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Vibration of Rings of Arbitrary Cross-Section

James Leonard Denoon Olmstead

*A thesis submitted to the Faculty of Graduate Studies and Research
in partial fulfillment of the requirements for the
Degree of Master of Engineering*

DEPARTMENT OF MECHANICAL AND AERONAUTICAL ENGINEERING
CARLETON UNIVERSITY
OTTAWA, ONTARIO
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ABSTRACT

The vibration of rings of asymmetrical cross-section in which in-plane and out of plane motions are coupled, is studied numerically and compared with various experimental results.

The equations of motion are first obtained for complete rings using a six degree of freedom (6 DOF) model which includes the effects of asymmetry, transverse shear, rotatory inertia and shear center offsets. Finite elements are then derived for ring segments to extend the applicability to partial rings and curved bar structures. These result in 22 DOF and 30 DOF elements for thin and thick rings respectively.

Numerical studies are compared with experimental data to assess the accuracy and efficiency of the methods

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NOMENCLATURE

A	area
C	St Venant torsion Coefficient
E	Young's modulus
G	shear modulus
K	stiffness matrix
K_{ij}	stiffness coefficient
L	element length
M	mass matrix
M_{ij}	mass coefficient
N	shape function
P	strain energy
R	centroidal radius of ring
R_s	shear center radius
S	length along element
T	kinetic energy
U	6 DOF element axial displacement
V	6 DOF element out of plane displacement
W	6 DOF element in-plane displacement
c	centroid
n	number of nodal diameters
q	modal vector

s	length along element centroid
u	axial displacement
$\bar{u}$	axial centroidal displacement
v	out of plane displacement
$\bar{v}$	out of plane shear center displacement
w	in-plane displacement
$\bar{w}$	in-plane shear center displacement
z	local cartesian coordinate
z_1, z_2	generalized coordinates
y	local cartesian coordinate
y_0	shear center offset (for y direction)
z	local cartesian coordinate
z_0	shear center offset (for z direction)
θ	generalized rotation coordinate
ϕ	generalized rotation coordinate
ψ	generalized rotation coordinate
α	shear coefficient
β	element half angle
ϵ	axial strain
θ_b	rotation due to out of plane bending
θ_s	rotation due to transverse shear out of plane
ξ	angle
r	twist of ring segment
ϕ	twist

ψ_b rotation due to in-plane bending.
 ψ_s rotation due to transverse shear in-plane
 ω frequency

Subscripts

s shear
 b bending

Superscripts

$()'$ derivative $\frac{d()}{dx}$
 $()$ time derivative, $\frac{d()}{dt}$

Chapter 1

Introduction

The vibration of complete rings has been the subject of much work in the past. Such rings form important elements of such structures as stiffened shells, gears and electrical machines. Many other structures of engineering importance contain curved arc segments or members—for example bridges and roadways. Almost all the work carried out has assumed cross-sections that are symmetrical with respect to the plane of the ring's curvature. This assumption uncouples in-plane and out of plane ring motions.

1.1 In-Plane Bending

Hoppe [22] gave the original solution for free in-plane vibrations of a thin ring. His equation

$$\omega_i^2 = \left(\frac{EI_m}{\rho AR^4} \right) \left[\frac{n^2(n^2 - 1)^2}{(n^2 + 1)} \right] \quad (1-1)$$

is generally referred to as the "classical solution". Because the effects of axial extension, shear and rotatory inertia are neglected, only low nodal diameter modes of thin rings are predicted well. Even with thin rings, careful experimental measurements [29,31] show that very substantial errors result when the number of nodal diameters becomes high. The effects of axial extension, transverse shear and rotatory inertia have been studied [19,32,38,42,40] in order to improve the classical solution. Federhofer [19] obtained an exact solution containing sets of Bessel functions which was somewhat cumbersome to use. Philipson [38] demonstrated that the effect of axial extension was negligible. The works of Seidel/Erdelyi [42] and Rao/Sundararajan [40] are perhaps of most interest in the approximate solutions. In reference [42] an energy approach is used with assumptions regarding non-linear variation of normal strain through the cross-section of the ring and an average shear angle. From this a cubic frequency equation is obtained. After the effect of mid-surface extension

was shown to be negligible, the cubic frequency equation was further reduced to a quadratic. In reference [40] inextensible motion and linear variation of normal strain through the section are assumed from the outset. This results in a quadratic frequency expression. Neither of these works [40,42] critically assess the accuracy of the solutions obtained. Both appear to make incorrect comparisons with the experimental work of Kuhl [29]. Thus, while these theoretical solutions agree within a few percent, the deviation from the experimental data exceeds 50 % for rings vibrating in only two nodal diameters.

Kirkhope [25] used an energy approach to formulate a very simple frequency expression for in-plane vibration of thick rings. For a ring vibrating in a mode of n nodal diameters,

$$\omega_n^2 = \omega_1^2 \left[\frac{1}{\left(1 + \frac{n^2 EI_{yy}}{G_0 AR^2}\right)} \right] \quad (1-2)$$

in which ω_1^2 is the classical value of the frequency for a thin ring, given by equation (1-1). The effects of mid-surface extension and non-linear variation of normal strain through the cross-section were neglected. Comparisons with the experimental data of references [29] and [31] demonstrated quite remarkable accuracy for very thick rings vibrating in high nodal diameter modes. In [27] this work is extended to include extension and non-linear variation of normal strain through the cross-section.

1.2 Out of Plane Bending and Torsion

Love [32] examined the early solutions for out of plane vibration. These solutions considered a thin ring with symmetrical cross-section about the plane of the ring. Each section was assumed to turn in its own plane about the centroidal axis of the section, while the axis itself was not displaced. Consequently the bending and torsional motions were considered uncoupled. The simple frequency expression for bending vibration is:

$$\omega_n^2 = \frac{EI_{xx}}{\rho AR^4} \frac{n^2(n^2 - 1)^2}{n^2 + \frac{EI_{xx}}{C}} \quad (1-3)$$

and for the torsional vibration:

$$\omega_n^2 = \frac{C}{\rho(I_{yy} + I_{xx})R^2} \left[n^2 + \frac{EI_{xx}}{C} \right] \quad (1-4)$$

Because transverse shear and rotatory inertia were neglected, the behavior of thick rings or even thin rings vibrating in high nodal diameter modes could not be predicted accurately. Rao [41] derived a cubic frequency equation for the frequencies of vibration of a thick ring taking into account these effects. He formulated a differential equation for the out of plane motion of the ring which reduces to Timoshenko's beam equation in the limiting case of infinite radius. His comparisons of calculated and experimental data, however, show substantial error in some cases. He did not examine the torsional modes of vibration of the ring. Volterra [31,44] also examined the application of the method of internal constraints for calculation of the frequencies of out of plane vibration of thick rings. He presented the results of very precise experimental measurements on rings of circular cross section. Other notable experimental studies have been carried out by Peterson [37] in connection with gear vibration and by Kuhl [29]. This body of carefully determined experimental data is useful in assessing the accuracy of theoretical frequency calculations.

In [26] Kirkhope derived dynamic stiffness matrices for complete circular rings. The effects of torsion and shear are included so that out of plane deflection, torsion and shear of the ring section are coupled. Solution accuracy was assessed by comparison of calculated frequencies with experimental data for thick rings of circular and rectangular cross-sections. Good results were obtained.

The energy functions of [25,26] could be used to formulate a curved sector finite element. Such an element would be limited to symmetric cross-sections and would reduce to a Timoshenko beam element as the radius of curvature of the element approaches infinity.

1.3 In-Plane and Out of Plane Coupled Vibration

Little work has been reported on the more general problem where the ring cross-section is arbitrary and asymmetrical with respect to the plane of the ring. In such cases the in-plane and out of plane motions are naturally coupled together.

Endo [18] applied the Rayleigh-Ritz method to asymmetrical cross-sections which were more representative of conical shells than compact rings and the method was somewhat unwieldy.

In 1977 Hawkings [20] presented a theory for inextensional vibration of circular rings of asymmetrical cross-section. Coupling between in-plane and out of plane motions was included as well as the effects of shear deformation and rotatory inertia. Hawkings' method resulted in a set of twelve governing linear simultaneous equations for each number of nodal diameters. He experimentally tested two rings of constant cross-sectional shape, measuring only the first five bending modes of the rings. The accuracy demonstrated for the theoretical method was within about 2.5 %. This was good but only a few low order modes were compared and no torsional modes were given or evaluated.

1.4 Incomplete Rings

1.4.1 Arches (In-Plane)

Lamb [30] was the first to study the motion of incomplete rings. He analysed the statics of incomplete rings with various boundary conditions and the dynamics of an incomplete free-free ring of small curvature. Den Hartog [17] applied the Rayleigh-Ritz method for finding the lowest natural frequency of circular arches with simply supported or clamped ends. Archer [4] derived a differential equation for inextensional vibration of a curved beam. He used it to calculate frequencies and mode shapes for incomplete rings with clamped ends. Nelson [35] employed the Rayleigh-Ritz method along with Lagrangian multipliers to form an infinite series solution for the natural frequencies and mode shapes of a simply supported arc. Both inextensional and extensional theories were considered. Rao and Sundararajan [40] derived and solved a differential equation for inextensional vibration of incomplete rings having fixed or hinged ends and complete rings with a number of radial supports. They included both shear and rotatory inertia.

The above methods are useful for preliminary design, but are not applicable to use in complex structures. Complex structures containing incomplete rings may be modelled by the finite element method. Martin [33] derived a stiffness matrix for in-plane extensional motion of a thin curved beam.

Much work was done by Ashwell et al [1,2,3] and Dawe [13,14,15,16] on curved arches. They tested various shape functions and formulations for curved beam elements in order to gain insight into the choice of suitable shape functions for curved shell elements. The general conclusion of [15] was that only quintic-quintic and assumed strain elements were generally satisfactory. In these tests, displacement, bending moments and forces were plotted over the element and compared with theory. In most of the elements studied axial extension was included. Because applications to shells was a primary goal, shear was not considered.

Meek [34] formulated an arc element of lower order than quintic but which demonstrated good performance. He assumed the axial strain given by $\frac{du}{ds} - \frac{v}{R}$, was zero (ie. the ring centerline was inextensible) or constant over the element. By utilizing the coupling relations inherent in the axial strain equation he was able to relax the element's continuity requirements while maintaining u as a quintic polynomial and w as a quartic polynomial. His element appeared to be efficient, but only very limited static evaluation was made.

Prathap and Bhashyam [39] investigated shear and in-plane locking effects in curved beams which used discrete Kirchhoff formulation. They demonstrate that certain rules must be followed in the formulation to ensure convergence to the correct solution for thin rings.

The development and testing of in-plane curved beam finite elements for dynamic analysis has been less common than that for static analysis. Davis et al. [11] developed an element including shear and rotatory inertia terms. Their analysis differs from the usual finite element formulations in that it does not commence with a displacement assumption. The displacement functions are obtained by an integration of the differential equation of an infinitesimal element in static equilibrium. Consequently the displacement functions are incompatible with other elements. Comparisons to theoretical, but not experimental results for complete rings were made.

Wang and Guilbert [45] also formulated a dynamic stiffness matrix for continuous circular curved beams. No comparison with theory is made.

It is evident that there remain some need to augment the small amount of work done so far.

1.4.2 Out of Plane Motion

Out of plane vibration of incomplete rings has received much less attention than in-plane vibration. Using the equations of motion derived by Love, Ojalvo [36] studied the coupled twist-bending vibrations of incomplete rings. Culver [9] solved the problem of out of plane vibration of a symmetrically curved beam simply supported but with torsional motion at the supports suppressed. More recently Wang et al. [46] derived a dynamic stiffness matrix for coupled out of plane vibration. No comparison of results with experiment was made. Davis et al. [12] seem to be the only workers who developed a finite element for out of plane vibration. The element included shear effects and rotatory inertia. It was derived similarly to the in-plane element [11]. Comparisons with experimental results for rectangular rings were made.

1.4.3 Coupled In-Plane and Out of Plane Motion

No partial arc finite elements that include the effects of shear, rotatory inertia, extension, coupling between in-plane and out of plane motions, and coupling between out of plane bending and torsion have been reported in the literature for this more general problem. The elements formulated in this thesis appear to be the first of their kind.

1.5 Proposed Work

This thesis will endeavor to augment and supplement the work done so far. Three finite elements for the analysis of rings are developed. They are all based upon a strain energy and a kinetic energy equation developed by integration of displacement over an arbitrary cross-section. The elements are a 6 degree of freedom axisymmetric element, a 22 degree of freedom partial arc element and a 30 degree of freedom partial arc element. The 6 and 30 degree of freedom elements include the effects of shear and rotatory inertia. Quite comprehensive comparisons with calculated frequencies and experimental data from Hawkings [20], Volterra [31] and Kirkhope [28] are made. Accuracy and limitations of the analysis are then discussed.

Chapter 2

Derivation of Strain and Kinetic Energy Equations For a Thick Ring

The equations of motion for the ring are most conveniently obtained by the Lagrangian Method using potential (strain) and kinetic energy functions. Derivation of suitable expressions for these energy functions is the subject of this chapter.

A ring segment of asymmetrical but uniform cross-section is considered. Figure 1 illustrates the deflections and rotations used to determine the ring deformation. Because of asymmetry of the cross-section, in-plane and out of plane motions will be coupled. Let us also assume that the flexural and torsional axes of the cross-section do not coincide. Thus, coupling between flexure and torsion results not only from the curvature of the segment, but also from the offsets of the shear center S from the centroid c . The effects of transverse shear and rotatory inertia are assumed to be significant in a ring of substantial cross-sectional thickness relative to the radius of curvature.

The problem of finding strain and kinetic energy expressions which include the above effects, can be solved by:

- (i) defining displacements over the cross-section of the ring;
- (ii) integrating the energies around the total ring.

Assuming plane sections remain planar, the deflections over the cross-section can be taken as the deflection of the centroid or shear center, plus contributions due to the rotation of the section.

$$\begin{aligned}u(y, z) &= \bar{u} \left(1 + \frac{z}{R}\right) - y\theta_b + z\psi_b \\v(y, z) &= \bar{v} - (z - z_0)\phi \\w(y, z) &= \bar{w} - (y - y_0)\phi\end{aligned}\tag{2-1}$$

In (2-1), $\bar{u}$ is the axial centroidal translation, $\bar{v}$ and $\bar{w}$ are the out of plane and in plane shear center translations respectively. y_0 and z_0 are the shear center offsets. R is the centroidal radius of the

ring. Note that the shear center radius, $R_s = R + z_0$. ϕ is the rotation of the cross-section about the normal to the cross-section.

Also in (2-1)

$$\begin{aligned}\theta_b &= \text{rotation due to bending only} \\ &= \frac{\partial v}{\partial s} - \theta_s\end{aligned}\quad (2-2)$$

where θ_s is the rotation caused by transverse shear, and $ds \equiv R d\xi$, i.e. is the incremental length equals the radius of the ring times the enclosed angle.

Similarly,

$$\psi_b = \frac{\partial w}{\partial s} - \psi_s \quad (2-3)$$

The axial strain ϵ at distance (y, z) from the centroid c is made up of a component due to axial extension, along with a component due to radial expansion.

$$\epsilon = \frac{\partial u}{\partial S} - \frac{w}{R+z} \quad (2-4)$$

where $S = (R+z)d\xi$. Substituting the displacement relations (2-1), (2-2), and (2-3) into the axial strain relation (2-4) yields

$$\epsilon = \frac{R}{R+z} \left[-y \frac{\partial \theta_b}{\partial s} + z \frac{\partial \psi_b}{\partial s} + \frac{R+z}{R} \frac{\partial \bar{u}}{\partial s} \right] - \left[\frac{\bar{w} - (y-y_0)\phi}{R+z} \right] \quad (2-5)$$

The strain energy P , is obtained by integrating over the cross section, thus

$$\begin{aligned}\frac{dP}{ds} &= \frac{E}{2} \int dA \left[\frac{R+z}{R} \right] \epsilon^2 \\ &\quad + \frac{G}{2} C r^2 \\ &\quad + \frac{G}{2} \alpha A [\theta_s^2 + \psi_s^2]\end{aligned}\quad (2-6)$$

In equation (2-6) the terms are due to axial strain, twist and transverse shear respectively. C is the torsional stiffness of the cross section and α the shear coefficient. Referring to Figure 4, the twist, r of a segment of a ring is given by a contribution from the torsional rotation, $\frac{1}{R_s} \frac{\partial \phi}{\partial s}$ and a contribution from the flexure due to bending, $\frac{\theta_b}{R_s}$. Using $(\cdot)'$ to denote $\frac{\partial}{\partial s}$ and substituting for θ_s , the twist is given by,

$$r = \phi' + \frac{\bar{v}'}{R_s} - \frac{\theta_s}{R_s} \quad (2-7)$$

In (2-6) the strain energy due to normal strain, ϵ may be expanded as follows

$$\begin{aligned} \int \frac{y^2}{R+z} dA R \theta_b'^2 &+ \int \frac{z^2}{R+z} dA R \psi_b'^2 &+ \int \frac{R+z}{R} dA u'^2 \\ + \int \frac{R}{R+z} dA \left(\frac{\bar{w}}{R}\right)^2 &+ \int \frac{y^2}{R+z} dA \frac{\bar{\phi}}{R} &+ \int \frac{y^2}{R+z} dA \frac{\bar{\phi}}{R} \\ + \int \frac{-2yz}{R+z} dA R \theta_b' \psi_b' &+ \int -2y dA \theta_b' u' &+ \int \frac{2yz}{R+z} dA \theta_b' \bar{w} \\ \int \left[\frac{R+z}{R} \right] \epsilon^2 dA = &+ \int \frac{-2y^2}{R+z} dA \theta_b' \phi &+ \int \frac{2yz}{R+z} dA y_0 \theta_b' \phi &+ \int 2z dA \psi_b' u' \\ + \int \frac{-2z^2}{R+z} dA \psi_b' \bar{w} &+ \int \frac{2yz}{R+z} dA \psi_b' \phi &+ \int \frac{-2z^2}{R+z} dA y_0 \psi_b' \phi \\ + \int \frac{2y}{R} dA u' \phi &+ \int \frac{-2}{R} dA u' \bar{w} &+ \int \frac{-2}{R} dA y_0 u' \phi \\ + \int \frac{-2yz}{R+z} dA \frac{\bar{w}}{R} &+ \int \frac{2yz}{R+z} dA \frac{y_0 \bar{\phi}}{R} &+ \int \frac{-2yz}{R+z} dA \frac{y_0 \bar{\phi}}{R} \end{aligned} \quad (2-8')$$

From the definition of the centroid of a section,

$$\int y dA = 0 \quad \int z dA = 0$$

Using these definitions and the following approximations obtained by expanding $(R+z)^{-1}$,

$$\begin{aligned} \int \frac{1}{R+z} dA &\approx \frac{A}{R} + \frac{I_{yy}}{R^3} \\ \int \frac{y}{R+z} dA &\approx \frac{-I_{yz}}{R^2} \quad \int \frac{z}{R+z} dA \approx \frac{-I_{yy}}{R^2} \\ \int \frac{y^2}{R+z} dA &\approx \frac{I_{xx}}{R} \quad \int \frac{yz}{R+z} dA \approx \frac{I_{yz}}{R} \quad \int \frac{z^2}{R+z} dA \approx \frac{I_{yy}}{R} \end{aligned}$$

the strain energy equation, (2-6) may be simplified to

$$\begin{aligned} \frac{dP}{ds} = \frac{E}{2} &\left(I_{xx} \left[\theta_b' - \frac{\phi}{R} \right]^2 + I_{yy} \left[\psi_b' + \frac{\bar{w}}{R^2} + \frac{y_0 \phi}{R^2} \right]^2 \right. \\ &- 2I_{yz} \left[\theta_b' - \frac{\phi}{R} \right] \left[\psi_b' + \frac{\bar{w}}{R^2} + \frac{y_0 \phi}{R^2} \right] \\ &\left. + A \left[u' - \frac{\bar{w}}{R} - \frac{y_0 \phi}{R} \right]^2 \right) \\ &+ \frac{GC}{2} \left[\phi' + \frac{u'}{R_s} - \frac{\theta_s}{R_s} \right]^2 + \frac{G\alpha A}{2} [\theta_s^2 + \psi_s^2] \end{aligned} \quad (2-8)$$

In equation (2-8) the expression $\left[\theta_b' - \frac{\phi}{R} \right]$ can be recognized as the flexural curvature out of plane. Similarly $\left[\psi_b' + \frac{\bar{w}}{R^2} + \frac{y_0 \phi}{R^2} \right]$ and $\left[u' - \frac{\bar{w}}{R} - \frac{y_0 \phi}{R} \right]$ are the flexural curvature in-plane and the axial strain at the centroid respectively.

The kinetic energy T , is also obtained by integrating over the cross-section. Taking dS as $\frac{R+z}{R} ds$ the kinetic energy term is

$$\frac{dT}{ds} = \frac{\rho}{2} \int dA \left[\frac{R+z}{R} \right] [u^2 + v^2 + w^2] \quad (2-9)$$

where u , v , and w are given in equation (2-1). If $\int y^m z^n dA \approx 0$, when $m+n \geq 3$ is assumed, then the above expression integrates to

$$\begin{aligned} \frac{2}{\rho} \frac{dT}{ds} = & A \left[\dot{u}^2 + \dot{v}^2 + \dot{w}^2 \right] + [(I_{xx} + z_0^2 A) + (I_{yy} + y_0^2 A)] \dot{\phi}^2 \\ & + \left[I_{xx} \dot{\theta}_b^2 + I_{yy} \left(\dot{\psi}_b + \frac{\dot{u}}{R} \right)^2 \right] + 2I_{yy} \left[\frac{\dot{u}}{R} \left(\dot{\psi}_b + \frac{\dot{u}}{R} \right) + \frac{\dot{\phi}}{R} \left(\dot{v} + \frac{z_0 \dot{\phi}}{R} \right) \right] \\ & - 2I_{yy} \left[\dot{\theta}_b \left(\dot{\psi}_b + \frac{2\dot{u}}{R} \right) + \frac{\dot{\phi}}{R} \left(\dot{w} + \frac{y_0 \dot{\phi}}{R} \right) \right] + 2A \left[z_0 \dot{v} \dot{\phi} + y_0 \dot{w} \dot{\phi} \right] \end{aligned} \quad (2-10)$$

In equation (2-10) the expressions $[\dot{u}^2 + \dot{v}^2 + \dot{w}^2]$ and $[(I_{xx} + z_0^2 A) + (I_{yy} + y_0^2 A)] \dot{\phi}^2$ are the translational and torsional inertias respectively. $[I_{xx} \dot{\theta}_b^2 + I_{yy} (\dot{\psi}_b + \frac{\dot{u}}{R})^2]$ is the rotatory inertia term.

The above expressions for the strain energy, (2-8), and kinetic energy (2-10), can now be used to obtain the stiffness and inertia properties of complete rings or partial ring segments. In Chapter 3 these properties are derived for a complete ring and in Chapters 4 and 5 to thin and thick ring segments respectively. Verification of the accuracy and efficiency of these finite element matrices is assessed by comparison with theoretical and experimental data for the vibration of rings obtained by other workers.

Chapter 3

Six Degree of Freedom Axisymmetric Element

3.1 Formulation of 6 Degree of Freedom [DOF] Axisymmetric Element

Given the potential energy function P , and the kinetic energy function T , it is possible to formulate a finite element by applying Lagrange's equations,

$$\begin{aligned} K_{ij} &= \int_0^L \frac{\partial^2}{\partial z_i \partial z_j} \left[\frac{dP}{ds} \right] ds \\ M_{ij} &= \int_0^L \frac{\partial^2}{\partial \dot{z}_i \partial \dot{z}_j} \left[\frac{dT}{ds} \right] ds \end{aligned} \quad (3-1)$$

where z_i and z_j are generalized end co-ordinates of the element, L is the element length and ds is the incremental element length. K_{ij} and M_{ij} are respectively the stiffness and mass coefficients for degrees of freedom z_i and z_j .

In order to evaluate the potential and kinetic energies in Lagrange's equation a displacement field must be assumed. Assuming the ring vibrates with frequency ω in a mode comprising n nodal diameters, then the following displacement field may be assumed; namely,

$$\begin{bmatrix} \bar{u} \\ \bar{v} \\ \bar{w} \\ \phi \\ \psi_s \\ \theta_s \end{bmatrix} = \begin{bmatrix} U \sin n\xi \\ V \cos n\xi \\ W \cos n\xi \\ \Phi \cos n\xi \\ \Psi_s \sin n\xi \\ \Theta_s \sin n\xi \end{bmatrix} e^{i\omega t} \quad (3-2)$$

where n equals the number of nodal diameters and ξ is the angle. The number of nodal diameters equals the number of nodes around a ring divided by two. Figure 5 illustrates in-plane flexural modes of 1, 2 and 3 nodal diameters.

At this point it is prudent to take advantage of the relationships between flexure due to bending and shear. Terms for the flexure due to either shear or bending may be substituted for in the potential and kinetic energy relations. As a result the energy equations can be expressed in terms of only 6 DOF instead of 8, by eliminating either θ_b and ψ_b , or θ_s and ψ_s .

For this element the shear-bending relationships, (2-2) and (2-3) were used to eliminate the flexure due to bending terms in the energy equations. The displacement field (3-2) was then substituted into the new energy equations and Lagrange's equations (3-1) were applied to obtain the equation of free vibration

$$[K] - \omega^2 [M] [q] = 0 \quad (3-3)$$

where $[q]$ is the vector,

$$[q]^T = [U \ V \ W \ \phi \ \psi \ \theta_s]$$

Details of these mathematical manipulations follow. Expressions for the stiffness and inertia coefficients are in Tables 3 and 4.

3.2 Integral of Strain Energy

The integral of the strain energy $\int_0^L \left(\frac{dP}{ds} \right) ds$, which is the inner part of Lagrange's equation $K_{ij} = \frac{\partial^2}{\partial x_i \partial x_j} \left[\int_0^L \left(\frac{dP}{ds} \right) ds \right]$, was calculated as a first step in obtaining stiffness coefficients, K_{ij} .

For the assumed displacement field (3-2), the equations relating flexure due to bending to the flexure due to shear are

$$\begin{aligned} \theta'_b &= \left[-\frac{n^2}{R^2} \bar{v} - \frac{n^2 z_0}{R^2} \phi - \frac{n}{R} \theta_s \right] \cos n\xi \\ \psi'_b &= \left[-\frac{n^2}{R^2} \bar{w} - \frac{n^2 y_0}{R^2} \phi - \frac{n}{R} \psi_s \right] \sin n\xi \end{aligned} \quad (3-4)$$

If the displacement field and the shear-bending relations are substituted into the strain energy equation (2-8) and the result integrated with respect to s , the following individual terms result:

For out of plane flexure:

$$\int_0^{2\pi R} \frac{E}{2} I_{ss} \left[\theta'_b - \frac{\phi}{R} \right]^2 ds = \frac{EI_{ss}}{2} \left[-\frac{n^2}{R^2} V - \left(\frac{1}{R} + \frac{n^2 z_0}{R^2} \right) \phi + \frac{n}{R} \theta_s \right]^2 \int_0^{2\pi} \cos^2 n\xi (R d\xi)$$

For in-plane flexure:

$$\begin{aligned} \int_0^{2\pi R} \frac{E}{2} I_{yy} \left[\psi'_b + \frac{\bar{w}}{R^2} + \frac{y_0}{R^2} \phi \right]^2 ds \\ = \frac{EI_{yy}}{2} \left[\left(\frac{1-n^2}{R^2} \right) W + \left(\frac{1-n^2}{R^2} \right) y_0 \Phi - \frac{n}{R} \Psi_* \right]^2 \int_0^{2\pi} \cos^2 n\xi (R d\xi) \end{aligned}$$

For coupling:

$$\begin{aligned} \int_0^{2\pi R} -EI_{yz} \left[\theta'_b - \frac{\phi}{R} \right] \left[\psi'_b + \frac{\bar{w}}{R^2} + \frac{y_0}{R^2} \phi \right] ds = EI_{yz} \left[\frac{n^2}{R^2} V + \left(\frac{1}{R} + \frac{n^2 z_0}{R^2} \right) \Phi - \frac{n}{R} \Theta_* \right] \\ \left[\left(\frac{1-n^2}{R^2} \right) W + \left(\frac{1-n^2}{R^2} \right) y_0 \Phi - \frac{n}{R} \Psi_* \right] \int_0^{2\pi} \cos^2 n\xi (R d\xi) \end{aligned}$$

For extension:

$$\int_0^{2\pi R} \frac{EA}{2} \left[u' - \frac{\bar{w}}{R} - \frac{y_0}{R} \phi \right]^2 ds = \frac{EA}{2} \left[\frac{n}{R} U - \frac{W}{R} - \frac{y_0}{R} \Phi \right]^2 \int_0^{2\pi} \cos^2 n\xi (R d\xi)$$

For torsion:

$$\int_0^{2\pi R} \frac{GC}{2} \left[\phi' + \frac{v'}{R_*} - \frac{\theta_*}{R_*} \right]^2 ds = \frac{GC}{2} \left[-\frac{n}{R} \Phi - \frac{n}{R_* R} V - \frac{\Theta_*}{R_*} \right]^2 \int_0^{2\pi} \sin^2 n\xi (R d\xi)$$

For shear:

$$\int_0^{2\pi R} \frac{G\alpha A}{2} [\theta_*^2 + \psi_*^2] ds = \frac{G\alpha A}{2} [\Theta_*^2 + \Psi_*^2] \int_0^{2\pi} \sin^2 n\xi (R d\xi)$$

Note that $\int_0^{2\pi} \cos^2 n\xi R d\xi = \int_0^{2\pi} \sin^2 n\xi R d\xi = \pi R$, $n \neq 0$.

For example, to get a stiffness term for shear, just apply Lagrange's equation to the shear equation above and the result will be:

$$\begin{aligned} K_{\Theta, \Theta} &= \frac{\partial^2}{\partial \Theta_* \partial \Theta_*} \left[\frac{G\alpha A}{2} (\Theta_*^2 + \Psi_*^2) \pi R \right] \\ &= G\alpha A [\pi R] \end{aligned}$$

Similarly, by further application of Lagrange's equation all the stiffness terms may be found. See Table 3 for resulting stiffnesses.

3.3 Integral of Kinetic Energy

In order to find the mass coefficients M_{ij} , the kinetic energy integral $\int_0^L (\dot{\mathbf{d}}^T) ds$, which is the inner part of Lagrange's equation $M_{ij} = \frac{\partial^2}{\partial \dot{x}_i \partial \dot{x}_j} \left[\int_0^L (\dot{\mathbf{d}}^T) ds \right]$, was calculated.

In order to eliminate the flexure due to bending the following equations relating bending to shear for the assumed displacement field were used

$$\begin{aligned}\dot{\theta}_b &= -\frac{n}{R} \dot{v} - \frac{nz_0}{R} \dot{\phi} - \dot{\theta}_s \\ \dot{\psi}_b &= -\frac{n}{R} \dot{w} - \frac{ny_0}{R} \dot{\phi} - \dot{\psi}_s\end{aligned}\quad (3-5)$$

The displacement field and the shear-bending relations were substituted into the kinetic energy equation (2-10) and the result was integrated with respect to s . The individual terms that resulted are:

For translation:

$$\int_0^{2\pi R} \frac{\rho A}{2} [\dot{u}^2 + \dot{v}^2 + \dot{w}^2] ds = \frac{\rho A}{2} [U^2 + V^2 + W^2] [-\omega^2 \pi R]$$

For torsion:

$$\int_0^{2\pi R} \frac{\rho}{2} [(I_{xx} + z_0^2 A) + (I_{yy} + y_0^2 A)] \dot{\phi}^2 ds = \frac{\rho}{2} [(I_{xx} + z_0^2 A) + (I_{yy} + y_0^2 A)] \Phi^2 [-\omega^2 \pi R]$$

For rotatory inertia:

$$\begin{aligned}\int_0^{2\pi R} \frac{\rho}{2} \left[I_{xx} \dot{\theta}_b^2 + I_{yy} \left(\dot{\psi}_b + \frac{\dot{u}}{R} \right)^2 \right] ds \\ = \frac{\rho}{2} \left[\left(-\frac{n}{R} V - \frac{nz_0}{R} \Phi - \Theta_s \right)^2 + \left(-\frac{n}{R} W - \frac{ny_0}{R} \Phi - \Psi_s + \frac{U}{R} \right)^2 \right] [-\omega^2 \pi R]\end{aligned}$$

For additional rotatory terms:

$$\begin{aligned} \int_0^{2\pi R} \frac{\rho}{2} 2I_{yy} \left[\frac{\dot{u}}{R} \left(\dot{\psi}_b + \frac{\dot{u}}{R} \right) + \frac{\dot{\phi}}{R} \left(\dot{v} + \frac{z_0}{R} \dot{\phi} \right) \right] ds \\ = \rho I_{yy} \left[\frac{U}{R} \left(-\frac{n}{R} W - \frac{n y_0}{R} \phi - \psi_s + \frac{U}{R} \right) + \frac{\dot{\phi}}{R} \left(V + \frac{z_0}{R} \dot{\phi} \right) \right] [-\omega^2 \pi R] \end{aligned}$$

For coupling:

$$\begin{aligned} \int_0^{2\pi R} -\frac{\rho}{2} 2I_{yz} \left[\dot{\psi}_b \left(\dot{\psi}_b + \frac{2}{R} \dot{u} \right) + \frac{\dot{\phi}}{R} \left(\dot{w} + \frac{y_0}{R} \dot{\phi} \right) \right] ds \\ = -\rho I_{yz} \left[\left(-\frac{n}{R} V - \frac{n z_0}{R} \phi - \theta_s \right) \left(-\frac{n}{R} W - \frac{n y_0}{R} \phi - \psi_s + \frac{2}{R} U \right) + \frac{\dot{\phi}}{R} \left(W + \frac{y_0}{R} \dot{\phi} \right) \right] [-\omega^2 \pi R] \end{aligned}$$

and

$$\int_0^{2\pi R} \rho A \left[z_0 \dot{v} \dot{\phi} + y_0 \dot{w} \dot{\phi} \right] = \rho A \left[z_0 V \dot{\phi} + y_0 W \dot{\phi} \right] [-\omega^2 \pi R]$$

For example, to get a mass term for extensional translation, just apply Lagrange's equation to the translation equation above and the result will be:

$$\begin{aligned} M_{UU} &= \frac{\partial^2}{\partial U \partial U} \left[\frac{\rho A}{2} (U^2 + V^2 + W^2) (\pi R) \right] \\ &= \frac{\rho A}{2} (\pi R) \end{aligned}$$

Similarly, by further application of Lagrange's equation all the mass terms may be found. See Table 4 for resulting mass coefficients.

3.4 Verification of Theory

The theoretical representation for curved ring vibration was compared with experimental results obtained by Volterra, Hawkings and Kirkhope. The radii of all rings tested were approximately five inches. Properties of the rings under consideration may be found in Tables 1 and 2. Scale diagrams of each of the cross-sections may be found in Figures 6 and 7.

A computer program was assembled to calculate theoretical natural frequencies and mode shapes. It consisted of the stiffness and mass matrices, along with a Jacobi eigenvalue solver. A

Jacobi solver was chosen after careful consideration of Bathe's and Wilson's paper [6] "Solution Methods for Eigenvalue Problems in Structural Mechanics". Comparisons between the natural frequencies measured and those predicted are tabulated in Tables 5 to 17.

3.5 Discussion

To ensure correctness of results, the eigenproblem was solved with a Jacobi solver along with a power iteration solver. The results obtained from both were the same.

The rings ordered 1 to 8 are listed in order of decreasing thickness. Ring thickness was taken as proportional to $\frac{I_{yy}}{AR^2}$.

Of primary interest are the lower frequencies. The higher frequencies are included as additional information. Since the higher frequencies are quite high for practical applications, the larger errors at these frequencies do not negate the applicability of the element to design situations.

Five toroidal rings with the same centroidal radius but decreasing cross-section, V1 to V5, were examined. This was in order to separate the effects of ring cross-sectional size on errors in frequency for both in-plane and out of plane modes. Since the toroidal rings were of symmetrical cross-section the in-plane and out of plane modes were uncoupled, and the shear center and centroidal axis coincided.

Examining Tables 10 through 15 it is evident that for the toroidal rings the errors in frequencies ranged from -2.9% to 0.8%. These results are good and probably result from the good compatibility of the shape functions used in the formulation. Prathap and Bhashyam in their paper "Reduced Integration and the Shear-Flexible Beam Element" [39] showed that within a squared energy term, for example $\left(\left[\psi'_b + \frac{\bar{w}}{R^2} + \frac{v_{\theta}\phi}{R^2} \right] \text{ in } \frac{EI_{xx}}{2} \left[\psi'_b + \frac{\bar{w}}{R^2} + \frac{v_{\theta}\phi}{R^2} \right]^2 \right)$; if all terms in the brackets are not of the same order then spurious constraints are introduced.

For example from the Prathap paper a linear beam element has the shape functions

$$\begin{aligned} \theta &= \theta_1 + [\theta_2 - \theta_1] \frac{x}{l} & O(\theta) &= 2 \\ w &= w_1 + [w_2 - w_1] \frac{x}{l} & O(w) &= 2 \\ O(\theta - w, x) &= 1 \end{aligned}$$

Then

$$\int_0^b (\theta - w_x)^2 dx = b \left[(\theta_2 + \theta_1 \frac{b}{2} - w_1 \frac{b}{b})^2 + (\theta_2 - \theta_1)^2 \right] \\ \approx b \left[(\theta - w_x)^2 + \frac{b^3}{12} \theta_{,x}^2 \right] \quad \text{by central difference formulae}$$

The first term $(\theta - w_x)$ is a valid constraint, but the second $\theta_{,x}^2$ is not. That is, $(\theta - w_x)$ may be zero but the approximation to the energy can be non-zero since $\theta_{,x}$ may not be zero. The total number of constraints equals the order of θ , but the number of spurious constraints equals the order of θ minus the order of w_x , which in this case is 1.

Because the shape functions (3-2), used in deriving the ring element are composed of the infinite order functions $\cos$ and $\sin$, spurious constraints do not arise in the formulation.

For the toroids, the in-plane bending errors are negative for the largest sections and gradually increase to small positive values for the smallest section. The underestimations of the natural frequency errors are approximately constant with respect to mode number for the individual rings. This would indicate the error is a dominant function of cross-sectional properties not mode number. The underestimation of the natural frequency could be the result of an overestimation of the mass terms or an underestimation of the stiffness terms. For symmetric sections all coupling terms given by $\int y^m z^n dA$ equal zero if either m or n are odd. This would indicate that the mass matrix specified was exact.

