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**LA THÈSE A ÉTÉ
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*-VALUATED DIVISION RINGS, ORDERINGS

AND ELLIPTIC HERMITIAN SPACES

by

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B.Sc. (Hons.), M.Sc.

A thesis submitted to the faculty of
Graduate Studies and Research in partial
fulfillment of the requirements for the degree of

Doctor of Philosophy

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ABSTRACT

This work is subdivided into three parts. In Chapter (I), we will offer a generalization of the notion of c -ordering from a usual involution to ε -involution (in the sense given by Gross). We will also investigate the more general notion of semi c -ordering.

In Chapter (II), we introduce an important simplification in the notion of strong ordering due to S.S. Holland (Jr.). This simplified type of ordering we will call a Jordan ordering. Of course, the ordering in question is defined in a division ring with involution. One can talk about the order subring of that ordering. In fact, one can attach to that subring a canonical valuation that behaves as for a c -ordering. Better still, one can show that this valuation verifies a certain law (congruence law). This, in turn, renders a Jordan ordering with two additional features: (1) A Jordan ordering is, in fact, a Baer ordering, (2) A Jordan ordering can be extended uniquely to a strong ordering.

In Chapter (III), we will deal with a hermitian space V over involutorial division ring D with valuation. We will be concerned with the ring of linear transformations of V . Of particular interest will be the notion of unitary transformation modulo the subgroup $1+J$ of multiplicative infinitesimal. It is shown that these transformations form a group containing the unitary group of V . A matrix criterion for a linear transformation to be unitary modulo $1+J$ is offered. The action of such linear transformations on a certain orthogonal basis will be investigated.

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INTRODUCTION

There have been recent works on division rings with involution equipped with order relations of certain types. For instance, S.S. Holland (Jr.) revived the notion of a $*$ -ordering due to R. Baer which he calls a Baer ordering. L-S. Shio, in his Ph.D. thesis, investigated the same ordering. Parallely to this, M. Chacron introduced the notion of c -ordering which behaves partially as a Baer ordering and partially as a Hilbert ordering of a division ring. As a special case of a c -ordering and/or a Baer ordering, Holland investigated the notion of strong ordering. Very interesting features of these orderings have been their associated order valuations. Examples of finite-dimensional Baer ordered division rings were found in [10]. Various finite-dimensional c -ordered division rings were obtained in [5]. A generalization of the notion of involution itself was attempted in Gross's "*Quadratic forms in infinite dimensional vector spaces*". Structure of hermitian spaces over division rings with involution and valuation were investigated independently in [10] and [4].

This work is three fold. In Chapter (I) we will expand on the notion of ϵ -involution in the sense given by Gross. As a focal point for this chapter we will be attempting a generalization of the notion of c -ordering from usual involution to ϵ -involution. In Chapter (II), we offer a very interesting simplification in the axiomatic of a strong ordering, which is picked at the level of the Jordan ring of a division ring. Finally, in Chapter (III), we shall be concerned mainly with a notion suggested to us by Professor M.

Chacron, namely the notion of unitary transformation modulo the group of multiplicative infinitesimals.

Concerning Chapter (I): An anti-automorphism $*$ of a division ring D is called ϵ -involution if there is some $\epsilon \in D$, such that for each $x \in D$ follows $x^{**} = \epsilon^{-1}x\epsilon$, where $\epsilon\epsilon^* = \epsilon^*\epsilon = 1$. In this chapter, we are interested in c -orderability of a division ring D with ϵ -involution $*$. Following M. Chacron, we call such division ring c -orderable if the $*$ -core of D (sums of products of norms) excludes 0. In fact, any c -orderable division ring D with ϵ -involution is shown to admit a c -ordering in the following sense. For a, b, c, d arbitrary elements of D , we have

- (1) $1 > 0$,
- (2) $a \geq b$ implies $a^* \geq b^*$,
- (3) $a \geq b$ and $c \geq d$ imply $a+c \geq b+d$,
- (4) $a < b$ or $a = b$ or $a > b$ whenever $a = \epsilon a^*$ and $b = \epsilon b^*$,
- (5) $a \geq b$ implies $ax\epsilon x^* \geq bx\epsilon x^*$ and $ax\epsilon^*x^* \geq bx\epsilon^*x^*$.

In Section (1.5), we introduce the notion of ϵ -valuation of the considered division ring. We shall establish that for any c -orderable division ring D , the bounded elements x at the c -ordering (e.g. $x\epsilon x^* \leq r_0$ for some rational r_0) form an ϵ -valuation subring. In the course of the proof of this result we will investigate the passage from $*$ to a co-gradient involution $\#$. The notion of a field ordering was generalized in [13]. In Section (1.7), we carry

over a similar generalization from c -ordering to semi c -ordering.

This is an order relation which satisfies axioms (1), (2), (3) and (5) in the preceding; but, not necessarily (4). We will establish that any semi c -ordering can be extended to some c -ordering. In fact, we will show that to a certain extent the treatment of semi c -ordering reduces to that of c -ordering.

Concerning Chapter (II): By Jordan ordering, we mean an order relation of D such that if π is the domain of positivity of that ordering, then:

- (0) $x > y$ if and only if $x - y \in \pi$;
- (1) π consists solely of symmetric elements $s = s^*$;
- (2) $0 \notin \pi$;
- (3) π is closed under arithmetical sum;
- (4) $xx^* \in \pi$, for each $0 \neq x \in D$;
- (5) if $s, d \in \pi$, then $sd + ds \in \pi$ (Jordan closure);

and

- (6) π contains s or $-s$ for every $0 \neq s = s^* \in D$.

It will take some work to show that this ordering verifies the Baer axiom:

- (7) If $s > 0$, then $xsx^* > 0$ for every $0 \neq x \in D$. This is done by stages using techniques of order valuations that were inspired from [5]. At later stages, we will establish the main result of this chapter to know that any Jordan ordering extends uniquely to a strong ordering of D . The knowledge of the commutator subgroup of D will

prove particularly relevant to this result. A congruence law will be verified that will force $\bar{D}$ to be a purely transcendental extension of its centre. Finally, we shall show that a change from $\ast$ to $\bar{d}$ cogradient involution $\#$ does not alter Jordan orderability of $\bar{D}$.

Concerning Chapter (III): In this concluding chapter, we will be dealing with a special case of an elliptic space V in the sense given in [4]. We will be concerned with the ring of linear transformations of the space V as compared to the ring of infinite matrices over D . Most of the results that will be carried out in the first section, were first obtained in [4]. However, the proofs that we will give here will make quicker use of the additional assumptions that we will enforce. For instance, this is done for the isomorphism between the ring L_0 of all linear transformations ϕ that increase lengths and the ring of all row-finite matrices over the valuation ring R .

In the second section of this chapter, we will investigate the notion of a unitary transformation modulo the group of multiplicative infinitesimals $1+J$. This is a linear transformation ϕ of V which is one-to-one, onto and such that for every $v \in V$ follows $v\phi \cdot v\phi \equiv v \cdot v \pmod{1+J}$. Of course, every unitary transformation is unitary modulo $1+J$. The set of all unitary transformations modulo $1+J$ is a group under multiplication. Unlike the situation for a unitary transformation, a unitary transformation modulo $1+J$ does not generally take orthogonal basis to an orthogonal basis. Subsequently, it is not generally true that $u\phi \cdot v\phi \equiv u \cdot v \pmod{1+J}$. Note, however, that

a unitary transformation modulo $1+J$ induces a unitary transformation on the residue space. In this connection we will establish a result that contains the preceding fact as a particular case (Theorem (3.2.10)). We also give an explicit matrix criterion for a linear transformation to be unitary modulo $1+J$ by looking at the matrix of that linear transformation.

CHAPTER I

C-ORDERABLE DIVISION RINGS WITH ϵ -INVOLUTION

1.1 Introduction

There have been extensive works done in the area of division rings endowed with involution. However, there are situations where quite often it is impossible to restrict ourselves to involutions (in the usual sense). This suggests a generalization of the notion of an involution which we will refer to as an ϵ -involution (refer to Definition (2.1.1), [8] and [19]).

In §1.2, we establish elementary properties and basic facts about the considered division rings. In the classical case of involutorial division ring D , the composite of an involution and an inner automorphism of D is an involution of D if and only if this inner automorphism is induced by a symmetric element. Unlike the classical case, the composite of an ϵ -involution and any inner automorphism of D is an ϵ -involution. It follows that every division ring D with ϵ -involution admits an involution provided characteristic $(D) \neq 2$.

In §1.3, we offer a generalization of the notion of c -ordering to the considered division ring D . As for usual c -ordered division ring we will attach to D its order subring, which is formed by all elements x in D such that xx^* is bounded by some rational number. This will be done in §1.4, where we will derive structure theorems that generalize the classical case.

In §1.5, we introduce the notion of ϵ -valuation of the considered division rings. Relationship between the division ring D , and the one obtained by changing $*$ to a co-gradient involution $\#$ will be established.

in §1.6, where it is shown that the order subring will not change. In the final section of this chapter, the notion of semi ϵ -ordering is investigated.

1.2 Definitions and basic properties

Let D be a division ring with centre Z . Throughout this chapter we assume that the characteristic of D is not 2 ($\text{char}(D) \neq 2$).

Definition (1.2.1): An anti-automorphism $x \mapsto x^*$ of the division ring D is called an ϵ -involution if there is $\epsilon \in D$ such that for each $x \in D$, $(x^*)^* = \epsilon^{-1}x\epsilon$; where $\epsilon\epsilon^* = \epsilon^*\epsilon = 1$.

As a first observation, when $\epsilon = 1$, then $*$ is an involution in the usual sense. More generally, if ϵ is in the centre of D , again $*$ is an involution, and if $\epsilon \neq \pm 1$ this involution is then an involution of the second kind (meaning that there is $z \in Z$, $z^* \neq z$).

Definition (1.2.2): [8] Let D be a division ring with ϵ -involution $*$. An element $x \in D$ is called successively a symmetric element if $\epsilon x^* = x$, a skew-symmetric element if $\epsilon x^* = -x$.

There are odd-dimensional division rings that do not admit ϵ -involutions. An example is provided by Dickson's 9-dimensional division algebra D over its centre $Z = \mathbb{Q}$, the rational field. For, as shown by Dickson (see also [8, page 54]) D has no anti-automorphism. Also, for commutative division ring D an ϵ -involution is, again, a usual involution. For $D = \mathbb{C}$, the complex field, the only involution different from the identity automorphism is the ϵ -involution with $\epsilon = \sqrt{-1}$.

Let $H = \mathbb{R}\{i, j \mid i^2 = -1, j^2 = -1, ij = -ji\}$ stand for the classical real quaternionic algebra $\{\alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k \mid k = ij, \alpha_i \in \mathbb{R}\}$. Take for $*$ the mapping defined by

$$(\alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k)^* = \alpha_0 - \alpha_1 i - \alpha_2 j + \alpha_3 k \quad (\alpha_i \in \mathbb{R}).$$

It is easy to check that $*$ is an anti-automorphism of H such that $x^{**} = \varepsilon^{-1} x \varepsilon$ for every $x \in H$, where $\varepsilon = i$. Thus $*$ is an ε -involution. Here the additive subgroup of symmetric elements is the 1-dimensional subspace $\mathbb{R}(1+i)$, the additive subgroup of skew symmetric elements is $\mathbb{R}(i-1) + \mathbb{R}(j-k) + \mathbb{R}(k-j)$.

Notation: Let S, K denote the additive subgroups of the division ring D formed by the symmetric elements and the skew-symmetric elements respectively. In symbols:

$$S = \{x \in D \mid \varepsilon x^* = x\}; \quad K = \{x \in D \mid \varepsilon x^* = -x\}.$$

Let T denote the subgroup of traces: $T = \{x + \varepsilon x^* \mid x \in D\}$.

Remark (1.2.3): (1) For each $x \in D$, $x = s+k$ where s is a symmetric element and k is a skew-symmetric element.

$$(2) \quad T = S.$$

Proof: (1) Since $\text{char}(D) \neq 2$ it follows that $x = \frac{1}{2}(x + \varepsilon x^*) + \frac{1}{2}(x - \varepsilon x^*) = s+k$ as wished

(2) Clearly $T \subset S$. Since each symmetric element s can be written in the form $s = (\frac{1}{2}s) + \varepsilon(\frac{1}{2}s)^*$, it follows that, $T = S$.

Remark (1.2.4): S and K are preserved under the 2-sided translation $a \rightarrow xax^*$, $x \in D$.

Proof: If a is a symmetric element, then $\varepsilon(xax^*)^* = \varepsilon x \varepsilon^{-1} x^* a^* x^* = x a^* x^* = a^*$, so that xax^* is a symmetric element. Similarly one can prove the result for a skew-symmetric element a .

Remark (1.2.5): Any ε -involution $*$ with $\varepsilon \neq 1$ is such that ε is not symmetric.

Proof: If ε is symmetric, then $\varepsilon \varepsilon^* = \varepsilon$, so that $\varepsilon = 1$, a contradiction.

Remark (1.2.6): Let k be a skew symmetric element and s a symmetric element of D . Then the following are symmetric elements:

$$k(1+\varepsilon)k^* = kk^* - k^2; \quad s(1+\varepsilon)s^* = ss^* + s^2.$$

The proof of Remark (1.2.6) is evident.

Notation: For $a, b \in D$, put $(a, b) = ab - ba$.

Lemma (1.2.7): Let D be a division ring with ε -involutions $*$, and suppose that $sk - ks = 0$ for all symmetric s and skew symmetric k in D . Then for all $s, x \in D$, follows

$$(s, (s, x)) = 0.$$

Proof: Firstly we claim that $\varepsilon \in \mathbb{Z}$. Since $1+\varepsilon$ is a symmetric element, it follows that $1+\varepsilon$ commutes with each skew symmetric element in D , as does ε . Also, since $1-\varepsilon$ is a skew symmetric element of D , it follows that $1-\varepsilon$ commutes with each symmetric element, as does ε . Thus ε commutes with all symmetries and skew symmetries in D . In view of Remark (1.2.3), ε is in the centre of D .

Now, let s and s' be two arbitrary symmetric elements of D . It is easy to check that $(1+\varepsilon^*)(s, s')$ is a skew symmetric element;

and consequently, this element commutes with s . Since $\epsilon \in Z$, it follows that $\epsilon^* \in Z$ so that $1+\epsilon^* \in Z$. Hence $(s, (s, s^*)) = 0$. Since we also have $(s, (s, k)) = 0$ for all skew symmetric elements $k \in D$, it follows that $(s, (s, x)) = 0$ for all $x \in D$. $\square$

Lemma (1.2.8) [9, page 5]: Let D be a division ring with $\text{char}(D) \neq 2$. If $(a, (s, x)) = 0$ for all $x \in D$, then a is in the centre of D .

The following corollary is an immediate consequence of the above two lemmas.

Corollary (1.2.9): If D is a division ring with ϵ -involution $*$ such that $sk - ks = 0$ for all symmetric s and skew symmetric k in D , then all the symmetric elements of D are in Z .

Theorem (1.2.10): Let D be a division ring with ϵ -involution $*$, where $\epsilon \neq 1$. Suppose that all the symmetric elements in D are in Z . Then D is a commutative field.

Proof: Since $1+\epsilon \in S \in Z$, it follows that $\epsilon \in Z$. Two cases to be considered:

(i) $\epsilon \neq -1$: Let $k_0 = \frac{1-\epsilon}{1+\epsilon}$. Then $k_0 \in Z$ and $\epsilon = \frac{1-k_0}{1+k_0}$. Also, $k_0^* = -k_0$. Now, we claim that each skew symmetric element σ is central. Because, $\epsilon(k_0\sigma)^* = \epsilon\sigma^*k_0^* = \sigma k_0 = k_0\sigma$ (where $k_0 \in Z$), so that $k_0\sigma$ is a symmetric element. Then $k_0\sigma \in Z$. Since $k_0 \in Z$, it follows that $\sigma \in Z$.

(ii) $\epsilon = -1$: In this case s is a symmetric element if and only if $s^* = -s$, and k is a skew symmetric element if and only if

$k^* = k$. Then each $s = -s^*$ is central. For a symmetric element s and a skew symmetric element k in D one has $(sk)^* = k^*s^* = -ks = -(sk)$. Thus $sk \in Z$ giving $k \in Z$.

Remark (1.2.11): Let D be a division ring with ε -involution $*$. If ρ is any non-zero element in D , then the formula

$$(1) \quad x^\# = \rho^{-1} x^* \rho$$

defines an ε_1 -involution $\#$ of D , where $\varepsilon_1 \in D$ can be found.

Proof: Clearly $\#$ is an anti-automorphism. Put $\varepsilon_1 = \varepsilon(\rho^{-1})^* \rho$. It is to be shown that $\#$ is an ε_1 -involution. For,

$$\begin{aligned} x^{\#\#} &= \rho^{-1} (\rho^{-1} x^* \rho)^* \rho \\ &= \rho^{-1} \rho^* \varepsilon^{-1} x \varepsilon (\rho^{-1})^* \rho \\ &= \varepsilon_1^{-1} x \varepsilon_1 \end{aligned}$$

One also has $\varepsilon_1 \varepsilon_1^\# = \varepsilon(\rho^{-1})^* \rho \rho^{-1} \rho^* \varepsilon^{-1} = 1$. Thus $\#$ is an ε_1 -involution.

By analogy with the usual notion of co-gradient involution ([1]), refer to $\#$ appearing in (1) to be a co-gradient ε -involution (ε need not be the same). It is appropriate to observe that while in the classical case one requires ρ to be a symmetric element, here we will dispense with this requirement. To find a co-gradient involution $\#$, all that is needed is that $\varepsilon_1 = \varepsilon(\rho^{-1})^* \rho = 1$, meaning that ρ^{-1} is a symmetric element relative to the initial ε -involution.

Theorem (1.2.12): If D is a division ring with ε -involution $*$, then D admits a usual involution.

Proof: If $x^* = x$ for each $x \in D$, then evidently D is commutative, in which case there is nothing more to prove. If, on the other hand, there is $x \neq x^*$, we contend that D contains a symmetric element λ . For, put $\lambda = x + \varepsilon x^*$. If $\lambda \neq 0$, then we are done. If, on the other hand, $\lambda = 0$ then $\varepsilon x^* = -x$, so that $\mu = xx^* - x^*$ is a non-zero symmetric element. Now, take ρ to be λ^{-1} . Define $\#$ by:

$$a^\# = \rho^{-1} a^* \rho \quad (a \in D).$$

Thus $\#$ is an involution. Note that if all the symmetric elements are central, then by Remark (1.2.10), D is commutative. $\square$

For the converse of Theorem (1.2.12), take $\#$ to be a usual involution of D . Define $*$ of D by

$$x^* = \mu^{-1} x^\# \mu \quad (\text{for some } \mu \in D, \mu \neq \pm \mu^\#);$$

Clearly $*$ is an anti-automorphism of D , and $x^{**} = \varepsilon^{-1} x^\#$ when $\varepsilon = (\mu^{-1})^\# \mu$. Also $\varepsilon \varepsilon^* = (\mu^{-1})^\# \cdot \mu \cdot \mu^{-1} \varepsilon^\# \mu = (\mu^{-1})^\# \mu \mu^{-1} \mu = 1$, as wished.

To illustrate, let $D = \mathbb{H}$ the classical real quaternionic algebra. The usual involution of D is defined by

$$(\alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k)^\# = \alpha_0 - \alpha_1 i - \alpha_2 j - \alpha_3 k.$$

With reference to the same ε -involution $*$ considered before on page

3, $(\alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k)^* = \alpha_0 - \alpha_1 i - \alpha_3 j + \alpha_2 k$, one can say that $*$ is a co-gradient ε -involution. For one can check that $x^* = \rho^{-1} x^\# \rho$ ($x \in D$), where $\rho = 1+i$. Changing $\rho = 1+i$ to $\rho = 1+j$ or $1+k$, this gives two additional ε -involutions of D .

In [15], W. Scharlau, making use of a theorem of Hilbert (see

[7, page 35]), established that if D is a finite dimensional central simple algebra with an anti-automorphism θ such that $\theta^2(x) = b^{-1}xb$ ($x \in D$), where b is such that $\theta(b) \cdot b = b \cdot \theta(b) = a\theta(a)$ for some central element a , then D possesses an involution.

From Scharlau's arguments we may restrict our attention to the case where D is a division ring. In this case the given anti-automorphism θ is, in fact, an ε -involution. For, if $\varepsilon = a^{-1}b$, then

$$\begin{aligned}\varepsilon\theta(\varepsilon) &= (a^{-1}b)\theta(a^{-1}b) \\ &= a^{-1}b\theta(b)\theta(a^{-1}) = 1.\end{aligned}$$

Also, $\varepsilon^{-1}x\varepsilon = a(b^{-1}xb)a^{-1} = a(\theta^2(x))a^{-1} = \theta^2(x)$.

Therefore, Scharlau's criterion is simply Theorem (1.2.12). We note that in Theorem (1.2.12) we do not use Hilbert's theorem.

The last topic of this section is about the symmetric and the skew-symmetric elements of $(D, *)$ and $(D, \#)$, where $\#$ is a co-gradient ε_1 -involution of D .

Theorem (1.2.13): Let S_1 and K_1 be the additive subgroups of $(D, \#)$ of symmetric elements and skew symmetric elements respectively.

Then

$$S_1 = S\rho; \text{ and } K_1 = K\rho.$$

Proof: Let $s_1 \in S_1$. Then $\varepsilon_1 s_1^\# = s_1$, so that

$$\varepsilon(\rho^*)^{-1} \rho \rho^{-1} s_1^* \rho = s_1.$$

$$\varepsilon(\rho^*)^{-1} s_1^* \rho = s_1.$$

Hence $\varepsilon(s_1 \rho^{-1})^* = (s_1 \rho^{-1})$, and hence, $s_1 \rho^{-1}$ is a symmetric element

with respect to $*$. Thus $S_1 \rho^{-1} \subseteq S_1$, that is, $S_1 \subseteq S \rho$.

Conversely, if $\epsilon s^* = s$, then

$$\begin{aligned} \epsilon_1(sp)^{\#} &= \epsilon_1 \rho^{\#} s^{\#} \\ &= \epsilon s^* \rho = sp. \end{aligned}$$

Hence $sp \in S_1$ so that, $S \rho \subseteq S_1$. Similarly, one can show that $K_1 = K \rho$. $\square$

1.3 C-orderability of a division ring with ϵ -involution

In this section, we assume that D is any division ring with ϵ -involution $*$. The $*$ -core P of D is the set of sums

$c = \sum_i p_{1i} p_{2i} \dots p_{ri}$, where each p_{ni} is of the form $a \epsilon a^*$ or $a \epsilon^* a^*$ for some $0 \neq a \in D$. There follow some facts about the $*$ -core P , listed from (1) to (10).

Fact (1): P is closed under sums and products.

Fact (2): P contains $a \epsilon a^*$ and $a \epsilon^* a^*$ for every nonzero $a \in D$.

