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'THE LAW OF THE WALL IN TWO- AND THREE-DIMENSIONAL TURBULENT BOUNDARY LAYERS

By

Dale William Ladoon

B.Sc. (Mechanical Engineering)

**A thesis submitted to the Faculty of Graduate Studies
and Research in Partial fulfillment
of the requirements for the degree of**

**Master of Engineering in
Aerospace Engineering**

**THE OTTAWA-CARLETON INSTITUTE FOR MECHANICAL AND
AEROSPACE ENGINEERING**

Department of Mechanical and Aerospace Engineering

Carleton University

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*The Law of the Wall in Two- and Three-Dimensional
Turbulent Boundary Layers*

submitted by

D.W. Ladoon, B.Sc.

in partial fulfilment of the requirements
for the degree of Master of Engineering

Chairman, Department of Mechanical
and Aerospace Engineering

Thesis Supervisor

Carleton University

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ABSTRACT

The existence of law-of-the-wall similarity for the mean flow of two- and three-dimensional turbulent boundary layers was investigated. Expressions for the logarithmic portion of the near-wall region were derived for the streamwise and crossflow velocity components of a planar and three-dimensional boundary layer. Moreover, logarithmic laws, similar to those of Yowakim (1985) and Kind et al. (1988), were applied to two- and three-dimensional annular flows. Near-wall similarity was observed for both the planar and annular flows. However, similarity plots for the pressure-driven and shear-driven secondary flows indicate that there are distinct similarity differences between these types of flows. Furthermore, the concept of law-of-the-wall similarity in three-dimensional flows does not seem to be as robust as that of two-dimensional flows. Nevertheless, the logarithmic laws were found to predict both velocity components for a shear-driven flow. However, for the pressure-driven three-dimensional flows, only the streamwise log-law relation was seen to correlate the data. An explanation for the success of the logarithmic laws according to turbulence equilibrium concepts is provided.

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TABLE OF CONTENTS

	Page
ABSTRACT	iii
ACKNOWLEDGEMENTS	iv
LIST OF APPENDICES	viii
LIST OF TABLES	ix
LIST OF FIGURES	x
NOMENCLATURE	xvii
 CHAPTER 1 INTRODUCTION	 1
1.1 THESIS OUTLINE	3
 CHAPTER 2 LITERATURE REVIEW	 5
2.1 THE TWO-DIMENSIONAL LAW OF THE WALL	5
2.1.1 Terminology for the Law-of-the-Wall	8
2.2 GENERAL FEATURES OF THREE-DIMENSIONAL TURBULENT BOUNDARY LAYERS	8
2.2.1 Mean Velocity Profiles in Three-Dimensional Turbulent Boundary Layers	12
2.3 LAW-OF-THE-WALL SIMILARITY MODELS FOR THREE- DIMENSIONAL FLOWS	14
2.3.1 A Note Concerning the Law-of-the-Wall Velocity Scales U^{**} and W^{**}	23
 CHAPTER 3 MATHEMATICAL FORMULATION	 26
3.1 COORDINATE SYSTEMS	26
3.1.1 Cylindrical System	26
3.1.2 Cartesian System	29

3.2	GOVERNING EQUATIONS	29
3.2.1	Cylindrical Equations	31
3.2.2	Cartesian Equations	32
3.3	THE EDDY-VISCOSITY CONCEPT	34
3.4	THE PRANDTL MIXING-LENGTH DEFINITION OF EDDY VISCOSITY	35
3.4.1	The Anisotropic Nature of the Turbulent Shear Stress in Three-Dimensional Boundary Layers	36
3.5	THE TWO-DIMENSIONAL LOG LAW AND THE EDDY-VISCOSITY/MIXING-LENGTH HYPOTHESIS	38
3.6	THE EDDY-VISCOSITY/MIXING-LENGTH MODEL IN THREE-DIMENSIONAL BOUNDARY LAYERS	40
3.7	COMPONENT LOGARITHMIC LAWS FOR A CYLINDRICAL COORDINATE SYSTEM	44
3.7.1	A Logarithmic Law for the Axial Velocity Component	44
3.7.2	Logarithmic Laws for the Tangential Velocity Component	45
3.8	COMPONENT LOGARITHMIC LAWS FOR A CARTESIAN COORDINATE SYSTEM	48
3.9	LOGARITHMIC LAW FOR THE RESULTANT VELOCITY	49
CHAPTER 4	TURBULENT BOUNDARY LAYERS ON PLANAR SURFACES	54
4.1	LAW-OF-THE-WALL SIMILARITY FOR TWO-DIMENSIONAL FLOWS	54
4.2	THREE-DIMENSIONAL TURBULENT BOUNDARY LAYER DATA	65
4.2.1	Description of Experiments	66
4.3	LAW-OF-THE-WALL SIMILARITY PLOTS FOR THREE-DIMENSIONAL FLOWS	79
4.4	COMPONENT LOG-LAW EQUATIONS IN THREE-DIMENSIONAL FLOWS	86
4.4.1	Interpreting k as a Vector	91
4.5	DETERMINATION OF k , B_x , AND B_z	93

