of 3.10. do not involve A , and the conclusion is valid in the absence of any control, i.e., when

$$dX^0(t) = \sqrt{\Sigma} dW(t), \quad X^0(0) = 0. \quad (3.31)$$

The solution of this equation is a cylindrical Brownian motion with covariance Σ and it is H_λ continuous iff $\sum \lambda_i^2 \sigma_i^2 < \infty$. Since the uncontrolled process is continuous it is not surprising that the controlled one is also.

We point out that 3.10. is only of interest when $\sum \lambda_i^2 \sigma_i^2 / 2\alpha_i = \infty$ or else 3.1. will apply to give continuity of the stationary process, and hence of $\{X^0(t)\}$. In particular, we are only interested in the case $\alpha_i \rightarrow 0$. Since $1 - e^{-2x}/2x$ is bounded away from zero on compacts, we see that when α_i is bounded,

$$\sum \lambda_i^2 \sigma_i^2 \leq K \sum \lambda_i^2 \sigma_i^2 / 2\alpha_i (1 - e^{-2\alpha_i t}) = K \sum \lambda_i^2 E(X_i^0(t))^2, \quad (3.32)$$

for some constant K , and so the condition $\sum \lambda_i^2 \sigma_i^2 < \infty$ is necessary as well. This means that a bounded control operator is useless as an aid to continuity in H_λ .

Proposition 3.11. *For fixed H_λ and m , there exist coefficients $\{\sigma_i^2\}$ and $\{\alpha_i\}$ such that $\{X^0(t)\}$ is H_λ continuous. Consequently, every translate $x + H_\lambda$ of H_λ is invariant under $\{X(t)\}$ and if $m(H_\lambda) = 0$, then $\text{Cap}(x + H_\lambda) = 0$ for all $x \in X$.*

Comment: Here we show that if $m(H_\lambda) = 0$, then H_λ can be made exceptional by choosing σ_i^2 small enough. This should be compared with Proposition 2.16. which showed that we can make $\text{Cap}(H_\lambda) > 0$ by taking σ_i^2 large enough.

Proof: Since m is fixed, so are the ratios $\sigma_i^2 / 2\alpha_i$ (see (1.54)). Choose $\sigma_i^2 \leq (\sigma_i^2 / 2\alpha_i) \wedge (1/i^2 \lambda_i^2)$ so that $\sum \lambda_i^2 \sigma_i^2 < \infty$ and $\{\alpha_i\}$ is bounded. Then by Proposition 3.10., $\{X^0(t)\}$

is H_λ continuous, and the equation $X^z(t) = (x_1 e^{-\alpha_1 t}, \dots, x_i e^{-\alpha_i t}, \dots) + X^0(t)$, shows that every translate of H_λ is also invariant. Here we used the boundedness of $\{\alpha_i\}$. Since $m(H_\lambda) = 0$, $m(x + H_\lambda) = 0$ for all x and we have seen earlier that an invariant set with no measure has no capacity. $\square$

From 3.11. we conclude that the layers spoken of earlier can be arbitrarily 'thin'. In fact, if x and y differ in infinitely many coordinates it is possible to separate them by an Ornstein-Uhlenbeck process with invariant measure m . Just take λ_i so large that $\sum \lambda_i^2 (x_i - y_i)^2 = \infty$ and σ_i^2 so small that $\sum \lambda_i^2 \sigma_i^2 < \infty$.

The following result is a byproduct of the fact, shown in Proposition 2.24., that invariant sets must have measure zero or one.

Proposition 3.12. *For any measurable $B \subset X$ and any H_λ , the set*

$$B + H_\lambda = \bigcup \{b + h : b \in B, h \in H_\lambda\} \quad (3.33)$$

has m measure zero or one.

3.3. Regular points and the Dirichlet problem

Let us consider the following analogue to the classical Dirichlet problem in R^d .

Dirichlet problem: *Let G be a bounded open set in ℓ^2 and φ a bounded continuous function on ∂G . Find f which is harmonic on G and such that*

$$\lim_{\substack{y \rightarrow x \\ y \in G}} f(y) = \varphi(x) \quad (3.34)$$

for all $x \in \partial G$.