The strain energy terms were formulated by employing certain truncated expansions of

$$(R+z)^{-1} = \frac{1}{R} \left[1 - \frac{z}{R} + \left(\frac{z}{R}\right)^2 - \left(\frac{z}{R}\right)^3 + \dots \right]$$

Generally all the coupling terms of odd order result in integrals of odd order which are zero for symmetric cross-sections. The terms remaining are the positive even order expressions. Since these are truncated the truncation error is positive and thus the strain energy and stiffness are underestimated. As a result the natural frequencies are underestimated by a constant determined by cross-sectional size.

For the out of plane bending results, errors are negative for all the toroidal rings except for small positive values for the smallest sectioned ring. The out of plane bending error is much smaller

in magnitude than the in-plane bending error. This is to be expected since the location of the out of plane neutral bending axis is known exactly while the in-plane neutral bending axis is known only approximately. The errors ranged from -1.1 to 0.8% and except for the largest section, error is fairly constant for each ring. The error is greatest for the medium area sectioned ring. This indicates that no linear relationship between section size and error exists.

Sensitivity tests on the geometric ring characteristics indicated that the matrices were stable and not very sensitive to small variations in the input coefficients.

The effect of shear center offset inclusion was much smaller than the existing error on predicted frequency. The additional complexity of shear center offset inclusion was not warranted because of the small gains in accuracy. Thus for the rings under test, 1 through 8, shear center offsets were set to zero and R_s was equated to R .

Ring 1, the only symmetric ring had the lowest overall error. This is probably the result of the non-existence of in-plane and out of plane coupling. There is coupling between the out of plane bending and torsion. This is evident by a positive increase in error with mode number for bending accompanied by a negatively increasing error for torsion.

For the rings the out of plane bending error increases positively with mode number from an initial negative value for n equal 2. As this happens the out of plane torsion seems to increase negatively. Ring 3 shows this trend very strongly. This indicates that the coupling between the torsion and bending is too strong. Torsional modes whose frequencies are higher than bending modes tend to increase the bending mode frequencies too much and the lower bending frequencies decrease the predicted torsional frequencies.

In fact all errors increased positively with mode number except in the torsional case.

The in-plane error has a smaller range than the out of plane error.

Comparison of Hawkings' rings (rings 5 and 7) is interesting. Their major axes are inclined at an angle of 30 degrees to the out of plane and in plane axis respectively. Both sections are of approximately the same shape. The direction with the largest moment of inertia has the largest error. This is consistent with the results for the toroidal rings. What is of interest, however, is

the change in relative magnitude of the errors with section orientation. The relative value of the in-plane and out of plane error seem to interchange as the sections are turned.

Chapter 4

22 Degree of Freedom Arc Element

4.1 Introduction

In view of the excellent accuracy demonstrated by the 6 DOF element, a 22 DOF element spanning an arc of angle 2β was formulated. This arc element would be more versatile in that incomplete ring segments could be analysed, while the 6 DOF model of Chapter 3 is necessarily restricted to complete rings.

Since the effect of shear center off-set was found to be of dubious advantage with the 6 DOF element, all terms containing the shear offsets y_0 and z_0 were eliminated by setting the offsets to zero. The 22 DOF element was further simplified by letting the bending flexures θ_b and ψ_b equal to the deflection slopes v' and w' . As a result the shear terms due to flexure, θ_s and ψ_s , were eliminated (set to zero). With these simplifications introduced into the potential and kinetic energy functions, an element was formulated. The element was a Hermitian quintic in u , v and w , and a Hermitian cubic in twist, ϕ . See Figure 8 for illustration of deflections and rotations.

4.2 Simplified Energy Equations

Assuming Lagrange's equations $K_{ij} = \int_0^L \frac{\partial^2}{\partial x_i \partial x_j} \left[\frac{dT}{ds} \right] ds$ and $M_{ij} = \int_0^L \frac{\partial^2}{\partial x_i \partial x_j} \left[\frac{dP}{ds} \right] ds$, the first step in formulating an element is to define strain and kinetic energy functions P and T . By letting the shear center offset terms y_0 and z_0 and the shear flexure terms θ_s and ψ_s equal zero, and equating θ_b with v' and ψ_b with w' , the potential and kinetic energy equations (2-8) and (2-10) become

$$\begin{aligned}
 \frac{dP}{ds} = & \frac{E}{2} \left(I_{xx} \left[v'' - \frac{\phi}{R} \right]^2 + I_{yy} \left[w'' + \frac{w}{R^2} \right]^2 \right. \\
 & - 2I_{yx} \left[v'' - \frac{\phi}{R} \right] \left[w'' + \frac{w}{R^2} \right] \\
 & \left. + A \left[u' - \frac{w}{R} \right]^2 \right) \\
 & + \frac{GC}{2} \left[\phi' + \frac{v'}{R} \right]^2
 \end{aligned} \tag{4-1}$$

and

$$\begin{aligned}
 \frac{2}{\rho} \frac{dT}{ds} = & A \left[\dot{u}^2 + \dot{v}^2 + \dot{w}^2 \right] \\
 & + I_{yy} \left[\left(\dot{w}' - \frac{\dot{u}}{R} \right) \left(\dot{w}' + \frac{3\dot{u}}{R} \right) - \frac{2\dot{v}\dot{\phi}}{R} \right] \\
 & + I_{xx} \left[\dot{v}' \right]^2 \\
 & + [I_{yy} + I_{xx}] \dot{\phi}^2 \\
 & - \frac{\rho I_{yx}}{2} \left[2\dot{v}'\dot{w}' + 4\frac{\dot{v}'\dot{u}}{R} + 2\frac{\dot{w}\dot{\phi}}{R} \right]
 \end{aligned} \tag{4-2}$$

4.3 Formulation of 22 DOF Element

With the strain and kinetic energy functions defined, a choice of shape functions for displacements within the element in terms of element nodal displacements had to be chosen. The element formulated was two noded. The number of degrees of freedom per node and the order of the shape functions used to describe displacements between nodes are closely related.

In order for the shape functions to be able to represent all possible deformation shapes, the polynomial shape functions must contain at least as many terms as the exact solution of the governing differential equations. Meek¹ [34] in his paper "An Accurate Polynomial Displacement

¹ Note in Meek's paper on page 267 that in the stiffness coefficient K_{27} , $\frac{45}{\rho^2}$ should be $\frac{45}{\rho^3}$.

Function For Finite Ring Elements " gives the in-plane differential equation for extensional deformation $\{ \epsilon = \frac{1}{R} \left(\frac{\partial u}{\partial \theta} - w \right) \}$ as a sixth order differential equation in u and a fifth order differential equation in w . This implies that the shape function for u should be at least of fifth order, and for w it should be at least of fourth order. Hermitian quintic shape functions meet these criteria, thus an element Hermitian quintic in u , v and w and Hermitian cubic in twist, ϕ , was formulated. Dawe [15] in his paper "Numerical Studies Using Circular Arch Finite Elements" indicate a strong preference for quintic functions over cubics for increased accuracy of solution of circular arch problems. Figure 9 illustrates Hermitian quintic and cubic shape functions. The nodal displacement vector for this element was

$$[u_1 \ u'_1 \ u''_1 \ v_1 \ v'_1 \ v''_1 \ w_1 \ w'_1 \ w''_1 \ \phi_1 \ \phi'_1 \ u_2 \ u'_2 \ u''_2 \ v_2 \ v'_2 \ v''_2 \ w_2 \ w'_2 \ w''_2 \ \phi_2 \ \phi'_2]$$

This displacement vector with its large number of degrees of freedom creates large energy terms and thus a formidable element formulation problem. The amount of mathematics involved is enormous but simple. To overcome this situation a computational algebra program was employed to do the extensive mathematics. The program used for this purpose was MACSYMA, a mathematical symbolic manipulation program developed at MIT over 10 years with DOE funding.

4.3.1 Details of Derivation

To obtain the stiffness and mass matrices one must:

- (i) define a displacement field over the element
- (ii) introduce this displacement field into the differential energy equations, (4-1) and (4-2)
- (iii) integrate the energy equations over the element
- (iv) apply Lagrange's equations, (3-1) to obtain the individual stiffness and mass coefficients.

The displacement field over the element can be expressed in terms of Hermitian cubic and quintic functions of the displacements at the ends of the element. The Hermitian functions are readily available in many books [21]. It is more efficient to form a matrix and solve for the coefficients of the quintic polynomials when employing a computer algebra program than to type the functions in directly.

4.3.2 Shape Functions

Given a quintic function which spans the domain of the element $-1 \leq \frac{s}{R\beta} \leq 1$ i.e.

$$(\) = a_1 + a_2 \left(\frac{s}{R\beta} \right) + a_3 \left(\frac{s}{R\beta} \right)^2 + a_4 \left(\frac{s}{R\beta} \right)^3 + a_5 \left(\frac{s}{R\beta} \right)^4 + a_6 \left(\frac{s}{R\beta} \right)^5$$

and its derivatives

$$\begin{aligned} \frac{\partial(\)}{\partial s} &= \frac{1}{R\beta} \left[a_2 + 2a_3 \left(\frac{s}{R\beta} \right) + 3a_4 \left(\frac{s}{R\beta} \right)^2 + 4a_5 \left(\frac{s}{R\beta} \right)^3 + 5a_6 \left(\frac{s}{R\beta} \right)^4 \right] \\ \frac{\partial^2(\)}{\partial s^2} &= \frac{1}{(R\beta)^2} \left[2a_3 + 6a_4 \left(\frac{s}{R\beta} \right) + 12a_5 \left(\frac{s}{R\beta} \right)^2 + 20a_6 \left(\frac{s}{R\beta} \right)^3 \right] \end{aligned}$$

and assuming shape functions of the form

$$N_i = \sum_j a_{ji} \left(\frac{s}{R\beta} \right)^{j-1}$$

are desired, the matrix of coefficients to solve for a_{ji} , of each quintic can be formed by letting the first row equal $(\)|_{(s/R\beta)=-1}$, the second equal $\frac{\partial(\)}{\partial(s/R\beta)}|_{(s/R\beta)=-1}$, the third equal $\frac{\partial^2(\)}{\partial(s/R\beta)^2}|_{(s/R\beta)=-1}$, the fourth equal $(\)|_{(s/R\beta)=1}$, the fifth equal $\frac{\partial(\)}{\partial(s/R\beta)}|_{(s/R\beta)=1}$, and the sixth equal $\frac{\partial^2(\)}{\partial(s/R\beta)^2}|_{(s/R\beta)=1}$, and the rhs matrix equal $\text{diag} [1 \ R\beta \ (R\beta)^2 \ 1 \ R\beta \ (R\beta)^2]$.

The resulting matrix equation is

$$\begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 & -4 & 5 \\ 0 & 0 & 2 & -6 & 12 & -20 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 2 & 6 & 12 & 20 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & R\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & (R\beta)^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & R\beta & 0 \\ 0 & 0 & 0 & 0 & 0 & (R\beta)^2 \end{bmatrix}$$

Since $R\beta$ and $(R\beta)^2$ are constants, to conserve operations it is easier to set the right hand side to $[I]$ now and obtain stiffness and mass coefficients for the displacement vector

$$\begin{bmatrix} u_1 & \frac{u'_1}{R\beta} & \frac{u''_1}{(R\beta)^2} & v_1 & \frac{v'_1}{R\beta} & \frac{v''_1}{(R\beta)^2} & \dots \end{bmatrix}$$

Later the stiffness and mass matrices can be transformed into the desired coordinate space by pre- and post-multiplication of $[K]$ and $[M]$ by the vector V , where

$$[V] = \text{diag} \begin{bmatrix} 1 \\ R\beta \\ (R\beta)^2 \\ \vdots \end{bmatrix} \begin{matrix} \rightarrow \text{zeroth derivative variable} \\ \rightarrow \text{first derivative variable} \\ \rightarrow \text{second derivative variable} \\ \text{etc.} \end{matrix}$$

and

$$\begin{aligned} [K] &= [V]^T [K] [V] \\ [M] &= [V]^T [M] [V] \end{aligned}$$

Now that a displacement field is defined, for example

$$\begin{aligned} u &= \left[\begin{array}{l} N_1 u_1 + N_2 \frac{du_1}{ds} + N_3 \frac{d^2 u_1}{ds^2} + N_4 u_2 + N_5 \frac{du_2}{ds} + N_6 \frac{d^2 u_2}{ds^2} \end{array} \right] \\ \frac{1}{R\beta} \frac{du}{ds} &= \frac{1}{R\beta} \left[\begin{array}{l} \frac{dN_1}{ds} u_1 + \frac{dN_2}{ds} \frac{du_1}{ds} + \frac{dN_3}{ds} \frac{d^2 u_1}{ds^2} + \frac{dN_4}{ds} u_2 + \frac{dN_5}{ds} \frac{du_2}{ds} + \frac{dN_6}{ds} \frac{d^2 u_2}{ds^2} \end{array} \right] \\ \frac{1}{(R\beta)^2} \frac{d^2 u}{ds^2} &= \frac{1}{(R\beta)^2} \left[\begin{array}{l} \frac{d^2 N_1}{ds^2} u_1 + \frac{d^2 N_2}{ds^2} \frac{du_1}{ds} + \frac{d^2 N_3}{ds^2} \frac{d^2 u_1}{ds^2} + \frac{d^2 N_4}{ds^2} u_2 + \frac{d^2 N_5}{ds^2} \frac{du_2}{ds} + \frac{d^2 N_6}{ds^2} \frac{d^2 u_2}{ds^2} \end{array} \right] \end{aligned}$$

all that is left to do is to substitute the displacement field into the energy equations and integrate the energy equation over the element

$$\int_0^{2R\beta} \frac{P(s)}{T(s)} ds = R\beta \int_{-1}^{+1} \frac{P(\xi)}{T(\xi)} d\xi$$

It can be noted that the constant $R\beta$ may be taken outside the integral. Also the element end displacements may be regarded as constants and taken outside the integral. This step effectively applies Lagrange's equations to the integral.

4.4 Testing of Element

To test the element several simple test cases were examined. A static analysis finite element program was assembled. The program used an active column solver. It was very similar to the one listed in Bathe's [5] book except that it was modified to allow for imposed displacement boundary conditions. The test cases examined were:

- (i) a curved cantilevered beam spanning 90 degrees, loaded out of plane and
- (ii) a curved cantilevered beam spanning 90 degrees, loaded in-plane.

The displacements at the free end were compared with theoretical solutions obtained by the unit load method. For the free end the maximum error on any displacement was 0.02% for the out of plane case and 0.21% for the in-plane case. These small errors indicated that the stiffness matrix was formulated correctly. Static test case results are available in Appendix C.

In order to check the mass matrix, a straight cantilevered beam of unit area and of length ten units ($R=10000$, $\beta = 0.0001$) was examined. A Jacobi solver with a Sturm sequence checker, and optional frequency shifter was attached to the finite element program. The I_{yy} and I_{zz} inertia terms due to rotation were omitted to allow comparison with simple theory from Blevins' book [8]. Five elements were used. No deviations from theory for the first five natural frequencies for in-plane bending, out of plane bending, axial extension and torsion vibration were obtained. These results confirmed that the Jacobi solver and the stiffness and mass matrices were correct for the limit R approaching infinity. The Sturm sequence checker further confirmed that the Jacobi solver was working.

4.5 Verification of Theory

As with the 6 DOF element, the 22 DOF element theoretical results were compared with experimental results obtained by Volterra, Hawkings and Kirkhope. The results for the 22 DOF cases were obtained by using four elements each spanning 45 degrees to make a 180 degree arc. The end nodes at 0 and 180 degrees had their out of plane displacement v , their in-plane displacement w , and their twist ϕ , suppressed. All other degrees of freedom were active. Since the Jacobi solver implemented could only handle positive eigenvalues and the suppressions listed still allowed 3 rigid body modes ($\text{freq}=0$) the stiffness and mass matrices had to be shifted up in frequency to allow for the system to be solved with rigid body modes included. Comparisons between the natural frequencies measured and those predicted are tabulated in Tables 18 to 30.

4.6 Discussion

Practical use of the element was demonstrated by using a ring model with only four elements. These four elements were not enough to provide convergence for very high order modes, but it was generally observed that modes of four nodal diameters or less could be considered convergent for engineering purposes. Still with only four elements a 49 DOF system of equations had to be

solved. This is much larger than the 6 DOF used in the previous element formulated, but is the cost of versatility.

The 22 DOF element had no shear terms. Shear effects increase the flexibility of structures. For rings, the larger the cross-section the larger the effect of shear and consequently the greater the reduction in flexibility. As a result error should be positive and increase with cross-sectional area for the rings under test. Examining the in-plane and out of plane bending results (Figures 40 and 41) shows that errors are directly related to section size. Furthermore error increases with mode number in an almost linear fashion. The effects of shear deformation become more important with increasing mode number. Positive errors should also be expected due to the incompatibility of shape function order in the energy terms in the formulation of the element.

All errors increased positively with mode number with the exception of ring 3's out of plane torsion. This is to be expected since the element deflection shapes do not exactly match the experimental mode shapes so according to Raleigh's principle, the natural frequencies will be overestimated because the energy functions are not at their minimum values. This seems to indicate that coupling between out of plane bending and torsion seems to have all but disappeared.

Generally, the smaller the section the smaller the error. The less complicated out of plane bending errors are generally less than the in-plane bending errors.

Chapter 5

30 Degree of Freedom Arc Element

1

5.1 Introduction

The axisymmetric 6 DOF ring element performed well, but for large cross-section rings the more general 22 DOF arc element did not perform well. This appeared to be a result of the neglect of the shear terms when the 22 DOF element was formulated. To investigate if this was the case, a 30 DOF element that included the effects of shear but excluded the effects of shear center offset was formulated. As with the 22 DOF element the shear center off-sets, y_0 and z_0 were set to zero and the shear center radius was taken as the centroidal radius. The flexural terms due to shear were eliminated by utilizing the shear-bending relations (2-2) and (2-3).

The element spanned an arc of 2β . It was a Hermitian quintic in u , v and w , and a Hermitian cubic in twist, ϕ and flexures due to bending. See Figure 45 for an illustration of deflections and rotations.

5.2 Simplified Energy Equation

The first step in formulating an element is to define the strain and kinetic energy functions P and T so that Lagrange's equations may be evaluated. By letting the shear center off-set terms y_0 and z_0 equal zero and using the shear bending relationships to express flexures due to shear in terms of bending flexures, the potential and kinetic energy equations (2-6) and (2-8) become

$$\begin{aligned}
 \frac{dP}{ds} = & \frac{E}{2} \left(I_{xx} \left[\theta'_b - \frac{\phi}{R} \right]^2 + I_{yy} \left[\psi'_b + \frac{w}{R^2} \right]^2 \right. \\
 & - 2I_{yz} \left[\theta'_b - \frac{\phi}{R} \right] \left[\psi'_b + \frac{w}{R^2} \right] \\
 & \left. + A \left[u' - \frac{\bar{w}}{R} \right]^2 \right) \\
 & + \frac{GC}{2} \left[\phi' + \frac{\theta_b}{R_s} \right]^2 \\
 & + \frac{G\alpha A}{2} \left[(v' - \theta_b)^2 + (\bar{w}' - \psi_b)^2 \right]
 \end{aligned} \tag{5-1}$$

and

$$\begin{aligned}
 \frac{2}{\rho} \frac{dT}{ds} = & A \left[\dot{u}^2 + \dot{v}^2 + \dot{w}^2 \right] \\
 & + I_{yy} \left[\left(\dot{\psi}_b - \frac{\dot{u}}{R} \right) \left(\dot{\psi}_b + \frac{3\dot{u}}{R} \right) - \frac{2\dot{v}\dot{\phi}}{R} \right] \\
 & + I_{xx} \left[\dot{\theta}_b \right]^2 \\
 & + [I_{yy} + I_{xx}] \dot{\phi}^2 \\
 & - \frac{\rho I_{yz}}{2} \left[2\dot{\theta}_b \dot{\psi}_b + 4 \frac{\dot{\theta}_b \dot{u}}{R} + 2 \frac{\dot{w} \dot{\phi}}{R} \right]
 \end{aligned} \tag{5-2}$$

5.3 Formulation of 30 DOF Element

The 30 DOF element was formulated along similar lines to the 22 DOF element. Hermitian quintic displacement functions in terms of the element nodal displacements were substituted into the potential and kinetic energy equations. The energy equations were integrated over the element and then they were differentiated with respect to the element end coordinates to get the stiffness and mass coefficients. The stiffness and mass coefficients may be found in Appendix B. The nodal displacement vector for this element was

$$[u_1 \ u'_1 \ u''_1 \ v_1 \ v'_1 \ v''_1 \ \theta_{b1} \ \theta'_{b1} \ w_1 \ w'_1 \ w''_1 \ \psi_{b1} \ \psi'_{b1} \ \phi_1 \ \phi'_1 \ u_2 \ u'_2 \ u''_2 \ v_2 \ v'_2 \ v''_2 \ \theta_{b2} \ \theta'_{b2} \ w_2 \ w'_2 \ w''_2 \ \psi_{b2} \ \psi'_{b2} \ \phi_2 \ \phi'_2]$$

5.4 Testing of the Element

The stiffness matrix was tested first to determine its correctness. The static analysis finite element program developed to examine the 22 DOF element was used. The test cases examined were

- (i) a curved cantilevered beam spanning 90 degrees, loaded out of plane and
- (ii) a curved cantilevered beam spanning 90 degrees, loaded in-plane.

The free end displacements were compared to theoretical results obtained by the method of unit loads. For the free end the greatest error on any displacement was 0.32 % for the out of plane case and -0.34 % for the in-plane case. These small errors confirmed that the stiffness matrix was formulated correctly. The maximum errors were in the bending slopes. This is probably the result of round-off error from the added complication of having degrees of freedom for both slopes including both shear and bending slopes together and slopes just including bending which must be coupled together. Static test case results may be found in Appendix C.

The mass matrix was checked by examining the vibration of a straight cantilevered shear beam. The error on frequency increased with increasing mode number (deflection shape complexity). The errors were greater than those calculated by simpler methods. The error for all practical purposes is small, ranging from 0.12 % for mode 1 to 0.89 % for mode 4.

5.5 Verification of Theory

The 30 DOF element theoretical results were compared with experimental results obtained by Volterra, Hawkings and Kirkhope. The results for the 30 DOF case were obtained by using 4 elements, spanning 180 degrees to model half a ring. At each end of the rings out of plane displacement v , in-plane displacement w and twist ϕ were suppressed. All other degrees of freedom were active. The stiffness and mass matrices were shifted up in frequency to allow for the system of equations to be solved with rigid body motions included. Comparisons between natural frequencies measured and those predicted are tabulated in Tables 31 to 43.

5.6 Discussion

Only 4 elements were used in the ring model to demonstrate feasibility. Four elements were not enough to provide convergence for higher order modes. As a result higher order modes were overestimated. Still with only 4 elements a 69 DOF system of equations had to be solved. This is a fairly large system.

The error increased positively with mode number for the toroidal rings. The larger the section the more positive the error. This is consistent with the effect of shear being underestimated. Out of plane error was greater than in-plane error

As could be expected due to energy considerations error increased positively with mode number. Out of plane bending was generally underestimated. For this reason the effect of coupling between out of plane bending and torsion was not as evident. The larger sections had their greatest error for out of plane torsion.

Chapter 6

Conclusions

6.1 Discussion of Elements Developed

Three finite elements have been developed for circular curved rings and ring segments of asymmetrical cross-section. One of the elements, the 6 DOF axisymmetric element, could only be used for complete rings. The 22 DOF and 30 DOF elements could be used for partial arcs. The 6 and 30 DOF elements included the effects of shear but the 22 DOF did not. The elements were evaluated by comparing their numerical predictions of frequencies and mode shapes with experimental results for rings of constant closed cross-section. Comparisons of element predictions with experimental results are shown in Figures 10 through 43.

For the range of shapes analysed for complete rings, the 6 DOF element performed best. This is due to the use of shape function which were compatible with each other and emulated the mode shapes best. The shape functions were sinusoidal. Another advantage of using the 6 DOF element for complete rings was that each set of 6 modes with a specific number of nodal diameters could be solved using only a 6x6 matrix. Use of the 22 or 30 DOF elements entailed solving a large system of equations with much pre-processing necessary to set up the equations for computation and a great deal of manual post-processing to determine the order and type of modes. Also no decision as to the number of elements to be used was necessary when using the 6 DOF element. This reduced the input data necessary to get results and frees the analyst from one more task.

In fact, it is interesting to note that for out of plane bending for the first few modes that the 6 DOF element gave more accurate results than Volterra's method even though Volterra's method was only formulated for toroidal rings. Volterra predicted in-plane bending frequencies for toroids

more accurately, with the exception of the largest ring where frequencies were predicted more accurately by the 6 DOF.

Comparison of the 6 DOF results with experimental and numerical predictions of Hawkings is more relevant since Hawkings examined rings of non-symmetric cross-section. Rings 5 and 7, Hawkings rings, are of rectangular cross-section with their sides inclined 30 degrees and 60 degrees to the plane of the ring. It is interesting to compare Hawkings and the 6 DOF element results. In the more flexible direction the 6 DOF element makes better predictions than Hawkings does, but in the stiffest direction Hawkings method predictions are slightly better.

From a designer point of view both methods yield excellent results, but since a designer often seeks the lowest natural frequency the 6 DOF element would be the preferred method. For complete rings the 6 DOF elements performance appears exemplary. Also the 6 DOF element is easier to use than Hawkings. Hawkings method involves solving 12 simultaneous equations and also eliminates axial extension modes by assuming centroid axial constraint equals zero and uses this condition as a constraint equation.

In order to analyse incomplete rings two two noded partial arc elements with 22 and 30 DOF were formulated. Some work by other investigators in this area has been done before. T.-M. Wang et al. [46] examined out of plane bending for horizontally curved beams of constant section. For their analysis the cross-sections were symmetric and no comparison of results to experimental data were made.

Dawe [15] investigated in-plane bending elements for static analysis for curved plates of constant thickness. He found quintic-quintic elements to provide the fastest convergence.

The 22 and 30 DOF elements were quintics. Dawe's study indicated that the stiffness matrices formulated could be used for static analysis with confidence. Dawe's elements did not include the effects of shear. Thus the 22 DOF element stiffness matrix was similar to Dawe's, but if one had a thick ring or a very short ring segment the 30 DOF element stiffness matrix would provide better results. Unlike Dawe's elements the 22 and 30 DOF elements can analyse out of plane motions as well as asymmetric cross-sections.

Several civil engineers interested in curved girders for bridges, highway entrance ramps, etc have presented analyses. Some like Culver [9] examined out of plane vibration for symmetric cross-sections. Dabrowski [10] developed a theory for the most general of cross-sections (asymmetric, open or closed with shear-center offset). It was very complicated and used higher order moments of inertia. No assessment of the accuracy of the theory was presented. Tabba and Turkstra [43] applied this method to predict the vibrations of curved box girders but again none of the data was compared to experimental data. This leaves only the experimental data of Kirkhope [28], Volterra [31] and Hawkings [20] for complete rings as the only experimental data that the 22 and 30 DOF elements can be tested against.

Generally the effects of shear deformation absent from the 22 DOF element are more important with increasing mode number, decreasing structure length and increasing structure cross-section. Thus if a curved structure is either of large cross-section or short length, and higher order modes need to be known accurately, the 30 DOF element should be used instead of the 22 DOF element.

If only a complete rings are to be examined, methods restricted to complete rings are more efficient and better.

As a last observation, for out of plane torsion for Ring 3, the experimental errors for 5 and 6 nodal diameters change abruptly from the previous values. This may indicate that the experimental values are wrong.

6.2 Recommendations for Future Work

While the elements developed performed well, some continuation of the present work to test, optimize and extend them would be valuable. The areas where additional work would be most useful are:

- i) Experimental data for full rings or partial-ring segments is desirable for further assessment of the 22 DOF and 30 DOF elements.
- ii) In order to optimize the implementation of the energy equations derived in chapter 2, elements could be formulated based on so called Discrete Kirchhoff

Theory. In this instance of the theory deflections and rotations are interpolated independantly. Such elements have exhibited a phenomenon called "shear locking" so that the correct solution for thin rings is not obtained. In order to avoid this a reduced integration or equivalent method such as described in the Prathap and Bhashyam paper [39] could used. This might lead to more efficient elements with no "locking" effects. In the course of this thesis work, subsequent to the development of the 22 DOF and 30 DOF elements, a 12 DOF linear reduced integration element was developed. It had the nodal displacement vector

$$[u_1 \ w_1 \ \psi_1 \ v_1 \ \theta_1 \ \phi_1 \ u_2 \ w_2 \ \psi_2 \ v_2 \ \theta_2 \ \phi_2]$$

and was very briefly investigated for accuracy.

The static test cases of Appendix C were done using five elements. The in-plane test case performed well, although only 15 DOF were used. For the out of plane case which is less complicated, the results were not as good. When the element was used in modal analysis (9 elements, 48 DOF) it was not very good for in-plane analysis and was poor for out of plane analysis. Future work on such elements, however, is worth pursuing. Perhaps a quadratic element formulated and checked carefully would provide a more efficient element for analysis of curved structures.

iii)

Finally, to aid in the design of light structures, an element which included the effects of warping and could analyse thin walled open section or hollow ring segments would be useful.

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Figure 2 Out of Plane Deflections

Figure 3 In-Plane Deflections

$$\begin{aligned}
 \tau &= \frac{(\phi + d\phi) - \phi + \theta_0 d\xi}{R_0 d\xi} \\
 &= \frac{1}{R_0} \frac{d\phi}{d\xi} + \frac{\theta_0}{R_0} \\
 &= \frac{1}{R_0} \frac{d\phi}{d\xi} + \frac{1}{R_0} \frac{d\bar{u}}{ds} + z_0 \frac{d\phi}{ds} - \frac{\theta_0}{R_0} \\
 &= \frac{1}{R} \frac{d\phi}{d\xi} + \frac{1}{R_0 R} \frac{d\bar{u}}{d\xi} - \frac{\theta_0}{R_0}
 \end{aligned}$$

Figure 4 Twist Deflection Components

i = number of nodal diameters. $i=1$ is a rigid body translation.

Figure 5 In-Plane Flexural Modes

RING 1

RING 2

RING 3

RING 4

RING 5

RING 6

Figure 6 Ring Cross-sections

Scale = 2 : 1

RING 7

RING 8

Volterra 1

Volterra 2

Volterra 3

Volterra 4

Volterra 5

Scale = 2 : 1

Figure 7 Ring Cross-sections

Figure 8.22 DOF Arc Element

Hermitian finite elements. (a) Cubic element (b) Quintic element.

Figure 3 Hermitian Shape Functions

FIGURE 10 OUT OF PLANE BENDING
FOR RING 1

FIGURE 11 IN-PLANE BENDING
FOR RING 1

FIGURE 12 OUT OF PLANE TORSION
FOR RING 1

FIGURE 13 EXTENSIONAL MODES
FOR RING 1

FIGURE 14 OUT OF PLANE BENDING
FOR RING 2

FIGURE 15 IN-PLANE BENDING
FOR RING 2

FIGURE 16 OUT OF PLANE TORSION
FOR RING 2

FIGURE 17 EXTENSIONAL MODES
FOR RING 2

FIGURE 28 OUT OF PLANE BENDING
FOR RING 6

FIGURE 29 IN-PLANE BENDING
FOR RING 6

Figure 44 Boundary Conditions for 22 DOF Element Mesh

Figure 45 30 DOF Arc Element

TABLE 1 CHARACTERISTIC PROPERTIES OF RINGS

RING	R	A	I _{YY}	I _{ZZ}	I _{YZ}	C	$\frac{I_{yy}}{AR^2}$
	² (in)	² (in)	⁴ (in)	⁴ (in)	⁴ (in)	⁴ (in)	⁻³ x 10
1	4.928	3.984	1.3227	1.3227	0.0	2.2317	13.67
2	4.595	1.888	0.4171	0.3763	-0.1981	0.4004	10.46
3	4.595	1.1565	0.2555	0.0865	-0.0743	0.1308	10.46
4	4.679	2.833	0.6277	0.8888	-0.2565	1.062	10.12
5	5.000	2.000	0.29167	0.54167	0.21651	0.45776	5.833
6	4.321	0.6792	0.05192	0.05060	-0.02563	0.05190	4.094
7	5.025	0.4	0.026333	0.01233	0.01212	0.015969	2.607
8	5.206	0.0625	0.000868	0.000217	-0.000217	0.0003516	0.5124
V1	5.000	3.1415	0.7854	0.7854	0.0	1.571	10.00
V2	5.000	1.6422	0.2146	0.2146	0.0	0.4292	5.227
V3	5.000	0.70584	0.03965	0.03965	0.0	0.07929	2.247
V4	5.000	0.19635	0.003068	0.003068	0.0	0.006136	0.625
V5	5.000	0.04909	0.0001917	0.0001917	0.0	0.0003835	0.1562

All rings were made of steel.

EXPERIMENTAL FREQUENCIES FOR RINGS 1, 2, 3, 4, 6, AND 8 WERE MEASURED BY KIRKHOPE [28].

EXPERIMENTAL FREQUENCIES FOR RINGS 5 AND 7 WERE MEASURED BY HAWKINGS [20].

EXPERIMENTAL FREQUENCIES FOR RINGS V1 THRU V5 WERE MEASURED BY VOLTERRA [31].

TABLE 2 CHARACTERISTIC PROPERTIES OF RINGS

RING	R	A	I _{YY}	I _{ZZ}	I _{YZ}	C	$\frac{I_{yy}}{AR^2}$
	² (mm)	² (mm)	⁴ (mm) ³ x 10	⁴ (mm) ³ x 10	⁴ (mm) ³ x 10	⁴ (mm) ³ x 10	⁻³ x 10
1	125.2	2570.	550.5	550.5	0.0	928.9	13.67
2	116.7	1218.	173.6	156.6	-82.46	166.7	10.46
3	116.7	746.1	106.3	36.00	-30.93	54.44	10.46
4	118.8	1828.	261.3	369.9	-106.8	442.0	10.12
5	127.0	1290.	121.4	225.5	90.12	190.5	5.833
6	109.8	438.2	21.61	21.06	-10.67	21.60	4.094
7	127.6	258.1	10.96	5.132	5.045	6.647	2.607
8	132.2	40.32	0.3613	0.09032	-0.09032	0.1463	0.5124
V1	127.0	2027.0	326.9	326.9	0.0	653.9	10.00
V2	127.0	1059.	89.32	89.32	0.0	178.6	5.227
V3	127.0	455.4	16.50	16.50	0.0	33.00	2.247
V4	127.0	126.7	1.277	1.277	0.0	2.554	0.625
V5	127.0	31.67	0.07979	0.07979	0.0	0.1596	0.1562

All rings were made of steel.

EXPERIMENTAL FREQUENCIES FOR RINGS 1, 2, 3, 4, 6, AND 8 WERE MEASURED BY KIRKHOPE [28].

EXPERIMENTAL FREQUENCIES FOR RINGS 5 AND 7 WERE MEASURED BY HAWKINGS [20].

EXPERIMENTAL FREQUENCIES FOR RINGS V1 THRU V5 WERE MEASURED BY VOLTERRA [31].