CHAPTER 5	ANNULAR FLOWS	101
5.1	DESCRIPTION OF EXPERIMENTS	101
5.2	LAW-OF-THE-WALL SIMILARITY FOR ANNULAR FLOWS	114
5.3	INFLUENCE OF CURVATURE AND ROTATION ON THE TANGENTIAL DIRECTION LOG-LAW EQUATION	141
CHAPTER 6	THE LOGARITHMIC LAWS AND THE TURBULENT TRANSPORT EQUATIONS	145
6.1	THE TURBULENT TRANSPORT EQUATIONS	145
6.2	STRUCTURAL SIMILARITY	147
6.3	MODELS FOR VISCOUS DISSIPATION	149
6.4	THE TWO-DIMENSIONAL LOGARITHMIC LAW AND THE TURBULENT KINETIC ENERGY TRANSPORT EQUATION	151
6.5	THE TWO-DIMENSIONAL LOGARITHMIC LAW AND THE $\overline{u^2}$ TRANSPORT EQUATION	156
6.6	THE NEAR-WALL EQUILIBRIUM CONDITION IN THREE- DIMENSIONAL BOUNDARY LAYERS	166
6.7	OTHER ISSUES CONCERNING THE LAW-OF-THE-WALL AND THE TURBULENT TRANSPORT EQUATIONS	169
6.8	$\overline{q} - \epsilon$ TURBULENCE MODELS	173
6.8.1	The Standard $\overline{q} - \epsilon$ Turbulence Model	174
6.8.2	A Two-Layer Low-Reynolds-Number Turbulence Model	177
6.8.3	A Near-Wall Turbulence Model for Three-Dimensional Turbulent Boundary Layers	180
CHAPTER 7	SUMMARY, CONCLUSIONS, AND RECOMMENDATIONS	183
7.1	SUMMARY	183
7.2	CONCLUSIONS	185
7.3	RECOMMENDATIONS	188
REFERENCES		191

APPENDICES

	Page
APPENDIX A VISCOUS SUBLAYER EQUATIONS	195
APPENDIX B UNCERTAINTY ANALYSIS	200
B.1 SAMPLE UNCERTAINTY CALCULATION FOR DATA STATION S45 0 6 1	205
APPENDIX C MODIFIED CLAUSER PLOTS TO DETERMINE THE SKIN FRICTION COEFFICIENTS C_f , C_{fx} , AND $C_{f\theta}$	209
C.1 MODIFIED CLAUSER PLOTS FOR TWO-DIMENSIONAL TURBULENT BOUNDARY LAYERS	209
C.2 MODIFIED CLAUSER PLOTS FOR THE COMPONENTS OF VELOCITY	212
APPENDIX D PROGRAM CODE FOR THE CALCULATION OF COMPONENT MODIFIED CLAUSER PLOTS	220
APPENDIX E DERIVATION OF THE SECOND INVARIANT, I_2	231

LIST OF TABLES

Table		Page
2.1	List of the Equivalent Velocities and Near-Wall Velocity Scales Used in the Six Simpler Similarity Models.	18
4.1	Calculated Skin Friction Values for a Sample of Two-Dimensional Flows.	59
5.1	Brief Summary of Annular Flow Experiments Indicating Some of the Measurements Taken.	104
5.2	Calculated Data for the Experiment of Yowakim (1985) $SN = 0.87$ Convex Wall.	121
5.3	Calculated Data for the Experiment of Yowakim (1985) $SN = 0.87$ Concave Wall.	122
5.4	Calculated Data for the Experiment of Lohmann (1976).	123
5.5	Calculated Data for the Experiment of Bissonnette and Mellor (1974).	124
B.1	A Partial List of Measurement Uncertainties from McAllister (1979).	203
B.2	A Partial List of the Measured Data at Station S45 0 6 1.	206
B.3	Calculated Values for the Data Station S45 0 6 1 at the Position $y = 0.81 \text{ mm}$.	207

