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A Coupled Thermo-Mechanical Analysis of Pipelines Buried in Freezing Ground

by

DAIYU WANG

**M. E. (Engineering Mechanics)
Tsinghua University, Beijing, 1981**

**A thesis submitted to
The Faculty of Graduate Studies and Research
In partial fulfillment of the requirements**

**For the degree of
Master of Engineering**

**Department of Civil and Environmental Engineering
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in Freezing Ground**

submitted by

DAIYU WANG

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For the degree of
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A.G. Razaqpur, Chair of Department of Civil
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A.G . Razaqpur, Thesis Supervisor

B. Rajani, External Examiner

Department of Civil and Environmental Engineering

Carleton University, Ottawa

May, 1994

Abstract

This thesis provides a coupled thermo-mechanical analysis of stresses and deformations of pipelines buried in non-uniformly freezing soil. The thesis consists of four parts: derivation of a generalized Clausius-Clapeyron equation, a simplified frost heave model, a nonlinear finite element model of the pipe on elastic foundation and a thermo-mechanical modeling of soil-pipeline interaction.

The proposed generalized form of the Clausius-Clapeyron equation is derived from basic thermodynamic principles. This equation describes not only the relation of water and ice pressure, and temperature, but also the effect of the location of the water table relative to the freezing front on the freezing process. The proposed equation is more general and practical than the commonly used form of the Clausius-Clapeyron equation derived by Kay et al..

The proposed simplified frost heave model for freezing soil is based on coupled heat and moisture transport. The ice pressure at the base of ice lens is assumed to be equal to the overburden pressure, the concomitant water pressure at this location is obtained using the foregoing general form of the Clausius-Clapeyron equation while the water pressure at the freezing front can be described as a boundary value. The rate of water flow across the frozen fringe is determined by Darcy's law. The principal advantage of the proposed model over existing models is the simplicity and ease of calculation and the relative transparency of the physics of the frost heave phenomenon.

In the proposed nonlinear finite element model of pipe, a buried pipe is simulated as a beam-column on elastic foundation with geometric and material nonlinear behaviour. A layered approach is applied to discretize the pipe through its depth, and the pipe material is assumed to have a trilinear stress-strain relation. A geometric stiffness matrix and a foundation matrix are introduced to describe the effects of the large displacements and the reaction of the foundation, respectively.

The proposed modeling of soil-pipeline interaction is based on the coupled analysis of the time-dependent nonlinear thermo-mechanical process which combines the frost heave in the soil and mechanical response of the pipe. In the model it is assumed that the soil below the pipeline is a multi-layer half space, and the interface of the pipeline and soil is the free surface of the half space. Using the elastic theory for the multi-layer half space, the time-dependent soil stiffness is determined. The moving boundary conditions of the pipeline are simulated by upward distributed loads acting on the portion of the pipe buried in the frost susceptible soil. The magnitude of the loads is proportional to the upward ground movement caused by frost heave and by creep effects in the frozen soil.

A complete computer program based on the above theoretical analysis is developed. This program basically consists of two parts. The first one is based on the simplified frost heave model for calculating the time-dependent variation of temperature profile in the freezing soil and the amount of frost heave, using the finite element method in both the space and time domains in conjunction with an iterative technique. The second part is based on the nonlinear finite element model of the pipe and the thermo-mechanical model of soil-pipeline interaction, which yields the time-dependent stresses and deformations of the pipe. In the geometrically and materially nonlinear analysis an incremental-iterative procedure which is combined with updated Lagrangian method. Numerical examples are presented which demonstrate reasonable agreement between the results from the proposed analysis and the existing theoretical and/or experimental results.

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List of Symbols

A	=cross-sectional area of the beam element or soil element
a	=experimental constant in SP model
a, b	=undetermined permeability parameters
a_1	=non-dimensional factor which describes the decrease of the distributed load with the distance to the interface
a_2	=non-dimensional factor which describes the increase of the distributed load near the end
B, n	=experimental creep parameters
$B(t)$	=width of frozen bulb
$\overline{B}$	=9.8 KPa/m
b_j	=the width of the j th layer
C_s	=thermal capacitance matrix at time t
$\bar{c}$	=specific heat capacity of soil
c	=heat capacity of soil
c_s	=heat capacity of soil
$[C_s]$	=heat capacity matrix of soil
c_w	=heat capacity of water

D	=the stress dependent material modulus
D_j	=the stress material modulus of the j th layer
$\{d\}$	=nodal displacement vector in local coordinate system
$\{\bar{d}_0\}$	=the displacement vector before the iterative procedure
$\{\bar{d}\}$	=nodal displacement vector in reference coordinate system
$\Delta\bar{d}_0$	=the incremental displacement at the beginning of load step r
$\Delta\bar{d}_i$	=the incremental displacement at iteration step i at load step r .
d_f	=the thickness of the frozen fringe
E	=elastic modulus of beam-column
E_{in}	=the energy entering the element
E_{out}	=the energy exiting of the element
E_{sh}	=strain-hardening modulus
E_{si}	=Young's modulus of soil layer i
E_T	=strain-hardening modulus
ΔE	=the change in stored energy within the element
e_j	=internal energy of soil system j
$\{F\}$	=nodal force vector in the local coordinate system
$\{\bar{F}\}$	=nodal force vector in the reference coordinate system
$\{F_{I1}\}$	=internal force vector
$\{F_{I2}\}$	=internal force vector

$\{F_{I3}\}$	=internal force vector
$\{\bar{F}_{i-1}\}$	=the total applied load vector at the end of the $(i - 1)$ th iteration
$\{\bar{F}_{Ii-1}\}$	=the total internal force vector at the end of the $(i - 1)$ th iteration.
F_m	=a generalized migration force
F_h^n	=vector of thermal loads due to latent heat and heat flux at time step n
F_h^{n+1}	=vector of thermal loads due to latent heat and heat flux at time step $n + 1$
$\text{grad } F_m$	=gradient of a generalized migration force
$\{f\}$	=applied nodal force vector
$\{f_g\}$	=equivalent nodal force vector due to body force
Δh_f^m	=the total amount of frost heave at the end of time step m
$\{f_Q\}$	=equivalent nodal heat vector
$\{f_q\}$	=nodal heat flux vector
$\{f_s\}$	=equivalent nodal force vector due to surface tractions
G	=temperature gradient
G_j	=free energy of soil system j
g	=gravitational acceleration of gravity
$\{g\}$	=vector of body force in the element
h_i	=thickness of soil layer i

h_s^{m-1}	=total free upward displacement of soil by the end of time step $m - 1$
dh	=the heat released into the soil due to the phase change during freezing
Δh	=differential upward movement between the two soils
Δh_a	=amount of frost heave due to the freezing of in situ pore water during time interval Δt
Δh_f	=amount of frost heave due to the freezing of migrating water during time interval Δt
Δh_t	=total amount of frost heave during time interval Δt
Δh_m	=increment of differential displacement during time step m
Δh_u	=upward movement of non-frost susceptible soil
I	=moment of inertia of the beam-column
I_w	=hydraulic gradient in the frozen fringe
i	=iteration step number
K	=permeability
$\overline{K}_f$	=overall permeability of freezing soil in the frozen fringe
$[K]$	=stiffness matrix in local coordinate system
$[\overline{K}]$	=stiffness matrix in reference system
$[K_f]$	=foundation matrix
$[K_G]$	=geometrical stiffness matrix
$[K_m]$	=conventional small displacement stiffness matrix

$[K_s]$	=thermal conductivity matrix of soil
K_s	=thermal conductance matrix corresponding to thermal conductivity values at a given time t
k_s	=thermal conductivity of soil
k_{s1}	=axial stiffness of the foundation
k_{s2}	=vertical stiffness of the foundation
L	=the element length
L_1	=the length of the element 1, or length of the portion of the pipe buried in the frost susceptible soil
L_2	=the length of the element 2
L_h	=latent heat of fusion
N	=axial force in the pipe cross-section
N_1, N_2	=linear shape
$[N_b]$	=the bending shape function matrix.
$[N_p]$	=the in-plane shape function functions, i.e.
N_{t1}, N_{t2}	=time-dependent shape function
P_e	=overburden pressure
P_i	=ice pressure
P_j	=applied pressure on soil system j
P_{is}	=the ice pressure at the coldest side of the frozen fringe
P_w	=water pressure
P_{ws}	=the water pressure at the coldest side of the frozen fringe

P_{wf}	=the water pressure at the freezing front
$P(x)$	=net pressure function in the simulation of pipe boundary condition
Q_L	=released latent heat in the element r
$\overline{Q}_{i-1}$	=the equivalent latent heat at node $(i - 1)$
$\overline{Q}_i^1$	=the equivalent latent heat at node i in element 1
$\overline{Q}_s$	=the amount of latent heat generated by the freezing of the migrated water
$\{q_s\}$	=vector of surface traction in the element
$q(x)$	=contact stress, $\bar{q}P_0$
q_z	=heat flux at surface z
q_{z1}	=heat flux at node 1
q_{z2}	=heat flux at node 2
R_p^{m-1}	=reaction of the pipeline acting on the soil surface at the end of time step $(m - 1)$
r	=load step number
r_{iw}	=radius of the ice-water interface
S	=the first moment of cross-sectional area of the beam-column
s_j	=entropy (J/k _j)
SP	=segregation potential
SP_0	=the value of SP under zero overburden pressure

T	=absolute temperature at a point
$[T]$	=transformation matrix
T_f	=absolute freezing temperature at the freezing front
T_{i-1}	=known temperature at node $i - 1$
T_i	=known temperature at node i
T_{i+1}	=known temperature at node $i + 1$
T_s	=absolute segregation freezing temperature
T_1	=the temperature at node 1
T_2	=the temperature at node 2 ($^{\circ}C$)
$\text{grad } T$	=gradient of temperature in the frozen fringe
$\mathbf{T}^n$	=vector of nodal temperature at time step n
$\mathbf{T}^{n+1}$	=vector of nodal temperature at time step $n + 1$
t	=time
U	=strain energy stored in the beam-column element
$U(t)$	=water intake flux
u	=axial displacement of the beam
u_a	=axial displacement of the beam due to axial force
u_b	=axial displacement of the beam due to small deflection theory bending
u_i	=displacement in x direction at node i
u_l	=axial displacement of the beam due to large deflection
$\bar{u}_i, \bar{v}_i$	=nodal displacements in reference coordinate system

v	=vertical displacement of the beam
v_i	=displacement in z direction at node i
v_j	=specific volume of soil system
V_m	=specific volume of water
v_p^{m-1}	=net upward displacement of pipe by the end of time step $m - 1$
v_w	=water flux
v_x	=water velocity in x direction
v_y	=water velocity in y direction
W	=the work of the external forces
W_f	=the work done by the foundation reaction
W_j	=weighting function
W_n	=the work done by all the external forces except the foundation reaction
$X(t)$	=depth of frozen bulb
x, y, z	=Cartesian coordinates
$\bar{Z}$	=water head of the freezing front
ΔZ	=advance of the freezing front during time increment Δt
$z_1 =$	=coordinate of the bottom of soil layer 1
$\frac{dZ}{dt}$	=the rate of advance of the freezing front
α_1	=a non-dimensional factor which describes the intensity of the applied load at the interface between the two soils

$\alpha_1 \bar{P}_0$	=pressure acting on the pipe at the interface between the two soils
α_2	=a non-dimensional factor which describes the intensity of the applied pressure at the free end of the pipe
α^{r-1}	=the angle of the element in load step r
β	=correction factor β to account for the transition from the elastic range to the plastic one
γ	=a factor related to weighting function W_j ,
ϵ_e	=elastic strain
ϵ_c	=creep strain of the frozen soil
$\frac{d\epsilon_c}{dt}$	=the creep rate
ϵ_0	=axial strain due to small displacement theory
ϵ_l	=axial strain due to large displacements
ϵ_{sh}	=shrinkage strain in the pipe displacement
ϵ_v	=volumetric expansion
ϵ_x	=longitudinal normal strain
ϵ_y	=yield strain
η	=the porosity of the soil
θ	=rotation of the normal to beam
θ_i	=volumetric ice fraction, or rotation at node i in the beam-column element
θ_w	=volumetric fraction of liquid water

λ_s^m	=vertical flexibility of soil at time step m
ν_i	=Poisson's ratio of soil layer i
Π	=potential energy of the beam-column
π	=osmotic pressure of dilute solutions
ρ_i	=density of ice
ρ_s	=density of soil
ρ_w	=density of water
σ_f	=frost heave force
σ_{iw}	=surface tension at ice-water interface
σ_{sg}	=soil stress due to overlying soil weight and/or overburden pressure
σ_{sh}	=shrinkage stress
σ_s^{m-1}	=total stress of frozen soil at the end of time step $(m - 1)$
σ_{sr}	=soil stress due to the reaction of the pipe acting on the soil at the interface of soil and the pipe
σ_y	=yield stress
σ_{z1}	=stress at the bottom of soil layer 1
σ_{z0}	=stress at the top of soil layer 1
$(\Delta\sigma_i^r)_j^*$	=the true incremental stress in the j th layer
$(\Delta\sigma_i^r)_{je}$	=the elastic component of the incremental stress
$(\Delta\sigma_i^r)_{jp}$	=the plastic component of the incremental stress
ϕ	=aspect ratio, $\frac{l}{b}$

Chapter 1

Introduction

1.1 General

Over the past two decades there has been much interest in building buried pipelines to transport chilled gas in the permafrost areas of North America, Europe and Asia. It is well known that oil and gas resources are found in regions such as Alaska in the United States, Northern Alberta in Canada, Siberia in Russia and Heilongjiang in China, which are located in the permafrost areas. The frost-related phenomena in frozen soils have been of much interest to pipeline engineers working on projects in permafrost regions. Some pipelines have been built above the ground, such as the elevated parts of the Trans-Alaska Pipeline system, to avoid the differential ground movements, and thermokarst caused by extensive thermal degradation of the frozen soil in the permafrost areas, phenomena which may lead to failure of pipelines. But elevated pipelines are raising environmental concerns because they affect wild life migration. Thus, generally, gas pipelines are preferred to be buried in the frozen soil.

The most dangerous phenomenon related to frost action on gas pipelines buried in permafrost areas is thawing of ground which is a result of melting of ice in frozen ground. The thawing of ground not only causes the ground to move downward, but also greatly reduces the bearing capacity of the soil because thawed soil has a looser structure and a much higher moisture content compared to its prefrozen state.

A convenient method for preventing the drawing of existing permafrost is transporting gas in the pipelines at below freezing temperature. Hence long distance chilled gas buried pipelines are widely used to transport natural gas passing through permafrost regions. The major problem created by chilled pipelines traversing a

discontinuous permafrost region is the differential upward ground movement due to frost heave which may cause damage to the pipelines. Frost heave is the result of the expansion of freezing water in the ground and the consequent ground surface movements. When the ground experiences sub-zero temperature, first the pore water freezes, followed by the formation of segregated ice. The segregated ice forms primarily due to the freezing of the water that migrates from the underlying ground water to the freezing front. This newly arriving water freezes and the resulting ice lenses push the surrounding soil, thus causing upward ground movements.

Three conditions are required for frost heave to occur: sub-zero temperature, sufficient water supply, and frost-susceptible soil. It is well known that frost heave is affected by the location of ground water table. If the water table is situated rather far from the frozen soil, frost heaving will not be significant because as stated earlier, appreciable frost heave requires continuous water migration to the frozen fringe. Frost susceptibility is dependent on the particle size distribution of the soil which also influences its capillary and permeability properties. In nonuniform soils with more than 3% of soil grains being smaller than 0.02 mm, and in very uniform soils with more than 10% of soils grains being smaller than 0.02 mm, frost heave will occur (Eranti and Lee 1986). Canadian criterion for frost susceptibility of soils is shown in Fig.1.1. On the other hand, the permeability of very-fine-grained soils is low and the water migration to the frozen fringe will thus be limited. Generally, silts are highly susceptible, sands are not susceptible, and clays are somewhere in between.

In design of chilled gas pipelines buried in freezing soil, evaluation of the behaviour of pipelines should consider the following processes: frost heave, mechanical responses of the pipe and the thermo-mechanical soil-pipeline interaction.

There are numerous laboratory test results and empirical relations available to predict the magnitude of frost heave and to determine the values of the various parameters associate with frost heave. But laboratory or field tests can not provide universally applicable data because it is practically impossible to measure or to

simulate all the possible variations of the numerous parameters which can affect frost heave. Consequently, there is need for a suitable model which can provide scientific insight and practical guidance to pipeline designers.

During the last two decades extensive research has been carried out to understand the principles of frost heave and to formulate frost heave models (Tsytovich 1975, Miller 1980, Hopke 1980, Konrad and Morgenstern 1980, 1981, Williams and Smith 1989, Ladanyi and Shen 1989,1991, Shah and Razaqpur 1993) for scientific and engineering purposes. The inherent complexity of the freezing process has rendered most of the existing presented models rather complex, hence it may still take some time before these models can be widely applied in practice. The development of a simple, yet sufficiently accurate model for practical applications, has been and still remains the objective of a number of investigations.

Differential ground movements due to frost heave and creep of frozen soil are resisted by the buried pipeline and the surrounding soil, which resistance causes displacements and stresses in the pipeline. The response of pipeline to frost heave has been researched for a long time. The pipes have been generally modeled by 1-D elastic beam (beam-column) element, or 3-D elastic shell element undergoing small displacements . A common deficiency of those models is generally the omission of the nonlinear responses of the pipe. Experimental results have shown that for long term soil freezing, the displacements of the pipe (excluding rigid body motions) due to frost heave and creep may be much larger than the diameter of the pipe, and the stresses and strains developed in the buried pipe may be beyond the yield stress and strain. Therefore an accurate evaluation of the response of the pipe should take into account the nonlinear effects of the pipe in both geometry and material.

For pipelines buried in frozen soil, the soil-pipeline interaction is a hydro-thermo-mechanical process which is dependent on three processes: (1) **Frost heave** which causes the differential ground movements and it is a time-dependent process. (2) **Variation of frost-related properties of soil** which depends on the penetration of frozen zone and which will affect the mechanical response of the pipe. (3)

Creep of frozen soil which will cause additional stresses and deformations in the pipe. Various techniques have been used for modeling soil-pipeline interactions. All existing soil-pipeline interaction models have used classical mechanical analysis or two independent processes, i.e. the frost heave process in the freezing soil and the mechanical process in the pipe, to simulate the soil-pipeline interaction during frost heaving. The classical or separate analysis of soil-pipeline interaction can not describe reliably the time-dependent coupling of the frost deformation of the freezing soil and the mechanical response of the pipe primarily due to the effects of creep redistribution of stresses. Therefore, it is necessary to develop a reliable modeling technique for soil-pipeline interaction based on coupled thermo-mechanical analysis.

1.2 Objectives and Scope

It is evident from the preceding discussion that the design of chilled gas pipelines can be very complex due to the complexity of frost heave and the complex thermo-mechanical processes in soil-pipeline interaction. Most of the existing frost heave models only include the coupling of heat and moisture transport in frozen soils. Only a few models consider the effect of overburden pressure, but cumbersome calculations make those models difficult to apply in practice. It is the desire of most researchers and designers to develop a simple and practical frost heave model. On the other hand, to my knowledge, none of the existing theoretical models about soil-pipeline interaction is based on coupled analysis of the thermo-mechanical process, and none of the existing models for the analysis of stresses and deformations of the pipe buried in freezing soil consider the nonlinear geometric and material properties of the pipe during soil freezing.

The goal of this study is to develop a model for the complete thermo-mechanical analysis of pipelines buried in freezing soils, including frost heave, soil-pipeline interaction, frozen soil creep and the nonlinear effects of the pipe. The proposed method will be relatively simple but robust enough to be used in most practical applications. The specific objectives of this thesis are:

(1) Development of a generalized Clausius-Clapeyron equation valid for common soil system including cases where the water table is not located at the frozen fringe,

(2) Development of a simplified frost heave model suitable for practical applications which considers the coupled heat and moisture transport in frozen soils as well as the effects of overburden pressure,

(3) Development of a general nonlinear model of the pipe which considers the effects of both large displacements and elastoplasticity in the pipe, and the development of the accompanying computational procedures within the of finite element framework analysis, including incremental and iterative solution procedures,

(4) Development of a coupled thermo-mechanical model for soil-pipeline interaction during frost heaving. The model shall include the interactions , the soil deformations caused by frost heave, the changing soil properties due to variable thermal and moisture regimes in the soil, the effect of soil deformations on the movement of the pipe on the soil mechanical response, including its creep response.

Fig. 1.1 Grain distribution limit for frost susceptibility according to Canadian Department of Transport (after Eranti and Lee, 1986)

Chapter 2

Literature Review

2.1 General

The processes and phenomena that are germane to the subject matter of this thesis are frost heave, ground movements, the interaction between the ground movements and the buried pipeline, and the stresses and deformations of the pipeline caused by the displacements being imposed on it by the moving ground. Frost heave is a complex phenomenon, involving heat and moisture transport, including phase change. Freezing is a function of the local pressure and temperature of the water, which are constantly changing during long term freezing cycles. Considering that pipelines behaviour is governed by so many different phenomena, in this chapter we will briefly review the important literature related to each of those aspects.

2.2 Frost Heave Mechanism

Understanding the frost heave mechanisms has been a very important research subject for a long time. Various concepts and theories about frost heave have been proposed by previous researchers. In the following section the three basic concepts of frost heave and three important frost heave theories will be reviewed.

2.2.1 Basic Concepts of Frost Heave

2.2.1.1 Water Migration

It is well known that frost heave is not only due to freezing of in-situ pore water, but also due to water migration. As indicated by Tsytovich (1975) the basic process in freezing soil involves the redistribution of moisture as a result of migration of water during freezing and accumulation of segregation ice.

The migration of water in a frozen soil can take place in three physical states: vapour, liquid, and solid. The migration in solid state is the process of redistribution of ice in such a way that first ice melts to water and immediately it re-freezes under slightly altered conditions of temperature and external load. This phenomenon is known as regelation. Admittedly there is some controversy about water migration in solid phase, i.e. by regelation. Some investigators(e.g. Vyalov 1959) have reported that this form of water migration is not significant for freezing soil, and it can be neglected. Other studies (O'Neill and Miller ,1982,1985; Ohrai and Yamamoto,1985; Shah and Razaqpur,1993), have shown that the regelation contributes significantly to frost heave.

Water transport by vapour migration in freezing soil is different from that in unfrozen soils. Firstly, in frozen soils water vapour not only converts to liquid, but it also converts directly to ice, causing an increase in the ice content of the soil. Secondly, at a given temperature vapour migration can occur due to differential vapour pressure since the vapour pressure over ice is lower than the vapour pressure over water. When moisture content is very low, moisture can be transported only in the vapour state.

Extensive research has shown that the amount of water migration increases with an increase in total water content, mainly due to water migration in liquid state. Dirkson and Miller(1966), Tsytovich(1975), Williams and Smith(1989) indicated that the amount of vapour diffusion is very small compared to water flow in the liquid phase in water-saturated soils. Because the presence of a saturated soil is a necessary condition for frost heave to occur, the effect of vapour migration is negligible for practical purpose. Accordingly, the the dominant mode of water migration is via the liquid phase.

2.2.1.2 Frozen Fringe

Frost heave mainly occurs within a region of the soil known as the frozen fringe. Miller(1972) defined this zone as the zone between the freezing front and the base of warmest ice lens. The frozen fringe is characterized by the temperature T_f at

the freezing front and the segregation-freezing temperature T_s , where T_f is the nucleation temperature.

In fine grained porous soils T_f is sub-zero because of the actions of surface absorption and capillary tension. In some frost heave models T_f is defined as the normal freezing temperature, but T_f varies with both the type of soil and overburden pressure, as was reported by Anderson and Tice(1972), and Williams(1989). The segregation-freezing temperature is also dependent on the type of soil and overburden pressure. Konrad and Morgenstern (1982) in their experiments showed that the segregation temperature drops from -0.24°C to -0.39°C , when the overburden pressure increases from 0.2MPa to 0.4MPa. Most of the present frost heave models do not consider the variations of T_f and T_s . Generally, the effects of the freezing point depression and the segregation freezing temperature depression are not obvious, but in some special cases, with significant change of overburden pressure, the effects of the depression of T_f and T_s need to be considered.

During soil freezing, water migrates from the unfrozen soil through the frozen fringe to the base of the ice lens and freezes therein; thus, in the frozen fringe, the unfrozen water content decreases rapidly from 100% to 0% of the total water content. Consequently, ice content increases from 0% to 100% of the total water content, and a large water suction is developed in the fringe in response to the decreasing unfrozen water content. In many frost heave models, the unfrozen water content is used as a parameter to measure frost heave.

The water suction gradient and the temperature gradient in the frozen fringe depend on the thickness of the frozen fringe . It was indicated by Holden (1985), Ma and Wu(1991) that the thickness of the frozen fringe increases in response to the temperature distribution during the course of the freezing process. Consequently, the rate of frost heave decreases in response to the reduction of water suction gradient.

It is commonly recognized that permeability, which is dependent on the thickness of the unfrozen water film, is one of the dominant factors which control water

migration. Since the unfrozen water content decreases from the freezing front to the base of the warmest ice lens, the permeability of freezing soil decreases rapidly in the frozen fringe. It was reported by Burt and Williams(1976) that the permeability of a given soil at -2°C is 0.1% to 0.01% of the permeability of the same soil in unfrozen state.

2.2.1.3 Frost Heave Criteria

Various frost heave criteria have been proposed to determine the start of the frost heave process. Sheppard(1978) assumed that the critical ice content is equal to the difference between the porosity of the soil and the unfrozen water. Taylor and Luthin(1978) gave a critical ice content which is not dependent on the unfrozen water content, but is equal to 85% of its porosity. O'Neill and Miller(1985) used a criterion based on the relationship between the neutral stress in the soil and the overburden pressure. Specifically, it was stated that if the neutral stress exceeds the overburden pressure the ice lens will form and frost heave will occur.

Although there are different frost heave criteria, as stated by Shen and Ladanyi (1989), the difference in heave criteria does not yield much difference in the amount of calculated frost heave.

2.2.2 Capillary Theory

Capillary theory is one of the earliest frost heave theories(Shtukenberg 1885), in which it was first proposed that frost heave is caused by water migration from the unfrozen soil to the freezing front. According to capillary theory, the driving force for the migrating water to the frozen fringe is the capillary suction, which can be expressed by Kelvin's equation as

$$P_i - P_w = \frac{2\sigma_{iw}}{r_{iw}} \quad (2.1)$$

where

P_i = ice pressure

P_w = water pressure

σ_{iw} = surface tension at ice-water interface

r_{iw} = radius of the ice-water interface

According to Miller(1972), the migration force is due to the difference between water pressure and ice pressure at the water-ice interface which is caused by both the surface tension at the interface. When ice goes to penetrate into small pores, capillary tension develops due to the curvature of ice water interface, and water is drawn toward the ice front.

The freezing of soil described by capillary theory is a mechanical process which does not depend on temperature, temperature gradient and the penetration of the frozen zone. The latter is not in agreement with the well-known fact that frost heave is a thermo-mechanical process, involving coupled heat and mass transport.

Holden et al.(1981) derived an improved capillary equation(Eq.2.2) by combining Eq.2.1 and the Clausius-Clapeyron equation, and proposed that it be used for modelling frost-heave. The equation was written as

$$\Delta T = \frac{2V_m\sigma_{iw}T_0}{r_{iw}L_h} \quad (2.2)$$

where

T = temperature

T_0 = freezing point of water

$\Delta T = T - T_0$

L_h = latent heat of fusion

V_m = specific volume of water

Williams and Smith(1989) have indicated that Eq.2.2 is only valid at a temperature of 0°C . More importantly, the accumulation of migrating water and the formation of ice lens occur mostly at the base of the ice lens, and not at the freezing front, a phenomenon which can not be explained by the capillary theory. It can therefore be concluded that the capillary theory is incapable of accounting for the

more complex thermodynamic processes that occur in a freezing soil.

2.2.3 Thermodynamic Theory (Clausius-Clapeyron Equation)

The freezing process, including heat transfer, mass transport and the effect of overburden pressure, can be explained by thermodynamics, as embodied by the so-called Clapeyron or Clausius-Clapeyron equation. This equation relates ice and water pressure, and temperature in the form of a thermodynamic equation.

A widely used form of the Clausius-Clapeyron equation was derived by Kay and Groenevelt(1974) as

$$\frac{p_w}{\rho_w} = \frac{p_i}{\rho_i} + L_h \ln \frac{T}{T_f} \quad (2.3)$$

where

T_f =freezing point of water

T = temperature at measured point

ρ_w = density of water

ρ_i = density of ice

It can be seen that this equation gives the relationship between water and ice pressure, as well as the effect of latent heat which is released during water freezing. This equation implicitly establishes the relation between water suction and the overburden pressure if a certain relation can be found between the ice pressure and the overburden pressure.

