Dynamic Stability Analysis of a Helicopter Blade with Adaptive Damper

by

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Honors Diploma (Mechanics)
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Thesis Co-Supervisor

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Abstract

This study investigates the aeroelastic stability of the elastic lead-lag and flap bending, torsion of a uniform, untwisted hingeless “smart” rotor blade in hover flight.

As a first attempt, no offsets are considered between the elastic, aerodynamic and tension axes and the pre-cone angle is taken to be zero. In order to perform a realistic stability analysis, configuration parameters of a reduced scale model of an actual blade of a Eurocopter BO105 helicopter are used in the present work.

The nonlinear partial differential equations of motion for such a blade are derived. First, by applying Galerkin’s method, the equilibrium deflections are defined. This is followed by the development of the perturbation equations, which describe the unsteady blade motion near the equilibrium operating condition. The stability is then determined using standard techniques.

The equations of motion are also modified to incorporate sinusoidal “smart” spring actuation. Actuating the spring, one can change the cross-sectional parameters of the blade, thus affecting the dynamic behaviour of the rotor. The effects of these changes are investigated in this study. The time-dependent linear differential perturbation equations of motion are solved for the eigenvalues using the Floquet method. The obtained results are compared with those found using a constantly actuated or non-actuated “smart” spring.

Results are obtained for a variety of blade configurations in order to provide “smart” rotor blade stability characteristics. It is shown that the “smart” spring technology could be used to eliminate instabilities, such as divergence and flutter, by sinusoidally actuating the spring.

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Nomenclature

A	Cross-section area
$[A(\psi)]$	Coefficient matrix for a periodic system solution
$A_i - O_i$	Integrated products of mode shape functions ignoring the “smart” spring
a	Airfoil lift-curve slope
$a, d, t_f, t_w, t_r, l_r, l_l, l_2, r_1, r_2, f, l_f$	Specification parameters of the “smart” blade cross-section
$B_{xij} - S_{xi}$	Integrated products of mode shape functions due to the inclusion of the “smart” spring, Eq. (3.10)
b	Number of blades
$b(T), b(U)$	Kinetic and strain energy boundary conditions
B_1^*	Blade cross-section integral
B_2^*	Blade cross-section integral
c	Blade chord
c_{d_0}	Airfoil profile drag coefficient
c_k	Coefficients for a periodic system solution
$[C]$	Modal damping matrix
$[C_c]$	Modal damping matrix pre-multiplied by $[U]$ and $[U]^T$
$[\Delta C]$	Modal damping matrix corresponding to the “smart” spring
C_1	Blade cross-section integral
C_1^*	Blade cross-section integral
E	Young’s modulus
E_a, E_L, E_T, E_N	Young’s modulus of aluminum, longitudinal and transverse Young’s moduli of graphite/epoxy

e	Mass centroid offset from elastic axis, positive when in front of the elastic axis
e_A	Tensile axis (area centroid) offset from elastic axis, positive when in front of the elastic axis
G	Shear modulus
G_a, G_{LT}, G_{LN}	Shear modulus of aluminum, and shear moduli of graphite/epoxy
$[G]$	Modal damping matrix without aerodynamic terms
$[\Delta H(x)]$	Heaviside function
$\bar{i}, \bar{j}, \bar{k}$	Unit vectors associated with undeformed blade coordinate system
$\bar{i}', \bar{j}', \bar{k}'$	Unit vectors associated with deformed blade coordinate system
$\bar{I}, \bar{J}, \bar{K}$	Unit vectors associated with undeformed blade coordinate systems, prior to undergoing a pre-cone displacement
$[I]$	Identity matrix
$I_{y'}, I_{z'}$	Cross-section area moments of inertia about y' and z'
J	Torsional stiffness constant
k_A	Cross-section polar radius of gyration
k_m	Mass radius of gyration
k_{m_1}, k_{m_2}	Principal mass radii of gyration
K	Non-dimensional parameter
$[K]$	Modal stiffness matrix
$[K_k], [K_k(\psi)]$	Modal stiffness matrices pre-multiplied by $[U]$ and $[U]^T$
$[K^s]$	Modal stiffness matrix due to the structural terms
$[K^a]$	Modal stiffness matrix due to the aerodynamic terms
$[\Delta K],$ $[\Delta K(\psi)]$	Modal stiffness matrices due to the “smart” spring

L_u, L_v, L_w	Generalized aerodynamic forces per unit length in the u , v and w directions, respectively
$[M]$	Modal mass matrix
M_ϕ	Generalized aerodynamic moment per unit length
$M_{x'}$	Twisting moment about x' axis
$M_{y'} M_{z'}$	Bending moments about y' and z' axes
$[M_m]$	Modal mass matrix made diagonal
$[\Delta M]$	Modal mass matrix due to the “smart” spring
m	Mass per unit length
m_1, m_2, m_s	Masses per unit length of tuning bars and the host spar
N	Number of non-rotating, uncoupled modes for flap, lead-lag bending and torsion
n	Frequency of sinusoidal actuation of the “smart” spring over one rotor revolution of the blade
O	A fixed point at the root of the blade
$[P]$, $[P^*]$, and $[P^*(\psi)]$	State space matrices in the perturbation equations
$P_{x'}$	Warp term in the strain energy expression
$[Q]$	Transition matrix
R	Length of the blade
$\mathcal{R}$	Flap-lag structural coupling parameter; also inertial frame of reference
r	Distance along the deformed elastic axis
$S_{x'}$	Twisting moment due to shear stress
T	Period of actuation, kinetic energy, and blade tension
$T_{x'}$	Twisting moment due to longitudinal stress

t	Time and blade thickness
U	Strain energy
$[U]$	Matrix of eigenvectors for free vibration case
u, v, w	Elastic axis displacements along x, y, z directions, respectively
v_i	Induced inflow velocity
$\vec{V}$	Velocity of a point in the blade
$V_{x'}$	Axial force (tension) in the x' direction
$V_{y'}, V_{z'}$	Shear forces in the y' and z' directions, respectively
$V_j, W_j,$ V_{0j}, W_{0j}	Lead-lag and flap bending generalized coordinates and equilibrium deflections, respectively
x, y, z	Undeformed coordinate system (x is the elastic axis)
x_l, y_l, z_l	Coordinates of the point in the deformed blade
x', y', z'	Deformed coordinate system
$\bar{x}_1, \bar{x}_2,$ x_1, x_2	Non-dimensional and dimensional starting and ending points of the “smart” spring along the blade length
x_p	Axis, which rotates about the Z axis with angular velocity Ω .
$\{X\}$	Vector of generalized perturbation modal coordinates
$\bar{Y}_u, \bar{Y}_v, \bar{Y}_w, \bar{Y}_\phi$	Notation for writing the energy expressions in a concise form
Z	Number of low frequency modes
$Z_u, \bar{Z}_u$	Notation for writing the energy expressions in a concise form
$Z_v, \bar{Z}_v, Z_{v'}, \bar{Z}_{v'}$	Notation for writing the energy expressions in a concise form
$Z_w, \bar{Z}_w, Z_{w'}, \bar{Z}_{w'}$	Notation for writing the energy expressions in a concise form
$Z_\phi, \bar{Z}_\phi$	Notation for writing the energy expressions in a concise form

$Z(x)$	Represent mass and cross-section properties as functions of the radial coordinate
Z_0, Z_{ss}	Represent mass and cross-section properties corresponding to the blade without the “smart” spring, and to the “smart” spring, respectively
$\alpha_j, \beta_j, \gamma_j$	Constants used in the expressions for assumed mode shapes
β_{pc}	Pre-cone angle
γ	Lock number
δW	Virtual work of the non-conservative forces
ε	Small parameter
$\varepsilon_{11}, \varepsilon_{12}, \varepsilon_{13},$ $\varepsilon_{22}, \varepsilon_{23}, \varepsilon_{33}$	Tensor strain components
$\varepsilon_{xx}, \varepsilon_{x\eta}, \varepsilon_{x\zeta}$	Engineering strain components
η, ζ	Principle coordinates of the blade cross-section
η_k	Complex characteristic numbers
$\hat{\eta}, \hat{\zeta}$	Equal to $\eta - \frac{\partial \lambda}{\partial \zeta}, \zeta + \frac{\partial \lambda}{\partial \eta}$, respectively
$\Theta(\bar{x})$	Non-rotating torsional mode shape
θ	Blade pitch angle
κ	Non-dimensional torsional rigidity
λ_k	Eigenvalues of the transition matrix $[Q]$
$\lambda(\eta, \zeta)$	Warp function
$\hat{\lambda}_{ss1}, \hat{\lambda}_{ss2},$ $\tilde{\lambda}_{ss1}, \tilde{\lambda}_{ss2},$ $\hat{h}_{ss}, \tilde{h}_{ss}$	Notation for writing the sinusoidal variation properties of the “smart” spring in a concise form
Λ_1, Λ_2	Non-dimensional bending stiffnesses

μ, μ_1, μ_2	Non-dimensional radii of gyration
ν	Poisson's ratio
$\nu_a, \nu_{LT}, \nu_{LN}, \nu_{TN}$	Poisson's ratios of aluminum, and graphite/epoxy
ρ	Structural mass density
$\rho_{lead}, \rho_a, \rho_g$	Mass density of lead, aluminum, and graphite/epoxy, respectively
ρ_∞	Air density
σ	Solidity, $bc/\pi R$
$\hat{\sigma}_k, \hat{\omega}_k$	Real and imaginary parts of the characteristic numbers; represent damping and frequency, respectively
$\sigma_{11}, \sigma_{12}, \sigma_{13},$ $\sigma_{22}, \sigma_{23}, \sigma_{33}$	Tensor stress components
$\sigma_{xx}, \sigma_{x\eta}, \sigma_{x\zeta}$	Engineering stress components
ϕ	Torsional displacement about the elastic axis
ϕ_{0j}	Equilibrium torsional displacements
ϕ_j	Torsional generalized coordinates
$\Psi_j(\bar{x})$	Non-rotating lead-lag and flap bending mode shapes
ψ	Azimuth angle (dimensionless time)
$\omega_v, \omega_w, \omega_\phi$	Fundamental lead-lag, flap, and torsion natural frequencies
Ω	Rotor angular velocity
$()'$	Derivative with respect to x or $\bar{x}$
$(\dot{ })$	Derivative with respect to t or ψ
$(\bar{ })$	Refers to non-dimensional quantities
$\Delta()$	Refers to perturbation quantities
$()^T$	Refers to matrix transpose operation
$()_{NR}$	Refers to non-rotating non-dimensional natural frequencies

$()_{ss}$	Refers to the properties of the “smart” spring
$()^{na}$	Refers to the case of non-actuated “smart” spring
$()^a$	Refers to the case of constantly actuated “smart” spring. Also refers to the aerodynamic stiffness matrix
$()_0$	Refers to the properties of the blade without the “smart” spring
$()_R$ $()_I$	Refers to the real and imaginary parts

Chapter 1

Introduction

1.1 Overview

A helicopter is defined as an aircraft, which uses rotating wings, called rotors, for lift, propulsion, and directional control. This allows the helicopter to hover relative to the ground without vertical and horizontal flight velocities. Also, the helicopter must be able to fly forward, climb, and cruise in order to transport its occupants and cargo. These abilities necessitate mechanical and aerodynamic complexity and higher power requirements than for a fixed-wing aircraft of the same weight and size.

One of the important factors that influences the control and stability of the helicopter is the arrangement and configuration of its rotors. To turn the rotor and lift the helicopter, power must be developed by the engine. Associated with this power is a resulting torque on the helicopter fuselage. Therefore, the problem is how to counter the main rotor that is forcing the fuselage to rotate in the opposite direction of the rotor rotation. One solution for balancing the torque is to introduce a tail rotor, which pushes the fuselage in the opposite direction of the torque, Fig. (1.1a). Balance is provided by the tail rotor

thrust, which acts on the arm about the main rotor shaft. The control of the helicopter about the vertical axis during hover flight is performed by the tail rotor.

1.1a

1.1b

1.1c

1.1d

Figure 1.1 Helicopter configurations, [1]

The other solution to counter the fuselage torque is the twin main rotor system, according to which two equal size and loading counter rotating rotors are utilized. A tandem rotor configuration in Fig. (1.1b), may serve as one example for this system. Due to the opposite rotation of the rotors, the torque produced by each of them is neutralized. Control about the vertical axis during hover flight is accomplished by bending the rotor discs against each other. Another example of the twin main rotor configuration is the one

that contains two rotors, “combing” into one another and rotating in the opposite directions, Fig. (1.1d). One more example of helicopter configuration is utilizing a coaxial rotor shown in Fig. (1.1c). This arrangement is such that one rotor is placed on the top of the other while rotating in opposite directions.

1.2 Helicopter Rotors

In order to achieve efficient aerodynamic lift and flight, large rotor blades with high aspect ratios are required. As the result, the blade behaves as a long, slender, and flexible rotating beam. In general, four types of displacements may be associated with such a structure:

- (i) flap: which refers to the out-of-plane displacement of the blade about the root,
- (ii) lead-lag: which refers to the rotation about the vertical axis at the root of the beam causing in-plane deflections,
- (iii) pitch: which refers to the rotation of the beam about its longitudinal elastic axis resulting in twisting of the blade, and
- (iv) axial displacement: which refers to the longitudinal deformation of the blade and, generally, is much smaller than the other three.

Obviously, the corresponding dynamic behaviour of such a rotating beam depends on its support conditions at the blade root. In the early development of helicopters, a frequently adopted design was to provide hinges at the blade root. The result of such an articulated blade design is that bending moments at the root are eliminated (Fig. (1.2)).

Figure 1.2 Schematic of an articulated rotor hub and root, [2]

The primary drawback of such a design is that it requires relatively complicated mechanical design at the hinge connection of the blade to the rotor hub. This leads to concern about fatigue failure, wear, and tear in various parts of the articulated joint and hence expensive maintenance and operational costs.

With developments in advanced materials and composites in recent years, the new trend in the design of rotor blades has been to develop hingeless blades by eliminating some or all of the hinges at the blade root connection to the hub. The advantage of the hingeless blade lies in the fact that it is mechanically simple with relatively fewer parts. However, since the loads become higher for the hingeless arrangement, the blade must be able to accommodate considerable structural deflections.

Figure 1.3 Primary elements of a typical hingeless rotor blade, [3]

A typical hingeless blade structure may consist of two main flexible segments: the inboard and the outboard segments connected by a pitch change bearing, as shown in Fig. (1.3). The inboard segment is attached to the hub at the root. The outboard segment rotates about a pitch change bearing by means of a pitch link represented as a spring element attached to a swash plate. As a result, rigid body pitching motion of the outboard blade segment can take place. The design can vary, for example, by replacing the inboard segment by a pitch bearing attached to the hub, leaving only the outboard segment.

1.3 Stability of a Hingeless Helicopter Blade

Aeroelastic stability of a helicopter rotor blade system, in general, is a complex problem involving structural characteristics associated with a long, slender, rotating beam that is subjected to aerodynamic as well as inertial loads. The basic concern of the stability of motion of a rotor blade system is to ensure that under the applied aerodynamic loads any “small” perturbation of the steady state, i.e., the so-called trim condition that

refers to the dynamic equilibrium of rotational motion of the rotor blades, will not grow unbounded. Thus, the tendency of the system to return to its trimmed condition after a small perturbation results in a stable system.

In the case of the articulated rotor blades, improper relative positioning of the centre of mass, elastic axis and aerodynamic centre in the chordwise direction of the blade may result in the instability that is usually characterized by coupling of flap deflections and torsion twist with a frequency near the torsion natural frequency.

On the other hand, in the case of the hingeless cantilever blades, structural coupling between bi-axial bending and torsion deflections can cause a further type of instability that is characterized by coupled flap, lead-lag, and torsion deflections of the blade with a frequency near the lead-lag natural frequency [3].

The inboard and outboard segments of a hingeless blade that were introduced in Section 1.2, both will deflect, and both will have large structural nonuniformities of bending stiffness and torsional rigidity that will contribute to the structural coupling between the flap, lead-lag and torsion deformations of the blade. Thus, the structural parameters of the hingeless blade are an important factor in their stability analysis.

As the result, in order to concentrate primarily on the basic structural aspects of the aeroelastic stability of either conventional or “smart” hingeless blades one may exclude the effects of other causes of instability, for example, by setting the mass, aerodynamic centre and tension axes offsets from the elastic axis of the blade equal to zero. Other parameters that have an important influence on rotor blade stability are pre-cone, droop, sweep, and torque offset. These blade configuration parameters are shown in Fig. (1.4).

Figure 1.4 Configuration parameters of a hingeless rotor blade, [3]

The structure of the blade also influences the behaviour of the rotor. It may contain spars, stiffeners, the skin, and balance weight components, or it may be constructed of composite materials. Numerous design arrangements are possible that result in the desired aerodynamic, strength, and stiffness properties of the blade.

1.4 Brief Review of Literature

The stability characteristics of conventional hingeless rotor blades are reasonably well understood, and considerable work has been devoted to this problem during the last four to five decades. But, investigation of the stability of so-called “smart” hingeless blades has become a topic of concern only in the last few years as a result of recent developments in the area of adaptive structures, in general, and the possibility of

incorporating the smart materials technology into the design and operation of “smart” rotor blades.

One of the earliest investigations of flap-lag-torsion stability of elastic hingeless blades was the work of Houbolt and Brooks [4], where the linear equations of motion were developed. Only the linear flap-lag coupling terms were considered in that work and the nonlinear bending-torsion coupling terms were not included. Young [5] showed that nonlinear flap-lag terms can cause instability resulting from the destabilizing inertia and aerodynamic coupling effects associated with the two bending degrees of freedom present in the problem.

A later treatment of the linear flap-lag problem was presented by Ormiston and Hodges [6]. They used a torsionally rigid blade approximation to show that the stability can be greatly influenced by the elastic coupling.

Along with the above analyses, the complete flap-lag-torsion stability problem has been investigated by Mil' et al. [7]. It was found that the nonlinear structural moments caused by flap and lead-lag bending greatly influence the torsion deflection of the hingeless blade. This so-called bending-torsion coupling was detected to be proportional to the product of the flap and lead-lag bending curvatures and the difference between two bending flexibilities.

Also, the effort to understand the nonlinear flap-lag-torsion coupling problem of an elastic rotating blade was carried out by Arcidiacono [8], who derived the nonlinear equations of motion in a detailed and complete way. However, some nonlinear terms were ignored in the derivation.

The other effort has been performed by Friedman and Tong [9] who approximated the torsional deflections by rigid body pitching motion, while considering elastic flap and lead-lag bending of a uniform blade. The results showed that it was important to consider torsion. Also, the stability characteristics were found to be susceptible to the number and types of mode shapes.

A trial to investigate the nonlinear flap-lag-torsion coupling of a flexible cantilever blade was also done by Hodges and Dowell [10]. Using two methods, namely the variational method based on Hamilton's principle, and the Newtonian approach based on the equilibrium of a differential blade element, they derived the nonlinear flap-lag-torsion equations of motion of a twisted nonuniform rotor blade. Hodges and Ormiston, [3], have utilized these equations for the stability analysis of a uniform, untwisted rotor blade. The research carried out has shown that stability was observed for most hingeless configurations with zero pre-cone. However, positive pre-cone was found to be destabilizing. Also, the main factor that determined stability was the nonlinear structural coupling between bending and torsion deflections, [3, 11].

Aerodynamic loads on a rotating helicopter rotor blade vary considerably, and are periodic in steady flight. Rotor forces and moments that cause fuselage vibration are transmitted to the hub from the blades and consist of aerodynamic and inertia forces resulting from blade flapping and lagging motion. Elimination and control of vibration of the helicopter is of particular interest. It is important to provide a comfortable environment for the crew and passengers, and minimize fatigue of the rotor and the airframe. Another factor in urban areas is noise; because it may limit helicopter operation.

As a means of active control of vibration and noise in rotary wings, two different approaches have been considered in the literature. The first is the blade pitch angle control introduced in the non-rotating frame through the swash plate, termed higher harmonic control (HHC). The second approach is an individual blade control (IBC) technique, which has been proposed by Kretz [12] and Ham [13]. In this technique, by mounting actuators either on the swash-plate or on the individual blade, each blade can be independently actuated on its flap, pitch, and lead-lag degrees of freedom. Incorporating smart structures technology into this technique can result in a highly effective means of reducing vibrations in helicopter blades.

To this effect, three main directions have been investigated in developing a “smart” rotor using adaptive structures. The first direction involves the incorporation of flaps actuated by piezoceramic beam elements [14], resulting in a change of the airfoil effective camber. The second approach uses distributed actuators attached under the blade skin or piezoelectric fibers in a composite blade to control the blade twist [15]. According to the third concept offered by Nitzsche, Lammering, and Breitbach [16], the “smart” actuators can be attached at the blade root in order to change the root boundary conditions.

The present work proposes controlling the dynamic impedance of individual blades. This can be achieved by incorporating a piezoelectric element as a “smart” spring in an appropriate location along each blade. By actuating this spring, one can change the cross-sectional parameters of the blade [17, 18], thus affecting the dynamic behaviour of the rotor. The main focus of this work is to investigate the aeroelastic stability of such a “smart” blade. Another task to accomplish is to derive the equations of motion for the

dynamically controlled “smart” unit by introducing the sinusoidal actuation and de-actuation of the spring. The stability characteristics of such a blade will be compared to the non-actuated and constantly actuated “smart” blade. The research will be limited to the hovering flight conditions. To this end, a reduced model of blade for the Eurocopter BO105 helicopter is designed and used throughout all simulations. The same reduced model is intended to be built and used in the future experimental work.

1.5 Thesis Overview

An introduction to hingeless rotor stability was presented in Chapter 1. The overview of several helicopter configurations is briefly outlined. The parameters, which affect rotor blade stability, are discussed with the necessary background literature presented afterwards.

Chapter 2 presents a brief review of the research done by Hodges and Dowell [10] and Hodges and Ormiston [3]. The nonlinear partial differential equations of motion for a rotating cantilever rotor blade are developed. These equations are next non-dimensionalized and transformed into a set of nonlinear ordinary differential equations of motion in terms of generalized coordinates by applying Galerkin’s method. The generalized coordinates are decomposed into steady equilibrium and perturbation quantities. The steady equilibrium quantities are then substituted into the nonlinear modal equations of motion, thus defining the first set of nonlinear algebraic equations.

The second set is obtained by substituting the sum of steady state quantities and perturbation quantities into the modal equations of motion. Subtracting the first set of equations from the second one and neglecting the nonlinear products of perturbation

quantities leads to the linear perturbation differential equations of motion. The steady equilibrium deflections are found from the first set of nonlinear algebraic equations resulting in the coefficients for the linearized perturbation equations. These equations define the unsteady blade motion about the equilibrium deflection state. The eigenvalues of the perturbation equations determine the stability of the unsteady blade motion. They are available by applying the principal coordinate transformation resulting in the simplification of the system of differential equations.

Chapter 3 introduces the concept of a “smart” spring and its application to investigate the aeroelastic stability of the blade. Therefore, modified equations of motion are derived in order to incorporate the “smart” spring contribution. These equations are then solved by Galerkin’s method resulting in linearized perturbation equations of motion about the equilibrium deflection state. Next, the equations of motion are modified to incorporate sinusoidal “smart” spring actuation to investigate the changes on the blade stability using periodic actuation as opposed to a constant or non-actuated spring. The linear differential equations of motion are time dependent, and therefore need a more complicated treatment for determining the eigenvalues. The Floquet method was chosen to solve for the eigenvalues in this case.

The blade cross-section design is developed in this chapter, and includes spar dimensions, tuning bars’ characteristics, as well as the choice of material for the spar and bars. The data from the Eurocopter BO105 helicopter is used for the blade design. The piezoelectric material is then included in the cross-section design and calculations of the blade properties.

Chapter 4 deals with the aeroelastic stability of the “smart” blade. The results of the stability analysis are studied and discussed for six regimes, namely: Conventional Baseline Blade, Smart Baseline Blade, Direct Current regimes, and three different sinusoidal regimes corresponding to the frequency of actuation. The placement of the piezoelectric unit and the presence of shear as well as the structural coupling parameter result in six design cases, which are presented in this chapter. Steady equilibrium deflections are obtained for each design case and, finally, the eigenvalues are determined, described, and presented in graphs and tables.

The results are summarized in Chapter 5, along with the discussion of the important findings from this work, and several conclusions are presented. Recommendations for future work are offered to conclude the chapter.

Chapter 2

Equations of Motion

For actively controlling the dynamics of a hingeless blade, the main objective is to investigate the effect of a “smart” spring in improving the stability of the blade or reducing its vibrations. To this end, an appropriate dynamic model needs to be incorporated first. Nonlinear equations of motion for a rotating nonuniform cantilever rotor blade were developed by Hodges and Dowell [10] using both the variational method based on Hamilton’s principle and Newton’s method based on force and moment equilibrium. A brief review of this work is presented in this chapter. Due to its rigorous and systematic approach, the variational method is summarized here. Equations of motion are valid to second order for long, slender, straight, homogeneous, isotropic blades undergoing small displacements, without sweep, droop, and torque offsets.

In general, the equations are derived for a blade with mass and area centroid axes offset from the elastic axis, nonuniform mass and stiffness properties, variable pre-twist and a small pre-cone angle. This is done by means of the ordering scheme restricting the squares of the bending slopes, the torsion deformation, the chord/radius, and the

thickness/radius ratios to be small compared to unity. These equations are then simplified by assuming no twist and no chordwise offsets between the mass, elastic, tension, and aerodynamic centre axes, and with uniform mass and stiffness.

The simplified equations are then written in the non-dimensional form. These non-linear, integro-partial differential equations are transformed into a system of nonlinear ordinary differential equations by applying Galerkin's method. The resulting differential equations of generalized coordinates are next linearized for perturbation motions about the equilibrium deflection state. The dynamic stability of helicopter blade motion can be determined by investigating the eigenvalues of these perturbation equations.

2.1 Coordinate Systems

The orthogonal axis system $OXYZ$, attached to the hub at the root of the blade, is fixed in the inertial frame $\mathcal{R}$. The orthogonal axes x_p , y , Z rotate with respect to $\mathcal{R}$ at constant angular velocity Ω . For an undeformed blade in Fig 2.1, a Cartesian coordinate system $Oxyz$ is chosen such that the x -axis coincides with the elastic axis of the beam and inclined to the x_p axis at the pre-cone angle β_{pc} .

Figure 2.1 Rotor blade coordinate systems [10].

The displacements of any point on the elastic axis in the x , y , and z directions are denoted by u , v , and w , respectively. During the deformation, the blade undergoes flap and lead-lag bending, extension, and torsional rotation ϕ . The blade cross-section, before and after deformation, is shown projected on the Oyz plane in Fig.2.2.

Figure 2.2 Rotor blade cross section before and after deflections [3].

The deformed blade is identified with the coordinate system $x' y' z'$. It is fixed to the deformed blade with x' tangent to the deformed elastic axis, and y' and z' identical to the principal axes η and ζ , respectively, that are inclined at the “built-in ” end with the pitch angle, θ .

Also shown in Fig. (2.2) is the blade rotated through a pitch angle, θ , before deformation. After deformation, the cross-section is subjected to an elastic angle of twist, ϕ , and the elastic axis is displaced by two transverse displacements v and w due to lead-lag and flap, respectively, and the axial displacement u – not shown in Fig. (2.2). It should be noted that the cross-section of the blade after deformation does not lie in the Oyz plane.

2.2 Ordering Scheme

In deriving the nonlinear equations of motion an ordering scheme was used to ignore the higher order terms in a systematic manner. This approach ensures that the resulting equations are self-adjoint while eliminating higher order terms to achieve simpler equations. To this end, the order of magnitude of the dimensionless deflections v/R and w/R are represented by a parameter ε . The angle of twist ϕ is considered to be a small angle so that $\sin\phi \cong \phi$ and $\cos\phi \cong 1$. Moreover, the axial coordinate x is considered to be of the same order of magnitude as the blade length R , and lateral coordinates of the cross-section are of the same order as the chord length c and thickness t , respectively, where both c and t are assumed to be of the same order of magnitude as v or w . Also, the warp function λ is of the order of ct . Based on these assumptions, the following orders of magnitude have been used for various parameters:

Table 2.1 Order of magnitude of various parameters

$\frac{v}{R}, \frac{w}{R}, \frac{c}{R} = O(\varepsilon)$	$\frac{u}{R} = O(\varepsilon^2)$
$\phi = O(\varepsilon)$	$\frac{\eta}{R} = O(\varepsilon)$
$\frac{x}{R} = O(1)$	$\frac{\zeta}{R} = O(\varepsilon)$
θ (aerodynamic forces only) $= O(\varepsilon)$	$\frac{\partial \lambda}{R \partial \eta} = O(\varepsilon)$
$\theta = O(1)$	$\frac{\partial \lambda}{R \partial \zeta} = O(\varepsilon)$
$\beta_{pc} = O(\varepsilon)$	$\frac{\lambda}{R^2} = O(\varepsilon^2)$
$\frac{v_i}{\Omega R} = O(\varepsilon)$	$\frac{EI_{y'}}{m\Omega^2 R^4}, \frac{EI_{z'}}{m\Omega^2 R^4}, \frac{GJ}{m\Omega^2 R^4} = O(1)$
$\frac{d}{\Omega dt}, R \frac{d}{dx} = O(1)$	$\frac{k_A}{R}, \frac{k_{m1}}{R}, \frac{k_{m2}}{R}, \frac{k_m}{R} = O(\varepsilon)$
$\frac{EA}{m\Omega^2 R^2} = O(\varepsilon^{-2})$	
$\frac{c_{d0}}{a} = O(\varepsilon^2)$	

In Table 2.1 the following are defined: v_i is the induced inflow velocity; Ω is the rotor blade angular velocity; m is mass per unit length; $EI_{y'}$, $EI_{z'}$ are the flapwise and chordwise bending stiffness; and GJ is the torsional rigidity; k_A is the blade cross-section polar radius of gyration; k_{m1} and k_{m2} are the principal mass radii of gyration; k_m is the blade cross-section mass radius of gyration; θ and β_{pc} are the pitch and pre-cone angles (β_{pc} is assumed to be zero in the present study), respectively; c_{d0} and a are the drag coefficient and the lift curve slope.

Within the equations, the terms with order of magnitude of $O(\varepsilon^2)$ and higher were ignored with respect to unity. However, some higher order terms, which contribute to damping, were retained in the equations. Moreover, as it is outlined in Section 2.3 the strain energy and kinetic energy will be combined into Hamilton's principal. In so doing,

one needs to relate the relative order of magnitude of the structural terms to those of the inertial terms. This is accomplished by introducing a dimensionless scaling parameter

$$\frac{m\Omega^2 R^2}{EA} \text{ whose order of magnitude is taken to be } O(\epsilon^2), [10].$$

2.3 Hamilton's Principle

The equations of motion and boundary conditions for a cantilever blade are derived from Hamilton's principle following the procedure outlined by Hodges and Dowell [10]. These equations are obtained based on the assumption that the rotor blade can be represented as a long, straight, slender, homogeneous, isotropic beam with small displacements and a cross-section that is symmetric with respect to the chordwise axis.

Hamilton's principle is expressed as:

$$\int_{t_1}^{t_2} [\delta(U - T) - \delta W] dt = 0 \quad (2.1)$$

where δU , δT , δW are the variations of the strain energy, kinetic energy and virtual work done by the external loads, respectively; and t_1 and t_2 , represent arbitrary initial and final times, respectively.

2.4 Strain Energy Contributions

The expression for strain energy is:

$$U = \frac{1}{2} \int_0^R \iiint_A (\sigma_{xx} \epsilon_{xx} + \sigma_{x\eta} \epsilon_{x\eta} + \sigma_{x\zeta} \epsilon_{x\zeta}) d\eta d\zeta dx \quad (2.2)$$

and the variation is:

$$\delta U = \int_0^R \iint_A (\sigma_{xx} \delta \varepsilon_{xx} + \sigma_{x\eta} \delta \varepsilon_{x\eta} + \sigma_{x\zeta} \delta \varepsilon_{x\zeta}) d\eta d\zeta dx \quad (2.3)$$

where

$$\begin{aligned} \sigma_{xx} &= E \varepsilon_{xx} \\ \sigma_{x\eta} &= G \varepsilon_{x\eta} \\ \sigma_{x\zeta} &= G \varepsilon_{x\zeta} \end{aligned}$$

denote engineering stresses and strains for the material with Young's modulus, E , and shear modulus, G .

According to Hodges and Dowell [10]:

$$\begin{aligned} \delta \varepsilon_{xx} &= \delta u' + v' \delta v' + w' \delta w' + (\eta^2 + \zeta^2)(\theta + \phi)' \delta \phi' - \lambda \delta \phi'' - \\ &\quad - [\eta \cos(\theta + \phi) - \zeta \sin(\theta + \phi)](\delta v'' + w'' \delta \phi) - \\ &\quad - [\eta \sin(\theta + \phi) + \zeta \cos(\theta + \phi)](\delta w'' - v'' \delta \phi) \\ \delta \varepsilon_{x\eta} &= -\hat{\zeta} \delta \phi' & \hat{\eta} &= \eta - \frac{\partial \lambda}{\partial \zeta} \\ \delta \varepsilon_{x\zeta} &= \hat{\eta} \delta \phi' & \hat{\zeta} &= \zeta + \frac{\partial \lambda}{\partial \eta} \end{aligned} \quad (2.4)$$

where λ represents the warp function. These expressions are important in the development of the nonlinear system of equations of motion since they represent the nonlinear strain-displacement relations accurate to the second order.

Substituting expressions (2.4) into Eq. (2.3), Hodges and Dowell obtained:

$$\begin{aligned} \delta U &= \int_0^R \{ V_{x'} (\delta u' + v' \delta v' + w' \delta w') + (S_{x'} + T_{x'}) \delta \phi' + P_{x'} \delta \phi'' + \\ &\quad + [M_{z'} \cos(\theta + \phi) + M_{y'} \sin(\theta + \phi)] (\delta v'' + w'' \delta \phi) + \\ &\quad + [M_{z'} \sin(\theta + \phi) - M_{y'} \cos(\theta + \phi)] (\delta w'' - v'' \delta \phi) \} dx \end{aligned} \quad (2.5)$$

where the moments and forces are:

$$\begin{aligned}
V_{x'} &\equiv \iint_A \sigma_{xx} d\eta d\zeta = \\
&= EA \left\{ u' + \frac{v'^2}{2} + \frac{w'^2}{2} + k_A^2 \theta' \phi' - e_A [v'' \cos(\theta + \phi) + w'' \sin(\theta + \phi)] \right\} \\
S_{x'} &\equiv \iint_A (\hat{\eta} \sigma_{x\zeta} - \hat{\zeta} \sigma_{x\eta}) d\eta d\zeta = GJ \phi' \\
T_{x'} &\equiv \iint_A (\theta + \phi)' \sigma_{xx} (\eta^2 + \zeta^2) d\eta d\zeta = EA k_A^2 (\theta + \phi)' \left(u' + \frac{v'^2}{2} + \frac{w'^2}{2} \right) + \\
&\quad + EB_1^* \theta'^2 \phi' - EB_2^* \theta' (v'' \cos \theta + w'' \sin \theta) \\
P_{x'} &\equiv - \iint_A \lambda \sigma_{xx} d\eta d\zeta = EC_1 \phi'' + EC_1^* (w'' \cos \theta - v'' \sin \theta) \\
M_{y'} &\equiv \iint_A \zeta \sigma_{xx} d\eta d\zeta = EI_{y'} [v'' \sin(\theta + \phi) - w'' \cos(\theta + \phi)] - EC_1^* \phi'' \\
M_{z'} &\equiv - \iint_A \eta \sigma_{xx} d\eta d\zeta = EI_{z'} [v'' \cos(\theta + \phi) + w'' \sin(\theta + \phi)] - \\
&\quad - EA e_A \left(u' + \frac{v'^2}{2} + \frac{w'^2}{2} \right) - EB_2^* \theta' \phi'
\end{aligned} \tag{2.6}$$

In Eqs. (2.6) the following cross-sectional properties are defined:

$$\begin{aligned}
A &= \iint_A d\eta d\zeta & Ae_A &= \iint_A \eta d\eta d\zeta \\
I_{y'} &= \iint_A \zeta^2 d\eta d\zeta & I_{z'} &= \iint_A \eta^2 d\eta d\zeta \\
A k_A^2 &= \iint_A (\eta^2 + \zeta^2) d\eta d\zeta & J &= \iint_A (\hat{\eta}^2 + \hat{\zeta}^2) d\eta d\zeta \\
B_1^* &= \iint_A (\eta^2 + \zeta^2)^2 d\eta d\zeta & B_2^* &= \iint_A \eta(\eta^2 + \zeta^2) d\eta d\zeta \\
C_1 &= \iint_A \lambda^2 d\eta d\zeta & C_1^* &= \iint_A \zeta \lambda d\eta d\zeta
\end{aligned} \tag{2.7}$$

In (2.7), A denotes the effective cross-sectional area under tension. e_A is the tension axis offset from the elastic axis; Ak_A^2 is the polar moment of inertia; $I_{z'}$ and $I_{y'}$ are the

chordwise and flapwise moments of inertia; J is the torsional constant, including cross-section warping; C_I is the warping rigidity; and B_I^* , B_2^* , and C_I^* are sectional constants.

Due to the anti-symmetry of the warping function, λ , and the symmetry of the cross-section about the η axis, the following integrals vanish in the derivation of Eqs. (2.6).

$$\begin{aligned} \iint_A \lambda \, d\eta \, d\zeta &= 0 & \iint_A \eta \lambda \, d\eta \, d\zeta &= 0 \\ \iint_A \eta \zeta \, d\eta \, d\zeta &= 0 & \iint_A \zeta \, d\eta \, d\zeta &= 0 \\ \iint_A \zeta (\eta^2 + \zeta^2) \, d\eta \, d\zeta &= 0 & \iint_A \lambda (\eta^2 + \zeta^2) \, d\eta \, d\zeta &= 0 \end{aligned} \quad (2.8)$$

Integrating Eq. (2.5) by parts, one obtains:

$$\delta U = \int_0^R (\bar{Y}_u \delta u + \bar{Y}_v \delta v + \bar{Y}_w \delta w + \bar{Y}_\phi \delta \phi) dx + b(U) \quad (2.9)$$

with

$$\begin{aligned} \bar{Y}_u &= -(V_{x'})' \\ \bar{Y}_v &= [M_{z'} \cos(\theta + \phi) + M_{y'} \sin(\theta + \phi)]'' - (V_{x'} v')' \\ \bar{Y}_w &= [M_{z'} \sin(\theta + \phi) - M_{y'} \cos(\theta + \phi)]'' - (V_{x'} w')' \\ \bar{Y}_\phi &= (P_{x'})'' - (S_{x'} + T_{x'})' - v'' [M_{z'} \sin(\theta + \phi) - M_{y'} \cos(\theta + \phi)] + \\ &\quad + w'' [M_{z'} \cos(\theta + \phi) + M_{y'} \sin(\theta + \phi)] \end{aligned} \quad (2.10)$$

and the boundary condition:

$$\begin{aligned} b(U) &= V_{x'} \delta u \Big|_0^R + \{V_{x'} v' - [M_{z'} \cos(\theta + \phi) + M_{y'} \sin(\theta + \phi)]'\} \delta v \Big|_0^R + \\ &\quad + [M_{z'} \cos(\theta + \phi) + M_{y'} \sin(\theta + \phi)] \delta v' \Big|_0^R + [M_{z'} \sin(\theta + \phi) - \\ &\quad - M_{y'} \cos(\theta + \phi)] \delta w' \Big|_0^R + \{V_{x'} w' - [M_{z'} \sin(\theta + \phi) - \\ &\quad - M_{y'} \cos(\theta + \phi)]'\} \delta w \Big|_0^R + [S_{x'} + T_{x'} - (P_{x'})'] \delta \phi \Big|_0^R + P_{x'} \delta \phi' \Big|_0^R \end{aligned} \quad (2.11)$$

2.5 Kinetic Energy Contributions

By considering the velocity $\vec{V}$ of an arbitrary point on the blade with respect to the inertial frame of reference $\mathfrak{R}$, the kinetic energy of the blade becomes, [10]:

$$T = \frac{1}{2} \int_0^R \iiint_A \rho \vec{V} \cdot \vec{V} d\eta d\zeta dx \quad (2.12)$$

where ρ is the mass density of the blade. Denoting the angular velocity of the rotor by Ω and the position of an arbitrary point after deformation of the blade by (x_1, y_1, z_1) in Eq. (2.12), one has:

$$\begin{aligned} \vec{V} = & (\dot{x}_1 - \Omega y_1 \cos \beta_{pc}) \vec{i} + (\dot{y}_1 + \Omega x_1 \cos \beta_{pc} - \Omega z_1 \sin \beta_{pc}) \vec{j} + \\ & + (\dot{z}_1 + \Omega y_1 \sin \beta_{pc}) \vec{k} \end{aligned} \quad (2.13)$$

Then, noting that the variation of the kinetic energy

$$\delta T = \int_0^R \iiint_A \rho \vec{V} \cdot \delta \vec{V} d\eta d\zeta dx \quad (2.14)$$

must be integrated in time according to Hamilton's principal, the product $\vec{V} \cdot \delta \vec{V}$ can be integrated by parts in time (with the arbitrary initial and final times taken as zero). Next, truncating the resulting expressions consistent with the ordering scheme, one obtains, [10]:

$$\delta T = \int_0^R \iiint_A (Z_u \delta u + Z_v \delta v + Z_w \delta w + Z_\phi \delta \phi + Z_{v'} \delta v' + Z_{w'} \delta w' + Z_{\phi'} \delta \phi') \rho d\eta d\zeta dx \quad (2.15)$$

where

$$\begin{aligned}
Z_u &= \Omega^2 x + 2\Omega \dot{v} \\
Z_v &= \Omega^2 [v + (y_1 - v)] - \ddot{v} + \ddot{\phi}(z_1 - w) + 2\Omega \beta_{pc} \dot{w} - 2\Omega [\dot{u} - \dot{v}'(y_1 - v) - \dot{w}'(z_1 - w)] \\
Z_w &= -\Omega^2 \beta_{pc} x - 2\Omega \beta_{pc} \dot{v} - \ddot{w} - \ddot{\phi}(y_1 - v) \\
Z_\phi &= -\Omega^2 x(y_1 - v)w' + \Omega^2 x(z_1 - w)v' - \Omega^2 [v + (y_1 - v)](z_1 - w) \\
&\quad + \ddot{v}(z_1 - w) - \ddot{w}(y_1 - v) - \Omega^2 \beta_{pc} x(y_1 - v) - \ddot{\phi}[(y_1 - v)^2 + (z_1 - w)^2] \\
Z_{v'} &= -(\Omega^2 x + 2\Omega \dot{v})(y_1 - v) \\
Z_{w'} &= -(\Omega^2 x + 2\Omega \dot{v})(z_1 - w) \\
Z_{\phi'} &= -\Omega^2 \lambda x
\end{aligned}$$

In the above expressions some higher order torsion inertia terms are important for small values of torsion rigidity and, therefore, have been retained.

After integrating Eq. (2.15) over the blade cross-section A , and integrating by parts with respect to x , the expression for the kinetic energy variation becomes:

$$\delta T = \int_0^R \left[\bar{Z}_u \delta u + (\bar{Z}_v - \bar{Z}_{v'}) \delta v + (\bar{Z}_w - \bar{Z}_{w'}) \delta w + \bar{Z}_\phi \delta \phi \right] dx + b(T) \quad (2.16)$$

where the kinetic boundary condition is:

$$b(T) = \bar{Z}_{v'} \delta v \Big|_0^R + \bar{Z}_{w'} \delta w \Big|_0^R \quad (2.17)$$

and

$$\begin{aligned}
\bar{Z}_u &= m(\Omega^2 x + 2\Omega \dot{v}) \\
\bar{Z}_v &= m\Omega^2 [v + e \cos(\theta + \phi)] + 2m\Omega(\beta_{pc} \dot{w} - \dot{u}) + \\
&\quad + 2me\Omega(\dot{v}' \cos \theta + \dot{w}' \sin \theta) - m\ddot{v} + me\ddot{\phi} \sin \theta \\
\bar{Z}_w &= -m\beta_{pc}(\Omega^2 x + 2\Omega \dot{v}) - m\ddot{w} - me\ddot{\phi} \cos \theta \\
\bar{Z}_\phi &= -mk_m^2 \ddot{\phi} - m\Omega^2 (k_{m2}^2 - k_{m1}^2) \cos(\theta + \phi) \sin(\theta + \phi) - \\
&\quad - me\Omega^2 x(w' \cos \theta - v' \sin \theta) - me\Omega^2 v \sin \theta - \\
&\quad - me\Omega^2 \beta_{pc} x \cos \theta + me(\ddot{v} \sin \theta - \ddot{w} \cos \theta) \\
\bar{Z}_{v'} &= -me[\Omega^2 x \cos(\theta + \phi) + 2\Omega \dot{v} \cos \theta] \\
\bar{Z}_{w'} &= -me[\Omega^2 x \sin(\theta + \phi) + 2\Omega \dot{v} \sin \theta]
\end{aligned}$$

In obtaining Eq. (2.16), the following are identified:

$$\iint_A \rho \zeta d\eta d\zeta = \iint_A \rho \eta \zeta d\eta d\zeta = \iint_A \rho \lambda d\eta d\zeta = 0$$

This is due to cross-sectional symmetry with respect to η axis and anti-symmetry of λ .

Also,

$$\begin{aligned} m &\equiv \iint_A \rho d\eta d\zeta & me &\equiv \iint_A \rho \eta d\eta d\zeta \\ mk_{m_1}^2 &\equiv \iint_A \rho \zeta^2 d\eta d\zeta & mk_{m_2}^2 &\equiv \iint_A \rho \eta^2 d\eta d\zeta \\ k_{m_1}^2 + k_{m_2}^2 &= k_m^2 \\ \iint_A \rho (y_1 - v) d\eta d\zeta &= me \cos(\theta + \phi) \\ \iint_A \rho (z_1 - w) d\eta d\zeta &= me \sin(\theta + \phi) \\ \iint_A \rho (y_1 - v)(z_1 - w) d\eta d\zeta &= m(k_{m_2}^2 - k_{m_1}^2) \cos(\theta + \phi) \sin(\theta + \phi) \\ \iint_A \rho [(y_1 - v)^2 + (z_1 - w)^2] d\eta d\zeta &= mk_m^2 \end{aligned} \quad (2.18)$$

where m is mass per unit length; the offset of the mass axis from the elastic axis is e ; mk_m^2 is torsional mass moment of inertia; $mk_{m_1}^2$ and $mk_{m_2}^2$ represent mass moment of inertia in the flapwise and chordwise directions, respectively.

2.6 Contributions from Non-Conservative Forces

The virtual work done by non-conservative forces may be expressed in a form:

$$\delta W = \int_0^R (L_u \delta u + L_v \delta v + L_w \delta w + M_\phi \delta \phi) dx \quad (2.19)$$

where L_u , L_v , L_w , and M_ϕ are distributed loads in the x , y , z directions, respectively, and distributed torque about the elastic axis. In general, the non-conservative forces include

aerodynamic or non-conservative mechanical loads. In this study the non-conservative forces are only due to the aerodynamic loads.

2.7 Differential Equations of Motion

By substituting the variation of kinetic and strain energy terms, Eqs. (2.16), (2.9), and the virtual work contributions, Equation (2.19), into Hamilton's principle, Hodges and Dowell obtained the variational equations together with the boundary conditions for the blade in terms of u , v , w , and ϕ . For arbitrary, admissible displacements δu , δv , δw , and $\delta \phi$, the coefficients of these variations in the integrand must vanish for all x from 0 to R . This leads to the following four nonlinear partial differential equations, [10]:

δu coefficient:

$$-T' - m(\Omega^2 x - 2\Omega \dot{v}) = L_u \quad (2.20.1)$$

δv coefficient:

$$\begin{aligned} & -(Tv')' + \left\{ -EAe_A \left(u' + \frac{v'^2}{2} + \frac{w'^2}{2} \right) \cos(\theta + \phi) - EB_2^* \theta' \phi' \cos \theta - \right. \\ & - EC_1^* \phi'' \sin \theta + [EI_{z'} \cos^2(\theta + \phi) + EI_{y'} \sin^2(\theta + \phi)] v'' + \\ & + (EI_{z'} - EI_{y'}) \cos(\theta + \phi) \sin(\theta + \phi) w'' \}'' + 2m\Omega \dot{u} + m\ddot{v} - \\ & - me\ddot{\phi} \sin \theta - 2me\Omega(\dot{v}' \cos \theta + \dot{w}' \sin \theta) - m\Omega^2[v + e \cos(\theta + \phi)] - \\ & - 2m\Omega\beta_{pc} \dot{w} - \{ me[\Omega^2 x \cos(\theta + \phi) + 2\Omega \dot{v} \cos \theta] \}' = L_v \end{aligned} \quad (2.20.2)$$

δw coefficient:

$$\begin{aligned}
 & -(Tw')' + \left\{ -EAe_A \left(u' + \frac{v'^2}{2} + \frac{w'^2}{2} \right) \sin(\theta + \phi) - EB_2^* \theta' \phi' \sin \theta + \right. \\
 & + EC_1^* \phi'' \cos \theta + (EI_{z'} - EI_{y'}) \cos(\theta + \phi) \sin(\theta + \phi) v'' + \\
 & + [EI_{z'} \sin^2(\theta + \phi) + EI_{y'} \cos^2(\theta + \phi)] w'' \}'' + m\ddot{w} + me\ddot{\phi} \cos \theta + \\
 & + 2m\Omega\beta_{pc}\dot{v} - \{ me[\Omega^2 x \sin(\theta + \phi) + 2\Omega\dot{v} \sin \theta] \}' = L_w - m\Omega^2 \beta_{pc} x
 \end{aligned} \tag{2.20.3}$$

$\delta\phi$ coefficient:

$$\begin{aligned}
 & -[Eak_A^2 (\theta + \phi)' \left(u' + \frac{v'^2}{2} + \frac{w'^2}{2} \right) + EB_1^* \theta'^2 \phi' - EB_2^* \theta' (v'' \cos \theta + \\
 & + w'' \sin \theta)]' - EAe_A \left(u' + \frac{v'^2}{2} + \frac{w'^2}{2} \right) (w'' \cos \theta - v'' \sin \theta) - (GJ\phi')' + \\
 & + [EC_1 \phi'' + EC_1^* (w'' \cos \theta - v'' \sin \theta)]'' + (EI_{z'} - EI_{y'}) [(w''^2 - v''^2) \cdot \\
 & \cdot \cos \theta \sin \theta + v'' w'' \cos(2\theta)] + mk_m^2 \ddot{\phi} + m\Omega^2 \phi (k_{m2}^2 - k_{m1}^2) \cos(2\theta) + \\
 & + me[\Omega^2 x (w' \cos \theta - v' \sin \theta) - (\ddot{v} - \Omega^2 v) \sin \theta + \ddot{w} \cos \theta] = \\
 & = M_\phi - m\Omega^2 (k_{m2}^2 - k_{m1}^2) \cos \theta \sin \theta - me\Omega^2 \beta_{pc} x \cos \theta
 \end{aligned} \tag{2.20.4}$$

where the tension, T is:

$$T \equiv V_x \equiv EA \left[u' + \frac{v'^2}{2} + \frac{w'^2}{2} + k_A^2 \theta' \phi' - e_A [v'' \cos(\theta + \phi) + w'' \sin(\theta + \phi)] \right] \tag{2.20.5}$$

The boundary conditions obtained from the strain and kinetic energies are given in Appendix A.