U

V

W

X

U

$$EA \frac{n^2}{R^2}$$

$$-EA \frac{n^2}{R^2}$$

$$-EA \frac{n^2}{R^2}$$

V

$$EI_{xx} \frac{n^2}{R^2}$$

$$GC \frac{n^2}{R^2 R_0}$$

$$-EI_{xx} \left[\frac{n^2 - n^2}{R^2} \right]$$

$$EI_{xx} \left[\frac{n^2}{R^2} + \frac{n^2}{R^2} \right]$$

$$-EI_{xx} \left[\frac{n^2 - n^2}{R^2} \right] R_0$$

$$GC \frac{n^2}{R^2 R_0}$$

W

$$EI_{xx} \left[\frac{1 - n^2}{R^2} \right]^2$$

$$EA \frac{n^2}{R^2}$$

$$EI_{xx} R_0 \left[\frac{1 - n^2}{R^2} \right]^2$$

$$EA \frac{n^2}{R^2}$$

$$-EI_{xx} \left[\frac{n^2}{R^2} + \frac{1}{R} \right] \left[\frac{n^2 - 1}{R^2} \right]$$

EI_{xx}

$$EI_{xx} \left[\frac{n^2}{R^2} + \frac{1}{R} \right]^2$$

$$EI_{xx} R_0^2 \left[\frac{1 - n^2}{R^2} \right]^2$$

$$-EI_{xx} R_0 \left[\frac{n^2}{R^2} + \frac{1}{R} \right] \left[\frac{n^2 - 1}{R^2} \right]$$

$$EA \frac{n^2}{R^2}$$

$$GC \frac{n^2}{R^2}$$

$$EI_{xx} \left[\frac{1}{R^2} \right]$$

$$-EI_{xx}$$

Table 3 Stiffness Coefficients for 6 DOF Element

$$-EI \frac{u^2}{R^3}$$

$$EI \frac{u^2}{R^3}$$

$$GC \frac{u}{RR_c}$$

$$I_{xx} \left[\frac{u^2 - u}{R^3} \right]$$

$$-EI_{xx} \left[\frac{u^2 - u}{R^3} \right]$$

$$I_{xx} \left[\frac{u^2 - u}{R^3} p_0 \right]$$

$$EI_{xx} \left[\frac{u^2 p_0}{R^3} + \frac{u}{R^3} \right]$$

$$-EI_{xx} \left[\frac{u^2 - u}{R^3} \right] p_0$$

$$GC \frac{u}{RR_c}$$

$$EI_{xx} \frac{u^2}{R^3}$$

$$-EI_{xx} \frac{u^2}{R^3}$$

$$EI_{xx} \frac{u^2}{R^3}$$

$$GC \frac{u}{RR_c}$$

$$GC$$

$$R_j^2$$

U

V

W

•

•

U

$$\frac{A}{R^2} \frac{1}{r^2}$$

$$\frac{2n}{R^2} \frac{1}{r^2}$$

$$-\frac{2n}{R^2} \frac{1}{r^2}$$

$$\begin{aligned} & -\frac{2np_0}{R^2} \frac{1}{r^2} \\ & \frac{2np_0}{R^2} \frac{1}{r^2} \end{aligned}$$

$$\frac{2}{R} \frac{1}{r^2}$$

V

$$\frac{A}{R^2} \frac{n^2}{r^2}$$

$$-\frac{n^2}{R^2} \frac{1}{r^2}$$

$$\begin{aligned} & \frac{A p_0}{R^2} \frac{n^2}{r^2} \frac{1}{r^2} \\ & -\frac{1}{R} \frac{1}{r^2} \\ & \frac{n^2 p_0}{R^2} \frac{1}{r^2} \end{aligned}$$

$$-\frac{n}{R} \frac{1}{r^2}$$

W

$$\frac{A}{R^2} \frac{n^2}{r^2} \frac{1}{r^2}$$

$$\begin{aligned} & \frac{A p_0}{R^2} \frac{n^2}{r^2} \frac{1}{r^2} \\ & \left[\frac{n^2 p_0}{R^2} + \frac{1}{R} \right] \left[-\frac{1}{r^2} \right] \end{aligned}$$

$$\frac{n}{R} \frac{1}{r^2}$$

•

$$\begin{aligned} & A \left[y_0^2 + z_0^2 \right] \\ & \left[\frac{1}{r^2} + \frac{1}{r^2} \right] \\ & \left[\frac{n^2 y_0^2}{R^2} - \frac{2z_0}{R} \right] \frac{1}{r^2} \\ & \frac{n^2 z_0^2}{R^2} \frac{1}{r^2} \\ & - \left[\frac{2n^2 p_0 z_0}{R^2} + \frac{p_0}{R} \right] \frac{1}{r^2} \end{aligned}$$

$$\begin{aligned} & \frac{n p_0}{R} \frac{1}{r^2} \\ & -\frac{n p_0}{R} \frac{1}{r^2} \end{aligned}$$

•

$\frac{1}{r^2}$

•

$$\frac{1}{R} I_{xx}$$

$$\frac{n}{R} I_{xx}$$

Table 4 Mass Coefficients for 6 DOF Element

$$\frac{M_{ij}}{\rho \pi R}$$

$$-\frac{n}{R} I_{xx}$$

$$\frac{n \Delta x}{R} I_{xx}$$

$$-\frac{n \Delta x}{R} I_{xx}$$

$$-I_{xx}$$

$$I_{xx}$$

TABLE 5 RING 1 RESULTS FOR 6 DOF ELEMENT

N	OUT OF PLANE BENDING			IN PLANE BENDING			OUT OF PLANE TORSION			EXTENSION		
	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %
2	1746.	1742.	-.2	1930.	1912.	-.9	9269.	8957.	-3.4	14036.	13991.	-.3
3	4545.	4587.	.9	4979.	4957.	-.4	12963.	12496.	-3.6	19588.	19681.	.5
4	7887.	8036.	1.9	8601.	8617.	.2	16840.	16220.	-3.7			
5	11500.	11820.	2.8	12516.	12614.	.8	20795.	20032.	-3.7			
6	15233.	15782.	3.6	16576.	16797.	1.3	25134.	23898.	-4.9			

MATERIAL PROPERTIES

A - AREA = 3.984000
 ASTAR = 3.984000
 E - YOUNG'S MODULUS = .30000000E+08
 G - SHEAR MODULUS = .11628000E+08
 IXX - TORSIONAL INERTIA = 2.645400
 IYZ - BENDING COUPLING INERTIA = 0.000000
 IYY - IN-PLANE BENDING INERTIA = 1.322700
 IZZ - OUT-PLANE BENDING INERTIA = 1.322700
 C - ST. VENANT TORSION = 2.231700
 RG - CENTROIDAL RADIUS = 4.928000
 RHO - DENSITY = .7324000E-03
 RS - SHEAR CENTER RADIUS = 4.928000
 Y0 - SHEAR CENTER OFF-SET = 0.000000
 Z0 - SHEAR CENTER OFF-SET = 0.000000

TABLE 6 RING 2 RESULTS FOR 6 DOF, ELEMENT

N	OUT OF PLANE BENDING			IN PLANE BENDING			OUT OF PLANE TORSION			EXTENSION		
	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %
2	1245.	1228.	-1.4	1804.	1895.	5.0	8376.	8072.	-3.6	15187.	15158.	-.2
3	3343.	3356.	.4	4773.	5054.	5.9	11469.	11001.	-4.1	21243.	21348.	.5
4	5973.	6131.	2.6	8174.	8840.	8.1	14879.	14206.	-4.5	27279.	27752.	1.7
5	8859.	9367.	5.7	11820.	12908.	9.2	18511.	17610.	-4.9	30405.	34221.	12.6
6	11843.	12915.	9.1	15539.	17052.	9.7	22329.	21227.	-4.9			

MATERIAL PROPERTIES

A	= AREA	= 1.888000
ASTAR	=	= 1.888000
E	= YOUNG'S MODULUS	= .30000000E+08
G	= SHEAR MODULUS	= .11628000E+08
IXX	= TORSIONAL INERTIA	= .7934000
IYZ	= BENDING COUPLING INERTIA	= -.1981000
IYY	= IN-PLANE BENDING INERTIA	= .4171000
IZZ	= OUT-PLANE BENDING INERTIA	= .3763000
C	= ST. VENANT TORSION	= .4004000
RG	= CENTROIDAL RADIUS	= 4.595000
RHO	= DENSITY	= .7324000E-03
RS	= SHEAR CENTER RADIUS	= 4.595000
Y0	= SHEAR CENTER OFF-SET	= 0.000000
Z0	= SHEAR CENTER OFF-SET	= 0.000000

TABLE 7 RING 3 RESULTS FOR 6 DOF ELEMENT

N	OUT OF PLANE BENDING			IN PLANE BENDING			OUT OF PLANE TORSION			EXTENSION		
	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %
2	870.	868.	-0.2	1775.	1805.	1.7	7002.	6651.	-5.0	15192.	15162.	-0.2
3	2356.	2408.	2.2	4675.	4810.	2.9	9785.	9247.	-5.5	21283.	21357.	0.3
4	4260.	4491.	5.4	8164.	8465.	3.7	12750.	12044.	-5.5	27441.	27768.	1.2
5	6411.	7020.	9.5	11928.	12406.	4.0	19174.	15045.	-21.5	33525.	34249.	2.2
6	8657.	9902.	14.4	15763.	16255.	3.1	26515.	18445.	-30.4	38346.	40742.	6.2

MATERIAL PROPERTIES

A = AREA = 1.156500
 ASTAR = 1.156500
 E = YOUNG'S MODULUS = 3.000000E+08
 G = SHEAR MODULUS = 1.162800E+08
 IXX = TORSIONAL INERTIA = 3.420000
 IYZ = BENDING COUPLING INERTIA = -7.430000E-01
 IYY = IN-PLANE BENDING INERTIA = 2.555000
 IZZ = OUT-PLANE BENDING INERTIA = 8.650000E-01
 C = ST. VENANT TORSION = 1.308000
 RG = CENTROIDAL RADIUS = 4.595000
 RHO = DENSITY = 7.324000E-03
 RS = SHEAR CENTER RADIUS = 4.595000
 Y0 = SHEAR CENTER OFF-SET = 0.000000
 Z0 = SHEAR CENTER OFF-SET = 0.000000

TABLE 8 RING 4 RESULTS FOR 6 DOF ELEMENT

N	OUT OF PLANE BENDING			IN PLANE BENDING			OUT OF PLANE TORSION			EXTENSION		
	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %
2	1533.	1508.	-1.6	1904.	1957.	2.8	9180.	9037.	-1.6	14945.	14904.	-.3
3	4063.	4043.	-.5	4992.	5173.	3.6	12569.	12366.	-1.6	20880.	20996.	.6
4	7182.	7254.	1.0	8650.	9009.	4.2	16216.	15942.	-1.7	26823.	27300.	1.8
5	10621.	10908.	2.7		13147.		20005.	19649.	-1.8			
6	14191.	14845.	4.6	16675.	17424.	4.5	23875.	23453.	-1.8			

MATERIAL PROPERTIES

A	= AREA	= 2.833000
ASTAR	=	= 2.833000
E	= YOUNG'S MODULUS	= .3000000E+08
G	= SHEAR MODULUS	= .1162800E+08
IXX	= TORSIONAL INERTIA	= 1.516500
IYZ	= BENDING COUPLING INERTIA	= -.2565000
IYY	= IN-PLANE BENDING INERTIA	= .6277000
IZZ	= OUT-PLANE BENDING INERTIA	= .8888000
C	= ST. VENANT TORSION	= 1.062000
RG	= CENTROIDAL RADIUS	= 4.679000
RHO	= DENSITY	= .7324000E-03
RS	= SHEAR CENTER RADIUS	= 4.679000
Y0	= SHEAR CENTER OFF-SET	= 0.000000
Z0	= SHEAR CENTER OFF-SET	= 0.000000

TABLE 9 RING 5 RESULTS FOR 6 DOF ELEMENT

N	OUT OF PLANE BENDING		IN PLANE BENDING	
	EXPT. [HZ]	THEORY [HZ]	EXPT. [HZ]	THEORY [HZ]
2	1560.	1533.	-1.7	970.
3	4280.	4202.	-1.8	2660.
4				4910.

MATERIAL PROPERTIES

A	= AREA	= 2.000000	ASTAR	=	= 1.666700
E	= YOUNG'S MODULUS	= .3000000E+08	IXX	= TORSIONAL INERTIA	= .8333400
G	= SHEAR MODULUS	= .1162800E+08	IYY	= IN-PLANE BENDING INERTIA	= .2916700
IYZ	= BENDING COUPLING INERTIA	= .2165100	C	= ST. VENANT TORSION	= .4577600
IZZ	= OUT-PLANE BENDING INERTIA	= .5416700	RHO	= DENSITY	= .7324000E-03
R	= CENTROIDAL RADIUS	= 5.000000

TABLE 10 RING 6 RESULTS FOR 6 DOF ELEMENT

N	OUT OF PLANE BENDING			IN PLANE BENDING			OUT OF PLANE TORSION			EXTENSION		
	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %
2	884.	864.	-2.3	1294.	1335.	3.2	8709.	8522.	-2.1	16568.	16458.	-.7
3	2463.	2420.	-1.7	3623.	3775.	4.2	11690.	11442.	-2.1	23370.	23244.	-.5
4	4582.	4548.	-.7	6617.	6972.	5.4	14965.	14611.	-2.4	30349.	30283.	-.2
5	7131.	7164.	.5	10029.	10682.	6.5	18407.	17913.	-2.7			
6	10019.	10191.	1.7	13656.	14726.	7.8	22006.	21315.	-3.1			

MATERIAL PROPERTIES

A - AREA = .6792000
 ASTAR = .6792000
 E = YOUNG'S MODULUS = .30000000E+08
 G = SHEAR MODULUS = .11628000E+08
 IXX = TORSIONAL INERTIA = .1025200
 IYZ = BENDING COUPLING INERTIA = -.2563000E-01
 IYY = IN-PLANE BENDING INERTIA = .5192000E-01
 IZZ = OUT-PLANE BENDING INERTIA = .5060000E-01
 C = ST. VENANT TORSION = .5190000E-01
 RG = CENTROIDAL RADIUS = 4.321000
 RHO = DENSITY = .7324000E-03
 RS = SHEAR CENTER RADIUS = 4.321000
 Y0 = SHEAR CENTER OFF-SET = 0.000000
 Z0 = SHEAR CENTER OFF-SET = 0.000000

1098mm
 4321mm

TABLE 11 RING 7 RESULTS FOR 6 DOF ELEMENT

N OUT OF PLANE BENDING		IN PLANE BENDING	
EXPT. [HZ]	THEORY [HZ]	EXPT. [HZ]	THEORY ERROR %
2	390.	395.	1.3
3	1090.	1106.	1.5
4	2070.	2099.	1.4

MATERIAL PROPERTIES

A	= AREA	= .4000000	ASTAR	=	.33333300
E	= YOUNG'S MODULUS	= .30000000E+08			
G	= SHEAR MODULUS	= .11628000E+08	IXX	= TORSIONAL INERTIA	= .3866600E-01
IYZ	= BENDING COUPLING INERTIA	= .1212440E-01	IYY	= IN-PLANE BENDING INERTIA	= .2633300E-01
IZZ	= OUT-PLANE BENDING INERTIA	= .1233300E-01	C	= ST. VENANT TORSION	= .1596900E-01
R	= CENTROIDAL RADIUS	= 5.025000	RHO	= DENSITY	= .7324000E-03

TABLE 12 RING 8 RESULTS FOR 6 DOF ELEMENT

N	OUT OF PLANE BENDING			IN PLANE BENDING			OUT OF PLANE TORSION			EXTENSION		
	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %
2	167.	155.	-7.2	391.	371.	-5.1	5198.	5194.	-.1	13880.	13814.	-.5
3	452.	440.	-2.7	1117.	1068.	-4.4	7100.	7154.	.8	19505.	19533.	.1
4	859.	845.	-1.6	2141.	2053.	-4.1	9294.	9223.	-.8			
5	1381.	1365.	-1.2	3444.	3307.	-4.0	11236.	11342.	.9			
6	2020.	2001.	-.9	5004.	4816.	-3.8	13566.	13487.	-.6			
7	2771.	2749.	-.8	6768.	6563.	-3.0	15752.	15648.	-.7			

MATERIAL PROPERTIES

A	-AREA	= .6250000E-01
ASTAR	=	.6250000E-01
E	= YOUNG'S MODULUS	= .3000000E+08
G	= SHEAR MODULUS	= .1162800E+08
IXX	= TORSIONAL INERTIA	= .1085000E-02
IYZ	= BENDING COUPLING INERTIA	= -.2170000E-03
IYY	= IN-PLANE BENDING INERTIA	= .8680000E-03
IZZ	= OUT-PLANE BENDING INERTIA	= .2170000E-03
C	= ST. VENANT TORSION	= .3516000E-03
RG	= CENTROIDAL RADIUS	= 5.206000
RHO	= DENSITY	= .7324000E-03
RS	= SHEAR CENTER RADIUS	= 5.206000
Y0	= SHEAR CENTER OFF-SET	= 0.000000
Z0	= SHEAR CENTER OFF-SET	= 0.000000

132.2mm
5.206in

TABLE 13 VI RESULTS FOR 6 DOF ELEMENT

N	OUT OF PLANE BENDING			IN PLANE BENDING		
	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %
2	1556.	1545.	-.7	1666.	1631.	-2.1
3	4119.	4100.	-.5	4387.	4284.	-2.3
4	7287.	7256.	-.4	7739.	7540.	-2.6
5	10820.	10775.	-.4	11459.	11151.	-2.7
6	14561.	14510.	-.4	15402.	14970.	-2.8
7	18428.	18371.	-.3	19463.	18908.	-2.9
8	22358.	22305.	-.2	23576.	22911.	-2.8
9	26305.	26277.	-.1	27706.	26948.	-2.7
10	30250.	30267.	.1	31822.	30998.	-2.6
11	34195.	34261.	.2	35908.	35050.	-2.4

MATERIAL PROPERTIES

A	= AREA	= 3.141500
ASTAR	=	= 2.782000
E	= YOUNG'S MODULUS	= .3000000E+08
G	= SHEAR MODULUS	= .1162800E+08
IXX	= TORSIONAL INERTIA	= 1.571000
IYZ	= BENDING COUPLING INERTIA	= 0.000000
IYY	= IN-PLANE BENDING INERTIA	= .7854000
IZZ	= OUT-PLANE BENDING INERTIA	= .7854000
C	= ST. VENANT TORSION	= 1.571000
RG	= CENTROIDAL RADIUS	= 5.000000
RHO	= DENSITY	= .7324000E-03
RS	= SHEAR CENTER RADIUS	= 5.000000
Y0	= SHEAR CENTER OFF-SET	= 0.000000
Z0	= SHEAR CENTER OFF-SET	= 0.000000

TABLE 14 V2 RESULTS FOR 6 DOF ELEMENT

N	OUT OF PLANE BENDING			IN PLANE BENDING		
	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %
2	1169.	1161.	-.7	1228.	1211.	-1.4
3	3201.	3179.	-.7	3333.	3281.	-1.6
4	5846.	5805.	-.7	6063.	5959.	-1.7
5	8936.	8866.	-.8	9244.	9073.	-1.8
6	12334.	12235.	-.8	12747.	12492.	-2.0
7	15953.	15819.	-.8	16466.	16124.	-2.1
8	19721.	19552.	-.9	20341.	19902.	-2.2
9	23595.	23387.	-.9	24321.	23779.	-2.2
10	27536.	27291.	-.9	28369.	27722.	-2.3
11	31512.	31240.	-.9	32435.	31708.	-2.2

MATERIAL PROPERTIES

A	= AREA	= 1.642200
ASTAR	=	= 1.454300
E	= YOUNG'S MODULUS	= .3000000E+08
G	= SHEAR MODULUS	= .1162800E+08
IXX	= TORSIONAL INERTIA	= .4292000
IYZ	= BENDING COUPLING INERTIA	= 0.000000
IYY	= IN-PLANE BENDING INERTIA	= .2146000
IZZ	= OUT-PLANE BENDING INERTIA	= .2146000
C	= ST. VENANT TORSION	= .4292000
RG	= CENTROIDAL RADIUS	= 5.000000
RHO	= DENSITY	= .7324000E-03
RS	= SHEAR CENTER RADIUS	= 5.000000
Y0	= SHEAR CENTER OFF-SET	= 0.000000
Z0	= SHEAR CENTER OFF-SET	= 0.000000

22
OF / DE

TABLE 15 V3 RESULTS FOR 6 DOF ELEMENT

N	OUT OF PLANE BENDING			IN PLANE BENDING		
	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %
2	788.	781.	-.9	816.	808.	-1.0
3	2206.	2191.	-.7	2264.	2240.	-1.1
4	4139.	4112.	-.7	4229.	4182.	-1.1
5	6505.	6458.	-.7	6629.	6549.	-1.2
6	9226.	9155.	-.8	9391.	9269.	-1.3
7	12242.	12138.	-.8	12449.	12274.	-1.4
8	15493.	15351.	-.9	15734.	15509.	-1.4
9	18931.	18746.	-1.0	19219.	18926.	-1.5
10	22519.	22287.	-1.0	22867.	22487.	-1.7
11	26230.	25942.	-1.1	26615.	26162.	-1.7

MATERIAL PROPERTIES

A	= AREA	= .7058400
ASTAR	=	= .6250800
E	= YOUNG'S MODULUS	= .3000000E+08
G	= SHEAR MODULUS	= .1162800E+08
IXX	= TORSIONAL INERTIA	= .7929000E-01
IYZ	= BENDING COUPLING INERTIA	= 0.000000
IYY	= IN-PLANE BENDING INERTIA	= .3965000E-01
IZZ	= OUT-PLANE BENDING INERTIA	= .3965000E-01
C	= ST. VENANT TORSION	= .7929000E-01
RG	= CENTROIDAL RADIUS	= 5.000000
RHO	= DENSITY	= .7324000E-03
RS	= SHEAR CENTER RADIUS	= 5.000000
Y0	= SHEAR CENTER OFF-SET	= 0.000000
Z0	= SHEAR CENTER OFF-SET	= 0.000000

TABLE 16 V4 RESULTS FOR 6 DOF ELEMENT

N	OUT OF PLANE BENDING			IN PLANE BENDING		
	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %
2	421.	418.	-.7	434.	430.	-.9
3	1197.	1191.	-.5	1217.	1210.	-.6
4	2289.	2277.	-.5	2316.	2302.	-.6
5	3676.	3655.	-.6	3708.	3685.	-.6
6	5339.	5305.	-.6	5376.	5340.	-.7
7	7253.	7208.	-.6	7306.	7248.	-.8
8	9408.	9343.	-.7	9462.	9388.	-.8
9	11777.	11690.	-.7	11850.	11741.	-.9
10	14345.	14231.	-.8	14431.	14288.	-1.0
11	17091.	16945.	-.9	17191.	17008.	-1.1

MATERIAL PROPERTIES

A	= AREA	= .1963500
ASTAR	=	= .1738800
E	= YOUNG'S MODULUS	= .3000000E+08
G	= SHEAR MODULUS	= .1162800E+08
IXX	= TORSIONAL INERTIA	= .6136000E-02
IYZ	= BENDING COUPLING INERTIA	= 0.000000
IYY	= IN-PLANE BENDING INERTIA	= .3068000E-02
IZZ	= OUT-PLANE BENDING INERTIA	= .3068000E-02
C	= ST. VENANT TORSION	= .6136000E-02
RG	= CENTROIDAL RADIUS	= 5.000000
RHO	= DENSITY	= .7324000E-03
RS	= SHEAR CENTER RADIUS	= 5.000000
Y0	= SHEAR CENTER OFF-SET	= 0.000000
Z0	= SHEAR CENTER OFF-SET	= 0.000000

TABLE 17 VS RESULTS FOR 6 DOF ELEMENT

N	OUT OF PLANE BENDING			IN PLANE BENDING		
	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %
2	210.	210.	.0	216.	216.	.0
3	598.	601.	.5	608.	610.	.3
4	1147.	1156.	.8	1162.	1166.	.3
5	1857.	1870.	.7	1869.	1881.	.6
6	2720.	2738.	.7	2734.	2751.	.6
7	3735.	3758.	.6	3747.	3771.	.6
8	4893.	4926.	.7	4916.	4940.	.5
9	6198.	6237.	.6	6221.	6252.	.5
10	7647.	7688.	.5	7668.	7705.	.5
11	9219.	9275.	.6	9252.	9292.	.4

MATERIAL PROPERTIES

A	= AREA	= .4908700E-01
ASTAR	=	= .4347100E-01
E	= YOUNG'S MODULUS	= .3000000E+08
G	= SHEAR MODULUS	= .1162800E+08
IXX	= TORSIONAL INERTIA	= .3835000E-03
IYZ	= BENDING COUPLING INERTIA	= 0.000000
IYY	= IN-PLANE BENDING INERTIA	= .1917000E-03
IZZ	= OUT-PLANE BENDING INERTIA	= .1917000E-03
C	= ST. VENANT TORSION	= .3835000E-03
RG	= CENTROIDAL RADIUS	= 5.000000
RHO	= DENSITY	= .7324000E-03
RS	= SHEAR CENTER RADIUS	= 5.000000
Y0	= SHEAR CENTER OFF-SET	= 0.000000
Z0	= SHEAR CENTER OFF-SET	= 0.000000

TABLE 18 RING 1 RESULTS FOR 22 DOF ELEMENT

	OUT OF PLANE BENDING			IN PLANE BENDING			OUT OF PLANE TORSION			EXTENSION		
	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %
2	1746.	1780.	1.9	1930.	2017.	4.5	9269.	9416.	1.6	14036.	14194.	1.1
3	4545.	4874.	7.2	4979.	5537.	11.2	12963.	13329.	2.8	19588.	20042.	2.3
4	7887.	8897.	12.8	8601.	10203.	18.6	16840.	17646.	4.8			
5	11500.	13475.	17.2	12516.	15737.	25.7	20795.	22316.	7.3			
6	15233.	18490.	21.4	16576.	21949.	32.4	25134.	27578.	9.7			

MATERIAL PROPERTIES

A = AREA
 ASTAR = 3.984000
 E = 3.984000
 G = .3000000E+08
 IXX = TORSIONAL INERTIA = 2.645400
 IYZ = BENDING COUPLING INERTIA = 0.000000
 IYY = IN-PLANE BENDING INERTIA = 1.322700
 IZZ = OUT-PLANE BENDING INERTIA = 1.322700
 C = ST. VENANT TORSION = 2.231700
 R = CENTROIDAL RADIUS = 4.928000
 RHO = DENSITY = .7324000E-03

125.2mm
 4.928in

TABLE 19 RING-2 RESULTS FOR 22 DOF ELEMENT

N	OUT OF PLANE BENDING			IN PLANE BENDING			OUT OF PLANE TORSION			EXTENSION		
	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %
2	1245.	1238.	-0.6	1804.	1976.	9.5	8376.	8404.	.3	15187.	15325.	.9
3	3343.	3420.	2.3	4773.	5552.	16.3	11469.	11704.	2.0	21243.	21635.	1.8
4	5973.	6350.	6.4	8174.	10111.	23.7	14879.	15722.	5.7	27279.	28194.	3.4
5	8859.	9900.	11.8	11820.	14847.	25.6	18511.	20753.	12.1	30405.	34871.	14.7
6	11843.	14081.	18.9	15539.	19640.	26.4	22329.	26866.	20.3			

MATERIAL PROPERTIES

A	-AREA	= 1.888000
ASTAR	=	1.888000
E	= YOUNG'S MODULUS	= .3000000E+08
G	= SHEAR MODULUS	= .1162800E+08
IXX	= TORSIONAL INERTIA	= .7934000
IYZ	= BENDING COUPLING INERTIA	= -.1981000
IYY	= IN-PLANE BENDING INERTIA	= .4171000
IZZ	= OUT-PLANE BENDING INERTIA	= .3763000
C	= ST. VENANT TORSION	= .4004000
R	= CENTROIDAL RADIUS	= 4.595000
RHO	= DENSITY	= .7324000E-03

TABLE 20 RING 3 RESULTS FOR 22 DOF ELEMENT

N	OUT OF PLANE BENDING			IN PLANE BENDING			OUT OF PLANE TORSION			EXTENSION		
	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %
2	870.	874.	.5	1775.	1880.	5.9	7002.	6774.	-3.3	15192.	15326.	.9
3	2356.	2435.	3.4	4675.	5248.	12.3	9785.	9520.	-2.7	21283.	21638.	1.7
4	4260.	4597.	7.9	8164.	9569.	17.2	12750.	12783.	.3	27441.	28200.	2.8
5	6411.	7294.	13.8	11928.	13801.	15.7	19174.	17182.	-10.4	33525.	34871.	4.0
6	8657.	10561.	22.0	15763.	17582.	11.5	26515.	23081.	-13.0	38346.	41714.	8.8

MATERIAL PROPERTIES

A - AREA
 A STAR = 1.156500
 E - YOUNG'S MODULUS
 G - SHEAR MODULUS
 IXX - TORSIONAL INERTIA
 IYZ - BENDING COUPLING INERTIA
 IYY - IN-PLANE BENDING INERTIA
 IZZ - OUT-PLANE BENDING INERTIA
 C - ST. VENANT TORSION
 R - CENTROIDAL RADIUS
 RHO - DENSITY

TABLE 21 RING 4 RESULTS FOR 22 DOF ELEMENT

N	OUT OF PLANE BENDING			IN PLANE BENDING			OUT OF PLANE TORSION			EXTENSION		
	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %
2	1533.	1537.	.3	1904.	2037.	7.0	9180.	9532.	3.8	14945.	15061.	.8
3	4063.	4218.	3.8	4992.	5682.	13.8	12569.	13327.	6.0	20880.	21271.	1.9
4	7182.	7789.	8.5	8650.	10383.	20.0	16216.	17713.	9.2	26823.	27726.	3.4
5	10621.	12067.	13.6		15525.		20005.	22703.	13.5			
6	14191.	17034.	20.0	16675.	17034.	2.2	23875.	28670.	20.1			

MATERIAL PROPERTIES

A	-AREA	= 2.833000
ASTAR	=	2.833000
E	= YOUNG'S MODULUS	= 3.000000E+08
G	= SHEAR MODULUS	= .1162800E+08
IXX	= TORSIONAL INERTIA	= 1.516500
IYZ	= BENDING COUPLING INERTIA	= -.2565000
IYY	= IN-PLANE BENDING INERTIA	= .6277000
IZZ	= OUT-PLANE BENDING INERTIA	= .8888000
C	= ST. VENANT TORSION	= 1.062000
R	= CENTROIDAL RADIUS	= 4.679000
RHO	= DENSITY	= .7324000E-03

TABLE 22 RING 5 RESULTS FOR 22 DOF ELEMENT

N	OUT OF PLANE BENDING		IN PLANE BENDING	
	EXPT. [Hz]	THEORY ERROR %	EXPT. [Hz]	THEORY ERROR %
2	1566.	1.5	970.	979. 0.9
3	4286.	4561. 6.6	2660.	2708. 1.8
4			4910.	5074. 3.3

MATERIAL PROPERTIES

A	= AREA	= 2.000000	ASTAR	=	= 1.666700
E	= YOUNG'S MODULUS	= .3000000E+08	IXX	= TORSIONAL INERTIA	= .8333400
G	= SHEAR MODULUS	= .1162800E+08	IYY	= IN-PLANE BENDING INERTIA	= .2916700
IY2	= BENDING COUPLING INERTIA	= .2165100	C	= ST. VENANT TORSION	= .4577600
IZZ	= OUT-PLANE BENDING INERTIA	= .5416700	RHO	= DENSITY	= .7324000E-03
R	= CENTROIDAL RADIUS	= 5.000000			

TABLE 23 RING 6 RESULTS FOR 22 DOF ELEMENT

N	OUT OF PLANE BENDING			IN PLANE BENDING			OUT OF PLANE TORSION			EXTENSION		
	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %
2	884.	870.	-1.6	1294.	1361.	5.2	8709.	8672.	-.4	16568.	16520.	-.3
3	2463.	2445.	-.7	3623.	3964.	9.4	11690.	11728.	.3	23370.	23347.	-.1
4	4592.	4641.	1.3	6617.	7593.	14.7	14965.	15162.	1.3	30349.	30430.	.3
5	7131.	7414.	4.0	10029.	12004.	19.7	18407.	18882.	2.6			
6	10019.	10012.	7.9	13656.	16983.	24.4	22006.	23164.	5.3			

MATERIAL PROPERTIES

A - AREA
 ASTAR -
 E - YOUNG'S MODULUS
 G - SHEAR MODULUS
 IXX - TORSIONAL INERTIA
 IYZ - BENDING COUPLING INERTIA
 IYY - IN-PLANE BENDING INERTIA
 IZZ - OUT-PLANE BENDING INERTIA
 C - ST. VENANT TORSION
 R - CENTROIDAL RADIUS
 RHO - DENSITY

100.8mm
 4.32in

TABLE 24 RING 7 RESULTS FOR 22 DOF ELEMENT

	OUT OF PLANE BENDING		IN PLANE BENDING	
	EXPT. (HZ)	THEORY ERROR %	EXPT. (HZ)	THEORY ERROR %
2	398.	482. 3.1	898.	888. -8.2
3	1898.	1316. 1.8	2548.	2698. 2.4
4	2878.	2112. 2.8

MATERIAL PROPERTIES

A	- AREA	- .4888888	ASTAR	-	- .3333330
E	- YOUNG'S MODULUS	- .3888888E+09	IXX	- TORSIONAL INERTIA	- .3866600E-01
G	- SHEAR MODULUS	- .1162888E+08	IYY	- IN-PLANE BENDING INERTIA	- .2633300E-01
IYI	- BENDING COUPLING INERTIA	- .1212448E-01	C	- ST. VENANT TORSION	- .1596900E-01
IS	- OUT-PLANE BENDING INERTIA	- .1233300E-01	RHO	- DENSITY	- .7324800E-03
R	- CENTROIDAL RADIUS	- 5.825888			

TABLE 25 RING 8 RESULTS FOR 22 DOF ELEMENT

N	OUT OF PLANE BENDING			IN PLANE BENDING			OUT OF PLANE TORSION			EXTENSION		
	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %
2	167.	171.	2.4	391.	379.	-3.1	5198.	5199.	.0	13880.	13820.	-.4
3	452.	446.	-1.3	1117.	1076.	-3.7	7100.	7165.	.9	19505.	19543.	.2
4	859.	850.	-1.0	2141.	2078.	-2.9	9294.	9286.	-.1			
5	1381.	1374.	-.5	3444.	3371.	-2.1	11236.	11467.	2.1			
6	2020.	2025.	.2	5004.	4961.	-.9	13566.	13873.	2.3			
7	2771.	2815.	1.6	6768.	6862.	1.4						

MATERIAL PROPERTIES

A - AREA
 ASTAR -
 E - YOUNG'S MODULUS
 G - SHEAR MODULUS
 IXX - TORSIONAL INERTIA
 IYZ - BENDING COUPLING INERTIA
 IYY - IN-PLANE BENDING INERTIA
 IZZ - OUT-PLANE BENDING INERTIA
 C - ST. VENANT TORSION
 R - CENTROIDAL RADIUS
 RHO - DENSITY

= .6250000E-01
 = .6250000E-01
 = .3000000E+08
 = .1162800E+08
 = .1085000E-02
 = .2170000E-03
 = .8680000E-03
 = .2170000E-03
 = .3516000E-03
 = 5.206000
 = .7324000E-03

12.70mm
 0.5in
 0.35mm
 0.25in

132.3mm
 5.206in

TABLE 26 V1 RESULTS FOR 22 DOF ELEMENT

N	OUT OF PLANE BENDING			IN PLANE BENDING		
	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %
2	1556.	1578.	1.4	1666.	1709.	2.6
3	4119.	4356.	5.8	4387.	4724.	7.7
4	7287.	8079.	10.9	7739.	8789.	13.6
5	10820.	12506.	15.6	11459.	13703.	19.6
6	14561.	17561.	20.6	15402.	19326.	25.5
7	18428.			19463.	25554.	31.3

MATERIAL PROPERTIES

A	= AREA	= 3.141500
ASTAR	=	= 2.782000
E	= YOUNG'S MODULUS	= .3000000E+08
G	= SHEAR MODULUS	= .1162800E+08
IXX	= TORSIONAL INERTIA	= 1.571000
IYZ	= BENDING COUPLING INERTIA	= 0.000000
IYY	= IN-PLANE BENDING INERTIA	= .7854000
IZZ	= OUT-PLANE BENDING INERTIA	= .7854000
C	= ST. VENANT TORSION	= 1.571000
R	= CENTROIDAL RADIUS	= 5.000000
RHO	= DENSITY	= .7324000E-03

TABLE 27 V2 RESULTS FOR 22 DOF ELEMENT

N	OUT OF PLANE BENDING			IN PLANE BENDING		
	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %
2	1169.	1176.	.6	1228.	1244.	1.3
3	3201.	3304.	3.2	3333.	3471.	4.1
4	5846.	6248.	6.9	6063.	6547.	8.0
5	8936.	9892.	10.7	9244.	10373.	12.2
6	12334.	14225.	15.3	12747.	14889.	16.8
7	15953.	19299.	21.0	16466.	20048.	21.8

MATERIAL PROPERTIES

A	= AREA	= 1.642200
ASTAR	=	= 1.454300
E	= YOUNG'S MODULUS	= .3000000E+08
G	= SHEAR MODULUS	= .1162800E+08
IXX	= TORSIONAL INERTIA	= .4292000
IYZ	= BENDING COUPLING INERTIA	= 0.000000
IYY	= IN-PLANE BENDING INERTIA	= .2146000
IZZ	= OUT-PLANE BENDING INERTIA	= .2146000
C	= ST. VENANT TORSION	= .4292000
R	= CENTROIDAL RADIUS	= 5.000000
RHO	= DENSITY	= .7324000E-03

TABLE 28 V3 RESULTS FOR 22 DOF ELEMENT

N	OUT OF PLANE BENDING			IN PLANE BENDING		
	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %
2	788.	788.	.0	816.	820.	.5
3	2206.	2234.	1.3	2264.	2300.	1.6
4	4139.	4276.	3.3	4229.	4379.	3.5
5	6505.	6867.	5.6	6629.	7018.	5.9
6	9226.	10003.	8.4	9391.	10208.	8.7
7	12242.	13790.	12.6	12449.	13948.	12.0

MATERIAL PROPERTIES

A	= AREA	= .7058400
ASTAR	=	= .6250800
E	= YOUNG'S MODULUS	= .3000000E+08
G	= SHEAR MODULUS	= .1162800E+08
IXX	= TORSIONAL INERTIA	= .7929000E-01
IYZ	= BENDING COUPLING INERTIA	= 0.000000
IYY	= IN-PLANE BENDING INERTIA	= .3965000E-01
IZZ	= OUT-PLANE BENDING INERTIA	= .3965000E-01
C	= ST. VENANT TORSION	= .7929000E-01
R	= CENTROIDAL RADIUS	= 5.000000
RHO	= DENSITY	= .7324000E-03

TABLE 29 V4 RESULTS FOR 22 DOF ELEMENT

N	OUT OF PLANE BENDING			IN PLANE BENDING		
	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %
2	421.	424.	.7	434.	438.	.9
3	1197.	1200.	.3	1217.	1222.	.4
4	2289.	2309.	.9	2316.	2336.	.9
5	3676.	3737.	1.7	3708.	3771.	1.7
6	5339.	5499.	3.0	5376.	5530.	2.9
7	7253.	7629.	5.2	7306.	7627.	4.4

MATERIAL PROPERTIES

A	= AREA	= .1963500
ASTAR	=	= .1738800
E	= YOUNG'S MODULUS	= .3000000E+08
G	= SHEAR MODULUS	= .1162800E+08
IXX	= TORSIONAL INERTIA	= .6136000E-02
IYZ	= BENDING COUPLING INERTIA	= 0.000000
IYY	= IN-PLANE BENDING INERTIA	= .3068000E-02
IZZ	= OUT-PLANE BENDING INERTIA	= .3068000E-02
C	= ST. VENANT TORSION	= .6136000E-02
R _c	= CENTROIDAL RADIUS	= 5.000000
RHO	= DENSITY	= .7324000E-03

TABLE 30 V5 RESULTS FOR 22 DOF ELEMENT

	OUT OF PLANE BENDING			IN PLANE BENDING		
	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %
2	210.	222.	5.7	216.	227.	5.1
3	598.	606.	1.3	608.	615.	1.2
4	1147.	1164.	1.5	1162.	1173.	.9
5	1857.	1886.	1.6	1869.	1899.	1.6
6	2720.	2781.	2.2	2734.	2791.	2.1
7	3735.	3868.	3.6	3747.	3862.	3.1

MATERIAL PROPERTIES

A	= AREA	= .4908700E-01
ASTAR	=	= .4347100E-01
E	= YOUNG'S MODULUS	= .3000000E+08
G	= SHEAR MODULUS	= .1162800E+08
IXX	= TORSIONAL INERTIA	= .3835000E-03
IYZ	= BENDING COUPLING INERTIA	= 0.000000
IYY	= IN-PLANE BENDING INERTIA	= .1917000E-03
IZZ	= OUT-PLANE BENDING INERTIA	= .1917000E-03
C	= ST. VENANT TORSION	= .3835000E-03
R	= CENTROIDAL RADIUS	= 5.000000
RHO	= DENSITY	= .7324000E-03

TABLE 31 RING 1 RESULTS FOR 30 DOF ELEMENT

N	OUT OF PLANE BENDING			IN PLANE BENDING			OUT OF PLANE TORSION			EXTENSION		
	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %
2	1746.	1615.	-7.5	1930.	1913.	-.9	9269.	10575.	14.1	14036.	13991.	-.3
3	4545.	4207.	-7.4	4979.	4962.	-.3	12963.	14930.	15.2	19588.	19682.	.5
4	7887.	7339.	-6.9	8601.	8622.	.2	16840.	19577.	16.3			
5	11500.	10849.	-5.7	12516.	12722.	1.6	20795.	24367.	17.2			
6	15233.	14770.	-3.0	16576.	17135.	3.4	25134.	29347.	16.8			