The Clausius-Clapeyron equation has been verified by many experiments (Vignes and Dijkema 1974, Biermans et. 1978, Konrad 1989). According to this equation, a large water suction must develop in the frozen fringe to satisfy Eq.2.3, a large difference between the water pressure and the ice pressure must exist, and the maximum ice pressure is only limited by the overburden pressure. The development of such large suction in the frozen fringe is in agreement with measurements

in the laboratory.

It should be mentioned that Eq.2.3 is only valid for specific soil systems, i.e. it is valid only when the freezing front is located at the water table, because the latter equation is based on the assumption that $p_w = p_i = 0$ at the freezing front. Van Loon(1991) derived a different form of the Clausius-Clapeyron equation for an open soil system in which the water pressure at the freezing front is negative, i.e., the freezing front is not at the water table. However, in Van Loon's analysis, an incorrect boundary condition of negative ice pressure at the freezing front was used, thus rendering his equation to be the same as Eq.2.3. In this thesis a general form of the Clausius-Clapeyron equation will be derived in Chapter 3.

2.2.4 Migration Force Theory

Tsyтовich(1975) proposed a migration force theory to explain the general principles of frost heave. According to his theory, the migrating water flux v_w can be expressed in a general form as

$$v_w = -\alpha.gradF_m \quad (2.4)$$

where

v_w =water flux

α =a coefficient characterizing the specific resistance given by the soil system to water migration

$gradF_m$ = gradient of a generalized migration force, F_m

In this theory, if the phase equilibrium between water and ice is disturbed, and the related gradients are developed, which may exist in various forms of gradients such as moisture,temperature, absorption-film, osmotic as well as other pressures, a migration force develops and causes water flow along the direction of the generalized force gradient. The water flux is directly proportional to the migration force. Tsyтовich noted that 'any of the migration forces can be taken as generalized force.'

As a general frost heave theory, Eq.2.4 provides a general relation between water flux and migration force, which can be used not only to describe frost heaving, but also it can be used to explain many frost heave phenomena. For example, for absorption (Delmatov 1957), the difference between the absorption force and the overburden pressure is taken as a generalized migration force, with the water flux being directly proportional to this force. In the thermodynamic approach proposed by Porkhayev (1964), the generalized migration force is the potential of the film moisture. The physiochemical process during soil freezing and other phenomena that occur during frost heave, such as, segregation potential (Konrad and Morgenstern 1980) can also be described by this theory. It should be noted, however, that the migration force theory provides a basic framework within which different models can be developed, depending on the various assumptions made. The simplified model in Chapter 4 of this thesis was developed within this general framework.

2.3 Frost Heave Models

Since the 1970's mathematical modeling of frost heave in freezing soils has become a very important subject of research. A number of frost heave models have been proposed based on various principles and assumptions. A few important frost heave models will be reviewed in the following section.

2.3.1 Hydrodynamic Model

One of the earliest hydrodynamic frost heave models which coupled heat and mass transport was proposed by Harlan(1973). The model was governed by the equation

$$c_s \frac{\partial T}{\partial t} + L_h \rho_i \frac{\partial \theta_i}{\partial t} = \frac{\partial}{\partial x} \left(k_s \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k_s \frac{\partial T}{\partial y} \right) - c_w \rho_a \left(\frac{\partial v_x T}{\partial x} - \frac{\partial v_y T}{\partial y} \right) \quad (2.5)$$

where

c_s = heat capacity of soil

c_w = heat capacity of water

θ_i = volumetric ice fraction

k_s = thermal conductivity of soil

v_x = water flow velocity in x direction

v_y = water flow velocity in y direction

The last two terms in Eq.2.5 represent heat transfer by convection which is deemed to have negligible effect on frost heave (Taylor and Luthin,1978). Using Eq.2.5, Harlan provided the variations of temperature, water suction, ice content and unfrozen water content in response to soil freezing, but he did not calculate the amount of frost heave.

Since 1973, other hydrodynamic frost heave models have been developed based on Harlan's original model. Taylor and Luthin(1978) developed a frost heave model based on the following simplified form of Eq.2.5

$$c_s \frac{\partial T}{\partial t} + L_h \rho_i \frac{\partial \theta_i}{\partial t} = \frac{\partial}{\partial x} \left(k_s \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k_s \frac{\partial T}{\partial y} \right) \quad (2.6)$$

In this model it was shown that the amount of frost heave could be calculated by introducing the criterion that frost heave would occur only when the ice content exceeds 85% of soil porosity. Jame and Norum(1980) used experimental results of unfrozen water content to solve Eq.2.6 and to evaluate the amount of frost heave.

Guymon et al.(1981, 1984) proposed a 1-D and a 2-D frost heave model. In their models they assumed that during soil freezing, a certain portion of water in the soil remains in unfrozen condition. Since water migrates continuously into the frozen fringe from the unfrozen soil, at some point the water content will exceed the maximum unfrozen water content and the extra water will freeze, causing frost heave. The maximum unfrozen water content was assumed to be the same as the soil porosity.

Based on the above models, several numerical simulations have been performed

using the time-dependent finite element method, or the finite difference method. An obvious drawback of these models is that they employ classical hydrodynamic analysis, combining heat and moisture transfer, to describe the soil freezing process which is in reality a hydrodynamic and thermodynamic process. Consequently they are not able to explain the thermodynamic equilibrium between water pressure and ice pressure during freezing of soils, and it follows that the very important effect of overburden pressure can not be included in those models

2.3.2 Segregation Potential Model

Konrad and Morgenstern(1980, 1981, 1982) proposed a frost heave model based on the following assumptions: (1) The frozen fringe can be characterized by segregation freezing temperature T_s , an overall permeability, and freezing temperature $T_f = 0$. (2) The Clausius-Clapeyron equation is valid at the base of the ice lens. (3) Water flow through the frozen fringe is continuous.

In this model, the frost heave is caused by migrating water which flows from the unfrozen soil to the frozen fringe and freezes at the base of the ice lens. The driving force for water migration is the suction which is developed at the interface of water and ice. The model employs the Clausius-Clapeyron equation to determine the water pressure under the condition of zero overburden pressure, expressed as

$$P_w = \frac{L_h \Delta T}{V_w T_f} \quad (2.7)$$

where

$$\Delta T = T - T_f$$

$$T_f = \text{freezing point of water}$$

$$V_w = \text{specific volume of water}$$

Konrad and Morgenstern assumed that the water intake velocity, v_w , at the time of the formation of the final ice lens is proportional to the temperature gradient in the frozen fringe (Fig.2.1). The principle feature of the model is the introduction

of an engineering parameter, SP , defined as a segregation potential, which couples heat and mass transfer via a linear relation between water flux and temperature gradient. To overcome the difficulties of locally measuring permeability and thermal properties of the freezing soil, as well as its unfrozen water content, a macroscopic analysis is used, assuming a linear distribution of water flux in the frozen fringe, with the water flux being expressed as

$$v_w = SP.gradT \quad (2.8)$$

where

SP = segregation potential

$gradT$ = gradient of temperature in the frozen fringe

It should be pointed out that this model can be explained by Tsytovich's migration force theory, the generalized migration force is the segregation potential and the temperature gradient is the proportionality factor.

To be able to consider the effects of overburden pressure, Konrad and Morgenstern(1984) proposed the relation

$$SP = SP_0.exp(-a.P_e) \quad (2.9)$$

where

SP_0 = the value of SP under zero overburden pressure

P_e = overburden pressure

a = experimental constant

This model has been used by a number of investigators to model frost heave and its effect on buried pipelines (Dallimore 1984, Nixon 1985, Konrad and Morgenstern 1984). The results have been found to be satisfactory for engineering purposes. The major problem with this model is the introduction of a macroscopic soil parameter, i.e., SP , which needs to be determined for each soil type and possibly local soil

conditions.

2.3.3 Rigid Ice Model(RIM)

To date, the rigid ice model(O'Neill and Miller 1982) is the only model which includes water transport in both the liquid and the solid phases. In RIM it is assumed that under suitable conditions of temperature gradient, the pore ice is closely connected to the warmer ice lens, forming a continuous rigid and intricate body of ice. The formation of continuous rigid ice can be explained as follows: In partially frozen soil, incoming water is attracted by existing ice which acts as a nucleus, and subsequently under suitable condition of temperature gradient the water converts to ice around the nucleus.

According to RIM, unfrozen water migrates from unfrozen soil into the frozen fringe, accumulates at the warm side of the ice body, and as a result of increase of water content, water pressure increases, then the equilibrium between water pressure and ice pressure is disturbed at the interface of water and ice on the warm side of ice body. To reestablish equilibrium some of the water freezes, releases latent heat, leading to ice pressure increase with increase of ice content. If the ice body is at rest, from statical equilibrium condition, the ice pressure on the cold side should increase, but water pressure is unchanged (water content is constant). The latent heat released by the freezing water on the warm side is absorbed by the ice on the cold side, causing some ice to melt, hence the equilibrium between ice and water pressure on the cold side is once again disturbed, and ice pressure decreases to satisfy the equilibrium condition at the interface of water and ice on the cold side of the ice body. As a result of the changes of ice pressure on both sides of the ice body, one increases and the other decreases, a driving force is developed to move the ice body from warmer side to the colder side(Fig.2.1).

When the driving force is large enough to move soil particles, the particles are pushed apart by the force, then the particle-free-zone is occupied by the water flowing across the fringe. This water subsequently freezes and forms a new ice lens under sub-zero temperature conditions.

In the rigid ice model, it is assumed that the ice body has a uniform ice velocity which is taken into account in frost heave calculations. Since the rigid ice body is surrounded by a water film in which water flows continuously from the unfrozen soil into froze fringe, the water migration in the liquid state is considered simultaneously with the ice velocity.

In RIM, the Clausius-Clapeyron equation is used to describe phase equilibrium, the governing equations consist of mass transport equation, energy transport equation, and ice velocity equation. The frost heave is determined by solving the governing equations under two prescribed boundary conditions, the conditions at the warm and cold surfaces of the ice body.

Shah and Razaqpur(1993) developed a two dimensional rigid ice model. In their model, the governing equations were solved by using the finite element method in space and the finite difference method in time. A numerical simulation of chilled gas pipeline was provided by combining the 2-D RIM with a nonlinear elasto-plastic stress model, and the results of the analysis were found to be in reasonable agreement with experimental results.

The principal advantage of the rigid ice model over most existing models is that it considers the effects of overburden pressure and regelation, its main disadvantage is the complexity of the calculation and the lack of experimental evidence for the existence of regelation. Simplified versions of this model would need to be developed, to reduce the complex calculations, before it can be widely used in practice.

2.3.4 Coupled Heat Moisture and Stress Fields

From the engineering point of view, analysis of the stresses and deformations of soils during frost heave is very important for evaluating the response of structures built in or on freezing soils because the stresses and strains of structures are directly influenced by the stresses and strains in the soils. Blanchard and Fremond(1985) proposed a coupled heat, moisture and stress fields model based on the principle of conservation of energy. This was the first frost heave model which included the

analysis of stresses in the soil, but it did not consider the effects of creep strain in frozen soils, and it did not use the Clausius-Clapeyron equation to describe the equilibrium between water pressure and ice pressure in the frozen fringe.

Another model for coupling heat, moisture and stress fields was provided by Shen and Ladanyi(1987) which includes the effects of creep strain and the volumetric strain in frozen soil due to the phase change of migrating water. In this model the governing equations consist of mass and heat transfer equations and the Clausius-Clapeyron equation. The mass transfer equation is

$$\frac{\partial \theta_w}{\partial t} + \frac{\rho_i}{\rho_w} \frac{\partial \theta_i}{\partial t} = \frac{1}{\rho_w \cdot g} \left[\frac{\partial}{\partial x} \left(K \frac{\partial P_w}{\partial x} \right) + \frac{\partial}{\partial y} \left(K \frac{\partial P_w}{\partial y} \right) \right] \quad (2.10)$$

where

θ_w = volumetric fraction of liquid water

K = permeability

x, y = coordinates

The frozen soil deformations were calculated using

$$\epsilon = \epsilon_e + \epsilon_c + \epsilon_v \quad (2.11)$$

where

ϵ_e = elastic strain

ϵ_c = creep strain

ϵ_v =volumetric expansion strain

Numerical simulations based on this model were performed by Shen and Ladanyi which used the finite difference method to solve the heat and mass transport equations, and the finite element method to calculate the stresses and strains in the frozen soil. Two numerical simulations were performed, one was the Penner's (1986) frost heave experiment, the other was the chilled gas pipeline experiment carried out in Caen, France. The calculated results were in good agreement with both

experimental results.

Peng et al.(1991) proposed a coupled heat, moisture and stress fields model by introducing an equation of the form

$$\begin{aligned} \frac{\rho_i \rho_w}{\rho_i + \rho_w} \frac{\partial}{\partial x} \left(K \frac{\gamma}{\rho_i} + \frac{K L_h}{T_k} \frac{\partial T}{\partial x} + \frac{K}{\rho_i} \frac{\partial \sigma_f}{\partial x} \right) \\ = \frac{\partial \theta_w}{\partial T} \frac{\partial T}{\partial t} + \frac{\rho_i}{\rho_w} \frac{\partial \theta_i}{\partial t} \end{aligned} \quad (2.12)$$

where

$$\gamma = P_e - P_i$$

$$\sigma_f = \text{frost heave force}$$

and

$$P_e = \text{overburden pressure}$$

In this model, the frost heave stresses can be obtained in a single calculation process by solving the above equation together with the heat transfer equation and the Clausius-Clapeyron equation. The preceding models are not suitable for practical applications, especially to chilled gas buried pipelines because first they are too complex and time consuming to apply and second they ignore the presence of the pipeline and its effect on the frost heave and soil behaviour.

2.3.5 Other Models

The first frost heave model which included the effects of overburden pressure, was proposed by Hopke(1980) based on capillary theory. In this model, it was assumed that the ice pressure is equal to zero at the freezing front, and to the mean value of overburden pressure at the base of the ice lens. The Clapeyron equation was used in this model to determine the water pressure. Although Hopke's frost heave model did not provide a good prediction of frost heave, his method for considering overburden pressure has been widely used in many subsequent models.

Another frost heave model based on energy equilibrium was provided by Miyata(1988). In this model, the ice segregation rate was derived from the balance of

hydraulic energy of the unfrozen pore water and the released mechanical energy due to the freezing of water. This model considers the heat and mass transfer as well as the effects of overburden pressure. It was shown that it can simulate well the microscopic phenomenon of frost heave. Finally, Fremond and Mikkola(1991) proposed a thermo-mechanical model of freezing soil. In this model, water suction was assumed to develop due to the phase change of freezing water in a porous medium. A thermodynamic analysis, combined with mechanical analysis, was carried out by choosing appropriate expressions for both the free energy and the dissipation potential to describe the suction, the mass and heat transfer and the frost heave caused by formation of ice. But no comparisons were made between the results based on this model and the corresponding experimental data. In fact, to the writer's knowledge this model has not been actually implemented.

2.4 Soil-Pipeline Interaction

In the analysis of soil-pipeline interaction, the model of beam on Winkler foundation and continuum mechanics approach have been generally employed.

Selvadurai (1983, 1985) developed some approximate analytical solutions of soil-pipeline interaction based on a Winkler model, which were derived by solving the governing differential equation of beam on elastic foundation. In his analysis, the loads on the pipe caused by ground movements were assumed to be given, hence the analysis included mainly the mechanical soil-pipeline interaction since it did not consider the variations of ground movements and soil properties during soil freezing. Later he provided several numerical simulations by the finite element method or coupled finite element-boundary element analysis which dealt with the geometric non-linearity of the pipe and material non-linearity of soil, but these are based on conventional finite element techniques. Razaqpur(1991) proposed an accurate finite element for analysis of a beam-column on Winkler foundation, which combines a fundamental solution of governing equation with finite element technique. This element can be conveniently used to simulate mechanical interaction of beam-column

and foundation with high accuracy. Another analytical solution was derived for soil-pipeline interaction by Rajani(1992) which considers the effects of creep in the soil.

A model of back-analyzing the results of observation made during test at Caen was provided by Ladanyi and Lemaire(1985), which was based on the finite difference method. Good agreement between their results and the experimental results was obtained, but this model can not give a continuous analysis of soil-pipeline interaction during frost heaving, and it does not consider the non-linear responses of the pipe. Using continuum mechanics, Nixon et al.(1983) proposed a model for frost heave-pipeline interaction. They assumed that the soil was an elastic or non-linear viscous continuum, and the pipe to be a structure embedded within it. In their study, the discontinuous movements at the interface between frost susceptible and non-frost susceptible soils were simulated by displacement and pressure boundary condition associated with the amount of frost heave. Selvadurai(1993) gave a modeling of time dependent soil-pipeline interaction in which the interaction was induced by the time-dependent growth of a frost bulb. His analysis was based on the empirical relationship provided by Nixon(1987), i.e.

$$X(t) = X_0(t)^{\eta A} \quad (2.14)$$

$$B(t) = B_0(t)^{\eta B} \quad (2.15)$$

where

$B(t)$ = width of bulb

$X(t)$ = depth of bulb

X_0 , B_0 , ηA and ηB are constants which depend on soil properties

In this analysis, it was assumed that the thermo-mechanical interaction can be considered as two independent processes, thermal process and mechanical process. This model dealt with the creep strain in the frozen soil and the volumetric strain

caused by frost heave, but it did not consider the nonlinearity of the pipe.

Shah and Razaqpur(1993) studied the stresses and deformations in the chilled pipeline buried in frozen soil. They used the 2-D rigid ice model to analyze the hydro-thermal processes during soil freezing, and the finite element method to calculate the mechanical responses of pipe and the soil. In the process of time dependent interaction, the effects of frost heave due to water migration in the soil phase were included in their model. The soil and the pipe were modeled as elasto-plastic media, and the pipe-soil interface behaviour was included in their analysis. The Caen pipeline was modelled and satisfactory results for displacements in the pipe were obtained.

A coupled heat moisture and stress fields model combined with the finite element method was used by Shen and Ladanyi(1991) to analyze the interaction during frost heaving. In this analysis, the creep of frozen soil and the expansion of freezing soil were considered. Due to the limitation of their modeling of soil-pipeline interaction, only the results in two limiting cases, the case of a free floating pipe and the case of a rigid fixed pipe were studied.

2.5 Summary

From the preceding review, it can be seen that a great deal has been achieved in the areas of frost-heave modelling and soil-pipeline interaction. Nevertheless, practical problems associated with stresses and deformations of pipelines in freezing soil have not been fully solved. Problems that still need further refinement and investigation include:

(1) Development of simple and practical frost heave models

Up to the present, most of the proposed frost heave models are complex. They can not be conveniently used in practice, even though some of them can predict the amount of frost heave very well. The SP model is the simplest, but, the two parameters, i.e. the segregation potential under zero overburden pressure, SP_0 ,

and the coefficient of applied load, α , require experimental data. For engineering purposes, an even simpler and easily implementable model is needed.

(2) Investigate the relationship between frost heave and the location of water table

It is well known that the frost heave is closely related to the location of the water table. Up to now most of the present frost heave models consider only the freezing process in saturated soils. Some models deal with the unsaturated soils, but do not account for the relation between frost heave and water table.

(3) Development of a generalized Clausius-Clapeyron equation

The form of the Clausius-Clapeyron equation derived by Kay et al.(1974) is only valid when the freezing front is located at the water table. Yet nearly all available models, which include the phenomena implied in the Clausius-Clapeyron equation, use that equation, despite the fact that the actual conditions may not be in compliance with the original assumptions of Kay et al. Accordingly, it is necessary to derive a generalized Clausius-Clapeyron equation for the freezing soils.

(4) Investigate frost heave mechanisms

Study of regelation, i.e. moisture transport through solid phase, is significant phenomenon in frost heave.

(5) Study the thermo-mechanical soil-pipeline interaction

To date no model of soil-pipeline interaction uses fully coupled thermo-mechanical analysis. Some advanced models analyze the time dependent process of interaction by using two independent processes, thermal process and mechanical process, but these simulations can not evaluate accurately the variations of frost-related soil properties and the effects of creep in frozen soils, and they can not be used in general cases because the analysis of mechanical interaction depends on the results of data from frost heave tests.

(6). Describe the non-linear response of pipe

For long term soil freezing, the material and geometrically non-linear behaviour

of the pipe is not negligible. None of the existing models of soil-pipeline interaction currently combine the nonlinear soil-pipeline interaction and the non-linearities involved in frost heave development. In fact, most existing sophisticated frost-heave models treat the soil-pipeline interaction in a freezing soil as a plane strain problem. From a practical view point the plane strain analysis is of limited use because insofar as the pipe is concerned, flexural action along the pipeline is the dominant mode of deformation. Thus, there is need for a model which treats the behaviour of the pipeline in the longitudinal direction rather than in the cross-sectional plane.

This thesis will attempt to address a number of these effects and it is intended to develop a clear and workable computer program for practical analysis of pipelines in freezing soils.

Ice segregation and frost heaving

Fig.2.1 Series-parallel transport in a frozen non-colloidal soil (after Miller et al., 1975)

Chapter 3

A General Clausius-Clapeyron Equation For Freezing Soils

3.1 General

The freezing of soil is a complex process involving heat transfer (including phase change), mass transport and their interaction. Edlefsen and Anderson (1943) pointed out that consideration of only the change of mechanical energy during soil freezing is too limited because the effect of temperature on the total energy change of a system must also be taken into account. The well known thermodynamic equilibrium equation for two phase systems, known as the Clausius-Clapeyron equation, can be used to explain the presence of the large suction which drives the mass transport in the freezing soil and the overall freezing process. This is a fundamental equation for the analysis of freezing of soil.

A commonly used form of the Clausius-Clapeyron equation, derived by Kay and Groenevelt (1974), is

$$P_w = \frac{\rho_w}{\rho_i} P_i + L_h \rho_w \ln \frac{T}{T_f} \quad (3.1)$$

where $P_w(P_i)$, $\rho_w(\rho_i)$ are the water(ice) pressure and density, T_f is the absolute temperature and L_h is the latent heat of fusion of water.

For the last twenty years, Eq.3.1 has been commonly used by many researchers to analyze the process of soil freezing, and its applicability has been verified experimentally by Vignes and Dijkema (1974), Biermans, Dijkema and Vries (1978), Hopke(1980) and Konrad(1989). More recently Van Loon (1991) derived a new form of the Clausius-Clapeyron equation. In Van Loon's formulation, the water pressure at the freezing front was assumed to be a function of the distance between

the freezing front and the water table. However, due to an error in the assumed boundary conditions, the model predicts the ice pressure to be equal to the water pressure at the freezing front. Otherwise, the form of Clausius-Clapeyron equation derived by Van Loon is the same as that by Kay and Groenevelt.

Generally speaking, Equation 3.1 is only valid under the condition that $P_w = P_i = 0$ at the freezing front. In other words, Eq.3.1 is only valid when the freezing front is located at the water table level. It was pointed out by Loch(1978) that this form of the Clausius-Clapeyron equation is not valid during penetration of the freezing front. Fons(1993) and author have noted the following phenomenon: if Eq.3.1 is used to study the freezing process in an open soil system in which the freezing front and the water table are separated by some distance, then a water suction profile should develop as shown in Fig.3.1. Figure 3.1, however, shows an anomalous situation in which the water pressure gradient between the freezing front and the water table is zero. As a consequence, no water can flow from the water source across the unfrozen soil into the frozen fringe in response to the zero driving force of migrating water in the unfrozen soil, notwithstanding a high water pressure gradient in the frozen fringe. Although it can be argued that in practice the water table in most cases is likely to be quite close to the freezing fringe, nevertheless it is useful to derive a more general relationship which can correct the foregoing anomaly.

In this chapter a general form of Clausius-Clapeyron equation is derived from the basic thermodynamical principles. This equation can be used in any common soil system, including closed and open soil system, and is believed to provide a reliable water suction profile from the base of ice lens to the water table.

3.2 Thermodynamic Equilibrium between Ice and Water in Frozen Soils

In studies of phase change, a property of the system, known as the Gibbs free energy, which is a measure of the ability of the system to perform work, is given by

Williams and Smith(1989) as

$$G_j = e_j + P_j v_j - T_j s_j \quad (3.2)$$

where the subscript j refers to system j and

G_j = free energy (J)

e_j =internal energy (J)

P_j =applied pressure(J/ m^3)

v_j =specific volume (m^3)

T_j =absolute temperature (k)

s_j =entropy (J/k)

According to the second law of thermodynamics, the increase in the entropy of system j , ds_j , can be related to its increment of heat intake, dh_j , by

$$ds_j = \frac{dh_j}{T_j} \quad (3.3)$$

A differential form of Eq. 3.2 given by Fons(1993) is

$$dG_j = de_j + v_j dP_j + P_j dv_j - T_j ds_j - s_j dT_j \quad (3.4)$$

where the prefix d denotes differential quantities

According to the first law of thermodynamics, the increment of internal energy of system j , de_j , can be expressed by

$$de_j = dh_j - dW_j = dh_j - P_j dv_j \quad (3.5)$$

where

dW_j =increment of work done by system j

Substituting Eq.3.5 into Eq.3.4, gives

$$dG_j = dh_j - P_j dv_j + v_j dP_j + P_j dv_j - T_j ds_j - s_j dT_j \quad (3.6)$$

or

$$dG_j = dh_j + v_j dP_j - T_j ds_j - s_j dT_j \quad (3.7)$$

Combining Eqs.3.3 and 3.7, yields

$$dG_j = v_j dP_j - T_j ds_j \quad (3.8)$$

From Eq.3.8 the increments of the free energy of water and ice can be expressed, respectively by

$$dG_w = v_w dP_w^* - s_w dT_w \quad (3.9)$$

$$dG_i = v_i dP_i - s_i dT_i \quad (3.10)$$

where

$$P_w^* = P_w - \pi$$

and

P_w^* =total water pressure

P_w =water pressure of pure water

π =osmotic pressure of dilute solutions

In the present analysis it is assumed that salt effects are negligible, thus π is zero. Consequently, Eq.3.9 can be written as

$$dG_w = v_w dP_w - T_w ds_w \quad (3.11)$$

As described by Kay et al. (1974), from the view point of thermodynamical equilibrium, at the interface of water and ice in a frozen soil, the free energy of water and the free energy of ice must be in equilibrium before and after a change of temperature. The thermodynamical expressions corresponding to those conditions are given by

$$G_w(T) = G_i(T) \quad (3.12)$$

and

$$G_w(T + dT) = G_i(T + dT) \quad (3.13)$$

From Eqs.3.12 and 3.13, the increment of free energy of water is equal to the increment of free energy of ice, i.e.

$$dG_w = dG_i \quad (3.14)$$

Substituting Eqs.3.9 and 3.10 into Eq.3.14, yields

$$v_w dP_w - s_w dT_w = v_i dP_i - s_i dT_i \quad (3.15)$$

Assuming that at the interface of water and ice

$$T_w = T_i = T \quad (3.16)$$

and

$$dT_w = dT_i = dT \quad (3.17)$$

Eq. 3.15 becomes

$$s_w - s_i = v_w \frac{dP_w}{dT} - v_i \frac{dP_i}{dT} \quad (3.18)$$

For the total system, including water and ice, the left term of Eq.3.18 is the increment of entropy of the frozen soil which can be expressed as

$$s_w - s_i = \Delta s = \frac{\Delta h}{T} \quad (3.19)$$

where dh = the heat released into the soil due to the phase change during freezing. But the latter quantity is actually the latent heat of fusion, hence

$$s_w - s_i = \Delta s = \frac{L_h}{T} \quad (3.20)$$

Combining Eqs.3.18 and 3.20 yields the well known Clausius-Clapeyron equation

$$v_w \frac{dP_w}{dT} - v_i \frac{dP_i}{dT} = \frac{L_h}{T} \quad (3.21)$$

Since $v_w = \frac{1}{\rho_w}$ and $v_i = \frac{1}{\rho_i}$, Eq.3.21 can be written as

$$\frac{dP_w}{\rho_w} - \frac{dP_i}{\rho_i} = L_h \frac{dT}{T} \quad (3.22)$$

The water pressure at the freezing front is generally known from prescribed boundary conditions as shown in Fig.3.2. Specifically, before the development of a high water suction gradient due to freezing, the pore water is in static equilibrium with the surrounding soil. When the temperature of the soil falls below the freezing temperature, a high water suction profile develops due to freezing of water, thus the water located at the freezing front migrates upward to the base of the ice lens, and the water is drawn from the unfrozen soil into the frozen fringe by the water suction at the freezing front. The water suction at the freezing front given by Van Loon is

$$P_w(T_f) = \overline{Z} \cdot \overline{B} \quad (3.23)$$

where

$\bar{Z}$ =water head of the freezing front

$\bar{B}$ =9.8 KPa/m

T_f =absolute temperature of normal freezing point

In fact, when water flows from the unfrozen zone into the frozen fringe, the water suction at the freezing front is larger than that given by Eq.3.23. $P_w(T_f)$ is the minimum water suction needed to draw water into the frozen fringe.

