FRACTION SYMBOLS AND THEIR RELATION TO CONCEPTUAL FRACTION
KNOWLEDGE FOR STUDENTS IN GRADES 4 AND 6

By

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Abstract

How do students make sense of fractions? Formal fraction knowledge begins when students start mapping fractions shown visually (e.g., area models), with symbols (e.g., $\frac{3}{4}$) and with words (e.g., three-quarters). Students were recruited from three schools that service rural and small-town communities. Participating students in grade 4 ($N=64$) and grade 6 ($N = 66$) completed measures of cognition, language, and three novel measures developed for this study: mathematical vocabulary, orthography (i.e., the conventions for writing symbolic math), and fraction mapping. Five months later, their conceptual fraction skills (i.e., mapping, word problems and number line) were measured. I used two analytical approaches to examine the role of fraction mapping as students acquire conceptual fraction knowledge.

In Study 1 (Chapters 4 and 5), I tested a path model in which mathematical vocabulary and orthography predicted fraction mapping, and fraction mapping predicted conceptual fraction skills. The model was largely supported for both grade 4 and grade 6. Moreover, mathematical vocabulary also predicted conceptual fraction skills for sixth graders. Thus, once students have sufficient knowledge of fraction mappings, other skills such as mathematical vocabulary may contribute more strongly to students' knowledge of fraction concepts.

In Study 2 (Chapter 6), I used latent profile analysis to group students based on their fraction number line estimation. Three groups emerged. Relational estimators had the most advanced fraction concepts because they viewed the fraction as a unit. Compared to the other groups, relational estimators were more likely to be in sixth grade, have better mapping skills and more accurate whole number line estimation. Whole-component and denominator estimators, respectively, interpreted the fraction based on the magnitudes of both components (i.e., the numerator and denominator) or just the denominator. Only fraction mapping skills

differentiated whole-component estimators from denominator estimators. Thus, students' knowledge of fraction mappings is a precursor to interpreting fractions as units rather than as composites, and therefore necessary for successfully placing those fractions on a number line.

In summary, this research shows that students who struggle to acquire fraction concepts in grades 4 and 6 have not mastered fraction mappings. Knowing how fraction symbols are connected to magnitude is foundational knowledge for fraction learning.

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Table of Contents

Abstract	ii
Acknowledgements	iv
Table of Contents	v
List of Tables	ix
List of Illustrations	x
List of Appendices	xi
CHAPTER 1: INTRODUCTION	12
Theories of Fraction Development	12
The Transition from Informal to Formal Fraction Knowledge.....	18
Early Fraction Concepts.....	18
Growth in Formal Fraction Knowledge.....	23
Advanced Fraction Concepts	23
The Current Dissertation.....	24
CHAPTER 2: LITERATURE REVIEW	26
Fraction Concepts	27
Language and Fractions	30
Covariates of Fraction Skills.....	44
Quantity/Arithmetic Knowledge and Fractions	44
Attention/Working Memory and Fractions.....	46

The Current Research	48
Hypotheses and Research Questions	48
CHAPTER 3: METHODS	52
Recruitment.....	52
Participants.....	53
Procedure	53
Measures	53
Whole Number Skills.....	55
Attention and Working Memory.....	56
Language Skills.....	58
Fraction Skills	60
Analysis Plan	65
CHAPTER 4: INDIVIDUAL DIFFERENCES IN FRACTION KNOWLEDGE PART 1	66
Descriptive Statistics.....	66
Whole Number Skills.....	68
Math Language Skills	68
Fraction Skills	69
Correlations.....	74
Fraction Skills and Language.....	74
Covariates of Fraction Skills.....	78
Chapter Summary	80
CHAPTER 5: INDIVIDUAL DIFFERENCES IN FRACTION KNOWLEDGE PART 2	81

Model Testing	84
Fraction Skills	84
Chapter Summary	92
CHAPTER 6: PROFILES OF FRACTION NUMBER LINE STRATEGIES	94
Number Line Estimation Strategies	96
Appropriate Fraction Number Line Strategies.....	96
Number Line Strategies based on Misconceptions	98
Skills that Relate to Successful Strategy Use	100
Knowledge of Fraction Symbols	100
Number Line Knowledge.....	101
Whole Number Arithmetic	101
Current “Study”	102
Method	103
Measures	103
Results.....	106
Descriptives.....	106
Latent Profile Analysis	109
Discussion.....	121
Predictors of Fraction Number Line Performance.....	124
Limitations and Future Research	126
Educational Implications	127
CHAPTER 7: GENERAL DISCUSSION	129
Individual Differences in Fraction Knowledge.....	133

Hypotheses 1 and 2	134
Hypothesis 3: Mathematical Vocabulary and Fraction Development	136
Hypothesis 4: Fraction Mapping and Fraction Development	136
Profiles of Fraction Number Line Skills	137
Theoretical Implications	139
Limitations and Future Research	141
Practical Implications.....	144
REFERENCES	147
APPENDICES	177

List of Tables

Table 3.1	54
Table 3.2	62
Table 3.3	63
Table 4.1	67
Table 4.2	74
Table 4.3	75
Table 4.4	79
Table 6.1	105
Table 6.2	107
Table 6.3	108
Table 6.4	109
Table 6.5	111
Table 6.6	115
Table 6.7	121

List of Illustrations

Figure 1.1	17
Figure 1.2	22
Figure 2.1	50
Figure 3.1	59
Figure 3.2	60
Figure 3.3	63
Figure 4.1	70
Figure 5.1	83
Figure 5.2	86
Figure 5.3	89
Figure 5.4	91
Figure 6.1	95
Figure 6.2	104
Figure 6.3	112
Figure 7.1	131
Figure 7.2	133
Figure 7.3	138
Figure 7.4	145

List of Appendices

Appendix A.....	177
Appendix B.....	180
Appendix C.....	189
Appendix D.....	193
Appendix E.....	199
Appendix F.....	205

CHAPTER 1: INTRODUCTION

Many children and adults experience difficulty with fractions (Brizuela, 2006; Hecht et al., 2007). These difficulties start in elementary school and persist into high school and beyond (Siegler & Lortie-Forgues, 2017; Siegler & Pyke, 2013). Educators are concerned about students' difficulties with fractions because fraction knowledge is a foundation for later success in algebra and for more advanced math learning (Booth & Newton, 2012; Hurst & Cordes, 2018a; Siegler et al., 2012). Moreover, people need fraction knowledge to successfully participate in STEM fields (i.e., science, technology, engineering and mathematics; Hansen et al., 2017). Thus, fractions are important, but why are fractions often difficult? To understand why people experience difficulty with fractions, it is important to understand how they acquire formal fraction knowledge and how that knowledge changes over time. In this dissertation, I examined students' developing fraction knowledge by considering the most basic element of formal fraction knowledge – the fraction symbol. I proposed and tested a model in which the fraction symbol is placed at the centre of fraction learning and provide evidence in favour of this model. More specifically, the goals of this dissertation were to determine which skills support students' ability to map the fraction symbol to its referent and how students' knowledge of fraction mapping supports the development of other fraction skills.

Theories of Fraction Development

Fractions are complex, in part, because they represent rational numbers and thus have features that are different than those of whole numbers. These differences are commonly cited as a source of fraction errors (McMullen et al., 2018; Ni & Zhou, 2005; Vamvakoussi & Vosniadou, 2004). For example, students may incorrectly reason that $\frac{2}{9}$ is greater than $\frac{1}{2}$ because 2 is greater than 1 and 9 is greater than 2; they are focusing on the whole number components of

the fraction rather than on the relation between those components (Fazio et al., 2016; Mazzocco & Devlin, 2008). As such, the dominant theories of fraction development focus on how students transition from reasoning about integers to reasoning about fractions. There are two views on this transition; one describes a discontinuous transition whereas the other describes a continuous transition.

The discontinuous view of fraction learning is focused on *conceptual change*. According to this view, whole number knowledge interferes with fraction learning and thus children need to restructure their concept of numbers to accommodate the concept of fractions (Ni & Zhou, 2005; Van Hoof et al., 2017). Proponents of the view that conceptual change is the mechanism for fraction learning argue that whole number concepts interfere with fraction learning. Many whole number concepts do not generalize to fractions, leading to a whole number bias (Ni & Zhou, 2005). For example, magnitude representations are different between whole numbers and fractions -- small integers represent small quantities whereas fraction magnitude depends on the relation between integers (Kainulainen et al., 2017; Van Hoof et al., 2017). Moreover, the counting successor function (i.e., $n + 1$ is the next number in a count sequence) does not apply to fractions -- there are an infinite number of fractions between any two fractions; McMullen et al., 2015; Van Hoof et al., 2017). The whole number bias leads to errors when children reason about fraction magnitudes (Fazio et al., 2016; Kainulainen et al., 2017; Mazzocco & Devlin, 2008). In summary, according to conceptual change, children need to overcome their whole number bias to learn and fraction concepts.

In contrast, continuous views of fraction learning are focused on the conceptual similarities between fractions and whole numbers (Sophian, 2017). On this view, fraction knowledge develops as an extension of whole number knowledge (Dewolf & Vosniadou, 2015;

Obersteiner et al., 2019; Siegler et al., 2013; Sophian, 2017; Steffe & Olive, 2010; Vamvakoussi & Vosniadou, 2004). There are two theories that describe fraction learning as continuous. First, according to the *integrated theory of numerical development* proposed by Siegler and colleagues, fraction knowledge develops as students adapt their knowledge of whole number magnitude to accommodate fractions (Siegler et al., 2011; Siegler & Braithwaite, 2017; Siegler & Lortie-Forgues, 2017). On this view, learning that fractions, like whole numbers, “have magnitudes that can be ordered and assigned a specific location on a number line” is integral for students’ developing fraction skills (Siegler, 2016; Siegler et al., 2011, p. 274, 2013).

The second theory of continuous learning is the *reorganization hypothesis* (Steffe, 2002; Steffe & Olive, 2010) which posits that fraction knowledge emerges as students adapt their integer counting scheme to accommodate fractions. According to this view, unit fractions are the concept that unifies whole number and fraction knowledge, that is, recognizing that unit fractions, like whole numbers can be used to count (e.g., $\frac{3}{5}$ is 3 instances of $\frac{1}{5}$; Sophian, 2017). In summary, theories that support fraction learning as an extension of whole number knowledge have different perspectives on the critical knowledge that links whole numbers to fractions. The integrated theory is focused on how higher-level skills, specifically number line estimation develops whereas the reorganization hypothesis is focused on how lower-level skills, specifically counting, is integrated into fraction concepts. Nevertheless, both of these views implicate whole number knowledge as the foundation of rational number understanding.

Conceptual change and conceptual integration are not mutually exclusive views. For example, Siegler and colleagues (Siegler et al., 2011; Siegler & Braithwaite, 2017; Siegler & Lortie-Forgues, 2017) argue that children need to recognize both the similarities and differences between whole numbers and fractions. Further, fraction errors based on inappropriate application

of whole-number reasoning may reflect an intuitive response (Obersteiner et al., 2013) whereby children use what they know about existing concepts to explain new concepts. Thus, it has been argued that if students are made aware of the conceptual differences between whole numbers and fractions (Obersteiner et al., 2013; Vamvakoussi et al., 2012), then they can successfully build on their existing whole number knowledge. Accordingly, whole number knowledge can be seen as a foundation for fraction learning as opposed to an obstacle (Sophian, 2017).

But what whole number knowledge is foundational? To answer this question, first consider the conceptual knowledge that underlies fraction understanding. As illustrated in Figure 1.1, six fraction concepts have been identified: multiple representations, equal partitions, magnitude, equivalence, density, and whole reference.

- (i) Fractions have multiple representations - they can be represented as part of a whole, a ratio, an operator, a quotient or a measure on a number line (Behr et al., 1983; Brousseau et al., 2004; Charalambous & Pitta-Pantazi, 2007). Thus, to be successful with fractions, children need to recognize for example, that 3 out of 8 balls is represented by the same fraction (i.e., $\frac{3}{8}$) as is 3 out of 8 slices of pizza or 3 out of 8 sections on a 0-1 number line (see Figure 1.1)
- (ii) The fraction represents equal partitions; that is, each slice of pizza from the last example has to be the same size (i.e., $\frac{1}{8}$).
- (iii) Now, magnitude – fractions can be ordered. For any two fractions it is possible to compare their magnitudes, say which is greater $\frac{1}{2} > \frac{3}{8}$.

- (iv) unless of course the fractions are equivalent. Unlike whole numbers, fractions with different integers can represent the same magnitude, $\frac{3}{4}$ for example represents the same magnitude as $\frac{6}{8}$ as shown in Figure 1.1.

Now, consider the conceptual knowledge involved in successfully placing fractions on a number line with zero on the left endpoint and 1 on right endpoint. This is a complex activity that potentially captures multiple fraction concepts. Most critically, students need to understand fraction representations, that is, what the fraction symbol stands for. They also need to reason about fraction magnitude sufficiently well to understand if the fraction is closer to 0 or closer to 1. Moreover, children need to understand and apply the concept of equal partitioning to estimate proportionally where to place the fraction (Barth & Paladino, 2011). Finally, students may need to recognize fraction equivalence if they are placing more than one fraction ($\frac{3}{4}$ and $\frac{6}{8}$ for example) on the same number line. In essence, number line estimation reflects advanced fraction knowledge but recognizing what the fraction symbol represents is at the core. Knowledge of fraction representations is necessary for more advanced fraction skills such as placing fractions on the number line, comparing fraction magnitudes, and understanding fraction equivalence. Thus, I argue that knowledge of fraction representations is foundational for acquiring fraction skills. Moreover, this knowledge is operationalized by fraction mapping skills.

Figure 1.1*The Conceptual Knowledge that Underlies Fraction Understanding*

How do students acquire knowledge about fraction mapping? Children enter school with informal fraction knowledge; that is, applied knowledge they have acquired through real-life experiences to solve real-life problems (Mack, 1990). For example, children can solve equal sharing problems (Empson, 1999), reason about proportions (Mix et al., 1999) and do basic fraction calculations (Mix et al., 1999). However, to progress, children must connect their informal knowledge about fractions to formal knowledge, specifically, the fraction symbol (Hiebert, 1988; Wearne & Hiebert, 1989). From grade 1 to 3, students build on their informal knowledge by exploring concrete, pictorial, and verbal representations of fractions. Students also learn about fractional quantities through language. For example, fractions are described using words (e.g., “two out of four”, “one-half”, “one-fourth”; Ontario Ministry of Education, 2005). This process of connecting symbols to their referents (i.e., concrete, pictorial and verbal representations) is typically described as mapping. Thus, understanding the skills that support

students' acquisition of fraction mapping is in an important first step for modelling how fraction knowledge develops.

The Transition from Informal to Formal Fraction Knowledge

Early Fraction Concepts

In general, knowledge of how mathematical symbols are mapped to meaning is assumed to be central to children's initial learning of formal mathematics (Brankaer et al., 2014; Hurst et al., 2017; Kolkman et al., 2013; Lyons et al., 2014). For example, preschooler's proficiency in mapping digits to words and to quantities predicts their digit comparison skills (i.e., what is greater 7 or 5; Hurst et al., 2017; Jiménez Lira et al., 2017). Digit knowledge in preschoolers also mediates the longitudinal relation between informal (non-symbolic) and formal mathematics (digit comparison, arithmetic) one year later (Purpura et al., 2013). Further, early digit skills such as number identification have been linked longitudinally to other symbolic math skills (Göbel et al., 2014; LeFevre et al., 2010; Nguyen et al., 2016; for further discussion also see Merkley & Ansari, 2016)). In summary, research supports the view that mapping of integers (i.e., connecting digits, quantities, and words) underlies more advanced numerical skills.

The importance of mapping formal symbols to their meanings may characterize mathematical learning at many levels. As new symbolic representations are introduced, students learn, for example, fraction, decimal, and percentage notations which need to be mapped to rational number magnitudes (Hurst & Cordes, 2018b). One way in which fraction notations are complex is that they combine integer notations to represent rational numbers. In acquiring fraction mappings, the first step is to connect the various external representations (i.e., pictures, words and symbols). As the symbols are reviewed and revisited, knowledge builds and the internal representation (i.e., the concept itself) becomes strengthened (e.g., symbols, images)

(Hiebert, 1988; Osana & Pitsoulantis, 2013). Thus, once children understand how the two integers in a fraction symbol represent the magnitude (e.g., that 3 out of 4 pieces is three-quarters of a whole) they can start to compare fractions (e.g., recognize that $\frac{3}{4}$ is greater than $\frac{2}{3}$), recognize that $\frac{3}{4}$ is located between $\frac{1}{2}$ and 1 on a number line and generally, develop a broader understanding of fraction magnitudes. What skills do students need and use to help them master this complex process? Insights from the literature on integer mappings can inform our understanding of the fraction mapping process.

One way in which children connect meaning to number symbols is through spoken language (Hurst et al., 2017; Jiménez Lira et al., 2017). Spoken language is often used in the informal learning experiences that provide foundational knowledge for children's mathematical learning. For example, when children understand cardinality (i.e., that the last word of a count sequence represents the quantity of the counted set), they can directly link number words to quantities. They also learn to associate number words to written symbols. Eventually, these mappings extend to direct relations between symbols and their meanings, as when children can do number comparison tasks (e.g., which is greater, 4 or 7?) by accessing stored associations directly.

Similarly, spoken language is important as students begin to connect meaning to fraction symbols. Teachers use general vocabulary and math-specific vocabulary to explain fractional quantities. For example, when mapping the symbol $\frac{3}{4}$ to an image showing a circle with 3 of 4 sections coloured, teachers may describe the images generally as "3 out of 4 equal portions of the circle." They may also use math-specific terms such as numerator and denominator. These fraction words may help students correctly write the fraction as $\frac{3}{4}$ versus $\frac{4}{3}$. Notably, in research on fractions, measures of domain-general spoken language skills (e.g., receptive vocabulary) are

often used to index individual differences in language that are relevant to mathematics (Bailey et al., 2017; Hansen et al., 2017; Rinne et al., 2017; Ye et al., 2016). However, other researchers have argued that math-specific language skills are more important than general language skills for understanding mathematical development (e.g., Hornburg et al., 2018; Powell et al., 2017; Purpura & Napoli, 2015). General vocabulary skills may support mathematical learning for a variety of reasons, however, math-specific language skills have been shown to mediate the relations between general vocabulary skills and digit knowledge in children aged 3 to 5 (Purpura et al., 2017; Toll & Van Luit, 2014) and word problem solving amongst second graders (Xu et al., in review). Thus, students with better math vocabulary may develop better fraction mapping skills or acquire those skills more easily.

Written language skills also may be important as students learn the rules and conventions for writing fraction symbols and begin to connect meaning to fraction symbols. In particular, students need to distinguish between the numerator and denominator and to recognize how to notate the fraction bar (i.e., $\frac{2}{3}$ and $\frac{2}{3}$ are conventional whereas $2\backslash 3$ is not.). Consider the challenges inherent in mathematical symbolism: different symbols can represent the same concept (e.g. division; $\frac{2}{3}$, $2 \div 3$, $3 \overline{)2}$), the same symbol can have different meanings (e.g., $-$; opposite, minus or negative), symbols may be implicit (e.g., $3 \frac{1}{2}$ means $3 + \frac{1}{2}$ whereas $3x$ means $3 \times x$) and the placement of symbols can change their meaning (e.g., $3x$, x^3) (Rubenstein & Thompson, 2001). Understanding how mathematics text is organized can help students focus on pertinent information for problem solving. Specifically, decoding math symbols, recognizing how symbols are ordered, and knowledge of the patterns for writing math text presumably all influence how students and adults attend to and understand mathematical text (Crooks & Alibali,

2013; McNeil & Alibali, 2005; Thomas et al., 2015). Thus, familiarity with the rules and conventions for writing mathematical text may be important for understanding fraction symbols.

Mathematical orthography knowledge is defined as the ability to recognize conventional mathematical symbols and know the rules for combining those symbols (Douglas et al., in press; Headley, 2016). Mathematical orthography has been linked to math achievement in adults and students. Specifically, performance on a measure of math orthography was related to general math achievement amongst seventh and eighth graders (Headley, 2016), fraction procedures in adults (Douglas et al., in press), and arithmetic and word problem solving amongst students in grades 2 and 3 (Xu et al., in review). However, researchers studying students' fraction learning have not considered the role that individual differences in students' knowledge of mathematical orthography may play in fraction learning. On the assumption mathematical orthography is important for understanding fraction symbols, I propose that students with better mathematical orthography knowledge in general will also have better mapping skills.

Do math-specific language skills (i.e., knowledge of math vocabulary and orthography) support students' ability to connect informal fraction knowledge to fraction symbols? Previous research into the relations between language skills and math learning has been focused on general vocabulary (Fuchs et al., 2006; Hansen et al., 2017; LeFevre et al., 2010; Vukovic et al., 2014). When math-specific vocabulary has been assessed, the focus has been predominately with preschool children (Hornburg et al., 2018; Purpura & Reid, 2016). Further, in the handful of studies that have considered mathematical orthography, the focus has been on either pre-algebra symbols (Douglas et al., in press; Headley, 2016) on integer mathematics (Xu et al., in review), or on the meaning and use of the symbols (Powell & Fluhler, 2018). I address this gap in the literature by assessing the relations among math vocabulary, mathematical orthography, and

fraction mapping. Specifically, I modelled the relations amongst mapping and other fraction skills for novice fraction learners (grade 4) and for more experienced fraction learners (grade 6) and discussed the different predictive patterns. The light-blue portion of Figure 1.2 captures the view that math-specific language skills are foundations for fraction mapping.

Figure 1.2

Central View that Math-Specific Language Skills are Foundational for Fraction Mapping and Fraction Mapping is a Foundation for Conceptual Fraction Knowledge

There is strong theoretical support for the view that mapping symbols to their referents is the central organizing knowledge for mathematics, more generally. Mathematical skills develop hierarchically, with higher-level skills building on lower-level skills (Cirino et al., 2016). In his theory of *Developing Mathematical Symbol Competence*, Hiebert (1988) positions symbol-to-referent mappings as the first stage in the mathematical skill hierarchy. He specifically describes skill development as a process of revisiting, elaborating and abstracting mathematical symbols.

The *Hierarchal Symbol Integration* (HSI) model (Xu et al., 2019; Xu & LeFevre in press) builds on the work of Hiebert and others (e.g., Case et al., 1996; Siegler & Chen, 2008; Werner,

1957) by describing mathematical knowledge as an increasingly integrated hierarchy of symbol associations. In terms of whole numbers, as mathematical skills increase, less advanced digit knowledge (e.g., understanding the cardinality and ordinality of digits) becomes integrated with more advanced symbol knowledge (e.g., understanding how addition and multiplication symbols describe operations; Xu et al., 2019). In the present research, I considered the fraction symbol the lowest indicator of formal fraction knowledge. I thus, tested a model of fraction development, positioning fraction mapping skills as a predictor of more advanced indicators of fraction knowledge. Specifically, I modelled the relations for novice fraction learners (grade 4) and more advanced fraction learners (grade 6). The view that fraction mapping is foundational symbol knowledge for developing fraction skills is captured in Figure 1.2.

Growth in Formal Fraction Knowledge

Advanced Fraction Concepts

Conceptual knowledge has been defined as “knowledge of concepts, which are abstract and general principles” and knowledge that is rich in connections (Rittle-Johnson et al., 2015, p. 588; Rittle-Johnson & Schneider, 2014). Conceptual fraction knowledge is the underlying knowledge that supports students as they compare and order fractions, recognize equivalent fractions, place fractions on a number line, estimate fraction sums, shade shapes and map symbols to visual representations (Hallett et al., 2012; Hansen et al., 2015; Hecht & Vagi, 2010; Jordan et al., 2013; Vukovic et al., 2014; Ye et al., 2016). In contrast, procedural fraction knowledge is described as the ability to execute action sequences (i.e., procedures) to solve problems (Rittle-Johnson & Schneider, 2014). Procedural assessments thus, involve familiar problem types and include tasks children have been taught (Hallett et al., 2010; Rittle-Johnson & Schneider, 2014). For example, procedural fraction knowledge supports students as they do

fraction arithmetic (Bailey et al., 2015; Hallett et al., 2010, 2012; Hecht & Vagi, 2010; Jordan et al., 2013) and apply known procedures to find equivalent fractions and convert between fractions and decimals. Both conceptual and procedural skills are necessary for people to use fractions in meaningful and correct ways (e.g., Rittle-Johnson & Schneider, 2014).

Once students can link the symbol to its referent, they can interpret the fraction symbol in terms of magnitude. Skills that tap into fraction magnitude knowledge include fraction comparisons (e.g., What is greater $\frac{1}{2}$ or $\frac{1}{3}$?) and ordering fraction magnitudes (e.g., Place these fractions in order from smallest to largest $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{2}{5}$). Importantly, knowledge of fraction magnitude is integral to successfully estimate the position of fractions on a fraction number line (Siegler et al., 2011; Torbeyns et al., 2015). Further, performance on the fraction number line is correlated with a range of other fraction skills (Bailey et al., 2017; Hecht et al., 2003; Siegler & Pyke, 2013) and overall math achievement (Resnick et al., 2016; Rodrigues et al., 2019; Schneider et al., 2018; Siegler et al., 2011). Thus, understanding the strategies students use to estimate fractions on the number line can provide insights into how students reason about fractional magnitude.

The Current Dissertation

In this dissertation I conducted a short-term longitudinal study to test the hypothesis that fraction mapping is the central organizing knowledge necessary for fraction learning. According to this view, fraction mapping skills are a necessary precursor for more advanced fraction knowledge. I predicted that students' math-specific language skills, specifically their math vocabulary and math orthography, would directly support their fraction mapping, and that fraction mapping, in turn, would directly support the development of conceptual knowledge about fractions. In particular, I assessed the relations amongst math language skills and fraction

mapping concurrently and fraction mapping and fraction outcomes longitudinally and then compared the patterns of relations for students in grade 4 and grade 6.

I developed three novel assessments for this research. First, the mathematical vocabulary task is a receptive vocabulary measure that covers a broad range of the math terms teachers use in the classroom. Second, the symbol decision task for math (SDT-Math), implemented as an iPad app, was designed to test students' ability to distinguish between conventional and non-conventional combinations of mathematical symbols. The SDT-Math operationalizes students' knowledge of mathematical orthography. Third, the fraction mapping task, implemented as an iPad app, requires students match fraction symbols with their verbal and visual referents. Task development is described in Chapter 2 and details on task validation are included in Appendices B, C and D. I have also provided access to these tasks for the broader community and thus, they can be used to support further research on the development of children's mathematical cognition.

This dissertation is organized as follows. In Chapter 2, I provide an overview of the literature on how students acquire fraction knowledge. In Chapter 3, I examine the reliability and validity of the measures used and describe the research methodology. In Chapters 4 and 5, I present the findings of Study 1 where the goal was to test a model that describes the skills that support fraction mapping and shows the relation between fraction mapping and developing fraction skills. In Chapter 6, I present Study 2, where the goal was to explore strategy use on a fraction number line to better understand how students reason about fractions and the magnitudes they represent. Finally, in Chapter 7 I provide an overview of the results and discuss the implications for theories of fraction learning and applications to fraction instruction.

CHAPTER 2: LITERATURE REVIEW

In this chapter, I provide an overview of the literature on students' fraction learning. First, I discuss fraction concepts and the research that describes the cognitive predictors of conceptual fraction skills. Second, I examine the links between math-specific language skills (i.e., mathematical vocabulary and orthography) and fraction learning. I then describe the cognitive covariates (i.e., whole number knowledge and working memory) that are related to fraction learning. Finally, I integrate the research and present a model that depicts the pattern of relations amongst the predictors of fraction skills (i.e., math-specific language skills, fraction mapping) for students in grade 4 and grade 6. Importantly, this model positions fraction mapping as the central organizing knowledge necessary for fraction development.

Important Definitions

Before beginning this chapter I provide a set of definitions. The first two have been adapted from Holloway and Ansari (2015). I will use these terms throughout the remainder of the thesis.

1. I describe two uses of the word **representation**. The term **mental representation** captures the semantic knowledge associated with the fraction and this information is held in the mind. The term **expressed representation** refers to how the mental representation is communicated thus it includes visual images, mathematical symbols, and lexical (oral or written) representations. When I use the term representation, I am typically referring to the expressed representation.
2. **Referent** is the information represented by a symbol. Thus, fraction symbols are associated with **semantic** referents (i.e., the quantitative meaning; most often symbolized

as a visually expressed representation) and **lexical** referents (i.e., the written or spoken name of the fraction).

3. **Mapping** describes the connection between expressed representations. For example, the word **three-fifths** maps onto the fraction symbol $\frac{3}{5}$.

Formal fraction learning begins when students are taught to connect the fraction symbol to its semantic (i.e., visual) and lexical (i.e., spoken) referents. Although students may have some exposure to fraction symbols outside of school, in Ontario, teaching of fractions symbols starts formally in grade 4 when students are approximately 7 years old (Ontario Ministry of Education, 2005). By grade 6, Ontario students are approximately 9 years old and they have been working with fraction symbols for two years. Students begin to develop competence with symbols as they construct connections or mappings between the symbol and its referents (Hiebert, 1988). Despite the importance of symbol knowledge, researchers have not fully outlined the skills students need to help them map fraction symbols to their referents or how this knowledge supports fraction learning. In this chapter I present empirical support for a model of fraction learning that positions fraction mapping as the key organizing knowledge for understanding fraction concepts. Importantly, math-specific language skills are included in the model as an important yet untested predictor of fraction learning.