MATERIAL PROPERTIES

A	- AREA	- 3.984000
ASTAR	-	- 3.984000
E	- YOUNG'S MODULUS	- .30000000E+08
G	- SHEAR MODULUS	- .11628000E+08
IXX	- TORSIONAL INERTIA	- 2.645400
IYZ	- BENDING COUPLING INERTIA	- 0.000000
IYY	- IN-PLANE BENDING INERTIA	- 1.322700
IZZ	- OUT-PLANE BENDING INERTIA	- 1.322700
C	- ST. VENANT TORSION	- 2.231700
R	- CENTROIDAL RADIUS	- 4.928000
D	- DENSITY	- .7324000E-03

TABLE 32 RING 2 RESULTS FOR 30 DOP ELEMENT

N	OUT OF PLANE BENDING			IN PLANE BENDING			OUT OF PLANE TORSION			EXTENSION		
	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %
2	1245.	1167.	-6.3	1884.	1889.	4.7	8376.	9172.	9.5	15187.	15202.	.1
3	3343.	3154.	-5.7	4773.	5049.	5.8	11469.	12637.	10.2	21243.	21369.	.6
4	5973.	5716.	-4.3	8174.	8873.	8.6	14879.	16456.	10.6	27279.	27761.	1.8
5	8859.	8753.	-1.2	11820.	13165.	11.4	18511.	20499.	10.7	30405.	34243.	12.6
6	11843.	12277.	3.7	15539.	17832.	14.8	22329.	24802.	11.1			

MATERIAL PROPERTIES

A = AREA
 ASTAR =
 E = YOUNG'S MODULUS
 G = SHEAR MODULUS
 IXX = TORSIONAL INERTIA
 IYZ = BENDING COUPLING INERTIA
 IYY = IN-PLANE BENDING INERTIA
 IZZ = OUT-PLANE BENDING INERTIA
 C = ST. VENANT TORSION
 R = CENTROIDAL RADIUS
 RHO = DENSITY

TABLE 33 RING 3 RESULTS FOR 30 DOF ELEMENT

	OUT OF PLANE BENDING			IN PLANE BENDING			OUT OF PLANE TORSION			EXTENSION		
	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %
2	870.	824.	-5.3	1775.	1808.	1.9	7002.	7711.	10.1	15192.	15185.	.0
3	2356.	2257.	-4.2	4675.	4831.	3.3	9785.	10817.	10.5	21283.	21369.	.4
4	4260.	4178.	-1.9	8164.	8539.	4.6	12750.	14199.	11.4	27441.	27775.	1.2
5	6411.	6553.	2.2	11928.	12750.	6.9	19174.	17768.	-7.3	33525.	34270.	2.2
6	8657.	9426.	8.9	15763.	17338.	10.0	26515.	21578.	-18.6	38346.	40713.	6.2

MATERIAL PROPERTIES

A = AREA = 1.156500
 ASTAR = 1.156500
 E = YOUNG'S MODULUS = 30000000E+08
 G = SHEAR MODULUS = 11628000E+08
 IXX = TORSIONAL INERTIA = 3420000
 IYZ = BENDING COUPLING INERTIA = -7430000E-01
 IYY = IN-PLANE BENDING INERTIA = 2555000
 IZZ = OUT-PLANE BENDING INERTIA = 8650000E-01
 C = ST. VENANT TORSION = 1308000
 R = CENTROIDAL RADIUS = 4.595000
 RHO = DENSITY = .7324000E-03

TABLE 34 RING 4 RESULTS FOR 30 DOF ELEMENT

N	OUT OF PLANE BENDING			IN PLANE BENDING			OUT OF PLANE TORSION			EXTENSION		
	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %
2	1533.	1455.	-5.1	1904.	1921.	.9	9180.	10097.	10.0	14945.	14931.	-.1
3	4063.	3872.	-4.7	4992.	5071.	1.6	12569.	13970.	11.1	20880.	21007.	.6
4	7182.	6906.	-3.8	8650.	8849.	2.3	16216.	18179.	12.1	26823.	27220.	1.5
5	10621.	10419.	-1.9		13070.		20005.	22565.	12.8			
6	14191.	14392.	1.4	16675.	17666.	5.9	23875.	27220.	14.0			

MATERIAL PROPERTIES

A = AREA
 ASTAR = 2.833000
 E = YOUNG'S MODULUS = 2.833000
 G = SHEAR MODULUS = .30000000E+08
 IXX = TORSIONAL INERTIA = .11628000E+08
 IYZ = BENDING COUPLING INERTIA = 1.516500
 IYY = IN-PLANE BENDING INERTIA = -.2565000
 IZZ = OUT-PLANE BENDING INERTIA = .6277000
 C = ST. VENANT TORSION = .8888000
 R = CENTROIDAL RADIUS = 1.062000
 RHO = DENSITY = 4.679000
 = .73240000E-03

TABLE 35 RING 5 RESULTS FOR 30 DOF ELEMENT

N	OUT OF PLANE BENDING		IN PLANE BENDING	
	EXPT. [HZ]	THEORY ERROR %	EXPT. [HZ]	THEORY ERROR %
2	1560.	1513. -3.6	970.	949. -2.2
3	4280.	4142. -3.2	2660.	2584. -2.9
4			4910.	4741. -3.4

MATERIAL PROPERTIES

A	= AREA	= 2.0000000	ASTAR	=	= 1.666700
E	= YOUNG'S MODULUS	= .30000000E+08	IXX	= TORSIONAL INERTIA	= .8333400
G	= SHEAR MODULUS	= .11628000E+08	IYY	= IN-PLANE BENDING INERTIA	= .2916700
IYZ	= BENDING COUPLING INERTIA	= .2165100	C	= ST. VENANT TORSION	= .4577600
IZZ	= OUT-PLANE BENDING INERTIA	= .5416700	RHO	= DENSITY	= .7324000E-03
R	= CENTROIDAL RADIUS	= 5.0000000

TABLE 36 RING 6 RESULTS FOR 30 DOF ELEMENT

N	OUT OF PLANE BENDING			IN PLANE BENDING			OUT OF PLANE TORSION			EXTENSION		
	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %
2	884.	851.	-3.7	1294.	1331.	2.9	8709.	8927.	2.5	16568.	16474.	-.6
3	2463.	2371.	-3.7	3623.	3758.	3.7	11690.	12057.	3.1	23370.	23253.	-.5
4	4582.	4446.	-3.0	6617.	6940.	4.9	14965.	15508.	3.6	30340.	30288.	-.2
5	7131.	7073.	-.8	10029.	10751.	7.2	18407.	19124.	3.9			
6	10019.	10283.	2.6	13656.	15134.	10.8	22006.	23004.	4.5			

MATERIAL PROPERTIES

A - AREA = .6792000
 ASTAR = .6792000
 E = YOUNG'S MODULUS = .3000000E+08
 G = SHEAR MODULUS = .1162800E+08
 IXX = TORSIONAL INERTIA = .1025200
 IYZ = BENDING COUPLING INERTIA = -.2563000E-01
 IYY = IN-PLANE BENDING INERTIA = .5192000E-01
 IZZ = OUT-PLANE BENDING INERTIA = .5060000E-01
 C = ST. VENANT TORSION = .5190000E-01
 R = CENTROIDAL RADIUS = 4.321000
 RHO = DENSITY = .7324000E-03

109.8mm
 4.321in

TABLE 37 RING 7 RESULTS FOR 30 DOF ELEMENT

OUT OF PLANE BENDING		IN PLANE BENDING	
EXPT. [HZ]	THEORY ERROR %	EXPT. [HZ]	THEORY ERROR %
2 390.	396. 1.5	890.	875. -1.7
3 1090.	1091. 0.01	2540.	2508. -1.3
4 2070.	2067. -.1		

MATERIAL PROPERTIES

A	= AREA	= .4000000	ASTAR	=	.3333300
E	= YOUNG'S MODULUS	= .3000000E+08	IXX	= TORSIONAL INERTIA	= .3866600E-01
G	= SHEAR MODULUS	= .1162800E+08	IYY	= IN-PLANE BENDING INERTIA	= .2633300E-01
IYZ	= BENDING COUPLING INERTIA	= .1212440E-01	C	= ST. VENANT TORSION	= .1596900E-01
IZZ	= OUT-PLANE BENDING INERTIA	= .1233300E-01	RHO	= DENSITY	= .7324000E-03
R	= CENTROIDAL RADIUS	= 5.025000

TABLE 38 RING 8 RESULTS FOR 30 DOF ELEMENT

N	OUT OF PLANE BENDING			IN PLANE BENDING			OUT OF PLANE TORSION			EXTENSION		
	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %
2	167.	170.	1.8	391.	378.	-3.3	5198.	5244.	.9	13880.	13815.	-.5
3	452.	445.	-1.5	1117.	1071.	-4.1	7100.	7230.	1.8	19505.	19534.	.1
4	859.	846.	-1.5	2141.	2056.	-4.0	9294.	9371.	.8			
5	1381.	1383.	.1	3444.	3353.	-2.6	11236.	11575.	3.0			
6	2020.	2000.	3.0	5004.	4991.	-.3	13566.	13972.	3.0			
7	2771.	3006.	8.5	6768.	7087.	4.7	15752.	16830.	6.8			

MATERIAL PROPERTIES

A - AREA = .6250000E-01
 ASTAR = .6250000E-01
 E = YOUNG'S MODULUS = .3000000E+08
 G = SHEAR MODULUS = .1162800E+08
 IXX = TORSIONAL INERTIA = .1085000E-02
 IYZ = BENDING COUPLING INERTIA = .2170000E-03
 IYY = IN-PLANE BENDING INERTIA = .8680000E-03
 IZZ = OUT-PLANE BENDING INERTIA = .2170000E-03
 C = ST. VENANT TORSION = .3516000E-03
 R = CENTROIDAL RADIUS = 5.206000
 RHO = DENSITY = .7324000E-03

TABLE 39 V1 RESULTS FOR 30 DOF ELEMENT

N	OUT OF PLANE BENDING			IN PLANE BENDING		
	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %
2	1556.	1460.	-6.2	1666.	1632.	-2.0
3	4119.	3848.	-6.6	4387.	4289.	-2.2
4	7287.	6799.	-6.7	7739.	7544.	-2.5
5	10820.	10175.	-6.0	11459.	11250.	-1.8
6	14561.	13977.	-4.0	15402.	15287.	-.7
7	18428.	18508.	.4	19463.	19699.	1.2

MATERIAL PROPERTIES

A	= AREA	= 3.141500
ASTAR	=	= 2.782000
E	= YOUNG'S MODULUS	= .3000000E+08
G	= SHEAR MODULUS	= .1162800E+08
IXX	= TORSIONAL INERTIA	= 1.571000
IYZ	= BENDING COUPLING INERTIA	= 0.000000
IYY	= IN-PLANE BENDING INERTIA	= .7854000
IZZ	= OUT-PLANE BENDING INERTIA	= .7854000
C	= ST. VENANT TORSION	= 1.571000
R	= CENTROIDAL RADIUS	= 5.000000
RHO	= DENSITY	= .7324000E-03

TABLE 40 V2 RESULTS FOR 30 DOF ELEMENT

N	OUT OF PLANE BENDING			IN PLANE BENDING		
	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %
2	1169.	1126.	-3.7	1228.	1213.	-1.2
3	3201.	3068.	-4.2	3333.	3285.	-1.4
4	5046.	5597.	-4.3	6063.	5963.	-1.6
5	8936.	8634.	-3.4	9244.	9167.	-.8
6	12334.	12177.	-1.3	12747.	12812.	.5
7	15953.	16478.	3.3	16466.	16960.	3.0

MATERIAL PROPERTIES

A	= AREA	= 1.642200
ASTAR	=	= 1.454300
E	= YOUNG'S MODULUS	= .3000000E+08
G	= SHEAR MODULUS	= .1162800E+08
IXX	= TORSIONAL INERTIA	= .4292000
IYZ	= BENDING COUPLING INERTIA	= 0.000000
IYY	= IN-PLANE BENDING INERTIA	= .2146000
IZZ	= OUT-PLANE BENDING INERTIA	= .2146000
C	= ST. VENANT TORSION	= .4292000
R	= CENTROIDAL RADIUS	= 5.000000
RHO	= DENSITY	= .7324000E-03

TABLE 41 V3 RESULTS FOR 30 DOF ELEMENT

N	OUT OF PLANE BENDING			IN PLANE BENDING		
	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %
2	788.	773.	-1.9	816.	811.	-.6
3	2206.	2158.	-2.2	2264.	2244.	-.9
4	4139.	4049.	-2.2	4229.	4185.	-1.0
5	6505.	6435.	-1.1	6629.	6627.	.0
6	9226.	9343.	1.3	9391.	9550.	1.7
7	12242.	12978.	6.0	12449.	13048.	4.8

MATERIAL PROPERTIES

A	= AREA	= .7058400
ASTAR	=	= .6250800
E	= YOUNG'S MODULUS	= .3000000E+08
G	= SHEAR MODULUS	= .1162800E+08
IXX	= TORSIONAL INERTIA	= .7929000E-01
IYZ	= BENDING COUPLING INERTIA	= 0.000000
IYY	= IN-PLANE BENDING INERTIA	= .3965000E-01
IZZ	= OUT-PLANE BENDING INERTIA	= .3965000E-01
C	= ST. VENANT TORSION	= .7929000E-01
R	= CENTROIDAL RADIUS	= 5.000000
RHO	= DENSITY	= .7324000E-03

TABLE 42 V4 RESULTS FOR 30 DOF ELEMENT

N	OUT OF PLANE BENDING			IN PLANE BENDING		
	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %
2	421.	422.	.2	434.	436.	.5
3	1197.	1189.	-.7	1217.	1214.	-.2
4	2289.	2272.	-.7	2316.	2305.	-.5
5	3676.	3694.	.5	3708.	3733.	.7
6	5339.	5501.	3.0	5376.	5524.	2.8
7	7253.	7848.	8.2	7306.	7783.	6.5

MATERIAL PROPERTIES

A	= AREA	= .1963500
ASTAR	=	= .1738800
E	= YOUNG'S MODULUS	= .3000000E+08
G	= SHEAR MODULUS	= .1162800E+08
IXX	= TORSIONAL INERTIA	= .6136000E-02
IYZ	= BENDING COUPLING INERTIA	= 0.000000
IYY	= IN-PLANE BENDING INERTIA	= .3068000E-02
IZZ	= OUT-PLANE BENDING INERTIA	= .3068000E-02
C	= ST. VENANT TORSION	= .6136000E-02
R	= CENTROIDAL RADIUS	= 5.000000
RHO	= DENSITY	= .7324000E-03

TABLE 43 V5 RESULTS FOR 30 DOF ELEMENT

N	OUT OF PLANE BENDING			IN PLANE BENDING		
	EXPT. [HZ]	THEORY [HZ]	ERROR %	EXPT. [HZ]	THEORY [HZ]	ERROR %
2	210.	221.	5.2	216.	227.	5.1
3	598.	605.	1.2	608.	614.	1.0
4	1147.	1159.	1.0	1162.	1169.	.6
5	1857.	1899.	2.3	1869.	1908.	2.1
6	2720.	2854.	4.9	2734.	2850.	4.2
7	3735.	4119.	10.3	3747.	4067.	8.5

MATERIAL PROPERTIES

A	= AREA	= .4908700E-01
ASTAR	=	= .4347100E-01
E	= YOUNG'S MODULUS	= .3000000E+08
G	= SHEAR MODULUS	= .1162800E+08
IXX	= TORSIONAL INERTIA	= .3835000E-03
IYZ	= BENDING COUPLING INERTIA	= 0.000000
IYY	= IN-PLANE BENDING INERTIA	= .1917000E-03
IZZ	= OUT-PLANE BENDING INERTIA	= .1917000E-03
C	= ST. VENANT TORSION	= .3835000E-03
R	= CENTROIDAL RADIUS	= 5.000000
RHO	= DENSITY	= .7324000E-03

Appendix A

22 DOF Stiffness and Mass Matrices

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SUBROUTINE ELSTIF(IEL)
  REAL K(22,22)
  COMMON/TWO/NDFPN,NDFPE,NNPE
  COMMON/TP/NEL,NNODE,NBNODE
  COMMON/BLK1/NODE(10,2),ID(15,10),NBN(10)
  COMMON/TX/LM(30)
  COMMON/BLK2/S(465),EM(465)
  COMMON/STIFFK/K
  JJ=0
  DO 40 I=1,22
    DO 30 J=I,22
      JJ=JJ+1
      S(JJ)=K(I,J)
30    CONTINUE
40  CONTINUE
  RETURN
  END
C*****
C***
C*** SUBROUTINE: STIFF [CREATES A 22 DOF CURVED BEAM FINITE
C*** ELEMENT]
C***
C*****
SUBROUTINE STIFF(ANGLE)
  REAL K(22,22)
  REAL V(22),L
  REAL
  1, A , ASTAR , CW , E , G
  2, IXX , IYZ , IYY , IZZ , KG
  2, R , RHO , RS , Y0 , Z0
  COMMON/SECTION/ A , ASTAR , CW , E , G
  1, IXX , IYZ , IYY , IZZ , KG
  2, R , RHO , RS , Y0 , Z0
  COMMON/STIFFK/K
  C=KG
  DO 8 I=1,22
    DO 5 J=I,22
      K(I,J)=0.0
5    CONTINUE
8    CONTINUE
  R2=R*R
  R3=R2*R
  R4=R3*R
  P=ANGLE
  P2=P*P
  P3=P2*P
  P4=P3*P
C*** OUT-PLANE BENDING
C*** (c36) FOR I:4 THRU 22 DO
C*** FOR J:I THRU 22 DO
C*** IF I<=6 OR I>=10 AND I<=11 OR I>=15 AND I<=17 OR I>=21 THEN
C*** IF J<=6 OR J>=10 AND J<=11 OR J>=15 AND J<=17 OR J>=21 THEN

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C***PRINT("      K(" , I , " , " , J , ") = K(" , I , " , " , J , ") + E*IZZ*( " ,
C***      STRING(K[I,J]) , " ) ) $
C***

K( 4 , 4 ) = K( 4 , 4 ) + E*IZZ*( 15.0/(7.0*P4*R4) )
K( 4 , 5 ) = K( 4 , 5 ) + E*IZZ*( 15.0/(7.0*P4*R4) )
K( 4 , 6 ) = K( 4 , 6 ) + E*IZZ*( 3.0/(14.0*P4*R4) )
K( 4 , 10 ) = K( 4 , 10 ) + E*IZZ*( 9.0/(14.0*P2*R3) )
K( 4 , 11 ) = K( 4 , 11 ) + E*IZZ*( 1.0/(7.0*P2*R3) )
K( 4 , 15 ) = K( 4 , 15 ) + E*IZZ*( -15.0/(7.0*P4*R4) )
K( 4 , 16 ) = K( 4 , 16 ) + E*IZZ*( 15.0/(7.0*P4*R4) )
K( 4 , 17 ) = K( 4 , 17 ) + E*IZZ*( -3.0/(14.0*P4*R4) )
K( 4 , 21 ) = K( 4 , 21 ) + E*IZZ*( -9.0/(14.0*P2*R3) )
K( 4 , 22 ) = K( 4 , 22 ) + E*IZZ*( 1.0/(7.0*P2*R3) )
K( 5 , 5 ) = K( 5 , 5 ) + E*IZZ*( 96.0/(35.0*P4*R4) )
K( 5 , 6 ) = K( 5 , 6 ) + E*IZZ*( 11.0/(35.0*P4*R4) )
K( 5 , 10 ) = K( 5 , 10 ) + E*IZZ*( 8.0/(7.0*P2*R3) )
K( 5 , 11 ) = K( 5 , 11 ) + E*IZZ*( 12.0/(35.0*P2*R3) )
K( 5 , 15 ) = K( 5 , 15 ) + E*IZZ*( -15.0/(7.0*P4*R4) )
K( 5 , 16 ) = K( 5 , 16 ) + E*IZZ*( 54.0/(35.0*P4*R4) )
K( 5 , 17 ) = K( 5 , 17 ) + E*IZZ*( -4.0/(35.0*P4*R4) )
K( 5 , 21 ) = K( 5 , 21 ) + E*IZZ*( -1.0/(7.0*P2*R3) )
K( 5 , 22 ) = K( 5 , 22 ) + E*IZZ*( -2.0/(35.0*P2*R3) )
K( 6 , 6 ) = K( 6 , 6 ) + E*IZZ*( 6.0/(35.0*P4*R4) )
K( 6 , 10 ) = K( 6 , 10 ) + E*IZZ*( 1.0/(70.0*P2*R3) )
K( 6 , 11 ) = K( 6 , 11 ) + E*IZZ*( 1.0/(21.0*P2*R3) )
K( 6 , 15 ) = K( 6 , 15 ) + E*IZZ*( -3.0/(14.0*P4*R4) )
K( 6 , 16 ) = K( 6 , 16 ) + E*IZZ*( 4.0/(35.0*P4*R4) )
K( 6 , 17 ) = K( 6 , 17 ) + E*IZZ*( 1.0/(35.0*P4*R4) )
K( 6 , 21 ) = K( 6 , 21 ) + E*IZZ*( -1.0/(70.0*P2*R3) )
K( 6 , 22 ) = K( 6 , 22 ) + E*IZZ*( -2.0/(105.0*P2*R3) )
K( 10 , 10 ) = K( 10 , 10 ) + E*IZZ*( 26.0/(35.0*R2) )
K( 10 , 11 ) = K( 10 , 11 ) + E*IZZ*( 22.0/(105.0*R2) )
K( 10 , 15 ) = K( 10 , 15 ) + E*IZZ*( -9.0/(14.0*P2*R3) )
K( 10 , 16 ) = K( 10 , 16 ) + E*IZZ*( 1.0/(7.0*P2*R3) )
K( 10 , 17 ) = K( 10 , 17 ) + E*IZZ*( -1.0/(70.0*P2*R3) )
K( 10 , 21 ) = K( 10 , 21 ) + E*IZZ*( 9.0/(35.0*R2) )
K( 10 , 22 ) = K( 10 , 22 ) + E*IZZ*( -13.0/(105.0*R2) )
K( 11 , 11 ) = K( 11 , 11 ) + E*IZZ*( 8.0/(105.0*R2) )
K( 11 , 15 ) = K( 11 , 15 ) + E*IZZ*( -1.0/(7.0*P2*R3) )
K( 11 , 16 ) = K( 11 , 16 ) + E*IZZ*( -2.0/(35.0*P2*R3) )
K( 11 , 17 ) = K( 11 , 17 ) + E*IZZ*( 2.0/(105.0*P2*R3) )
K( 11 , 21 ) = K( 11 , 21 ) + E*IZZ*( 13.0/(105.0*R2) )
K( 11 , 22 ) = K( 11 , 22 ) + E*IZZ*( -2.0/(35.0*R2) )
K( 15 , 15 ) = K( 15 , 15 ) + E*IZZ*( 15.0/(7.0*P4*R4) )
K( 15 , 16 ) = K( 15 , 16 ) + E*IZZ*( -15.0/(7.0*P4*R4) )
K( 15 , 17 ) = K( 15 , 17 ) + E*IZZ*( 3.0/(14.0*P4*R4) )
K( 15 , 21 ) = K( 15 , 21 ) + E*IZZ*( 9.0/(14.0*P2*R3) )
K( 15 , 22 ) = K( 15 , 22 ) + E*IZZ*( -1.0/(7.0*P2*R3) )
K( 16 , 16 ) = K( 16 , 16 ) + E*IZZ*( 96.0/(35.0*P4*R4) )
K( 16 , 17 ) = K( 16 , 17 ) + E*IZZ*( -11.0/(35.0*P4*R4) )
K( 16 , 21 ) = K( 16 , 21 ) + E*IZZ*( -8.0/(7.0*P2*R3) )
K( 16 , 22 ) = K( 16 , 22 ) + E*IZZ*( 12.0/(35.0*P2*R3) )
K( 17 , 17 ) = K( 17 , 17 ) + E*IZZ*( 6.0/(35.0*P4*R4) )

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      K( 17 , 21 ) = K( 17 , 21 ) + E*IZZ*( 1.0/(70.0*P2*R3) )
      K( 17 , 22 ) = K( 17 , 22 ) + E*IZZ*( -1.0/(21.0*P2*R3) )
      K( 21 , 21 ) = K( 21 , 21 ) + E*IZZ*( 26.0/(35.0*R2) )
      K( 21 , 22 ) = K( 21 , 22 ) + E*IZZ*( -22.0/(105.0*R2) )
      K( 22 , 22 ) = K( 22 , 22 ) + E*IZZ*( 8.0/(105.0*R2) )
C***  IN-PLANE BENDING STIFFNESS
C***  (C23) FOR I:7 THRU 20 DO
C***    FOR J:I THRU 20 DO
C***      IF I<=9 OR I>=18 THEN
C***      IF J<=9 OR J>=18 THEN
C***        PRINT("      K(",I,"",",J,"") = K(",I,"",",J,"") + E*IYY*(",
C***          STRING(K[I,J]),")" )$
C***
      K( 7 , 7 ) = K( 7 , 7 ) + E*IYY*(
1 -10.0/(7.0*P2*R4)+15.0/(7.0*P4*R4)+181.0/(231.0*R4) )
      K( 7 , 8 ) = K( 7 , 8 ) + E*IYY*(
1 -10.0/(7.0*P2*R4)+15.0/(7.0*P4*R4)+311.0/(1155.0*R4) )
      K( 7 , 9 ) = K( 7 , 9 ) + E*IYY*(
1 -1.0/(21.0*P2*R4)+3.0/(14.0*P4*R4)+281.0/(6930.0*R4) )
      K( 7 , 18 ) = K( 7 , 18 ) + E*IYY*(
1 10.0/(7.0*P2*R4)-15.0/(7.0*P4*R4)+50.0/(231.0*R4) )
      K( 7 , 19 ) = K( 7 , 19 ) + E*IYY*(
1 -3.0/(7.0*P2*R4)+15.0/(7.0*P4*R4)-151.0/(1155.0*R4) )
      K( 7 , 20 ) = K( 7 , 20 ) + E*IYY*(
1 1.0/(21.0*P2*R4)-3.0/(14.0*P4*R4)+181.0/(6930.0*R4) )
      K( 8 , 8 ) = K( 8 , 8 ) + E*IYY*(
1 -32.0/(35.0*P2*R4)+96.0/(35.0*P4*R4)+416.0/(3465.0*R4) )
      K( 8 , 9 ) = K( 8 , 9 ) + E*IYY*(
1 -2.0/(15.0*P2*R4)+11.0/(35.0*P4*R4)+23.0/(1155.0*R4) )
      K( 8 , 18 ) = K( 8 , 18 ) + E*IYY*(
1 3.0/(7.0*P2*R4)-15.0/(7.0*P4*R4)+151.0/(1155.0*R4) )
      K( 8 , 19 ) = K( 8 , 19 ) + E*IYY*(
1 2.0/(35.0*P2*R4)+54.0/(35.0*P4*R4)-38.0/(495.0*R4) )
      K( 8 , 20 ) = K( 8 , 20 ) + E*IYY*(
1 -4.0/(105.0*P2*R4)-4.0/(35.0*P4*R4)+52.0/(3465.0*R4) )
      K( 9 , 9 ) = K( 9 , 9 ) + E*IYY*(
1 -8.0/(315.0*P2*R4)+6.0/(35.0*P4*R4)+4.0/(1155.0*R4) )
      K( 9 , 18 ) = K( 9 , 18 ) + E*IYY*(
1 1.0/(21.0*P2*R4)-3.0/(14.0*P4*R4)+181.0/(6930.0*R4) )
      K( 9 , 19 ) = K( 9 , 19 ) + E*IYY*(
1 4.0/(105.0*P2*R4)+4.0/(35.0*P4*R4)-52.0/(3465.0*R4) )
      K( 9 , 20 ) = K( 9 , 20 ) + E*IYY*(
1 -4.0/(315.0*P2*R4)+1.0/(35.0*P4*R4)+2.0/(693.0*R4) )
      K( 18 , 18 ) = K( 18 , 18 ) + E*IYY*(
1 -10.0/(7.0*P2*R4)+15.0/(7.0*P4*R4)+181.0/(231.0*R4) )
      K( 18 , 19 ) = K( 18 , 19 ) + E*IYY*(
1 10.0/(7.0*P2*R4)-15.0/(7.0*P4*R4)-311.0/(1155.0*R4) )
      K( 18 , 20 ) = K( 18 , 20 ) + E*IYY*(
1 -1.0/(21.0*P2*R4)+3.0/(14.0*P4*R4)+281.0/(6930.0*R4) )
      K( 19 , 19 ) = K( 19 , 19 ) + E*IYY*(
1 -32.0/(35.0*P2*R4)+96.0/(35.0*P4*R4)+416.0/(3465.0*R4) )
      K( 19 , 20 ) = K( 19 , 20 ) + E*IYY*(
1 2.0/(15.0*P2*R4)-11.0/(35.0*P4*R4)-23.0/(1155.0*R4) )

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      K( 20 , 20 )= K( 20 , 20 )+E*IYY*(
1      -8.0/(315.0*P2*R4)+6.0/(35.0*P4*R4)+4.0/(1155.0*R4))
C*** COUPLING STIFFNESS TERMS
C*** (c75) FOR I:4 THRU 22 DO
C***   FOR J:I THRU 22 DO
C***     IF I<12 OR I>14 THEN
C***     IF J<12 OR J>14 THEN
C***     IF K[I,J] # 0 THEN
C*** PRINT("      K(",I,"",J,"")= K(",I,"",J,"")-E*IYZ*(",
C***                                     STRING(K[I,J]),")")$
      IYZ=2.0*IYZ
      K(4 , 7)= K(4 , 7)-E*IYZ*(15.0/(14.0*P4*R4)-5.0/(14.0*P2*R4))
      K(4 , 8)= K(4 , 8)-E*IYZ*(15.0/(14.0*P4*R4)-3.0/(28.0*P2*R4))
      K(4 , 9)= K(4 , 9)-E*IYZ*(3.0/(28.0*P4*R4)-1.0/(84.0*P2*R4))
      K(4 , 18)= K(4 , 18)-E*IYZ*(5.0/(14.0*P2*R4)-15.0/(14.0*P4*R4))
      K(4 , 19)= K(4 , 19)-E*IYZ*(15.0/(14.0*P4*R4)-3.0/(28.0*P2*R4))
      K(4 , 20)= K(4 , 20)-E*IYZ*(1.0/(84.0*P2*R4)-3.0/(28.0*P4*R4))
      K(5 , 7)= K(5 , 7)-E*IYZ*(15.0/(14.0*P4*R4)-17.0/(28.0*P2*R4))
      K(5 , 8)= K(5 , 8)-E*IYZ*(48.0/(35.0*P4*R4)-8.0/(35.0*P2*R4))
      K(5 , 9)= K(5 , 9)-E*IYZ*(11.0/(70.0*P4*R4)-1.0/(30.0*P2*R4))
      K(5 , 18)= K(5 , 18)-E*IYZ*(3.0/(28.0*P2*R4)-15.0/(14.0*P4*R4))
      K(5 , 19)= K(5 , 19)-E*IYZ*(1.0/(70.0*P2*R4)+27.0/(35.0*P4*R4))
      K(5 , 20)= K(5 , 20)-E*IYZ*(-1.0/(105.0*P2*R4)-2.0/(35.0*P4*R4))
      K(6 , 7)= K(6 , 7)-E*IYZ*(3.0/(28.0*P4*R4)-1.0/(84.0*P2*R4))
      K(6 , 8)= K(6 , 8)-E*IYZ*(11.0/(70.0*P4*R4)-1.0/(30.0*P2*R4))
      K(6 , 9)= K(6 , 9)-E*IYZ*(3.0/(35.0*P4*R4)-2.0/(315.0*P2*R4))
      K(6 , 18)= K(6 , 18)-E*IYZ*(1.0/(84.0*P2*R4)-3.0/(28.0*P4*R4))
      K(6 , 19)= K(6 , 19)-E*IYZ*(1.0/(105.0*P2*R4)+2.0/(35.0*P4*R4))
      K(6 , 20)= K(6 , 20)-E*IYZ*(1.0/(70.0*P4*R4)-1.0/(315.0*P2*R4))
      K(7 , 10)= K(7 , 10)-E*IYZ*(9.0/(28.0*P2*R3)-8.0/(21.0*R3))
      K(7 , 11)= K(7 , 11)-E*IYZ*(1.0/(14.0*P2*R3)-3.0/(28.0*R3))
      K(7 , 15)= K(7 , 15)-E*IYZ*(5.0/(14.0*P2*R4)-15.0/(14.0*P4*R4))
      K(7 , 16)= K(7 , 16)-E*IYZ*(15.0/(14.0*P4*R4)-3.0/(28.0*P2*R4))
      K(7 , 17)= K(7 , 17)-E*IYZ*(1.0/(84.0*P2*R4)-3.0/(28.0*P4*R4))
      K(7 , 21)= K(7 , 21)-E*IYZ*(-9.0/(28.0*P2*R3)-5.0/(42.0*R3))
      K(7 , 22)= K(7 , 22)-E*IYZ*(1.0/(14.0*P2*R3)+5.0/(84.0*R3))
      K(8 , 10)= K(8 , 10)-E*IYZ*(4.0/(7.0*P2*R3)-11.0/(84.0*R3))
      K(8 , 11)= K(8 , 11)-E*IYZ*(6.0/(35.0*P2*R3)-1.0/(21.0*R3))
      K(8 , 15)= K(8 , 15)-E*IYZ*(3.0/(28.0*P2*R4)-15.0/(14.0*P4*R4))
      K(8 , 16)= K(8 , 16)-E*IYZ*(1.0/(70.0*P2*R4)+27.0/(35.0*P4*R4))
      K(8 , 17)= K(8 , 17)-E*IYZ*(-1.0/(105.0*P2*R4)-2.0/(35.0*P4*R4))
      K(8 , 21)= K(8 , 21)-E*IYZ*(-1.0/(14.0*P2*R3)-29.0/(420.0*R3))
      K(8 , 22)= K(8 , 22)-E*IYZ*(1.0/(30.0*R3)-1.0/(35.0*P2*R3))
      K(9 , 10)= K(9 , 10)-E*IYZ*(1.0/(140.0*P2*R3)-5.0/(252.0*R3))
      K(9 , 11)= K(9 , 11)-E*IYZ*(1.0/(42.0*P2*R3)-1.0/(126.0*R3))
      K(9 , 15)= K(9 , 15)-E*IYZ*(1.0/(84.0*P2*R4)-3.0/(28.0*P4*R4))
      K(9 , 16)= K(9 , 16)-E*IYZ*(1.0/(105.0*P2*R4)+2.0/(35.0*P4*R4))
      K(9 , 17)= K(9 , 17)-E*IYZ*(1.0/(70.0*P4*R4)-1.0/(315.0*P2*R4))
      K(9 , 21)= K(9 , 21)-E*IYZ*(-1.0/(140.0*P2*R3)-17.0/(1260.0*R3))
      K(9 , 22)= K(9 , 22)-E*IYZ*(2.0/(315.0*P2*R3)-1.0/(105.0*P2*R3))
      K(10 , 18)= K(10 , 18)-E*IYZ*(-9.0/(28.0*P2*R3)-5.0/(42.0*R3))
      K(10 , 19)= K(10 , 19)-E*IYZ*(1.0/(14.0*P2*R3)+29.0/(420.0*R3))
      K(10 , 20)= K(10 , 20)-E*IYZ*(-1.0/(140.0*P2*R3)-17.0/(1260.0*R3))

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K(11 , 18) = K(11 , 18) - E* IYZ* (-1.0/(14.0*P2*R3) - 5.0/(84.0*R3))
K(11 , 19) = K(11 , 19) - E* IYZ* (1.0/(30.0*R3) - 1.0/(35.0*P2*R3))
K(11 , 20) = K(11 , 20) - E* IYZ* (1.0/(105.0*P2*R3) - 2.0/(315.0*R3))
K(15 , 18) = K(15 , 18) - E* IYZ* (15.0/(14.0*P4*R4) - 5.0/(14.0*P2*R4))
K(15 , 19) = K(15 , 19) - E* IYZ* (3.0/(28.0*P2*R4) - 15.0/(14.0*P4*R4))
K(15 , 20) = K(15 , 20) - E* IYZ* (3.0/(28.0*P4*R4) - 1.0/(84.0*P2*R4))
K(16 , 18) = K(16 , 18) - E* IYZ* (17.0/(28.0*P2*R4) - 15.0/(14.0*P4*R4))
K(16 , 19) = K(16 , 19) - E* IYZ* (48.0/(35.0*P4*R4) - 8.0/(35.0*P2*R4))
K(16 , 20) = K(16 , 20) - E* IYZ* (1.0/(30.0*P2*R4) - 11.0/(70.0*P4*R4))
K(17 , 18) = K(17 , 18) - E* IYZ* (3.0/(28.0*P4*R4) - 1.0/(84.0*P2*R4))
K(17 , 19) = K(17 , 19) - E* IYZ* (1.0/(30.0*P2*R4) - 11.0/(70.0*P4*R4))
K(17 , 20) = K(17 , 20) - E* IYZ* (3.0/(35.0*P4*R4) - 2.0/(315.0*P2*R4))
K(18 , 21) = K(18 , 21) - E* IYZ* (9.0/(28.0*P2*R3) - 8.0/(21.0*R3))
K(18 , 22) = K(18 , 22) - E* IYZ* (3.0/(28.0*R3) - 1.0/(14.0*P2*R3))
K(19 , 21) = K(19 , 21) - E* IYZ* (11.0/(84.0*R3) - 4.0/(7.0*P2*R3))
K(19 , 22) = K(19 , 22) - E* IYZ* (6.0/(35.0*P2*R3) - 1.0/(21.0*R3))
K(20 , 21) = K(20 , 21) - E* IYZ* (1.0/(140.0*P2*R3) - 5.0/(252.0*R3))
K(20 , 22) = K(20 , 22) - E* IYZ* (1.0/(126.0*R3) - 1.0/(42.0*P2*R3))
IYZ = IYZ/2.0
C*** (c48) FOR I:1 THRU 20 DO
C***   FOR J:I THRU 20 DO
C***     IF I<=3 OR I>=7 AND I<=9 OR I>=12 AND I<=14 OR I>=18 THEN
C***       IF J<=3 OR J>=7 AND J<=9 OR J>=12 AND J<=14 OR J>=18 THEN
C***         PRINT("      K(",I,",",J,") = K(",I,",",J,") + E*A*(",
C***           STRING(K(I,J)),")")$
K( 1 , 1 ) = K( 1 , 1 ) + E*A*( 5.0/(7.0*P2*R2) )
K( 1 , 2 ) = K( 1 , 2 ) + E*A*( 3.0/(14.0*P2*R2) )
K( 1 , 3 ) = K( 1 , 3 ) + E*A*( 1.0/(42.0*P2*R2) )
K( 1 , 7 ) = K( 1 , 7 ) + E*A*( 1.0/(2.0*P*R2) )
K( 1 , 8 ) = K( 1 , 8 ) + E*A*( 11.0/(42.0*P*R2) )
K( 1 , 9 ) = K( 1 , 9 ) + E*A*( 1.0/(21.0*P*R2) )
K( 1 , 12 ) = K( 1 , 12 ) + E*A*( -5.0/(7.0*P2*R2) )
K( 1 , 13 ) = K( 1 , 13 ) + E*A*( 3.0/(14.0*P2*R2) )
K( 1 , 14 ) = K( 1 , 14 ) + E*A*( -1.0/(42.0*P2*R2) )
K( 1 , 18 ) = K( 1 , 18 ) + E*A*( 1.0/(2.0*P*R2) )
K( 1 , 19 ) = K( 1 , 19 ) + E*A*( -11.0/(42.0*P*R2) )
K( 1 , 20 ) = K( 1 , 20 ) + E*A*( 1.0/(21.0*P*R2) )
K( 2 , 2 ) = K( 2 , 2 ) + E*A*( 16.0/(35.0*P2*R2) )
K( 2 , 3 ) = K( 2 , 3 ) + E*A*( 1.0/(15.0*P2*R2) )
K( 2 , 7 ) = K( 2 , 7 ) + E*A*( -11.0/(42.0*P*R2) )
K( 2 , 8 ) = K( 2 , 8 ) + ( 0 )
K( 2 , 9 ) = K( 2 , 9 ) + E*A*( 1.0/(126.0*P*R2) )
K( 2 , 12 ) = K( 2 , 12 ) + E*A*( -3.0/(14.0*P2*R2) )
K( 2 , 13 ) = K( 2 , 13 ) + E*A*( -1.0/(35.0*P2*R2) )
K( 2 , 14 ) = K( 2 , 14 ) + E*A*( 2.0/(105.0*P2*R2) )
K( 2 , 18 ) = K( 2 , 18 ) + E*A*( 11.0/(42.0*P*R2) )
K( 2 , 19 ) = K( 2 , 19 ) + E*A*( -13.0/(105.0*P*R2) )
K( 2 , 20 ) = K( 2 , 20 ) + E*A*( 13.0/(630.0*P*R2) )
K( 3 , 3 ) = K( 3 , 3 ) + E*A*( 4.0/(315.0*P2*R2) )
K( 3 , 7 ) = K( 3 , 7 ) + E*A*( -1.0/(21.0*P*R2) )
K( 3 , 8 ) = K( 3 , 8 ) + E*A*( -1.0/(126.0*P*R2) )
K( 3 , 9 ) = K( 3 , 9 ) + ( 0 )
K( 3 , 12 ) = K( 3 , 12 ) + E*A*( -1.0/(42.0*P2*R2) )