Equation 3.22 is a differential equation which gives the local relations between the increments of water pressure, ice pressure and temperature. The relationship between water pressure, ice pressure and temperature can be obtained by integrating Eq.3.22

$$\int_{T_f}^T \frac{dP_w}{\rho_w} - \int_{T_f}^T \frac{dP_i}{\rho_i} = L_h \int_{T_f}^T \frac{dT}{T} \quad (3.24)$$

or

$$\frac{P_w(T) - P_w(T_f)}{\rho_w} - \frac{P_i(T) - P_i(T_f)}{\rho_i} = L_h \ln \frac{T}{T_f} \quad (3.25)$$

Considering that the ice pressure at the freezing front is zero because water just begins to freeze there and the ice content is zero, the water suction at the freezing front can be seen as a boundary value which can be obtained using general geomechanical method for unfrozen soil and it is related to the distance between the freezing front and water table. Accordingly, the boundary conditions at the freezing front are

$$P_i(T_f) = 0 \quad (3.26)$$

$$P_w(T_f) = P_{wf} \quad (3.27)$$

where

P_{wf} =water pressure at the freezing front

Substituting Eqs.2.26, 2.27 into Eq.3.25, the general form of Clausius-Clapeyron equation for frozen soil can be written as

$$\frac{P_w}{\rho_w} - \frac{P_{wf}}{\rho_w} = \frac{P_i}{\rho_i} + L_h \ln \frac{T}{T_f} \quad (3.28)$$

or

$$P_w = \frac{\rho_w P_i}{\rho_i} + L_h \rho_w \ln \frac{T}{T_f} + P_{wf} \quad (3.29)$$

where

P_w =water pressure at temperature T

P_i =ice pressure at temperature T

In the case in which the freezing front is located at the water table, one can substitute $P_{wf} = 0$ in to Eq.3.29, which would cause Eq.3.29 to revert to the conventional Clausius-Clapeyron equation given by Eq.3.1.

3.3 Summary

In this chapter a modified form of the Clausius-Clapeyron equation was derived from thermodynamic considerations which include the effect of the location of the water table relative to the freezing front on the freezing process in porous media. The equation shows that the freezing of soil does not occur in a close system and that it is affected by the water suction at the frozen fringe. Thus, the water suction within the frozen fringe varies not only with the overburden pressure, temperature gradient, but also with the suction of the freezing front. In other words, and the freezing process is affected by the location of the water table. The Clausius-Clapeyron equation derived by Kay and Groenevelt is only a special form of the more general equation derived herein and it applies only to those cases in which the freezing front is located at the water table level.

The shape of water suction profile from the proposed Clausius-Clapeyron equation is shown in Fig.3.3, which is in agreement with the shape of the suction profile

experimentally obtained by Williams(1982), Konrad and Morgenstern (1982), Konrad (1989).

Fig. 3.1 Water suction profile at frozen fringe given by Clausius-Clapeyron equation derived by Kay & Groenevelt

Fig.3.2 Water suction

Fig.3.3 Water suction profile given by the proposed Clausius-Clapeyron equation

Chapter 4

A Simplified Frost Heave Model

4.1 General

Since the end of the nineteenth century numerous investigators have studied frost heave phenomena because the upward movements of ground caused by frost heave, damage structures built in the regions in which seasonally frozen ground occurs. During the last two decades a number of frost heave models, based on different assumptions, have been proposed for predicting the magnitude of frost heave and soil stresses during freezing.

As is well known, frost heave is an extremely complex process being affected by various factors such as moisture migration, heat transfer, overburden pressure, soil type, etc. The complexity of the frost heave process causes difficulties in developing simple yet practical frost heave models. To use most of the existing models, local temperature, unfrozen water content, ice content, and soil properties have to be measured. The large in-situ variability in the magnitude of those properties make practical applications of the complex models difficult.

For engineering purposes, a frost heave model should have two basic attributes: the model should be easy to understand and apply, and it should provide a simple method to calculate the amount of frost heave with acceptable accuracy. Based on those considerations, an engineering oriented frost heave model was proposed by Konrad and Morgenstern(1980,1981,1982) in which it was assumed that the three macroscopic parameters which characterize a freezing soil are the average temperature, the average permeability, and the average suction in the frozen fringe. According to Konrad and Morgenstern, " the first two parameters can be obtained quite simply from a conventional freezing test. But since the average suction is strongly related to the shape of suction profile which in turn depends on the actual

shape of the permeability profile, it is impossible to determine with any satisfactory degree of accuracy the value of that average suction." Thus in that model an engineering parameter termed segregation potential, SP , in the frozen fringe was introduced, which coupled mass transfer to heat flow. It was suggested by the latter investigators that SP was the property of a given soil, and it could be determined from laboratory freezing tests. The total amount of water intake flux, u , in the frozen fringe was related to SP by the formula $u = SP \text{grad}(T)$, where grad is the gradient operator. Although the segregation potential model has proven useful, the notion that SP is an intrinsic soil property is debatable and it perhaps needs further theoretical justification.

As an alternative to the SP model, in this chapter a one dimensional frost heave model is developed based on Hu's hypothesis (1993) which states that frost heave occurs mainly due to the freezing of the water migrating through the frozen fringe, and the driving force for moisture transfer is only the water potential. The proposed model is macroscopic and is designed to overcome the difficulty of determining the actual shape of the suction profile in the frozen fringe. The model requires that only the water pressure at the coldest side of the frozen fringe be calculated. It is assumed that the ice pressure at the coldest side of the frozen fringe is equal to the mean value of the local overburden pressure. The assumption is based on the hypothesis that the soil on the coldest side of the frozen fringe behaves as a solid and its pressure can be determined from statical equilibrium conditions. The water pressure at the coldest side of the frozen fringe is obtained using the general Clausius-Clapeyron equation developed in Chapter 2. The water suction at the frozen front is specified as a given boundary condition. Finally, the intake water flux across the frozen fringe is assumed to depend on the gradient of the water potential and on the overall permeability of the frozen fringe and can be determined by Darcy's law.

In the analysis for the unsteady heat flow in soil, it is assumed that the latent heat associated with the amount of freezing water is released at the coldest side

of the freezing fringe which means that the coupling between heat and moisture transfer occurs only within the frozen fringe .

The proposed model is employed in a finite element program, using a time-marching scheme, in conjunction with an iterative technique, to calculate the time-dependent variation of the amount of frost heave. Some numerical examples are solved to ascertain the accuracy of the method.

4.2 Basic Assumptions

The proposed simplified frost heave model is based on the following basic assumptions:

1. The water flow across the frozen fringe is continuous.
2. The Clausius-Clapeyron equation is valid at the coldest side of the frozen fringe.
3. The freezing fringe can be characterized by a particular segregation freezing temperature T_s and a general freezing temperature T_f which can be zero or sub-zero temperatures.
4. The effect of salt, and other solute within the soil, on the freezing process is negligible.
5. The volume of soil particles is unchanged during the freezing process.

4.3 The Freezing Process

It is well recognized that for significant frost heave to occur, the following three conditions are required: (1) The soil temperature must be sub-zero °C; (2) The soil must be frost susceptible; (3) The water table must be near the freezing front to supply a continuous water flow into the freezing fringe.

The freezing process mostly occurs in the zone between the freezing front and the growing ice lens, referred to as the frozen fringe (Miller 1972). The freezing fringe can be characterized by a segregation freezing temperature T_s , i.e. the

warmest temperature at which ice can grow, and the normal freezing point of water T_f as shown in Fig.4.1.

Konrad and Morgenstern(1981) described the freezing process from the equilibrium of free energy between water and ice. When the ground surface experiences a sudden sustained cold temperature, unsteady heat flow develops in the soil. A temperature gradient in the soil develops which leads to the removal of heat from the warm soil to the ground surface. Once the temperature reaches the freezing temperature at a location in the soil, the pore water freezes in situ, but liquid water still exists in the form of absorbed water film surrounding the soil particles. Since the free energy of the unfrozen water film varies with the change in the temperature, a water suction gradient is produced in response to temperature gradient. The water suction drives the water migration from the unfrozen soil through the continuous unfrozen water film into the frozen fringe.

The water flow is governed by both the local permeability and the water suction in the soil. Note that for the same soil the permeability changes during the frost heave process. The thermodynamic processes involved may be described as follows: the incoming water accumulates in the film surrounding the soil particles; at this location the free energy of water becomes higher than that of the adjacent ice and the thermodynamic equilibrium is disturbed. To restore the equilibrium between water and ice, some of the water freezes. The unfrozen water content in the frozen fringe and the thickness of the unfrozen water film decreases with decreasing negative temperature (Fig.4.2). Consequently, the permeability of the frozen soil decreases because the permeability is dependent on the thickness of the unfrozen water film. During rapid penetration of the freezing front there is not enough time for the water to accumulate at one level in the soil and to create a continuous ice lens. After the drop in soil temperature, a nonlinear suction profile develops in the frozen fringe in response to the decreasing permeability of the freezing soil (Fig.4.1b).

Increasing amounts of migrating water accumulate near the base of the ice lens at the frozen fringe which is not able to migrate upward continuously because of

the very low soil permeability at that location. The accumulated water experiences segregation freezing temperature, leading to segregation and ice lens growth. Fig.4.3 shows an idealization of the formation of the segregation ice. Considering the foregoing processes, a simplified yet sufficiently rigorous, frost heave model is developed in this chapter.

4.4 Moisture Migration

In the proposed model a macroscopic approach is employed for mass transfer in order to avoid the difficulties associated with measuring the local physical properties of the unfrozen water content in the freezing soil.

The frozen fringe is defined by the segregation freezing temperature T_s and the freezing temperature T_f . In the *SP* model the T_f is specified to be 0°C which is valid in most cases, but Holden (1985) and Williams (1993) indicated that the freezing temperature T_f is sub-zero in fine grained porous soil. In the proposed model T_f is considered to be a generalized temperature at the freezing front which may be zero or sub-zero.

Considering the freezing process described in Section 4.1, even though water begins freezing in situ, migrating water remains liquid at the freezing front. The latter water freezes and forms segregation ice at the coldest side of the frozen fringe. Consequently, the ice pressure at the freezing front, P_{if} , is equal to zero due to the zero ice content, but it is difficult to determine the ice pressure at the coldest side of the frozen fringe. In this chapter it is hypothesized that the ice between soil particles is highly stressed, and is bonded to the soil grains. Consequently, the soil-ice composite on the coldest side of the frozen fringe is assumed to behave as a solid, and thus according to the static equilibrium conditions, the ice pressure at the location of the segregation freezing temperature is equal to the mean value of the overburden pressure. Having determined the ice pressure at both the coldest side of the frozen fringe and at the freezing front, the water pressure at the base of the ice lens can be obtained using the Clausius-Clapeyron equation

$$P_{ws} = \frac{\rho_w}{\rho_i} P_{is} + L_h \rho_w \ln \frac{T_s}{T_f} + P_{wf} \quad (4.1)$$

where

P_{ws} = the water pressure at the coldest side of the frozen fringe (Pa)

P_{wf} = the water pressure at the freezing front (Pa)

P_{is} = the ice pressure at the coldest side of the frozen fringe (Pa)

ρ_i = the density of ice (kg/m^3)

ρ_w = the density of water (kg/m^3)

L_h = latent heat of fusion (J/kg)

T_s = absolute segregation freezing temperature (K)

T_f = absolute freezing temperature at the freezing front (K)

The water suction profile (Fig.2.3) developed based on Eq.4.1 is similar to the water suction profile (Fig.4.5) given by Williams (1982). The simplified water pressure profile according to this assumption is shown in Fig.4.4 which is same with Shen and Ladanyi's model (1991). According to Darcy's law for continuous water flow in saturated soils, the water flux is directly due to the gradient of the water potential. Therefore, the rate of water flow across the freezing fringe can be expressed by

$$U(t) = \frac{P_{wf} - P_{ws}}{\rho_w g d_f} \bar{K}_f \quad (4.2)$$

where

$U(t)$ = water intake flux (m/s)

d_f = the thickness of the frozen fringe (m)

g = gravitational acceleration of gravity (m/sec^2)

$\bar{K}_f$ = overall permeability of freezing soil in the frozen fringe (m/s)

I_w = hydraulic gradient in the froze fringe (m/m)

Substituting for P_{ws} from Eq.4.1 into Eq.4.2, we obtain

$$U(t) = I_w \bar{K}_f \quad (4.3)$$

where

$$I_w = -\frac{1}{gd_f} \left(\frac{P_{is}}{\rho_i} + L_h \ln \frac{T_s}{T_f} \right) \quad (4.4)$$

The latter equation implies that the hydraulic gradient of the frozen fringe is not affected by the water suction at the freezing front if the water flows continuously from the unfrozen soil into the frozen fringe. Eq.4.3 shows that the rate of water flow $U(t)$ is proportional to the overall permeability which decreases rapidly with the decrease in the temperature of the soil. The determination of the frozen fringe overall permeability will be discussed in Section 4.6. For the unsteady heat flow, the thickness of the frozen fringe d_f varies with the variation of temperature distribution in the freezing soil. At any given time d_f can be determined from the corresponding temperature field as will be shown in Section 4.7. The amount of frost heave, Δh_f , due to the freezing of the migrating water during the time interval Δt is obtained by integrating Eq.4.4

$$\Delta h_f = 1.09U(t)\Delta t \quad (4.5)$$

The multiplier 1.09 represents the volumetric expansion of freezing water. Considering the additional amount of frost heave, Δh_a due to the freezing of in situ pore water during Δt , the total amount of frost heave Δh_t is given by

$$\Delta h_t = \Delta h_f + \Delta h_a \quad (4.6)$$

where

$$\Delta h_a = 0.09\eta\Delta Z$$

and

η = the porosity of the soil

ΔZ = advance of the freezing front during Δt (m)

4.5 Heat Transfer

For the analysis of the unsteady heat flow in the freezing soil, the coupling between heat and moisture transfer happens only within the freezing fringe. The coupling exists because the amount of the latent heat is a function of the amount of freezing water on the coldest side of the freezing fringe where migrated water freezes and ice lens forms. Considering the element of the soil column as in Fig.4.6, the energy balance equation is

$$E_{in} + Q_L = \Delta U + E_{out} \quad (4.7)$$

or

$$q_z A dt + Q_L = \Delta U + q_{z+dz} A dt \quad (4.8)$$

Eq.4.7 does not include heat transfer by convection which is negligible compared to transfer by conduction. In the latter relations

E_{in} = the energy entering the element (J or W.h)

E_{out} = the energy exiting from the element (J or W.h)

Q_L = released latent heat within the element (J or W.h)

ΔU = the change in stored energy within the element (J or W.h)

q_z = heat flux at surface z (W/m^2)

q_{z+dz} = heat flux at surface $z + dz$ (W/m^2)

A = cross-section area of the element (m^2)

t = time (s)

T = temperature ($^{\circ}\text{C}$)

The stored energy is given by

$$\Delta E = \bar{c}\rho_s A dz dT$$

or

$$\Delta E = c A dz dT \quad (4.9)$$

where

$\bar{c}$ = specific heat capacity of soil ($\text{J}/(\text{Kg} \cdot ^{\circ}\text{C})$)

c = heat capacity of soil ($\text{J}/(\text{m}^3 \cdot ^{\circ}\text{C})$)

ρ_s = density of soil (Kg/m^3)

The heat flux is given by Fourier's law of heat conduction as

$$q_z = -k_z \frac{dT}{dz} \quad (4.10)$$

where

k_z = thermal conductivity in the z direction ($\text{W}/(\text{m} \cdot ^{\circ})$)

$\frac{dT}{dz}$ = thermal gradient ($^{\circ}\text{C}/\text{m}$)

Using a two term Taylor series, q_{z+dz} is expressed as

$$q_{z+dz} = -\left[k_z \frac{dT}{dz} + \frac{d}{dz}\left(k_z \frac{dT}{dz}\right)dz\right] \quad (4.11)$$

On substituting Eqs.4.8, 4.9 and 4.10 into Eq.4.7, dividing by $A dz dt$, the one dimensional heat conduction equation is obtained as

$$k_z \frac{\partial^2 T}{\partial z^2} + \bar{Q}_L = c \frac{\partial T}{\partial t} \quad (4.12)$$

where

$\overline{Q}_L$ = latent heat per unit area of cross-section (J/m^3s)

Eq.4.12 is based on the assumption that k_z is not a function of z . Since both thermal conductivity and heat capacity in the freezing soil vary with the temperature as shown in Figs.4.7 and 4.8, an approximation will be employed to capture the variation of soil thermal properties. The soil will be divided into three zones with different specified thermal properties (Fig.4.9) as follows:

In the unfrozen zone:

$$k_z = k_{zu}$$

$$c = c_u$$

In the frozen fringe:

$$k_z = k_{zf1}$$

$$c = c_{f1}$$

In the frozen zone:

$$k_z = k_{zf2}$$

$$c = c_{f2}$$

The latent heat $\overline{Q}_L$ is liberated at two locations: (1) where T is equal to the segregation freezing temperature T_s , and (2) where T is equal to the in situ freezing temperature T_f .

At $T = T_s$

$$\overline{Q}_L = U(t)L_h\rho_w \quad (4.13)$$

At $T = T_f$

$$\overline{Q}_L = \eta L_h \rho_w \frac{dz}{dt} \quad (4.14)$$

where

$\frac{dz}{dt}$ = the rate of advance of the freezing front (m/s)

4.6 Frozen Fringe Overall Permeability

In the proposed model an average or overall permeability is used, which is the average frozen fringe permeability as defined by Konrad and Morgenstern (1980).

The variation of frozen soil permeability is very complex. As described by Harlan(1973), the frozen soil permeability is related to the unfrozen water content which depends on the temperature and water suction. In the frozen fringe the unfrozen water content decreases with the decrease of negative temperature. Consequently, the permeability decreases significantly from the location of T_o to the location of T_s , and develops a nonlinear profile as shown in Fig.4.10 and Fig.4.11. The two common characteristics of the permeability profiles in the frozen fringe for various soils under zero applied load are shown in Figs.4.10 and 4.11. It can be observed that the magnitude of the permeability decrease from 10^{-8} – 10^{-9} at the freezing front to 10^{-12} – 10^{-13} at the coldest side of the frozen fringe. Considering Figs. 4.11 and 4.12, the shapes of the permeability profiles can be approximated by an exponential function. Horiguchi and Miller (1983) have proposed the following relationship for the permeability of freezing soils under zero overburden pressure condition

$$K_f(T) = aT^b \quad (4.15)$$

where a and b are constant and T is the temperature. This function can describe in general the variation of permeability within the frozen fringe, but it leads to wrong results at the freezing front because $K_f(T_f) = 0$, if $T_f = 0$. Consequently, this function can not be used to calculate the overall permeability by integrating the permeability function over the whole frozen fringe and employing the known permeability of the unfrozen soil as a boundary value.

In this chapter an exponential freezing soil permeability function under zero overburden pressure is proposed as follows:

$$K_f(T) = ae^{bT} \quad (4.16)$$

where

T = temperature in degree Celsius($^{\circ}C$)

$K_f(T)$ = soil permeability at T temperature

a, b = undetermined permeability parameters

Hence the average frozen fringe permeability can now be expressed as

$$\overline{K}_f = \frac{\int_{T_f}^{T_s} k_f(T) dT}{\int_{T_f}^{T_s} dT} \quad (4.17)$$

Substituting Eq.4.16 into Eq.4.17, gives

$$\overline{K}_f = \frac{\int_{T_f}^{T_s} ae^{bT} dT}{\int_{T_f}^{T_s} dT} = \left(\frac{a}{b}\right) \frac{1}{(T_s - T_f)} (e^{bT_s} - e^{bT_f}) \quad (4.18)$$

In Eq.4.18 the undetermined constants a and b can be obtained from freezing soil tests for the particular soil concerned. If the $K_f(T_f)$ and $K_f(T_s)$ are known, where $K_f(T_f)$ is the freezing front or unfrozen soil permeability, and $K_f(T_s)$ is the frozen soil permeability at the coldest side of the frozen fringe, then the overall frozen fringe permeability $\overline{K}_f$ can be directly calculated. Substituting $K_f(T_f)$ and $K_f(T_s)$ into Eq.4.16, the following two equations are obtained

$$K_f(T_f) = ae^{bT_f} \quad (4.19)$$

$$K_f(T_s) = ae^{bT_s} \quad (4.20)$$

Dividing Eq.4.19, by Eq.4.20 leads to

$$\frac{K_f(T_f)}{K_f(T_s)} = e^{b(T_f - T_s)} \quad (4.21)$$

Taking the natural logarithm of Eq.4.21 yields

$$b(T_f - T_s) = \ln \frac{K_f(T_f)}{K_f(T_s)} \quad (4.22)$$

Consequently, the constant b can be obtained as

$$b = \frac{1}{(T_f - T_s)} \ln \frac{K_f(T_f)}{K_f(T_s)} \quad (4.23)$$

Combining Eq.4.23 with Eq.4.19 or Eq.4.20, the constant a is determined

$$a = K_f(T_f)e^{-bT_f} = K_f(T_f)e^{-T_f(T_f - T_s)^{-1} \ln \frac{K_f(T_f)}{K_f(T_s)}} \quad (4.24)$$

or

$$a = K_f(T_s)e^{-bT_s} = K_f(T_s)e^{-T_s(T_s - T_f)^{-1} \ln \frac{K_f(T_f)}{K_f(T_s)}} \quad (4.25)$$

If $T_f = 0$, Eq.4.22 gives

$$a = K_f(0) \quad (4.26)$$

and

$$b = \frac{-1}{T_s} \ln \frac{K_f(0)}{K_f(T_s)}$$

4.7 Time-Dependent Finite Element Analysis

Frost heave is a spatial and temporal phenomenon. Therefore, the numerical solution of the governing equations of the problem require both spatial and temporal discretization. In this thesis the finite element method will be used for both spatial and time discretization. The temporal discretization is analogous to the more popular finite difference approach and it yields the same results as the later method.

4.7.1 Finite Element Formulation in Space

A one dimensional time-dependent finite element is used to solve the governing heat transfer equation (Eq.4.12).

The two node finite element is shown in Fig.4.12. The temperature field within the element can be approximated by

$$T = N_1 T_1 + N_2 T_2 \quad (4.27)$$

or

$$T = [N_1 \quad N_2] \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} \quad (4.28)$$

which can be written symbolically as

$$T = [N]\{T\} \quad (4.29)$$

where

T_1 = the temperature at node 1 ($^{\circ}C$)

T_2 = the temperature at node 2 ($^{\circ}C$)

N_1 and N_2 are linear shape functions, i.e.

$$N_1 = 1 - \frac{z}{L} \quad (4.30)$$

$$N_2 = \frac{z}{L} \quad (4.31)$$

where L = the element length and z is a spatial coordinate measured along the element length from node 1.

The temperature gradient, G , is given by

$$G = \frac{dT}{dz} = [B]\{T\} \quad (4.32)$$

where

$$[B] = \left[\frac{dN_1}{dz} \quad \frac{dN_2}{dz} \right] = \left[\frac{-1}{L} \quad \frac{1}{L} \right] \quad (4.33)$$

Galerkin's weighted residual method will be employed to formulate the finite element equations. In the weighted residual method, it is required that the weighted value of the residual be a minimum over the whole region.

Let us begin by rewriting Eq.4.12 as

$$k_z \frac{d^2 T}{dz^2} - c \frac{dT}{dz} + \bar{Q}_L = 0 \quad (4.34)$$

In the Galerkin's method Eq.4.34 and shape functions $[N]$ are defined as the residual and weighting functions, respectively. Applying Galerkin's criterion, the weighted integral of the residual is given by

$$\int_0^L [N]^T \left(k_z \frac{d^2 T}{dz^2} - c \frac{dT}{dz} + \bar{Q}_L \right) dz = 0 \quad (4.35)$$

Using integration by parts, the first term of Eq.3.35 becomes

$$\int_0^L [N]^T \left(k_z \frac{d^2 T}{dz^2} \right) dz = k_z [N]^T \frac{dT}{dz} \Big|_0^L - k_z \frac{d[N]^T}{dz} \frac{dT}{dz} \quad (4.36)$$

where

$$k_z [N]^T \frac{dT}{dz} \Big|_0^L = k_z \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} \frac{dT}{dz} \Big|_0^L \quad (4.37)$$

Considering the boundary conditions of the element:

$$N_1 = 1 \text{ and } N_2 = 0, \text{ at } z = 0;$$

$$N_1 = 0 \text{ and } N_2 = 1, \text{ at } z = L.$$

Eq.4.37 becomes

$$k_z [N]^T \frac{dT}{dz} \Big|_0^L = \begin{Bmatrix} q_{z1} \\ q_{z2} \end{Bmatrix} \quad (4.38)$$

where

q_{z1} = heat flux at node 1

q_{z2} = heat flux at node 2

Substituting Eqs.4.36 and 4.38 into Eq.4.35, We obtain

$$\int_0^L k_z \frac{d[N]^T}{dz} \frac{dT}{dz} dz + \int_0^L c [N]^T \frac{dT}{dt} dz - \int_0^L [N]^T \bar{Q}_L dz = \begin{Bmatrix} q_{z1} \\ q_{z2} \end{Bmatrix} \quad (4.39)$$

But

$$\frac{dT}{dz} = [B] \{T\}$$

$$\begin{aligned} \frac{dT}{dt} &= \frac{d}{dt} \{N_1 T_1 + N_2 T_2\} \\ &= N_1 \frac{dT_1}{dt} + N_2 \frac{dT_2}{dt} \\ &= [N] \frac{d}{dt} \{T\} \end{aligned}$$

Substituting the preceding two relations into Eq.4.39, yields

$$\left(\int_0^L [B]^T k_z [B] dz \right) \{T\} + \left(\int_0^L [N]^T c [N] dz \right) \frac{d\{T\}}{dt} - \int_0^L [N]^T Q_L dz = \begin{Bmatrix} q_{z1} \\ q_{z2} \end{Bmatrix}$$

or

$$[K_s] \{T\} + [C_s] \frac{d\{T\}}{dt} - \{f_Q\} = \{f_q\} \quad (4.40)$$

where

$[K_s]$ = thermal conductivity matrix of soil

$[C_s]$ = heat capacity matrix of soil

$\{f_Q\}$ = equivalent nodal heat vector

$\{f_q\}$ = nodal heat flux vector

and

$$\begin{aligned} [K_s] &= [B]^T k_z [B] dz \\ &= k_z \int_0^L \left\{ \frac{-1}{L} \right\} \left[\frac{-1}{L} \quad \frac{1}{L} \right] dz \\ &= \frac{k_z}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} [C_s] &= c \int_0^L [N]^T [N] dz \\ &= \int_0^L c \left\{ 1 - \frac{z}{L} \right\} \left[\frac{1-z}{L} \quad \frac{z}{L} \right] dz \\ &= \frac{cL}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \end{aligned}$$

$$\{f_Q\} = \left\{ \frac{\bar{Q}_1}{\bar{Q}_2 q} \right\}$$

$$\{f_q\} = \left\{ \begin{matrix} q_{z1} \\ q_{z2} \end{matrix} \right\}$$

The vector $\{f_Q\}$ can be determined from Eqs. 4.13 and 4.14 as follows:

If node j is located at the coldest side of the frozen fringe, then

$$\bar{Q}_j = U(t) L_h \rho_w$$

If node j is located at the freezing fringe, then

$$\bar{Q}_j = \eta L_h \rho_h \frac{dz}{dt} \quad .$$

Finally, if node j is located at any other location, the determination of the equivalent nodal latent heat will be obtained as described in Section 4.7.3

Eq. 4.40 can be written more succinctly as

$$[K_s]\{T\} + [C_s]\frac{d\{T\}}{dt} = \{F_h\} \quad (4.41)$$

where

$$\{F_h\} = \{f_Q\} + \{f_q\} \quad (4.42)$$

The last term in the above equation, $\{f_q\}$, usually exists only at the free end of a system modeled by finite elements. When the elements are assembled, the heat fluxes q_{z1} and q_{z2} are equal but opposite at the node common to the two elements, unless there is an internal concentrated heat flux in the system. For soil column with an insulated end and without internal concentrated heat flux, the quantities q_{z1} and q_{z2} are zero.

4.7.2 Time Marching Scheme

Eq.4.41 can be cast in the form of a generalized unsteady heat flow equation

$$K_s T + C_s \frac{dT}{dt} = F_h \quad (4.43)$$

where ,

K_s = thermal conductance matrix corresponding to thermal conductivity values at a given time t

C_s = thermal capacitance matrix at time t

For the unsteady heat transfer defined by Eq.4.43, the temperature is a function of time t , therefore the solution will be time dependent. In the analysis an element associated with time t is employed as shown in Fig.4.13. The temperature at any time within the time interval Δt is approximated by the interpolation function

$$T(t) = N_{t1} T^n + N_{t2} T^{n+1} \quad (4.44)$$

or

$$\mathbf{T}(t) = [N_t]\{\Phi\} \quad (4.45)$$

where

$\mathbf{T}^n$ = vector of nodal temperature at time step n

$\mathbf{T}^{n+1}$ = vector of nodal temperature at time step $n + 1$,

while N_{t1}, N_{t2} = time-dependent shape functions

$$N_{t1} = 1 - \frac{t}{\Delta t} \quad (4.46)$$

$$N_{t2} = \frac{t}{\Delta t} \quad (4.47)$$

while

$$[N_t] = [N_{t1} \quad N_{t2}]$$

and

$$\{\Phi\} = \left\{ \begin{matrix} \mathbf{T}^n \\ \mathbf{T}^{n+1} \end{matrix} \right\} \quad (4.48)$$

and Δt is the time increment from time n to $n + 1$.