Fraction Concepts

There are many definitions of conceptual knowledge. Rittle-Johnson and colleagues describe conceptual knowledge as “knowledge of concepts, which are abstract and general principles” (Rittle-Johnson et al., 2015, p. 588) and knowledge that has a “richness of connections...that increases with expertise” (Rittle-Johnson & Schneider, 2014, p. 1119). In contrast, procedural knowledge has been defined as “the ability to execute action sequences (i.e.,

procedures) to solve problems” (Rittle-Johnson & Schneider, 2014). There is strong evidence that learning concepts and procedures is iterative, one type of knowledge supports learning the other type of knowledge (Rittle-Johnson et al., 2015; Rittle-Johnson & Alibali, 1999; Rittle-Johnson & Schneider, 2014; Schneider & Stern, 2010).

Fraction skills have been classified as either predominantly conceptual or predominantly procedural (Hallett et al., 2010, 2012). Examples of conceptual fraction skills include shading fraction figures to indicate quantity, comparing fraction quantities, identifying equivalent fractions, and placing fractions on a number line (Bailey et al., 2015; Hallett et al., 2010, 2012; Jordan et al., 2013). In contrast, procedural skills include tasks “judged to be easily solved by applying an algorithm or procedure taught in school” (Hallett et al., 2010, p. 398). Examples of procedural skills include fraction arithmetic, creating equivalent fractions, and converting fractions to decimals and percentages (Bailey et al., 2015; Hallett et al., 2010, 2012; Jordan et al., 2013). However, because procedures refer to taught algorithms, skill classification may differ based on a student’s experiences. For example, fraction addition would be classified as predominantly conceptual if it had not been taught whereas it would be classified predominantly procedural if it had been taught and practiced. A review of curriculum expectations thus can help researchers distinguish between conceptual and procedural skills.

Fraction instruction in Ontario begins with a focus on fraction concepts (Ontario Ministry of Education, 2005). In fourth grade, students learn to map concrete and verbal fraction representations to the fraction symbol, they compare fraction quantities and describe equivalent fractions using concrete materials and drawings. Students in fourth grade do not use procedures to solve fraction problems. Although sixth grade students continue to use concrete materials and drawings to order and compare fractions, they also begin to use procedures to solve fraction

problems. These procedures include converting between fractions and decimals, converting between improper fractions and mixed numbers and finding equivalent fractions. Fraction arithmetic operations are not formally introduced until grade 7 (Ontario Ministry of Education, 2005)¹. Researchers studying fraction development often describe learning in terms of both procedural and conceptual skill. However, because early fraction learning in Ontario is primarily centred on concepts, the focus of this dissertation is on the development of conceptual knowledge of fractions.

Developing fraction skills involves learning challenging new concepts. One of the most challenging concepts is understanding fraction magnitude (Dewolf & Vosniadou, 2015; Siegler et al., 2011; Steffe & Olive, 2010; Vamvakoussi & Vosniadou, 2004). For example, students often struggle to see the fraction as a unit rather than as two independent integers (Rinne et al., 2017). Students also need to distinguish the multiple ways fractions can be interpreted, such as part of a whole, a ratio, an operator, a quotient, or a measure on a number line (Behr et al., 1983; Brousseau et al., 2004; Charalambous & Pitta-Pantazi, 2007). Understanding these challenging concepts does not happen quickly. Thus, comparing the skills needed to learn fraction concepts amongst early fraction learners (grade 4) compared to more experienced fraction learners (grade 6) can provide insights into how students' conceptual fraction knowledge develops.

Researchers have developed a variety of tasks to assess conceptual fraction knowledge. These tasks have included fraction mapping (Hecht & Vagi, 2010; Jordan et al., 2013), estimating the position of a fraction on a number line (Bailey et al., 2017; Siegler & Pyke, 2013), comparing fraction magnitudes (i.e., what fraction is greater $\frac{1}{2}$ or $\frac{1}{4}$?) (Hecht & Vagi, 2010; Siegler & Pyke, 2013) and identifying equivalent fraction images (e.g., what picture shows $\frac{3}{4}$ =

¹ Notably however, this is changing; in the 2020/2021 Ontario math curriculum children will begin learning fraction arithmetic operations in grade 5.

6/8?; Jordan et al., 2013). Often, several tasks are combined into a broad measure of concept knowledge (Hallett et al., 2010, 2012; Hecht et al., 2012; Hecht & Vagi, 2010). This general approach precludes a more nuanced analysis of the development the different aspects of conceptual fraction knowledge (e.g., multiple representations versus magnitude). However, because fraction learning is built on existing skills, I review the research on cognitive skills that support fraction concept knowledge to provide insights into the skills that support fraction mapping.

Language and Fractions

Students learn about mathematics using many different cognitive skills. However, because spoken and written language are used to communicate mathematical ideas and define mathematical terminology, there are multiple possible relations between math and language (Cirino, 2011; Gjicali et al., 2019; Kleemans et al., 2014; Vukovic & Lesaux, 2013). Oral language skills relevant to learning mathematics include vocabulary, phonological awareness, oral comprehension, receptive syntax, grammar, rapid automatic naming, and sentence comprehension (Chow & Ekholm, 2019; Chow & Jacobs, 2016; Peng et al., 2020). Written language skills that support math learning may include print knowledge such as letter identification, word discrimination (Purpura et al., 2011; Purpura & Napoli, 2015; X. Zhang et al., 2014), and lexical orthography (X. Zhang & Lin, 2015). In studies of mathematical learning, there has been greater emphasis on the role of oral language skills. Thus, considering the role of both oral language and written language skills in fraction learning may provide a more complete analysis of developing fraction skills.

General Vocabulary Skills

The Pathways to Mathematics Model describes three cognitive precursor skills (language, quantity knowledge, and attention/working memory) that contribute to math learning (LeFevre et al., 2010; Sowinski et al., 2015). The language path (indexed with receptive vocabulary and phonological awareness) consistently predicted individual differences in performance for a range of mathematical outcomes measured two years later (LeFevre et al., 2010; Sowinski et al., 2015). Consistent with the model, general vocabulary skills are linked to number skills (i.e., number line and arithmetic; Sowinski et al., 2015), word-problem solving skills (Fuchs et al., 2006; LeFevre et al., 2010; Méndez et al., 2019) and knowledge of fraction concepts (Hansen et al., 2017; Vukovic et al., 2014). Thus, general language skills and vocabulary in particular are important skills students may use as they attempt to develop an understanding of fractions and the quantities they represent.

Evidence on the relation between vocabulary and fraction concepts comes from research on fraction naming. Educators recommend describing the fractional parts with precise words such as numerator and denominator and describing the fractional unit with one label (i.e., “three-fourths”) because imprecise language such as describing the fraction $\frac{3}{4}$ as “3 over 4” or describing 3 as the “top number” can reinforce the misconception that the fraction is two distinct numbers (Hughes et al., 2016; Powell et al., 2019). Moreover, research has suggested that congruent fraction labels support improved fraction mapping (Hurst & Cordes, 2019; Mix & Paik, 2008). Specifically, when the fraction quantity $\frac{3}{4}$ was labelled with one word (“blick”) compared to a numerator-focused label (“3 blicks”) or a structural label (“three-out-of-four blicks”), students aged 6 to 7 years were better able to recognize images showing equivalent proportions. Support for congruent naming as a useful teaching tool was also observed in a study

of first and second graders' fraction mapping skills (Mix & Paik, 2008). Mix and Paik (2008) demonstrated that teaching transparent labels for fraction symbols (i.e., $\frac{3}{4}$ means "of four parts three") improved students' performance on a task of mapping symbols to images. They concluded that imprecise language hinders conceptual fraction understanding whereas precise naming supports conceptual fraction understanding.

Additional support that general vocabulary skills are related to students' knowledge of fraction concepts come from longitudinal research (Hansen et al., 2017; Vukovic et al., 2014). Hansen et al. (2017) used latent growth analysis to identify patterns of growth in fraction concept knowledge as students progressed from grade 3 to grade 6. They compared students who started with low skills and made minimal progress (low growth) to students who started low but showed improvement (some growth). Students with weak vocabulary skills were 1.8 times more likely to be in the low-growth class compared to the some-growth class, supporting the view that language skills are important for developing concept knowledge.

Similarly, Vukovic et al. (2014) found that students' vocabulary and listening recall skills, measured in grade one, predicted their knowledge of fraction concepts three years later. Interestingly, the predictive path between language and fraction concepts was indirect through whole number skills (i.e., number line estimation and arithmetic measured in grade 2) which suggests that general vocabulary may be important for learning fraction concepts because vocabulary skills support earlier number knowledge. Thus, general vocabulary skills are important for developing knowledge of fraction concepts.

Math-Specific Language Skills

Conceptual knowledge of mathematics, including fractions, is built through three forms of mathematical communication: (i) oral and written mathematical vocabulary, (ii) mathematical

symbols, and (iii) visual representations (O'Halloran, 2015; Schleppegrell, 2007). On this view, students' ability to map fraction symbols to their lexical (oral and written) and visual representations should be related to their knowledge of fraction concepts. Moreover, mathematical vocabulary, students' knowledge of mathematical symbols, and knowledge of fraction representations should be related to growth in their knowledge of fraction concepts.

Mathematical Vocabulary. Mathematical vocabulary refers to the lexicon of oral and written “words or phrases that express mathematical concepts or procedures” (Hebert & Powell, 2016, p. 1515). There is considerable support for the view that mathematical vocabulary is a mediating factor in the relations between general language skills and math learning. For preschool and kindergarten students, mathematical vocabulary knowledge is related to early number skills (Hornburg et al., 2018; Purpura & Reid, 2016; Toll & Van Luit, 2014), mediates relations between vocabulary and early number skills (Purpura & Reid, 2016; Toll & Van Luit, 2014), and accounts for growth in early number skills (Toll & Van Luit, 2014). Similarly, with school-aged students, mathematical vocabulary knowledge is related to number skills for first through fifth graders (Peng & Lin, 2019; Powell et al., 2017; Powell & Nelson, 2017) and mediates some of the relation between vocabulary and word problem solving for second, third and fifth graders (Xu et al. , in review; Peng & Lin, 2019). Thus, these findings suggest that fraction learning will also be related to students' general and math-specific vocabulary knowledge. However, the relation between mathematical vocabulary and fraction skills has not been tested.

The relation between math vocabulary and math learning may change with developing expertise. Some researchers have suggested that during elementary school, students' math-specific language skills become more predictive of math performance than their general

language skills (Powell et al., 2017; Purpura & Reid, 2016; Toll & Van Luit, 2014). For example, Powell et al. (2017) compared the extent to which general vocabulary and arithmetic predicted mathematical vocabulary for third and fifth graders. As mathematical vocabulary skills increased, the predictive power of general vocabulary decreased; the reliance on general language skills decreased with age. Extending these findings to fraction skills, the relation between mathematical vocabulary and fraction skills may be stronger for more experienced fraction learners.

Measuring Mathematical Vocabulary. Mathematical vocabulary skills have been measured extensively in preschool children. In an early study, Toll and Van Luit (2014) assessed math vocabulary using a subset of 22 math-related words that were part of an existing measure of general receptive vocabulary. In this test, children were asked to select a picture that showed a given math word. The math word was included as part of a sentence and was either a quantitative word (half, more, equal) or a spatial word (behind, between, opposite). One potential limitation of this task was that the math words were limited to what was available as part of an existing assessment of general vocabulary.

The Preschool Assessment of the Language of Mathematics (PALM; Purpura & Logan, 2015) was designed to assess children's knowledge of quantitative and spatial math words. However, because the PALM was developed specifically to assess mathematical vocabulary, the terms were chosen from preschool curriculum documents. Item response theory was used to ensure that the chosen words discriminated a range of skill levels; thus, the measure was well-suited to assess preschool mathematical vocabulary. Like the Toll and Van Luit measure, the 16-item PALM assessed receptive vocabulary with the exception of one quantitative item which measures expressive vocabulary. The PALM measure has been used extensively to assess

mathematical vocabulary in preschool-aged children (e.g., Hornburg et al., 2018; Kung et al., 2019; Purpura & Reid, 2016; Schmitt et al., 2019).

Current measures of mathematical vocabulary skills amongst school-aged children typically tap into knowledge of multiple math domains and test both receptive and expressive math vocabulary. Powell and colleagues have developed math vocabulary assessments for students in grades 1 through 8 (Hughes et al., 2020; Powell et al., 2017; Powell & Nelson, 2017). In each assessment, math words were selected based on curriculum documents and math textbooks. Notably, for the grade 7 and 8 measure, teachers also ranked word importance to provide further validation of word choice (Hughes et al., 2020). All measures were pencil-and-paper tests administered in the classroom. The Grade 1 assessment included 64 math terms and three levels of response: recall, comprehension and application (Powell & Nelson, 2017). Similarly, the Grade 3 and 5 assessment included 133 terms and multiple question types (multiple choice, word/definition matching, short answer etc.) (Forsyth & Powell, 2017; Powell et al., 2017). Finally, the Grade 7 and 8 measure included 57 items and one question type; all questions were multiple choice and students were expected to choose the answer that best defined the math word. Powell and colleagues' measures have been used extensively to assess mathematical vocabulary in school-aged children (Powell et al., 2020). In several studies, these measures have shown good internal reliability and validity, in the sense that they been used in studies with different groups of students (e.g., students with learning disabilities and English language learners) (Forsyth & Powell, 2017; Powell et al., 2020), with different math measures (e.g., arithmetic, equation solving and word problems) and have been similarly related to general vocabulary ($r=.61$ to $.70$) across studies.

The existing math vocabulary tools have both strengths and weaknesses. There are several strengths. First, curriculum documents and (where applicable) math textbooks were referenced when selecting target words for the PALM and Powell assessments; teacher input was also used in the Hughes et al. (2020) assessment. Second, item response theory was used to narrow word-choice and ensure word-choice captured a range of skill levels in the PALM and in Powell and colleagues' grades 7 and 8 test (Hughes et al., 2020). Third, words from multiple math domains were included in Powell and Hughes' assessments, thus providing a broad assessment of math vocabulary. However, these measures also have some limitations. First, the target age group was limited in the PALM assessment. Second, test administration for the grade 1 to 5 Powell assessments was long (30 minutes) and involved pencil and paper responses needing post-test coding. Third, different kinds of questions were used in the Powell tests. Although this approach provides useful details for educators, different question types makes comparisons across questions difficult (i.e., word/definition matching may be easier than short answer questions). Finally, I was unable to use any of these measures because there was no version suitable for students in grades 4 and 6.

I developed a new measure of math vocabulary for students in grades 4 and 6, attempting to capitalize on the strengths of the prior measures. Task development is described in detail in Appendix B. Briefly, curriculum documents and math textbooks were referenced to select math words from multiple domains. Item response theory was applied to ensure word choice discriminated across a range of skill levels. Further, the task was modelled on an extremely well known measure of general receptive vocabulary, the Peabody Picture Vocabulary Test (PPVT; Dunn & Dunn, 2007). In short, this measure of mathematical vocabulary task is quick to

administer, all questions were structured the same, it shows good internal reliability, and has good face and construct validity as shown in Appendix B.

Mathematical Orthography. Written mathematical symbols are the main text-based tools used to communicate mathematical concepts. Skemp (1982) distinguishes between two levels of written mathematical language: the surface structure of symbolic text (i.e., the physical characteristics and syntax), which I will refer to as mathematical orthography, and deep structures of the symbols (i.e., semantic characteristics). Mathematical orthography includes rules about the *material symbol* (i.e., what it looks like, size, spatial register) and the *symbol syntax* (i.e., how symbols are written in mathematical strings) (Bardini et al., 2015; O'Halloran, 2005; Quinnell & Carter, 2013; Rubenstein & Thompson, 2001). The term *mathematical orthography* is borrowed from literacy research – the word orthography is derived from the Greek roots *orthos*, meaning correct, and *graphein*, meaning to write. Hence, mathematical orthography means correctly writing mathematical text. For example, it is conventional to write subtraction strings with the minus sign spatially centred after the minuend (i.e., $7 - 5$) rather than at the bottom of the spatial register (i.e., $7 _ 5$) or preceding the minuend (i.e., $- 7 \ 5$). Similarly, it is conventional to write mixed fractions with the whole number preceding the fraction (i.e., $3 \frac{1}{2}$ vs. $\frac{1}{2} \ 3$). In summary, mathematical orthography refers to the body of knowledge of mathematical symbols and the conventions for combining those symbols into expressions and equations (Douglas et al., in press; Headley, 2016).

Evidence that mathematical orthography supports math learning comes from research on students' symbol knowledge. Powell and Fluhler (2018) developed the Mathematics Symbol Measure to assess students' understanding of the symbols used in the elementary grades. Specifically they tested students in grades 1, 3 and 5 by measuring their understanding of 23

common symbols (e.g., =, >, %, \$). Students completed a written test where they were asked to name, use and define each symbol. Across grades, students' symbol knowledge predicted their arithmetic skills. Although symbol orthography was not explicitly part of the Mathematics Symbol Measure, students needed an understanding of orthography to correctly use the symbols. For example, writing 3.00\$ is an incorrect use of the \$ sign, whereas writing \$3.00 is correct. The finding that symbol knowledge was a significant predictor of arithmetic across grades 1, 3 and 5 supports the view that orthography is related to math skills.

Evidence that mathematical orthography supports math learning also comes from research focused on single symbols, specifically, the equal sign (=) and the negative symbol (-). Students in elementary and middle school often interpret the equal sign as an operator that means “put the answer here” (Crooks & Alibali, 2014; Powell & Fuchs, 2010; Rittle-Johnson & Alibali, 1999). Thus, students may judge equations such as $3 + 5 = 2 + 6$, as incorrect or meaningless (Li, 2008; Steinberg et al., 1991) and subsequently struggle to solve equations with missing numbers (Powell & Fuchs, 2010; Sherman & Bisanz, 2009). Interestingly, drawing attention to the orthographic features related to the equal sign – that is, identifying sides of an equation, noticing the location of the equal sign, and recognizing the order of the mathematical symbols in an equation – resulted in improved performance solving equations (Alibali et al., 2018; McNeil et al., 2011). These findings support the view that knowledge of mathematical orthography is related to mathematical skills.

Students have difficulty interpreting the orthography of the negative symbol (Herscovics & Linchevski, 1994; Vlassis, 2008). Middle school students for example, made errors solving equations with adjacent minus/negative symbols (e.g., $5 - (-1)$) (Vlassis, 2008) and solving algebraic equations where the negative symbol came before the variable such as $12 - x = 5$;

solve for x . A common error was incorrectly writing $12 = 5 - x$ or $x = 5 - 12$ (Herscovics & Linchevski, 1994; Vlassis, 2008). However, because the students in these studies had demonstrated a conceptual understanding of the negative symbol, their errors reflected poor understanding of the conventions for notating the negative symbol. These findings provide further support for a link between mathematical orthography and mathematical skills.

Perceptual differences among of fraction symbols may be one way in which mathematical orthography is related to students' developing fraction skills. For example, there are multiple ways to notate fractions (i.e., $\frac{2}{3}$, $2/3$, $2/3$) and the fraction bar can have multiple meanings (the slash as a date 2/3/2020 or unit m/s; Quinnell & Carter, 2013; Rubenstein & Thompson, 2001). Thus, students who are more attentive to the perceptual cues associated with fraction conventions (i.e., the orthography) may also have a better understanding of fraction symbols. This view is supported by research showing that attention to perceptual cues in equations is related to equation solving (Hoch & Dreyfus, 2004; Landy et al., 2014; Landy & Goldstone, 2007; Papadopoulos & Gunnarsson, 2020). For example, the equation $3 + 5 \times 4$ can also be written $3 + (5 \times 4)$. The brackets are unnecessary, but they helped second and third graders attend to the order of operations and successfully solve the equations (Marchini & Papadopoulos, 2011, as cited in Papadopoulos & Gunnarsson, 2020). Similarly, narrowly spacing precedent operations within equations, for example, $3 + 5 \times 4$ versus $3 + 5 \times 4$, helped students attend to the order of operations and thus better discriminate between the correct (i.e., 23) and incorrect (i.e., 32) solutions (Landy & Goldstone, 2007). Finally, attending to the orthographic features of the equal sign was positively related to equation solving (Alibali et al., 2018; McNeil et al., 2011). Therefore, I predicted that students who are better able to attend to

the orthographic features of fraction symbols, as evidenced by their knowledge of mathematical orthography, will be more successful fraction learners.

Is the relation between mathematical orthography and fraction skills stable over development? Findings are mixed. Research with the Symbol Knowledge Measure indicates that the relation between symbol knowledge and math skills changes over development. In their report that described developing and validating the Symbol Knowledge Measure (SKM), Powell and Fluhler (2018), found that symbol knowledge accounted for more variance in arithmetic skills amongst younger students compared to older students (i.e., grade 1 compared to grade 3, grade 3 compared to grade 5).. In contrast, research with the Symbol Decision Task (SDT-Math) indicates that the relation between symbol knowledge (orthography specifically) and math skills is stable over development. Performance on the SDT-Math was related to general math skills for students in grades 7 and 8 (Headley, 2016), accounted for unique variance (after controlling for cognitive and number skills) in fraction and algebra arithmetic for adults (Douglas et al., in press) and was related to arithmetic skills and accounted for unique variance in word problem solving for students in grades 2 and 3 (Xu et al., in review). The SKM, by design is tapping into both the surface structures (i.e., appearance, syntax; students write symbol expressions) and the deep structures (i.e., semantics; students describe what the symbol means) of symbols knowledge (Skemp, 1982). Moreover, because the SKM is a written test, other skills such as reading and writing may account for some of the symbol knowledge and arithmetic relations – particularly for the younger students (e.g., Grimm, 2008). Thus, task differences between the SKM and the SDT may account for mixed findings across grades. In summary, because the SDT is a more direct measure of orthography than the SKM, I am basing my hypothesis on the previous work with SDT. Extending the findings from work with the SDT task to fraction skills, I predict the

relation between mathematical orthography and fraction mapping will be similar for students in grades 4 and 6.

Measuring Mathematical Orthography. To assess students' mathematical orthography, I developed a measure that assessed students' ability to discriminate between well-formed mathematical text (e.g., $7 - 5$) and poorly formed mathematical text (e.g., $7 _ 5$). This measure is called the **symbol decision task for math** (i.e., SDT-Math). Task validation is described in Appendix C. The SDT-Math is based on a similar symbol decision task that was focused on knowledge of pre-algebra symbols and was used in research with adults and youth (Douglas et al., in press; Headley, 2016). Task validity of the original SDT-Math task was supported by the finding that grade 7 and 8 students with higher mathematical orthography scores performed better on standardized math assessments than students with lower mathematical orthography scores (Headley, 2016). Similarly, adults with higher mathematical orthography scores performed better on measures of algebraic and fraction computations than those with lower scores (Douglas et al., in press). Finally, students in grades 2 and 3 who had higher scores on a similar mathematical orthography measure performed better on measures of arithmetic and word-problem solving than students with lower mathematical orthography scores (Xu et al., in review). Together, these findings show that a relation between mathematical orthography and math skills can be captured with the SDT-Math.

In summary, I propose that mathematical vocabulary and mathematical orthography are key skills that help students develop their understanding of fraction symbols. For example, knowing the term *equal* may help students understand that $\frac{2}{3}$ represents two out of three *equal* portions. Further, recognizing the conventions for writing the fraction bar may help students distinguish between $\frac{2}{3}$ and $\frac{3}{2}$. Importantly, the relations between math vocabulary,

mathematical orthography, and knowledge of fraction symbols have not been tested. Thus, I developed measures of mathematical language skills (mathematical vocabulary and orthography) and fraction symbol knowledge and tested a set of hypotheses about the contributions of math-specific language skills to fraction learning.

Visual Representations. The terms used to describe visual fraction representations can overlap and the definitions are not always used consistently. To clarify, the term part-whole fraction construct has been defined as partitioning of both continuous and discrete quantities; “a continuous quantity is partitioned into equal size [parts] (e.g., dividing a cake into equal parts), and partitioning would be the same with a set of discrete objects (e.g., distributing the same amount of sweets among a group of children)” (Gabriel et al., 2013, p. 2). Thus, to describe the visual representations, it is necessary for researchers to distinguish between fractions of continuous versus discrete quantities. Visual representations of continuous quantities have been referred to as area models (e.g., Fuchs et al., 2017; Roesslein & Coddington, 2019), discretized models (Begolli et al., 2020) continuous models (Newstead & Murray, 1998) and part-whole models (Misquitta, 2011). The area model label is limited because it precludes representations of linear or volumetric quantities. The discretized model label is a new term and thus not well recognized. Finally, the continuous model label is limited because it conflicts with terminology used to identify non-partitioned quantities.

To avoid confusion, I have included definitions of the various representations. These terms will be used throughout the dissertation. (i) **Part-whole** fraction representations depict the proportion of area or volume - typically partitioned and shaded - in a whole shape. For example a circle showing 3 out of 4 sections shaded or a beaker that is filled to the 3rd of 4 tick marks both refer to the fraction $\frac{3}{4}$. **Discrete** fraction representations depict the proportion of separated

objects in a group. For example, a picture showing 3 hats as part of a set of 4 hats, refers to the fraction $\frac{3}{4}$. Notably both discrete and part-whole fraction representations are examples of the part-whole fraction construct. (iii) **Continuous** fraction representations by contrast depict the area or volume of a shape or distance on a line where there are no partitions. A beaker partially filled with no partitioned tick marks or a spot marked on number line with no other partitioned tick marks are examples of continuous fraction representations.

Some visual fraction representations are more difficult for students to name than others (Behr et al., 1983; Brousseau et al., 2004; Charalambous & Pitta-Pantazi, 2007). For example, students aged 7 to 12 were more successful identifying equivalent fractions when they compared part-whole representations (i.e., 3 out of 4 sections of a rectangle coloured) than comparing discrete representations (i.e., 3 out of 4 balls) (Begolli et al., 2020). Students in fourth and fifth grade were more successful mapping visual representations of unit fractions presented as part-whole area models (rectangles and circles) than those presented as continuous number line models (Tunç-Pekkan, 2015). Further, an over-reliance on circle representations has been identified as a limiting factor in students' developing fraction knowledge (Yearley & Bruce, 2014). The finding that students have better understanding of some fraction representations compared to others suggests that researchers may learn more about how fraction skills evolve by studying students' fraction mapping skills.

There is also empirical support for the proposal that students' knowledge of visual fraction representations, and thus their fraction mapping skill, is related to their conceptual fraction knowledge. For example, students who experience difficulty correctly identifying visual representations of the part-whole construct also experience difficulty with more advanced fractional thinking such as comparing symbolic fraction magnitudes (i.e., what is greater $\frac{2}{5}$ or $\frac{3}{7}$)

(Lewis, 2016; Mazzocco et al., 2013) and placing fractions on a number line (Tunç-Pekkan, 2015). The findings that basic knowledge of part-whole fraction representations is linked to fraction comparison and number line estimation skills supports the view that fraction mapping skills are related to knowledge of fraction concepts.

Covariates of Fraction Skills

The Pathways Model provides a framework to organize the general cognitive skills that are related to individual differences in mathematics generally (LeFevre et al., 2010; Sowinski et al., 2015; Träff et al., 2018) and fraction knowledge specifically (Vukovic et al., 2014).

Reviewing the literature on fraction development, I examined the cognitive predictors of conceptual fraction skills in terms of three precursor pathways: language, number/quantity knowledge, and attention/working memory.

Quantity/Arithmetic Knowledge and Fractions

A range of measures have been used to assess early number skills in math learning. The Pathways Model described math development from preschool to grade 2 (LeFevre et al., 2010). In this first model, quantity knowledge was captured with a preschool subitizing task, that is children enumerated dot quantities between 1 and 3. The quantitative pathway was then refined to extend the model to students in grade 3 (Sowinski et al., 2015). Quantity knowledge thus reflected a broader range of number skills, that is, counting, subitizing, and digit comparison. Cardinal skills, as measured by digit comparison, are consistently correlated with students' arithmetic skills (for a review see: Schneider et al., 2017). Ordinal skills (i.e., Is 2 4 3 in increasing order?) are even stronger predictors of arithmetic performance, especially for older students and adults (Lyons et al., 2016; Lyons & Beilock, 2011; Xu et al., 2019). However, there

is limited research examining the relations between these basic number skills and fraction concepts (Gilmore et al., 2018).

On the assumption that there is a hierarchy of number skills (Cirino et al., 2016; Hiebert, 1988; Xu et al., 2019), digit skills such as digit comparison and simple arithmetic should be related to fraction learning. This view is supported through longitudinal research. For example, following students from grade 1 to grade 4, Vukovic et al., (2014) found that students' addition skills in grade 1 predicted their more advanced arithmetic skills in grade 2 which in turn predicted their knowledge of fraction concepts in grade 4. Hansen et al., (2017) found a similar longitudinal relation with older students whereby students' addition skills in grade 3 predicted and their knowledge of fraction concepts in grade 6. Concurrent research also supports a link between digit skills and fraction learning. For example, number comparison skills in sixth grade predicted students' performance on a combined measure of fraction procedures and concepts concurrently through double-digit arithmetic (Cirino et al., 2016). Further, addition fluency (Hansen et al., 2017) and combined addition and multiplication fluency (Hansen et al., 2015; Hecht et al., 2003; Hecht & Vagi, 2010) directly predicted fraction concepts and fraction arithmetic concurrently for students in grade 5 (Hecht et al., 2003). Together, these findings suggest that basic number knowledge forms a foundation for arithmetic skills which in turn support developing fraction concepts. Thus, to model the skills that support knowledge of fraction concepts, whole number arithmetic will be used to characterize the quantitative pathway.