Fact (3): P contains ϵ and $\epsilon^{-1} = \epsilon^*$.

Fact (4): P contains $a^* a$ for every non-zero $a \in D$.

Proof: Since $a^* a = a^* \epsilon (\epsilon^{-1} a \epsilon) \epsilon^*$

$$\begin{aligned} &= a^* \epsilon a^* \epsilon^* \\ &= (a^*) \epsilon (a^*)^* \epsilon^*, \end{aligned}$$

it follows that $a^* a \in P$ for every $a \in D$. $\square$

Fact (5): P is closed under conjugation in D , ($x \in P$ implies $b x b^{-1} \in P$).

Proof: Let $a, b \in D$. Then

$$\begin{aligned} b(a\epsilon a^*)b^{-1} &= b(a\epsilon a^*)(b^*b^{*-1})b^{-1} \\ &= (ba)\epsilon(ba)^* \cdot (b^{-1})^*(b^{-1}) \in P. \end{aligned}$$

Also, $b(a\epsilon^*a^*)b^{-1} = (ba)\epsilon^*(ba)^* \cdot (b^{-1})^*(b^{-1}) \in P$. Since P is generated by elements of the preceding forms, it follows that P is preserved under conjugation. $\square$

Fact (6): P is preserved under all 2-sided translations: $a \in P$ implies $xax^* \in P^c$; where $0 \neq x \in D$.

Proof: If $p \in P$ is of the form $p = a\epsilon a^*$, then

$$b(a\epsilon a^*)b^* = (ba)\epsilon(ba)^* \in P.$$

If $p \in P$ is a product of two norms $(a_1\epsilon a_1^*)(a_2\epsilon a_2^*)$, then by Facts (1) and (5) follows

$$b(a_1\epsilon a_1^*)(a_2\epsilon a_2^*)b^* = b(a_1\epsilon a_1^*)b^* \cdot (b^*)^{-1}(a_2\epsilon a_2^*)b^* \in P,$$

For general $p \in P$, we repeat the above step enough times to get that $bpb^* \in P$. $\square$

Fact (7): P contains aa^* for every non-zero $a \in D$.

Proof: Since $\epsilon \in P$, it follows by Facts (5) and (6) that,

$a^*\epsilon(a^*)^{-1} \in P$, so that $a(a^*\epsilon(a^*)^{-1})a^* \in P$. Then $aa^*\epsilon \in P$ and, hence, $aa^* = aa^*\epsilon \cdot \epsilon^{-1} \in P$. $\square$

Fact (8): $P^* = P$.

Proof: Let $x \in P$. If x is of the form $x = a\epsilon a^*$, then

$$\begin{aligned} (a\epsilon a^*)^* &= (a^*)^*\epsilon^*a^* \\ &= \epsilon^{-1}a\epsilon\epsilon^*a^* = \epsilon^{-1}aa^* \in P. \end{aligned}$$

Similarly,

$$\begin{aligned}(ae^*a^*)^* &= a^{**}ea^* \\ &= \varepsilon^{-1}ae^2a^* \\ &= \varepsilon^{-1} \cdot (aea^*)(a^{*-1}ea^*) \in P.\end{aligned}$$

For general x , the result follows from the preceding and Fact (1). $\square$

Fact (9): P contains inverses.

Proof: If $x \in P$, then $x^* \in P$. Thus by Facts (1) and (7), $x^{-1}(x^{-1})^*(x^{-1})^*x^* \in P$ follows. $\square$

Fact (10): P is preserved under all 2-sided translations:

$c \in P$ implies $x \in ex^* \in P$, where $0 \neq x \in D$.

Proof: For $c = ae^*a^*$, one has

$$\begin{aligned}x(ae^*a^*)ex^* &= (xae^*a^*) \cdot x^*x^{*-1} \cdot ex^* \\ &= (xa) \varepsilon (xa)^* \cdot (x^*)^{-1} ex^* \in P.\end{aligned}$$

Similarly, for $c = ae^*a^*$ one has $x(ae^*a^*)ex^* \in P$. If $c = p_1 p_2$, $p_1 \in P$, and p_1 is of the form $p_1 = ae^*a^*$ (or ae^*a^*), for some $a \in D$, then

$$\begin{aligned}x(p_1 p_2)ex^* &= xp_1(ex^*x^{*-1} \varepsilon^{-1})p_2ex^* \\ &= xp_1ex^* \cdot x^{*-1}(\varepsilon^{-1}p_2\varepsilon)x^* \in P.\end{aligned}$$

It is now easy to deduce the result for general $c \in P$. $\square$

Definition (1.3.1): If $0 \notin P$, we shall call D a c-orderable division ring.

Lemma (1.3.2): D is c-orderable if and only if $-1 \notin P$.

Proof: Assume $-1 \in P$. Then $0 = 1 + (-1) \in P$, which is a

contradiction. Conversely, suppose that $0 \in P$. Then there is an element p in P such that $p = \sum p_{1i} p_{2i} \cdots p_{ni}$, where each $p_{ij} = aa^*$ or a^*a^* for some nonzero $a \in D$. Thus $0 = p_{n1}^{-1} p_{(n-1)1}^{-1} \cdots p_{11}^{-1} p = 1 + p'$ where $p' \in P$. Hence, $-1 \in P$, which is a contradiction. $\square$

As a side observation, in the case where $*$ is a usual involution one can characterize non c -orderability as follows:

Lemma (1.3.3): For a division ring D with usual involution $*$, to say that D is not c -orderable is the same as saying that $S \subset P$, where S is the additive subgroup of symmetric elements.

Proof: If $0 \notin P$, then clearly $S \not\subset P$. Conversely, assume that $S \subset P$, but $-1 \notin P$. Then, for each symmetric $s \neq -1$, follows $s = \frac{1}{2}[(s+1)^2 + (-1)s^2 + (-1)] \in P$. Hence, $S \subset P$, which is a contradiction. $\square$

Definition (1.3.4): By c -ordering of the division ring D , we mean an order relation (notation: $a \geq b$) such that:

- (i) $1 > 0$,
- (ii) $a \geq b$ implies $ea^* \geq eb^*$,
- (iii) $a \geq b$, $c \geq d$ imply $a+c \geq b+d$,
- (iv) $s < d$, $s = d$ or $s > d$ for each $s = as^*$, $d = ed^*$, and
- (v) $a \geq b$ implies $ax \in x^* \geq bx \in x^*$ and $ax \in x^* \geq bx \in x^*$.

One can give the following equivalent formulation of a c -ordering:

There is some subset D^+ of D (the domain of positivity of $\geq$) such that:

$$(C1) \quad \epsilon(D^+)^* = D^+,$$

$$(C2) \quad D^+ + D^+ \subseteq D^+,$$

$$(C3) \quad D^+ \cup (-D^+) \subseteq \{x \in D/0 \mid x = \epsilon x^*\},$$

$$(C4) \quad 0 \notin D^+, \quad 1 \in D^+; \quad a \in D^+ \Rightarrow ax\epsilon x^* \text{ and } ax\epsilon^* x^* \in D^+ \text{ for}$$

each $x \neq 0$; and

$$(C5) \quad a \leq b \text{ if and only if } a-b \in D^+ \cup \{0\}.$$

Theorem (1.3.5): ~~In order~~ for the division ring D to have some c -ordering it is necessary and sufficient that D be c -orderable.

Proof: If D has some c -ordering D^+ , it is clear that $x\epsilon x^*$ and $x\epsilon^* x^*$ are in D^+ , for every $0 \neq x \in D$. By (C4) follows $x\epsilon x^* \cdot y\epsilon^* y^* \in D^+$ for any non-zero elements x and y in D . By induction $p_{i_1} p_{i_2} \dots p_{i_r} \in D^+$ for $p_{i_j} = x\epsilon x^*$ or $x\epsilon^* x^*$, and for some $0 \neq x \in D$. Thus $P \subseteq D^+$, and $0 \notin P$ follows.

Conversely, suppose that D is any c -orderable division ring. Using Zorn's lemma, we can find a maximal P -additive subset $D^+ = \epsilon(D^+)^*$ excluding 0 . We proceed to show that D^+ satisfies axiom (C3) in the above definition. Let $0 \neq s = \epsilon s^* \in D$, and suppose that $-s \notin D^+$. Let $D_s^+ = D^+ \cup sP \cup (sP + D^+)$. We will show that D_s^+ satisfies (C1), (C2) and (C4). It is clear that D_s^+ is additive, $0 \notin D_s^+$ and $PD_s^+ \subseteq D_s^+$. To show that $\epsilon(D_s^+)^* = (D_s^+)$ proceed as follows. Suppose $x \in D_s^+$. We consider two cases: $x \in D^+$ and $x \in sP$. If $x \in sP$, then $x = sy$, for some $y \in P$. Hence $x^* = y^* s^* = y^* \epsilon^{-1} s^* = (y^* \epsilon^{-1}) s \in P \cdot sP \subseteq P \cdot D_s^+ \subseteq D_s^+$. Then $x^* \in D_s^+$. Also, if $x \in D^+$, then $\epsilon x^* \in \epsilon(D^+)^* = D^+ \subseteq D_s^+$.

Now, by maximality of D^+ , $D^+ = D_s^+$ follows, that is, $s \in D^+$.

We conclude that the relation defined by $a \geq b$ if and only if $a-b \in D^+ \cup \{0\}$, is a c -ordering on D .

Theorem (1.3.6): Let D be any c -orderable division ring. Then:

- (i) r is positive rational if and only if r is positive in D ,
- (ii) for $a \in D$, $a \geq 0$ if and only if $a^{-1} \geq 0$; and
- (iii) for $a \geq b \geq 0$ and c, d and $c-d \in P$ follows $ac \geq bd$.

Proof: (i) Since $P \subset D^+$, it follows that each positive rational r belongs to D^+ . The converse is clear.

(ii) If $a \geq 0$, then $a \cdot a^{-1} \in (a^{-1})^* \geq 0$, so that $(a^{-1})^* \cap D^+ = D^+$. Hence $a^{-1} \in D^+$.

(iii) From $a-b \in D^+$ and $d \in P$, follows $ad-bd \in D^+$. Also $a \in D^+$ and $c-d \in P$ together imply that $ac-ad \in D^+$. Thus $ac-bd \in D^+$.

Remark (1.3.7): Since $1 \in D^+$, then $\underbrace{1+1+\dots+1}_{n \text{ times}} > 0$ for any positive integer n . Thus any c -orderable division ring is with characteristic 0.

Definition (1.3.8): Let $\geq$ be a c -ordering of the division ring D . Then:

(1) If $r_0 \leq x \leq r_1$ for some rationals r_1 and r_2 , then call x a unit.

(2) If $x \leq r$ for each rational r , then call x an infinitesimal.

(3) If x is a unit or an infinitesimal, call x a bounded element.

Definition (1.3.9): Let $\geq$ be a fixed ordering on the division ring D . Let a, b be positive elements, and let c be an element with sign (e.g., $c > 0$, $c = 0$, $c < 0$).

- (1) Define $|c| = c$ if $c \geq 0$, and $|c| = -c$ if $c < 0$.
- (2) Write $a \ll b$ if $a < rb$ for each rational r .
- (3) Write $a \approx b$ if $a < (1+r)b$ and $b < (1+r)a$ for each rational r .
- (4) Write $a \sim b$ if $a \approx b$ and $b \approx a$ (archimedean equivalent pair).

Convention: Throughout this work, we will assume that $a \ll r$ for some rational r , that is, a is a bounded element of D .

Remark (1.3.10): For $a, b \in D$, follows

$$(a+b)\varepsilon(a+b)^* \leq 2(a\varepsilon a^* + b\varepsilon b^*).$$

Proof: Use the equality

$$(a+b)\varepsilon(a+b)^* + (a-b)\varepsilon(a-b)^* = 2(a\varepsilon a^* + b\varepsilon b^*),$$

noting that $x\varepsilon x^* > 0$ for each $0 \neq x \in D$.

Remark (1.3.11): For $s = \varepsilon s^*$, $k = -\varepsilon k^*$ in D , follows

$$(sk - ks) \leq s^2 - k^2.$$

Proof: The remark readily follows from the equality

$$(s+k)\varepsilon(s+k)^* = (s^2 - k^2) - (sk - ks).$$

Remark (1.3.12): For $s = \varepsilon s^*$ in D , and r any rational follows, $-r \leq s \leq r$ implies $s^2 \leq r^2$. Conversely, $s^2 \leq r^2$ implies $-r' \leq s \leq r'$ (for some rational r').

Proof: If $-r < s < r$ then the left side of the following equality is positive:

$$2r(r^2 - s^2)\epsilon = (r-s) \cdot (r+s)\epsilon(r+s)^* + (r+s) \cdot (r-s)\epsilon(r-s)^*$$

Thus $r^2 - s^2 > 0$, that is $s^2 < r^2$.

For the converse, one has

$$(1+\epsilon)r + s = \frac{[(1+\epsilon)r^2 - s^2]}{2(1+\epsilon)r} + \frac{[(1+\epsilon)r + s]^2}{2(1+\epsilon)r}$$

Since $(1+\epsilon)r + s$ is symmetric, it follows that $[(1+\epsilon)r + s]^2$ is positive. Also, since $s^2 < r^2$, it follows that $s^2 \leq r^2(1+\epsilon)^2$ (where $1 < 1+\epsilon < P$). Thus the right side of the above equality is positive and, hence,

$$s \leq (1+\epsilon)r < r + r\epsilon = r'$$

Also,

$$s \geq -(1+\epsilon)r > -(r + r\epsilon) = -r'$$

Remark (1.3.13): For $s = \epsilon s^*$ any symmetric element such that $r_0 \leq s \leq r_0 < 1$ for some rational r_0 , follows $s^2 < |s| + r$ for each rational r .

Proof: We subdivide the proof into two cases:

Case (i): If $|s| \leq r < 1$ implies $|s| \leq r^2$ for each rational r . In this case, $s \leq r_0 < 1$ implies $|s| \leq r_0^{2^n}$, $n=1, 2, \dots$. Then, $|s|$ is an infinitesimal. From Remark (1.2.12), s^2 is also an infinitesimal. Thus, $s^2 < |s| + r$ for each rational r .

Case (ii): Suppose there is some rational r_1 with $|s| \leq r_1$ but $|s| \not\leq r_1^2$. Then $|s| > r_1^2$ follows. Thus $r_1^2 < |s| \leq r_1 < r_1^2$

and, by Remark (1.3.12), $s^2 \leq r_1 < s$. From this, $s^2 < |s| + r$ for each rational r . $\square$

Remark (1.3.14): For $s = \varepsilon s^*$, $k = -\varepsilon k^*$ follows $+(sk-ks) \leq s^2 - k^2$ provided s^2 and $-k^2$ are not archimedean equivalent.

Proof: Since s^2 and $-k^2$ are not archimedean equivalent ($s^2 \not\sim (-k^2)$), it follows that $s^2 \ll -k^2$ or $-k^2 \ll s^2$, that is $s^2 < -rk^2$ or $-k^2 < rs^2$ for each rational r . Let us suppose for instance, $s^2 \ll -k^2$, that is, $s^2 < -rk^2$ for each rational r . Then $s^2 \ll -(rk)^2$ for each rational r . Now, $+r(sk-ks) = +[s(rk) - (rk)s] = s^2 - (rk)^2$ (by Remark (1.3.11) where rk is a skew symmetric element). However, $s^2 \ll -(rk)^2$ and hence $+r(sk-ks) \leq -2r^2 k^2$. Thus $+(sk-ks) \leq -2rk^2$ which gives $+(sk-ks) \ll -2k^2 \leq s^2 - k^2$. Therefore, $+(sk-ks) \leq s^2 - k^2$. $\square$

Remark (1.3.15): Generally, $+(sk-ks) \ll s^2 - k^2$ if and only if $x\varepsilon x^* \approx \varepsilon x^* x$ where $x = s+k$ and $s = \varepsilon s^*$, $k = -\varepsilon k^*$ in $\mathcal{D}$.

Proof: Since $x\varepsilon x^* = (s^2 - k^2) + (ks - sk)$, and $\varepsilon x^* x = (s^2 - k^2) + (sk - ks)$, it follows that $x\varepsilon x^* = \varepsilon x^* x + 2(ks - sk)$. Now, if $+(sk-ks) \ll s^2 - k^2$, then $(sk-ks) \leq r(s^2 - k^2)$ for each rational r . Hence,

$$\begin{aligned} x\varepsilon x^* &= \varepsilon x^* x + (ks - sk) + (ks - sk) \\ &< \varepsilon x^* x + (ks - sk) + r(s^2 - k^2) \\ &< \varepsilon x^* x + r(\varepsilon x^* x) \\ &= (1+r)(\varepsilon x^* x). \end{aligned}$$

1.4. The bounded elements

In this section, we will establish that if V is the set of bounded elements of the ϵ -orderable division ring D , then V is a $\ast$ -closed subring of D , which is a valuation ring in D , in the sense that it contains x or x^{-1} for each $0 \neq x \in D$.

Remark (1.4.1)

(1) The set V of bounded elements is a $\mathbb{Q}$ -additive subgroup of D ($\mathbb{Q}$ is the field of rationals).

(2) The subset J of infinitesimals is a $\mathbb{Q}$ -subgroup of D .

Proof: (2) If $xx^* \leq r$ for each rational r , and $yy^* \leq t$ for each rational t , then by Remark (1.3.10), $(x+y)\epsilon(x+y)^* \leq 2(r+t)$ follows. Therefore $x+y$ is an infinitesimal. The proof of (1) is similar.

Remark (1.4.2): Let $s = es^*$ be any positive symmetric element in D . Then: (1) s is bounded if and only if $s \leq r$ for some rational r .

(2) s is a unit if and only if $r_1 \leq s \leq r_2$ for some rationals r_1 and r_2 .

(3) s is an infinitesimal if and only if $s^* \leq r$ for each rational r .

Proof: (2) If s is a unit, then $r_1 \leq s^2 \leq r_2$, which implies that $s^2 \leq r_2$. Thus $s \leq r$ for some rational r (Remark (1.3.12)).

Conversely, assume that $-r \leq s \leq r$. From Remark (1.3.12), follows

$0 \leq ses^* = s^2 \leq r'$, for some rational r' and, hence, s is a unit,

The proofs of (1) and (3) are similar. $\square$

Remark (1.4.3): Let $x \in D$. Then x is bounded if and only if $xx^* + x\epsilon x^*$ is bounded.

Proof: Assume that $xx^* + x\epsilon x^*$ is bounded. Since $xx^* + x\epsilon x^*$ is a symmetric element, it follows by Remark (1.4.2), that $xx^* + x\epsilon x^* < r$, for some rational r . Since xx^* and $x\epsilon x^* \in P \cap D^+$ (= the domain of positivity of the ϵ -ordering of D), it follows that $xx^* < r$, that is, x is bounded.

Conversely, if x is bounded, then $x\epsilon x^* < r$ for some rational r . Hence $\epsilon(x\epsilon x^*)^* < \epsilon r^*$, that is, $xx^* < \epsilon r < r_2$ for some rational r_2 . Thus $xx^* + x\epsilon x^* < r_2 + r$. Therefore, by Remark (1.4.2), $xx^* + x\epsilon x^*$ is bounded. $\square$

Remark (1.4.4): For x any bounded element in D the elements $x\epsilon$, ϵx , $x\epsilon^*$ and $\epsilon^* x$ are all bounded.

Proof: From $x \in V$, follows that for some rational r

$$(x\epsilon)(x\epsilon)^* + (x\epsilon)\epsilon(x\epsilon)^* = xx^* + x\epsilon x^* < r.$$

By Remark (1.4.3), $x\epsilon \in V$ follows. Similarly,

$$(\epsilon x)(\epsilon x)^* + (\epsilon x)\epsilon(\epsilon x)^* = \epsilon(xx^* + x\epsilon x^*)\epsilon^*$$

$$\leq \epsilon r \epsilon^* = r.$$

Hence ϵx is bounded. For the boundedness of the remaining elements $\epsilon^* x$ and $x\epsilon^*$ use similar arguments. $\square$

We proceed to show that the additive subgroup V of bounded

elements is a $*$ -closed valuation ring in D . The following lemmas will be used.

Lemma (1.4.5): V is $*$ -closed.

Proof: Since $(1-x)\varepsilon(1-x)^* + (1-x)(1-x)^* > 0$, it follows that, $1 + (xx^* + x\varepsilon x^*) > (x + \varepsilon x^*) + (x^* + x\varepsilon)$. Suppose that $x \in V$. Since $xx^* + x\varepsilon x^* < r_1$ for some rational r_1 ; $(x + \varepsilon x^*) + (x^* + x\varepsilon) < 1 + r_1 + r_1$ follows (where $\varepsilon < r_1$). Similarly, from $(1+x)\varepsilon(1+x)^* + (1+x)(1+x)^* > 0$, it follows that $(x + \varepsilon x^*) + (x^* + x\varepsilon) > -(1 + r_1 + r_1)$. Since $(x + \varepsilon x^*) + (x^* + x\varepsilon)$ is a symmetric element, it follows by Remark (1.3.12) that $(x + \varepsilon x^* + x^* + x\varepsilon)^2 \leq (1 + r_1 + r_1)^2$. Hence $x + \varepsilon x^* + x^* + x\varepsilon \in V$. Since x and $x\varepsilon$ are in V , and since V is an additive $\mathbb{Q}$ -subgroup, it follows that $\varepsilon x^* + x^*$ is in V . By Remark (1.4.3) $aa^* + a\varepsilon a^* \in V$ follows, where $a = \varepsilon x^* + x^*$. Then $aa^* + a\varepsilon a^* < r$ for some rational r . Replacing a with $\varepsilon x^* + x^*$ in this inequality, this gives $x^* \varepsilon x^* + x^* x \varepsilon < r$. However, $b = x^*(x^*)^* + x^* \varepsilon (x^*)^* = x^* \varepsilon x^* + x^* x \varepsilon < r$. Hence, b is in V (note that b is a symmetric element). Thus, by Remark (1.4.3), x^* is in V . $\square$

Lemma (1.4.6): Let x and y be bounded elements in D . Then the elements $x \cdot y y^*$ and $x \cdot y^* y^*$ are bounded

Proof: By Remark (1.4.3), follows $0 < y y^* + y \varepsilon y^* < r$ for some rational r . By Remark (1.3.12), follows $(y y^* + y \varepsilon y^*)^2 \leq r^2$. Expanding the left side of this inequality and noting that all the terms are positive, this gives $y \varepsilon y^* \cdot y y^* \leq r^2$ and $y y^* \cdot y \varepsilon y^* \leq r^2$. Now, if $x \in V$, then by Lemma (1.4.5), $x^* \in V$ follows. Thus $(x^* \varepsilon x + x^* x)^2 =$

$x^*(x^*)^* + x^*e(x^*)^* < r'$ for some rational r' and, hence,

$b = x^*e^*x + x^*x < r_1^{-1} < r_1$ for some rational r_1 . Now,

$$(yey^*)b(yy^*) \leq (yey^*)r_1(yy^*)$$

$$\leq r_1 r^2$$

However, $(yey^* \cdot x^*)(yey^* \cdot x^*)^* + (yey^* \cdot x^*)e(yey^* \cdot x^*)^*$

$$= (yey^*)b(yy^*) ,$$

which is in V . Thus $yey^* \cdot x^*$ is in V (Remark (1.4.3)). By similar arguments one can show that $yy^* \cdot x^*$ is in V . Hence, by Lemma (1.4.5), $(yy^* \cdot x^*)^*$ and $(yey^* \cdot x^*)^*$ are in V . Therefore, $x \cdot yy^*$ and $x \cdot yey^*$ are in V . $\square$

Corollary (1.4.7): Let x be a bounded element in D . Then xx^* , x^2 , xex^* , x^*x , x^*ex , $xx^* \cdot xex^*$ and $xex^* \cdot xx^*$ are all bounded elements.