If a natural time coordinate $\zeta = \frac{t}{\Delta t}$ is introduced, then the load term $\mathbf{F}_h$ in Eq.4.43 can be expressed as

$$\mathbf{F}_h = [N_t] \left\{ \begin{matrix} \mathbf{F}_h^n \\ \mathbf{F}_h^{n+1} \end{matrix} \right\} \quad (4.49)$$

or

$$\mathbf{F}_h = (1 - \zeta)\mathbf{F}^n + \zeta\mathbf{F}^{n+1} = [N_t]\{\mathbf{F}_t\} \quad (4.50)$$

where

$\mathbf{F}_h^n$ = vector of thermal loads due to latent heat and heat flux at time step n

$\mathbf{F}_h^{n+1}$ = vector of thermal loads due to latent heat and heat flux at time step $n + 1$

Substituting Eqs.4.45 and 4.50 into Eq.4.43, we obtain

$$\mathbf{K}_s[N_t]\{\Phi\} + \mathbf{C}_s \frac{d[N_t]}{dt} \{\Phi\} - [N_t]\{\mathbf{F}_t\} = 0 \quad (4.51)$$

where

$$\begin{aligned} \frac{d[N_t]}{dt} &= \left[\frac{dN_{t1}}{dt} \quad \frac{dN_{t2}}{dt} \right] \\ &= \left[\frac{-1}{\Delta t} \quad \frac{1}{\Delta t} \right] \end{aligned}$$

The appropriate residual is given by Eq.4.51. Applying Galerkin's method, Eq.4.51 becomes

$$\int_0^1 W_j (\mathbf{K}_s[N_t]\{\Phi\} + \mathbf{C}_s \frac{d[N_t]}{dt} \{\Phi\} - [N_t]\{\mathbf{F}_t\}) d\zeta = 0 \quad (4.52)$$

where

W_j = weighting function

Substituting Eqs.4.48, 4.49 and 4.50 into Eq.4.52, the weighted residual equation is expressed by

$$\begin{aligned} \mathbf{K}_s \int_0^1 W_j \{(1 - \zeta)\mathbf{T}^n + \zeta\mathbf{T}^{n+1}\} d\zeta + \mathbf{C}_s \int_0^1 W_j \left\{ \frac{-1}{\Delta t} \mathbf{T}^n + \frac{1}{\Delta t} \mathbf{T}^{n+1} \right\} d\zeta \\ - \int_0^1 W_j (1 - \zeta) \mathbf{F}_h^n d\zeta - \int_0^1 W_j \zeta \mathbf{F}_h^{n+1} d\zeta = 0 \end{aligned} \quad (4.53)$$

or

$$\begin{aligned} (\mathbf{K}_s \int_0^1 W_j (1 - \zeta) d\zeta - \mathbf{C}_s \int_0^1 W_j \frac{1}{\Delta t} d\zeta) \mathbf{T}^n \\ + (\mathbf{K}_s \int_0^1 W_j \zeta d\zeta + \mathbf{C}_s \int_0^1 \frac{W_j}{\Delta t} d\zeta) \mathbf{T}^{n+1} \\ - \int_0^1 W_j (1 - \zeta) \mathbf{F}_h^n d\zeta - \int_0^1 W_j \zeta \mathbf{F}_h^{n+1} d\zeta = 0 \end{aligned} \quad (4.54)$$

Dividing by $\int_0^1 W_j d\zeta$, Eq.4.54 becomes

$$[K_s(1 - \gamma) - C_s \frac{1}{\Delta t}]T^n + (K_s\gamma + \frac{C_s}{\Delta t})T^{n+1} - F_h^n(1 - \gamma) - F_h^{n+1}\gamma = 0 \quad (4.55)$$

or

$$[K_s(1 - \gamma) - C_s \frac{1}{\Delta t}]T^n + (K_s\gamma + \frac{C_s}{\Delta t})T^{n+1} = F_h^n(1 - \gamma) + F_h^{n+1}\gamma \quad (4.56)$$

where

$$\gamma = \frac{\int_0^1 \zeta W_j d\zeta}{\int_0^1 W_j d\zeta} \quad (4.57)$$

Since γ is a factor related to weighting function W_j , it can be seen as an average weighting function. Various numerical integration schemes result, depending on the specific choice of the weighting function W_j . For example, setting

$$W_j = \delta_j$$

where

$$\delta_j = \begin{cases} 1 & \text{at } \zeta = 0; \\ 0 & \text{at } \zeta \neq 0 \end{cases}$$

Eq.4.57 gives $\gamma = 0$, which is defined as the forward difference, or Euler's difference method, in numerical methods. On the other hand, letting

$$W_j = \delta_j$$

where

$$\delta_j = \begin{cases} 1 & \text{at } \zeta = \frac{1}{2}; \\ 0 & \text{at } \zeta \neq \frac{1}{2}. \end{cases}$$

Eq.4.57 yields $\gamma = 0.5$, which is known as the central difference, or Crank-Nicholson's difference method.

By setting

$$W_j = \delta_j$$

where

$$\delta_j = \begin{cases} 1 & \text{at } \zeta = 1; \\ 0 & \text{at } \zeta \neq 1. \end{cases}$$

Eq.4.57 gives $\gamma = 1$, which is known as the backward difference method. Alternatively, $W_j = \zeta$ gives

$$\gamma = \frac{\int_0^1 \zeta^2 d\zeta}{\int_0^1 \zeta d\zeta} = \frac{2}{3}$$

Finally, by setting

$$W_j = 1 - \zeta$$

we obtain

$$\gamma = \frac{\int_0^1 \zeta(1 - \zeta) d\zeta}{\int_0^1 (1 - \zeta) d\zeta} = \frac{1}{3}$$

$\gamma = \frac{2}{3}$ and $\frac{1}{3}$ are defined as the Galerkin's difference methods.

Generally, any value of the weighting function W_i is valid for residual equation 4.52 . But in the case of $\gamma < \frac{1}{2}$, the solution is conditionally stable, provided a suitable value of Δt is chosen to avoid oscillation. For the cases of $\gamma \geq \frac{1}{2}$, the solution is unconditionally stable.

In this study the central difference method is used to solve the time-dependent problem. Using the central difference scheme, by setting $\gamma = 0.5$, Eq.4.56 becomes

$$\left(\frac{1}{2}\mathbf{K}_s - \frac{\mathbf{C}_s}{\Delta t}\right)\mathbf{T}^n + \left(\frac{1}{2}\mathbf{K}_s + \frac{\mathbf{C}_s}{\Delta t}\right)\mathbf{T}^{n+1} = \frac{1}{2}\mathbf{F}_h^n + \frac{1}{2}\mathbf{F}_h^{n+1} \quad (4.58)$$

or

$$\mathbf{K}_t \mathbf{T}^{n+1} = \mathbf{C}_t \mathbf{T}^n + \mathbf{F}_t \quad (4.59)$$

$$\mathbf{K}_t = \mathbf{K}_s + \frac{2\mathbf{C}_s}{\Delta t} \quad (4.60)$$

$$\mathbf{C}_t = \frac{2\mathbf{C}_s}{\Delta t} - \mathbf{K}_s \quad (4.61)$$

$$\mathbf{F}_t = \mathbf{F}_h^n + \mathbf{F}_h^{n+1} \quad (4.62)$$

Eqs.4.60 to 4.62 may be written in matrix form as

$$\mathbf{K}_t = \frac{k_z}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{cL}{3\Delta t} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\mathbf{C}_t = \frac{cL}{3\Delta t} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \frac{k_z}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\{\mathbf{F}_t\} = \left\{ \begin{array}{c} \overline{Q}_1 \\ \overline{Q}_2 \end{array} \right\}$$

For small time intervals, it can be assumed that $\mathbf{F}_h^n = \mathbf{F}_h^{n+1}$.

Eq.4.59 can be solved in step-by-step fashion. Given an initial temperature $\mathbf{T}^0$ at $t = 0$ and a selected time step Δt , Eq.4.59 can be solved for $\mathbf{T}^1$, corresponding to $t = \Delta t$. Then using $\mathbf{T}^1$, $\mathbf{T}^2$ can be determined at $t = 2\Delta t$ and so on. In the calculation $\mathbf{K}_t$, $\mathbf{C}_t$, and $\mathbf{F}_t$ vary with the change of location of the frozen zone in the soil which is associated with the variation in temperature. For the unsteady heat transfer all the preceding quantities are functions of time t . In the analysis, at

time step t^n , K_t , C_t , and F_t are treated as constants within the small time interval Δt , and their values can be obtained from the known temperature distribution in the frozen soil at the end of previous time step T^{n-1}

4.7.3 Equivalent Nodal Latent Heat

In the calculation it should be noted that one of the load terms in Eqs.4.49, 4.50 and 4.62 is the latent heat, which is released by the freezing of the migrated water at the location of temperature T_s , and is dependent on the amount of the migrated water. The other load term is the latent heat, which is released by the freezing of the in situ or the pore water at the location of temperature T_o , and is dependent on the penetration of the freezing front. Therefore the locations of the frozen front and the coldest side of the frozen fringe must be determined before calculating the amount of latent heat at every time step. On the other hand, Eq.4.4 shows that the amount of the frost heave depends on the thickness of the frozen fringe, which can also be obtained after determining the locations of T_s and T_f . Generally, the locations of T_s and T_f do not coincide with the location of element nodes. Hence one must find their locations. The following two common cases need to be considered: (1) T_s and T_f are located in two adjacent elements (see Fig.4.14); (2) they are located within one element (see Fig.4.15).

At any given time step, the locations of the freezing front and the coldest side of the frozen fringe and the equivalent nodal latent heat can be determined from the known temperature distribution as follows. For case (1):

$$|T_{i-1}| < |T_s|$$

$$|T_f| < |T_i| < |T_s|$$

$$|T_f| < |T_{i-1}|$$

where

$$T_{i-1} = \text{known temperature at node } i-1$$

T_i = known temperature at node i

T_{i+1} = known temperature at node $i + 1$

In element 1, shown in Fig.4.15, the relationship between T_s and nodal temperatures T_{i-1} and T_i is given by

$$T_s = N_1 T_{i-1} + N_2 T_i \quad (4.63)$$

Substituting Eqs.4.29 and 4.30 into Eq.4.63, gives

$$T_s = (1 - \frac{z_1}{L_1}) T_{i-1} + \frac{z_1}{L_1} T_i \quad (4.64)$$

where z_1 = coordinate of the point in element 1, measured from node $(i - 1)$

at which $T = T_s$

L_1 = the length of element 1

Solving Eq.4.64, z_1 is obtained as

$$z_1 = \frac{L_1(T_s - T_{i-1})}{T_i - T_{i-1}} \quad (4.65)$$

Thus, the equivalent nodal latent heats in element 1 are given by

$$\bar{Q}_{i-1} = \bar{Q}_s \frac{z_1}{L_1} \quad (4.66)$$

$$\bar{Q}_i^1 = \bar{Q}_s \frac{L_1 - z_1}{L_1} \quad (4.67)$$

where

$\bar{Q}_s$ = the amount of latent heat generated by the freezing of the migrated water

$\bar{Q}_{i-1}$ = the equivalent latent heat at node $(i - 1)$

$\bar{Q}_i^1$ = the equivalent latent heat at node i in element 1

Consider element 2

$$T_f = N_1 T_i + N_2 T_{i+1} \quad (4.68)$$

$$z_2 = \frac{L_2(T_f - T_i)}{T_{i+1} - T_i} \quad (4.69)$$

where

z_2 = coordinate of the point in element 2, measured from node i at which $T = T_f$

L_2 = the length of the element 2

The equivalent nodal latent heats in element 2 can now be calculated as

$$\bar{Q}_i^2 = \bar{Q}_f \frac{z_2}{L_2} \quad (4.70)$$

$$\bar{Q}_{i+1} = \bar{Q}_f \frac{L_2 - z_2}{L_2} \quad (4.71)$$

where

$\bar{Q}_f$ = the amount of latent heat due to the freezing of the in situ or pore water
(see Eq.4.14)

The frost penetration rate can be determined using

$$\frac{dz}{dt} = z_2^n - z_2^{n-1}$$

where

z_2^n = the location of T_f at time step n

z_2^{n-1} = the location of T_f at time step $n - 1$

$\bar{Q}_{i+1}$ = the equivalent latent heat at node $i + 1$

$\bar{Q}_i^j$ = the equivalent latent heat at node i in element j

The total equivalent latent heat at node i is

$$\bar{Q}_i = \bar{Q}_i^1 + \bar{Q}_i^2 \quad (4.72)$$

Then the thickness of the frozen fringe is given by

$$d_f = (L_1 - z_1) + z_2 \quad (4.73)$$

The same procedure applies to the Case 2 (see Fig.4.16), and the results are given by

$$z_1 = \frac{L(T_s - T_i)}{T_{i+1} - T_i} \quad (4.74)$$

$$z_2 = \frac{L(T_f - T_i)}{T_{i+1} - T_i} \quad (4.75)$$

$$d_f = z_2 - z_1 = \frac{L(T_f - T_i)}{T_{i+1} - T_i} \quad (4.76)$$

$$\bar{Q}_i = \bar{Q}_s \frac{z_1}{L} + \bar{Q}_f \frac{z_2}{L} \quad (4.77)$$

$$\bar{Q}_{i+1} = \bar{Q}_s \frac{(L - z_1)}{L} + \bar{Q}_f \frac{(L - z_2)}{L} \quad (4.78)$$

$$\frac{dz}{dt} = (z_2^n) - (z_2^{n-1})$$

where $\bar{Q}_s$ and $\bar{Q}_f$ are given by Eqs.4.13 and 4.14

4.8 Implementation Procedure For Proposed Frost Heave Model

The amount of frost heave and the temperature distribution are obtained in step-by-step fashion. The total freezing time is subdivided into a number of increments. At a given time, the temperature profile is determined using the time-dependent finite element method described in Section 4.7.2. In the calculation, the

freezing fringe overall permeability, the freezing temperature T_f and the segregation freezing temperature T_s are assumed to be constants for a given soil. At the beginning of time step n , according to the determined temperature profile at the end of time step $n - 1$, the locations of the base of ice lens and the freezing front are known, then the thickness of the frozen fringe is given by Eq.4.73 or 4.76, and the ice pressure at the base of the ice lens can be calculated by the summation of the given external load acting on the soil surface and the soil weight above the location of the ice lens. Knowing the hydraulic gradient of the frozen fringe, I_w , given by Eq.4.4, then the water intake flux $U(t)$ can be obtained using Eq.4.3. Thus, the increment of frost heave during the n th time interval Δt is given by Eq.4.5, and the total amount of frost heave at the end of time step n is given by the summation of the n th increment of frost heave and the amount of frost heave achieved at the end of the previous time step. On the other hand, the equivalent nodal latent heat due to the freezing of migrating water during time Δt can be determined by Eq.4.72 or Eq.4.77.

4.9 Numerical Examples

Three examples are given to demonstrate the capabilities of the proposed model. The first one simulates Penner's (1986) laboratory experiments on frost heave (1986), the second example analyzes the effects of overburden pressure on frost heave process, which was studied by Holden (1985), using a numerical simulation. The third example calculates the non-homogeneous temperature distribution in the freezing ground which was investigated by Mohan (1975).

4.9.1 Example 1

Penner (1986) experimentally measured the amount of frost heave, the growth of ice lens and the penetration of frozen zone during freezing of an open soil system. The frost cell which was used in Penner's experiments is shown in Fig.4.16, the soil samples were cylinders 100 mm long and 100 mm in diameter. The water content

of the soil samples after consolidation was about 20% by soil weight, which was achieved as follows: adding enough water into air-dry soils to get the moisture content to be just above the predetermined water limit; storing the wet soil in a container for several days; placing it in layers in the cell and consolidating it in steps to the pressure of 500 KPa; reducing the pressure to 50 KPa.

In Penner's test, freezing of the samples was developed from the bottom of the samples upward. Inside the frost cell a constant temperature chamber was held close to the temperature of samples, and temperature baths for supplying liquid coolant to heat exchanges at the ends of the sample were placed outside the constant temperature chamber. The external water supply was held level with the porous plate at the top of the frost cell. Penner used a multi-tasking HP minicomputer to control the temperature of the liquid coolant and the amount of both the water intake and frost heave. The temperature boundary conditions were changed from the initial temperature of -0.35°C at the cold end and the temperature of 0.55°C at the warm end at the rate of $-0.002^{\circ}\text{C}/\text{day}$, as shown in Fig.4.17.

In the numerical simulation here, the thermal conductivities used are k_{zu} (unfrozen soil) $= 1.55(\text{W}/\text{m}^{\circ}\text{C})$ and k_{zf} (frozen soil) $= 2.00(\text{W}/\text{m}^{\circ}\text{C})$, the heat capacities used are c_u (unfrozen soil) $= 2.60 \times 10^6(\text{J}/\text{m}^3)^{\circ}\text{C})$ and c_f (frozen soil) $= 2.34 \times 10^6(\text{J}/\text{m}^3)^{\circ}\text{C})$. The above thermal properties of the soil sample were determined from the following formulas (Kay et al., 1977)

$$c = \sum c_j \lambda_j$$

$$k_z = \prod (k_z)_j \lambda_j$$

where c_j , $(k_z)_j$ and λ_j are the heat capacity, the thermal conductivity and the volumetric fraction of each phase in the soil, respectively. The overall permeability of the frozen fringe is $3.58 \times 10^{-11}(\text{m}/\text{s})$ which is obtained from Eq.4.18 by using $k_{zu} = 2.25 \times 10^{-10}(\text{m}/\text{s})$ given by Shah (1990) and the assumed value of $k_{zf} = 1.2 \times 10^{-13}(\text{m}/\text{s})$.

In the calculation, the soil sample was divided into 20 elements and 100 elements, respectively. The calculated penetration of the freezing front is shown in Fig.4.18, the calculated frost heave and water intake are shown in Fig.4.19 and Fig.4.20, respectively. From the results for the 100 element discretization, as illustrated in Figs 4.19 and 4.20, it is seen that more than 95 percent of the frost heave is caused by the freezing of the migrated water at the coldest side of the frozen fringe, which is in reasonable agreement with the experiment.

Note that at the beginning of the freezing process, the temperature of the soil near the cold end of the soil sample decreases from the initial temperature of 0.55°C to the normal freezing temperature, T_f , leading to rapid penetration of the freezing front. After this transient state, the variation of the temperature distribution becomes more steady.

The penetration of the freezing front for 20 element discretization was too fast, and the amount of water intake and frost heave was much less compared to Penner's results. It is believed that the 20 element mesh is too coarse to yield sufficiently accurate results. This phenomenon can be explained by the loss of latent heat. At the beginning of the freezing process, the soil freezing occurs within the element with the cold boundary node. In the finite element analysis, the latent heat is treated as equivalent nodal latent heat given by Eqs.4.72 and 4.77. When the temperature boundary conditions are introduced to solve the unsteady heat flow Eq.4.59, the equivalent nodal latent heat of the cold boundary is disregarded because the temperature at that location is a prescribed boundary condition. If the segregation freezing temperature, T_s , is located within the first element, some latent heat must be omitted, and the longer the size of the first element, the more the omitted amount of the latent heat.

4.9.2 Example 2

Holden (1985) proposed a computer simulation based on the rigid ice model to investigate the effect of overburden pressure during soil freezing. In Holden's study, the length of the soil column was 150 mm, the uniform initial temperature of 4°C

was assumed in the soil column, the top surface of the column was cooled at a constant rate down to -6°C , and then held constant. The thermal conductivities were assumed to be $k_{zu} = 3 \text{ (W/m}^{\circ}\text{C)}$, $k_{zf} = 4 \text{ (W/m}^{\circ}\text{C)}$, the heat capacities were assumed to be $c_u = 3.2 \times 10^6 \text{ (J/m}^3\text{)}^{\circ}\text{C}$ and $c_f = 2.95 \times 10^6 \text{ (J/m}^3\text{)}^{\circ}\text{C}$, the porosity of the soil $\eta = 0.4$. In his simulation he studied the effect of the overburden pressure on the soil freezing characteristics. He assumed pressure of 25 kPa, 50 kPa, 75 kPa, 100 kPa and 150 kPa.

In the proposed simulation, all the above values of the soil parameters, the temperature conditions and the overburden pressure were used. The overall permeability of the frozen soil was assumed to be $5 \times 10^{-9} \text{ (m/s)}$. The calculated results are shown in Fig.4.21. It is seen from Fig.4.21 that The calculated results based proposed model and the previous results given by Holden are in good agreement with each other.

Fig.4.21 shows that the variation of the frost heave with the overburden pressure is significant. According to the proposed frost heave model, the amount of water intake flux depends on the hydraulic gradient of the frozen fringe, which is given by the equation

$$I_w = \frac{-1}{gd_f} \left(\frac{p_{is}}{\rho_i} + l_h \ln \frac{T_s}{T_f} \right)$$

By assuming the segregation freezing temperature T_s and the freezing in situ temperature T_{kf} to be constants for a given frozen soil, and considering $\ln \frac{T_{ks}}{T_{kf}} < 0$, the last term of the above equation is a positive constant, the last term of the equation is negative because the ice pressure is equal to the positive overburden pressure. So the hydraulic potential of the frozen fringe is reduced by an amount which is equal to $\frac{1}{gd_f} \left(\frac{p_{is}}{\rho_i} \right)$. Therefore, the hydraulic potential of the frozen fringe decreases with the increase of overburden pressure, consequently, the amount of frost heave decreases with the increasing overburden pressure.

It is believed that the difference between the calculated results and Holden's results is due to the assumed values of the overall permeability of the frozen soil. In

2

PM-1 3½"x4" PHOTOGRAPHIC MICROCOPY TARGET
NBS 1010a ANSI/ISO #2 EQUIVALENT

PRECISIONSM RESOLUTION TARGETS

$$\begin{aligned}
\{F_{I3}\} &= \begin{Bmatrix} F_{3(x1)} \\ F_{3(y1)} \\ M_{3(1)} \\ F_{3(x2)} \\ F_{3(y2)} \\ M_{3(2)} \end{Bmatrix} \\
&= \begin{bmatrix} 140k_{s1} & & & & & \\ 0 & 156k_{s2} & & & & \\ 0 & 22lk_{s2} & 4l^2k_{s2} & & & \\ 70ks1 & 0 & 0 & 140k_{s1} & & \\ 0 & 54k_{s2} & 13lk_{s2} & 0 & 156k_{s2} & \\ 0 & -13lk_{s2} & -3l^2k_{s2} & 0 & -22lk_{s2} & 4l^2k_{s2} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{Bmatrix} \quad \text{Symmetric} \quad (5.63)
\end{aligned}$$

Note that in the presence of material nonlinearity, the internal forces vector $\{F_{I1}\}$ can not be written in closed-form.

The contribution of each layer to the internal forces is obtained by integration over the thickness of that layer only. The internal forces in a cross-section are then found from the summation of the contributions of all the layers in the element. Gaussian integration will be used to calculate the internal forces. Accordingly, the internal force nodal vector $\{F_{I1}\}$ is given by:

$$\begin{aligned}
\{F_{I1}\} &= [K_m]\{d\} \\
&= \int_v [B_0]^T [D] [B_0] \{d\} dv \\
&= \int_v [B_0]^T \{\sigma_0\} dv
\end{aligned} \quad (5.64)$$

where σ_0 is longitudinal normal stress in the pipe according to small displacement theory. Note that in the present case vectors $\{\sigma_0\}$ and $\{\epsilon_0\}$ have only one element.

$$\{\sigma_0\} = [D]\{\epsilon_0\}$$

$$\{\epsilon_0\} = [B_0]\{d\}$$

and from Eq.5.39, $[B_0]$ can be expressed as

of latent heat is released. The latent heat delays the penetration of the frozen front, thus the non-homogeneous temperature distribution is developed and a high temperature gradient is formed. Note that the thermal process of soil freezing is different with normal thermal process of unsteady heat flow since it is directly influenced by the water phase change.

4.10 Summary

The proposed simplified one dimensional frost heave model is based on macroscopic analysis of the freezing process. It does not consider the local soil freezing phenomena, the post-frost heave phenomena, as well as the capillary phenomena.

This model considers the dominant phenomena during soil freezing, including the water migration, the latent heat release, the ice lens formation, and the effect of overburden pressure. Numerical examples demonstrate reasonable agreement between the results from the proposed analysis and the existing theoretical and/or experimental results.

The concepts behind of the proposed model are transparent, and are easy to understand, and the proposed calculations are simple. This model provides a powerful tool to predict frost heave for cold region engineering. The proposed model is valid if the water flow is continuous from the unfrozen soil into the frozen fringe, wherever the water table may be located, at the freezing fringe or somewhere else.

It should be noted that in the finite element discretization enough large number of soil elements are need to be used in order to reduce the loss of latent head at the cold boundary node. The proposed model implies that the frost heave in an unsaturated soil is not significant if the frozen fringe is located far from the water table. The rate of water flow from the water table across the unfrozen soil to the freezing front is dependent on the hydraulic gradient between the freezing front and the water table, which will decrease with the increase of the distance between them. When the rate of water flow decreases to less than the required rate of continuous water flow in the freezing fringe, given by the proposed model, water flow will be

discontinuous in the unfrozen soil and in the frozen fringe. Consequently, the rate of water flow in the freezing fringe will decrease, and the amount of frost heave will diminish.

Fig. 4.1 Characteristics of the frozen fringe
 (a) Simplified (b) Actual shape
 (after Konrad and Morgenstern, 1981)

Fig. 4.2 Variation of unfrozen water content with temperature for freezing soil (after Burt and Williams, 1976)

Fig. 4.3 Schematic rhythmic ice lens formation
(after Konrad and Morgenstern, 1980)

Fig. 4.4 Distribution of water pressure within frozen fringe

Fig.4.5 Theoretical and experimental relationship
between temperature and suction
(after Williams 1982)

Fig. 4.6 Heat balance condition

Fig. 4.7 Thermal conductivity as a function of temperature for frozen Caen silt at different degree of compaction (after Williams 1991)

Fig.4.8 Apparent volumetric heat capacity of a frozen soil, Caen silt, as a function of temperature for two initial dry densities (after Williams 1991)

Fig. 4.9 The three layers in a freezing soil

Fig. 4.10 Permeability of frozen soil as a function of temperature
(after Burt and Williams, 1976)

Fig. 4.11 Hydraulic conductivities of eight soils as functions of bath temperature (after Horiguchi and miller 1983)

Fig.4.12 Temperature element in space field

Fig. 4.13 Temperature element for time domain

Fig 4.14 Equivalent latent heat case 1

Fig 4.15 Equivalent latent heat case 2

Fig. 4.16 Frost cell used in Penner's experiments
(after Penner, 1986)

Fig.4.17 Temperature boundary condition.
in Penner(1986) experiment

Fig. 4.18 Comparison of calculated frost penetration with that measured by Penner (1986)

Fig.4.19 Comparison of calculated amount of water intake with experimental data by Penner (1986)

Fig.4.20 Comparison of calculated amount of frost heave with experimental data by Penner (1986)

Fig. 4.21 Frost heave versus time curves
for Example 2

Fig 4.22 Temperature distribution profile
in frozen soil for example 3

Chapter 5

A Pipe on Elastic Foundation Finite Element for the Large Displacement Inelastic Analysis of Pipelines

5.1 General

When a chilled gas pipeline crosses a transition zone between two initially unfrozen soils with different frost heave characteristics, frost heave causes differential movement along the pipeline, and such movement produces deformations and stresses in the pipeline. The induced stresses if sufficiently large can cause the pipe to fail.

The behaviour of pipelines buried in freezing ground has been studied in the past. Some typical investigations were conducted by Nixon et al.(1983), Ladanyi et al.(1985,1991), Selvadurai (1983,1993). Most of the above studies are based on the assumption that the displacements of the pipeline are small and the pipe remains elastic. But actually the displacements in a pipeline due to frost heave can be much larger than its diameter and the stresses in the pipe due to frost heave can surpass the yield stress in some cases. To obtain an accurate assessment of the stresses, the effects of both geometrical and material nonlinearities should be considered.

Although the analysis of a buried pipeline within the soil medium is a three dimensional problem, for a long distance pipeline the frost heave effect manifests itself mostly along the length of the pipeline because generally soils with different susceptibilities are distributed along the axis of the pipe. Thus the amount of differential heave produced in the other two directions is not as important. Accordingly a one dimensional beam-column model is adequate for engineering purposes.