Using the view that fraction mapping is the critical organizing knowledge for fraction learning, there is a need to clarify the skill hierarchy among arithmetic, fraction mapping, and more advanced fraction concepts (Cirino et al., 2016, p. 112). The hierarchical symbol integration (HSI) model was developed to describe the skill hierarchy for whole number relations

(Xu et al., 2019). The HSI describes whole number knowledge as an integrated network of increasingly complex symbol-symbol associations. The hierarchy is built from cardinal associations (What is greater 7 or 4?), ordinal associations (Are these numbers in increasing order 1 5 3?), and arithmetic associations. The model refers to integration, because associations that are learned earlier (e.g., that 3 4 have a particular ordinal relation to 5) have to be available in counting tasks whereas addition relations, learned later (e.g., $3 + 4 = 7$), need to be accessible in arithmetic tasks. Extending the HSI model to describe fraction skills, students must first learn to link the fraction symbol to the quantity it represents in order to compare and add fractions. Thus, I predict that fraction mapping knowledge will support more advanced knowledge of fraction concepts such as comparing fraction magnitudes and placing fractions on a number line.

Attention/Working Memory and Fractions

There is some evidence that attention and working memory predict fraction learning (Bailey et al., 2014; Hansen et al., 2015; Hecht & Vagi, 2010; Jordan et al., 2013; Vukovic et al., 2014; see also Peng et al., 2016 for a review). I am using the Baddeley and Hitch multi-component model of working memory (WM) (Baddeley, 2000) as a theoretical framework to describe the general cognitive processes related to fraction learning. According to this model, working memory is made up of the central executive, the phonological loop, the visual-spatial sketchpad and the episodic buffer. The central executive coordinates and updates information from the phonological loop and visual-spatial sketchpad. The phonological loop is responsible for temporary storage of verbal information and the visual-spatial sketchpad is responsible for temporary storage of both visual and spatial information (Baddeley, 2003). Verbal working memory draws on the phonological loop and central executive resources whereas spatial working

memory draws on the visual-spatial sketchpad and central executive resources. Finally, the episodic buffer is where the information is simultaneously maintained and manipulated.

Working memory and attention may be important for fraction learning because doing fraction tasks such as arithmetic, comparing magnitudes and placing fractions on a number line entails attending to two sources of information – the numerator and the denominator (Peng et al., 2016). Studies on working memory and fractions, however, often include only one working memory domain. Verbal working memory has been included in multiple studies of fraction skills (Bailey et al., 2014, 2017; Hansen et al., 2017; Hecht et al., 2003; Hecht & Vagi, 2010; Ye et al., 2016). It is moderately correlated with fraction concepts (Peng et al., 2016) and may predict fraction concepts indirectly through shared whole-number knowledge (Hecht et al., 2003; Hecht & Vagi, 2010). Spatial working memory by contrast, has been included in few studies of fraction skills (Cirino et al., 2016) but was found to be related to fraction learning indirectly through arithmetic skills (Cirino et al., 2016).

Findings on working memory and fraction performance are mixed. For example, for students from fourth to sixth grade, Hansen et al. (2017) found that verbal working memory did not influence growth in fraction concepts. Comparing the influence of working memory on fraction learning, Fuchs et al. (2014) found that students with poor verbal working memory skills (operationalized by sentence listening recall) learned fractions better with conceptual activities whereas students with better working memory skills learned fractions better with procedural activities. Finally, Peng et al. (2016) suggest that there are too few studies to characterize the role of the various working memory components with different fraction outcomes. Thus, in the current study, I included measures of both verbal and visual-spatial working memory.

The Current Research

The primary goal of this dissertation was to examine students' developing fraction knowledge to test a model in which fraction mapping skills are the central organizing feature for developing conceptual fraction knowledge. Importantly, math-specific language is a necessary skill that supports connecting fraction symbols and fraction representations. Students in grades 4 and 6 were compared because previous research has shown that this is a critical time period for students to consolidate their fraction knowledge (Barbieri et al., 2019; Hansen et al., 2017; Jordan et al., 2017). In addition, the students in the present research were first taught fraction symbols in grade 4; by grade 6, students were expected to have had a reasonable amount of practice using fraction symbols and thus their fraction knowledge should be more advanced.

Three novel tasks were specifically developed for this thesis. The new tasks included two measures of math-specific language skills: mathematical vocabulary and mathematical orthography. By testing aspects of verbal (vocabulary) and written (orthography) math language skills I was able to analyse the effect of math language in general and to explore in detail individual differences in fraction-related language. The fraction mapping task was a novel iPad app developed to allow students to show how accurately they could connect the three forms of mathematical discourse: visual representations, symbols, and words. To control for any effect of representation type, the mapping task included an equal number of part-whole models (e.g., area and volume representations) and discrete models (Begolli et al., 2020; Bruce et al., 2013; Hebert & Powell, 2016; Rau & Matthews, 2017).

Hypotheses and Research Questions

In Chapter 5, I tested the following hypotheses. Based on the assumptions that (a) knowledge becomes more domain-specific with developing expertise (Powell et al., 2017) and

(b) conceptual math knowledge is built through math-specific discourse (O'Halloran, 2015), I hypothesized that math-specific language skills would predict fraction mapping skills differently for grade 4 students compared to grade 6 students. These hypotheses are illustrated in Figure 2.1.

Specifically:

H1. Based on the finding that mathematical vocabulary (i.e., a math-specific language skill) is more strongly related to outcomes in older, versus younger students (Powell et al., 2017), I hypothesized that mathematical vocabulary would directly predict fraction mapping skills in sixth graders (H1b) but not in fourth graders (H1a).

H2. Based findings that mathematical orthography is related to math outcomes in multiple age groups (Douglas, in press; Headley, 2016; Xu et al., in review) mathematical orthography will directly predict fraction mapping skills for fourth graders (H2a) and sixth graders (H2b).

H3. Because the reliance on general vocabulary skills decreases with mathematical expertise (Powell, 2017), mathematical vocabulary will directly predict fraction outcomes for advanced fraction learners, that is sixth graders (H3b) but not in fourth graders (H3a).

H4. Based on the assumptions that conceptual math knowledge is built through math-specific discourse (O'Halloran, 2015) and connecting symbols to their referents is the first level of symbol expertise (Hiebert, 1988), Time 1 fraction mapping skills will directly predict Time 2 fraction outcomes for both fourth graders (H4a) and sixth graders (H4b).

Figure 2.1

Path Models Capturing the Hypotheses Predicting Fraction Skills in Fourth and Sixth Graders

In Chapter 6, I took an exploratory approach to examine more closely how students reason about fraction magnitudes. Thus, in Chapter 6, I examined two research questions.

Q1. What strategies do students use to estimate fractions on a number line?

Q2. What skills differentiate students based on their fraction estimation profiles?

The main goal of this dissertation was to characterize the skills that support students' ability to map the fraction symbol to its referent and to describe how individual differences in students' knowledge of fraction mapping relates to their developing fraction skills. Thus, in Study 1, the concurrent and longitudinal relations between math-specific language skills, fraction mapping and subsequent fraction knowledge were modelled, and patterns of relations were compared between fourth and sixth graders. In Study 2, to better understand how students reason about fractional quantities, performance of the fraction number line was examined in greater detail. Supporting literature for the fraction number line study is reviewed in Chapter 6, which was written as a free-standing paper.

CHAPTER 3: METHODS

In this thesis, I proposed and tested a model in which fraction mapping skills are the central organizing element for developing fraction skills. I assumed that math-specific language skills support fraction mapping. To compare the relations between math-specific language skills, fraction mapping and developing fraction knowledge, students in grades 4 and 6 completed a battery of assessments. Known predictors (cognitive, numeric and linguistic) of fraction skills were assessed and students were tested at two time points, Fall 2018 and Spring 2019. The study was correlational and longitudinal over approximately five months. Students' cognitive skills and fraction mapping were assessed in the fall; in the Spring, fraction mapping was measured again, as were several different fraction outcomes. The project was designed to address the following research questions:

1. What cognitive skills support students' ability to map the fraction symbol to its referent?
2. How does students' knowledge of fraction mapping support their developing fraction skills?

Recruitment

The study was approved by the Carleton University and Research Ethics Board (file #108058) and subsequently, the research steering committee of an Ontario school board. As per board protocol, the board sent out study information to schools in the study catchment area. Interested principals then contacted the board and I set up information meetings with these principals and their teachers. After meeting with the teachers, those who wanted students in their classes to participate sent home information letters. Parents provided written consent allowing their child to participate. Ten teachers with 12 classes of students agreed to participate and thus approximately 250 students received the letter and consent forms.

Participants

One hundred and thirty-two students from grade 4 ($N=65$, 31 boys, $M_{\text{age}}=113$ months) and grade 6 ($N=66$, 38 boys, $M_{\text{age}}=137$ months) were recruited from three schools serving Kindergarten to grade 6 students. Of these students, one student was excluded from the study because this student did not complete the second wave of testing and one student chose not to participate. All parents reported English as the home language and parents were well educated; 66% of fathers and 84% of mothers were college or university graduates. The schools are in areas that serve both town and rural students. No information about the specific situation of individual students was collected.

Procedure

Students participated in three testing sessions. The first two sessions were in November 2018 (TIME 1) and the final session was in April 2019 (Time 2). Each session lasted approximately 25 minutes. Prior to each testing session, we described the testing protocol to students and the students gave oral assent. All students were tested individually at their school by a research assistant (RA) who was previously trained by the lead researcher. Researcher training included a practice session (2 to 3 hours) where the RA practiced the testing and scoring procedures and was instructed on working with children. Two to four students were tested at a time, depending on the availability of RAs. Testing took place in a quiet area at the school such as the library or an empty classroom. The lead researcher was present during all testing times and thus, could observe and respond to any questions or concerns raised by the RAs.

Measures

The test battery and timing are shown in Table 3.1. Additional measures that are not part of the current analyses are described in Appendix A.

Table 3.1

Test Battery Including Administration Times and Sessions

Tasks	Classification	Time (min)	Time 1	Time 2
Whole-number skills				
• Number comparison	cardinal skills	3	✓	
• Order judgement	ordinal skills	3	✓	
• Number line estimation	whole number skills	5	✓	
• Single-digit arithmetic fluency	whole number skills	3	✓	
Cognitive measures				
• Forward digit span	verbal short-term memory	3	✓	
• Backwards digit span	verbal working memory	3	✓	
• Spatial span	spatial memory	5	✓	
• Black and white Stroop*	inhibitory control	3	✓	
• WISC matrices*	non-verbal reasoning			
General language				
• WIAT*	word reading	5	✓	
• PPVT	general vocabulary	5	✓	
Math language				
• Mathematical vocab.	math vocabulary	5	✓	
• Symbol decision task-Math	math orthography	4	✓	✓
Math outcomes				
• Key Math Problem solving*	general math knowledge	5	✓	
• Double-digit arithmetic *	computational skill	3		✓
Fraction measures				
• Fraction mapping	concepts	5	✓	✓
• Fraction word problems	concepts	5		✓
• Fraction worksheet	concepts and procedures	5		✓
• Fraction number line	concepts	5		✓
Total testing time (min.)			63	27

Note. * Measures not included in subsequent analyses. For details see Appendix A.

Whole Number Skills

Students completed four tasks designed to assess their basic whole number knowledge.

Symbolic Number Comparison Task. Students see two single-digit numbers on an iPad and tap the larger number as fast as possible (e.g., What is larger 7 or 9?) (Big Number; <https://carleton.ca/cacr/math-lab/apps/bigger-number-app/>). There are 26 trials, and the ratio and the distance between the two numbers is manipulated (Bugden & Ansari, 2011). Each ratio is repeated once in random order, and each number is counterbalanced for the side of presentation. Scoring is the linear integration of speed and accuracy score (LISAS) calculated using the equation:

$$\text{LISAS} = \text{RT} + \frac{\text{SD}_{\text{RT}}}{\text{SD}_{\text{PE}}} \times \text{PE}$$

where RT is mean response time, SD is the standard deviation and PE is the participant's proportion of errors (Vandierendonck, 2017). Thus, the final score was an adjusted response time that was corrected to reflect potentially trade-offs between accuracy and response times.

Reliability based on response times for all trials was high (Cronbach's $\alpha = .82$).

Number Order Judgement. In this iPad app, students see three digits (e.g., 1 2 3) and press a checkmark if the digits are correctly ordered or press an x if the digits are unordered. Stimuli include single digits from 1 to 9. Stimuli are balanced between ordered sequences, non-ordered sequences, counting sequences (e.g., 5 6 7, 6 5 7) and non-counting sequences (2 4 8, 8 2 4). There are 24 sequences. Scoring was calculated using the linear integration of speed and accuracy score (LISAS). Reliability based on response times for all trials was high (Cronbach's $\alpha = .92$).

Number line Estimation. Students estimate the position of a target number on a 0-1000 number line (Hume & Hume, 2014). To eliminate potential ceiling effects, the number line range

was 0-1000. Scoring was the percent of absolute error (PAE) between the placement of each number compared to the actual location of that number. PAE was calculated by taking $100 * (|target - estimate|) / 1000$. Reliability comparing percent absolute error for each item was high (Cronbach's $\alpha = .92$).

Arithmetic Fluency. In this paper and pencil task, students are given one minute to complete one sheet of addition questions (sums to 18), one minute to complete one sheet of subtraction questions (inverse of addition questions) and one minute to complete one sheet of single-digit multiplication questions (2,3,4, and 5 times tables up to 10 and the 6,7,8, and 9 times tables to 5). There are 60 problems per page, arranged in 3 columns. Total score was the number of correct responses. Reliability comparing totals for each operation/question sheet was high (Cronbach's $\alpha = .87$).

Attention and Working Memory

All tasks have been used in previous studies and are part of an ongoing study exploring math and language. Spans were combined to create an attention and working memory factor score (see Chapter 4).

Forward Digit Span. This task is part of the WISC-V test battery (Wechsler, 2014). Students hear a series of pre-recorded numbers and repeat the numbers back to the experimenter in the same order they were heard. For example, if trial 1 of the 3-digit span is “3-5-7”, the student repeats “3-5-7”. There are two trials for each digit span and digit spans were pre-recorded to ensure stimuli are consistent. Students started at a digit span of two numbers. Testing was discontinued after both trials for a given span length were repeated incorrectly. Scoring was the total number of correct trials. Reliability comparing sub-scores for first trials to sub-scores for second trails was acceptable (Cronbach's $\alpha = 0.69$)

Backwards Digit Span. This task is part of the WISC-V test battery (Wechsler, 2014). On each trial of this task, students hear a sequence of pre-recorded numbers and respond by repeating the numbers in reverse order. For example, if they hear “2-7-9” the student should answer “9-7-2”. There are two practice trials (span length of 2) followed by two trials for every span length (i.e., 3, 4, 5, 6, and 7). Students started at a digit span of 2. Testing was discontinued when the student responded incorrectly on both trials for a given span length are repeated incorrectly. Scoring was the total number of correct trials. Reliability comparing sub-scores for first trials to sub-scores for second trials was good (Cronbach’s $\alpha = 0.77$)

Spatial Span. In the *PathSpan* iPad app (Hume & Hume, 2014), nine green dots are shown on the iPad screen. Dots light up one by one in a sequence. Sequence length starts with a span of two dots and increases every three trials (i.e., span lengths of 2, 3, 4, 5, 6, ...). For each trial, students were asked to touch the dots in the same order in which they lit up. At the beginning of the task, the experimenter demonstrated a practice trial. During the practice trial, students were asked to watch and remember a sequence of two locations. Then, students were given three more trials of sequences of two locations without any feedback. If they correctly reproduced at least one of those sequences, then the task proceeded to the next level (i.e., three trials with sequences of three locations). However, if students made errors on all three sequences for a span, the task was terminated. This task has been used extensively to index visual-spatial working memory processes in children (Astle et al., 2013; LeFevre et al., 2010; Xu & LeFevre, 2016). The score of the spatial span task was the total number of sequences completed correctly. Reliability comparing sub-scores for first trials to sub-scores for second and third trials was acceptable (Cronbach’s $\alpha = 0.78$).

Language Skills

General Vocabulary. General vocabulary was measured with the Peabody Picture Vocabulary Test Third Edition (PPVT-III; Dunn & Dunn, 1997). In this domain-general measure of receptive vocabulary, students see four pictures on a page and the researcher says a word that describes one of the pictures. The student's task is to identify the picture that corresponds with the given word. Words are organized into sets. Students in grade 4 began testing with word set 7; while students in grade 6 began testing with word set 9. As per standardized procedure, testing was discontinued after a student incorrectly identified 8 words in a set of 12 words. Scoring was the total number of words correctly identified. Note, for consistency between grades, the vocabulary scores reported are for sets 9 to 14 and have been standardized by grade. Reliability was calculated using the 72 items in sets 9 to 14 (Cronbach's $\alpha=.91$).

Math Vocabulary. In the mathematical vocabulary task, like the PPVT, students are presented with four pictures on a page and the researcher says a math word or phrase that describes one of the pictures. The student's task is to identify the picture that corresponds with the given word. A sample image for the target word *perimeter* is shown in Figure 3.1. The task includes 36 words. Scoring is the proportion of correctly identified words. Reliability across the 36 test items was good (Cronbach's $\alpha=.74$). See Appendix B for a detailed description of task development and validation using item response analysis.

Figure 3.1

Stimuli for Math Vocabulary Word “perimeter”

Mathematical Orthography. The *children’s symbol decision task for math (SDT-Child)* is a revised version of the youth/adult task developed by Headley (2016). In this task, students see conventional (e.g., $3 - 1 = 2$) and non-conventional (e.g., $3 _ 1 = 2$) symbolic math stimuli. If the student judges the stimulus “readable/looks right” they press the check mark (see example trial in Figure 3.2); if they judge the stimulus “non-readable” they press the X.

Conventional stimuli were based on curricular expectations from grades 2 through 6. Non-conventional stimuli were created by transforming conventional stimuli in various ways. Following Headley’s (2016) protocol, transformations were either character substitutions (e.g., a $+$ sign replaced with a $\oplus$ symbol), a change in symbol order (e.g., see Figure 3.2), or a change in symbol orientation (e.g., a minus symbol $-$ moved to the bottom of the spatial register $_$).

To ensure understanding, students were given six practice trials. Feedback was given on the practice trials. The 54 experimental trials (27 conventional) were presented in a different random order for each student. Decisions and response times were recorded. Reliability across the 54 items was acceptable (Cronbach’s $\alpha = .68$) but not high because of the variability of the item types. See Appendix C for a detailed description of task development and validation using item response analysis.

Figure 3.2

Screenshot of the Children's Symbol Decision Task – Math.

Note. In this non-conventional example, the + symbol and the digits are ordered incorrectly (i.e., $2 + 1 = 3$).

Fraction Skills

Fraction Mapping. The mapping task captures students' ability to connect formal fraction notation with words and images. In this iPad task, students see a fraction represented as either a number (e.g., $\frac{2}{3}$), a picture, or a fraction word (e.g., two-thirds). As shown in Table 3.2, three possible fraction alternatives are shown below the fraction representation. The student selects the appropriate response and their accuracy and response time are automatically recorded.

Mapping sets include word-to-number, number-to-picture, picture-to-number, and word-to-picture. There are 10 trials for each set of mappings. Set order is random but the type of trial is blocked within a set (i.e., in some instances, the first 10 trials will be word-to-number) and the order of the 10 trials is random within the set. Prior to each set of 10 trials, a message appears identifying the mapping direction. The position of the correct response is randomized across trials. The task begins with four practice trials all with the target fraction one-half.

Picture representations include an equal mix of part-whole models (e.g., 3 out of 4 sections of circle shaded) and discrete models (e.g., 2 out of 5 balls circled) to control for

differences in representation formats. Target fractions all have denominators of 3, 4, 5, 8, or 10. Foils are based on common mistakes that students make, such as confusing the numerator and the denominator (e.g. $\frac{3}{2}$ vs. $\frac{2}{3}$), or confusing ratio representations with fraction representations (e.g., for 2 out of 5 balls circled one foil is $\frac{2}{3}$ because 3 is the number of balls not circled). Scoring is based on accuracy. See Appendix D for task stimuli and detailed task validation. Reliability based on item accuracy of the 40 items was high (Cronbach's $\alpha = .88$).

Fraction Number Line. In this iPad task, students estimate the position of a fraction on a 0 to 1 number line (see Figure 3.3). A fraction is shown at the top of the screen. Students then tap the number line and a red hatch mark appears which the student then slides to estimate the location of the fraction on the number line. Students begin the task with 4 trials where they are prompted to explain their strategy for placing the number on the number line. Details on the four strategy trials and strategy coding are described in Chapter 6. There are 27 experimental trials. Following the protocol used by Torbeyns et al. (2015) trials include all non-reducible proper fractions with single-digit denominators (see Table 3.3). Scoring was the percentage of absolute error (PAE) between the placement of each fraction compared to the actual location of that fraction. PAE was calculated using decimal values and by taking $100 * (|\text{target} - \text{estimate}|)$. Reliability across the 27 items was high (Cronbach's $\alpha = 0.91$).

Table 3.2

Mapping Formats and Corresponding Screenshots of the Fraction Mapping App

Representations	Mapping Formats			
	Word to image	Image to number	Number to image	Word to number
Part-whole
Discrete
Symbolic only
Total Stimuli	10	10	10	10

Figure 3.3

Screenshot of Fraction Number Line Estimation App

Table 3.3

Target Fractions for the Fraction Number line App

Denominator							
2	3	4	5	6	7	8	9
$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}^*$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$	$\frac{1}{9}$
	$\frac{2}{3}$		$\frac{2}{5}$		$\frac{2}{7}$		$\frac{2}{9}$
		$\frac{3}{4}$	$\frac{3}{5}$		$\frac{3}{7}^*$		
			$\frac{4}{5}^*$		$\frac{4}{7}$		$\frac{4}{9}$
				$\frac{5}{6}$	$\frac{5}{7}$	$\frac{5}{8}$	$\frac{5}{9}$
					$\frac{6}{7}$		
						$\frac{7}{8}$	$\frac{7}{9}^*$
							$\frac{8}{9}$

Note. The * denotes the fractions for which students reported an estimation strategy (i.e., Trials 1 to 4).

Fraction Word Problems. In this measure of fraction concept knowledge, the tester shows the student an image and reads a question based on the image (see Figure 3.4). The six questions tap into common fraction concepts such as fraction density and quantity estimation. Questions were based on a variety of sources (Education Quality and Accountability Office, 2017; Hallett et al., 2010, 2012; Morrow et al., 2004). Details are included in Appendix E. Reliability based on item accuracy was acceptable (Cronbach's $\alpha = .73$).

Figure 3.4

Questions 1 and 2 from the Fraction Word-Problems Task

Fraction Worksheet. The fraction worksheet included 3 types of questions. Magnitude comparisons required students compare two fractions and circle the greater quantity (e.g., $\frac{1}{4}$ or $\frac{2}{3}$; $n = 6$). On transcoding trials, students convert fractions to decimals and vice versa (tenths, hundredths, quarters and 25ths), mixed numbers to improper fractions and vice versa, or solve for equivalent fractions ($n = 20$). On fraction arithmetic trials ($n = 6$), students solve fraction arithmetic questions, three with common denominators (e.g., $\frac{3}{7} + \frac{2}{7}$) and three that are easily converted to common denominators (e.g., $\frac{1}{6} + \frac{1}{3}$).

To ensure students felt comfortable trying the arithmetic task, students were told that we are interested in how they approach fraction arithmetic before formal instruction. In Ontario, fraction arithmetic is introduced in grade 7. All students completed the fraction magnitude

comparison and the fraction arithmetic tasks. For each type of transcoding task, students were shown a practice problem. If they were unfamiliar with the task (some items were only in the grade 6 curriculum) they were given the option to “try it” or “skip it”. Where feasible, questions have been taken from Ontario text books (e.g., Math Makes Sense), curricular support documents (e.g., Jump Math Student Workbooks), standardized test booklets (i.e., Educational Quality and Accountability Office of Ontario; www.EQAO.com) and previously used fraction assessment tasks (Hallett et al., 2010, 2012). Reliability on fraction arithmetic was high (Cronbach’s $\alpha = .91$). Reliability across the two transcoding tasks, equivalent fractions and number comparison scales was also high (Cronbach’s $\alpha = .83$). The complete set of questions is included in Appendix E.

Analysis Plan

The data was analysed in two “studies”. In Study 1 (Chapter 4), I examined the descriptives and scoring distributions by grade for each applicable measure. Based on these findings, I confirmed which measures could be compared across grades and thus included in the hypothesis testing. Next, I compared the patterns of correlations by grade identifying any unexpected patterns. Based on the patterns of correlations, subsequent model specifications were confirmed. Finally, I used multi-group path analysis with *MPlus* Version 8 (Muthen & Muthen, 1988-2019) to test the patterns of relations captured by the hypotheses (see Chapter 5).

In Study 2 (Chapter 6), performance of the fraction number line task was explored in greater detail. Latent profile analysis was used to group students based on their item-by-item estimation accuracy. Subsequently, multinomial logistic regression was used to determine which skills differentiated the more-skilled from the less-skilled fraction estimators.

CHAPTER 4: INDIVIDUAL DIFFERENCES IN FRACTION KNOWLEDGE PART 1

I begin this chapter by first comparing task performance by grade thus confirming that students in grade 6 outperformed students in grade 4 on cognitive skills, number skills, and language skills (see Table 4.1), and that grade 6 students had more advanced fraction knowledge than grade 4 students. Second, I reviewed the scoring distributions and identified tasks at floor or ceiling levels. Finally, I compared the correlational patterns by grade to provide support for either constraining (i.e., estimating paths to be similar across grades) or freely estimating (i.e., assuming paths to be different across grades) the predicted model pathways. Based on the correlational analyses, details of the model testing were specified. The models were tested in Chapter 5.

Descriptive Statistics

Task performance is reported for all outcomes in Table 4.1. Students in grade 6 outperformed students in grade 4 on most tasks with only two exceptions. For digit span forward, the difference was not significant, $p = .06$. For fraction addition, the difference was not significant, $p = .59$. Notably, neither group of students had been taught fraction arithmetic and thus scores were at floor for both groups (the median score was 0 for both groups of students). The fraction arithmetic task was therefore excluded from model development and testing.

Table 4.1*Mean Performance by Grade on All Tasks*

	<u>Grade 4</u>					<u>Grade 6</u>					<u>Group comparisons</u>	
	<i>M</i>	<i>SD</i>	<i>Min</i>	<i>Max</i>	<i>Skew</i>	<i>M</i>	<i>SD</i>	<i>Min</i>	<i>Max</i>	<i>Skew</i>	<i>t</i>	<i>d</i>
<i>Whole Number Skills</i>												
Digit comparison ⁴	1.01	0.17	.73	1.63	4.63	0.88	0.16	.63	1.41	4.11	4.81**	0.84
Order judgement ⁴	2.41	0.54	1.30	4.27	2.79	1.98	0.49	1.09	3.36	1.76	4.74**	0.83
0-1000 number-line ³	11.96	6.51	3.25	30.41	3.80	7.36	2.98	3.99	17.45	5.35	5.21**	0.91
Arithmetic Total ¹	30.25	13.36	3	75	3.46	44.76	18.52	13	97	3.06	-5.11**	0.90
Addition ¹	14.69	5.19	3	35	2.71	20.00	7.55	3	39	1.96	-4.66**	0.82
Subtraction ¹	8.77	4.76	0	22	3.86	12.42	5.93	3	29	2.44	-3.87**	0.68
Multiplication ¹	6.80	6.32	0	30	6.03	12.33	6.88	1	37	5.10	-4.78**	0.84
<i>Attention/Memory</i>												
Digit forward ¹	7.09	1.53	3	11	0.37	7.68	1.99	4	12	1.34	-1.89 ⁺	0.33
Digit back ¹	5.98	1.48	3	10	0.91	6.92	1.66	3	11	1.41	-3.41**	0.60
Spatial span ¹	11.06	3.00	3	24	3.08	12.60	2.95	2	20	-1.98	-2.92**	0.52
<i>General Vocabulary</i>												
PPVT (sets9-14) ¹	37.59	10.70	15	65	0.83	47.53	9.76	22	68	-1.10	-5.51**	0.97
<i>Mathematical language</i>												
Vocabulary ²	0.54	0.13	.22	.78	-1.67	.62	.12	.30	.97	0.52	-3.79**	0.67
Orthography (SDT) ⁵	1.89	0.64	0.49	3.31	-0.14	2.34	0.69	0.00	3.87	-0.86	-3.92**	0.68
<i>Outcomes (Time 1)</i>												
Fraction mapping ²	0.64	0.18	.30	.88	-1.02	0.77	0.14	.38	.98	-2.26	-4.61**	0.81
<i>Outcomes (Time 2)</i>												
Fraction mapping ²	0.70	0.18	.23	.98	-2.07	0.82	0.13	.30	1.00	-5.37	-4.31**	0.76
0-1 number-line ³	16.79	11.33	2.95	43.95	1.96	10.13	7.06	2.23	30.04	4.15	4.03**	0.71
Word problems ¹	1.64	1.26	0	4	1.42	2.47	1.68	0	6	0.18	-3.17**	0.56
Fraction comparison ¹	3.86	1.13	0	6	-1.12	4.58	1.08	2	6	-2.72	-3.70**	0.64
Fraction transcoding ¹	3.15	3.39	0	14	4.89	10.60	6.52	0	20	0.10	-7.87**	1.43
Fraction addition ¹	1.12	1.56	0	6	3.04	1.30	2.05	0	6	4.44	-0.53	0.10

Notes. ¹ total correct; ² proportion correct (i.e. trials correct/total trials); ³ percent absolute error; ⁴LISAS adjusted response times (s) per item; ⁵ dPrime sensitivity score, PPVT = General Vocabulary Test. * $p < .05$, ** $p < .01$, **bolded** skew indicates z-skew > |3.29|

Most of the measures were normally distributed. The fraction tasks were novel for grade 4 students and thus their performance was poor on some measures. Comparing differences in scoring distributions between grades helped inform the decision on which fraction tasks to include in subsequent analyses by grade.