The equations of motion presented above are valid for a blade with variable stiffness and mass properties, chordwise offsets of the tension axis and mass centroid from the elastic axis, a small pre-cone angle, and variable built-in axial twist.

2.8 Simplified Equations of Motion

To simplify the equations of motion, linear expansion of $\cos(\theta + \phi)$ and $\sin(\theta + \phi)$ are used in the corresponding equations and boundary conditions, i.e.,

$$\begin{aligned}\cos(\theta + \phi) &= \cos \theta - \phi \sin \theta \\ \sin(\theta + \phi) &= \sin \theta + \phi \cos \theta\end{aligned}$$

The generalized aerodynamic force in the radial direction, L_u is neglected according to the ordering scheme, as it includes higher order terms, [10]. Moreover, the following simplifications are assumed in the blade model: no chordwise offsets between the elastic, mass, tension, and aerodynamic centres on the blade cross-section, the unloaded blade is assumed to be free of twist along the axial direction, and the pre-cone is assumed to be absent. Applying these simplifications, reduces Eqs. (2.20) to:

δu coefficient:

$$-T' - m(\Omega^2 x + 2\Omega \dot{v}) = 0 \quad (2.21.1)$$

δv coefficient:

$$\begin{aligned}-(Tv')' + \{(EI_{z'} - EI_{y'})[\frac{\sin(2\theta)}{2} + \phi \cos(2\theta)]w'' + \\ + [EI_{z'}(\cos^2 \theta - \phi \sin(2\theta)) + EI_{y'}(\sin^2 \theta + \phi \sin(2\theta))]v''\}'' + \\ + m(2\Omega \dot{u} + \ddot{v} - \Omega^2 v) = L_v\end{aligned} \quad (2.21.2)$$

δw coefficient:

$$\begin{aligned}-(Tw')' + \{(EI_{z'} - EI_{y'})[\frac{\sin(2\theta)}{2} + \phi \cos(2\theta)]v'' + \\ + [EI_{z'}(\sin^2 \theta + \phi \sin(2\theta)) + EI_{y'}(\cos^2 \theta - \\ - \phi \sin(2\theta))]w''\}'' + m\ddot{w} = L_w\end{aligned} \quad (2.21.3)$$

$\delta\phi$ coefficient:

$$\begin{aligned}
 & -(GJ\phi')' - (k_A^2 T\phi')' + (EI_z - EI_{y'})[(w'^2 - v'^2)\frac{\sin(2\theta)}{2} + \\
 & + v''w''\cos(2\theta)] + mk_m^2 \ddot{\phi} + m\Omega^2 \phi (k_{m2}^2 - k_{m1}^2)\cos(2\theta) = \\
 & = M_\phi - m\Omega^2 (k_{m2}^2 - k_{m1}^2)\frac{\sin(2\theta)}{2}
 \end{aligned} \tag{2.21.4}$$

where the tension is simplified to the following expression:

$$T \equiv EA(u' + \frac{v'^2}{2} + \frac{w'^2}{2}) \tag{2.21.5}$$

These equations contain the mass and structural properties of the blade cross-section such as the flapwise and chordwise bending stiffnesses $EI_{y'}$ and EI_z , respectively, the torsional rigidity GJ , the radial stiffness EA , and the polar radius of gyration k_A .

Since a detailed discussion of the significance of various terms in Eqs. (2.21) is available in reference [10], only a brief review of the more important terms in these equations is presented here. In the δu equation, the term $2m\Omega\dot{v}$ represents a centrifugal force due to lead-lag motion. When the tension, is combined with the nonlinear terms $(Tv')'$ and $(Tw')'$ in the lead-lag and flap equations, respectively, it results in Coriolis coupling. Also, the $2m\Omega\dot{u}$ term representing the Coriolis force due to radial velocity in the lead-lag equation produces nonlinearity when solving the δu equation for u and substituting $\dot{u}$ in the lead-lag equation term $2m\Omega\dot{u}$. The two inertia terms discussed above should be included into the system of equations since they represent centrifugal and Coriolis coupling between radial extension, flap and lead-lag displacements. The torsion inertia in the $\delta\phi$ equation is represented by $mk_m^2\ddot{\phi}$. The “tennis racket” terms

$m\Omega^2\phi(k_{m2}^2 - k_{m1}^2)\cos 2\theta$ and $m\Omega^2(k_{m2}^2 - k_{m1}^2)\frac{\sin 2\theta}{2}$ cause stiffening of the rotating blade.

Structural coupling terms are important for hingeless rotor stability analysis because they depend on several configuration parameters and, therefore, influence the stability of the rotor blade. The nonlinear bending–torsion term in the $\delta\phi$ equation is proportional to the difference between the lead-lag and flap bending stiffnesses, $(EI_{z'} - EI_{y'})$, and the products of the bending curvatures. Similar nonlinear bending-torsion and linear flap-lag structural coupling terms are present in the lead-lag and flap equations. The former terms couple flap, lead-lag bending and torsion. While the latter terms couple the lead-lag bending and flap bending at various pitch angles $\theta \neq 0$.

In general, as the blade pitch varies, the principal elastic axes of the outboard blade segment rotate through an angle θ while the inboard segment principal axes do not. This affects the flap and lead-lag structural coupling of the blade so that the degree of this coupling depends on the relative stiffness of the blade segments inboard and outboard of the pitch bearing. Although the stability of the blade is influenced by variations in these coupling effects, they are not easy to be modelled exactly without complicating the governing equations of motion of the blade. An approximate representation of these effects has been suggested by Hodges and Ormiston [3]. The approach assumes the average inclination of the principal elastic axes of a nonuniform rotor to be equal to some fraction of the inclination of the principal axes of a uniform single segment blade. As a result, the angle θ is replaced by $\mathcal{R}\theta$ in the structural terms in the equations. The

parameter, $\Re$, representing the degree of coupling, can take any arbitrary value between 0 and 1, defining a totally decoupled case versus a completely coupled one, respectively.

Next, the axial displacement u , can be eliminated as an independent variable, resulting in a system of three equations with three independent variables, v , w , and ϕ . To this end, one obtains u' from Eqs. (2.21.1) and (2.21.5). After integrating u' in x and differentiating with respect to time, the resulting expression for $\dot{u}$ is substituted into Eq. (2.21.2). The obtained three equations read as follows:

Lead-lag equation:

$$\begin{aligned}
 & [EI_{z'} - (EI_{z'} - EI_{y'}) \sin^2(\Re\theta)] v'''' + (EI_{z'} - EI_{y'}) \frac{\sin(2\Re\theta)}{2} w'''' + \\
 & + (EI_{z'} - EI_{y'}) \left[\cos(2\Re\theta) (\phi w'')'' - \sin(2\Re\theta) (\phi v'')'' \right] - \frac{m\Omega^2}{2} \left[v'(R^2 - x^2) \right]' - \\
 & - 2m\Omega \left(v' \int_x^R \dot{v} dx \right)' - 2m\Omega \int_0^x (v' \dot{v}' + w' \dot{w}') dx + m(\ddot{v} - \Omega^2 v) = L_v
 \end{aligned} \tag{2.22.1}$$

Flap equation:

$$\begin{aligned}
 & [EI_{y'} + (EI_{z'} - EI_{y'}) \sin^2(\Re\theta)] w'''' + (EI_{z'} - EI_{y'}) \frac{\sin(2\Re\theta)}{2} v'''' + \\
 & + (EI_{z'} - EI_{y'}) \left[\cos(2\Re\theta) (\phi v'')'' + \sin(2\Re\theta) (\phi w'')'' \right] - \frac{m\Omega^2}{2} \left[w'(R^2 - x^2) \right]' - \\
 & - 2m\Omega \left(w' \int_x^R \dot{v} dx \right)' + m\ddot{w} = L_w
 \end{aligned} \tag{2.22.2}$$

Torsion equation:

$$\begin{aligned}
 & -GJ\phi'' - \frac{m\Omega^2}{2} k_A^2 \left[\phi' (R^2 - x^2) \right]' + mk_m^2 \ddot{\phi} + m\Omega^2 (k_{m2}^2 - k_{m1}^2) \phi \cos(2\theta) + \\
 & + (EI_{z'} - EI_{y'}) \left[v'' w'' \cos(2\Re\theta) + (w''^2 - v''^2) \frac{\sin(2\Re\theta)}{2} \right] = \\
 & = M_\phi - m\Omega^2 (k_{m2}^2 - k_{m1}^2) \frac{\sin(2\theta)}{2}
 \end{aligned} \tag{2.22.3}$$

2.9 Aerodynamic Forces

As the initial phase of a feasibility study, the hovering flight conditions are considered in the present work. Based on Greenberg's extension of Theodorsen's theory [3], for a two-dimensional airfoil undergoing sinusoidal motion in pulsating incompressible flow, Hodges and Ormiston [3] have determined the corresponding aerodynamic loads for hover flight conditions to be:

$$\begin{aligned}
 L_v \cong \frac{\rho_\infty ac}{2} \{ & v_i^2 - \Omega^2 x^2 \frac{c_{d0}}{a} - \Omega x v_i (\theta + \phi) - [2\Omega x \frac{c_{d0}}{a} + (\theta + \phi) v_i] \dot{v} + \\
 & + [2v_i - \Omega x (\theta + \phi)] \dot{w} \}
 \end{aligned} \tag{2.23.1}$$

$$\begin{aligned}
 L_w \cong \frac{\rho_\infty ac}{2} \{ & -\Omega x v_i + \Omega^2 x^2 (\theta + \phi + \int_0^x v' w'' dx) - \Omega^2 x v (\beta_{pc} + w') + \\
 & + \Omega^2 \frac{xc}{2} (\beta_{pc} + w') + [2\Omega x (\theta + \phi) - v_i] \dot{v} - \Omega x \dot{w} + \frac{3}{4} c \Omega x \dot{\phi} - \frac{c}{4} \ddot{w} \}
 \end{aligned} \tag{2.23.2}$$

$$M_\phi = M \cong -\frac{\rho_\infty ac}{2} \left(\frac{c^2}{8} \Omega x \dot{\phi} \right) \tag{2.23.3}$$

where the following are defined: c_{d0} is a constant profile drag coefficient; ρ_∞ is air density; a is airfoil lift curve slope.

In Eqs. (2.23), v_i is the induced inflow velocity assumed steady and uniform along the blade. This velocity is taken to be equal to the value of nonuniform inflow given by the

blade element momentum theory at 75% of the blade length. The blade angle at this point is taken to be the blade collective pitch plus equilibrium elastic twist ϕ_0 at $x = 0.75R$.

Then:

$$v_i = \text{sgn}[\theta + \phi_0(0.75R)] \Omega R \frac{\pi \sigma}{8} \left(\sqrt{1 + \frac{12}{\pi \sigma} |\theta + \phi_0(0.75R)|} - 1 \right) \quad (2.24)$$

where $\sigma = bc/\pi R$ is the blade solidity; b is the number of blades; $\phi_0(0.75R)$ is the equilibrium elastic twist at 75% of the blade length.

2.10 Non-dimensional Blade Equations of Motion

Now, the aerodynamic loads given by Eqs. (2.23) can be substituted into the Eqs. (2.22). The resulting equations are non-dimensionalized by dividing the lead-lag and flap equations by $m\Omega^2 R$ and the torsion equation by $m\Omega^2 R^2$. The final blade equations of motion in non-dimensional form are:

Lead-lag equation:

$$\begin{aligned} & -(\bar{v}' \int_{\bar{x}}^1 (\bar{x} + 2\bar{v}) d\bar{x})' + \Lambda_2 \bar{v}'''' - (\Lambda_2 - \Lambda_1) \sin^2(\Re \theta) \bar{v}'''' - 2 \int_0^{\bar{x}} \bar{v}' \bar{v}' d\bar{x} - 2 \int_0^{\bar{x}} \bar{w}' \bar{w}' d\bar{x} + \\ & + (\Lambda_2 - \Lambda_1) \left[\frac{\sin(2\Re \theta)}{2} \bar{w}'''' - \sin(2\Re \theta) (\phi \bar{v}'')'' + \cos(2\Re \theta) (\phi \bar{w}'')'' \right] + \ddot{\bar{v}} - \bar{v} + \\ & + \frac{\gamma}{6} \{ \bar{x} \bar{v}_i \phi + [2 \frac{c_{d0}}{a} \bar{x} + (\theta + \phi) \bar{v}_i] \dot{\bar{v}} - [2 \bar{v}_i - \bar{x}(\theta + \phi)] \dot{\bar{w}} \} = \\ & = \frac{\gamma}{6} \{ \bar{v}_i^2 - \frac{c_{d0}}{a} \bar{x}^2 - \bar{x} \bar{v}_i \theta \} \end{aligned} \quad (2.25.1)$$

Flap equation:

$$\begin{aligned}
 & -(\bar{w}' \int_{\bar{x}}^1 (\bar{x} + 2\bar{v}) d\bar{x})' + \Lambda_1 \bar{w}''' + (\Lambda_2 - \Lambda_1) \sin^2(\Re \theta) \bar{w}''' + \ddot{\bar{w}} (1 + \frac{\gamma \bar{c}}{24}) + \\
 & + (\Lambda_2 - \Lambda_1) [\frac{\sin(2\Re \theta)}{2} \bar{v}''' + \sin(2\Re \theta) (\phi \bar{w}'')' + \cos(2\Re \theta) (\phi \bar{v}'')'] + \\
 & + \frac{\gamma}{6} \{-\bar{x}^2 (\phi + \int_0^{\bar{x}} \bar{v}' \bar{w}'' d\bar{x}) + \bar{x} \bar{v} \bar{w}' - \frac{\bar{c}}{2} \bar{x} \bar{w}' - [2\bar{x}(\theta + \phi) - \bar{v}_i] \dot{\bar{v}} + \bar{x} \dot{\bar{w}} - \frac{3}{4} \bar{c} \bar{x} \dot{\phi}\} = \\
 & = \frac{\gamma}{6} (-\bar{x} \bar{v}_i + \bar{x}^2 \theta)
 \end{aligned} \tag{2.25.2}$$

Torsion equation:

$$\begin{aligned}
 & -\frac{\mu^2 K}{2} \{(1 - \bar{x}^2) \phi'\}' - \kappa \phi'' + (\Lambda_2 - \Lambda_1) [\frac{\bar{w}''^2 - \bar{v}''^2}{2} \sin(2\Re \theta) + \bar{v}'' \bar{w}'' \cos(2\Re \theta)] + \\
 & + \mu^2 \ddot{\phi} + (\mu_2^2 - \mu_1^2) \phi \cos(2\theta) + \frac{\gamma \bar{c}^2}{48} \bar{x} \dot{\phi} = -(\mu_2^2 - \mu_1^2) \frac{\sin(2\theta)}{2}
 \end{aligned} \tag{2.25.3}$$

where the dimensionless parameters used are as follows:

$$\begin{aligned}
 K &= \frac{k_A^2}{k_m^2} & \bar{x} &= \frac{x}{R} & \bar{v} &= \frac{v}{R} \\
 \mu_1 &= \frac{k_{m1}}{R} & \Lambda_1 &= \frac{EI_{y'}}{m\Omega^2 R^4} & \bar{w} &= \frac{w}{R} \\
 \mu_2 &= \frac{k_{m2}}{R} & \Lambda_2 &= \frac{EI_{z'}}{m\Omega^2 R^4} & \bar{v}_i &= \frac{v_i}{\Omega R} \\
 \mu &= \frac{k_m}{R} & \kappa &= \frac{GJ}{m\Omega^2 R^4} & (') &= \frac{\partial}{\partial \bar{x}} () \\
 \bar{c} &= \frac{c}{R} & \gamma &= \frac{3\rho_\infty a c R}{m} & (\dot{}) &= \frac{\partial}{\partial \psi} = \frac{\partial}{\Omega \partial t} ()
 \end{aligned} \tag{2.26}$$

2.11 Solution Methodology

This section outlines the solution procedure to evaluate the stability of small perturbation motions about the equilibrium operating conditions [3].

To this end, one may start by expressing the dimensionless flap, lead-lag and torsional displacements in terms of a series of generalized coordinates and mode shape functions:

$$\begin{aligned}
\bar{v} &= \sum_{j=1}^N V_j(\psi) \Psi_j(\bar{x}) \\
\bar{w} &= \sum_{j=1}^N W_j(\psi) \Psi_j(\bar{x}) \\
\phi &= \sum_{j=1}^N \phi_j(\psi) \Theta_j(\bar{x}) \quad , \quad j = 1, \dots, N
\end{aligned} \tag{2.27}$$

where $\psi = \Omega t$ and $\bar{x} = x/R$; N is the number of modes; and $\Psi_j(\bar{x})$ and $\Theta_j(\bar{x})$ are the mode shapes for the bending and torsion of a uniform straight cantilever beam:

$$\begin{aligned}
\Psi_j(\bar{x}) &= \cosh(\beta_j \bar{x}) - \cos(\beta_j \bar{x}) - \alpha_j [\sinh(\beta_j \bar{x}) - \sin(\beta_j \bar{x})] \\
\Theta_j(\bar{x}) &= \sqrt{2} \sin(\gamma_j \bar{x})
\end{aligned} \tag{2.28}$$

The constants α_j , β_j , and γ_j are given in reference 19.

Substituting Eq. (2.27) into (2.25) and applying Galerkin's method to the resulting set of equations yields a set of $3N$ constant coefficient nonlinear, nonhomogeneous, ordinary differential equations of motion in terms of the generalized coordinates V_j , W_j , and ϕ_j :

Lead-lag equations:

$$\begin{aligned}
&\sum_{j=1}^N \left\{ V_j \{ D_{ij} + [\Lambda_2 - (\Lambda_2 - \Lambda_1) \sin^2(\Re \theta)] \beta_j^4 \delta_{ij} - \delta_{ij} \} + \right. \\
&+ W_j \frac{\sin(2\Re \theta)}{2} (\Lambda_2 - \Lambda_1) \beta_j^4 \delta_{ij} + 2 \sum_{k=1}^N (F_{ikj} - F_{jki}) V_k \dot{V}_j - \\
&- 2 \sum_{k=1}^N F_{jki} W_k \dot{W}_j + (\Lambda_2 - \Lambda_1) \sum_{k=1}^N K_{jki} \phi_j [W_k \cos(2\Re \theta) - V_k \sin(2\Re \theta)] + \\
&+ \delta_{ij} \ddot{V}_j + \frac{\gamma}{6} [\bar{v}_i I_{ij} \phi_j + \left(2 \frac{c_{d0}}{a} E_{ij} + \theta \bar{v}_i \delta_{ij} + \bar{v}_i \sum_{k=1}^N \phi_k H_{ijk} \right) \dot{V}_j - \\
&- \left(2 \bar{v}_i \delta_{ij} - \theta E_{ij} - \sum_{k=1}^N \phi_k G_{ijk} \right) \dot{W}_j] \} = \frac{\gamma}{6} [\bar{v}_i^2 A_i - \frac{c_{d0}}{a} C_i - \bar{v}_i \theta B_i], \\
&i = 1, \dots, N
\end{aligned} \tag{2.29.1}$$

Flap equations:

$$\begin{aligned}
 & \sum_{j=1}^N \left\langle V_j (\Lambda_2 - \Lambda_1) \frac{\sin(2\Re\theta)}{2} \beta_j^4 \delta_{ij} + W_j \{D_{ij} + [\Lambda_1 + (\Lambda_2 - \Lambda_1) \sin^2(\Re\theta)] \beta_j^4 \delta_{ij}\} + \right. \\
 & + (\Lambda_2 - \Lambda_1) \sum_{k=1}^N K_{jki} \phi_j [W_k \sin(2\Re\theta) + V_k \cos(2\Re\theta)] + 2 \sum_{k=1}^N F_{ikj} W_k \dot{V}_j + \\
 & + \delta_{ij} \left(1 + \frac{\gamma\bar{c}}{24} \right) \ddot{W}_j + \frac{\gamma}{6} [-J_{ij} \phi_j + \sum_{k=1}^N L_{ijk} V_j W_k - (2\theta E_{ij} + 2 \sum_{k=1}^N \phi_k G_{ijk} - \bar{v}_i \delta_{ij}) \dot{V}_j - \\
 & - \frac{\bar{c}}{2} O_{ij} W_j + E_{ij} \dot{W}_j - \frac{3\bar{c}}{4} I_{ij} \phi_j] \rangle = \frac{\gamma}{6} [-\bar{v}_i B_i + \theta C_i], \\
 & i = 1, \dots, N
 \end{aligned} \tag{2.29.2}$$

Torsion equations:

$$\begin{aligned}
 & \sum_{j=1}^N \left\langle \phi_j \left\{ \mu^2 K N_{ij} + [\kappa \gamma_j^2 + (\mu_2^2 - \mu_1^2) \cos(2\theta)] \delta_{ij} \right\} + (\Lambda_2 - \Lambda_1) \cdot \right. \\
 & \cdot \sum_{k=1}^N K_{ijk} [\cos(2\Re\theta) V_j W_k + \mu^2 \delta_{ij} \ddot{\phi}_j + \frac{\sin(2\Re\theta)}{2} (W_j W_k - V_j V_k)] + \\
 & + \frac{\gamma\bar{c}^2}{48} M_{ij} \dot{\phi}_j \rangle = -(\mu_2^2 - \mu_1^2) \frac{\sin(2\theta)}{\sqrt{2}\gamma_i}, \quad i = 1, \dots, N
 \end{aligned} \tag{2.29.3}$$

where,

$$\begin{aligned}
 A_i &= \int_0^1 \Psi_i d\bar{x} & J_{ij} &= \int_0^1 \bar{x}^2 \Psi_i \Theta_j d\bar{x} \\
 B_i &= \int_0^1 \bar{x} \Psi_i d\bar{x} & K_{ijk} &= \int_0^1 \Theta_i \Psi_j'' \Psi_k'' d\bar{x} \\
 C_i &= \int_0^1 \bar{x}^2 \Psi_i d\bar{x} & L_{ijk} &= \int_0^1 \bar{x} \Psi_i \Psi_j \Psi_k' d\bar{x} - \\
 & & & - \int_0^1 \bar{x}^2 \Psi_i \left(\int_0^{\bar{x}} \Psi_j'(\tilde{x}) \Psi_k''(\tilde{x}) d\tilde{x} \right) d\bar{x} \\
 D_{ij} &= \int_0^1 \frac{1 - \bar{x}^2}{2} \Psi_i' \Psi_j' d\bar{x} & M_{ij} &= \int_0^1 \bar{x} \Theta_i \Theta_j d\bar{x} \\
 E_{ij} &= \int_0^1 \bar{x} \Psi_i \Psi_j d\bar{x} & N_{ij} &= \int_0^1 \frac{1 - \bar{x}^2}{2} \Theta_i' \Theta_j' d\bar{x} \\
 F_{ijk} &= -\frac{1}{\beta_k^4} \int_0^1 \Psi_i' \Psi_j' \Psi_k''' d\bar{x}
 \end{aligned} \tag{2.30}$$

$$\begin{aligned}
G_{ijk} &= \int_0^1 \bar{x} \Psi_i \Psi_j \Theta_k d\bar{x} & O_{ij} &= \int_0^1 \bar{x} \Psi_i \Psi_j' d\bar{x} \\
H_{ijk} &= \int_0^1 \Psi_i \Psi_j \Theta_k d\bar{x} & \delta_{ij} &= \int_0^1 \Psi_i \Psi_j d\bar{x} = \int_0^1 \Theta_i \Theta_j d\bar{x} \\
I_{ij} &= \int_0^1 \bar{x} \Psi_i \Theta_j d\bar{x} & &= \begin{cases} 0; & i \neq j \\ 1; & i = j \end{cases}
\end{aligned}$$

The generalized coordinates are then decomposed into steady equilibrium quantities and small time dependent unsteady perturbation quantities as:

$$\begin{aligned}
V_j(\psi) &= V_{0j} + \Delta V_j(\psi) \\
W_j(\psi) &= W_{0j} + \Delta W_j(\psi) \\
\phi_j(\psi) &= \phi_{0j} + \Delta \phi_j(\psi)
\end{aligned} \tag{2.31}$$

Next, a set of $3N$ nonlinear algebraic equilibrium equations is obtained by substituting only the steady quantities into Eqs. (2.29). The resulting set of equations defines the trim conditions of the rotating blade, i.e., the equilibrium deflections V_{0j} , W_{0j} , and ϕ_{0j} :

Lead-lag equilibrium equations:

$$\begin{aligned}
&\sum_{j=1}^N \left\langle V_{0j} \{ D_{ij} + [\Lambda_2 - (\Lambda_2 - \Lambda_1) \sin^2(\Re \theta)] \beta_j^4 \delta_{ij} - \delta_{ij} \} + W_{0j} \frac{\sin(2\Re \theta)}{2} \right. \\
&\cdot (\Lambda_2 - \Lambda_1) \beta_j^4 \delta_{ij} + (\Lambda_2 - \Lambda_1) \sum_{k=1}^N K_{jki} \phi_{0j} [W_{0k} \cos(2\Re \theta) - V_{0k} \sin(2\Re \theta)] + \\
&\left. + \frac{\gamma}{6} [\bar{v}_i I_{ij} \phi_{0j}] \right\rangle = \frac{\gamma}{6} [\bar{v}_i^2 A_i - \frac{c_{d0}}{a} C_i - \bar{v}_i \theta B_i], \quad i = 1, \dots, N
\end{aligned} \tag{2.32.1}$$

Flap equilibrium equations:

$$\begin{aligned}
 & \sum_{j=1}^N \left\langle V_{0j} (\Lambda_2 - \Lambda_1) \frac{\sin(2\Re\theta)}{2} \beta_j^4 \delta_{ij} + W_{0j} \{ D_{ij} + [\Lambda_1 + (\Lambda_2 - \Lambda_1) \sin^2(\Re\theta)] \beta_j^4 \delta_{ij} \} + \right. \\
 & + (\Lambda_2 - \Lambda_1) \sum_{k=1}^N K_{jki} \phi_{0j} [W_{0k} \sin(2\Re\theta) + V_{0k} \cos(2\Re\theta)] + \frac{\gamma}{6} [-J_{ij} \phi_{0j} + \\
 & + \sum_{k=1}^N L_{ijk} V_{0j} W_{0k} - \frac{\bar{c}}{2} O_{ij} W_{0j}] \rangle = \frac{\gamma}{6} [-\bar{v}_i B_i + \theta C_i], \\
 & i = 1, \dots, N
 \end{aligned} \tag{2.32.2}$$

Torsion equilibrium equations:

$$\begin{aligned}
 & \sum_{j=1}^N \left\langle \phi_{0j} \left\{ \mu^2 K N_{ij} + [\kappa \gamma_j^2 + (\mu_2^2 - \mu_1^2) \cos(2\theta)] \delta_{ij} \right\} + \right. \\
 & + (\Lambda_2 - \Lambda_1) \sum_{k=1}^N K_{ijk} [\cos(2\Re\theta) V_{0j} W_{0k} + \frac{\sin(2\Re\theta)}{2} (W_{0j} W_{0k} - V_{0j} V_{0k})] \rangle = \\
 & = -(\mu_2^2 - \mu_1^2) \frac{\sin(2\theta)}{\sqrt{2} \gamma_i}, \quad i = 1, \dots, N
 \end{aligned} \tag{2.32.3}$$

Next, a second set of $3N$ equations can be obtained by substituting Eqs. (2.31) into the modal differential equations of motion (2.29), subtracting the equilibrium equations (2.32), and neglecting all nonlinear terms of the perturbation quantities. The resulting equations are constant coefficient, homogeneous, ordinary differential equations of perturbation that are linearized about the trim conditions. In matrix form these equations can be expressed as:

$$[M]\{\ddot{X}\} + [C]\{\dot{X}\} + [K]\{X\} = \{0\} \tag{2.33}$$

where the $3N \times 1$ perturbation vector is:

$$\{X\} = \begin{Bmatrix} \Delta V_1 \\ \vdots \\ \Delta V_N \\ \Delta W_1 \\ \vdots \\ \Delta W_N \\ \Delta \phi_N \\ \vdots \\ \Delta \phi_N \end{Bmatrix} \quad (2.34)$$

The coefficients $[M]$, $[C]$, and $[K]$ that are functions of the equilibrium solution, V_{0j} , W_{0j} , ϕ_{0j} , represent the mass, damping, and stiffness matrices defined as:

$$[M] = \begin{bmatrix} \delta_{ij} & 0 & 0 \\ 0 & \delta_{ij} \left(1 + \frac{\gamma \bar{c}}{24} \right) & 0 \\ 0 & 0 & \mu^2 \delta_{ij} \end{bmatrix} \quad (2.35.1)$$

$$[C] = 2 \begin{bmatrix} \sum_{k=1}^N (F_{ikj} - F_{jki}) V_{0k} & -\sum_{k=1}^N F_{jki} W_{0k} & 0 \\ \sum_{k=1}^N F_{ikj} W_{0k} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \quad (2.35.2)$$

$$+ \frac{\gamma}{6} \begin{bmatrix} (2 \frac{c_{d0}}{a} E_{ij} + \theta \bar{v}_i \delta_{ij} + \bar{v}_i \sum_{k=1}^N H_{ijk} \phi_{0k}) & (-2 \bar{v}_i \delta_{ij} + \theta E_{ij} + \sum_{k=1}^N G_{ijk} \phi_{0k}) & 0 \\ (-2 \theta E_{ij} + \bar{v}_i \delta_{ij} - 2 \sum_{k=1}^N G_{ijk} \phi_{0k}) & E_{ij} & -\frac{3 \bar{c}}{4} I_{ij} \\ 0 & 0 & \frac{\bar{c}^2}{8} M_{ij} \end{bmatrix}$$

$$[K] = \begin{bmatrix} K_{11}^s & K_{12}^s & K_{13}^s \\ K_{21}^s & K_{22}^s & K_{23}^s \\ K_{31}^s & K_{32}^s & K_{33}^s \end{bmatrix} + \frac{\gamma}{6} \begin{bmatrix} K_{11}^a & K_{12}^a & K_{13}^a \\ K_{21}^a & K_{22}^a & K_{23}^a \\ K_{31}^a & K_{32}^a & K_{33}^a \end{bmatrix} \quad (2.35.3)$$

where superscripts $()^s$ and $()^a$ denote the stiffness matrices due to the structural and aerodynamic contribution, respectively, such that:

$$\begin{aligned} (K_{11}^s)_{ij} &= D_{ij} + \left[\Lambda_2 - (\Lambda_2 - \Lambda_1) \sin^2(\Re \theta) \right] \beta_j^4 \delta_{ij} - \\ &\quad - (\Lambda_2 - \Lambda_1) \sum_{k=1}^N K_{kji} \phi_{0k} \sin(2\Re \theta) \\ (K_{12}^s)_{ij} &= (\Lambda_2 - \Lambda_1) \frac{\sin(2\Re \theta)}{2} \beta_j^4 \delta_{ij} + (\Lambda_2 - \Lambda_1) \sum_{k=1}^N K_{kji} \phi_{0k} \cos(2\Re \theta) \\ (K_{13}^s)_{ij} &= (\Lambda_2 - \Lambda_1) \sum_{k=1}^N K_{jki} [\cos(2\Re \theta) W_{0k} - \sin(2\Re \theta) V_{0k}] \\ (K_{21}^s)_{ij} &= (\Lambda_2 - \Lambda_1) \frac{\sin(2\Re \theta)}{2} \beta_j^4 \delta_{ij} + (\Lambda_2 - \Lambda_1) \sum_{k=1}^N K_{kji} \phi_{0k} \cos(2\Re \theta) \\ (K_{22}^s)_{ij} &= D_{ij} + \left[\Lambda_1 + (\Lambda_2 - \Lambda_1) \sin^2(\Re \theta) \right] \beta_j^4 \delta_{ij} + \\ &\quad + (\Lambda_2 - \Lambda_1) \sum_{k=1}^N K_{kji} \phi_{0k} \sin(2\Re \theta) \\ (K_{23}^s)_{ij} &= (\Lambda_2 - \Lambda_1) \sum_{k=1}^N K_{jki} [\cos(2\Re \theta) V_{0k} + \sin(2\Re \theta) W_{0k}] \\ (K_{31}^s)_{ij} &= (\Lambda_2 - \Lambda_1) \sum_{k=1}^N K_{ijk} [-\sin(2\Re \theta) V_{0k} + \cos(2\Re \theta) W_{0k}] \\ (K_{32}^s)_{ij} &= (\Lambda_2 - \Lambda_1) \sum_{k=1}^N K_{ijk} [\cos(2\Re \theta) V_{0k} + \sin(2\Re \theta) W_{0k}] \\ (K_{33}^s)_{ij} &= \mu^2 K N_{ij} + [\kappa \gamma_j^2 + (\mu_2^2 - \mu_1^2) \cos(2\theta)] \delta_{ij} \end{aligned} \quad (2.36)$$

$$(K_{11}^a)_{ij} = (K_{12}^a)_{ij} = (K_{31}^a)_{ij} = (K_{32}^a)_{ij} = (K_{33}^a)_{ij} = [0]$$

$$(K_{13}^a)_{ij} = \bar{v}_i I_{ij}$$

$$(K_{21}^a)_{ij} = \beta_{pc} E_{ij} + \sum_{k=1}^N L_{ijk} W_{0k}$$

$$(K_{22}^a)_{ij} = -\frac{\bar{c}}{2} O_{ij} + \sum_{k=1}^N L_{ikj} V_{0k}$$

$$(K_{23}^a)_{ij} = -J_{ij}$$

The mass matrix $[M]$ is a symmetric, $3N \times 3N$ matrix. It contains both the aerodynamic and inertial terms. The aerodynamic apparent mass term $\gamma \bar{c} \delta_{ij}/24$, usually has small order of magnitude and, therefore, is neglected in the present analysis. The damping matrix $[C]$ is an asymmetric, $3N \times 3N$ matrix that contains gyroscopic and aerodynamic damping terms. Finally, the matrix $[K]$ has stiffness contributions due to structural and aerodynamic terms.

In order to perform the stability analysis of the perturbation motion about the equilibrium, the eigenvalues of the $6N \times 6N$ matrix $[P]$ should be determined from the following equation:

$$\begin{Bmatrix} \dot{X} \\ \ddot{X} \end{Bmatrix} = \begin{bmatrix} [0] & [I] \\ -[M]^{-1}[K] & -[M]^{-1}[C] \end{bmatrix} \begin{Bmatrix} X \\ \dot{X} \end{Bmatrix} = [P] \begin{Bmatrix} X \\ \dot{X} \end{Bmatrix} \quad (2.37)$$

Since the main objective of this study is to investigate the stability of the blade at its dominant lower frequencies (the 1st lead-lag, 1st and 2nd flap, and 1st torsion frequencies), there is a number of non-rotating modes, N for which the eigenvalues converge. Hodges and Ormiston suggested $N = 5$ for a hingeless helicopter blade to obtain a convergence of lower eigenvalues [3].

The size of matrix $[P]$ can be reduced without significantly affecting the eigenvalues. To achieve this, one may apply the modal coordinates transformation as outlined in the

following. First, consider the free vibration of the hingeless helicopter blade about the equilibrium deflection state in the absence of aerodynamic loads. The corresponding equation of motion is:

$$[M]\{\ddot{X}\} + [G]\{\dot{X}\} + [K^s]\{X\} = \{0\} \quad (2.38)$$

where the symmetric matrix $[K^s]$ is structural stiffness matrix in Eq. (2.35.3). Also, $[G]$ is the damping matrix given in Eq. (2.35.2) with the aerodynamic terms set to zero. In order to avoid the complex eigenvalues and eigenvectors, Eq. (2.38) is written:

$$[M]\{\ddot{X}\} + [K^s]\{X\} = \{0\} \quad (2.39)$$

The matrix of eigenvectors, $[U]$ from Eq. (2.39) is orthogonal with respect to $[M]$:

$$[U]^T[M][U] = [M_m] \quad (2.40)$$

where $[M_m]$ is a diagonal mass matrix. The superscript $[]^T$ defines matrix transpose operation.

A principal coordinate transformation [20], can be applied to Eq. (2.33) by replacing $\{X\}$ by $[U]\{X\}$ and premultiplying Eq. (2.33) by $[U]^T$. In view of Eq. (2.40) one obtains:

$$\begin{Bmatrix} \dot{X} \\ \ddot{X} \end{Bmatrix} = \begin{bmatrix} [0] & [I] \\ -[M_m]^{-1}[U]^T[K][U] & -[M_m]^{-1}[U]^T[C][U] \end{bmatrix} \begin{Bmatrix} X \\ \dot{X} \end{Bmatrix} = [P^*] \begin{Bmatrix} X \\ \dot{X} \end{Bmatrix} \quad (2.41)$$

The eigenvectors in matrix $[U]$ corresponding to the higher frequencies can be removed, thus reducing the $6N \times 6N$ matrix $[P^*]$ to the size of $2Z \times 2Z$, where Z is the number of low frequency modes that are retained.

Chapter 3

“Smart” Spring Implementation

The equations of motion of a conventional hingeless blade were determined in Chapter 2. In this chapter a set of modified equations of motion will be developed for the case of the “smart” helicopter rotor. The modifications are based on the fact that the flap, lag, and torsional stiffnesses, the radii of gyration as well as mass and area of the blade cross-section, are now the functions of the radial coordinate, x . Next, the Galerkin’s method is applied and equilibrium deflections are found. Finally, the perturbation equations are developed for the “smart” blade in order to compare the stability of the dynamic behaviour of the controlled rotor to that of the uncontrolled one.

In general, it is proposed that the “smart” spring is implemented within the blade along the radial distance from x_1 to x_2 as shown in Fig. (3.1).

Figure 3.1 Side view of the “smart” blade

The design of this “smart” spring should not affect the cross-section symmetry about the η axis, as assumed in Chapter 2. The design of the “smart” blade is discussed further in Section 3.7.

3.1 Governing Differential Equations of Motion of the “Smart” Blade

The equations of motion of the blade with the “smart” spring are based on the general simplified Eqs. (2.21) which imply a uniform, untwisted, cantilever rotor blade with coincident mass, tension, elastic, and aerodynamic axes.

The configuration parameters are not represented as constant quantities any longer. An important feature of the mass and cross-sectional properties is that they now vary along the radial coordinate x , i.e.:

$$Z(x) = Z_0 + \Delta H(x) Z_{ss} \quad (3.1)$$

where the subscript $_0$ corresponds to the configuration parameters of the main structure only, while subscript $()_{ss}$ refers to the “smart” spring contribution. $\Delta H(x)$ in the above expression is the Heaviside function, so that, Fig. (3.2),:

$$\Delta H(x) = \begin{cases} 0, & \text{for } x < x_1 \text{ or } x > x_2 \\ 1, & \text{for } x_1 \leq x \leq x_2 \end{cases} \quad (3.2)$$

Figure 3.2 Configuration parameters variation along the spanwise direction of the blade.

$Z(x)$ in the Eq. (3.1) refers to any of the following necessary mass and cross-section parameters:

$$\begin{aligned}
m(x) &= m_0 + \Delta H(x)m_{ss} & m_{ss} &= \iint_{A_{ss}} \rho_{ss} d\eta d\zeta \\
A(x) &= A_0 + \Delta H(x)A_{ss} & A_{ss} &= \iint_{A_{ss}} d\eta d\zeta \\
I_{y'}(x) &= (I_{y'})_0 + \Delta H(x)(I_{y'})_{ss} & (I_{y'})_{ss} &= \iint_{A_{ss}} \zeta^2 d\eta d\zeta \\
I_{z'}(x) &= (I_{z'})_0 + \Delta H(x)(I_{z'})_{ss} & (I_{z'})_{ss} &= \iint_{A_{ss}} \eta^2 d\eta d\zeta \\
J(x) &= J_0 + \Delta H(x)J_{ss} & J_{ss} &= (I_{y'})_{ss} + (I_{z'})_{ss} \\
k_A^2(x) &= (k_A^2)_0 + \Delta H(x)(k_A^2)_{ss} & (k_A^2)_0 &= \frac{1}{A_0} \iint_{A_0} (\eta^2 + \zeta^2) d\eta d\zeta \\
& & & (3.3) \\
& & k_A^2 &= \frac{1}{A_0 + A_{ss}} \iint_{A_0 + A_{ss}} (\eta^2 + \zeta^2) d\eta d\zeta \\
k_{m1}^2(x) &= (k_{m1}^2)_0 + \Delta H(x)(k_{m1}^2)_{ss} & (k_{m1}^2)_0 &= \frac{1}{m_0} \iint_{A_0} \rho \zeta^2 d\eta d\zeta \\
& & k_{m1}^2 &= \frac{1}{m_0 + m_{ss}} \iint_{A_0 + A_{ss}} \rho \zeta^2 d\eta d\zeta \\
k_{m2}^2(x) &= (k_{m2}^2)_0 + \Delta H(x)(k_{m2}^2)_{ss} & (k_{m2}^2)_0 &= \frac{1}{m_0} \iint_{A_0} \rho \eta^2 d\eta d\zeta \\
& & k_{m2}^2 &= \frac{1}{m_0 + m_{ss}} \iint_{A_0 + A_{ss}} \rho \eta^2 d\eta d\zeta \\
(k_m^2(x))_{ss} &= (k_{m1}^2)_{ss} + (k_{m2}^2)_{ss}
\end{aligned}$$

Substituting the above into Eqs. (2.21), one may obtain equations of motion of the “smart” blade:

δu equation:

$$-T' - m_0(\Omega^2 x + 2\Omega \dot{v}) - \Delta H(x)m_{ss}(\Omega^2 x + 2\Omega \dot{v}) = 0 \quad (3.4.1)$$

δv equation:

$$\begin{aligned}
 & -(Tv')' + \{((EI_{z'})_0 - (EI_{y'})_0) \left[\frac{\sin(2\Re\theta)}{2} + \phi \cos(2\Re\theta) \right] w'' + \\
 & + \Delta H(x)((EI_{z'})_{ss} - (EI_{y'})_{ss}) \left[\frac{\sin(2\Re\theta)}{2} + \phi \cos(2\Re\theta) \right] w'' + \\
 & + [(EI_{z'})_0 (\cos^2(\Re\theta) - \phi \sin(2\Re\theta)) + (EI_{y'})_0 (\sin^2(\Re\theta) + \\
 & + \phi \sin(2\Re\theta))] v'' + \Delta H(x)[(EI_{z'})_{ss} (\cos^2(\Re\theta) - \phi \sin(2\Re\theta)) + \\
 & + (EI_{y'})_{ss} (\sin^2(\Re\theta) + \phi \sin(2\Re\theta))] v'' \}'' + (m_0 + \Delta H(x)m_{ss}) 2\Omega \dot{u} + \\
 & + (m_0 + \Delta H(x)m_{ss})(\ddot{v} - \Omega^2 v) = L_v
 \end{aligned} \tag{3.4.2}$$

δw equation:

$$\begin{aligned}
 & -(Tw')' + \{((EI_{z'})_0 - (EI_{y'})_0) \left[\frac{\sin(2\Re\theta)}{2} + \phi \cos(2\Re\theta) \right] v'' + \\
 & + \Delta H(x)((EI_{z'})_{ss} - (EI_{y'})_{ss}) \left[\frac{\sin(2\Re\theta)}{2} + \phi \cos(2\Re\theta) \right] v'' + \\
 & + [(EI_{z'})_0 (\sin^2(\Re\theta) + \phi \sin(2\Re\theta)) + (EI_{y'})_0 (\cos^2(\Re\theta) - \\
 & - \phi \sin(2\Re\theta))] w'' + \Delta H(x)[(EI_{z'})_{ss} (\sin^2(\Re\theta) + \phi \sin(2\Re\theta)) + \\
 & + (EI_{y'})_{ss} (\cos^2(\Re\theta) - \phi \sin(2\Re\theta))] w'' \}'' + (m_0 + \Delta H(x)m_{ss}) \ddot{w} = L_w
 \end{aligned} \tag{3.4.3}$$

$\delta \phi$ equation:

$$\begin{aligned}
 & -\{(GJ)_0 \phi' + \Delta H(x)(GJ)_{ss} \phi'\}' - \left\{ \left((k_A^2)_0 + \Delta H(x)(k_A^2)_{ss} \right) T \phi' \right\}' + \\
 & + \left((EI_{z'})_0 - (EI_{y'})_0 \right) [(w''^2 - v''^2) \frac{\sin(2\Re\theta)}{2} + v'' w'' \cos(2\Re\theta)] + \\
 & + \Delta H(x) \left((EI_{z'})_{ss} - (EI_{y'})_{ss} \right) [(w''^2 - v''^2) \frac{\sin(2\Re\theta)}{2} + v'' w'' \cos(2\Re\theta)] + \\
 & + (m_0 + \Delta H(x)m_{ss}) \left((k_m^2)_0 + \Delta H(x)(k_m^2)_{ss} \right) \ddot{\phi} + (m_0 + \Delta H(x)m_{ss}) \cdot \\
 & \cdot \Omega^2 \phi \left[(k_{m2}^2)_0 - (k_{m1}^2)_0 + \Delta H(x) \left((k_{m2}^2)_{ss} - (k_{m1}^2)_{ss} \right) \right] \cos(2\theta) = \\
 & = M_\phi - (m_0 + \Delta H(x)m_{ss}) \Omega^2 [(k_{m2}^2)_0 - (k_{m1}^2)_0 + \\
 & + \Delta H(x) \{ (k_{m2}^2)_{ss} - (k_{m1}^2)_{ss} \}] \frac{\sin(2\theta)}{2}
 \end{aligned} \tag{3.4.4}$$

where

$$T \equiv ((EA)_0 + \Delta H(x)(EA)_{ss}) \left(u' + \frac{v'^2}{2} + \frac{w'^2}{2} \right) \quad (3.4.5)$$

These equations have been derived similar what was done in reference 26. The obtained system of four equations and four unknowns, u , v , w , and ϕ may then be reduced resulting in a system of three equations with three unknown variables, v , w , and ϕ . This simplification can be done by applying the following procedure outlined in [10]:

- The δu equation is integrated with respect to x to express the tension, T as:

$$\begin{aligned} T(x) = & \frac{m_0 \Omega^2}{2} (R^2 - x^2) + m_{ss} \Omega^2 \int_x^R \Delta H(x) x dx + \\ & + 2m_0 \Omega \int_x^R \dot{v} dx + 2m_{ss} \Omega \int_x^R \Delta H(x) \dot{v} dx \end{aligned} \quad (3.5.1)$$

The tension is then substituted into δv and δw equations, Eqs. (3.4.2) and (3.4.3).

When substituting into $\delta \phi$ equation, Eq. (3.4.4), the third and the fourth terms in Eq. (3.5.1) are neglected, as they result in higher order terms, $O(\epsilon^3)$.

- Next, from the Eqs. (3.4.5) and (3.5.1) one has:

$$\begin{aligned} u' = & \frac{m_0 \Omega^2 R^2}{2EA(x)} \left(1 - \frac{x^2}{R^2} \right) + \frac{m_{ss} \Omega^2}{EA(x)} \int_x^R \Delta H(x) x dx + \\ & + \frac{2m_0 \Omega^2 R^2}{EA(x)} \int_x^R \frac{\dot{v}}{\Omega R^2} dx + \frac{2m_{ss} \Omega^2 R^2}{EA(x)} \int_x^R \frac{\Delta H(x) \dot{v}}{\Omega R^2} dx - \frac{v'^2}{2} - \frac{w'^2}{2} \end{aligned} \quad (3.5.2)$$

which after integrating with respect to x and differentiating with respect to time, yields:

$$\begin{aligned} \dot{u}(x) = & \int_0^x \frac{2m_0 \Omega^2 R^2}{(EA)_0 + \Delta H(x)(EA)_{ss}} \int_x^R \frac{\ddot{v}}{\Omega R^2} dx dx + \\ & + \int_0^x \frac{2m_{ss} \Omega^2 R^2}{(EA)_0 + \Delta H(x)(EA)_{ss}} \int_x^R \frac{\Delta H(x) \ddot{v}}{\Omega R^2} dx dx - \int_0^x (\dot{v}'v' + \dot{w}'w') dx \end{aligned} \quad (3.5.3)$$

The first and the second terms in the above expression can be neglected as higher order terms. Then $\dot{u}(x)$ is substituted into the $2(m_0 + \Delta H(x)m_{ss})\Omega\dot{u}$ Coriolis term in the lead-lag equation, Eq. (3.4.2).