In order to construct a stochastic solution to this problem, we take an Ornstein-Uhlenbeck process with ℓ^2 continuous sample-paths. For this we may use either of Propositions 3.1. or 3.3., but we must have at least that $\sum \sigma_i^2/2\alpha_i < \infty$. If we now define the function

$$f(y) = E_y(\varphi(X_{r_{\partial G}})) = H^{\partial G} \varphi(y) \quad (3.35)$$

we see that it has the property $f = H^{X \setminus G} f$ q.e. If $f \in \mathcal{D}(\mathcal{E})$, then by Proposition 2.21. we see that it satisfies the conditions of Definition 2.20. and is harmonic on G . We are interested in whether or not f satisfies (3.34). Following Gross [9], we call a set *strongly regular* if each of its boundary points is regular for the interior of the complement of the set. The following is a modification for the Ornstein-Uhlenbeck process of Gross' result on the Wiener process.

Proposition 3.13. *If G is strongly regular, then f in (3.35) satisfies (3.34).*

Proof: We define the process begun at the origin $\{X^0(t)\}$ on (Ω, P) and then let

$$X^y(t, \omega) = ye^{-\alpha t} + X^0(t, \omega), \quad (3.36)$$

where $ye^{-\alpha t}$ is shorthand for the vector $(y_1 e^{-\alpha_1 t}, \dots, y_n e^{-\alpha_n t}, \dots)$. For $y \in X$, we let $\tau_y(\omega) = \inf\{t > 0 : X^y(t, \omega) \notin G\}$. We wish to show that $\tau_y \rightarrow 0$ P -a.s. as $y \rightarrow x$ where $y \in G$ and $x \in \partial G$. Let Ω_0 be the set of ω such that $t \mapsto X^0(t, \omega)$ is continuous and $\inf\{t > 0 : X^x(t, \omega) \notin \text{int}(X \setminus G)\} = 0$. By supposition $P(\Omega_0) = 1$ and if $\omega \in \Omega_0$, then for any $\epsilon > 0$ there is a $t_0 \leq \epsilon$ so that $xe^{-\alpha t_0} + X^0(t_0, \omega)$ is in the interior of $X \setminus G$. In particular, the distance from $xe^{-\alpha t_0} + X^0(t_0, \omega)$ to G is greater than some positive δ . If $y \in G$ and $|y - x| < \delta/2$, then $|ye^{-\alpha t_0} + X^0(t_0, \omega) - xe^{-\alpha t_0} + X^0(t_0, \omega)| \leq |(x - y)e^{-\alpha t_0}| \leq |x - y| < \delta/2$ and so $ye^{-\alpha t_0} + X^0(t_0, \omega) \notin G$. Therefore $\tau_y(\omega) \leq \epsilon$ for all $|y - x| < \delta/2$. Now $f(y) = E(\varphi(ye^{-\alpha \tau_y} + X_{\tau_y}))$ and letting $y \rightarrow x$ gives $\tau_y \rightarrow 0$ a.s. and so $ye^{-\alpha \tau_y} + X_{\tau_y} \rightarrow x$ a.s. By continuity of φ and bounded convergence we get $f(y) \rightarrow \varphi(x)$. $\square$

In order to apply this result we need to be able to give conditions under which a set is strongly regular. In considering regularity of points let us begin with the simplest possible set, the half-space. We give conditions under which x is regular for the half-space $\{y : \langle y, x \rangle > \|x\|^2\}$. Since the Ornstein-Uhlenbeck process has a drift towards the origin, one sees, by the following lemma for example, that x is always regular for the other half, i.e., $\{y : \langle y, x \rangle < \|x\|^2\}$. The following well-known result is crude, but works in many reasonable situations. For example, in finite dimensions, it gives us Poincaré's cone condition.