Whole Number Skills

The whole number skill measures included number comparison, number ordering, arithmetic, and the 0-1000 number line (also called the integer number line). Arithmetic fluency was a combined score of single-digit addition, subtraction and multiplication. All sub-scores are included in Table 4.1. Although performance was somewhat skewed on the combined fluency measure for fourth graders, the median performance was very near the mean ($Med = 28.00$, $M=30.25$), and included a wide range of scores, thus, the combined fluency measure was used in subsequent analyses. Performance on the number line task was skewed for both groups of students which is typical in number line estimation because there is an absolute floor for this task (i.e., minimum of zero).

Math Language Skills

Mathematical Vocabulary. Performance was normally distributed, and details of item-based responses are described in the technical report in Appendix B.

Mathematical Orthography. There was one extreme outlier scoring more than 3 standard deviations (SD) below the mean which prompted me to inspect this student's performance more closely. This student correctly classified 78% of the conventional stimuli yet only 22% of the unconventional stimuli indicating a response bias; the student pressed "yes" 78% of the time. To minimize the influence of this outlier, the outlier score was made less

extreme by reducing it to next lowest score, 1.6 SD below the mean. Although this reduction led to minor changes in correlations, model testing results were unchanged.

Fraction Skills

Although students in both grades completed the same fraction tasks, some tasks were too difficult for the fourth graders and some tasks were not sufficiently variable for statistical analyses. To better understand these differences, I briefly describe the performance on the fraction tasks separately.

Fraction Mapping. Time 1 fraction mapping was normally distributed for both groups of students whereas Time 2 fraction mapping was skewed for sixth graders. Some fourth graders scored at or near chance levels (i.e., 33.3% because there were three alternatives). In contrast, no sixth grader scored below 40%. As shown in Figure 4.1, the Time 2 fraction mapping performance was high for most sixth graders however a few students continued to have very poor fraction mapping skills. Importantly, there was some variability even in grade 6 and performance was not at ceiling level. Thus Time 2 fraction mapping skills are included in subsequent model analysis.

Fraction Number Line. Performance on the fraction number line (also called the 0-1 number line) was skewed amongst grade 6 students. Many grade 6 students were skilled estimators thus their scores were precise with low errors (i.e., concentrated close to 0), yet a few grade 6 students appeared to be inaccurate estimators evidenced by the scores at the tail of the distribution (see Figure 4.2). The relatively flat distribution of scores for grade 4 students indicates that although some students were accurate estimators, many students were not successful fraction estimators. In summary, there was variability in the scoring, thus, the measure was included in subsequent model testing.

Figure 4.1

Distribution of Scoring by Grade on the Fraction Mapping Task

Note. Accuracy scoring is the proportion of correct responses.

Figure 4.2

Distribution of Scoring by Grade on the Fraction Number Line Task

Note. PAE = percent absolute error.

Fraction Word Problems. Students answered word problems designed to assess their knowledge of fraction concepts and any potential fraction misconceptions. The word problems included different fraction concepts (e.g., density, magnitude, knowledge of the whole), some more challenging than others; thus, reliability comparing accuracy across the six problems was low (Cronbach's $\alpha=0.66$). This task was difficult for both groups of students as reflected in the accuracy scores shown in the first pair of columns in Figure 4.3. In spite of the low scores, overall task performance was normally distributed for both grades. Importantly, there was variability in the scoring thus the fraction word problems were included in subsequent model testing.

Fraction Worksheet. The fraction worksheet included magnitude comparison, transcoding, and fraction addition questions. Task performance on the worksheet problems (i.e., compare, transcode, add) is also shown in Figure 4.3.

Fraction Number Comparison. The fraction number comparison measure did not capture sufficient variability in performance to include in model testing. The Grade 6 students scored near ceiling with a median score of 5 out of 6. Further, task reliability was very low for both grade 4 students and grade 6 students (Cronbach's $\alpha = 0.07$ and 0.25 respectively) presumably reflecting the lack of variability.

Figure 4.3

Performance on the Fraction Word Problems and the Fraction Worksheet Problems by Grade

Note. Error bars represent 95% confidence intervals on the mean. ** $p < .01$, *** $p < .001$.

Fraction Transcoding. The transcoding measure was too difficult for fourth graders. Students answered three sets of transcoding questions; transcoding equivalent fractions, transcoding fractions and decimals, and transcoding mixed numbers and improper fractions. The first two sets of transcoding questions are part of the grade 4 curriculum, however, mixed numbers and improper fractions are not practiced until grade 5 (Ontario Ministry of Education, 2005). I did not expect the grade 4 students to complete the mixed number portion of the task, however, if they expressed interest in trying the questions, we guided them through a practice question. Students were told they could skip any questions they were unfamiliar with and thus we do not have responses to all questions from all students (as shown in Table 4.2). Because the proportion of grade 4 students completing the tasks was so low, the task could not be included in hypothesis testing by grade.

Fraction Addition. Scores on the addition task were not significantly different between fourth and sixth graders. Both groups did poorly on the addition problems and the median scores for both groups were 0. Fraction addition was not included in further analyses.

In summary, based on the descriptive review, three fraction outcomes were excluded from model testing because of limited variability in scoring or incomplete responses: magnitude comparison, transcoding, and fraction addition. More importantly, the outcomes included in model testing were fraction mapping skills (Time 1 and Time 2), the fraction number line estimation task, and the fraction word problems.

Table 4.2*Percentage of Students in Each Grade who Responded to Each Transcoding Question, by Type*

Question Type	Grade	Items (in each set)							
		a	b	c	d	e	f	g	h
Equivalent	4	81.3	78.1	54.7	62.5	54.7	48.4		
fractions	6	90.9	90.9	87.9	90.9	89.4	89.4		
Fractions and	4	51.6	39.1	32.8	29.7	40.6	28.1	21.9	23.4
decimals	6	78.8	74.2	72.7	68.2	74.2	71.2	74.2	74.2
Mixed numbers and	4	51.6	53.1	60.9	56.3	37.5	40.6		
improper fractions	6	72.7	78.8	80.3	78.8	68.2	71.2		

Correlations***Fraction Skills and Language***

The current research involved a number of fraction tasks; if the tasks are tapping into similar aspects of fraction knowledge, they should be correlated with each other. As shown in Table 4.3, fraction skills (i.e., mapping skills, number line, word problems) were all correlated with each other (shown shaded blue in the table). That is, students who performed well on one fraction measure also performed well on the other fraction measures and vice versa. There was, however, some range in correlations (.29 to .57). Nonetheless, the similar patterns of correlation indicate a level of concurrent validity for the novel and adapted fraction measures.

Table 4.3

Correlations Amongst Variables. Numbers Above the Diagonal are for Grade 4 students, Below the Diagonal are Grade 6 Students

	<u>Attention/Memory</u>						<u>Time 1 Whole Number and Language</u>					<u>Time 2 Fraction Skills</u>					
	1. SDT	2. MV	3. FMP1	4. DF	5. DB	6. SS	7. GV	8. NC	9. OJ	10. FLU	11. INL	12. FNL	13. FWP	14. FNC	15. FTC	16. FADD	17. FMP2
1		.35**	.44**	.26*	.06	.19	.26*	.08	-.07	.23	-.26*	-.31*	.32**	.36**	.31*	.38**	.42**
2	.29*		.42**	.38**	.30*	-.03	.59**	-.10	-.10	.28*	-.33**	-.29*	.43**	.29*	.33*	.31**	.54**
3	.43**	.47**		.18	.23	.06	.42**	.01	-.14	.24+	-.52**	-.59**	.41**	.28*	.37**	.25	.55**
4	.14	.23	.29*		.16	.26*	.31*	-.18	-.30*	.26*	-.23	-.26*	.02	.14	.21	.37**	.30*
5	.51**	.23	.43**	.30*		.24+	.38**	-.10	-.35**	.51**	-.28*	-.40**	.19	.17	.29*	.26	.38**
6	.24	.22	.17	.27*	.29*		-.03	-.12	-.24	.22	-.38**	-.19	.13	.19	.19	.07	.04
7	.27*	.48**	.41**	.39**	.32**	.38**		.12	-.01	.24	-.23	-.29*	.40**	.35**	.37**	.31*	.58**
8	-.21	-.06	-.13	.01	-.29*	-.17	.03		.58**	-.36**	.29*	.11	-.03	-.09	.02	.14	-.02
9	-.20	-.15	-.06	-.08	-.25*	-.25*	-.12	.57**		-.45**	.28*	.19	-.12	-.25*	-.24	-.16	-.22
10	.29*	.52**	.45**	.20	.35**	.24	.22	-.48**	-.37**		-.51**	-.37**	.24	.29*	.49**	.34**	.35**
11	-.11	-.34**	-.07	-.06	-.08	-.36**	-.21	.34**	.22	-.26*		.57**	-.21	-.27*	-.36**	-.10	-.55**
12	-.25*	-.60**	-.43**	-.24**	-.31*	-.35**	-.40**	.24	.27*	-.46**	.59**		-.29*	-.53**	-.48**	-.34**	-.61**
13	.23	.49**	.46**	.41**	.37**	.33**	.47**	-.24	-.26*	.34**	-.40**	-.55**		.27*	.39**	.14	.39**
14	.12	.43**	.30*	.14	.18	.20	.24+	-.14	-.28*	.33**	-.42**	-.50**	.36**		.49**	.36**	.35**
15	.29*	.62**	.50**	.26*	.34**	.34**	.36**	-.23	-.33*	.50**	-.31*	-.56**	.51**	.51**		.52**	.45**
16	.04	.33**	.30*	.08	.04	.22	.23	-.22	-.17	.20	-.11	-.25+	.25*	.07	.32*		.40**
17	.17	.56**	.57**	.19	.33**	.21	.36**	-.08	-.15	.37**	-.25*	-.63**	.35**	.46**	.52**	.20	

Notes. SDT = math orthography (i.e., symbol decision task), MV = math vocabulary, FMP= fraction mapping, DB= digit backwards, DF = digit forward, SS = spatial span, GV = general vocabulary, NC = number comparison, OJ= order judgement, FLU = single digit arithmetic, INL= integer number line accuracy, FNL = fraction number line, FWP = fraction word problems, FNC = fraction number comparison, FTC = fraction transcoding, FADD = fraction addition. Gray shading indicates math language/outcome correlations. Blue shading indicates mapping/outcome correlations. * $p < .05$, ** $p < .01$.

Math Language. Across grades, math language skills (mathematical orthography and math vocabulary), fraction mapping, and fraction outcomes were highly correlated (see Table 4.3). As expected, students with better fraction mapping skills at Time 1 also had stronger fraction skills at Time 2. Moreover, students with stronger math language skills were more likely to have better fraction mapping skills both concurrently and five months later. These findings support the view that math language skills and fraction mapping skills are relevant to consider when assessing fraction development.

Reviewing the correlations, I compared the differences in patterns of relations between math language skills and fraction outcomes across grades (shaded in dark grey in Table 4.3). I hypothesized that math language skills (vocabulary and orthography) would relate differently to fraction mapping for fourth and sixth grade students. For students in grade 4, performance on both of the math language tasks was positively related to their Time 1 and Time 2 fraction mapping skills, whereas for students in grade 6, performance on both of the math language tasks was positively related to their Time 1 fraction mapping skills but orthography was not related to Time 2 fraction mapping performance.

I also hypothesized that math language skills would relate differently to fraction outcomes for fourth and sixth grade students. These correlations are shown highlighted in gray in Table 4.3. These predictions were supported. For students in grade 4, performance on both of the math language tasks was moderately related to their fraction outcomes, with correlations ranging from .29 to .43. For students in grade 6, mathematical vocabulary was related to all of the fraction outcomes with correlations ranging from .33 to -.60. Orthography, however, was not significantly correlated with fraction word problems for sixth grade students. To summarize, the correlational patterns between math language skills and fraction outcomes suggest that the

relations between math language skills and fraction skills change as students gain more knowledge. Accordingly, to test my hypothesis related to the differences in relations between mathematical vocabulary, mathematical orthography and fraction skills, the paths will be freely estimated across grades.

I hypothesized that fraction mapping skills would predict fraction outcomes for both fourth and sixth grade students. Providing support for this view, patterns of correlation between Time 1 fraction mapping skills and the target fraction outcomes (fraction number line, fraction word problems and fraction mapping) were similar across grades. Further, the correlations were strong at 0.59, 0.41 and 0.55 respectively for the fourth graders and 0.45, 0.46 and 0.56 for the sixth graders (see Table 4.3). To test the hypothesis that fraction mapping skills directly predict fraction outcomes, the relations between fraction mapping and subsequent fraction outcomes will be freely estimated across grades.

General Vocabulary. The relations between general vocabulary and both math language skills, and between general vocabulary and fraction outcomes were similar across grades. Students with a stronger general vocabulary scored better on mathematical vocabulary and mathematical orthography, they were more skilled matching fraction pictures, numbers and words, and performed better on the fraction number line and solved more fraction word problems correctly than students with weaker general vocabulary. Because I am interested in seeing how math language skills, general vocabulary, and fraction mapping are related, the concurrent path (i.e., general vocabulary predicting Time 1 fraction mapping) will be freely estimated in the path models.

Covariates of Fraction Skills

Working Memory. The patterns of correlation between each working memory variable and each fraction outcome is similar. Thus, the three span tasks (i.e., digit forward, digit backward, and spatial span) were reduced to create a composite working memory variable. The reduction was done by grade. For the grade 6 students, the working memory factor accounted for 52.4% of the variance in working memory measures (digit forward, digit back and spatial span) and factor loadings were 0.72, 0.73, and 0.71 respectively. For the grade 4 students, the working memory factor accounted for 48.3% of the variance in working memory measures and factor loadings were 0.68, 0.65, and 0.75 respectively. The correlations for the working memory factor and fraction skills are shown in Table 4.4.

The patterns of correlations between the working memory factor score and fraction skills are similar between grades with the exception of the correlations between working memory and word problems which were significantly correlated for sixth graders but not for fourth graders (see Table 4.4). In summary, the relation between working memory and the fraction outcomes is mostly similar by grade thus can be constrained to be equal in model testing. However, when modelling the path between working memory and the concept problems, the path will be freely estimated by grade.

Table 4.4

Correlations by Grade Between the Working Memory Factor and (a) Language Skills, and (b) Fraction Skills

	Math Language Skills				Fraction Skills		
	General Vocabulary	SDT Math	Math Vocab.	Time 1 Mapping	Time 2 Mapping	Number line	Word problems
Grade 4	.31*	.24 [†]	.33**	.23	.34**	-.40**	.17
Grade 6	.50**	.39**	.31*	.41**	.33**	-.41**	.51**

Note. PPVT = Peabody picture vocabulary test, SDT = symbol decision task-math, vocab. = vocabulary. [†] $p \leq .06$, * $p < .05$, ** $p < .01$.

Number Skills. Whole number knowledge is assumed to be relevant to fraction knowledge (e.g., Siegler et al., 2011; Sophian, 2017; Steffe & Olive, 2010). However, there are few comprehensive assessments of a range of number skills and their relations to fraction performance. Thus, to reduce the number of variables in the model, the correlations were used to identify which number skills to include in model testing. Number comparison and order judgement were not related to math language skills or to either Time 1 or Time 2 fraction mapping for either grade. Thus, they were not included in the models of fraction performance. Performance on the more advanced number skills (arithmetic and 0-1000 number line) were both related to math language skills and fraction outcomes across grades with one exception: Performance on the 0-1000 number line was not related to fraction mapping skills for sixth graders. However, 0-1000 number line performances is relevant to fraction number line performance (Resnick et al., 2016) and thus will be included in the fraction number line model. Because whole number skills are relevant to fraction performance and the correlational patterns

between arithmetic and fraction outcomes were similar between grades, the arithmetic and fraction outcome paths will be constrained across grades.

Chapter Summary

In summary, based on the descriptive analyses, variables were selected for hypothesis testing and, based on the correlational analyses, decisions about the path analyses were made prior to the actual analyses. In essence, because the sample size is moderate for these multi-group path analyses, decisions were made to reduce the total number of different paths that were tested.

First, model testing will involve three Time 2 fraction outcomes: fraction mapping, fraction number line estimation, and fraction word problems. By also including Time 1 fraction mapping, the results will test for growth in fraction mapping skills (i.e., Time 2 fraction mapping, controlling for Time 1 fraction mapping). Second, general vocabulary, working memory, and number skills (arithmetic fluency and where applicable 0-1000 number line) will be included in the models because they are skills that, according to the literature, are relevant to fraction knowledge. Notably, because correlations between the working memory and number skills were similar across grades, the paths related to these variables will be constrained to be equal across grades with one exception. Working memory skills will be freely estimated in the model predicting fraction word problems. Third, to test the hypothesis that the relations between math language skills and fraction knowledge differ with expertise, the relations among mathematical vocabulary, mathematical orthography, and the fraction outcomes will be freely estimated across grade. Finally, to test the hypothesis that Time 1 fraction mapping skills predict all fraction outcomes, the path from Time 1 fraction mapping skills to the fraction outcomes will be freely estimated across grade.

CHAPTER 5: INDIVIDUAL DIFFERENCES IN FRACTION KNOWLEDGE PART 2

In this Chapter I describe and compare sources of individual differences in fraction skills amongst novice (grade 4 students) and more experienced (grade 6 students) fraction learners. Specifically, I focus on fraction mapping skills and the influence of math-specific language skills on students' developing knowledge of fraction concepts. Students' cognitive skills (receptive vocabulary, working memory), math-specific language skills (mathematical vocabulary and orthography), number skills (arithmetic, 0-1000 number line) and fraction mapping skills were measured in the first round of testing. Five months later, students completed a battery of fraction assessments (fraction mapping, fraction word problems, and fraction number line).

The analyses were organized to address multiple hypotheses as outlined in Chapter 2 (see page 38) and illustrated in Figure 5.1. First, I proposed that after accounting for general language skills, math-specific language skills (i.e., mathematical vocabulary and mathematical orthography) would predict Time 1 fraction mapping skills for students in both grades. Specifically, based on the finding that mathematical vocabulary (i.e., a math-specific language skill) is more strongly related to outcomes in older, versus younger students (Powell et al., 2017), I hypothesized that mathematical vocabulary would directly predict fraction mapping skills in sixth graders (H1b) but not in fourth graders (H1a).

Hypothesis 2 was that, after accounting for general vocabulary, mathematical orthography would directly predict fraction mapping skills for both groups of students. Specifically, based on the findings that mathematical orthography is related to math outcomes in multiple age groups (Douglas et al., in press; Headley et al., 2016; Xu et al., in review), mathematical orthography will directly predict fraction mapping skills for fourth graders (H2a) and sixth graders (H2b).

Hypothesis 3 was that, because the reliance on general vocabulary skills decreases with mathematical expertise (Powell, 2017), mathematical vocabulary will directly predict fraction outcomes for advanced fraction learners, sixth graders (H3b) but not in fourth graders (H3a).

Finally, based on the assumptions that conceptual math knowledge is built through math-specific discourse (O'Halloran, 2015), and that connecting symbols to their referents is the first level of symbol expertise (Hiebert, 1988), I hypothesized that Time 1 fraction mapping skills would directly predict Time 2 fraction outcomes for both fourth graders (H4a) and sixth graders (H4b).

Figure 5.1

Path Models Capturing the Hypotheses Predicting Fraction Skills in Fourth and Sixth Graders

Note. This figure is also included as Figure 2.1 in Chapter 2. It has been included here to illustrate the hypotheses in the context of model testing.

Model Testing

I used multi-group path analysis with *MPlus* Version 8 (Muthen & Muthen, 1988-2019), to test the four hypotheses that are captured in Figure 5.1. In the current research, the mathematical vocabulary measure and the symbol decision task for math (SDT-Math) are assumed to capture students' verbal and written math language skills. Thus, to address Hypotheses 1 and 2, I tested whether these two measures of math-specific language skills would be related differently to fraction mapping for novice fraction learners (grade 4 students) and more experienced fraction learners (grade 6 students). These hypotheses are captured in the "Time 1" portion of the path models. To address Hypotheses 3 and 4, I tested the relations predicting the fraction outcomes that are captured in the "Time 2" portion of the models.

Each fraction outcome was modelled separately. In general, 1) the paths involving number skills and working memory were constrained to be equal across groups unless otherwise noted and 2) the path coefficients of interest were estimated independently for each group. The differences in predictive paths are described for each outcome. A Wald test of parameter constraints was also used as a supplementary test to compare whether the path coefficients differed statistically by grade. However, the Wald test may lack sufficient power to detect a significant effect if a) the difference in path strength is small, and b) the sample size is small (Kline, 2016). For this reason, the Wald test is referred to but not necessarily discussed.

Fraction Skills

Fraction Mapping. The model describing the relations between general vocabulary, math language skills, fraction mapping and growth in fraction mapping skills is shown in Figure 5.2. To simplify the model diagram, the covariate paths amongst number skills, working memory and the Time 1 measures are not shown. The predictive paths from orthography, math

vocabulary and general vocabulary to Time 1 fraction mapping were estimated independently by grade as were the paths between math vocabulary, Time 1 fraction mapping and Time 2 fraction mapping. Model fit was strong, $\chi^2(14) = 17.068$, $p = 0.253$, SRMR = 0.048, CFI = .974, RMSEA = 0.058, 95% CI [.000, 0.140]. Wald tests of model constraints were not significant ($p = 0.98$).

Hypotheses 1 and 2. Hypothesis 1 was supported. Mathematical vocabulary directly predicted Time 1 fraction mapping for sixth grade students (H1b) whereas the path was not significant for fourth grade students (H1a) (see Figure 5.2). This pattern is consistent with other research in which knowledge of mathematical vocabulary is more strongly related to math outcomes in older than in younger students (Powell et al., 2017). In grade 4, general and math-specific vocabulary jointly predict variance in the fraction mapping task at Time 1.

Hypothesis 2 was supported. Mathematical orthography directly predicted fraction mapping skills for both fourth grade students (H2a) and sixth grade students (H2b). This finding is consistent with previous research with younger student (i.e., grades 3 and 4; Xu et al., in review) older students (i.e., grade 8; Headley, 2016) and adults (Douglas et al., in press), that has shown positive relations between outcomes and mathematical orthography. Interestingly, the simple correlation between orthography and mapping skills at Time 2 for sixth grade students was not significant (see Table 4.3; $r = .17$), suggesting that orthography and mapping relation may change with developing expertise. This finding will be discussed further in Chapter 7.

Figure 5.2

Path Model Predicting Growth in Fraction Mapping Skills in a) Grade 4 and b) Grade 6

Notes. Values shown are the standardized coefficients. * $p < .05$, ** $p < .01$, *** $p < .001$. Dotted lines indicate non-significant paths and faded font indicates paths unrelated to hypotheses.

Hypotheses 3 and 4. The relations predicting growth in fraction mapping were comparable between grades. Hypothesis 3 was partially supported in that mathematical vocabulary directly predicted growth in fraction mapping skills for sixth graders (H3b). However, I had predicted that mathematical vocabulary would be less strongly related to fraction outcomes in fourth graders (H3a) compared to sixth graders - the relations were comparable. The pattern of modelled relations was the same across grades. Not surprisingly, Hypothesis 4 was supported: Mapping skills at Time 1 directly predicted growth in fraction mapping skills for both groups of students. In summary, growth in fraction mapping skills were similarly predicted for students in both fourth and sixth grades.

Fraction word problems. The fraction concepts task was an oral assessment of students' conceptual fraction knowledge and included questions on fraction density, equal partitioning, and part-whole relations. For the fraction concept model the relation between working memory and fraction word problems was freely estimated across grades because the correlations were different between grades (0.17 and 0.51 respectively).

The model predicting fraction concepts is shown in Figure 5.3. As with the previous model, for ease of understanding, the illustrated model has been simplified such that covariate paths are not shown and the predictive paths from arithmetic and working memory are in faded font. Model fit was excellent, $\chi^2(13) = 11.477, p = 0.571$, SRMR = 0.042, CFI = 1.000, RMSEA = 0.000, 95% CI [.000, 0.110].

Hypothesis 3. The path analysis provided full support for Hypothesis 3. After controlling for vocabulary, working memory and arithmetic skills, mathematical vocabulary directly predicted knowledge of fraction concepts for sixth graders (H3b) as hypothesized. For fourth graders, although the direct math vocabulary/fraction concepts path was not significant (H3a) the

path coefficients were comparable. Further, the combined effect of math vocabulary directly and indirectly through fraction mapping was significant for both fourth and sixth graders ($\beta = .271$, $p = .027$; $\beta = .280$, $p = .014$ respectively). These findings suggest that math vocabulary skills are important for both more and less experienced fraction learners when the outcome has a strong verbal component.

Hypothesis 4. Hypothesis 4 was supported, mapping skills at Time 1 directly predicted fraction concepts for both fourth and sixth graders; however, the relation was not significant for sixth graders. A Wald test of parameter constraints suggests the fraction mapping and fraction concepts relation was not statistically different between fourth and sixth graders, $W_1(1) = 0.24$, $p = .62$. However, as stated previously, with the small sample size the Wald test may be underpowered to distinguish significant differences (Kline, 2016).

One interesting observation is that the models accounted for more variance in fraction concepts for sixth graders compared to fourth graders (see Figure 5.3, $R^2 = .414$ versus $.251$). This finding suggests that other skills, not included in testing may be implicated in fourth graders performance. Further, working memory predicted fraction concepts for sixth graders but not for fourth graders. Together these findings indicate that the fraction concepts task was more cognitively demanding for sixth graders presumably because they had sufficient skills to tackle the more difficult questions. In summary, mathematical vocabulary supported oral fraction concept skills directly for sixth graders and indirectly for both fourth and sixth graders whereas fraction mapping skills were more strongly implicated in fraction concepts for fourth graders compared to sixth graders.

Figure 5.3

Path Models Predicting Fraction Concept Word Problems in a) Grade 4 and b) Grade 6

Notes. Values shown are the standardized coefficients. + $p=.05$, * $p<.05$, ** $p<.01$, *** $p<.001$. Dotted lines indicate non-significant paths and faded font are paths unrelated to the hypotheses.

Fraction Number Line. In the fraction number line task, students estimated the position of target fractions on a horizontal line that ranged from 0 on the left to 1 on the right. To control for number skills, whole number line estimation was included in the model. There were two reasons for this choice: (1) the two number line measures were strongly correlated for fourth and sixth graders (0.57 and 0.59 respectively), and (2) whole number line estimation has been shown to predict growth in fraction number skills (Resnick et al., 2016). The path models predicting fraction number line are shown in Figure 5.4. Model fit was acceptable $\chi^2(13) = 21.539$, $p = 0.063$, SRMR = 0.058, CFI = .941, RMSEA = 0.101, 95% CI [.000, 0.173]. The Wald test of parameter constraints indicated that the predicative paths from math vocabulary and fraction mapping to fraction number line were significantly different for grade 4 students compared to grade 6 students $W_T(2) = 7.27$, $p = .026$.

Hypothesis 3. After accounting for domain-general cognitive skills and number skills, mathematical vocabulary directly predicted fraction number line for sixth graders but not for fourth graders. Hypothesis 3 was supported.

Hypothesis 4. Hypothesis 4 was supported. Fraction mapping skills directly predicted fraction number line for both fourth and sixth graders. The strength of relation was, however, weaker for sixth graders as shown in Figure 5.4.

Figure 5.4*Path Models Predicting Fraction Number Line*

Notes. Values shown are the standardized coefficients. $+p=.05$, $*p<.05$, $**p<.01$, $***p<.001$. Dotted lines indicate non-significant paths and faded font are paths unrelated to the hypotheses.

Chapter Summary

In this chapter I tested four hypotheses using path analysis. The first two hypotheses related to the math-specific language skills that support students' concurrent knowledge of fraction mapping. Hypotheses 1 and Hypothesis 2 were supported. After controlling for general vocabulary, mathematical vocabulary skills directly predicted fraction mapping skills in sixth graders (H1) and mathematical orthography directly predicted fraction mapping skills in fourth graders and sixth graders (H2). These findings confirm that math-specific language skills account for individual differences in students' accuracy mapping visual fraction representations to abstract fraction symbols.

The last two hypotheses described the Time 2 (5 months later) relations between fraction mapping and mathematical vocabulary and students' developing knowledge of fraction concepts. Developing knowledge of fraction concepts was assessed with three measures: Time 2 fraction mapping skills, word problems that tapped into knowledge of fraction concepts and fraction number line estimation.

In support of Hypothesis 3, mathematical vocabulary directly predicted all fraction outcomes for sixth graders. Notably, mathematical vocabulary skills also predicted growth in fraction mapping and was the combined effect of mathematical vocabulary (direct and indirect through mapping skills) was significant for fourth graders. Thus, although I predicted mathematical vocabulary would be strongly linked to all fraction outcomes for sixth graders, math vocabulary also accounted for individual differences in the outcomes that had a strong language component for fourth graders. Together, the findings indicate that mathematical vocabulary skills account for individual differences in developing fraction knowledge differently for fourth and sixth graders. For less experienced learners (fourth graders), mathematical

vocabulary was implicated in fraction outcomes with a strong verbal component whereas for more experienced learners (sixth graders), mathematical vocabulary was directly implicated in all fraction outcomes.

Supporting Hypothesis 4, Time 1 fraction mapping skills predicted all outcomes amongst fourth graders. However, for sixth graders performance on the fraction concept word problems was not predicted by fraction mapping. Interestingly, the strength of the relation between fraction mapping and the fraction number line task was weaker for sixth grade students than fourth grade students. These findings suggest that although fraction mapping skills are central for developing knowledge of fraction concepts, the relation may differ for more advanced fraction learners. That is, once students have sufficient fraction mapping skills, other skills such as number knowledge, may contribute more strongly to their developing fraction knowledge. In summary, the concurrent and longitudinal findings support the view that fraction mapping skills are the central organizing knowledge for fraction learning and math-specific language skills are a key piece in the fraction learning puzzle.