Proof: By Lemma (1.4.6), follows that xx^* and xex^* are in V . Now, xx^* and $x \in V$ together imply that $xx^* \cdot xex^* \in V$. Similarly, $xex^* \cdot xx^*$ is in V . Again, from x and $1-x \in V$, follows that $x(1-x)(1-x^*)$ is in V . Hence one can deduce that x^2 is in V . Finally, since x^* is in V , it follows that x^*e^*x and x^*x are in V . $\square$

Lemma (1.4.8): x is bounded if and only if xx^* is bounded.

Proof: If x is bounded, the result readily follows from Corollary (1.4.7). Conversely, if xx^* is bounded, then $(xx^*)^*$ is

bounded, that is, $xx^* \in V$. Hence $xx^* + xex^* \in V$, which implies by Remark (1.4.3), that $x \in V$. $\square$

Lemma (1.4.9): V is closed under product.

Proof: Let x and y be in V . We claim first that $xx^* \cdot y^*y \in V$. Since $y^*y \in V$ (Corollary (1.4.7)), it follows that $y^*y \cdot xx^* \in V$. Now, by Remark (1.4.4) follows that

$$\begin{aligned} xx^* \cdot y^*y &= e(xx^* \cdot y^*y)^{**e-1} \\ &= y^*y \cdot xx^* \in V. \end{aligned}$$

Next, we claim that $xy+yx \in V$. Since $x+y$ belongs to V , it follows that $(x+y)^2 \in V$ (Corollary (1.4.7)). Then, $x^2+y^2+xy+yx \in V$. However, since x^2 and y^2 are in V , it follows that $xy+yx \in V$.

Finally, since y and $xx^* \cdot y^*$ are in V , it follows by the preceding claim that, $(xx^* \cdot y^*)y + y(xx^* \cdot y^*) \in V$. Thus $yxx^*y^* \in V$, that is, $(yx)(yx)^* \in V$, which implies by Lemma (1.4.8) that $yx \in V$. $\square$

Theorem (1.4.10): Let D be any c -orderable division ring, let V be the set of bounded elements, and let J be the subset of infinitesimals. Then:

- (1) V is a $*$ -closed valuation subring of D .
- (2) The set of non-units of the valuation ring V coincides with the subset J of infinitesimals, so that J is the unique maximal right and 2-sided ideal of the ring V .

Proof: (1) By Lemmas (1.4.5) and (1.4.9), all that is left to

show is that: if $x \notin V$, then $x^{-1} \in V$. For such x , $xx^* > r$ for all rationals r . Hence $((x^*)^{-1} \varepsilon^*(x^{-1}))(xx^*) > ((x^*)^{-1} \varepsilon^*(x^{-1})) \cdot r$, that is, $(x^*)^{-1} \varepsilon^*(x^{-1}) > \frac{r}{x}$, for each rational r . Then, $(x^{-1})^* \varepsilon(x^{-1})^* = x^* x^{-1} \varepsilon = ((x^*)^{-1} \varepsilon^* x^{-1})^* < \frac{1}{r}$ for each rational r . Thus $(x^{-1})^* \in V$ and by Remark (1.4.5), $x^{-1} \in V$ follows.

(2) Let x be a unit (e.g. $r_2 \leq xx^* \leq r_1$ for some rationals r_1 and r_2). From $xx^* \leq r_1$, $x \in V$ follows. Also $xx^* \geq r_2$ implies $x^{-1} \in V$ (as in the proof of (1)). Thus x is a unit of the ring V .

Conversely, let x be a unit of V . From $x \in V$, it follows that $xx^* < r_1$ for some rational r_1 . Also, since $x^{-1} \in V$, it follows that $(x^{-1})^* \in V$. Reversing the arguments in the proof of (1), this gives $xx^* \geq r_2$, for some rational r_2 . Therefore x is a unit. $\square$

Lemma (1.4.11): For each $x \in D$, the elements $\frac{1}{1+xx^*}$ and $(\frac{1}{1+xx^*})x$ are bounded.

Proof: Assume that x is bounded. Then $1+xx^*$ is bounded, and hence, $1 < 1+xx^* \leq r$ for some rational r . Thus, $r^{-1} = (1+xx^*)^{-1} < 1$, and hence $\frac{1}{1+xx^*}$ is bounded. Then $(\frac{1}{1+xx^*})x$ is bounded.

Now, assume that x is not bounded. It follows that $(1+xx^*)$ is not bounded. Since V is a valuation ring, $\frac{1}{1+xx^*}$ is then a bounded element. Assume to the contrary, that $(\frac{1}{1+xx^*})x$ is not bounded. It follows that $x^{-1}+x^* = x^{-1}(1+xx^*)$ is bounded. However, from x not bounded, follows x^{-1} is bounded. Thus x^* is bounded and hence $x \in V$, a contradiction. $\square$

Corollary (1.4.12): The centre Z_b of the ring V is a valuation domain of the field Z (the centre of D).

Proof: Let $a \in Z_b$. Then a commutes with $\frac{1}{1+xx^*}$ and $(\frac{1}{1+xx^*})x$, for each $x \in D$. Thus a commutes with x , placing a in Z . It is now evident that Z_b is a $*$ -closed valuation ring of the field Z . $\square$

1.5. ϵ -valuation

Let D be a division ring with ϵ -involution $*$ and let $D^\times$ be the multiplicative group of non-zero elements of D . By ϵ -valuation we mean a mapping $w : D^\times \rightarrow G$, where G is a linearly ordered additive group, such that:

Axiom (1): $w(xy) = w(x) + w(y)$ ($x, y \in D^\times$);

Axiom (2): $w(x+y) \geq \min(w(x), w(y))$, $x+y \neq 0$;

Axiom (3): w maps $D^\times$ onto G ; and

Axiom (4): $w(\epsilon x^*) = w(x)$.

Remark (1.5.1): Any ϵ -valuation w is a $*$ -valuation (in the sense given by Holland [10]) such that $w(\epsilon) = 0$.

Proof: Let w be any ϵ -valuation of D . From Axioms (1) and (4), follows $w(\epsilon) + w(x^*) = w(x)$. Replacing x by 1 , this gives $w(\epsilon) = 0$. Then $w(x) = w(\epsilon x^*) = w(\epsilon) + w(x^*) = w(x^*)$ and, hence, w is a $*$ -valuation. $\square$

As a corollary to the preceding remark it follows that the valuation w is abelian, that is, its value group G is an abelian group. Let R stand for the set of elements $x \in D$ such that $w(x) \geq 0$ or

$x = 0$:

$$R = \{x \in D^{\times} / w(x) \neq 0\} \cup \{0\}.$$

Evidently R is a subring of D , R is total (in the sense given by P.M. Cohn, that for each $0 \neq x \in D$, x or $x^{-1} \in R$), and

Fact (1): R contains every $\epsilon x^* x^{-1}$ for all $x \in D$.

Proof: Since $w(\epsilon x^* x^{-1}) = w(\epsilon x^*) + w(x^{-1}) = w(x) + w(x^{-1}) = 0$, it follows that $\epsilon x^* x^{-1} \in R$ for every $x \in D$. $\square$

Definition (1.5.2): Any subring R of D which is total and is as in Fact (1) is called an ϵ -valuation ring.

Theorem (1.5.3): Given an ϵ -valuation subring R of a division ring D with ϵ -involution, then there exist a linearly ordered abelian group G , and an ϵ -valuation $w : D \rightarrow G$ such that R coincides with the valuation ring of w .

Proof: Let U denote the multiplicative group of invertible elements of R . Then $\epsilon x^* x^{-1} \in U$ for every $x \in D$, because $(\epsilon x^* x^{-1})^{-1} = x(x^*)^{-1} \epsilon^{-1} = \epsilon(\epsilon x^*)^*(\epsilon x^*)^{-1} \in R$. Observe that ϵ and ϵ^* are in U . Also, since U is multiplicative, it follows that $x^* x^{-1} = \epsilon^* \epsilon x^* x^{-1} \in U$, for every $x \in D$.

We now claim that $xyx^{-1}y^{-1} \in U$ for every $x, y \in D$. For, one has the following identity (similar to the one in [10, page 221]):

$$xyx^{-1}y^{-1} = [\epsilon(x^*)^*(x^*)^{-1}] [(\epsilon y^* x)^*(\epsilon y^* x)^{-1}] [\epsilon y^* y^{-1}].$$

Hence, U contains the commutator subgroup $[D, D]$ of $D^{\times}$. Thus U is a normal subgroup of $D^{\times}$, and, consequently, the factor group

$G = D^{\times}/U$ is abelian. Define $w : D^{\times} \rightarrow G$ to be the canonical mapping, and order G by setting

$$a \in D^{\times}, aU \geq 1 \text{ if and only if } w(a) \geq 0.$$

Switching from the multiplicative linearly ordered group G to the same ordered system where the multiplication is written addition, one can quickly check Axioms (1) to (4), so that, w is an ε -valuation of D with value group precisely G and with valuation ring evidently R . $\square$

Definition (1.5.4). Any subring R of D is called symmetric if R contains $\varepsilon x^* x^{-1}$ for every $x \in D$.

In the course of the proof of Theorem (1.5.3) we showed the

Remark (1.5.5): Every symmetric subring R contains $\varepsilon, \varepsilon^*$ and $x^* x^{-1}$, for every $x \in D$.

Lemma (1.5.6): Any symmetric subring R of a division ring D has the following properties:

- (1) it is $*$ -closed ($x \in R \Rightarrow x^* \in R$);
- (2) $[D, D] \subseteq R$, and hence R is preserved under conjugation ($y^{-1} R y = R$ for all $y \in D$);
- (3) Each ideal Δ in R is $*$ -closed, is two sided, and is such that $xy \in D$ implies $yx \in D$.

Proof: (1) If $x \in R$, then $x^* = \varepsilon^* \cdot \varepsilon x^* x^{-1} \cdot x \in R$. Hence $R^* = R$.

(2) See the proof of Theorem (1.5.3).

(3) If Δ is a left ideal of R , and if $x \in \Delta$, then $x^* = (x^* x^{-1})x \in \Delta$, where $x^* x^{-1} \in R$. Also, if Δ is a right ideal of R , then from $x \in \Delta$ follows $x^* = x(x^{-1} x^*) \in \Delta$, where $x^{-1} x^* \in R$.

(because $x^{-1}(x^*x^{-1})x \in R$).

Now, let Δ be a left ideal of R , and let $x \in \Delta$. Thus, if $y \in R$, then $x^* \in \Delta$ and $\varepsilon y^* \in R$ together imply that $\varepsilon y^* x^* \in \Delta$, that is, $(xy\varepsilon^*)^* \in \Delta$. Hence $\varepsilon^{-1}xy = (xy\varepsilon^*)^{**} \in \Delta$. However $x \in R$, and Δ is a left ideal of R , so that $xy \in \Delta$. Thus Δ is also a right ideal of R .

Finally, if $xy \in \Delta$, then $x^*y^* = (x^*x^{-1})(xy)(y^{-1}y^*) \in \Delta$, where x^*x^{-1} and $y^{-1}y^* \in R$. Thus $yx = \varepsilon(x^*y^*)^*\varepsilon^{-1} \in \Delta$. $\square$

The following lemma is evident:

Lemma (1.5.7): The set J of non-invertible elements in R is a proper ideal of R that contains every other ideal. One has

$$J = \{x \in R / w(x) > 0\} \cup \{0\}.$$

Definition (1.5.8): By residue division ring $\bar{D}$ we simply mean the division ring R/J , where J is the maximal ideal of non invertible elements in R . Then the map $*$ of $\bar{D}$ defined by $(x+J)^* = x^* + J$ ($x \in R$) is an ε -involution of $\bar{D}$ (the residue ε -involution).

In [6], a necessary and sufficient condition is given for extending an abelian valuation from a division ring D to the over division ring D_1 . We will solve here a $*$ -version of this problem, where valuations are replaced with ε -valuations.

Let D be a division ring with ε -involution $*$. Denote by $[D, D]$, the commutator subgroup of the multiplicative group of D . Along the lines of the solution to the problem given in [6], we shall consider pairs (V, J) , where V is a $*$ -closed subring of D such that

$V \supset [D, D]$, and J is a $*$ -closed proper ideal of V . We note that, since $[D, D]$ is a group, every element of $[D, D]$ is a unit in V . Therefore $[D, D] \cap J = \emptyset$. We shall write $(V, J) \leq (V', J')$ and say that (V', J') dominates (V, J) , if $V \subseteq V'$ and $J \subseteq J'$. Clearly this is a partial ordering for the collection of all such pairs (V, J) .

Lemma (1.5.9): Let D be a division ring with ε -involution $*$, R a $*$ -closed subring containing $[D, D]$ and M a $*$ -closed proper ideal of R . Then there exist a $*$ -closed subring V containing $[D, D]$, and a $*$ -closed ideal J such that (V, J) is maximal among pairs dominating (R, M) . Further, any such maximal pair (V, J) is made up of the ε -valuation ring V in D with maximal 2-sided ideal precisely J .

Proof: Clearly the pairs (V, J) dominating (R, M) form an inductive system. By Zorn's lemma, one can find a maximal member. Let (V, J) be any such member. Since V is $*$ -closed and preserved under conjugation, it follows that V is a $*$ -valuation ring provided V is a total subring: From the maximality follows that J is a maximal ideal of V , and by the localization at J , we can enlarge V to a local ring. Then by maximality, V is in fact a local ring with maximal ideal J . Now, we claim that J is also closed under conjugation. Assume $a \in J$, $b \in D^\times$. If $b^{-1}ab \notin J$, then $b^{-1}ab$ is a unit of V , that is $b^{-1}a^{-1}b \in V$. Hence $a^{-1} \in V$, and this contradicts the fact that $a \in J$.

To complete the proof we must show that $c \in V$ or $c^{-1} \in V$ for every $c \in D$. Assume that $c \notin V$. Then $V[c] \supset V$. If the ideal J' generated by J in $V[c]$ is proper, then $(V[c], J') \geq (V, J)$ and this

contradicts the maximality of (V, J) . Hence $J' = V[c]$, that is, we have an equation

$$(1) \quad 1 = a_0 + a_1 c + \dots + a_m c^m, \quad a_i \in J.$$

Similarly, if $c^{-1} \notin V$, we get an equation

$$(2) \quad 1 = b_0 + b_1 c^{-1} + \dots + b_n c^{-n}, \quad b_j \in J.$$

We may assume that m, n are chosen to be as small as possible, and by symmetry we may take $m \geq n$. Multiplying equation (2) by c^m and noting that $cJ = Jc$, we get

$$(3) \quad (1 - b_0)c^m = b_1 c^{m-1} + \dots + b_n c^{m-n}.$$

If we multiply equation (1) by $(1 - b_0)$ and substitute for $(1 - b_0)c^m$ from equation (3), we obtain an equation of the form

$$1 = a'_0 + a'_1 c + \dots + a'_\ell c^\ell, \quad \ell < m.$$

This is a contradiction. This shows that either c or c^{-1} lies in V . $\square$

Theorem (1.5.10): Let $D \subseteq D_1$ be any division ring extension.

Given any ϵ -valuation w on D with ϵ -valuation subring V and maximal ideal J , there is an extension ϵ -valuation w_1 of w to D_1 if and only if $J[D_1, D_1]$ is a proper ideal in $V[D_1, D_1]$, that is, if and only if there is no equation of the form:

$$(4) \quad \sum_i a_i c_i = 1, \quad a_i \in J, \quad c_i \in [D_1, D_1].$$

Proof: If there is an extension w_1 of w , and (4) holds, then

$$0 = w_1(1) \geq \min_i \{w(a_i) + w_1(c_i)\}.$$

Since $a_i \in J$, $w(a_i) > 0$, and $w_1(\dot{c}_i) = 0$ (for, c_i is a product of commutators), it follows that the right hand side of the preceding inequality is positive, a contradiction. Hence no equation (4) holds

true. If there is an equation of the form (4), then $J[D_1, D_1]$ is a proper ideal in $V[D_1, D_1]$.

To prove the converse, we first note that $J[D_1, D_1] = [D_1, D_1]J$, and $V[D_1, D_1] = [D_1, D_1]V$ (for, $r(xy x^{-1} y^{-1}) = (rxy r r^{-1} x^{-1} y^{-1} r^{-1})r$).

Since both V and $[D_1, D_1]$ are $*$ -closed, it follows that $(V[D_1, D_1])^* = [D_1, D_1]^* V^* = [D_1, D_1]V = V[D_1, D_1]$. Similarly, $J[D_1, D_1]$ is $*$ -closed. Now, clearly $V[D_1, D_1] \supset [D_1, D_1]$ and, hence, by Lemma (1.5.9), there is a maximal pair (V_1, J_1) dominating (V, J) . Thus, V_1 is an ϵ -valuation ring of D_1 , which defines the desired extension. $\square$

For examples of a division ring extension where the equation (4) holds, see [6, page 30] and [16, page 34].

1.6. Scaling by a core element:

We say that a property of a division ring D scales if, whenever it holds in D with a given ϵ -involution $*$, it also holds in D with any ϵ_1 -involution $\#$ co-gradient to $*$. In this section we shall show a weakened version of this requirement. We prove

Theorem (1.6.1): Let D be a division ring with ϵ -involution $*$. If D is c -orderable, and if ρ is in the $*$ -core P of D , then a change from $*$ to the co-gradient ϵ_1 -involution $\# : x \mapsto x^\# = \rho^{-1} x^* \rho$ does not alter c -orderability. Equivalently $(D, \#)$ is c -orderable.

Proof: Let D^+ be the domain of positivity of the ϵ -ordering of $(D, *)$. We shall show that D^+ is a domain of positivity for a ϵ -ordering of $(D, \#)$. Firstly, we show that $(D^+)^{\#} = D^+$. Assume $x \in D^+$. Since $x^* \in D^+$, $\rho \in P$ and $PD^+P \subset D^+$, it follows that $x^{\#} \in D^+$. By Lemma (1.2.12) follows, $S_1 = Sp$. If now $x \in S_1$, then $x = yp$ for some $y \in S$. Since $y \in D^+ \cup (-D^+)$, it follows that $x \in D^+P \cup (-D^+P) \subset D^+ \cup (-D^+)$. It remains to show that $x \in D^+$ implies $x \cdot y \cdot x^{\#} \in D^+$ for each $y \in D$. (Recall that $\epsilon_1 = \epsilon(\rho^{-1})^*\rho$.) Since ϵ_1 and ρ are in P , it follows by Section (1.3) Fact (6) that, $y\epsilon_1y^{\#} \in P$. Thus $x \cdot y \cdot \epsilon_1y^{\#} \in D^+P \subset D^+$. This completes the proof. $\square$

Remark (1.6.2): Under the hypothesis of Theorem (1.6.1), if P_1 is the $\#$ -core of D , then $P_1 \subseteq P$.

Proof: Since $\epsilon_1 = \epsilon(\rho^*)^{-1}\rho \in P$, it follows that $x\epsilon_1x^{\#} = x\epsilon_1\rho^{-1}x^*\rho \in P$. Similarly $x\epsilon_1x^{\#} \in P$ for each $x \in D$. Thus $P_1 \subseteq P$. $\square$

As a corollary to Remark (1.6.2), one can show that ϵ -orderability is preserved via the element ρ in the $*$ -core P of D . Note, however, that the proof of the preceding theorem proves something more, namely, that both ϵ -orderings have the same domain of positivity.

Corollary (1.6.3): Let D be a ϵ -orderable division ring with ϵ -involution $*$. Let $\#$ be the co-gradient involution $x \mapsto x^{\#} = \rho^{*-1}x^*\rho^*$, where $\rho = \epsilon\rho^*$ is in the $*$ -core P of $(D, *)$. Then $(D, \#)$ is ϵ -orderable. $\square$

Corollary (1.6.4): Let D be a ϵ -orderable division ring with involution $\#$. Let $*$ be the co-gradient ϵ -involution $x^* = \rho^{-1}x^{\#}\rho$,

where $u \neq u^\#$ is in the $\#$ -core of $(D, \#)$. Then $(D, *)$ is c -orderable.

Theorem (1.6.5): Under the hypothesis of Corollary (1.6.3), if V and V_1 denote the rings of bounded elements at the c -orderings of $(D, *)$ and $(D, \#)$ respectively, then $V_1 \subseteq V$.

Proof: For, let $x \in V_1$, we first claim that $e \in V_1$. Since $e < r$ for some rational r , it follows that $e^\# = \rho^{*-1} e^* \rho^* < r$, that is, $e^\# < r^2$. Thus $e \in V_1$. We next show that $x^* \in V_1$. Since V_1 is $\#$ -closed subring which is closed under conjugation ([3, Theorem 7]), it follows that $x^* = \rho^* x^\# (\rho^*)^{-1} \in V_1$. Thus $xx^* \in V_1$ and the $\#$ -symmetric element $(xx^*) + (xx^*)^\# \in V_1$. Hence $xx^* + (xx^*)^\# < r_1$ for some rational r_1 , that is, $xx^* + \rho^{-1} x e x^* \rho^* < r_1$. Similarly, $x e x^* \in V_1$ implies that $x e x^* + \rho^{-1} x x^* \rho^* < r_2$ for some rational r_2 . Therefore $(xx^* + x e x^*) + \rho^{-1} (xx^* + x e x^*) \rho^* < r_1 + r_2$. Thus $xx^* + x e x^* < r_1 + r_2$, and x is $*$ -bounded. (Remark (1.4.3)). $\square$

Corollary (1.6.6): V is closed under conjugation.

Proof: Since $x^{-1} x^\# \in V_1$ for each $x \in D$ ([3, Lemma (8)]), then it follows by Lemma (1.5.6) that $[D, D] \subseteq V_1$. Therefore $[D, D] \subseteq V$, and V is closed under conjugation. $\square$

Corollary (1.6.7): V is an ϵ -valuation subring.