As a result, the governing nonlinear integro-partial differential equations for the blade motion with the embedded “smart” spring follow as:

δv equation:

$$\begin{aligned}
 & -\frac{m_0\Omega^2}{2}\left[(R^2-x^2)v'\right]' - m_{ss}\Omega^2\left[v'\int_x^R\Delta H(x)xdx\right]' - 2m_0\Omega\left[v'\int_x^R\dot{v}dx\right]' - \\
 & -2m_{ss}\Omega\left[v'\int_x^R\Delta H(x)\dot{v}dx\right]' + \left\{((EI_{z'})_0 - (EI_{y'})_0)\left[\frac{\sin(2\Re\theta)}{2} + \phi\cos(2\Re\theta)\right]w'' + \right. \\
 & + \Delta H(x)((EI_{z'})_{ss} - (EI_{y'})_{ss})\left[\frac{\sin(2\Re\theta)}{2} + \phi\cos(2\Re\theta)\right]w'' + [(EI_{z'})_0(\cos^2(\Re\theta) - \\
 & - \phi\sin(2\Re\theta)) + (EI_{y'})_0(\sin^2(\Re\theta) + \phi\sin(2\Re\theta))]v'' + \Delta H(x)[(EI_{z'})_{ss}(\cos^2(\Re\theta) - \\
 & - \phi\sin(2\Re\theta)) + (EI_{y'})_{ss}(\sin^2(\Re\theta) + \phi\sin(2\Re\theta))]v''\}'' - 2\Omega(m_0 + \Delta H(x)m_{ss}) \cdot \\
 & \cdot \int_0^x (\dot{v}'v' + \dot{w}'w')dx + (m_0 + \Delta H(x)m_{ss})(\ddot{v} - \Omega^2v) = L_v
 \end{aligned} \tag{3.6.1}$$

δw equation:

$$\begin{aligned}
 & -\frac{m_0\Omega^2}{2}\left[(R^2-x^2)w'\right]' - m_{ss}\Omega^2\left[w'\int_x^R\Delta H(x)xdx\right]' - 2m_0\Omega\left[w'\int_x^R\dot{v}dx\right]' - \\
 & -2m_{ss}\Omega\left[w'\int_x^R\Delta H(x)\dot{v}dx\right]' + \left\{((EI_{z'})_0 - (EI_{y'})_0)\left[\frac{\sin(2\Re\theta)}{2} + \phi\cos(2\Re\theta)\right]v'' + \right. \\
 & + \Delta H(x)((EI_{z'})_{ss} - (EI_{y'})_{ss})\left[\frac{\sin(2\Re\theta)}{2} + \phi\cos(2\Re\theta)\right]v'' + [(EI_{z'})_0(\sin^2(\Re\theta) + \\
 & + \phi\sin(2\Re\theta)) + (EI_{y'})_0(\cos^2(\Re\theta) - \phi\sin(2\Re\theta))]w'' + \Delta H(x)[(EI_{z'})_{ss}(\sin^2(\Re\theta) + \\
 & + \phi\sin(2\Re\theta)) + (EI_{y'})_{ss}(\cos^2(\Re\theta) - \phi\sin(2\Re\theta))]w''\}'' + (m_0 + \Delta H(x)m_{ss})\ddot{w} = L_w
 \end{aligned} \tag{3.6.2}$$

$\delta\phi$ equation:

$$\begin{aligned}
& -\{(GJ)_0\phi' + \Delta H(x)(GJ)_{ss}\phi'\}' - \left\{ \left((k_A^2)_0 + \Delta H(x)(k_A^2)_{ss} \right) \phi' \frac{m_0 \Omega^2}{2} (R^2 - x^2) \right\}' + \\
& + \left\{ \left((k_A^2)_0 + \Delta H(x)(k_A^2)_{ss} \right) \phi' m_{ss} \Omega^2 \int_x^R \Delta H(x) x dx \right\}' + \left((EI_{z'})_0 - (EI_{y'})_0 \right) [(w''^2 - \\
& - v''^2) \frac{\sin(2\Re\theta)}{2} + v'' w'' \cos(2\Re\theta)] + \Delta H(x) \left((EI_{z'})_{ss} - (EI_{y'})_{ss} \right) [(w''^2 - v''^2) \cdot \\
& \cdot \frac{\sin(2\Re\theta)}{2} + v'' w'' \cos(2\Re\theta)] + (m_0 + \Delta H(x)m_{ss}) \left((k_m^2)_0 + \Delta H(x)(k_m^2)_{ss} \right) \ddot{\phi} + \\
& + (m_0 + \Delta H(x)m_{ss}) \Omega^2 \phi \left[(k_{m2}^2)_0 - (k_{m1}^2)_0 + \Delta H(x) \left((k_{m2}^2)_{ss} - (k_{m1}^2)_{ss} \right) \right] \cos(2\theta) \\
& = M_\phi - (m_0 + \Delta H(x)m_{ss}) \Omega^2 [(k_{m2}^2)_0 - (k_{m1}^2)_0 + \Delta H(x) \{ (k_{m2}^2)_{ss} - \\
& - (k_{m1}^2)_{ss} \}] \frac{\sin(2\theta)}{2}
\end{aligned} \tag{3.6.3}$$

3.2 Non-dimensional Form of the “Smart” Blade Equations of Motion

As in the previous chapter, non-conservative aerodynamic loads defined by Eqs. (2.23) are substituted into Eqs. (3.6). Similar to Section 2.10, the resulting equations are non-dimensionalized, such that:

Lead-lag equation:

$$\begin{aligned}
& -\frac{1}{2} \left((1 - \bar{x}^2) \bar{v} \right)' - \bar{m}_{ss} \left[\bar{v}' \int_{\bar{x}}^1 \Delta H(\bar{x}) \bar{x} d\bar{x} \right]' - 2 \left[\bar{v}' \int_{\bar{x}}^1 \dot{\bar{v}} d\bar{x} \right]' - 2 \bar{m}_{ss} \left[\bar{v}' \int_{\bar{x}}^1 \Delta H(\bar{x}) \dot{\bar{v}} d\bar{x} \right]' + \\
& + \left(\Lambda_{02} - (\Lambda_{02} - \Lambda_{01}) \sin^2(\Re\theta) \right) \bar{v}'''' + \left(\Delta H(\bar{x}) (\Lambda_{ss2} - (\Lambda_{ss2} - \Lambda_{ss1}) \sin^2(\Re\theta)) \right) \bar{v}'''' + \\
& + (\Lambda_{02} - \Lambda_{01}) \frac{\sin(2\Re\theta)}{2} \bar{w}'''' + (\Delta H(\bar{x}) (\Lambda_{ss2} - \Lambda_{ss1}) \frac{\sin(2\Re\theta)}{2} \bar{w}'''' + \\
& + (\Lambda_{02} - \Lambda_{01}) [\cos(2\Re\theta) (\phi \bar{w}'')'' - \sin(2\Re\theta) (\phi \bar{v}'')''] + (\Delta H(\bar{x}) (\Lambda_{ss2} - \Lambda_{ss1}) \cdot \\
& \cdot \cos(2\Re\theta) \phi \bar{w}'')'' - (\Delta H(\bar{x}) (\Lambda_{ss2} - \Lambda_{ss1}) \sin(2\Re\theta) \phi \bar{v}'')'' - 2(1 + \Delta H(\bar{x}) \bar{m}_{ss}) \cdot \\
& \cdot \int_0^{\bar{x}} (\bar{v}' \dot{\bar{v}} + \bar{w}' \dot{\bar{w}}) d\bar{x} + (1 + \Delta H(\bar{x}) \bar{m}_{ss}) (\ddot{\bar{v}} - \bar{v}) + \frac{\gamma}{6} \{ \bar{x} \bar{v}_i \phi + [2 \frac{c_{d0}}{a} \bar{x} + \\
& + (\theta + \phi) \bar{v}_i] \dot{\bar{v}} - [2 \bar{v}_i - \bar{x}(\theta + \phi)] \dot{\bar{w}} \} = \frac{\gamma}{6} (\bar{v}_i^2 - \frac{c_{d0}}{a} \bar{x}^2 - \bar{x} \bar{v}_i \theta)
\end{aligned} \tag{3.7.1}$$

Flap equation:

$$\begin{aligned}
& -(\bar{w}' \frac{1-\bar{x}^2}{2})' - \bar{m}_{ss} \left[\bar{w}' \int_{\bar{x}}^1 \Delta H(\bar{x}) \bar{x} d\bar{x} \right]' - 2 \left[\bar{w}' \int_{\bar{x}}^1 \dot{\bar{v}} d\bar{x} \right]' - 2 \bar{m}_{ss} \left[\bar{w}' \int_{\bar{x}}^1 \Delta H(\bar{x}) \dot{\bar{v}} d\bar{x} \right]' + \\
& (\Lambda_{01} + (\Lambda_{02} - \Lambda_{01}) \sin^2(\Re \theta)) \bar{w}'''' + (\Delta H(\bar{x}) (\Lambda_{ss1} + (\Lambda_{ss2} - \Lambda_{ss1}) \sin^2(\Re \theta)) \bar{w}'')'' + \\
& + (\Lambda_{02} - \Lambda_{01}) \frac{\sin(2\Re \theta)}{2} \bar{v}'''' + (\Delta H(\bar{x}) (\Lambda_{ss2} - \Lambda_{ss1}) \frac{\sin(2\Re \theta)}{2} \bar{v}'')'' + \\
& + (\Lambda_{02} - \Lambda_{01}) \sin(2\Re \theta) (\phi \bar{w}'')'' + (\Delta H(\bar{x}) (\Lambda_{ss2} - \Lambda_{ss1}) \sin(2\Re \theta) \phi \bar{w}'')'' + \\
& + (\Lambda_{02} - \Lambda_{01}) \cos(2\Re \theta) (\phi \bar{v}'')'' + (\Delta H(\bar{x}) (\Lambda_{ss2} - \Lambda_{ss1}) \cos(2\Re \theta) \phi \bar{v}'')'' + \\
& (1 + \Delta H(\bar{x}) \bar{m}_{ss}) \ddot{\bar{w}} + \frac{\gamma \bar{c}}{24} \ddot{\bar{w}} + \frac{\gamma}{6} \left\{ -\bar{x}^2 (\phi + \int_0^{\bar{x}} \bar{v}' \bar{w}'' d\bar{x}) + \bar{x} \bar{v} \bar{w}' - \right. \\
& \left. - \frac{\bar{c}}{2} \bar{x} \bar{w}' - [2\bar{x}(\theta + \phi) - \bar{v}_i] \dot{\bar{v}} + \bar{x} \dot{\bar{w}} - \frac{3}{4} \bar{c} \bar{x} \dot{\phi} \right\} = \frac{\gamma}{6} (-\bar{x} \bar{v}_i + \bar{x}^2 \theta)
\end{aligned} \tag{3.7.2}$$

Torsion equation:

$$\begin{aligned}
& -\kappa_0 \phi'' - (\Delta H(\bar{x}) \kappa_{ss} \phi')' - \left[\phi' (\mu_0^2 K_0 + \Delta H(\bar{x}) \mu_{ss}^2 K_{ss}) \int_{\bar{x}}^1 (1 + \Delta H(\bar{x}) \bar{m}_{ss}) \bar{x} d\bar{x} \right]' + \\
& + (\Lambda_{02} - \Lambda_{01} + \Delta H(\bar{x}) (\Lambda_{ss2} - \Lambda_{ss1})) [0.5(\bar{w}''^2 - \bar{v}''^2) \sin(2\Re \theta) + \bar{v}'' \bar{w}'' \cos(2\Re \theta)] + \\
& + (1 + \Delta H(\bar{x}) \bar{m}_{ss}) (\mu_0^2 + \Delta H(\bar{x}) \mu_{ss}^2) \ddot{\phi} + (1 + \Delta H(\bar{x}) \bar{m}_{ss}) (\mu_{02}^2 - \mu_{01}^2 + \\
& + \Delta H(\bar{x}) (\mu_{ss2}^2 - \mu_{ss1}^2)) \phi \cos(2\theta) + \frac{\gamma \bar{c}^2}{48} \bar{x} \dot{\phi} = -(1 + \Delta H(\bar{x}) \bar{m}_{ss}) (\mu_{02}^2 - \mu_{01}^2 + \\
& + \Delta H(\bar{x}) (\mu_{ss2}^2 - \mu_{ss1}^2)) \frac{\sin(2\theta)}{2}
\end{aligned} \tag{3.7.3}$$

where in addition to the parameters specified in Section 2.10 the following non-dimensional parameters are introduced in Eqs. (3.7):

$$\begin{aligned}
K_0 &= \frac{(k_A^2)_0}{(k_m^2)_0} & K_{ss} &= \frac{(k_A^2)_{ss}}{(k_m^2)_{ss}} \\
\mu_{01} &= \frac{(k_{m1})_0}{R} & \Lambda_{01} &= \frac{(EI_{y'})_0}{m_0 \Omega^2 R^4} \\
\mu_{02} &= \frac{(k_{m2})_0}{R} & \Lambda_{02} &= \frac{(EI_{z'})_0}{m_0 \Omega^2 R^4} \\
\mu_0 &= \frac{(k_m)_0}{R} & \mu_{ss} &= \frac{(k_m)_{ss}}{R} \\
\gamma &= \frac{3\rho_\infty acR}{m_0} & \bar{m}_{ss} &= \frac{m_{ss}}{m_0} \\
\mu_{ss1} &= \frac{(k_{m1})_{ss}}{R} & \Lambda_{ss1} &= \frac{(EI_{y'})_{ss}}{m_0 \Omega^2 R^4} \\
\mu_{ss2} &= \frac{(k_{m2})_{ss}}{R} & \Lambda_{ss2} &= \frac{(EI_{z'})_{ss}}{m_0 \Omega^2 R^4} \\
\kappa_0 &= \frac{(GJ)_0}{m_0 \Omega^2 R^4} & \kappa_{ss} &= \frac{(GJ)_{ss}}{m_0 \Omega^2 R^4}
\end{aligned} \tag{3.8}$$

3.3 Solution Methodology

The nonlinear, integro-partial differential Eqs. (3.7) which describe the motion of the “smart” blade, may be reduced to ordinary nonlinear differential equations by means of Galerkin’s method. The resulting nonlinear, ordinary differential equations in terms of generalized coordinates are obtained as:

Lead-lag equations:

$$\begin{aligned}
& \sum_{j=1}^N \langle V_j \{ D_{ij} + \bar{m}_{ss} D_{xij} + [\Lambda_{02} - (\Lambda_{02} - \Lambda_{01}) \sin^2(\Re \theta)] \beta_j^4 \delta_{ij} - (\delta_{ij} + \bar{m}_{ss} \delta_{xij1}) + \\
& + [\Lambda_{ss2} - (\Lambda_{ss2} - \Lambda_{ss1}) \sin^2(\Re \theta)] \Delta_{xij1} \} + W_j \frac{\sin(2\Re \theta)}{2} \{ (\Lambda_{02} - \Lambda_{01}) \beta_j^4 \delta_{ij} + \\
& + (\Lambda_{ss2} - \Lambda_{ss1}) \Delta_{xij1} \} + (\Lambda_{02} - \Lambda_{01}) \sum_{k=1}^N K_{jki} \phi_j [W_k \cos(2\Re \theta) - V_k \sin(2\Re \theta)] + \\
& + (\Lambda_{ss2} - \Lambda_{ss1}) \sum_{k=1}^N K_{xjki} \phi_j [W_k \cos(2\Re \theta) - V_k \sin(2\Re \theta)] + \\
& + 2 \sum_{k=1}^N (F_{ikj} - F_{jki} + \bar{m}_{ss} (F_{xikj} - F_{xjki})) V_k \dot{V}_j - \\
& - 2 \sum_{k=1}^N (F_{jki} + \bar{m}_{ss} F_{xjki}) W_k \dot{W}_j + (\delta_{ij} + \bar{m}_{ss} \delta_{xij1}) \ddot{V}_j + \frac{\gamma}{6} [\bar{v}_i I_{ij} \phi_j + \\
& + \left(2 \frac{c_{d0}}{a} E_{ij} + \theta \bar{v}_i \delta_{ij} + \bar{v}_i \sum_{k=1}^N \phi_k H_{ijk} \right) \dot{V}_j - \left(2 \bar{v}_i \delta_{ij} - \theta E_{ij} - \sum_{k=1}^N \phi_k G_{ijk} \right) \dot{W}_j] \rangle = \\
& = \frac{\gamma}{6} [\bar{v}_i^2 A_i - \frac{c_{d0}}{a} C_i - \bar{v}_i \theta B_i], \quad i=1, \dots, N
\end{aligned} \tag{3.9.1}$$

Flap equations:

$$\begin{aligned}
& \sum_{j=1}^N \langle V_j \frac{\sin(2\Re \theta)}{2} [(\Lambda_{02} - \Lambda_{01}) \beta_j^4 \delta_{ij} + (\Lambda_{ss2} - \Lambda_{ss1}) \Delta_{xij1}] + W_j \{ D_{ij} + \bar{m}_{ss} D_{xij} + \\
& + [\Lambda_{01} + (\Lambda_{02} - \Lambda_{01}) \sin^2(\Re \theta)] \beta_j^4 \delta_{ij} + [\Lambda_{ss1} + (\Lambda_{ss2} - \Lambda_{ss1}) \sin^2(\Re \theta)] \Delta_{xij1} \} + \\
& + (\Lambda_{02} - \Lambda_{01}) \sum_{k=1}^N K_{jki} \phi_j [V_k \cos(2\Re \theta) + W_k \sin(2\Re \theta)] + \\
& + (\Lambda_{ss2} - \Lambda_{ss1}) \sum_{k=1}^N K_{xjki} \phi_j [V_k \cos(2\Re \theta) + W_k \sin(2\Re \theta)] + \\
& + 2 \sum_{k=1}^N (F_{ikj} + \bar{m}_{ss} F_{xikj}) W_k \dot{V}_j + \ddot{W}_j (\delta_{ij} + \bar{m}_{ss} \delta_{xij1}) + \frac{\gamma \bar{c}}{24} \ddot{W}_j \delta_{ij} + \frac{\gamma}{6} [-J_{ij} \phi_j + \\
& + \sum_{k=1}^N L_{ijk} V_j W_k - \frac{\bar{c}}{2} O_{ij} W_j - (2\theta E_{ij} + 2 \sum_{k=1}^N \phi_k G_{ijk} - \bar{v}_i \delta_{ij}) \dot{V}_j + E_{ij} \dot{W}_j - \\
& - \frac{3\bar{c}}{4} I_{ij} \dot{\phi}_j] \rangle = \frac{\gamma}{6} [-\bar{v}_i B_i + \theta C_i], \quad i=1, \dots, N
\end{aligned} \tag{3.9.2}$$

Torsion equations:

$$\begin{aligned}
& \sum_{j=1}^N \left\langle \phi_j [\mu_0^2 K_0 N_{ij} + \mu_0^2 \bar{m}_{ss} K P_{xij} + \mu_{ss}^2 K_{ss} Q_{xij} + \mu_{ss}^2 \bar{m}_{ss} K_{ss} R_{xij} + \right. \\
& + [\kappa_0 \gamma_j^2 + (\mu_{02}^2 - \mu_{01}^2) \cos(2\theta)] \delta_{ij} + \kappa_{ss} \gamma_j^2 \delta_{xij2} + (\mu_{ss2}^2 - \mu_{ss1}^2 + \\
& + \bar{m}_{ss} (\mu_{02}^2 - \mu_{01}^2) + \bar{m}_{ss} (\mu_{ss2}^2 - \mu_{ss1}^2)) \cos(2\theta) \delta_{xij2}] + \\
& + (\Lambda_{02} - \Lambda_{01}) \sum_{k=1}^N K_{ijk} [\cos(2\Re\theta) V_j W_k + \frac{\sin(2\Re\theta)}{2} (W_j W_k - V_j V_k)] + \\
& + (\Lambda_{ss2} - \Lambda_{ss1}) \sum_{k=1}^N K_{xijk} [\cos(2\Re\theta) V_j W_k + \frac{\sin(2\Re\theta)}{2} (W_j W_k - V_j V_k)] + \\
& + \frac{\gamma_c^2}{48} M_{ij} \ddot{\phi}_j + (\mu_0^2 \delta_{ij} + \mu_{ss}^2 \delta_{xij2} + \bar{m}_{ss} \mu_0^2 \delta_{xij2} + \bar{m}_{ss} \mu_{ss}^2 \delta_{xij2}) \ddot{\phi}_j \rangle = \\
& = -S_{xi} (\mu_{ss2}^2 - \mu_{ss1}^2 + \bar{m}_{ss} (\mu_{02}^2 - \mu_{01}^2) + \bar{m}_{ss} (\mu_{ss2}^2 - \mu_{ss1}^2)) \frac{\sin(2\theta)}{2} - \\
& - (\mu_{02}^2 - \mu_{01}^2) \frac{\sin(2\theta)}{\sqrt{2}\gamma_i}, \quad i=1, \dots, N
\end{aligned} \tag{3.9.3}$$

In addition to the integrals specified in Eqs. (2.30), new integrals of the products of shape functions appear in the above equations as coefficients. These are due to the “smart” spring contribution and are as follows:

$$\begin{aligned}
\delta_{xij1} &= \int_{\bar{x}_1}^{\bar{x}_2} \Psi_i \Psi_j d\bar{x} & K_{xijk} &= \int_{\bar{x}_1}^{\bar{x}_2} \Theta_i \Psi_j \Psi_k d\bar{x} \\
\delta_{xij2} &= \int_{\bar{x}_1}^{\bar{x}_2} \Theta_i \Theta_j d\bar{x} & S_{xi} &= \int_{\bar{x}_1}^{\bar{x}_2} \Theta_i d\bar{x} \\
\Delta_{xij1} &= \int_{\bar{x}_1}^{\bar{x}_2} \Psi_i \Psi_j d\bar{x} & B_{xi} &= \int_{\bar{x}_1}^{\bar{x}_2} \bar{x} \Psi_i d\bar{x} \\
P_{xij} &= \int_{\bar{x}_1}^{\bar{x}_2} \frac{\bar{x}_2^2 - \bar{x}^2}{2} \Theta_i' \Theta_j' d\bar{x} + \int_0^{\bar{x}_1} \frac{\bar{x}_2^2 - \bar{x}_1^2}{2} \Theta_i' \Theta_j' d\bar{x} \\
Q_{xij} &= \int_{\bar{x}_1}^{\bar{x}_2} \frac{1 - \bar{x}^2}{2} \Theta_i' \Theta_j' d\bar{x} \\
R_{xij} &= \int_{\bar{x}_1}^{\bar{x}_2} \frac{\bar{x}_2^2 - \bar{x}^2}{2} \Theta_i' \Theta_j' d\bar{x}
\end{aligned} \tag{3.10}$$

$$D_{xij} = \int_{\bar{x}_1}^{\bar{x}_2} \frac{\bar{x}_2^2 - \bar{x}^2}{2} \Psi_i' \Psi_j' d\bar{x} + \int_0^{\bar{x}_1} \frac{\bar{x}_2^2 - \bar{x}_1^2}{2} \Psi_i' \Psi_j' d\bar{x}$$

$$F_{xijk} = -\frac{1}{\beta_k^4} \left[\int_{\bar{x}_1}^{\bar{x}_2} \Psi_i' \Psi_j' \Psi_k''' d\bar{x} + \Psi_k'''(\bar{x}_2) \int_0^{\bar{x}_2} \Psi_i' \Psi_j' d\bar{x} - \Psi_k'''(\bar{x}_1) \int_0^{\bar{x}_1} \Psi_i' \Psi_j' d\bar{x} \right]$$

Decomposing the generalized coordinates into steady equilibrium quantities and small unsteady perturbation quantities as given in Eq. (2.31), and substituting the equilibrium steady quantities into Eqs. (3.9) one obtains the following nonlinear algebraic equations from which the steady equilibrium quantities, $V_{0j}, W_{0j}, \phi_{0j}$ can be determined:

Lead-lag equilibrium equations:

$$\begin{aligned} & \sum_{j=1}^N \left\langle V_{0j} \{ D_{ij} + \bar{m}_{ss} D_{xij} + [\Lambda_{02} - (\Lambda_{02} - \Lambda_{01}) \sin^2(\Re\theta)] \beta_j^4 \delta_{ij} + \right. \\ & + [\Lambda_{ss2} - (\Lambda_{ss2} - \Lambda_{ss1}) \sin^2(\Re\theta)] \Delta_{xij1} - (\delta_{ij} + \bar{m}_{ss} \delta_{xij1}) \} + \\ & + W_{0j} \frac{\sin(2\Re\theta)}{2} \{ (\Lambda_{02} - \Lambda_{01}) \beta_j^4 \delta_{ij} + (\Lambda_{ss2} - \Lambda_{ss1}) \Delta_{xij1} \} + \\ & + (\Lambda_{02} - \Lambda_{01}) \sum_{k=1}^N K_{jki} \phi_{0j} [W_{0k} \cos(2\Re\theta) - V_{0k} \sin(2\Re\theta)] + \\ & + (\Lambda_{ss2} - \Lambda_{ss1}) \sum_{k=1}^N K_{xjki} \phi_{0j} [W_{0k} \cos(2\Re\theta) - V_{0k} \sin(2\Re\theta)] - \\ & \left. + \frac{\gamma}{6} \bar{v}_i I_{ij} \phi_{0j} \right\rangle = \frac{\gamma}{6} [\bar{v}_i^2 A_i - \frac{c_{d0}}{a} C_i - \bar{v}_i \theta B_i], \quad i=1, \dots, N \end{aligned} \quad (3.11.1)$$

Flap equilibrium equations:

$$\begin{aligned}
& \sum_{j=1}^N \left\langle V_{0j} \frac{\sin(2\Re\theta)}{2} [(\Lambda_{02} - \Lambda_{01})\beta_j^4 \delta_{ij} + (\Lambda_{ss2} - \Lambda_{ss1})\Delta_{xij1}] + \right. \\
& + W_{0j} \{D_{ij} + \bar{m}_{ss} D_{xij} + [\Lambda_{01} + (\Lambda_{02} - \Lambda_{01})\sin^2(\Re\theta)]\beta_j^4 \delta_{ij} + \\
& + [\Lambda_{ss1} + (\Lambda_{ss2} - \Lambda_{ss1})\sin^2(\Re\theta)]\Delta_{xij1}\} + \\
& + (\Lambda_{02} - \Lambda_{01}) \sum_{k=1}^N K_{jki} \phi_{0j} [V_{0k} \cos(2\Re\theta) + W_{0k} \sin(2\Re\theta)] + \\
& + (\Lambda_{ss2} - \Lambda_{ss1}) \sum_{k=1}^N K_{xjki} \phi_{0j} [V_{0k} \cos(2\Re\theta) + W_{0k} \sin(2\Re\theta)] + \\
& + \frac{\gamma}{6} [-J_{ij} \phi_{0j} + \sum_{k=1}^N L_{ijk} V_{0j} W_{0k} - \frac{\bar{c}}{2} O_{ij} W_{0j}] \rangle = \frac{\gamma}{6} [-\bar{v}_i B_i + \theta C_i], \\
& i = 1, \dots, N
\end{aligned} \tag{3.11.2}$$

Torsion equilibrium equations:

$$\begin{aligned}
& \sum_{j=1}^N \left\langle \phi_{0j} [\mu_0^2 K_0 N_{ij} + \mu_0^2 \bar{m}_{ss} K P_{xij} + \mu_{ss}^2 K_{ss} Q_{xij} + \mu_{ss}^2 \bar{m}_{ss} K_{ss} R_{xij} + \right. \\
& + [\kappa_0 \gamma_j^2 + (\mu_{02}^2 - \mu_{01}^2) \cos(2\theta)] \delta_{ij} + \kappa_{ss} \gamma_j^2 \delta_{xij2} + (\mu_{ss2}^2 - \mu_{ss1}^2 + \\
& + \bar{m}_{ss} (\mu_{02}^2 - \mu_{01}^2) + \bar{m}_{ss} (\mu_{ss2}^2 - \mu_{ss1}^2)) \cos(2\theta) \delta_{xij2}] + \\
& + (\Lambda_{02} - \Lambda_{01}) \sum_{k=1}^N K_{ijk} [\cos(2\Re\theta) V_{0j} W_{0k} + \frac{\sin(2\Re\theta)}{2} (W_{0j} W_{0k} - V_{0j} V_{0k})] + \\
& + (\Lambda_{ss2} - \Lambda_{ss1}) \sum_{k=1}^N K_{xijk} [\cos(2\Re\theta) V_{0j} W_{0k} + \frac{\sin(2\Re\theta)}{2} (W_{0j} W_{0k} - V_{0j} V_{0k})] \rangle = \\
& = -S_{xi} (\mu_{ss2}^2 - \mu_{ss1}^2 + \bar{m}_{ss} (\mu_{02}^2 - \mu_{01}^2) + \bar{m}_{ss} (\mu_{ss2}^2 - \mu_{ss1}^2)) \frac{\sin(2\theta)}{2} - \\
& - (\mu_{02}^2 - \mu_{01}^2) \frac{\sin(2\theta)}{\sqrt{2}\gamma_i}, \quad i = 1, \dots, N
\end{aligned} \tag{3.11.3}$$

As outlined in Chapter 2, finally, a set of $3N$ constant coefficient, homogeneous, ordinary differential equations of perturbation motion is derived:

$$\begin{Bmatrix} \dot{X} \\ \ddot{X} \end{Bmatrix} = \begin{bmatrix} [0] & [I] \\ -[M + \Delta M]^{-1}[K + \Delta K] & -[M + \Delta M]^{-1}[C + \Delta C] \end{bmatrix} \begin{Bmatrix} X \\ \dot{X} \end{Bmatrix} \tag{3.12}$$

where the $3N \times 1$ perturbation vector is:

$$\{X\} = \begin{Bmatrix} \Delta V_1 \\ \vdots \\ \Delta V_N \\ \Delta W_1 \\ \vdots \\ \Delta W_N \\ \Delta \phi_1 \\ \vdots \\ \Delta \phi_N \end{Bmatrix} \quad (3.13)$$

Matrices $[M]$, $[C]$, and $[K]$ have the same elements as determined in Section 2.11, while matrices $[\Delta M]$, $[\Delta C]$, and $[\Delta K]$ have a size $3N \times 3N$ and are the mass, damping, and stiffness matrices, respectively, due to the “smart” spring, as follows:

$$[\Delta M] = \begin{bmatrix} \bar{m}_{ss} \delta_{xij1} & 0 & 0 \\ 0 & \bar{m}_{ss} \delta_{xij1} & 0 \\ 0 & 0 & (\mu_{ss}^2 + \bar{m}_{ss} \mu_0^2 + \bar{m}_{ss} \mu_{ss}^2) \delta_{xij2} \end{bmatrix} \quad (3.14.1)$$

$$[\Delta C] = 2 \begin{bmatrix} \bar{m}_{ss} \sum_{k=1}^N (F_{xikj} - F_{xjki}) V_{0k} & -\bar{m}_{ss} \sum_{k=1}^N F_{xjki} W_{0k} & 0 \\ \bar{m}_{ss} \sum_{k=1}^N F_{xikj} W_{0k} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (3.14.2)$$

$$[\Delta K] = \begin{bmatrix} \Delta K_{11} & \Delta K_{12} & \Delta K_{13} \\ \Delta K_{21} & \Delta K_{22} & \Delta K_{23} \\ \Delta K_{31} & \Delta K_{32} & \Delta K_{33} \end{bmatrix} \quad (3.14.3)$$

with the elements of $[\Delta K]$ matrix, each having the size of $N \times N$, and defined as:

$$\begin{aligned}
(\Delta K_{11})_{ij} &= \bar{m}_{ss}(D_{xij} - \delta_{xij1}) + [\Lambda_{ss2} - (\Lambda_{ss2} - \Lambda_{ss1})\sin^2(\Re\theta)]\Delta_{xij1} - \\
&\quad - (\Lambda_{ss2} - \Lambda_{ss1}) \sum_{k=1}^N K_{xkji} \phi_{0k} \sin(2\Re\theta) \\
(\Delta K_{12})_{ij} &= (\Lambda_{ss2} - \Lambda_{ss1}) \sum_{k=1}^N K_{xkji} \phi_{0k} \cos(2\Re\theta) + (\Lambda_{ss2} - \Lambda_{ss1}) \Delta_{xij1} \frac{\sin(2\Re\theta)}{2} + \\
(\Delta K_{13})_{ij} &= (\Lambda_{ss2} - \Lambda_{ss1}) \sum_{k=1}^N K_{xkji} [W_{0k} \cos(2\Re\theta) - V_{0k} \sin(2\Re\theta)] \\
(\Delta K_{21})_{ij} &= (\Lambda_{ss2} - \Lambda_{ss1}) \sum_{k=1}^N K_{xkji} \phi_{0k} \cos(2\Re\theta) + (\Lambda_{ss2} - \Lambda_{ss1}) \Delta_{xij1} \frac{\sin(2\Re\theta)}{2} \\
(\Delta K_{22})_{ij} &= (\Lambda_{ss2} - \Lambda_{ss1}) \sum_{k=1}^N K_{xkji} \phi_{0k} \sin(2\Re\theta) + \bar{m}_{ss} D_{xij} + \\
&\quad + [\Lambda_{ss1} + (\Lambda_{ss2} - \Lambda_{ss1})\sin^2(\Re\theta)]\Delta_{xij1} \\
(\Delta K_{23})_{ij} &= (\Lambda_{ss2} - \Lambda_{ss1}) \sum_{k=1}^N K_{xkji} [V_{0k} \cos(2\Re\theta) + W_{0k} \sin(2\Re\theta)] \\
(\Delta K_{31})_{ij} &= (\Lambda_{ss2} - \Lambda_{ss1}) \sum_{k=1}^N K_{xijk} [-V_{0k} \sin(2\Re\theta) + W_{0k} \cos(2\Re\theta)] \\
(\Delta K_{32})_{ij} &= (\Lambda_{ss2} - \Lambda_{ss1}) \sum_{k=1}^N K_{xijk} [V_{0k} \cos(2\Re\theta) + W_{0k} \sin(2\Re\theta)] \\
(\Delta K_{33})_{ij} &= \mu_0^2 \bar{m}_{ss} K P_{xij} + \mu_{ss}^2 K_{ss} Q_{xij} + \mu_{ss}^2 \bar{m}_{ss} K_{ss} R_{xij} + \\
&\quad + \kappa_{ss} \gamma_j^2 \delta_{xij2} + (\mu_{ss2}^2 - \mu_{ss1}^2 + \bar{m}_{ss}(\mu_{02}^2 - \mu_{01}^2) + \bar{m}_{ss}(\mu_{ss2}^2 - \mu_{ss1}^2)) \cos(2\theta) \delta,
\end{aligned} \tag{3.14.4}$$

As previously, the size of Eq. (3.12) can be reduced by applying the principal coordinate transformation discussed in Section 2.11. This leads to the following equation:

$$\begin{Bmatrix} \dot{X} \\ \ddot{X} \end{Bmatrix} = \begin{bmatrix} [0] & [I] \\ -[M_m]^{-1}[K_k] & -[M_m]^{-1}[C_c] \end{bmatrix} \begin{Bmatrix} X \\ \dot{X} \end{Bmatrix} = [P^*] \begin{Bmatrix} X \\ \dot{X} \end{Bmatrix} \tag{3.15}$$

where

$$\begin{aligned}
[M_m] &= [U]^T [M + \Delta M] [U] \\
[C_c] &= [U]^T [C + \Delta C] [U] \\
[K_k] &= [U]^T [K + \Delta K] [U]
\end{aligned} \tag{3.16}$$

and $[P^*]$ is of the size $2Z \times 2Z$; Z defines the number of low frequency modes; $[U]$ is the matrix of the eigenvectors.

3.4 General Equations of Motion with Periodic Coefficients

In this section the equations of motion will be modified to incorporate periodic “smart” spring actuation. The purpose is to investigate the changes on the blade stability using periodic spring actuation as opposed to using a constant or non-actuated spring. With a periodic actuation, the “smart” spring is activated at certain time period intervals, and the equations of motion will have periodic coefficients. When actuated, the “smart” spring changes the cross-sectional blade parameters, thus, significantly affecting the stability.

The coefficients Λ_{1ss} , Λ_{2ss} , κ_{ss} , and K_{ss} in Eqs. (3.9), previously assumed constant, are modified to include a sinusoidal variation in time with a period $2\pi/n\Omega$, where Ω is the rotor spin velocity, and n is the number of actuations and de-actuations over one rotation of the blade. Using dimensionless time, $\psi=\Omega t$, i.e., the azimuth angle, the dimensionless period of actuation becomes $T=2\pi/n$. Fig. (3.3) shows the sinusoidal variation of the above-mentioned parameters represented by $Z_{ss}(t)$ versus the azimuth angle. Shown in Fig. (3.3), Z_{ss}^a corresponds to the case of constant actuation of the “smart” spring, while Z_{ss}^{na} refers to the case of non-actuated spring.

Figure 3.3 Sinusoidal variation in configuration parameters.

The mathematical representation of the time varying non-dimensional configuration parameters given by $\Lambda_{ss1}, \Lambda_{ss2}, \kappa_{ss}$, and K_{ss} is as follows:

$$\begin{aligned}
 \Lambda_{ss1,2}(t) &= 0.5(\Lambda_{ss1,2}^a + \Lambda_{ss1,2}^{na}) + 0.5(\Lambda_{ss1,2}^a - \Lambda_{ss1,2}^{na}) \sin(n\psi) \\
 \kappa_{ss}(t) &= 0.5(\kappa_{ss}^a + \kappa_{ss}^{na}) + 0.5(\kappa_{ss}^a - \kappa_{ss}^{na}) \sin(n\psi) \\
 K_{ss}(t) &= K_{ss1} + K_{ss2} \sin(n\psi)
 \end{aligned} \tag{3.17}$$

or

$$\begin{aligned}
 \Lambda_{ss1,2}(t) &= \tilde{\lambda}_{ss1,2} + \tilde{\lambda}_{ss1,2} \sin(n\psi) \\
 \kappa_{ss}(t) &= \tilde{\hbar}_{ss} + \tilde{\hbar}_{ss} \sin(n\psi) \\
 K_{ss}(t) &= K_{ss1} + K_{ss2} \sin(n\psi)
 \end{aligned} \tag{3.18}$$

where the following is introduced:

$$\begin{aligned}
\tilde{\lambda}_{ss1,2} &= 0.5(\Lambda_{ss1,2}^a + \Lambda_{ss1,2}^{na}) & K_{ss1} &= \frac{(k_A^2)_{ss}^a + (k_A^2)_{ss}^{na}}{2(k_m^2)_{ss}} \\
\tilde{\lambda}_{ss1,2} &= 0.5(\Lambda_{ss1,2}^a - \Lambda_{ss1,2}^{na}) & K_{ss2} &= \frac{(k_A^2)_{ss}^a - (k_A^2)_{ss}^{na}}{2(k_m^2)_{ss}} \\
\tilde{h}_{ss} &= 0.5(\kappa_{ss}^a + \kappa_{ss}^{na}) \\
\tilde{h}_{ss} &= 0.5(\kappa_{ss}^a - \kappa_{ss}^{na})
\end{aligned} \tag{3.19}$$

Substituting Eq. (3.18) into Eqs. (3.9), the governing differential equations of the “smart” rotor are obtained as a set of nonlinear ordinary differential equations with variable time coefficients:

Lead-lag equations:

$$\begin{aligned}
& \sum_{j=1}^N \langle V_j \{ D_{ij} + \bar{m}_{ss} D_{xij} + [\Lambda_{02} - (\Lambda_{02} - \Lambda_{01}) \sin^2(\Re \theta)] \beta_j^4 \delta_{ij} + \\
& + [\tilde{\lambda}_{ss2} + \tilde{\lambda}_{ss2} \sin(n\psi) - (\tilde{\lambda}_{ss2} - \tilde{\lambda}_{ss1} + (\tilde{\lambda}_{ss2} - \tilde{\lambda}_{ss1}) \sin(n\psi)) \sin^2(\Re \theta)] \Delta_{xij1} - \\
& - (\delta_{ij} + \bar{m}_{ss} \delta_{xij1}) \} + W_j \frac{\sin(2\Re \theta)}{2} \{ (\Lambda_{02} - \Lambda_{01}) \beta_j^4 \delta_{ij} + \\
& + (\tilde{\lambda}_{ss2} - \tilde{\lambda}_{ss1} + (\tilde{\lambda}_{ss2} - \tilde{\lambda}_{ss1}) \sin(n\psi)) \Delta_{xij1} \} + \\
& + (\Lambda_{02} - \Lambda_{01}) \sum_{k=1}^N K_{jki} \phi_j [W_k \cos(2\Re \theta) - V_k \sin(2\Re \theta)] + \\
& + (\tilde{\lambda}_{ss2} - \tilde{\lambda}_{ss1} + (\tilde{\lambda}_{ss2} - \tilde{\lambda}_{ss1}) \sin(n\psi)) \sum_{k=1}^N K_{xjki} \phi_j [W_k \cos(2\Re \theta) - \\
& - V_k \sin(2\Re \theta)] + 2 \sum_{k=1}^N (F_{ikj} - F_{jki} + \bar{m}_{ss} (F_{xikj} - F_{xjki})) V_k \dot{V}_j - \\
& - 2 \sum_{k=1}^N (F_{jki} + \bar{m}_{ss} F_{xjki}) W_k \dot{W}_j + (\delta_{ij} + \bar{m}_{ss} \delta_{xij1}) \ddot{V}_j + \frac{\gamma}{6} [\bar{v}_i I_{ij} \phi_j + \\
& + \left(2 \frac{c_{d0}}{a} E_{ij} + \theta \bar{v}_i \delta_{ij} + \bar{v}_i \sum_{k=1}^N \phi_k H_{ijk} \right) \dot{V}_j - \left(2 \bar{v}_i \delta_{ij} - \theta E_{ij} - \sum_{k=1}^N \phi_k G_{ijk} \right) \dot{W}_j] = \\
& = \frac{\gamma}{6} [\bar{v}_i^2 A_i - \frac{c_{d0}}{a} C_i - \bar{v}_i \theta B_i], \quad i = 1, \dots, N
\end{aligned} \tag{3.20.1}$$

Flap equations:

$$\begin{aligned}
& \sum_{j=1}^N \left\langle V_j \frac{\sin(2\Re\theta)}{2} [(\Lambda_{02} - \Lambda_{01}) \beta_j^4 \delta_{ij} + (\tilde{\lambda}_{ss2} - \tilde{\lambda}_{ss1} + (\tilde{\lambda}_{ss2} - \tilde{\lambda}_{ss1}) \sin(n\psi)) \Delta_{xij1}] + \right. \\
& + W_j \{ D_{ij} + \bar{m}_{ss} D_{xij} + [\Lambda_{01} + (\Lambda_{02} - \Lambda_{01}) \sin^2(\Re\theta)] \beta_j^4 \delta_{ij} + [\tilde{\lambda}_{1ss} + \tilde{\lambda}_{1ss} \sin(n\psi) + \\
& + (\tilde{\lambda}_{ss2} - \tilde{\lambda}_{ss1} + (\tilde{\lambda}_{ss2} - \tilde{\lambda}_{ss1}) \sin(n\psi)) \sin^2(\Re\theta)] \Delta_{xij1} \} + \\
& + (\Lambda_{02} - \Lambda_{01}) \sum_{k=1}^N K_{jki} \phi_j [V_k \cos(2\Re\theta) + W_k \sin(2\Re\theta)] + \\
& + (\tilde{\lambda}_{ss2} - \tilde{\lambda}_{ss1} + (\tilde{\lambda}_{ss2} - \tilde{\lambda}_{ss1}) \sin(n\psi)) \sum_{k=1}^N K_{xjki} \phi_j [V_k \cos(2\Re\theta) + \\
& + W_k \sin(2\Re\theta)] + 2 \sum_{k=1}^N (F_{ikj} + \bar{m}_{ss} F_{xikj}) W_k \dot{V}_j + \frac{\gamma \bar{c}}{24} \ddot{W}_j \delta_{ij} + \\
& \ddot{W}_j (\delta_{ij} + \bar{m}_{ss} \delta_{xij1}) + \frac{\gamma}{6} [-J_{ij} \phi_j - \frac{\bar{c}}{2} O_{ij} W_j + \\
& + \sum_{k=1}^N L_{ijk} V_j W_k - (2\theta E_{ij} + 2 \sum_{k=1}^N \phi_k G_{ijk} - \bar{v}_i \delta_{ij}) \dot{V}_j + E_{ij} \dot{W}_j - \frac{3\bar{c}}{4} I_{ij} \dot{\phi}_j] = \\
& = \frac{\gamma}{6} [-\bar{v}_i B_i + \theta C_i], \quad i=1, \dots, N
\end{aligned} \tag{3.20.2}$$

Torsion equations:

$$\begin{aligned}
& \sum_{j=1}^N \left\langle \phi_j [\mu_0^2 K_0 N_{ij} + \mu_0^2 \bar{m}_{ss} K_0 P_{xij} + \mu_{ss}^2 Q_{xij} (K_{ss1} + K_{ss2} \sin(n\psi)) + \mu_{ss}^2 \bar{m}_{ss} R_{xij} (K_{ss1} + \right. \\
& + K_{ss2} \sin(n\psi)) + [\kappa_0 \gamma_j^2 + (\mu_{02}^2 - \mu_{01}^2) \cos(2\theta)] \delta_{ij} + (\tilde{h}_{ss} + \tilde{h}_{ss} \sin(n\psi)) \gamma_j^2 \delta_{xij2} + \\
& + (\mu_{ss2}^2 - \mu_{ss1}^2 + \bar{m}_{ss} (\mu_{02}^2 - \mu_{01}^2) + \bar{m}_{ss} (\mu_{ss2}^2 - \mu_{ss1}^2)) \cos(2\theta) \delta_{xij2}] + \\
& + (\Lambda_{02} - \Lambda_{01}) \sum_{k=1}^N K_{ijk} [\cos(2\Re\theta) V_j W_k + \frac{\sin(2\Re\theta)}{2} (W_j W_k - V_j V_k)] + \\
& + (\tilde{\lambda}_{ss2} - \tilde{\lambda}_{ss1} + (\tilde{\lambda}_{ss2} - \tilde{\lambda}_{ss1}) \sin(n\psi)) \sum_{k=1}^N K_{xijk} [\cos(2\Re\theta) V_j W_k + \\
& + \frac{\sin(2\Re\theta)}{2} (W_j W_k - V_j V_k)] + \frac{\gamma \bar{c}^2}{48} M_{ij} \dot{\phi}_j + (\mu_0^2 \delta_{ij} + \mu_{ss}^2 \delta_{xij2} + \bar{m}_{ss} \mu_0^2 \delta_{xij2} + \\
& + \bar{m}_{ss} \mu_{ss}^2 \delta_{xij2}) \ddot{\phi}_j \rangle = -S_{xi} (\mu_{ss2}^2 - \mu_{ss1}^2 + \bar{m}_{ss} (\mu_{02}^2 - \mu_{01}^2) + \bar{m}_{ss} (\mu_{ss2}^2 - \mu_{ss1}^2)) \frac{\sin(2\theta)}{2} - \\
& - (\mu_{02}^2 - \mu_{01}^2) \frac{\sin(2\theta)}{\sqrt{2} \gamma_i}, \quad i=1, \dots, N
\end{aligned} \tag{3.20.3}$$

3.5 Perturbation Equations of Motion with Periodic Coefficients

In Section 2.11, the generalized coordinates in Eqs. (2.31) were decomposed into steady equilibrium quantities and unsteady perturbation quantities. Following the procedure presented in Section 2.11, the solution methodology for obtaining the equations describing perturbation motions were outlined in Section 3.3. The resulting homogeneous ordinary differential equations with periodic matrix coefficients are (see Eqs. (3.12)):

$$\begin{Bmatrix} \dot{X} \\ \ddot{X} \end{Bmatrix} = \begin{bmatrix} [0] & [I] \\ -[M + \Delta M]^{-1} [K + \Delta K(\psi)] & -[M + \Delta M]^{-1} [C + \Delta C] \end{bmatrix} \begin{Bmatrix} X \\ \dot{X} \end{Bmatrix} = [P(t)] \begin{Bmatrix} X \\ \dot{X} \end{Bmatrix} \quad (3.21)$$

where matrices $[M]$, $[K]$, and $[C]$ are the same as specified in Eqs. (2.35), (2.36); $[\Delta M]$ and $[\Delta C]$ are given in Eqs. (3.14.1) and (3.14.2), whereas $[\Delta K(\psi)]$ matrix is the function of azimuth angle, ψ , representing actuation and de-actuation of the “smart” spring, such that (see Eq. (3.14.3)):

$$[\Delta K(\psi)] = \begin{bmatrix} \Delta K_{11} & \Delta K_{12} & \Delta K_{13} \\ \Delta K_{21} & \Delta K_{22} & \Delta K_{23} \\ \Delta K_{31} & \Delta K_{32} & \Delta K_{33} \end{bmatrix} \quad (3.22.1)$$

In view of Eqs. (3.14.4), due to a periodic actuation of the “smart” spring, the elements of $[\Delta K(\psi)]$ matrix become:

$$\begin{aligned}
(\Delta K_{11})_{ij} &= \bar{m}_{ss} (D_{xij} - \delta_{xij1}) + [\tilde{\lambda}_{2ss} + \tilde{\lambda}_{2ss} \sin(n\psi) - \\
&\quad - (\tilde{\lambda}_{2ss} + \tilde{\lambda}_{2ss} \sin(n\psi) - \tilde{\lambda}_{1ss} - \tilde{\lambda}_{1ss} \sin(n\psi)) \sin^2(\Re\theta)] \Delta_{xij1} - \\
&\quad - (\tilde{\lambda}_{2ss} + \tilde{\lambda}_{2ss} \sin(n\psi) - \tilde{\lambda}_{1ss} - \tilde{\lambda}_{1ss} \sin(n\psi)) \sum_{k=1}^N K_{xkji} \phi_{0k} \sin(2\Re\theta) \\
(\Delta K_{12})_{ij} &= (\tilde{\lambda}_{2ss} + \tilde{\lambda}_{2ss} \sin(n\psi) - \tilde{\lambda}_{1ss} - \tilde{\lambda}_{1ss} \sin(n\psi)) \sum_{k=1}^N K_{xkji} \phi_{0k} \cos(2\Re\theta) + \\
&\quad + (\tilde{\lambda}_{2ss} + \tilde{\lambda}_{2ss} \sin(n\psi) - \tilde{\lambda}_{1ss} - \tilde{\lambda}_{1ss} \sin(n\psi)) \Delta_{xij1} \frac{\sin(2\Re\theta)}{2} \\
(\Delta K_{13})_{ij} &= (\tilde{\lambda}_{2ss} + \tilde{\lambda}_{2ss} \sin(n\psi) - \tilde{\lambda}_{1ss} - \tilde{\lambda}_{1ss} \sin(n\psi)) \sum_{k=1}^N K_{xkji} [W_{0k} \cos(2\Re\theta) - \\
&\quad - V_{0k} \sin(2\Re\theta)] \\
(\Delta K_{21})_{ij} &= (\tilde{\lambda}_{2ss} + \tilde{\lambda}_{2ss} \sin(n\psi) - \tilde{\lambda}_{1ss} - \tilde{\lambda}_{1ss} \sin(n\psi)) \sum_{k=1}^N K_{xkji} \phi_{0k} \cos(2\Re\theta) + \\
&\quad + (\tilde{\lambda}_{2ss} + \tilde{\lambda}_{2ss} \sin(n\psi) - \tilde{\lambda}_{1ss} - \tilde{\lambda}_{1ss} \sin(n\psi)) \Delta_{xij1} \frac{\sin(2\Re\theta)}{2} \\
(\Delta K_{22})_{ij} &= (\tilde{\lambda}_{2ss} + \tilde{\lambda}_{2ss} \sin(n\psi) - \tilde{\lambda}_{1ss} - \tilde{\lambda}_{1ss} \sin(n\psi)) \sum_{k=1}^N K_{xkji} \phi_{0k} \sin(2\Re\theta) + \\
&\quad + \bar{m}_{ss} D_{xij} + [\tilde{\lambda}_{1ss} + \tilde{\lambda}_{1ss} \sin(n\psi) + [\tilde{\lambda}_{2ss} + \tilde{\lambda}_{2ss} \sin(n\psi) - \tilde{\lambda}_{1ss} - \\
&\quad - \tilde{\lambda}_{1ss} \sin(n\psi)] \sin^2(\Re\theta)] \Delta_{xij1} \\
(\Delta K_{23})_{ij} &= (\tilde{\lambda}_{2ss} + \tilde{\lambda}_{2ss} \sin(n\psi) - \tilde{\lambda}_{1ss} - \tilde{\lambda}_{1ss} \sin(n\psi)) \cdot \\
&\quad \cdot \sum_{k=1}^N K_{xkji} [V_{0k} \cos(2\Re\theta) + W_{0k} \sin(2\Re\theta)] \\
(\Delta K_{31})_{ij} &= (\tilde{\lambda}_{2ss} + \tilde{\lambda}_{2ss} \sin(n\psi) - \tilde{\lambda}_{1ss} - \tilde{\lambda}_{1ss} \sin(n\psi)) \cdot \\
&\quad \cdot \sum_{k=1}^N K_{xijk} [-V_{0k} \sin(2\Re\theta) + W_{0k} \cos(2\Re\theta)] \\
(\Delta K_{32})_{ij} &= (\tilde{\lambda}_{2ss} + \tilde{\lambda}_{2ss} \sin(n\psi) - \tilde{\lambda}_{1ss} - \tilde{\lambda}_{1ss} \sin(n\psi)) \cdot \\
&\quad \cdot \sum_{k=1}^N K_{xijk} [V_{0k} \cos(2\Re\theta) + W_{0k} \sin(2\Re\theta)] \\
(\Delta K_{33})_{ij} &= \mu_0^2 \bar{m}_{ss} K_0 P_{xij} + \mu_{ss}^2 Q_{xij} (K_{ss1} + K_{ss2} \sin(n\psi)) + \\
&\quad + \mu_{ss}^2 \bar{m}_{ss} R_{xij} (K_{ss1} + K_{ss2} \sin(n\psi)) + (\hbar_{ss} + \tilde{\hbar}_{ss} \sin(n\psi)) \gamma_j^2 \delta_{xij2} + \\
&\quad + (\mu_{2ss}^2 - \mu_{1ss}^2 + \bar{m}_{ss} (\mu_{20}^2 - \mu_{10}^2) + \bar{m}_{ss} (\mu_{2ss}^2 - \mu_{1ss}^2)) \cos(2\theta) \delta_{xij2}
\end{aligned} \tag{3.22.2}$$

By applying the principal coordinate transformation, Eq. (3.21) can then be reduced to the following:

$$\begin{Bmatrix} \dot{X} \\ \ddot{X} \end{Bmatrix} = \begin{bmatrix} [0] & [I] \\ -[M_m]^{-1} [K_k(\psi)] & -[M_m]^{-1} [C_c] \end{bmatrix} \begin{Bmatrix} X \\ \dot{X} \end{Bmatrix} = [P^*(\psi)] \begin{Bmatrix} X \\ \dot{X} \end{Bmatrix} \quad (3.23)$$

or:

$$\{\dot{x}\} = [P^*(\psi)] \{x\} \quad (3.24)$$

where matrix $[P^*]$ is periodic with period $T=2\pi/n$, matrices $[M_m]$ and $[C_c]$ are the same as in Eqs. (3.16), and

$$\begin{aligned} [K_k(\psi)] &= [U]^T [K + \Delta K(\psi)] [U] \\ \{x\} &= \begin{Bmatrix} X \\ \dot{X} \end{Bmatrix} \end{aligned} \quad (3.25)$$

3.6 Implementation of Floquet Method

As can be seen from Eq. (3.24), the perturbation equations have periodic coefficients. Consequently, this periodic system requires a more complicated treatment for stability analysis. Some methods that solve equations with periodic coefficients are the Floquet method, multi-blade coordinate and harmonic balance method. A good discussion of these methods can be found in reference 21 and 25. In this study the Floquet method was used due to its simplicity and computational ease.