Lemma 3.14. *If $\limsup_{t \rightarrow 0} P_x(X(t) \in B) > 0$, then x is regular for B .*

Proof: $P_x(\tau_B > t) \leq P_x(X(t) \notin B) = 1 - P_x(X(t) \in B)$, so that if $P_x(X(t) \in B)$ fails to converge to zero, then $P_x(\tau_B > t)$ does not converge to 1 as $t \rightarrow 0$. But $\lim_{t \rightarrow 0} P_x(\tau_B > t) = P_x(\tau_B > 0)$ so $P_x(\tau_B = 0) > 0$. By the zero-one law, $P_x(\tau_B = 0) = 1$. $\square$

Under P_x the random variable $\langle x, X(t) \rangle = \sum x_i X_i(t)$ is distributed as

$$N\left(\sum x_i^2 e^{-\alpha_i t}, \sum x_i^2 (\sigma_i^2 / 2\alpha_i) (1 - e^{-2\alpha_i t})\right). \quad (3.37)$$

Therefore,

$$P_x(\langle x, X(t) \rangle > \|x\|^2) = P\left(Z > \frac{\sum x_i^2 (1 - e^{-\alpha_i t})}{\sqrt{\sum x_i^2 (\sigma_i^2 / 2\alpha_i) (1 - e^{-2\alpha_i t})}}\right), \quad (3.38)$$

where Z is $N(0, 1)$ under P . In order to apply the above lemma we require that

$$\frac{\sum x_i^2 (1 - e^{-\alpha_i t})}{\sqrt{\sum x_i^2 (\sigma_i^2 / 2\alpha_i) (1 - e^{-2\alpha_i t})}} \quad (3.39)$$

does not go to infinity as $t \rightarrow 0$. Since both the numerator and the denominator go to zero as $t \rightarrow 0$, it is a question of with what rates.

Proposition 3.15. *If $\sum x_i^2(2\alpha_i/\sigma_i^2) < \infty$ or if $\sum x_i^2(1 - e^{-\alpha_i t})$ is $O(\sqrt{t})$, then x is regular for $\{y : \langle y, x \rangle > \|x\|^2\}$.*

Proof: For the first part, we use the Cauchy-Schwarz inequality to obtain,

$$\begin{aligned} \sum x_i^2(1 - e^{-\alpha_i t}) &\leq \sqrt{\sum \frac{x_i^2 \sigma_i^2}{2\alpha_i} (1 - e^{-2\alpha_i t})} \sqrt{\sum \frac{2x_i^2 \alpha_i}{\sigma_i^2} \frac{(1 - e^{-\alpha_i t})^2}{(1 - e^{-2\alpha_i t})}} \\ &= \sqrt{\sum \frac{x_i^2 \sigma_i^2}{2\alpha_i} (1 - e^{-2\alpha_i t})} \sqrt{\sum \frac{2x_i^2 \alpha_i}{\sigma_i^2} \frac{(1 - e^{-\alpha_i t})}{(1 + e^{-\alpha_i t})}} \\ &\leq \sqrt{\sum \frac{x_i^2 \sigma_i^2}{2\alpha_i} (1 - e^{-2\alpha_i t})} \sqrt{\sum \frac{2x_i^2 \alpha_i}{\sigma_i^2}}. \end{aligned} \quad (3.40)$$

Therefore the ratio (3.39) is bounded if $\sum x_i^2(2\alpha_i/\sigma_i^2)$ is finite. On the other hand, we may bound this ratio directly by throwing away everything in the sum in the denominator but the first term. We are then left with a constant times $\sqrt{1 - e^{-2\alpha_1 t}}$ which is $O(\sqrt{t})$ and this gives us the result. $\square$

Comment: This proposition gives us two things. Firstly, we find that there is a collection $\{x : \sum x_i^2(2\alpha_i/\sigma_i^2) < \infty\}$ of points which are regular, regardless of the size of the drift. Secondly, we get a condition on the drift under which all points are regular. If $\{\alpha_i\}$ is bounded, for example, then

$$\sum x_i^2(1 - e^{-\alpha_i t}) \leq \sum x_i^2 \alpha_i t \leq ct, \quad (3.41)$$

for any x , where c is a constant. Clearly the conditions of 3.15. are satisfied so that every x is regular.