Proof: We shall show that $x^{-1} x^* \in V$, for every $x \in D$. Since V is a valuation ring it follows that $x^{-1} x^* \in V$, or $x^{*-1} x \in V$. Assume that $x^{*-1} x \in V$, for some $x \in D$. By Corollary (1.6.6), $xx^{*-1} = x^*(x^{*-1} x) x^{*-1} \in V$. Since V is $*$ -closed, it follows that

$x^{-1} \varepsilon x^* = \varepsilon (xx^{*-1})^* \in V$. Thus $x^* x^{-1} = x^* (x^{-1} \varepsilon x^*) (x^*)^{-1} \in V$. Therefore $x^{-1} x^* = x^{-1} (x^* x^{-1}) x \in V$. $\square$

Using the fact that V is closed under conjugation we can show the

Lemma (1.6.8): $V = V_1$.

Proof: If $x \in V$, then $x^* \in V$, which gives $x^\# = \rho^{*-1} x^* \rho^* \in V$. Hence, $xx^\# \in V$. By Remark (1.4.2) follows $xx^\# + \varepsilon (xx^\#)^* < r$, for some rational r . Thus $xx^\# < r$, and $x \in V_1$. $\square$

Let J be the ideal of $*$ -infinitismals of the ring V , and let J_1 be the ideal of $\#$ -infinitismals. One has the following lemma:

Lemma (1.6.9): $J = J_1$.

Proof: Assume $x \in J_1$. Then $x \in V_1 = V$, which implies that $x^* \in V$. Since J_1 is an ideal of V , it follows that $x \varepsilon x^* \in J_1$. Hence $x \varepsilon x^* + (x \varepsilon x^*)^\#$ is $\#$ -symmetric in J_1 . By Remark (1.4.2), follows $(x \varepsilon x^*) + (x \varepsilon x^*)^\# < r$, for each rational r . Thus $x \varepsilon x^* < r$ for each r , and $x \in J$ follows.

Conversely, if $x \in J$, then $xx^\# \in J$. Thus the $*$ -symmetric element $-(xx^\#) + \varepsilon (xx^\#)^*$ is an infinitesimal, so that, $xx^\# < r$ for each rational r . Thus $x \in J_1$. $\square$

To close this section let us take a look at some special instances of division rings with ε -involution.

Remark (1.6.10): c -orderability is not generally scaling.

Proof: Let $H = \mathbb{R}\{i, j \mid i^2 = -1, j^2 = -1, ij = -ji\}$ be the classical real quaternionic algebra with the involution $\#$ with fixed set

precisely $\mathbb{R}$, the centre of H . Now, as shown in Section (1.2), if

$*$ is defined by

$$(\alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k)^* = \alpha_0 - \alpha_1 i - \alpha_2 j + \alpha_3 k$$

where $\alpha_i \in \mathbb{R}$ and $k = ij$; then $*$ is an ε -involution where $\varepsilon = i$, which is co-gradient to $\#$,

$$x^* = (1+i)^{-1} \# x (1+i), \quad x \in H.$$

We claim that $(H, *)$ is not c -orderable. For, assume to the contrary that there is a c -ordering of $(H, *)$ with domain of positivity H^+ . By construction, $1+\varepsilon = 1+i$ is in the $*$ -core of H . By Theorem (1.6.1), it follows that H^+ is a domain of positivity for $(H, \#)$, which must be $\mathbb{R}$, the core of positive real numbers. However, as shown in Section (1.2) (page 3), each symmetric element relative to $*$ is in $\mathbb{R}(1+i)$. This contradicts the fact that $H^+ = \mathbb{R}^+$. $\square$

Remark (1.6.11): There is a non-normal quaternionic division ring D , which is c -ordered.

Proof: Take any usually c -ordered division ring D with usual involution $\#$ and with anti-commuting generators $s = s^\#, d = d^\#$. Put $\mu = (1+s)^2(1+d)^2$. Since $sd \neq ds$, it follows that $1+s^2+2s$ does not commute with $1+d^2+2d$ ($s^2, d^2 \in Z$), so that $\mu \neq \mu^\#$. Evidently $\mu \notin Z$. Pass to the co-gradient ε -involution induced by μ . Clearly μ is in the usual core of D relative to $\#$. By Theorem (1.6.1), D is now c -orderable relative to $*$. $\square$

Remark (1.6.12): There is an infinite-dimensional division ring D which is c -ordered.

Proof: Along the lines of the example given in [3, page 520], let F be any field with involution, which is naturally orderable. Let $R = F[x, y]/(xy - yx - 1)$ with the involution at which $x = x^\#$ and $y = -y^\#$. Let D be the division ring of quotients of R with the extended involution $\#$. Then D is usually ϵ -orderable. To define an ϵ -involution of D , it suffices to find some non-central non symmetric element in the core of D . Take $\mu = -y^2 x^2$. Then $\mu^\# = -x^2 y^2 \neq \mu$. Hence μ is non-symmetric non-central and $\ast : a \mapsto \mu^{-1} a \mu$, ($a \in D$) is an ϵ -involution of D . Since $\mu = yy^\# : xx^\#$ is in the $\#$ -core of D , it follows by Theorem (1.6.1), that D is ϵ -orderable relative to $\ast$. $\square$

It is appropriate to observe that as soon as a division ring D with involution $\ast$ is not normal and $\text{char}(D) \neq 2$, then there must be an element a of the form $a = xx^\ast yy^\ast$ with $a \neq a^\ast$. Therefore any usually ϵ -ordered division ring with involution can be turned to a ϵ -orderable division ring with ϵ -involution, as this was done in the two preceeding remarks.

1.7. Semi ϵ -ordering:

Let D be a division ring with ϵ -involution $\ast$.

Definition (1.7.1): By semi ϵ -ordering, we mean any order relation of the division ring D such that if $M = \{x \in D \mid x > 0\}$ then:

Axiom (0): $x > y \Leftrightarrow x - y \in M$.

Axiom (I): $0 \notin M$, $1 \in M$;

Axiom (II): $a, b \in M \Rightarrow a + b \in M$;

Axiom (III): $a \in M \Rightarrow a^\ast \in M$;

Axiom (IV): $a \in M$, $x \neq 0 \Rightarrow ax \in x^\ast$ and $ax \in x^\ast M$.

Any subset M verifying Axioms (I) to (IV), we will refer to as a semi-domain of positivity. Any ϵ -ordering of D is clearly a semi ϵ -ordering.

Also, any intersection of c -orderings of D is a semi c -ordering of D .

If D is c -orderable, that is $0 \notin P$ ($=$ the $\star$ -core of D), then the

$\star$ -core P is a semi-domain of positivity, with the following features:

$P \cap M$ and $PM = MP = M$, for each semi-domain of positivity M . Throughout the rest of this section, we will assume that $0 \notin P$.

Theorem (1.7.2): If M is a semi domain of positivity, and if s is a symmetric element such that $s \notin M$, then there is a semi-domain of positivity M' containing M , with $-s \in M'$.

Proof Let $M' = M \cup -sP \cup (M+(-s)P)$. Since $1 \in P$, it follows that $-s \in M'$. We will check Axioms (I) to (IV) for M' .

Axiom (I): Since $M^* = M$ and $(-sP)^* = P^*(-s) = P(+s) = (-s)P$, it follows that $M'^* = M'$.

Axiom (II): Since M and P are additive, it follows that $M' + M' \subseteq M'$.

Axiom (III): Clearly $1 \in M'$. Since $0 \notin M$ and $0 \notin (-sP)$, to show that $0 \notin M'$, it suffices to show that $0 \notin M+(-s)P$. Suppose to the contrary that $0 \in M+(-s)P$. Then $m-sp = 0$, for some $m \in M$ and $p \in P$. Hence $s = mp^{-1} \in MP = M$, which is a contradiction. As for Axiom (IV), this is evident. $\square$

Theorem (1.7.3): Any semi-domain of positivity M_0 of D can be extended to some domain of positivity M .

Proof: By Zorn's lemma, the set of all semi-domain of positivity extending M_0 contains some maximal semi-domain of positivity M . If $s \notin M$ for some symmetric s , it follows by Theorem (1.7.2) that there is a semi domain of positivity M' containing M , and such that $-s \in M'$. From maximality of M , follows $M' = M$, so that $-s \in M$. Thus M is a domain of positivity. $\square$

Corollary (1.7.4): A semi-domain of positivity M is maximal (with respect to set theoretic inclusion) if and only if M is a domain of positivity.

The proof of Corollary (1.7.4) is evident.

Notation: By $\text{Sym}(A)$ we mean the subset of symmetric elements of A .

Corollary (1.7.5): $\text{Sym}(P) = \text{Sym}(\bigcap_i M_i)$, where the intersection runs over all domains of positivity M_i of D .

Proof: Clearly $\text{Sym}(P) \subseteq \text{Sym}(\bigcap_i M_i)$. Conversely, we show that $s = s^* \notin P$ implies $s \notin M$ for some domain of positivity M . Since P is a semi-domain of positivity, it follows by Theorem (1.7.2) that, there is a semi-domain of positivity M_0 containing P such that $-s \in M_0$. Now, by Theorem (1.7.3), M_0 can be extended to some domain of positivity M . Since $-s \in M_0 \subset M$, and M is a domain of positivity, it follows that $s \notin M$. $\square$

Corollary (1.7.6): Let M_0 be any semi-domain of positivity. Then $\text{Sym}(M_0) = \text{Sym}(\bigcap_i M_i)$, where the intersection runs over all domains of positivity M_i containing M_0 .

The proof of Corollary (1.7.6) is similar to that of Corollary (1.7.5).

Lemma (1.7.7): Let M_1 and M_2 be any two domains of positivity. If $M_1 \subset M_2$, then $\text{Sym}(M_1) = \text{Sym}(M_2)$.

Proof: For if there is $s = s^* \in M_2 \setminus M_1$ then from $s \notin M_1$ follows

$-s \in M_1 \cap M_2$, so that both s and $-s$ are in M_2 which is nonsense.

[1]

Lemma (1.7.8): Let D_1 be any division ring extension of D .

Let M be a domain of positivity of D . Let M_1 be the set of elements in D , which are expressible as arithmetical sums of elements b_j of the form $b_j = \prod_i a_{ji} p_{ji}$ ($a_{ji} \in M$ and $p_{ji} \in P_1$ (= the $*$ -core of D_1)). If $0 \notin M_1$, then M_1 can be enlarged to some domain of positivity M_2 of D_1 .

Proof: Since $0 \notin M_1$, it follows that $0 \notin P_1$, and D_1 is c -orderable. We claim that M_1 is a semi-domain of positivity of D_1 . For clearly $M_1 + M_1 \subseteq M_1$ and $a \in M_1$ implies that $ax \in x^*$ and $ax \in x^*x$ are in M_1 , $0 \neq x \in D_1$. From $(\prod_i a_i p_i)^* = \prod_i (1 \cdot p_i^*) (a_i^* \cdot 1) \in M_1$, it follows that $M_1^* = M_1$. Thus, by Theorem (1.7.3), M_1 can be enlarged to some domain of positivity $M_2 \supset M_1 \supset M$. $\square$

Exactly as for c -ordering, one can define the bounded elements, the infinitesimals and the units at a given semi c -ordering of the division ring D . For M_0 , the semi-domain of positivity of this semi c -ordering, let R_0, J_0 denote the sets of all bounded elements and infinitesimals respectively.

Theorem (1.7.9): Let $(M_i)_{i \in I}$ be the family of domains of positivity containing M_0 , where M_0 is a given semi-domain of positivity of D . Let R_i, J_i be the subring of bounded elements and the ideal of infinitesimals respectively, attached to the c -ordering induced by M_i . Let R_i^x stand for the group of units of the ring R_i .

Then

- (i) $\bigcap_i R_i = R_0$;
 (ii) $\bigcap_i J_i = J_0$;
 (iii) $J_0 \subset \text{Jac}(R_0)$; and
 (iv) $\bigcap_i R_i^x = R_0^x$.

Proof: (i) $\bigcap_i R_i = R_0$.

Assume $x \in R_0$. Since $M_0 \subset M_i$ for every $i \in I$, it follows that $x \in R_i$ for each $i \in I$. Thus $x \in \bigcap_i R_i$. Conversely, if $x \notin R_0$, we show that $x \notin R_i$, for some i . Put $s = xx^* + x_0^*$. From $x \notin R_0$, follows $s \notin R_0$ (Remark (1.4.3)). Then $s \notin r$ for every positive rational r , that is, $r-s \notin M_0^+$, for every rational r . Let $M_r = M_0 \cup (s-r)P \cup (M_0 + (s-r)P)$, where P is the $*$ -core of D . Let $M_0' = \bigcup_r M_r$. We claim that M_0' is a semi-domain of positivity. We check the

Axiom (1): $1 \in M_0^+$ and $0 \notin M_0'$.

Clearly $1 \in M_0 \subset M_0'$. Since $0 \notin M_0^+$, and $0 \notin (s-r)P$, to show that $0 \notin M_0'$, it remains thus to show that $0 \notin M_0 + (s-r)P$, for every rational r . Assume that $m - (r-s)p = 0$ for some $m \in M_0$ and $p \in P$. Then $r-s = mp^{-1} \in M_0 P \subset M_0^+$, which is a contradiction.

Axiom (2): $M_0' + M_0' \subseteq M_0'$.

Suppose that $x = x_1 - (r_1 - s)y_1$, $y = x_2 - (r_2 - s)y_2$ are arbitrary elements in M_0' where $x_1, x_2 \in M_0$; $y_1, y_2 \in P$; and r_1, r_2 are rationals. Assume $r_1 \geq r_2$. Then

$$\begin{aligned}
x+y &= (x_1+x_2) - r_1(y_1+y_2) + s(y_1+y_2) + (r_1-r_2)y_2 \\
&= (x_1+x_2) + (r_1-r_2)y_2 - (r_1-s)(y_1+y_2) \\
&\in M_0 - (r_1-s)P \in M'_0,
\end{aligned}$$

for r_1-r_2 is a positive rational and, hence, $(r_1-r_2)y_2$ is positive in M_0 .

$$\text{Axiom (3): } (M'_0)^* = M'_0.$$

Since $M_0^* = M_0$, it suffices to show that $((s-r)p)^* \in M'_0$, where $p \in P$. Now $((s-r)p)^* = p^*(s-r) = p^{*-1}(s-r) = p'(s-r)$, where $p' = p^{*-1} \in P$. From $\varepsilon < r_\varepsilon$ follows $p'(r_\varepsilon r - \varepsilon r) > 0$, that is, $p'(r_\varepsilon r - \varepsilon r) \in M_0$. Since $p'(s-r_\varepsilon r) \in M'_0$, it follows that $((s-r)p)^* = p'(s-r_\varepsilon r) + p'(r_\varepsilon r - \varepsilon r) \in M'_0$.

Axiom (4): $M'_0 x \in x^*$ and $M'_0 x \in x^* x$ are both contained in M'_0 .

This is evident.

Thus M'_0 is a semi-domain of positivity containing M_0 ; and $s-r$ for every rational r . By Theorem (1.7.3), M'_0 can be extended to some domain of positivity M . Clearly $M_0 \subset M$, and since $s-r \in M$, then $s > r$ for every rational r . Thus s is not bounded in M . Hence x is not bounded in M (Remark (1.4.3)). Therefore $x \notin \bigcap_i R_i$, as desired.

$$(ii). \quad J_0 = \bigcap_{i \in I} J_i :$$

Assume $x \in J_0$. Put $s = xx^* + x \varepsilon x^*$. Then s is an infinitesimal, that is, $s < r$ for every rational r . Hence $(r-s) \in M_0$ for every

rational r . Since $M_0 \subset M_i$, it follows that $(r-s) \in M_i$ for every $i \in I$, and for every rational r . Thus $s \in \bigcap_i J_i$, and $x \in \bigcap_i J_i$ follows. Then $J_0 \subset \bigcap_i J_i$.

Conversely, if $x \in \bigcap_i J_i$, then $s = xx^* + x \in \bigcap_i J_i$, and $r-s \in M_i$ for every $i \in I$, and for every rational r . Since M_i is a domain of positivity, for every i , it follows that $(r-s)^* \in M_i$. Hence the symmetric element $er+r-2s$ is in M_i for every i . By Corollary (1.7.6), $er+r-2s \in M_0$ follows and, hence, $s \in \frac{r+r}{2}$ for every rational r . Thus s is an infinitesimal in M_0 , and $x \in J_0$ follows.

$$(iii) \quad \underline{J_0 \subset \text{Jac}(R_0)}.$$

We use here the well-known fact that $x \in \text{Jac}(R_0)$ if and only if $1-tx$ is invertible in R_0 for every $t \in R_0$. Assume $x \in J_0$. Then $x \in J_i = \text{Jac}(R_i)$ for every i . If $t \in R_0$, then $t \in R_i$ for every $i \in I$. Thus $1-tx$ is invertible in R_i for every $i \in I$. Hence $1-tx$ is invertible in R_0 . Since this is true for every $t \in R_0$, it follows that $x \in \text{Jac}(R_0)$.

$$(iv) \quad \underline{\bigcap_i R_i^x = R_0^x}.$$

Clearly $R_0^x \subset \bigcap_i R_i^x$. Conversely, assume that $x \in \bigcap_{i \in I} R_i^x$. Put $s = xx^* + x \in \bigcap_i R_i^x$ (Remark (1.4.3)). Since $x \in \bigcap_i R_i^x \subset \bigcap_i R_i = R_0$, it follows that $s \in R_0$, that is, $q-s \in M_0$ for some positive rational q . We contend that $s-r_0 \in M_0$ for some positive rational r_0 . For, assume to the contrary that $s-r \notin M_0$ for each positive

rational r . Set $M_0^{(r)} = M_0 \cup (r-s)P \cup (M_0 + (r-s)P)$, and let $M_0^{(\infty)} = \bigcup_r M_0^{(r)}$, where the union runs over all positive rationals. As, in the proof of (i), one can show that $M_0^{(\infty)}$ is a semi-domain of positivity containing M_0 , and containing $(r-s)$ for every positive rational r . By Theorem (1.7.3), follows that $M_0^{(\infty)}$ is contained in some domain of positivity M . Thus, $r-s \in M$ for every rational r , a contradiction. This shows that $s-r_0 \in M_0$ for some positive rational r_0 and, hence, s is a unit in R_0 . It remains to show that x is a unit in R_0 . For, from $s = xx^* + xcx^* \in r_0$, follows $xx^* \in r_0$. Then $1 \in (x^*)^{-1} \in (x^{-1}) \cdot r_0$, so that, $(x^{-1})^* \in (x^{-1*})^* = (x^{-1*} \in x^{-1})^* \in \frac{1}{r_0}$. Hence $(x^{-1})^* \in R_0$. By Lemma (1.4.5) follows, $x^{-1} \in R_0$. $\square$

To a certain extent Theorem (1.7.9) reduces the treatment of semi c -ordering to that of c -ordering. For instance one has the

Corollary (1.7.10): For any semi-domain of positivity M_0 of D , the subring R_0 of bounded elements is preserved under conjugation.

Proof: This is just a combination of Theorem (1.7.9), and the fact that each R_i is preserved under conjugation. $\square$

CHAPTER II

JORDAN ORDERING OF A DIVISION RING WITH INVOLUTION

2.1. Introduction, definitions and basic properties:

In this chapter, we introduce a new type of ordering for a division ring with involution, which we will call a Jordan ordering. Basic properties of this ordering are studied. Also, we investigate the structure of the bounded subring that is associated with a given Jordan ordering.

We will establish that this subring is a $*$ -valuation ring. In §2.2, we will take a look at multiplicative commutators and establish that, if one of the arguments is symmetric then the commutator is congruent to 1 modulo J , where J is the ideal of non-units of the bounded subring.

In §2.3, we deduce that the Jordan ordering can be extended to an ordering which verifies the axioms of a Baer ordering and a $*$ -ordering plus the closure under product (strong ordering).

Let D be a division ring with involution $*$.

Definition (2.1.1): By Jordan ordering, we mean any order relation of the division ring D such that if $\pi = \{s \in D / s > 0\}$ is the domain of positivity of D , then:

Axiom (0): $x > y$ if and only if $x - y \in \pi$;

Axiom (1): π consists solely of symmetric elements

Axiom (2): $0 \notin \pi$

Axiom (3): π is closed under addition (arithmetical addition);

Axiom (4): $xx^* \in \pi$ for each non-zero $x \in D$;

Axiom (5): if $s, d \in \pi$, then $sd + ds \in \pi$ (Jordan closure);

Axiom (6): D contains s or $-s$, for every non-zero symmetric s in D .

When $*$ = identity, then D is commutative, and the definition of a Jordan ordering reduces to that of a classical Artin-Schreier ordering. The only thing that needs to be checked is this: $s > 0$ and $d > 0$ together imply $sd > 0$. In fact, since $2sd = sd + ds > 0$, and since $sd > 0$ or $sd < 0$, it follows that $sd > 0$.

Following Holland [11], by strong ordering of a division ring D with involution $*$ we mean an order relation of D such that its domain of positivity M is a $*$ -closed normal subgroup of the group $D^\times$, M is closed under sums, M contains every xx^* , where $0 \neq x \in D$, and, for each non-zero symmetric, $0 \neq s = s^* \in D$, $s \in M$ or $-s \in M$.

Remark (2.1.2): Every strong ordering of D gives rise to a Jordan ordering.

Proof: If M is the domain of positivity of a strong ordering, and if S is the additive subgroup of symmetric elements of D , then, clearly $M \cap S$ is a domain of positivity of a Jordan ordering. $\square$

Conversely, one might ask whether the given of a Jordan ordering of the division ring D is enough to produce a strong ordering of D . As a matter of fact we will show that this is always the case. But, this is a highlight of our work which we will be able to demonstrate only at later stages. On the other hand, because of this very equivalence of the notions of Jordan ordering and strong ordering, it is then clear that one could not find new examples of Jordan ordered division rings apart, of course, from the examples of strongly ordered division rings.

For a revision of these examples reference is given to [3, page 520] and [11]. We proceed to two observations about the considered ordering.

Firstly, since by construction $\underbrace{xx^* + \cancel{xx^*} + \dots + xx^*}_{n\text{-times}} > 0$ for each

non zero x in D , it follows that D is with characteristic zero.

Secondly, a technical observation: the notation $s > 0$ in D implies $s = s^*$, since s is in the domain of positivity, which consists solely of symmetric elements.

Theorem (2.1.3): Let D be any Jordan ordered division ring.

Then:

- (1) If $s > 0$, $d > 0$ and $sd = ds$, then $sd > 0$;
- (2) $s > 0$ if and only if $s^{-1} > 0$;
- (3) If $s > 0$ and r is a positive rational, then $rs > 0$;
- (4) If $0 < s < 1$, then $0 < s^2 < s < 1$, and if $0 < s < r$, then $0 < s^2 < rs < r^2$;
- (5) If $s > 0$, then $s^2 > s \geq 1$, and if $s > r$, then $s^2 > rs > r^2$; and
- (6) If $s > 1$, then $s^{-1} < 1$.

Proof: (1) If $s > 0$, $d > 0$ and $sd = ds$, then $2sd = sd + ds > 0$.