The Floquet theorem states that for a system of ordinary differential equations with periodic coefficients, for example Eq. (3.24), over the period $T=2\pi/n$ the transient solution has the form:

$$\{x(\psi)\} = [A(\psi)] \{c_k e^{\eta_k \psi}\} \quad (3.26)$$

where $[A(\psi)]$ is periodic with period T ; η_k are complex characteristic numbers; c_k are constants found from the initial conditions:

$$\{c_k\} = [A(0)]^{-1} \{x(0)\} \quad (3.27)$$

Then, the stability of the system can be determined from the values η_k .

The Floquet transition matrix $[Q]$ of the system is defined by:

$$\{x(T)\} = \begin{bmatrix} x^{(1)} & x^{(2)} & \dots & x^{(2Z)} \end{bmatrix} \begin{Bmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_{2Z}(0) \end{Bmatrix} = [Q] \{x(0)\} \quad (3.28)$$

Here, $[Q]$ has the size of $2Z \times 2Z$, $x^{(1)}$ is the solution at $\psi=T$ for initial conditions $x_1(0)=1$ and all remaining $x_i(0)=0$; $x^{(2)}$ is the solution at $\psi=T$ for initial conditions $x_2(0)=1$ and all remaining $x_i(0)=0$; etc., [21]. The transition matrix, $[Q]$, was obtained using an “n-pass method” [22], according to which the equations of motion were integrated $2Z$ times in order to cover all initial conditions.

From Eq. (3.26):

$$\begin{aligned} \{x(0)\} &= [A(0)] \{c_k\} \\ \{x(T)\} &= [A(0)] [\lambda_k] \{c_k\} \end{aligned} \quad (3.29)$$

where $[A(0)] = [A(T)]$ and $\lambda_k = e^{\eta_k T}$. Substituting Eq. (3.29) into Eq. (3.28), yields:

$$[Q][A(0)] \{c_k\} - [A(0)] [\lambda_k] \{c_k\} = \{0\} \quad (3.30)$$

Since the coefficients c_k are arbitrary, one must have:

$$([Q] - \lambda_i [I]) \{a_0\}_i = \{0\} \quad (3.31)$$

where $\{a_0\}_i$ is the i^{th} column of matrix $[A(0)]$, such that $\{a_0\}_i \neq \{0\}$, $i=1,\dots,2Z$.

Then,

$$|[Q]-\lambda_i[I]|=0 \quad (3.32)$$

Thus, λ_k are the eigenvalues of matrix $[Q]$ and, in general, they are complex numbers.

Once λ_k are found, one can determine η_k as:

$$\eta_k = \frac{1}{T} \ln(\lambda_k) = \hat{\sigma}_k + i\hat{\omega}_k \quad (3.33.1)$$

where,

$$\begin{aligned} \hat{\sigma}_k &= \frac{1}{T} \ln |\lambda_k| = \frac{1}{2T} \ln \left[(\lambda_k)_R^2 + (\lambda_k)_I^2 \right] \\ \hat{\omega}_k &= \frac{1}{T} \arctan \left[(\lambda_k)_I / (\lambda_k)_R \right] \pm j \frac{2\pi}{T}, \quad j \in N \end{aligned} \quad (3.33.2)$$

The subscripts $()_R$ and $()_I$ define the real and imaginary parts, respectively. The real part of the characteristic number, $\hat{\sigma}_k$, is the growth or decay of the response, while $\hat{\omega}_k$ represents the frequency. Values of $\hat{\sigma}_k > 0$ indicate instability. Further discussion of possible instabilities will be presented in Chapter 4.

3.7 “Smart” Blade Design

A realistic stability analysis should be performed using configuration parameters of an actual blade. To this end, a Eurocopter BO105 helicopter blade is considered in the present work. Moreover, since the actual testing of a model rotor blade is part of the future phase of this investigation, a reduced scale model of Eurocopter BO105 blade, with a two-meter span, was considered for the present investigation. The basic dimensions of the reduced scale model are given in Table (3.1).

Table 3.1 Basic dimensions of the reduced scale model of Eurocopter BO105 helicopter blade.

Parameter	Description	Dimensions	Value
R	rotor radius	m	2
c	chord length	m	0.121
b	number of blades		4

This model is planned to be built and used in the next phase for experimental verification.

The reduced scale model was designed to include a single spar that is embedded in non-structural foam. A non-structural skin cover applied to the foam completes the airfoil cross-section of the model.

The reduced scale model of the blade should properly replicate the dynamic characteristics, i.e., natural frequencies, of the full-size blade. On the other hand, these natural frequencies, which in turn serve as rotor blade configuration parameters, are determined mainly by the structural and inertial parameters of the design. The uncoupled, non-rotating natural frequencies of a BO105 blade when normalized with respect to the blade rotational speed of $\Omega = 110$ rad/sec give values of 0.514, 0.143 and 3.711 as the uncoupled, non-dimensional, non-rotating natural frequencies for the first lead-lag, first flap, and first torsion modes, respectively, [23].

In general, the structural and inertial parameters cannot be determined in terms of the blade natural frequencies using explicit expressions, and hence these parameters should be determined iteratively. In this work, the structural and inertial parameters of the reduced model are determined using the uncoupled, non-rotating natural frequencies of the BO105 blade by means of the following expressions [3]:

$$\begin{aligned}
\frac{EI_{z'}}{m\Omega^2 R^4} &= \frac{(\omega_{v_{NR}} / \Omega)^2}{\beta_1^4} \\
\frac{EI_{y'}}{m\Omega^2 R^4} &= \frac{(\omega_{w_{NR}} / \Omega)^2}{\beta_1^4} \\
\frac{GJ}{m\Omega^2 k_m^2 R^2} &= \frac{(\omega_{\phi_{NR}} / \Omega)^2}{\gamma_1^2}
\end{aligned} \tag{3.34}$$

where the subscript ()_{NR} refers to the frequency of the non-rotating blade. Eqs. (3.34) therefore impose additional limiting conditions on the spar structure dimensions, material choice, and blade design in general.

The following criteria were considered in order to design an appropriate reduced scale model blade with the required parameters:

1. The skin and the embedded foam do not contribute markedly to the bending and torsional stiffnesses and, therefore, were not considered in the structural design of the blade.
2. The blade cross-section conforms to that of the NACA 0012 symmetric airfoil sections.
3. The size and placing of the spar were designed so that to ensure the location of the elastic centre axis to be at the quarter chord from the leading edge.
4. The mass centre of the cross-section was also placed at the quarter chord from the leading edge by adding lead bars near the leading and trailing edges of the airfoil. This provides no offsets between the elastic and mass axes. These lead bars change the mass and inertial properties of the blade without affecting the stiffnesses.

These criteria limit the choice of possible spar cross-sections. A computer code was written in order to consider different material choices, and possible cases of spar design

dimensions and lead bar masses. The optimal dimensions that satisfy the conditions discussed above were then found. In this study, two design cases were considered: an aluminum spar blade and a composite spar blade, Figs. (3.4) and (3.5).

Figure 3.4 The structure of the “smart” blade cross-section using aluminum spar.

Figure 3.5 The structure of the “smart” blade cross-section using composite spar.

For the first design case the material chosen for the optimal spar structure is aluminum 2024-T3, [24]. Shown in Figs. (3.4) and (3.6), the spar represents a uniform hollow rectangular cross-section with the rectangular ribs on the under sides. The dimensions for the blade structure and material properties are shown in Table 3.2.

Figure 3.6 Aluminum spar in the “smart” blade design

Table 3.2 Specifications of the “smart” blade for the aluminum spar

Parameter	Description	Dimensions	Value
ρ_{lead}	density of lead	kg/m^3	11340
E_a	Young's modulus of aluminum 2024-T3	GPa	73.1
ν_a	Poisson's ratio of aluminum 2024-T3		0.33
ρ_a	density of aluminum 2024-T3	kg/m^3	2770
a	chord-wise length of the spar	mm	44
d	flap-wise length of the spar	mm	9
t_f	thickness of the flanges	mm	2×0.5
t_w	thickness of the webs	mm	2×2
t_r	thickness of the top and bottom ribs	mm	2×1
l_r	length of the top and bottom ribs	mm	2×13
l_1	distance to the first lead bar from the leading edge	mm	4.6
l_2	distance to the second lead bar from the leading edge	mm	112
m_1	mass of the first lead bar	kg/mm	8×10^{-5}
m_2	mass of the second lead bar	kg/mm	2.5×10^{-5}
m_s	mass of the spar	kg/mm	3×10^{-4}
r_1	radius of the first lead bar	mm	1.5
r_2	radius of the second lead bar	mm	0.8

For the second design case a composite spar structure made of graphite/epoxy composite was used. The spar was designed to have a hollow rectangular shape with two horizontal flanges at the sides employing 10 angle plies oriented at $[\pm 15^\circ]$. The cross-section corresponding to this case is shown in Figs. (3.5) and (3.7), and the dimensions and material properties are listed in Table 3.3.

Table 3.3 Specifications of the “smart” blade for the composite spar.

Parameter	Description	Dimensions	Value
E_L	longitudinal Young’s modulus of graphite/epoxy	GPa	142
E_T	1 st transverse Young’s modulus of graphite/epoxy	GPa	9.6
E_N	2 nd transverse Young’s modulus of graphite/epoxy	GPa	9.6
G_{LT}	longitudinal shear modulus of graphite/epoxy	GPa	4.8
G_{LN}	shear modulus of graphite/epoxy	GPa	4.8
ν_{LT}	Poisson’s ratio of graphite/epoxy composite		0.3
ν_{LN}	Poisson’s ratio of graphite/epoxy composite		0.3
ν_{TN}	Poisson’s ratio of graphite/epoxy composite		0.42
t_p	Ply thickness	mm	0.127
ρ_g	density of graphite-epoxy	kg/m ³	1800
a	chord-wise length of the spar	mm	35
d	flap-wise length of the spar	mm	9
f	thickness of the side flanges	mm	2×1.27
l_f	length of the side flanges	mm	2×4
t_f	thickness of the flanges	mm	2×1.27
t_w	thickness of the webs	mm	2×1.27
l_1	distance to the first lead bar from the leading edge	mm	6
l_2	distance to the second lead bar from the leading edge	mm	54.5
m_1 and m_2	masses of the first and second lead bars	Kg/mm	2×1×10 ⁻⁴
m_s	mass of the spar	kg/mm	2×10 ⁻⁴
r_1 and r_2	radii of the first and second lead bars	mm	2×1.9

Figure 3.7 Composite spar in the "smart" blade design

Due to possibly easier construction of the aluminum spar during the future experimental phase, this model was considered for further stability analysis in the present work. The "smart" spring was placed inside the hollow tube of the spar at the root of the blade. Two different lengths of the smart blade were considered: one being of 15% of the blade length and the second one being of 5% of the blade length.

In order to investigate the effective position of the "smart" spring, a beam having a cross-section shown in Fig. (3.4) and subjected to a uniform distributed load was considered. The displacement of the tip of the beam was calculated as a function of the position of the "smart" spring relative to the root of the blade. It was found that as the "smart" spring is placed closer to the root of the blade, the blade tip deflection is lower. The plotted tip displacement versus the position of the "smart" spring from the blade root is presented in Fig. (3.8) in Appendix B.1. In view of this result, it is suggested to locate the "smart" spring directly at the root of the blade.

During actuation, the "smart" spring expands and closes the gap between it and the spar structure resulting in a change in stiffness and area properties of the blade. In this

work, a piezoelectric material was chosen to represent the “smart” spring due to its strength and deformation properties upon application of an electric field. The material properties of the piezoelectric material are given in Table 3.4.

Table 3.4 Piezoelectric material properties

E_{ss}	Young’s modulus of piezoelectric material	Gpa	71.4
G_{ss}	shear modulus of piezoelectric material	Gpa	27.5
ρ_{ss}	density of piezoelectric material	kg/m^3	7500

Further configuration parameters that are necessary to be defined are aerodynamic, coupling, and control parameters. These are discussed below.

The aerodynamic parameters represent the drag coefficient, $c_{d0} = 0.01$, and the lift curve slope, $a = 2\pi/rad$. The number of blades b , is four, and the ratio of the chord length to the radius, c/R is 6%. Therefore, the solidity ratio $\sigma = bc/(R\pi)$ was found to be 0.077. The geometric pre-cone angle, β_{pc} , was chosen to represent the baseline value of zero degrees.

The control parameter defining the rotor operating condition is the pitch angle θ . The pitch angle has a significant influence on the stability, and thus a range of values was investigated. The pitch angle was chosen to vary from its baseline value of 0 radians to 0.3 radians.

A structural coupling parameter, $\mathcal{H}$ representing the coupling between the inboard and outboard blade segments was assigned to range from its totally decoupled value of zero ($\mathcal{H} = 0$) to the completely coupled case with the baseline value $\mathcal{H} = 1$.

Moreover, there are two sets of cross-section parameters, one corresponding to the non-actuated “smart” spring, and the other representing the “smart ” spring being constantly actuated. These two “extreme” regimes, namely the Smart Baseline Blade (SBB), and the Direct Current (DC) regimes, respectively, will be discussed later in Chapter 4. For the case of the Smart Baseline Blade regime the “smart” spring does not carry any load, and therefore, does not contribute to the stiffness and area properties of the blade, but contributes to the mass. Thus, structural properties of the blade were considered to be only due to the host spar for this regime. When actuated, the piezoelectric material fills the gap between it and the host spar, allowing for the contribution of the “smart” spring. In this case the structural properties are calculated by including the “smart” spring in the load path (Appendix B.2).

Chapter 4

Stability Analysis

The results from the aeroelastic stability analysis of the “smart” rotor blade are presented in this chapter. In order to investigate any changes in the blade stability resulting from the implementation of the “smart” spring, various actuation scenarios of the spring are considered. In this regard, the effect of the following parameters are also considered: a) the length of the “smart” spring, b) the structural coupling parameter, $\mathfrak{R}$, and c) the inclusion of shear strain energy in the “smart” rotor blade model.

4.1 Actuation Regimes

Several scenarios (regimes) corresponding to various blade structure configurations and number of actuations of the “smart” spring per rotor revolution have been considered in order to compare the stability of the controlled blade versus an uncontrolled one.

These regimes are briefly summarized below:

Conventional Baseline Blade (CBB) represents only the blade structure, which includes the aluminum spar with two lead bars at the leading and trailing edges of the blade without the presence of a “smart” spring.

Smart Baseline Blade (SBB) represents the CBB structure together with a non-actuated “smart” spring.

Direct Current (DC) regime refers to the SBB system in which the “smart” spring is subjected to a DC current.

In **Alternating Current (AC)** regime the “smart” spring in the SBB is actuated sinusoidally.

During rotor rotation helicopters experience increasing vibration levels due to the periodic aerodynamic loads on the rotor blades at harmonics of the rotational speed Ω . These transmit to the non-rotating system as vibratory hub loads at harmonics of the frequency $b\Omega$, where b is the number of blades. The vibrations result in significant passenger and crew discomfort and increased component fatigue. In order to achieve the vibration control, it is envisaged that actuation and de-actuation of the “smart” blade will be enforced at $(b-1)\Omega$, $b\Omega$ and $(b+1)\Omega$ frequencies per rotor revolution. This is expected to affect the stability of the blade. Therefore, the actuation frequencies were taken to be 3Ω , 4Ω and 5Ω . The actuation regimes **AC3**, **AC4** and **AC5** each refer to three, four, and five cycles of actuation and de-actuation of the “smart” spring per one rotor revolution, respectively.

For Conventional Baseline Blade the dynamic equations of motion used to find the equilibrium deflections are Eqs. (2.32). These are nonlinear, constant coefficient algebraic equations. The corresponding eigenvalues were determined from the

perturbation equations (2.41) in order to analyze the stability of the motion of the uncontrolled blade about the equilibrium state.

For the Smart Baseline Blade and DC regimes, the corresponding nonlinear algebraic equations with constant coefficients defining the trim deflections are Eqs. (3.11). As outlined in Section 3.7, the actuation of the “smart” spring changes the cross-sectional parameters of the Smart Baseline Blade. For both cases the eigenvalues were subsequently found from Eqs. (3.15).

In contrast to the previous regimes, for the AC regimes the periodic actuation of the “smart” spring results in governing differential equations with periodic coefficients and hence periodic matrix coefficient in Eq. (3.24). In order to investigate the stability of the controlled blade, the eigenvalues were then found using the Floquet method.

For all the cases, the eigenvalues appear in complex conjugate pairs. As it was discussed in Section 3.6, the real part of an eigenvalue represents the measure of growth or decay of the oscillatory response, whereas the imaginary part is the oscillation frequency.

An eigenvalue with negative real part represents a stable motion, while roots with positive real parts correspond to an unstable motion. The stability boundary is thus represented by a root on the imaginary axis in the root loci plane (s-plane). Therefore, in order to make a system unstable, a real root should cross the stability boundary along the real axis, or as a complex conjugate roots at a finite value of frequency.

The type of instability corresponding to the case when the real root goes along the real axis into the right half-plane is called divergence. Divergence represents a static instability, because there are no velocity or acceleration forces associated with zero

frequency. Flutter is an instability occurring when a complex conjugate pair of roots crosses the stability boundary at a finite frequency. The unbounded oscillatory motion of the system is associated with this type of instability.

4.2 Parametric Study

In addition to the regimes specified above, a parametric investigation was also carried out. The parameters that were chosen in order to analyze their effect in causing the unstable behaviour of the blade were: the length of the “smart” spring, the structural coupling parameter, $\mathcal{R}$, and the pitch angle, θ . These were expected to have a significant influence on the stability, and thus a range of the values was investigated. By varying the values of these parameters, different system responses can be obtained.

The pitch angle was taken to range from its baseline value of zero radians to 0.3 radians. It should be noted that a typical operational pitch angle is considerably lower than 0.3 radians, but this investigation is aimed to cover the stability of a hingeless blade for a complete array of practical applications.

The structural coupling parameter $\mathcal{R}$ was chosen to be either zero, corresponding to the decoupled case, or unity, defining a completely coupled one. Two different lengths of the “smart” spring were considered. These were 15% and 5 % of the blade length. This gives the non-dimensional starting and ending points of the “smart” spring, $\bar{x}_1$ and $\bar{x}_2$, the values of 0 and 0.15 (or 0.05), respectively. Various combinations of these parameters therefore resulted in the following design cases:

Design Case 1. $\mathcal{R} = 0$, $\bar{x}_1 = 0$, $\bar{x}_2 = 0.15$, including shear strain energy.

Design Case 2. $\mathcal{R} = 1$, $\bar{x}_1 = 0$, $\bar{x}_2 = 0.15$, including shear strain energy.

Design Case 3. $\mathcal{R} = 0$, $\bar{x}_1 = 0$, $\bar{x}_2 = 0.15$, neglecting shear strain energy.

Design Case 4. $\mathcal{R} = 1$, $\bar{x}_1 = 0$, $\bar{x}_2 = 0.15$, neglecting shear strain energy.

Design Case 5. $\mathcal{R} = 0$, $\bar{x}_1 = 0$, $\bar{x}_2 = 0.05$, including shear strain energy.

Design Case 6. $\mathcal{R} = 0$, $\bar{x}_1 = 0$, $\bar{x}_2 = 0.05$, neglecting shear strain energy.

The parametric study was performed using a MATLAB numerical computer code developed for this investigation (see Appendix E). The program performs the numerical integration of the products of the shape functions, calculates the steady equilibrium deflections, and assembles the mass, damping, and stiffness matrices. It also solves for the eigenvalues of the differential perturbation equations for all the regimes discussed above.

In this program, the input parameters $\bar{x}_1$, $\bar{x}_2$, $\mathcal{R}$ and activation frequency for the AC regimes should be specified by the user in the MainProgram1.m (MainProgram2.m) routine. A User Manual, which contains a detailed explanation of each routine used by the MainProgram1.m, (MainProgram2.m) is also presented in Appendix E.

4.3 Code Verification

The accuracy of the code was verified against data that has been published by Hodges and Ormiston [3]. They carried out a stability analysis of an elastic, uniform, untwisted hingeless blade, ignoring chordwise offsets between the elastic, mass, aerodynamic, and tension axes of the blade. The structural, inertia, geometric, and aerodynamic parameters that were used in their work are summarized in Table 4.1 below.

Table 4.1 Values of operating parameters used by Hodges and Ormiston, [3]

Parameters	Baseline values	Additional Value
ω_v	0.7, 1.5	Variable
ω_w	1.15	-----
ω_ϕ	2.5, 5.0, 8.0	Variable
$(k_A/k_m)^2$	1.5	-----
θ , rad	0.0	-----
β_{pc} , rad	0.0	Variable 0.0-0.5
$\mathcal{R}$	0.0, 1.0	Variable
k_m/R	0.025	Variable 0.0-1.0
k_{m1}/k_{m2}	0.0	-----
$\gamma = (3\rho_\infty acR/m)$	5	-----
σ	0.1	-----
c/R	$\pi/40$	-----
B	4	-----
c_{do}/a	$0.01/2\pi$	-----

The flap, lag and torsion stability characteristics obtained using the author's code are shown in Figs. (4.1 - 4.4) in a format different than that presented by Hodges and Ormiston. These represent the loci of the roots of the 1st lead-lag, 1st flap and 1st torsion modes as the blade pitch angle θ is increased from 0 to 0.5 radians. The natural rotating frequencies of this blade are $\omega_v = 0.7$, $\omega_w = 1.15$, $\omega_\phi = 2.5, 5.0, 8.0$, respectively. The pre-cone angle, β_{pc} and structural coupling parameter, $\mathcal{R}$, were taken to be zero.

These results are in perfect agreement with the ones obtained by Hodges and Ormiston [3], thus validating the accuracy of the author's code. Therefore, this code was used to find the stability characteristics of the "smart" rotor blade, presented in the following sections.

Figure 4.1 Loci of roots of the 1st lead-lag mode for a soft inplane rotor blade: $\omega_v = 0.7$, $\omega_w = 1.15$, $\omega_\phi = 2.5, 5.0, 8.0$, $\beta_{pc} = 0.0$, $\mathcal{R} = 0.0$

Figure 4.2 Loci of roots of the 1st flap mode for a soft inplane rotor blade: $\omega_v = 0.7$, $\omega_w = 1.15$, $\omega_\phi = 2.5, 5.0, 8.0$, $\beta_{pc} = 0.0$, $\mathcal{R} = 0.0$

Figure 4.3 Loci of roots of the 1st torsion mode for a soft inplane rotor blade: $\omega_v = 0.7$, $\omega_w = 1.15$, $\omega_\phi = 2.5$, $\beta_{pc} = 0.0$, $\mathcal{R} = 0.0$

Figure 4.4 Loci of roots of the 1st torsion mode for a soft inplane rotor blade: $\omega_v = 0.7$, $\omega_w = 1.15$, $\omega_\phi = 5.0, 8.0$, $\beta_{pc} = 0.0$, $\mathcal{R} = 0.0$

4.4 Equilibrium Deflections

The algebraic equations for determining the steady state equilibrium deflections are nonlinear. Therefore, they should be solved iteratively to obtain V_0 , W_0 , and ϕ_0 , which are then used as the coefficients of the corresponding linear perturbation differential equations. The operating condition of the rotor is also defined by the equilibrium deflections. The rotor thrust in hover is dependent on the total blade angle, which is the sum of the blade pitch angle θ and the equilibrium elastic twist, ϕ_0 .

The dimensionless tip lead-lag, flap, and torsional deflections, V_{tip} , W_{tip} , ϕ_{tip} , respectively, can be expressed in terms of the modal equilibrium deflections V_{0j} , W_{0j} , and ϕ_{0j} as:

$$\begin{aligned} V_{tip} &= 2 \sum_{j=1}^N V_{0j} (-1)^{j+1} \\ W_{tip} &= 2 \sum_{j=1}^N W_{0j} (-1)^{j+1} \\ \phi_{tip} &= \sqrt{2} \sum_{j=1}^N \phi_{0j} (-1)^{j+1} \end{aligned} \quad (4.1)$$

A set of three graphs each comprising three regimes, namely CBB, SBB, and DC corresponding to Design Case 1, is given in Figs. (4.5.1 – 4.5.3). Shown in these figures are the dimensionless tip lead-lag, flap, and torsional deflections, V_{tip} , W_{tip} , ϕ_{tip} , respectively, plotted versus the pitch angle, θ that varies from 0 to 0.3 radians.

Figure 4.5.1 Equilibrium tip lead-lag deflections of Design Case 1

Figure 4.5.2 Equilibrium tip flap deflections of Design Case 1

Figure 4.5.3 Equilibrium tip torsion deflections of Design Case 1

It can be seen from the figures that, in general, increasing θ leads to an increase in the aerodynamic loads, and thus to amplification in the magnitude of the equilibrium deflections. Also, the blade tends to bend in the direction normal to the airfoil chord so that negative V_{tip} deflections are associated with positive W_{tip} deflections. The equilibrium deflections for the other Design Cases are presented in Appendix C.

4.5 Results of Stability Analysis

The comprehensive overview of the “smart” rotor blade flap-lag-torsion stability is presented by six groups of graphs and tables, corresponding to six design cases outlined in Section 4.2. Each group consists of four graphs representing the variations in damping and frequency plotted versus the pitch angle for the 1st lead-lag, 1st and 2nd flap and 1st

torsion modes. The tabular data associated with the above figures is listed in Appendix D, so that Tables (D.1 – D.6) correspond to the Design Cases 1 through 6, respectively. All damping and frequency graphs are plotted for a step increment of 0.05 radians in the pitch angle.

Individual Design Cases are discussed in the following subsections.

4.5.1 Design Case 1 ($\mathcal{R}=0$, $\bar{x}_1=0$, $\bar{x}_2=0.15$, including shear strain energy)

The results for this case are given in Figs. (4.6.1 – 4.6.4) and tabulated in Appendix D (Tables (D.1.1 – D.1.4)).

4.5.1.1 1st lead-lag mode

Figs. (4.6.1) and Table (D.1.1) show the responses of the various regimes in terms of damping and frequency corresponding to the 1st lead-lag mode of the hingeless blade. It can be observed that all of the regimes start at high damping at zero pitch angle. For the CBB, SBB, and DC regimes, root separation occurs at 0.1 radians associated with zero frequency. However, the presence of the negative roots on the real axis of the Laplace domain indicates stability at this pitch angle for these regimes. Nevertheless, these regimes experience divergence starting at 0.15 radians continuing up to the 0.3 radians of pitch angle since the roots are split on the real axis on the left and right halves of the Laplace domain. Also, for the AC regimes the oscillation frequency decreases to zero at pitch angle of 0.15 radians. The roots are split on the left half-plane of the Laplace domain, i.e., the negative damping region, thus resulting in stable motion. An interesting trend is exhibited by the AC5 regime, where the system experiences divergence at 0.25

radians, for the real roots are separated by the stability boundary. It can be seen that AC3 and AC4 promote stability for the unstable regimes for this mode.

Figure 4.6.1a Frequency of the 1st lead-lag mode of Design Case 1

Figure 4.6.1b Damping of the 1st lead-lag mode of Design Case 1

4.5.1.2 1st flap mode

Figs. (4.6.2) and Table (D.1.2) show the eigenvalues corresponding to the 1st flap mode. As the pitch angle is increased from 0.15 to 0.3 radians, the CBB and DC regimes undergo flutter instability. This is associated with positive damping and oscillatory motion of the system. Similarly, SBB encounters flutter at 0.3 radians of pitch angle. As observed, AC4 exhibits a negative root separation at 0.15 and 0.3 radians, i.e. stability is observed at these pitch angles for this regime. Also, the roots are split on the real axis on the left half-plane of the Laplace domain for the AC 5 regime at 0.3 radians of pitch angle. Again, a sinusoidal actuation of the “smart” spring (the AC regimes) obviously stabilizes the motion of the blade throughout the whole range of the pitch angles for this mode.

Figure 4.6.2a Frequency of the 1st flap mode of Design Case 1

Figure 4.6.2b Damping of the 1st flap mode of Design Case 1

4.5.1.3 2nd flap mode

In general, an increase in damping of the 2nd flap mode is seen in Figs. (4.6.3) and Table (D.1.3) for all systems, except for the AC3 and AC4 regimes. The response for the AC3 regime shows an increase in the magnitude of damping at pitch angles from 0 to 0.05 radians and from 0.25 to 0.3 radians, while changing the response towards the stability boundary for the pitch angles from 0.05 to 0.25 radians. The AC5 regime for this mode is weakly damped.

Figure 4.6.3a Frequency of the 2nd flap mode of Design Case 1 Figure 4.6.3b Damping of the 2nd flap mode of Design Case 1

4.5.1.4 1st torsion mode

For the 1st torsion mode, the DC and AC5 regimes start at high damping and a non-zero oscillation frequency at zero pitch angle, and continue with a decrease in damping, as shown in Figs. (4.6.4) and Table (D.1.4). An increase in the magnitude of the damping for this mode is observed for the weakly damped AC3 and AC4 regimes as the pitch angle θ is increased from 0 to 0.3 radians. An interesting trend is exhibited by SBB, where the system has a poorly damped response at zero pitch angle and highly damped responses as the pitch angle increases. CBB is highly damped throughout the given range of the pitch angles.

Figure 4.6.4a Frequency of the 1st torsion mode of Design Case 1

Figure 4.6.4b Damping of the 1st torsion mode of Design Case 1

4.5.2 Design Case 2 ($\mathcal{R}=1$, $\bar{x}_1=0$, $\bar{x}_2=0.15$, including shear strain energy)

This design case is similar to Design Case 1, but with the structural coupling parameter $\mathcal{R}$ being equal to unity and representing full coupling. The results given in Figs. (4.7.1 – 4.7.4) and Tables (D.2.1 – D.2.4) show the damping and frequency as a function of the pitch angle θ for the 1st lead-lag, 1st and 2nd flap, and 1st torsion modes.

4.5.2.1 1st lead-lag mode

Figs. (4.7.1) and Table (D.2.1) demonstrate the response behaviour of the various regimes for the 1st lead-lag mode. Divergence is experienced by the CBB and DC as well as by the AC3, AC4, and AC5 regimes starting at approximately 0.2 radians. In addition, for AC4 roots are represented by real negative values at 0.1 radians, as well as for DC at

0.15 radians, CBB from 0.1 to 0.15 radians, and SBB from 0.15 to 0.3 radians. Hence, the sinusoidal actuation of the “smart” spring does not tend to stabilize the blade motion.

Figure 4.7.1a Frequency of the 1st lead-lag mode of Design Case 2

Figure 4.7.1b Damping of the 1st lead-lag mode of Design Case 2

4.5.2.2 1st flap mode

Figs. (4.7.2) and Table (D.2.2) show that for this mode, in contrast to the corresponding mode in Design Case 1, flutter is experienced only by the DC regime from 0.2 to 0.3 radians of the pitch angle. Its frequency increases as it does for the CBB and SBB regimes. In spite of the fact that the frequency of the AC3, AC4, and AC5 regimes drops to zero at 0.25 and 0.3 (AC3), 0.2 and 0.25 (AC4), and 0.3 (AC5) radians, stable behaviour is indicative since the real roots are separated in the left half-plane of the Laplace domain. It can be noted that the AC3, AC4, and AC5 regimes, all stabilize an unstable DC regime.

Figure 4.7.2a Frequency of the 1st flap mode of Design Case 2

Figure 4.7.2b Damping of the 1st flap mode of Design Case 2

4.5.2.3 2nd flap mode

As shown in Figs. (4.7.3) and Table (D.2.3) the system responds by an increase in damping from 0 to 0.2 radians for the CBB, SBB, and DC regimes, while changing the response towards the stability boundary for pitch angles from 0.2 to 0.3 radians. Initially, a decrease in frequency is observed for these regimes followed by an increase. AC regimes exhibit large variations in damping as well as in frequency. This, however, does not lead to instabilities in this mode, such as flutter and divergence.

Figure 4.7.3a Frequency of the 2nd flap mode of Design Case 2 Figure 4.7.3b Damping of the 2nd flap mode of Design Case 2

4.5.2.4 1st torsion mode

Figs. (4.7.4) and Table (D.2.4) show that, as in Design Case 1, the CBB, DC, and AC5 regimes are highly damped. Also, SBB has a tendency to rapidly increase to high values of damping magnitude as the pitch angle ranges from 0 to 0.05 radians. For AC3 and AC4 the system also responds by an increase in damping magnitude as the pitch angle increases from 0 to 0.3 radians.

Figure 4.7.4a Frequency of the 1st torsion mode of Design Case 2

Figure 4.7.4b Damping of the 1st torsion mode of Design Case 2

4.5.3 Design Case 3 ($\mathcal{R}=0$, $\bar{x}_1=0$, $\bar{x}_2=0.15$, excluding shear strain energy)

This case is different from Design Cases 1 and 2 in that in determining the structural parameters of the actuated blade along the segment of the blade which contains the “smart” spring, the strain energy due to shear is neglected. The results for this case are given in Figs. (4.8.1 - 4.8.4) and tabulated in Tables (D.3.1 – D.3.4).

4.5.3.1 1st lead-lag mode

Similar to Design Case 1, the real roots are split on the left half-plane of the Laplace domain for the 1st lead-lag mode for the CBB, SBB, and DC systems at the pitch angle of 0.1 radians, Figs. (4.8.1) and Table (D.3.1). Again, these regimes experience divergence at pitch angles higher than 0.15 radians, since the roots are separated on the real axis on the left and right halves of the Laplace domain. AC regimes for this design case

experience a zero oscillation frequency at pitch angles from 0.2 to 0.3 radians with negative real roots representing damping, i.e., the stability is provided at these pitch angles. Yet, divergence instability is evident for the AC4 and AC5 at 0.15 radians of pitch angle. Therefore, only the AC3 regime gives the stability improvement to the unstable CBB, SBB, and DC regimes.

Figure 4.8.1a Frequency of the 1st lead-lag mode of Design Case 3

Figure 4.8.1b Damping of the 1st lead-lag mode of Design Case 3

4.5.3.2 1st flap mode

Figs. (4.8.2) and Table (D.3.2) demonstrate that the CBB, SBB, and DC regimes experience flutter, characterized by a positive real part in the eigenvalue throughout the pitch angles from 0.15 to 0.3 radians, thus representing an unstable behaviour of the blade. As can be seen, all AC regimes have highly damped regions and show varying stable responses. The damping magnitude of AC3 as well as AC4 increases for the range of pitch angles from 0 to 0.05 radians and from 0.2 to 0.25 radians and decreases throughout the remaining pitch angle intervals. The magnitude of damping for AC5 rapidly decreases as the pitch angle increases from 0.1 to 0.2 radians. The frequency of the AC4 regime decreases to zero at 0.2 radians and the real roots are negative. Hence, the stability is still present at this pitch angle. Thus, it is evident that the application of the

sinusoidally varying actuation of the “smart” spring significantly improves the stability of the blade when actuated three, four, and five times per rotor revolution.

Figure 4.8.2a Frequency of the 1st flap mode of Design Case 3

Figure 4.8.2b Damping of the 1st flap mode of Design Case 3

4.5.3.3 2nd flap mode

Similar to the 2nd flap mode in Design Case 1, Figs. (4.8.3) and Table (D.3.3) indicate that in this case damping increases for CBB, SBB, and DC with an increase in the pitch angle. A highly damped AC3 response at zero radians experiences a significant decrease in the damping as the pitch angle ranges from 0 to 0.3 radians. One may also observe that the AC4 regime also experiences a sharp decrease in its damping magnitude for the pitch angles from 0 to 0.05 radians followed by an increase in damping on the interval from 0.05 to 0.15 radians and continues with a decrease in damping magnitude for the rest of the pitch angles. The AC5 regime shows an increase from its weakly damped values at low pitch angles to the highly damped ones at higher pitch angles.

Figure 4.8.3a Frequency of the 2nd flap mode of Design Case 3

Figure 4.8.3b Damping of the 2nd flap mode of Design Case 3

4.5.3.4 1st torsion mode

For the range of pitch angles from 0 to 0.3 radians, the CBB, SBB, DC, and AC5 regimes are highly damped (Figs. (4.8.4) and Table (D.3.4)). The SBB, DC, and AC5 regimes also exhibit a sharp decrease in the magnitude of damping at low pitch angles. AC3 and AC4 demonstrate a weakly damped behaviour for the pitch angles ranging from 0 to 0.3 radians.

Figure 4.8.4a Frequency of the 1st torsion mode of Design Case 3 Figure 4.8.4b Damping of the 1st torsion mode of Design Case 3

4.5.4 Design Case 4 ($\mathcal{R} = 1$, $\bar{x}_1 = 0$, $\bar{x}_2 = 0.15$, excluding shear strain energy)

This design case is similar to Design Case 3, but with the structural coupling parameter $\mathcal{R}$ being equal to unity. The results for this case are presented in Figs. (4.9.1-4.9.4) and Tables (D.4.1 – D.4.4).

4.5.4.1 1st lead-lag mode

An increase in θ greatly influences the stability of the CBB and DC regimes, Figs. (4.9.1) and Table (D.4.1). These regimes are stable initially, but are subject to divergence associated with zero frequency and real roots separation on the left and right halves of the Laplace domain at pitch angles from 0.2 to 0.3 radians. Moreover, flutter and divergence are observed for AC3 at 0.25 and 0.3 radians, respectively. The divergence is detected for the AC4 and AC5 regimes at 0.3 radians. Therefore, neither constant nor sinusoidal excitation of the spring stabilizes the “smart” blade for this mode at higher pitch angles.

Figure 4.9.1a Frequency of the 1st lead-lag mode of Design Case 4

Figure 4.9.1b Damping of the 1st lead-lag mode of Design Case 4

4.5.4.2 1st flap mode

As shown in Figs. (4.9.2) and Table (D.4.2), the frequency increases for the CBB, SBB, and DC regimes for the given range of pitch angles. Flutter is observed for DC for pitch angles from 0.2 to 0.3 radians as the damping is represented by positive values at finite frequency. AC4 is characterized by a decrease in frequency for pitch angles from 0 to 0.2 radians, where it then reaches a zero value. At this pitch angle the stability is provided by the negative real roots. However, flutter occurs at 0.25 radians of pitch angle at a non-zero frequency. For AC3 and AC5, the frequency reaches its zero value at 0.3 radians together with the presence of the negative real roots. Thus, AC3 and AC5 are stable regimes throughout the given range of the pitch angles.

Figure 4.9.2a Frequency of the 1st flap mode of Design Case 4 Figure 4.9.2b Damping of the 1st flap mode of Design Case 4

4.5.4.3 2nd flap mode

This mode is represented in Figs. (4.9.3) and Table (D.4.3). In general, a decrease in damping is observed for the AC4 and AC3 regimes, especially on the pitch angle intervals from 0 to 0.05 and from 0 to 0.2 radians, respectively. No rapid damping and frequency variations are present for the CBB, SBB, and DC regimes. As previously in Design Case 2, the AC5 regime is weakly damped at low pitch angles and has a high damping magnitude at 0.3 radians of pitch angle.

Figure 4.9.3a Frequency of the 2nd flap mode of Design Case 4

Figure 4.9.3b Damping of the 2nd flap mode of Design Case 4

4.5.4.4 1st torsion mode

Similar to the corresponding mode in Design Case 3 the SBB, DC, and AC5 regimes exhibit a decrease in damping from the highest value at zero radians, while for the CBB, AC3 and AC4 the damping is increased, Figs. (4.9.4) and Table (D.4.4). This mode is characterized by relatively small frequency variations for all the regimes.

Figure 4.9.4a Frequency of the 1st torsion mode of Design Case 4 Figure 4.9.4b Damping of the 1st torsion mode of Design Case 4

4.5.5 Design Case 5 ($\mathcal{R} = 0$, $\bar{x}_1 = 0$, $\bar{x}_2 = 0.05$, including shear strain energy)

In order to investigate the dependence of the stability of the blade on the length of the “smart” spring, it was determined to examine the stability when the “smart” spring is 5% of the blade length. The results for this case are presented in Figs. (4.10.1-4.10.4) and Tables (D.5.1 – D.5.4).

4.5.5.1 1st lead-lag mode

The 1st lead-lag mode for this design case is depicted in Figs. (4.10.1) and Table (D.5.1). The system experiences a stable behaviour for the CBB and DC regimes on the interval from 0 to 0.1 radians of pitch angle. This is followed by divergence instability for the range of the pitch angles from 0.15 to 0.3 for these regimes. SBB undergoes divergence starting at 0.2 radians. AC3 and AC4 regimes, being stable throughout the range of pitch angles from 0 to 0.3 radians, behave in a non-oscillatory manner starting at 0.15 radians. This is characterized by zero frequency, while damping is represented by two negative roots, which promote the stability for these regimes at this range of pitch angles. AC5 shows a different response from the rest of the regimes - it is damped without oscillation at all pitch angles except for the interval from 0.05 to 0.15 radians. It can be concluded that for this mode the actuation of the “smart” spring three, four, and five times per one rotor revolution is beneficial in providing the stability for the blade at a given range of the pitch angles.

Figure 4.10.1a Frequency of the 1st lead-lag mode of Design Case 5 Figure 4.10.1b Damping of the 1st lead-lag mode of Design Case 5

4.5.5.2 1st flap mode

As can be observed in Figs. (4.10.2) and Table (D.5.2) an increase in the pitch angle leads to flutter in the CBB regime starting at 0.15 radians. SBB and DC regimes are weakly damped with no instabilities present. AC regimes are highly damped except for the AC3 for which the magnitude of damping drops sharply from its highest value at zero radians to its lowest value at 0.3 radians. It is evident that the inclusion of the “smart” spring and its sinusoidal actuation definitely stabilize the behaviour of CBB for this mode.

Figure 4.10.2a Frequency of the 1st flap mode of Design Case 5

Figure 4.10.2b Damping of the 1st flap mode of Design Case 5

4.5.5.3 2nd flap mode

Highly damped CBB, SBB and DC regimes show little variation in damping, as can be seen in Figs. (4.10.3) and Table (D.5.3). A very rapid decrease in damping towards the stability boundary for the AC4 regime, while its frequency has relatively constant behaviour, may possibly lead to flutter at higher pitch angles. AC3 and AC5 also have variable responses in damping magnitude.

Figure 4.10.3a Frequency of the 2nd flap mode of Design Case 5 Figure 4.10.3b Damping of the 2nd flap mode of Design Case 5

4.5.5.4 1st torsion mode

The stability of this mode is associated with low damping for the DC, SBB, AC3, AC4, and AC5 regimes at zero pitch angle as shown in Figs. (4.10.4) and Table (D.5.4). It is also obvious that an increase in the pitch angle results in an increase in damping for these regimes. The CBB regime is highly damped at all pitch angles.

Figure 4.10.4a Frequency of the 1st torsion mode of Design Case 5

Figure 4.10.4b Damping of the 1st torsion mode of Design Case 5

4.5.6 Design Case 6 ($\mathcal{H}=0$, $\bar{x}_1=0$, $\bar{x}_2=0.05$, excluding shear strain energy)

The results for this case are given in Figs. (4.11.1 - 4.11.4) and listed in Tables (D.6.1 – D.6.4).

4.5.6.1 1st lead-lag mode

The 1st lead-lag mode for this design case is depicted in Figs. (4.11.1) and Table (D.6.1). Similar to Design Case 5, the responses of the system demonstrate that negative root separation takes place for the CBB, SBB, DC regimes at a pitch angle of 0.1 radians. At the pitch angles higher than 0.15 radians, the system starts to experience divergence - the roots represent a pair of positive and negative real values for these regimes. Again, the roots are split on the real axis on the left half-plane of the Laplace domain for the AC

regimes from 0.15 to 0.3 radians, hence promoting stable motion for this mode when the sprig is actuated three, four, five times per one revolution.

Figure 4.11.1a Frequency of the 1st lead-lag mode of Design Case 6

Figure 4.11.1b Damping of the 1st lead-lag mode of Design Case 6

4.5.6.2 1st flap mode

Figs. (4.11.2) and Table (D.6.2) show the eigenvalues corresponding to the 1st flap mode of the hingeless blade where flutter is observed for the CBB, SBB, and DC regimes starting at 0.15 radians of pitch angle. The AC4 and AC5 regimes indicate that the frequencies tend to decrease to zero value at 0.15 and 0.3 radians, respectively, and at the same time show quite a variable response. However, the stability is still present at these pitch angles since both roots, i.e., damping, are negative for these regimes. The stability improvement in the CBB, SBB, and DC regimes is therefore provided by the AC regimes for this mode.

Figure 4.11.2a Frequency of the 1st flap mode of Design Case 6

Figure 4.11.2b Damping of the 1st flap mode of Design Case 6

4.5.6.3 2nd flap mode

In this case, as similarly observed in the corresponding mode in Design Case 5, the stability is characterized by highly damped CBB, SBB, and DC regimes, Figs. (4.11.3) and Table (D.6.3). As in the corresponding mode in Design Case 5 the magnitude of damping varies significantly for the AC3 regime, while frequency is nearly constant. In contrast, AC5 has low damping magnitudes.

Figure 4.11.3a Frequency of the 2nd flap mode of Design Case 6 Figure 4.11.3b Damping of the 2nd flap mode of Design Case 6

4.5.6.4 1st torsion mode

Figs. (4.11.4) and Table (D.6.4) indicate that the trends of the 1st torsion mode for the CBB, DC, SBB, and AC5 regimes are characterized by a decrease in the damping magnitude as θ increases. The damping magnitude increases in general for the AC3 and AC4 with an increase in the pitch angle.

Figure 4.11.4a Frequency of the 1st torsion mode of Design Case 6

Figure 4.11.4b Damping of the 1st torsion mode of Design Case 6

Chapter 5

Conclusions and Recommendations

5.1 Conclusions

An aeroelastic stability analysis of a hingeless, “smart” helicopter rotor blade has been studied in this thesis. This work contributed to developing a “smart” rotor blade using smart structure technology with accompanying theoretical dynamic modelling.

Proposed by Hodges and Dowell [10] and Hodges and Ormiston [3], the governing equations of motion and the governing equations of perturbation of the Conventional Baseline Blade regime were used to determine the eigenvalues of the hingeless blade. The development of these equations was based on a uniform, long, straight, slender, homogeneous, isotropic, untwisted blade without chordwise offsets between the mass, tension, elastic, and aerodynamic axes, undergoing small displacements.

Consequently, a reduced-scale model of the Eurocopter BO105 helicopter blade was designed and used as a basis of the study and modelling of a realistic helicopter blade cross-section. The dimensions of the model blade cross-section were determined based on the dynamic characteristics of the full-size Eurocopter BO105 blade

A new set of dynamic equations of motion was derived to account for the implementation of the “smart” spring in the Smart Baseline Blade with the constant actuation of the spring resulting in the Direct Current regime. The “smart” spring was also simulated for periodic actuation using a sinusoidal control scheme (Alternating Current regimes). The eigenvalues were determined next using the Floquet method. The aeroelastic stability analysis of the “smart” blade was then completed to yield damping and oscillation frequencies. This was done by means of considering the lowest modes and analyzing the eigenvalues of the perturbation equations of the 1st lead-lag, 1st and 2nd flap, and the 1st torsion modes of the CBB, SBB, DC, and various AC regimes.