Generally, a nonlinear finite element analysis may be considered to have three components: an element model (the interpolation functions, strain-displacement relation and stress-strain relation), a member coordinate system and a solution procedure for the nonlinear equilibrium equations. In this thesis the well known Hamilton beam-column on elastic foundation element will be developed, and a layered approach will be applied to capture the variation of material properties within the height of the pipe, assuming a trilinear stress-strain relation. A combined incremental-iterative method will be employed to solve the equilibrium equations. The nonlinear effects of both large displacements and inelastic behaviour will be determined by the combined use of up-dated stiffness matrix and coordinate transformation at each load step.

5.2 Theory of Beam-Column on Elastic Foundation

The following analysis of a beam-column on Winkler foundation (Fig.5.1a) is based on the classical Euler-Bernoulli theory which assumes that plane sections remain plane after bending. This assumption implies that shear deformations are negligible, which is valid for pipeline because they are thin-walled and very long.

For beam-column problems, the deformation coupling between in-plane and bending actions is taken into account as shown in Fig.5.2. For the beam in Fig.5.1, the axial displacement u , consists of three parts: u_a , due to axial forces, u_b , due to small deflection theory bending, the additional axial displacement u_l , due to large deflection as shown in Fig.5.3. Thus the axial displacement, $u(x)$, at any point located at a distance x from the left end can be expressed by

$$u(x) = u_a + u_b + u_l \quad (5.1)$$

According to conventional beam theory

$$u_b = -z\theta = -z \frac{dv}{dx} \quad (5.2)$$

where

v = vertical displacement of the beam

θ = rotation of the normal to beam axis

The additional displacement u_l can be derived from Fig.5.3 as follows:

$$\begin{aligned} du_l &= ds - dx \\ &= \sqrt{1 + \left(\frac{dv}{dx}\right)^2} dx - dx \end{aligned}$$

Using Taylor's expansion of the term under the square root, we obtain

$$du_l = \left(1 + \frac{1}{2}\left(\frac{dv}{dx}\right)^2 - \frac{1}{8}\left(\frac{dv}{dx}\right)^4 + \dots\right) dx - dx \quad (5.3)$$

where it is recognized that

$$ds = \sqrt{1 + \left(\frac{dv}{dx}\right)^2} dx$$

Neglecting the high order terms

$$du_l = \frac{1}{2}\left(\frac{dv}{dx}\right)^2 dx$$

Integrating the latter equation

$$u_l = \int_0^x du_l = \int_0^x \frac{1}{2}\left(\frac{dv}{dx}\right)^2 dx \quad (5.4)$$

Substitute Eqs.5.2 and 5.4 into 5.1

$$u(x) = u_a - z \frac{dv}{dx} + \int_0^x \frac{1}{2}\left(\frac{dv}{dx}\right)^2 dx \quad (5.5)$$

The longitudinal normal strain ϵ_x , referred to henceforth as the axial strain, is given by

$$\epsilon_x = \frac{du}{dx}$$

or

$$\epsilon_x = \frac{du_a}{dx} - z \frac{d^2 v}{dx^2} + \frac{1}{2} \left(\frac{dv}{dx} \right)^2 \quad (5.6)$$

or

$$\epsilon_x = \epsilon_0 + \epsilon_l \quad (5.7)$$

where

$$\epsilon_0 = \frac{du_a}{dx} - z \frac{d^2 v}{dx^2}$$

$$\epsilon_l = \frac{1}{2} \left(\frac{dv}{dx} \right)^2$$

ϵ_0 = axial strain due to small displacement theory

ϵ_l = axial strain due to large displacement.

The stress-strain relation is given as usual by

$$\sigma_x = D \epsilon_x$$

where D is the material modulus which depends on the level of stress.

The strain energy stored in the beam-column element is

$$\begin{aligned} U &= \int_v \frac{1}{2} \sigma \epsilon dv \\ &= \int_v \frac{D}{2} \epsilon^2 dv \end{aligned} \quad (5.8)$$

Substituting Eq.5.6 into Eq.5.8

$$U = \int_v \frac{D}{2} (\epsilon_0 + \epsilon_l) dv = \int_v \frac{D}{2} (\epsilon_0^2 + 2\epsilon_0 \epsilon_l + \epsilon_l^2) dv \quad (5.9)$$

with v being the volume of the beam element. Further substitution from Eq. 5.6 into Eq. 5.9 gives

$$U = \frac{D}{2} \int_L \int_A \left\{ \left(\frac{du}{dx} \right)^2 - 2z \frac{du}{dx} \frac{d^2v}{dx^2} + z^2 \left(\frac{d^2v}{dx^2} \right)^2 + \frac{du}{dx} \left(\frac{dv}{dx} \right)^2 - z \frac{d^2v}{dx^2} \left(\frac{dv}{dx} \right) + \frac{1}{4} \left(\frac{dv}{dx} \right)^4 \right\} dA dx \quad (5.10)$$

If the beam material remains elastic, then one can neglect $\left(\frac{dv}{dx} \right)^4$ and also the inner integrals can be simplified as

$$\int_A dA = A$$

$$\int_A z dA = 0$$

$$\int_A z^2 dA = I$$

$$\int_A D \frac{du}{dx} dA = N$$

where A and I are the beam cross-section and its moment of inertia, respectively, and N is its axial force.

In view of those simplifications, U can be written as

$$U = \frac{DA}{2} \int_0^l \left(\frac{du}{dx} \right)^2 dx + \frac{DI}{2} \int_0^l \left(\frac{d^2v}{dx^2} \right)^2 dx + \frac{N}{2} \int_0^l \left(\frac{dv}{dx} \right)^2 dx \quad (5.11)$$

Eq. 5.11 is the well-known strain energy expression for an elastic beam-column undergoing large deflection

The strain energy given by Eq.5.10 can also be expressed as

$$\begin{aligned}
 U = & \frac{D}{2} \int_v \left(\left(\frac{du}{dx} \right)^2 - 2z \frac{dv}{dx} \frac{d^2v}{dx^2} + z^2 \left(\frac{d^2v}{dx^2} \right)^2 \right) du \\
 & + \frac{D}{2} \int_v \left(\frac{du}{dx} - z \frac{d^2v}{dx^2} + \frac{1}{2} \left(\frac{dv}{dx} \right)^2 \right) \left(\frac{1}{2} \left(\frac{dv}{dx} \right)^2 - \frac{1}{8} \left(\frac{dv}{dx} \right)^4 \right) dv
 \end{aligned} \quad (5.12)$$

For elastoplastic beam-column

$$\int_A dA = A$$

$$\int_A z dA = S$$

$$\int_A z^2 dA = I$$

where S is the first moment of cross-sectional area. S will only be zero if the neutral axis is also an axis of material symmetry for the given cross-section

$$\int_A D \left(\frac{du}{dx} - z \frac{d^2v}{dx^2} + \frac{1}{2} \left(\frac{dv}{dx} \right)^2 \right) dA = \int_A d\epsilon_x dA \int_A \sigma_x dA = N$$

By discarding the higher order term $\left(\frac{dv}{dx} \right)^4$, Eq. 5.12 may now be written as

$$U = \int_0^l \frac{DA}{2} \left(\frac{du}{dx} \right)^2 dx - \int_0^l DS \left(\frac{du}{dx} \right) \left(\frac{d^2v}{dx^2} \right) dx + \int_0^l \frac{DI}{2} \left(\frac{d^2v}{dx^2} \right)^2 dx + \frac{N}{2} \int_0^l \left(\frac{dv}{dx} \right)^2 dx \quad (5.13)$$

The potential energy, Π , of the beam-column on Winkler foundation is

$$\Pi = U - W \quad (5.14)$$

where W is the work of the external forces, including the reactions of the foundation.

The W can be generally expressed as

$$W = W_n + W_f \quad (5.15)$$

where

W_n = the work done by all the external forces except the foundation reaction

W_f = the work done by the foundation reaction.

The work done by the foundation reaction is given by

$$W_f = \int_0^l dW_f dx \quad (5.16)$$

with

$$dW_f = -\frac{1}{2}f_{s1}u(x) - \frac{1}{2}f_{s2}v(x) \quad (5.17)$$

where as shown in Fig.5.1b f_{s1} and f_{s2} are the axial and vertical foundation reactions, respectively. Note that the negative sign signifies that the foundation reactions are in the opposite direction to the corresponding displacements.

Using Eq.5.15, Eq.5.14 becomes

$$\Pi = U - W_f - W_n \quad (5.18)$$

The axial displacement $u(x)$ in Eq.5.17 is given by Eq.5.5

$$u(x) = u_a - z \frac{dv}{dx} + \int_0^x \frac{1}{2} \left(\frac{dv}{dx} \right)^2 dx$$

In the present analysis, the axial foundation reaction is calculated based on the beam axial displacement $u(x) = u(x)|_{z=0}$, at the centroid level. Hence $z \frac{dv}{dx} = 0$, and neglecting the term $\int_0^x \frac{1}{2} \left(\frac{dv}{dx} \right)^2 dx$ is justified because in the incremental up-dated Lagrangian approach adopted herein because it is assumed that displacements are small at every load step.

Accordingly,

$$u(x) = u_a$$

Now Eq.5.17 can be written as

$$dW_f = -\frac{1}{2}f_{s1}u_a - \frac{1}{2}f_{s2}v(x) \quad (5.19)$$

and

$$f_{s1} = k_{s1}u_a \quad (5.20)$$

$$f_{s2} = k_{s2}v \quad (5.21)$$

in which

k_{s1} =axial modulus of the foundation

k_{s2} =vertical modulus of the foundation

It should be pointed out that in large deflection problem both moduli will be needed for proper representation of the soil-pipeline interaction.

Substituting Eq.5.19, 5.20, and 5.21 into Eq.5.16

$$W_f = -\frac{1}{2} \int_0^l (k_{s1}u_a^2 + k_{s2}v^2) dx \quad (5.22)$$

According to the principle of stationary potential energy

$$\delta\Pi = 0 \quad (5.23)$$

where δ as usual represents a small variation.

Using Eq.5.18 , and considering Eq.5.23

$$\delta U - \delta W_f = \delta W_n \quad (5.25)$$

where δU and δW_f can be obtained by the differentiation of Eqs.5.13 and 5.22

$$\begin{aligned} \delta U = & \int_0^l DA \left(\frac{du}{dx} \right) \delta \left(\frac{du}{dx} \right) dx - \int_0^l DS \left(\left(\delta \frac{du}{dx} \right) \left(\frac{d^2v}{dx^2} \right) + \left(\frac{du}{dx} \right) \left(\delta \frac{d^2v}{dx^2} \right) \right) dx \\ & + \int_0^l \frac{DI}{2} \left(\frac{d^2v}{dx^2} \right) \left(\delta \frac{d^2v}{dx^2} \right) dx + N \int_0^l \left(\frac{dv}{dx} \right) \left(\delta \frac{dv}{dx} \right) dx \end{aligned} \quad (5.26)$$

$$\delta W_f = - \int_0^l (k_{s1} u_a \delta u_a + k_{s2} v \delta v) dx \quad (5.27)$$

We shall use Eqs.5.26 and 5.27 develop an appropriate finite element.

5.3 Finite Element Formulation

The pipeline is discretized into an assemblage of finite two node beam-column elements, each with the degrees of freedom shown in Fig.5.4. The degrees of freedom consist of axial displacement u , vertical displacement v and rotation θ .

5.3.1 Element Displacements

The axial and vertical displacements (in the x and z directions) at any point within the element are approximated using linear and cubic interpolation functions, respectively. That is

$$u_a = N_1 u_1 + N_4 u_2$$

$$v = N_2 v_1 + N_3 \theta_1 + N_5 v_2 + N_6 \theta_2$$

or

$$u_a = [N_p] \{d\} \quad (5.28)$$

$$v = [N_b] \{d\} \quad (5.29)$$

where

v_i = displacement in z direction at node i .

u_i = displacement in x direction at node i

θ_i = rotation at node i

Functions N_i are shape functions given by

$$N_1 = 1 - \frac{x}{l}$$

$$N_4 = \frac{x}{l}$$

$$N_2 = 1 + 2\left(\frac{x}{l}\right)^3 - 3\left(\frac{x}{l}\right)^2$$

$$N_3 = x - 2\frac{x^2}{l} + \frac{x^3}{l^2}$$

$$N_5 = 3\left(\frac{x}{l}\right)^2 - 2\left(\frac{x}{l}\right)^3$$

$$N_6 = -\frac{x^2}{l} + \frac{x^3}{l^2}$$

$$[N_p] = [N_1 \quad 0 \quad 0 \quad N_4 \quad 0 \quad 0] \quad (5.30)$$

$$[N_b] = [0 \quad N_2 \quad N_3 \quad 0 \quad N_5 \quad N_6] \quad (5.31)$$

in which l is the element length. Note that $[N_p]$ is the in-plane shape function matrix, $[N_b]$ is the bending shape function matrix.

Let $\{d\}$ be the nodal displacement vector

$$\{d\} = [u_1 \quad v_1 \quad \theta_1 \quad u_2 \quad v_2 \quad \theta_2]^T$$

Then the displacement vector for any point within the element is given by

$$\{q\} = \begin{Bmatrix} u_a \\ v \end{Bmatrix} = [N]\{d\}$$

where

$$[N] = \begin{bmatrix} N_1 & 0 & 0 & N_4 & 0 & 0 \\ 0 & N_2 & N_3 & 0 & N_5 & N_6 \end{bmatrix}$$

The derivations of the displacements, which are needed later for strain calculations, can be written as follows:

$$\frac{du}{dx} = \frac{d[N_p]}{dx} \{d\} = [B_p] \{d\} \quad (5.32)$$

$$\frac{dv}{dx} = \frac{d[N_b]}{dx} \{d\} = [B_b] \{d\} \quad (5.33)$$

$$\frac{d^2v}{dx^2} = \frac{d[B_b]}{dx} \{d\} = [G] \{d\} \quad (5.34)$$

$$\begin{aligned} [B_p] &= \frac{d[N_p]}{dx} \\ &= \begin{bmatrix} \frac{dN_1}{dx} & 0 & 0 & \frac{dN_4}{dx} & 0 & 0 \end{bmatrix} \\ &= [B_{p1} \quad 0 \quad 0 \quad B_{p4} \quad 0 \quad 0] \end{aligned} \quad (5.35)$$

$$\begin{aligned} [B_b] &= \frac{d[N_b]}{dx} \\ &= [0 \quad B_2 \quad B_3 \quad 0 \quad B_5 \quad B_6] \end{aligned} \quad (5.36)$$

$$\begin{aligned} [G] &= \frac{d^2[N_b]}{dx^2} \\ &= [0 \quad G_2 \quad G_3 \quad 0 \quad G_5 \quad G_6] \end{aligned} \quad (5.37)$$

and

$$\begin{Bmatrix} B_{p1} \\ B_{p4} \end{Bmatrix} = \frac{1}{l} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

$$\begin{Bmatrix} B_2 \\ B_3 \\ B_5 \\ B_6 \end{Bmatrix} = \begin{Bmatrix} \frac{dN_2}{dx} \\ \frac{dN_3}{dx} \\ \frac{dN_5}{dx} \\ \frac{dN_6}{dx} \end{Bmatrix} = \begin{Bmatrix} \frac{6x^2}{l^3} - \frac{6x}{l^3} \\ 1 - \frac{4x}{l} + \frac{3x^2}{l^2} \\ -\frac{6x^2}{l^3} + \frac{6x}{l^3} \\ -\frac{2x}{l} + \frac{3x^2}{l^2} \end{Bmatrix}$$

$$\begin{Bmatrix} G_2 \\ G_3 \\ G_5 \\ G_6 \end{Bmatrix} = \begin{Bmatrix} \frac{dB_2}{dx} \\ \frac{dB_3}{dx} \\ \frac{dB_5}{dx} \\ \frac{dB_6}{dx} \end{Bmatrix} = \begin{Bmatrix} \frac{12x}{l^3} - \frac{6}{l^2} \\ -\frac{4}{l} + \frac{6x}{l^2} \\ \frac{6}{l^2} - \frac{12x}{l^3} \\ -\frac{2}{l} + \frac{6x}{l^2} \end{Bmatrix}$$

Note: In matrix algebra, the square of a matrix $[A]$ is written as $[A]^T[A]$

5.3.2 Element Strains, Stresses and Stiffness

By substituting Eqs.5.32, 5.33, 5.34 into Eq.5.6, then the strain at any point in the element may be calculated as

$$\epsilon_x = [B_p]\{d\} - z[G]\{d\} + \frac{1}{2}\{d\}^T[B_b]^T[B_b]\{d\} \quad (5.38)$$

$$\epsilon_0 = ([B_p] - z[G])\{d\} = [B_0]\{d\} \quad (5.39)$$

$$\epsilon_l = \frac{1}{2}\{d\}^T[B_b]^T[B_b]\{d\} \quad (5.40)$$

From Eq.5.28, 5.29, 5.32, 5.33 and 5.34

$$\delta u_a = [N_p]\{\delta d\} \quad (5.41)$$

$$\delta v = [N_b]\{\delta d\} \quad (5.42)$$

$$\delta \frac{du}{dx} = [B_p] \{\delta d\} \quad (5.43)$$

$$\delta \frac{dv}{dx} = [B_b] \{\delta d\} \quad (5.44)$$

$$\delta \left(\frac{d^2 v}{dx^2} \right) = [G] \{\delta d\} \quad (5.45)$$

Substituting Eqs.5.32, 5.33, 5.34, 5.43, 5.44 and 5.45 into Eq.5.26

$$\begin{aligned} \delta U = & \{\delta d\}^T \left(\int_0^l DA[B_p]^T[B_p]dx - \int_0^l DS([B_p]^T[G] + [G]^T[B_p])dx \right) \{d\} \\ & + \{\delta d\}^T \left(\int_0^l DI[G]^T[G]dx + \int_0^l N[B_b]^T[F_b]dx \right) \{d\} \end{aligned} \quad (5.46)$$

Substituting Eq.5.28, 5.29, 5.41 and 5.42 into Eq.5.27

$$\delta W_f = -\{\delta d\}^T \left(\int_0^l (k_{s1}[N_p]^T[N_p] + k_{s2}[N_b]^T[N_b])dx \right) \{d\} \quad (5.47)$$

Substitute Eqs.5.46, 5.47 and $\delta W_n = \{\delta d\}^T \{F\}$ into Eq.5.25, then the equilibrium equation will be obtained as

$$[K]\{d\} = \{F\} \quad (5.48)$$

where $\{F\}$ is the total nodal force vector, and $[K]$ is the element stiffness matrix which may be decomposed as

$$[K] = [K_m] + [K_G] + [K_f] \quad (5.49)$$

in which

$[K_m]$ = conventional small displacement stiffness matrix, which depends on the dimensions and material properties of the element

$[K_G]$ = geometrical stiffness matrix ,which depends on the magnitude of the axial force in the element

$[K_f]$ = foundation matrix ,which depends on the modulus of the foundation material surrounding the element.

Elements of the foregoing stiffness matrices are given explicitly by

$$\begin{aligned}
 [K_m] &= \int_v [B_0]^T D [B_0] dv \\
 &= \int_0^l (DA[B_p]^T [B_p] - DS([B_p]^T [G] + [G]^T [B_p]) + DI[G]^T [G]) dx \\
 &= \int_0^l \begin{bmatrix} AB_{p1}^2 & & & & & \\ SB_{p1}G_2 & IG_2^2 & & & & \\ SB_{p1}G_3 & IG_2G_3 & IG_3^2 & & & \\ AB_{p1}B_{p4} & SG_2B_{p4} & SG_3B_{p4} & AB_{p4}^2 & & \\ SB_{p1}G_5 & IG_2G_5 & IG_3G_5 & SB_{p4}G_5 & IG_5^2 & \\ SB_{p1}G_6 & IG_2G_6 & IG_3G_6 & SB_{p4}G_6 & IG_5G_6 & IG_6^2 \end{bmatrix} dx \quad \text{Symmetric} \quad (5.50)
 \end{aligned}$$

or

$$[K_m] = \begin{bmatrix} k_{m11} & k_{m12} & \dots & k_{m16} \\ k_{m21} & k_{m22} & \dots & k_{m26} \\ \vdots & \vdots & \ddots & \vdots \\ k_{m61} & k_{m62} & \dots & k_{m66} \end{bmatrix}$$

where for a general pipe element with variable D, A, S and I

$$k_{m11} = \int_0^l \frac{DA}{l^2} dx$$

$$k_{m12} = \int_0^l \frac{DS}{l} \left(-\frac{6}{l^2} + 12\frac{x}{l^3} \right) dx$$

$$k_{m13} = \int_0^l \frac{DS}{l} \left(-\frac{4}{l} + 6\frac{x}{l^2} \right) dx$$

$$k_{m14} = \int_0^l \frac{-DA}{l^2} dx$$

$$k_{m15} = \int_0^l \frac{DS}{l} \left(+\frac{6}{l^2} - 12\frac{x}{l^3} \right) dx$$

$$\begin{aligned}
k_{m16} &= \int_0^l \frac{DS}{l} \left(-\frac{2}{l} + 6\frac{x}{l^2} \right) dx \\
k_{m22} &= \int_0^l \frac{DI}{l^3} \left(36 - 144\frac{x}{l} + 144\frac{x^2}{l^2} \right) dx \\
k_{m23} &= \int_0^l \frac{DI}{l^3} \left(24 - 84\frac{x}{l} + 172\frac{x^2}{l^2} \right) dx \\
k_{m24} &= \int_0^l \frac{DS}{l} \left(-\frac{6}{l^2} + 12\frac{x}{l^3} \right) dx \\
k_{m25} &= \int_0^l \frac{-DI}{l^3} \left(36 + 144\frac{x}{l} + 144\frac{x^2}{l^2} \right) dx \\
k_{m26} &= \int_0^l \frac{DI}{l^3} \left(12 - 60\frac{x}{l} + 72\frac{x^2}{l^2} \right) dx \\
k_{m33} &= \int_0^l \frac{DI}{l^2} \left(16 - 48\frac{x}{l} + 36\frac{x^2}{l^2} \right) dx \\
k_{m34} &= \int_0^l \frac{-DS}{l} \left(-\frac{4}{l} + 6\frac{x}{l^2} \right) dx \\
k_{m35} &= \int_0^l \frac{-DI}{l^3} \left(24 - 84\frac{x}{l} + 72\frac{x^2}{l^2} \right) dx \\
k_{m36} &= \int_0^l \frac{DI}{l^2} \left(8 - 36\frac{x}{l} + 36\frac{x^2}{l^2} \right) dx \\
k_{m44} &= \int_0^l \frac{DA}{l^2} dx \\
k_{m45} &= \int_0^l \frac{-DS}{l} \left(\frac{6}{l^2} - 12\frac{x}{l^3} \right) dx \\
k_{m46} &= \int_0^l \frac{-DS}{l} \left(-\frac{2}{l} + 6\frac{x}{l^2} \right) dx \\
k_{m55} &= \int_0^l \frac{DI}{l^4} \left(36 - 144\frac{x}{l} + 144\frac{x^2}{l^2} \right) dx \\
k_{m56} &= \int_0^l \frac{DI}{l^3} \left(12 - 60\frac{x}{l} + 72\frac{x^2}{l^2} \right) dx \\
k_{m66} &= \int_0^l \frac{DI}{l^2} \left(14 - 24\frac{x}{l} + 36\frac{x^2}{l^2} \right) dx
\end{aligned} \tag{5.51}$$

where k_{m11} , k_{m14} , k_{m41} and k_{m44} are elements of the in-plane stiffness matrix, while the remaining elements of the first and fourth rows and columns represent the coupling effect between the in-plane and bending actions. Elements of rows and columns 2, 3, 5 and 6 are bending stiffness coefficients.

$$\begin{aligned}
[K_G] &= N \int_0^l [B_b]^T [B_b] dx \\
&= \int_0^l N \begin{bmatrix} 0 & & & & & \\ 0 & B_2^2 & & & & \\ 0 & B_3 B_2 & B_3^2 & & & \\ 0 & 0 & 0 & 0 & & \\ 0 & B_5 B_2 & B_5 B_3 & 0 & B_5^2 & \\ 0 & B_6 B_2 & B_6 B_3 & 0 & B_6 B_5 & B_6^2 \end{bmatrix} dx
\end{aligned} \tag{5.52}$$

$$= \frac{N}{10l} \begin{bmatrix} 0 & & & & & \\ 0 & 12 & & \text{Symmetric} & & \\ 0 & l & \frac{4l^2}{3} & & & \\ 0 & 0 & 0 & 0 & & \\ 0 & -12 & -l & 0 & 12 & \\ 0 & l & -\frac{l^2}{3} & 0 & -l & \frac{4l^2}{3} \end{bmatrix}$$

$$\begin{aligned}
[K_f] &= \int_0^l (k_{s1} [N_p]^T [N_p] + k_{s2} [N_b]^T [N_b]) dx \\
&= \int_0^l k_{s2} \begin{bmatrix} \frac{k_{s1}}{k_{s2}} N_1^2 & & & & & \\ 0 & N_2^2 & 0 & \text{Symmetric} & & \\ 0 & N_2 N_3 & N_3^2 & & & \\ \frac{k_{s1}}{k_{s2}} N_1 N_4 & 0 & 0 & \frac{k_{s1}}{k_{s2}} N_4^2 & & \\ 0 & N_2 N_5 & N_3 N_5 & 0 & N_5^2 & \\ 0 & N_2 N_6 & N_3 N_6 & 0 & N_5 N_6 & N_6^2 \end{bmatrix} dx
\end{aligned} \tag{5.53}$$

$$= \frac{l}{420} \begin{bmatrix} 140k_{s1} & & & & & \\ 0 & 156k_{s2} & & \text{Symmetric} & & \\ 0 & 22lk_{s2} & 4l^2k_{s2} & & & \\ 70k_{s1} & 0 & 0 & 140k_{s1} & & \\ 0 & 54k_{s2} & 13lk_{s2} & 0 & 156k_{s2} & \\ 0 & -13lk_{s2} & -3l^2k_{s2} & 0 & -22lk_{s2} & 4l^2k_{s2} \end{bmatrix}$$

in Eq.5.52 the tensile axial force is positive.

In the general case the total nodal force vector is given by

$$\{F\} = \{f\} + \{f_g\} + \{f_s\}$$

where

$\{f\}$ =applied nodal force vector

$\{f_g\}$ =equivalent nodal force vector due to body forces

$\{f_s\}$ =equivalent nodal force vector due to surface tractions

The later two set of nodal forces can be calculated using

$$\{f_g\} = \int_v [N]^T \{g\} dv$$

$$\{f_s\} = \int_v [N]^T \{q_s\} dv$$

in which

$\{g\}$ =vector of body force per unit volume of the element

$\{q_s\}$ = vector of surface tractions in the element

5.4 Elastoplastic Behaviour

5.4.1 Trilinear Model

To account for in-elasticity, the material will be assumed to exhibit a trilinear stress-strain behaviour as shown in Fig.5.5. The stress-strain relation in Fig.5.5 can be used to model most common materials. For steel, the modulus $E_T = 0$, and E_{sh} will represent the strain-hardening modulus. If an elastoplastic behaviour is assumed, then $E_{sh} = 0$.

5.4.2 Layered Elastoplastic Element Stiffness Matrix

The total stiffness matrix consists of the stiffness matrix $[K_m]$, the geometrical stiffness matrix $[K_G]$ and the foundation stiffness matrix $[K_f]$, as shown in Eq.5.49. Matrices $[K_G]$ and $[K_f]$ can be obtained in closed-form by integrating Eqs.5.51 and 5.52, respectively because the the geometric stiffness matrix depends only on the internal forces in the element, while the foundation stiffness matrix depends only on the modulus of the foundation material.

On the other hand, exact integration of the stiffness matrix $[K_m]$ will be difficult because $[K_m]$ generally depends on material properties. To account for variations in material properties within the pipe cross-section, a layered approach will be used to determine $[K_m]$.

In the layered approach the pipe is subdivided into a number of layers through its depth(Fig.5.6). In the finite element solution, it is assumed that as soon as the stress in the middle of any layer has reached the yield stress, then the particular layer become plastic, while the rest of the layers remain elastic as shown in Fig.5.7. After yield, the value of stress may increase, depending on whether strain-hardening can occur. In calculating the stiffness matrix, for each layer the modulus will be adjusted according to the stress level.