Since sd is a symmetric element it follows that $sd > 0$ or $sd < 0$.

If $sd < 0$, then $sd + ds = 2sd < 0$ follows a contradiction. Thus $sd > 0$.

(2) From (1), if $s > 0$ it follows that $s^{-1} = (s(s^{-2}))^{-1} > 0$.

(3) Let $r = \frac{m}{n}$, where m and n are integers. Then for $s > 0$, follows $(\frac{m}{n})s = (n(ms)^{-1})^{-1} > 0$.

(4) and (5) Applying (1) to s and $1-s$ this gives $s-s^2 = s(1-s) > 0$, so that $1 > s > s^2 > 0$. For the rest of (4), and for (5) use similar argument.

(6) For $s > 1$, one gets $s^{-1} > 0$ (using (1)). Then $1-s^{-1} = s^{-2}(s-1) + (1-s^{-1})^2 > 0$.

Remark (2.1.4): (1) For each pair a and b in D , follows $(a+b)(a+b)^* < 2(aa^* + bb^*)$.

(2) For each $s = s^*$ and $k = k^*$ in D , follows $|sk - ks| \leq s^2 - k^2$.

Proof: (1) Use the equality

$$(a+b)(a+b)^* + (a-b)(a-b)^* = 2(aa^* + bb^*),$$

and note that $xx^* > 0$ for each non-zero x in D .

(2) This is clear by making use of the equality

$$(s+k)(s+k)^* = (s^2 - k^2) - (sk - ks).$$

Remark (2.1.5): For each $s = s^*$ in D , and $r \in \mathbb{Q}^+$ (= the cone of positive rationals) follows: $-r \leq s \leq r$ if and only if $s^2 \leq r^2$.

Proof: Let $s^2 \leq r^2$. Then

$$r \pm s = \frac{1}{2r}(r \pm s)^2 + \frac{1}{2r}(r^2 - s^2) > 0$$

so that $r \pm s > 0$. Hence $-r \leq s \leq r$.

Conversely, if $-r \leq s \leq r$, it follows by Theorem (2.1.3), that

$$2r(r^2 - s^2) = (r-s)((r+s)(r+s)^*) + (r+s)((r-s)(r-s)^*) > 0. \text{ Thus } s^2 \leq r^2. \quad \square$$

Let D be any Jordan ordered division ring with involution $*$.

We call $b \in D$ bounded, if $bb^* \leq r$ for some $r \in Q^+$. If $bb^* \leq r$ for every $r \in Q^+$, we call b an infinitesimal. Let V denote the set of all bounded elements in D , and let J denote the set of all infinitesimals in D . When $J = \{0\}$, call the ordering archimedean. Our next goals will be: (1) To establish that V is a $*$ -closed valuation ring in D ; (2) To establish that the multiplicative group $V^\times$ of invertible elements in V is formed by precisely those b such that $r_1 \leq bb^* \leq r_2$, for some positive rationals r_1 and r_2 ; (we call these elements units).

Two more remarks will be needed. For the proof use similar arguments as given in the proof of Remarks (1.4.1) and (1.4.2).

Remark (2.1.6): (1) The set V of bounded elements is an additive Q -subgroup of D . (2) The set J of infinitesimals is an additive Q -subgroup.

Remark (2.1.7): Let $s = s^*$ be positive and let $x \in D$, then:

- (1) s is bounded if and only if $s \leq r_0$ for some $r_0 \in Q^+$;
- (2) s is a unit if and only if $r_1 \leq s \leq r_2$ for some $r_1, r_2 \in Q^+$;
- (3) s is an infinitesimal if and only if $s \leq r$ for every $r \in Q^+$; and

- (4) x is bounded (resp. unit, infinitesimal) if and only if xx^* is bounded (resp. unit, infinitesimal).

Theorem (2.1.8): Let D be a Jordan ordered division ring. Then:

- (1) V is a $*$ -closed valuation subring of D ;
- (2) The set of non-units of the ring V is precisely the ideal of infinitesimals, and, consequently, J is a $*$ -closed ideal of V .

Proof: (1) Since $(1-x)(1-x)^* \geq 0$, it follows that $(1+xx^*) - (x+x^*) \geq 0$, so that $1+xx^* \geq x+x^*$. Also, from $(1+x)(1+x)^* > 0$, follows $x+x^* > -(1+xx^*)$. Now, if $x \in V$, it follows that $xx^* < r$, for some $r_0 \in 0^+$, so that

$$1+r_0 \geq x+x^* \geq -(1+r_0).$$

By Remark (2.1.7), follows $(x+x^*)^2 \leq (1+r_0)^2$. Then $x+x^*$ is bounded. Hence, $x^* \in V$, and V is $*$ -closed.

Next, we claim that if s and x^* are in V , with $s = s^* > 0$, then $xsx^* \in V$. From $x \in V$, follows $x^* \in V$, so that $x^*x \leq r_0$, for some r_0 . Also, $s \in V$ implies that $s < r_1$ for some r_1 . Now, $(r_0 - x^*x)s + s(r_0 - x^*x) > 0$, and, hence,

$$0 < x^*xs + sx^*x < 2r_0s < 2r_0r_1 = r.$$

Thus $x^*xs + sx^*x \in V$. Also

$$(r - (x^*xs + sx^*x))s + s(r - (x^*xs + sx^*x)) > 0,$$

so that $x^*xs^2 + s^2x^*x + 2sx^*xs < 2rs < 2rr_1$. However, since

$x^*xs^2 + s^2x^*x$ is bounded, it follows that $(sx^*)(sx^*)^* = sx^*xs \in V$.

Hence, $sx^* \in V$, and $xs = (sx^*)^* \in V$. Thus $xs^2x^* = (xs)(xs)^* \in V$.

From $(s+1) \in V$, follows that $x(s+1)^2x^* = xs^2x^* + 2xsx^* + xx^*$ is

bounded. This gives $xsx^* \in V$, as wished. Now, let $x, y \in V$. Then

x and s are both in V , where $s = yy^*$ is a positive symmetric element, as wished. Hence $(xy)(xy)^* = xyy^*x^* = xsx^* \in V$, so that $xy \in V$. Therefore, V is a subring.

We are left with the assertion that V is a valuation ring. If $a \notin V$, then $aa^* > r$, for some rational r . By Theorem (2.1.3),

follows $(aa^*-r)(a^{*-1}a^{-1}) > 0$. Hence $(a^{-1})^* a^{-1} \geq \frac{1}{r}$, which shows that $(a^{-1})^* \in V$. Thus $a^{-1} \in V$.

(2) If x is a unit then by construction $x \in V$. As shown in the above, from $xx^* \geq r$ for some positive rational r , follows $x^{-1} \in V$ and, hence, x is invertable in the ring V . Conversely, the process can be reversed showing thus that any invertable in V is a unit. Therefore J coincides with the set of elements x in V which are non-invertable. $\square$

In the quotient division ring $\bar{D} = V/J$, endowed with the residue involution $(x+J)^* = x^* + J$, $x \in V$, one can define a Jordan ordering that makes $\bar{D}$ an archimedean Jordan-ordered division ring, where the symmetric elements are central. To establish this result we will need the following remarks (for the proof use similar arguments as given in the proof of Remarks (1.3.13) and (1.3.14)).

Remark (2.1.9): For each $s = s^*$ such that $-r_0 \leq s \leq r_0 < 1$ for some $r_0 \in \mathbb{Q}^+$, follows $s^2 < |s| + r$ for every $r \in \mathbb{Q}^+$.

Remark (2.1.10): For each $s = s^*$ and $k = -k^*$ in D , follows $|sk - ks| \ll s^2 - k^2$, provided s^2 and $-k^2$ are not archimedean equivalent (in the sense given in Definition (1.3.9)).

Lemma (2.1.11): For each $s = s^*$ and $k = -k^*$ in V , follows $sk - ks \in J$.

Proof: If s^2 and $-k^2$ are not archimedean equivalent then, by Remark (2.1.10), $|sk - ks| \ll s^2 - k^2$, that is $|sk - ks| < r(s^2 - k^2)$ for every $r \in \mathbb{Q}^+$. Since s and k are in V , it follows that $s^2 = ss^* < r$,

and $-k^2 = kk^* < r_2$ for some rationals r_1 and r_2 . Thus $|sk-ks| < r(r_1+r_2)$ for every $r \in \mathbb{Q}^+$ and, hence, $sk-ks \in J$. Assume, on the other hand, that s^2 and $-k^2$ form an archimedean equivalent pair, meaning that there exist rational numbers r_1 and r_2 such that $s^2 > -r_1 k^2$ and $-k^2 > r_2 s^2$. If s or k is a non-unit, then there is nothing more to show. If on the other hand, both s and k are units, then by Theorem (2.1.13), follows $|s|$ and $-k^2$ are archimedean equivalent. Thus, one can find a rational q_0 with $0 < |s| + q_0 k^2 < -k^2$. If $|s|$ is replaced by $|s| + q_0 k^2$ this does not change the commutator $sk-ks$ nor does it affect the assumption on s and k . If, further, we suppose that $-k^2 \leq \frac{1}{2}$ then,

$$\begin{aligned}
 |sk-ks| &= |(|s| + q_0 k^2)k - k(|s| + q_0 k^2)| \\
 &\leq (|s| + q_0 k^2)^2 - k^2 && \text{(Remark (2.1.4))} \\
 &\leq (|s| + q_0 k^2) + \varepsilon - k^2 && \text{(Remark (2.1.9))} \\
 &\leq -k^2 + \varepsilon - k^2 \leq -3k^2
 \end{aligned}$$

where ε is any rational smaller than $-k^2$. Replacing k by rk with r any rational between 0 and 1 and eliminating shows that $|sk-ks| < -k^2$. Hence, $sk-ks \in J$. For general k , it suffices to pass to $k/2r$, where r is any rational such that $-k^2 \leq r^2$. $\square$

Since by the preceding lemma each symmetric element in the residue division ring $\bar{D}$ commutes with each skew-symmetric element, it follows that all the symmetric elements are central and, hence, either $\bar{D}$ is a field or a normal quaternions. This is the

Corollary (2.1.12): The residue division ring $\bar{D} = V/J$ is either a field or a normal quaternionic division ring (normality means that each symmetric element is central).

Theorem (2.1.13): In any Jordan ordered division ring D , the residue division ring $\bar{D} = V/J$ carries a Jordan ordering which is archimedean.

Proof: Let $\bar{\pi} = \{\bar{s} = s+J/s \in V-J, s = s^* > 0 \text{ in } D\}$. We claim that $\bar{\pi}$ is a Jordan domain of positivity of $\bar{D}$. That axioms (1), (2), (4) and (5) hold true is evident. We are left with axioms (3) and (6).

Axiom (3): $\bar{\pi}$ is closed under addition.

Let $\bar{s}, \bar{d} \in \bar{\pi}$. Then $\bar{s} + \bar{d} = \overline{(s+d)}$, and $s+d > 0$ in D . We claim that $s+d \notin J$. For since s and d are not in J , it follows that $s > r_1$ and $d > r_2$ for some r_1 and r_2 in $\mathbb{Q}^+$. Then $s+d > r_1+r_2$, that is $s+d \notin J$. Therefore $\bar{s} + \bar{d} > 0$ in $\bar{D}$.

Axiom (6): To show that if $0 \neq \bar{s} = \bar{s}^*$ in $\bar{D}$, either $\bar{s} \in \bar{\pi}$ or $-\bar{s} \in \bar{\pi}$. First we verify that every symmetric $\bar{a}$ in $\bar{V}$ comes from some symmetric s in V . From $a+J = a^*+J$, follows $a-a^* \in J$. Take $s = \frac{1}{2}(a+a^*)$, then s is a symmetric element and $a-s = \frac{1}{2}(a-a^*) \in J$. Hence $a+J = s+J$, as wished. Since $-s > 0$ or $-s > 0$ in D the result follows.

It remains to show that $\bar{D}$ contains no non-zero infinitesimals.

For, if $s = s^* \in D$ is such that $0 < s+J < r+J$, for every $r \in \mathbb{Q}^+$, then $0 < s < r$ for every such r , so that $s \in J$, a contradiction.

Our next objective is to establish that V is a $*$ -valuation ring in the sense given in §1.5. Here is to begin with one additional notation.

Notation: Throughout the rest of this chapter, let $[a,b]$ stand for the multiplicative commutator of a and b . In symbols;

$$[a,b] = aba^{-1}b^{-1}.$$

Theorem (2.1.14): In any Jordan ordered division ring D , the following holds true:

- (1) if $s > 0$ and $d \neq d^*$ in D , then $dsd > 0$;
- (2) if $s > 0$ and $d = d^*$ in D , then $dsd^{-1} + d^{-1}sd > 0$;
- (3) if $s = s^*$ and $d = d^*$ in D , then $(sd)^2 + (ds)^2 > 0$; and
- (4) if $s = s^*$ and $d = d^*$ in D , then $[s,d] + [s,d]^* > 0$.

Proof: (1) Since dsd is a symmetric element, then either $dsd > 0$ or $dsd < 0$. Assume that $dsd < 0$. If $d > 0$, then from $-dsd > 0$, follows $(-dsd)d^{-1} + d^{-1}(-dsd) > 0$, so that $ds + sd < 0$ for $s, d > 0$. This is a contradiction. If, on the other hand $d < 0$, then replacing d by $-d$ will not change dsd , so that $dsd > 0$, for every $d = d^*$ in D .

(2) From $dsd > 0$, and $d^{-2} > 0$ follows that $d^{-1}sd + dsd^{-1} = d^{-2}(dsd) + (dsd)d^{-2} > 0$, as wished.

(3) Assume that $s > 0$. Then $dsd > 0$, which implies $(ds)^2 + (sd)^2 = (dsd)s + s(ds d) > 0$. Replacing s by $-s$ will not change $(ds)^2 + (sd)^2$, so that $(ds)^2 + (sd)^2 > 0$ for every s and d in D .

(4) From (1) and (3) follows $[s,d] + [s,d]^* = d^{-1}\{s^{-1}((sd)^2 + (ds)^2)s^{-1}\}d^{-1} > 0$, as wished. $\square$

Throughout the rest of this section D will consistently denote a Jordan ordered division ring.

Lemma (2.1.15): For each non-zero symmetric ϕ in D , and each $0 < s = s^* \in J$, follows $dsd^{-1} + d^{-1}sd \in J$.

Proof: Since $s \in J$, it follows that $s < r$ for every $r \in Q^+$. Hence, by Theorem (2.1.14), follows $d(r-s)d^{-1} + d^{-1}(r-s)d > 0$, so that $dsd^{-1} + d^{-1}sd < 2r$ for every r . Therefore $dsd^{-1} + d^{-1}sd \in J$. $\square$

Lemma (2.1.16): For each unit (resp. infinitesimal) $s = s^*$ in V , and each $d = d^*$ in D , follows dsd^{-1} is a unit (resp. an infinitesimal) in V .

Proof: Let s be a unit of V , and assume that $dsd^{-1} \notin V$. Then $ds^{-1}d^{-1} \in J$, so that $ds^{-1}d^{-1} + d^{-1}s^{-1}d \in J$. Since s^{-1} is also a unit, it follows that $s^{-1} \geq r$ for some rational r . Then, by Theorem (2.1.14), $d(s^{-1}-r)d^{-1} + d^{-1}(s^{-1}-r)d > 0$, so that $ds^{-1}d^{-1} + d^{-1}s^{-1}d > 2r$, a contradiction. Hence $dsd^{-1} \in V$.

Now, if s is an infinitesimal in V , it follows by Lemma (2.1.15), that $dsd^{-2}sd + d^{-1}sd^2sd^{-1} = (dsd^{-1} + d^{-1}sd)^2 - (ds^2d^{-1} + d^{-1}s^2d)$ is in J . Hence $xx^* + x^*x \in J$, where $x = dsd^{-1}$. Since $xx^* + x^*x$ is symmetric, and since xx^*, x^*x are positive, it follows that $xx^* \in J$. Therefore $x \in J$. $\square$

Lemma (2.1.17): For each unit (resp. infinitesimal) $k = -k^*$ in V , and each $d = d^*$ in D , follows dkd^{-1} is a unit (resp. an infinitesimal) in V .

Proof: If k is a unit, then k^2 is a symmetric unit. Hence, by Lemma (2.1.16), follows $(dkd^{-1})^2 = dk^2d^{-1}$ is a unit. Suppose that $dkd^{-1} \in J$. Then $(dkd^{-1})^2 \in J$, which is a contradiction. Also, if $dkd^{-1} \notin V$, it follows that $(dkd^{-1})^{-1} \in V$, so that $dkd^{-1} =$

$(dkd^{-1})^{-1} \cdot (dkd^{-1})^2 \in V$. Hence dkd^{-1} is a bounded element, which is not an infinitesimal. Therefore dkd^{-1} is a unit. For k an infinitesimal, use similar argument. $\square$

Lemma (2.1.18): For each $0 \neq x \in D$, the element $x^{-1}x^*$ is a unit of V .

Proof: We have, for $x+x^* \neq 0$,

$$\begin{aligned} (x+x^*)x^{-1}x^*(x+x^*)^{-1} &= x(1+x^{-1}x^*)x^{-1}x^*(1+x^{-1}x^*)^{-1}x^{-1} \\ &= xx^{-1}x^*x^{-1} = x^*x^{-1}. \end{aligned}$$

If then $x^{-1}x^* \in J$, clearly $x+x^* \neq 0$ follows, so that $x^*x^{-1} \in (x+x^*)J(x+x^*)^{-1} \subseteq J$ (Lemma (2.1.16) and (2.1.17)). However, $(x^{-1}x^*)^{-1} = (x^*)^{-1}x = (x^*x^{-1})^*$. Thus $((x^{-1}x^*)^{-1})^* = x^*x^{-1} \in J$, and consequently, $(x^{-1}x^*)^{-1} \in J$, which contradicts the relation $x^{-1}x^* \notin J$. This shows that $x^{-1}x^* \notin J$. Similarly $(x^{-1}x^*)^{-1} = (x^*)^{-1}x = (x^*)^{-1}(x^*)^* \notin J$ follows. Therefore $x^{-1}x^*$ is a unit of V . $\square$

Theorem (2.1.19): Let D be a Jordan ordered division ring with involution $*$. Then the bounded subring V is closed under conjugation. In fact, V is a $*$ -valuation subring.

Proof: As observed in [10], $aba^{-1}b^{-1}$ is a product of at most three elements of the form x^*x^{-1} :

$$aba^{-1}b^{-1} = (a^*)^*(a^*)^{-1} \cdot (b^*a)^*(b^*a)^{-1} \cdot b^*b^{-1}.$$

Then, by Lemma (2.1.18), it follows that V contains all multiplicative commutators. Now, if $y \in V$, then $(xyx^{-1}y^{-1})y \in V$, or $xyx^{-1} \in V$, for every $x \in D$. $\square$

Since the bounded subring V of a Jordan ordered division ring D is a $*$ -valuation subring of D , it follows by Theorem (1.5.3), that there exists a $*$ -valuation w with valuation ring precisely the bounded ring V ; we call such valuation, an order valuation. We will show, in fact, that w is compatible with the ordering according to the following definition.

Definition (2.1.20): Call a $*$ -valuation w compatible with the Jordan ordering if for each pair of non-zero symmetrics s and d ,

$$0 < s < d \Rightarrow w(s) \geq w(d).$$

Theorem (2.1.21): For every Jordan ordered division ring D , the order valuation w is compatible with the ordering.

Proof: Let s, d be a pair of non zero symmetrics such that $d > s > 0$. Then, $d-s > 0$, so that, $s(d-s)+(d-s)s > 0$, and, hence, $sd+ds > 2s^2$. Also $d(d-s)+(d-s)d > 0$, which gives $2d^2 > sd+ds$. Thus, $2d^2 > sd+ds > 2s^2$. Hence, $1 > d^{-1}s^2d^{-1} = (d^{-1}s)(d^{-1}s)^*$ (use Theorem (2.1.14)), so that $d^{-1}s \in V$. Therefore, $w(d^{-1}s) \geq 0$ or $w(s) \geq w(d)$. $\square$

Corollary (2.1.22): For every Jordan ordered division ring D , if s and d are symmetric elements with $s > 0$ and $w(d) > w(s)$, then $s+d > 0$.

Proof: Since $s+d$ is a symmetric element it follows that either $s+d > 0$ or $s+d < 0$. Assume $s+d < 0$. Then $0 < s < -d$, so that, $w(s) \geq w(-d) = w(d)$ (Theorem (2.1.21)), a contradiction. Therefore, $s+d > 0$. $\square$

2.2 Congruence law:

Throughout this section we will be dealing with a Jordan ordered division ring D with involution $*$. The objective of this section is to establish the cited congruence law in the introduction.

The ideal J of non-units of the valuation ring V gives rise to the normal subgroup $1+J$ of the group $D^\times$ of non-zero elements of D . The congruence of this group determined by the normal subgroup $1+J$, we will consistently denote by $\equiv$. In symbols,

$$\begin{aligned} a \equiv b &\iff ab^{-1} \in 1+J \\ &\iff a = b(1+j), \text{ some } j \in J. \end{aligned}$$

In the case a or b are units of the valuation ring V , then evidently $a \equiv b \iff a \equiv b \pmod{J} \iff a-b \in J$. It is also appropriate to add that $\equiv$ is compatible with $*$ in that $a \equiv b \implies a^* \equiv b^*$.

To be precise, we will establish in this section that if $s = s^*$, and $x \in D^\times$, then $sxs^{-1} \equiv x \pmod{J}$.

Theorem (2.2.1): For each $s = s^* \in V$, and each $d = d^* \in D$, $dsd^{-1} \equiv s \pmod{J}$ follows.

Proof: Let s be a unit in V . This means that $r_1 \geq s \geq r_2$ for some positive rationals r_1 and r_2 . Then, by Theorem (2.1.14), $r_1 \geq \frac{1}{2}(dsd^{-1} + d^{-1}sd) \geq r_2$ follows.