LIST OF FIGURES

Figure		Page
2.1	Terminology for the Near-Wall Region of a 2D Turbulent Boundary Layer.	9
2.2	Isometric Plot of the Velocity Profile for a Typical Three-Dimensional Boundary Layer.	11
2.3	Velocity Component Representation of a Unidirectional Three-Dimensional Boundary Layer.	13
2.4	Velocity Component Representation of a Bidirectional Three-Dimensional Boundary Layer.	13
2.5	Typical Hodograph Plot for a Unidirectional Three-Dimensional Turbulent Boundary Layer.	15
2.6	Typical Hodograph used to Determine the Equivalent Velocity and the Law-of-the-Wall Velocity Scale, V_{eq} and V_{eq}^* , for the 3D Similarity Model of Johnston (1960).	24
3.1	Coordinate System for Flows Along the Axis of A Cylinder.	27
3.2	Coordinate System for Annular Flows.	27
3.3	Cartesian Coordinate System for Planar Flows.	31
4.1	Modified Clauser Plot for the Determination of the Skin Friction Coefficient, C_f .	57
4.2	Modified Clauser Plot for the Determination of the Skin Friction Coefficient, C_f .	58
4.3	Semi-Logarithmic Plot for the Sample of Two-Dimensional Velocity Profiles.	60
4.4	Nomenclature for a Coordinate System Rotation in the X-Z Plane.	61
4.5	Effect of Coordinate System Rotation on the X-Direction Velocity Profiles for the Streamwise Station $x = 6.917$ ft.: Experiment of Bradshaw and Ferriss (1965).	63
4.6	Effect of Coordinate System Rotation on the Z-Direction Velocity Profiles for the Streamwise Station $x = 6.917$ ft.: Experiment of Bradshaw and Ferriss (1965).	63
4.7	Effect of Coordinate System Rotation on the X-Direction Velocity Profiles Using the Law-of-the-Wall Length Scale of Yowakim (1985) and Kind et al. (1989).	64
4.8	Effect of Coordinate System Rotation on the Z-Direction Velocity Profiles Using the Law-of-the-Wall Length Scale of Yowakim (1985) and Kind et al. (1989).	64

4.9	Wind Tunnel Configuration Used by McAllister (1979).	67
4.10	Teardrop Body Used to Create the Pressure Driven Flow.	68
4.11	Swept Step Used to Create the Shear Driven Flows, $\alpha_0 = 45^\circ$ and 30° .	68
4.12	Pressure Gradient Map for One Quadrant of the Teardrop Body.	69
4.13	Wall Shear and Velocity Vector Map for the Teardrop Body.	69
4.14	Pressure Gradient Map Behind the 45° Step.	70
4.15	Wall Shear and Velocity Vector Map Behind the 45° Step.	70
4.16	Pressure Gradient Map Behind the 30° Step.	71
4.17	Wall Shear and Velocity Vector Map Behind the 30° Step.	71
4.18	Typical Streamwise Velocity Profiles for the Teardrop Flow of McAllister (1979).	73
4.19	Typical Crossflow Velocity Profiles for the Teardrop Flow of McAllister (1979).	73
4.20	Distribution of Flow Angle, ϕ , for a Few Data Stations. Teardrop Flow of McAllister (1979).	74
4.21	Typical Streamwise Velocity Profiles for the 45° Step Flow of McAllister (1979).	75
4.22	Typical Crossflow Velocity Profiles for the 45° Step Flow of McAllister (1979).	75
4.23	Distribution of Flow Angle, ϕ , for a Few Data Stations. 45° Step Flow of McAllister (1979).	76
4.24	Typical Streamwise Velocity Profiles for the 30° Step Flow of McAllister (1979).	77
4.25	Typical Crossflow Velocity Profiles for the 30° Step Flow of McAllister (1979).	77
4.26	Distribution of Flow Angle, ϕ , for a Few Data Stations. 30° Step Flow of McAllister (1979).	78
4.27	Worst Near-Wall Similarity Agreement for the Streamwise and Crossflow Velocity Profiles: Teardrop Flow Experiment of McAllister (1979).	80
4.28	Typical Good Near-Wall Similarity Agreement for the Streamwise and Crossflow Velocity Profiles: Teardrop Flow Experiment of McAllister (1979).	80
4.29	Worst Near-Wall Similarity Agreement for the Streamwise and Crossflow Velocity Profiles: 45° Step Flow Experiment of McAllister (1979).	81
4.30	Typical Good Near-Wall Similarity Agreement for the Streamwise and Crossflow Velocity Profiles: 45° Step Flow Experiment of McAllister (1979).	81
4.31	Worst Near-Wall Similarity Agreement for the Streamwise and Crossflow Velocity Profiles: 30° Step Flow Experiment of McAllister (1979).	82