When material nonlinearity is considered, numerical integration will be employed to obtain the stiffness matrix. The integration will be performed using three points Gauss quadrature. The cross-section will be assumed to consist of a number of layers, as indicated in Eq.Fig.5.8. To be able to perform the integration, natural coordinate ξ will be introduced as follows:

$$x = \frac{l}{2}(\xi + 1)$$

and

$$d\xi = \frac{2}{l}dx$$

Accordingly, the stiffness matrix $[K_m]$ becomes

$$\begin{aligned} [K_m] &= \int_A \int_0^l [B_0]^T D [B_0] dx dA \\ &= \int_A \int_{-1}^1 [B_0]^T D [B_0] \frac{l}{2} d\xi dA \end{aligned}$$

or

$$\begin{aligned} [K_m] &= \int_0^l \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} [B_0(x, y, z)]^T D [B_0(x, y, z)] dy dz dx \\ &= \int_{-1}^1 \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} [B_0(\xi, y, z)]^T D [B_0(\xi, y, z)] \frac{l}{2} dy dz d\xi \end{aligned} \quad (5.54)$$

where b and t represent the width and height of the element cross-section. For pipelines, b and t will be equal to the diameter of the pipe. The pipeline cross-section will be idealized by the layering scheme shown in Fig.5.6. Assuming the material properties to be constant across the layers, the numerical integration of the stiffness matrix can be performed as follows:

$$\begin{aligned}
 [K_m] &= \sum_{i=1}^3 \sum_{j=1}^{\lambda} W_i [B_0(\xi_i, z_j)]^T D_{ij} [B_0(\xi_i, z_j)] \frac{l}{2} \\
 &= \begin{bmatrix} k_{m11} & k_{m12} & \dots & k_{m16} \\ k_{m21} & k_{m22} & \dots & k_{m26} \\ \vdots & \vdots & \ddots & \vdots \\ k_{m61} & k_{m62} & \dots & k_{m66} \end{bmatrix}
 \end{aligned} \tag{5.55}$$

where λ is the total number of layers and weight factors W_i for three point scheme are:

$$[W_1 \quad W_2 \quad W_3] = \left[\frac{5}{9} \quad \frac{8}{9} \quad \frac{5}{9} \right]$$

The natural co-ordinates of the three Gauss points are:

$$[\xi_1 \quad \xi_2 \quad \xi_3] = [-\sqrt{0.6} \quad 0 \quad \sqrt{0.6}]$$

The local co-ordinates of Gauss points are:

$$[x_1 \quad x_2 \quad x_3] = [(0.5 - \sqrt{0.15})l \quad 0.5l \quad (0.5 + \sqrt{0.15})l]$$

Using the foregoing values, the elements of the stiffness matrix (5.55) may be written as

$$\begin{aligned}
 k_{m11} &= \sum_{i=1}^3 \sum_{j=1}^{\lambda} W_i D_j A_j \frac{1}{2l} \\
 k_{m12} &= \sum_{i=1}^3 \sum_{j=1}^{\lambda} W_i \left(-6 + 12 \frac{x_i}{l}\right) D_j S_j \frac{1}{2l^2}
 \end{aligned}$$

$$k_{m13} = \sum_{i=1}^3 \sum_{j=1}^{\lambda} W_i(-4 + 6\frac{x_i}{l})D_j S_j \frac{1}{2l}$$

$$k_{m14} = \sum_{i=1}^3 \sum_{j=1}^{\lambda} -W_i D_j A_j \frac{1}{2l}$$

$$k_{m15} = \sum_{i=1}^3 \sum_{j=1}^{\lambda} W_i(6 - 12\frac{x_i}{l})D_j S_j \frac{1}{2l^2}$$

$$k_{m16} = \sum_{i=1}^3 \sum_{j=1}^{\lambda} W_i(-2 + 6\frac{x_i}{l})D_j S_j \frac{1}{2l}$$

$$k_{m22} = \sum_{i=1}^3 \sum_{j=1}^{\lambda} W_i(36 - 144\frac{x_i}{l} + 144\frac{x_i^2}{l^2})D_j I_j \frac{1}{2l^3}$$

$$k_{m23} = \sum_{i=1}^3 \sum_{j=1}^{\lambda} W_i(24 - 84\frac{x_i}{l} + 72\frac{x_i^2}{l^2})D_j I_j \frac{1}{2l^2}$$

$$k_{m24} = \sum_{i=1}^3 \sum_{j=1}^{\lambda} -W_i(-6 + 12\frac{x_i}{l})D_j S_j \frac{1}{2l^2}$$

$$k_{m25} = \sum_{i=1}^3 \sum_{j=1}^{\lambda} -W_i(36 - 144\frac{x_i}{l} + 144\frac{x_i^2}{l^2})D_j I_j \frac{1}{2l^3}$$

$$k_{m26} = \sum_{i=1}^3 \sum_{j=1}^{\lambda} W_i(12 - 60\frac{x_i}{l} + 72\frac{x_i^2}{l^2})D_j I_j \frac{1}{2l^2}$$

$$k_{m33} = \sum_{i=1}^3 \sum_{j=1}^{\lambda} W_i(16 - 48\frac{x_i}{l} + 36\frac{x_i^2}{l^2})D_j I_j \frac{1}{2l}$$

$$k_{m34} = \sum_{i=1}^3 \sum_{j=1}^{\lambda} -W_i(-4 + 6\frac{x_i}{l})D_j S_j \frac{1}{2l}$$

$$k_{m35} = \sum_{i=1}^3 \sum_{j=1}^{\lambda} -W_i(24 - 84\frac{x_i}{l} + 72\frac{x_i^2}{l^2})D_j I_j \frac{1}{2l^2}$$

$$k_{m36} = \sum_{i=1}^3 \sum_{j=1}^{\lambda} W_i(8 - 36\frac{x_i}{l} + 36\frac{x_i^2}{l^2})D_j I_j \frac{1}{2l}$$

$$k_{m44} = k_{11}$$

$$k_{m45} = -k_{15}$$

$$k_{m46} = -k_{16}$$

$$k_{m55} = k_{22}$$

$$k_{m56} = -k_{26}$$

$$k_{m66} = \sum_{i=1}^3 \sum_{j=1}^{\lambda} W_i \left(4 - 24 \frac{x_i}{l} + 36 \frac{x_i^2}{l^2} \right) D_j I_j \frac{1}{2l}$$

$$k_{mij} = k_{mji}$$

Note that for any given layer j

$$A_j = b_j t_j$$

$$S_j = b_j t_j z_j$$

$$I_j = b_j t_j z_j^2$$

i = sampling point number

j = layer number

b_j = the width of the j th layer

t_j = the thickness of the j th layer

z_j = the z -coordinate of the middle of the j th layer

D_j = the material modulus of the j th layer which can assume the following values:

$$D_j = \begin{cases} E, & \text{for } \epsilon \leq \epsilon_y; \\ E_T, & \text{for } \epsilon_y \leq \epsilon \leq \epsilon_{sh}; \\ E_{sh}, & \text{for } \epsilon \leq \epsilon_{sh}. \end{cases} \quad (5.56)$$

5.4.3 Internal force Vector

For an elastic pipe element the internal nodal forces are given directly by

$$\{F_I\} = [K]\{d\} \quad (5.57)$$

For an elastoplastic pipe element internal forces can be obtained from Eq.5.48 and 5.49

$$\begin{aligned}
 \{F_I\} &= [K]\{d\} \\
 &= ([K_m] + [K_G] + [K_f])\{d\} \\
 &= \{F_{I1}\} + \{F_{I2}\} + \{F_{I3}\}
 \end{aligned} \tag{5.58}$$

where

$$\{F_{I1}\} = [K_m]\{d\} \tag{5.59}$$

$$\{F_{I2}\} = [K_G]\{d\} \tag{5.60}$$

$$\{F_{I3}\} = [K_f]\{d\} \tag{5.61}$$

Substituting Eq.5.52, 5.53 into Eq.5.60, 5.61, respectively, we get the internal force vectors $\{F_{I2}\}$ and $\{F_{I3}\}$

$$\begin{aligned}
 \{F_{I2}\} &= \begin{Bmatrix} F_{2(x1)} \\ F_{2(y1)} \\ M_{2(1)} \\ F_{2(x2)} \\ F_{2(y2)} \\ M_{2(2)} \end{Bmatrix} \\
 &= \frac{-N}{10l} \begin{bmatrix} 0 & & & & & \\ 0 & 12 & & & & \\ 0 & l & \frac{4l^2}{3} & & & \\ 0 & 0 & 0 & 0 & & \\ 0 & -12 & -l & 0 & 12 & \\ 0 & l & \frac{-4l^2}{3} & 0 & -l & \frac{4l^2}{3} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{Bmatrix}
 \end{aligned} \tag{5.62}$$

$$\{F_{I3}\} = \begin{Bmatrix} F_{3(x1)} \\ F_{3(y1)} \\ M_{3(1)} \\ F_{3(x2)} \\ F_{3(y2)} \\ M_{3(2)} \end{Bmatrix}$$

$$= \begin{bmatrix} 140k_{s1} & 0 & 156k_{s2} & 4l^2k_{s2} & \text{Symmetric} & 0 \\ 0 & 22lk_{s2} & 0 & 0 & 140k_{s1} & 0 \\ 70ks1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 54k_{s2} & 13lk_{s2} & 0 & 156k_{s2} & -22lk_{s2} \\ 0 & -13lk_{s2} & -3l^2k_{s2} & 0 & -22lk_{s2} & 4l^2k_{s2} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{Bmatrix} \quad (5.63)$$

Note that in the presence of material nonlinearity, the internal forces vector $\{F_{I1}\}$ can not be written in closed-form.

The contribution of each layer to the internal forces is obtained by integration over the thickness of that layer only. The internal forces in a cross-section are then found from the summation of the contributions of all the layers in the element. Gaussian integration will be used to calculate the internal forces. Accordingly, the internal force nodal vector $\{F_{I1}\}$ is given by:

$$\begin{aligned} \{F_{I1}\} &= [K_m]\{d\} \\ &= \int_v [B_0]^T [D] [B_0] \{d\} dv \\ &= \int_v [B_0]^T \{\sigma_0\} dv \end{aligned} \quad (5.64)$$

where σ_0 is longitudinal normal stress in the pipe according to small displacement theory. Note that in the present case vectors $\{\sigma_0\}$ and $\{\epsilon_0\}$ have only one element.

$$\{\sigma_0\} = [D]\{\epsilon_0\}$$

$$\{\epsilon_0\} = [B_0]\{d\}$$

and from Eq.5.39, $[B_0]$ can be expressed as

$$[B_0] == \left[-\frac{1}{l} \quad z\left(-\frac{6}{l} + \frac{12x}{l^3}\right) \quad z\left(-\frac{4}{l} + \frac{6x}{l^2}\right) \quad \frac{1}{l} \quad z\left(\frac{6}{l^2} - \frac{12x}{l^3}\right) \quad z\left(-\frac{2}{l} + \frac{6x}{l^2}\right) \right] \quad (5.65)$$

The terms of internal force nodal vector $\{F_{I1}\}$ in Eq.5.64 are given by

$$F_{1(x1)} = \sum_{i=1}^3 \sum_{j=1}^{\lambda} -W_i(A_j\sigma_{0j})_i \frac{1}{2} \quad (5.66)$$

$$F_{1(y1)} = \sum_{i=1}^3 \sum_{j=1}^{\lambda} W_i(-6 + 12\frac{x_i}{l})(S_j\sigma_{0j})_i \frac{1}{2l} \quad (5.67)$$

$$M_{1(1)} = \sum_{i=1}^3 \sum_{j=1}^{\lambda} W_i(-4 + 6\frac{x_i}{l})(-I_j\sigma_{0j})_i \frac{1}{2} \quad (5.68)$$

$$F_{1(x2)} = \sum_{i=1}^3 \sum_{j=1}^{\lambda} W_i(A_j\sigma_{0j})_i \frac{1}{2} \quad (5.69)$$

$$F_{1(y2)} = \sum_{i=1}^3 \sum_{j=1}^{\lambda} W_i(6 - 12\frac{x_i}{l})(S_{0j}\sigma_{0j})_i \frac{1}{2l} \quad (5.70)$$

$$M_{1(2)} = \sum_{i=1}^3 \sum_{j=1}^{\lambda} W_i(-2 + 6\frac{x_i}{l})(-I_j\sigma_{0j})_i \frac{1}{2} \quad (5.71)$$

The symbols in the latter equations represent the same quantities as defined in Section 5.3.2. For an elastoplastic pipe element, the internal force vector should be replaced by the internal force vector $\{F_I\}$ in Eq.5.58. In the preceding analysis we should note that the value of the material modulus of a given layer is determined from the value of the total strain in the layer, which is given by Eq.5.38.

5.5 Coordinate Update

An updated Lagrangian procedure will be employed in the analysis to account for large displacements. It is assumed that the displacements remain small at every load step in spite of the geometric nonlinearity of the structure as a whole. In fact, if suitable incremental loads(including the loads caused by prescribed displacements) are applied at every step, the small displacement theory can be used. As shown

in Fig.5.9, two coordinate systems are introduced, one the local coordinate system $x - y$ and the other a reference (or global) coordinate $\bar{x} - \bar{y}$.

From Fig.5.9 the nodal displacements u_i and v_i in local coordinates are related to the corresponding $\bar{u}_i$ and $\bar{v}_i$ in reference coordinates by

$$u_i = \bar{u}_i \cos \alpha + \bar{v}_i \sin \alpha$$

$$v_i = -\bar{u}_i \sin \alpha + \bar{v}_i \cos \alpha$$

or more generally

$$\begin{Bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{Bmatrix} = \begin{bmatrix} c & s & 0 & & & \\ -s & c & 0 & & 0 & \\ 0 & 0 & 1 & & & \\ & & & c & s & 0 \\ & 0 & & -s & c & 0 \\ & & & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \bar{u}_1 \\ \bar{v}_1 \\ \bar{\theta}_1 \\ \bar{u}_2 \\ \bar{v}_2 \\ \bar{\theta}_2 \end{Bmatrix}$$

or

$$\{d\} = [T]\{\bar{d}\} \quad (5.72)$$

where

$$c = \cos \alpha$$

$$s = \sin \alpha$$

α = the angle of orientation of the element (angle between the local x and reference $\bar{x}$)

measured counterclockwise from the reference to the local axis

$\{d\}$ = nodal displacement vector in local coordinate system

$\{\bar{d}\}$ = nodal displacement vector in reference coordinate system

$[T]$ = transformation matrix

The nodal force vectors in the two systems are related by

$$\{F\} = [T]\{\bar{F}\} \quad (5.73)$$

where

$\{F\}$ = nodal force vector in the local coordinate system

$\{\bar{F}\}$ = nodal force vector in the reference coordinate system

Substituting Eq.5.72 and 5.73 into Eq.5.48 we obtain

$$[K][T]\{\bar{d}\} = [T]\{\bar{F}\} \quad (5.74)$$

The equilibrium equations in reference coordinate system is as usual given by

$$[T]^T[K][T]\{d\} = \{\bar{F}\} \quad (5.75)$$

or

$$[\bar{K}]\{d\} = \{\bar{F}\} \quad (5.76)$$

$[\bar{K}]$ can be written as

$$\begin{aligned} [\bar{K}] &= [T]^T[K][T] \\ &= \begin{bmatrix} c & -s & 0 & 0 \\ s & c & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} k_{11} & k_{12} & \dots & k_{16} \\ k_{21} & k_{22} & \dots & k_{26} \\ \vdots & \vdots & \ddots & \vdots \\ k_{61} & k_{62} & \dots & k_{66} \end{bmatrix} \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \bar{k}_{11} & \bar{k}_{12} & \dots & \bar{k}_{16} \\ \bar{k}_{21} & \bar{k}_{22} & \dots & \bar{k}_{26} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{k}_{61} & \bar{k}_{62} & \dots & \bar{k}_{66} \end{bmatrix} \end{aligned}$$

where

$$\bar{k}_{11} = (k_{11}c - k_{12}s)c - (k_{12}c - k_{22}s)s$$

$$\bar{k}_{12} = (k_{11}c - k_{12}s)s + (k_{12}c - k_{22}s)c$$

$$\bar{k}_{13} = k_{13}c - k_{23}s$$

$$\bar{k}_{14} = (k_{14}c - k_{24}s)c - (k_{15}c - k_{25}s)s$$

$$\bar{k}_{15} = (k_{14}c - k_{24}s)s + (k_{15}c - k_{25}s)c$$

$$\bar{k}_{16} = k_{16}c - k_{26}s$$

$$\bar{k}_{22} = (k_{11}s + k_{12}c)s + (k_{12}s + k_{22}c)c$$

$$\bar{k}_{23} = k_{13}s + k_{23}c$$

$$\bar{k}_{24} = (k_{14}s + k_{24}c)c - (k_{15}s + k_{25}c)s$$

$$\bar{k}_{25} = (k_{14}s + k_{24}c)s + (k_{15}s + k_{25}c)c$$

$$\bar{k}_{26} = k_{16}s + k_{26}c$$

$$\bar{k}_{33} = k_{33}$$

$$\bar{k}_{34} = k_{34}c - k_{35}s$$

$$\bar{k}_{35} = k_{34}s + k_{35}c$$

$$\bar{k}_{36} = k_{36}$$

$$\bar{k}_{44} = (k_{44}c - k_{45}s)c - (k_{45}c - k_{55}s)s$$

$$\bar{k}_{45} = (k_{44}c - k_{45}s)s + (k_{45}c - k_{55}s)c$$

$$\bar{k}_{46} = k_{46}c - k_{56}s$$

$$\bar{k}_{55} = (k_{44}s + k_{45}c)s + (k_{45}s + k_{55}c)c$$

$$\bar{k}_{56} = k_{46}s + k_{56}c$$

$$\bar{k}_{66} = k_{66}$$

$$\bar{k}_{ij} = k_{ji}$$

5.6 Solution Of Nonlinear Equations

5.6.1 Incremental-Iterative Method

The complete structure stiffness matrix $[\bar{K}_T]$ and nodal load vector $\{\bar{F}_T\}$ are obtained using the standard method of assembly. In linear problems the stiffness matrix $[\bar{K}_T]$ is constant, thus Eq.5.76 represents a set of linear equations which can be solved by traditional methods of linear equation solvers. In nonlinear problems, the stiffness matrix $[\bar{K}_T]$ is dependent on the displacements, $\{\bar{d}\}$. Accordingly; Eq.5.76 becomes

$$[\bar{K}_T(\{\bar{d}\})]\{\bar{d}\} = \{\bar{F}\} \quad (5.77)$$

where $\{\bar{d}\}$ within the square brackets signifies that $\bar{K}_T$ is a function of $\{\bar{d}\}$.

Eq.5.77 gives a set of nonlinear equations whose solution can be very complex.

The solution of nonlinear equations can be achieved through a series of incrementally linear solutions. The procedure commences by dividing the total load vector load vector $\{\bar{F}_T\}$ into n load increments such that

$$\{\bar{F}_T\} = \sum_{r=1}^n \{\Delta \bar{F}_T^r\}$$

where r denotes the load increment(step) number.

The incremental form of the equilibrium equation 5.77 is given by

$$[\bar{K}_T^{-1}]\{\Delta \bar{d}^r\} = \{\Delta \bar{F}_T^r\} \quad (5.78)$$

During load step r a constant stiffness matrix, which is evaluated at the end of the previous load step, is assumed. Thus

$$[\bar{K}_T^{r-1}] = [\bar{K}_T^{-1}(\{\bar{d}\})]$$

In both geometrical and material nonlinear problems Eq.5.55 indicates that $[\bar{K}_T^{r-1}]$ is dependent on the values of the moduli D^{r-1} , which are functions of the level of stress in each layer. The value of angle α^{r-1} can be found from the element nodal deformations at the end of the previous load step,

$$\tan(\alpha)^{r-1} = \frac{(\bar{z}_2)^{r-1} - (\bar{z}_1)^{r-1}}{(\bar{x}_2)^{r-1} - (\bar{x}_1)^{r-1}} \quad (5.79)$$

where $(\bar{x}_i, \bar{z}_i)$ are the coordinates of node i .

The total displacement vector at the end of load step r is given by

$$\{\bar{d}^{r-1}\} = \{\bar{d}^{r-2}\} + \{\Delta\bar{d}^{r-1}\} \quad (5.80)$$

where $\Delta\bar{d}^{r-1}$ = the increment of nodal displacement caused by the load increment during load step $(r - 1)$

In Eq.5.79 the global coordinates of node i at the end of step $(r - 1)$ are given by

$$(\bar{x}_i)^{r-1} = (\bar{x}_i)^{r-2} + \Delta(\bar{u}_i)^{r-1}$$

$$(\bar{y}_i)^{r-1} = (\bar{y}_i)^{r-2} + \Delta(\bar{v}_i)^{r-1} \quad (5.81)$$

A better approximation, which is termed as midpoint tangent incremental approach, will be applied to improve the results. The method replaces the stiffness matrix $[\bar{K}_T^{r-1}]$ at the end of step $(r - 1)$ by the midpoint stiffness matrix $[\bar{K}_T^{r-\frac{1}{2}}]$ as shown in Fig.5.9. In this scheme two cycles of analysis are performed for each load step. In the first cycle $\frac{1}{2}\{\Delta\bar{F}_T^r\}$ is applied and the corresponding displacement increment $\{\Delta\bar{d}^{r-\frac{1}{2}}\}$ is calculated by

$$[\bar{K}_T^{r-1}]\{\Delta\bar{d}^{r-\frac{1}{2}}\} = \frac{1}{2}\{\Delta\bar{F}_T^r\} \quad (5.82)$$

The displacement vector and the angle α at half the increment $\{\bar{d}^{r-\frac{1}{2}}\}$ are obtained from

$$\{\bar{d}^{r-\frac{1}{2}}\} = \{\bar{d}^{r-1}\} + \{\Delta\bar{d}^{r-\frac{1}{2}}\}$$

$$\tan(\alpha)^{r-\frac{1}{2}} = \frac{(\bar{z}_2)^{r-\frac{1}{2}} - (\bar{z}_1)^{r-\frac{1}{2}}}{(\bar{x}_2)^{r-\frac{1}{2}} - (\bar{x}_1)^{r-\frac{1}{2}}}$$

and then the stiffness matrix is updated to get the full displacement increment $\{\Delta\bar{d}^r\}$

$$[\bar{K}_T^{r-\frac{1}{2}}]\{\Delta\bar{d}^{r-\frac{1}{2}}\} = \{\Delta\bar{F}_T^r\} \quad (5.83)$$

Eq.5.83 can be written in conventional form as

$$[\bar{K}_T^{r-1}]\{\Delta\bar{d}^{r-\frac{1}{2}}\} = \{\Delta\bar{F}_T^r\} \quad (5.84)$$

where stiffness matrix $[\bar{K}_T^{r-1}]$ is the midpoint tangent stiffness matrix $[\bar{K}_T^{r-\frac{1}{2}}]$. It is interesting to examine the errors introduced by the midpoint tangent incremental procedure and the simple incremental procedure.

Let us denote the load and displacement by f and d , respectively. The true displacement vector at the end of step r may be expressed by a four-term Taylor series

$$\begin{aligned} d^r(f) &= d(f^{r-1} + \Delta f) \\ &= d^{r-1}(f) + (d^{r-1}(f))' \Delta f + \frac{1}{2}(d^{r-1}(f))'' \Delta f^2 + \frac{1}{6}(d^{r-1}(f))''' \Delta f^3 + \dots \end{aligned} \quad (5.85)$$

This indicates that the simple linear incremental approach introduces during each increment an error whose dominant terms are

$$\varepsilon = \frac{1}{2}(d^{r-1}(f))'' \Delta f^2 + \frac{1}{6}(d^{r-1}(f))''' \Delta f^3 \quad (5.86)$$

The approximate displacement vector at the end of step r derived by midpoint tangent incremental approach is

$$\begin{aligned}
 d^r(f) &= d^{r-1}(f) + (d^{r-1}(f))' \Delta f \\
 &= d^{r-1}(f) + (d(f^{r-1} + \frac{1}{2}\Delta f))' \delta f \\
 &= d^{r-1}(f) + (d^{r-1}(f))' \Delta f + \frac{1}{2}(d^{r-1}(f))'' \Delta f^2 + \frac{1}{8}(d^{r-1}(f))''' \Delta f^3
 \end{aligned} \tag{5.87}$$

Comparing Eq.5.85 and 5.87, the dominant term in the latter case is seen to be

$$\epsilon = \frac{1}{24}(d^{r-1}(f))''' \Delta f^3 \tag{5.88}$$

Comparing the two error terms in Eqs.5.86 and 5.87, it is clear that the latter is expected to be much smaller than the former. Fig .5.10 also shows that the results obtained by this modification are much better than those obtained by the ordinary incremental approach.

Fig.5.11 shows that typically the equilibrium equations are not satisfied at every load step because an approximate constant stiffness matrix is applied at every load step. Hence an iterative procedure will be employed to minimize the error during every load step.

Using the incremental load iterative procedure, iterations are performed until the incremental equilibrium conditions are satisfied. For every iteration, the residual force (unbalanced force) to be applied as nodal load vector is given by

$$\{\Delta \bar{R}_i\} = \{\bar{F}_{i-1}\} - \{\bar{F}_{Ii-1}\}$$

where

$\{\bar{F}_{i-1}\}$ = the total applied load vector at the end of the $(i-1)th$ iteration

$\{\bar{F}_{Ii-1}\}$ = the total internal force vector at the end of the $(i-1)th$ iteration.

The displacement increment $\{\Delta \bar{d}_i\}$ corresponding to $\{\Delta \bar{R}_i\}$ is then calculated by

$$[\bar{K}_{i-1}]\{\Delta\bar{d}_i\} = \{\Delta\bar{F}_i\}$$

and the total displacement after the i th iteration is given by

$$\{\bar{d}_i\} = \{\bar{d}_0\} + \sum_{j=1}^i \{\Delta\bar{d}_j\} \quad (5.89)$$

where

$\{\bar{d}_0\}$ = the displacement vector before the iterative procedure

$[\bar{K}_{i-1}]$ = up-dated stiffness matrix

5.6.2 Solution Procedure

The total load vector $\{\bar{F}\}$ is subdivided into n load increments $\Delta\bar{f}^r$

$$\{\bar{F}\} = \sum_{r=1}^n \{\Delta\bar{f}^r\}$$

For the first load step, an initial stiffness matrix $[\bar{K}^0]$ is obtained based on the initial moduli, and then the incremental displacements $\{\bar{d}^1\}$ are calculated from

$$[\bar{K}^0]\{\Delta\bar{d}^1\} = \{\Delta\bar{f}^1\}$$

The internal forces $\{\bar{F}_I^1\}$ are obtained from Eq. 5.58, the equilibrium conditions are then checked to see whether

$$\{\bar{F}_I^1\} = \{\bar{F}^1\} \quad (5.90)$$

If Eq. 5.90 is satisfied, then we go to the next load step and in this fashion the analysis continues.

For the r th load step if the equilibrium conditions are not satisfied, then the iterative procedure is used to improve the results. For the i th iteration, the incremental equilibrium equation is

$$[\bar{K}_{i-1}^r]\{\Delta\bar{d}_i^r\} = \{\Delta\bar{R}_i^r\} \quad (5.91)$$

where

r = load step number

i = iteration step number,

The stiffness matrix is evaluated based on the displacements calculated in the previous iteration

$$\{\bar{d}_{i-1}^r\} = \{\bar{d}_{i-2}^r\} + \{\Delta\bar{d}_{i-1}^r\}$$

and the angle α_{i-1}^r is calculated using

$$\alpha_{i-1}^r = \frac{(\bar{z}_2^r)_{i-1} - (\bar{z}_1^r)_{i-1}}{(\bar{x}_2^r)_{i-1} - (\bar{x}_1^r)_{i-1}}$$

The incremental displacements in global coordinates, which are obtained from Eq.5.83, are transformed to incremental displacements in local coordinate system by

$$\{\Delta d_i^r\} = [T]\{\Delta\bar{d}_i^r\}$$

Using the incremental displacements, the incremental strain and stress in any element can be obtained using Eq.5.38

$$\{\Delta\epsilon_i^r\} = [B_p]\{\Delta d\} - z[G]\{\Delta d\} + \frac{1}{2}\{\Delta d\}^T[B_b]^T[B_b]\{\Delta d\}$$

and

$$\{\Delta\sigma_i^r\} = [D]\{\Delta\epsilon_i^r\} \quad (5.92)$$

The total strain and stress in any element at the end of the i th iteration are given by

$$\{\epsilon_i^r\}^e = \{\epsilon_{i-1}^r\} + \{\Delta\epsilon_i^r\}$$

and

$$\{\sigma_i^r\}^e = \{\sigma_{i-1}^r\} + \{\Delta\sigma_i^r\}$$

In a layered element the stress and strain in any layer j at the end of the i th iteration is given by

$$(\Delta\sigma_i^r)_j = (D_i^r)_j(\Delta\epsilon_i^r)_j$$

The value of modulus $(D_i^r)_j$ is given by Eq.5.56, except for the following cases:

(1) For the elasto-plastic transition zone, i.e. when

$$(\sigma_{i-1}^r)_j < \sigma_y \quad \text{and} \quad (\sigma_i^r)_j > \sigma_y$$

as shown in Fig 5.12, the true incremental stress can be obtained from

$$(\Delta\sigma_i^r)_j^* = (\Delta\sigma_i^r)_{je} + (\Delta\sigma_i^r)_{jp}$$

where

$(\Delta\sigma_i^r)_j^*$ = the true incremental stress in the j th layer

$(\Delta\sigma_i^r)_{je}$ = the elastic component of the incremental stress

$(\Delta\sigma_i^r)_{jp}$ = the plastic component of the incremental stress

If we introduce a correction factor β to account for the transition from the elastic range to the plastic one, i.e.

$$\beta = \frac{\sigma_y - (\sigma_{i-1}^r)_j}{(\Delta\sigma_i^r)_j}$$

then $(\Delta\sigma_i^r)_{je}$ and $(\Delta\sigma_i^r)_{jp}$ can be expressed as

$$(\Delta\sigma_i^r)_{je} = \beta(\Delta\sigma_i^r)_j$$

$$(\Delta\sigma_i^r)_{jp} = (1 - \beta)(E_T)_j(\Delta\epsilon_i^r)_j$$

In the plastic to strain-hardening zone where

$$\sigma_y < (\sigma_{i-1}^r)_j < \sigma_{sh} \quad \text{and} \quad (\sigma_i^r)_j > \sigma_{sh}$$

the true incremental stress is obtained from the relationship

$$(\Delta\sigma_i^r)_j = \beta(\Delta\sigma_i^r)_j + (1 - \beta)(E_{sh})_j(\Delta\epsilon_i^r)_j$$

The element internal force vector, which is evaluated from Eq.5.58, 5.62, 5.63, and 5.64, can be transformed from local to global coordinate by means of

$$\{\bar{F}_I\} = [T]^T \{F_I\}$$

The total nodal internal force is given by

$$\{\bar{F}_I\} = \sum \{\bar{F}_I\}$$

In nonlinear structural analysis problems, to check the convergence of the solution, generally three convergence criteria are employed, and they are the displacement, the residual force and the internal energy norms. In the present analysis the displacement convergence criterion is applied. Specifically, it is assumed that convergence has been satisfied when

$$\frac{\sum \Delta \bar{d}_i^r}{\sum \Delta \bar{d}_0^r} \leq 0.005$$

where

$\Delta \bar{d}_0^r$ = the incremental displacement at the beginning of load step r

$\Delta \bar{d}_i^r$ = the incremental displacement after iteration step i at load step r . The 0.5% difference between the displacement increments of two consecutive iterations

is deemed to be adequate for practically all problems of the kind to be solved by this program.