Hence s and $\frac{1}{2}(dsd^{-1} + d^{-1}sd)$ are symmetric bounded elements which admit the same rational upper and lower bounds, so that by [3, Theorem 5],

$$\frac{1}{2}(dsd^{-1} + d^{-1}sd) \equiv s \pmod{J}.$$

Now, since the symmetric elements of V/J are central, then the product $s_1 s_2$ of two symmetric bounded elements s_1 and s_2 is symmetric. Replacing s by $s_1 s_2$, this gives

$$\begin{aligned} 2s_1 s_2 &\equiv ds_1 s_2 d^{-1} + d^{-1} s_1 s_2 d \\ &= (ds_1 d^{-1})(ds_2 d^{-1}) + (d^{-1} s_1 d)(d^{-1} s_2 d) \\ &\quad + 2(ds_1 d^{-1})(d^{-1} s_2 d) - 2(ds_1 d^{-1})(d^{-1} s_2 d) \\ &= (ds_1 d^{-1})(ds_2 d^{-1} + d^{-1} s_2 d) + (d^{-1} s_1 d + ds_1 d^{-1})d^{-1} s_2 d \\ &\quad - 2(ds_1 d^{-1})(d^{-1} s_2 d) \\ &\equiv (ds_1 d^{-1} \cdot 2s_2 + 2s_1 \cdot d^{-1} s_2 d - 2(ds_1 d^{-1})(d^{-1} s_2 d)) \end{aligned}$$

$$\text{Thus } s_1 s_2 \equiv ds_1 d^{-1} s_2 + s_1 d^{-1} s_2 d - (ds_1 d^{-1})(d^{-1} s_2 d)$$

$$\text{so that } (s_1 - ds_1 d^{-1})s_2 + (ds_1 d^{-1} - s_1)(d^{-1} s_2 d) \equiv 0$$

$$(s_1 - ds_1 d^{-1})(s_2 - d^{-1} s_2 d) \equiv 0$$

Since s_1 and s_2 are arbitrary symmetric elements, it follows that $dsd^{-1} \equiv s$. $\square$

Lemma (2.2.2): For each $s = s^*$ and $d = d^*$ in D , follows $[s, d] \not\equiv -1 \pmod{J}$.

Proof: Suppose that $[s, d] \equiv -1$. Then $[s, d] = -1 + j$ for some $j \in J$. By Theorem (2.1.14), follows $[s, d] + [s, d]^* > 0$, so that $-2 + j + j^* > 0$. Hence $j + j^* > 2$. Since $j + j^*$ is a symmetric element, this contradicts the fact that $j + j^*$ is an infinitesimal. Thus $[s, d] \not\equiv -1$. $\square$

Remark (2.2.3): [5] For $a, b \in D^X$, if $[a^2, b] \equiv 1$ and $[a, b] \neq 1$, then $[a, ab-ba] = -1$.

Proof: Let b^a stand for the conjugate of b , $b^a = aba^{-1}$.

Since $[xy, z] = [y, z]^X [x, z]$, it follows that

$$1 = [a^2, b] = [a, b]^a [a, b].$$

Put $t = [a, b]$, then $t^a = t^{-1}$. Now,

$$ab-ba = (aba^{-1}b^{-1}-1)ba$$

so that, $[a, ab-ba] = [a, (aba^{-1}b^{-1}-1)ba]$.

$$= [a, (t-1)ba]$$

$$= [a, t-1][a, ba]^{(t-1)}$$

$$= [a, t-1][a, b]^{(t-1)}[a, a]^{t-1} \quad (\text{where } [x, yz] = [x, y][x, z]^y)$$

$$= a(t-1)a^{-1}(t-1)^{-1}(t-1)t(t-1)^{-1}$$

$$= (ata^{-1}-1)t(t-1)^{-1}$$

$$= (t^{-1}-1)t(t-1)^{-1}$$

$$= (1-t)(t-1)^{-1} = -1.$$

Lemma (2.2.4): For each $s = s^*$ and $k = -k^*$ in D , $[s, k] \neq 1$ follows.

Proof: If $[s, k] \equiv -1$, then by Remark (2.2.3), $[s, sk-ks] \equiv -1$ follows, which contradicts Lemma (2.2.2). $\square$

Lemma (2.2.5): For each $s = s^*$ and $k = -k^* \in D^X$, where k is a unit, $[s, k] \neq 1$ follows.

Proof: Since k^2 is a symmetric Bounded element, it follows by Theorem (2.2.1) that, $[s, k^2] \equiv 1$. Assume that $[s, k] \neq 1$. Then, by Remark (2.2.3), $[k, sk - ks] \equiv -1$ follows, which contradicts Lemma (2.2.4). Therefore $[s, k] \equiv 1$.

Theorem (2.2.6): For each $s = s^*$ and $x \in D^X$ where x is a unit, $[s, x] \equiv 1$ follows.

Proof: Assume that $x = d + k$ is a unit in V , where $d = d^*$ and $k = -k^*$. Then d and k are bounded elements. If k is an infinitesimal, then $x \equiv d \pmod{J}$ and by Theorem (2.2.1) follows

$$sxs^{-1} = sds^{-1} + sks^{-1} \equiv d \equiv x;$$

that is, $[s, x] \equiv 1$. Also, if d is an infinitesimal the result will follow from Lemma (2.2.5). Finally, if d and k are units, then

$$sxs^{-1} = sds^{-1} + sks^{-1} \equiv d + k \equiv x.$$

Theorem (2.2.7): Each commutator $[a, b] = aba^{-1}b^{-1}$, where $a, b \in D^X$, is a unitary element modulo J .

Proof: Since $[a, b] = (a^*)^*(a^*)^{-1}(b^*a)^*(b^*a)^{-1}b^*b^{-1}$, it suffices then to show the theorem for $[a, b]$ replaced with x^*x^{-1} , where $x \in D^X$. Now, if $x + x^* \neq 0$, then

$$\begin{aligned} (x+x^*)x^{-1}x^*(x+x^*)^{-1} &= x(1+x^{-1}x^*)x^{-1}x^*(1+x^{-1}x^*)^{-1}x^{-1} \\ &= xx^{-1}x^*x^{-1} = x^*x^{-1}. \end{aligned}$$

Since $x^{-1}x^*$ is a unit, it follows from Lemma (2.2.6) that $x^*x^{-1} = (x+x^*)x^{-1}x^*(x+x^*)^{-1} \equiv x^{-1}x^*$. Hence $x^*x^{-1}x^*x^{-1} \equiv 1$, that is, $(x^*x^{-1})(x^*x^{-1})^* \equiv 1$, or x^*x^{-1} is a unitary element modulo J .

If, on the other hand, $x+x^* = 0$, then $x^* = -x$ so that $x^*x^{-1} = -1$ a unitary element. $\square$

Lemma (2.2.8): For every natural number n , and for each $s = s^*$ and $d = d^*$ in D^* , $[s,d]^{2^n} + ([s,d]^*)^{2^n} > 0$ follows.

Proof: Since $[s,d]$ is a unit, it follows by Theorem (2.2.6) that $[s^2,d] = [s,d]^s[s,d] = [s,d][s,d] = [s,d]^2$, that is, $[s^2,d] = [s,d]^2$. However, by Theorem (2.1.16), follows $[s^2,d] + [s^2,d]^* > 0$, so that, $[s,d]^2 + ([s,d]^*)^2 > 0$. By induction, $[s^{2^n},d] = [s,d]^{2^n}$. Since $[s^{2^n},d] + [s^{2^n},d]^* > 0$, it follows that $[s,d]^{2^n} + ([s,d]^*)^{2^n} > 0$. $\square$

Theorem (2.2.9): For each $s = s^*$ and $d = d^*$ in D , $[s,d]$ follows.

Proof: Let $c = [s,d] = a+b$, where $a = a^*$ and $b = -b^*$. By Theorem (2.2.7), c is a unitary element modulo $\mathcal{U}$, that is, $cc^* \equiv 1$. Then

$$\begin{aligned} 1 \equiv cc^* &= a^2 - b^2 + (ba-ab) \\ &\equiv a^2 - b^2. \end{aligned}$$

If $a^2 \equiv 1$, then $-b^2 \equiv 0$ giving $c \equiv a \equiv \pm 1$. Since $a = \frac{1}{2}(c+c^*) > 0$, it follows that $c \equiv 1$, as wished. Suppose now that $a^2 \not\equiv 1$. Then $-b^2 \not\equiv 0$. From $1 \equiv a^2 - b^2$ follows that $a^2 \leq \frac{1}{1+r}$ for some positive rational r . Put $q = \frac{1}{1+r}$. From $q - a^2 \geq 0$ and Theorem (2.1.16) follows $a(q - a^2)a \geq 0$, that is, $qa^2 - a^{2^2} \geq 0$. Then $a^{2^2} \leq qa^2 < q^2$. By induction on n follows $a^{2^n} \leq q^{2^n}$. Accordingly there is no positive rational q_0 such that $a^{2^n} > q_0$ for each natural

number n . We proceed to show that this is a contradictory (to the assumption $a^2 \neq 1$). For put $c^{2^n} = a_n^2 + b_n^2$, where n is a natural number, $a_n^* = a_n$ and $b_n^* = -b_n$. From $c^{2^n} + (c^{2^n})^* > 0$, (Lemma (2.2.8)), it follows that $a_{n+1}^2 = a_n^2 + b_n^2 \leq a_n^2$. By induction on n follows $a_n \leq a^{2^n}$. Since $a_{n+1} > 0$, it follows that $a_n^2 + b_n^2 > 0$ so that $a_n^2 > -b_n^2$. Since $c^{2^n} = [s, d]^{2^n} \equiv [s^{2^n}, d]$ (see the proof of Lemma (2.2.8)), it follows that c^{2^n} is a unitary modulo J . From $c^{2^n} (c^*)^{2^n} \equiv 1$ follows $a_n^2 - b_n^2 \equiv 1$ giving $2a_n^2 > a_n^2 - b_n^2 \equiv 1$ and, hence, $2a_n^2 > 1 - \varepsilon$ for any positive rational ε . Thus $a_n^2 > \frac{1}{4}$. From $a_n > 0$ and $a_n^2 > \frac{1}{4}$ follows $a_n > \frac{1}{2}$ giving $a^{2^n} \geq a_n > \frac{1}{2}$, which is a contradiction. $\square$

Lemma (2.2.10): For each $s = s^*$ and $k = -k^*$ in D , $[s, k] \equiv 1$ follows.

Proof: Since k^2 is a symmetric element, it follows by Theorem (2.2.9) that, $[s, k^2] \equiv 1$. Assume that $[s, k] \neq 1$. Then by Remark (2.2.3), follows $[k, sk - ks] \equiv -1$, which contradicts Lemma (2.2.4). Thus $[s, k] \equiv 1$. $\square$

Theorem (2.2.11). Let D be any Jordan ordered division ring. For each $s = s^*$ and $x \in D^X$ follows $[s, x] \equiv 1$.

Proof: From $x(1+x^{-1}x^*) = x+x^*$, $x \in D^X$, follows $x = (x+x^*)(1+x^{-1}x^*)^{-1}$. Similarly, $x = (x-x^*)(1-x^{-1}x^*)^{-1}$. By Lemma (2.1.20), $x^{-1}x^*$ is a unit, so that one of the elements $1+x^{-1}x^*$, $1-x^{-1}x^*$ is a unit of V . Then one of the elements $(1+x^{-1}x^*)^{-1}$, $(1-x^{-1}x^*)^{-1}$ is a unit of V . Hence x can be written either as a product of a symmetric element and a unit, or as a product of a skew symmetric element

and a unit. Since $[s, x] \in 1$ for x a unit, a symmetric and a skew symmetric element, it follows that $[s, x] \in 1^2$ for all $x \in D^*$. $\square$

On the positive side, the congruence law $[s, x] \in 1 \pmod{1+J}$, where $s = s^*$ and x are arbitrary in D , opens the way to a number of interesting results, some of which will be established in the next section. On the negative side, this congruence law shows that there exist no finite dimensional division rings that can be Jordan ordered apart, of course, from the normal quaternionic division ring and fields. This is a consequence of the following theorem which is due to Professor Chacron.

Theorem (2.2.12): [5]: Let D be any division ring with $*$ -valuation w such that $w(n) = 0$ for each integer n . Assume that for each pair of non-zero symmetrics s and d , $[s, d] \in 1 \pmod{1+J}$. Then D is one of the following types:

- (i) D is a field;
- (ii) D is a normal quaternionic division ring,
- (iii) D is a purely transcendental extension of its centre.

Corollary (2.2.13): Any Jordan ordered division ring D contains no algebraic elements over its centre, except when D is a field or D is 4-dimensional and every symmetric element is central.

2.3 Extending a Jordan ordering

In [10] and [11], Holland studied two types of orderings of a division ring D with involution $*$, Baer and strong orderings. In this section we investigate the connection between the Jordan ordering

and these types of orderings. We shall establish that any Jordan ordering is a Baer ordering. We shall also establish that any Jordan ordering can be extended to a strong ordering. In fact, we show that there is a largest strong ordering extending the given Jordan ordering.

Definition (2.3.1) [10]: A division ring D with involution $*$, is called Baer ordered if it contains a subset π such that: (1) π consists solely of symmetric elements, (2) π contains 1 but not 0, (3) π is closed under sum, (4) Along with s , π contains every xsx^* where $0 \neq x \in D$ and, (5) π contains either s or $-s$, for every non-zero symmetric s in D .

Lemma (2.3.2): Let D be a Jordan ordered division ring with involution and let w be the order valuation. If $s = s^*$ and $0 \neq x \in D$, then

$$(1) \quad w(sx+xs) = w(x) + w(s), \text{ and}$$

$$(2) \quad w(sx-xs) > w(x) + w(s).$$

Proof: (1) By construction,

$$w(sx+xs) \geq w(s) + w(x).$$

Assume that $w(sx + xs) > w(s) + w(x)$. Then

$$w(sx + xs) - w(s) - w(x) > 0$$

$$w(sx + xs) + w(s^{-1}) + w(x^{-1}) > 0$$

$$w((sx + xs)s^{-1}x^{-1}) > 0$$

$$w([s, x] + 1) > 0,$$

that is, $[s, x] + 1 \in J$ and, hence, $[s, x] \equiv -1 \pmod{1+J}$, a contra-

diction. Therefore $w(sx+xs) = w(s) + w(x)$.

(2) Since $[s,x] \in J$ (Theorem (2.2.11)), it follows that $[s,x]-1 \in J$, so that, $w([s,x]-1) > 0$. Then $w((sx-xs)s^{-1}x^{-1}) > 0$ which gives $w(sx-xs) > w(s) + w(x)$. 7

Theorem (2.3.3): Let $\geq$ be a Jordan ordering of a division ring D with involution $*$. Then $\geq$ satisfies the Baer axiom: if $s > 0$, then $x*sx > 0$, $0 \neq x \in D$. Hence, $\geq$ is a Baer ordering of D .

Proof: Assume $x*sx < 0$ for some $s > 0$. Since $sx = xs[s^{-1},x^{-1}]$, it follows that $x*xs[s^{-1},x^{-1}] < 0$. By Theorem (2.2.11), it follows $[s^{-1},x^{-1}] = 1+j$ for some $j \in J$. Hence $x*xs(1+j) < 0$, so that, $x*xs(1+j) + (1+j*)sx*x < 0$. Thus $(x*xs + sx*x) + (x*xsj + j*sx*x) < 0$, that is,

$$0 < x*xs + sx*x < -(x*xsj + j*sx*x).$$

Let w be the $*$ -valuation associated to $\geq$. By Theorem (2.1.22), it follows $w(x*xsj + j*sx*x) \leq w(x*xs + sx*x)$. Since $w(x*xs + sx*x) = 2w(x) + w(s)$ (Lemma (2.3.3)), it follows that $2w(x) + w(s) + w(j) \leq 2w(x) + w(s)$ and, hence, $w(j) \leq 0$, a contradiction. Thus $x*sx > 0$. []

The converse of Theorem (2.3.3) is not true in general, since not every Baer ordering is a Jordan ordering. For instance, in [10], Holland constructed odd-dimensional division rings which are Baer orderable. We know from Corollary (2.2.13), that such division rings cannot be Jordan ordered. We proceed to establish the converse of Remark (2.1.2). This is the

Theorem (2.3.4): Let D be a Jordan ordered division ring, and let w be the order valuation. Let P be the subset of D formed by all elements a of the form $a = s+k$, where $s = s^* > 0$, $k = -k^*$ and $w(k) > w(s)$. Then P induces a strong ordering of D , which extends the given Jordan ordering.

Proof: Clearly the subset P contains the domain of positivity of the given Jordan ordering. Hence P contains every xx^* , $0 \neq x \in D$, and, for each non-zero symmetric s , P contains either s or $-s$. We need to show that P is a $*$ -closed normal subgroup of the multiplicative group of D , that is closed under sum.

(1) P is $*$ -closed: For let $a \in P$. Then $a^* = s-k$, where $s > 0$ and $w(-k) = w(k) > w(s)$. Hence $a^* \in P$.

(2) P is closed under sum: Let $a_1, a_2 \in P$ where $a_1 = s_1 + k_1$ and $a_2 = s_2 + k_2$. Then $s_1, s_2 > 0$, $w(k_1) > w(s_1)$ and $w(k_2) > w(s_2)$. Now, $a_1 + a_2 = (s_1 + s_2) + (k_1 + k_2)$ where $s_1 + s_2 > 0$. Then it remains to show that $w(k_1 + k_2) > w(s_1 + s_2)$. Since $s_1 + s_2 > s_1 > 0$, and since w is compatible with the Jordan ordering, it follows that $w(s_1) \leq w(s_1 + s_2)$. Similarly $w(s_2) \leq w(s_1 + s_2)$ so that,

$$w(s_1 + s_2) \leq \min \{w(s_1), w(s_2)\}.$$

By construction,

$$w(s_1 + s_2) \geq \min \{w(s_1), w(s_2)\}$$

and hence,

$$w(s_1 + s_2) = \min \{w(s_1), w(s_2)\}.$$

Thus, $w(k_1 + k_2) \geq \min \{w(k_1), w(k_2)\}$

$$> \min \{w(s_1), w(s_2)\} = w(s_1 + s_2).$$

(3) P is closed under product: Let $a_1, a_2 \in P$ as in (2). Then,

$$\begin{aligned} a_1 a_2 &= (s_1 + k_1)(s_2 + k_2) \\ &= (s_1 s_2 + s_1 k_2 + k_1 s_2 + k_1 k_2) \end{aligned}$$

The symmetric part of $a_1 a_2$ is $s = \frac{1}{2}\{(s_1 s_2 + s_2 s_1) + (s_1 k_2 + k_2 s_1) + (k_1 s_2 + s_2 k_1) + (k_1 k_2 + k_2 k_1)\}$, and the skew symmetric part is $k = \frac{1}{2}\{(s_1 s_2 - s_2 s_1) + (s_1 k_2 + k_2 s_1) + (k_1 s_2 + s_2 k_1) + (k_1 k_2 - k_2 k_1)\}$.

First, we show that $s > 0$. By Lemma (2.3.2), follows $w(s_1 k_2 + k_2 s_1) > w(s_1) + w(k_2) > w(s_1) + w(s_2) = w(s_1 s_2 + s_2 s_1)$. Since $s_1 s_2 + s_2 s_1 > 0$, it follows by Corollary (2.1.22), that $(s_1 s_2 + s_2 s_1) + (s_1 k_2 + k_2 s_1) > 0$. Repeating this process for the remaining two terms in s , we get $s > 0$. Next, we claim that $w(k) > w(s)$. For,

$$\begin{aligned} w(k) &> \min\{w(s_1 s_2 - s_2 s_1), w(s_1 k_2 + k_2 s_1), \\ &\quad w(k_1 s_2 + s_2 k_1), w(k_1 k_2 + k_2 k_1)\} \\ &> \min\{w(s_1) + w(s_2), w(s_1) + w(k_2), \\ &\quad w(k_1) + w(s_2), w(k_1) + w(k_2)\} \end{aligned}$$

Since each term on the right hand side is strictly greater than $w(s_1) + w(s_2)$, it follows that $w(k) > w(s_1) + w(s_2)$. Finally, we show that $w(s) = w(s_1) + w(s_2)$. Since

$$w(s_1 s_2 + s_2 s_1) = w(s_1) + w(s_2) ,$$

$$w(s_1 k_2 + k_2 s_1) > w(s_1) + w(k_2) > w(s_1) + w(s_2) ,$$

$$w(k_1 s_2 + s_2 k_1) > w(k_1) + w(s_2) > w(s_1) + w(s_2) ,$$

and $w(k_1 k_2 + k_2 k_1) > w(k_1) + w(k_2) > w(s_1) + w(s_2) ,$

it follows that $w(s) = \text{Min} \{w(s_1 s_2 + s_2 s_1), w(s_1 k_2 - k_2 s_1), w(k_1 s_2 - s_2 k_1), w(k_1 k_2 + k_2 k_1)\} = w(s_1 s_2 + s_2 s_1) = w(s_1) + w(s_2)$.

(4) P contains inverses: If $a \in P$, then $a^* \in P$. Since $(a^{-1})^*(a^{-1}) \in P$, and P is closed under product, it follows that $a^{-1} = a^* \cdot ((a^{-1})^*(a^{-1})) \in P$.

(5) P is preserved under conjugation: Let $a = s+k \in P$. Since $xsx^* > 0$, for each $0 \neq x \in D$, (Theorem 2.3.3), and $w(xkx^*) = 2w(x) + w(k) > 2w(x) + w(s) = w(xsx^*)$, it follows that $xax^* = xsx^* + xkx^* \in P$. Now, since $(x^{-1})^* x^{-1} \in P$ for every $0 \neq x \in D$, and P is closed under product, it follows that

$$xax^{-1} = (xax^*)((x^{-1})^* x^{-1}) \in P,$$

for every $0 \neq x \in D$. □

One might ask whether the change from the given Jordan ordering to the extended strong ordering can affect the order valuation. Since the ring of bounded elements V is defined in terms of symmetric elements only (the norms), and since the orderings agree on the additive subgroup of symmetric elements, it follows that the order valuations of the Jordan ordering and the extended strong ordering are one and the same valuation. This is the

Lemma (2.3.5): Let D be a Jordan ordered division ring. Then, the order valuations of the given Jordan ordering and the extended strong ordering are the same.

Theorem (2.3.6): Let D be a strongly ordered division ring with involution $*$. For every positive element a , $a \equiv a^* \pmod{1+J}$

follows.

Proof: From $a > 0$, it follows that a^{-1} and $a^* > 0$, so that $c = a^*a^{-1} > 0$. Thus, $c^{2^n} + (c^{2^n})^* > 0$, for every positive integer n . Also c is a unitary element modulo J (see the proof of Theorem (2.2.7)). Using similar arguments as in the proof of Theorem (2.2.9), one can prove that $c \equiv 1$, that is $a \equiv a^* \pmod{1+J}$. $\square$

Corollary (2.3.7): In a strongly ordered division ring D , each positive element a is of the form $a = s+k$, where $s = s^* > 0$, $k = -k^*$ and $w(k) > w(s)$.

Proof: Since $2s = a+a^* > 0$, it follows that $s > 0$. From Theorem (2.3.6), $a^*a^{-1}-1 \in J$ follows, that is, $w(a^*a^{-1}-1) > 0$. Since $a^*a^{-1}-1 = (a^*-a)a^{-1} = -2ka^{-1}$, it follows that $w(k) = w(a) = w(-2ka^{-1}) = w(a^*a^{-1}-1) > 0$, so that $w(k) > w(a)$. If $w(s) \geq w(k)$, then $w(a) \geq \min\{w(s), w(k)\} = w(k)$, which is a contradiction. Thus $w(s) < w(k)$. $\square$

From Corollary (2.3.7) and Theorem (2.3.4) follows that any Jordan ordering extends uniquely to a strong ordering, which is the largest strong ordering that contains the given Jordan ordering.

To conclude this section we show that Jordan orderability scales in the sense given in §1.6.