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K( 3 , 13 ) = K( 3 , 13 ) + E*A*( -2.0/(105.0*P2*R2) )
K( 3 , 14 ) = K( 3 , 14 ) + E*A*( 2.0/(315.0*P2*R2) )
K( 3 , 18 ) = K( 3 , 18 ) + E*A*( 1.0/(21.0*P*R2) )
K( 3 , 19 ) = K( 3 , 19 ) + E*A*( -13.0/(630.0*P*R2) )
K( 3 , 20 ) = K( 3 , 20 ) + E*A*( 1.0/(315.0*P*R2) )
K( 7 , 7 ) = K( 7 , 7 ) + E*A*( 181.0/(231.0*R2) )
K( 7 , 8 ) = K( 7 , 8 ) + E*A*( 311.0/(1155.0*R2) )
K( 7 , 9 ) = K( 7 , 9 ) + E*A*( 281.0/(6930.0*R2) )
K( 7 , 12 ) = K( 7 , 12 ) + E*A*( -1.0/(2.0*P*R2) )
K( 7 , 13 ) = K( 7 , 13 ) + E*A*( 11.0/(42.0*P*R2) )
K( 7 , 14 ) = K( 7 , 14 ) + E*A*( -1.0/(21.0*P*R2) )
K( 7 , 18 ) = K( 7 , 18 ) + E*A*( 50.0/(231.0*R2) )
K( 7 , 19 ) = K( 7 , 19 ) + E*A*( -151.0/(1155.0*R2) )
K( 7 , 20 ) = K( 7 , 20 ) + E*A*( 181.0/(6930.0*R2) )
K( 8 , 8 ) = K( 8 , 8 ) + E*A*( 416.0/(3465.0*R2) )
K( 8 , 9 ) = K( 8 , 9 ) + E*A*( 23.0/(1155.0*R2) )
K( 8 , 12 ) = K( 8 , 12 ) + E*A*( -11.0/(42.0*P*R2) )
K( 8 , 13 ) = K( 8 , 13 ) + E*A*( 13.0/(105.0*P*R2) )
K( 8 , 14 ) = K( 8 , 14 ) + E*A*( -13.0/(630.0*P*R2) )
K( 8 , 18 ) = K( 8 , 18 ) + E*A*( 151.0/(1155.0*R2) )
K( 8 , 19 ) = K( 8 , 19 ) + E*A*( -38.0/(495.0*R2) )
K( 8 , 20 ) = K( 8 , 20 ) + E*A*( 52.0/(3465.0*R2) )
K( 9 , 9 ) = K( 9 , 9 ) + E*A*( 4.0/(1155.0*R2) )
K( 9 , 12 ) = K( 9 , 12 ) + E*A*( -1.0/(21.0*P*R2) )
K( 9 , 13 ) = K( 9 , 13 ) + E*A*( 13.0/(630.0*P*R2) )
K( 9 , 14 ) = K( 9 , 14 ) + E*A*( -1.0/(315.0*P*R2) )
K( 9 , 18 ) = K( 9 , 18 ) + E*A*( 181.0/(6930.0*R2) )
K( 9 , 19 ) = K( 9 , 19 ) + E*A*( -52.0/(3465.0*R2) )
K( 9 , 20 ) = K( 9 , 20 ) + E*A*( 2.0/(693.0*R2) )
K( 12 , 12 ) = K( 12 , 12 ) + E*A*( 5.0/(7.0*P2*R2) )
K( 12 , 13 ) = K( 12 , 13 ) + E*A*( -3.0/(14.0*P2*R2) )
K( 12 , 14 ) = K( 12 , 14 ) + E*A*( 1.0/(42.0*P2*R2) )
K( 12 , 18 ) = K( 12 , 18 ) + E*A*( -1.0/(2.0*P*R2) )
K( 12 , 19 ) = K( 12 , 19 ) + E*A*( 11.0/(42.0*P*R2) )
K( 12 , 20 ) = K( 12 , 20 ) + E*A*( -1.0/(21.0*P*R2) )
K( 13 , 13 ) = K( 13 , 13 ) + E*A*( 16.0/(35.0*P2*R2) )
K( 13 , 14 ) = K( 13 , 14 ) + E*A*( -1.0/(15.0*P2*R2) )
K( 13 , 18 ) = K( 13 , 18 ) + E*A*( -11.0/(42.0*P*R2) )
K( 13 , 19 ) = K( 13 , 19 ) + ( 0 )
K( 13 , 20 ) = K( 13 , 20 ) + E*A*( 1.0/(126.0*P*R2) )
K( 14 , 14 ) = K( 14 , 14 ) + E*A*( 4.0/(315.0*P2*R2) )
K( 14 , 18 ) = K( 14 , 18 ) + E*A*( 1.0/(21.0*P*R2) )
K( 14 , 19 ) = K( 14 , 19 ) + E*A*( -1.0/(126.0*P*R2) )
K( 14 , 20 ) = K( 14 , 20 ) + ( 0 )
K( 18 , 18 ) = K( 18 , 18 ) + E*A*( 181.0/(231.0*R2) )
K( 18 , 19 ) = K( 18 , 19 ) + E*A*( -311.0/(1155.0*R2) )
K( 18 , 20 ) = K( 18 , 20 ) + E*A*( 281.0/(6930.0*R2) )
K( 19 , 19 ) = K( 19 , 19 ) + E*A*( 416.0/(3465.0*R2) )
K( 19 , 20 ) = K( 19 , 20 ) + E*A*( -23.0/(1155.0*R2) )
K( 20 , 20 ) = K( 20 , 20 ) + E*A*( 4.0/(1155.0*R2) )

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C*** (c62) FOR I:4 THRU 22 DO

C*** FOR J:I THRU 22 DO

C*** IF I<=6 OR I>=10 AND I<=11 OR I>=15 AND I<=17 OR I>=21 THEN

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C*** IF J<=6 OR J>=10 AND J<=11 OR J>=15 AND J<=17 OR J>=21 THEN
C*** PRINT(" K(",I,"",J,"") = K(",I,"",J,"")+G*C*(",
C*** STRING(K[I,J]),")")$
C*** TWIST STIFFNESS FOR HERMITIAN QUADRATIC ELEMENT
C***

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K( 4 , 4 ) = K( 4 , 4 ) + G*C*( 5.0/(7.0*R2) )/R2/P2
K( 4 , 5 ) = K( 4 , 5 ) + G*C*( 3.0/(14.0*R2) )/R2/P2
K( 4 , 6 ) = K( 4 , 6 ) + G*C*( 1.0/(42.0*R2) )/R2/P2
K( 4 , 10 ) = K( 4 , 10 ) + G*C*( 9.0/(14.0*R) )/R2/P2
K( 4 , 11 ) = K( 4 , 11 ) + G*C*( 1.0/(7.0*R) )/R2/P2
K( 4 , 15 ) = K( 4 , 15 ) + G*C*( -5.0/(7.0*R2) )/R2/P2
K( 4 , 16 ) = K( 4 , 16 ) + G*C*( 3.0/(14.0*R2) )/R2/P2
K( 4 , 17 ) = K( 4 , 17 ) + G*C*( -1.0/(42.0*R2) )/R2/P2
K( 4 , 21 ) = K( 4 , 21 ) + G*C*( -9.0/(14.0*R) )/R2/P2
K( 4 , 22 ) = K( 4 , 22 ) + G*C*( 1.0/(7.0*R) )/R2/P2
K( 5 , 5 ) = K( 5 , 5 ) + G*C*( 16.0/(35.0*R2) )/R2/P2
K( 5 , 6 ) = K( 5 , 6 ) + G*C*( 1.0/(15.0*R2) )/R2/P2
K( 5 , 10 ) = K( 5 , 10 ) + G*C*( 1.0/(7.0*R) )/R2/P2
K( 5 , 11 ) = K( 5 , 11 ) + G*C*( 12.0/(35.0*R) )/R2/P2
K( 5 , 15 ) = K( 5 , 15 ) + G*C*( -3.0/(14.0*R2) )/R2/P2
K( 5 , 16 ) = K( 5 , 16 ) + G*C*( -1.0/(35.0*R2) )/R2/P2
K( 5 , 17 ) = K( 5 , 17 ) + G*C*( 2.0/(105.0*R2) )/R2/P2
K( 5 , 21 ) = K( 5 , 21 ) + G*C*( -1.0/(7.0*R) )/R2/P2
K( 5 , 22 ) = K( 5 , 22 ) + G*C*( -2.0/(35.0*R) )/R2/P2
K( 6 , 6 ) = K( 6 , 6 ) + G*C*( 4.0/(315.0*R2) )/R2/P2
K( 6 , 10 ) = K( 6 , 10 ) + G*C*( 1.0/(70.0*R) )/R2/P2
K( 6 , 11 ) = K( 6 , 11 ) + G*C*( 1.0/(21.0*R) )/R2/P2
K( 6 , 15 ) = K( 6 , 15 ) + G*C*( -1.0/(42.0*R2) )/R2/P2
K( 6 , 16 ) = K( 6 , 16 ) + G*C*( -2.0/(105.0*R2) )/R2/P2
K( 6 , 17 ) = K( 6 , 17 ) + G*C*( 2.0/(315.0*R2) )/R2/P2
K( 6 , 21 ) = K( 6 , 21 ) + G*C*( -1.0/(70.0*R) )/R2/P2
K( 6 , 22 ) = K( 6 , 22 ) + G*C*( -2.0/(105.0*R) )/R2/P2
K( 10 , 10 ) = K( 10 , 10 ) + G*C*( 3.0/5.0 )/R2/P2
K( 10 , 11 ) = K( 10 , 11 ) + G*C*( 1.0/10.0 )/R2/P2
K( 10 , 15 ) = K( 10 , 15 ) + G*C*( -9.0/(14.0*R) )/R2/P2
K( 10 , 16 ) = K( 10 , 16 ) + G*C*( 1.0/(7.0*R) )/R2/P2
K( 10 , 17 ) = K( 10 , 17 ) + G*C*( -1.0/(70.0*R) )/R2/P2
K( 10 , 21 ) = K( 10 , 21 ) + G*C*( -3.0/5.0 )/R2/P2
K( 10 , 22 ) = K( 10 , 22 ) + G*C*( 1.0/10.0 )/R2/P2
K( 11 , 11 ) = K( 11 , 11 ) + G*C*( 4.0/15.0 )/R2/P2
K( 11 , 15 ) = K( 11 , 15 ) + G*C*( -1.0/(7.0*R) )/R2/P2
K( 11 , 16 ) = K( 11 , 16 ) + G*C*( -2.0/(35.0*R) )/R2/P2
K( 11 , 17 ) = K( 11 , 17 ) + G*C*( 2.0/(105.0*R) )/R2/P2
K( 11 , 21 ) = K( 11 , 21 ) + G*C*( -1.0/10.0 )/R2/P2
K( 11 , 22 ) = K( 11 , 22 ) + G*C*( -1.0/15.0 )/R2/P2
K( 15 , 15 ) = K( 15 , 15 ) + G*C*( 5.0/(7.0*R2) )/R2/P2
K( 15 , 16 ) = K( 15 , 16 ) + G*C*( -3.0/(14.0*R2) )/R2/P2
K( 15 , 17 ) = K( 15 , 17 ) + G*C*( 1.0/(42.0*R2) )/R2/P2
K( 15 , 21 ) = K( 15 , 21 ) + G*C*( 9.0/(14.0*R) )/R2/P2
K( 15 , 22 ) = K( 15 , 22 ) + G*C*( -1.0/(7.0*R) )/R2/P2
K( 16 , 16 ) = K( 16 , 16 ) + G*C*( 16.0/(35.0*R2) )/R2/P2
K( 16 , 17 ) = K( 16 , 17 ) + G*C*( -1.0/(15.0*R2) )/R2/P2
K( 16 , 21 ) = K( 16 , 21 ) + G*C*( -1.0/(7.0*R) )/R2/P2

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K( 16 , 22 )= K( 16 , 22 )+G*C*( 12.0/(35.0*R) )/R2/P2
K( 17 , 17 )= K( 17 , 17 )+G*C*( 4.0/(315.0*R2) )/R2/P2
K( 17 , 21 )= K( 17 , 21 )+G*C*( 1.0/(70.0*R) )/R2/P2
K( 17 , 22 )= K( 17 , 22 )+G*C*( -1.0/(21.0*R) )/R2/P2
K( 21 , 21 )= K( 21 , 21 )+G*C*( 3.0/5.0 )/R2/P2
K( 21 , 22 )= K( 21 , 22 )+G*C*( -1.0/10.0 )/R2/P2
K( 22 , 22 )= K( 22 , 22 )+G*C*( 4.0/15.0 )/R2/P2
L=2.0*R*ANGLE
V(1)=1.0
V(2)=0.5*L
V(3)=0.25*L*L
V(4)=1.0
V(5)=0.5*L
V(6)=0.25*L*L
V(7)=1.0
V(8)=0.5*L
V(9)=0.25*L*L
V(10)=1.0
V(11)=0.5*L
V(12)=1.0
V(13)=0.5*L
V(14)=0.25*L*L
V(15)=1.0
V(16)=0.5*L
V(17)=0.25*L*L
V(18)=1.0
V(19)=0.5*L
V(20)=0.25*L*L
V(21)=1.0
V(22)=0.5*L
DO 10 I=1,22
  DO 20 J=I,22
    K(I,J)=K(I,J)*R*ANGLE*V(I)*V(J)
20  CONTINUE
10  CONTINUE
RETURN
END

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```

SUBROUTINE ELMASS(IEL)
  REAL M(22,22)
  COMMON/TWO/NDFPN,NDFPE,NNPE
  COMMON/TP/NEL,NNODE,NBNODE
  COMMON/BLK1/NODE(10,2),ID(15,10),NBN(10)
  COMMON/TX/LM(30)
  COMMON/BLK2/S(465),EM(465)
  COMMON/STIFFM/M
  JJ=0
  DO 40 I=1,22
    DO 30 J=I,22
      JJ=JJ+1
30      EM(JJ)=M(I,J)
40      CONTINUE
  RETURN
  END

C*****
C***
C*** SUBROUTINE:    MASS  [CREATES A 22 DOF CURVED BEAM FINITE
C***                      ELEMENT]
C***
C*****

SUBROUTINE MASS(ANGLE)
  REAL M(22,22)
  REAL V(22),L
  REAL
  1,      A      , ASTAR , CW      , E      , G
  2,      IXX    , IYZ   , IYY    , IZZ    , KG
  2,      R      , RHO   , RS     , Y0     , Z0
  COMMON/SECTION/ A      , ASTAR , CW      , E      , G
  1,      IXX    , IYZ   , IYY    , IZZ    , KG
  2,      R      , RHO   , RS     , Y0     , Z0
  COMMON/STIFFM/M
C*** C=KG
  DO 8 I=1,22
    DO 5 J=I,22
      M(I,J)=0.0
5      CONTINUE
8      CONTINUE
  R2=R*R
C*** R3=R2*R
C*** R4=R3*R
  P=ANGLE
  P2=P*P
C*** P3=P2*P
C*** P4=P3*P
C*** (c36) FOR I:1 THRU 14 DO
C***   FOR J:I THRU 14 DO
C***     IF I<=3 OR I>=12 THEN
C***       IF J<=3 OR J>=12 THEN
C***         PRINT("      M('I','J')= M('I','J')+RHO*A*(",

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C***                               STRING(K[I,J]),")" )$
C***
C***      AXIAL MASS [U]
M( 1 , 1 )= M( 1 , 1 )+RHO*A*( 181.0/231.0 )
M( 1 , 2 )= M( 1 , 2 )+RHO*A*( 311.0/1155.0 )
M( 1 , 3 )= M( 1 , 3 )+RHO*A*( 281.0/6930.0 )
M( 1 , 12 )= M( 1 , 12 )+RHO*A*( 50.0/231.0 )
M( 1 , 13 )= M( 1 , 13 )+RHO*A*( -151.0/1155.0 )
M( 1 , 14 )= M( 1 , 14 )+RHO*A*( 181.0/6930.0 )
M( 2 , 2 )= M( 2 , 2 )+RHO*A*( 416.0/3465.0 )
M( 2 , 3 )= M( 2 , 3 )+RHO*A*( 23.0/1155.0 )
M( 2 , 12 )= M( 2 , 12 )+RHO*A*( 151.0/1155.0 )
M( 2 , 13 )= M( 2 , 13 )+RHO*A*( -38.0/495.0 )
M( 2 , 14 )= M( 2 , 14 )+RHO*A*( 52.0/3465.0 )
M( 3 , 3 )= M( 3 , 3 )+RHO*A*( 4.0/1155.0 )
M( 3 , 12 )= M( 3 , 12 )+RHO*A*( 181.0/6930.0 )
M( 3 , 13 )= M( 3 , 13 )+RHO*A*( -52.0/3465.0 )
M( 3 , 14 )= M( 3 , 14 )+RHO*A*( 2.0/693.0 )
M( 12 , 12 )= M( 12 , 12 )+RHO*A*( 181.0/231.0 )
M( 12 , 13 )= M( 12 , 13 )+RHO*A*( -311.0/1155.0 )
M( 12 , 14 )= M( 12 , 14 )+RHO*A*( 281.0/6930.0 )
M( 13 , 13 )= M( 13 , 13 )+RHO*A*( 416.0/3465.0 )
M( 13 , 14 )= M( 13 , 14 )+RHO*A*( -23.0/1155.0 )
M( 14 , 14 )= M( 14 , 14 )+RHO*A*( 4.0/1155.0 )
C*** (c38) FOR I:4 THRU 17 DO
C***   FOR J:I THRU 17 DO
C***     IF I<=6 OR I>=15 THEN
C***       IF J<=6 OR J>=15 THEN
C***         PRINT("      M(",I,",",J,")= M(",I,",",J,")+RHO*A*(",
C***           STRING(K[I,J]),")" )$
C***
C***      AXIAL INERTIA [V]
M( 4 , 4 )= M( 4 , 4 )+RHO*A*( 181.0/231.0 )
M( 4 , 5 )= M( 4 , 5 )+RHO*A*( 311.0/1155.0 )
M( 4 , 6 )= M( 4 , 6 )+RHO*A*( 281.0/6930.0 )
M( 4 , 15 )= M( 4 , 15 )+RHO*A*( 50.0/231.0 )
M( 4 , 16 )= M( 4 , 16 )+RHO*A*( -151.0/1155.0 )
M( 4 , 17 )= M( 4 , 17 )+RHO*A*( 181.0/6930.0 )
M( 5 , 5 )= M( 5 , 5 )+RHO*A*( 416.0/3465.0 )
M( 5 , 6 )= M( 5 , 6 )+RHO*A*( 23.0/1155.0 )
M( 5 , 15 )= M( 5 , 15 )+RHO*A*( 151.0/1155.0 )
M( 5 , 16 )= M( 5 , 16 )+RHO*A*( -38.0/495.0 )
M( 5 , 17 )= M( 5 , 17 )+RHO*A*( 52.0/3465.0 )
M( 6 , 6 )= M( 6 , 6 )+RHO*A*( 4.0/1155.0 )
M( 6 , 15 )= M( 6 , 15 )+RHO*A*( 181.0/6930.0 )
M( 6 , 16 )= M( 6 , 16 )+RHO*A*( -52.0/3465.0 )
M( 6 , 17 )= M( 6 , 17 )+RHO*A*( 2.0/693.0 )
M( 15 , 15 )= M( 15 , 15 )+RHO*A*( 181.0/231.0 )
M( 15 , 16 )= M( 15 , 16 )+RHO*A*( -311.0/1155.0 )
M( 15 , 17 )= M( 15 , 17 )+RHO*A*( 281.0/6930.0 )
M( 16 , 16 )= M( 16 , 16 )+RHO*A*( 416.0/3465.0 )
M( 16 , 17 )= M( 16 , 17 )+RHO*A*( -23.0/1155.0 )
M( 17 , 17 )= M( 17 , 17 )+RHO*A*( 4.0/1155.0 )

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C***
C*** (c39) FOR I:7 THRU 20 DO
C***   FOR J:I THRU 20 DO
C***     IF I<=9 OR I>=18 THEN
C***       IF J<=9 OR J>=18 THEN
C***         PRINT("      M(",I,"",",J,"")= M(",I,"",",J,"")+RHO*A*("
C***                                           STRING(K[I,J]),")" )$
C***
C***   AXIAL INERTIA [W]
C***     M( 7 , 7 )= M( 7 , 7 )+RHO*A*( 181.0/231.0 )
C***     M( 7 , 8 )= M( 7 , 8 )+RHO*A*( 311.0/1155.0 )
C***     M( 7 , 9 )= M( 7 , 9 )+RHO*A*( 281.0/6930.0 )
C***     M( 7 , 18 )= M( 7 , 18 )+RHO*A*( 50.0/231.0 )
C***     M( 7 , 19 )= M( 7 , 19 )+RHO*A*( -151.0/1155.0 )
C***     M( 7 , 20 )= M( 7 , 20 )+RHO*A*( 181.0/6930.0 )
C***     M( 8 , 8 )= M( 8 , 8 )+RHO*A*( 416.0/3465.0 )
C***     M( 8 , 9 )= M( 8 , 9 )+RHO*A*( 23.0/1155.0 )
C***     M( 8 , 18 )= M( 8 , 18 )+RHO*A*( 151.0/1155.0 )
C***     M( 8 , 19 )= M( 8 , 19 )+RHO*A*( -38.0/495.0 )
C***     M( 8 , 20 )= M( 8 , 20 )+RHO*A*( 52.0/3465.0 )
C***     M( 9 , 9 )= M( 9 , 9 )+RHO*A*( 4.0/1155.0 )
C***     M( 9 , 18 )= M( 9 , 18 )+RHO*A*( 181.0/6930.0 )
C***     M( 9 , 19 )= M( 9 , 19 )+RHO*A*( -52.0/3465.0 )
C***     M( 9 , 20 )= M( 9 , 20 )+RHO*A*( 2.0/693.0 )
C***     M( 18 , 18 )= M( 18 , 18 )+RHO*A*( 181.0/231.0 )
C***     M( 18 , 19 )= M( 18 , 19 )+RHO*A*( -311.0/1155.0 )
C***     M( 18 , 20 )= M( 18 , 20 )+RHO*A*( 281.0/6930.0 )
C***     M( 19 , 19 )= M( 19 , 19 )+RHO*A*( 416.0/3465.0 )
C***     M( 19 , 20 )= M( 19 , 20 )+RHO*A*( -23.0/1155.0 )
C***     M( 20 , 20 )= M( 20 , 20 )+RHO*A*( 4.0/1155.0 )
C*** (c94) FOR I:7 THRU 20 DO
C***   FOR J:I THRU 20 DO
C***     IF I<=9 OR I>=18 THEN
C***       IF J<=9 OR J>=18 THEN
C***         PRINT("      M(",I,"",",J,"")= M(",I,"",",J,"")+RHO*IYY*("
C***                                           STRING(K[I,J]),")" )$
C***
C***   INPLANE INERTIA
C***     M( 7 , 7 )= M( 7 , 7 )+RHO*IYY*( 5.0/7.0 )/R2/P2
C***     M( 7 , 8 )= M( 7 , 8 )+RHO*IYY*( 3.0/14.0 )/R2/P2
C***     M( 7 , 9 )= M( 7 , 9 )+RHO*IYY*( 1.0/42.0 )/R2/P2
C***     M( 7 , 18 )= M( 7 , 18 )+RHO*IYY*( -5.0/7.0 )/R2/P2
C***     M( 7 , 19 )= M( 7 , 19 )+RHO*IYY*( 3.0/14.0 )/R2/P2
C***     M( 7 , 20 )= M( 7 , 20 )+RHO*IYY*( -1.0/42.0 )/R2/P2
C***     M( 8 , 8 )= M( 8 , 8 )+RHO*IYY*( 16.0/35.0 )/R2/P2
C***     M( 8 , 9 )= M( 8 , 9 )+RHO*IYY*( 1.0/15.0 )/R2/P2
C***     M( 8 , 18 )= M( 8 , 18 )+RHO*IYY*( -3.0/14.0 )/R2/P2
C***     M( 8 , 19 )= M( 8 , 19 )+RHO*IYY*( -1.0/35.0 )/R2/P2
C***     M( 8 , 20 )= M( 8 , 20 )+RHO*IYY*( 2.0/105.0 )/R2/P2
C***     M( 9 , 9 )= M( 9 , 9 )+RHO*IYY*( 4.0/315.0 )/R2/P2
C***     M( 9 , 18 )= M( 9 , 18 )+RHO*IYY*( -1.0/42.0 )/R2/P2
C***     M( 9 , 19 )= M( 9 , 19 )+RHO*IYY*( -2.0/105.0 )/R2/P2
C***     M( 9 , 20 )= M( 9 , 20 )+RHO*IYY*( 2.0/315.0 )/R2/P2

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M( 18 , 18 )= M( 18 , 18 )+RHO*IYY*( 5.0/7.0 )/R2/P2
M( 18 , 19 )= M( 18 , 19 )+RHO*IYY*( -3.0/14.0 )/R2/P2
M( 18 , 20 )= M( 18 , 20 )+RHO*IYY*( 1.0/42.0 )/R2/P2
M( 19 , 19 )= M( 19 , 19 )+RHO*IYY*( 16.0/35.0 )/R2/P2
M( 19 , 20 )= M( 19 , 20 )+RHO*IYY*( -1.0/15.0 )/R2/P2
M( 20 , 20 )= M( 20 , 20 )+RHO*IYY*( 4.0/315.0 )/R2/P2
C*** IN-PLANE, 3*(U/R)**2
M( 1 , 1 )= M( 1 , 1 )+RHO*IYY*( 181.0/(77.0*R2) )
M( 1 , 2 )= M( 1 , 2 )+RHO*IYY*( 311.0/(385.0*R2) )
M( 1 , 3 )= M( 1 , 3 )+RHO*IYY*( 281.0/(2310.0*R2) )
M( 1 , 12 )= M( 1 , 12 )+RHO*IYY*( 50.0/(77.0*R2) )
M( 1 , 13 )= M( 1 , 13 )+RHO*IYY*( -151.0/(385.0*R2) )
M( 1 , 14 )= M( 1 , 14 )+RHO*IYY*( 181.0/(2310.0*R2) )
M( 2 , 2 )= M( 2 , 2 )+RHO*IYY*( 416.0/(1155.0*R2) )
M( 2 , 3 )= M( 2 , 3 )+RHO*IYY*( 23.0/(385.0*R2) )
M( 2 , 12 )= M( 2 , 12 )+RHO*IYY*( 151.0/(385.0*R2) )
M( 2 , 13 )= M( 2 , 13 )+RHO*IYY*( -38.0/(165.0*R2) )
M( 2 , 14 )= M( 2 , 14 )+RHO*IYY*( 52.0/(1155.0*R2) )
M( 3 , 3 )= M( 3 , 3 )+RHO*IYY*( 4.0/(385.0*R2) )
M( 3 , 12 )= M( 3 , 12 )+RHO*IYY*( 181.0/(2310.0*R2) )
M( 3 , 13 )= M( 3 , 13 )+RHO*IYY*( -52.0/(1155.0*R2) )
M( 3 , 14 )= M( 3 , 14 )+RHO*IYY*( 2.0/(231.0*R2) )
M( 12 , 12 )= M( 12 , 12 )+RHO*IYY*( 181.0/(77.0*R2) )
M( 12 , 13 )= M( 12 , 13 )+RHO*IYY*( -311.0/(385.0*R2) )
M( 12 , 14 )= M( 12 , 14 )+RHO*IYY*( 281.0/(2310.0*R2) )
M( 13 , 13 )= M( 13 , 13 )+RHO*IYY*( 416.0/(1155.0*R2) )
M( 13 , 14 )= M( 13 , 14 )+RHO*IYY*( -23.0/(385.0*R2) )
M( 14 , 14 )= M( 14 , 14 )+RHO*IYY*( 4.0/(385.0*R2) )
C*** IN-PLANE 4( )
C*** (c68) FOR I:1 THRU 20 DO
C***   FOR J:1 THRU 20 DO
C***     IF I<=3 OR I>=7 AND I<=9 OR I>=12 AND I<=14 OR I>=18 THEN
C***       IF J<=3 OR J>=7 AND J<=9 OR J>=12 AND J<=14 OR J>=18 THEN
C***         IF K[I,J] # 0 THEN
C***           PRINT("      M(",I,",",J,")= M(",I,",",J,")+RHO*IYY*(",
C***             STRING(K[I,J]),")")$
C***
M( 1 , 7 )= M( 1 , 7 )+RHO*IYY*( -1.0/(2.0*P*R2) )
M( 1 , 8 )= M( 1 , 8 )+RHO*IYY*( 11.0/(42.0*P*R2) )
M( 1 , 9 )= M( 1 , 9 )+RHO*IYY*( 1.0/(21.0*P*R2) )
M( 1 , 18 )= M( 1 , 18 )+RHO*IYY*( 1.0/(2.0*P*R2) )
M( 1 , 19 )= M( 1 , 19 )+RHO*IYY*( -11.0/(42.0*P*R2) )
M( 1 , 20 )= M( 1 , 20 )+RHO*IYY*( 1.0/(21.0*P*R2) )
M( 2 , 7 )= M( 2 , 7 )+RHO*IYY*( -11.0/(42.0*P*R2) )
M( 2 , 9 )= M( 2 , 9 )+RHO*IYY*( 1.0/(126.0*P*R2) )
M( 2 , 18 )= M( 2 , 18 )+RHO*IYY*( 11.0/(42.0*P*R2) )
M( 2 , 19 )= M( 2 , 19 )+RHO*IYY*( -13.0/(105.0*P*R2) )
M( 2 , 20 )= M( 2 , 20 )+RHO*IYY*( 13.0/(630.0*P*R2) )
M( 3 , 7 )= M( 3 , 7 )+RHO*IYY*( -1.0/(21.0*P*R2) )
M( 3 , 8 )= M( 3 , 8 )+RHO*IYY*( -1.0/(126.0*P*R2) )
M( 3 , 18 )= M( 3 , 18 )+RHO*IYY*( 1.0/(21.0*P*R2) )
M( 3 , 19 )= M( 3 , 19 )+RHO*IYY*( -13.0/(630.0*P*R2) )
M( 3 , 20 )= M( 3 , 20 )+RHO*IYY*( 1.0/(315.0*P*R2) )

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M( 7 , 12 ) = M( 7 , 12 ) + RHO*IYY*( -1.0/(2.0*P*R2) )
M( 7 , 13 ) = M( 7 , 13 ) + RHO*IYY*( 11.0/(42.0*P*R2) )
M( 7 , 14 ) = M( 7 , 14 ) + RHO*IYY*( -1.0/(21.0*P*R2) )
M( 8 , 12 ) = M( 8 , 12 ) + RHO*IYY*( -11.0/(42.0*P*R2) )
M( 8 , 13 ) = M( 8 , 13 ) + RHO*IYY*( 13.0/(105.0*P*R2) )
M( 8 , 14 ) = M( 8 , 14 ) + RHO*IYY*( -13.0/(630.0*P*R2) )
M( 9 , 12 ) = M( 9 , 12 ) + RHO*IYY*( -1.0/(21.0*P*R2) )
M( 9 , 13 ) = M( 9 , 13 ) + RHO*IYY*( 13.0/(630.0*P*R2) )
M( 9 , 14 ) = M( 9 , 14 ) + RHO*IYY*( -1.0/(315.0*P*R2) )
M( 12 , 18 ) = M( 12 , 18 ) + RHO*IYY*( 1.0/(2.0*P*R2) )
M( 12 , 19 ) = M( 12 , 19 ) + RHO*IYY*( 11.0/(42.0*P*R2) )
M( 12 , 20 ) = M( 12 , 20 ) + RHO*IYY*( -1.0/(21.0*P*R2) )
M( 13 , 18 ) = M( 13 , 18 ) + RHO*IYY*( -11.0/(42.0*P*R2) )
M( 13 , 20 ) = M( 13 , 20 ) + RHO*IYY*( 1.0/(126.0*P*R2) )
M( 14 , 18 ) = M( 14 , 18 ) + RHO*IYY*( 1.0/(21.0*P*R2) )
M( 14 , 19 ) = M( 14 , 19 ) + RHO*IYY*( -1.0/(126.0*P*R2) )

C*** INPLANE, [V*PHI]/R
C***
C*** (c83) FOR I:4 THRU 22 DO
C***   FOR J:I THRU 22 DO
C***     IF I<=6 OR I>=10 AND I<=11 OR I>=15 AND I<=17 OR I>=21 THEN
C***       IF J<=6 OR J>=10 AND J<=11 OR J>=15 AND J<=17 OR J>=21 THEN
C***         IF K[I,J] # 0 THEN
C***           PRINT("      M(",I,"",J,"") = M(",I,"",J,"") - RHO*IYY*(",
C***             STRING(K[I,J]),")") S
C***
C***   IYY=IYY*2.0
C***   M( 4 , 10 ) = M( 4 , 10 ) - RHO*IYY*( 8.0/(21.0*R) )
C***   M( 4 , 11 ) = M( 4 , 11 ) - RHO*IYY*( 3.0/(28.0*R) )
C***   M( 4 , 21 ) = M( 4 , 21 ) - RHO*IYY*( 5.0/(42.0*R) )
C***   M( 4 , 22 ) = M( 4 , 22 ) - RHO*IYY*( -5.0/(84.0*R) )
C***   M( 5 , 10 ) = M( 5 , 10 ) - RHO*IYY*( 11.0/(84.0*R) )
C***   M( 5 , 11 ) = M( 5 , 11 ) - RHO*IYY*( 1.0/(21.0*R) )
C***   M( 5 , 21 ) = M( 5 , 21 ) - RHO*IYY*( 29.0/(420.0*R) )
C***   M( 5 , 22 ) = M( 5 , 22 ) - RHO*IYY*( -1.0/(30.0*R) )
C***   M( 6 , 10 ) = M( 6 , 10 ) - RHO*IYY*( 5.0/(252.0*R) )
C***   M( 6 , 11 ) = M( 6 , 11 ) - RHO*IYY*( 1.0/(126.0*R) )
C***   M( 6 , 21 ) = M( 6 , 21 ) - RHO*IYY*( 17.0/(1260.0*R) )
C***   M( 6 , 22 ) = M( 6 , 22 ) - RHO*IYY*( -2.0/(315.0*R) )
C***   M( 10 , 15 ) = M( 10 , 15 ) - RHO*IYY*( 5.0/(42.0*R) )
C***   M( 10 , 16 ) = M( 10 , 16 ) - RHO*IYY*( -29.0/(420.0*R) )
C***   M( 10 , 17 ) = M( 10 , 17 ) - RHO*IYY*( 17.0/(1260.0*R) )
C***   M( 11 , 15 ) = M( 11 , 15 ) - RHO*IYY*( 5.0/(84.0*R) )
C***   M( 11 , 16 ) = M( 11 , 16 ) - RHO*IYY*( -1.0/(30.0*R) )
C***   M( 11 , 17 ) = M( 11 , 17 ) - RHO*IYY*( 2.0/(315.0*R) )
C***   M( 15 , 21 ) = M( 15 , 21 ) - RHO*IYY*( 8.0/(21.0*R) )
C***   M( 15 , 22 ) = M( 15 , 22 ) - RHO*IYY*( -3.0/(28.0*R) )
C***   M( 16 , 21 ) = M( 16 , 21 ) - RHO*IYY*( -11.0/(84.0*R) )
C***   M( 16 , 22 ) = M( 16 , 22 ) - RHO*IYY*( 1.0/(21.0*R) )
C***   M( 17 , 21 ) = M( 17 , 21 ) - RHO*IYY*( 5.0/(252.0*R) )
C***   M( 17 , 22 ) = M( 17 , 22 ) - RHO*IYY*( -1.0/(126.0*R) )
C***   IYY=IYY/2.0
C*** (c91) FOR I:4 THRU 17 DO