The results of this study are summarized in the following paragraphs.

- The motion of the blade is considered stable if all the modes that are considered, are stable. If at least one becomes unstable, no blade stability is present. A stable behaviour is observed for CBB, SBB and DC regimes for the pitch angle, θ , from 0 to 0.1 radians in Design Cases 1, 3, and 6, and anywhere from 0 to 0.3 in Design Cases 2, 4, and 5. For example, SBB in Design Case 2 is stable up to $\theta = 0.3$ but in Design Case 5 stable only up to $\theta = 0.15$. Beyond these pitch angles one can observe flutter or divergence instabilities in the 1st lead-lag and 1st flap modes. For all the regimes and design cases, the 2nd flap and the 1st torsion modes behave in a stable manner throughout the range of pitch angles that was considered.
- Up to the pitch angles of stability in Design Cases 1, 3, 5, and 6 only, the AC regimes do not experience instabilities. Therefore, the main observation that emerges from the present study is that for the stable range of pitch angles, the inclusion of the “smart”

spring and its periodic actuation do not cause instabilities in the blade motion of an already stable blade for Design Cases 1, 3, 5, and 6.

- As θ increases beyond the stable range of pitch angles discussed above, instabilities may appear in the 1st lead-lag or 1st flap modes for CBB, SBB, and DC regimes. However, upon sinusoidal actuation (three, four, and five times per rotor revolution) of the “smart” spring, the stability of the CBB, SBB and DC regimes at these high pitch angles was improved by certain AC regimes. In Design Cases 5 and 6 all AC regimes, i.e., AC3, AC4, and AC5, improve the stability. Stability was also improved by AC3, and AC4 in Design Case 1. In Design Cases 3 and 4 the stability was provided by means of only AC3 and AC5 regimes, respectively. Nevertheless, the application of the AC voltage to actuate the “smart” spring may not always be beneficial to the blade stability as can be seen from Design Case 2, where none of the proposed actuation regimes improves stability. In fact, the AC regimes in Design Case 2 destabilized a stable SBB regime.
- The results of the stability analysis for various Design Cases indicate that, in general, as the structural coupling parameter, $\mathcal{N}$ increases, the system becomes unstable at higher pitch angles for all AC regimes.
- By comparing the damping and frequency plots for Design Cases 1 and 3, where the length of the “smart” spring was 15% of the blade length, it can be seen that for $\mathcal{N} = 0$ the responses for the CBB, SBB, and DC regimes are not strongly influenced by neglecting the shear strain energy in determining the blade stiffness (except for the SBB regime, which is weakly damped in Design Cases 1 and 2 and highly damped in Design Cases 3 and 4 at zero pitch angle in the 1st torsion mode). However, this is not

the case for the AC regimes. Similar conclusions are applicable to Design Cases 2 and 4, for which complete structural coupling was considered ($\mathcal{R} = 1$).

- Design Cases 5 and 6 with the “smart” spring length of 5% of the blade length represent the cases with and without shear effects, respectively. The results demonstrate that the responses in the 1st torsion mode change when shear effects are considered. In contrast to the highly damped SBB, DC, and AC5 regimes in Design Case 6, the corresponding responses for Design Case 5 start at low damping at zero pitch angle and vary at moderate damping values. Again, comparing the results from Design Case 5 to those from Design Case 6 indicate that neglecting the shear strain energy influences the behaviour of the AC responses. Therefore, it appears that shear effects become more significant as the length of the “smart” spring becomes smaller. It is concluded that for shorter lengths of the “smart” spring, it is important to consider the deformation due to shear when modelling that length of the blade along which the “smart” spring is implemented.

5.2 Recommendations

The following recommendations are proposed to further guide this research towards a fully realizable “smart” hingeless helicopter blade.

- The pre-cone angle of a hingeless helicopter blade significantly affects the dynamics and stability of the blade, as suggested by Hodges and Ormiston [3]. A zero pre-cone angle is assumed in this study. However, various angles should be considered in future analyses.

- Configuration effects of a helicopter blade, such as droop, sweep, and torque offsets can be accounted for in the development of more complicated dynamic equations of motion. Also, the effect of nonuniform blade properties such as mass, bending stiffness, torsional rigidity can be studied.
- The next natural step would be the investigation of the “smart” blade operation in forward flight, where periodic coefficients in the perturbation equations of motion will be present.
- Finally, there is a substantial amount of work ahead in designing an optimal feedback control system to improve system response. Ideally, all results should be validated by experimental data from a properly designed and built model of the “smart” blade that is subjected to proper wind tunnel testing.

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Appendix A

Boundary Conditions

The boundary conditions obtained from the strain and kinetic energies are:

$$\begin{aligned}
 & b(U) - b(T) = \\
 & T \delta u|_0^k + (Tv' - \{[EI_x \cos^2(\theta + \phi) + EI_y \sin^2(\theta + \phi)]v'' \\
 & + (EI_x - EI_y) \cos(\theta + \phi) \sin(\theta + \phi) w'' - EAe_x \left(u' + \frac{v'^2}{2} + \frac{w'^2}{2}\right) \cos(\theta + \phi) - EB_z \theta' \phi' \cos \theta \\
 & - EC_i \phi'' \sin \theta\}' + me [\Omega^2 x \cos(\theta + \phi) + 2\Omega \dot{v} \cos \theta]) \delta v|_0^k + \{[EI_x \cos^2(\theta + \phi) + EI_y \sin^2(\theta + \phi)]v'' \\
 & + (EI_x - EI_y) \cos(\theta + \phi) \sin(\theta + \phi) w'' - EAe_x \left(u' + \frac{v'^2}{2} + \frac{w'^2}{2}\right) \cos(\theta + \phi) - EB_z \theta' \phi' \cos \theta \\
 & - EC_i \phi'' \sin \theta\}' \delta v'|_0^k + (Tw' - \{(EI_x - EI_y) \cos(\theta + \phi) \sin(\theta + \phi) v'' + [EI_x \sin^2(\theta + \phi) \\
 & + EI_y \cos^2(\theta + \phi)]w'' - EAe_x \left(u' + \frac{v'^2}{2} + \frac{w'^2}{2}\right) \sin(\theta + \phi) - EB_z \theta' \phi' \sin \theta + EC_i \phi'' \cos \theta\}' \\
 & + me [\Omega^2 x \sin(\theta + \phi) + 2\Omega \dot{v} \sin \theta]) \delta w|_0^k + \{(EI_x - EI_y) \cos(\theta + \phi) \sin(\theta + \phi) v'' + [EI_x \sin^2(\theta + \phi) \\
 & + EI_y \cos^2(\theta + \phi)]w'' - EAe_x \left(u' + \frac{v'^2}{2} + \frac{w'^2}{2}\right) \sin(\theta + \phi) - EB_z \theta' \phi' \sin \theta + EC_i \phi'' \cos \theta\}' \delta w'|_0^k \\
 & + \{GJ\phi' + EAk_x (\theta + \phi)'\left(u' + \frac{v'^2}{2} + \frac{w'^2}{2}\right) + EB_z \theta' \phi' - EB_z \theta' (v'' \cos \theta + w'' \sin \theta) - [EC_i \phi'' \\
 & + EC_i (w'' \cos \theta - v'' \sin \theta)]'\} \delta \phi|_0^k + [EC_i \phi'' + EC_i (w'' \cos \theta - v'' \sin \theta)] \delta \phi'|_0^k
 \end{aligned}$$

Appendix B

**Effective Position of the “Smart” Spring and Equivalent
Stiffness Calculation**

B1. Effective Position of the “Smart” spring

A beam with the cross-section shown in Fig. (3.4) in Section 3.7 is subjected to a uniform distributed load f . The “smart” unit is situated at a distance l from the free end of the beam. The length of the “smart” spring was taken to be 15% of the beam length R , as shown in Fig. (B.1.1) below.

Figure B.1.1 Effective position of the “smart” spring.

The moment at any cross-section can be found:

$$M = -\frac{fx^2}{2} \quad (\text{B.1.1})$$

where x is the distance from the free end of the beam.

The relationship between bending stress, moment, and geometry of deformation is:

$$\sigma = \frac{Mz}{I_y} \quad (\text{B.1.2})$$

where z -axis is taken perpendicular to the chord of the blade and y -axis is along the chord.

When the spring is actuated becoming an integral part of the beam structure, the strain energy due to bending is:

$$U = \int_0^l \int_{A_0} \frac{\sigma^2}{2E_0} dA dx + \int_l^{l+0.15R} \int_{A_{eq}} \frac{\sigma^2}{2E_{eq}} dA dx + \int_{l+0.15L}^L \int_{A_0} \frac{\sigma^2}{2E_0} dA dx \quad (B.1.3)$$

Substituting Eq. (B.1.1) and (B.1.2) into (B.1.3) and calculating integrals one obtains:

$$U = \frac{f^2 R^5}{40(EI_y)_0} [c^5 + 1 - (c + 0.15)^5] + \frac{f^2 R^5}{40((EI_y)_0 + (EI_y)_{ss})} [(c + 0.15)^5 - c^5] \quad (B.1.4)$$

In the above expression, subscripts ()₀ and ()_{ss} correspond to the properties of the host spar and the piezoelectric material, respectively; A_{eq} is the total cross-section area due to the host spar and the “smart” spring, respectively; E_{eq} is the equivalent Young’s modulus. Also, the position of the piezoelectric material from the free end, l , was taken to be a percentage of the total blade length by:

$$l = cR \quad (B.1.5)$$

The external work due to loading is:

$$W = \frac{1}{5} fR\delta \quad (B.1.6)$$

where δ is the deflection due to bending at the free end.

Equating the elastic strain energy and the external work, one can find the relationship between the free end displacement and the distance from the free end of the beam to the starting point of the “smart” spring, $l = cR$, as follows:

$$\delta = \frac{fR^4}{8} \left[\frac{(EI_y)_0 + (c^5 + 1 - (c + 0.15)^5)(EI_y)_{ss}}{(EI_y)_0((EI_y)_0 + (EI_y)_{ss})} \right] \quad (B.1.7)$$

Shown in Fig. (B.1.2) is the variation of the beam tip displacement versus the position of the “smart” spring from the free end.

Figure B.1.2 Variation of the beam tip displacement versus the position of the “smart” spring relative to the free end

It can be seen that as the “smart” spring is placed closer to the root of the beam, the beam tip displacement decreases.

B.2 Equivalent Stiffness Calculation

Two cases will be considered here. The first is the SBB regime, where the “smart” spring is not actuated and therefore does not contribute to the stiffness of the blade. The second is the DC regime, which is associated with a constantly actuated “smart” spring. These two regimes produce different stiffness values, which will be derived here.

A beam with the cross-section shown in Fig. (3.4) is subjected to a load P applied at the free end of the beam. The “smart” unit was taken to be located throughout the whole length of the beam.

B.2.1 SBB Regime Stiffness

For this design case two options are possible: including shear strain energy in the stiffness calculations and ignoring it. These will be discussed in this section.

The blade is subjected to two types of displacements: due to bending and due to shear. The beam displacement due to bending is:

$$\delta^b = \frac{PR^3}{3(EI_y)_0} \quad (\text{B.2.1})$$

where the subscript $()_0$ corresponds to the properties of the host spar.

For the host beam, the shear is taken by the two webs and the shear stress is given by the shear force divided by the area:

$$\tau \approx \frac{P}{2t_w(d - 2t_f)} \quad (\text{B.2.2})$$

Considering a strip of width t_w , length dx , and height of dz in each web, the strain energy in the two strips is:

$$dU = \frac{1}{2} \frac{\tau^2}{G_0} (2t_w) dx dz \quad (\text{B.2.3})$$

or

$$dU = \frac{1}{4} \frac{P^2}{t_w (d - 2t_f)^2 G_0} dx dz \quad (\text{B.2.4})$$

The shear strain energy for the whole host beam is then obtained by integrating dU over the cross-section area and the length of the beam:

$$U = \frac{1}{4} \frac{P^2}{t_w (d - 2t_f)^2 G_0} \int_0^R dx \int_{-(d-2t_f)/2}^{(d-2t_f)/2} dz = \frac{5}{8} \frac{P^2 R}{t_w (d - 2t_f) E_0} \quad (\text{B.2.5})$$

The work done by the load is:

$$W = \frac{1}{2} P \delta^s \quad (\text{B.2.6})$$

Equating the shear strain energy to the work gives:

$$\delta^s = \frac{5}{4} \frac{PR}{E_0 t_w (d - 2t_f)} \quad (\text{B.2.7})$$

The total displacement due to bending and shear becomes:

$$\begin{aligned} \delta &= \delta^s + \delta^b = \\ &= \frac{PR^3}{3(EI_y)_0} \left[1 + \frac{15}{4} \frac{I_{y0}}{R^2 t_w (d - 2t_f)} \right] = \\ &= \frac{PR^3}{3(EI_y)_{eq}} \end{aligned} \quad (\text{B.2.8})$$

Therefore, the equivalent stiffness was found to be:

$$(EI_y)_{eq} = \frac{E_0 I_{y0}}{1 + C_1} \quad (\text{B.2.9})$$

where

$$C_1 = \frac{15}{4} \frac{I_{y0}}{R^2 t_w (d - 2t_f)} \quad (\text{B.2.10})$$

When ignoring the shear strain energy in the derivations, the equivalent stiffness becomes:

$$(EI)_{eq} = E_0 I_{y0} \quad (\text{B.2.11})$$

Similarly the stiffness can be found for bending about the z-axis.

B.2.2 DC Regime Stiffness

When attached, the piezoelectric material fills the gap between it and the host spar such that the host spar and piezoelectric material deform as a single beam. Therefore, in this case the cross-section which consists of two materials is transformed to an equivalent cross-section of only one of the two materials (aluminum). The procedure of finding the stiffnesses for this case will be outlined in this section.

The beam deflection due to bending is:

$$\delta^b = \frac{PR^3}{3(EI_y)_{transformed}} \quad (\text{B.2.12})$$

where

$$(EI_y)_{transformed} = E_0 (I_{y0} + nI_{yss}), \quad (\text{B.2.13})$$

$$n = \frac{E_{ss}}{E_0}$$

and subscript ()_{ss} corresponds to the properties of the “smart” spring.

The displacement of the transformed beam due to shear can be found using:

$$\tau = \frac{6P}{d^3(n(a-2t_w)+2t_w)} \left(\frac{d^2}{4} - z^2 \right) \quad (\text{B.2.14})$$

The strain energy in a strip of $(n(a-2t_w)+2t_w) \times dz \times dx$ is:

$$dU = \frac{1}{2} \frac{\tau^2}{G_0} (n(a-2t_w)+2t_w) dx dz \quad (\text{B.2.15})$$

Substituting Eq. (B.2.14) into Eq. (B.2.15) and integrating over the area and the length, the shear strain energy for the transformed beam is:

$$U = \frac{3}{2} \frac{P^2 R}{(n(a-2t_w)+2t_w) d E_0} \quad (\text{B.2.16})$$

Equating the work and the strain energy, the deflection of the transformed beam becomes:

$$\delta^s = \frac{3PR}{E_0(n(a-2t_w)+2t_w)d} \quad (\text{B.2.17})$$

Next, the total deflection is:

$$\begin{aligned} \delta &= \delta^s + \delta^b = \\ &= \frac{PR^3}{3E_0(I_{y0}+nI_{yss})} \left[1 + \frac{9(I_{y0}+nI_{yss})}{R^2 d(n(a-2t_w)+2t_w)} \right] = \\ &= \frac{PR^3}{3(EI_y)_{eq}} \end{aligned} \quad (\text{B.2.18})$$

Therefore,

$$(EI_y)_{eq} = \frac{E_0(I_{y0}+nI_{yss})}{1+C_2} \quad (\text{B.2.19})$$

where

$$C_2 = \frac{9}{R^2} \frac{(I_{y0} + nI_{yss})}{(n(a - 2t_w) + 2t_w)d} \quad (\text{B.2.20})$$

The stiffness due to the “smart” spring should be found for the DC regimes.

When accounting for the shear strain energy, the stiffness of the “smart” spring can be found from the following:

$$(EI_y)_{ss} = [(EI_y)_{eq}]_{DC} - [(EI_y)_{eq}]_{SBB} \quad (\text{B.2.21})$$

where

$$\begin{aligned} [(EI_y)_{eq}]_{DC} &= \frac{E_0(I_{y0} + nI_{yss})}{1 + C^2} \\ C_2 &= \frac{9}{R^2} \frac{(I_{y0} + nI_{yss})}{(n(a - 2t_w) + 2t_w)d} \end{aligned} \quad (\text{B.2.22})$$

and

$$[(EI_y)_{eq}]_{SBB} = \frac{E_0 I_{y0}}{1 + C_1}, \quad C_1 = \frac{15}{4} \frac{I_{y0}}{R^2 t_w (d - 2t_f)} \quad (\text{B.2.23})$$

For the case when the shear strain energy is not included, the stiffness due to the “smart” spring becomes:

$$(EI_y)_{ss} = [(EI_y)_{transformed}]_{DC} - [(EI_y)_{eq}]_{SBB} \quad (\text{B.2.24})$$

where $[(EI_y)_{transformed}]_{DC}$ is the stiffness of the transformed beam cross-section, while the

$[(EI_y)_{eq}]_{SBB}$ is:

$$[(EI_y)_{eq}]_{SBB} = E_0 I_{y0} \quad (\text{B.2.25})$$

Similarly, stiffnesses in bending about the z-axis can be found.

Appendix C

Equilibrium Deflections

Figure C.1 Equilibrium tip lead-lag deflections of Design Case 1

Figure C.2 Equilibrium tip flap deflections of Design Case 1

Figure C.3 Equilibrium tip torsion deflections of Design Case 1

Figure C.4 Equilibrium tip lead-lag deflections of Design Case 2

Figure C.5 Equilibrium tip flap deflections of Design Case 2

Figure C.6 Equilibrium tip torsion deflections of Design Case 2

Figure C.7 Equilibrium tip lead-lag deflections of Design Case 3

Figure C.8 Equilibrium tip flap deflections of Design Case 3

Figure C.9 Equilibrium tip torsion deflections of Design Case 3

Figure C.10 Equilibrium tip lead-lag deflections of Design Case 4

Figure C.11 Equilibrium tip flap deflections of Design Case 4

Figure C.12 Equilibrium tip torsion deflections of Design Case 4

Figure C.13 Equilibrium tip lead-lag deflections of Design Case 5

Figure C.14 Equilibrium tip flap deflections of Design Case 5

Figure C.15 Equilibrium tip torsion deflections of Design Case 5

Figure C.16 Equilibrium tip lead-lag deflections of Design Case 6

Figure C.17 Equilibrium tip flap deflections of Design Case 6

Figure C.18 Equilibrium tip torsion deflections of Design Case 6

Appendix D

Tables

Table D.1.1 Eigenvalues of the 1st lead-lag mode of Design Case 1: $\mathcal{R} = 0$, $\bar{x}_1 = 0$, $\bar{x}_2 = 0.15$, including shear effects, $\theta = 0$ to 0.3

θ (rad)	CBB	SBB	DC	AC3	AC4	AC5
0	-0.9050+ 0.3505i	-0.9012+ 0.3487i	-0.9091+ 0.4149i	-1.6353+ 0.4622i	-1.6252+ 0.3199i	-0.6968+ 0.0312i
0.05	-0.9083+ 0.2330i	-0.8971+ 0.3065i	-0.9006+ 0.3786i	-0.8547+ 0.5122i	-1.3509+ 0.3343i	-0.6733+ 0.0636i
0.1	-0.4049; -1.4472	-0.4670; -1.3135	-0.5884; -1.2146	-1.1006+ 0.4861i	-1.0658+ 0.3314i	-0.7504+ 0.1516i
0.15	0.0303; -2.1942	0.0389; -2.0368	0.0467; -2.0795	-1.7111; -0.1405	-0.5949; -0.1082	-0.0966; -1.8529
0.2	0.0305; -2.1256	0.0441; -2.0101	0.1059; -2.1833	-1.3590; -0.4624	-0.4010; -1.6484	-0.4010; -1.6475
0.25	0.0306; -2.0682	0.0493; -1.9949	0.1028; -2.1656	-2.0994; -0.0088	-0.0050; -2.1011	0.0003; -2.0943
0.3	0.0306; -2.0195	0.0548; -1.9886	0.1021; -2.1574	-1.5373; -0.1783	-0.1773; -0.6577	-0.1625; -1.4753

Table D.1.2 Eigenvalues of the 1st flap mode of Design Case 1: $\mathcal{R} = 0$, $\bar{x}_1 = 0$, $\bar{x}_2 = 0.15$, including shear effects, $\theta = 0$ to 0.3

θ (rad)	CBB	SBB	DC	AC3	AC4	AC5
0	-0.0028+ 0.6555i	-0.0028+ 0.6431i	-0.0029+ 0.8135i	-0.8972+ 0.5173i	-0.6801+ 0.4605i	-0.8505+ 0.5035i
0.05	-0.0095+ 0.6600i	-0.0103+ 0.6463i	-0.0073+ 0.8180i	-0.6516+ 0.5652i	-0.6607+ 0.4304i	-0.8361+ 0.5073i
0.1	-0.0304+ 0.7003i	-0.0398+ 0.6854i	-0.0208+ 0.8522i	-0.6565+ 0.6376i	-0.7442+ 0.3505i	-0.8268+ 0.4913i
0.15	0.0091+ 0.8394i	-0.0027+ 0.8549i	0.0392+ 1.0073i	-0.8157+ 0.2752i	-0.8931; -1.8644	-0.7622+ 0.5555i
0.2	0.0128+ 0.8456i	-0.0014+ 0.8615i	0.0668+ 1.0364i	-1.1917+ 0.4296i	-0.5125+ 0.0126i	-0.5236+ 0.4665i
0.25	0.0162+ 0.8503i	-0.0008+ 0.8663i	0.0678+ 1.0422i	-1.0633+ 0.5151i	-0.4036+ 0.2227i	-0.4077+ 0.2697i
0.3	0.0194+ 0.8539i	0.0004+ 0.8697i	0.0690+ 1.0475i	-0.8194+ 0.4144i	-1.0105; -1.4713	-0.7792; -0.9952

Table D.1.3 Eigenvalues of the 2nd flap mode of Design Case 1: $\mathcal{R}=0$, $\bar{x}_1=0$, $\bar{x}_2=0.15$, including shear effects, $\theta=0$ to 0.3

θ (rad)	CBB	SBB	DC	AC3	AC4	AC5
0	-0.6993+ 2.4637i	-0.6017+ 2.4110i	-0.6488+ 2.4687i	-0.6251+ 0.5439i	-0.8523+ 0.5108i	-0.0029+ 0.7439i
0.05	-0.7101+ 2.4481i	-0.6565+ 2.3316i	-0.7001+ 2.4216i	-1.3548+ 0.6285i	-0.8492+ 0.4968i	-0.0047+ 0.7336i
0.1	-0.7438+ 2.3816i	-0.7096+ 2.2067i	-0.7428+ 2.2790i	-0.8818+ 0.6553i	-0.8266+ 0.5489i	-0.0126+ 0.6909i
0.15	-0.7878+ 2.1459i	-0.7438+ 1.9352i	-0.7432+ 1.9046i	-0.7436+ 0.5885i	-0.7464+ 0.2703i	-0.0940+ 0.7192i
0.2	-0.7937+ 2.2411i	-0.7596+ 2.0093i	-0.7673+ 1.9044i	-0.4411+ 0.5601i	-0.9810+ 0.3881i	-0.0401+ 0.3677i
0.25	-0.7885+ 2.3289i	-0.7701+ 2.0817i	-0.7903+ 1.9692i	-0.3613+ 0.7334i	-1.0458+ 0.4646i	-0.0845+ 0.7317i
0.3	-0.7725+ 2.4084i	-0.7758+ 2.1549i	-0.8113+ 2.0326i	-0.8286+ 0.7215i	-0.8382+ 0.5318i	-0.1054+ 0.6674i

Table D.1.4 Eigenvalues of the 1st torsion mode of Design Case 1: $\mathcal{R}=0$, $\bar{x}_1=0$, $\bar{x}_2=0.15$, including shear effects, $\theta=0$ to 0.3

θ (rad)	CBB	SBB	DC	AC3	AC4	AC5
0	-2.0539+ 3.4072i	-0.0021+ 3.8941i	-2.0236+ 3.4118i	-0.0029+ 0.7444i	-0.0029+ 0.7441i	-1.6103+ 1.0382i
0.05	-2.0314+ 3.4172i	-0.7889+ 3.7673i	-1.4538+ 3.6130i	-0.0049+ 0.7369i	-0.0052+ 0.7360i	-1.3519+ 1.1423i
0.1	-1.9563+ 3.4695i	-0.8355+ 3.7059i	-0.9981+ 3.6774i	-0.0110+ 0.6961i	-0.0133+ 0.6928i	-1.0601+ 1.1116i
0.15	-1.8084+ 3.6841i	-0.8103+ 3.7350i	-0.7374+ 3.6535i	-0.0765+ 0.7405i	-0.0848+ 0.7383i	-0.7305+ 1.2366i
0.2	-1.8285+ 3.6192i	-0.7944+ 3.6948i	-0.7256+ 3.6329i	-0.0267+ 0.4704i	-0.0519+ 0.4053i	-0.9822+ 1.1335i
0.25	-1.8520+ 3.5517i	-0.7806+ 3.6408i	-0.7348+ 3.5926i	-0.1075+ 0.7488i	-0.0837+ 0.7370i	-1.0470+ 1.0366i
0.3	-1.8793+ 3.4814i	-0.7703+ 3.5696i	-0.7469+ 3.5294i	-0.0978+ 0.6821i	-0.1070+ 0.6733i	-0.7922+ 0.9549i

Table D.2.1 Eigenvalues of the 1st lead-lag mode of Design Case 2: $\mathcal{R}=1$, $\bar{x}_1=0$, $\bar{x}_2=0.15$, including shear effects, $\theta=0$ to 0.3

θ (rad)	CBB	SBB	DC	AC3	AC4	AC5
0	-0.9050+ 0.3505i	-0.9012+ 0.3487i	-0.9091+ 0.4149i	-1.6353+ 0.4622i	-1.6252+ 0.3199i	-0.6968+ 0.0312i
0.05	-0.8913+ 0.3048i	-0.8450+ 0.4355i	-0.8550+ 0.4725i	-0.9237+ 0.5254i	-1.3154+ 0.2802i	-0.6790+ 0.0405i
0.1	-0.5843; -1.0917	-0.7380+ 0.3722i	-0.7587+ 0.4435i	-0.7785+ 0.5843i	-1.3890; -0.8933	-0.7161+ 0.0731i
0.15	-0.0856; -1.5581	-0.1587; -1.1144	-0.4609; -0.8260	-0.2006+ 0.5368i	-0.7723+ 0.3081i	-0.7759+ 0.1695i
0.2	0.0126; -1.6102	-0.0125; -1.2152	0.0636; -1.4340	0.0611; -0.6543	-0.7136; 0.2750	0.3377; -0.6037
0.25	0.0078; -1.4368	-0.0194; -1.0745	0.1170; -1.4438	0.4076; -0.4451	-0.9925; 0.4929	0.5561; -0.7466
0.3	0.0027; -1.2868	-0.0253; -0.9664	0.0440; -1.1582	0.4129; -0.4450	-0.7766; 0.4839	0.5135; -0.7653

Table D.2.2 Eigenvalues of the 1st flap mode of Design Case 2: $\mathcal{R}=1$, $\bar{x}_1=0$, $\bar{x}_2=0.15$, including shear effects, $\theta=0$ to 0.3

θ (rad)	CBB	SBB	DC	AC3	AC4	AC5
0	-0.0028+ 0.6555i	-0.0028+ 0.6431i	-0.0029+ 0.8135i	-0.8972+ 0.5173i	-0.6801+ 0.4605i	-0.8505+ 0.5035i
0.05	-0.0145+ 0.6590i	-0.0149+ 0.6445i	-0.0146+ 0.8175i	-0.6110+ 0.5498i	-0.6936+ 0.4585i	-0.8433+ 0.5403i
0.1	-0.0614+ 0.6969i	-0.0729+ 0.6735i	-0.0634+ 0.8513i	-0.5734+ 0.6473i	-0.7159+ 0.4167i	-0.9785+ 0.5061i
0.15	-0.0803+ 0.8799i	-0.1199+ 0.8883i	-0.1209+ 1.0077i	-1.0445+ 0.5680i	-1.0618+ 0.3509i	-0.1792+ 0.4909i
0.2	-0.0527+ 0.9983i	-0.0768+ 1.0391i	0.0702+ 1.3091i	-0.8265+ 0.2889i	-0.8884; -0.5488	-0.7766+ 0.5015i
0.25	-0.0628+ 1.0512i	-0.0908+ 1.0957i	0.1829+ 1.4251i	-0.6724; -1.2427	-0.9709; -0.6803	-0.7625+ 0.2694i
0.3	-0.0697+ 1.1080i	-0.1018+ 1.1533i	0.1132+ 1.4795i	-0.9148; -0.9468	-0.8046+ 0.4763i	-0.6884; -0.9283

Table D.2.3 Eigenvalues of the 2nd flap mode of Design Case 2: $\mathcal{R}=1$, $\bar{x}_1=0$, $\bar{x}_2=0.15$, including shear effects, $\theta=0$ to 0.3

θ (rad)	CBB	SBB	DC	AC3	AC4	AC5
0	-0.6993+ 2.4637i	-0.6017+ 2.4110i	-0.6488+ 2.4687i	-0.6251+ 0.5439i	-0.8523+ 0.5108i	-0.0029+ 0.7440i
0.05	-0.7037+ 2.4588i	-0.6325+ 2.3856i	-0.6736+ 2.4519i	-1.3258+ 0.7055i	-0.8510+ 0.5402i	-0.0101+ 0.7253i
0.1	-0.7185+ 2.4381i	-0.6681+ 2.3533i	-0.7140+ 2.4026i	-1.2900+ 0.6916i	-0.7806+ 0.7476i	-0.0321+ 0.6434i
0.15	-0.7732+ 2.3331i	-0.7403+ 2.2434i	-0.7841+ 2.2963i	-0.8595+ 0.6778i	-0.5858+ 0.7613i	-0.6121+ 0.7617i
0.2	-0.7812+ 2.3594i	-0.7696+ 2.2637i	-0.9939+ 1.9598i	-0.6027+ 0.4461i	-1.0312+ 0.0892i	-0.6534+ 1.0272i
0.25	-0.7147+ 2.4866i	-0.7086+ 2.4282i	-1.1359+ 2.0079i	-1.2724+ 0.5500i	-1.0777+ 0.3994i	-0.7344+ 1.1129i
0.3	-0.6256+ 2.5806i	-0.6058+ 2.5683i	-1.1229+ 2.4404i	-0.7265+ 0.4875i	-1.0997+ 0.6008i	-0.8080+ 0.7977i

Table D.2.4 Eigenvalues of the 1st torsion mode of Design Case 2: $\mathcal{R}=1$, $\bar{x}_1=0$, $\bar{x}_2=0.15$, including shear effects, $\theta=0$ to 0.3

θ (rad)	CBB	SBB	DC	AC3	AC4	AC5
0	-2.0539+ 3.4072i	-0.0021+ 3.8941i	-2.0236+ 3.4118i	-0.0029+ 0.7444i	-0.0029+ 0.7441i	-1.6103+ 1.0382i
0.05	-2.0537+ 3.4035i	-0.8659+ 3.7952i	-1.5220+ 3.6441i	-0.0102+ 0.7285i	-0.0107+ 0.7264i	-1.3383+ 1.2063i
0.1	-2.0515+ 3.3958i	-1.0072+ 3.7391i	-1.1506+ 3.6900i	-0.0272+ 0.6559i	-0.0316+ 0.6487i	-0.9426+ 1.1807i
0.15	-2.0115+ 3.4185i	-1.0754+ 3.7062i	-0.9781+ 3.6072i	-0.5008+ 0.6635i	-0.1855+ 0.5037i	-1.0383+ 1.1599i
0.2	-2.0631+ 3.3589i	-1.1473+ 3.6789i	-0.8562+ 3.4919i	-0.8731+ 0.6358i	-0.6297+ 0.8253i	-1.0359+ 0.8866i
0.25	-2.1986+ 3.2452i	-1.2736+ 3.6099i	-0.8735+ 3.3918i	-0.5462+ 0.7414i	-0.5488+ 0.7991i	-1.0375+ 1.1588i
0.3	-2.3446+ 3.1555i	-1.4383+ 3.5383i	-0.9626+ 3.1482i	-0.9952+ 0.5157i	-0.6178+ 0.4843i	-0.9263+ 1.1098i

Table D.3.1 Eigenvalues of the 1st lead-lag mode of Design Case 3: $\mathcal{R} = 0$, $\bar{x}_1 = 0$, $\bar{x}_2 = 0.15$, shear effects are not included, $\theta = 0$ to 0.3

θ (rad)	CBB	SBB	DC	AC3	AC4	AC5
0	-0.9050+ 0.3504i	-0.9015+ 0.3552i	-0.9101+ 0.4120i	-0.9112+ 0.3877i	-0.9118+ 0.3872i	-0.6092+ 0.0579i
0.05	-0.9083+ 0.2327i	-0.8970+ 0.3006i	-0.9027+ 0.3684i	-0.8577+ 0.5141i	-1.4654+ 0.3896i	-0.7214+ 0.0437i
0.1	-0.4045; -1.4478	-0.4453; -1.3416	-0.5609; -1.2507	-0.7442+ 0.5937i	-1.2143+ 0.2276i	-0.6760+ 0.1598i
0.15	0.0303; -2.1945	0.0420; -2.0601	0.1091; -2.2249	-0.4710; -0.8176	-1.7144; 0.1105	-1.6600; 0.1223
0.2	0.0305; -2.1258	0.0434; -2.0205	0.1039; -2.1944	-0.5446+ 0.1905i	-1.6201; -0.6709	-1.9130; -0.2023
0.25	0.0307; -2.0684	0.0491; -2.0022	0.1016; -2.1756	-0.3204; -1.4571	-1.5976; -0.2795	-1.4030; -0.4910
0.3	0.0306; -2.0196	0.0550; -1.9924	0.1015; -2.1660	-0.4195; -1.6894	-1.6433; -0.4380	-1.6324; -0.4188

Table D.3.2 Eigenvalues of the 1st flap mode of Design Case 3: $\mathcal{R} = 0$, $\bar{x}_1 = 0$, $\bar{x}_2 = 0.15$, shear effects are not included, $\theta = 0$ to 0.3

θ (rad)	CBB	SBB	DC	AC3	AC4	AC5
0	-0.0028+ 0.6558i	-0.0028+ 0.6555i	-0.0029+ 0.8229i	-0.6072+ 0.5645i	-0.6081+ 0.4394i	-0.9114+ 0.3873i
0.05	-0.0095+ 0.6603i	-0.0102+ 0.6594i	-0.0072+ 0.8276i	-0.7021+ 0.5458i	-0.7169+ 0.4339i	-0.8563+ 0.4793i
0.1	-0.0030+ 0.7007i	-0.0382+ 0.7016i	-0.0194+ 0.8634i	-0.6470+ 0.6083i	-0.6738+ 0.3478i	-0.8505+ 0.4736i
0.15	0.0091+ 0.8396i	0.0032+ 0.8646i	0.0670+ 1.03246i	-0.5877+ 0.5074i	-0.4181+ 0.1516i	-0.2344+ 0.4963i
0.2	0.0128+ 0.8458i	0.0032+ 0.8693i	0.0685+ 1.03931i	-0.6063+ 0.3128i	-0.4038; -0.3066	-0.0786+ 0.5822i
0.25	0.0162+ 0.8506i	0.0041+ 0.8742i	0.0701+ 1.04536i	-1.0628+ 0.4355i	-0.6164+ 0.0979i	-0.0501+ 0.4372i
0.3	0.0194+ 0.8542i	0.0046+ 0.8777i	0.0717+ 1.05096i	-1.0067+ 0.4184i	-0.5635+ 0.1511i	-0.0406+ 0.4486i

Table D.3.3 Eigenvalues of the 2nd flap mode of Design Case 3: $\mathcal{R}=0$, $\bar{x}_1=0$, $\bar{x}_2=0.15$, shear effects are not included, $\theta=0$ to 0.3

θ (rad)	CBB	SBB	DC	AC3	AC4	AC5
0	-0.6993+ 2.4637i	-0.6157+ 2.4085i	-0.6488+ 2.4678i	-2.0476+ 0.4150i	-2.0461+ 0.5795i	-0.0029+ 0.7541i
0.05	-0.7101+ 2.4480i	-0.6719+ 2.3426i	-0.6983+ 2.4252i	-1.4722+ 0.5615i	-0.8494+ 0.4917i	-0.0047+ 0.7430i
0.1	-0.7438+ 2.3815i	-0.7248+ 2.2152i	-0.7483+ 2.2805i	-1.3581+ 0.6441i	-0.8606+ 0.5848i	-0.0112+ 0.6982i
0.15	-0.7877+ 2.1459i	-0.7531+ 1.9462i	-0.7478+ 1.8450i	-1.3577+ 0.5101i	-1.1926+ 0.2375i	-0.4455+ 0.7296i
0.2	-0.7936+ 2.2412i	-0.7679+ 2.0323i	-0.7743+ 1.9173i	-1.4325+ 0.7170i	-1.0715+ 0.5962i	-0.4840+ 0.7876i
0.25	-0.7885+ 2.3291i	-0.7748+ 2.1142i	-0.7975+ 1.9858i	-0.6136+ 0.6484i	-1.0017+ 0.4165i	-0.6129+ 0.3937i
0.3	-0.7724+ 2.4086i	-0.7740+ 2.1996i	-0.8182+ 2.0544i	-0.5871+ 0.6947i	-1.0261+ 0.5308i	-0.5499+ 0.2684i

Table D.3.4 Eigenvalues of the 1st torsion mode of Design Case 3: $\mathcal{R}=0$, $\bar{x}_1=0$, $\bar{x}_2=0.15$, shear effects are not included, $\theta=0$ to 0.3

θ (rad)	CBB	SBB	DC	AC3	AC4	AC5
0	-2.0539+ 3.4072i	-2.0261+ 3.3991i	-2.0226+ 3.4132i	-0.0029+ 0.7453i	-0.0029+ 0.7542i	-2.0453+ 0.9199i
0.05	-2.0313+ 3.4172i	-1.1490+ 3.6905i	-1.5352+ 3.5806i	-0.0045+ 0.7446i	-0.0048+ 0.7435i	-1.4541+ 1.0865i
0.1	-1.9561+ 3.4696i	-1.0052+ 3.6759i	-1.0708+ 3.6575i	-0.0133+ 0.7079i	-0.0139+ 0.7021i	-1.2249+ 1.0858i
0.15	-1.8082+ 3.6841i	-0.9040+ 3.7305i	-0.7688+ 3.6605i	-0.0391+ 0.4712i	-0.2162+ 0.5140i	-1.1802+ 1.2421i
0.2	-1.8283+ 3.6191i	-0.9010+ 3.6793i	-0.7775+ 3.6329i	-0.0620+ 0.7024i	-0.0732+ 0.6322i	-1.0251+ 0.6095i
0.25	-1.8519+ 3.5516i	-0.8990+ 3.6132i	-0.7894+ 3.5845i	-0.0975+ 0.6218i	-0.1059+ 0.5936i	-1.0526+ 1.0536i
0.3	-1.8792+ 3.4811i	-0.8996+ 3.5265i	-0.8027+ 3.5107i	-0.0300+ 0.5194i	-0.0479+ 0.4707i	-1.0620+ 0.9249i

Table D.4.1 Eigenvalues of the 1st lead-lag mode of Design Case 4: $\mathcal{R} = 1$, $\bar{x}_1 = 0$, $\bar{x}_2 = 0.15$, shear effects are not included, $\theta = 0$ to 0.3

θ (rad)	CBB	SBB	DC	AC3	AC4	AC5
0	-0.9050+ 0.3504i	-0.9015+ 0.3552i	-0.9101+ 0.4120i	-0.9112+ 0.3877i	-0.9118+ 0.3872i	-0.6092+ 0.0579i
0.05	-0.8913+ 0.3046i	-0.8416+ 0.4322i	-0.8575+ 0.4641i	-0.9189+ 0.5487i	-1.5135+ 0.2034i	-0.6669+ 0.0473i
0.1	-0.5838; -1.0923	-0.7346+ 0.3495i	-0.7584+ 0.4294i	-0.7336+ 0.5831i	-1.2963+ 0.2690i	-0.6770+ 0.0961i
0.15	-0.0850; -1.5594	-0.1222; -1.1804	-0.4043; -0.8891	-0.1380+ 0.4392i	-1.3991; -0.8279	-0.6817+ 0.2059i
0.2	0.0126; -1.6102	-0.0107; -1.2248	0.0624; -1.4331	-0.0970; -1.1579	-1.0540; -0.0686	-1.8476; -0.0053
0.25	0.0078; -1.4366	-0.0167; -1.0822	0.1187; -1.4448	0.0768+ 0.4029i	-0.9521+ 0.1437i	-1.0008+ 0.1747i
0.3	0.0027; -1.2865	-0.0221; -0.9724	0.0459; -1.1556	0.4137; -0.6113	-0.6831; 0.4995	-0.7092; 0.3916

Table D.4.2 Eigenvalues of the 1st flap mode of Design Case 4: $\mathcal{R} = 1$, $\bar{x}_1 = 0$, $\bar{x}_2 = 0.15$, shear effects are not included, $\theta = 0$ to 0.3

θ (rad)	CBB	SBB	DC	AC3	AC4	AC5
0	-0.0028+ 0.6558i	-0.0028+ 0.6555i	-0.0029+ 0.8229i	-0.6072+ 0.5645i	-0.6081+ 0.4394i	-0.9114+ 0.3873i
0.05	-0.0145+ 0.6593i	-0.0153+ 0.6577i	-0.0150+ 0.8274i	-0.5338+ 0.5680i	-0.6655+ 0.4595i	-0.8910+ 0.4860i
0.1	-0.0614+ 0.6973i	-0.0745+ 0.6915i	-0.0644+ 0.8642i	-0.6879+ 0.6458i	-0.6715+ 0.4032i	-0.7834+ 0.6019i
0.15	-0.0801+ 0.8805i	-0.1063+ 0.9185i	-0.1155+ 1.0282i	-0.7418+ 0.7266i	-0.7084+ 0.3127i	-0.1932+ 0.4003i
0.2	-0.0527+ 0.9986i	-0.0734+ 1.0498i	0.0697+ 1.3145i	-0.2019+ 0.3602i	-0.6315; -1.5010	-0.5534+ 0.5399i
0.25	-0.0628+ 1.0516i	-0.0879+ 1.1078i	0.1847+ 1.4320i	-0.9941+ 0.2040i	0.0667+ 0.3679i	-0.0035+ 0.2264i
0.3	-0.0697+ 1.1084i	-0.0996+ 1.1669i	0.1180+ 1.4896i	-0.8407; -1.1140	-0.7621+ 0.4021i	-0.6378; -0.7225

Table D.4.3 Eigenvalues of the 2nd flap mode of Design Case 4: $\mathcal{R}=1$, $\bar{x}_1=0$, $\bar{x}_2=0.15$, shear effects are not included, $\theta=0$ to 0.3

θ (rad)	CBB	SBB	DC	AC3	AC4	AC5
0	-0.6993+ 2.4637i	-0.6157+ 2.4085i	-0.6488+ 2.4678i	-2.0476+ 0.4150i	-2.0461+ 0.5795i	-0.0029+ 0.7541i
0.05	-0.7037+ 2.4587i	-0.6370+ 2.3907i	-0.6737+ 2.4524i	-1.5765+ 0.6644i	-0.8498+ 0.5661i	-0.0127+ 0.7411i
0.1	-0.7185+ 2.4381i	-0.6699+ 2.3588i	-0.7177+ 2.4029i	-1.3119+ 0.6773i	-0.7624+ 0.6225i	-0.0544+ 0.6979i
0.15	-0.7734+ 2.3326i	-0.7483+ 2.2310i	-0.7942+ 2.2892i	-1.3243+ 0.6711i	-0.6798+ 0.8765i	-0.6368+ 0.7850i
0.2	-0.7812+ 2.3595i	-0.7609+ 2.2784i	-1.0010+ 1.9735i	-0.7872+ 0.3849i	-0.8567+ 0.3301i	-0.2158+ 1.1335i
0.25	-0.7146+ 2.4868i	-0.6865+ 2.4408i	-1.1458+ 2.0227i	-0.8918+ 0.4364i	-1.0503+ 0.4016i	-0.7758+ 1.0242i
0.3	-0.6254+ 2.5807i	-0.5736+ 2.5710i	-1.1277+ 2.4960i	-1.0466+ 0.1890i	-1.1991+ 0.6276i	-1.1421+ 0.9236i

Table D.4.4 Eigenvalues of the 1st torsion mode of Design Case 4: $\mathcal{R}=1$, $\bar{x}_1=0$, $\bar{x}_2=0.15$, shear effects are not included, $\theta=0$ to 0.3

θ (rad)	CBB	SBB	DC	AC3	AC4	AC5
0	-2.0539+ 3.4072i	-2.0261+ 3.3991i	-2.0226+ 3.4132i	-0.0029+ 0.7453i	-0.0029+ 0.7542i	-2.0453+ 0.9199i
0.05	-2.0536+ 3.4035i	-1.2405+ 3.7274i	-1.6009+ 3.6114i	-0.0121+ 0.7446i	-0.0126+ 0.7421i	-1.4707+ 1.1758i
0.1	-2.0514+ 3.3958i	-1.1982+ 3.7098i	-1.2293+ 3.6684i	-0.0510+ 0.7168i	-0.0542+ 0.7054i	-1.2696+ 1.1933i
0.15	-2.0112+ 3.4188i	-1.1813+ 3.6929i	-1.0348+ 3.5935i	-0.4804+ 0.7341i	-0.1829+ 0.4367i	-1.1730+ 1.1621i
0.2	-2.0631+ 3.3588i	-1.2525+ 3.6580i	-0.9023+ 3.4876i	-1.0459+ 0.5475i	-0.1782+ 0.8927i	-0.9669+ 1.1762i
0.25	-2.1987+ 3.2450i	-1.3928+ 3.5831i	-0.9160+ 3.3798i	-0.6728+ 0.6074i	-0.6828+ 0.6376i	-0.9461+ 1.1800i
0.3	-2.3449+ 3.1552i	-1.5640+ 3.5105i	-1.0169+ 3.0877i	-0.6055+ 0.5449i	-0.6751+ 0.5662i	-0.7470+ 1.0288i

Table D.5.1 Eigenvalues of the 1st lead-lag mode of Design Case 5: $\mathcal{R}=0$, $\bar{x}_1=0$, $\bar{x}_2=0.05$, including shear effects, $\theta=0$ to 0.3

θ (rad)	CBB	SBB	DC	AC3	AC4	AC5
0	-0.9050+ 0.3505i	-0.8945+ 0.3712i	-0.8978+ 0.3898i	-0.8928+ 0.4415i	-0.0022+ 0.3198i	-0.7219; -0.6973
0.05	-0.9083+ 0.2330i	-0.8938+ 0.3186i	-0.8941+ 0.3436i	-0.9607+ 0.4430i	-0.1216+ 0.3624i	-0.7219; -0.6926
0.1	-0.4049; -1.4472	-0.5948; -1.1667	-0.5555; -1.1939	-0.9180+ 0.4595i	-0.7982+ 0.2823i	-0.7285+ 0.0755i
0.15	0.0303; -2.1942	-0.0720; -1.7238	0.1116; -1.9561	-1.8310; -0.2348	-1.8299; -0.2435	-1.8253; -0.2454
0.2	0.0305; -2.1256	0.0432; -1.9292	0.1365; -1.9860	-1.3963; -0.1057	-1.4479; -0.0696	-1.4678; -0.0630
0.25	0.0306; -2.0682	0.0443; -1.9163	0.1340; -1.9724	-1.4779; -0.7723	-1.4648; -0.1606	-1.2483; -0.4282
0.3	0.0306 ; -2.0195	0.0454; -1.9081	0.1332; -1.9674	-1.4076; -0.3884	-1.3507; -0.2903	-1.3097; -0.2981

Table D.5.2 Eigenvalues of the 1st flap mode of Design Case 5: $\mathcal{R}=0$, $\bar{x}_1=0$, $\bar{x}_2=0.05$, including shear effects, $\theta=0$ to 0.3

θ (rad)	CBB	SBB	DC	AC3	AC4	AC5
0	-0.0028+ 0.6555i	-0.0028+ 0.5705i	-0.0029+ 0.6537i	-0.6735+ 0.5089i	-0.8576+ 0.4675i	-0.8567+ 0.4579i
0.05	-0.0095+ 0.6600i	-0.0089+ 0.5718i	-0.0092+ 0.6553i	-0.6267+ 0.4948i	-0.8561+ 0.4548i	-0.8563+ 0.4425i
0.1	-0.0303+ 0.7003i	-0.0380+ 0.5925i	-0.0393+ 0.6792i	-0.6463+ 0.5370i	-0.7121+ 0.4433i	-0.8509+ 0.3206i
0.15	0.0091+ 0.8394i	-0.0605+ 0.7362i	-0.0225+ 0.8637i	-0.4828+ 0.5241i	-0.5151+ 0.2805i	-0.5234+ 0.2126i
0.2	0.0128+ 0.8456i	-0.0269+ 0.8134i	-0.0156+ 0.8760i	-0.2047+ 0.5652i	-0.9858+ 0.0133i	-0.9795+ 0.4697i
0.25	0.0162+ 0.8503i	-0.0258+ 0.8168i	-0.0177+ 0.8746i	-0.1141+ 0.3118i	-0.6904+ 0.3829i	-0.7330+ 0.1337i
0.3	0.0194+ 0.8539i	-0.0248+ 0.8191i	-0.0189+ 0.8733i	-0.0630+ 0.4131i	-0.6795+ 0.4471i	-0.6832+ 0.0612i

Table D.5.3 Eigenvalues of the 2nd flap mode of Design Case 5: $\mathcal{R} = 0$, $\bar{x}_1 = 0$, $\bar{x}_2 = 0.05$, including shear effects, $\theta = 0$ to 0.3