Theorem (2.3.8): Let D be a Jordan ordered division ring with domain of positivity π . Let $x^\# = \rho x^* \rho^{-1}$, where ρ is a non-zero symmetric element in D . Then, relative to the involution $\#$, D has a Jordan ordering with domain of positivity $\rho\pi$.

Proof: Without loss of generality, we may assume that $\rho \in \pi$.

We proceed to check the axioms in Definition (2.1.1).

Axioms (1) and (6): Since the additive subgroup of symmetric elements of D relative to $\#$ is ρS , where S is the additive subgroup of usual symmetric elements (Theorem (1.2.13)), evidently $\rho\pi$ consists solely of symmetric elements relative to $\#$. By the same token, if d is a non-zero symmetric element relative to $\#$, then $d \in \rho\pi$ or $d \in -\rho\pi$.

Axiom (2): This is evident.

Axiom (3): This is also evident.

Axiom (4): Since $\rho^{-1} \in \pi$, it follows by Theorem (2.3.3), that $a * \rho^{-1} a \in \pi$, $0 \neq a \in D$. Then $a \# a = \rho a * \rho^{-1} a \in \pi$, $0 \neq a \in D$.

Axiom (5): Let $s, d \in \pi$. Since $\rho \in \pi$ it follows that $\rho s d + d \rho s$ is a positive element at the extended strong ordering. However, $\rho s d + d \rho s$ is a symmetric element and, hence, this element is in π . Thus, for ρs and ρd in $\rho\pi$, follows $(\rho s)(\rho d) + (\rho d)(\rho s) = \rho(\rho s d + d \rho s) \in \rho\pi$, as wished. $\square$

CHAPTER III

ELLIPTIC HERMITIAN SPACES

3.1. Linear transformations of a strongly elliptic space:

In this concluding chapter we will be dealing with a special case of an elliptic space V in the sense given in [4], where the ground valuation w is in fact the order valuation that is associated with a Jordan ordering and where the constant of ellipticity of the elliptic space V (see [4, page 8]), is equal ~~to~~ zero. In this way, we will be dealing with a hermitian space V verifying a generalized Cauchy-Schwarz inequality relative to the ground valuation. We will concentrate on the ring of linear transformations ϕ of the space V that increase lengths in the sense given in [4]. Most of the results that will be carried out in the first section are not new results, they are in fact derived in the referred work making use of specific techniques. However, the arguments that will be given here will make quicker use of the considered richer set up.

In the second section of this chapter, we will investigate something new; namely, the notion of a unitary transformation modulo the group of multiplicative infinitesimals. We give an explicit matrix criterion for a linear transformation to be unitary modulo the group of multiplicative infinitesimals.

We begin with recalling the meaning of some of the terms that will be used in this chapter. Hereafter, D stands for a division ring with $\text{char}(D) \neq 2$. Let $x \rightarrow x^*$ be any fixed involution of D . By

*-valuation we mean a mapping $w : D \setminus \{0\} \rightarrow G$, where G is a linearly ordered additive group, such that

- (1) $w(xy) = w(x) + w(y) \quad (x, y \in D).$
- (2) $w(x+y) \geq \min(w(x), w(y)).$
- (3) w maps $D \setminus \{0\}$ onto G .
- (4) $w(x^*) = w(x).$

Let

- (5) $R = \{x \in D / w(x) \geq 0\};$
- (6) $J = \{x \in D / w(x) > 0\}.$

Then R is the valuation ring in D . Indeed, R is a subring of D , which is preserved under conjugation. Every one sided ideal of R is 2-sided. $J = \text{Jac}(R)$ is the largest nonzero ideal $\neq R$ in R .

Inequality (2) carries over to algebraic sums:

$$(7) \quad w\left(\sum_{i=1}^n x_i\right) \geq \min_{i=1,2,\dots,n} (w(x_i)).$$

In the special case $w(x_i) \neq w(x_j)$ for every pair $\{i \neq j\}$, one has the equality

$$(8) \quad w\left(\sum_{i=1}^n (x_i)\right) = \min_{i=1,2,\dots,n} (w(x_i)).$$

Definition (3.1.1): Let V be any non-degenerate hermitian space over D . V is called elliptic space relative to the valuation w , if

$$(9) \quad 2w(u \cdot v) \geq w(u \cdot u) + w(v \cdot v)$$

for all $u, v \in V$. An elliptic space V is called strongly elliptic if it possesses some orthogonal basis $\{e_i\}_{i=1}^n$ such that $w(e_i \cdot e_i) = 0$

for all $i \in I$.

Example (3.1.2) (of a strongly elliptic space): Let I be the set of all integers. Let $V = D^I$ be the discrete direct sum formed by all functions $u : I \rightarrow D$ such that $u_i = u(i)$ is zero for all but a finite number of indices i , where addition and multiplication are defined as follows:

$$(u+v)(i) = u_i + v_i$$

$$(uv)(i) = u_i v_i$$

In particular, V contains the standard functions written $e_{(i)}$ where $e_{(i)}(j) = \delta_{ij} = 1$ or 0 depending on whether $j = i$ or $j \neq i$.

For inner product on V take the coordinate inner product

$$u \cdot v = \sum_i u_i v_i^*$$

Clearly this is a hermitian form on V for which $\{e_{(i)}\}$ is an orthonormal basis. We contend that V is elliptic in the special case

where w is the order valuation associated to a Jordan ordering of D .

Since $u \cdot u = \sum_i u_i u_i^* > 0$ (in D) for each non-zero element $u \in V$;

then as noted in [10, page 235], one has

$$0 < (u \cdot v)(v \cdot v)^{-1}(v \cdot u) \leq u \cdot u$$

for all non-zero $u, v \in V$. Taking w and using compatibility this gives

$$w(u \cdot v) - w(v \cdot v) + w(v \cdot u) \geq w(u \cdot u),$$

or

$$2w(u \cdot v) \geq w(u \cdot u) + w(v \cdot v)$$

Therefore V is a strongly elliptic space.

Example (3.1.3) (Diagonal space): Let $V = M_{FV}^0(D)$ be the ring of matrices having only finitely many non-zero entries. We define

$$\underline{u} \cdot \underline{v} = \sum_{i,j,k} u_{ik} v_{kj}^*,$$

where $\underline{u} = (u_{ij})_{i,j}$ and $\underline{v} = (v_{ij})_{i,j}$ in V . This is a hermitian form of V over D with involution $*$. Clearly V has an orthonormal basis $\{E_{ij}\}$, where E_{ij} is the standard matrix with entry 1 in the (i,j) position and zero elsewhere. Let V_1 be the subspace of V of all diagonal matrices in V . If D is Jordan orderable, then as in Example (3.1.2), $\underline{u} \cdot \underline{u} = \sum u_i u_i^* > 0$ for each non-zero $\underline{u} \in V_1$, and V_1 is strongly elliptic space.

Let V be any non-degenerate hermitian space. Let L be the ring of linear transformations of V . The adjoint ϕ^* (if any) of $\phi \in L$ is defined to be the linear transformation ϕ^* such that

$$\underline{u} \cdot \underline{v}\phi = \underline{u}\phi^* \cdot \underline{v} \quad (\underline{u}, \underline{v} \in V).$$

If $\phi_1, \phi_2 \in L$ have adjoints in L , then so must $\phi_1^* = \phi_1 + \phi_2^*$ and $\phi_1\phi_2^*$. Then $\phi_1^{**} = \phi_1$, $(\phi_1 + \phi_2)^* = \phi_1^* + \phi_2^*$ and $(\phi_1\phi_2)^* = \phi_2^*\phi_1^*$. If $\phi \in L$ is such that $V\phi$ is finite dimensional, then ϕ is of finite rank. Let

$$FL = \{\phi \in L / \dim V\phi < \infty\}.$$

Then FL is a 2-sided ideal of the ring L . Let A be the set of linear transformations $\phi \in L$ having adjoints. Clearly A is a subring

which is the domain of the involution $\phi \rightarrow \phi^*$. As is well known, every linear transformation in FL has an adjoint in FL, so that $FL = A$.

Our objective in this section is to establish a parallel between linear transformations of a strongly elliptic space V and infinite matrices over D .

Let $M_\rho(D)$ be the ring of all row-finite matrices (i.e., all but a finite number of the elements of each row are zero). Let $M_V(D)$ be the ring of all column-finite matrices and let $M_{\rho V} = M_\rho(D) \cap M_V(D)$.

The standard matrix involution tr.^* is the involution of the ring $M_{\rho V}(D)$ of all row-finite column-finite matrices $\phi = (\phi_{ij})_{i,j \in I}$ over D ,

$$\phi^{\text{tr.}^*} = (\phi_{ji}^*)_{i,j \in I}$$

The matrix of inner products $S = \text{Diag} \{s_i\}_{i \in I}$ ($s_i = e_i \cdot e_i$ where $\{e_i\}$ is the orthogonal basis of V) induces a new involution (s) of $M_{\rho V}(D)$:

$$\phi^{(s)} = S \phi^{\text{tr.}^*} S^{-1}$$

For $\phi \in L$, we associate the matrix of ϕ ; $\text{Mtrx}(\phi) \in M_\rho(D)$ determined by the equations:

$$\text{Row}_i(\text{Mtrx}(\phi)) = (\phi_{ij})_{j \in I} \Leftrightarrow e_i \phi = (\phi_{ij})_{j \in I} \quad (i \in I).$$

Then, clearly $L \subseteq M_\rho(D)$. Denote by $M_{\rho V}^0(D)$ the ring of all matrices each of which has only a finite number of its elements non-zero (matrices of finite type).

Lemma (3.1.4) (i) $A \approx M_{\rho\nu}(D)$

(ii) ${}^1FL \approx M_{\rho\nu}^0(D)$

Proof: (i) First we claim that, if $\phi \in A$ has adjoint ϕ^* , then $\text{Mtrx } \phi^* = (\text{Mtrx } \phi)^{(s)} = S(\text{Mtrx } \phi)^{\text{tr.}*} S^{-1}$. By definition,

$$(10) \quad e_i \cdot e_j \phi = e_i \phi^* \cdot e_j$$

If $\text{Mtrx } \phi = (\phi_{ij})_{i,j}$ and $\text{Mtrx } \phi^* = (\phi'_{ij})_{i,j}$, then

$$e_j \phi = \sum_k \phi_{jk} e_k \quad \text{and} \quad e_i \phi^* = \sum_l \phi'_{il} e_l$$

Hence, for all $i, j \in I$, follows from (10) that

$$(e_i \cdot e_i) \phi_{ji}^* = \phi'_{ij} (e_j \cdot e_j)$$

that is,

$$\phi'_{ij} = s_i \phi_{ji}^* s_j^{-1}$$

Therefore, $\text{Mtrx } \phi^* = S(\text{Mtrx } \phi)^{\text{tr.}*} S^{-1}$

Now, if $\phi \in A$, then both $\text{Mtrx}(\phi)$ and $\text{Mtrx}(\phi^*)$ are row finite, so that $\text{Mtrx } \phi \in M_{\rho\nu}(D)$. Conversely, if $\phi \in M_{\rho\nu}(D)$, then there is $\phi' \in L$ such that $\text{Mtrx } \phi' = \phi$. We claim that $\phi \in A$. Since $\phi \in M_{\rho\nu}(D)$, it follows that $\phi \in M_{\rho\nu}^0(D)$, so that, there is $\phi' \in L$ such that $\text{Mtrx } \phi' = \phi^{(s)}$. We claim that $u \cdot v \phi = u \phi' \cdot v$ for all $u, v \in V$

(this gives $\phi' = \phi^*$ by the uniqueness of the adjoint). Since

$(\text{Mtrx } \phi)^{(s)} = \text{Mtrx } \phi'$, it follows as above that $e_i \cdot e_j \phi = e_i \phi' \cdot e_j$.

Now, if $u = \sum_i u_i e_i$ and $v = \sum_j v_j e_j$ are arbitrary in V , then

$$u \cdot v \phi = \sum_i u_i e_i \cdot \sum_j v_j (e_j \phi) = \sum_{i,j} u_i (e_i \cdot e_j \phi) v_j^*$$

and

$$u\phi' \cdot v = \sum_i u_i e_i \phi' \cdot \sum_j v_j e_j = \sum_{i,j} u_i (e_i \phi' e_j) v_j^*$$

Hence $u \cdot v\phi = u\phi' \cdot v$ and ϕ' is the adjoint of ϕ . Therefore $\phi \in A$.

(ii) If $\phi \in M_{\rho\nu}^0(D)$, then there is $\phi \in L$ such that $\text{Mtrx } \phi = \Phi$.

Clearly $e_i \phi = 0$ for all but a finite number of integers i . Thus ϕ is of finite rank. Conversely, let $\phi \in FL$. Since $\{e_i \phi\}_{i \in I}$ spans the image of ϕ , it follows that $e_i \phi = 0$ for all but a finite number of indices i . Thus $\Phi = \text{Mtrx } (\phi) \in M_{\rho\nu}^0(D)$. $\square$

Definition (3.1.5): Let V be any non-degenerate elliptic space.

Let L be the ring of linear transformations of V . Then $\phi \in L$ is called a linear transformation increasing lengths if

$$(11) \quad 2w(u \cdot v\phi) \geq w(u \cdot u) + w(v \cdot v) \quad (u \text{ or } v \neq 0).$$

If

$$(12) \quad 2w(u \cdot v\phi) > w(u \cdot u) + w(v \cdot v) \quad (u \neq 0 \text{ or } v \neq 0)$$

we refer to such a ϕ as to a linear transformation strictly increasing lengths.

By setting $u = v\phi$ in (11), it follows

$$(13) \quad w(u\phi \cdot u\phi) \geq \underline{w(u \cdot u)} \quad (u \in V).$$

Conversely, by the ellipticity of V , (13) implies back (11). Also, for an elliptic space V , (12) is equivalent to

$$(14) \quad w(u\phi \cdot u\phi) > w(u \cdot u) \quad (u \in V).$$

Let L_0 (respect. L_0^+) be the set of all linear transformations

(respect. strictly) increasing lengths.

Theorem (3.1.6) [4]: Let V be any non-degenerate elliptic space.

Then

(1) The set L_0 is a subring of L which is $\star$ -closed.

(2) L_0^+ is a two-sided ideal of L_0 .

Now, we shall show that $\phi \in L_0$ has matrix $\Phi = (\phi_{ij})_{i,j}$ where $\phi_{ij} \in D$ is such that $w(\phi_{ij}) \geq 0$. First we need the following lemma.

Lemma (3.1.7): If $v_1, v_2, \dots, v_r$ are non-zero orthogonal vectors in an elliptic space V , then

$$(15) \quad w\left(\sum_{i=1}^r v_i \cdot v_i\right) = \min_i w(v_i \cdot v_i).$$

Proof: From (11), follows

$$w\left(\sum_{i=1}^r (v_i \cdot v_i)\right) \geq \min_i w(v_i \cdot v_i)$$

$$\begin{aligned} \text{Now,} \quad 2w(v_i \cdot v_i) &= 2w\left(v_i \cdot \sum_{j=1}^r v_j\right) \\ &\geq w(v_i \cdot v_i) + w\left(\sum_{j=1}^r v_i \cdot v_j\right). \end{aligned}$$

Hence $w(v_i \cdot v_i) \geq w(\sum_{j=1}^r v_i \cdot v_j)$ so that

$$\min_i w(v_i \cdot v_i) \geq w(\sum_{i=1}^r v_i \cdot v_i),$$

which completes the proof. $\square$

Theorem (3.1.8): Let V be any non-degenerate strongly elliptic space. If $\phi \in L_0$ and $\Phi = \text{Mtr}(\phi) = (\phi_{ij})_{i,j}$, then $w(\phi_{ij}) \geq 0$ for all i, j . Also, $\phi \in L_0^+$ implies that $w(\phi_{ij}) > 0$ for all i, j .

Proof: Let $\{e_i\}$ be the orthogonal basis of V , where $w(e_i \cdot e_i) = 0$. Let $e_i \phi = \sum_j \phi_{ij} e_j$. Then $\{\phi_{ij} e_j\}_j$ is a finite subset of orthogonal vectors. Hence by Lemma (3.1.7) follows:

$$\begin{aligned} w(e_i \phi \cdot e_i \phi) &= w(\sum_j \phi_{ij} e_j \cdot \sum_j \phi_{ij}^* e_j) \\ &= \min_j w(\phi_{ij} (e_j \cdot e_j) \phi_{ij}^*) \\ &= \min_j 2w(\phi_{ij}) \end{aligned}$$

Now, since $\phi \in L_0$, it follows that $w(u\phi \cdot u\phi) \geq w(u \cdot u)$ for every $u \in V$. If $u = e_i$, then $w(e_i \phi \cdot e_i \phi) \geq w(e_i \cdot e_i) = 0$. Hence $\min_j 2w(\phi_{ij}) \geq 0$, so that $w(\phi_{ij}) \geq 0$ for every $j \in I$. Since $i \in I$ is arbitrary, it follows that $w(\phi_{ij}) \geq 0$ for all $i, j \in I$. $\square$

Corollary (3.1.9): If ϕ is in the unitary group $U = \{\phi \in L/\phi^* = \phi^{-1}\}$, and $\Phi = \text{Mtrx}(\phi) = (\phi_{ij})_{i,j}$ then $w(\phi_{ij}) \geq 0$.

Proof: We show that $\phi \in L_0$ and the result will follow by Theorem (3.1.8). From $\phi\phi^* = 1$, follows

$$u\phi \cdot u\phi = u\phi\phi^* \cdot u = u \cdot u \quad (\text{for every } u \in V).$$

Hence $w(u\phi \cdot u\phi) = w(u \cdot u)$. This shows that $\phi \in L_0$. $\square$

Corollary (3.1.10): If $\phi \in L_0$, then $\Phi = \text{Mtrx}(\phi) \in M_0(R)$ where R is the valuation subring of the ground division ring D .

Proof: If $\phi \in L_0$, then $w(\phi_{ij}) \geq 0$ for all $i, j \in I$. Therefore, $\phi_{ij} \in R$ for all $i, j \in I$. $\square$

Corollary (3.1.11): If $\phi \in L_0^+$ then $\Phi = \text{Mtrx } \phi \in M_\rho(J)$, where J is the ideal of infinitesimals of the valuation subring R .

Proof: This is evident.

Theorem (3.1.12): Let V be any non-degenerate strongly elliptic space. Then $M_\rho(R) \approx L_0$ and $M_\rho(J) \approx L_0^+$.

Proof: By Corollaries (3.1.10) and (3.1.11), all that is left to show is that: $M_\rho(R) \subset \text{Mtrx}(L_0)$ and $M_\rho(J) \subset \text{Mtrx}(L_0^+)$. Let $\phi \in M_\rho(R)$. Then there is $\phi \in L$ such that $\text{Mtrx}(\phi) = \phi$. We show that $\phi \in L_0$. Let $u = \sum_{i \in I} u_i e_i$ be an arbitrary element in V . There is a finite subset I_1 of I such that $u = \sum_{i \in I_1} u_i e_i$. Hence

$$\begin{aligned} w(u \cdot u) &= w\left(\sum_{i,j} u_i e_i \cdot u_j e_j\right) \\ &= w\left(\sum_i (u_i e_i \cdot u_i e_i)\right) \\ &= \min_i w(u_i e_i \cdot u_i e_i) \\ &= \min_i 2w(u_i) \end{aligned}$$

Let $e_i \phi = \sum_{j \in I} \phi_{ij} e_j = \sum_{j \in I_2} \phi_{ij} e_j$, where I_2 is a finite subset of I .

Then

$$\begin{aligned} w(u\phi \cdot u\phi) &= w\left(\sum_{i \in I_1} u_i (e_i \phi \cdot e_i \phi) u_i^*\right) \\ &= w\left[\sum_{i,j \in I_1} u_i \left(\sum_{k \in I_2} \phi_{ik} (e_k \cdot e_k) \phi_{jk}^*\right) u_j^*\right] \end{aligned}$$

$$\begin{aligned}
&= w \left\{ \sum_{i,j,k} u_i \phi_{ik} (e_k \cdot e_k) \phi_{jk}^* u_j^* \right\} \\
&\geq \min_{i,j,k} w(u_i \phi_{ik} (e_k \cdot e_k) \phi_{jk}^* u_j^*) \\
&= \min_{i,j} [2w(u_i) + 2w(\phi_{ij})] \\
&\geq \min_i 2w(u_i) \quad (\text{where } \phi_{ij} \in R) \\
&= w(u \cdot u)
\end{aligned}$$

If $\phi \in M_\rho(J)$, then $\phi_{ij} \in J$ and, hence,

$$w(u\phi \cdot u\phi) > w(u \cdot u) \quad \square$$

Corollary (3.1.13): (i) $L_0 \cap FL \approx M_{\rho\nu}^0(R)$
(ii) $L_0 \cap A \approx M_{\rho\nu}(R)$
(iii) $L_0^+ \cap FL \approx M_{\rho\nu}^0(J)$
(iv) $L_0^+ \cap A \approx M_{\rho\nu}(J)$.

The proof of Corollary (3.1.13) is immediate from Theorem (3.1.12) and Lemma (3.1.4).

From Theorem (3.1.10) and Corollary (3.1.13), one can inter-relate the ring L_0 and the ideal L_0^+ . We recall that a ring S is right vanishing [14] if, given any sequence $\{a_i\}_{i \geq 0}$ of elements of S , there is an integer n (depending on the a_1) such that $a_0 a_1 a_2 \dots a_n = 0$.

Theorem (3.1.14) [14]:

(1) Let M be one of the rings of matrices M_ρ , M_ν and $M_{\rho\nu}$, over a ring S , then

$$\text{Jac}(M(S)) \subset M(\text{Jac}(S));$$

$$(2) \quad \text{Jac}(M_{\rho\nu}^0(S)) = M_{\rho\nu}^0(\text{Jac}(S)) \quad \text{for all rings } S.$$

(3) For each of the rings M_ρ , M_ν and $M_{\rho\nu}$ follows that $\text{Jac}(M(S)) = M(\text{Jac}(S))$ if and only if $\text{Jac}(S)$ is right vanishing.

Corollary (3.1.15): (i) $\text{Jac}(L_0 \cap FL) = L_0^+ \cap FL$.

$$(ii) \quad \text{Jac}(L_0 \cap A) \subsetneq L_0^+ \cap A.$$

$$(iii) \quad \text{Jac}(L_0) \subsetneq L_0^+.$$

$$(iv) \quad \text{If } V \text{ is finite dimensional, then } \text{Jac}(L_0) = L_0^+.$$

$$(v) \quad L_0^+ \cap FL \subset \text{Jac}(L_0) \subset L_0^+.$$

Proof: (i) From Corollary (3.1.13) and Theorem (3.1.14), follows

$$\begin{aligned} \text{Jac}(L_0 \cap FL) &= \text{Jac}(M_{\rho\nu}^0(R)) \\ &= M_{\rho\nu}^0(\text{Jac}(R)) \\ &= M_{\rho\nu}^0(J) = L_0^+ \cap FL. \end{aligned}$$

(ii) and (iii) Since J is not nilpotent and, hence, not right-vanishing, it follows that $\text{Jac}(M_{\rho\nu}^0(R)) \neq M_{\rho\nu}^0(J)$ and $\text{Jac}(M_\rho(R)) \neq M_\rho(J)$.