5.6.3 Program Structure

A finite element program called NABF (Nonlinear Analysis of Beam-Column on Foundation) was developed based on the preceding methods. NABF is coded in FORTRAN 77, and comprises a main program and nine subroutines which amount to approximately 1600 statements. The program is quite general and it can be used to perform the geometric and material nonlinear analysis of pipeline structures, beam-columns on Winkler foundation and general in-plane frame structures, with and without elastic foundation. The flowchart of the algorithm used for nonlinear analysis in NABF is shown in Fig.5.13.

5.7 Numerical Examples

To check the correct implementation of the procedure described earlier in this chapter, and to determine the degree of accuracy that can be obtained by the proposed procedure, four numerical examples are solved which involve material and/or geometric nonlinearities. The results are compared with available analytical, numerical and/or experimental results.

5.7.1 The Elastic Problem

Determination of the elastic deflection curve of a tip loaded cantilever beam subjected to large displacements is referred to as the elastic problem in theoretical mechanics. The exact solution of this problem can be found in the form of elliptic integrals (Frisch-Fay 1962). The problem has also been solved previously by means of numerical methods (Yang 1973), with the results being practically the same as exact solution. To check the implementation here, a typical cantilever beam with the dimensions and loadings shown in Fig.5.14 will be solved, and the results will be compared with the theoretical solution presented by Yang(1973).

The problem was modelled by ten finite elements, and the load was applied in 1000 increments. The results of the analysis are shown as free end deflection versus applied load curve in Fig.5.14. The same figure also shows Yang's exact solution, and it can be observed that the two curves are in close agreement. The maximum tip deflection at a load of $P = 1.33\text{MN}$ was $7.9\text{ m}(0.79L)$, where L is the span). Obviously this is a rather large deflection. It should be pointed out that agreement between the present solution and the exact solution can be improved by adopting a more stringent convergence criterion but such an improvement will have little practical significance.

The good agreement in this example demonstrates the correct implementation of the large displacement calculation procedures in the program.

5.7.2 Continuous Beam on Winkler Foundation

This example is intended to verify the correct implementation of the procedures associated with beam on elastic foundation. As stated before, a buried pipeline can be modelled as a beam on elastic foundation.

The continuous beam in Fig.5.15, subjected to the load indicated in the figure, was analyzed by Razaqpur(1989) using a beam-column on elastic foundation finite element with exact shape functions. The beam properties and loadings are shown in Fig.5.15.

Figures 5.16(a) and (b), show the deflected shape and the bending moment variation along the beam. The curve also shows Razaqpur's results for the same problem, with good agreement between the two. Note that in the present analysis each span was discretized into six finite elements while Razaqpur's exact solution used only one element per span. However, Razaqpur's solution is restricted to linear elastic problems while the proposed element is general.

5.7.3 Elasto-Plastic Bending of A Simple Beam

Mitri and Redwood(1986) analyzed a simply supported beam over the complete loading range, including plastic bending, but assuming small deflection theory. the

beam is shown in Fig.5.17, together with its geometric and material properties.

Figure 5.17 shows the normalized load versus midspan deflection of the beam up to failure. Mitri and Redwood's results are also shown, with close agreement between the two sets of results. Notice that the load and deflection are normalized to the yield load and the corresponding deflection. In obtaining the results in Fig. 5.17, the beam was discretized into 8 finite elements, and the load was applied in 1500 steps. It is recognized that practically the same results can be obtained using far fewer load steps. It should also be pointed out that the nonlinear response of this beam is mainly due to in-elasticity rather than large displacements. In fact the maximum displacement is only 55mm (2.75 times the yield deflection) and $0.16h$, where h is the beam height.

The preceding example verifies the correct implementation of the materially nonlinear models in the program.

5.7.4 Gable Frame

Figure 5.18(a) shows the elevation of half of a symmetric and symmetrically loaded gable frame. The frame is constructed of steel I-sections, with their typical cross-section given in Fig.5.18(b). This frame was tested and analyzed by Espion(1985). The reported material properties are: $E_s = 26.7$ psi, strain-hardening modulus $E_{sh} = 414 \times 10^3$ psi, $\sigma_y = 60$ KPa and $\epsilon_p = 0.018$, where ϵ_p is the strain at which strain hardening begins. The frame was subjected to uniform gravity loads on the inclined beams. Note that imperial units are used in this problem in compliance with the original author's data.

The foregoing half frame was modelled by 8 finite elements, three elements in the column and five elements in the beam. The load was applied in 100 equal increments. It is, however, understood that in practice the load increments may be larger during the early stages of loading and they will become gradually smaller as the load is increased.

The load versus the frame hip deflection is plotted in Fig.5.18(c). The figure

also shows the corresponding experimental values and the results of a first order analysis performed by Espion(1985). We observe that inclusion of geometric non-linearity in the present analysis improved the predicted results over those obtained by Epsion, which included material nonlinearity only. For practical purposes, the present analysis gives results that are in very close agreement with experimental data. This close comparison confirms the correct implementation of the combined geometric and material nonlinear procedures in the program.

5.8 Summary

In this chapter a general finite element for the geometrical and material nonlinear analysis of beam-column on elastic foundation was presented. This element is intended for the analysis of surface or buried pipelines.

To describe the soil-pipe interaction, this element takes into account the effects of the soil reactions in both the vertical and the horizontal directions by introducing two soil parameters, the vertical stiffness k_{s2} and the horizontal stiffness k_{s1} . Both k_{s2} and k_{s1} can be treated as functions of spatial coordinate x , and temporal coordinate t . The latter will be discussed in detail in Chapter 6. It should be noted that the horizontal soil reaction which is proportional to the horizontal displacement is significant when the deflection of the pipeline due to frost heave becomes large. It is therefore expected that the results provided by this model would be better than those obtained from models which consider only the vertical soil reaction.

In the proposed analysis an incremental procedure is used for treating geometric and material nonlinearities and the simulation of the modified boundary conditions of the pipe. Generally, incremental procedures involve either applying increments of nodal loads or the increments of prescribed nodal displacements. The latter is used in the current analysis of pipelines in frozen ground. The buried pipeline is resting on the soil, the deformations of the pipe are caused by the incremental soil frost heave, but the position of the pipe changes with the advance of frost heave. Insofar as the pipe is concerned, the amount of soil frost heave can be treated as a

prescribed displacement imposed on the pipe, while the new pipe position can be established using coordinate transformation.

The layered method is widely used way for plastic analysis and is easy to implement. In the analysis the number of layer can be chosen according to the user's judgement, but the maximum number of layer is limited to ten.

The computer program based on the procedures described in this chapter can be applied to analyze numerous engineering structures such as pipeline in frozen ground, bending of beam-columns on foundation, the buckling of beam-columns and/or pipelines, soil-pipeline interaction and general in-plane frames with or without elastic foundation.

Fig. 5.1 (a) Beam-column on Winkler foundation
(b) Axial and vertical foundation reactions

Fig 5.2 Distribution of axial displacement

Fig. 5.3 Additional displacement

Fig 5.4 Beam-column element

Fig. 5.5 Trilinear Material Model

Fig . 5.6 Actual versus idealized pipe cross-section

Fig. 5.7 Distribution of stresses on the Cross-Section

Fig. 5.8 Location Of Gaussian Points

Fig 5.9 Coordinate Transformation

Fig. 5.10 Errors of an ordinary incremental step and a midpoint incremental step

Fig. 5.11 Incremental-iterative method

Fig. 5.12 Incremental stress and strain

Fig. 5.13 Algorithm of incremental stress analysis

Fig. 5.14 Load-large deflection curve of a cantilever beam

Fig. 5.15 Beam-column on Winkler foundation

Fig. 5.16 (a) Deflection of beam-column on Winkler foundation
(b) Bending moment of beam-column on Winkler foundation

Fig. 5.17 Normalized load versus midspan deflection of elasto-plastic simple beam

Fig. 5.18 Results of Example 4

Chapter 6

Thermo-Mechanical Modeling of Soil-Pipeline Interaction

6.1 General

When a chilled gas pipeline crosses from a non-frost susceptible soil to a frost susceptible soil or from a prefrozen soil to a freezing soil, or vice versa, the pipeline is subjected to differential movements caused by differential frost heave at the interface of the two soils, as schematically illustrated in Fig.6.1. The frost heave is resisted by the pipe and the overlying frozen soil. Consequently, deformations and curvature changes occur in the buried pipe and stresses and strains in both the pipe and the soil are developed. The actual magnitude of the deformations and stresses will depend on the complex soil-pipeline interactions involved.

The three fundamental differences between this thermo-mechanical soil-pipeline interaction and the usual mechanical soil-pipeline interaction are: (1) The differential movements of soils are due to the frost heave of freezing soil, which is the result of heat transfer, moisture migration and water phase changes in the freezing soil. (2) The mechanical behaviour of soil varies with the penetration of the frozen zone in the soil because the frozen soil stiffness is much higher than the unfrozen soil stiffness. As indicated by Tsytovich(1975), for the same soil the Young's modulus in the frozen state may be ten or possibly hundreds of times larger than those of unfrozen soil, and the Poisson's ratio may be 2 or 3 times less than that in the unfrozen state. (3) Frozen soil undergoes significant creep or cryogenic deformations compared to unfrozen soils. The creep strains lead to additional movements of the frozen soil, resulting in significant changes of stresses and strains in both the pipeline and the surrounding soil. Creep strains are dependent on the magnitude of the soil stresses which are proportional to the reactive forces exerted by the pipeline

on surrounding soil. The above characteristics signify that freezing soil-mechanical interaction is a complex time-dependent, thermo-mechanical problem.

A comprehensive analysis of the frost susceptible soil-pipeline interaction should consider the following thermo-mechanical processes: (1) Unsteady heat flow in both the frozen and unfrozen soils, moisture migration, and formation of segregation ice lens in the frozen soil. (2) The variation of the mechanical and cryogenic (creep) properties of the freezing soil during frost heave. (3) The material and geometrically nonlinear behaviour of the buried pipeline. (4) Moving boundary conditions associated with the growth of frost heave (development of frost bulb).

Since the complete coupled analysis of such complex time-dependent nonlinear mechanical processes is very difficult, most of the existing soil-pipeline interaction models treat the problem initially as basically two independent processes, the process of the frost heave and the process of the mechanical soil-pipeline interaction, in which the pipe is generally modelled as a flexible elastic structure. But from the preceding discussion it is clear that such models have three noticeable limitations. Firstly, they can not describe accurately the time-dependent mechanical behaviour of the soil because it continuously varies during the frost heave process. Secondly, they are unable to predict with sufficient accuracy the effects of creep in the frozen soil because as stated creep is directly linked to the freezing process. More specifically, the depth of the creeping zone of the soil increases with the penetration of the freezing front and the creep strains are also a function of the reaction of the pipe which, in turn, depends on the amount of frost heave. Thirdly, they generally do not consider the geometrically and materially nonlinear behaviour of the pipe, but in reality the nonlinear effects of the pipe would be significant if the differential frost heave between the two soils were appreciably large, which is often the case after several years of frost heave in the field.

In light of the foregoing considerations, in this chapter a model is presented for the coupled analysis of the thermo-mechanical soil-pipeline interaction based on the simplified frost heave model presented in Chapter 4, and the nonlinear finite

element model of the pipe on Winkler foundation presented in Chapter 5. This interaction model considers the soil to be a one-dimensional visco-elastic medium, comprising three layers, the frozen layer, the freezing fringe and the unfrozen layer, each with its own mechanical and thermal properties. The thickness of each layer is assumed to vary with the penetration of the freezing front.

For shallow buried pipelines, a convenient simplified analysis is used in which the soil below the pipeline is treated as a multi-layer half space and the interface of the pipeline and the soil is assumed to be situated at the free surface of the half space. Consequently, the reaction forces of the pipeline would act as a distributed load on this surface. Using the elastic theory for the multi-layer half space, the time-dependent soil stiffness in the interface region of pipeline and soil, and the stresses in the freezing soil are determined.

In the latter analysis, the moving boundary conditions of the pipeline are simulated by upward distributed loads acting on the portion of the pipe buried in the frost susceptible soil. The magnitude of the load is proportional to the amount of the corresponding frost heave and the soil stiffness. The creep response of the frozen soil is described by Norton's flow law.

6.2 Basic Assumptions

The proposed modeling of soil-pipeline interaction during frost heaving is based on the following assumptions:

1. The soils in the frozen and the unfrozen regions exhibit elastic behaviour and can be characterized as homogeneous isotropic linear visco-elastic materials.
2. The buried pipeline is in full contact with the surrounding soil during frost heaving
3. The shape of the cross-section of the pipe remains circular, i.e. the effects of ovalisation of the pipeline are ignored
4. In the vertical direction, the stiffness of the frozen soil at the interfaces with

its top and bottom of the pipeline is assumed to be the same

5. The porosity of the unfrozen soil is assumed constant, i.e. the volumetric change in unfrozen soil due to change of porosity engendered by the process of soil-pipeline interaction is ignored.

6.3 Frost-Related Soil Stiffness

In the proposed Winkler type soil model, the soil-pipeline interaction during freezing is represented by the localized soil stiffness at the interface of the soil and pipeline and the associated boundary loads. Clearly, a comprehensive analysis of freezing soil-pipeline interaction must be based on measured values of the stiffness of the frozen soil which is a function of both time and space. In this study, the soil stiffness is determined from the soil flexibility which can be obtained from the analysis of a multi-layer half space medium.

Generally, chilled gas pipelines are buried at shallow depths, hence it is sufficient to assumed that the soil below the pipeline consists of a three layer half space, as shown in Fig.6.2. The reaction of the pipeline on the contiguous soil can be treated as a distributed line load acting on the free surface of the half space.

Applying a concentrated vertical load P at point o on the surface of the half space(Fig.6.3), one can use the Boussinesq's solution to obtain the soil stresses at any point with Cartesian coordinates x , y and z

$$\sigma_x = \frac{3P}{2\pi} \left\{ \frac{x^2 z}{R^5} + \frac{1-2\nu}{3} \left[\frac{R^2 - Rz - z^2}{R^3(r+z)} - \frac{x^2(2R+z)}{R^3(R+z)^2} \right] \right\} \quad (6.1)$$

$$\sigma_y = \frac{3P}{2\pi} \left\{ \frac{y^2 z}{R^5} + \frac{1-2\nu}{3} \left[\frac{R^2 - Rz - z^2}{R^3(r+z)} - \frac{y^2(2R+z)}{R^3(R+z)^2} \right] \right\} \quad (6.2)$$

$$\sigma_z = \frac{3P}{2\pi} \frac{z^3}{R^5} \quad (6.3)$$

where

$$R = \sqrt{x^2 + y^2 + z^2}$$

ν = Poisson's ratio of the medium

If we consider the half plane xz , the stresses of the soil in the z direction ($x = 0$), under a vertical concentrated load P acting at point o , can be obtained by setting

$$x = y = 0$$

$$R = z$$

which leads to

$$\sigma_x = \sigma_y = \frac{P(2\nu - 1)}{4\pi z^2} \quad (6.4)$$

$$\sigma_z = \frac{3P}{2\pi z^2} \quad (6.5)$$

For uniform elastic soil, the vertical strain can be obtained using

$$\epsilon_z = \frac{1}{E_s} [\sigma_z - \nu(\sigma_x + \sigma_y)] \quad (6.6)$$

Substituting Eqs. 6.4 and 6.5 into Eq. 6.6, gives

$$\epsilon_z = \frac{P[3 - \nu(2\nu - 1)]}{2E_s \pi z^2} \quad (6.7)$$

Integrating Eq 6.7, the vertical displacement at any point along the z -axis will be given by

$$\begin{aligned} v_z &= \int_z^\infty \epsilon_z dz \\ &= P \int_z^\infty \frac{3 - 2\nu^2 + \nu}{2E_s \pi z^2} dz \\ &= \frac{P}{2E_s \pi z} (3 + \nu - 2\nu^2) \end{aligned} \quad (6.8)$$

For multi-layer soil, the vertical displacement can be obtained by superposition as

$$\begin{aligned}
 v_z &= P \sum_{i=1}^n \int_{z_{i-1}}^{z_i} \frac{3 - 2\nu_i^2 + \nu_i}{2E_{si}\pi z^2} dz \\
 &= \sum_{i=1}^n \frac{P}{2E_{si}\pi} (3 + \nu_i - 2\nu_i^2) \left(\frac{1}{z_{i-1}} - \frac{1}{z_i} \right)
 \end{aligned} \tag{6.9}$$

where

i = layer number

n = total number of layers

E_{si} = Young's modulus of layer i

ν_i = Poisson's ratio of layer i

Eq.6.8 shows that the theoretical displacement directly under the load is infinity. This is not in agreement with the actual observed displacement as shown in Fig.6.4. To overcome the limitation of the theoretical solution, a commonly used approximate method is employed for calculating the vertical displacement in the first layer, i.e.

$$\begin{aligned}
 v_{z1} &= \int_{z_0}^{z_1} \epsilon_z dz \\
 &\approx \int_{z_0}^{z_1} \frac{\sigma_z}{E_{s1}} dz \\
 &\approx \frac{\sigma_{z1} z_1 - \sigma_{z0} z_0}{E_{s1}}
 \end{aligned} \tag{6.10}$$

where with reference to Fig.6.3

v_{z1} = vertical displacement in layer 1

σ_{z1} = stress at the bottom of layer 1

σ_{z0} = stress at the top of layer 1

E_{s1} = Young's modulus of layer 1

z_1 = coordinate of the bottom of layer 1

Setting $z_0 = 0$, Eq.6.10 becomes

$$v_{z1} = \frac{\sigma_{z1} z_1}{E_{s1}}$$

Thus, for three layered soil Eq.6.9 can be written as

$$\begin{aligned} v_z &= v_{z1} + \sum_{i=2}^3 v_{zi} \\ &= \frac{\sigma_{z1} z_1}{E_{s1}} + \sum_{i=2}^3 \frac{P}{2E_{si}\pi} (3 + \nu_i - 2\nu_i^2) \left(\frac{1}{z_{i-1}} - \frac{1}{z_i} \right) \end{aligned} \quad (6.11)$$

The time-dependent flexibility of the soil at a given location on the surface of the half space is defined as the deflection of the soil under unit load acting at the same point and in the same direction as the measured displacement which can be obtained by setting $P = 1$ in Eq.6.11. The latter will yield

$$\lambda_s = \frac{\bar{\sigma}_{z1} z_1}{E_{s1}} + \sum_{i=2}^3 \frac{3 + \nu_i - 2\nu_i^2}{2E_{si}\pi} \left(\frac{1}{z_{i-1}} - \frac{1}{z_i} \right) \quad (6.12)$$

in which

$\bar{\sigma}_{z1}$ = stress at the bottom of layer 1 produced by a unit load, i.e.

$$\bar{\sigma}_{z1} = \frac{3}{2 \cdot \pi \cdot z_1^2} \quad (6.13)$$

For three layered soil, the time-dependent vertical flexibility of the soil at time step m can be expressed as

$$\lambda_s^m = \frac{\bar{\sigma}_{z1} z_1^{m-1}}{E_{s1}} + \sum_{i=2}^3 \frac{3 + \nu_i - 2\nu_i^2}{2E_{si}\pi} \left(\frac{1}{z_{i-1}^{m-1}} - \frac{1}{z_i^{m-1}} \right) \quad (6.14)$$

where

λ_s^m = vertical flexibility of soil at time step m

E_1, E_2, E_3 = Young's modulus of the frozen soil, freezing fringe and unfrozen soil respectively

ν_1, ν_2, ν_3 = Poisson's ratio of the frozen soil, freezing fringe and unfrozen soil,

respectively

and

z_1^{m-1} = location of the segregation temperature T_s by the end of time step $m-1$

z_2^{m-1} = location of the frozen front T_f by the end of time step $m-1$

Knowing the soil flexibility, its corresponding stiffness k_{s2}^m is as usual

$$k_{s2}^m = \frac{1}{\lambda_s^m} \quad (6.15)$$

Eq.6.14 shows that the flexibility of the soil decreases with decreasing temperature and with the penetration of frozen front, and Eq.6.15 gives the time-dependent soil stiffness which increases with the development of the frozen zone.

It should be noted that the singular displacement resulting from the theoretical solution is avoided by using Eq.6.10, but when the frozen zone is very thin, i.e. $z_1 \ll 1$, a spurious large stress $\bar{\sigma}_{z1}$ will be given by Eq.6.13. Then an unrealistically large flexibility will be given by Eq.6.14 which will lead to an unreasonably small soil stiffness. Consequently, during the early stages of the soil freezing, Eq.6.14 is not valid. In the present analysis a criterion for determining the valid range of Eq.6.14 is proposed as follows: If $z_1 > \beta$ times D_p , Eq.6.14 is valid, where β is a non-dimensional factor, and D_p is the diameter of the pipe. Another simple approximate method has been suggested (Tsytoich 1975) to get a reasonable estimate for soil stiffness by introducing an average elastic modulus of multi-layer freezing soil, E_{sv} , as follows:

$$E_{sv} = \frac{\sum_{i=1}^n h_i}{\sum_{i=1}^n \frac{h_i}{E_{si}}} \quad (6.16)$$

where

h_i = thickness of soil layer i

In this fashion the multi-layer soil can be treated as a uniform soil with the average elastic modulus given by Eq.6.16. The stiffness of uniform soil has been also

investigated by Fletcher and Herrmann(1971) and Selvadurai(1979). According to Fletcher and Herrmann(1971), for near surface buried pipelines, the soil stiffness is

$$k_{s2} = 0.65(1 + \nu^2)E_{sv} \quad (6.17)$$

From Eqs.6.16 and 6.17, the approximate time-dependent soil stiffness can be expressed as

$$k_{s2} = 0.65(1 + \nu^2) \frac{L_s}{\sum_{i=1}^3 \frac{z_i^{m-1} - z_{i-1}^{m-1}}{E_{si}}} \quad (6.18)$$

where

L_s = total thickness of the soil under consideration

In this study both methods are used to calculate the soil stiffness. At the beginning of the soil freezing process, Eqs.6.14 and 6.15 are used, but when $z_1 > \beta D_p$ (β is chosen to be 0.8) Eq.6.18 is used .

6.4 Creep Displacement

In this analysis, the creep behaviour of the frozen soil is represented by Norton's law

$$\frac{d\epsilon_c}{dt} = B\sigma_s^n \quad (6.19)$$

where

ϵ_c = creep strain of the frozen soil

$\frac{d\epsilon_c}{dt}$ = the creep rate

B, n = experimental creep parameters

the values of B and n are dependent on the type of frozen soil. The stress σ_s at any point in the frozen zone consists of the initial stress due to soil weight and any

additional stress due to the reaction of pipe, i.e.

$$\sigma_s = \sigma_{sg} + \sigma_{sr} \quad (6.20)$$

where

σ_{sg} = soil stress due to overlying soil weight and/or overburden

σ_{sr} = soil stress due to the reaction of the pipe acting on the soil at the interface of soil and the pipe

The initial soil stress, σ_{sg} , can be calculated using

$$\sigma_{sg} = \rho \cdot g \cdot z \quad (6.21)$$

while σ_{sr} , can be calculated using

$$\sigma_{sr} = \frac{3R_p}{2\pi z^2} \quad (6.22)$$

where

ρ = density of the soil

R_p = reaction of pipeline acting on the soil at the interface of the soil and the pipeline

For the time-dependent problem, the increment of the vertical creep displacement of frozen soil, dv_m , during time step m can be written as

$$dv_{cm} = \int_{h_0}^{h_1^{m-1}} d\epsilon_{cm} dz \quad (6.23)$$

The increment of creep strain during time step m can be written as

$$d\epsilon_{cm} = \frac{d\epsilon_{cm}}{dt} \Delta t_m \quad (6.24)$$

Substituting Eq.6.19 into Eq.6.24, the rate of creep strain during time step m is

$$\frac{d\epsilon_{cm}}{dt} = B(\sigma_s^{m-1})^n \quad (6.25)$$

where

σ_s^{m-1} = total stress of frozen soil at the end of time step $(m - 1)$

From Eq.6.22 the stress due to the reaction of the pipeline at the end of time step $(m - 1)$ can be written as

$$\sigma_{sr}^{m-1} = \frac{3R_p^{m-1}}{2\pi z^2} \quad (6.26)$$

Thus from Eqs.6.20, 6.21 and 6.25 the total soil stress at the end of time step $(m - 1)$ is

$$\begin{aligned} \sigma_s^{m-1} &= \rho_s g z + \sigma_{sr}^{m-1} \\ &= \rho_s g z + \frac{3R_p^{m-1}}{2\pi z^2} \end{aligned} \quad (6.27)$$

where

R_p^{m-1} = reaction of the pipeline acting on the soil surface at the end of time step $(m - 1)$

Substituting Eq.6.27 into Eq.6.25, leads to

$$\begin{aligned} \frac{d\epsilon_{cm}}{dt} &= B\left(\rho_s g z + \frac{3R_p^{m-1}}{2\pi z^2}\right)^n \\ &= B\left[(\rho_s g z)^n + n(\rho_s g z)^{n-1} \frac{3R_p^{m-1}}{2\pi z^2} + \cdots + \left(\frac{3R_p^{m-1}}{2\pi z^2}\right)^n\right] \end{aligned} \quad (6.28)$$

Assuming $n = 3$, Eq.6.28 becomes

$$d\epsilon_{cm} = B\left(\rho_s^3 g^3 z^3 + \frac{9\rho_s^2 g^2}{2\pi} R_p^{m-1} + \frac{27}{4} \frac{\rho_s g (R_p^{m-1})^2}{\pi^2 z^3} + \frac{27}{8} \frac{(R_p^{m-1})^3}{\pi^3 z^6}\right) \Delta t \quad (6.29)$$

Combining Eqs.6.23, 6.24 and 6.29, gives

$$\begin{aligned}
dv_{cm} &= \int_{z_0}^{z_1^{m-1}} B(\rho_s^3 g^3 z^3 + \frac{9\rho_s^2 g^2}{2\pi} R_p^{m-1} + \frac{27\rho_s g(R_p^{m-1})^2}{4\pi^2 z^3} + \frac{27(R_p^{m-1})^3}{8\pi^3 z^6}) \Delta t dz \\
&= B\{\frac{\rho_s^3 g^3}{4}(\bar{z}_1^4 - z_0^4) + \frac{9\rho_s^2 g^2}{2\pi} R_p^{m-1}(\bar{z}_1 - z_0) \\
&\quad + \frac{27\rho_s g(R_p^{m-1})^2}{8\pi^2}(\frac{1}{z_0^2} - \frac{1}{\bar{z}_1^2}) + \frac{27(R_p^{m-1})^3}{40\pi^3}(\frac{1}{z_0^5} - \frac{1}{\bar{z}_1^5})\} \Delta t
\end{aligned} \tag{6.30}$$

where

$$\bar{z}_1 = z_1^{m-1}$$

The total creep displacement of the frozen soil at the end of time step m can be determined by

$$v_c^m = \sum_{j=1}^m dv_{cj} \tag{6.31}$$

Substituting Eq.6.30 into Eq.6.31, we have

$$\begin{aligned}
v_c^m &= \sum_{j=1}^m B\{\frac{\rho_s^3 g^3}{4}(\bar{z}_1^4 - z_0^4) + \frac{9\rho_s^2 g^2}{2\pi} R_p^{m-1}(\bar{z}_1 - z_0) \\
&\quad + \frac{27\rho_s g(R_p^{m-1})^2}{8\pi^2}(\frac{1}{z_0^2} - \frac{1}{\bar{z}_1^2}) + \frac{27(R_p^{m-1})^3}{40\pi^3}(\frac{1}{z_0^5} - \frac{1}{\bar{z}_1^5})\} \Delta t
\end{aligned} \tag{6.32}$$

Thus the total upward displacement of the frozen soil at the end of time step m is given by

$$h_s^m = \Delta h_f^m + v_c^m \tag{6.33}$$

where Δh_f^m = the total amount of frost heave given by Eq.4.7 at the end of time step m .