Corollary 3.16. *If $\{\alpha_i\}$ are bounded, then the unit ball $\{y : \|y\| < 1\}$ is strongly regular.*

Proof: The unit ball supports the hyperplane $\{y : \langle y, x \rangle = 1\}$ at each of its boundary points x . The result now follows from the above proposition. $\square$

3.4. Measures of finite energy integral

We have seen that fixing $m = \prod_{i=1}^{\infty} N(0, \sigma_i^2/2\alpha_i)$ and letting α_i or σ_i^2 grow very quickly, forces the irregularity of the process $\{X(t)\}$ or equivalently (see (1.12)) forces the regularity of the functions in $\mathcal{D}(\mathcal{E})$. For example, Propositions 2.3. and 3.2. give results of this type. The result in this section can be regarded as a measure of the regularity of $f \in \mathcal{D}(\mathcal{E})$, when α_i are large.

Definition 3.17. A positive Radon measure μ is said to be of *finite energy integral* when the linear map

$$f \mapsto \int f(x) \mu(dx), \quad f \in \mathcal{D}(\mathcal{E}) \cap C_0(X) \quad (3.42)$$

is continuous on the Hilbert space $(\mathcal{D}(\mathcal{E}), \mathcal{E}_1)$. In this case, there is an element $U_1\mu \in \mathcal{D}(\mathcal{E})$, called the *1-potential* of μ , which represents the map (3.42) in the sense that

$$\mathcal{E}_1(U_1\mu, f) = \int f(x) \mu(dx), \quad f \in \mathcal{D}(\mathcal{E}) \cap C_0(X). \quad (3.43)$$

$\mathcal{E}_1(U_1\mu, U_1\mu)$ is called the *energy integral* of μ .

The importance of measures of finite energy integral comes from Theorem 3.3.2. of [7], where Fukushima has generalized the result of classical potential theory which states that a Borel set has positive capacity if and only if it supports such a measure.

In this section we shall consider those measures which are obtained by shifting m by a vector z . We are interested in determining when the measure $\mu(\cdot) = m(z + \cdot)$ is of finite energy integral. Of course, when $\mu \sim m$ that is, when $\sum z_i^2(2\alpha_i/\sigma_i^2) < \infty$, then μ is of finite energy integral since it has an L^2 density with respect to m . If $\mu \perp m$ but nevertheless μ is of finite energy integral, then by the result cited above, μ charges no set

of zero capacity. Thus we can find sets of positive capacity that have no m measure, and therefore we can draw interesting conclusions about the behavior of the process.

Let us begin with one dimension and calculate the energy of $\mu(\cdot) = m(z + \cdot)$ under the Ornstein-Uhlenbeck process with drift α and diffusion σ^2 . The energy integral is given⁴ by

$$\int_0^\infty e^{-t} \int \int o(t, x, y) \mu(dx) \mu(dy) dt \quad (3.44)$$

where $o(t, x, y)$ is the kernel from (1.41). Thus, if $m(x)$ is the density of m , then $o(t, x, y)m(x)$ is the density of $N(ye^{-\alpha t}, (\sigma^2/2\alpha)(1 - e^{-2\alpha t}))$. Also the density of μ with respect to m is given by

$$\frac{d\mu}{dm}(x) = \exp(-1/2(2\alpha/\sigma^2)z^2) \exp(1/2(2\alpha/\sigma^2)2zx). \quad (3.45)$$

Recall that if X has distribution $N(\mu, \sigma^2)$, then $E(e^{tX}) = \exp(\sigma^2 t^2/2 + \mu t)$. To calculate (3.44), we start by finding $\phi(y) = \int o(t, x, y) \mu(dx)$. But

$$\begin{aligned} \phi(y) &= \int o(t, x, y) \mu(dx) \\ &= \int o(t, x, y) m(x) \frac{d\mu}{dm}(x) dx \\ &= E\left(\frac{d\mu}{dm}(X)\right) \end{aligned} \quad (3.46)$$

where X has a $N(ye^{-\alpha t}, \sigma^2/2\alpha(1 - e^{-2\alpha t}))$ distribution. From (3.45) and the Laplace transform cited above we see that

$$\phi(y) = \exp(1/2(2\alpha/\sigma^2)z^2(-e^{-2\alpha t})) \exp(e^{-\alpha t}(2\alpha/\sigma^2)zy). \quad (3.47)$$