(iv) If V is finite-dimensional, then $FL = L$ and the result follows from (i).

(v) Since $M_{\rho\nu}^0(J) \subset \text{Jac}(M_\rho(R)) \subset M_\rho(J)$ (see [14]), it follows that $L_0^+ \cap FL \subset \text{Jac}(L_0) \subset L_0^+.$ $\square$

3.2. Unitary transformations modulo the group of multiplicative infinitesimals:

Recall that in a classical hermitian space V the linear transformation ϕ of V is unitary if ϕ preserves lengths. In the preceding section we looked at a linear transformation ϕ that strictly increases length, which is a different requirement altogether. In this section, we will be concerned with a type of linear transformation that generalizes the unitary transformation. We make the

Definition (3.2.1): Any linear transformation $\phi \in L$ is said to be unitary modulo $1+J$, if ϕ is one-to-one and onto mapping such that for every $v \in V$,

$$(16) \quad v\phi \cdot v\phi \equiv v \cdot v \pmod{1+J}.$$

If ϕ is a unitary transformation, then for each $v \in V$, follows $v\phi \cdot v\phi = v \cdot v$ and, hence, ϕ is also unitary modulo $1+J$. If V is finite-dimensional non-isotropic space, then any transformation ϕ verifying (16) is one-to-one, because $v\phi \cdot v\phi = 0$ implies $v \cdot v = 0$ and by the non-isotropy of V , $v = 0$ follows. Hence ϕ is also onto.

Example (3.2.2): Let D be a Jordan ordered division ring with involution $*$ and centre Z . Let V be the left vector space ${}_Z D$, relative to the coordinate form $x \cdot y = xy^*$. Then the linear transformation $\phi : x \rightarrow axa^{-1}$, ($x, a \in D$) is unitary modulo $1+J$. For,

$$\begin{aligned} x\phi \cdot x\phi &= (x\phi)(x\phi)^* \\ &= ax(a^{-1}a^{-1*})x^*a^* \end{aligned}$$

$$\equiv axx^*a^{-1}$$

$$\equiv xx^* = x \cdot x$$

for every $x \in D$ (where all symmetries are central modulo $1+J$).

Example (3.2.3): Let D be a Jordan ordered division ring with involution $*$. Let V be the left vector space ${}_D D$, relative to the coordinate form $x \cdot y = xsy^*$ for some $s = s^* \in D$. To say that $\phi \in L$ is unitary modulo $1+J$ is to say that $\phi = x_R$, the right translation of D induced by x , where x is a unitary element modulo $1+J$. For, let $xx^* = 1+j$, for some $j \in J$. Then $saa^*xx^* = saa^*(1+j)$ so that, $axsx^*a^* \equiv asa^*$ for every $a \in D$. Hence $a\phi \cdot a\phi \equiv a \cdot a$ for every $a \in V$.

Definition (3.2.4): [4, page 33] Call $\phi \in L$ medial or say ϕ preserves lengths, if

$$(17) \quad w(v\phi \cdot v\phi) = w(v \cdot v),$$

for every $v \in V$. Let M denote the set of all medial transformations.

Hereafter, V stands for any non-degenerate hermitian space, ellipticity of V not required unless otherwise stated.

Lemma (3.2.5): A unitary transformation modulo $1+J$ is medial.

Proof: Let ϕ be a unitary transformation modulo $1+J$, that is, $v\phi \cdot v\phi \equiv v \cdot v \pmod{1+J}$ for every $v \in V$. Then $v\phi \cdot v\phi = v \cdot v (1+j)$, for some $j \in J$. Since

$$w((v \cdot v)j) = w(v \cdot v) + w(j) > w(v \cdot v)$$

it follows, by (8), that

$$\begin{aligned}
 w(v\phi \cdot v\phi) &= w((v.v) + (v.v)j) \\
 &= \text{Min } \{w(v.v), w((v.v)j)\} \\
 &= w(v.v)
 \end{aligned}$$

A unitary transformation modulo $1+J$ is one which preserves essentially all the structure of V , its addition, its multiplication by scalars and its inner product modulo $1+J$. From Lemma (3.2.5), a unitary transformation modulo $1+J$ preserves length in the sense given in Definition (3.2.4). One might ask whether the converse holds true. The answer is no in general, as one can show in Example (3.2.3), that if x is a unit which is not unitary modulo $1+J$, then $\phi = x_R$ preserves length but not unitary modulo $1+J$.

Lemma (3.2.6): For each $\phi \in L_0^+$, if $\sigma = 1+\phi$, then σ is a unitary transformation modulo $1+J$.

Proof: From $\phi \in L_0^+$, follows that for each $v \in V$,

$$(18) \quad w(v\phi \cdot v\phi) > w(v.v) \quad \text{and}$$

$$(19) \quad w(v\phi \cdot v) = w(v.v\phi) > w(v.v)$$

From (18), follows

$$w(v\phi.v\phi) + w((v.v)^{-1}) > 0$$

$$w((v\phi \cdot v\phi)(v.v)^{-1}) > 0,$$

that is, $v\phi.v\phi = j_1(v.v)$, for some $j_1 \in J$.

Similarly, from (19) follows that for each $v \in V$,

$$v\phi.v = j_2(v.v)$$

and

$$v \cdot v\phi = j_3(v \cdot v)$$

Hence

$$\begin{aligned} v\phi \cdot v\phi &\equiv (v + v\phi) \cdot (v + v\phi) \\ &= (v \cdot v) + (v \cdot v\phi) + (v\phi \cdot v) + (v\phi \cdot v\phi) \\ &= (v \cdot v) + (v \cdot v)(j_1 + j_2 + j_3) \\ &= (v \cdot v)(1+j), \end{aligned}$$

where $j = j_1 + j_2 + j_3 \in J$. Therefore,

$$v\phi \cdot v\phi \equiv v \cdot v \pmod{1+J}$$

for every $v \in V$, as desired. $\square$

Theorem (3.2.7): The set C of all unitary transformations modulo $1+J$ has the following features:

(1) For each $0 \neq z \in Z$ (= centre of D), $z \cdot 1 \in C$ if and only if z is a unitary element modulo $1+J$.

(2) C is a multiplicative group.

Proof: (1) To say that $z \cdot 1$ is unitary modulo $1+J$ is to say that $zv \cdot zv \equiv v \cdot v \pmod{1+J}$, for each $v \in V$, if and only if $zz^*(v \cdot v) \equiv v \cdot v$, or $zz^* \equiv 1$ (if $v \neq 0$).

(2) If ϕ_1 and ϕ_2 are unitary transformations modulo $1+J$, then

$$\begin{aligned} v(\phi_1\phi_2) \cdot v(\phi_1\phi_2) &= (v\phi_1)\phi_2 \cdot (v\phi_1)\phi_2 \\ &\equiv v\phi_1 \cdot v\phi_1 \\ &\equiv v \cdot v, \end{aligned}$$

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for every $v \in V$. Hence $\phi_1 \phi_2 \in C$.

Also if $\phi \in C$, then

$$v.v = (v\phi^{-1})\phi \cdot (v\phi^{-1})\phi$$

$$\equiv v\phi^{-1} \cdot v\phi^{-1}$$

for every $v \in V$. Hence $\phi^{-1} \in C$. $\square$

It is known that a unitary transformation on V takes an orthonormal basis of V into an orthonormal basis of V . One might ask whether the same holds true for unitary transformation modulo $1+J$. The answer is no in general. For example, let V be any 2-dimensional hermitian space. Let $\{e_1, e_2\}$ be an orthonormal basis of V . Let $A = \begin{bmatrix} 1 & b \\ a & 1 \end{bmatrix}$, where $a, b \in J$ with $a \neq -b$. View A to be the matrix of some linear transformation ϕ of V . Since $A \in 1+M_0(J)$, it follows that $\phi \in C$ (Lemma (3.2.6)). Now, $e_1\phi = e_1 + ae_2$ and $e_2\phi = be_1 + e_2$, so that $e_1\phi \cdot e_2\phi = b+a \neq 0$. This shows that unitary transformations modulo $1+J$ do not generally take orthogonal basis to orthogonal basis.

Now, what can one say about the action of a unitary transformation modulo $1+J$ on an orthogonal basis of V . To answer this question we will first recall a construction of the so-called residue space $\bar{V}$. Call v a finite vector if for each $u \in V$

$$2w(u.v) \geq w(u.u)$$

Denote by V_0 , the set of all finite vectors. Call $v \in V_0$ infinitesimal, if

$$2w(u.v) \gg w(u.u) \quad (u.u \neq 0).$$

The subset of all infinitesimals v in V is denoted by V_0^+ . It is known that V_0 is an R -module, where R is the valuation ring in the valued division ring D . Also, V_0^+ is a submodule of V_0 .

Remark (3.2.8) [10, Theorem 5.4]: Let V be any hermitian space over the $*$ -valuated division ring D . Then the factor R -module $\bar{V} = V_0/V_0^+$ can be turned into a non-isotropic hermitian space over the residue division ring $\bar{D} = R/J$, relative to the form

$$u + V_0^+ \cdot v + V_0^+ = u \cdot v + J.$$

Now, let ϕ be a unitary transformation modulo $1+J$ of V . Then $u\phi \cdot u\phi \equiv u \cdot u \pmod{1+J}$ and, if u is a finite vector, that is, $w(u \cdot u) > 0$, then $u\phi \cdot u\phi = u \cdot u + j$ for some $j \in J$. Hence,

$$\begin{aligned} u + V_0^+ \cdot u + V_0^+ &= u \cdot u + J \\ &= u\phi \cdot u\phi + J \\ &= u\phi + V_0^+ \cdot u\phi + V_0^+, \end{aligned}$$

so that ϕ is a unitary transformation of $\bar{V}$. Let $\{e_i\}_{i \in I}$ be an orthogonal basis of V with $w(e_i \cdot e_i) = 0$ for every i . Then $\{e_i + V_0^+\}$ is an orthogonal basis of $\bar{V}$. Since ϕ is a unitary transformation of $\bar{V}$, it follows that $\{e_i\phi + V_0^+\}$ is an orthogonal basis of $\bar{V}$, that is $e_i\phi \cdot e_j\phi \in J$ for every $i \neq j$. Thus one has the

Lemma (3.2.9): If ϕ is a unitary transformation modulo $1+J$ of a hermitian space V , and if $\{e_i\}_{i \in I}$ is an orthogonal basis of V such that $w(e_i \cdot e_i) = 0$ for every i , then $e_i\phi \cdot e_j\phi \in J$ for $i \neq j$.

For the general case, one has the following theorem:

Theorem (3.2.10): Let ϕ be a unitary transformation modulo $1+J$

of the hermitian space V . Let u and v be any pair of orthogonal vectors in V and put $u\phi \cdot v\phi = a$. Assume that the element a does not induce the residue involution (that is, $a^{-1}\lambda a$ is not always congruent to λ^* , where λ is a general valuation unit), then

$$w(u\phi \cdot v\phi) > \text{Min} \{w(u \cdot u), w(v \cdot v)\}.$$

Proof: By definition there are α, β and γ in J such that $u\phi \cdot u\phi = (u \cdot u)(1+\alpha)$, $v\phi \cdot v\phi = (v \cdot v)(1+\beta)$, and $(u+v)\phi \cdot (u+v)\phi = (u+v \cdot u+v)(1+\gamma)$.

Expanding the last equality and eliminating gives

$$u\phi \cdot v\phi + v\phi \cdot u\phi = (u \cdot u)(\gamma - \alpha) + (v \cdot v)(\gamma - \beta).$$

Taking w , this gives

$$w(u\phi \cdot v\phi + v\phi \cdot u\phi) \geq \text{Min} \{w((u \cdot u)(\gamma - \alpha)), w((v \cdot v)(\gamma - \beta))\}.$$

Since $w(\gamma - \alpha) > 0$ and $w(\gamma - \beta) > 0$, it follows that

$$w(u\phi \cdot v\phi + v\phi \cdot u\phi) > \text{Min} \{w(u \cdot u), w(v \cdot v)\}.$$

Now, if $w(u\phi \cdot v\phi) \geq w(u\phi \cdot v\phi + v\phi \cdot u\phi)$, then clearly $w(u\phi \cdot v\phi) = w(u\phi \cdot v\phi + v\phi \cdot u\phi) > \text{Min} \{w(u \cdot u), w(v \cdot v)\}$. If, on the other hand, $w(u\phi \cdot v\phi) < w(u\phi \cdot v\phi + v\phi \cdot u\phi)$, it is clear that

$$u\phi \cdot v\phi \equiv -v\phi \cdot u\phi \pmod{1+J}.$$

In the preceding congruence replace u by λu , where λ is a valuation unit in $\bar{D}$. If $(\lambda u)\phi \cdot v\phi \equiv -v\phi \cdot (\lambda u)\phi$ for every valuation unit $\lambda \in \bar{D}$, then $\lambda(u\phi \cdot v\phi) \equiv -(v\phi \cdot u\phi)\lambda^* \equiv (u\phi \cdot v\phi)\lambda^*$, that is, $(u\phi \cdot v\phi)^{-1}\lambda(u\phi \cdot v\phi) = \lambda^*$. This is true for every valuation unit λ , which contradicts the hypothesis.

If, on the other hand, there is a valuation unit $\lambda \in \bar{D}$ such that $(\lambda u)\phi \cdot v\phi \not\equiv -v\phi \cdot (\lambda u)\phi$, then as before $w((\lambda u)\phi \cdot v\phi) = w((\lambda u)\phi \cdot v\phi + v\phi \cdot (\lambda u)\phi)$

$> \text{Min } \{w(\lambda u, \lambda u), w(v, v)\}$. Since $w(\lambda) = 0$, it follows that

$$w(u\phi, u\phi) > \text{Min } \{w(u, u), w(v, v)\}$$

as desired. $\square$

Corollary (3.2.11): If the residue division ring $\bar{D} = R/J$ has dimension at least 4, over its centre, it follows that:

$$w(\bar{u}\phi, \bar{v}\phi) > \text{Min } \{w(u, u), w(v, v)\}.$$

Proof: If there is an element in D that induces the involution modulo $1+J$, then $\bar{D}$ would admit an automorphism which is an anti-automorphism and, hence, $\bar{D}$ is commutative. This contradicts our hypothesis. It suffices then to quote the preceding theorem. $\square$

We proceed to give an explicit matrix criterion for a linear transformation to be unitary modulo $1+J$. Here is one additional lemma.

Lemma (3.2.12): Let ϕ be any linear transformation of a hermitian space V , that has adjoint ϕ^* . If $A = (a_{ij})_{i,j}$ is the matrix of ϕ relative to an orthogonal basis $\{e_i\}_{i \in I}$ of V , then

$$AA^* = ((e_i \phi \cdot e_j \phi)(e_j \cdot e_i)^{-1})_{i,j}.$$

Proof: Put $s_i = e_i \cdot e_i$, $i \in I$. Then

$$\begin{aligned} e_i \phi \cdot e_j \phi &= \left(\sum_k a_{ik} e_k \right) \cdot \left(\sum_l a_{jl} e_l \right) \\ &= \sum_{k,l} a_{ik} (e_k \cdot e_l) a_{jl}^* \\ &= \sum_k a_{ik} (e_k \cdot e_k) a_{jk}^* \\ &= \sum_k a_{ik} s_k a_{jk}^* \end{aligned}$$

From Lemma (3.1.6), $A^* = S(a_{ji}^*)S^{-1}$ follows, where S is the matrix of inner product $S = \text{Diag } \{s_i\}_{i \in I}$. Now, if $AA^* = (c_{ij})_{i,j}$, then

$$\begin{aligned} c_{ij} &= \sum_k a_{ik} s_k a_{jk}^* s_j^{-1} \\ &= (\sum_k a_{ik} s_k a_{jk}^*) s_j^{-1} \\ &= (e_i \phi \cdot e_j \phi)(e_j \cdot e_j)^{-1} \quad || \end{aligned}$$

Theorem (3.2.13): If ϕ is a unitary transformation modulo $1+J$, that has an adjoint ϕ^* , and if A is the matrix of ϕ relative to an orthogonal basis $\{e_i\}_{i \in I}$ of V such that $\omega(e_i \cdot e_i) = 0$, then $AA^* \in 1 + M_\rho(J)$.

Proof: From Lemma (3.2.12), follows that if $AA^* = (c_{ij})_{i,j}$, then $c_{ij} = (e_i \phi \cdot e_j \phi)(e_j \cdot e_j)^{-1}$. Since ϕ is unitary modulo $1+J$, it follows that $c_{ii} \equiv 1 \pmod{1+J}$. By Lemma (3.2.9), follows that $c_{ij} \in J$, for $i \neq j$. This completes the proof. $\square$

Note that, from Lemma (3.2.6), if the matrix A of a linear transformation ϕ is in $1+M_\rho(J)$, then $\phi \in C$. Thus, by Theorem (3.2.13), $\phi \in C$ implies that $AA^* \in 1+M_\rho(J)$, so that, $\phi\phi^* \in C$. Now, since ϕ is invertible, then $\phi^{-1} \cdot \phi\phi^* \in C$, that is $\phi^* \in C$. This is the

Lemma (3.2.14): If $\phi \in C$, then $\phi^* \in C$.

Theorem (3.2.15): Let ϕ be any linear transformation of an elliptic space V and let A be the matrix of ϕ relative to an orthogonal basis $\{e_i\}_{i \in I}$ of V . If $AA^* = \text{Diag}_i(1+\alpha_i)$, $\alpha_i \in J$, then ϕ is a unitary transformation modulo $1+J$.

Proof: Let $AA^* = (c_{ij})_{i,j}$. Then by Lemma (3.2.13), $c_{ij} = (e_i \phi \cdot e_j \phi)(e_j \cdot e_j)^{-1}$ follows. Since $AA^* = \text{Diag}_i(1+\alpha_i)$, $\alpha_i \in J$, it follows that $c_{ij} = 0$ (for $i \neq j$) and $c_{ii} \equiv 1$ for every $i \in I$. Then $e_i \phi \cdot e_i \phi \equiv e_i \cdot e_i \pmod{1+J}$ for every $i \in I$, and $e_i \phi \cdot e_j \phi = 0$ for $i \neq j$. Now, we claim that $v \phi \cdot v \phi \equiv v \cdot v \pmod{1+J}$, for every $v \in V$. Since $e_i \phi \cdot e_i \phi \equiv e_i \cdot e_i \pmod{1+J}$, it follows that

$$e_i \phi \cdot e_i \phi = (e_i \cdot e_i)(1+\varepsilon_i), \quad \varepsilon_i \in J.$$

Let $v = \sum_{i \in I} v_i e_i$ and let $s_i = e_i \cdot e_i$. Then

$$\begin{aligned} w\left(\sum_{i \in I} v_i s_i \varepsilon_i v_i^*\right) &\geq \min_i w(v_i s_i \varepsilon_i v_i^*) \\ &= \min_i (2w(v_i) + w(s_i) + w(\varepsilon_i)) \\ &> \min_i (2w(v_i) + w(s_i)) \\ &= \min_i w(v_i s_i v_i^*) \\ &= w\left(\sum_{i \in I} v_i s_i v_i^*\right), \end{aligned}$$

where the last equality comes from Lemma (3.1.6). Thus

$$\sum v_i s_i v_i^* + \sum v_i s_i \varepsilon_i v_i^* \equiv \left(\sum v_i s_i v_i^*\right)(1+j)$$

for some $j \in J$, that is

$$\sum v_i s_i (1+\varepsilon_i) v_i^* \equiv \sum v_i s_i v_i^* \pmod{1+J}$$

or,

$$\sum v_i (e_i \phi \cdot e_i \phi) v_i^* \equiv \sum v_i (e_i \cdot e_i) v_i^*.$$

Since $e_i \phi . e_j \phi = 0$ and $e_i . e_j = 0$ for $i \neq j$, it follows that

$$\sum_{i,j} v_i (e_i \phi . e_j \phi) v_j^* \equiv \sum v_i (e_i . e_j) v_j^* .$$

Therefore,

$$v \phi . v \phi \equiv v . v^* \pmod{1+J} ,$$

as desired. $\square$

REFERENCES

- [1] A.A. Albert, Structure of algebras, Amer. Math. Soc. Colloq. Publ. (1961).
- [2] M. Chacron, Unitaries in simple artinian rings, Can. J. Math., Vol. XXI, No. 3, 1979, p. 512-527.
- [3] M. Chacron, C-orderable division rings with involution, J. of Algebra, Vol. 75, No. 2 (1982), pp. 495-521.
- [4] M. Chacron, Non-isotropic Unitary Spaces and Modules with Cauchy-Schwarz inequalities, Pacific J. of Math., Vol. 119, No. 1 (1985), pp. 1-87.
- [5] M. Chacron, Structure of a c-valuated division ring with application to c-ordering (to appear).
- [6] P.M. Cohn and M. Mahdavi-Hezavehi, Extensions of valuations on skew fields, Lecture Notes in Mathematics, 825, Springer-Verlag, Berlin, 1980, 38-41.
- [7] P.K. Draxl, Skew fields, London Math. Soc. Lecture Note Series, 81, 1983.
- [8] H. Gross, Quadratic forms on infinite dimensional vector spaces, Progression Mathematics, Birkhäuser 1979.
- [9] I.N. Herstein, Topic in Ring Theory, University of Chicago Press (1967, Chicago).
- [10] S.S. Holland (Jr.), *-valuations and ordered *-fields, Trans. Amer. Math. Soc. Vol. 262, no. 1, 1980, p. 219-243.

- [11] S.S. Holland (Jr.), Strong ordering of $\ast$ -field (to appear in J. of Algebra).
- [12] I. Kaplansky, Linear Algebra and Geometry, Allyn and Bacon Inc.
- [13] T.Y. Lam, Orderings, Valuations and Quadratic forms; Amer. Math. Soc. Publ. V.52 (1983).
- [14] E.M. Patterson, On the radicals of rings of row finite matrices, Proc. Roy. Soc. Edin., A66 (1962), pp. 42-46.
- [15] W. Scharlau, Zur Existenz von Involutionen auf einfachen Algebren, Math. Z. 145 (1975), p. 25-32.
- [16] O.G.S. Schilling, The theory of Valuations, Mathematical Surveys No. 4 (1950).
- [17] T. Szele, On ordered skew fields, Proc. Amer. Math. Soc. 3 (1952), pp. 410-413.
- [18] J. Van Geel, Places and valuations in non-commutative ring theory, Dekker Lecture Notes Vol. 71, 1981.
- [19] C.T.C. Wall, On the axiomatic foundations of the theory of Hermitean forms, Proc. Camb. Phil. Soc. 67 (1970), 243-250.

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