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C***   FOR J:1 THRU 17 DO
C***       IF I<=6 OR I>=15 THEN
C***       IF J<=6 OR J>=15 THEN
C***           PRINT("      M(",I,"","J,")= M(",I,"","J,")+RHO*IZZ*("
C***                                   STRING(K[I,J]),")" )$
C***
C***   OUT-PLANE INERTIA TERM
M( 4 , 4 )= M( 4 , 4 )+RHO*IZZ*( 5.0/7.0 )/R2/P2
M( 4 , 5 )= M( 4 , 5 )+RHO*IZZ*( 3.0/14.0 )/R2/P2
M( 4 , 6 )= M( 4 , 6 )+RHO*IZZ*( 1.0/42.0 )/R2/P2
M( 4 , 15 )= M( 4 , 15 )+RHO*IZZ*( -5.0/7.0 )/R2/P2
M( 4 , 16 )= M( 4 , 16 )+RHO*IZZ*( 3.0/14.0 )/R2/P2
M( 4 , 17 )= M( 4 , 17 )+RHO*IZZ*( -1.0/42.0 )/R2/P2
M( 5 , 5 )= M( 5 , 5 )+RHO*IZZ*( 16.0/35.0 )/R2/P2
M( 5 , 6 )= M( 5 , 6 )+RHO*IZZ*( 1.0/15.0 )/R2/P2
M( 5 , 15 )= M( 5 , 15 )+RHO*IZZ*( -3.0/14.0 )/R2/P2
M( 5 , 16 )= M( 5 , 16 )+RHO*IZZ*( -1.0/35.0 )/R2/P2
M( 5 , 17 )= M( 5 , 17 )+RHO*IZZ*( 2.0/105.0 )/R2/P2
M( 6 , 6 )= M( 6 , 6 )+RHO*IZZ*( 4.0/315.0 )/R2/P2
M( 6 , 15 )= M( 6 , 15 )+RHO*IZZ*( -1.0/42.0 )/R2/P2
M( 6 , 16 )= M( 6 , 16 )+RHO*IZZ*( -2.0/105.0 )/R2/P2
M( 6 , 17 )= M( 6 , 17 )+RHO*IZZ*( 2.0/315.0 )/R2/P2
M( 15 , 15 )= M( 15 , 15 )+RHO*IZZ*( 5.0/7.0 )/R2/P2
M( 15 , 16 )= M( 15 , 16 )+RHO*IZZ*( -3.0/14.0 )/R2/P2
M( 15 , 17 )= M( 15 , 17 )+RHO*IZZ*( 1.0/42.0 )/R2/P2
M( 16 , 16 )= M( 16 , 16 )+RHO*IZZ*( 16.0/35.0 )/R2/P2
M( 16 , 17 )= M( 16 , 17 )+RHO*IZZ*( -1.0/15.0 )/R2/P2
M( 17 , 17 )= M( 17 , 17 )+RHO*IZZ*( 4.0/315.0 )/R2/P2
C*** (c71) FOR I:10 THRU 22 DO
C***   FOR J:1 THRU 22 DO
C***       IF I<=11 OR I>=21 THEN
C***       IF J<=11 OR J>=21 THEN
C***           PRINT("      M(",I,"","J,")= M(",I,"","J,")+RHO*(IYY+IZZ)*("
C***                                   STRING(K[I,J]),")" )$
C***
C***   TWIST INERTIA
M( 10 , 10 )= M( 10 , 10 )+RHO*(IYY+IZZ)*( 26.0/35.0 )
M( 10 , 11 )= M( 10 , 11 )+RHO*(IYY+IZZ)*( 22.0/105.0 )
M( 10 , 21 )= M( 10 , 21 )+RHO*(IYY+IZZ)*( 9.0/35.0 )
M( 10 , 22 )= M( 10 , 22 )+RHO*(IYY+IZZ)*( -13.0/105.0 )
M( 11 , 11 )= M( 11 , 11 )+RHO*(IYY+IZZ)*( 8.0/105.0 )
M( 11 , 21 )= M( 11 , 21 )+RHO*(IYY+IZZ)*( 13.0/105.0 )
M( 11 , 22 )= M( 11 , 22 )+RHO*(IYY+IZZ)*( -2.0/35.0 )
M( 21 , 21 )= M( 21 , 21 )+RHO*(IYY+IZZ)*( 26.0/35.0 )
M( 21 , 22 )= M( 21 , 22 )+RHO*(IYY+IZZ)*( -22.0/105.0 )
M( 22 , 22 )= M( 22 , 22 )+RHO*(IYY+IZZ)*( 8.0/105.0 )
C***   COUPLING -(PRIME[V]*PRIME[W])
C***
C*** (c102) FOR I:4 THRU 20 DO
C***   FOR J:1 THRU 20 DO
C***       IF I<=9 OR I>=15 THEN
C***       IF J<=9 OR J>=15 THEN
C***           IF K[I,J] # 0 THEN

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C***PRINT("      M(",I,"",",J,"")= M(",I,"",",J,"")-RHO* IYZ*(" ,
C***      STRING(K[I,J]),",")")$
C***
      IYZ=2.0* IYZ
      M( 4 , 7 )= M( 4 , 7 )-RHO* IYZ*( 5.0/(14.0*P2*R2) )
      M( 4 , 8 )= M( 4 , 8 )-RHO* IYZ*( 3.0/(28.0*P2*R2) )
      M( 4 , 9 )= M( 4 , 9 )-RHO* IYZ*( 1.0/(84.0*P2*R2) )
      M( 4 , 18 )= M( 4 , 18 )-RHO* IYZ*( -5.0/(14.0*P2*R2) )
      M( 4 , 19 )= M( 4 , 19 )-RHO* IYZ*( 3.0/(28.0*P2*R2) )
      M( 4 , 20 )= M( 4 , 20 )-RHO* IYZ*( -1.0/(84.0*P2*R2) )
      M( 5 , 7 )= M( 5 , 7 )-RHO* IYZ*( 3.0/(28.0*P2*R2) )
      M( 5 , 8 )= M( 5 , 8 )-RHO* IYZ*( 8.0/(35.0*P2*R2) )
      M( 5 , 9 )= M( 5 , 9 )-RHO* IYZ*( 1.0/(30.0*P2*R2) )
      M( 5 , 18 )= M( 5 , 18 )-RHO* IYZ*( -3.0/(28.0*P2*R2) )
      M( 5 , 19 )= M( 5 , 19 )-RHO* IYZ*( -1.0/(70.0*P2*R2) )
      M( 5 , 20 )= M( 5 , 20 )-RHO* IYZ*( 1.0/(105.0*P2*R2) )
      M( 6 , 7 )= M( 6 , 7 )-RHO* IYZ*( 1.0/(84.0*P2*R2) )
      M( 6 , 8 )= M( 6 , 8 )-RHO* IYZ*( 1.0/(30.0*P2*R2) )
      M( 6 , 9 )= M( 6 , 9 )-RHO* IYZ*( 2.0/(315.0*P2*R2) )
      M( 6 , 18 )= M( 6 , 18 )-RHO* IYZ*( -1.0/(84.0*P2*R2) )
      M( 6 , 19 )= M( 6 , 19 )-RHO* IYZ*( -1.0/(105.0*P2*R2) )
      M( 6 , 20 )= M( 6 , 20 )-RHO* IYZ*( 1.0/(315.0*P2*R2) )
      M( 7 , 15 )= M( 7 , 15 )-RHO* IYZ*( 5.0/(14.0*P2*R2) )
      M( 7 , 16 )= M( 7 , 16 )-RHO* IYZ*( 3.0/(28.0*P2*R2) )
      M( 7 , 17 )= M( 7 , 17 )-RHO* IYZ*( 1.0/(84.0*P2*R2) )
      M( 8 , 15 )= M( 8 , 15 )-RHO* IYZ*( 3.0/(28.0*P2*R2) )
      M( 8 , 16 )= M( 8 , 16 )-RHO* IYZ*( 8.0/(35.0*P2*R2) )
      M( 8 , 17 )= M( 8 , 17 )-RHO* IYZ*( 1.0/(30.0*P2*R2) )
      M( 9 , 15 )= M( 9 , 15 )-RHO* IYZ*( 1.0/(84.0*P2*R2) )
      M( 9 , 16 )= M( 9 , 16 )-RHO* IYZ*( 1.0/(30.0*P2*R2) )
      M( 9 , 17 )= M( 9 , 17 )-RHO* IYZ*( 2.0/(315.0*P2*R2) )
      M( 15 , 18 )= M( 15 , 18 )-RHO* IYZ*( -5.0/(14.0*P2*R2) )
      M( 15 , 19 )= M( 15 , 19 )-RHO* IYZ*( 3.0/(28.0*P2*R2) )
      M( 15 , 20 )= M( 15 , 20 )-RHO* IYZ*( -1.0/(84.0*P2*R2) )
      M( 16 , 18 )= M( 16 , 18 )-RHO* IYZ*( -3.0/(28.0*P2*R2) )
      M( 16 , 19 )= M( 16 , 19 )-RHO* IYZ*( -1.0/(70.0*P2*R2) )
      M( 16 , 20 )= M( 16 , 20 )-RHO* IYZ*( -1.0/(105.0*P2*R2) )
      M( 17 , 18 )= M( 17 , 18 )-RHO* IYZ*( -1.0/(84.0*P2*R2) )
      M( 17 , 19 )= M( 17 , 19 )-RHO* IYZ*( -1.0/(105.0*P2*R2) )
      M( 17 , 20 )= M( 17 , 20 )-RHO* IYZ*( 1.0/(315.0*P2*R2) )
      IYZ= IYZ/2.0
C*** COUPLING 4*[PRIME(V)*U/R]
C***
C*** (c103) FOR I:1 THRU 17 DO
C***   FOR J:I THRU 17 DO
C***     IF I<=6 OR I>=12 THEN
C***       IF J<=6 OR J>=12 THEN
C***         K[I,J]:=INTEGRATE( N9[I,1]*N4[J,1]+N9[J,1]*N4[I,1],T,-1,1)
C***       ELSE K[I,J]:=0 $
C***
C*** (c106) FOR I:1 THRU 17 DO
C***   FOR J:I THRU 17 DO
C***     IF I<=6 OR I>=12 THEN

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C***      IF J<=6 OR J>=12 THEN
C***      IF K[I,J] # 0 THEN
C***      PRINT("      M(",I,"",",J,"")= M(",I,"",",J,"")-RHO*IYZ*(",
C***      STRING(K[I,J]),",")") $
C***

M( 1 , 4 )= M( 1 , 4 )-RHO*IYZ*( -1.0/(2.0*P*R2) )
M( 1 , 5 )= M( 1 , 5 )-RHO*IYZ*( 11.0/(42.0*P*R2) )
M( 1 , 6 )= M( 1 , 6 )-RHO*IYZ*( 1.0/(21.0*P*R2) )
M( 1 , 15 )= M( 1 , 15 )-RHO*IYZ*( -1.0/(2.0*P*R2) )
M( 1 , 16 )= M( 1 , 16 )-RHO*IYZ*( 11.0/(42.0*P*R2) )
M( 1 , 17 )= M( 1 , 17 )-RHO*IYZ*( 1.0/(21.0*P*R2) )
M( 2 , 4 )= M( 2 , 4 )-RHO*IYZ*( -11.0/(42.0*P*R2) )
M( 2 , 6 )= M( 2 , 6 )-RHO*IYZ*( 1.0/(126.0*P*R2) )
M( 2 , 15 )= M( 2 , 15 )-RHO*IYZ*( -11.0/(42.0*P*R2) )
M( 2 , 17 )= M( 2 , 17 )-RHO*IYZ*( 1.0/(126.0*P*R2) )
M( 3 , 4 )= M( 3 , 4 )-RHO*IYZ*( -1.0/(21.0*P*R2) )
M( 3 , 5 )= M( 3 , 5 )-RHO*IYZ*( -1.0/(126.0*P*R2) )
M( 3 , 15 )= M( 3 , 15 )-RHO*IYZ*( -1.0/(21.0*P*R2) )
M( 3 , 16 )= M( 3 , 16 )-RHO*IYZ*( -1.0/(126.0*P*R2) )
M( 4 , 12 )= M( 4 , 12 )-RHO*IYZ*( -1.0/(2.0*P*R2) )
M( 4 , 13 )= M( 4 , 13 )-RHO*IYZ*( 11.0/(42.0*P*R2) )
M( 4 , 14 )= M( 4 , 14 )-RHO*IYZ*( -1.0/(21.0*P*R2) )
M( 5 , 12 )= M( 5 , 12 )-RHO*IYZ*( -11.0/(42.0*P*R2) )
M( 5 , 13 )= M( 5 , 13 )-RHO*IYZ*( 13.0/(105.0*P*R2) )
M( 5 , 14 )= M( 5 , 14 )-RHO*IYZ*( -13.0/(630.0*P*R2) )
M( 6 , 12 )= M( 6 , 12 )-RHO*IYZ*( -1.0/(21.0*P*R2) )
M( 6 , 13 )= M( 6 , 13 )-RHO*IYZ*( 13.0/(630.0*P*R2) )
M( 6 , 14 )= M( 6 , 14 )-RHO*IYZ*( -1.0/(315.0*P*R2) )
M( 12 , 15 )= M( 12 , 15 )-RHO*IYZ*( -1.0/(2.0*P*R2) )
M( 12 , 16 )= M( 12 , 16 )-RHO*IYZ*( -11.0/(42.0*P*R2) )
M( 12 , 17 )= M( 12 , 17 )-RHO*IYZ*( -1.0/(21.0*P*R2) )
M( 13 , 15 )= M( 13 , 15 )-RHO*IYZ*( 11.0/(42.0*P*R2) )
M( 13 , 16 )= M( 13 , 16 )-RHO*IYZ*( 13.0/(105.0*P*R2) )
M( 13 , 17 )= M( 13 , 17 )-RHO*IYZ*( 13.0/(630.0*P*R2) )
M( 14 , 15 )= M( 14 , 15 )-RHO*IYZ*( -1.0/(21.0*P*R2) )
M( 14 , 16 )= M( 14 , 16 )-RHO*IYZ*( -13.0/(630.0*P*R2) )
M( 14 , 17 )= M( 14 , 17 )-RHO*IYZ*( -1.0/(315.0*P*R2) )

C***      COUPLING [W]*[PHI]/R
C***      (c95) FOR I:7 THRU 22 DO
C***      FOR J:I THRU 22 DO
C***      IF I<=11 OR I>=18 THEN
C***      IF J<=11 OR J>=18 THEN
C***      IF K[I,J] # 0 THEN
C***      PRINT("      M(",I,"",",J,"")= M(",I,"",",J,"")-RHO*IYZ*(",
C***      STRING(K[I,J]),",")") $
C***      IYZ=2.0*IYZ
M( 7 , 10 )= M( 7 , 10 )-RHO*IYZ*( 8.0/(21.0*R) )
M( 7 , 11 )= M( 7 , 11 )-RHO*IYZ*( 3.0/(28.0*R) )
M( 7 , 21 )= M( 7 , 21 )-RHO*IYZ*( 5.0/(42.0*R) )
M( 7 , 22 )= M( 7 , 22 )-RHO*IYZ*( -5.0/(84.0*R) )
M( 8 , 10 )= M( 8 , 10 )-RHO*IYZ*( 11.0/(84.0*R) )
M( 8 , 11 )= M( 8 , 11 )-RHO*IYZ*( 1.0/(21.0*R) )
M( 8 , 21 )= M( 8 , 21 )-RHO*IYZ*( 29.0/(420.0*R) )

```

```

M( 8 , 22 ) = M( 8 , 22 ) - RHO* IYZ*( -1.0/(30.0*R) )
M( 9 , 10 ) = M( 9 , 10 ) - RHO* IYZ*( 5.0/(252.0*R) )
M( 9 , 11 ) = M( 9 , 11 ) - RHO* IYZ*( 1.0/(126.0*R) )
M( 9 , 21 ) = M( 9 , 21 ) - RHO* IYZ*( 17.0/(1260.0*R) )
M( 9 , 22 ) = M( 9 , 22 ) - RHO* IYZ*( -2.0/(315.0*R) )
M( 10 , 18 ) = M( 10 , 18 ) - RHO* IYZ*( 5.0/(42.0*R) )
M( 10 , 19 ) = M( 10 , 19 ) - RHO* IYZ*( -29.0/(420.0*R) )
M( 10 , 20 ) = M( 10 , 20 ) - RHO* IYZ*( 17.0/(1260.0*R) )
M( 11 , 18 ) = M( 11 , 18 ) - RHO* IYZ*( 5.0/(84.0*R) )
M( 11 , 19 ) = M( 11 , 19 ) - RHO* IYZ*( -1.0/(30.0*R) )
M( 11 , 20 ) = M( 11 , 20 ) - RHO* IYZ*( 2.0/(315.0*R) )
M( 18 , 21 ) = M( 18 , 21 ) - RHO* IYZ*( 8.0/(21.0*R) )
M( 18 , 22 ) = M( 18 , 22 ) - RHO* IYZ*( -3.0/(28.0*R) )
M( 19 , 21 ) = M( 19 , 21 ) - RHO* IYZ*( -11.0/(84.0*R) )
M( 19 , 22 ) = M( 19 , 22 ) - RHO* IYZ*( 1.0/(21.0*R) )
M( 20 , 21 ) = M( 20 , 21 ) - RHO* IYZ*( 5.0/(252.0*R) )
M( 20 , 22 ) = M( 20 , 22 ) - RHO* IYZ*( -1.0/(126.0*R) )
IYZ=IYZ/2.0
L=2.0*R*ANGLE
V(1)=1.0
V(2)=0.5*L
V(3)=0.25*L*L
V(4)=1.0
V(5)=0.5*L
V(6)=0.25*L*L
V(7)=1.0
V(8)=0.5*L
V(9)=0.25*L*L
V(10)=1.0
V(11)=0.5*L
V(12)=1.0
V(13)=0.5*L
V(14)=0.25*L*L
V(15)=1.0
V(16)=0.5*L
V(17)=0.25*L*L
V(18)=1.0
V(19)=0.5*L
V(20)=0.25*L*L
V(21)=1.0
V(22)=0.5*L
DO 10 I=1,22
  DO 20 J=I,22
    M(I,J)=M(I,J)*R*ANGLE*V(I)*V(J)
  20 CONTINUE
10 CONTINUE
RETURN
END

```

Appendix B

30 DOF Stiffness and Mass Matrices

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SUBROUTINE ELSTIF(IEL)
  REAL K(30,30)
  COMMON/TWO/NDFPN,NDFPE,NNPE
  COMMON/TP/NEL,NNODE,NBNODE
  COMMON/BLK1/NODE(10,2),ID(15,10),NBN(10)
  COMMON/TX/LM(30)
  COMMON/BLK2/S(465),EM(465)
  COMMON/STIFFK/K
  JJ=0
  DO 40 I=1,30
    DO 30 J=I,30
      JJ=JJ+1
30      S(JJ)=K(I,J)
40      CONTINUE
  RETURN
END

C*****
C***
C*** SUBROUTINE: STIFF [CREATES A 30 DOF CURVED BEAM FINITE
C*** ELEMENT]
C***
C*****

SUBROUTINE STIFF(ANGLE)
  REAL K(30,30)
  REAL V(30),L
  REAL
  1, A , ASTAR , CW , E , G
  2, IXX , IYZ , IYY , IZZ , KG
  2, R , RHO , RS , Y0 , Z0
  COMMON/SECTION/ A , ASTAR , CW , E , G
  1, IXX , IYZ , IYY , IZZ , KG
  2, R , RHO , RS , Y0 , Z0
  COMMON/STIFFK/K
  GC=G*KG
  GA=G*ASTAR
  DO 8 I=1,30
    DO 5 J=I,30
      K(I,J)=0.0
5      CONTINUE
8      CONTINUE
  R2=R*R
  R3=R2*R
  R4=R3*R
  P=ANGLE
  P2=P*P
C*** P3=P2*P
C***
C*** OUTPLANE STIFFNESS [ PRIME(THETA)-PHI/R ]**2.0
C***
C*** (c38) FOR I:7 THRU 30 DO
C*** FOR J:I THRU 30 DO

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```

C*** IF I>=7 AND I<=8 OR I>=14 AND I<=15 OR I>=22 AND I<=23 OR
C*** I>=29 AND I<=30 THEN
C*** IF J>=7 AND J<=8 OR J>=14 AND J<=15 OR J>=22 AND J<=23 OR
C*** J>=29 AND J<=30 THEN
C*** PRINT("      K(",I,"",J,"")= K(",I,"",J,"")+ E*IZZ*("
C***      STRING(K[I,J]),")" )$
C***
K( 7 , 7 )= K( 7 , 7 )+ E*IZZ*( 3.0/(5.0*P*R2) )
K( 7 , 8 )= K( 7 , 8 )+ E*IZZ*( 1.0/(10.0*P*R2) )
K( 7 , 14 )= K( 7 , 14 )+ E*IZZ*( 1.0/(2.0*P*R2) )
K( 7 , 15 )= K( 7 , 15 )+ E*IZZ*( 1.0/(5.0*P*R2) )
K( 7 , 22 )= K( 7 , 22 )+ E*IZZ*( -3.0/(5.0*P*R2) )
K( 7 , 23 )= K( 7 , 23 )+ E*IZZ*( 1.0/(10.0*P*R2) )
K( 7 , 29 )= K( 7 , 29 )+ E*IZZ*( 1.0/(2.0*P*R2) )
K( 7 , 30 )= K( 7 , 30 )+ E*IZZ*( -1.0/(5.0*P*R2) )
K( 8 , 8 )= K( 8 , 8 )+ E*IZZ*( 4.0/(15.0*P*R2) )
K( 8 , 14 )= K( 8 , 14 )+ E*IZZ*( -1.0/(5.0*P*R2) )
K( 8 , 15 )= K( 8 , 15 )+ E*IZZ*( 0.0 )
K( 8 , 22 )= K( 8 , 22 )+ E*IZZ*( -1.0/(10.0*P*R2) )
K( 8 , 23 )= K( 8 , 23 )+ E*IZZ*( -1.0/(15.0*P*R2) )
K( 8 , 29 )= K( 8 , 29 )+ E*IZZ*( 1.0/(5.0*P*R2) )
K( 8 , 30 )= K( 8 , 30 )+ E*IZZ*( -1.0/(15.0*P*R2) )
K( 14 , 14 )= K( 14 , 14 )+ E*IZZ*( 26.0/(35.0*R2) )
K( 14 , 15 )= K( 14 , 15 )+ E*IZZ*( 22.0/(105.0*R2) )
K( 14 , 22 )= K( 14 , 22 )+ E*IZZ*( -1.0/(2.0*P*R2) )
K( 14 , 23 )= K( 14 , 23 )+ E*IZZ*( 1.0/(5.0*P*R2) )
K( 14 , 29 )= K( 14 , 29 )+ E*IZZ*( 9.0/(35.0*R2) )
K( 14 , 30 )= K( 14 , 30 )+ E*IZZ*( -13.0/(105.0*R2) )
K( 15 , 15 )= K( 15 , 15 )+ E*IZZ*( 8.0/(105.0*R2) )
K( 15 , 22 )= K( 15 , 22 )+ E*IZZ*( -1.0/(5.0*P*R2) )
K( 15 , 23 )= K( 15 , 23 )+ E*IZZ*( 1.0/(15.0*P*R2) )
K( 15 , 29 )= K( 15 , 29 )+ E*IZZ*( 13.0/(105.0*R2) )
K( 15 , 30 )= K( 15 , 30 )+ E*IZZ*( -2.0/(35.0*R2) )
K( 22 , 22 )= K( 22 , 22 )+ E*IZZ*( 3.0/(5.0*P*R2) )
K( 22 , 23 )= K( 22 , 23 )+ E*IZZ*( -1.0/(10.0*P*R2) )
K( 22 , 29 )= K( 22 , 29 )+ E*IZZ*( -1.0/(2.0*P*R2) )
K( 22 , 30 )= K( 22 , 30 )+ E*IZZ*( 1.0/(5.0*P*R2) )
K( 23 , 23 )= K( 23 , 23 )+ E*IZZ*( 4.0/(15.0*P*R2) )
K( 23 , 29 )= K( 23 , 29 )+ E*IZZ*( -1.0/(5.0*P*R2) )
K( 23 , 30 )= K( 23 , 30 )+ E*IZZ*( 0.0 )
K( 29 , 29 )= K( 29 , 29 )+ E*IZZ*( 26.0/(35.0*R2) )
K( 29 , 30 )= K( 29 , 30 )+ E*IZZ*( -22.0/(105.0*R2) )
K( 30 , 30 )= K( 30 , 30 )+ E*IZZ*( 8.0/(105.0*R2) )

C***
C*** INPLANE STIFFNESS IYY*[ PRIME(Psi)+W/R/R ]**2.0
C***
C*** (c42) FOR I:9 THRU 28 DO
C***   FOR J:I THRU 28 DO
C***     IF I<=13 OR I>=24 AND I<=28 THEN
C***       IF J<=13 OR J>=24 AND J<=28 THEN
C***     PRINT("      K(",I,"",J,"")= K(",I,"",J,"")+ E*IIY*("
C***       STRING(K[I,J]),")" )$
C***

```

```

K( 9 , 9 ) = K( 9 , 9 ) + E*IIY*( 181.0/(231.0*R4) )
K( 9 , 10 ) = K( 9 , 10 ) + E*IIY*( 311.0/(1155.0*R4) )
K( 9 , 11 ) = K( 9 , 11 ) + E*IIY*( 281.0/(6930.0*R4) )
K( 9 , 12 ) = K( 9 , 12 ) + E*IIY*( -1.0/(2.0*P*R3) )
K( 9 , 13 ) = K( 9 , 13 ) + E*IIY*( 3.0/(14.0*P*R3) )
K( 9 , 24 ) = K( 9 , 24 ) + E*IIY*( 50.0/(231.0*R4) )
K( 9 , 25 ) = K( 9 , 25 ) + E*IIY*( -151.0/(1155.0*R4) )
K( 9 , 26 ) = K( 9 , 26 ) + E*IIY*( 181.0/(6930.0*R4) )
K( 9 , 27 ) = K( 9 , 27 ) + E*IIY*( 1.0/(2.0*P*R3) )
K( 9 , 28 ) = K( 9 , 28 ) + E*IIY*( -3.0/(14.0*P*R3) )
K( 10 , 10 ) = K( 10 , 10 ) + E*IIY*( 416.0/(3465.0*R4) )
K( 10 , 11 ) = K( 10 , 11 ) + E*IIY*( 23.0/(1155.0*R4) )
K( 10 , 12 ) = K( 10 , 12 ) + E*IIY*( -17.0/(70.0*P*R3) )
K( 10 , 13 ) = K( 10 , 13 ) + E*IIY*( 1.0/(210.0*P*R3) )
K( 10 , 24 ) = K( 10 , 24 ) + E*IIY*( 151.0/(1155.0*R4) )
K( 10 , 25 ) = K( 10 , 25 ) + E*IIY*( -38.0/(495.0*R4) )
K( 10 , 26 ) = K( 10 , 26 ) + E*IIY*( 52.0/(3465.0*R4) )
K( 10 , 27 ) = K( 10 , 27 ) + E*IIY*( 17.0/(70.0*P*R3) )
K( 10 , 28 ) = K( 10 , 28 ) + E*IIY*( -19.0/(210.0*P*R3) )
K( 11 , 11 ) = K( 11 , 11 ) + E*IIY*( 4.0/(1155.0*R4) )
K( 11 , 12 ) = K( 11 , 12 ) + E*IIY*( -3.0/(70.0*P*R3) )
K( 11 , 13 ) = K( 11 , 13 ) + E*IIY*( -1.0/(210.0*P*R3) )
K( 11 , 24 ) = K( 11 , 24 ) + E*IIY*( 181.0/(6930.0*R4) )
K( 11 , 25 ) = K( 11 , 25 ) + E*IIY*( -52.0/(3465.0*R4) )
K( 11 , 26 ) = K( 11 , 26 ) + E*IIY*( 2.0/(693.0*R4) )
K( 11 , 27 ) = K( 11 , 27 ) + E*IIY*( 3.0/(70.0*P*R3) )
K( 11 , 28 ) = K( 11 , 28 ) + E*IIY*( -1.0/(70.0*P*R3) )
K( 12 , 12 ) = K( 12 , 12 ) + E*IIY*( 3.0/(5.0*P2*R2) )
K( 12 , 13 ) = K( 12 , 13 ) + E*IIY*( 1.0/(10.0*P2*R2) )
K( 12 , 24 ) = K( 12 , 24 ) + E*IIY*( -1.0/(2.0*P*R3) )
K( 12 , 25 ) = K( 12 , 25 ) + E*IIY*( 17.0/(70.0*P*R3) )
K( 12 , 26 ) = K( 12 , 26 ) + E*IIY*( -3.0/(70.0*P*R3) )
K( 12 , 27 ) = K( 12 , 27 ) + E*IIY*( -3.0/(5.0*P2*R2) )
K( 12 , 28 ) = K( 12 , 28 ) + E*IIY*( 1.0/(10.0*P2*R2) )
K( 13 , 13 ) = K( 13 , 13 ) + E*IIY*( 4.0/(15.0*P2*R2) )
K( 13 , 24 ) = K( 13 , 24 ) + E*IIY*( -3.0/(14.0*P*R3) )
K( 13 , 25 ) = K( 13 , 25 ) + E*IIY*( 19.0/(210.0*P*R3) )
K( 13 , 26 ) = K( 13 , 26 ) + E*IIY*( -1.0/(70.0*P*R3) )
K( 13 , 27 ) = K( 13 , 27 ) + E*IIY*( -1.0/(10.0*P2*R2) )
K( 13 , 28 ) = K( 13 , 28 ) + E*IIY*( -1.0/(15.0*P2*R2) )
K( 24 , 24 ) = K( 24 , 24 ) + E*IIY*( 181.0/(231.0*R4) )
K( 24 , 25 ) = K( 24 , 25 ) + E*IIY*( -311.0/(1155.0*R4) )
K( 24 , 26 ) = K( 24 , 26 ) + E*IIY*( 281.0/(6930.0*R4) )
K( 24 , 27 ) = K( 24 , 27 ) + E*IIY*( 1.0/(2.0*P*R3) )
K( 24 , 28 ) = K( 24 , 28 ) + E*IIY*( 3.0/(14.0*P*R3) )
K( 25 , 25 ) = K( 25 , 25 ) + E*IIY*( 416.0/(3465.0*R4) )
K( 25 , 26 ) = K( 25 , 26 ) + E*IIY*( -23.0/(1155.0*R4) )
K( 25 , 27 ) = K( 25 , 27 ) + E*IIY*( -17.0/(70.0*P*R3) )
K( 25 , 28 ) = K( 25 , 28 ) + E*IIY*( -1.0/(210.0*P*R3) )
K( 26 , 26 ) = K( 26 , 26 ) + E*IIY*( 4.0/(1155.0*R4) )
K( 26 , 27 ) = K( 26 , 27 ) + E*IIY*( 3.0/(70.0*P*R3) )
K( 26 , 28 ) = K( 26 , 28 ) + E*IIY*( -1.0/(210.0*P*R3) )
K( 27 , 27 ) = K( 27 , 27 ) + E*IIY*( 3.0/(5.0*P2*R2) )

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K( 27 , 28 )= K( 27 , 28 )+ E* IYY*( -1.0/(10.0*P2*R2) )
K( 28 , 28 )= K( 28 , 28 )+ E* IYY*( 4.0/(15.0*P2*R2) )

C***
C***   AXIAL STIFFNESS      A*[ PRIME(U)-W/R ]**2.0
C***
C*** (c14) FOR I:1 THRU 26 DO
C***   FOR J:I THRU 26 DO
C***   IF I<=3 OR I>=9 AND I<=11 OR I>=16 AND I<=18 OR I>=24 THEN
C***   IF J<=3 OR J>=9 AND J<=11 OR J>=16 AND J<=18 OR J>=24 THEN
C***   PRINT("      K(",I,"",",J,"")= K(",I,"",",J,"")+ E*A*(",
C***                                     STRING(K[I,J]),",")")$
C***
K( 1 , 1 )= K( 1 , 1 )+ E*A*( 5.0/(7.0*P2*R2) )
K( 1 , 2 )= K( 1 , 2 )+ E*A*( 3.0/(14.0*P2*R2) )
K( 1 , 3 )= K( 1 , 3 )+ E*A*( 1.0/(42.0*P2*R2) )
K( 1 , 9 )= K( 1 , 9 )+ E*A*( 1.0/(2.0*P*R2) )
K( 1 , 10 )= K( 1 , 10 )+ E*A*( 11.0/(42.0*P*R2) )
K( 1 , 11 )= K( 1 , 11 )+ E*A*( 1.0/(21.0*P*R2) )
K( 1 , 16 )= K( 1 , 16 )+ E*A*( -5.0/(7.0*P2*R2) )
K( 1 , 17 )= K( 1 , 17 )+ E*A*( 3.0/(14.0*P2*R2) )
K( 1 , 18 )= K( 1 , 18 )+ E*A*( -1.0/(42.0*P2*R2) )
K( 1 , 24 )= K( 1 , 24 )+ E*A*( 1.0/(2.0*P*R2) )
K( 1 , 25 )= K( 1 , 25 )+ E*A*( -11.0/(42.0*P*R2) )
K( 1 , 26 )= K( 1 , 26 )+ E*A*( 1.0/(21.0*P*R2) )
K( 2 , 2 )= K( 2 , 2 )+ E*A*( 16.0/(35.0*P2*R2) )
K( 2 , 3 )= K( 2 , 3 )+ E*A*( 1.0/(15.0*P2*R2) )
K( 2 , 9 )= K( 2 , 9 )+ E*A*( -11.0/(42.0*P*R2) )
K( 2 , 10 )= K( 2 , 10 )+ E*A*( 0.0 )
K( 2 , 11 )= K( 2 , 11 )+ E*A*( 1.0/(126.0*P*R2) )
K( 2 , 16 )= K( 2 , 16 )+ E*A*( -3.0/(14.0*P2*R2) )
K( 2 , 17 )= K( 2 , 17 )+ E*A*( -1.0/(35.0*P2*R2) )
K( 2 , 18 )= K( 2 , 18 )+ E*A*( 2.0/(105.0*P2*R2) )
K( 2 , 24 )= K( 2 , 24 )+ E*A*( 11.0/(42.0*P*R2) )
K( 2 , 25 )= K( 2 , 25 )+ E*A*( -13.0/(105.0*P*R2) )
K( 2 , 26 )= K( 2 , 26 )+ E*A*( 13.0/(630.0*P*R2) )
K( 3 , 3 )= K( 3 , 3 )+ E*A*( 4.0/(315.0*P2*R2) )
K( 3 , 9 )= K( 3 , 9 )+ E*A*( -1.0/(21.0*P*R2) )
K( 3 , 10 )= K( 3 , 10 )+ E*A*( -1.0/(126.0*P*R2) )
K( 3 , 11 )= K( 3 , 11 )+ E*A*( 0.0 )
K( 3 , 16 )= K( 3 , 16 )+ E*A*( -1.0/(42.0*P2*R2) )
K( 3 , 17 )= K( 3 , 17 )+ E*A*( -2.0/(105.0*P2*R2) )
K( 3 , 18 )= K( 3 , 18 )+ E*A*( 2.0/(315.0*P2*R2) )
K( 3 , 24 )= K( 3 , 24 )+ E*A*( 1.0/(21.0*P*R2) )
K( 3 , 25 )= K( 3 , 25 )+ E*A*( -13.0/(630.0*P*R2) )
K( 3 , 26 )= K( 3 , 26 )+ E*A*( 1.0/(315.0*P*R2) )
K( 9 , 9 )= K( 9 , 9 )+ E*A*( 181.0/(231.0*R2) )
K( 9 , 10 )= K( 9 , 10 )+ E*A*( 311.0/(1155.0*R2) )
K( 9 , 11 )= K( 9 , 11 )+ E*A*( 281.0/(6930.0*R2) )
K( 9 , 16 )= K( 9 , 16 )+ E*A*( -1.0/(2.0*P*R2) )
K( 9 , 17 )= K( 9 , 17 )+ E*A*( 11.0/(42.0*P*R2) )
K( 9 , 18 )= K( 9 , 18 )+ E*A*( -1.0/(21.0*P*R2) )
K( 9 , 24 )= K( 9 , 24 )+ E*A*( 50.0/(231.0*R2) )
K( 9 , 25 )= K( 9 , 25 )+ E*A*( -151.0/(1155.0*R2) )

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K( 9 , 26 )= K( 9 , 26 )+ E*A*( 181.0/(6930.0*R2) )
K( 10 , 10 )= K( 10 , 10 )+ E*A*( 416.0/(3465.0*R2) )
K( 10 , 11 )= K( 10 , 11 )+ E*A*( 23.0/(1155.0*R2) )
K( 10 , 16 )= K( 10 , 16 )+ E*A*( -11.0/(42.0*P*R2) )
K( 10 , 17 )= K( 10 , 17 )+ E*A*( 13.0/(105.0*P*R2) )
K( 10 , 18 )= K( 10 , 18 )+ E*A*( -13.0/(630.0*P*R2) )
K( 10 , 24 )= K( 10 , 24 )+ E*A*( 151.0/(1155.0*R2) )
K( 10 , 25 )= K( 10 , 25 )+ E*A*( -38.0/(495.0*R2) )
K( 10 , 26 )= K( 10 , 26 )+ E*A*( 52.0/(3465.0*R2) )
K( 11 , 11 )= K( 11 , 11 )+ E*A*( 4.0/(1155.0*R2) )
K( 11 , 16 )= K( 11 , 16 )+ E*A*( -1.0/(21.0*P*R2) )
K( 11 , 17 )= K( 11 , 17 )+ E*A*( 13.0/(630.0*P*R2) )
K( 11 , 18 )= K( 11 , 18 )+ E*A*( -1.0/(315.0*P*R2) )
K( 11 , 24 )= K( 11 , 24 )+ E*A*( 181.0/(6930.0*R2) )
K( 11 , 25 )= K( 11 , 25 )+ E*A*( -52.0/(3465.0*R2) )
K( 11 , 26 )= K( 11 , 26 )+ E*A*( 2.0/(693.0*R2) )
K( 16 , 16 )= K( 16 , 16 )+ E*A*( 5.0/(7.0*P2*R2) )
K( 16 , 17 )= K( 16 , 17 )+ E*A*( -3.0/(14.0*P2*R2) )
K( 16 , 18 )= K( 16 , 18 )+ E*A*( 1.0/(42.0*P2*R2) )
K( 16 , 24 )= K( 16 , 24 )+ E*A*( -1.0/(2.0*P*R2) )
K( 16 , 25 )= K( 16 , 25 )+ E*A*( 11.0/(42.0*P*R2) )
K( 16 , 26 )= K( 16 , 26 )+ E*A*( -1.0/(21.0*P*R2) )
K( 17 , 17 )= K( 17 , 17 )+ E*A*( 16.0/(35.0*P2*R2) )
K( 17 , 18 )= K( 17 , 18 )+ E*A*( -1.0/(15.0*P2*R2) )
K( 17 , 24 )= K( 17 , 24 )+ E*A*( -11.0/(42.0*P*R2) )
K( 17 , 25 )= K( 17 , 25 )+ E*A*( 0.0 )
K( 17 , 26 )= K( 17 , 26 )+ E*A*( 1.0/(126.0*P*R2) )
K( 18 , 18 )= K( 18 , 18 )+ E*A*( 4.0/(315.0*P2*R2) )
K( 18 , 24 )= K( 18 , 24 )+ E*A*( 1.0/(21.0*P*R2) )
K( 18 , 25 )= K( 18 , 25 )+ E*A*( -1.0/(126.0*P*R2) )
K( 18 , 26 )= K( 18 , 26 )+ E*A*( 0.0 )
K( 24 , 24 )= K( 24 , 24 )+ E*A*( 181.0/(231.0*R2) )
K( 24 , 25 )= K( 24 , 25 )+ E*A*( -311.0/(1155.0*R2) )
K( 24 , 26 )= K( 24 , 26 )+ E*A*( 281.0/(6930.0*R2) )
K( 25 , 25 )= K( 25 , 25 )+ E*A*( 416.0/(3465.0*R2) )
K( 25 , 26 )= K( 25 , 26 )+ E*A*( -23.0/(1155.0*R2) )
K( 26 , 26 )= K( 26 , 26 )+ E*A*( 4.0/(1155.0*R2) )

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C***
C*** COUPLING STIFFNESS [ PRIME(THETA)-PHI/R ]*[ PRIME(PSI)+W/R/R ]
C***
C*** (c47) FOR I:7 THRU 30 DO
C***   FOR J:I THRU 30 DO
C***     IF I<=15 OR I>=22 THEN
C***       IF J<=15 OR J>=22 THEN
C***         IF K[I,J] # 0 THEN
C***           PRINT("      K(",I,"","J,")= K(",I,"","J,")- E*IYZ*(",
C***             STRING(K[I,J]),")" )$
C***