4.32	Typical Good Near-Wall Similarity Agreement for the Streamwise and Crossflow Velocity Profiles: 30° Step Flow Experiment of McAllister (1979).	82
4.33	Typical Uncertainty in the Near-Wall Similarity Variables: Teardrop Flow of McAllister (1979).	84
4.34	Typical Uncertainty in the Near-Wall Similarity Variables: 45° Step Flow of McAllister (1979).	84
4.35	Typical Uncertainty in the Near-Wall Similarity Variables: 30° Step Flow of McAllister (1979).	85
4.36	Semi-Logarithmic Plot of the Streamwise and Crossflow Velocity Profiles for the 45° Step Flow Experiment of McAllister (1979).	87
4.37	Semi-Logarithmic Plot of the Streamwise and Crossflow Velocity Profiles for the 30° Step Flow Experiment of McAllister (1979).	87
4.38	Comparison of the Logarithmic-Law Lines Using the Standard Two-Dimensional Constants With Those Determined from the Step Flows of McAllister (1979).	88
4.39	Calculated Best Fit Line for the Data Station S45 0 14 2.	89
4.40	Effect of an X-Z Coordinate System Rotation on the Streamwise Velocity Profiles for the Data Station S30 0 14 2: Experiment of McAllister (1979).	94
4.41	Effect of an X-Z Coordinate System Rotation on the Crossflow Velocity Profiles for the Data Station S30 0 14 2: Experiment of McAllister (1979).	94
4.42	Plot to Correlate the Scalar Slope, k , of the Logarithmic-Law Line Based on the Scalar Variable $\beta_w - \bar{\phi}$.	95
4.43	Plot to Correlate the Scalar Slope, k , of the Logarithmic-Law Line Based on the Scalar Variable $2I_{20}/I_{10}^2$.	95
4.44	Plot to Correlate the Intercept, B_x , of the Streamwise Logarithmic-Law Line.	96
4.45	Plot to Correlate the Intercept, B_z , of the Crossflow Logarithmic-Law Line.	96
4.46	Hypothetical Plot to Determine the Slope, k , of the Logarithmic-Law for the Streamwise Direction.	100
4.47	Hypothetical Plot to Determine the Slope, k , of the Logarithmic-Law for the Crossflow Direction.	100
5.1	Schematic Figure of the Experimental Arrangement of Yowakim (1985).	102
5.2	Schematic Figure of the Experimental Apparatus of Lohmann (1976).	102