CHAPTER 6: PROFILES OF FRACTION NUMBER LINE STRATEGIES

Students often experience difficulty making sense of fractions. Consider Jordan, a fictional grade 4 student who is just learning about fractions. When asked if $\frac{1}{3}$ is greater than $\frac{1}{2}$, Jordan said “yes, 3 is greater than 2 so $\frac{1}{3}$ has to be greater than $\frac{1}{2}$.” Jordan has a misconception that a greater denominator means a greater fraction. Fraction number lines have been recommended as teaching tools to help minimize fraction misconceptions like Jordan’s because they may help students to understand relative fraction magnitudes (e.g., Bruce et al., 2013; Common Core State Standards Initiative, 2010; Gersten et al., 2017; Ontario Ministry of Education, 2005; Siegler et al., 2010; Torbeyns et al., 2015). However, focused research on the relations between developing fraction knowledge and students’ performance placing fractions on number lines is scarce. In the previous chapter, I examined the influence of math-specific language skills and fraction mapping on an averaged measure of fraction number line accuracy. In the present chapter, I extended the findings from Chapter 5 by (1) analysing in detail students’ patterns of fraction estimation and (2) identifying the prerequisite number and fraction skills that differentiated students based on their patterns of fraction estimation. Results of these analyses can provide educators with insights into how students reason about fraction magnitude and knowledge of the skills that differentiate students like Jordan from more successful fraction learners,

Number lines are two-dimensional, horizontal lines with labelled reference points on either end; typically 0 is at the left endpoint and a multiple of 10 (e.g., 10, 100, 1000) is at the right endpoint (Siegler & Opfer, 2003) (see Figure 6.1a). In the fraction number line task, 0 is at the left endpoint and 1, 2, or 5 is at the right endpoint (see Figure 6.1b; Hansen et al., 2015; Resnick et al., 2016; Rodrigues et al., 2019; Ye et al., 2016). Students can be asked to estimate

the position of proper fractions, mixed numbers, and improper fractions on fraction number lines. Their performance on fraction number lines is correlated with their knowledge of other fraction skills (Bailey et al., 2017; Hecht et al., 2003; Siegler & Pyke, 2013) and overall math achievement (Resnick et al., 2016; Rodrigues et al., 2019; Schneider et al., 2018; Siegler et al., 2011). However, research into how students place fractions on a fraction number line is limited (Siegler et al., 2011; D. Zhang et al., 2017).

Figure 6.1

Number Line Estimation Task: a) 0-1000 Whole Number Number Line and b) 0-1 Fraction Number Line

What number skills differentiate students who perform poorly on fraction number line estimation from students who perform well? Resnick et al. (2016) assessed the growth in students' fraction number line skills as they progressed from grade 4 to grade 6 and showed that students' whole-number arithmetic skills and whole number line estimation skills predicted the transition from inaccurate fraction estimator to accurate fraction estimator. Knowledge of fraction concepts may also support fraction number line estimation (Cirino et al., 2016; Gunderson et al., 2019). Thus, a second goal of the present research was to identify the fraction skills that support students' success on the fraction number line. Understanding how students perform the fraction number line task and the related skills that predict individual differences in

performance will help researchers develop better theories of fraction skills and support recommendations for how teachers can use fraction number lines in classrooms.

Number Line Estimation Strategies

Estimation strategies tell us how students approach the number line task; students' accuracy on the number line task reveals whether those estimation strategies are successful. For example, when they are asked to place whole numbers on number lines, older students and adults rely on both implicit (e.g., the midpoint) and explicit reference points (e.g., labelled endpoints) to guide their number line estimates (Ashcraft & Moore, 2012; Barth & Paladino, 2011; Di Lonardo et al., 2019; Slusser & Barth, 2017). Notably, when students rely on implicit reference points they also use proportional reasoning skills (e.g., 45 is a little smaller than 50 so it should be placed just to the left of the midpoint on a 0-100 number line) (Ashcraft & Moore, 2012; Barth & Paladino, 2011; Newman & Berger, 1984; Peeters et al., 2016; Schneider et al., 2018). Notably, when estimation errors are graphed, grade 3 students who use the midpoint to anchor their proportional reasoning strategy show an "M"-shaped pattern in which the most accurate estimates occurred around the midpoint and endpoints (Ashcraft & Moore, 2012). Adults also use the midpoint as a reference but are more accurate for target numbers further away from reference points, resulting in a flattened "M"-shaped pattern (Di Lonardo et al., 2020; Luwel et al., 2018). In summary, patterns of estimation 'error' have been used successfully to describe the strategies used by students and adults to place whole numbers on number lines.

Appropriate Fraction Number Line Strategies

What estimation strategies do students use to successfully place fractions on a number line? In the one paper that systematically described appropriate estimation strategies, Siegler et al. (2011) identified two strategies, segmentation and numerical transformation. According to

Siegler et al., the segmentation strategy involves solvers generating internal landmarks on the number line to help locate target fractions. This description is very similar to the proportional strategy based on implicit reference points used with whole numbers (Barth & Paladino, 2011; Slusser & Barth, 2017). Segmentation examples included dividing the number line into equal sections based on common fractions such as halves and quarters, or defining segments based on the denominator (e.g., dividing the number line into seven segments to place $\frac{4}{7}$) (Siegler et al., 2011).

The numerical transformation strategy as defined by Siegler et al. (2011) involves changing the fraction into a more convenient number through rounding, reducing, or converting the fraction to a decimal or percentage (e.g., a student might “round” $\frac{4}{9}$ to $\frac{1}{2}$ and place it accordingly). The transformation strategy could also be used on whole number lines, for example, to place 7,231 on a 0-10000 number line, participants might round the number to 7,500 because it is a more familiar proportion. Students in grade 8 reported using a transformation strategy more frequently than those in grade 6 indicating that strategy selection changes as students’ fraction skills develop (Braithwaite & Siegler, 2018; Liu, 2018; Siegler et al., 2011).

Using only two strategies may be insufficient to describe how students place fractions on a number line. Thus, a more fine-grained description than transformation and segmentation may better capture differences in how students’ fraction knowledge is used on the number line task. Transformation strategies include rounding and converting (Siegler et al., 2011), however rounding $\frac{4}{9}$ to $\frac{1}{2}$ is a different strategy than converting $\frac{4}{9}$ to 44%; rounding can involve less precise use of reference points. Similarly segmenting $\frac{4}{5}$ into 5 equal portions is a different strategy than using an endpoint reference, for example, noting that $\frac{4}{5}$ is close to one. I suggest including

reference points - explicit endpoints and implicit midpoints - in the list of estimation strategies for a more detailed analysis of how students estimate fractions.

Number Line Strategies based on Misconceptions

Students who have weak fraction knowledge may use strategies for placing fractions on number lines that are based on their misconceptions. Zhang et al. (2017) examined grade 6 and 8 students' flawed fraction estimation strategies. They found that some students estimated fraction magnitudes based on the whole number magnitudes of either the numerator or the denominator, thus incorrectly extending whole-number properties to fractions and showing a *whole number bias* (Braithwaite & Siegler, 2018; Dewolf & Vosniadou, 2015; Ni & Zhou, 2005; Stafylidou & Vosniadou, 2004; Van Dooren et al., 2015). For example, students using numerator-based estimations erroneously stated that a fraction with a numerator greater than 1, such as $\frac{2}{4}$, could not be placed on a 0-1 number line because 2 (i.e., the numerator) is greater than the endpoint (Zhang et al., 2017), or incorrectly segmented the number line based on the magnitude of the numerator (Deringöl, 2019). Similarly, students using denominator-based estimations erroneously placed fractions with large denominators, such as $\frac{4}{9}$, near the endpoint 1 because 9 is a large number (Braithwaite & Siegler, 2018). These flawed strategies occurred because students relied on the magnitude of the numerator or denominators and failed to consider a fraction as a whole entity. These faulty strategies provide insights into the weaknesses in students' fraction knowledge.

Even when students use both numerator and denominator magnitudes, they may still make errors on placing fractions on number lines. Braithwaite and Siegler (2018) examined developmental changes in the whole number bias by asking students in grades 4 through 8 to estimate the position of equivalent fractions (e.g., $\frac{8}{12}$, $\frac{2}{3}$) on a number line. Compared to eighth

graders, fourth graders relied more on the magnitudes of both whole number components in comparison to the relation between components when placing fractions. That is, fourth graders placed large component fractions (e.g., $\frac{8}{12}$) closer to 1 than equivalent small component fractions (e.g., $\frac{2}{3}$). These students were possibly influenced by an endpoint strategy; thinking large component fractions are closer to the endpoint than smaller component fractions. Thus, whole-component estimations can negatively influence students' estimation accuracy and this effect is more pronounced in younger students who are less experienced with fractions (Braithwaite & Siegler, 2018).

To effectively identify fraction misconceptions within the number line task, performance on individual trials must be considered. However, item-based performance is usually not described in fraction number line studies (e.g., Siegler et al., 2011; Zhang et al., 2017). Even when item-based data is available, group means rather than individual patterns of performance are typically reported and thus, the relation between strategy selection and performance cannot be inferred (e.g., Resnick et al., 2016). How would item-based performance inform our understanding of fraction estimation? Consider the fractions $\frac{1}{9}$ and $\frac{8}{9}$. These fractions have large denominators. If students estimate the position of the fraction based primarily on the magnitude of the denominator, their accuracy placing $\frac{1}{9}$ would be low but their accuracy placing $\frac{8}{9}$ would be high (Resnick et al., 2016). Similarly, if students consider the magnitudes of both the numerator and denominator, they might place $\frac{2}{3}$ nearer to the lower endpoint because the numerator and denominator are small numbers (Braithwaite & Siegler, 2018) whereas they may place $\frac{8}{12}$ nearer to the upper endpoint because 8 and 12 are relatively larger numbers. Alternately, a student might place $\frac{2}{3}$ close to the upper endpoint because there is only a gap of one unit between

2 and 3 (Deringöl, 2019; Pearn & Stephens, 2004). By examining performance on individual trials, estimation patterns can be observed and thus allow researchers to draw inferences about students' fraction misconceptions.

In summary, fraction misconceptions may influence estimation patterns in multiple ways, but few attempts have been made to categorize patterns of performance. Students use a number of strategies on fraction number line tasks, such as dividing the number line into segments, transforming the fraction into a more convenient format (e.g., decimal, percent, equivalent fraction), and using endpoints and midpoint references to place estimates. The effectiveness of these strategies is influenced by the knowledge and experience of the student.

Skills that Relate to Successful Strategy Use

Knowledge of Fraction Symbols

In addition to effective strategies, students need an understanding of fraction symbols to successfully estimate the position of fractions on a number line. Presumably, if students do not understand how the fraction symbol maps to its quantitative referent, placing the fraction on a number line will be challenging. The knowledge of how whole numbers map to quantities is a foundational numeracy skill that predicts more advanced mathematical skills (Brankaer et al., 2014; Jiménez Lira et al., 2017; Lyons & Ansari, 2015; Salminen et al., 2018). Thus, it is reasonable to assume – and indeed, it was shown in Chapter 5 – that the ability to map fraction symbols to quantities would be a foundational skill for the fraction number line. However, fraction mapping is typically included in general measures of fraction concept knowledge (e.g., Hallett et al., 2010, 2012; Hansen et al., 2017; Hecht et al., 2003; Rodrigues et al., 2019) and thus has not been explicitly assessed as a predictor of more advanced fraction skills such as fraction number line estimation.

Number Line Knowledge

In addition to understanding fraction symbols, students need conceptual understanding of number lines to make accurate estimates (Resnick et al., 2016). Students with conceptual knowledge of the number line know that numbers can be represented in a proportionally-spaced ordered sequence along a line with one number on the left and a larger number on the right (Siegler & Opfer, 2003). Skill at estimating the position of whole-numbers on a number line are related concurrently (Bailey et al., 2014) and longitudinally to performance on fraction number lines (Resnick et al., 2016). Resnick et al. (2016) followed students' fraction estimation skills from fourth to sixth grade during which time, students completed two fraction number line tasks (0-1 and 0-2) at five time points. Latent transition analysis was used to classify students based on their mean estimation accuracy (as opposed to item accuracy) on each number line task. Three performance profiles emerged: (a) students who started accurately and ended accurately, (b) students who started inaccurately and ended accurately, and (c) students who started inaccurately and ended inaccurately. Resnick et al. found that after controlling for in-class attention and reading fluency, whole number estimation (i.e., 0-1000 number line) and multiplication fluency skills predicted the transition from inaccurate estimator to accurate estimator. Thus, whole-number-line knowledge supports skilled performance on the fraction number line task.

Whole Number Arithmetic

The numerical transformation strategy described earlier requires that solvers use arithmetic skills. For example, to change $\frac{4}{5}$ to a percentage entails dividing ($4 \div 5 = 0.8$) and multiplying ($0.8 \times 100 = 80\%$). Multiplication fluency is related longitudinally to accurate fraction number line estimation (Resnick et al., 2016) and measures of arithmetic calculation are related both concurrently (Namkung & Fuchs, 2016) and longitudinally (Ye et al., 2016) to

measures of fraction concept knowledge that include fraction number line estimation. These findings indicate that arithmetic fluency is related to accurate placement of fractions on number lines. Presumably, because transformation algorithms are not taught until grade 5 (Ontario Ministry of Education, 2005) and transformation strategies are used more by older than younger students (Siegler et al., 2011), arithmetic skills may relate more strongly to fraction number line performance amongst students in higher grades.

Current “Study”

The goal of the current chapter was to explore strategy use on fraction number line to better understand how students reason about fractions and the quantities they represent. As well, I also examined the skills that differentiated students based on their strategy selection and implementation.

First, latent profile analysis was used to classify students into different profiles based on their estimation accuracy on all fraction number line trials. This methodology has been used successfully to analyze number line estimation of whole numbers in children from preschool up to grade 2 (Bouwmeester & Verkoeijen, 2012; Xu, 2019; Xu et al., under review). In the present study, students estimated the position of 27 target fractions on a 0-1 number line. The percent absolute error (PAE) between the target fractions and the students’ estimates were used as indicators of the students’ number line estimation profiles (Xu, 2019). The number of estimation profiles was determined by comparing the fit statistics of the latent models and referencing theoretical support for the profiles. Students’ oral strategy reports were used to validate the estimation profiles (Xu, 2019) and provide additional insights into fraction estimation strategies

(Siegler et al., 2011; D. Zhang et al., 2017). Because this study is the first to assess estimation patterns for the fraction number line, our approach was exploratory.²

Next, multinomial logistic regression was used to link profile assignment with students' Time 1 math skills. The goal was to identify the skills that differentiate the groups defined in the latent profile analysis. Weak fraction mapping skills were expected to be associated with the least successful estimation patterns on the assumption that these students were still developing their conceptual understanding of the magnitudes represented by the fraction symbol. In contrast, better whole number estimation and arithmetic skills were expected to predict membership in more skilled profiles.

Method

Participants, procedures and measures are described in Chapter 3. Methodological details applicable to the current analyses are described below.

Measures

Fraction Number Line Estimation. In this iPad task, students estimated the position of a fraction on a 0-1 number line (see Figure 6.2). In the app, the student sees a numeric fraction at the top of the screen. The student is instructed to “place” the fraction on the number line by touching the number line and moving the red cursor to where they think the fraction should be located. When the student is happy with their estimate, they tap done and the next trial begins. Students began the task with four trials where they were prompted to explain their estimation strategy. These four trials ($\frac{1}{4}, \frac{3}{7}, \frac{4}{5}, \frac{7}{9}$) were presented in order at the beginning of the task and were chosen to allow a range of possible strategies. For example, students may use a midpoint

² Resnick et al. (2016) used latent profile analysis, however, their profiles were based on mean accuracy for two fraction number lines as opposed to individual item accuracy.

strategy to place $\frac{1}{4}$ recognizing that $\frac{1}{4}$ is half of half; they might segment the number line into 7 sections to place $\frac{3}{7}$; $\frac{4}{5}$ might be transformed to a decimal or a percentage; and students might use an endpoint reference to place $\frac{7}{9}$.

There were 27 experimental trials. Trials included all non-reducible proper fractions with single-digit denominators (i.e., $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{2}{3}, \frac{2}{5}, \frac{2}{7}, \frac{2}{9}, \frac{3}{4}, \frac{3}{5}, \frac{3}{7}, \frac{3}{8}, \frac{4}{5}, \frac{4}{7}, \frac{4}{9}, \frac{5}{6}, \frac{5}{7}, \frac{5}{8}, \frac{5}{9}, \frac{6}{7}, \frac{7}{8}, \frac{7}{9}, \frac{8}{9}$) (Torbeyns et al., 2015). Scoring was the percent of absolute error (PAE) between the placement of each fraction compared to the actual location of that fraction.

Figure 6.2

Fraction Number Line Prompt and Request for Strategy Self-Report

Coding of Strategies on the Fraction Number Line. Students reported their strategies for four target numbers on the fraction number line. Prior to testing, a coding scheme was developed based on previous strategies reported for fraction number lines (Siegler et al., 2011) and fraction magnitude comparisons (Dewolf & Vosniadou, 2015; Fazio et al., 2016; Reys et al., 1999). To provide a full description of the students' thinking we created a more detailed categorization scheme for strategy descriptions than had been used previously (i.e., Siegler et al., 2011).

Descriptions are shown in Table 6.1. Each comment was transcribed and coded during data collection. After data collection, codes and transcriptions were independently reviewed by two researchers. The inter-rater reliability was 84.1%. Discrepancies were discussed and resolved.

Table 6.1

Fraction Estimation Strategies

Strategy	Description	Example
Midpoint reference	estimating the fraction in reference to $\frac{1}{2}$.	$\frac{3}{7}$ is a bit less than one half
Endpoint reference	estimating the fraction in reference to either 0 or 1	$\frac{7}{9}$ is close to 1
Segmentation	dividing the number line into segments based on the denominator	I divided the line into 7 sections and counted to $\frac{3}{7}$
Transformation	transforming the fraction into a different fraction, decimal or percentage	$\frac{4}{5}$ is the same as $\frac{8}{10}$, 0.80, or 80%
I knew	Reports knowing without describing	I just know
I don't know	Reports guessing or not knowing	Not sure
Other	Any explanation that is not described above	I counted $\frac{1}{2}$, $\frac{1}{3}$ etc.

Predictors of Profile Assignment

Fraction Mapping. The mapping task captures students' ability to connect formal fraction numbers with words and images. In this iPad task, students saw a fraction represented as either a number (e.g., $\frac{2}{3}$), a picture, or a fraction word (e.g., two-thirds). Below the fraction

representation were three possible fraction matches. The student was instructed to select the appropriate response (e.g., What is the fraction of items/area that is/are circled?) and their accuracy and response time were automatically recorded. A more detailed description of the task is included in Chapter 3.

Arithmetic Fluency. In this paper and pencil task, students were given one minute to complete single-digit addition questions (sums to 18), one minute to complete subtraction questions (inverse of addition questions) and one minute to complete single-digit multiplication questions (2×2 to 5×9). Total score was the sum of correct responses for all three operations.

Number Line Estimation with Whole Numbers. Students estimated the position of a target number on a 0-1000 number line using an iPad app (Hume & Hume, 2014). To familiarize students with the app, we began with three practice trials where students tapped a green target on the number line. Students then completed 26 experimental trials. Scoring was the mean percent of absolute error (PAE) between the placement of each number compared to the actual location of that number.

Results

Descriptives

The descriptive data is shown in Tables 6.2 and 6.3 and scoring distributions are described in Chapter 4. A review of the item-based performance on the fraction number line indicates that the mean PAEs for all target numbers were less than 20% whereas maximum estimation errors were all greater than 40% indicating that some students had difficulty with the task (see Table 6.3). The mean PAE for the task and the individual items was skewed as is typical in number line estimation. An independent samples *t*-test was used to compare mean performance between grades. As predicted, sixth graders were significantly better fraction

estimators than fourth graders $t(128) = 4.03, p < .001$, Cohen's $d = 0.71$. Consistent with previous longitudinal (Resnick et al., 2016) and cross-sectional research (Siegler et al., 2011), there was a medium-to-large effect of grade on estimation fractions on a number line.

Table 6.2

Mean Performance by Grade on Tasks of Interest

	Grade 4				Grade 6			
	<i>M</i>	<i>SD</i>	Min	Max	<i>M</i>	<i>SD</i>	Min	Max
Fraction mapping ¹	63.53	17.65	30.00	87.50	76.50	14.02	37.50	97.50
Arithmetic fluency ²	30.25	13.36	3.00	75.00	44.76	18.52	13.00	97.00
0-1000 Number line ³	11.96	6.51	3.25	30.41	7.36	2.98	3.99	17.45
Fraction number line ³	16.79	11.33	2.95	43.95	10.13	7.06	2.23	30.04

Note. ¹ Percent accuracy, ² Total correct, ³ Percentage absolute error (PAE)

Table 6.3*Descriptive Statistics by Item for the Fraction Number Line*

Fraction	<i>M</i>	<i>SD</i>	<i>Z</i> _{Skew}	Max	Fraction	<i>M</i>	<i>SD</i>	<i>Z</i> _{Skew}	Max
$\frac{1}{2}$	11.62	17.47	6.05	50.00	$\frac{3}{7}$	16.46	12.50	3.76	52.41
$\frac{1}{3}$	11.49	12.90	7.52	62.67	$\frac{3}{8}$	13.49	12.49	7.43	55.04
$\frac{1}{4}$	9.18	13.61	11.95	75.00	$\frac{4}{5}$	16.46	18.59	5.14	65.86
$\frac{1}{5}$	11.55	16.50	11.52	78.30	$\frac{4}{7}$	13.67	10.57	3.14	43.49
$\frac{1}{6}$	14.82	21.28	8.95	83.33	$\frac{4}{9}$	13.18	12.84	7.95	53.31
$\frac{1}{7}$	14.46	21.96	8.29	81.16	$\frac{5}{6}$	15.43	17.46	7.38	83.33
$\frac{1}{8}$	15.50	24.52	7.86	84.59	$\frac{5}{7}$	12.25	11.65	7.38	58.31
$\frac{1}{9}$	17.21	27.73	7.90	88.22	$\frac{5}{8}$	12.33	10.51	6.62	54.85
$\frac{2}{3}$	17.33	18.30	4.71	61.27	$\frac{5}{9}$	11.65	11.55	5.90	44.47
$\frac{2}{5}$	12.23	11.15	7.95	60.00	$\frac{6}{7}$	12.01	13.97	8.48	69.45
$\frac{2}{7}$	11.60	14.12	9.24	59.11	$\frac{7}{8}$	11.26	17.92	13.81	84.53
$\frac{2}{9}$	15.64	21.83	8.24	76.20	$\frac{7}{9}$	10.70	11.45	14.24	76.08
$\frac{3}{4}$	15.87	18.67	5.29	62.74	$\frac{8}{9}$	10.15	16.03	14.14	87.25
$\frac{3}{5}$	13.96	12.95	4.48	48.77					

Note: All skew values were significant at $p < .05$.

The correlations among the tasks are shown in Table 6.4. The patterns of correlations across grades are similar with the exception of the fraction mapping and 0-1000 number line task. The two measures were correlated for fourth graders but not for sixth graders. A closer inspection of the mapping/0-1000 number line scatter plots and the scoring distributions suggests this difference was not driven by an outlier. Instead, this correlational pattern suggests that the proportional reasoning skills captured in the number line task may support students' initial understanding of fraction representations.

Table 6.4

Correlations Amongst All Tasks by Grade

	1	2	3	4
1. Fraction mapping		-.52**	.24	-.59**
2. 0-1000 number line	-.07		-.51**	.57**
3. Arithmetic fluency	.45**	-.26*		-.37**
4. Fraction number line	-.43**	.59**	-.46**	

Note. Correlations for grade 4 are above the diagonal and for grade 6 are below the diagonal.

* $p < .05$, ** $p < .01$

Latent Profile Analysis

Students were grouped based on their item-by-item PAEs using a latent profile analysis in *MPlus* (Muthén & Muthén, 1998–2012). Each model was tested with multiple sets of random start values exceeding 1000, with 50 initial stage iterations (Geiser, 2013). The best log likelihood value was replicated, suggesting that the optimal set of parameter estimates in the mathematical space is trustworthy. Assumptions for the latent profile analysis are described in Appendix F.

Step 1. Determining Profiles

Model Selection. To determine the best fitting profile, we compared the model fit statistics and the interpretability of the data for 1- through 5-profile models (see Table 6.5). Although the BIC values decreased across the profiles, with the 5-profile solution having the lowest BIC, the BIC value began to level out at the 3-profile solution (i.e., BIC differences of 840, 451, 218, and 24 from profiles 1-2, 2-3, 3-4 and 4-5, respectively) suggesting partial support for the 3-profile solution. In contrast, the Lo–Mendell–Rubin likelihood ratio test was significant for the 2-profile solution suggesting support for the 2-profile solution. Other fit statistics (classification probabilities, BLRT, and entropy) were comparable across the profiles. Overall, the fit statistics (BIC, classification probabilities, LMRL, BLRT, and entropy) did not suggest a clear choice between the two- and three-profile solutions.

Examination of the estimation patterns of the groups defined by the 2- versus 3-profile solutions suggested that the 3-profile solution was more interpretable. The 2-profile model simply divided participants into skilled and unskilled estimators; one profile had a relatively flat estimation pattern, indicating consistently accurate performance, whereas the other profile was erratic with high rates of error (>20%) on most fractions. In contrast, the 3-profile model split the unskilled estimators into two groups; one group was moderately accurate, presumably a transitional group, and one group was mostly inaccurate. The 4-profile model did not provide further helpful classifications – it split the mostly inaccurate participants into two mostly inaccurate groups. In summary, the 3-profile model was the most parsimonious model that differentiated unskilled estimators and provided insights into strategy selection.

Support for a 3-profile model for grouping students in relation to their fraction strategies comes from a study by Rinne et al. (2017) in which they used latent class analysis with students

in grades 4 through 6. Students were grouped according to their fraction magnitude comparison performance (i.e., what fraction is greater $\frac{1}{4}$ or $\frac{1}{2}$?). Rinne et al. found support for a similar 3-class model with groups whose performance could be interpreted as consistently accurate, transitional, and mostly inaccurate. Therefore, the 3-profile model was chosen in the present study.

Table 6.5

Fit Statistics for Latent Profile Analysis Models with One to Five Profile Solutions

Solution	LL ^a	BIC ^b	Classification probabilities ^c	LMR-LRT ^d <i>p</i> value	BLRT ^e <i>p</i> value	Entropy
One profile	1.60	29,013	1.00	-	-	-
Two profiles	1.76	28,173	.999	.048	<.001	.996
Three profiles	1.81	27,722	.998	.525	<.001	.995
Four profiles	2.01	27,504	.997	.713	<.001	.995
Five profiles	1.94	27,480	.997	.729	<.001	.996

Note. ^aScaled log likelihood values corrected for maximum likelihood estimation; ^bBayesian information criteria; ^cAverages of the classification probabilities; ^dLo–Mendell–Rubin likelihood ratio test; ^eBootstrap likelihood ratio test. Values in bold font are indicative of the better fitting model.

Figure 6.3

Mean PAE for Each Profile Group in the 3-Profile Solution. Error bars are standard errors of the mean.

Note: Target fractions are presented in order of increasing magnitude; * indicates target fractions where students provided a strategy.

Model Interpretation. Mean PAE for the students in each profile for the 27 target fractions is shown in Figure 6.3. Students in Profile 1 ($N = 85$, 34 fourth graders, 51 sixth graders) had low levels of error across all of the target fractions ($M = 7.17$, $SD = 3.39$). The dip at $\frac{1}{2}$ ($M_{PAE \frac{1}{2}} = 1.99$, $SD = 3.14$) suggests that $\frac{1}{2}$ is a familiar fraction. Students in Profile 1 were labeled *relational estimators* because of their relatively flat and accurate estimation pattern. This pattern suggests that students estimated target fractions based on the relation between the numerator and denominator.

Students in Profile 2 ($N = 24$; 14 fourth graders, 10 sixth graders) had high error rates on most fractions ($M = 20.59$, $SD = 5.15$). As shown in Figure 6.3, the estimation pattern reflected a range of performance across items. Students in this profile made more accurate estimates for small magnitude fractions located at the left end of the number line (e.g., $\frac{1}{7}$, $\frac{2}{9}$, $\frac{2}{7}$) and less accurate estimates for fractions with similar magnitude numerators and denominators that were located nearer to the right end of the number line (e.g., $\frac{1}{2}$, $\frac{2}{3}$, $\frac{6}{7}$). Based on their estimation patterns, we inferred that these students were using the magnitudes of both the numerator and denominator to place fractions on the number line and thus labelled this profile *whole-component estimators*. These patterns will be described more fully in the discussion.

Compared to whole-component estimators, students in Profile 3 ($N = 21$; 16 fourth graders, 5 sixth graders) had very high error rates on target fractions below the midpoint. For fractions above the midpoint, however, their estimates were similar to those of the whole-component profile. This group was labelled the *denominator estimators* because they over-estimated fractions with small numerators and large denominators such as $\frac{1}{9}$, $\frac{1}{8}$, $\frac{2}{7}$ (see Figure 6.3), placing these fractions close to the right endpoint. The finding that 8% of sixth graders were classified as denominator estimators suggests that, even though they had been working with

symbolic fractions since grade 4, some students still did not have a good grasp of how number symbols reflect fraction magnitudes. Although Profile 3 consisted of students who were the least accurate estimators overall ($M = 29.79$, $SD = 6.15$), on occasion these students made more accurate estimates than students in Profile 2. Specifically, students in Profile 3 were more accurate for fractions with similar magnitude numerators and denominators, such as $\frac{2}{3}$, $\frac{3}{4}$ and $\frac{5}{6}$. This difference indicates that both whole-component and denominator estimators have different misconceptions about fraction magnitude.