```

```

K( 7 , 9 )= K( 7 , 9 )- E*IYZ*( -1.0/(2.0*P*R3) )
K( 7 , 10 )= K( 7 , 10 )- E*IYZ*( -17.0/(70.0*P*R3) )
K( 7 , 11 )= K( 7 , 11 )- E*IYZ*( -3.0/(70.0*P*R3) )
K( 7 , 12 )= K( 7 , 12 )- E*IYZ*( 3.0/(5.0*P2*R2) )
K( 7 , 13 )= K( 7 , 13 )- E*IYZ*( 1.0/(10.0*P2*R2) )

```


$K(7, 24) = K(7, 24) - E* IYZ * (-1.0 / (2.0 * P * R3))$
 $K(7, 25) = K(7, 25) - E* IYZ * (17.0 / (70.0 * P * R3))$
 $K(7, 26) = K(7, 26) - E* IYZ * (-3.0 / (70.0 * P * R3))$
 $K(7, 27) = K(7, 27) - E* IYZ * (-3.0 / (5.0 * P^2 * R2))$
 $K(7, 28) = K(7, 28) - E* IYZ * (1.0 / (10.0 * P^2 * R2))$
 $K(8, 9) = K(8, 9) - E* IYZ * (3.0 / (14.0 * P * R3))$
 $K(8, 10) = K(8, 10) - E* IYZ * (1.0 / (210.0 * P * R3))$
 $K(8, 11) = K(8, 11) - E* IYZ * (-1.0 / (210.0 * P * R3))$
 $K(8, 12) = K(8, 12) - E* IYZ * (1.0 / (10.0 * P^2 * R2))$
 $K(8, 13) = K(8, 13) - E* IYZ * (4.0 / (15.0 * P^2 * R2))$
 $K(8, 24) = K(8, 24) - E* IYZ * (-3.0 / (14.0 * P * R3))$
 $K(8, 25) = K(8, 25) - E* IYZ * (19.0 / (210.0 * P * R3))$
 $K(8, 26) = K(8, 26) - E* IYZ * (-1.0 / (70.0 * P * R3))$
 $K(8, 27) = K(8, 27) - E* IYZ * (-1.0 / (10.0 * P^2 * R2))$
 $K(8, 28) = K(8, 28) - E* IYZ * (-1.0 / (15.0 * P^2 * R2))$
 $K(9, 14) = K(9, 14) - E* IYZ * (-16.0 / (21.0 * R3))$
 $K(9, 15) = K(9, 15) - E* IYZ * (-3.0 / (14.0 * R3))$
 $K(9, 22) = K(9, 22) - E* IYZ * (1.0 / (2.0 * P * R3))$
 $K(9, 23) = K(9, 23) - E* IYZ * (-3.0 / (14.0 * P * R3))$
 $K(9, 29) = K(9, 29) - E* IYZ * (-5.0 / (21.0 * R3))$
 $K(9, 30) = K(9, 30) - E* IYZ * (5.0 / (42.0 * R3))$
 $K(10, 14) = K(10, 14) - E* IYZ * (-11.0 / (42.0 * R3))$
 $K(10, 15) = K(10, 15) - E* IYZ * (-2.0 / (21.0 * R3))$
 $K(10, 22) = K(10, 22) - E* IYZ * (17.0 / (70.0 * P * R3))$
 $K(10, 23) = K(10, 23) - E* IYZ * (-19.0 / (210.0 * P * R3))$
 $K(10, 29) = K(10, 29) - E* IYZ * (-29.0 / (210.0 * R3))$
 $K(10, 30) = K(10, 30) - E* IYZ * (1.0 / (15.0 * R3))$
 $K(11, 14) = K(11, 14) - E* IYZ * (-5.0 / (126.0 * R3))$
 $K(11, 15) = K(11, 15) - E* IYZ * (-1.0 / (63.0 * R3))$
 $K(11, 22) = K(11, 22) - E* IYZ * (3.0 / (70.0 * P * R3))$
 $K(11, 23) = K(11, 23) - E* IYZ * (-1.0 / (70.0 * P * R3))$
 $K(11, 29) = K(11, 29) - E* IYZ * (-17.0 / (630.0 * R3))$
 $K(11, 30) = K(11, 30) - E* IYZ * (4.0 / (315.0 * R3))$
 $K(12, 14) = K(12, 14) - E* IYZ * (1.0 / (2.0 * P * R2))$
 $K(12, 15) = K(12, 15) - E* IYZ * (1.0 / (5.0 * P * R2))$
 $K(12, 22) = K(12, 22) - E* IYZ * (-3.0 / (5.0 * P^2 * R2))$
 $K(12, 23) = K(12, 23) - E* IYZ * (1.0 / (10.0 * P^2 * R2))$
 $K(12, 29) = K(12, 29) - E* IYZ * (1.0 / (2.0 * P * R2))$
 $K(12, 30) = K(12, 30) - E* IYZ * (-1.0 / (5.0 * P * R2))$
 $K(13, 14) = K(13, 14) - E* IYZ * (-1.0 / (5.0 * P * R2))$
 $K(13, 22) = K(13, 22) - E* IYZ * (-1.0 / (10.0 * P^2 * R2))$
 $K(13, 23) = K(13, 23) - E* IYZ * (-1.0 / (15.0 * P^2 * R2))$
 $K(13, 29) = K(13, 29) - E* IYZ * (1.0 / (5.0 * P * R2))$
 $K(13, 30) = K(13, 30) - E* IYZ * (-1.0 / (15.0 * P * R2))$
 $K(14, 24) = K(14, 24) - E* IYZ * (-5.0 / (21.0 * R3))$
 $K(14, 25) = K(14, 25) - E* IYZ * (29.0 / (210.0 * R3))$
 $K(14, 26) = K(14, 26) - E* IYZ * (-17.0 / (630.0 * R3))$
 $K(14, 27) = K(14, 27) - E* IYZ * (-1.0 / (2.0 * P * R2))$
 $K(14, 28) = K(14, 28) - E* IYZ * (1.0 / (5.0 * P * R2))$
 $K(15, 24) = K(15, 24) - E* IYZ * (-5.0 / (42.0 * R3))$
 $K(15, 25) = K(15, 25) - E* IYZ * (1.0 / (15.0 * R3))$
 $K(15, 26) = K(15, 26) - E* IYZ * (-4.0 / (315.0 * R3))$
 $K(15, 27) = K(15, 27) - E* IYZ * (-1.0 / (5.0 * P * R2))$

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K( 15 , 28 )= K( 15 , 28 )- E* IYZ*( 1.0/(15.0*P*R2) )
K( 22 , 24 )= K( 22 , 24 )- E* IYZ*( 1.0/(2.0*P*R3) )
K( 22 , 25 )= K( 22 , 25 )- E* IYZ*( -17.0/(70.0*P*R3) )
K( 22 , 26 )= K( 22 , 26 )- E* IYZ*( 3.0/(70.0*P*R3) )
K( 22 , 27 )= K( 22 , 27 )- E* IYZ*( 3.0/(5.0*P2*R2) )
K( 22 , 28 )= K( 22 , 28 )- E* IYZ*( -1.0/(10.0*P2*R2) )
K( 23 , 24 )= K( 23 , 24 )- E* IYZ*( 3.0/(14.0*P*R3) )
K( 23 , 25 )= K( 23 , 25 )- E* IYZ*( -1.0/(210.0*P*R3) )
K( 23 , 26 )= K( 23 , 26 )- E* IYZ*( -1.0/(210.0*P*R3) )
K( 23 , 27 )= K( 23 , 27 )- E* IYZ*( -1.0/(10.0*P2*R2) )
K( 23 , 28 )= K( 23 , 28 )- E* IYZ*( 4.0/(15.0*P2*R2) )
K( 24 , 29 )= K( 24 , 29 )- E* IYZ*( -16.0/(21.0*R3) )
K( 24 , 30 )= K( 24 , 30 )- E* IYZ*( 3.0/(14.0*R3) )
K( 25 , 29 )= K( 25 , 29 )- E* IYZ*( 11.0/(42.0*R3) )
K( 25 , 30 )= K( 25 , 30 )- E* IYZ*( -2.0/(21.0*R3) )
K( 26 , 29 )= K( 26 , 29 )- E* IYZ*( -5.0/(126.0*R3) )
K( 26 , 30 )= K( 26 , 30 )- E* IYZ*( 1.0/(63.0*R3) )
K( 27 , 29 )= K( 27 , 29 )- E* IYZ*( -1.0/(2.0*P*R2) )
K( 27 , 30 )= K( 27 , 30 )- E* IYZ*( 1.0/(5.0*P*R2) )
K( 28 , 29 )= K( 28 , 29 )- E* IYZ*( -1.0/(5.0*P*R2) )

C***
C***  TORSION  GC*[ PRIME(PHI)+THETA/R ]
C***
C***  (c51) FOR I:7 THRU 30 DO
C***    FOR J:I THRU 30 DO
C***  IF I<=8 OR I>=14 AND I<=15 OR I>=22 AND I<=23 OR I>=29 THEN
C***  IF J<=8 OR J>=14 AND J<=15 OR J>=22 AND J<=23 OR J>=29 THEN
C***  PRINT("      K(",I,"","J,"")= K(",I,"","J,"")+ GC*(",
C***                                     STRING(K[I,J]),")" )$
C***

K( 7 , 7 )= K( 7 , 7 )+ GC*( 26.0/(35.0*R2) )
K( 7 , 8 )= K( 7 , 8 )+ GC*( 22.0/(105.0*R2) )
K( 7 , 14 )= K( 7 , 14 )+ GC*( -1.0/(2.0*P*R2) )
K( 7 , 15 )= K( 7 , 15 )+ GC*( 1.0/(5.0*P*R2) )
K( 7 , 22 )= K( 7 , 22 )+ GC*( 9.0/(35.0*R2) )
K( 7 , 23 )= K( 7 , 23 )+ GC*( -13.0/(105.0*R2) )
K( 7 , 29 )= K( 7 , 29 )+ GC*( 1.0/(2.0*P*R2) )
K( 7 , 30 )= K( 7 , 30 )+ GC*( -1.0/(5.0*P*R2) )
K( 8 , 8 )= K( 8 , 8 )+ GC*( 8.0/(105.0*R2) )
K( 8 , 14 )= K( 8 , 14 )+ GC*( -1.0/(5.0*P*R2) )
K( 8 , 15 )= K( 8 , 15 )+ GC*( 0.0 )
K( 8 , 22 )= K( 8 , 22 )+ GC*( 13.0/(105.0*R2) )
K( 8 , 23 )= K( 8 , 23 )+ GC*( -2.0/(35.0*R2) )
K( 8 , 29 )= K( 8 , 29 )+ GC*( 1.0/(5.0*P*R2) )
K( 8 , 30 )= K( 8 , 30 )+ GC*( -1.0/(15.0*P*R2) )
K( 14 , 14 )= K( 14 , 14 )+ GC*( 3.0/(5.0*P2*R2) )
K( 14 , 15 )= K( 14 , 15 )+ GC*( 1.0/(10.0*P2*R2) )
K( 14 , 22 )= K( 14 , 22 )+ GC*( -1.0/(2.0*P*R2) )
K( 14 , 23 )= K( 14 , 23 )+ GC*( 1.0/(5.0*P*R2) )
K( 14 , 29 )= K( 14 , 29 )+ GC*( -3.0/(5.0*P2*R2) )
K( 14 , 30 )= K( 14 , 30 )+ GC*( 1.0/(10.0*P2*R2) )
K( 15 , 15 )= K( 15 , 15 )+ GC*( 4.0/(15.0*P2*R2) )
K( 15 , 22 )= K( 15 , 22 )+ GC*( -1.0/(5.0*P*R2) )

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K( 15 , 23 )= K( 15 , 23 )+ GC*( 1.0/(15.0*P*R2) )
K( 15 , 29 )= K( 15 , 29 )+ GC*( -1.0/(10.0*P2*R2) )
K( 15 , 30 )= K( 15 , 30 )+ GC*( -1.0/(15.0*P2*R2) )
K( 22 , 22 )= K( 22 , 22 )+ GC*( 26.0/(35.0*R2) )
K( 22 , 23 )= K( 22 , 23 )+ GC*( -22.0/(105.0*R2) )
K( 22 , 29 )= K( 22 , 29 )+ GC*( 1.0/(2.0*P*R2) )
K( 22 , 30 )= K( 22 , 30 )+ GC*( 1.0/(5.0*P*R2) )
K( 23 , 23 )= K( 23 , 23 )+ GC*( 8.0/(105.0*R2) )
K( 23 , 29 )= K( 23 , 29 )+ GC*( -1.0/(5.0*P*R2) )
K( 23 , 30 )= K( 23 , 30 )+ GC*( 0.0 )
K( 29 , 29 )= K( 29 , 29 )+ GC*( 3.0/(5.0*P2*R2) )
K( 29 , 30 )= K( 29 , 30 )+ GC*( -1.0/(10.0*P2*R2) )
K( 30 , 30 )= K( 30 , 30 )+ GC*( 4.0/(15.0*P2*R2) )

C***
C*** SHEAR STIFFNESS [ PRIME(V)-THETA ]**2.0
C***
C*** (c60) FOR I:4 THRU 23 DO
C***   FOR J:I THRU 23 DO
C***   IF I<=8 OR I>=19 AND I<=23 THEN
C***   IF J<=8 OR J>=19 AND J<=23 THEN
C***   PRINT("      K(",I,"",J,"")= K(",I,"",J,"")+ GC*(",
C***                                     STRING(K[I,J]),")" )S
C***

K( 4 , 4 )= K( 4 , 4 )+ GA*( 5.0/(7.0*P2*R2) )
K( 4 , 5 )= K( 4 , 5 )+ GA*( 3.0/(14.0*P2*R2) )
K( 4 , 6 )= K( 4 , 6 )+ GA*( 1.0/(42.0*P2*R2) )
K( 4 , 7 )= K( 4 , 7 )+ GA*( 1.0/(2.0*P*R) )
K( 4 , 8 )= K( 4 , 8 )+ GA*( 3.0/(14.0*P*R) )
K( 4 , 19 )= K( 4 , 19 )+ GA*( -5.0/(7.0*P2*R2) )
K( 4 , 20 )= K( 4 , 20 )+ GA*( 3.0/(14.0*P2*R2) )
K( 4 , 21 )= K( 4 , 21 )+ GA*( -1.0/(42.0*P2*R2) )
K( 4 , 22 )= K( 4 , 22 )+ GA*( 1.0/(2.0*P*R) )
K( 4 , 23 )= K( 4 , 23 )+ GA*( -3.0/(14.0*P*R) )
K( 5 , 5 )= K( 5 , 5 )+ GA*( 16.0/(35.0*P2*R2) )
K( 5 , 6 )= K( 5 , 6 )+ GA*( 1.0/(15.0*P2*R2) )
K( 5 , 7 )= K( 5 , 7 )+ GA*( -17.0/(70.0*P*R) )
K( 5 , 8 )= K( 5 , 8 )+ GA*( 1.0/(210.0*P*R) )
K( 5 , 19 )= K( 5 , 19 )+ GA*( -3.0/(14.0*P2*R2) )
K( 5 , 20 )= K( 5 , 20 )+ GA*( -1.0/(35.0*P2*R2) )
K( 5 , 21 )= K( 5 , 21 )+ GA*( 2.0/(105.0*P2*R2) )
K( 5 , 22 )= K( 5 , 22 )+ GA*( 17.0/(70.0*P*R) )
K( 5 , 23 )= K( 5 , 23 )+ GA*( -19.0/(210.0*P*R) )
K( 6 , 6 )= K( 6 , 6 )+ GA*( 4.0/(315.0*P2*R2) )
K( 6 , 7 )= K( 6 , 7 )+ GA*( -3.0/(70.0*P*R) )
K( 6 , 8 )= K( 6 , 8 )+ GA*( -1.0/(210.0*P*R) )
K( 6 , 19 )= K( 6 , 19 )+ GA*( -1.0/(42.0*P2*R2) )
K( 6 , 20 )= K( 6 , 20 )+ GA*( -2.0/(105.0*P2*R2) )
K( 6 , 21 )= K( 6 , 21 )+ GA*( 2.0/(315.0*P2*R2) )
K( 6 , 22 )= K( 6 , 22 )+ GA*( 3.0/(70.0*P*R) )
K( 6 , 23 )= K( 6 , 23 )+ GA*( -1.0/(70.0*P*R) )
K( 7 , 7 )= K( 7 , 7 )+ GA*( 26.0/35.0 )
K( 7 , 8 )= K( 7 , 8 )+ GA*( 22.0/105.0 )
K( 7 , 19 )= K( 7 , 19 )+ GA*( -1.0/(2.0*P*R) )

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K( 7 , 20 ) = K( 7 , 20 ) + GA*( 17.0/(70.0*P*R) )
K( 7 , 21 ) = K( 7 , 21 ) + GA*( -3.0/(70.0*P*R) )
K( 7 , 22 ) = K( 7 , 22 ) + GA*( 9.0/35.0 )
K( 7 , 23 ) = K( 7 , 23 ) + GA*( -13.0/105.0 )
K( 8 , 8 ) = K( 8 , 8 ) + GA*( 8.0/105.0 )
K( 8 , 19 ) = K( 8 , 19 ) + GA*( -3.0/(14.0*P*R) )
K( 8 , 20 ) = K( 8 , 20 ) + GA*( 19.0/(210.0*P*R) )
K( 8 , 21 ) = K( 8 , 21 ) + GA*( -1.0/(70.0*P*R) )
K( 8 , 22 ) = K( 8 , 22 ) + GA*( 13.0/105.0 )
K( 8 , 23 ) = K( 8 , 23 ) + GA*( -2.0/35.0 )
K( 19 , 19 ) = K( 19 , 19 ) + GA*( 5.0/(7.0*P2*R2) )
K( 19 , 20 ) = K( 19 , 20 ) + GA*( -3.0/(14.0*P2*R2) )
K( 19 , 21 ) = K( 19 , 21 ) + GA*( 1.0/(42.0*P2*R2) )
K( 19 , 22 ) = K( 19 , 22 ) + GA*( -1.0/(2.0*P*R) )
K( 19 , 23 ) = K( 19 , 23 ) + GA*( 3.0/(14.0*P*R) )
K( 20 , 20 ) = K( 20 , 20 ) + GA*( 16.0/(35.0*P2*R2) )
K( 20 , 21 ) = K( 20 , 21 ) + GA*( -1.0/(15.0*P2*R2) )
K( 20 , 22 ) = K( 20 , 22 ) + GA*( -17.0/(70.0*P*R) )
K( 20 , 23 ) = K( 20 , 23 ) + GA*( -1.0/(210.0*P*R) )
K( 21 , 21 ) = K( 21 , 21 ) + GA*( 4.0/(315.0*P2*R2) )
K( 21 , 22 ) = K( 21 , 22 ) + GA*( 3.0/(70.0*P*R) )
K( 21 , 23 ) = K( 21 , 23 ) + GA*( -1.0/(210.0*P*R) )
K( 22 , 22 ) = K( 22 , 22 ) + GA*( 26.0/35.0 )
K( 22 , 23 ) = K( 22 , 23 ) + GA*( -22.0/105.0 )
K( 23 , 23 ) = K( 23 , 23 ) + GA*( 8.0/105.0 )

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C***

C*** SHEAR STIFFNESS [PRIME(W)-PSI]

C***

C*** (c63) FOR I:9 THRU 28 DO

C*** FOR J:I THRU 28 DO

C*** IF I<=13 OR I>=24 THEN

C*** IF J<=13 OR J>=24 THEN

C*** PRINT(" K(",I,"",J,"") = K(",I,"",J,"") + GA*(",STRING(K[I,J]),"

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K( 9 , 9 ) = K( 9 , 9 ) + GA*( 5.0/(7.0*P2*R2) )
K( 9 , 10 ) = K( 9 , 10 ) + GA*( 3.0/(14.0*P2*R2) )
K( 9 , 11 ) = K( 9 , 11 ) + GA*( 1.0/(42.0*P2*R2) )
K( 9 , 12 ) = K( 9 , 12 ) + GA*( 1.0/(2.0*P*R) )
K( 9 , 13 ) = K( 9 , 13 ) + GA*( 3.0/(14.0*P*R) )
K( 9 , 24 ) = K( 9 , 24 ) + GA*( -5.0/(7.0*P2*R2) )
K( 9 , 25 ) = K( 9 , 25 ) + GA*( 3.0/(14.0*P2*R2) )
K( 9 , 26 ) = K( 9 , 26 ) + GA*( -1.0/(42.0*P2*R2) )
K( 9 , 27 ) = K( 9 , 27 ) + GA*( 1.0/(2.0*P*R) )
K( 9 , 28 ) = K( 9 , 28 ) + GA*( -3.0/(14.0*P*R) )
K( 10 , 10 ) = K( 10 , 10 ) + GA*( 16.0/(35.0*P2*R2) )
K( 10 , 11 ) = K( 10 , 11 ) + GA*( 1.0/(15.0*P2*R2) )
K( 10 , 12 ) = K( 10 , 12 ) + GA*( -17.0/(70.0*P*R) )
K( 10 , 13 ) = K( 10 , 13 ) + GA*( 1.0/(210.0*P*R) )
K( 10 , 24 ) = K( 10 , 24 ) + GA*( -3.0/(14.0*P2*R2) )
K( 10 , 25 ) = K( 10 , 25 ) + GA*( -1.0/(35.0*P2*R2) )
K( 10 , 26 ) = K( 10 , 26 ) + GA*( 2.0/(105.0*P2*R2) )
K( 10 , 27 ) = K( 10 , 27 ) + GA*( 17.0/(70.0*P*R) )
K( 10 , 28 ) = K( 10 , 28 ) + GA*( -19.0/(210.0*P*R) )
K( 11 , 11 ) = K( 11 , 11 ) + GA*( 4.0/(315.0*P2*R2) )

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$K(11, 12) = K(11, 12) + GA * (-3.0 / (70.0 * P * R))$
 $K(11, 13) = K(11, 13) + GA * (-1.0 / (210.0 * P * R))$
 $K(11, 24) = K(11, 24) + GA * (-1.0 / (42.0 * P^2 * R^2))$
 $K(11, 25) = K(11, 25) + GA * (-2.0 / (105.0 * P^2 * R^2))$
 $K(11, 26) = K(11, 26) + GA * (2.0 / (315.0 * P^2 * R^2))$
 $K(11, 27) = K(11, 27) + GA * (3.0 / (70.0 * P * R))$
 $K(11, 28) = K(11, 28) + GA * (-1.0 / (70.0 * P * R))$
 $K(12, 12) = K(12, 12) + GA * (26.0 / 35.0)$
 $K(12, 13) = K(12, 13) + GA * (22.0 / 105.0)$
 $K(12, 24) = K(12, 24) + GA * (-1.0 / (2.0 * P * R))$
 $K(12, 25) = K(12, 25) + GA * (17.0 / (70.0 * P * R))$
 $K(12, 26) = K(12, 26) + GA * (-3.0 / (70.0 * P * R))$
 $K(12, 27) = K(12, 27) + GA * (9.0 / 35.0)$
 $K(12, 28) = K(12, 28) + GA * (-13.0 / 105.0)$
 $K(13, 13) = K(13, 13) + GA * (8.0 / 105.0)$
 $K(13, 24) = K(13, 24) + GA * (-3.0 / (14.0 * P * R))$
 $K(13, 25) = K(13, 25) + GA * (19.0 / (210.0 * P * R))$
 $K(13, 26) = K(13, 26) + GA * (-1.0 / (70.0 * P * R))$
 $K(13, 27) = K(13, 27) + GA * (13.0 / 105.0)$
 $K(13, 28) = K(13, 28) + GA * (-2.0 / 35.0)$
 $K(24, 24) = K(24, 24) + GA * (5.0 / (7.0 * P^2 * R^2))$
 $K(24, 25) = K(24, 25) + GA * (-3.0 / (14.0 * P^2 * R^2))$
 $K(24, 26) = K(24, 26) + GA * (1.0 / (42.0 * P^2 * R^2))$
 $K(24, 27) = K(24, 27) + GA * (-1.0 / (2.0 * P * R))$
 $K(24, 28) = K(24, 28) + GA * (3.0 / (14.0 * P * R))$
 $K(25, 25) = K(25, 25) + GA * (16.0 / (35.0 * P^2 * R^2))$
 $K(25, 26) = K(25, 26) + GA * (-1.0 / (15.0 * P^2 * R^2))$
 $K(25, 27) = K(25, 27) + GA * (-17.0 / (70.0 * P * R))$
 $K(25, 28) = K(25, 28) + GA * (-1.0 / (210.0 * P * R))$
 $K(26, 26) = K(26, 26) + GA * (4.0 / (315.0 * P^2 * R^2))$
 $K(26, 27) = K(26, 27) + GA * (3.0 / (70.0 * P * R))$
 $K(26, 28) = K(26, 28) + GA * (-1.0 / (210.0 * P * R))$
 $K(27, 27) = K(27, 27) + GA * (26.0 / 35.0)$
 $K(27, 28) = K(27, 28) + GA * (-22.0 / 105.0)$
 $K(28, 28) = K(28, 28) + GA * (8.0 / 105.0)$
 $L = 2.0 * R * \text{ANGLE}$
 $V(1) = 1.0$
 $V(2) = 0.5 * L$
 $V(3) = 0.25 * L * L$
 $V(4) = 1.0$
 $V(5) = 0.5 * L$
 $V(6) = 0.25 * L * L$
 $V(7) = 1.0$
 $V(8) = 0.5 * L$
 $V(9) = 1.0$
 $V(10) = 0.5 * L$
 $V(11) = 0.25 * L * L$
 $V(12) = 1.0$
 $V(13) = 0.5 * L$
 $V(14) = 1.0$
 $V(15) = 0.5 * L$
 $V(16) = 1.0$
 $V(17) = 0.5 * L$

```
V(18)=0.25*L*L
V(19)=1.0
V(20)=0.5*L
V(21)=0.25*L*L
V(22)=1.0
V(23)=0.5*L
V(24)=1.0
V(25)=0.5*L
V(26)=0.25*L*L
V(27)=1.0
V(28)=0.5*L
V(29)=1.0
V(30)=0.5*L
DO 10 I=1,30
  DO 20 J=1,30
    K(I,J)=K(I,J)*R*ANGLE*V(I)*V(J)
  CONTINUE
10 CONTINUE
RETURN
END
```

```

SUBROUTINE ELMASS(IEL)
  REAL M(30,30)
  COMMON/TWO/NDFPN,NDFPE,NNPE
  COMMON/TP/NEL,NNODE,NBNODE
  COMMON/BLK1/NODE(10,2),ID(15,10),NBN(10)
  COMMON/TX/LM(30)
  COMMON/BLK2/S(465),EM(465)
  COMMON/STIFFM/M
  JJ=0
  DO 40 I=1,30
    DO 30 J=1,30
      JJ=JJ+1
30      EM(JJ)=M(I,J)
40      CONTINUE
  RETURN
END

C*****
C***
C*** SUBROUTINE:      MASS  [CREATES A 30 DOF CURVED BEAM FINITE
C***                  ELEMENT]
C***
C*****
SUBROUTINE MASS(ANGLE)
  REAL M(30,30)
  REAL V(30),L
  REAL
1,      A      , ASTAR , CW      , E      , G
2,      IXX    , IYZ   , IYY    , IZZ    , KG
  R      , RHO   , RS     , Y0     , Z0
  COMMON/SECTION/ A      , ASTAR , CW      , E      , G
1,      IXX    , IYZ   , IYY    , IZZ    , KG
2,      R      , RHO   , RS     , Y0     , Z0
  COMMON/STIFFM/M
C*** C=KG
  DO 8 I=1,30
    DO 5 J=1,30
      M(I,J)=0.0
5      CONTINUE
8      CONTINUE
  R2=R*R
C*** R3=R2*R
C*** R4=R3*R
C*** P=ANGLE
C*** P2=P*P
C*** AXIAL EXTENSION MASS TERM  [U]*[U]
C***
C***
C*** (c70) FOR I:1 THRU 18 DO
C***   FOR J:1 THRU 18 DO
C***     IF I<=3 OR I>=16 THEN
C***       IF J<=3 OR J>=16 THEN