In Fig.6.5 aa is the initial location of the pipe while $a'a'$ is the location of the pipe after the occurrence Δ^{m-1} . Let us consider the deformation of the soil column AB

which is in contact with the pipe at its top section. After time Δ_{m-1} the column AB will have an upward movement Δh_s^{m-1} due to frost heave and creep. If column AB can freely move, B will move to B'' , but the upward movement is resisted by the pipeline reaction, R_p^{m-1} , thus the final location of B after time Δ^{m-1} will be B' , and the soil shortening caused by R_p^{m-1} will be $B'B''$. According to Winkler's model, the reaction of the pipeline acting on the soil is proportional to the soil deformation in the direction of the reaction. Thus the reaction of pipeline acting on the soil is

$$\begin{aligned} R_p^{m-1} &= k_{s2}^{m-1} \Delta h_s^{m-1} \\ &= k_{s2}^{m-1} (h_s^{m-1} - v_p^{m-1}) \end{aligned} \quad (6.34)$$

where

h_s^{m-1} = total free upward displacement of soil by the end of time step $m - 1$
(given by Eq.6.33)

v_p^{m-1} = net upward displacement of pipe by the end of time step $m - 1$

6.5 Behaviour of Pipe on Freezing Foundation

The nonlinear behaviour of the pipeline on Winkler's foundation was discussed in Chapter 5. The pipe element stiffness matrix was given by Eq.5.49 as

$$[K] = [K_m] + [K_G] + [K_f]$$

For this time-dependent problem, the foundation matrix $[K_f]$ varies in response to the variations of the mechanical properties of the frozen soil. In the step by step calculations adopted in the present thesis, the value of k_{s2} is replaced by k_{s2}^m , which is obtained from Eq.6.15 or Eq.6.18 as appropriate. Thus Eq.5.53 becomes

$$[K_f] = \frac{l}{420} \begin{bmatrix} 140k_{s1} & 0 & 156k_{s2}^m & 4l^2k_{s2}^m & \text{Symmetric} & 0 & 0 & 0 \\ 0 & 70k_{s1} & 22lk_{s2}^m & 0 & 0 & 140k_{s1} & 0 & 0 \\ 0 & 0 & 54k_{s2}^m & 13lk_{s2}^m & 0 & 0 & 156k_{s2}^m & -22lk_{s2}^m \\ 0 & 0 & -13lk_{s2}^m & -3l^2k_{s2}^m & 0 & 0 & -22lk_{s2}^m & 4l^2k_{s2}^m \end{bmatrix} \quad (6.35)$$

The longitudinal soil stiffness k_{s1} can be assumed to be a constant because the change in the mechanical properties of the soil in this direction is not as critical. In practice loss of contact between the pipe and the soil in the longitudinal direction will cause slip.

6.6 Moving Load Boundary Condition

The stresses and deformations of the pipeline are caused by the differential upward movements between two types of soils, in which the movements are produced by differential frost heave and creep displacements. When the movements are resisted by the pipeline and the soil above it, the pipeline will be subjected to reactive pressure caused by these movements. It is clear that the load acting on the pipeline will depend on the amount of differential upward movements between two soils caused by frost heave and creep. In this study, the moving load boundary condition of the pipeline associated with the growth of frost heave is simulated by a distributed load acting on the pipe which is a moving boundary condition because the distributed load varies with the time-dependent frost heave and creep.

The problem of a finite beam resting on an elastic foundation has been investigated by many researchers. Gorbunov-Posadov and Serebjanyi (1961) provided results of 3-D analysis of a finite beam on an elastic half space medium subjected to a concentrated load (Fig.6.6). In their study the relative flexibility, R_p , of the soil-beam system is

$$R = \frac{\pi E_s L^3 b}{32 E_p I (1 - \nu^2)} \quad (6.36)$$

where b , l , I and E_s are respectively the beam width, length, moment of inertia and Young's modulus. As shown in Fig.6.6

$$q(x) = \bar{q}P_0$$

$$\phi = \frac{l}{b}$$

where $q(x)$ is the contact stress and ϕ is the aspect ratio.

From Fig.6.6, it can be seen that the variation of contact stress profile with relative flexibility is significant. For a rigid beam ($R = 0$), the coefficient $\bar{q} = 0.81$ at the end of the beam, for a flexible beam ($R > 5$), $\bar{q} < 0.1$ at the end of the beam. For a usual beam ($1 \leq R \leq 2$), $\bar{q}$ decreases with the distance from the applied load, but it increases near the end of the beam (Gorbunov-Posadov and Serebjanyi 1961).

In the frost heave-induced soil-pipeline interaction, the load acting on the pipe will be distributed along the length of that portion of the pipeline which is buried in frost susceptible soil because the differential frost heave occurs within a wide zone adjacent to the two soils interface. It is accepted that the density of the pressure at the interface between the two soils will be large because of the discontinuous upward movement of the soils, but it will decrease rapidly with the distance from the interface. Accordingly, the pressure distribution may be described by a function of the form $a_1 \frac{x}{l_1}$ ($a_1 > 1$).

This large distributed load within a relatively small region may be treated as a concentrated load applied on the pipe at the interface. Although the Gorbunov-Posadov and Serebjanyi's solutions are not completely suitable for the soil-pipeline system because the soil above the pipeline which has a finite thickness is not truly a half space, nevertheless there are similarities between the two systems, i.e. an increase of contact stress at the end of the pipe will happen because of its small relative flexibility, which will lead to a downward displacement of that point. The pressure distribution at the soil-pipe interface is quite complex, but the proposed function is a reasonable approximation for practical purposes. However, the pres-

sure actually does not monotonically decrease with the distance from the two soils interface, but it tends to increase near the free end of the pipe. Consequently, the original function may be corrected by adding to it the function a_2^{-x} . Fig.6.7 shows a reasonable approximation for the pressure distribution load function which can be obtained by combining the above two functions as

$$P(x) = \alpha_1 \bar{P}_0 (a_1^{-x} + \alpha_2 a_2^{-x}) \quad (6.37)$$

$$(0 \leq x \leq L_1)$$

where

$P(x)$ = net pressure function

L = total length of the pipe

L_1 = length of the portion of the pipe buried in the frost susceptible soil

$\alpha_1 \bar{P}_0$ = pressure acting on the pipe at the interface between the two soils

$$\bar{P}_0 = k_{s2} \Delta h$$

and

$$\Delta h = \Delta h_f - \Delta h_u$$

in which

Δh = differential upward movement between the two soils

Δh_f = upward movement of frost susceptible soil

Δh_u = upward movement of non-frost susceptible soil

α_1 = a non-dimensional factor which describes the intensity of applied load at the interface between the two soils

a_1 = a non-dimensional factor which describes the decrease of the distributed load with the distance to the interface

α_2 = a non-dimensional factor which describes the intensity of the applied

pressure at the free end of the pipe

α_2 = non-dimensional factor which describes the increase of the distributed load near the end

For the problem of pipeline surrounded by two different soils, all the above factors are dependent not only on the relative stiffness of soil-pipe system, but also on the relative rigidity of the two soils. If the portion of the pipeline buried in the frost susceptible soil is fixed, the load intensity will become equal to P_0 , but generally the values of those factors are:

$$0 \leq \alpha_1 \leq 1$$

$$0.05 \leq \alpha_2 \leq 0.8$$

$$\alpha_1 = \kappa L_1$$

$$\alpha_2 = \kappa L$$

where

$$0.8 \leq \kappa \leq 1.2$$

The above ranges for those factors were determined by trial and error and they will need further investigation for general applications. In the time dependent process of soil-pipeline interaction, the moving pressure boundary condition, Eq.6.36, can be expressed by

$$P(x)^m = \alpha_1 \bar{P}_0^m (a_1^{\frac{x}{L_1}-1} + \alpha_2 a_2^{-x}) \quad (6.38)$$

where $P(x)^m$ = load function at time step m

and

$$\bar{P}_0^m = k_{s2}^m \Delta h_m \quad (6.39)$$

where Δh_m = increment of differential displacement during time step m

3 of/de 3

PM-1 3½"x4" PHOTOGRAPHIC MICROCOPY TARGET
NBS 1010a ANSI/ISO #2 EQUIVALENT

PRECISIONSM RESOLUTION TARGETS

6.7 Calculating Procedure

In this thesis time-dependent analysis of the coupled frost heave-mechanical interaction problem is carried out in an incremental or step-by-step fashion.

The flow chart in Fig.6.8 gives the basic computational procedure used in the present analysis. Of course, the actual procedure involves additional sub-steps for achieving convergence and establishing equilibrium at each step. Briefly, at time step m , using the time-dependent finite element analysis of unsteady heat flow described in Chapter 4, the distribution of the temperature in the frozen soil is obtained. Based on the temperature profile in the freezing soil, the thickness of the frozen fringe and the extent of the frozen zone are determined. Consequently, the increment of frost heave during time step m is calculated according to the simplified frost heave model presented in Chapter 4, and the soil stiffness is modified according to Eq.6.15 or Eq. 6.18, using the previously determined changes in the frozen zone. Knowing the current soil stiffness, the foundation matrix, $[K_f]$, in the pipe model is modified using Eq.6.34, and the increment of additional creep displacements are obtained using the results at the end of the previous step in Eq.6.30. The creep displacements are added to the non-time-dependent displacement and the total increment of upward movements of the soils is obtained.

Next the boundary conditions for the step are determined using Eq.6.38. Once the new boundary conditions are known, the system of nonlinear equations pertaining to the mechanical response of the pipeline are solved, and the deformations and stresses of the pipe are determined. At the beginning of the next time step, i.e. $(m + 1)$, the thermal properties of the frozen soil are modified according to the temperature profile in the frozen soil and the penetration of the frozen zone by the end of time step m , and the above procedure is repeated. The calculating flowchart is shown in Fig.6.8.

6.8 Numerical Example

The proposed model of soil-pipeline interaction is used to simulate the famous Caen experiment, which is a large-scale experiment of freezing around a chilled gas pipeline in Caen, France, under a joint Canada-France program. On the Canadian side, the Geotechnical Science Laboratories of Carleton University are the principal participants in this endeavour.

The experimental facility consists of an 18 m long, 8 m wide and 5 m high refrigerated hall (Fig.6.9). The base or trough of the hall is 1.7 m deep, and is thermally and hydraulically insulated. One half of the pipeline is buried in a frost susceptible silt and the other half is buried in a non-frost susceptible sand. The pipeline has a diameter 0.273 m, a wall thickness of 5 mm, a Young's modulus 210 MPa and a yield stress of 230 MPa.

In the experiment the temperature of the hall was lowered to -0.75°C , and the average pipe temperature was -2°C . Initially the water table was maintained at 300 mm depth, but due to excessive heave of the pipe and only limited frost penetration, after three months the water table was lowered to 900 mm depth. Complete details of this experiment are given by Ref.[6].

In the proposed numerical simulation the pipe was discretized by 16 finite elements along its length, with each element being divided into 6 layers along the pipe depth (Fig.6.10). The soil was divided into 40 finite elements (Fig.6.11), and a time step of one minute was used in the time dependent calculations. The soil properties that were used are shown in Table 6.1 and the factors of load boundary condition which were used in the simulation are shown in Table 6.2. In the simulation, the effects of frost heave of the soil located above the pipeline were not taken into account because those upward movements do not affect the movements of the pipe.

Figs. 6.12 to 6.14 show the temperature profiles in Caen silt given by the proposed simulation and the corresponding experimental values which are in rea-

sonable agreement with each other. The calculated temperature profiles in sand are shown in Fig. 6.15. The evolution of the freezing front within both silt and sand are shown in Fig.6.16, where it can be noted that the calculated values match the experimental results reasonably well.

Figs. 6.17 and 6.18 show the time-dependent variation of pipeline vertical displacements. After 70 days of soil freezing, the pipeline displacements calculated by the proposed model were 12% larger than the corresponding experimental data, while after 98 days the calculated displacements were approximately 3.5% smaller than the experimental results. It is believed that the differences are mainly due to the selected values of the overall permeability of the soil. In the numerical simulation we could not compare the calculated amount of frost heave with the corresponding experimental values because of lack of measured data about the amount of free frost heave in the soil. The maximum displacement of the pipe after 252 days from the beginning of the experiment was calculated to be 103 mm, which is in good agreement with the experimental value of 107 mm.

From the pipeline designers point of view the accurate evaluation of stresses in the pipe during frost heave is of great importance. Figs.6.19 and 6.20 show the variation of the time-dependent distribution of longitudinal bending stresses in the pipe, which are again in reasonable agreement with the corresponding experimental results. It can be seen that the maximum stresses in the pipe occurred in the part of the pipe buried in sand and a distance of 1.1m from the interface of the two soils. After 259 days of soil freezing, the calculated maximum stress was 116 MPa, compared to the experimental value of 105 MPa. The variation of bending moment developed during frost heave is shown in Fig.6.21, the shape of bending moment profiles are in agreement with the results provided by Ladanyi(1985), Rajani(1992) and Selvadurai(1993).

The creep effects during soil-pipeline interaction are shown in Figs. 6.22 and 6.23. The figures show the calculated pipe vertical displacements and longitudinal normal stresses after 252 days of frost heave, with and without consideration of creep

effects in the soil. The figures also show the corresponding experimental data. It is evident that ignoring creep effects in the calculations can lead to significant error (36%) in the values of the predicted stresses and deformations. Fig.6.24 shows the time-dependent variation of soil stiffness for both silt and sand, and it may be noted that the variation of soil stiffness with the progress of frost heave can be significant. With the penetration of the frozen zone, the silt stiffness increased from 0.48 MPa/m to 0.60 MPa/m, while the stiffness of sand increases from 1.1 MPa/m to 2.2 MPa/m during the same time.

Figs.6.25 and 6.26 show the effect of geometric nonlinearity and large deformations on the pipe behaviour. It is interesting that the pipe displacements given by the linear and the nonlinear theory are different by a maximum of 6.3%, but the stresses differ by a maximum of 42%. This phenomenon is due to the fact that using linear theory causes large reactive forces acting on the pipe which in turn causes large creep strain in the soil. These creep strains tend to increase the amount of upward soil displacements and the consequent stresses in the pipe.

6.9 Summary

In this chapter the thermo-mechanical interaction of a chilled gas pipeline was modelled. The model accounted for heat and moisture transfer during soil freezing and the interaction between the soil and the pipe. The soil model included the time-dependent variation of soil stiffness and the effect of creep strains. Due to the dependency of pipe deformations on soil pressure and stiffness, the effect of the foregoing phenomena on pipeline stresses and displacements were considered. For the pipeline, the influence of large deformations on its stiffness and stresses was also included.

The proposed model was validated by simulating the large scale experiment conducted on a chilled air pipeline in Caen, France. The simulation yielded soil thermal profile, and pipe displacements and stresses. The analytical results were found to be in reasonable agreement with the corresponding experimental data for

various time intervals after the start-up of the experiment. It is therefore concluded that the assumptions made in the development of the proposed model are reasonable and acceptable for practical purposes.

Fig.6.1 Pipeline subjected to differential frost heave

Fig.6.2 Model of multi-layer half space

Fig.6.3 Half space with three layers under concentrated load P

Fig.6.4 Vertical displacement on the surface of half space under concentrated load P

Fig.6.5 Soil deformation due to reactive pressure of the pipe

Fig. 6.6 Contact stresses for finite beam on elastic foundation subjected to a central load (after Gorbunov-Posadov and Serebrjanyi, 1961)

Fig.6.7 Moving load boundary condition

HTTR----subroutine for analysis of unsteady heat transfer in soil

NABC----subroutine for nonlinear analysis of pipeline on Winkler foundation

Fig. 6.8 Flowchart of algorithm of thermal-mechanical soil-pipeline interaction

(a)

(b)

Fig 6.9 schematic diagram of Caen-experiment
 (a) longitudinal section of trough
 (b) cross section of trough
 (after Dallimore et al. 1984)

Fig.6.10 Finite element mesh of pipe
 (a) Longitudinal element
 (b) layered element in cross-section

Fig. 6.11 Finite element mesh of soil

Fig. 6.12 Temperature profile beneath centreline of pipe in Caen silt during freezing

Fig.6.13 Temperature profile beneath centreline of pipe in Caen silt during freezing

Fig 6.14 Temperature profile beneath centreline of pipe in Caen silt during freezing

Fig.6.15 Temperature profile beneath centreline of pipe in sand during freezing

Fig.6.16 Evolution of frost front

Vertical displacement(m)

Fig.6.17 Vertical movement of pipe during soil dreezing

Fig.6.18 Time dependent deformation of pipe

Fig. 6.19 Distribution of bending longitudinal stresses in the top fibre of the pipe

Fig.6.20 Distribution of longitudinal bending stresses in the top fibre of the pipe

Fig.6.21 Distribution of bending moment along the pipeline

Fig.6.22 Deformation of pipe (252 days after start-up)

Pipe stress(MPa)

Fig.6.23 Distribution of longitudinal bending stresses in the pipe (259 days after start-up)

Fig.6.24 Time-dependent variation of soil stiffness during frost heave

Fig 6.25 Deformation of pipe given by linear and nonlinear pipe models
(900 days after start-up, $B=9.5 \times 10^{-25} \text{ (Pa}^{-3} \text{sec}^{-1})$)

Fig 6.26 Distribution of bending stress in pipe given by linear and nonlinear pipe models (900 days after start-up, $B=9.5 \times 10^{-25} \text{ (Pa}^{-3} \text{sec}^{-1})$)

Table 6.1 Thermal, hydraulic and mechanical properties of Caen silt and sand

Property	Caen Silt *		Sand # *	
	Unfrozen	Frozen	Unfrozen	frozen
k_z (J/m C)	1.97	3.30	2.30	4.30
$cx10^6$ (J/m C)	3.15	2.35	3.6	2.95
E_s (Mpa)	0.60	1.60	1.65	3.20
$\overline{K}_f \times 10^{-10}$ (m/s)	5.5		2.1	
ρ_s (kg/m)	1850		1850	
n (porosity)	0.31		0.15	
μ	0.3		0.3	

* Williams (1993)

Approximate from Ladanyi et al. (1984)

Table 6.2 Factors of load boundary condition

α_1	α_2	a_1	a_2
0.7	0.1	10	15

Chapter 7

Conclusions and Recommendations

7.1 Conclusions

The soil freezing process can not be explained solely by the change of mechanical energy because during freezing heat transfer (including phase change), mass transport and their interaction occur due to the change of the temperature-related total energy or free energy of a soil system. The Clausius-Clapeyron equation provides the relation among water pressure and ice pressure, and temperature in a freezing soil, and it can be used to explain the development of large suction within the frozen fringe. In author's opinion, any advanced frost heave model which can reasonably describe the frost heave process must include the phenomena implicit in the Clausius-Clapeyron equation.

The proposed general form of the Clausius-Clapeyron equation overcomes the shortage of the commonly used Clausius-Clapeyron equation derived by Kay and Groenevelt (1974), which is only valid when the freezing front is located at the water table level, because the proposed Clausius-Clapeyron equation includes the effect of the location of the water table in relative to the position of the freezing front. According to the proposed Clausius-Clapeyron equation, the water suction developed during soil freezing depends not only on the overburden pressure and temperature gradient, but also on the value of suction at the freezing front. In the other words, the freezing process is directly affected by the location of water table. It is shown that available shapes of water suction profiles can be obtained from the proposed Clausius-Clapeyron equation which are in agreement with the shapes of water suction given by previous researchers. The proposed Clausius-Clapeyron equation also provides a fundamental tool for explaining the freezing process in unsaturated soils.

Development of a simple and practical frost heave model has been the goal of researchers for a long time. But the complexity of the frost heave process makes it difficult to develop a simple model. By using macroscopic analysis and by avoiding measuring local temperature, unfrozen water content, ice content and soil properties, the proposed model is one of the most simple frost heave models.

The proposed Darcy type model shows that the driving force which migrates water flow through the frozen fringe is the hydraulic gradient, and the water flow is mainly along the direction of the maximum hydraulic gradient, i.e. the direction perpendicular to the water table level. Thus, generally speaking, 1-D frost heave model is practical and available for engineering problems. It is not necessary to develop 2-D and 3-D frost heave models except in some special cases.

The proposed model considers the dominant phenomena during soil freezing including the water migration, the latent heat release, the ice lens formation and the effect of overburden pressure, and it only requires that the water pressure at the base of the ice lens in the frozen fringe be calculated. Hence, the proposed model satisfies the basic two requirements of an engineering oriented frost heave model, i.e. the model is easy to understand and apply, and it provides a simple method for calculating the amount of frost heave.

It should be noted that the proposed frost heave model is valid for saturated soils, regardless of whether or not the water table is located at the freezing front. In other words, the model is valid when the water flows continuously from the water table through the unfrozen soil into the frozen fringe.

The principal advantage of the proposed modeling of soil-pipeline interaction over existing soil-pipeline interaction models is that it is based on a complete coupled analysis of the time-dependent nonlinear thermo-mechanical process associated with chilled gas pipelines. In this model, the mechanical process in both the pipe and the soil, the thermal process of frost heave, and the frozen soil creep are related by the introduction of a moving load boundary conditions for the pipe. The boundary condition is introduced in such a way that the distributed upward loads

caused by frost heave, and acting on the pipe, are proportional to the amount of the frost heave and the frozen soil creep displacements.

In the process of soil-pipeline interaction during soil freezing, the effect of soil creep in a frozen zone is significant. Evaluation of the effect of the frozen soil creep must consider the coupling of frost heave and the mechanical response of the pipe because the extent of the frozen zone in soil varies with the variation of temperature during soil freezing, and the creep strains depend on the magnitude of the soil stresses which are dependent on the reactive force exerted by the pipe on its surrounding soil.

For long term soil freezing, the nonlinear effects on the pipe are significant. This conclusion is based not only on common knowledge about the nonlinear behaviour of the pipe subjected to a large differential frost heave, but also on the frost induced soil-pipeline interaction. Compared to linear analysis, the nonlinear analysis provides smaller stresses and larger deformations in the pipe under the same frost heave conditions. According to the proposed model, larger deformations of the pipe caused by a given amount of frost heave induce smaller soil deformations due to the higher reactive pressure of the pipe. Note that the net soil deformations are equal to the difference between the amount of free frost heave and the lateral displacement of the pipe. During soil freezing, the actual stresses in the pipe result from the superposition of stresses from classical nonlinear theory and the stresses from the soil-pipeline interaction. It would appear that the nonlinear effects of the pipe in soil-pipeline interaction during soil freezing are more significant than in conventional nonlinear analysis of structures.

Soil stiffness in the interface region of the pipeline and soil increases with the penetration of the frozen zone. For long term soil freezing the increase of the soil stiffness should be considered because the mechanical response of the pipe is directly related to the soil stiffness.

Finally, it should be mentioned that the proposed complete analysis of the pipe buried in freezing soil considers longitudinal bending of the pipeline. The latter

is different from those in the most existing models which treat the soil-pipeline interaction in freezing soil as a plane strain problem . From a practical view point the proposed analysis is believed to be better because the dominant mode of the pipe deformation is flexural and not plane strain as assumed by many previous models.

7.2 Recommendations for Future Study

1. The proposed Clausius-Clapeyron equation includes the effect of the location of the water table in the soil freezing, but the proposed frost heave model is only valid for saturated soils. Although frost heave models for saturated soils are most widely used in engineering, a frost heave model which can be used to explain the frost heave process in unsaturated soils is also important. Specially, a frost heave model which considers the effect of the location of water table is needed to study the case of discontinuous water flow from the water table across the unfrozen soil into the frozen fringe.

2. The proposed frost heave model shows that the magnitude of frost heave is proportional to the overall permeability in the frozen fringe. In this thesis the frozen fringe overall permeability is assumed to be an exponential function which yields an approximate value of the overall permeability. In reality, measuring the overall permeability can be extremely complex because it is affected by many factors, such as the unfrozen water content, the overburden pressure, the freezing temperature, the soil type, etc. Experimental and theoretical studies of determining the overall permeability are needed.

3. In the nonlinear finite element model of the pipe, the effects of large displacements are included in the geometric stiffness matrix of the pipe, but the effects on the foundation stiffness is not included. For most practical problems the above treatment is valid because the geometric matrix can account for the main effects of large displacements, and the use of the incremental-iterative method can limit the calculating errors. In some cases, however the geometrically nonlinearity in the

foundation response may need to be considered.

4. In the proposed model of soil-pipeline interaction, the creep strains of the frozen soil are considered and soil is treated as an elastic medium. Generally, this model will be adequate for practical applications because the plastic strains in the soil are not comparable to the creep strains, but there may be cases in which the soil plastic strains need to be considered. Thus introduction of an elasto-plastic soil model will enhance the modeling aspect of the soil-pipeline interaction problem.

5. The moving boundary conditions of the pipe are simulated by an assumed load function including factors α_1 , α_2 , a_1 and a_2 . In the present investigation, the given ranges of these factors were determined by trial and error. In fact, the determination of the values of these factors is complex because they are dependent on the size of the pipe, the mechanical properties of the pipe and the soils, the relative stiffness of soil-pipeline system and the relative rigidity of the two soils. More accurate determination of these values for various cases will be of practical significance.

6. The proposed analysis of the soil-pipeline interaction is based on the assumption that the soil surrounding the buried pipeline is in full contact with the pipeline during soil freezing. In practice, the loss of contact between the pipe and the soil will cause a change in the mechanical response of the pipe as well as a change in amount of creep in the soil. This aspect of the present model can be improved.

7. Investigation of the effect of ovalisation of the pipeline was not dealt with in the present research. However, in some cases involving a long term soil freezing and large deformations, the shape of the pipe's cross-section may not remain circular, thus the effect of ovalisation may need to be explored.

8. Further detailed parametric studies should be conducted to determine the response of a pipe buried for 10 to 20 years in a freezing soil. Such analysis may reveal other important parameters which effect pipeline stresses and deformations and the interaction of the pipeline with the surrounding soil.

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effects in the soil. The figures also show the corresponding experimental data. It is evident that ignoring creep effects in the calculations can lead to significant error (36%) in the values of the predicted stresses and deformations. Fig.6.24 shows the time-dependent variation of soil stiffness for both silt and sand, and it may be noted that the variation of soil stiffness with the progress of frost heave can be significant. With the penetration of the frozen zone, the silt stiffness increased from 0.48 MPa/m to 0.60 MPa/m, while the stiffness of sand increases from 1.1 MPa/m to 2.2 MPa/m during the same time.

Figs.6.25 and 6.26 show the effect of geometric nonlinearity and large deformations on the pipe behaviour. It is interesting that the pipe displacements given by the linear and the nonlinear theory are different by a maximum of 6.3%, but the stresses differ by a maximum of 42%. This phenomenon is due to the fact that using linear theory causes large reactive forces acting on the pipe which in turn causes large creep strain in the soil. These creep strains tend to increase the amount of upward soil displacements and the consequent stresses in the pipe.

6.9 Summary

In this chapter the thermo-mechanical interaction of a chilled gas pipeline was modelled. The model accounted for heat and moisture transfer during soil freezing and the interaction between the soil and the pipe. The soil model included the time-dependent variation of soil stiffness and the effect of creep strains. Due to the dependency of pipe deformations on soil pressure and stiffness, the effect of the foregoing phenomena on pipeline stresses and displacements were considered. For the pipeline, the influence of large deformations on its stiffness and stresses was also included.

The proposed model was validated by simulating the large scale experiment conducted on a chilled air pipeline in Caen, France. The simulation yielded soil thermal profile, and pipe displacements and stresses. The analytical results were found to be in reasonable agreement with the corresponding experimental data for