Now we want to find $\int \phi(y) \mu(dy)$, which we do by repeating the same argument with Y distributed as $N(z, \sigma^2/2\alpha)$. This gives

$$\begin{aligned} &\int \phi(y) \mu(dy) \\ &= \exp(1/2(2\alpha/\sigma^2)z^2(-e^{-2\alpha t})) \exp(1/2(\sigma^2/2\alpha)e^{-2\alpha t}(2\alpha/\sigma^2)^2 z^2) \exp(e^{-\alpha t}(2\alpha/\sigma^2)z^2) \\ &= \exp(e^{-\alpha t}(2\alpha/\sigma^2)z^2). \end{aligned} \quad (3.48)$$

So the energy integral is

$$\int_0^\infty e^{-t} \exp\left(e^{-\alpha t} \frac{2\alpha}{\sigma^2} z^2\right) dt. \quad (3.49)$$

This is also the energy integral when m and $\{X(t)\}$ are infinite dimensional as long as the shift z is only in one coordinate. By independence of coordinates, we see that if $z = (z_1, z_2, \dots, z_d, 0, 0, \dots)$, then the energy integral of μ is given by

$$\int_0^\infty e^{-t} \exp\left(\sum_{i=1}^d e^{-\alpha_i t} \frac{2\alpha_i}{\sigma_i^2} z_i^2\right) dt. \quad (3.50)$$

Now if μ_d is m shifted by $(z_1, z_2, \dots, z_d, 0, 0, \dots)$, then $\mu_d \Rightarrow \mu$ and one can use (3.43) to show that if the energies of μ_d are bounded, then μ is of finite energy with energy less than this bound. In other words, we get the following:

Proposition 3.18. *Let $\phi(t) = \exp(\sum_{i=1}^\infty e^{-\alpha_i t} \frac{2\alpha_i}{\sigma_i^2} z_i^2)$. If ϕ is locally integrable at zero, then $\mu(\cdot) = m(z + \cdot)$ is of finite energy integral.*

Comment: If $\sum_{i=1}^\infty 2z_i^2 \alpha_i / \sigma_i^2 < \infty$, then ϕ is bounded and so μ is of finite energy integral. This confirms what was mentioned earlier, that if $\mu \sim m$, then μ has finite energy.

The function ϕ is non-increasing and is continuous as soon as it is finite, by dominated convergence. It is finite at zero iff $\sum_{i=1}^\infty (2\alpha_i / \sigma_i^2) z_i^2 < \infty$. When $\sum_{i=1}^\infty (2\alpha_i / \sigma_i^2) z_i^2 = \infty$, then ϕ blows up at the origin and the relevant question is how fast. Presumably one can choose $\{\alpha_i\}$ so large that the energy integral is finite, but it is not clear how to do this.

As a sample calculation let us take $(2\alpha_i / \sigma_i^2) z_i^2 = 1$ for all i . Then

$$\int_0^\infty \sum_{i=1}^\infty e^{-\alpha_i t} dt = \sum_{i=1}^\infty 1/\alpha_i, \quad (3.51)$$

and

$$\int_0^\infty \left(\sum_{i=1}^\infty e^{-\alpha_i t} \right)^k dt = \sum_{i_1, \dots, i_k} 1/(\alpha_{i_1} + \dots + \alpha_{i_k}), \quad k \in \mathbb{N}, \quad (3.52)$$

so

$$\int_0^\infty e^{-t} \exp \left(\sum_{i=1}^\infty e^{-\alpha_i t} \right) dt \leq 1 + \sum_{k=1}^\infty \frac{1}{k!} \sum_{i_1, \dots, i_k} 1/(\alpha_{i_1} + \dots + \alpha_{i_k}). \quad (3.53)$$

The coefficients α_i must be very large to make the right hand side of (3.53) finite.