θ (rad)	CBB	SBB	DC	AC3	AC4	AC5
0	-0.6993+ 2.4637i	-0.6654+ 2.4588i	-0.6739+ 2.4987i	-0.0028+ 0.6220i	-0.7086+ 0.5048i	-0.0028+ 0.6219i
0.05	-0.7101+ 2.4481i	-0.6672+ 2.4206i	-0.6822+ 2.4491i	-0.0043+ 0.6145i	-0.7043+ 0.5073i	-0.0047+ 0.6144i
0.1	-0.7438+ 2.3816i	-0.6785+ 2.3087i	-0.6936+ 2.3414i	-0.0158+ 0.5997i	-0.3031+ 0.4793i	-0.0124+ 0.5887i
0.15	-0.7878+ 2.1459i	-0.7132+ 2.0579i	-0.6924+ 2.0207i	-0.0006+ 0.4679i	-0.0857+ 0.3929i	-0.0356+ 0.3999i
0.2	-0.7937+ 2.2411i	-0.7420+ 1.9858i	-0.7145+ 2.0675i	-0.0523+ 0.5905i	-0.1379+ 0.3877i	-0.1116+ 0.5551i
0.25	-0.7885+ 2.3289i	-0.7572+ 2.0390i	-0.7355+ 2.1314i	-0.0726+ 0.5631i	-0.1166+ 0.4046i	-0.0414+ 0.3194i
0.3	-0.7725+ 2.4084i	-0.7707+ 2.0867i	-0.7543+ 2.1838i	-0.3998+ 0.4521i	-0.0725+ 0.3740i	-0.0717+ 0.3530i

Table D.5.4 Eigenvalues of the 1st torsion mode of Design Case 5: $\mathcal{R} = 0$, $\bar{x}_1 = 0$, $\bar{x}_2 = 0.05$, including shear effects, $\theta = 0$ to 0.3

θ (rad)	CBB	SBB	DC	AC3	AC4	AC5
0	-2.0539+ 3.4072i	-0.0022+ 3.5499i	-0.0022+ 3.7684i	-0.0022+ 0.6800i	-0.0028+ 0.6219i	-0.0022+ 1.1797i
0.05	-2.0314+ 3.4172i	-0.0554+ 3.5545i	-0.1466+ 3.7115i	-0.0949+ 0.6529i	-0.0046+ 0.6149i	-0.1183+ 1.1350i
0.1	-1.9563+ 3.4695i	-0.1679+ 3.5727i	-0.2397+ 3.6249i	-0.2500+ 0.5325i	-0.0167+ 0.5946i	-0.2384+ 1.0396i
0.15	-1.8084+ 3.6841i	-0.3176+ 3.6239i	-0.3229+ 3.5186i	-0.4792+ 0.6950i	-0.3580+ 0.5056i	-0.4011+ 0.9923i
0.2	-1.8285+ 3.6192i	-0.3195+ 3.6398i	-0.2878+ 3.5237i	-0.9991+ 0.5621i	-0.1247+ 0.6050i	-0.1507+ 1.0874i
0.25	-1.8520+ 3.5517i	-0.2705+ 3.6194i	-0.2496+ 3.5302i	-0.6682+ 0.7143i	-0.3603+ 0.7186i	-0.3673+ 0.9006i
0.3	-1.8793+ 3.4814i	-0.2237+ 3.5980i	-0.2150+ 3.5343i	-0.5953+ 0.4190i	-0.3836+ 0.6592i	-0.3973+ 0.8598i

Table D.6.1 Eigenvalues of the 1st lead-lag mode of Design Case 6: $\mathcal{R}=0$, $\bar{x}_1=0$, $\bar{x}_2=0.05$, shear effects are not included, $\theta=0$ to 0.3

θ (rad)	CBB	SBB	DC	AC3	AC4	AC5
0	-0.9050+ 0.3504i	-0.9039+ 0.3502i	-0.9076+ 0.3668i	-2.0595+ 0.4013i	-0.8510+ 0.4566i	-0.7181+ 0.0064i
0.05	-0.9083+ 0.2327i	-0.8963+ 0.2998i	-0.8961+ 0.3147i	-0.9051+ 0.4020i	-1.4268+ 0.3757i	-0.6954+ 0.0150i
0.1	-0.4045; -1.4478	-0.4555; -1.3238	-0.4058; -1.3635	-0.9079+ 0.1411i	-0.8930+ 0.1370i	-0.6869+ 0.0858i
0.15	0.0303; -2.1945	0.0338; -2.0336	0.1297; -2.0707	-1.4038; -0.0744	-0.6767; -0.0528	-1.4558; -0.0412
0.2	0.0305; -2.1258	0.0376; -2.0160	0.1283; -2.0518	-1.0869; -0.3771	-0.2574; -1.1562	-1.2293; -0.1922
0.25	0.0307; -2.0684	0.0415; -2.0115	0.1295; -2.0512	-1.0237; -0.6195	-0.3634; -1.4095	-1.4365; -0.3099
0.3	0.0306; -2.0196	0.0455; -2.0178	0.1328; -2.0674	-1.1570; -0.3839	-0.3191; -1.3387	-1.3339; -0.3388

Table D.6.2 Eigenvalues of the 1st flap mode of Design Case 6: $\mathcal{R}=0$, $\bar{x}_1=0$, $\bar{x}_2=0.05$, shear effects are not included, $\theta=0$ to 0.3

θ (rad)	CBB	SBB	DC	AC3	AC4	AC5
0	-0.0028+ 0.6558i	-0.0028+ 0.6558i	-0.0029+ 0.7280i	-0.8760+ 0.4642i	-0.7223+ 0.4875i	-0.8486+ 0.4604i
0.05	-0.0095+ 0.6603i	-0.0101+ 0.6598i	-0.0098+ 0.7321i	-1.4293+ 0.5751i	-0.8759+ 0.3900i	-0.8731+ 0.3904i
0.1	-0.0303+ 0.7007i	-0.0368+ 0.7017i	-0.0363+ 0.7748i	-1.2004+ 0.4745i	-1.1341+ 0.3876i	-0.8863+ 0.1626i
0.15	0.0091+ 0.8396i	0.0048+ 0.8635i	0.0086+ 0.9205i	-0.7477+ 0.4251i	-0.8642; -1.5457	-0.7941+ 0.5488i
0.2	0.0128+ 0.8458i	0.0072+ 0.8698i	0.0075+ 0.9212i	-1.0794+ 0.4272i	-0.8373+ 0.1951i	-0.7942+ 0.3178i
0.25	0.0162+ 0.8506i	0.0092+ 0.8742i	0.0071+ 0.9209i	-1.0920+ 0.3439i	-0.0591+ 0.3587i	-0.8102+ 0.1562i
0.3	0.0194+ 0.8542i	0.0108+ 0.8771i	0.0077+ 0.9198i	-1.0072+ 0.3176i	-0.0727+ 0.3346i	-0.7666; -1.0191

Table D.6.3 Eigenvalues of the 2nd flap mode of Design Case 6: $\mathcal{R} = 0$, $\bar{x}_1 = 0$, $\bar{x}_2 = 0.05$, shear effects are not included, $\theta = 0$ to 0.3

θ (rad)	CBB	SBB	DC	AC3	AC4	AC5
0	-0.6993+ 2.4637i	-0.6804+ 2.4524i	-0.6894+ 2.4920i	-0.6888+ 0.5123i	-2.0510+ 0.6012i	-0.0029+ 0.6988i
0.05	-0.7101+ 2.4480i	-0.7423+ 2.3945i	-0.7465+ 2.4275i	-0.6535+ 0.5461i	-0.6852+ 0.4777i	-0.0046+ 0.6861i
0.1	-0.7438+ 2.3815i	-0.8037+ 2.2594i	-0.7780+ 2.2735i	-0.5848+ 0.6679i	-0.6635+ 0.4201i	-0.0111+ 0.6276i
0.15	-0.7877+ 2.1459i	-0.8266+ 1.9792i	-0.7749+ 1.9961i	-0.9997+ 0.5201i	-0.8967+ 0.3445i	-0.1392+ 0.6220i
0.2	-0.7936+ 2.2412i	-0.8600+ 2.0511i	-0.8094+ 2.0764i	-0.0988+ 0.4479i	-0.9390+ 0.3500i	-0.1501+ 0.4133i
0.25	-0.7885+ 2.3291i	-0.8913+ 2.1190i	-0.8427+ 2.1460i	-0.0377+ 0.3799i	-0.7448+ 0.3718i	-0.0671+ 0.3329i
0.3	-0.7724+ 2.4086i	-0.9217+ 2.1865i	-0.8762+ 2.2110i	-0.0657+ 0.3631i	-0.8747+ 0.4027i	-0.0671+ 0.3252i

Table D.6.4 Eigenvalues of the 1st torsion mode of Design Case 6: $\mathcal{R} = 0$, $\bar{x}_1 = 0$, $\bar{x}_2 = 0.05$, shear effects are not included, $\theta = 0$ to 0.3

θ (rad)	CBB	SBB	DC	AC3	AC4	AC5
0	-2.0539+ 3.4072i	-2.0336+ 3.3961i	-2.0327+ 3.3959i	-0.0029+ 0.6989i	-0.0029+ 0.6988i	-2.0577+ 0.9018i
0.05	-2.0313+ 3.4172i	-1.3767+ 3.6266i	-1.2445+ 3.6722i	-0.0050+ 0.6877i	-0.0049+ 0.6867i	-1.4198+ 1.1236i
0.1	-1.9561+ 3.4696i	-1.1204+ 3.6618i	-0.8457+ 3.6961i	-0.0107+ 0.6386i	-0.0131+ 0.6309i	-1.1195+ 1.1135i
0.15	-1.8082+ 3.6841i	-0.9770+ 3.7410i	-0.6843+ 3.6656i	-0.1147+ 0.6745i	-0.1350+ 0.6447i	-0.9194+ 1.1676i
0.2	-1.8283+ 3.6191i	-0.9612+ 3.6906i	-0.6786+ 3.6461i	-0.7076+ 0.6656i	-0.1347+ 0.4394i	-0.9627+ 1.0733i
0.25	-1.8519+ 3.5516i	-0.9479+ 3.6265i	-0.6831+ 3.6085i	-0.6949+ 0.5605i	-0.9557+ 0.5761i	-0.8956+ 0.9572i
0.3	-1.8792+ 3.4811i	-0.9378+ 3.5430i	-0.7020+ 3.5419i	-0.8447+ 0.5311i	-0.9118+ 0.6799i	-0.8918+ 0.8497i

Appendix E

User Manual and Computer Code

E.1 User Manual

- Open a **MATLAB** window and change to the directory, which contains the present code.
- Set an appropriate path by choosing: **File/Set Path..../**.
- Specify the non-dimensional ending location of the “smart” spring, **x2**, structural coupling parameter, **Rc**, pitch angles, **Tei**, and actuation frequency of the spring, **na**.
- Also, specify if the shear should be present or ignored by setting the function **InputData1.m** or **InputData2.m**
- Run **MainProgram1.m** to calculate the following:

X	Equilibrium deflections.
V	Steady state lead-lag deflection at the tip for particular pitch angle.
W	Steady state flap deflection at the tip for particular pitch angle.
T	Steady state torsion deflection at the tip for particular pitch angle.
vi	Induced inflow velocity.
Eigenvalues	Eigenvalues of the SBB regime.
Eigenvalues_s	Eigenvalues of the DC regime.
Eigenvalues_f	Eigenvalues of the AC regime.

- By running the code **MainProgram2.m** one may obtain eigenvalues for the CBB regime.

E.2 Flow Charts

The following is the flow chart for the MainProgram1.m in order to calculate equilibrium and modal deflections, tip deflections, induced inflow velocity, and eigenvalues of the SBB, DC, and AC regimes:

The following is the flow chart for the MainProgram2.m. Run MainProgram2.m to calculate the following:

X	Equilibrium deflections.
V	Steady state lead-lag deflection at the tip for particular pitch angle.
W	Steady state flap deflection at the tip for particular pitch angle.
T	Steady state torsion deflection at the tip for particular pitch angle.
vi	Induced inflow velocity.
Eigen_values	Eigenvalues of the CBB regime.

E.3 Code Description

EqSolver.m : Sets up the nonlinear algebraic equations of motion in order to solve for the equilibrium deflections for the SBB and DC regimes.

EqSolverCBB.m : Sets up the nonlinear algebraic equations of motion in order to solve for the equilibrium deflections for the CBB regime.

EqSolverFloquet.m : Sets up the nonlinear algebraic equations of motion in order to solve for the equilibrium deflections for the AC regimes.

floquet.m : sets up the state space matrix for the Floquet analysis.

InputData1.m: calculates the cross-section parameters of the smart blade for both actuated and non-actuated “smart” spring cases including shear effects.

InputData2.m: calculates the cross-section parameters of the smart blade for both actuated and non-actuated “smart” spring cases ignoring shear.

InputData1CBB.m: calculates the cross-section parameters of aluminum blade (CBB regime) ignoring shear effects.

InputData2CBB.m: calculates the cross-section parameters aluminum blade (CBB regime) including shear effects.

MCKassembly.m: assembles mass, stiffness, and damping matrices for the SBB and DC regimes.

MCKassemblyCBB.m: assembles mass, stiffness, and damping matrices for the CBB regime.

MCKassemblyFloquet.m: assembles mass, stiffness, and damping matrices for the AC regimes.

MainProgram1.m calculates the equilibrium modal deflections, blade tip deflections, induced inflow velocity, and eigenvalues for the SBB, DC and AC regimes.

MainProgram2.m calculates the equilibrium modal deflections, blade tip deflections, induced inflow velocity, and eigenvalues for the CBB regime.

ShapeFunctions.m: performs numerical integration of the products of the shape functions.

ShapeFunctionsCBB.m: performs numerical integration of the products of the shape functions not including the integrals due to the “smart” spring (CBB regime).

SortFrequencies.m: sorts eigenvalues in the ascending order according to the frequencies.

SysSolver.m: calculates the steady state equilibrium deflections, induced inflow velocity, and the blade tip deflections for the SBB regime.

SysSolverCBB.m: calculates the steady state equilibrium deflections, induced inflow velocity, and the blade tip deflections for the CBB regime.

SysSolverFloquet.m: calculates the steady state equilibrium deflections, induced inflow velocity, and blade tip deflections for the AC regimes.

SysSolverSmart.m: Calculates the steady state equilibrium deflections, induced inflow velocity, and blade tip deflections for the DC regime.

```

%MainProgram1.m
%Calculates the equilibrium modal deflections, blade tip deflections,
%induced inflow velocity, and eigenvalues for the SBB, DC and AC
%regimes.
%-----
clear all;
global bej gaj Ai Bi Ci Dij Eij Iij Jij Mij deij Nij Gamma
global Hij Fijk Fikj Fjki Gijk Hijk Kijk Kjki Kkji Lijk Likj
global Txi Bxi dexij1 dexij2 deltaxij1 Dxij Nx1ij Nx2ij Nx3ij Fxijk
Kxijk
global sb
global Ga Si Cd Bp ce Mu Mu1 Mu2 k1 Num_b Sty Stz Stj Ww Wv Wt
global Rc gamma
global Te x0 x0smart x0f
global V W T X vi x1 x2
global case_number
global mss Sty_p Stz_p Stj_p k1_p Mu2_p Mu1_p Mu_p
global P Psmart
global P_floquet
global omega1
global index10 Om
global tt
global Xelss Xelss_bar Xe2ss Xe2ss_bar
global Sy Sz
global x1 x2
global K1ss K2ss Stj_p_a Stj_p_na kappa_ss KKss

format long e;

% Define pitch angles
Tei=[0.0,0.05,0.1,0.15,0.2,0.25,0.3];

% case_number = 4 for the constantly actuated spring
case_number = 4;

%Define fequency of actuation
na=3;% values: 3,4,5

%Define starting and ending points of the piezoelectric material
x1 = 0.0;
x2 = 0.05;

Rc = 0.0;% structural coupling parameter
Bp = 0.0;% pre-cone angle
sb = 8;% number of mode shapes
Om = 110.0;% spin velocity of rotor
Q = zeros(8);%initial transition matrix
Period = 2*pi/na;% period of time actuation
omega1 = na*Om;% frequency of time actuation
case_number = 4;

%Calculate the integrals of shape functions due to the main structure
%and to the "smart" spring:
[bej,gaj,Ai,Bi,Ci,Dij,Eij,Iij,Jij,Mij,deij,Nij,Hij,Fijk,Fikj,Fjki,Gijk,
Hijk,Kijk,Kjki,Kkji,Lijk,Likj,Gamma,Txi,Bxi,dexij1,dexij2,deltaxij1,Dxi
j,Nx1ij,Nx2ij,Nx3ij,Fxijk,Kxijk] = ShapeFunctions;

```

```

[sb,sb1]=size(bej);

%Define the initial approximation for the equilibrium deflections.
%This initial approximation will be used in order to solve for the
%equilibrium deflections.

x0 = zeros(3*sb+2,1);% for the case when the "smart" spring is not
%actuated but is present in the blade structure
x0smart = zeros(3*sb+2,1);%for the case of "smart" spring being
%actuated
x0f = zeros(3*sb+2,number_intervals);% for the case of periodic
%actuation
number_intervals = 10;
delta = Period/number_intervals;
for k=0:number_intervals
    time(k+1,1) = k*delta;
end
for j=1:sb
    gamma(j)=pi*(j-0.5);
end
%-----
% Start loop on pitch angles, Tei
for uuu=1:length(Tei)

    %-----
    % SMART ROTOR BASELINE REGIME
    %-----

    case_number = 3; % corresponds to this regime

    % Calculate the configuration parameters of the rotor blade for this
    % regime:

    [Ga,Si,Cd,ce,Mu,Mu1,Mu2,k1,Num_b,Sty,Stz,Stj,mss,Sty_p,Stz_p,Stj_p,k
    1_p,Mu2_p,Mu1_p,Mu_p,ka_p,km_p]=InputData1;

    %-----
    % This section will be used in the Floquet analysis
    lambdass12 = Sty_p;
    lambdass22 = Stz_p;
    ka_p_na = ka_p;
    Stj_p_na = Stj_p;
    %-----

    Te=Tei(uuu);

    % Solve for the equilibrium deflections, tip deflections, induced
    inflow velocity:
    [X,vi,T,V,W]=SysSolver;
    Tip=[V W T vi];
    Tips(uuu,:)=Tip;

    % Assemble mass, stiffness and damping matrices:
    [Ms,Mt,Ct1,Ct,Kt1,Kt,Kvac1,deltaK1]=MCKAssembly(X,vi);

    %Find the eigenvalues for the free vibration (vacuum) case:
    P0 =inv(Mt)*Kvac1;

```



```

[Vc0, Va0] = eig(P0);% Va is the squared eigenvalue
Va0=diag(Va0);
[Va01,v]=sort(Va0); % Sort the eigenvalues in ascending order

%Sort the eigenvectors in according to the eigenvalues
JJ=3*sb;
for k=1:JJ
    Vc01(:,k) = Vc0(:,v(k));
end
%Select first four eigenvectors:1stlead-lag mode,1st and 2nd flap
%modes,1st torsion
for k=1:4
    U(:,k) = Vc01(:,k);
end

%Assemble the state space matrix [P]:
Mm=U'*Mt*U;
P11 = zeros(4);
P12 = eye(4);
P21 = -inv(Mm)*U'*Kt*U;
P22 = -inv(Mm)*U'*Ct*U;
P = [P11 P12;P21 P22];

% Find the eigenvalues and eigenvectors of [P]
[Vc,Va]=eig(P);
Va=diag(Va);
[Va,eigen_vector]=SortFrequencies(Va,Vc);
Eigen_values(:,uuu) = Va;

%-----
%-----
%DC REGIME
%-----

clear X vi T V W Ms Mt Ct1 Ct Kt1 Kt
clear Ga Si Cd ce Mu Mu1 Mu2 k1 Num_b Sty Stz Stj mss Sty_p Stz_p
Stj_p k1_p
clear Mu2_p Mu1_p Mu_p ka_p
case_number = 4;% corresponds to this regime

% Calculate the configuration parameters of the rotor blade for this
% regime:
[Ga,Si,Cd,ce,Mu,Mu1,Mu2,k1,Num_b,Sty,Stz,Stj,mss,Sty_p,Stz_p,Stj_p,k
1_p,Mu2_p,Mu1_p,Mu_p,ka_p,km_p]=InputData1;

%-----
% The parameters of this section will be used in the Floquet
% analysis
lambdass13 = Sty_p;
lambdass23 = Stz_p;
ka_p_a = ka_p;
Stj_p_a = Stj_p;
%-----

% Solve for the equilibrium deflections, tip deflections, induced
%inflow velocity:
[X,vi,T,V,W]=SysSolverSmart;

```

```

Tip_smart=[V W T vi];
Tips_smart(uuu,:)=Tip_smart;

% Assemble mass, stiffness and damping matrices:
[Ms,Mt,Ct1,Ct,Kt1,Kt,Kvac2,deltaK2]=MCKassembly(X,vi);

%Find the eigenvalues for the free vibration (vacuum) case:
P00 =inv(Mt)*Kvac2;
[Vc00, Va00] = eig(P00);% Va is the squared eigenvalue
Va00=diag(Va00);
[Va001,v1]=sort(Va00); % Sort the eigenvalues in ascending order

%Sort the eigenvectors in ascending order
JJ=3*sb;
for k=1:JJ
    Vc001(:,k) = Vc00(:,v1(k));
end

%Select first four eigenvectors:
%1stlead-lag mode,1st and 2nd flap modes,1st torsion mode
for k=1:4
    U1(:,k) = Vc001(:,k);
end

%Assemble the state space matrix [P]:
Mm1=U1'*Mt*U1;
Psmart11 = zeros(4);
Psmart12 = eye(4);
Psmart21 = -inv(Mm1)*U1'*Kt*U1;
Psmart22 = -inv(Mm1)*U1'*Ct*U1;
Psmart = [Psmart11 Psmart12;Psmart21 Psmart22];

% Find the eigenvalues and eigenvectors of [P]
[Vcsmart,Vasmart]=eig(Psmart);
Vasmart=diag(Vasmart);
[Vasmart,eigen_vector_smart]=SortFrequencies(Vasmart,Vcsmart);

Eigen_values_smart(:,uuu) = Vasmart;

%-----
%-----
%AC REGIMES
%-----

% The following section uses the Floquet method to solve for the
% eigenvalues.
clear x
clear X vi T V W Ms Mt Ct1 Ct Kt1 Kt
clear Ga Si Cd ce Mu Mu1 Mu2 k1 Num_b Sty Stz Stj mss Sty_p Stz_p
Stj_p k1_p
clear Mu2_p Mu1_p Mu_p

%-----
% Define time varying cross sectional configuration parameters:
Xelss = (lambdass13+lambdass12)/2;
Xelss_bar = (lambdass13-lambdass12)/2;

```

```

Xe2ss = (lambdass23+lambdass22)/2;
Xe2ss_bar = (lambdass23-lambdass22)/2;

K1ss = ((ka_p_a)^2 + (ka_p_na)^2)/2/(km_p)^2;
K2ss = ((ka_p_a)^2 - (ka_p_na)^2)/2/(km_p)^2;

%-----
% This part is needed in order to calculate the eigenvalues for the
% AC3, AC4, and AC5 regimes. Particularly, the transition matrix is
% calculated and the eigenvalues are found according to the Eqs.
% (3.33) in the thesis, Chapter 3
for i=1:8
x0_floquet=zeros(8,1);% initial approximation of the generalized
% coordinates to find the equilibrium deflections
    x0_floquet(i,1)=1;
    for index=1:number_intervals
        tspan = [time(index,1) time(index+1,1)];
        tt = time(index,1);
        %-----
        %Find the equilibrium deflections, and induced inflow
        %velocity:
        [X,vi,X1]=SysSolverFloquet(x0f(:,index));
        x0f(:,index) = X1;

        % Assemble mass, stiffness and damping matrices:
        [Ms,Mt,Ct1,Ct,Kt1,Kt,Kvac_floquet,deltaK_floquet]=
        MCKassemblyFloquet(X,vi);

        %Find the eigenvalues for the free vibration (vacuum) case:
        Pf =inv(Mt)*Kvac_floquet;
        [Vcf, Vaf] = eig(Pf);% Va are the squared eigenvalues
        Vaf=diag(Vaf);
        [Vaf1,vf]=sort(Vaf);% Sort the eigenvalues in ascending order
        % Sort the eigenvectors in ascending order
        JJ=3*sb;
        for k=1:JJ
            Vcf1(:,k) = Vcf(:,vf(k));
        end

        %Select the first four eigenvectors:
        for k=1:4
            Uf(:,k) = Vcf1(:,k);
        end

        %Assemble the state space matrix [P]:
        Mmf=Uf'*Mt*Uf;
        P11f = zeros(4);
        P12f = eye(4);
        P21f = -inv(Mmf)*Uf'*Kt*Uf;
        P22f = -inv(Mmf)*Uf'*Ct*Uf;
        P_floquet = [P11f P12f;P21f P22f];

        %Find the transition matrix [Q]
        [t,x]=ode45('floquet',tspan,x0_floquet);
        [a1,b1]=size(x);
        x0_floquet=x(a1,:);
    end % index

```

```

        Q(:,i)=x0_floquet;% Transition matrix
    end % i

    [Vc_floquet, Va_floquet] = eig(Q);%Eigenvalues and
    %eigenvectors of the transition matrix

    Va_floquet=diag(Va_floquet);

    for ii=1:8
        % Find damping:
        p_re2(ii,1) = 1/2/Period*log((imag(Va_floquet(ii,1)))^2+...
            (real(Va_floquet(ii,1)))^2);

        % Find frequency:
        p_im2(ii,1) =
            1/Period*atan(imag(Va_floquet(ii,1))/real(Va_floquet(ii,1)));

        %The complex eigenvalue is:
        p2(ii,1)=p_re2(ii,1)+p_im2(ii,1)*sqrt(-1);

    end % ii

    p3 = SortFrequencies1(p2);% sorts eigenvalues according to
                                % the magnitude of the frequency
    p3r(:,uuu)=real(p3);
    p3i(:,uuu)=imag(p3);
    p4(:,uuu) = p3;

end %uuu

% Display the obtained eigenvalues on the screen (SBB, DC and AC
% regimes):
for i=1:8
    for j=1:length(Tei)
        Eigenvalues(i,j) = Eigen_values(i,j);% SRB Regime
        Eigenvalues_s(i,j) = Eigen_values_smart(i,j);% DC Regime
        Eigenvalues_f(i,j) = p4(i,j);% AC Regime
    end
end
disp(Eigenvalues);
disp(Eigenvalues_s);
disp(Eigenvalues_f);

```

```

function[bej,gaj,Ai,Bi,Ci,Dij,Eij,Iij,Jij,Mij,deij,Nij,Hij,Fijk,Fikj,Fj
ki,Gijk,Hijk,Kijk,Kjki,Kkji,Lijk,Likj,Gamma,Txi,Bxi,dexij1,dexij2,delta
xij1,Dxij,Nx1ij,Nx2ij,Nx3ij,Fxijk,Kxijk] = ShapeFunctions
% This subroutine calculates integrals of products of shape functions
% for both the main structure and the "smart" spring contribution
%-----
global Alfa Beta x2 x1 case_number
format long e

% Beta and Alfa (associated with the natural frequencies)
%for a fixed - free beam they are:
Beta=[1.875104068712 4.694091132974 7.854757438238 10.99554073488
14.13716839105...
      17.27875953209 20.42035225104 23.56194490181].';
Alfa= -[0.7340955137589 1.018467318759 0.9992244965174
1.0000335532520.9999985501087 1.000000062656 0.999999972924 ...
1.000000000117].';

bej = Beta; gaj = Alfa;

number_modes = 8;% number of modes
number_elmts=100;% number of elements for the numerical integration

% Find 1-D integrals: Ai, Bi, Ci:
for ii=1:number_modes
    Gamma = pi/2+(ii-1)*pi;
    for jj=1:(number_elmts+1)
        XX=(jj-1)/number_elmts;
        F_Ai(jj) = sff(XX,0,ii);
        F_Bi(jj) = XX*sff(XX,0,ii);
        F_Ci(jj) = XX*XX*sff(XX,0,ii);
    end

    [Ai(ii)]=tr_rule(number_elmts+1,F_Ai);
    [Bi(ii)]=tr_rule(number_elmts+1,F_Bi);
    [Ci(ii)]=tr_rule(number_elmts+1,F_Ci);

end %ii

% Find 2-D integrals: Dij, Eij, Hij, Iij, Jij, Mij, Nij:
for ii1=1:number_modes
    for ii2=1:number_modes
        for jj=1:(number_elmts+1)
            XX=(jj-1)/number_elmts;
            F_Dij(jj)=(1-XX*XX)/2*sff(XX,1,ii1)*sff(XX,1,ii2);
            F_Eij(jj)=XX*sff(XX,0,ii1)*sff(XX,0,ii2);
            F_Hij(jj)=XX*sff(XX,0,ii1)*sff(XX,1,ii2);
            F_Iij(jj)=XX*sff(XX,0,ii1)*sfs(XX,0,ii2);
            F_Jij(jj)=XX*XX*sff(XX,0,ii1)*sfs(XX,0,ii2);
            F_Mij(jj)=XX*sfs(XX,0,ii1)*sfs(XX,0,ii2);
            F_Nij(jj)=(1-XX*XX)/2*sfs(XX,1,ii1)*sfs(XX,1,ii2);
        end
        [Dij(ii1,ii2)]=tr_rule(number_elmts+1,F_Dij);
        [Eij(ii1,ii2)]=tr_rule(number_elmts+1,F_Eij);
        [Hij(ii1,ii2)]=tr_rule(number_elmts+1,F_Hij);
        [Iij(ii1,ii2)]=tr_rule(number_elmts+1,F_Iij);
        [Jij(ii1,ii2)]=tr_rule(number_elmts+1,F_Jij);
    end
end

```

```

[Mij(ii1,ii2)]=tr_rule(number_elmts+1,F_Mij);
[Nij(ii1,ii2)]=tr_rule(number_elmts+1,F_Nij);

end %ii2
end %ii1

disp('2-D DONE');
deij = eye(number_modes);

% Find 3-D integrals: Fijk, Gijk, Hijk, Kijk, Kjki, Kkji
for ii1=1:number_modes
    for ii2=1:number_modes
        for ii3=1:number_modes
            for jj=1:(number_elmts+1)
                XX=(jj-1)/number_elmts;
                F_Fijk(jj)=-(1/Beta(ii3))^4*sff(XX,1,ii1)*sff(XX,1,ii2)*...
                    sff(XX,3,ii3);
                F_Gijk(jj)=XX*sff(XX,0,ii1)*sff(XX,0,ii2)*sfs(XX,0,ii3);
                F_Hijk(jj)=sff(XX,0,ii1)*sff(XX,0,ii2)*sfs(XX,0,ii3);
                F_Kijk(jj)=sfs(XX,0,ii1)*sff(XX,2,ii2)*sff(XX,2,ii3);
                F_Kjki(jj)=sfs(XX,0,ii2)*sff(XX,2,ii3)*sff(XX,2,ii1);
                F_Kkji(jj)=sfs(XX,0,ii3)*sff(XX,2,ii2)*sff(XX,2,ii1);

            end
            [Fijk(ii1,ii2,ii3)]=tr_rule(number_elmts+1,F_Fijk);
            [Gijk(ii1,ii2,ii3)]=tr_rule(number_elmts+1,F_Gijk);
            [Hijk(ii1,ii2,ii3)]=tr_rule(number_elmts+1,F_Hijk);
            [Kijk(ii1,ii2,ii3)]=tr_rule(number_elmts+1,F_Kijk);
            [Kjki(ii2,ii3,ii1)]=tr_rule(number_elmts+1,F_Kjki);
            [Kkji(ii3,ii2,ii1)]=tr_rule(number_elmts+1,F_Kkji);

        end
    end
end %ii1

% Write them into file:
fid = fopen('Gijk.txt','w');
for k=1:8
    for i=1:8
        fprintf(fid,'\n');
        for j=1:8
            fprintf(fid,'%20.8d',Gijk(i,j,k));
        end
    end
end

fclose(fid);

%Find Lijk, Likj:
for ii1=1:number_modes
    for ii2=1:number_modes
        for ii3=1:number_modes
            for jj=1:(number_elmts+1)
                XX=(jj-1)/number_elmts;
                F_1(jj)=sff(XX,1,ii2)*sff(XX,2,ii3);
                F_2(jj)=sff(XX,1,ii3)*sff(XX,2,ii2);
            end
        end
    end
end

```

```

        DX=1/number_elmts;
        G1(1)=0.0;
        G1(2)= (F_1(1)+F_1(2))*DX*0.5;
        G2(1)=0.0;
        G2(2)=(F_2(1)+F_2(2))*DX*0.5;
        for jj=3:(number_elmts+1)
            [G1(jj)]=tr_rule(jj,F_1);
            [G2(jj)]=tr_rule(jj,F_2);
        end
        for jj=1:(number_elmts+1)
            XX=(jj-1)/number_elmts;
            F_Lijk(jj)=-G1(jj)*XX*XX*sff(XX,0,ii1)+...
                XX*sff(XX,0,ii1)*sff(XX,0,ii2)*sff(XX,1,ii3);
            F_Likj(jj)=-G2(jj)*XX*XX*sff(XX,0,ii1)+...
                XX*sff(XX,0,ii1)*sff(XX,0,ii3)*sff(XX,1,ii2);
        end
        [Lijk(ii1,ii2,ii3)]=tr_rule(number_elmts+1,F_Lijk);
        [Likj(ii1,ii3,ii2)]=tr_rule(number_elmts+1,F_Likj);

    end %ii3
end %ii2
end %ii1

disp('3-D DONE');

% The following section is written to find the integrals of
% the shape functions due to the "smart" spring:

% Set the matrices to zero:
Txi = zeros(1,number_modes);
Bxi = zeros(1,number_modes);
dexij1 = zeros(number_modes,number_modes);
dexij2 = zeros(number_modes,number_modes);
deltaxij1 = zeros(number_modes,number_modes);
Dxij = zeros(number_modes,number_modes);
Nx1ij = zeros(number_modes,number_modes);
Nx2ij = zeros(number_modes,number_modes);
Nx3ij = zeros(number_modes,number_modes);
Fxijk = zeros(number_modes,number_modes,number_modes);
Kxijk = zeros(number_modes,number_modes,number_modes);

% Find 1-D integrals Txi, Bxi:
for ii=1:number_modes
    for jj=1:(number_elmts+1)
        XX=x1+(jj-1)*(x2-x1)/number_elmts;
        F_Txi(jj) = sfs(XX,0,ii);
        F_Bxi(jj) = XX*sff(XX,0,ii);
    end
    [Txi(ii)]=integral_smart(x1,x2,number_elmts+1,F_Txi);
    [Bxi(ii)]=integral_smart(x1,x2,number_elmts+1,F_Bxi);
end %ii

% Find 2-D integrals dexij1, dexij2, deltaxij1, Dxij, Nx1ij, Nx2ij,
% Nx3ij:
for ii1=1:number_modes
    for ii2=1:number_modes
        for jj=1:(number_elmts+1)

```

```

Upperlimit=x2;
Lowerlimit=x1;

XX=Lowerlimit+(jj-1)*(Upperlimit-Lowerlimit)/number_elmts;
F_dexij1(jj)=sff(XX,0,ii1)*sff(XX,0,ii2);
F_dexij2(jj)=sfs(XX,0,ii1)*sfs(XX,0,ii2);
F_deltaxij1(jj)=sff(XX,2,ii1)*sff(XX,2,ii2);
F1_Dxij(jj)=sff(XX,1,ii1)*sff(XX,1,ii2)*(x2^2-XX^2)/2.0;
F1_Nx1ij(jj)=sfs(XX,1,ii1)*sfs(XX,1,ii2)*(x2^2-XX^2)/2.0;
F_Nx2ij(jj)=(1-XX*XX)/2*sfs(XX,1,ii1)*sfs(XX,1,ii2);
F_Nx3ij(jj)=(x2^2-XX*XX)/2*sfs(XX,1,ii1)*sfs(XX,1,ii2);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
Upperlimit=x1;
Lowerlimit=0;
XX=Lowerlimit+(jj-1)*(Upperlimit-Lowerlimit)/number_elmts;
F2_Dxij(jj)=sff(XX,1,ii1)*sff(XX,1,ii2)*(x2^2-x1^2)/2.0;
F2_Nx1ij(jj)=sfs(XX,1,ii1)*sfs(XX,1,ii2)*(x2^2-x1^2)/2.0;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
F_Dxij(jj)= F1_Dxij(jj)+ F2_Dxij(jj);
F_Nx1ij(jj) = F1_Nx1ij(jj) + F2_Nx1ij(jj);
end

[dexij1(ii1,ii2)]=integral_smart(x1,x2,number_elmts+1,F_dexij1);
[dexij2(ii1,ii2)]=integral_smart(x1,x2,number_elmts+1,F_dexij2);

[deltaxij1(ii1,ii2)]=integral_smart(x1,x2,number_elmts+1,F_deltaxij1);
[Dxij(ii1,ii2)]=integral_smart(x1,x2,number_elmts+1,F1_Dxij)+...
    integral_smart(0,x1,number_elmts+1,F2_Dxij);

[Nx1ij(ii1,ii2)]=integral_smart(x1,x2,number_elmts+1,F1_Nx1ij)+...
    .integral_smart(0,x1,number_elmts+1,F2_Nx1ij);
[Nx2ij(ii1,ii2)]=integral_smart(x1,x2,number_elmts+1,F_Nx2ij);
[Nx3ij(ii1,ii2)]=integral_smart(x1,x2,number_elmts+1,F_Nx3ij);

end %ii2
end %ii1

% Find 3-D integrals Fxijk, Kxijk:
for ii1=1:number_modes
    for ii2=1:number_modes
        for ii3=1:number_modes
            for jj=1:(number_elmts+1)
                Upperlimit=x2;
                Lowerlimit=x1;
                XX=Lowerlimit+(jj-1)*(Upperlimit-Lowerlimit)/number_elmts;
                F_Fx1ijk(jj)=(1/Beta(ii3))^4*sff(XX,1,ii1)*sff(XX,1,ii2)*...
                    .sff(XX,3,ii3);
                F_Kxijk(jj)=sfs(XX,0,ii1)*sff(XX,2,ii2)*sff(XX,2,ii3);
                %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
                Upperlimit=x2;
                Lowerlimit=0;
                XX=Lowerlimit+(jj-1)*(Upperlimit-Lowerlimit)/number_elmts;
                F_Fx2ijk(jj)=(1/Beta(ii3))^4*sff(XX,1,ii1)*sff(XX,1,ii2);
            
```



```

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
    Upperlimit=x1;
    Lowerlimit=0;
    XX=Lowerlimit+(jj-1)*(Upperlimit-Lowerlimit)/number_elmts;
    F_Fx3ijk(jj)=-(1/Beta(ii3))^4*sff(XX,1,ii1)*sff(XX,1,ii2);
    end

    [Fx1ijk(ii1,ii2,ii3)]=integral_smart(x1,x2,number_elmts+1,F_
_Fx1ijk);

    [Fx2ijk(ii1,ii2,ii3)]=integral_smart(0,x2,number_elmts+1,F_
Fx2ijk);

    [Fx3ijk(ii1,ii2,ii3)]=integral_smart(0,x1,number_elmts+1,F_
Fx3ijk);
    Fxijk(ii1,ii2,ii3)= Fx1ijk(ii1,ii2,ii3) + ...
    sff(x2,3,ii3)*Fx2ijk(ii1,ii2,ii3) + ...
    sff(x1,3,ii3)*Fx3ijk(ii1,ii2,ii3);

    [Kxijk(ii1,ii2,ii3)]=integral_smart(x1,x2,number_elmts+1,F_
Kxijk);

    end %ii3
    end %ii2
end %ii1

% Write the matrices into file
fid = fopen('Kxijk.txt','w');
for k=1:8
    for i=1:8
        fprintf(fid,'\n');
        for j=1:8
            fprintf(fid,'%20.8d',Kxijk(i,j,k));
        end
    end
end
fclose(fid);
disp('3-DX DONE');

```

```

function[Ga,Si,Cd,ce,Mu,Mu1,Mu2,k1,Num_b,Sty,Stz,Stj,mss,Sty_p,Stz_p,Stj_p,k1_p,Mu2_p,Mu1_p,Mu_p,ka_p,km_p]=InputData1

%This subroutine provides with cross-sectional parameters used by the
%MainProgram1.m.
%InputData1.m calculates these parameters for the cases of non-actuated
%smart spring and constantly actuated smart spring
% case_number = 3 for non-actuated case
% case_number = 4 for actuated case
%This program includes both shear and bending in the calculation of the
%cross-sectional parameters
%-----

global Ga Si Cd ce Mu Mu1 Mu2 k1 Num_b Sty Stz Stj Ww Wv Wt gamma
global mss Sty_p Stz_p Stj_p k1_p Mu2_p Mu1_p Mu_p
global case_number
global x1 x2

format long e;

if case_number==3
    Wv = 0.514;%non-dimensional, non-rotating lead-lag natural frequency
    Ww = 0.143;%non-dimensional, non-rotating flap natural frequency
    Wt = 3.77;%non-dimensional, non-rotating torsion natural frequency
    Bbe = 1.8575104;%constant from the assumed mode shape
    Gga = pi/2;%constant from the assumed mode shape
    Om = 110;% spin velocity, rad/sec
    R = 2.0*10^3;% blade radius, mm
    Ro_lead = 11340*10^(-9);%lead density, kg/mm^3

    chord = 121.0;% chord length of the blade, mm
    E_p = 71.4*10^3;% Young's modulus of piezoelectric, N/mm^2
    nu_p = 0.3;% Poisson's ratio of the piezoelectric
    G_p = E_p/2/(1+nu_p);% Shear modulus of the piezoelectric, N/mm^2
    Ro_p = 7.5*10^(-6);% Piezoelectric density, kg/mm^3
    E1 = 73.1*10^3;% Young's modulus of the aluminum, N/mm^2
    Ro1 = 2770*10^(-9);%Aluminum density, kg/mm^3
    nu = 0.33;% Poisson's ratio of the aluminum
    G1=E1/2/(1+nu);%Shear modulus of the aluminum, N/mm^2

    %Baseline data
    percentage = 25.0;%defines the location of the shear center of the
    %blade
    c = 44.4;% horizontal dimension of the spar, mm
    alpha = 0.0501;%cross-sectional constant
    kappa = 0.0501;%cross-sectional constant
    ratio = 0.12;%cross-sectional constant
    r2 = percentage*chord/100;% distance from the leading edge to the
    %spar
    delta = 0.3;%cross-sectional constant

    coef1 = (1 - (1-2*alpha)^3*(1-2*kappa) + 2*delta*(ratio)^3 +...
    12*2*delta*ratio*(1/2 + ratio/2)^2)/12;% coefficient to define the
    %moment of inertia
    coef2 = (1 - (1-2*kappa)^3*(1-2*alpha) + 2*delta^3*ratio)/12;%
    %coefficient to define the moment of inertia

```

```

BB = Wv/Ww*sqrt(coef1/coef2);
d = c/BB;%vertical dimension of the spar, mm

K = 2*kappa*alpha*d*c*(c-kappa*c)^2*(d-alpha*d)^2/...
(c*kappa*c + d^2*alpha - alpha^2*d^2 - kappa^2*c^2);% torsional
%constant of the spar, mm^4

Iy = d^3*c*coef1;% moment of inertia about y axis,mm^4
Iz = c^3*d*coef2;% moment of inertia about z axis, mm^4
J = d^3*c*coef1 + c^3*d*coef2;% polar moment of inertia of the spar,
%mm^4
A2 = c*d - (c-2*kappa*c)*(d-2*alpha*d) + 2*ratio*d*delta*c;% area of
%the spar, mm^2
m2 = Ro1*A2;% mass per unit length of the spar, kg/mm
m = E1*10^3*Bbe^4*coef2*BB^3*d^4/(Om^2*R^4*Wv^2);% mass per unit
%length of the blade, kg/mm
r3 = 112.0;% distance from the leading edge to the second tuning
%bar, mm
r1= (K*G1*10^3*Gga^2/(Om^2*R^2*Wt^2)-Ro1*J - (m-m2)*(r3-r2)^2)/...
((m - m2)*(r2-r3)) + 2*r2 - r3;% distance from the leading edge to
%the 1st tuning bar, mm

m1 = (m - m2)*(r2-r3)/(r1-r3);%mass per unit length of the 1st
%tuning bar, kg/mm
m3 = m-m1-m2;% mass per unit length of the second tuning bar, kg/mm

A_p = (c - 2*kappa*c)*(d - 2*alpha*d);% area of the piezoelectric
%material,mm^2
m_p = Ro_p*A_p;% mass of the piezoelectric material per unit length,
%kg/m

% Define mass radii of gyration:
km = sqrt((J*Ro1 + m1*(r2-r1)^2 + m3*(r3-r2)^2)/m );% mm
km1 = sqrt(Iy*Ro1/m);% mm
km2 = sqrt((Iz*Ro1+m1*(r2-r1)^2 + m3*(r3-r2)^2)/m);%mm
ka = sqrt(J/A2);% area radius of gyration

Num_b = 4;% number of blades
ce = chord/R;% chors length
rho_inf = 1.225*10^(-9); % air density in kg/mm^3
Ga = 3*rho_inf*2*pi*0.0605*2000^2/m;
Si = ce*Num_b/pi;% Solidity ratio
Cd = 1.59e-3;% drag coefficient

% Define non-dimensional radii of gyration:
Mu1 = km1/R;
Mu2 = km2/R;
Mu = sqrt(Mu1^2+Mu2^2);

k1 = (ka/km)^2;

% Define non-dymansional stiffnesses and torsional rigidity, Sty,
%Stz, Stj:
% For both shear and bending.
cc1 = 3*(1+nu)*Iy/R^2/(x2-x1)^2/(kappa*c)/(d-2*alpha*d);%
%coefficient
bb1 = 3*(1+nu)*Iz/R^2/(x2-x1)^2/(alpha*d)/(c-2*kappa*c);%coefficient

```

```

EIyo = E1*Iy/(1+cc1);% stiffness
EIzo = E1*Iz/(1+bb1);% stiffness

% The non-dimensional stiffnesses are:
Sty = EIyo/m/Om^2/R^4*10^3;
Stz = EIzo/m/Om^2/R^4*10^3;
Stj = G1*K/m/Om^2/R^4*10^3;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Define configuration parameters due to piezoelectric material:
%Moments of inertia:
IY_p = (c-2*kappa*c)*(d-2*alpha*d)^3/12;% mm^4
Iz_p = (c-2*kappa*c)^3*(d-2*alpha*d)/12;% mm^4
J_p = Iz_p + IY_p;% polar moment of inertia, mm^4

mss = m_p/m;% needed in the equations
m_f = m + m_p;% total mass per unit length, kg/mm

%Define total parameters for the "smart" blade:
IY_f = IY_p + IY;
Iz_f = Iz_p + Iz;
J_f = Iz_f + IY_f;
km1_f = sqrt((IY*Ro1 + IY_p*Ro_p)/m_f);
km2_f = sqrt((Iz*Ro1 + Iz_p*Ro_p + m1*(r2-r1)^2 + m3*(r3-
r2)^2)/m_f);
km_f = sqrt((J*Ro1 + IY_p*Ro_p + Iz_p*Ro_p + m1*(r2-r1)^2 + m3*(r3-
r2)^2)/m_f);

%Define parameters due to the piezoelectric material only:
km1_p = sqrt(km1_f^2 - km1^2); %mm
km2_p = sqrt(km2_f^2 - km2^2); %mm
km_p = sqrt(km_f^2 - km^2);

ka_p = 0.0;
k1_p = 0.0;
Mu1_p = km1_p/R;
Mu2_p = km2_p/R;
Mu_p = km_p/R;

%Define non-dimensional stiffnesses and rigidity for piezoelectric
%material only: %Sty_p, Stz_p, Stj_p:
Sty_p = 0.0;
Stz_p = 0.0;
Stj_p = 0.0;

end% case 3

if case_number==4
Wv = 0.514;%non-dimensional, non-rotating lead-lag natural frequency
Ww = 0.143;%non-dimensional, non-rotating flap natural frequency
Wt = 3.77;%non-dimensional, non-rotating torsionnatural frequency
Bbe = 1.8575104;%constant from the assumed mode shape
Gga = pi/2;%constant from the assumed mode shape
Om = 110;% spin velocity, rad/sec
R = 2.0*10^3;% blade radius, mm
Ro_lead = 11340*10^(-9);%lead density, kg/mm^3

```

```

chord = 121.0; % chord length of the blade, mm
E_p = 71.4*10^3; % Young's modulus of piezoelectric, N/mm^2
nu_p = 0.3; % Poisson's ratio of the piezoelectric
G_p = E_p/2/(1+nu_p); % Shear modulus of the piezoelectric, N/mm^2
Ro_p = 7.5*10^(-6); % Piezoelectric density, kg/mm^3
E1 = 73.1*10^3; % Young's modulus of the aluminum, N/mm^2
Ro1 = 2770*10^(-9); % Aluminum density, kg/mm^3
nu = 0.33; % Poisson's ratio of the aluminum
G1 = E1/2/(1+nu); % Shear modulus of the aluminum, N/mm^2

percentage = 25.0; % defines the location of the shear center of the
% blade
c = 44.4; % horizontal dimension of the spar, mm
alpha = 0.0501; % cross-sectional constant
kappa = 0.0501; % cross-sectional constant
ratio = 0.12; % cross-sectional constant
r2 = percentage*chord/100; % distance from the leading edge to the
% spar
delta = 0.3; % cross-sectional constant

coef1 = (1 - (1-2*alpha)^3*(1-2*kappa) + 2*delta*(ratio)^3 + ...
12*2*delta*ratio*(1/2 + ratio/2)^2)/12; % coefficient to define the
% moment of inertia
coef2 = (1 - (1-2*kappa)^3*(1-2*alpha) + 2*delta^3*ratio)/12; %
% coefficient to define the moment of inertia
BB = Wv/Ww*sqrt(coef1/coef2);
d = c/BB; % vertical dimension of the spar, mm

K = 2*kappa*alpha*d*c*(c-kappa*c)^2*(d-alpha*d)^2/...
(c*kappa*c + d^2*alpha - alpha^2*d^2 - kappa^2*c^2);
% torsional constant of the spar, mm^4

Iy = d^3*c*coef1; % moment of inertia about y axis, mm^4
Iz = c^3*d*coef2; % moment of inertia about z axis, mm^4
J = d^3*c*coef1 + c^3*d*coef2; % polar moment of inertia of the spar,
% mm^4
A2 = c*d - (c-2*kappa*c)*(d-2*alpha*d) + 2*ratio*d*delta*c; % area of
% the spar, mm^2

m2 = Ro1*A2; % mass per unit length of the spar, kg/mm
m = E1*10^3*Bbe^4*coef2*BB^3*d^4/(Om^2*R^4*Wv^2); % mass per unit
% length of the blade, kg/mm

r3 = 112.0; % distance from the leading edge to the second tuning
% bar, mm
r1 = (K*G1*10^3*Gga^2/(Om^2*R^2*Wt^2)-Ro1*J - (m-m2)*(r3-r2)^2)/...
((m - m2)*(r2-r3)) + 2*r2 - r3; % distance from the leading edge to
% the 1st tuning bar, mm
m1 = (m - m2)*(r2-r3)/(r1-r3); % mass per unit length of the 1st
% tuning bar, kg/mm
m3 = m-m1-m2; % mass per unit length of the 2nd tuning bar, kg/mm
A_p = (c - 2*kappa*c)*(d - 2*alpha*d); % area of the piezoelectric
% material, mm^2
m_p = Ro_p*A_p; % mass of the piezoelectric material per unit length,
% kg/m