5.3	Schematic Figure of the Experimental Apparatus of Bissonnette and Mellor (1976).	103
5.4	Radial Distribution of the Axial Velocity for $SN = 0.87$: Experimental of Yowakim (1985).	106
5.5	Radial Distribution of the Tangential Velocity for $SN = 0.87$: Experimental of Yowakim (1985).	106
5.6	Radial Distribution of Flow Angle, ϕ , at Different Axial Stations for $SN = 0.87$: Experiment of Yowakim (1985).	107
5.7	Radial Distribution of Flow Angle, ϕ , at Different Axial Stations for $SN = 0.87$: Experiment of Yowakim (1985).	107
5.8	Experimental Axial Velocity Profiles from Lohmann (1976).	110
5.9	Experimental Tangential Velocity Profiles from Lohmann (1976).	110
5.10	Distribution of the Flow Angle, ϕ , at Different Axial Stations for the Experiment of Lohmann (1976).	111
5.11	Distribution of the Flow Angle, ϕ , at Different Axial Stations for the Experiment of Lohmann (1976).	111
5.12	Experimental Axial Velocity Profiles from Bissonnette and Mellor (1974).	112
5.13	Experimental Tangential Velocity Profiles from Bissonnette and Mellor (1974).	112
5.14	Distribution of the Flow Angle, ϕ , at Different Axial Stations for the Experiment of Bissonnette and Mellor (1974).	113
5.15	Distribution of the Flow Angle, ϕ , at Different Axial Stations for the Experiment of Bissonnette and Mellor (1974).	113
5.16	Modified Clauser Plot for Determining the Skin Friction Coefficient C_{fx} at the Axial Station $x = 22.0$ in. for $SN = 0.87$: Experiment of Yowakim (1985).	117
5.17	Modified Clauser Plot for Determining the Skin Friction Coefficient $C_{f\theta}$ at the Axial Station $x = 22.0$ in. for $SN = 0.87$: Experiment of Yowakim (1985).	117
5.18	Modified Clauser Plot for Determining the Skin Friction Coefficient C_{fx} at the Axial Station $x = 10.0$ in.: Experiment of Lohmann (1976).	118
5.19	Modified Clauser Plot for Determining the Skin Friction Coefficient $C_{f\theta}$ at the Axial Station $x = 10.0$ in.: Experiment of Lohmann (1976).	118
5.20	Modified Clauser Plot for Determining the Skin Friction Coefficient C_{fx} at the Axial Station $x = 4.0$ in.: Experiment of Bissonnette and Mellor (1974).	119

5.21	Modified Clauser Plot for Determining the Skin Friction Coefficient C_{f0} at the Axial Station $x = 4.0$ in.: Experiment of Bissonnette and Mellor (1974).	119
5.22	Modified Clauser Plot for Determining the Skin Friction Coefficient C_{f0} at the Axial Station $x = 1.0$ in.: Experiment of Lohmann (1976).	120
5.23	Modified Clauser Plot for Determining the Skin Friction Coefficient C_{f0} at the Axial Station $x = 1.0$ in.: Experiment of Bissonnette and Mellor (1974).	120
5.24	Semi-Logarithmic Plot of the Convex Wall Axial Velocity Profiles for $SN = 0.87$: Experiment of Yowakim (1985).	126
5.25	Semi-Logarithmic Plot of the Convex Wall Tangential Velocity Profiles for $SN = 0.87$: Experiment of Yowakim (1985).	126
5.26	Semi-Logarithmic Plot of the Concave Wall Axial Velocity Profiles for $SN = 0.87$: Experiment of Yowakim (1985).	127
5.27	Semi-Logarithmic Plot of the Concave Wall Tangential Velocity Profiles for $SN = 0.87$: Experiment of Yowakim (1985).	127
5.28	Semi-Logarithmic Plot of the Convex Wall Tangential Velocity Profiles at the Station $x = 10.0$ in. for $SN = 0.23$: Experiment of Yowakim (1985).	128
5.29	Semi-Logarithmic Plot of the Convex Wall Tangential Velocity Profiles at the Station $x = 10.0$ in. for $SN = 0.87$: Experiment of Yowakim (1985).	128
5.30	Semi-Logarithmic Plot of the Axial Velocity Profiles for the Experiment of Lohmann (1976).	130
5.31	Semi-Logarithmic Plot of the Tangential Velocity Profiles for the Experiment of Lohmann (1976).	130
5.32	Semi-Logarithmic Plot of the Axial Velocity Profiles for the Experiment of Bissonnette and Mellor (1974).	131
5.33	Semi-Logarithmic Plot of the Tangential Velocity Profiles for the Experiment of Bissonnette and Mellor (1974).	131
5.34	Semi-Logarithmic Plot of the Axial Velocity Profiles for the Experiment of Bissonnette and Mellor (1974).	132
5.35	Semi-Logarithmic Plot of the Tangential Velocity Profiles for the Experiment of Bissonnette and Mellor (1974).	132, 170
5.36	Distribution of Axial Turbulent Shear Stress at Different Axial Stations: Experiment of Yowakim (1985) $SN = 0.87$ Convex Wall.	135