3.5. The law of the iterated logarithm (LIL)

In this section we obtain results on the LIL behavior at zero of our infinite dimensional Ornstein-Uhlenbeck process begun at zero. In other words, we will consider the behavior as $t \rightarrow 0$ of the net of random vectors $\{X^0(t, \omega)/L(t) : 0 < t \leq 1/3\}$. Here $L(t) = \sqrt{2t \log \log(1/t)}$ and $\{X^0(t)\}$ is the mild solution of

$$dX^0(t) = -AX^0(t)dt + \sqrt{\Sigma}dW(t), \quad X^0(0) = 0 \quad (3.54)$$

where $A = \text{diag}(\alpha_i)$, $\Sigma = \text{diag}(\sigma_i^2)$ and $\{W(t)\}$ is a cylindrical Brownian motion. We have

$$X^0(t) = \int_0^t T_{t-s} \sqrt{\Sigma} dW(s) \quad (3.55)$$

where T_t denotes the operator e^{-tA} . We regard $\{X^0(t)\}$ and $\{\sqrt{\Sigma}W(t)\}$ as H_λ valued processes for suitable values of $\{\lambda_i\}$. For comparison purposes, let us recall the LIL at zero for the process $\{\sqrt{\Sigma}W(t)\}$ (see [13], [15]).

Proposition 3.19. *Let $\{\lambda_i\}$ be chosen so that $\{\sqrt{\Sigma}W(t)\}$ is H_λ continuous, i.e., such that $\sum \lambda_i^2 \sigma_i^2 < \infty$. Then, almost surely, the vector $\sqrt{\Sigma}W(t, \omega)/L(t)$ converges to $K = \{x : \sum x_i^2 / \sigma_i^2 \leq 1\}$ and clusters at every point thereof.*

Comment: We note that the limit set K is independent of the choice of $\{\lambda_i\}$. The conclusion in the above proposition means that $\sqrt{\Sigma}W(t, \omega)/L(t)$ is eventually within any

ϵ -neighborhood of K , and it visits each ϵ -neighborhood of every $x \in K$ infinitely often as $t \rightarrow 0$. The neighborhoods are taken in the Hilbert topology of H_λ .

It seems reasonable to expect the Ornstein-Uhlenbeck process has the same LIL as $\{\sqrt{\Sigma}W(t)\}$. The results which follow may be considered as a partially successful attempt to show that this is true.

Proposition 3.20. *In finite dimensions, $\{X^0(t)\}$ has the same LIL as $\{\sqrt{\Sigma}W(t)\}$. That is, $X^0(t, \omega)/L(t)$ almost surely satisfies the conclusion of the previous proposition.*

Proof: Let P_W (resp. P_X) denote the measure on $C([0, \infty); R^d)$ induced by $\{\sqrt{\Sigma}W(t)\}$ (resp. $\{X^0(t)\}$). By Girsanov's formula [4; p. 82], $P_X \sim P_W$ on $\mathcal{F}_t$ for all $t > 0$. In particular, they are equal on $\mathcal{F}_{0+}$ which is a trivial field under either measure. Therefore for any Borel B , the following holds a.s.

$$\bigcap_{t>0} \overline{\{X^0(s, \omega)/L(s) : 0 < s \leq t\}} \cap B = \bigcap_{t>0} \overline{\{\sqrt{\Sigma}W(s, \omega)/L(s) : 0 < s \leq t\}} \cap B, \quad (3.56)$$

where $\overline{A}$ means the closure of A in H_λ . By the previous proposition, the set on the right hand side is a.s. empty when B is the complement of an ϵ -neighborhood of K . Letting $\epsilon \rightarrow 0$ through a sequence shows that $X^0(s, \cdot)/L(s)$ converges to K a.s. Similarly, the set on the right hand side is non-empty when B is a neighborhood of a point in K . Taking a countable dense subset of K , and at each point a countable system of neighborhoods converging to that point, we conclude that $X^0(s, \cdot)/L(s)$ clusters at each $x \in K$ almost surely. $\square$

In the following proposition we prove one half of the LIL in the infinite dimensional case.

Proposition 3.21. *Let $\{\lambda_i\}$ be so that $\sum \lambda_i^2 \sigma_i^2 < \infty$. Then, almost surely, the*

vector $X^0(t, \omega)/L(t)$ clusters at every $x \in K$.