Mean estimation accuracy was compared between profiles using a one-way ANOVA. Unequal variances were assumed. There was an effect of estimation accuracy based on profile $F_{w(2, 33.69)} = 188.82$, $p < .001$ and Games Howell post hoc analyses confirmed that students in Profile 1 were more accurate estimators, than students in Profile 2 who in turn, were more accurate estimators than students in Profile 3 ($ps < .001$).

In summary, the pattern of performance revealed in the profile analysis showed three different groups. Relational estimators have a good understanding of fraction magnitudes and were consistently accurate performance across the number line whereas whole-component and denominator estimators both have misconceptions which interfere with their number line performance. However, these misconceptions differ between groups. The whole-component estimators showed somewhat better performance because they considered both elements of the fraction whereas the denominator estimators focused on denominator magnitudes. In the next section, these profiles were validated in relation to the strategy reports that students provided on the first four trials of the task.

Step 2. Strategy Validation

Students described their estimation strategies for four target fractions: $\frac{1}{4}$, $\frac{3}{7}$, $\frac{4}{5}$ and $\frac{7}{9}$. In the previous section, we used the latent profile analysis to make inferences about the strategies students used based on the number line estimation patterns that emerged for each profile. In this section, the students' strategy descriptions were used to validate these inferences and to provide a better understanding of the profile differences. The frequencies of reported estimation strategies are shown by fraction and by profile in Table 6.6. Strategies were coded into four key categories: endpoint reference, midpoint reference, segmentation, and transformation. The remaining explanations (just knew, guessed and other) were not codable using this scheme and are not reported. However, the majority of strategy reports (81, 71, 82, and 81 % across the fractions 1 2 3 4) were coded using this scheme. Thus, the coding scheme was an accurate reflection of most strategy reports. Note however that the strategy reports provided by students in the relational profile were more likely to be codable according to this scheme than those of students in the other two groups for all four fractions. Thus, the coding scheme captures both the strategy selection for the strategies that would lead to accurate performance but may not capture performance of students who have misconceptions about fractions such that their performance was less accurate.

As shown in Table 6.6, strategy selection varied depending on the target fraction and the profile designation. Notably, only students in the relational profile used a transformation strategy (i.e., 2 to 4 students per target fraction).

Table 6.6*Percent of Students in Each Profile Reporting A Given Strategy*

Profile	Strategy	Percent of Students Reports			
		$\frac{1}{4}$	$\frac{3}{7}$	$\frac{4}{5}$	$\frac{7}{9}$
Relational					
	Endpoint	1.1	4.8	23.2	25.6
	Midpoint	42.8	24.1	8.5	9.8
	Segmentation	41.7	48.2	58.5	53.7
	Transformation	2.4	2.4	3.5	2.4
	Total	87.0	79.5	93.7	91.5
Whole-component					
	Endpoint	0	29.2	12.5	54.2
	Midpoint	54.2	12.5	37.5	4.2
	Segmentation	16.7	8.3	0.0	12.5
	Total	70.9	50.0	50.0	70.9
Denominator					
	Endpoint	9.5	9.5	9.5	38.1
	Midpoint	42.9	14.3	42.9	4.8
	Segmentation	4.8	19.0	4.8	4.8
	Total	57.2	42.8	57.2	47.7

Note. Totals do not add up to 100% because “other” strategies (e.g., counting, I don’t know, I just knew) are not included in the table.

Were students’ self-reports reactive? The first fraction students placed was $\frac{1}{4}$; students in the whole-component and denominator profiles were more accurate than might be expected,

given their performance on other fractions. Recall that students began the estimation task by describing their strategies for placing four target fractions. As shown in Figure 6.3, the whole number estimators and denominator estimators more accurately placed $\frac{1}{4}$ than $\frac{1}{2}$ on the number line. This finding was unexpected because we anticipated students would accurately estimate both $\frac{1}{4}$ and $\frac{1}{2}$ because they are familiar fractions (Liu, 2018). As expected, the students in the relational profile were very accurate at placing $\frac{1}{2}$. One possible explanation of the pattern is that students may have been more thoughtful about their fraction placement when they were describing their strategy to the experimenter than later on in the task. Because $\frac{1}{2}$ was presented randomly in the sequence, some students may have struggled to inhibit their whole-number reasoning thus estimating $\frac{1}{2}$ based on the components as opposed to the relation between the components (Avgerinou & Tolmie, 2019; Dewolf & Vosniadou, 2015; Shtulman & Valcarcel, 2012). In contrast, accuracy on the other three fractions for which strategy descriptions were given were consistent with the profile patterns suggesting that bias from describing strategies was limited to $\frac{1}{4}$.

Relational Estimators. Students labeled as relational estimators described strategies that supported the assumption that they understood that fraction magnitude is based on the relation between the numerator and denominator. For example, using an endpoint strategy to place $\frac{7}{9}$ one student reasoned “ $\frac{9}{9}$ is one whole and $\frac{7}{9}$ is approximately $\frac{1}{3}$ off a whole.” These students frequently reported using a segmentation strategy (42% to 58% per target fraction; see Table 6.6). Successful segmentation strategies were typically described as follows, “I divided it [the number line] into four and took one part.” Another student demonstrated relational thinking when she explained a midpoint strategy while placing the fraction $\frac{3}{7}$, “out of 7, 3 is less than half

of 7”; the research assistant noted the student then placed $\frac{3}{7}$ to the left of $\frac{1}{2}$. These oral reports suggest that, when relational estimators placed fractions on a number line, they used a range of strategies but most of the strategies were based on successful application of correct fraction knowledge.

Whole-component Estimators. Whole component estimators relied mainly on endpoint and midpoint strategies (see Table 6.6) and their oral reports reflected a partial understanding of fraction magnitudes. For example, one student’s report that “one half is in the middle and it $<\frac{1}{4}>$ is less than half” reflects an understanding that for unit fractions, a large denominator means a small fraction. Whereas midpoint reports for $\frac{4}{5}$, such as, “five is the middle of ten and four is just under five” and “not in the middle, five-fifths would be in the middle, four-fifths is a bit farther than half” suggest some students envisioned a number line to ten then looked at the difference between the numerator and denominator to place the fraction. Notably, students were inconsistent in their strategizing. Different fractions elicited different rationales. Placing $\frac{7}{9}$ for example, some students relied on the magnitude of the numerator as evidenced by their endpoint reports: “1 is 100 and 7 is like 70 and it’s closer to 100”, “close to 1 because the top number is higher than 5”, “5 is in the middle, this is a bit more than 5”. The whole-component estimators articulated faulty magnitude reasoning, and based on their comments, these students typically considered either the numerator or the numerator and denominator as whole components instead of relational components.

Denominator Estimators. Students classified as denominator estimators primarily described “other” strategies and reference point strategies that were focused on the magnitude of the denominator. For example, one “other” strategy was counting: “I counted 0, $\frac{1}{2}$, $\frac{1}{3}$ then $\frac{1}{4}$ ”, “

$\frac{1}{2}$ and then $\frac{1}{3}$ then $\frac{1}{4}$.” This counting pattern indicates that these students thought the unit fraction got larger as the denominator got larger. In contrast to whole number estimators, denominator estimators did not understand that a larger denominator means a lesser unit fraction. Students’ midpoint reports for $\frac{4}{5}$ were also focused on the denominator “near the middle ‘cause it’s five.” Similar denominator thinking was described by students who estimated $\frac{3}{7}$ using an endpoint reference: “It’s close to ten like a seven is close to ten”, “close, seven is almost at nine.” In addition, endpoints strategy reports for $\frac{7}{9}$, included “nine is close to the end” “the end is 10 so a little over 9” and “pretty close to 1 whole”. Notably, the magnitude of the numerator was ignored in these endpoint strategy reports. Based on the oral reports and the observed estimation patterns, we suggest that, along with some of the faulty reasoning whole number estimators demonstrated, denominator estimators may be counting and misapplying endpoint and midpoint strategies to estimate fraction magnitudes.

Step 3. Influence of Profile Predictors

We hypothesized that arithmetic, whole number estimation and fraction mapping skills would differentiate latent profile membership. Using multinomial logistic regression, we examined the odds of profile assignment based on students’ Time 1 math skills (see Table 6.7). Because there were an unequal number of students in each profile, significance testing was based on 1,000 bootstrapped samples. We compared the probabilities of profile membership in relational and denominator estimators to whole-component estimators because we were specifically interested in the skills that differentiated the moderately successful fraction estimators from the weaker estimators and more proficient estimators.

All predictors (i.e., arithmetic fluency, grade, whole-number estimation and fraction mapping) were initially entered into the model and backward elimination was used to remove

the predictors that did not influence profile membership. The resulting model included fraction mapping, 0-1000 number line, and grade. The logistic regression model fit the data significantly better than the null model, $\chi^2(6) = 69.470$, $p < .001$. Notably, the χ^2 values for the Pearson and deviance goodness of fit statistics were non-significant ($p = .84$ and $p = .99$, respectively) providing evidence that the regression model fit the data well.

What skills differentiated profile membership? Compared to whole-component estimators, relational estimators were more likely to be in sixth grade, have better fraction mapping skills and be more accurate placing whole numbers on a 0-1000 number line (see Table 6.7). In contrast, compared to denominator estimators, whole-component estimators were more likely to have stronger fraction mapping skills. Grade and 0-1000 number line estimation did not differentiate whole component estimators from denominator estimators. In fact, students with strong mapping skills were almost twice as likely to be classified as whole-component estimators compared to denominator estimators. That is, the odds of being classified as a whole-component estimator increased as fraction mapping skills increased ($e^b = 1.92$). In summary, fraction mapping skills differentiated profile membership at all levels whereas, 0-1000 number line estimation accuracy only differentiated the most successful estimators. Presumably, because both whole-component and denominator estimators had fraction misconceptions, their proportional reasoning skills captured in the 0-1000 number line were not used to their fraction estimation.

Profile membership for the fraction number line task was related to students' fraction mapping skills and whole number estimation skills. Arithmetic skills did not differentiate profile membership as hypothesized. These findings suggest that fraction mapping skills support students as they progress from estimating fractions based on whole number magnitudes (i.e., denominator and whole-component estimators) to using a relational understanding of fractions.

Further, whole number estimation skills may become more relevant once students have an understanding of fraction notation.

Table 6.7

Multinomial Regression Predicting Probabilities of Profile Membership

Profile Comparisons	<i>B</i>	<i>S.E.</i>	<i>p</i>	Odds ratio $e^{(B)}$	95% CI for $e^{(B)}$	
Relational compared to whole-component						
Intercept	2.00	0.52	.001			
Fraction mapping	0.60	0.33	.035	1.82	1.02	3.24
0-1000 number line	-1.53	0.44	.001	0.22^a	0.12	0.42
Grade	-1.46	0.73	.015	0.23^a	0.07	0.73
Denominator compared to whole-component						
Intercept	-1.21	1.67	.050			
Fraction mapping	-0.65	0.36	.020	0.52^a	0.27	1.00
0-1000 number line	0.12	0.40	.691	1.13	0.62	2.05
Grade	0.85	1.68	.226	2.35	0.59	9.41

Note. $R^2 = .419$ (Cox & Snell), $.506$ (Nagelkerke). Model $\chi^2(6) = 66.470$, $p < .001$.

Parameter estimates are based on 1000 bootstrap samples. ^aFor ease of interpretation, in the text we refer to the inverse of the odds ratios ($1/e^{(B)}$) for odd ratios that are less than one. For example, students with better fraction mapping skills had 1.92 greater odds ($1/0.52$) of being classified as whole-component estimators than denominator estimators.

Discussion

Placing fractions on a number line has been described as a skill that can foster understanding of rational numbers (e.g., Booth & Newton, 2012; Bruce et al., 2018; Obersteiner et al., 2019; Siegler et al., 2011) yet research describing how students place fractions on the number line is limited (Schneider et al., 2018). Researchers have used qualitative analyses to describe students' misconceptions about fraction magnitudes (Deringöl, 2019; D. Zhang et al., 2017) whereas quantitative researchers have primarily focused on overall task performance (Liu,

2018; Resnick et al., 2016; Siegler et al., 2011; Torbeyns et al., 2015) and its relation to later math skills (Rodrigues et al., 2019; Siegler & Pyke, 2013; Torbeyns et al., 2015). Thus, the goals of the present research were to better understand the strategies students use to place fractions on the number line and determine what mathematical skills differentiate students based on their estimation strategies.

One challenge in understanding number line estimation is that students may use different estimation strategies for different target numbers (Bouwmeester & Verkoeijen, 2012; Xu, 2019; Authors, in review). Thus, examining performance averaged across all trials precludes a detailed understanding of students' estimation strategies. Latent profile analysis is useful for identifying groups of students who show similar patterns of performance across trials (Oberski, 2016). In the present study, we used percentage of absolute error on each number line trial to classify students into estimation profiles (Xu, 2019; Xu et al., in review). Students also provided verbal reports describing their estimation strategies for four target numbers. Integrating the error patterns with students' verbal strategy reports provided clear descriptions of the strategies used by students in the observed profiles, that is, relational estimators, whole-component estimators, and denominator estimators.

Relational estimators were the largest group: 53% of grade 4 and 77% of grade 6 students. Students in this profile were uniformly accurate estimators across all fractions. Based on the estimation patterns and students' strategy reports, we concluded that relational estimators considered the fraction as a whole unit and then estimated its magnitude in relation to equal segments or implicit reference points. Relational estimators thus had both a solid conceptual understanding of the fraction symbol and an accurate representation of the magnitude of the

fraction symbol. This knowledge led to accurate placement of the fractions on a visual number line.

Whole-component estimators included 22% of grade 4 and 15 % of grade 6 students.

They were moderately accurate, but based on their estimation patterns and strategy reports, we inferred that whole-component estimators failed to consider fractions as one relational unit. Instead, they considered the numerator and denominator independently. Students in this profile accurately estimated the magnitude of unit fractions indicating that they had some understanding that lesser denominators can lead to greater fraction values (Dewolf & Vosniadou, 2015; Rinne et al., 2017). However, whole-component estimators were inaccurate when estimating small-component fractions (i.e., small numerator, small denominator) such as $\frac{2}{3}, \frac{3}{4}, \frac{3}{5}$. Students' lower accuracy on these fractions suggests that they may be using the difference between the numerator and the denominator to estimate fraction magnitude, that is they may place $\frac{2}{3}$ close to the right endpoint because there is only a gap of one unit between the numerator and denominator; an error described as gap thinking by some researchers (Gonzalez-Forte et al., 2019; Pearn & Stephens, 2004). Alternatively, these students may incorrectly assume that small component fractions represent small magnitudes (Braithwaite & Siegler, 2018) irrespective of the relation between the numerator and denominator. Whole-component estimators reported using the endpoints and midpoint to place the four target fractions but because they had misconceptions about fraction magnitudes, these placements were inaccurate. Whole component estimators were in a transition phase; they had a partial understanding that the characteristic features of whole numbers differed from the characteristic features of fractions (Rinne et al., 2017; Siegler, 2016; Siegler et al., 2011; Torbeyns et al., 2015) however, they did not view the fraction as one relational unit.

Denominator estimators included 25% of grade 4 and 8% of grade 6 students. These students showed the highest average errors on fractions with small numerators and large denominators such as $\frac{1}{9}$, $\frac{1}{8}$, and $\frac{2}{9}$. Their estimation patterns and strategy reports suggested that they judged the fraction magnitude based primarily on the magnitude of the denominator. Students in this group placed $\frac{3}{7}$ near the right endpoint, presumably because 7 is a large number. They reported using counting, that is, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ to estimate the position of $\frac{1}{4}$. And although these students reported using explicit reference points (i.e., endpoints) and implicit reference points (i.e., the midpoint) to place the four target fractions, their placements were wildly inaccurate because they lacked conceptual understanding of the magnitude reflected in the fraction symbol. In this sense, therefore, they did not understand fraction notation. Because 8% of sixth graders were classified as denominator estimators it is clear that for some students, weak conceptual knowledge of fraction symbols is a significant hindrance to their further fraction development (Jordan et al., 2017; Resnick et al., 2016).

Predictors of Fraction Number Line Performance

Because fraction number line skills are linked to more advanced mathematics (Mou et al., 2016; Schneider et al., 2018; Torbeyns et al., 2015), the second goal of this research was to identify the skills that differentiated among the estimation profiles. To accurately place fractions on a fraction number line, students need to integrate information from the numerator and the denominator to determine the fraction magnitude, then they need to locate the magnitude on a number line (Schneider et al., 2018). Consistent with this view, we found that a conceptual understanding of fraction symbols (Hansen et al., 2015) and strong whole number line estimation skills (Jordan et al., 2017; Liu, 2018; Resnick et al., 2016) were related to fraction number line accuracy. Specifically, relational estimators had stronger fraction mapping skills and more

precise whole number line estimation skills than whole-component estimators. This pattern supports the conclusion that relational estimators had stronger conceptual knowledge of fraction symbols than whole-component estimators. Understanding fraction magnitudes also entails knowledge of partitioning (Mack, 1990; Wilkins & Norton, 2018). Presumably the proportional reasoning skills captured in the whole number line task (Barth & Paladino, 2011) reflect the precision required to accurately partition the number line into equal segments. Accordingly, relational estimators frequently reported using a segmentation strategy. Thus, consistent with previous research, we found that students' proportional reasoning skills (as indexed by their whole number estimation) support fraction estimation (Hansen et al., 2015; Möhring et al., 2016). Not surprisingly, older students were more likely to be relational estimators than younger students. Together these findings indicate that the most skilled fraction estimators have conceptual knowledge of fraction symbols and sufficient proportional reasoning skills to accurately estimate the position of the fraction on a 0-1 number line.

Whole-component estimators were differentiated from denominator estimators by their fraction mapping skills. That is, whole-component estimators were better able to map fraction symbols to their referents than denominator estimators. However, performance on the whole-number number line did not differentiate their profile membership because these students had some fraction misconceptions. For example, if a whole-component estimator thinks $\frac{2}{3}$ is greater than $\frac{5}{7}$ because there is a gap of one compared to a gap of two between the numerator and denominator (Deringöl, 2019; Pearn & Stephens, 2004), their fraction placement will be inaccurate. Similarly, if a denominator estimator thinks $\frac{1}{9}$ is close to 1 because 9 is a large number, their fraction placement will be inaccurate. Proportional reasoning skills were not

relevant in differentiating profile membership because magnitude knowledge was inaccurate for both whole-component and denominator estimators.

In other research, strong arithmetic skills (Jordan et al., 2017; Resnick et al., 2016) were also related to fraction number line estimation. Students with stronger arithmetic skills may have more integrated whole-number knowledge (Siegler, 2016; Siegler et al., 2011; Xu et al., 2019) which in turn, could support fraction development. Further, students with stronger arithmetic skills may be more successful at transforming fractions into a number representation that is easier to estimate [e.g., transforming $\frac{1}{5}$ into a decimal (0.20) or a percentage (20%)] resulting in more successful number line estimation (Siegler et al., 2011). In contrast, in the present research, arithmetic skills did not differentiate between estimation profiles. Resnick et al. (2016) found that whole number knowledge (whole number line estimation and multiplication fluency) predicted the transition from inaccurate to accurate fraction number line estimation. I offer two explanations for the differences between their work and the present research. First, because multiplication fluency performance was poor for our fourth-grade participants, we used a combined measure of addition, subtraction and multiplication in the regressions. That said, I ran the regressions with the multiplication scores and the results were unchanged. Thus, a more likely explanation is strategy use. Strategy reports indicated that few students used arithmetic to transform fractions when placing them on the number line. Based on the curriculum expectations, our participants had limited exposure to transformation algorithms, hence, arithmetic may not differentiate their estimation patterns.

Limitations and Future Research

The present study provides valuable information about how students approach the fraction number line task. Nonetheless, the study is not without its limitations. First, although

this study provided a detailed account of students' fraction number line estimation, there was no measure of students' fraction comparison skills. Charalambous et al. (2007) suggested that fraction comparison skills are part of a knowledge continuum goes from understanding fractions as part-whole relations to understanding fractions as units of measure that can be placed on a number line. Recognizing for example, that $\frac{2}{3} > \frac{1}{4}$ may help students accurately place $\frac{2}{3}$ on the number line. Thus, future work could include fraction comparison skills as a predictor of estimation profiles. Second, I only examined students' fraction estimation at one time point. Thus, could not provide insights into the skills that predict changes in estimation profiles. In the future, statistical analyses such as latent transition analysis could be used to determine which skills predict the transition from denominator to whole-component estimator or whole-component to relational estimator. Further, longitudinal research could also provide insights into the consistency of fraction estimation patterns (e.g., are the number and types of estimation profiles stable over grades). Thus, future longitudinal research that examines changes in estimation profiles could provide a more complete picture of students' estimation strategies and their understanding of fractions.

Educational Implications

The current research is the first study, to our knowledge, to examine item-based PAE fraction number line estimation profiles amongst students in fourth and sixth grades. Previous research shows that fraction number lines are challenging for students (Liu, 2018; Schneider et al., 2018; Siegler et al., 2011; Torbeyns et al., 2015; D. Zhang et al., 2017) in part, because students may have misconceptions about fraction symbols and their representations (Braithwaite & Siegler, 2018; Mack, 1995; Ni & Zhou, 2005; Tunç-Pekkan, 2015; Vamvakoussi, 2015). Consistent with previous research, the findings of the current study show larger gaps in

conceptual fraction knowledge amongst younger students compared to older students (Braithwaite & Siegler, 2018). However, some older students continue to struggle with basic knowledge of fractions and their representations. Specifically, understanding fractions as a relational unit as opposed to two independent numbers. Importantly, students require sufficient understanding of the relations between fraction symbols and their referents (Liu, 2018) to develop their fraction estimation skills. Thus, helping struggling students learn to connect fraction symbols with their referents may be a critical first step in students' development of fraction magnitude knowledge.

CHAPTER 7: GENERAL DISCUSSION

In this dissertation, I proposed that fraction mapping is an important factor in conceptual fraction knowledge. The development of mathematical skills are hierarchical whereby earlier skills are the building blocks for higher-level skills (Cirino et al., 2016; Hiebert, 1988; Siegler & Lortie-Forgues, 2017; Xu et al., 2019). Previous research has suggested a fraction skill hierarchy such that whole-number skills support students' understanding of fraction magnitudes which, in turn, allows them to acquire other fraction skills (Siegler, 2016; Siegler et al., 2011; Siegler & Braithwaite, 2017). However, this hierarchy does not account for basic symbol knowledge; that is, knowledge of what the fraction symbol represents. A student cannot reason about fractions or place fractions on a number line without knowing what the fraction stands for, that is, linking the symbol to the magnitude it represents. Thus, the present research was built upon the view that in the mathematical skill hierarchy, the basic building block for developing formal fraction skills is the fraction symbol (Hiebert, 1988).

In any symbolic system, knowledge of what the symbol represents; that is, being able to connect the symbol to its referents, is foundational. For example, learning to read involves mapping letters to their sounds. Early readers need to learn that letters and letter combinations are linked to specific sounds and then apply that knowledge to interpret text (Castles et al., 2018). Similarly, learning to play music from symbols involves mapping musical notation to the keys on a piano. To acquire mathematical knowledge, learners first need to map numbers to their referents (Hurst et al., 2017; Jiménez Lira et al., 2017). Math symbols are also linked to words and importantly, are grounded in meaning; numbers represent magnitudes. Early math learners recognize that numbers are linked to magnitudes and that knowledge is foundational for arithmetic skills (Hiebert, 1988; Vanbinst et al., 2016; Xu et al., 2019). Thus, connecting the

fraction symbol to its referent, by extension, is foundational for understanding and acquiring knowledge about fractions.

I defined fraction mapping as connecting the fraction symbol (i.e., $\frac{3}{4}$) to its spoken lexical (e.g., three-fourths) and pictorial/visual representations, or referents. These connections are illustrated in Figure 7.1. Children learn about math through oral language (Chow & Ekholm, 2019; Chow & Jacobs, 2016; Peng et al., 2020) and written language (Purpura et al., 2011; Purpura & Napoli, 2015; X. Zhang et al., 2014; X. Zhang & Lin, 2015). For example, understanding terms such as equal sharing, equivalent, and one-fourth presumably support learning about fraction representations. Similarly, knowledge of how math symbols are written (e.g., size, spatial register) and combined (e.g., $3\frac{1}{2}$ versus $\frac{1}{2}3$) presumably supports learning about fraction representations. Thus, math-specific language skills should support students' knowledge of fraction mapping. In this dissertation, I focused on two math-specific language skills that are relevant to fraction learning, mathematical vocabulary and orthography. Mathematical vocabulary is defined as the lexicon of oral and written “words or phrases that express mathematical concepts or procedures” (Hebert & Powell, 2016, p. 1515) and mathematical orthography is defined as knowledge of mathematical symbols and the conventions for combining those symbols into expressions and equations (Douglas et al., in press; Headley, 2016). I proposed that fraction mapping is supported by math-specific language skills (i.e., vocabulary and orthography) and mapping is, in turn, supports developing knowledge of other fraction concepts.

Figure 7.1

Mapping the Numeric Fraction Symbol to Lexical and Visual Referents

Note. Three-fourths is the lexical referent and the pictures depicting discrete and part-whole fraction magnitudes are the visual referents.

The main goals of the current research were to characterize the skills that support students' ability to map the fraction symbol to its referent and to describe how individual differences in students' knowledge of fraction mapping relates to their developing fraction concepts. Thus, I modelled the relations between math-specific language skills, fraction mapping and measures of fraction skills for students in grades 4 and 6. Grades 4 and 6 students were chosen because in Ontario, students first learn formal fraction symbols (e.g., $\frac{2}{3}$, $\frac{5}{7}$, $\frac{3}{8}$) in Grade 4, thus, these students are novice fraction symbol users. Grade 6 students represent more experienced fraction symbol users because they have worked with fraction symbols for 2 years (Ontario Ministry of Education, 2005). By testing both novice and experienced fraction symbol users I could draw conclusions about how the relations among math-specific language skills, fraction mapping, and fraction concepts differ with expertise.

Three novel measures were developed to operationalize fraction mapping and math-specific language skills: i) the fraction mapping task, implemented as an iPad app, requires students to match fraction symbols with their visual and verbal referents; ii) the mathematical vocabulary task, a receptive vocabulary measure covers a broad range of the mathematical terminology used in the classroom and iii) the symbol decision task for math (SDT-Math), implemented as an iPad app, tests a student's ability to distinguish between conventional and unconventional mathematical orthography³. The SDT-Math operationalizes students' mathematical orthography.

In the first phase of testing, students completed the novel measures, measures of number skills (e.g., arithmetic, 0-1000 number line) and cognitive skills (e.g., working memory and general vocabulary). Five months later, students' fraction skills were assessed. Time 2 fraction tasks included a) fraction mapping (again), b) fraction number line, and c) fraction word problems. A second goal of the current research was to better understand how students reason about fractions and the quantities they represent. Thus, a more detailed analysis of the fraction number line task was conducted in which students' estimation strategies and the skills that related to their strategy selection were described.

Data analyses were organized into two "studies". In Study 1, *Individual Differences in Fraction Knowledge*, the relations between math language skills, fraction mapping, and subsequent fraction concept knowledge were modelled and patterns of relations for fourth and sixth graders were compared. In Study 2, *Profiles of Fraction Number Line Skills*, latent profile analysis was used to group students based on their estimation accuracy for each fraction number

³ I extended a version of the SDT-Math (Douglas et al., in press; Headley, 2016) which I initially developed for children in grades 1 to 3 (Xu et al., in review) to children in grades 4 to 6.

line trial. The number skills that predicted group membership were identified. Below, I discuss the results of each study in relation to the model of fraction learning that I proposed in Chapter 1.

Individual Differences in Fraction Knowledge

In the current research, math-specific language skills were captured with two tasks, mathematical vocabulary and the middle-school version of the symbol decision task for math (SDT-Math). These tasks were correlated – students with better mathematical vocabulary were better at discriminating between conventional and unconventional mathematical orthography than students with weaker mathematical vocabulary. As well, students with better math-specific language skills more accurately mapped fractions to their referents than students with weaker math-specific language skills.

Figure 7.2

Summary Model

Note. Dotted lines indicate the paths that are not significant.

The summary model shown in Figure 7.2 captures the overall pattern of relations among the key variables in this thesis. The coloured lines indicate pathways that differ across grades. As

predicted, fraction mapping skills capture variance in the fraction outcome measures, beyond all the various controls. There was one exception: For grade 6 students, word problems were not predicted by fraction mapping. For both grades, mathematical orthography predicted fraction mapping at Time 1. As hypothesized, for grade 6 students, math vocabulary predicted fraction mapping, and all fraction outcomes. For grade 4 students however, math vocabulary only predicted growth in fraction mapping whereas general vocabulary and math-specific vocabulary predicted shared variance in fraction mapping at Time 1. Thus, between grades 4 and 6, students' knowledge of math-specific vocabulary started to mediate the relations between general vocabulary knowledge and fraction outcomes. These results are very similar to those reported by Powell et al. (2017) whereby older learners rely less on general vocabulary and more on mathematical vocabulary to solve arithmetic problems. Moreover, these results extend Powell's findings to include fraction skills. These findings support the view that math-specific language skills are relevant to fraction learning. The results are also consistent with my original view that fraction mapping skills are important for students' development of other fraction skills. Even in grade 6, fraction mapping skills measured early in the school year predicted students' fraction outcomes towards the end of the school year.

Hypotheses 1 and 2.