```

```

% Define mass radii of gyration:
km = sqrt((J*Ro1 + m1*(r2-r1)^2 + m3*(r3-r2)^2)/m);
km1 = sqrt(Iy*Ro1/m);
km2 = sqrt((Iz*Ro1+m1*(r2-r1)^2 + m3*(r3-r2)^2)/m);
ka = sqrt(J/A2);

%The following parameters have the same meaning as in the
%case_number = 3
Num_b = 4;
ce = chord/R;
rho_inf = 1.225*10^(-9); % air density in kg/mm^3
Ga = 3*rho_inf*2*pi*0.0605*2000^2/m;
Si = ce*Num_b/pi;
Cd = 1.59e-3;
Mu1 = km1/R;
Mu2 = km2/R;
Mu = sqrt(Mu1^2+Mu2^2);
k1 = (ka/km)^2;

% Find Sty, Stz, Stj:
cc1 = 3*(1+nu)*Iy/(x2-x1)^2/R^2/(kappa*c)/(d-2*alpha*d);
bb1 = 3*(1+nu)*Iz/(x2-x1)^2/R^2/(alpha*d)/(c-2*kappa*c);
EIyo = E1*Iy/(1+cc1);
EIzo = E1*Iz/(1+bb1);
% Find the non-dimensional stiffnesses and torsional rigidity
Sty = EIyo/m/Om^2/R^4*10^3;
Stz = EIzo/m/Om^2/R^4*10^3;
Stj = G1*K/m/Om^2/R^4*10^3;

% Define configuration parameters due to piezoelectric material:
%Moments of inertia:
Iy_p = (c-2*kappa*c)*(d-2*alpha*d)^3/12;
Iz_p = (c-2*kappa*c)^3*(d-2*alpha*d)/12;
J_p = Iz_p + Iy_p;

mss = m_p/m; % needed in the equations
m_f = m + m_p;% total mass per unit length, kg/mm

%Define total parameters for the "smart" blade:
A_f = A_p + A2;
Iy_f = Iy_p + Iy;
Iz_f = Iz_p + Iz;
J_f = Iz_f + Iy_f;

km1_f = sqrt((Iy*Ro1 + Iy_p*Ro_p)/m_f);
km2_f = sqrt((Iz*Ro1 + Iz_p*Ro_p + m1*(r2-r1)^2 + m3*(r3-
r2)^2)/m_f);

km_f = sqrt((J*Ro1 + Iy_p*Ro_p + Iz_p*Ro_p + m1*(r2-r1)^2 + m3*(r3-
r2)^2)/m_f);

%Define parameters due to the piezoelectric material only:
km1_p = sqrt(km1_f^2 - km1^2); %mm
km2_p = sqrt(km2_f^2 - km2^2); %mm
km_p = sqrt(km_f^2 - km^2);
ka_f = sqrt(J_f/A_f);
ka_p = sqrt((ka_f)^2 - ka^2);

```

```

k1_p = (ka_p/km_p)^2;
Mu1_p = km1_p/R;
Mu2_p = km2_p/R;
Mu_p = km_p/R;

%Define non-dimensional stiffnesses and rigidity due to the
%piezoelectric material only:
%Find Sty_p:
nn = E_p/E1;

cc2=36/5*(1+nu)*(Iy+nn*Iy_p)/(x2-x1)^2/R^2/d/(2*kappa*c + nn*(c-
2*kappa*c));
EIy_eq_bond = E1*(Iy + Iy_p*nn)/(1+cc2);
Sty_p = (EIy_eq_bond - EIyo)/m/Om^2/R^4*10^3;

%Find Stz_p:
bb2=36/5*(1+nu)*(Iz+nn*Iz_p)/(x2-x1)^2/R^2/c/(nn*(d-
2*alpha*d)+2*alpha*d);
EIz_eq_bond = E1*(Iz + Iz_p*nn)/(1+bb2);
Stz_p = (EIz_eq_bond - EIzo)/m/Om^2/R^4*10^3;

%Find Stj_p:
K_p = (c-2*kappa*c)*(d-2*alpha*d)^3*(1/3 - 0.21*(d-2*alpha*d)/...
(c-2*kappa*c)*(1 - (d-2*alpha*d)^4/(12*(c-2*kappa*c)^4)));
GK_eq = G1*K+G_p*K_p;
Stj_p = (G_p*K_p)/m/Om^2/R^4*10^3;
end

```

```
Function[Ga,Si,Cd,ce,Mu,Mu1,Mu2,k1,Num_b,Sty,Stz,Stj,mss,Sty_p,Stz_p,Stj_p,k1_p,Mu2_p,Mu1_p,Mu_p,ka_p,km_p]=InputData2
```

```
%This subroutine provides with cross-sectional parameters used by the %MainProgram1.m.
```

```
%InputData2.m calculates these parameters for the cases of non-actuated %"smart" spring and constantly actuated smart spring
```

```
% case_number = 3 corresponds to the non-actuated case
```

```
% case_number = 4 corresponds to the actuated case
```

```
%This program does not include shear strain energy calculation into %account.
```

```
% All the configuration parameters are analogous to those defined in %the InputData1.m
```

```
%-----
```

```
global Ga Si Cd ce Mu Mu1 Mu2 k1 Num_b Sty Stz Stj Ww Wv Wt gamma
```

```
global mss Sty_p Stz_p Stj_p k1_p Mu2_p Mu1_p Mu_p
```

```
global case_number
```

```
global x1 x2
```

```
format long e;
```

```
if case_number==3
```

```
Wv = 0.514;
```

```
Ww = 0.143;
```

```
Wt = 3.77;
```

```
Bbe = 1.8575104;
```

```
Gga = pi/2;
```

```
Om = 110;
```

```
R = 2.0*10^3;
```

```
Ro_lead = 11340*10^(-9);
```

```
chord = 121.0;
```

```
E_p = 71.4*10^3;
```

```
nu_p = 0.3;
```

```
G_p = E_p/2/(1+nu_p);
```

```
Ro_p = 7.5*10^(-6);
```

```
E1 = 73.1*10^3;
```

```
Ro1 = 2770*10^(-9);
```

```
nu = 0.33;
```

```
G1=E1/2/(1+nu);
```

```
percentage = 25.0;
```

```
c = 44.4;
```

```
alpha = 0.0501;
```

```
kappa = 0.0501;
```

```
ratio = 0.12;
```

```
r2 = percentage*chord/100;
```

```
delta = 0.3;
```

```
coef1 = (1 - (1-2*alpha)^3*(1-2*kappa) + 2*delta*(ratio)^3 + ...  
12*2*delta*ratio*(1/2 + ratio/2)^2)/12;
```

```
coef2 = (1 - (1-2*kappa)^3*(1-2*alpha) + 2*delta^3*ratio)/12;
```

```
BB = Wv/Ww*sqrt(coef1/coef2);
```

```
d = c/BB;
```

```
K = 2*kappa*alpha*d*c*(c-kappa*c)^2*(d-alpha*d)^2/...
```

```
(c*kappa*c + d^2*alpha - alpha^2*d^2 - kappa^2*c^2);
```

```
Iy = d^3*c*coef1;
```

```
Iz = c^3*d*coef2;
```

```
J = d^3*c*coef1 + c^3*d*coef2;
```

```
A2 = c*d - (c-2*kappa*c)*(d-2*alpha*d) + 2*ratio*d*delta*c;
```

```
m2 = Ro1*A2;
```

```
m = E1*10^3*Bbe^4*coef2*BB^3*d^4/(Om^2*R^4*Wv^2);
```

```
r3 = 112.0;
```

```
r1= (K*G1*10^3*Gga^2/(Om^2*R^2*Wt^2)-Ro1*J - (m-m2)*(r3-r2)^2)/...
```



```

        ((m - m2)*(r2-r3)) + 2*r2 - r3;
m1 = (m - m2)*(r2-r3)/(r1-r3);
m3 = m-m1-m2;

A_p = (c - 2*kappa*c)*(d - 2*alpha*d);
m_p = Ro_p*A_p;% kg/m

km = sqrt((J*Ro1 + m1*(r2-r1)^2 + m3*(r3-r2)^2)/m );
km1 = sqrt(Iy*Ro1/m);
km2 = sqrt((Iz*Ro1+m1*(r2-r1)^2 + m3*(r3-r2)^2)/m);
ka = sqrt(J/A2);

Num_b = 4;
ce = chord/R;
rho_inf = 1.225*10^(-9); % air density in kg/mm^3
Ga = 3*rho_inf*2*pi*0.0605*2000^2/m;
Si = ce*Num_b/pi;
Cd = 1.59e-3;
Mu1 = km1/R;
Mu2 = km2/R;
Mu = sqrt(Mu1^2+Mu2^2);
k1 = (ka/km)^2;

% Find Sty, Stz, Stj:
EIyo = E1*Iy;
EIzo = E1*Iz;

Sty = EIyo/m/Om^2/R^4*10^3;
Stz = EIzo/m/Om^2/R^4*10^3;
Stj = G1*K/m/Om^2/R^4*10^3;

Iy_p = (c-2*kappa*c)*(d-2*alpha*d)^3/12;
Iz_p = (c-2*kappa*c)^3*(d-2*alpha*d)/12;
J_p = Iz_p + Iy_p;

mss = m_p/m; % needed in the equations
m_f = m + m_p;
Iy_f = Iy_p + Iy;
Iz_f = Iz_p + Iz;
J_f = Iz_f + Iy_f;

km1_f = sqrt((Iy*Ro1 + Iy_p*Ro_p)/m_f);
km2_f = sqrt((Iz*Ro1 + Iz_p*Ro_p + m1*(r2-r1)^2 + m3*(r3-
r2)^2)/m_f);
km_f = sqrt((J*Ro1 + Iy_p*Ro_p + Iz_p*Ro_p + m1*(r2-r1)^2 + m3*(r3-
r2)^2)/m_f);

km1_p = sqrt(km1_f^2 - km1^2);
km2_p = sqrt(km2_f^2 - km2^2);
km_p = sqrt(km_f^2 - km^2);

ka_p = 0.0;
k1_p = 0.0;
Mu1_p = km1_p/R;
Mu2_p = km2_p/R;
Mu_p = km_p/R;

```

```

%Find Sty, Stz:
Sty_p = 0.0;
Stz_p = 0.0;
Stj_p = 0.0;
end% case 3

if case_number==4
    Wv = 0.514;
    Wt = 3.77;
    Gga = pi/2;
    R = 2.0*10^3;
    chord = 121.0;
    E_p = 71.4*10^3;
    G_p = E_p/2/(1+nu_p);
    E1 = 73.1*10^3;
    nu = 0.33;

    Ww = 0.143;
    Bbe = 1.8575104;
    Om = 110;
    Ro_lead = 11340*10^(-9); %kg/mm^3

    nu_p = 0.3;
    Ro_p = 7.5*10^(-6);
    Ro1 = 2770*10^(-9);
    G1 = E1/2/(1+nu);

    percentage = 25.0;
    c = 44.4;
    alpha = 0.0501;
    kappa = 0.0501;
    ratio = 0.12;
    r2 = percentage*chord/100;
    delta = 0.3;

    coef1 = (1 - (1-2*alpha)^3*(1-2*kappa) + 2*delta*(ratio)^3 + ...
        12*2*delta*ratio*(1/2 + ratio/2)^2)/12;
    coef2 = (1 - (1-2*kappa)^3*(1-2*alpha) + 2*delta^3*ratio)/12;
    BB = Wv/Ww*sqrt(coef1/coef2);
    d = c/BB;

    K = 2*kappa*alpha*d*c*(c-kappa*c)^2*(d-alpha*d)^2/...
        (c*kappa*c + d^2*alpha - alpha^2*d^2 - kappa^2*c^2);

    Iy = d^3*c*coef1;
    Iz = c^3*d*coef2;
    J = d^3*c*coef1 + c^3*d*coef2;
    A2 = c*d - (c-2*kappa*c)*(d-2*alpha*d) + 2*ratio*d*delta*c;
    m2 = Ro1*A2;
    m = E1*10^3*Bbe^4*coef2*BB^3*d^4/(Om^2*R^4*Wv^2);
    r3 = 112.0;
    r1 = (K*G1*10^3*Gga^2/(Om^2*R^2*Wt^2)-Ro1*J - (m-m2)*(r3-r2)^2)/...
        ((m - m2)*(r2-r3)) + 2*r2 - r3;
    m1 = (m - m2)*(r2-r3)/(r1-r3);
    m3 = m-m1-m2;

    A_p = (c - 2*kappa*c)*(d - 2*alpha*d);
    m_p = Ro_p*A_p;

    km = sqrt((J*Ro1 + m1*(r2-r1)^2 + m3*(r3-r2)^2)/m);
    km1 = sqrt(Iy*Ro1/m);
    km2 = sqrt((Iz*Ro1+m1*(r2-r1)^2 + m3*(r3-r2)^2)/m);
    ka = sqrt(J/A2);
    Num_b = 4;
    ce = chord/R;
    rho_inf = 1.225*10^(-9); % air density in kg/mm^3
    Ga = 3*rho_inf*2*pi*0.0605*2000^2/m;% Lock number

```

```

Si = ce*Num_b/pi;
Cd = 1.59e-3;
Mu1 = km1/R;
Mu2 = km2/R;
Mu = sqrt(Mu1^2+Mu2^2);
k1 = (ka/km)^2;

% Find Sty, Stz, Stj:
EIyo = E1*Iy;
EIzo = E1*Iz;
Sty = EIyo/m/Om^2/R^4*10^3;
Stz = EIzo/m/Om^2/R^4*10^3;
Stj = G1*K/m/Om^2/R^4*10^3;

Iy_p = (c-2*kappa*c)*(d-2*alpha*d)^3/12;
Iz_p = (c-2*kappa*c)^3*(d-2*alpha*d)/12;
J_p = Iz_p + Iy_p;

mss = m_p/m; % needed in the equations
m_f = m + m_p;

A_f = A_p + A2;
Iy_f = Iy_p + Iy;
Iz_f = Iz_p + Iz;
J_f = Iz_f + Iy_f;

km1_f = sqrt((Iy*Ro1 + Iy_p*Ro_p)/m_f);
km2_f = sqrt((Iz*Ro1 + Iz_p*Ro_p + m1*(r2-r1)^2 + m3*(r3-
r2)^2)/m_f);

km_f = sqrt((J*Ro1 + Iy_p*Ro_p + Iz_p*Ro_p + m1*(r2-r1)^2 + m3*(r3-
r2)^2)/m_f);

km1_p = sqrt(km1_f^2 - km1^2);
km2_p = sqrt(km2_f^2 - km2^2);
km_p = sqrt(km_f^2 - km^2);

ka_f = sqrt(J_f/A_f);
ka_p = sqrt((ka_f)^2 - ka^2);
k1_p = (ka_p/km_p)^2;

Mu1_p = km1_p/R;
Mu2_p = km2_p/R;
Mu_p = km_p/R;

%Find Sty_p:
Sty_p = E_p*Iy_p/m/Om^2/R^4*10^3;
Stz_p = E_p*Iz_p/m/Om^2/R^4*10^3;

%Find Stj:
K_p = (c-2*kappa*c)*(d-2*alpha*d)^3*(1/3 - 0.21*(d-2*alpha*d)/...
(c-2*kappa*c)*(1 - (d-2*alpha*d)^4/(12*(c-2*kappa*c)^4)));
GK_eq = G1*K+G_p*K_p;
Stj_p = (G_p*K_p)/m/Om^2/R^4*10^3;

end

```

```

function [X,vi,T,V,W]=SysSolver
% This subroutine solves for the equilibrium deflections, as well as
% for the tip deflections and inflow velocity for the SBB regime
%-----

global x0 sb Te
format long e;
% Set the options used in the fsolve subroutine
options=optimset('Display', 'iter','LargeScale', 'off','TolX',1e-8);
disp('THETA'); disp(Te);

% Solve for the equilibrium quantities:
X = fsolve('EqSolver',x0,options);

x0=X;

% Steady state lead-lag deflection at the tip:
V=0;
for i=1:sb
    V=V+X(i)*(-1)^(i+1);
end
V=2*V;

% Steady state flap deflection at the tip:
W=0;
for i=1:sb
    W=W+X(sb+i)*(-1)^(i+1);
end
W=2*W;

% Steady state torsion deflection at the tip:
T=0;
for i=1:sb
    T=T+X(2*sb+i)*(-1)^(i+1);
end
T=sqrt(2)*T;

% Inflow velocity:
vi = X(3*sb+2);

X=X(1:3*sb);

```

```

function [Ms,Mt,Ct1,Ct,Kt1,Kt,Kvac,deltaK]=MCKassembly(X,vi)
%This function assembles mass, stiffness, and damping matrices for
%the SBB and DC regimes. It also uses the equilibrium deflections
%which were found previously
%-----

global Stz Sty Stj Rc Te Bp ce Si Mu Mu1 Mu2 k1
global Ga Cd
global bej gaj deij Dij Ai Bi Ci Hij Jij Nij Lijk Likj Kkji Kijk Kjki
global Iij Eij Mij Gijk Hijk Fikj Fjki Fijk Gamma
global Txi Bxi dexij1 dexij2 deltaxij1 Dxij Nxlij Nx2ij Nx3ij Fxijk
Kxijk
global sb
global gamma case_number
global mss Sty_p Stz_p Stj_p k1_p Mu2_p Mu1_p Mu_p
format long e
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Redefine the equilibrium deflections
V0=X(1:sb);
W0=X((sb+1):(2*sb));
PHI0=X((2*sb+1):(3*sb));

% Mass matrix construction.
M11 = deij;
M22 = deij;
M33 = deij*Mu^2;
M12 = zeros(sb);
M13 = zeros(sb);
M21 = zeros(sb);
M23 = zeros(sb);
M31 = zeros(sb);
M32 = zeros(sb);
Ms = [M11 M12 M13;M21 M22 M23; M31 M32 M33];

% Structural and aerodynamics mass matrix
Mt11=M11;
Mt22=M22+deij*Ga*ce/24;
Mt33=M33;
Mt = [Mt11 M12 M13;M21 Mt22 M23;M31 M32 Mt33];

% Mass matrix construction due to the "smart" spring.
dM11 = mss*dexij1;
dM22 = mss*dexij1;
dM33 = dexij2*(Mu_p^2+mss*Mu^2+mss*Mu_p^2);
dM12 = zeros(sb);
dM13 = zeros(sb);
dM21 = zeros(sb);
dM23 = zeros(sb);
dM31 = zeros(sb);
dM32 = zeros(sb);
deltaM = [dM11 dM12 dM13;dM21 dM22 dM23; dM31 dM32 dM33];
Mt = Mt + deltaM;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Write it in file
fid = fopen('mass_tot.txt','w');
```

```

for i=1:sb*3
    fprintf(fid,'\n');
    for j=1:sb*3
        fprintf(fid,'%20.8d',Mt(i,j));
    end
end

fclose(fid);

% Damping matrix construction
% Element (1,1)
Cv111 = zeros(sb);
for i=1:sb
    for j=1:sb
        for k=1:sb
            Cv111(i,j)=Cv111(i,j)+2*(Fijk(i,k,j)-Fijk(j,k,i))*X(k);
        end
    end
end
Cv{1,1}=Cv111;

% Element (1,2)
Cv121 = -2*Bp*deij;
Cv122 = zeros(sb);
for i=1:sb
    for j=1:sb
        for k=1:sb
            Cv122(i,j)=Cv122(i,j)-2*Fijk(j,k,i)*X(k+sb);
        end
    end
end
Cv{1,2} = Cv121 + Cv122;

% Element (1,3)
Cv{1,3} = zeros(sb);

% Element (2,1)
Cv211 = 2*Bp*deij;
Cv212 = zeros(sb);
for i=1:sb
    for j=1:sb
        for k=1:sb
            Cv212(i,j)=Cv212(i,j)+2*Fijk(i,k,j)*X(k+sb);
        end
    end
end
Cv{2,1} = Cv211 + Cv212;

% Element (2,2)
Cv{2,2} = zeros(sb);

% Element (2,3)
Cv{2,3} = zeros(sb);

% Element (3,1)
Cv{3,1} = zeros(sb);

```

```

% Element (3,2)
Cv{3,2} = zeros(sb);

% Element (3,3)
Cv{3,3} = zeros(sb);

% Aerodynamic damping matrix
% Element (1,1)
Ca111 = Ga/6*2*Cd*Eij;
Ca112 = Ga/6*Te*vi*deij;
Ca113 = zeros(sb);
for i=1:sb
    for j=1:sb
        for k=1:sb
            Ca113(i,j) = Ca113(i,j) + Ga/6*vi*Hijk(i,j,k)*X(k+2*sb);
        end
    end
end
Ca{1,1}=Ca111+Ca112+Ca113;

% Element (1,2)
Ca121 = Ga/6*Te*Eij;
Ca122 = -Ga/6*2*vi*deij;

Ca123 = zeros(sb);

for i=1:sb
    for j=1:sb
        for k=1:sb
            Ca123(i,j) = Ca123(i,j) + Ga/6*Gijk(i,j,k)*X(k+2*sb);
        end
    end
end
Ca{1,2}=Ca121+Ca122+Ca123;

% Element (1,3)
Ca{1,3} = zeros(sb);

% Element (2,1)
Ca211 = -Ga/6*2*Te*Eij;
Ca212 = Ga/6*vi*deij;
Ca213 = zeros(sb);

for i=1:sb
    for j=1:sb
        for k=1:sb
            Ca213(i,j) = Ca213(i,j) - 2*Ga/6*Gijk(i,j,k)*X(k+2*sb);
        end
    end
end
Ca{2,1}=Ca211+Ca212+Ca213;

% Element (2,2)
Ca{2,2} = Ga/6*Eij;

% Element (2,3)

```

```

Ca{2,3} = -0.75*Ga/6*ce*Iij;

% Element (3,1)
Ca{3,1} = zeros(sb);

% Element (3,2)
Ca{3,2} = zeros(sb);

% Element (3,3)
Ca{3,3} = Ga/6*ce^2/8*Mij;

%Aerodynamic damping matrix
Ca = [Ca{1,1} Ca{1,2} Ca{1,3};Ca{2,1} Ca{2,2} Ca{2,3}; Ca{3,1} Ca{3,2}
Ca{3,3}];
%Structural damping matrix
Cs = [Cv{1,1} Cv{1,2} Cv{1,3};Cv{2,1} Cv{2,2} Cv{2,3}; Cv{3,1} Cv{3,2}
Cv{3,3}];

Ct1=Ca+Cs;

% Stiffness matrix construction
% Structural stiffness matrix
% Element (1,1)
for i=1:sb
    for j=1:sb
        Ks111(i,j) = (Stz-(Stz-Sty)*(sin(Rc*Te))^2)*bej(j)^4*deij(i,j);
    end % j
end % i
Ks112 = zeros(sb);
for i=1:sb
    for j=1:sb
        for k=1:sb
            Ks112(i,j) = Ks112(i,j) + (Stz-
Sty)*sin(2*Rc*Te)*Kijk(k,j,i)*X(k+2*sb);
        end
    end
end
Ks11 = Dij+Ks111-deij-Ks112;

% Element (1,2)
for i=1:sb
    for j=1:sb
        Ks121(i,j) = (Stz-Sty)*(sin(2*Rc*Te))/2.0*bej(j)^4*deij(i,j);
    end % j
end % i
Ks122 = zeros(sb);
for i=1:sb
    for j=1:sb
        for k=1:sb
            Ks122(i,j)=Ks122(i,j) + (Stz-
Sty)*cos(2*Rc*Te)*Kijk(k,j,i)*X(k+2*sb);
        end
    end
end
Ks12 = Ks121+Ks122;

% Element (1,3)

```



```

Ks131 = zeros(sb);
for i=1:sb
    for j=1:sb
        for k=1:sb
            Ks131(i,j)=Ks131(i,j) + (Stz-
Sty)*Kijk(j,k,i)*(X(k+sb)*cos(2*Rc*Te)-X(k)*sin(2*Rc*Te));
            end % k loop
        end
    end
Ks13 = Ks131;

% Element (2,1)
Ks21 = Ks12;

% Element (2,2)
for i=1:sb
    for j=1:sb
        Ks221(i,j) = (Sty+(Stz-Sty)*(sin(Rc*Te))^2)*bej(j)^4*deij(i,j);
    end % j
end % i
Ks222 = Ks112;
Ks22 = Dij+Ks221+Ks222;

% Element (2,3)
Ks231 = zeros(sb);
for i=1:sb
    for j=1:sb
        for k=1:sb
            Ks231(i,j)=Ks231(i,j) + (Stz-
Sty)*Kijk(j,k,i)*(X(k+sb)*sin(2*Rc*Te)+X(k)*cos(2*Rc*Te));
            end % k
        end
    end
Ks23 = Ks231;

% Element (3,1)
Ks311 = zeros(sb);
for i=1:sb
    for j=1:sb
        for k=1:sb
            Ks311(i,j)=Ks311(i,j) + (Stz-
Sty)*Kijk(i,j,k)*(X(k+sb)*cos(2*Rc*Te)-...
            X(k)*sin(2*Rc*Te));
            end % k loop
        end
    end
Ks31 = Ks311;

% Element (3,2)
Ks321 = zeros(sb);
for i=1:sb
    for j=1:sb
        for k=1:sb
            Ks321(i,j)=Ks321(i,j) + (Stz-
Sty)*Kijk(i,j,k)*(X(k+sb)*sin(2*Rc*Te)+...
            X(k)*cos(2*Rc*Te));
            end % k

```

```

    end
end
Ks32 = Ks321;

% Element (3,3)
Ks331=Mu^2*k1*Nij+(Mu2^2-Mu1^2)*cos(2*Te)*deij;
for i=1:sb
    for j=1:sb
        Ks332(i,j) = Stj*gamma(j)^2*deij(i,j);
    end % j
end % i
Ks33=Ks331+Ks332;

Ks = [Ks11 Ks12 Ks13;Ks21 Ks22 Ks23; Ks31 Ks32 Ks33];

% Construct aerodynamic stiffness matrix, Kar
% Element (1,1)
Ka11 = zeros(sb);

% Element (1,2)
Ka12 = zeros(sb);

% Element (1,3)
Ka13 = Ga/6*vi*Iij;

% Element (2,1)
Ka211 = Ga/6*Bp*Eij;
Ka212 = zeros(sb);
for i=1:sb
    for j=1:sb
        for k=1:sb
            Ka212(i,j) = Ka212(i,j) + Ga/6*Lijk(i,j,k)*X(k+sb);
        end % k
    end
end
end

Ka21 = Ka211+Ka212;

% Element (2,2)
Ka221 = Ga/6*(-ce/2*Hij);
Ka222 = zeros(sb);
for i=1:sb
    for j=1:sb
        for k=1:sb
            Ka222(i,j) = Ka222(i,j) + Ga/6*Lijk(i,k,j)*X(k);
        end % k
    end
end
end

Ka22 = Ka221+Ka222;

% Element (2,3)
Ka23 = Ga/6*(-Jij);

% Element (3,1)
Ka31 = zeros(sb);

% Element (3,2)

```

```

Ka32 = zeros(sb);

% Element (3,3)
Ka33 = zeros(sb);

% Aerodynamics stiffness matrix, Ka
Ka = [Ka11 Ka12 Ka13; Ka21 Ka22 Ka23; Ka31 Ka32 Ka33];

% Total stiffness matrix
Kt1 = Ks+Ka;

disp('DONE matrices');

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Define damping matrix due to the piezo
dC111 = zeros(sb);

% Element (1,1)
for i=1:sb
    for j=1:sb
        for k=1:sb
            dC111(i,j)=dC111(i,j)+2*mss*(Fxiijk(i,k,j)-Fxiijk(j,k,i))*X(k);
        end
    end
end
dC{1,1}=dC111;

% Element (1,2)
dC121 = -2*mss*Bp*dexij1;
dC122 = zeros(sb);
for i=1:sb
    for j=1:sb
        for k=1:sb
            dC122(i,j)=dC122(i,j)-2*mss*Fxiijk(j,k,i)*X(k+sb);
        end
    end
end
dC{1,2} = dC121 + dC122;

% Element (1,3)
dC{1,3} = zeros(sb);

% Element (2,1)
dC211 = 2*mss*Bp*dexij1;
dC212 = zeros(sb);
for i=1:sb
    for j=1:sb
        for k=1:sb
            dC212(i,j)=dC212(i,j)+2*Fxiijk(i,k,j)*X(k+sb);
        end
    end
end
dC{2,1} = dC211 + dC212;

% Element (2,2)
dC{2,2} = zeros(sb);

```

```

% Element (2,3)
dC{2,3} = zeros(sb);

% Element (3,1)
dC{3,1} = zeros(sb);

% Element (3,2)
dC{3,2} = zeros(sb);

% Element (3,3)
dC{3,3} = zeros(sb);

%Damping matrix due to piezo is:

deltaC = [dC{1,1} dC{1,2} dC{1,3};dC{2,1} dC{2,2} dC{2,3}; dC{3,1}
dC{3,2} dC{3,3}];

Ct = Ct1 + deltaC;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Stiffness matrix due to the piezoelectric material
% Element (1,1)
for i=1:sb
    for j=1:sb
        dK111(i,j) = (Stz_p-(Stz_p-Sty_p)*(sin(Rc*Te))^2)*deltaxij1(i,j);
    end % j
end % i
dK112 = zeros(sb);

for i=1:sb
    for j=1:sb
        for k=1:sb
            dK112(i,j) = dK112(i,j) + (Stz_p-
Sty_p)*sin(2*Rc*Te)*Kxijk(k,j,i)*X(k+2*sb);
        end
    end
end
dK11 = mss*(Dxij-dexij1)+dK111-dK112;

% Element (1,2)
for i=1:sb
    for j=1:sb
        dK121(i,j) = (Stz_p-Sty_p)*(sin(2*Rc*Te))/2.0*deltaxij1(i,j);
    end % j
end % i
dK122 = zeros(sb);
for i=1:sb
    for j=1:sb
        for k=1:sb
            dK122(i,j)=dK122(i,j) + (Stz_p-
Sty_p)*cos(2*Rc*Te)*Kxijk(k,j,i)*X(k+2*sb);
        end
    end
end
dK12 = dK121+dK122;

% Element (1,3)

```

```

dK131 = zeros(sb);
for i=1:sb
    for j=1:sb
        for k=1:sb
            dK131(i,j)=dK131(i,j) + (Stz_p-
Sty_p)*Kijk(j,k,i)*(X(k+sb)*cos(2*Rc*Te)-X(k)*sin(2*Rc*Te));
            end % k
        end
    end
end
dK13 = dK131;

% Element (2,1)
dK21 = dK12;

% Element (2,2)
for i=1:sb
    for j=1:sb
        dK221(i,j) = (Sty_p+(Stz_p-Sty_p)*(sin(Rc*Te))^2)*deltaxij1(i,j);
    end % j
end % i
dK222 = dK112;
dK22 = mss*Dxij+dK221+dK222;

% Element (2,3)
dK231 = zeros(sb);
for i=1:sb
    for j=1:sb
        for k=1:sb
            dK231(i,j)=dK231(i,j) + (Stz_p-
Sty_p)*Kxijk(j,k,i)*(X(k+sb)*sin(2*Rc*Te)+X(k)*cos(2*Rc*Te));
            end % k
        end
    end
end
dK23 = dK231;

% Element (3,1)
dK311 = zeros(sb);
for i=1:sb
    for j=1:sb
        for k=1:sb
            dK311(i,j)=dK311(i,j) + (Stz_p-
Sty_p)*Kxijk(i,j,k)*(X(k+sb)*cos(2*Rc*Te)-...
            X(k)*sin(2*Rc*Te));
            end % k
        end
    end
end
dK31 = dK311;

% Element (3,2)
dK321 = zeros(sb);
for i=1:sb
    for j=1:sb
        for k=1:sb
            dK321(i,j)=dK321(i,j) + (Stz_p-
Sty_p)*Kxijk(i,j,k)*(X(k+sb)*sin(2*Rc*Te)+...
            X(k)*cos(2*Rc*Te));
            end % k
        end
    end
end

```

```

    end
end
dK32 = dK321;

% Element (3,3)
dK331=Mu^2*mss*k1*Nx1ij+Mu_p^2*k1_p*Nx2ij+Mu_p^2*mss*k1_p*Nx3ij+Stj_p*d
exij2+...
    cos(2*Te)*dexij2*(Mu2_p^2-Mu1_p^2 + mss*(Mu2^2-Mu1^2) +
mss*(Mu2_p^2-Mu1_p^2));
for i=1:sb
    for j=1:sb
        dK332(i,j) = Stj_p*gamma(j)^2*dexij2(i,j);
    end % j
end % i
dK33=dK331+dK332;

deltaK = [dK11 dK12 dK13;dK21 dK22 dK23; dK31 dK32 dK33];

Kvac = Ks + deltaK;
Kt = Kt1 + deltaK;

```

```
function [A1,B1] = SortFrequencies(A,B)
% A - vector of eigenvalues
% B - matrix of eigenvectors
% This function sorts the imaginary part of the eigenvalue vector in
% the ascending order and puts the sorted eigenvalues into A1
% B1 is the matrix of eigenvectors obtained from B such that
% the i-th eigenvector corresponds to the i-th eigenvalue

[Aim,index1] = sort(abs(imag(A)));
[aa11,bb11] = size(A);
for i=1:aa11
    A1(i,1) = A(index1(i,1),1);
    B1(:,i) = B(:,index1(i));
end
```

```

function [X,vi,T,V,W]=SysSolverSmart
% This subroutine solves for the equilibrium deflections, as well as
% for the tip deflections and inflow velocity for the DC regime
%-----
global x0 sb Te x0smart
format long e;

% Set the options used in the fsolve subroutine
options=optimset('Display', 'iter','LargeScale', 'off','TolX',1e-8);
disp('THETA'); disp(Te);

% Solve for the equilibrium quantities:
X = fsolve('EqSolver',x0smart,options);

x0smart=X;

% Steady state lead-lag deflection at the tip
V=0;
for i=1:sb
    V=V+X(i)*(-1)^(i+1);
end
V=2*V;

% Steady state flap deflection at the tip
W=0;
for i=1:sb
    W=W+X(sb+i)*(-1)^(i+1);
end
W=2*W;

% Steady state torsion deflection at the tip
T=0;
for i=1:sb
    T=T+X(2*sb+i)*(-1)^(i+1);
end
T=sqrt(2)*T;

% Inflow velocity
vi = X(3*sb+2);

X=X(1:3*sb);

```



```
function [X,vi,X1]=SysSolverFloquet(xOff)
% This subroutine solves for the equilibrium deflections for the case
% of the periodic actuation regimes
%-----

global sb Te
format long e;

% Set the options used in the fsolve subroutine
options=optimset('Display', 'iter','LargeScale', 'off','TolX',1e-8);
disp('THETA'); disp(Te);

% Solve for the equilibrium quantities:
X = fsolve('EqSolverFloquet',xOff,options);
X1 = X;

% Inflow velocity
vi = X(3*sb+2);

X=X(1:3*sb);% Equilibrium deflections
```

```

function [Ms,Mt,Ct1,Ct,Kt1,Kt,Kvac,deltaK]=MCKassemblyFloquet(X,vi)
% This function assembles mass, stiffness, and damping matrices for
%the AC regimes. It also uses the equilibrium deflections, which were
%found previously
%-----
global Stz Sty Stj Rc Te Bp ce Si Mu Mu1 Mu2 k1 % Structural and
%geometric parameters w.o. piezo
global Ga Cd % aerodynamics parameters
global bej gaj deij Dij Ai Bi Ci Hij Jij Nij Lijk Likj Kkji Kijk Kjki
global Iij Eij Mij Gijk Hijk Fikj Fjki Fijk Gamma
global Txi Bxi dexij1 dexij2 deltaxij1 Dxij Nxlij Nx2ij Nx3ij Fxijk
Kxijk
global sb
global gamma case_number
global mss Stj_p k1_p Mu2_p Mu1_p Mu_p
global Sy Sz
global Klss K2ss Stj_p_a Stj_p_na kappa_ss KKss
format long e

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Redefine the equilibrium deflections
V0=X(1:sb);
W0=X((sb+1):(2*sb));
PHI0=X((2*sb+1):(3*sb));

% Mass matrix construction.
M11 = deij;
M22 = deij;
M33 = deij*Mu^2;
M12 = zeros(sb);
M13 = zeros(sb);
M21 = zeros(sb);
M23 = zeros(sb);
M31 = zeros(sb);
M32 = zeros(sb);
Ms = [M11 M12 M13;M21 M22 M23; M31 M32 M33];

% Structural and aerodynamics Mass matrix
Mt11=M11;
Mt22=M22+deij*Ga*ce/24;
Mt33=M33;
Mt = [Mt11 M12 M13;M21 Mt22 M23;M31 M32 Mt33];

% Mass matrix construction due to the "smart" spring.
dM11 = mss*dexij1;
dM22 = mss*dexij1;
dM33 = dexij2*(Mu_p^2+mss*Mu^2+mss*Mu_p^2);
dM12 = zeros(sb);
dM13 = zeros(sb);
dM21 = zeros(sb);
dM23 = zeros(sb);
dM31 = zeros(sb);
dM32 = zeros(sb);
deltaM = [dM11 dM12 dM13;dM21 dM22 dM23; dM31 dM32 dM33];

Mt = Mt + deltaM;

```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
% Damping matrix construction
% Structural damping
% Element (1,1)
Cv111 = zeros(sb);

for i=1:sb
    for j=1:sb
        for k=1:sb
            Cv111(i,j)=Cv111(i,j)+2*(Fijk(i,k,j)-Fijk(j,k,i))*X(k);
        end
    end
end
Cv{1,1}=Cv111;

% Element (1,2)
Cv121 = -2*Bp*deij;
Cv122 = zeros(sb);

for i=1:sb
    for j=1:sb
        for k=1:sb
            Cv122(i,j)=Cv122(i,j)-2*Fijk(j,k,i)*X(k+sb);
        end
    end
end
Cv{1,2} = Cv121 + Cv122;

% Element (1,3)
Cv{1,3} = zeros(sb);

% Element (2,1)
Cv211 = 2*Bp*deij;
Cv212 = zeros(sb);
for i=1:sb
    for j=1:sb
        for k=1:sb
            Cv212(i,j)=Cv212(i,j)+2*Fijk(i,k,j)*X(k+sb);
        end
    end
end
Cv{2,1} = Cv211 + Cv212;

% Element (2,2)
Cv{2,2} = zeros(sb);

% Element (2,3)
Cv{2,3} = zeros(sb);

% Element (3,1)
Cv{3,1} = zeros(sb);

% Element (3,2)
Cv{3,2} = zeros(sb);

% Element (3,3)
```

```

Cv{3,3} = zeros(sb);

% Aerodynamic damping matrix
% Element (1,1)
Ca111 = Ga/6*2*Cd*Eij;
Ca112 = Ga/6*Te*vi*deij;
Ca113 = zeros(sb);
for i=1:sb
    for j=1:sb
        for k=1:sb
            Ca113(i,j) = Ca113(i,j) + Ga/6*vi*Hijk(i,j,k)*X(k+2*sb);
        end
    end
end
Ca{1,1}=Ca111+Ca112+Ca113;

% Element (1,2)
Ca121 = Ga/6*Te*Eij;
Ca122 = -Ga/6*2*vi*deij;

Ca123 = zeros(sb);

for i=1:sb
    for j=1:sb
        for k=1:sb
            Ca123(i,j) = Ca123(i,j) + Ga/6*Gijk(i,j,k)*X(k+2*sb);
        end
    end
end
Ca{1,2}=Ca121+Ca122+Ca123;

% Element (1,3)
Ca{1,3} = zeros(sb);

% Element (2,1)
Ca211 = -Ga/6*2*Te*Eij;
Ca212 = Ga/6*vi*deij;
Ca213 = zeros(sb);

for i=1:sb
    for j=1:sb
        for k=1:sb
            Ca213(i,j) = Ca213(i,j) - 2*Ga/6*Gijk(i,j,k)*X(k+2*sb);
        end
    end
end
Ca{2,1}=Ca211+Ca212+Ca213;

% Element (2,2)
Ca{2,2} = Ga/6*Eij;

% Element (2,3)
Ca{2,3} = -0.75*Ga/6*ce*Iij;

% Element (3,1)
Ca{3,1} = zeros(sb);

```

```

% Element (3,2)
Ca{3,2} = zeros(sb);

% Element (3,3)
Ca{3,3} = Ga/6*ce^2/8*Mij;

%Aerodynamic damping matrix
Ca = [Ca{1,1} Ca{1,2} Ca{1,3};Ca{2,1} Ca{2,2} Ca{2,3}; Ca{3,1} Ca{3,2}
Ca{3,3}];
%Structural damping matrix
Cs = [Cv{1,1} Cv{1,2} Cv{1,3};Cv{2,1} Cv{2,2} Cv{2,3}; Cv{3,1} Cv{3,2}
Cv{3,3}];

Ct1=Ca+Cs;

% Stiffness matrix construction
% Structural stiffness matrix
% Element (1,1)
for i=1:sb
    for j=1:sb
        Ks111(i,j) = (Stz-(Stz-Sty)*(sin(Rc*Te))^2)*bej(j)^4*deij(i,j);
    end
end % i
Ks112 = zeros(sb);
for i=1:sb
    for j=1:sb
        for k=1:sb
            Ks112(i,j) = Ks112(i,j) + (Stz-
Sty)*sin(2*Rc*Te)*Kijk(k,j,i)*X(k+2*sb);
        end
    end
end
Ks11 = Dij+Ks111-deij-Ks112;

% Element (1,2)
for i=1:sb
    for j=1:sb
        Ks121(i,j) = (Stz-Sty)*(sin(2*Rc*Te))/2.0*bej(j)^4*deij(i,j);
    end % j
end % i
Ks122 = zeros(sb);
for i=1:sb
    for j=1:sb
        for k=1:sb
            Ks122(i,j)=Ks122(i,j) + (Stz-
Sty)*cos(2*Rc*Te)*Kijk(k,j,i)*X(k+2*sb);
        end
    end
end
Ks12 = Ks121+Ks122;

% Element (1,3)
Ks131 = zeros(sb);
for i=1:sb

```

```

    for j=1:sb
        for k=1:sb
            Ks131(i,j)=Ks131(i,j) + (Stz-
Sty)*Kijk(j,k,i)*(X(k+sb)*cos(2*Rc*Te)-X(k)*sin(2*Rc*Te));
            end % k
        end
    end
    Ks13 = Ks131;

% Element (2,1)
Ks21 = Ks12;

% Element (2,2)
for i=1:sb
    for j=1:sb
        Ks221(i,j) = (Sty+(Stz-Sty)*(sin(Rc*Te))^2)*bej(j)^4*deij(i,j);
    end % j
end % i
Ks222 = Ks112;
Ks22 = Dij+Ks221+Ks222;

% Element (2,3)
Ks231 = zeros(sb);
for i=1:sb
    for j=1:sb
        for k=1:sb
            Ks231(i,j)=Ks231(i,j) + (Stz-
Sty)*Kijk(j,k,i)*(X(k+sb)*sin(2*Rc*Te)+X(k)*cos(2*Rc*Te));
            end % k
        end
    end
    Ks23 = Ks231;

% Element (3,1)
Ks311 = zeros(sb);
for i=1:sb
    for j=1:sb
        for k=1:sb
            Ks311(i,j)=Ks311(i,j) + (Stz-
Sty)*Kijk(i,j,k)*(X(k+sb)*cos(2*Rc*Te)-...
X(k)*sin(2*Rc*Te));
            end % k
        end
    end
    Ks31 = Ks311;

% Element (3,2)
Ks321 = zeros(sb);
for i=1:sb
    for j=1:sb
        for k=1:sb
            Ks321(i,j)=Ks321(i,j) + (Stz-
Sty)*Kijk(i,j,k)*(X(k+sb)*sin(2*Rc*Te)+...
X(k)*cos(2*Rc*Te));

```

```

        end % k
    end
end
Ks32 = Ks321;

% Element (3,3)
Ks331=Mu^2*k1*Nij+(Mu2^2-Mu1^2)*cos(2*Te)*deij;
for i=1:sb
    for j=1:sb
        Ks332(i,j) = Stj*gamma(j)^2*deij(i,j);
    end % j
end % i
Ks33=Ks331+Ks332;

Ks = [Ks11 Ks12 Ks13;Ks21 Ks22 Ks23; Ks31 Ks32 Ks33];

% Construct aerodynamic stiffness matrix, Ka
% Element (1,1)
Ka11 = zeros(sb);

% Element (1,2)
Ka12 = zeros(sb);

% Element (1,3)
Ka13 = Ga/6*vi*Iij;

% Element (2,1)
Ka211 = Ga/6*Bp*Eij;
Ka212 = zeros(sb);
for i=1:sb
    for j=1:sb
        for k=1:sb
            Ka212(i,j) = Ka212(i,j) + Ga/6*Lijk(i,j,k)*X(k+sb);
        end % k
    end
end
Ka21 = Ka211+Ka212;

% Element (2,2)
Ka221 = Ga/6*(-ce/2*Hij);
Ka222 = zeros(sb);
for i=1:sb
    for j=1:sb
        for k=1:sb
            Ka222(i,j) = Ka222(i,j) + Ga/6*Lijk(i,k,j)*X(k);
        end % k
    end
end
Ka22 = Ka221+Ka222;

% Element (2,3)
Ka23 = Ga/6*(-Jij);

% Element (3,1)
Ka31 = zeros(sb);

```

```

% Element (3,2)
Ka32 = zeros(sb);

% Element (3,3)
Ka33 = zeros(sb);

% Aerodynamics stiffness matrix, Ka
Ka = [Ka11 Ka12 Ka13; Ka21 Ka22 Ka23; Ka31 Ka32 Ka33];

% Total stiffness matrix
Kt1 = Ks+Ka;

disp('DONE matrices');

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Define damping matrix due to the "smart" spring
dC111 = zeros(sb);

% Element (1,1)
for i=1:sb
    for j=1:sb
        for k=1:sb
            dC111(i,j)=dC111(i,j)+2*mss*(Fxiijk(i,k,j)-Fxiijk(j,k,i))*X(k);
        end
    end
end
dC{1,1}=dC111;

% Element (1,2)
dC121 = -2*mss*Bp*dexij1;
dC122 = zeros(sb);
for i=1:sb
    for j=1:sb
        for k=1:sb
            dC122(i,j)=dC122(i,j)-2*mss*Fxiijk(j,k,i)*X(k+sb);
        end
    end
end
dC{1,2} = dC121 + dC122;

% Element (1,3)
dC{1,3} = zeros(sb);

% Element (2,1)
dC211 = 2*mss*Bp*dexij1;
dC212 = zeros(sb);
for i=1:sb
    for j=1:sb
        for k=1:sb
            dC212(i,j)=dC212(i,j)+2*Fxiijk(i,k,j)*X(k+sb);
        end
    end
end
dC{2,1} = dC211 + dC212;

```



```

% Element (2,2)
dC{2,2} = zeros(sb);

% Element (2,3)
dC{2,3} = zeros(sb);

% Element (3,1)
dC{3,1} = zeros(sb);

% Element (3,2)
dC{3,2} = zeros(sb);

% Element (3,3)
dC{3,3} = zeros(sb);

%Damping matrix due to piezo is:

deltaC = [dC{1,1} dC{1,2} dC{1,3};dC{2,1} dC{2,2} dC{2,3}; dC{3,1}
dC{3,2} dC{3,3}];

Ct = Ct1 + deltaC;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Stiffness matrix due to the "smart" spring
% Element (1,1)
for i=1:sb
    for j=1:sb
        dK111(i,j) = (Sz-(Sz-Sy)*(sin(Rc*Te))^2)*deltaxij1(i,j);
    end % j
end % i
dK112 = zeros(sb);

for i=1:sb
    for j=1:sb
        for k=1:sb
            dK112(i,j) = dK112(i,j) + (Sz-
Sy)*sin(2*Rc*Te)*Kxijk(k,j,i)*X(k+2*sb);
        end
    end
end
dK11 = mss*(Dxij-dexij1)+dK111-dK112;

% Element (1,2)
for i=1:sb
    for j=1:sb
        dK121(i,j) = (Sz-Sy)*(sin(2*Rc*Te))/2.0*deltaxij1(i,j);
    end % j
end % i
dK122 = zeros(sb);
for i=1:sb
    for j=1:sb
        for k=1:sb
            dK122(i,j)=dK122(i,j) + (Sz-
Sy)*cos(2*Rc*Te)*Kxijk(k,j,i)*X(k+2*sb);
        end
    end
end

```

```

        end
    end
end
dK12 = dK121+dK122;

% Element (1,3)
dK131 = zeros(sb);
for i=1:sb
    for j=1:sb
        for k=1:sb
            dK131(i,j)=dK131(i,j) + (Sz-
Sy)*Kijk(j,k,i)*(X(k+sb)*cos(2*Rc*Te)-...
            X(k)*sin(2*Rc*Te));
        end % k
    end
end
dK13 = dK131;

% Element (2,1)
dK21 = dK12;

% Element (2,2)
for i=1:sb
    for j=1:sb
        dK221(i,j) = (Sy+(Sz-Sy)*(sin(Rc*Te))^2)*deltaxij1(i,j);
    end % j
end % i
dK222 = dK112;
dK22 = mss*Dxij+dK221+dK222;

% Element (2,3)
dK231 = zeros(sb);
for i=1:sb
    for j=1:sb
        for k=1:sb
            dK231(i,j)=dK231(i,j) + (Sz-
Sy)*Kxijk(j,k,i)*(X(k+sb)*sin(2*Rc*Te)+...
            X(k)*cos(2*Rc*Te));
        end % k
    end
end
dK23 = dK231;