Proof: We begin by noting that $X_i^0(t)/\sqrt{t}$ has smaller variance than $\sigma_i^2 W_i(1)$ for all small t , because

$$E \left(X_i^0(t)/\sqrt{t} \right)^2 = \frac{\sigma_i^2}{2\alpha_i} (1 - e^{-2\alpha_i t})/t \leq \sigma_i^2. \quad (3.57)$$

Let Q_k be the map which sends $(x_1, x_2, \dots, x_k, x_{k+1}, \dots)$ to $(x_1, x_2, \dots, x_k, 0, 0, \dots)$. Since $Q_k x \rightarrow x$ a.s. in H_λ , we may use Fernique's result (cf. [15; p. 105]) to find k so large that

$$E \exp \left(\gamma \left\| \sqrt{\Sigma} W(1) - Q_k \sqrt{\Sigma} W(1) \right\|_\lambda^2 \right) < \infty \quad (3.58)$$

for any fixed $\gamma > 0$. Now for any $\epsilon > 0$, choose γ so that $2\gamma\epsilon^2 > r^2 > 1$. We get the following inequality for small t ,

$$\begin{aligned} P \left(\left\| X^0(t) - Q_k X^0(t) \right\|_\lambda \geq \epsilon \sqrt{2t \log \log(1/t)} \right) \\ \leq P \left(\left\| \sqrt{\Sigma} W(1) - Q_k \sqrt{\Sigma} W(1) \right\|_\lambda \geq \epsilon \sqrt{2 \log \log(1/t)} \right) \\ \leq E \exp \left(\gamma \left\| \sqrt{\Sigma} W(1) - Q_k \sqrt{\Sigma} W(1) \right\|_\lambda^2 \right) \exp(-2\gamma\epsilon^2 \log \log(1/t)) \\ \leq e^{-r^2 \log \log(1/t)} \end{aligned} \quad (3.59)$$

Now fix $\epsilon > 0$ and $x \in K$. Choose k so large that $\|x - Q_k x\|_\lambda < \epsilon$ and that (3.59) holds for this ϵ . Now by the proof of Strassen's LIL in finite dimensions (cf. [13; p. 261]), we know that if c is large enough, then

$$P \left(\left\| Q_k(\sqrt{\Sigma} W(\lfloor c^n \rfloor)/L(1/\lfloor c^n \rfloor)) - x \right\|_\lambda \leq \epsilon \text{ i.o.} \right) = 1. \quad (3.60)$$

where $\lfloor c^n \rfloor$ is the integer part of c^n .

Using Girsanov's formula as in Proposition 3.20., we get the same result for the Ornstein-Uhlenbeck process, i.e.,

$$P \left(\left\| Q_k(X^0(1/\lfloor c^n \rfloor)/L(1/\lfloor c^n \rfloor)) - x \right\|_\lambda \leq \epsilon \text{ i.o.} \right) = 1. \quad (3.61)$$

Replacing t by $1/[c^n]$ in (3.59) we see that the right hand side is summable in n , so that Borel-Cantelli gives us

$$P \left(\| (X^0(1/[c^n]) - Q_*(X^0(1/[c^n])) / L(1/[c^n])) \|_\lambda \leq \epsilon, n \text{ large enough} \right) = 1. \quad (3.62)$$

Combining (3.60) and (3.62) gives us the required result. $\square$

Finally let us consider the LIL behavior of the real valued process

$$\langle x, X^0(t) \rangle = \sum_{i=1}^{\infty} x_i X_i^0(t). \quad (3.63)$$

Note that $\langle x, \sqrt{\Sigma} W(t) \rangle$ is a Brownian motion with variance parameter $\sum \sigma_i^2 x_i^2$. When x is zero for all but finitely many coordinates, Girsanov's formula may be used to show that $\langle x, X^0(t) \rangle$ and $\langle x, \sqrt{\Sigma} W(t) \rangle$ have the same LIL. In general we may obtain the following result which is weaker by a factor of $\sqrt{2}$.