The first two hypotheses described the concurrent relations between math-specific language skills and fraction mapping for fourth graders and sixth graders. Based on the theory that conceptual math knowledge is built through math discourse (O'Halloran, 2015), that is, math-specific oral, written and visual communication, I predicted that mathematical vocabulary and mathematical orthography would account for individual differences in fraction mapping skills. Specifically, based on findings suggesting that, as math skills increase, math vocabulary

becomes more predictive of math learning than general vocabulary (Powell et al., 2017; Purpura et al., 2017; Toll & Van Luit, 2014), I hypothesized that mathematical vocabulary would be directly related to fraction mapping in 6th graders but not in 4th graders (H1). As hypothesized, after controlling for general vocabulary skills, mathematical vocabulary directly predicted fraction mapping skills for sixth graders but not for fourth graders (see Figure 7.2, blue path). Mathematical vocabulary and general vocabulary are not differentiated in grade 4 – both are related to fraction mapping.

Hypothesis 2 was that mathematical orthography would directly predict fraction mapping skills for fourth graders and sixth graders. This hypothesis was based on previous findings that showed a relation between orthography and math skills in children and adults (Douglas et al., in press; Headley, 2016; Xu et al., in review). Hypothesis 2 was supported (see Figure 7.2). Interestingly, other research examining students' developing symbol knowledge has found that symbol knowledge is more predictive of arithmetic skills in younger students (grade 1) compared to older students (grade 3 and 5) (Powell & Fluhler, 2018). I offer two possible explanations for these discrepant findings. First, Powell and Fluhler used a more advanced measure of symbol knowledge than the orthography task; students were asked to name, use and explain the meaning of each symbol. Thus, they were tapping into different aspect of symbol knowledge. Second, students in grades 1, 3 and 5 were tested on the same set of symbols potentially limiting the variability in scores for the older students – potentially resulting in less predictive relations. The symbol decision tasks used to assess orthography in previous studies (Headley, 2016; Xu et al., in review) and the current study were designed to assess grade-specific mathematical orthography. Mathematical orthography, a measure of students' knowledge of the conventions

for writing mathematical symbols, is similarly related to fraction mapping for students in grades 4 and 6.

Hypothesis 3: Mathematical Vocabulary and Fraction Development

Hypothesis 3 was that mathematical vocabulary would directly predict fraction outcomes for sixth graders but not for fourth graders. This hypothesis was based on findings that, with developing expertise, math vocabulary becomes more predictive of math performance than general language skills (Powell et al., 2017; Purpura & Reid, 2016; Toll & Van Luit, 2014). As expected, the relations between mathematical vocabulary and later fraction skills differed between novice (grade 4 students) and more experienced fraction learners (grade 6 students) as shown by the blue paths on Figure 7. 2. In support of Hypothesis 3, mathematical vocabulary directly predicted all fraction outcomes for sixth graders even after controlling for Time 1 fraction mapping skills and general vocabulary. Further supporting Hypothesis 3, mathematical vocabulary did not directly predict fraction outcomes for fourth graders. There was however one exception; mathematical vocabulary directly predicted growth in fraction mapping skills for fourth graders. Before completing the Time 2 fraction mapping task, fourth graders had focused instruction on fractions and their representations. Their mapping expertise had increased; thus it is reasonable to assume that, fourth grader's knowledge of math vocabulary could predict these learning gains.

Hypothesis 4: Fraction Mapping and Fraction Development

The relations between fraction mapping and later fraction skills differed between novice (grade 4 students) and more experienced fraction learners (grade 6 students) as shown by the red path in Figure 7.2. In support of Hypothesis 4, Time 1 fraction mapping skills directly predicted all fraction outcomes. There was one exception. For sixth graders, Time 1 fraction mapping

skills did not directly performance on the fraction word problems. Moreover, the strength of the relation between Time 1 fraction mapping and the fraction number line outcome (although significant for both groups) was stronger for fourth graders than sixth graders. These findings suggest that although fraction mapping skills are central for developing knowledge of fraction concepts, the relation between fraction mapping and fraction concept skills may change with expertise.

Profiles of Fraction Number Line Skills

The fraction number line has been recommended as a teaching tool to improve students' understanding of fractions and the quantities they represent (e.g., Bruce et al., 2013; Common Core State Standards Initiative, 2010; Gersten et al., 2017; Ontario Ministry of Education, 2005; Siegler et al., 2010; Torbeyns et al., 2015). Thus, understanding how students estimate fractions on the number line can provide insights into their understanding fractions and the quantities they represent. For example, students may incorrectly place $\frac{6}{8}$ closer to the right endpoint on a 0-1 number line than $\frac{3}{4}$ because $\frac{6}{8}$ has larger components (i.e., 6 and 8 are greater than 3 and 4) in spite of the fractions being equivalent (Braithwaite & Siegler, 2018). Or students may incorrectly place $\frac{1}{8}$ near the right endpoint because 8 is a large number indicating that the student is only considering the value of the denominator (Ni & Zhou, 2005). Both these examples reflect poor understanding of fraction representations. The goal for Study 2 was to explore strategy use on fraction number line to better understand how students reason about fractions and the quantities they represent.

In Study 2, latent profile analysis was used to categorize groups of students based on their patterns of performance on the fraction number line. Students' verbal strategy reports supported the profile analysis. Relational estimators accurately placed fractions on the number line and

described strategies that reflected an understanding that fraction magnitude is based on the relation between the numerator and denominator. Whole-component estimators were moderately accurate estimators, but inaccurately placed fractions when the numerator and denominators were similar in magnitude (e.g., $\frac{1}{3}$, $\frac{2}{3}$, $\frac{5}{6}$). Whole-component estimators articulated faulty magnitude reasoning, and based on their comments, these students typically considered either the numerator or the numerator and denominator as whole components instead of relational components. Denominator estimators were the least accurate estimators and had the highest error rates. Students in this profile were least accurate placing fractions with small numerators and large denominators (e.g., $\frac{1}{7}$, $\frac{1}{8}$, $\frac{2}{9}$). Although denominator estimators had difficulty describing their strategies, the strategies they described tended to focus on the magnitude of the denominator, for example, counting $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ to estimate $\frac{1}{4}$. In summary, three estimation profiles emerged, one skilled profile and two less-skilled profiles.

Figure 7.3

Skills that Predict Probabilities of Profile Membership

Note. Solid lines indicate odds ratio predicting profile membership was significantly different than 1. Dotted lines indicate the odds ratio predicting profile membership was not significant

What skills differentiated profile membership? Results are summarized in Figure 7.3. Compared to whole component estimators, relational estimators were more likely to be in sixth grade, have better fraction mapping skills and have more accurate whole number line estimation skills. Arithmetic skills did not differentiate these groups. Compared to denominator estimators, whole component estimators did not differ on arithmetic, whole number estimation, or grade. The only significant differentiator of profile membership was fraction mapping skills; whole-component estimators were more likely to have better fraction mapping skills than denominator estimators. There are two conclusions we can draw from these findings. First, fraction misconceptions can be robust. Grade did not differentiate membership between denominator and whole-component estimators because a small portion of sixth graders had the same fraction misconceptions as some fourth graders. Second, and most importantly, regardless of grade, fraction mapping skills were necessary to successfully place fractions on the number line. Thus, the present findings suggest that fraction mapping is essential knowledge for students' developing fraction skills.

Theoretical Implications

Current theories of fraction learning focus on the transition from whole number knowledge to fraction knowledge. The *conceptual change* view is that whole number knowledge can hinder fraction learning (Ni & Zhou, 2005; Vamvakoussi et al., 2012; Vamvakoussi & Vosniadou, 2004). Evidence for this view comes from research on whole number bias - errors attributed to inappropriately applying whole number reasoning to fractions (Fazio et al., 2016; Kainulainen et al., 2017; Mazzocco & Devlin, 2008; Ni & Zhou, 2005). In contrast, according to

continuous views of fraction learning, whole number knowledge provides the basic knowledge required for fraction learning. More specifically, the *integrated theory of numerical development* is based on the view that magnitude knowledge acts as unifying concept from whole numbers to fractions (Lortie-Forgues et al., 2015; Siegler, 2016; Siegler et al., 2011) whereas the *reorganization hypothesis* is that counting knowledge acts as the unifying concept (Steffe, 2002; Steffe & Olive, 2010). In this thesis I took a step back and focused on the fraction symbol itself. Rather than discarding theories on the transition from whole number knowledge and fraction knowledge, the fraction mapping view I have proposed and tested extends those theories.

Previous research with young students provides evidence that mapping digits to their referents is foundational for mathematical learning (Göbel et al., 2014; Jiménez Lira et al., 2017; Merkley & Ansari, 2016; Nguyen et al., 2016). The current research extends this theoretical perspective to fractions. To illustrate, fourth and sixth grader's skill at mapping fractions to visual representations directly predicted their performance on fraction outcomes (number line estimation, growth in fraction mapping). Further, mapping skills differentiated how students reasoned about fraction magnitudes as evidenced by the estimation profiles described in the number line estimation study. Thus, the current research positions the fraction symbol as the unifying concept and fraction mapping as the foundational skill for fraction learning.

The findings from the current research support the view that fraction mapping skills are most relevant when students are developing their understanding of fraction concepts. With whole numbers, once children have acquired sufficient knowledge of the underlying quantity the digit represents, the symbol becomes abstracted; children can manipulate the symbols without accessing the quantity referent (Hiebert, 1988; Merkley & Ansari, 2016; Siegler & Lortie-Forgues, 2014). Mapping skills are no longer directly relevant (Szűcs et al., 2014). Instead, more

complex associations become the proximal predictors of advanced skills (Lyons et al., 2014; Sasanguie & Vos, 2018; Xu et al., 2019). Similarly, I assumed that fraction mapping skills are most relevant when students are first acquiring formal fraction knowledge. The finding that fraction mapping skills were more strongly linked to fraction outcomes for fourth graders compared to sixth graders supports this assumption. Moreover, the finding that fraction mapping skills, not grade, differentiated fraction estimation amongst the least skilled estimators suggests that fraction mapping skills are relevant for learners who are struggling to acquire fraction knowledge. Based on these findings, educational practices that emphasize linking fraction symbols with visual representations may help new and struggling fraction learners to succeed.

As students first learn about how conventional fraction symbols are interpreted, their understanding of mathematical orthography (i.e. their knowledge of the conventions for notating mathematical symbols) supports mapping fraction symbols to their visual representations. For more experienced fraction learners, knowledge of mathematical vocabulary also supports fraction mapping. Importantly, as expertise builds, math vocabulary becomes a stronger predictor of fraction skills than knowledge of fraction mapping. Together these findings emphasize the importance of how teachers write about and talk about fractions in the classroom.

Limitations and Future Research

The studies described in this dissertation provide valuable insights into how math language skills and fraction mapping describe individual differences in developing knowledge of fractions concepts. Nonetheless, the studies have some limitations which could be addressed with further research. First, Study 1 included math-specific language skills (i.e., mathematical vocabulary and mathematical orthography) whereas general vocabulary was the only domain-general language skill included in the study. Other domain-general language skills may be

relevant, especially for younger students. For example, receptive syntax has been linked to mathematical performance amongst students in first and second grade (Chow & Ekholm, 2019). Thus, future studies should include domain-general language measures, for example of orthography, to control for domain-general language skills.

Another limitation of this work is that only the most basic fraction symbol knowledge was assessed. Based on the view that earlier symbol skills are the building blocks for higher-level symbol skills (Cirino et al., 2016; Hiebert, 1988; Siegler & Lortie-Forgues, 2017; Xu et al., 2019), it would be interesting to test the predictive relations amongst fraction mapping and more advanced indices of fraction symbol knowledge such as fraction comparison and fraction ordering. Although fraction comparison was included in this study, testing time was limited and thus, there were too few comparison questions and response time was not measured thus efficiency of access could not be assessed. More extensive longitudinal research on fraction mapping and more advanced indices of fraction symbol knowledge are needed to draw conclusions about the relations between fraction mapping and fraction development.

A third limitation is that causality cannot be inferred from the correlational approach used in this thesis. My findings indicate that many students lack understanding of fraction mappings. Intervention studies focused on practices that promote linking the fraction symbol to its referents could provide further evidence on the causal relation between fraction mapping and fraction skills. In other work for my thesis, I developed an online tutor that could be used to support fraction learning (Di Lonardo Burr et al., in review). A wider variety of methods and approaches are necessary to help educators understand and apply research to instructional situations. Moreover, the data collected for this research includes item-level responses for a range of

fraction measures. Future research could include an in-depth analysis of student errors to help characterize the patterns of errors observed in students with varying levels of fraction skill.

Can these results be generalized beyond the sample of students who participated? I consider three potential levels of generalization: Extending to students in other school boards in Ontario, extending to students in other English-speaking countries with similar school systems, and extending to countries with other languages and/or other instructional systems. The sample population in this study was drawn from a school board that draws students from both rural areas and small towns. These children came from predominantly English-speaking families and, based on parent education, they represented a range of socioeconomic conditions from moderate to upper middle-class status. However, Ontario more generally has a culturally diverse population, especially in the large population centres like Ottawa and Toronto. Studies which tap into this more diverse group of students are necessary to establish whether the results generalize beyond English-speaking families. I do note that Ontario received an “A” rating from the Organization for Economic Co-operation and Development (OECD) on equity in math learning outcomes, specifically, “the language spoken at home by immigrant students does not necessarily affect their scores on the PISA math test” (Conference Board of Canada, 2014, para.1). Further support that my findings can be extended across the province comes from findings that mathematical performance is similar for younger (grades 2 and 3) Ontario students, both first and second language learners (Xu et al., in review). However, it is nevertheless important to further test for similar findings in diverse groups.

Do my results generalize to other Canadian provinces and to the United States? Education is a provincial responsibility in Canada and thus, although in Ontario fraction symbols are introduced in grade 4, in other provinces and in the United States, students start to use

fraction symbols in Grade 3. Thus, novice fraction learners in other provinces and countries may be younger than the students in the current study. Although I anticipate that a similar pattern of relations amongst novice and more experienced fraction learners would occur, this hypothesis needs to be tested with younger students.

Finally, further research is necessary to determine if these results can be extended across cultures which have different languages and different instructional systems. For example, fraction naming is more transparent in East Asian languages than in English; “one-half” roughly translates to “of two parts one” in Korean and thus verbal labels are very transparent in their mapping to the symbols and the magnitudes. In English, consider that we use one quarter and three-fourths whereas in Korean these would have the same structure, “of four parts one”, and “of four parts three”. This more-transparent labelling helped children more accurately match fraction symbols to their visual referents (Miura et al., 1999) and promoted knowledge of fraction proportions (Mix & Paik, 2008). Thus, linguistic differences may influence the relations among math-language skills, fraction mapping, and fraction outcomes. To summarize, the current findings can be extended across the province and potentially across the country if consideration is given to the age of participants and differences in the curriculum. Further research is needed to generalize the results across cultures.

Practical Implications

Findings from the current research indicate that knowledge of mathematical vocabulary is related to students’ understanding of fraction concepts. Language plays an important role in fraction learning. Teachers communicate ideas using through language and children share ideas through language. General vocabulary skills for example are related to knowledge of fraction concepts (Hansen et al., 2017; Vukovic et al., 2014). Misunderstanding or misapplying academic

language however, can hinder students' understanding of fractions (Riccomini, 2016; Riccomini et al., 2015). Further, limited or imprecise mathematical vocabulary can be a source of mathematical errors (Hughes et al., 2016; Hurst & Cordes, 2019; Powell et al., 2019; Rubenstein & Thompson, 2002; Thompson & Rubenstein, 2014). As an example, when I was working with a student, I described a fraction as “1 over 5”, to my chagrin, the student misunderstood my meaning and instead of writing the fraction $\frac{1}{5}$, the student wrote exactly what I had said (see Figure 7.4). Supporting students as they learn new concepts by using consistent, concise and appropriate terminology is one suggestion teachers can try to help students better understand new concepts (Powell et al., 2019).

Figure 7.4

Example Illustrating the Effect of Inappropriate Terminology

My findings also suggest that a focus on students' knowledge of fraction symbols and the quantities they represent may help both struggling students and typically-developing students as they begin formal fraction instruction. In a review of 22 fraction intervention studies, Hwang et al. (2019) found that interventions that used multiple types of fraction representations led to gains in students' conceptual understanding of fractions. Further, having students work with multiple fraction representations such as area models (e.g., 3 out of 8 sections of a rectangle shaded), discrete models (e.g., 3 out of 8 balls selected), and fraction number lines (e.g., the 3rd

tick on a 0-1 number divided into eighths) is also recommended teaching practice (Ontario Ministry of Education, 2005; Siegler et al., 2010). The present findings suggest that helping students to understand why the symbolic version of these quantities is the same, even when the models differ, may be important in helping them to acquire integrated representations.

There are two main contributions of the present research. First, this research is the first to identify fraction mapping as core knowledge that students may need before they can acquire further fraction concepts. Second, I developed and tested three new measures, each with strong internal reliability and face and construct validity. These measures can be used by researchers to replicate and extend the current findings. The fraction mapping task for example, can be used in studies on the cognitive development of fraction skills. Moreover, the mathematical vocabulary and orthography tasks can be used to examine the role of math-specific language skills in multiple math domains. In my commitment to the Open Science movement, I have made these measures available through <https://osf.io/h47c2/> and <https://carleton.ca/cacr/math-lab/apps/>.

In summary, my research shows that students who experience difficulty acquiring fraction concepts in the mid-to late-elementary grades have not mastered the mappings among the symbolic, visual, and verbal fraction representations. The links between written symbols and their visual representations are critical. These important connections are also supported by the two other ways that magnitudes and symbols are linked, that is, between the written symbol and the verbal description, and between the verbal description and the visual magnitude. For teachers, targeting instruction to help students master these mappings is recommended. The development of fully integrated representations which can be leveraged to support a variety of tasks that involve fractions rests on students' knowledge of how fraction symbols are connected to magnitudes.

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APPENDICES

Appendix A

Additional Measures Not Included in the Final Analyses

Domain General Measures

Black/White Stroop Task. The Stroop task is used as an indicator of inhibitory control. In this version of the Stroop, children see one black square and one white square on the iPad screen. They hear a color name (black or white) and then quickly press the same (congruent) or different (incongruent) coloured square. There are four trial blocks (5 trials each) alternating between congruent blocks (e.g., if they hear “white”, they are instructed to press the “white” square) and incongruent blocks (e.g., if they hear “white”, they are instructed to press the “black” square). Each child completes a total of 20 trials. The score is the difference between RT (reaction time) on congruent and incongruent trials. This is a measure of interference cost. Reliability based on the RTs for the 4 trial blocks was acceptable (Cronbach’s $\alpha = 0.74$)

Matrix Reasoning. Matrix reasoning taps into fluid reasoning; a child's skill at grasping relationships amongst visual objects (e.g., shapes, designs, visuospatial patterns) and applying their reasoning skills to identify and apply rules to complete visual patterns (Keith et al., 2006). This task is part of the Weschler Intelligence Scale for Children (WISC-5) test battery.

In this task the child sees an incomplete 4 by 4 matrix with one missing item. The child selects one of five options to complete the matrix. The test-set for this age group includes two practice trials followed by test items 5 to 35. Testing is discontinued after 3 consecutive errors. Scoring is based on the total correct. Reliability based on accuracy across all items was high (Cronbach’s $\alpha = 0.91$).

WIAT Word Reading (Weschler, 2005). In this task, children are shown a set of increasingly difficult words (total words 40) and asked to read the words aloud, one at a time. Children are encouraged to try each word even if they are unsure of the pronunciation. As per standardized administration procedures, testing is discontinued after three consecutive errors. Reliability across the 40 words was high (Cronbach's $\alpha = .93$).

Math Measures

Word Problem Solving. (KeyMath 3; Connolly, 2000). In this standardized task, children solve a series of progressively more difficult math word problems. The researcher reads a question to the child and shows a corresponding picture. The child responds orally. A subset of thirteen questions were chosen to capture a range of skills appropriate for grades 4 to grade 6. The items were selected based on the standardized means for the age-range and the standard deviations from the mean. Testing was discontinued after 3 consecutive errors. Scoring is the total number of correct responses. An example of the types of questions posed is shown in Figure A1. Reliability across the 13 responses was acceptable (Cronbach's $\alpha = .79$).

Figure A1

Example of the Type of Word-problem Solving Question Used

Note: Due to copyright, actual test questions cannot be shown. For this sample question, the researcher would state, “There are 15 steps on the ladder. Martha is on step two. How many more steps does Martha have to climb to reach the top?”

Double-digit Arithmetic. In this shortened version of the Calculation Fluency Task (Sowinski et al., 2014) the first 30 questions for each addition and subtraction were selected from the complete set of stimuli. Children are timed as they complete a page each of 30 double-digit addition questions and double-digit subtraction questions. Students are given 90 seconds per page. Total score is the number of correct responses. Reliability comparing scores for addition and subtraction was high (Cronbach’s $\alpha = .81$).

Appendix B

Mathematical Vocabulary Technical Report - Grades 3 to 6

Test Development

We designed a receptive math vocabulary measure to assess students' knowledge of math-specific words used in the classroom. Math vocabulary assessments that are currently available are designed to assess early vocabulary or tap into both receptive and productive math vocabulary. For example, many measures are targeted at a younger age group (e.g., Purpura & Reid, 2016; Toll & Van Luit, 2014) or they involve using manipulatives or providing written responses (e.g., Powell, Driver, Roberts, & Fall, 2017; Powell & Nelson, 2017). Although these measures inform our understanding of the role of math vocabulary in math learning, we sought to create a task that is quick to administer while covering a range of mathematical terms. Receptive vocabulary assessments are quick and easy to administer so many words can be tested.

Words were chosen to reflect the terminology used in math classrooms in Ontario. A master list of over 200 words was created using curriculum documents and text glossaries (Common Core State Standards Initiative, 2010; Manitoba Education, 2013; Ontario Ministry of Education, 2005). This list was organized by grade and the five math strands described in the Ontario curriculum. (i.e., numeration, measurement, geometry and spatial sense, patterning and algebra, data management and probability). The list was narrowed down to 40 common terms by selecting key words from each of the strands giving a heavier emphasis on numeration words. The final word list was reviewed by teacher educators.

What does the Mathematical Vocabulary Test measure?

The mathematical vocabulary test is a measure of receptive math vocabulary. Two versions of the task were created, one for children in grades 2 and 3 (Math vocabulary, primary

grades) and a subsequent version for children in grade 3 to grade 6 (Math vocabulary, junior grades). The first task was modelled on the pre-school math vocabulary task developed by Purpura and Reid (2016) and was designed to extend the age range and breadth of terms included in of the Purpura task. The second task was modelled on a measure of general receptive vocabulary (PPVT, Peabody Picture Vocabulary Test; Dunn & Dunn, 2007) and was designed to further extend the testing age range using the well-established test format of the PPVT. We report on the final version of the task.

Method

Description of Test

In this math vocabulary task, the participant chooses a picture that corresponds with a given math word or expression. The researcher says a math word aloud while the student sees an iPad screen with 4 images labelled a,b,c, or d (1 correct response, 3 foils). The student points or names the appropriate response (see Figure B1). Word selection taps into the math strands described in the Ontario math curriculum; geometry and spatial sense (e.g., triangular prism), numeration (e.g., hundreds column, factor), measurement (e.g., perimeter), patterning and algebra (e.g., ascending order) and data management and probability (e.g. axes). There are 40 questions in all.

Figure B1

Target vocabulary term “perimeter”

Procedure

Administration of the test was approved as part of a larger study involving children in the Ottawa region. The study was approved by the Carleton University Research Ethics Board and school board research committees (Ottawa Carleton Research Ethics Committee, Research Steering Committee of the Upper Canada District School Board). Teachers at participating schools sent information packages to parents and those whose children wanted to participate provided informed consent. Children provided verbal assent prior to testing.

Children were tested one-on-one in a quiet area of the school. The tester began by telling the child that they (the researcher) would read a word or sentence (e.g., “dime”, “which lines are parallel”) and the child would then select the corresponding picture from the 4 options. The tester told the child that there were a number of words, some of which the child may not have learned yet in school. But the child’s job was to just try their best. Children were encouraged to “make a best guess” and not to skip questions. Anecdotal observations from research assistants indicated that children guessed accurately on items even though the child lacked confidence in their response. Children completed additional measures not described here. We report on two standardized measures included in the test battery, the Key Math Applied Problems Solving Subset (Connolly, 2007) and the Peabody Picture Vocabulary Test (Dunn & Dunn, 2007).

Sample

Children ($N=217$, $N_{\text{Girls}}=106$, $M_{\text{age}} = 9\text{year:9month}$) were tested in the fall or winter of the 2018-2019 school year. Children were taught math in English and English was the first language for 83% ($N=181$) of children. The most common non-English first languages spoken were Mandarin ($N= 9$, 4 %,) and Arabic ($N= 9$, 4 %). All children were tested in English. Parents were well educated. Only 3% of fathers and 1% of mothers did not complete high school.

Whereas, 75% of fathers and 88% of mothers reported completing post-secondary diplomas or degrees (i.e. college, university or post-graduate).

Results

Student performance across all 40 items is shown in Table B1. Scores were analysed in a 3 group (grade: 3, 4, 6) ANOVA. As expected, student scores differed across grades $F(2, 138) = 19.12, p < .001$ and Turkey post hoc analyses confirmed that students in grade 6 outperformed students in both grades 3 ($p < .001$) and 4 ($p < .01$) and grade 4 students outperformed students in grade 3 ($p < .05$). Scores were normally distributed such that skew, and kurtosis were not significant, and medians were similar to means.

Table B1

Performance on the Math Vocabulary Measure by Grade

Grade	<i>N</i>	<i>Min</i>	<i>Max</i>	<i>Mdn.</i>	<i>M</i>	<i>S.D</i>	<i>Skew_z</i>	<i>Kurtosis_z</i>
3	87	9	30	19.00	19.02	5.02	0.07	-1.38
4	64	9	31	22.00	20.95	4.76	-1.32	-.042
6	66	12	39	24.00	23.95	4.78	0.85	1.80
Total	217							

Item response analysis can provide a nuanced analysis of task quality (Kimberlin & Winterstein, 2008; Reise et al., 1993). We thus tested an item response model to better assess how the individual test items relate to the latent trait, math vocabulary. A two-parameter logistic (2-PL) item response analysis was run using Stata/SE 15.1 (StataCorp, 2017). Two task properties are estimated in a 2-PL model, item discrimination and item difficulty. Item discrimination estimates success on an individual item based on overall math vocabulary (i.e.,

the latent trait estimate). A high positive discrimination level indicates that the item differentiates well between low and high levels of math vocabulary. In contrast, a negative discrimination level indicates that high scoring students are less likely to choose the correct item than low scoring students. The difficulty parameter is an indication of where the item fits in terms of ability. If the difficulty parameter is low, the probability of success on the item is high. The higher the difficulty, the lower the probability of success on the item. Reviewing these properties we can select test items with a range of difficulty that positively discriminate between low and high levels of math vocabulary knowledge.

Based on the item response model, 4 test items were discarded. First, we examined discrimination values. The terms unit fraction, variable, perpendicular and integer all had negative discrimination values (see Table B2) thus were excluded from further analysis. Further, scores on these 4 items did not correlate with total performance. Next we looked at difficulty values. We wanted to include items with a range of difficulty, and we were most interested in identifying performance of typical students, not differentiating struggling or gifted students. Difficulty ratings for the remaining 36 items ranged broadly from easy (cylinder, $b=-9.02$) to difficult (equivalent fraction, $b=4.02$). To determine the range of skill best differentiated by the math vocabulary measure we refer to the test information function (TIF) graph (see Figure B2). Using the TIF we can determine the range of performance (indicated by the θ value, the latent trait math vocabulary) where the information we predict exceeds error associated with the estimates. As shown in Figure B.2, our 36-item measure provides the most information for students located near $\theta=0$ and information exceeds error on a broad range of performance levels ($-3.0 < \theta < 1.8$). This finding suggests our task captures a range math vocabulary skill and can be used to differentiate vocabulary performance amongst low to above average performing students

in grades 3 to 6. Finally, internal consistency based on responses to the reduced word set was good (Cronbach's $\alpha = 0.76$).

Figure B.2

Test Information Function for the 36-item Math Vocabulary Measure.

Note. Theta is predicted value of the latent trait math vocabulary

Table B2*Item Accuracy, Correlations, Discrimination and Difficulty*

Item	Item detail	Grade	Accuracy <i>M</i>	Item to total (<i>r</i>)	Discrimination (<i>a</i>)	Difficulty (<i>b</i>)
1	Dime	1	.85	0.31***	.96	-2.14
2	Tens column	2	.80	0.41***	1.23	-1.42
3	cylinder	1	.94	0.12	.30	-9.02
4	One-quarter	2	.77	0.31***	.70	-1.89
5	Hundreds column	2	.84	0.21**	.61	-2.97
6	Less than	2	.47	0.22**	.23	.45
7	Mass	3	.65	0.39***	.78	-.90
8	sum	3	.45	0.35***	.58	.36
9	ascending		.49	0.31***	.51	.09
10	difference	3	.47	0.31***	.48	.30
11	array	3	.54	0.24***	.35	-.46
12	horizontal	3	.39	0.37***	.73	.67
13	product	3	.40	0.33***	.64	.71
14	perimeter	3	.81	0.40***	1.28	-1.44
15	axes	3	.49	0.29***	.51	.11
16	Triangular prism	3	.91	0.37***	1.67	-1.93
17	column	4	.55	0.23***	.30	-.66
18	factor	4	.39	0.38***	.73	.67
19	key	4	.68	0.32***	.65	-1.23
20	denominator	4	.49	0.55***	2.25	.02
21	numerator	4	.45	0.49***	1.73	.17
22	Unit fraction	4	.15	-0.06	-.39	-4.48
23	decimal	4	.80	0.46***	1.65	-1.20
24	Hundredths column	4	.10	0.28***	1.07	2.48
25	parallel	5	.78	0.42***	1.11	-1.40
26	quotient	6	.49	0.22**	.24	.20
27	Equivalent fraction	5	.22	0.19**	.30	4.26
28	multiple	5	.34	0.14*	.21	3.18
29	Variable	5	.61	0.00	-.26	1.68
30	Acute angle	5	.57	0.45***	1.10	-.30
31	clockwise	3	.86	0.25***	.79	-2.51
32	Obtuse angle	5	.60	0.37***	.99	-.49
33	Equilateral triangle	6	.56	0.31***	.47	-.51
34	ratio	6	.21	0.20**	.43	3.26
35	Negative number	7	.67	0.40***	.90	-.94
36	Prime number	6	.28	0.24***	.49	1.98
37	Perpendicular	7	.35	0.11	-.01	-61.35
38	radius	8	.33	0.23**	.28	2.60
39	Square root	7	.38	0.39***	.78	.71
40	Integer	7	.10	0.05	-.16	-15.45

Correlation is significant at following levels, * $p < .05$, ** $p < .01$, *** $p < .001$

Construct Validity

Math vocabulary skills are positively related to general vocabulary (Powell & Nelson, 2017; Purpura et al., 2017; Toll & Van Luit, 2014) and word problem-solving skills (Peng & Lin, 2019). Students who perform better on math vocabulary tests outperform their lower scoring peers on measures of general vocabulary and word problem-solving. To determine if our task captured math vocabulary skills across the three grades, we tested the predicted correlations. As expected, math vocabulary scores correlated positively with both general vocabulary and word problem-solving skills in each of the grades we tested. These correlational findings suggest our math vocabulary task is indeed measuring knowledge of mathematical vocabulary.