% Element (3,1)
dK311 = zeros(sb);
for i=1:sb
    for j=1:sb
        for k=1:sb
            dK311(i,j)=dK311(i,j) + (Sz-
Sy)*Kxijk(i,j,k)*(X(k+sb)*cos(2*Rc*Te)-...
            X(k)*sin(2*Rc*Te));
        end % k
    end
end

```

```

    end
end
dK31 = dK311;

% Element (3,2)
dK321 = zeros(sb);
for i=1:sb
    for j=1:sb
        for k=1:sb
            dK321(i,j)=dK321(i,j) + (Sz-
Sy)*Kxijk(i,j,k)*(X(k+sb)*sin(2*Rc*Te)+...
X(k)*cos(2*Rc*Te));
        end % k
    end
end
dK32 = dK321;

% Element (3,3)
dK331=Mu^2*mss*k1*Nx1ij+Mu_p^2*KKss*Nx2ij+Mu_p^2*mss*KKss*Nx3ij+kappa_s
s*dexij2+...
cos(2*Te)*dexij2*(Mu2_p^2-Mu1_p^2 + mss*(Mu2^2-Mu1^2) +
mss*(Mu2_p^2-Mu1_p^2));
for i=1:sb
    for j=1:sb
        dK332(i,j) = Stj_p*gamma(j)^2*dexij2(i,j);
    end % j
end % i
dK33=dK331+dK332;

deltaK = [dK11 dK12 dK13;dK21 dK22 dK23; dK31 dK32 dK33];

Kvac = Ks + deltaK;
Kt = Kt1 + deltaK;

```

```
function xdot=floquet(t,x)
% This fuction is needed to define the system of perturbation equation
%that is used in the Floquet analysis
global P_floquet
format long e;
xdot=P_floquet*x;
```

```

function S=EqSolver(X);
% In this subroutine the nonlinear algebraic equations of motion are
% defined in order to calculate equilibrium deflections for the SBB and
% DC regimes
%-----
global bej gaj Ai Bi Ci Dij Eij Iij Jij Mij deij Nij
global Hij Fjki Fikj Fjki Gijk Hijk Kijk Kjki Kkji Lijk Likj
global Txi Bxi dexij1 dexij2 deltaxij1 Dxij Nxlij Nx2ij Nx3ij Fxijk
Kxijk
global Ga Si Cd Bp ce Mu Mu1 Mu2 k1
global Rc gamma
global Te
global Stz Sty Stj
global sb case_number
global mss Sty_p Stz_p Stj_p k1_p Mu2_p Mu1_p Mu_p
format long e;

% Define the twist at the station of 75% of the blade length

T75=0;
for zzz=1:sb
    T75=T75+sqrt(2)*X(2*sb+zzz)*sin(0.75*(zzz-0.5)*pi);
end
T75=T75-X(3*sb+1);

% Define the inflow velocity
vi= sign(Te+X(3*sb+1))*pi*Si/8.0*(sqrt(1.0+12.0/(pi*...
    Si)*abs(Te+X(3*sb+1)))-1.0)-X(3*sb+2);

% In this section the equations of motion are defined in order to find
% the equilibrium deflections
if (case_number==4)|(case_number==3)
    P=zeros(sb,1);
    Z=zeros(sb,1);
    D=zeros(sb,1);
    % Due to the smart spring:
    Ps=zeros(sb,1);
    Zs=zeros(sb,1);
    Ds=zeros(sb,1);

    Ptotal=zeros(sb,1);
    Ztotal=zeros(sb,1);
    Dtotal=zeros(sb,1);

    %Equation for dVoi, dWoi, dPhioi
    for i = 1:sb
        for j = 1:sb
            Q=0;U1=0;U2=0;E=0;
            Qs=0;U1s=0;Es=0;
            for k =1:sb
                Q=Q+Kijk(j,k,i)*X(2*sb+j)*(-X(k)*sin(2.0*Rc*Te)+...
                    X(sb+k)*cos(2.0*Rc*Te));
                U1=U1+Kijk(j,k,i)*X(2*sb+j)*(X(k)*cos(2*Rc*Te)+...
                    X(sb+k)*sin(2*Rc*Te));
                U2=U2+Lijk(i,j,k)*X(j)*X(sb+k);
            end
        end
    end
end

```

```

E=E+Kijk(i,j,k)*(sin(2*Rc*Te)/2.0*(X(sb+j)*X(sb+k)...
-X(j)*X(k))+cos(2*Rc*Te)*X(j)*X(sb+k));
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% The terms due to the smart spring:
Qs=Qs+Kxijk(j,k,i)*X(2*sb+j)*(-X(k)*sin(2.0*Rc*Te)+...
X(sb+k)*cos(2.0*Rc*Te));
Uls=Uls+Kxijk(j,k,i)*X(2*sb+j)*(X(k)*cos(2*Rc*Te)+...
X(sb+k)*sin(2*Rc*Te));
Es=Es+Kxijk(i,j,k)*(sin(2*Rc*Te)/2.0*(X(sb+j)*X(sb+k)...
-X(j)*X(k))+cos(2*Rc*Te)*X(j)*X(sb+k));
end %k
% Lead-lag eqn:
P(i) = P(i) + X(j)*(Dij(i,j)+(Stz-(Stz-
Sty)*(sin(Rc*Te))^2)*...
bej(j)^4*deij(i,j)-deij(i,j))+
X(sb+j)*sin(2.0*Rc*Te)/2.0*(Stz-...
Sty)*bej(j)^4*deij(i,j)+(Stz-Sty)*Q+...
Ga/6.0*X(3*sb+2)*Iij(i,j)*X(2*sb+j);
% Terms due to the smart spring:
Ps(i) = Ps(i) + X(j)*(mss*Dxij(i,j)+(Stz_p-(Stz_p-
Sty_p)*(sin(Rc*Te))^2)*...
deltaxij1(i,j)-mss*dexij1(i,j))+
X(sb+j)*sin(2.0*Rc*Te)/2.0*(Stz_p-...
Sty_p)*deltaxij1(i,j)+(Stz_p-Sty_p)*Qs;

Ptotal(i)=P(i)+Ps(i);

% Flap eqn:
Z(i) = Z(i)+X(j)*sin(2.0*Rc*Te)/2.0*(Stz-Sty)*bej(j)^4*...
deij(i,j)+X(sb+j)*(Dij(i,j)+(Sty+(Stz-Sty)*(sin(Rc*Te))^2)*...
bej(j)^4*deij(i,j)+(Stz-Sty)*U1+Ga/6.0*(Bp*...
Eij(i,j)*X(j)-ce/2.0*Hij(i,j)*X(sb+j)-Jij(i,j)*X(2*sb+j)+U2);
% Terms due to the smart spring:
Zs(i) = Zs(i)+X(j)*sin(2.0*Rc*Te)/2.0*(Stz_p-
Sty_p)*deltaxij1(i,j)+...
X(sb+j)*(mss*Dxij(i,j)+(Sty_p+(Stz_p-
Sty_p)*(sin(Rc*Te))^2)*deltaxij1(i,j))+...
(Stz_p-Sty_p)*Uls;

Ztotal(i) = Z(i)+Zs(i);

% Torsion eqn:
D(i) = D(i) + X(2*sb+j)*(Mu^2*k1*Nij(i,j)+(Stj*gamma(j)^2+...
(Mu2^2-Mu1^2)*cos(2*Te))*deij(i,j)+(Stz-Sty)*E;
% Terms due to the smart spring:
Ds(i) = Ds(i) +
X(2*sb+j)*(Mu^2*mss*k1*Nx1ij(i,j)+Mu_p^2*k1_p*Nx2ij(i,j)+...
Mu_p^2*mss*k1_p*Nx3ij(i,j) + ...
(Stj_p*(gamma(j))^2+(Mu2_p^2-Mu1_p^2+mss*(Mu2^2-Mu1^2)+ ...
mss*(Mu2_p^2-Mu1_p^2))*cos(2*Te))*dexij2(i,j)+(Stz_p-
Sty_p)*Es;

Dtotal(i) = D(i) + Ds(i);

end %j
Ptotal(i)=Ptotal(i)-Ga/6.0*((X(3*sb+2))^2*Ai(i)-Cd*Ci(i)-...
X(3*sb+2)*Te*Bi(i));

```

```

Ztotal(i)=Ztotal(i)+ Bp*(Bi(i)+mss*Bxi(i))-Ga/6.0*(Te*Ci(i)-
X(3*sb+2)*Bi(i)+ ce/2.0*Bp*Bi(i));

Dtotal(i)=Dtotal(i)+(Mu2^2-Mu1^2)*sin(2*Te)/(sqrt(2)*gamma(i)) +
Txi(i)*(Mu2_p^2-Mu1_p^2 + mss*(Mu2^2-Mu1^2) + mss*(Mu2_p^2-
Mu1_p^2))*sin(2.0*Te)/2.0;

end %i
% Define the system of equations:
S=[T75; vi; Ptotal; Ztotal; Dtotal];
end %if

```

```

function S=EqSolverFloquet(X);
% In this subroutine the nonlinear algebraic equations of motion are
%defined in order to calculate equilibrium deflections for the AC
%regimes
%-----
global bej gaj Ai Bi Ci Dij Eij Iij Jij Mij deij Nij
global Hij Fjki Fikj Fjki Gijk Hijk Kijk Kjki Kkji Lijk Likj
global Txi Bxi dexij1 dexij2 deltaxij1 Dxij Nxlij Nx2ij Nx3ij Fxijk
Kxijk
global Ga Si Cd Bp ce Mu Mu1 Mu2 k1
global Rc gamma
global Te
global Stz Sty Stj
global sb case_number
global mss Stj_p k1_p Mu2_p Mu1_p Mu_p
global tt
global Xelss Xelss_bar Xe2ss Xe2ss_bar
global Sy Sz omegal
global Klss K2ss Stj_p_a Stj_p_na kappa_ss KKss

format long e;

Om = 110.0;% spin velocity
Sy = Xelss + Xelss_bar*sin(omegal*tt/Om);% time varying parameter
Sz = Xe2ss + Xe2ss_bar*sin(omegal*tt/Om);% time varying parameter

KKss = Klss + K2ss*sin(omegal*tt/Om);% time varying parameter
kappa_ss = 0.5*(Stj_p_a + Stj_p_na) + 0.5*(Stj_p_a -
Stj_p_na)*sin(omegal*tt/Om);% time varying parameter

T75=0;

% Define the twist at the station of 75% of the blade length
for zzz=1:sb
    T75=T75+sqrt(2)*X(2*sb+zzz)*sin(0.75*(zzz-0.5)*pi);
end
T75=T75-X(3*sb+1);

% Define the inflow velocity
vi= sign(Te+X(3*sb+1))*pi*Si/8.0*(sqrt(1.0+12.0/(pi*...
    Si)*abs(Te+X(3*sb+1)))-1.0)-X(3*sb+2);

% In this section the equations of motion are defined in order to
%find the equilibrium deflections

P=zeros(sb,1);
Z=zeros(sb,1);
D=zeros(sb,1);
% Due to the smart spring:
Ps=zeros(sb,1);
Zs=zeros(sb,1);
Ds=zeros(sb,1);

Ptotal=zeros(sb,1);
Ztotal=zeros(sb,1);
Dtotal=zeros(sb,1);

```



```

%Equation for dVoi, dWoi, dPhioi
for i = 1:sb
    for j = 1:sb
        Q=0;U1=0;U2=0;E=0;
        Qs=0;U1s=0;Es=0;
        for k =1:sb
            Q=Q+Kijk(j,k,i)*X(2*sb+j)*(-X(k)*sin(2.0*Rc*Te)+...
                X(sb+k)*cos(2.0*Rc*Te));
            U1=U1+Kijk(j,k,i)*X(2*sb+j)*(X(k)*cos(2*Rc*Te)+...
                X(sb+k)*sin(2*Rc*Te));
            U2=U2+Lijk(i,j,k)*X(j)*X(sb+k);
            E=E+Kijk(i,j,k)*(sin(2*Rc*Te)/2.0*(X(sb+j)*X(sb+k)...
                -X(j)*X(k))+cos(2*Rc*Te)*X(j)*X(sb+k));
            %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
            % Terms due to the smart spring:
            Qs=Qs+Kxijk(j,k,i)*X(2*sb+j)*(-X(k)*sin(2.0*Rc*Te)+...
                X(sb+k)*cos(2.0*Rc*Te));
            U1s=U1s+Kxijk(j,k,i)*X(2*sb+j)*(X(k)*cos(2*Rc*Te)+...
                X(sb+k)*sin(2*Rc*Te));
            Es=Es+Kxijk(i,j,k)*(sin(2*Rc*Te)/2.0*(X(sb+j)*X(sb+k)...
                -X(j)*X(k))+cos(2*Rc*Te)*X(j)*X(sb+k));
        end %k
    % Lead lag eqn:
    P(i) = P(i) + X(j)*(Dij(i,j)+(Stz -(Stz -
        Sty)*(sin(Rc*Te))^2)*...
        bej(j)^4*deij(i,j)-deij(i,j))+ X(sb+j)*sin(2.0*Rc*Te)/2.0*(Stz-
            Sty)*bej(j)^4*deij(i,j)+(Stz-Sty)*Q+...
            Ga/6.0*X(3*sb+2)*Iij(i,j)*X(2*sb+j);
    % Terms due to the smart spring:
    Ps(i) = Ps(i) + X(j)*(mss*Dxij(i,j)+(Sz -(Sz -
        Sy)*(sin(Rc*Te))^2)*...
        deltaxij1(i,j)-mss*dexij1(i,j))+ X(sb+j)*sin(2.0*Rc*Te)/2.0*(Sz-
        Sy)*deltaxij1(i,j)+(Sz-Sy)*Qs;

    Ptotal(i)=P(i)+Ps(i);

    % Flap eqn:
    Z(i) = Z(i)+X(j)*sin(2.0*Rc*Te)/2.0*(Stz-Sty)*bej(j)^4*...
        deij(i,j)+X(sb+j)*(Dij(i,j)+(Sty+(Stz-Sty)*(sin(Rc*Te))^2)*...
        bej(j)^4*deij(i,j))+ (Stz-Sty)*U1+Ga/6.0*(Bp*...
        Eij(i,j)*X(j)-ce/2.0*Hij(i,j)*X(sb+j)-Jij(i,j)*X(2*sb+j)+U2);
    % Terms due to the smart spring:
    Zs(i) = Zs(i)+X(j)*sin(2.0*Rc*Te)/2.0*(Sz-Sy)*deltaxij1(i,j)+...
        X(sb+j)*(mss*Dxij(i,j)+(Sy+(Sz-
        Sy)*(sin(Rc*Te))^2)*deltaxij1(i,j))+...
        (Sz-Sy)*U1s;

    Ztotal(i) = Z(i)+Zs(i);

    % Torsion eqn:
    D(i) = D(i) + X(2*sb+j)*(Mu^2*k1*Nij(i,j)+(Stj*gamma(j)^2+...
        (Mu2^2-Mu1^2)*cos(2*Te))*deij(i,j))+ (Stz-Sty)*E;
    % Terms due to the smart spring:
    Ds(i) = Ds(i) +
        X(2*sb+j)*(Mu^2*mss*k1*Nx1ij(i,j)+Mu_p^2*KKss*Nx2ij(i,j)+...
        Mu_p^2*mss*KKss*Nx3ij(i,j) + ...
        (kappa_ss*(gamma(j))^2+(Mu2_p^2-Mu1_p^2 + mss*(Mu2^2-Mu1^2) + ...

```

```

mss*(Mu2_p^2-Mu1_p^2))*cos(2*Te))*dexij2(i,j))+(Sz-Sy)*Es;

Dtotal(i) = D(i) + Ds(i);

end %j

Ptotal(i)=Ptotal(i)-Ga/6.0*((X(3*sb+2))^2*Ai(i)-Cd*Ci(i)-...
X(3*sb+2)*Te*Bi(i));

Ztotal(i)=Ztotal(i)+ Bp*(Bi(i)+mss*Bxi(i))-Ga/6.0*(Te*Ci(i)-
X(3*sb+2)*Bi(i)+...
ce/2.0*Bp*Bi(i));

Dtotal(i)=Dtotal(i)+(Mu2^2-Mu1^2)*sin(2*Te)/(sqrt(2)*gamma(i)) + ...
Txi(i)*(Mu2_p^2-Mu1_p^2 + mss*(Mu2^2-Mu1^2) + mss*(Mu2_p^2-
Mu1_p^2))*...
sin(2.0*Te)/2.0;

end %i

% Define the system of equations:
S=[T75; vi; Ptotal; Ztotal; Dtotal];

```

```

%MainProgram2.m
%Calculates the equilibrium modal deflections, blade tip deflections,
%induced inflow velocity, and eigenvalues for the CBB regime.
%-----
global bej gaj Ai Bi Ci Dij Eij Iij Jij Mij deij Nij Gamma
global Hij Fijk Fikj Fjki Gijk Hijk Kijk Kjki Kkji Lijk Likj
global Txi Bxi dexij1 dexij2 deltaxij1 Dxij Nxlij Nx2ij Nx3ij Fxijk
Kxijk
global sb
global Ga Si Cd Bp ce Mu Mu1 Mu2 k1 Num_b Sty Stz Stj Ww Wv Wt
global Rc gamma
global Te x0
global V W T X vi x1 x2
global case_number
global mss Sty_p Stz_p Stj_p k1_p Mu2_p Mu1_p Mu_p
format long;

% Define pitch angles:
Tei=[0.0,0.05,0.1,0.15,0.2,0.25,0.3,0.35,0.4];
Rc = 1.0;% structural coupling parameter
Bp = 0.0;% pre-cone angle
sb = 8;% number of mode shapes

% Define the constant gamma(j) for a fixed-free beam:
for j=1:sb
    gamma(j)=pi*(j-0.5);
end

% Calculate the configuration parameters of the rotor blade for this
%regime:
[Ga,Si,Cd,ce,Mu,Mu1,Mu2,k1,Num_b,Sty,Stz,Stj]=InputData1CBB;
%Calculate the integrals of shape functions due to the main structure.
[bej,gaj,Ai,Bi,Ci,Dij,Eij,Iij,Jij,Mij,deij,Nij,Hij,Fijk,Fikj,Fjki,Gijk,
Hijk,Kijk,Kjki,Kkji,Lijk,Likj,Gamma] = ShapeFunctionsCBB;

[sb,sb1]=size(bej);

%Define the initial approximation for the equilibrium deflections.
%This initial approximation will be used in order to solve for the
%equilibrium deflections.
x0 = zeros(3*sb+2,1);

% Start loop on pitch angles, Thetai
for uuu=1:length(Tei)
    Te=Tei(uuu);
    % Solve for the equilibrium deflections, tip deflections, induced
    %inflow velocity:
    [X,vi,T,V,W]=SysSolverCBB;
    Tip=[V W T vi];
    Tips(uuu,:)=Tip;

    % Assemble mass, stiffness and damping matrices:
    [Ms,Mt,C,Ks,Kt]=MCKAssemblyCBB(X,vi);

    %Assemble the state space matrix [Pnew]:
    Pnew11 = zeros(24);
    Pnew12 = eye(24);

```

```
Pnew21 = -inv(Mt)*Kt;
Pnew22 = -inv(Mt)*C;
Pnew = [Pnew11 Pnew12;Pnew21 Pnew22];

% Find the eigenvalues and eigenvectors of [Pnew]
[Vcnew,Va2]=eig(Pnew);
Va2=diag(Va2);
[Va2,j]=SortFrequencies(Va2,Vcnew); %Sort the eigenvalues in
%ascending order

Lead_lag(uuu,1)=Va2(2,1);% Defines 1st lead-lag mode
Flap_1(uuu,1)=Va2(4,1);% Defines 1st flap mode
Flap_2(uuu,1)=Va2(6,1);% 2nd flap mode
Torsion(uuu,1)=Va2(8,1);% Defines 1st torsion mode

Eigen_values(:,uuu) = Va2;

end %uuu

% Display the obtained eigenvalues on the screen:
disp(Eigen_values);
```

```

function[bej,gaj,Ai,Bi,Ci,Dij,Eij,Iij,Jij,Mij,deij,Nij,Hij,Fijk,Fikj,Fj
ki,Gijk,Hijk,Kijk,Kjki,Kkji,Lijk,Likj,Gamma] = ShapeFunctionsCBB
% This subroutine calculates integrals of products of shape functions
% for the CBB regime
%-----
global Alfa Beta x2 x1 case_number
format long e

% Beta and Alfa (associated with the natural frequencies) for
% fixed - free beam are:
Beta=[1.875104068712 4.694091132974 7.854757438238 10.99554073488
14.13716839105...
      17.27875953209 20.42035225104 23.56194490181].';
Alfa= -[0.7340955137589 1.018467318759 0.9992244965174
1.000033553252...
      0.9999985501087 1.000000062656 0.9999999972924 1.000000000117].';

bej = Beta; gaj = Alfa;

number_modes = 8;% number of modes
number_elmts=100;% number of elements for the numerical integration

% Find 1-D integrals: Ai, Bi, Ci:
for ii=1:number_modes
    Gamma = pi/2+(ii-1)*pi;
    for jj=1:(number_elmts+1)
        XX=(jj-1)/number_elmts;
        F_Ai(jj) = sff(XX,0,ii);
        F_Bi(jj) = XX*sff(XX,0,ii);
        F_Ci(jj) = XX*XX*sff(XX,0,ii);
    end

    [Ai(ii)]=tr_rule(number_elmts+1,F_Ai);
    [Bi(ii)]=tr_rule(number_elmts+1,F_Bi);
    [Ci(ii)]=tr_rule(number_elmts+1,F_Ci);

end %ii

% Find 2-D integrals: Dij, Eij, Hij, Iij, Jij, Mij, Nij:
for ii1=1:number_modes
    for ii2=1:number_modes
        for jj=1:(number_elmts+1)
            XX=(jj-1)/number_elmts;
            F_Dij(jj)=(1-XX*XX)/2*sff(XX,1,ii1)*sff(XX,1,ii2);
            F_Eij(jj)=XX*sff(XX,0,ii1)*sff(XX,0,ii2);
            F_Hij(jj)=XX*sff(XX,0,ii1)*sff(XX,1,ii2);
            F_Iij(jj)=XX*sff(XX,0,ii1)*sfs(XX,0,ii2);
            F_Jij(jj)=XX*XX*sff(XX,0,ii1)*sfs(XX,0,ii2);
            F_Mij(jj)=XX*sfs(XX,0,ii1)*sfs(XX,0,ii2);
            F_Nij(jj)=(1-XX*XX)/2*sfs(XX,1,ii1)*sfs(XX,1,ii2);
        end
        [Dij(ii1,ii2)]=tr_rule(number_elmts+1,F_Dij);
        [Eij(ii1,ii2)]=tr_rule(number_elmts+1,F_Eij);
        [Hij(ii1,ii2)]=tr_rule(number_elmts+1,F_Hij);
        [Iij(ii1,ii2)]=tr_rule(number_elmts+1,F_Iij);
        [Jij(ii1,ii2)]=tr_rule(number_elmts+1,F_Jij);
        [Mij(ii1,ii2)]=tr_rule(number_elmts+1,F_Mij);
    end
end

```

```

[Nij(ii1,ii2)]=tr_rule(number_elmts+1,F_Nij);

    end %ii2
end %ii1

disp('2-D DONE');

deij = eye(number_modes);

% Find 3-D integrals: Fijk, Gijk, Hijk, Kijk, Kjki, Kkji:
for ii1=1:number_modes
    for ii2=1:number_modes
        for ii3=1:number_modes
            for jj=1:(number_elmts+1)
                XX=(jj-1)/number_elmts;
                F_Fijk(jj)=-(1/Beta(ii3))^4*sff(XX,1,ii1)*sff(XX,1,ii2)*...
                    sff(XX,3,ii3);
                F_Gijk(jj)=XX*sff(XX,0,ii1)*sff(XX,0,ii2)*sfs(XX,0,ii3);
                F_Hijk(jj)=sff(XX,0,ii1)*sff(XX,0,ii2)*sfs(XX,0,ii3);
                F_Kijk(jj)=sfs(XX,0,ii1)*sff(XX,2,ii2)*sff(XX,2,ii3);
                F_Kjki(jj)=sfs(XX,0,ii2)*sff(XX,2,ii3)*sff(XX,2,ii1);
                F_Kkji(jj)=sfs(XX,0,ii3)*sff(XX,2,ii2)*sff(XX,2,ii1);
            end
            [Fijk(ii1,ii2,ii3)]=tr_rule(number_elmts+1,F_Fijk);
            [Gijk(ii1,ii2,ii3)]=tr_rule(number_elmts+1,F_Gijk);
            [Hijk(ii1,ii2,ii3)]=tr_rule(number_elmts+1,F_Hijk);
            [Kijk(ii1,ii2,ii3)]=tr_rule(number_elmts+1,F_Kijk);
            [Kjki(ii2,ii3,ii1)]=tr_rule(number_elmts+1,F_Kjki);
            [Kkji(ii3,ii2,ii1)]=tr_rule(number_elmts+1,F_Kkji);
        end
    end %ii3
end %ii2
end %ii1

%Find Lijk, Likj:
for ii1=1:number_modes
    for ii2=1:number_modes
        for ii3=1:number_modes
            for jj=1:(number_elmts+1)
                XX=(jj-1)/number_elmts;
                F_1(jj)=sff(XX,1,ii2)*sff(XX,2,ii3);
                F_2(jj)=sff(XX,1,ii3)*sff(XX,2,ii2);
            end
            DX=1/number_elmts;
            G1(1)=0.0;
            G1(2)=(F_1(1)+F_1(2))*DX*0.5;
            G2(1)=0.0;
            G2(2)=(F_2(1)+F_2(2))*DX*0.5;
            for jj=3:(number_elmts+1)
                [G1(jj)]=tr_rule(jj,F_1);
                [G2(jj)]=tr_rule(jj,F_2);
            end
            for jj=1:(number_elmts+1)
                XX=(jj-1)/number_elmts;
                F_Lijk(jj)=-G1(jj)*XX*XX*sff(XX,0,ii1)+...
                    XX*sff(XX,0,ii1)*sff(XX,0,ii2)*sff(XX,1,ii3);
            end
        end
    end
end

```

```
F_Likj(jj)=-G2(jj)*XX*XX*sff(XX,0,ii1)+...
          XX*sff(XX,0,ii1)*sff(XX,0,ii3)*sff(XX,1,ii2);
end
[Lijk(ii1,ii2,ii3)]=tr_rule(number_elmts+1,F_Lijk);
[Likj(ii1,ii3,ii2)]=tr_rule(number_elmts+1,F_Likj);

    end %ii3
  end %ii2
end %ii1
disp('3-D DONE');
```

```

function [X,vi,T,V,W]=SysSolverCBB
% This subroutine solves for the equilibrium deflections, as well as
% for the tip deflections and inflow velocity for the CBB regime
%-----
global x0 sb Te
% Set the options used in the fsolve subroutine
options=optimset('Display', 'iter','LargeScale','off','TolX',1e-8);
disp('THETA'); disp(Te);

% Solve for the equilibrium quantities:
X = fsolve('EqSolver5',x0,options);

x0=X;

% Steady state lead-lag deflection at the tip
V=0;
for i=1:sb
    V=V+X(i)*(-1)^(i+1);
end
V=2*V;

% Steady state flap deflection at the tip
W=0;
for i=1:sb
    W=W+X(sb+i)*(-1)^(i+1);
end
W=2*W;

% Steady state torsion deflection at the tip
T=0;
for i=1:sb
    T=T+X(2*sb+i)*(-1)^(i+1);
end
T=sqrt(2)*T;
% Inflow velocity
vi = X(3*sb+2);
X=X(1:3*sb);

```



```

function [Ms,Mt,C,Ks,Kt]=MCKassemblyCBB(X,vi)
% This function assembles mass, stiffness, and damping matrices for the
%CBB regime.
% It also uses the equilibrium deflections which were found
%previously.
%-----
% Defining the adjustable parameters
global Stz Sty Stj Rc Te Bp ce Si Mu Mu1 Mu2 k1
global Ga Cd
global bej gaj deij Dij Ai Bi Ci Hij Jij Nij Lijk Likj Kkji Kijk Kjki
global Iij Eij Mij Gijk Hijk Fikj Fjki Fijk Gamma
global Txi Bxi dexij1 dexij2 deltaxij1 Dxij Nxlij Nx2ij Nx3ij Fxijk
Kxijk
global sb
global gamma case_number
global mss Sty_p Stz_p Stj_p k1_p Mu2_p Mu1_p Mu_p

format long e

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%
% Redefine the equilibrium deflections
V0=X(1:sb);
W0=X((sb+1):(2*sb));
PHI0=X((2*sb+1):(3*sb));

% Mass matrix construction.
M11 = deij;
M22 = deij;
M33 = deij*Mu^2;
M12 = zeros(sb);
M13 = zeros(sb);
M21 = zeros(sb);
M23 = zeros(sb);
M31 = zeros(sb);
M32 = zeros(sb);
Ms = [M11 M12 M13;M21 M22 M23; M31 M32 M33];

% Structural and aerodynamics Mass matrix
Mt11=M11;
Mt22=M22+deij*Ga*ce/24;
Mt33=M33;
Mt = [Mt11 M12 M13;M21 Mt22 M23;M31 M32 Mt33];

% Damping matrix construction
% Structural damping
% Element (1,1)
Cv111 = zeros(sb);
for i=1:sb
    for j=1:sb
        for k=1:sb
            Cv111(i,j)=Cv111(i,j)+2*(Fijk(i,k,j)-Fijk(j,k,i))*X(k);
        end
    end
end
end
Cv{1,1}=Cv111;

```

```

% Element (1,2)
Cv121 = -2*Bp*deij;
Cv122 = zeros(sb);

for i=1:sb
    for j=1:sb
        for k=1:sb
            Cv122(i,j)=Cv122(i,j)-2*Fijk(j,k,i)*X(k+sb);
        end
    end
end
Cv{1,2} = Cv121 + Cv122;

% Element (1,3)
Cv{1,3} = zeros(sb);

% Element (2,1)
Cv211 = 2*Bp*deij;
Cv212 = zeros(sb);
for i=1:sb
    for j=1:sb
        for k=1:sb
            Cv212(i,j)=Cv212(i,j)+2*Fijk(i,k,j)*X(k+sb);
        end
    end
end
Cv{2,1} = Cv211 + Cv212;

% Element (2,2)
Cv{2,2} = zeros(sb);

% Element (2,3)
Cv{2,3} = zeros(sb);

% Element (3,1)
Cv{3,1} = zeros(sb);

% Element (3,2)
Cv{3,2} = zeros(sb);

% Element (3,3)
Cv{3,3} = zeros(sb);

% Aerodynamic damping matrix
% Element (1,1)
Ca111 = Ga/6*2*Cd*Eij;
Ca112 = Ga/6*Te*vi*deij;
Ca113 = zeros(sb);
for i=1:sb
    for j=1:sb
        for k=1:sb
            Ca113(i,j) = Ca113(i,j) + Ga/6*vi*Hijk(i,j,k)*X(k+2*sb);
        end
    end
end
Ca{1,1}=Ca111+Ca112+Ca113;

```

```

% Element (1,2)
Ca121 = Ga/6*Te*Eij;
Ca122 = -Ga/6*2*vi*deij;

Ca123 = zeros(sb);
for i=1:sb
    for j=1:sb
        for k=1:sb
            Ca123(i,j) = Ca123(i,j) + Ga/6*Gijk(i,j,k)*X(k+2*sb);
        end
    end
end
Ca{1,2}=Ca121+Ca122+Ca123;

% Element (1,3)
Ca{1,3} = zeros(sb);

% Element (2,1)
Ca211 = -Ga/6*2*Te*Eij;
Ca212 = Ga/6*vi*deij;
Ca213 = zeros(sb);
for i=1:sb
    for j=1:sb
        for k=1:sb
            Ca213(i,j) = Ca213(i,j) - 2*Ga/6*Gijk(i,j,k)*X(k+2*sb);
        end
    end
end
Ca{2,1}=Ca211+Ca212+Ca213;

% Element (2,2)
Ca{2,2} = Ga/6*Eij;

% Element (2,3)
Ca{2,3} = -0.75*Ga/6*ce*Iij;

% Element (3,1)
Ca{3,1} = zeros(sb);

% Element (3,2)
Ca{3,2} = zeros(sb);

% Element (3,3)
Ca{3,3} = Ga/6*ce^2/8*Mij;

%Aerodynamic damping matrix
Ca = [Ca{1,1} Ca{1,2} Ca{1,3};Ca{2,1} Ca{2,2} Ca{2,3}; Ca{3,1} Ca{3,2}
Ca{3,3}];
%Structural damping matrix

Cs = [Cv{1,1} Cv{1,2} Cv{1,3};Cv{2,1} Cv{2,2} Cv{2,3}; Cv{3,1} Cv{3,2}
Cv{3,3}];

% Total damping matrix
C=Ca+Cs;

% Stiffness matrix construction

```

```

% Structural stiffness matrix
% Element (1,1)
for i=1:sb
    for j=1:sb
        Ks111(i,j) = (Stz-(Stz-Sty)*(sin(Rc*Te))^2)*bej(j)^4*deij(i,j);
    end % j
end % i
Ks112 = zeros(sb);
for i=1:sb
    for j=1:sb
        for k=1:sb
            Ks112(i,j) = Ks112(i,j) + (Stz-
Sty)*sin(2*Rc*Te)*Kijk(k,j,i)*X(k+2*sb);
        end
    end
end
Ks11 = Dij+Ks111-deij-Ks112;

% Element (1,2)
for i=1:sb
    for j=1:sb
        Ks121(i,j) = (Stz-Sty)*(sin(2*Rc*Te))/2.0*bej(j)^4*deij(i,j);
    end % j
end % i
Ks122 = zeros(sb);
for i=1:sb
    for j=1:sb
        for k=1:sb
            Ks122(i,j)=Ks122(i,j) + (Stz-
Sty)*cos(2*Rc*Te)*Kijk(k,j,i)*X(k+2*sb);
        end
    end
end
Ks12 = Ks121+Ks122;

% Element (1,3)
Ks131 = zeros(sb);
for i=1:sb
    for j=1:sb
        for k=1:sb
            Ks131(i,j)=Ks131(i,j) + (Stz-
Sty)*Kijk(j,k,i)*(X(k+sb)*cos(2*Rc*Te)-X(k)*sin(2*Rc*Te));
        end % k
    end
end
Ks13 = Ks131;

% Element (2,1)
Ks21 = Ks12;

% Element (2,2)
for i=1:sb
    for j=1:sb
        Ks221(i,j) = (Sty+(Stz-Sty)*(sin(Rc*Te))^2)*bej(j)^4*deij(i,j);
    end % j
end

```

```

end % i
Ks222 = Ks112;
Ks22 = Dij+Ks221+Ks222;

% Element (2,3)
Ks231 = zeros(sb);
for i=1:sb
    for j=1:sb
        for k=1:sb
            Ks231(i,j)=Ks231(i,j) + (Stz-
Sty)*Kijk(j,k,i)*(X(k+sb)*sin(2*Rc*Te)+X(k)*cos(2*Rc*Te));
        end % k
    end
end
Ks23 = Ks231;

% Element (3,1)
Ks311 = zeros(sb);
for i=1:sb
    for j=1:sb
        for k=1:sb
            Ks311(i,j)=Ks311(i,j) + (Stz-
Sty)*Kijk(i,j,k)*(X(k+sb)*cos(2*Rc*Te)-...
X(k)*sin(2*Rc*Te));
        end % k
    end
end
Ks31 = Ks311;

% Element (3,2)
Ks321 = zeros(sb);
for i=1:sb
    for j=1:sb
        for k=1:sb
            Ks321(i,j)=Ks321(i,j) + (Stz-
Sty)*Kijk(i,j,k)*(X(k+sb)*sin(2*Rc*Te)+...
X(k)*cos(2*Rc*Te));
        end % k
    end
end
Ks32 = Ks321;

% Element (3,3)
Ks331=Mu^2*k1*Nij+(Mu2^2-Mu1^2)*cos(2*Te)*deij;
for i=1:sb
    for j=1:sb
        Ks332(i,j) = Stj*gamma(j)^2*deij(i,j);
    end % j
end % i
Ks33=Ks331+Ks332;

Ks = [Ks11 Ks12 Ks13;Ks21 Ks22 Ks23; Ks31 Ks32 Ks33];

% Construct aerodynamic stiffness matrix, Kar
% Element (1,1)
Ka11 = zeros(sb);

```

```

% Element (1,2)
Ka12 = zeros(sb);

% Element (1,3)
Ka13 = Ga/6*vi*Iij;

% Element (2,1)
Ka211 = Ga/6*Bp*Eij;
Ka212 = zeros(sb);
for i=1:sb
    for j=1:sb
        for k=1:sb
            Ka212(i,j) = Ka212(i,j) + Ga/6*Lijk(i,j,k)*X(k+sb);
        end % k
    end
end
Ka21 = Ka211+Ka212;

% Element (2,2)
Ka221 = Ga/6*(-ce/2*Hij);
Ka222 = zeros(sb);
for i=1:sb
    for j=1:sb
        for k=1:sb
            Ka222(i,j) = Ka222(i,j) + Ga/6*Lijk(i,k,j)*X(k);
        end % k
    end
end
Ka22 = Ka221+Ka222;

% Element (2,3)
Ka23 = Ga/6*(-Jij);

% Element (3,1)
Ka31 = zeros(sb);

% Element (3,2)
Ka32 = zeros(sb);

% Element (3,3)
Ka33 = zeros(sb);

% Aerodynamics stiffness matrix, Ka
Ka = [Ka11 Ka12 Ka13; Ka21 Ka22 Ka23; Ka31 Ka32 Ka33];

% Total stiffness matrix
Kt = Ks+Ka;

disp('DONE matrices');

```

```

function [Ga,Si,Cd,ce,Mu,Mu1,Mu2,k1,Num_b,Sty,Stz,Stj,]=InputData1CBB
%This subroutine provides with cross-sectional parameters used by the
%MainProgram2.m.
%This program includes the stiffness calculations due to bending only.
%-----
global Ga Si Cd ce Mu Mu1 Mu2 k1 Num_b Sty Stz Stj Ww Wv Wt gamma
global mss Sty_p Stz_p Stj_p k1_p Mu2_p Mu1_p Mu_p
global case_number
format long

Wv = 0.514;%non-dimensional, nonrotating lead-lag natural frequency
Ww = 0.143;%non-dimensional, nonrotating flap natural frequency
Wt = 3.77;%non-dimensional, nonrotating torsionnatural frequency
Bbe = 1.8575104;%constant from the assumed mode shape
Gga = pi/2;%constant from the assumed mode shape
Om = 110;% spin velocity, rad/sec
R = 2.0*10^3;% blade radius, mm
Ro_lead = 11340*10^(-9);%lead density, kg/mm^3

chord = 121.0;% chord length of the blade, mm
E1 = 73.1*10^3;% Young's modulud of the aluminum, N/mm^2
Ro1 =2770*10^(-9);%Aluminum density, kg/mm^3
nu = 0.33;% Poisson's ratio of the aluminum
G1=E1/2/(1+nu);%Shear modulus of the aluminum, N/mm^2

percentage = 25.0;%defines the location of the shear center of the
blade
c = 44.4;% horizontal dimension of the spar, mm
alpha = 0.0501;%cross-sectional constant
kappa = 0.0501;%cross-sectional constant
ratio = 0.12;%cross-sectional constant
r2 = percentage*chord/100;% distance from the leading edge to the
spar
delta = 0.3;%cross-sectional constant

coef1 = (1 - (1-2*alpha)^3*(1-2*kappa) + 2*delta*(ratio)^3 +...
12*2*delta*ratio*(1/2 + ratio/2)^2)/12;% coefficient to define
the moment of inertia
coef2 = (1 - (1-2*kappa)^3*(1-2*alpha) + 2*delta^3*ratio)/12;%
coefficient to define the moment of inertia
BB = Wv/Ww*sqrt(coef1/coef2);
d = c/BB;%vertial dimension of the spar, mm

K = 2*kappa*alpha*d*c*(c-kappa*c)^2*(d-alpha*d)^2/...
(c*kappa*c + d^2*alpha - alpha^2*d^2 - kappa^2*c^2);%
torsional constant of the spar, mm^4

Iy = d^3*c*coef1;% moment of inertia about y axis,mm^4
Iz = c^3*d*coef2;% moment of inertia about z axis, mm^4
J = d^3*c*coef1 + c^3*d*coef2;% polar moment of inertia of the spar,
mm^4
A2 = c*d - (c-2*kappa*c)*(d-2*alpha*d) + 2*ratio*d*delta*c;% area of
the spar, mm^2
m2 = Ro1*A2;% mass per unit lengh of the spar, kg/mm
m = E1*10^3*Bbe^4*coef2*BB^3*d^4/(Om^2*R^4*Wv^2);% mass per unit
length of the blade, kg/mm

```

```

    r3 = 112.0;% distance from the leading edge to the second tuning
bar, mm
    r1= (K*G1*10^3*Gga^2/(Om^2*R^2*Wt^2)-Ro1*J - (m-m2)*(r3-r2)^2)/...
        ((m - m2)*(r2-r3)) + 2*r2 - r3;% distance from the leading edge
to the 1st tuning bar, mm

    m1 = (m - m2)*(r2-r3)/(r1-r3);%mass per unit length of the 1st
tuning bar, kg/mm
    m3 = m-m1-m2;% mass per unit length of the second tuning bar, kg/mm

    % Define mass radii of gyration:
    km = sqrt((J*Ro1 + m1*(r2-r1)^2 + m3*(r3-r2)^2)/m );% mm
    km1 = sqrt(Iy*Ro1/m);% mm
    km2 = sqrt((Iz*Ro1+m1*(r2-r1)^2 + m3*(r3-r2)^2)/m);%mm
    ka = sqrt(J/A2);% area radius of gyration

    Num_b = 4;% number of blades
    ce = chord/R;% chors length
    rho_inf = 1.225*10^(-9); % air density in kg/mm^3
    Ga = 3*rho_inf*2*pi*0.0605*2000^2/m;%Lock number
    Si = ce*Num_b/pi;% Solidity ratio
    Cd = 1.59e-3;% drag coefficient

    % Define non-dimensional radii of gyration:
    Mu1 = km1/R;
    Mu2 = km2/R;
    Mu = sqrt(Mu1^2+Mu2^2);

    k1 = (ka/km)^2;

    % Define non-dimensional stiffnesses and torsional rigidities:
    Sty = E1*Iy/m/Om^2/R^4*10^3;
    Stz = E1*Iz/m/Om^2/R^4*10^3;
    Stj = G1*K/m/Om^2/R^4*10^3;

```



```

function [Ga,Si,Cd,ce,Mu,Mu1,Mu2,k1,Num_b,Sty,Stz,Stj]=InputData2CBB
%This subroutine provides with cross-sectional parameters used by the
%MainProgram2.m.
%This program includes stiffness calculations due to shear and bending.
%-----
global Ga Si Cd ce Mu Mu1 Mu2 k1 Num_b Sty Stz Stj Ww Wv Wt gamma
global mss Sty_p Stz_p Stj_p k1_p Mu2_p Mu1_p Mu_p
global case_number
format long
% The following characteristics are analogous to the ones defined in
%InputData1CBB.
    Wv = 0.514;                Ww = 0.143;
    Wt = 3.77;                Bbe = 1.8575104;
    Gga = pi/2;               Om = 110;
    R = 2.0*10^3;              Ro_lead = 11340*10^(-9);

    chord = 121.0;
    E1 = 73.1*10^3;            Ro1 = 2770*10^(-9);
    nu = 0.33;                 G1=E1/2/(1+nu);

    percentage = 25.0;
    c = 44.4;
    alpha = 0.0501;
    kappa = 0.0501;
    ratio = 0.12;
    r2 = percentage*chord/100;
    delta = 0.3;

    coef1 = (1 - (1-2*alpha)^3*(1-2*kappa) + 2*delta*(ratio)^3 + ...
             12*2*delta*ratio*(1/2 + ratio/2)^2)/12;
    coef2 = (1 - (1-2*kappa)^3*(1-2*alpha) + 2*delta^3*ratio)/12;
    BB = Wv/Ww*sqrt(coef1/coef2);
    d = c/BB;

    K = 2*kappa*alpha*d*c*(c-kappa*c)^2*(d-alpha*d)^2/...
        (c*kappa*c + d^2*alpha - alpha^2*d^2 - kappa^2*c^2);
    Iy = d^3*c*coef1;
    Iz = c^3*d*coef2;
    J = d^3*c*coef1 + c^3*d*coef2;
    A2 = c*d - (c-2*kappa*c)*(d-2*alpha*d) + 2*ratio*d*delta*c;
    m2 = Ro1*A2;
    m = E1*10^3*Bbe^4*coef2*BB^3*d^4/(Om^2*R^4*Wv^2);
    r3 = 112.0;
    r1 = (K*G1*10^3*Gga^2/(Om^2*R^2*Wt^2)-Ro1*J - (m-m2)*(r3-r2)^2)/...
        ((m - m2)*(r2-r3)) + 2*r2 - r3;
    m1 = (m - m2)*(r2-r3)/(r1-r3);
    m3 = m-m1-m2;

    % Define mass radii of gyration:
    km = sqrt((J*Ro1 + m1*(r2-r1)^2 + m3*(r3-r2)^2)/m);
    km1 = sqrt(Iy*Ro1/m);
    km2 = sqrt((Iz*Ro1+m1*(r2-r1)^2 + m3*(r3-r2)^2)/m);
    ka = sqrt(J/A2);

    Num_b = 4;
    ce = chord/R;
    rho_inf = 1.225*10^(-9); % air density in kg/mm^3

```

```
Ga = 3*rho_inf*2*pi*0.0605*2000^2/m;% Lock number
Si = ce*Num_b/pi;
Cd = 1.59e-3;

% Define non-dimensional radii of gyration:
Mu1 = km1/R;
Mu2 = km2/R;
Mu = sqrt(Mu1^2+Mu2^2);
k1 = (ka/km)^2;

% Define stiffnesses and torsional rigidities:
cc1 = 3*E1*Iy/(2*R^2*G1*kappa*c*d*(1-2*alpha));
EIyo =E1*Iy/(1+cc1);

bb1 = 3*E1*Iz/(2*R^2*G1*alpha*d*c*(1-2*kappa));
EIzo =E1*Iz/(1+bb1);

% Define non-dimensional stiffnesses and torsional rigidities:
Sty = EIyo/m/Om^2/R^4*10^3;
Stz = EIzo/m/Om^2/R^4*10^3;
Stj = G1*K/m/Om^2/R^4*10^3;
```

```

function S=EqSolverCBB(X);
% In this subroutine the nonlinear algebraic equations of motion are
% defined in order to calculate equilibrium deflections for the CBB
% regime.
%-----
global bej gaj Ai Bi Ci Dij Eij Iij Jij Mij deij Nij
global Hij Fjki Fikj Fjki Gijk Hijk Kijk Kjki Kkji Lijk Likj
global Txi Bxi dexij1 dexij2 deltaxij1 Dxij Nx1ij Nx2ij Nx3ij Fxijk
Kxijk
global Ga Si Cd Bp ce Mu Mu1 Mu2 k1
global Rc gamma
global Te
global Stz Sty Stj
global sb case_number
global mss Sty_p Stz_p Stj_p k1_p Mu2_p Mu1_p Mu_p
format long;
% Define the twist at the station of 75% of the blade length
T75=0;
for zzz=1:sb
    T75=T75+sqrt(2)*X(2*sb+zzz)*sin(0.75*(zzz-0.5)*pi);
end
T75=T75-X(3*sb+1);
% Define the inflow velocity
vi= sign(Te+X(3*sb+1))*pi*Si/8.0*(sqrt(1.0+12.0/(pi*...
    Si)*abs(Te+X(3*sb+1)))-1.0)-X(3*sb+2);

% In this section the equations of motion are defined in order to find
the
% equilibrium deflections
P=zeros(sb,1);
Z=zeros(sb,1);
D=zeros(sb,1);
%Equation for dVoi, dWoi, dPhioi
for i = 1:sb
    for j = 1:sb
        Q=0;U1=0;U2=0;E=0;
        for k =1:sb
            Q=Q+Kijk(j,k,i)*X(2*sb+j)*(-X(k)*sin(2.0*Rc*Te)+...
                X(sb+k)*cos(2.0*Rc*Te));
            U1=U1+Kijk(j,k,i)*X(2*sb+j)*(X(k)*cos(2*Rc*Te)+...
                X(sb+k)*sin(2*Rc*Te));
            U2=U2+Lijk(i,j,k)*X(j)*X(sb+k);
            E=E+Kijk(i,j,k)*(sin(2*Rc*Te)/2.0*(X(sb+j)*X(sb+k)...
                -X(j)*X(k))+cos(2*Rc*Te)*X(j)*X(sb+k));
        end %k
        % Lead lag eqn:
        P(i) =P(i) + X(j)*(Dij(i,j)+(Stz-(Stz-Sty)*(sin(Rc*Te))^2)*...
            bej(j)^4*deij(i,j)-deij(i,j))+
            X(sb+j)*sin(2.0*Rc*Te)/2.0*(Stz-...
            Sty)*bej(j)^4*deij(i,j)+(Stz-Sty)*Q+...
            Ga/6.0*X(3*sb+2)*Iij(i,j)*X(2*sb+j);
        % Flap eqn:
        Z(i) = Z(i)+X(j)*sin(2.0*Rc*Te)/2.0*(Stz-Sty)*bej(j)^4*...
            deij(i,j)+X(sb+j)*(Dij(i,j)+(Sty+(Stz-
            Sty)*(sin(Rc*Te))^2)*...

```

```

        bej(j)^4*deij(i,j))+(Stz-Sty)*U1+Ga/6.0*(Bp*...
        Eij(i,j)*X(j)-ce/2.0*Hij(i,j)*X(sb+j)-
        Jij(i,j)*X(2*sb+j)+U2);
    % Torsion eqn:
    D(i) = D(i) + X(2*sb+j)*(Mu^2*k1*Nij(i,j)+(Stj*gamma(j)^2+...
        (Mu2^2-Mu1^2)*cos(2*Te))*deij(i,j))+(Stz-Sty)*E;

end %j

P(i)=P(i)-Ga/6.0*((X(3*sb+2))^2*Ai(i)-Cd*Ci(i)-...
    X(3*sb+2)*Te*Bi(i));

Z(i)=Z(i)+ Bp*Bi(i)-Ga/6.0*(Te*Ci(i)-X(3*sb+2)*Bi(i)+...
    ce/2.0*Bp*Bi(i));

D(i)=D(i)+(Mu2^2-Mu1^2)*sin(2*Te)/(sqrt(2)*gamma(i));

end %i
% Define the system of equations:
S=[T75; vi; P; Z; D];

```