Proposition 3.22. $P \left(\limsup_{t \rightarrow 0} \langle x, X^0(t) \rangle / L(t) \leq \sqrt{2 \sum \sigma_i^2 x_i^2} \right) = 1.$

Proof: We recall that the Ornstein-Uhlenbeck process $\{X_i^0(t)\}$ has smaller variance and smaller increment variance than $\{\sigma_i^2 W_i(t)\}$. That is,

$$E(X_i^0(t))^2 = \sigma_i^2 / 2\alpha_i (1 - e^{-2\alpha_i t}) \leq \sigma_i^2 t$$

and

$$\begin{aligned} E(X_i^0(t) - X_i^0(s))^2 &= \sigma_i^2 / 2\alpha_i \left(2(1 - e^{-\alpha_i |t-s|}) - (e^{-\alpha_i t} - e^{-\alpha_i s})^2 \right) \\ &\leq 2\sigma_i^2 / 2\alpha_i (1 - e^{-\alpha_i |t-s|}) \\ &\leq \sigma_i^2 |t-s|. \end{aligned} \quad (3.64)$$

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Therefore we get the following inequality:

$$\begin{aligned}
 & E(X_i^0(t)/L(t) - X_i^0(s)/L(s))^2 \\
 &= E(X_i^0(t)/L(t) - X_i^0(s)/L(t) + X_i^0(s)/L(t) - X_i^0(s)/L(s))^2 \\
 &\leq 2E(X_i^0(t) - X_i^0(s)/L(t))^2 + 2E(X_i^0(s)(1/L(t) - 1/L(s)))^2 \\
 &\leq 2E(\sigma_i^2 W_i(t) - \sigma_i^2 W_i(s)/L(t))^2 + 2E(\sigma_i^2 W_i(s)(1/L(t) - 1/L(s)))^2 \\
 &= 2E(\sigma_i^2 W_i(t)/L(t) - \sigma_i^2 W_i(s)/L(s))^2.
 \end{aligned} \tag{3.65}$$

In the final equality above, we used the independent increments of the Wiener process. From (3.65) we see that the real Gaussian process $\{(x, X^0(t)/L(t))\}$ has smaller increment variance than $\{(x, \sqrt{2\Sigma}W(t)/L(t))\}$. It therefore has smaller maximum in the sense that

$$E \sup_{0 < s \leq t} \langle x, X^0(s)/L(s) \rangle \leq E \sup_{0 < s \leq t} \langle x, \sqrt{2\Sigma}W(s)/L(s) \rangle \tag{3.66}$$

for any $t > 0$, (see [11]). As t goes to 0, the random variables under the expectation above each converge to a constant almost surely. By the LIL for Brownian motion, on the right hand side we get $\sqrt{2 \sum \sigma_i^2 x_i^2}$. Therefore

$$\limsup_{t \rightarrow 0} \langle x, X^0(t)/L(t) \rangle \leq \sqrt{2 \sum \sigma_i^2 x_i^2} \text{ a.s.} \tag{3.67}$$

□

3.6. Some further questions

We have considered the properties of the infinite dimensional Ornstein-Uhlenbeck process from several viewpoints in this chapter. Our intention was to convey thereby some of the interesting possibilities that await a researcher in this developing field. We now conclude with some unanswered questions raised by the material in the previous five sections.

For finite dimensional diffusions, the drift is negligible when one considers the local behaviour of the process. We have seen that this is not the case in infinite dimensions and one would like to get a better understanding of its effect on say, sample path continuity or LIL behaviour. One wonders, for example,

- (i) Is it possible to find necessary and sufficient conditions for continuity in ℓ^2 in terms of $\{\alpha_i\}$ and $\{\sigma_i^2\}$? Is it true that ℓ^2 discontinuity can only be caused by leaving the space, that is, does $\text{Cap}(X \setminus \ell^2) = 0$ entail ℓ^2 -continuity?
- (ii) Can one prove the LIL for $\{X(t)\}$, not only in H_λ (3.19) but also in (topologies of) spaces which support $\{X(t)\}$ but not the driving Wiener process $\{\sqrt{t}W(t)\}$?

It appears that the tools of Fukushima's theory are well suited to attacking some of these questions. It would be useful to develop simple criteria based on this theory for addressing the problem of negligible sets.

- (iii) Can one find simple geometric rules for determining non-vanishing capacity, analogous to those in $\mathbb{R}^d$ in terms of Hausdorff dimension? Can measures of finite energy integral be made to reveal sets of positive capacity that cannot be touched by our methods?

Lastly, we point out that potential theory on Hilbert space awaits a more complete development. Results concerning regularity of points and the Martin boundary would be of considerable interest.

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