Table B3

Correlations by Grade for Math Vocabulary, General Vocabulary (PPVT) and Word-Problem Solving

Grade	<i>N</i>	Math vocabulary/ PPVT	Math vocabulary/ KM	PPVT/KM
3	87	0.45**	.60**	.25*
4	63	0.66**	.50**	.41**
6	66	0.51**	.53**	.39**

Discussion

The goal of the present research was to validate a measure of receptive math vocabulary. The goal was met. The math vocabulary task included a broad range of math words some introduced as early as grade 1 and others that children may not experience until grade 7. Using item response analysis, we reduced the trial word list from 40 to 36 terms that covered a range of

math concepts and terminology. Further, we confirmed that the items varied in difficulty and could differentiate performance amongst low to high performing students. The task however is not designed to identify remedial or gifted students. Students in grades 3, 4 and 6 successfully completed the task. As anticipated, the older students scored higher than the younger students and scores were normally distributed within grades indicating the task was neither too easy nor too hard for the students tested. The task had good internal consistency and based on correlations with general vocabulary and math problem solving we conclude that the task also has strong construct validity.

Appendix C

Item Analysis for the Extended Version of the Children's Symbol Decision Task

To better understand and validate the extended version of the Symbol Decision Task (SDT-Child) we analysed item accuracy and estimated the discrimination and difficulty of each item using an IRT model. Using the findings from item analyses we determined which if any items should be discarded.

Students accurately accepted most conventional stimuli and rejected most non-conventional stimuli (see Table C1). Exceptions included the formula for area (item 53, $A = l \times w$) which was consistently rejected indicating students were unfamiliar with this symbolic text. Further, the mean accuracy on the area item was not correlated with total mean accuracy on conventional items suggesting it may be problematic. Similarly, students mistakenly classified items 48 and 60 (20% and 0.25¢) as correct indicating a lack of familiarity with these items. Further, mean accuracy on these two non-conventional items did not correlate with mean non-conventional item accuracy. Reviewing item accuracy, we identified three potentially problematic stimuli.

To determine if the three low-accuracy items should be excluded from the final score, we assessed the discrimination value and difficulty of the items using an IRT model. If the low accuracy items have either a negative or low discrimination value, we can conclude that they do not help differentiate student performance and can potentially be excluded from the total score. If, however the discrimination values are positive and strong, we can conclude that the items should be included in the total score. To evaluate the difficulty ratings, large negative values represent the least difficult items whereas large positive values represent the most difficult items. Ideally, we want the scales (conventional, non-conventional) to reflect a range of difficulty to

ensure they are neither too easy nor too hard. First, we examined the conventional items. Item 53 had a low discrimination value ($a = .05$) and was very difficult ($b = 36.8$). We therefore excluded this item and its corresponding non-conventional transformation item 54. Next we considered the non-conventional items. Both item 48 (20%) and 60 (0.25¢) had low discrimination values ($a = .01$, $a = -.14$ respectively) and extreme difficulty ratings ($b = 13.3$ and $b = -13.5$ respectively). All three low-accuracy items and their paired conventional/non-conventional matches were thus excluded from the final score.

Table C1*Mean Item Accuracy and Item Response Parameters for the Symbol Decision Task*

Item #	Stimuli	Conventional Stimuli					Item #	Stimuli	Non-Conventional Stimuli				
		Mean Acc.	S.D.	Item to scale r	Disc. <i>a</i>	Diff. <i>b</i>			Mean Acc.	S.D.	Item to scale r	Disc. <i>a</i>	Diff. <i>b</i>
1	430	.79	.41	.39**	.64	-2.22	2	026	.66	.48	.28**	0.36	-1.88
3	$\frac{1}{2}$	.89	.32	.37**	1.40	-1.92	4	$\frac{1}{2}$	.77	.42	.36**	0.91	-1.56
5	$\frac{1}{3}$	.82	.39	.55**	1.71	-1.28	6	$\frac{2}{3}$	.88	.33	.38**	1.02	-2.29
7	\$3.00	.94	.24	.07	-0.15	18.28	8	\$2;00	.91	.29	.29**	0.88	-2.94
9	4>2	.72	.45	.32**	0.72	-1.46	10	3 ∨ 2	.93	.25	.39**	1.30	-2.51
11	2 + 2 = 4	.97	.17	.10	-0.17	20.37	12	2 + 2 ≡ 4	.88	.33	.39**	1.24	-2.00
13	3<5	.71	.45	.42**	0.96	-1.12	14	4<<5	.96	.19	.24**	0.91	-3.98
15	2<3	.80	.40	.40**	1.40	-1.34	16	1∧3	.92	.28	.37**	0.96	-2.87
17	3 + 2 = 5	.94	.24	.10	0.32	-8.60	18	3 + 2 ≡ 5	.64	.48	.29**	0.34	-2.87
19	2 + 1 = 3	.95	.21	.48**	2.75	-2.03	20	+ 2 1 = 3	.94	.24	.37**	0.34	-1.67
21	3 – 1 = 2	.81	.39	.29**	0.24	-6.24	22	4_1=3	.86	.34	.18*	0.42	-4.58
23	2 x 3 = 6	.91	.29	.22**	0.43	-5.56	24	2 x 4 = 8	.55	.50	.29**	0.41	-0.47
25	3x2=6	.93	.25	.14	0.25	-10.56	26	4~2=8	.89	.31	.39**	0.95	-2.58
27	3=2+1	.52	.50	.20*	-0.11	0.84	28	4=3⊕1	.89	.32	.36**	1.11	-2.23
29	8÷2=4	.86	.34	.34**	0.54	-3.55	30	8·12 = 4	.92	.28	.29**	0.99	-2.80
31	7 + 2 =9	.95	.21	.19*	0.75	-4.39	32	1+2 = 3	.91	.29	.31**	0.89	-2.92
33	2 = 3 - 1	.46	.50	.24**	-0.01	-13.35	34	2 = 3 ∩ 1	.95	.22	.50**	2.96	-1.85
35	35¢	.94	.24	.17+	0.31	-9.01	36	3¢5	.90	.30	.20*	0.56	-4.18
37	15¢	.96	.19	.23**	1.17	-3.28	38	20¢	.90	.30	.44**	2.14	-1.65
39	2:30	.91	.29	.27**	0.44	-5.45	40	13:0	.74	.44	.23**	0.43	-2.55
41	1:00	.87	.34	.44**	0.80	-2.66	42	3..00	.90	.30	.21*	0.65	-3.66

Conventional Stimuli							Non-Conventional Stimuli						
Item #	Stimuli	Mean Acc.	S.D.	Item to scale r	Disc. <i>a</i>	Diff. <i>b</i>	Item #	Stimuli	Mean Acc.	S.D.	Item to scale r	Disc. <i>a</i>	Diff. <i>b</i>
43	15°C	.86	.34	.17+	0.29	-6.50	44	2°0C	.75	.44	.30**	0.61	-1.94
45	80%	.90	.30	.36**	1.24	-2.22	46	%80	.74	.44	.37**	0.60	-1.89
47	20%	.88	.33	.30**	0.41	-5.00	48	20%	.20	.40	.17+	0.10	13.30
49	$3\frac{1}{2}$	.55	.50	.48**	1.60	-0.19	50	$\frac{1}{2}3$	.77	.42	.25**	0.33	-3.61
51	$2\frac{3}{4}$	.48	.50	.45**	1.53	0.06	52	2 3 4	.91	.29	.26**	0.57	-4.24
53	A = 1 x w	.13	.34	.15	0.05	36.77	54	A = 1 x _w	.89	.31	.14	0.23	-9.20
55	1.5	.86	.34	.43**	1.26	-1.85	56	1:5	.92	.27	.14	0.46	-5.59
57	2.3	.86	.35	.40**	1.26	-1.79	58	23.	.54	.50	.46**	1.03	-0.18
59	25¢	.97	.17	.17	0.99	-3.95	60	0.25¢	.27	.44	.13	-0.07	-13.51

Note. Disc. = discrimination, Diff = difficulty

Appendix D

Fraction Mapping Task

Target Stimuli and Foils

Target stimuli are listed in Table D1. A variety of foil-types were chosen to reflect the types of errors children make in labelling and naming fractions. Foils-types have been coded according to the descriptions that follow Table D1.

Table D1

Stimuli Set for the Fraction Mapping Task

Mapping direction	Trial	Target	Foil 1	Foil type	Foil 2	Foil type	Fraction type
Word to image	P1	$\frac{1}{2}$	$\frac{1}{4}$		1		Part-whole
Word to number	P2	$\frac{1}{2}$	$\frac{1}{4}$		$\frac{1}{3}$		Proper
Number to image	P3	$\frac{1}{2}$	$\frac{1}{4}$		1		Part-whole
Image to number	P4	$\frac{1}{2}$	$\frac{1}{4}$		$\frac{1}{3}$		Part-whole
Word to image	1	$\frac{3}{4}$	$\frac{3}{7}$	1	$\frac{4}{7}$	2	Discrete
	2	$\frac{2}{3}$	$\frac{2}{4}$	6	$\frac{2}{5}$	1	Discrete
	3	$\frac{2}{5}$	$\frac{5}{7}$	2	$\frac{2}{7}$	1	Discrete
	4	$\frac{5}{8}$	$\frac{5}{13}$	1	$\frac{3}{8}$	4	Discrete
	5	$\frac{3}{10}$	$\frac{3}{13}$	1	$\frac{7}{10}$	4	Discrete
	6*	$\frac{3}{4}$	$\frac{2}{8}$	11	$\frac{3}{8}$	5	Part-whole
	7	$\frac{4}{5}$	$\frac{4}{6}$	6	$\frac{3}{5}$	5	Part-whole
	8	$\frac{3}{8}$	$\frac{5}{8}$	4	$\frac{3}{6}$	5	Part-whole
	9	$\frac{2}{3}$	$\frac{2}{4}$	6	$\frac{3}{5}$	2	Part-whole
	10	$\frac{7}{10}$	$\frac{6}{10}$	5	$\frac{3}{10}$	4	Part-whole
Word to number	11	$\frac{3}{8}$	$\frac{3}{24}$	7	$\frac{8}{3}$	3	Proper
	12	$\frac{3}{4}$	$\frac{3}{25}$	8	$\frac{4}{3}$	3	Proper
	13	$\frac{4}{5}$	$\frac{5}{4}$	3	$\frac{4}{9}$	1	Proper
	14	$\frac{3}{10}$	$\frac{3}{13}$	1	$\frac{30}{10}$	3	Proper
	15	$\frac{2}{3}$	$\frac{3}{2}$	3	$\frac{2}{1}$	9	Proper
	16	$4\frac{3}{4}$	$\frac{43}{25}$	8	$\frac{43}{4}$	10	Mixed/improper
	17	$\frac{8}{3}$	$8\frac{1}{3}$	14	$\frac{3}{8}$	3	Mixed/improper
	18	$2\frac{4}{5}$	$\frac{6}{5}$	14	$\frac{24}{5}$	10	Mixed/improper
	19	$\frac{20}{10}$	$\frac{10}{20}$	3	200	7	Mixed/improper
	20	$\frac{9}{8}$	$\frac{9}{72}$	7	$\frac{8}{9}$	3	Mixed/improper
Number to image	21	$\frac{2}{5}$	$\frac{5}{7}$	2	$\frac{2}{7}$	1	Discrete
	22	$\frac{3}{4}$	$\frac{3}{7}$	1	$\frac{4}{7}$	2	Discrete
	23*	$\frac{2}{3}$	$\frac{2}{6}$	6	$\frac{2}{5}$	1	Discrete

	24	5/8	3/8	4	5/13	1	Discrete
	25	1/10	1/11	2	10/10	5	Discrete
	26	6/8	2/4	3	2/3	3	Part-whole
	27	3/5	uneven	12	uneven	12	Part-whole
	28	7/10	3/10	4	9/12	11	Part-whole
	29	1/3	1/2	11	2/3	4	Part-whole
	30	1/4	uneven	12	uneven	12	Part-whole
Image to number	31	2/5	1/3	11	2/3	9	Discrete
	32	3/8	1/3	11	3/5	9	Discrete
	33	1/4	1/2	6	1/3	9	Discrete
	34*	1/3	2/4	14	6/2	14	Discrete
	35	2/10	2/5	11	1/4	11	Discrete
	36	1/4	3/1	13	1/3	9	Part-whole
	37	3/8	5/3	13	3/5	9	Part-whole
	38	2/5	2/3	9	3/5	4	Part-whole
	39	3/10	3/5	6	3/7	9	Part-whole
	40*	2/3	6/3	14	3/2	14	Part-whole

Note. * denotes either a target fraction or foil that is an equivalent fraction. For example, for item 6, the target fraction is “three-fourths” and the correct answer is an image of a rectangle partitioned into 8 sections, 6 of which are shaded.

Description of Foil Types.

1. Ratio foils. The denominator of the foil is either the sum or difference of the numerator and denominator of the target fraction. E.g., for a target of $\frac{3}{4}$, the foil is $\frac{3}{7}$, for a target of $\frac{2}{5}$, the foil is $\frac{2}{3}$.
2. Combined ratio and numerator foil. The denominator of the foil is the sum of the numerator and denominator of the target fraction and the numerator is the denominator of the target fraction. E.g., for a target fraction of $\frac{3}{4}$, the foil is $\frac{4}{7}$.
3. Inverse foils. The foil is the inverse of the target fraction. E.g., for a target of three-eighths, the foil is $\frac{8}{3}$.
4. Numerator/difference foils. The foil has an incorrect numerator that reflects the other part of the fraction e.g., for a target of $\frac{5}{8}$, the foil is $\frac{3}{8}$.
5. Numerator foil. The foil has an incorrect numerator e.g., for a target of $\frac{4}{5}$ the foil is $\frac{3}{5}$.
6. Denominator foils. The foil has an incorrect denominator e.g., for a target of $\frac{3}{8}$, the foil is $\frac{3}{6}$.
7. Product foil. The foil denominator is a product e.g. for a target of “three-eighths” the foil is $\frac{3}{24}$.
8. Fourth/quarter foil. The denominator is 25 for a target denominator of “quarter” e.g. For a target of “three quarters” the foil is $\frac{3}{25}$.
9. Denominator/difference foils. The foil has an incorrect denominator that reflects the other part of the fraction e.g., for a target of “two-thirds”, the foil is $\frac{2}{1}$ – this is a type of ratio error.
10. Numerator/Mixed number foil. The foil numerator is two-digit number. E.g. for “four and three quarters” the numerator is 43.
11. Numerator and denominator are incorrect e.g. for the target $\frac{6}{8}$, the foil is $\frac{2}{3}$.

12. Uneven segments – the foil is portioned unevenly
13. Combined ratio and inverse
14. Other

Task Validation

To validate the fraction mapping task and identify possible problematic test items we first assessed internal consistency across all 40 items. Internal reliability was strong (Cronbach's $\alpha = 0.88$) indicating the students responded consistently to the items.

Next we looked at item responses and conducted a two-parameter logistic (2-PL) item response analysis using Stata/SE 15.1 (StataCorp, 2017). The 2PL model estimates discrimination (a) and difficulty (b) levels for each test item. Item discrimination (a) estimates the probability of success as skill-level changes. A test item that negatively discriminates ($a < 0$) is problematic because the probability of success on that item increases as skill level decreases. All items had positive a -values indicating that each item positively discriminated between skilled and un-skilled levels of performance (see Table D2). Item difficulty captures the probability of success on an individual item and is a standardized normal score. Thus, negative b -values are relatively easy items whereas positive b -values are relatively hard. We wanted to include items with a range of difficulty levels and as shown in Table D2, we were successful as item difficulty ranged from $b = -7.01$ to $b = 4.9$. Further, as indicated by the test information function graph (see Figure D1), our measure provides the most information (task information $>$ task error) for students with latent trait scores between $\theta = -2$ to $\theta = 1$ which corresponds to test scores of 13 to 34 (see Figure D2). With actual scores ranging from 12 to 39 and a mean score of 27.9, the task differentiates fraction mapping performance amongst low to above average performing students. Based on the internal consistency and item response analysis, all 40 items will be included in the total performance score.

Table D2*Fraction Mapping Response Accuracy and Item Response Model Parameters*

Item	Accuracy (<i>M</i>)	Item-total (<i>r</i>)	Discrimination (<i>a</i>)	Difficulty (<i>b</i>)
1	.64	.56**	1.68	-0.50
2	.70	.50**	1.34	-0.84
3	.58	.62**	1.85	-0.28
4	.60	.67**	2.28	-0.34
5	.67	.57**	1.53	-0.65
6	.38	.26**	0.43	1.19
7	.65	.44**	0.99	-0.76
8	.70	.48**	1.24	-0.88
9	.78	.50**	1.38	-1.21
10	.87	.31**	0.88	-2.44
11	.94	.21*	0.59	-4.83
12	.91	.27**	0.72	-3.58
13	.95	.09	0.24	-11.78
14	.89	.26**	0.64	-3.51
15	.91	.15	0.32	-7.06
16	.89	.23**	0.73	-3.15
17	.77	.27**	0.45	-2.79
18	.85	.44**	1.33	-1.72
19	.88	.27**	0.78	-2.87
20	.87	.21**	0.30	-6.39
21	.62	.59**	1.83	-0.42
22	.65	.63**	2.01	-0.52
23	.09	.16	0.50	4.91
24	.52	.63**	2.04	-0.06
25	.62	.46**	1.06	-0.57
26	.92	.34**	1.31	-2.38
27	.47	.17	0.20	0.55
28	.78	.35**	0.74	-1.93
29	.71	.36**	0.83	-1.26
30	.53	.26**	0.40	-0.28
31	.73	.64**	2.34	-0.77
32	.69	.59**	1.83	-0.69
33	.79	.66**	2.49	-1.01
34	.18	.33**	0.99	1.83
35	.89	.30**	0.86	-2.77
36	.77	.55**	1.71	-1.05
37	.76	.55**	1.65	-1.03
38	.79	.63**	2.68	-0.99
39	.78	.54**	1.90	-1.04
40	.22	.34**	0.97	1.51

Note. *N* = 130. Shaded items indicate below or near chance (*M* < 0.33) performance.

Figure D1

Test Information Graph for the 40-item Fraction Mapping Task.

Note. Theta (θ) is the predicted value of the latent trait fraction mapping.

Figure D2

Test Characteristic Curve Showing the Predicted Fraction Mapping Scores Across the Range of Theta (Latent Trait Values).

Note. Predicted test scores for latent trait values of $\theta = -2$ and $\theta = 1$ are marked.

Interestingly, students scored below or near chance on four test items (see Table D2). The four items involved matching equivalent fractions. For example, in question 6 students were asked to match the fraction “three-fourths” with a picture of a rectangle that showed 6 out of 8 sections shaded. These items were the most difficult ($b=1.19$ to 4.91) but they positively

discriminated performance ($a = 0.43$ to 0.99) thus were included in the final task scoring.

However, because these 4-items were not equally balanced across the mapping formats and representation types, they were not included in analyses comparing performance across different mapping formats and representations.

Appendix E

Fraction Skills Assessment

A number of resources were referenced to develop this test measure. To ensure questions reflected what students were learning, questions were adapted from the Ontario Math Makes Sense text and practice booklets and the Grade 6 EQAO (Education Quality and Accountability Office) standardized test. Jump Math support documents were also referenced. To align with previous fraction research, questions were included from the Fraction Test Battery (Fuchs et al., 2017) and two versions of the Conceptual Procedural Fraction Measure (Hallett et al., 2010, 2012). These two tests (Fraction test battery and Conceptual and procedural fraction measure) could not be used in their entirety since many of the questions involved procedures such as fraction multiplication and division that are introduced in grades 7 and 8 in Ontario.

The underlying purpose of the measure was to target students' knowledge of fraction concepts and procedures. Conceptual items captured knowledge of fraction magnitude and fraction facts whereas procedural items involved fraction operations.

Table E1*Fraction Concepts Word Problem Solving Questions and Sources*

Question	Reference
1. Mary and John both have some pocket money. Mary spends $\frac{1}{4}$ of hers, John spends $\frac{1}{2}$ of his. a) Is it possible that Mary spent more than John? 2. b) Explain	Hallett, 2010,2012
3. Estimate the sum of $\frac{12}{13} + \frac{7}{8}$. Which number is closest to the answer? a) 1 b) 2 c) 19 d) 21	Hallett, 2012
4. How many $\frac{1}{2}$ cup measures in $2\frac{1}{2}$ cups of sugar?	MMS, grade 4 teacher book
5. Your teacher plants 4 tulips and 6 roses. What fraction of the flowers are tulips?	Modified grade 6 EQAO (question was about ratios)
6. How many possible fractions are between $\frac{1}{4}$ and $\frac{1}{2}$?	Hallett, 2012

In the second part of the fraction assessment, students completed a worksheet that included a variety of fraction questions. The references for the questions are shown in Table E2 and the worksheet follows in Figure E1.

Table E2*Fraction Worksheet Questions and Sources*

Skill	References
Fraction Magnitude comparison	Hallet et al., 2010, 2012 Grade 6 EQAO Grade 4 Math Makes Sense Textbook
Creating equivalent fractions	Fuchs et al., 2017
Convert improper fractions/mixed numbers	Grade 6 Math Makes Sense Textbook
Convert fractions/decimals	Modified EQAO questions, the original question was to convert fractions to percentages. I changed to decimals, so it also aligned with the grade 4 curriculum.
Fraction arithmetic	These questions all come from the Jump Math Student Essentials Workbook(B) * John Mighton developed a fraction unit for young children to help promote confidence in math. Although fraction addition is grade 7 curriculum it is included in the Jump Math resource booklet for students in grade 3.

Figure E1*Fraction Worksheet*

Date: _____

Student ID _____

Fraction Practice

1. Circle the
- larger fraction**
- in the pair.

a) $\frac{3}{7}$ $\frac{5}{7}$

d) $\frac{3}{2}$ $1\frac{3}{4}$

b) $\frac{3}{5}$ $\frac{3}{4}$

e) $\frac{4}{5}$ $\frac{4}{10}$

c) $\frac{4}{5}$ $\frac{5}{6}$

f) $\frac{15}{9}$ $\frac{9}{15}$

2. Multiply to find the missing number so the fractions are
- equivalent**
- .

PRACTICE

$$\frac{1}{2} = \frac{?}{4} \quad ? = \underline{\quad}$$

a) $\frac{1}{3} = \frac{?}{6} \quad ? = \underline{\quad}$

d) $\frac{1}{2} = \frac{6}{?} \quad ? = \underline{\quad}$

b) $\frac{2}{5} = \frac{?}{10} \quad ? = \underline{\quad}$

e) $\frac{?}{6} = \frac{2}{3} \quad ? = \underline{\quad}$

c) $\frac{4}{6} = \frac{8}{?} \quad ? = \underline{\quad}$

f) $\frac{?}{10} = \frac{4}{5} \quad ? = \underline{\quad}$

Write fractions as **decimals**.

a) $\frac{3}{10} =$

b) $\frac{4}{100} =$

c) $\frac{25}{100} =$

d) $\frac{1}{25} =$

Write decimals as **fractions**.

e) $0.8 = \text{—}$

f) $0.03 = \text{—}$

g) $0.125 = \text{—}$

h) $0.12 = \text{—}$

Write as **improper fractions**.

PRACTICE $1 \frac{1}{7} = \text{—}$

a) $3 \frac{2}{9} =$

b) $4 \frac{3}{4} =$

c) $2 \frac{1}{4} =$

Write as **mixed numbers**.

a) $\frac{7}{5} =$

b) $\frac{25}{6} =$

c) $\frac{11}{3} =$

Fraction Challenge

Add the following fractions.

a) $\frac{3}{7} + \frac{2}{7} =$

b) $\frac{4}{9} + \frac{3}{9} =$

c) $\frac{1}{6} + \frac{1}{3} =$

d) $\frac{1}{2} + \frac{1}{8} =$

e) $\frac{2}{11} + \frac{3}{11} =$

f) $\frac{1}{4} + \frac{3}{8} =$

Appendix F

Latent Profile Analysis Assumptions

Latent profile analysis assumes local independence. That is, the values of the indicators (i.e., the trial accuracies) within each profile should be independent or uncorrelated after profile membership is established. Prior to assigning profile membership, however, the indicator values across all participants should be dependent or correlated (Tein et al., 2013). Correlations between the trials were thus examined to test the assumption of local independence. As shown in Table F1, most of the PAEs were correlated with the exception of a few target fractions.

To determine if non-significant correlations would influence profile membership, we identified outlier PAE scores that could potentially influence correlations. Using a conservative approach, we first standardized all PAEs (approximately 3500) and any greater than the 3 standard deviations above or below the mean were identified as outliers PAE ($N = 48$). Next, we identified outliers based on individual PAEs ($N = 176$). For example, if a participant had a PAE trial that was 2 SD above their mean PAE (e.g., $M = 9.00$, $SD = 11.31$, $PAE_{\text{trial}} = 34.00$) that trial was considered an outlier score. Finally, we identified any score that was classified as an outlier using both approaches ($N = 9$). Model fit was unchanged by adding the outliers thus we concluded that the assumption of local independence was suitably met.

Table F1*Correlations amongst PAEs for all 27 target fractions on the fraction number line*

	1/3	1/4	1/5	1/6	1/7	1/8	1/9	2/3	2/5	2/7	2/9	3/4	3/5	3/7	3/8	4/5	4/7	4/9	5/6	5/7	5/8	5/9	6/7	7/8	7/9	8/9
1/2	.61	.23	.49	.47	.43	.46	.38	.69	.43	.36	.38	.64	.48	.25	.35	.39	.14	.22	.57	.26	.20	.22	.35	.31	.38	.28
1/3	-	.38	.54	.41	.34	.35	.32	.53	.45	.27	.43	.51	.50	.25	.35	.39	.22	.27	.56	.38	.31	.32	.45	.56	.50	.52
1/4		-	.26	.22	.20	.23	.13	.22	.30	.22	.30	.19	.25	.30	.28	.13	.02	.18	.21	.03	.04	.21	.17	.19	.09	.21
1/5			-	.65	.63	.74	.70	.30	.62	.58	.65	.25	.18	.25	.45	.34	.08	.53	.39	.23	.26	.34	.34	.38	.42	.39
1/6				-	.57	.68	.67	.34	.51	.59	.70	.28	.23	.27	.50	.34	.22	.43	.47	.23	.32	.47	.33	.38	.50	.37
1/7					-	.66	.58	.31	.36	.68	.74	.27	.09	.19	.52	.26	.12	.49	.35	.16	.27	.51	.35	.34	28	.30
1/8						-	.77	.44	.54	.68	.72	.29	.23	.27	.48	.29	.03	.54	.38	.21	.30	.47	.35	.29	.31	.21
1/9							-	.36	.50	.67	.69	.25	.13	.22	.45	.43	.06	.60	.33	.15	.22	.44	.40	.26	.39	.19
2/3								-	.46	.30	.34	.81	.61	.22	.27	.43	.17	.20	.57	.35	.22	.25	.55	.32	.28	.27
2/5									-	.44	.42	.40	.44	.20	.22	.32	.10	.30	.34	.28	.20	.19	.28	.27	.38	.30
2/7										-	.81	.23	.04	.42	.62	.21	.00	.60	.23	.02	.02	.37	.30	.13	.18	.13
2/9											-	.27	.09	.34	.62	.24	.02	.58	.34	.12	.20	.57	.35	.37	.35	.28
3/4												-	.52	.17	.25	.45	.23	.24	.65	.40	.24	.18	.49	.38	.28	.31
3/5													-	.17	.10	.41	.30	.00	.47	.32	.28	.18	.48	.31	.25	.27
3/7														-	.39	.11	.00	.39	.19	.04	.02	.26	.16	.06	.14	.02
3/8															-	.22	.14	.63	.32	.04	.04	.25	.15	.08	.10	.14
4/5																-	.15	.25	.38	.27	.25	.14	.46	.37	.29	.30
4/7																	-	.10	.11	.19	.26	.07	.13	.12	.10	.12
4/9																		-	.14	.02	.14	.39	.24	.07	.14	.11
5/6																			-	.57	.31	.28	.57	.53	.54	.55
5/7																				-	.36	.18	.43	.45	.39	.53
5/8																					-	.42	.29	.41	.33	.40
5/9																						-	.35	.40	.33	.42
6/7																							-	.52	.33	.49
7/8																								-	.64	.78
7/9																									-	.63

Note: Significant relations at $p \leq .05$ are **bolded**, negative correlations are *italicized*