```

```

C***PRINT("      M(" , I , " , " , J , ") = M(" , I , " , " , J , ") + RHO * A * (" ,
C***      STRING(M[I,J]), " ) " ) $
C***
M( 1 , 1 ) = M( 1 , 1 ) + RHO * A * ( 181.0/231.0 )
M( 1 , 2 ) = M( 1 , 2 ) + RHO * A * ( 311.0/1155.0 )
M( 1 , 3 ) = M( 1 , 3 ) + RHO * A * ( 281.0/6930.0 )
M( 1 , 16 ) = M( 1 , 16 ) + RHO * A * ( 50.0/231.0 )
M( 1 , 17 ) = M( 1 , 17 ) + RHO * A * ( -151.0/1155.0 )
M( 1 , 18 ) = M( 1 , 18 ) + RHO * A * ( 181.0/6930.0 )
M( 2 , 2 ) = M( 2 , 2 ) + RHO * A * ( 416.0/3465.0 )
M( 2 , 3 ) = M( 2 , 3 ) + RHO * A * ( 23.0/1155.0 )
M( 2 , 16 ) = M( 2 , 16 ) + RHO * A * ( 151.0/1155.0 )
M( 2 , 17 ) = M( 2 , 17 ) + RHO * A * ( -38.0/495.0 )
M( 2 , 18 ) = M( 2 , 18 ) + RHO * A * ( 52.0/3465.0 )
M( 3 , 3 ) = M( 3 , 3 ) + RHO * A * ( 4.0/1155.0 )
M( 3 , 16 ) = M( 3 , 16 ) + RHO * A * ( 181.0/6930.0 )
M( 3 , 17 ) = M( 3 , 17 ) + RHO * A * ( -52.0/3465.0 )
M( 3 , 18 ) = M( 3 , 18 ) + RHO * A * ( 2.0/693.0 )
M( 16 , 16 ) = M( 16 , 16 ) + RHO * A * ( 181.0/231.0 )
M( 16 , 17 ) = M( 16 , 17 ) + RHO * A * ( -311.0/1155.0 )
M( 16 , 18 ) = M( 16 , 18 ) + RHO * A * ( 281.0/6930.0 )
M( 17 , 17 ) = M( 17 , 17 ) + RHO * A * ( 416.0/3465.0 )
M( 17 , 18 ) = M( 17 , 18 ) + RHO * A * ( -23.0/1155.0 )
M( 18 , 18 ) = M( 18 , 18 ) + RHO * A * ( 4.0/1155.0 )
C*** OUTPLANE TRANSLATIONAL INERTIA [V]*[V]
C***
C*** (c74) FOR I:4 THRU 21 DO
C***   FOR J:I THRU 21 DO
C***     IF I<=6 OR I>=19 THEN
C***       IF J<=6 OR J>=19 THEN
C***PRINT("      M(" , I , " , " , J , ") = M(" , I , " , " , J , ") + RHO * A * (" ,
C***      STRING(M[I,J]), " ) " ) $
C***
M( 4 , 4 ) = M( 4 , 4 ) + RHO * A * ( 181.0/231.0 )
M( 4 , 5 ) = M( 4 , 5 ) + RHO * A * ( 311.0/1155.0 )
M( 4 , 6 ) = M( 4 , 6 ) + RHO * A * ( 281.0/6930.0 )
M( 4 , 19 ) = M( 4 , 19 ) + RHO * A * ( 50.0/231.0 )
M( 4 , 20 ) = M( 4 , 20 ) + RHO * A * ( -151.0/1155.0 )
M( 4 , 21 ) = M( 4 , 21 ) + RHO * A * ( 181.0/6930.0 )
M( 5 , 5 ) = M( 5 , 5 ) + RHO * A * ( 416.0/3465.0 )
M( 5 , 6 ) = M( 5 , 6 ) + RHO * A * ( 23.0/1155.0 )
M( 5 , 19 ) = M( 5 , 19 ) + RHO * A * ( 151.0/1155.0 )
M( 5 , 20 ) = M( 5 , 20 ) + RHO * A * ( -38.0/495.0 )
M( 5 , 21 ) = M( 5 , 21 ) + RHO * A * ( 52.0/3465.0 )
M( 6 , 6 ) = M( 6 , 6 ) + RHO * A * ( 4.0/1155.0 )
M( 6 , 19 ) = M( 6 , 19 ) + RHO * A * ( 181.0/6930.0 )
M( 6 , 20 ) = M( 6 , 20 ) + RHO * A * ( -52.0/3465.0 )
M( 6 , 21 ) = M( 6 , 21 ) + RHO * A * ( 2.0/693.0 )
M( 19 , 19 ) = M( 19 , 19 ) + RHO * A * ( 181.0/231.0 )
M( 19 , 20 ) = M( 19 , 20 ) + RHO * A * ( -311.0/1155.0 )
M( 19 , 21 ) = M( 19 , 21 ) + RHO * A * ( 281.0/6930.0 )
M( 20 , 20 ) = M( 20 , 20 ) + RHO * A * ( 416.0/3465.0 )
M( 20 , 21 ) = M( 20 , 21 ) + RHO * A * ( -23.0/1155.0 )

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      M( 21 , 21 )= M( 21 , 21 )+RHO*A*( 4.0/1155.0 )
C*** INPLANE TRANSLATIONAL INERTIA [W]*[W]
C***
C*** (c76) FOR I:9 THRU 26 DO
C***   FOR J:I THRU 26 DO
C***     IF I<=11 OR I>=24 THEN
C***     IF J<=11 OR J>=24 THEN
C*** PRINT("      M(",I,"",J,"")= M(",I,"",J,"")+RHO*A*(",
C***                                     STRING(M[I,J]), " )" )$
C***
      M( 9 , 9 )= M( 9 , 9 )+RHO*A*( 181.0/231.0 )
      M( 9 , 10 )= M( 9 , 10 )+RHO*A*( 311.0/1155.0 )
      M( 9 , 11 )= M( 9 , 11 )+RHO*A*( 281.0/6930.0 )
      M( 9 , 24 )= M( 9 , 24 )+RHO*A*( 50.0/231.0 )
      M( 9 , 25 )= M( 9 , 25 )+RHO*A*( -151.0/1155.0 )
      M( 9 , 26 )= M( 9 , 26 )+RHO*A*( 181.0/6930.0 )
      M( 10 , 10 )= M( 10 , 10 )+RHO*A*( 416.0/3465.0 )
      M( 10 , 11 )= M( 10 , 11 )+RHO*A*( 23.0/1155.0 )
      M( 10 , 24 )= M( 10 , 24 )+RHO*A*( 151.0/1155.0 )
      M( 10 , 25 )= M( 10 , 25 )+RHO*A*( -38.0/495.0 )
      M( 10 , 26 )= M( 10 , 26 )+RHO*A*( 52.0/3465.0 )
      M( 11 , 11 )= M( 11 , 11 )+RHO*A*( 4.0/1155.0 )
      M( 11 , 24 )= M( 11 , 24 )+RHO*A*( 181.0/6930.0 )
      M( 11 , 25 )= M( 11 , 25 )+RHO*A*( -52.0/3465.0 )
      M( 11 , 26 )= M( 11 , 26 )+RHO*A*( 2.0/693.0 )
      M( 24 , 24 )= M( 24 , 24 )+RHO*A*( 181.0/231.0 )
      M( 24 , 25 )= M( 24 , 25 )+RHO*A*( -311.0/1155.0 )
      M( 24 , 26 )= M( 24 , 26 )+RHO*A*( 281.0/6930.0 )
      M( 25 , 25 )= M( 25 , 25 )+RHO*A*( 416.0/3465.0 )
      M( 25 , 26 )= M( 25 , 26 )+RHO*A*( -23.0/1155.0 )
      M( 26 , 26 )= M( 26 , 26 )+RHO*A*( 4.0/1155.0 )
C*** INPLANE INERTIA [PSI]**2.0
C*** FOR I:12 THRU 28 DO
C***   FOR J:I THRU 28 DO
C***     IF I<=13 OR I>=27 THEN
C***     IF J<=13 OR J>=27 THEN
C*** PRINT("      M(",I,"",J,"")= M(",I,"",J,"")+ RHO*IYY*(",
C***                                     STRING(M[I,J]), " )" )$
C***
      M( 12 , 12 )= M( 12 , 12 )+ RHO*IYY*( 26.0/35.0 )
      M( 12 , 13 )= M( 12 , 13 )+ RHO*IYY*( 22.0/105.0 )
      M( 12 , 27 )= M( 12 , 27 )+ RHO*IYY*( 9.0/35.0 )
      M( 12 , 28 )= M( 12 , 28 )+ RHO*IYY*( -13.0/105.0 )
      M( 13 , 13 )= M( 13 , 13 )+ RHO*IYY*( 8.0/105.0 )
      M( 13 , 27 )= M( 13 , 27 )+ RHO*IYY*( 13.0/105.0 )
      M( 13 , 28 )= M( 13 , 28 )+ RHO*IYY*( -2.0/35.0 )
      M( 27 , 27 )= M( 27 , 27 )+ RHO*IYY*( 26.0/35.0 )
      M( 27 , 28 )= M( 27 , 28 )+ RHO*IYY*( -22.0/105.0 )
      M( 28 , 28 )= M( 28 , 28 )+ RHO*IYY*( 8.0/105.0 )
C*** IN-PLANE INERTIA TERM 3*[U]*[U]/R/R
C***
C***
C*** (c78) FOR I:1 THRU 18 DO

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C***   FOR J:I THRU 18 DO
C***   IF I<=3 OR I>=16 THEN
C***   IF J<=3 OR J>=16 THEN
C***PRINT("      M(",I,"","J,")= M(",I,"","J,")+RHO*IYY*("
C***PRINT                                STRING(M[I,J]), " " )$
C***
      M( 1 , 1 )= M( 1 , 1 )+RHO*IYY*( 181.0/(77.0*R2) )
      M( 1 , 2 )= M( 1 , 2 )+RHO*IYY*( 311.0/(385.0*R2) )
      M( 1 , 3 )= M( 1 , 3 )+RHO*IYY*( 281.0/(2310.0*R2) )
      M( 1 , 16 )= M( 1 , 16 )+RHO*IYY*( 50.0/(77.0*R2) )
      M( 1 , 17 )= M( 1 , 17 )+RHO*IYY*( -151.0/(385.0*R2) )
      M( 1 , 18 )= M( 1 , 18 )+RHO*IYY*( 181.0/(2310.0*R2) )
      M( 2 , 2 )= M( 2 , 2 )+RHO*IYY*( 416.0/(1155.0*R2) )
      M( 2 , 3 )= M( 2 , 3 )+RHO*IYY*( 23.0/(385.0*R2) )
      M( 2 , 16 )= M( 2 , 16 )+RHO*IYY*( 151.0/(385.0*R2) )
      M( 2 , 17 )= M( 2 , 17 )+RHO*IYY*( -38.0/(165.0*R2) )
      M( 2 , 18 )= M( 2 , 18 )+RHO*IYY*( 52.0/(1155.0*R2) )
      M( 3 , 3 )= M( 3 , 3 )+RHO*IYY*( 4.0/(385.0*R2) )
      M( 3 , 16 )= M( 3 , 16 )+RHO*IYY*( 181.0/(2310.0*R2) )
      M( 3 , 17 )= M( 3 , 17 )+RHO*IYY*( -52.0/(1155.0*R2) )
      M( 3 , 18 )= M( 3 , 18 )+RHO*IYY*( 2.0/(231.0*R2) )
      M( 16 , 16 )= M( 16 , 16 )+RHO*IYY*( 181.0/(77.0*R2) )
      M( 16 , 17 )= M( 16 , 17 )+RHO*IYY*( -311.0/(385.0*R2) )
      M( 16 , 18 )= M( 16 , 18 )+RHO*IYY*( 281.0/(2310.0*R2) )
      M( 17 , 17 )= M( 17 , 17 )+RHO*IYY*( 416.0/(1155.0*R2) )
      M( 17 , 18 )= M( 17 , 18 )+RHO*IYY*( -23.0/(385.0*R2) )
      M( 18 , 18 )= M( 18 , 18 )+RHO*IYY*( 4.0/(385.0*R2) )

C***
C***   INPLANE INERTIA      2.0*[ PSI ]*[ U/R ]
C***
C***   (c25) FOR I:1 THRU 28 DO
C***   FOR J:I THRU 28 DO
C***   IF I<=3 OR I>=12 AND I<=13 OR I>=16 AND I<=18 OR I>=27 THEN
C***   IF J<=3 OR J>=12 AND J<=13 OR J>=16 AND J<=18 OR J>=27 THEN
C***   IF K[I,J] # 0 THEN
C***PRINT("      M(",I,"","J,")= M(",I,"","J,")+ RHO*IYY*("
C***PRINT                                STRING(K[I,J]), " " )$
C***
      M( 1 , 12 )= M( 1 , 12 )+ RHO*IYY*( 32.0/(21.0*R) )
      M( 1 , 13 )= M( 1 , 13 )+ RHO*IYY*( 3.0/(7.0*R) )
      M( 1 , 27 )= M( 1 , 27 )+ RHO*IYY*( 10.0/(21.0*R) )
      M( 1 , 28 )= M( 1 , 28 )+ RHO*IYY*( -5.0/(21.0*R) )
      M( 2 , 12 )= M( 2 , 12 )+ RHO*IYY*( 11.0/(21.0*R) )
      M( 2 , 13 )= M( 2 , 13 )+ RHO*IYY*( 4.0/(21.0*R) )
      M( 2 , 27 )= M( 2 , 27 )+ RHO*IYY*( 29.0/(105.0*R) )
      M( 2 , 28 )= M( 2 , 28 )+ RHO*IYY*( -2.0/(15.0*R) )
      M( 3 , 12 )= M( 3 , 12 )+ RHO*IYY*( 5.0/(63.0*R) )
      M( 3 , 13 )= M( 3 , 13 )+ RHO*IYY*( 2.0/(63.0*R) )
      M( 3 , 27 )= M( 3 , 27 )+ RHO*IYY*( 17.0/(315.0*R) )
      M( 3 , 28 )= M( 3 , 28 )+ RHO*IYY*( -8.0/(315.0*R) )
      M( 12 , 16 )= M( 12 , 16 )+ RHO*IYY*( 10.0/(21.0*R) )
      M( 12 , 17 )= M( 12 , 17 )+ RHO*IYY*( -29.0/(105.0*R) )
      M( 12 , 18 )= M( 12 , 18 )+ RHO*IYY*( 17.0/(315.0*R) )

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M( 13 , 16 )= M( 13 , 16 )+ RHO*IYY*( 5.0/(21.0*R) )
M( 13 , 17 )= M( 13 , 17 )+ RHO*IYY*( -2.0/(15.0*R) )
M( 13 , 18 )= M( 13 , 18 )+ RHO*IYY*( 8.0/(315.0*R) )
M( 16 , 27 )= M( 16 , 27 )+ RHO*IYY*( 32.0/(21.0*R) )
M( 16 , 28 )= M( 16 , 28 )+ RHO*IYY*( -3.0/(7.0*R) )
M( 17 , 27 )= M( 17 , 27 )+ RHO*IYY*( -11.0/(21.0*R) )
M( 17 , 28 )= M( 17 , 28 )+ RHO*IYY*( 4.0/(21.0*R) )
M( 18 , 27 )= M( 18 , 27 )+ RHO*IYY*( 5.0/(63.0*R) )
M( 18 , 28 )= M( 18 , 28 )+ RHO*IYY*( -2.0/(63.0*R) )
C*** INPLANE INERTIA      -[V]*[PHI/R]
C*** IF M[I,J] # 0 THEN
C*** PRINT("      M(",I,"","J,")= M(",I,"","J,")- RHO*IYY*(",
C***                                STRING(M[I,J]),")" )$
M( 4 , 14 )= M( 4 , 14 )- RHO*IYY*( 16.0/21.0 )
M( 4 , 15 )= M( 4 , 15 )- RHO*IYY*( 3.0/14.0 )
M( 4 , 29 )= M( 4 , 29 )- RHO*IYY*( 5.0/21.0 )
M( 4 , 30 )= M( 4 , 30 )- RHO*IYY*( -5.0/42.0 )
M( 5 , 14 )= M( 5 , 14 )- RHO*IYY*( 11.0/42.0 )
M( 5 , 15 )= M( 5 , 15 )- RHO*IYY*( 2.0/21.0 )
M( 5 , 29 )= M( 5 , 29 )- RHO*IYY*( 29.0/210.0 )
M( 5 , 30 )= M( 5 , 30 )- RHO*IYY*( -1.0/15.0 )
M( 6 , 14 )= M( 6 , 14 )- RHO*IYY*( 5.0/126.0 )
M( 6 , 15 )= M( 6 , 15 )- RHO*IYY*( 1.0/63.0 )
M( 6 , 29 )= M( 6 , 29 )- RHO*IYY*( 17.0/630.0 )
M( 6 , 30 )= M( 6 , 30 )- RHO*IYY*( -4.0/315.0 )
M( 14 , 19 )= M( 14 , 19 )- RHO*IYY*( 5.0/21.0 )
M( 14 , 20 )= M( 14 , 20 )- RHO*IYY*( -29.0/210.0 )
M( 14 , 21 )= M( 14 , 21 )- RHO*IYY*( 17.0/630.0 )
M( 15 , 19 )= M( 15 , 19 )- RHO*IYY*( 5.0/42.0 )
M( 15 , 20 )= M( 15 , 20 )- RHO*IYY*( -1.0/15.0 )
M( 15 , 21 )= M( 15 , 21 )- RHO*IYY*( 4.0/315.0 )
M( 19 , 29 )= M( 19 , 29 )- RHO*IYY*( 16.0/21.0 )
M( 19 , 30 )= M( 19 , 30 )- RHO*IYY*( -3.0/14.0 )
M( 20 , 29 )= M( 20 , 29 )- RHO*IYY*( -11.0/42.0 )
M( 20 , 30 )= M( 20 , 30 )- RHO*IYY*( 2.0/21.0 )
M( 21 , 29 )= M( 21 , 29 )- RHO*IYY*( 5.0/126.0 )
M( 21 , 30 )= M( 21 , 30 )- RHO*IYY*( -1.0/63.0 )
C***
C*** OUTPLANE INERTIA    [THETA]**2.0
C***
C*** (c66) FOR I:7 THRU 23 DO
C***   FOR J:I THRU 23 DO
C***     IF I<=8 OR I>=22 THEN
C***       IF J<=8 OR J>=22 THEN
C***     PRINT("      M(",I,"","J,")= M(",I,"","J,")+ RHO*IZZ*(",
C***                                   STRING(M[I,J]),")" )$
C***
M( 7 , 7 )= M( 7 , 7 )+ RHO*IZZ*( 26.0/35.0 )
M( 7 , 8 )= M( 7 , 8 )+ RHO*IZZ*( 22.0/105.0 )
M( 7 , 22 )= M( 7 , 22 )+ RHO*IZZ*( 9.0/35.0 )
M( 7 , 23 )= M( 7 , 23 )+ RHO*IZZ*( -13.0/105.0 )
M( 8 , 8 )= M( 8 , 8 )+ RHO*IZZ*( 8.0/105.0 )
M( 8 , 22 )= M( 8 , 22 )+ RHO*IZZ*( 13.0/105.0 )

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M( 8 , 23 )= M( 8 , 23 )+ RHO*IZZ*( -2.0/35.0 )
M( 22 , 22 )= M( 22 , 22 )+ RHO*IZZ*( 26.0/35.0 )
M( 22 , 23 )= M( 22 , 23 )+ RHO*IZZ*( -22.0/105.0 )
M( 23 , 23 )= M( 23 , 23 )+ RHO*IZZ*( 8.0/105.0 )
C*** TWIST INERTIA [PHI]**2.0
C*** (c75) FOR I:14 THRU 30 DO
C***   FOR J:I THRU 30 DO
C***   IF I<=15 OR I>=29 THEN
C***   IF J<=15 OR J>=29 THEN
C*** PRINT("      M(",I,"","J,")= M(",I,"","J,")+ RHO*(IYY+IZZ)*("
C***                                     STRING(M[I,J]),")" )$
C***
C***
C***
M( 14 , 14 )= M( 14 , 14 )+ RHO*(IYY+IZZ)*( 26.0/35.0 )
M( 14 , 15 )= M( 14 , 15 )+ RHO*(IYY+IZZ)*( 22.0/105.0 )
M( 14 , 29 )= M( 14 , 29 )+ RHO*(IYY+IZZ)*( 9.0/35.0 )
M( 14 , 30 )= M( 14 , 30 )+ RHO*(IYY+IZZ)*( -13.0/105.0 )
M( 15 , 15 )= M( 15 , 15 )+ RHO*(IYY+IZZ)*( 8.0/105.0 )
M( 15 , 29 )= M( 15 , 29 )+ RHO*(IYY+IZZ)*( 13.0/105.0 )
M( 15 , 30 )= M( 15 , 30 )+ RHO*(IYY+IZZ)*( -2.0/35.0 )
M( 29 , 29 )= M( 29 , 29 )+ RHO*(IYY+IZZ)*( 26.0/35.0 )
M( 29 , 30 )= M( 29 , 30 )+ RHO*(IYY+IZZ)*( -22.0/105.0 )
M( 30 , 30 )= M( 30 , 30 )+ RHO*(IYY+IZZ)*( 8.0/105.0 )
C***
C*** COUPLING INERTIA [THETA]*[PSI]
C***
C*** (c62) FOR I:7 THRU 28 DO
C***   FOR J:I THRU 28 DO
C***   IF I<=8 OR I>=12 AND I<=13 OR I>=22 AND I<=23 OR I>=27 THEN
C***   IF J<=8 OR J>=12 AND J<=13 OR J>=22 AND J<=23 OR J>=27 THEN
C***   IF M[I,J] # 0 THEN
C*** PRINT("      M(",I,"","J,")= M(",I,"","J,")- RHO*IYZ*(("
C***                                     STRING(M[I,J]),")" )$
C***
C***
M( 7 , 12 )= M( 7 , 12 )- RHO*IYZ*( 26.0 /35.0 )
M( 7 , 13 )= M( 7 , 13 )- RHO*IYZ*( 22.0 /105.0 )
M( 7 , 27 )= M( 7 , 27 )- RHO*IYZ*( 9.0 /35.0 )
M( 7 , 28 )= M( 7 , 28 )- RHO*IYZ*( -13.0 /105.0 )
M( 8 , 12 )= M( 8 , 12 )- RHO*IYZ*( 22.0 /105.0 )
M( 8 , 13 )= M( 8 , 13 )- RHO*IYZ*( 8.0 /105.0 )
M( 8 , 27 )= M( 8 , 27 )- RHO*IYZ*( 13.0 /105.0 )
M( 8 , 28 )= M( 8 , 28 )- RHO*IYZ*( -2.0 /35.0 )
M( 12 , 22 )= M( 12 , 22 )- RHO*IYZ*( 9.0 /35.0 )
M( 12 , 23 )= M( 12 , 23 )- RHO*IYZ*( -13.0 /105.0 )
M( 13 , 22 )= M( 13 , 22 )- RHO*IYZ*( 13.0 /105.0 )
M( 13 , 23 )= M( 13 , 23 )- RHO*IYZ*( -2.0 /35.0 )
M( 22 , 27 )= M( 22 , 27 )- RHO*IYZ*( 26.0 /35.0 )
M( 22 , 28 )= M( 22 , 28 )- RHO*IYZ*( -22.0 /105.0 )
M( 23 , 27 )= M( 23 , 27 )- RHO*IYZ*( -22.0 /105.0 )
M( 23 , 28 )= M( 23 , 28 )- RHO*IYZ*( 8.0 /105.0 )
C***
C*** COUPLING INERTIA 2.0*[ PRIME(THETA) ]*[ U/R ]
C***

```

```

C*** (c28) FOR I:1 THRU 23 DO
C***   FOR J:1 THRU 23 DO
C***   IF I<=3 OR I>=7 AND I<=8 OR I>=16 AND I<=18 OR I>=22 THEN
C***   IF J<=3 OR J>=7 AND J<=8 OR J>=16 AND J<=18 OR J>=22 THEN
C***   IF K[I,J] # 0 THEN
C***   PRINT("      M(",I,"",J,"")= M(",I,"",J,"")- RHO* IYZ*(",
C***                                     STRING(K[I,J]),")" )$
      M( 1 , 7 )= M( 1 , 7 )- RHO* IYZ*( 32.0/(21.0*R) )
      M( 1 , 8 )= M( 1 , 8 )- RHO* IYZ*( 3.0/(7.0*R) )
      M( 1 , 22 )= M( 1 , 22 )- RHO* IYZ*( 10.0/(21.0*R) )
      M( 1 , 23 )= M( 1 , 23 )- RHO* IYZ*( -5.0/(21.0*R) )
      M( 2 , 7 )= M( 2 , 7 )- RHO* IYZ*( 11.0/(21.0*R) )
      M( 2 , 8 )= M( 2 , 8 )- RHO* IYZ*( 4.0/(21.0*R) )
      M( 2 , 22 )= M( 2 , 22 )- RHO* IYZ*( 29.0/(105.0*R) )
      M( 2 , 23 )= M( 2 , 23 )- RHO* IYZ*( -2.0/(15.0*R) )
      M( 3 , 7 )= M( 3 , 7 )- RHO* IYZ*( 5.0/(63.0*R) )
      M( 3 , 8 )= M( 3 , 8 )- RHO* IYZ*( 2.0/(63.0*R) )
      M( 3 , 22 )= M( 3 , 22 )- RHO* IYZ*( 17.0/(315.0*R) )
      M( 3 , 23 )= M( 3 , 23 )- RHO* IYZ*( -8.0/(315.0*R) )
      M( 7 , 16 )= M( 7 , 16 )- RHO* IYZ*( 10.0/(21.0*R) )
      M( 7 , 17 )= M( 7 , 17 )- RHO* IYZ*( -29.0/(105.0*R) )
      M( 7 , 18 )= M( 7 , 18 )- RHO* IYZ*( 17.0/(315.0*R) )
      M( 8 , 16 )= M( 8 , 16 )- RHO* IYZ*( 5.0/(21.0*R) )
      M( 8 , 17 )= M( 8 , 17 )- RHO* IYZ*( -2.0/(15.0*R) )
      M( 8 , 18 )= M( 8 , 18 )- RHO* IYZ*( 8.0/(315.0*R) )
      M( 16 , 22 )= M( 16 , 22 )- RHO* IYZ*( 32.0/(21.0*R) )
      M( 16 , 23 )= M( 16 , 23 )- RHO* IYZ*( -3.0/(7.0*R) )
      M( 17 , 22 )= M( 17 , 22 )- RHO* IYZ*( -11.0/(21.0*R) )
      M( 17 , 23 )= M( 17 , 23 )- RHO* IYZ*( 4.0/(21.0*R) )
      M( 18 , 22 )= M( 18 , 22 )- RHO* IYZ*( 5.0/(63.0*R) )
      M( 18 , 23 )= M( 18 , 23 )- RHO* IYZ*( -2.0/(63.0*R) )

C***
C***   COUPLING INERTIA   [W]*[PHI/R]
C***
C*** (c75) FOR I:9 THRU 30 DO
C***   FOR J:1 THRU 30 DO
C***   IF I<=11 OR I>=14 AND I<=15 OR I>=24 AND I<=26 OR I>=29 THEN
C***   IF J<=11 OR J>=14 AND J<=15 OR J>=24 AND J<=26 OR J>=29 THEN
C***   IF M[I,J] # 0 THEN
C***   PRINT("      M(",I,"",J,"")= M(",I,"",J,"")- RHO* IYZ*(",
C***                                     STRING(M[I,J]),")" )$
C***
      M( 9 , 14 )= M( 9 , 14 )- RHO* IYZ*( 16.0/21.0 )
      M( 9 , 15 )= M( 9 , 15 )- RHO* IYZ*( 3.0/14.0 )
      M( 9 , 29 )= M( 9 , 29 )- RHO* IYZ*( 5.0/21.0 )
      M( 9 , 30 )= M( 9 , 30 )- RHO* IYZ*( -5.0/42.0 )
      M( 10 , 14 )= M( 10 , 14 )- RHO* IYZ*( 11.0/42.0 )
      M( 10 , 15 )= M( 10 , 15 )- RHO* IYZ*( 2.0/21.0 )
      M( 10 , 29 )= M( 10 , 29 )- RHO* IYZ*( 29.0/210.0 )
      M( 10 , 30 )= M( 10 , 30 )- RHO* IYZ*( -1.0/15.0 )
      M( 11 , 14 )= M( 11 , 14 )- RHO* IYZ*( 5.0/126.0 )
      M( 11 , 15 )= M( 11 , 15 )- RHO* IYZ*( 1.0/63.0 )
      M( 11 , 29 )= M( 11 , 29 )- RHO* IYZ*( 17.0/630.0 )

```

```

M( 11 , 30 ) = M( 11 , 30 ) - RHO* IYZ*( -4.0/315.0 )
M( 14 , 24 ) = M( 14 , 24 ) - RHO* IYZ*( 5.0/21.0 )
M( 14 , 25 ) = M( 14 , 25 ) - RHO* IYZ*( -29.0/210.0 )
M( 14 , 26 ) = M( 14 , 26 ) - RHO* IYZ*( 17.0/630.0 )
M( 15 , 24 ) = M( 15 , 24 ) - RHO* IYZ*( 5.0/42.0 )
M( 15 , 25 ) = M( 15 , 25 ) - RHO* IYZ*( -1.0/15.0 )
M( 15 , 26 ) = M( 15 , 26 ) - RHO* IYZ*( 4.0/315.0 )
M( 24 , 29 ) = M( 24 , 29 ) - RHO* IYZ*( 16.0/21.0 )
M( 24 , 30 ) = M( 24 , 30 ) - RHO* IYZ*( -3.0/14.0 )
M( 25 , 29 ) = M( 25 , 29 ) - RHO* IYZ*( -11.0/42.0 )
M( 25 , 30 ) = M( 25 , 30 ) - RHO* IYZ*( 2.0/21.0 )
M( 26 , 29 ) = M( 26 , 29 ) - RHO* IYZ*( 5.0/126.0 )
M( 26 , 30 ) = M( 26 , 30 ) - RHO* IYZ*( -1.0/63.0 )

```

```

L=2.0*R*ANGLE

```

```

V(1)=1.0

```

```

V(2)=0.5*L

```

```

V(3)=0.25*L*L

```

```

V(4)=1.0

```

```

V(5)=0.5*L

```

```

V(6)=0.25*L*L

```

```

V(7)=1.0

```

```

V(8)=0.5*L

```

```

V(9)=1.0

```

```

V(10)=0.5*L

```

```

V(11)=0.25*L*L

```

```

V(12)=1.0

```

```

V(13)=0.5*L

```

```

V(14)=1.0

```

```

V(15)=0.5*L

```

```

V(16)=1.0

```

```

V(17)=0.5*L

```

```

V(18)=0.25*L*L

```

```

V(19)=1.0

```

```

V(20)=0.5*L

```

```

V(21)=0.25*L*L

```

```

V(22)=1.0

```

```

V(23)=0.5*L

```

```

V(24)=1.0

```

```

V(25)=0.5*L

```

```

V(26)=0.25*L*L

```

```

V(27)=1.0

```

```

V(28)=0.5*L

```

```

V(29)=1.0

```

```

V(30)=0.5*L

```

```

DO 10 I=1,30

```

```

    DO 20 J=I,30

```

```

        M(I,J)=M(I,J)*R*ANGLE*V(I)*V(J)

```

```

20    CONTINUE

```

```

10    CONTINUE

```

```

    RETURN

```

```

    END

```

Appendix C

Theoretical Tests For 22 DOF/30 DOF Elements

Appendix C

90° Curved Cantilever Subject to Out of Plane Load

Analysis type: Static

Problem:

For a curved cantilever beam of radius 10 units and length spanning an angle of $\frac{\pi}{2}$ find the out of plane deflections and twist when a unit out of plane load is applied at its free end. Assume $E = 30 \times 10^6$, $I_{yy} = I_{zz} = 0.04909$, $A = 0.7854$, $A' = 0.7854$ and $GC = 0.09817$.

Theoretical Solution: Use the method of unit loads.

Let M' represent the moment due to a unit load applied at α . Let T' represent the torque due to a unit load applied at α . Let S' represent the shear due to a unit load applied at α . Let M , T and S be respectively the moment, torque and shear due to force P . Then:

$$M = PR \sin \xi$$

$$M' = R \sin (\xi - \alpha)$$

$$T = PR(1 - \cos \xi)$$

$$T' = R(1 - \cos [\xi - \alpha])$$

$$S = P$$

$$S' = 1$$

Therefore by the method of unit loads,

$$\begin{aligned} v(\alpha) &= \int_{\alpha}^{\frac{\pi}{2}} \frac{M M'}{EI} R d\xi + \int_{\alpha}^{\frac{\pi}{2}} \frac{T T'}{GC} R d\xi + \int_{\alpha}^{\frac{\pi}{2}} \frac{S S'}{GA'} R d\xi \\ &= \int_{\alpha}^{\frac{\pi}{2}} \frac{PR^3}{EI} \sin \xi \sin (\xi - \alpha) d\xi + \int_{\alpha}^{\frac{\pi}{2}} \frac{PR^3}{GC} (1 - \cos \xi)(1 - \cos [\xi - \alpha]) d\xi + \int_{\alpha}^{\frac{\pi}{2}} \frac{PR}{GA'} d\xi \end{aligned}$$

$$\begin{aligned} v(\alpha) &= \frac{PR^3}{EI} \left[\frac{(\pi - 2\alpha) \cos \alpha}{4} \right] \\ &\quad + \frac{PR^3}{GC} \left[\frac{4 \sin \alpha + (\pi - 2\alpha - 4) \cos \alpha + 2\pi - 4\alpha - 4}{4} \right] \\ &\quad + \frac{PR}{GA'} \left[\frac{(\pi - 2\alpha)}{2} \right] \end{aligned}$$

$$\begin{aligned} \frac{dv(\alpha)}{-R d\alpha} = & \frac{PR^2}{EI} \left[\frac{(\pi - 2\alpha) \sin \alpha + 2 \cos \alpha}{4} \right] \\ & + \frac{PR^2}{GC} \left[\frac{(\pi - 2\alpha - 4) \sin \alpha - 2 \cos \alpha + 4}{4} \right] \\ & + \frac{P}{GA^*} \end{aligned}$$

$$\begin{aligned} \frac{d^2v(\alpha)}{R^2 d^2\alpha} = & \frac{PR}{EI} \left[\frac{4 \sin \alpha - (\pi - 2\alpha) \cos \alpha}{4} \right] \\ & - \frac{PR}{GC} \left[\frac{(\pi - 2\alpha - 4) \cos \alpha}{4} \right] \end{aligned}$$

$$\begin{aligned} \phi(\alpha) = & \int_{\alpha}^{\frac{1}{2}\pi} -\frac{PR^2}{EI} \sin \xi \sin(\xi - \alpha) d\xi \\ & + \int_{\alpha}^{\frac{1}{2}\pi} \frac{PR^2}{GC} \cos(\xi - \alpha) (1 - \cos \xi) d\xi \end{aligned}$$

$$\begin{aligned} \phi(\alpha) = & -\frac{PR^2}{EI} \left[\frac{(\pi - 2\alpha) \cos \alpha}{4} \right] \\ & - \frac{PR^2}{GC} \left[\frac{(\pi - 2\alpha - 4) \cos \alpha}{4} \right] \end{aligned}$$

$$\begin{aligned} \frac{d\phi(\alpha)}{-R d\alpha} = & \frac{-PR}{EI} \left[\frac{(\pi - 2\alpha) \sin \alpha + 2 \cos \alpha}{4} \right] \\ & - \frac{PR}{GC} \left[\frac{(\pi - 2\alpha - 4) \sin \alpha + 2 \cos \alpha}{4} \right] \end{aligned}$$

22 DOF Out of Plane Static Test (No Shear)

Freedom at free end	FEM	Theory	Error %
v	845.3×10^{-3}	845.3×10^{-3}	0
v'	77.75×10^{-6}	77.75×10^{-6}	0
v''	-3.453×10^{-6}	-3.453×10^{-6}	0
ϕ	-34.53×10^{-6}	-34.53×10^{-6}	0
ϕ'	-7.777×10^{-6}	-7.775×10^{-6}	0.02

30 DOF Out of Plane Static Test

Freedom at free end	FEM	Theory	Error %
v	847.1×10^{-6}	847.1×10^{-6}	0
$\theta_a = v' - \theta_b$	109.5×10^{-9}	109.5×10^{-9}	0
$\theta'_a = v'' - \theta'_b$	4.900×10^{-20}	0	-
θ_b	77.75×10^{-6}	77.75×10^{-6}	0
θ'_b	-3.464×10^{-6}	-3.453×10^{-6}	0.32
ϕ	-34.53×10^{-6}	-34.53×10^{-6}	0
ϕ'	-7.778×10^{-6}	-7.775×10^{-6}	0.04

90° Curved Cantilever Subject to In-Plane Load

Analysis type Static
Problem

A curved cantilever beam of radius 10 units is subject to a unit axial load at its free end. Find the resultant displacement at the free end. Use 5 elements and assume the following properties. $E = 30.0 \times 10^6$, $A = 0.7854$, $G = 11.628 \times 10^6$, $A^* = 0.7854$, and $I_{yy} = I_{zz} = 0.04909$

Theoretical Solution Use the method of unit loads

Let M' represent the moment due to a unit load applied at α . Let F' represent the axial load due to a unit load applied at α . Let S' represent the shear due to a unit load applied at α . Let M , F and S be respectively the moment, axial force and shear due to force P . Then

For w	$M = PR(1 - \cos \xi)$	For u	
	$M' = R \sin(\xi - \alpha)$		$M' = R[1 - \cos(\xi - \alpha)]$
	$F = P \cos \xi$		
	$F' = -\sin(\xi - \alpha)$		$F' = \cos(\xi - \alpha)$
	$S = P \sin \xi$		
	$S' = \cos(\xi - \alpha)$		$S' = \sin(\xi - \alpha)$

By the method of unit loads:

$$\begin{aligned}
 w(\alpha) = & \int_0^1 \frac{-PR}{EA} \cos(\xi) \sin(\xi - \alpha) d\xi \\
 & + \int_0^1 \frac{PR^3}{EI} (1 - \cos \xi) \sin(\xi - \alpha) d\xi \\
 & + \int_0^1 \frac{PR}{GA^*} \sin \xi \cos(\xi - \alpha) d\xi
 \end{aligned}$$

$$w(\alpha) = \frac{PR}{EA} \left[\frac{(\pi - 2\alpha) \sin \alpha + 2 \cos \alpha}{4} \right] \\ + \frac{PR^3}{EI} \left[\frac{(\pi - 2\alpha - 4) \sin \alpha - 2 \cos \alpha + 4}{4} \right] \\ + \frac{PR}{GA^*} \left[\frac{(\pi - 2\alpha) \sin \alpha + 2 \cos \alpha}{4} \right]$$

$$\frac{dw}{-R d\alpha} = \frac{P}{EA} \left[\frac{(2\alpha - \pi) \cos \alpha}{4} \right] \\ + \frac{PR^2}{EI} \left[\frac{(\pi - 2\alpha - 4) \cos \alpha}{4} \right] \\ + \frac{P}{GA^*} \left[\frac{4 \sin \alpha - (\pi - 2\alpha) \cos \alpha}{4} \right]$$

$$\frac{d^2 w}{R^2 d^2 \alpha} = \frac{P}{EAR} \left[\frac{(2\alpha - \pi) \sin \alpha - 2 \cos \alpha}{4} \right] \\ + \frac{PR}{EI} \left[\frac{(4 + 2\alpha - \pi) \sin \alpha - 2 \cos \alpha}{4} \right] \\ + \frac{P}{GA^* R} \left[\frac{(2\alpha - \pi) \sin \alpha - 6 \cos \alpha}{4} \right]$$

$$u(\alpha) = \int_{\alpha}^{\frac{\pi}{2}} \frac{PR}{EA} \cos \xi \cos(\xi - \alpha) d\xi \\ + \frac{PR^3}{EI} \int_{\alpha}^{\frac{\pi}{2}} (1 - \cos \xi)(1 - \cos(\xi - \alpha)) d\xi \\ + \frac{PR}{GA^*} \int_{\alpha}^{\frac{\pi}{2}} \sin \xi \sin(\xi - \alpha) d\xi$$

$$u(\alpha) = \frac{PR}{EA} \left[\frac{(\pi - 2\alpha) \cos \alpha}{4} \right] \\ + \frac{PR^3}{EI} \left[\frac{4 \sin \alpha + (\pi - 2\alpha - 4) \cos \alpha + 2\pi - 4\alpha - 4}{4} \right] \\ + \frac{PR}{GA^*} \left[\frac{(\pi - 2\alpha) \cos \alpha}{4} \right]$$

$$\frac{du}{-R d\alpha} = \frac{P}{EA} \left[\frac{(\pi - 2\alpha) \sin \alpha + 2 \cos \alpha}{4} \right] \\ + \frac{PR^2}{EI} \left[\frac{(\pi - 2\alpha - 4) \sin \alpha - 2 \cos \alpha + 4}{4} \right] \\ + \frac{P}{GA^*} \left[\frac{(\pi - 2\alpha) \sin \alpha + 2 \cos \alpha}{4} \right]$$

$$\begin{aligned}
 \frac{d^2 u}{R^2 d^2 \alpha} = & \frac{P}{EAR} \left[\frac{4 \sin \alpha - (\pi - 2\alpha) \cos \alpha}{4} \right] \\
 & + \frac{PR}{EI} \left[\frac{(4 + 2\alpha - \pi) \cos \alpha}{4} \right] \\
 & + \frac{P}{GA^* R} \left[\frac{4 \sin \alpha - (\pi - 2\alpha) \cos \alpha}{4} \right]
 \end{aligned}$$

22 DOF In-Plane Static Test (No Shear)

Freedom at free end	FEM	Theory	Error %
u	242.5×10^{-6}	242.2×10^{-6}	0.12
u'	33.99×10^{-6}	33.97×10^{-6}	0.06
u''	1.457×10^{-6}	1.454×10^{-6}	0.21
w	339.5×10^{-6}	339.3×10^{-6}	0.06
w'	14.57×10^{-6}	14.54×10^{-6}	0.21
w''	-3.395×10^{-5}	-3.397×10^{-5}	-0.06

30 DOF In-Plane Static Test

Freedom at free end	FEM	Theory	Error %
u	243.4×10^{-6}	243.1×10^{-6}	0.12
u'	34.05×10^{-6}	34.08×10^{-6}	-0.09
u''	1.449×10^{-6}	1.445×10^{-6}	0.28
w	340.1×10^{-6}	340.8×10^{-6}	-0.21
$\psi_a = w' - \psi_b$	-18.77×10^{-6}	0	-
$\psi'_a = w'' - \psi'_b$	-10.95×10^{-6}	-10.95×10^{-6}	0.0
ψ_b	14.49×10^{-6}	14.54×10^{-6}	-0.34
ψ'_b	-3.402×10^{-5}	-3.397×10^{-5}	0.15

Straight Cantilever Shear Beam

Analysis type Modal
Problem

For a cantilever beam of length 10 units find the first 4 natural frequencies using four 30 DOF elements. Assume: $\rho = 0.001$, $I_{yy} = I_{zz} = 0.05$, $A = 1.0$, $E = 10 \times 10^6$, $G = 3.846 \times 10^6$, $C = 0.1$, and $R = 100000.0$.

Theoretical Solution.

Kapur [24] solved the specific case of a cantilever beam using equations developed by Huang [23] which were based on Timoshenko's shear beam theory.

30 DOF Shear Cantilever Beam Test

Mode Number	Exact Solution	FEM O/P	Error %	FEM I/P	Error %
1	3.284	3.288	0.12	3.288	0.12
2	15.488	15.569	0.52	15.569	0.52
3	34.301	34.541	0.70	34.541	0.70
4	53.652	54.130	0.89	54.130	0.89

Appendix D

Convergence Test for 22 and 30 DOF Element

The following are results of convergence tests done with the 22 DOF element using Volterra's ring number 5. The number of elements to model a half circle is given followed by the percentage error from theory.

OUT OF PLANE BENDING

N	EXPT.	2EL	%	4EL	%	6EL	%	8EL	%
2	210.	223.	6.2	222.	5.7	222.	5.7	222.	5.7
3	598.	617.	3.2	606.	1.3	606.	1.3	606.	1.3
4	1147.	1179.	2.8	1164.	1.5	1162.	1.3	1161.	1.2
5	1857.			1886.	1.6	1881.	1.3	1880.	1.2
6	2720.			2781.	2.2	2763.	1.6	2759.	1.4
7	3735.			3868.	3.6	3803.	1.8	3798.	1.7
8	4893.			5039.	3.0	5008.	2.4	4998.	2.1
9	6198.					6384.	3.0	6350.	2.5
10	7647.					7939.	3.8	7866.	2.7
11	9219.					9686.	5.1	9543.	3.5

IN-PLANE BENDING

N	EXPT.	2EL	%	4EL	%	6EL	%	8EL	%
2	216.	228.	5.6	227.	5.1	227.	5.1	227.	5.1
3	608.	629.	3.5	615.	1.2	615.	1.2	615.	1.2
4	1162.	1286.		1173.	0.9	1173.	0.9	1173.	0.9
5	1869.			1899.	1.6	1893.	1.3	1893.	1.3
6	2734.			2791.	2.1	2775.	1.5	2773.	1.4
7	3747.			3862.	3.1	3817.	1.9	3813.	1.8
8	4916.			5083.	3.4	5023.	2.2	5013.	2.0
9	6221.					6395.	2.8	6369.	2.4
10	7668.					7941.	3.6	7885.	2.8
11	9252.					9670.	4.5	9563.	3.4

The following are results of convergence tests done with the 30 DOF element using Volterra's ring number 1. The number of elements to model a half circle is given followed by the percentage error from theory.

OUT OF PLANE BENDING

N	EXPT.	2EL	%	4EL	%	6EL	%	8EL	%
2	1556.	1475.	-5.2	1461.	-6.1	1460.	-6.2	1460.	-6.2
3	4119.	4097.	-0.5	3849.	-6.6	3844.	-6.7	3843.	-6.7
4	7287.			6802.	-6.6	6783.	-6.9	6779.	-7.0
5	10820.			10180.	-5.9	10055.	-7.1	10042.	-7.2
6	14561.			13986.	-3.9	13520.	-7.1	13496.	-7.3
7	18428.			18523.	0.5	17157.	-6.9	17061.	-7.4
8	22358.					20928.	-6.4	20681.	-7.5
9	26305.					24923.	-7.4	24371.	-7.4
10	30250.					29304.	-3.1	28101.	-7.1
11	34195.					34234.	0.1	33397.	-2.3

IN-PLANE BENDING

N	EXPT.	2EL	%	4EL	%	6EL	%	8EL	%
2	1666.	1634.	-1.9	1633.	-2.0	1632.	-2.0	1632.	-2.0
3	4387.	4426.	0.8	4291.	-2.2	4287.	-2.3	4287.	-2.3
4	7739.	8924.	15.3	7549.	-2.5	7548.	-2.5	7545.	-2.5
5	11459.			11259.	-1.7	11172.	-2.5	11162.	-2.6
6	15402.			15301.	-0.7	14989.	-2.7	14992.	-2.7
7	19463.			19720.	1.3	19015.	-2.3	18947.	-2.7
8	23576.					23134.	-1.9	22944.	-2.7
9	27706.					27390.	-1.1	27062.	-2.3
10	31822.					31815.	-0.02	31187.	-2.0
11	35908.					36372.	1.3	35359.	-1.5

END

112.06.186

FIN