5.37	Distribution of Tangential Turbulent Shear Stress at Different Axial Stations: Experiment of Yowakim (1985) $SN = 0.87$ Convex Wall.	135
5.38	Distribution of Axial Turbulent Shear Stress at Different Axial Stations: Experiment of Yowakim (1985) $SN = 0.87$ Concave Wall.	136
5.39	Distribution of Tangential Turbulent Shear Stress at Different Axial Stations: Experiment of Yowakim (1985) $SN = 0.87$ Concave Wall.	136
5.40	Distribution of Axial Turbulent Shear Stress at Different Axial Stations: Experiment of Lohmann (1976).	137
5.41	Distribution of Tangential Turbulent Shear Stress at Different Axial Stations: Experiment of Lohmann (1976).	137
5.42	Distribution of Axial Turbulent Shear Stress at Different Axial Stations: Experiment of Bissonnette and Mellor (1974).	138
5.43	Distribution of Tangential Turbulent Shear Stress at Different Axial Stations: Experiment of Bissonnette and Mellor (1974).	138
5.44	Comparison of the Tangential Log-Law Line, Equation (3.65), and the Planar Log-Law Line, Equation (3.67), for a Convex Wall.	144
5.45	Comparison of the Tangential Log-Law Line, Equation (3.65), and the Planar Log-Law Line, Equation (3.65), for a Concave Wall.	144
6.1	Turbulent Kinetic Energy Balance in a Two-dimensional Boundary Layer Along a Smooth Wall With Zero Pressure Gradient.	153
6.2	Terms in the Budget of $\overline{u_1 u_1}$ in Wall Coordinates.	161
6.3	Split of the Velocity Pressure-Gradient Term, Π_{11} , into a Pressure Transport Transport Term, $(\overline{u_1 u_1} / 2\bar{q}) \Pi_{11}$, and a Pressure-Strain Term, $\Pi_{11} - (\overline{u_1 u_1} / 2\bar{q}) \Pi_{11}$.	161
6.4	Terms in the Budget of Turbulent Kinetic Energy in Wall Coordinates.	162
6.5	Semi-Logarithmic Plot of the Tangential Velocity Profiles for the Last Three Data Stations of Bissonnette and Mellor (1974).	170
C.1	Traditional Clauser Plot for Determining C_f .	211
C.2	Modified Clauser Plot for Determining C_f .	211
C.3	Modified Clauser Plot for Determining the Skin Friction Coefficient $C_{f\tau}$ at the Axial Station $x = 46.0$ in. for $SN = 0.87$: Experiment of Yowakim (1985).	215

C.4	Modified Clauser Plot for Determining the Skin Friction Coefficient $C_{f\theta}$ at the Axial Station $x = 46.0$ in. for $SN = 0.87$: Experiment of Yowakim (1985).	215
C.5	Modified Clauser Plot for Determining the Skin Friction Coefficient C_{fx} at the Axial Station $x = 8.0$ in.: Experiment of Lohmann (1976).	216
C.6	Modified Clauser Plot for Determining the Skin Friction Coefficient $C_{f\theta}$ at the Axial Station $x = 8.0$ in.: Experiment of Lohmann (1976).	216
C.7	Modified Clauser Plot for Determining the Skin Friction Coefficient C_{fx} at the Axial Station $x = 8.0$ in.: Experiment of Bissonnette and Mellor (1974).	217
C.8	Modified Clauser Plot for Determining the Skin Friction Coefficient $C_{f\theta}$ at the Axial Station $x = 8.0$ in.: Experiment of Bissonnette and Mellor (1974).	217
C.9	Modified Clauser Plot for Determining the Streamwise Direction Skin Friction Coefficient, C_{fx}, at the Data Station S30 0 14 2: Experiment of McAllister (1979).	219
C.10	Modified Clauser Plot for Determining the Crossflow Direction Skin Friction Coefficient, C_{fx}, at the Data Station S30 0 14 2: Experiment of McAllister (1979).	219

NOMENCLATURE

A, A_1, A_2	= constants
a, a_1, a_3	= constants
b_{11}, b_{22}, b_{33}	= ratio of dissipation components, ϵ_{11} , ϵ_{22} , and ϵ_{33} , to TKE dissipation, ϵ , respectively
B	= logarithmic law intercept
C, C', C'_1, C'_2	= constants
C_i	= influence coefficient, Eqn. (B.3)
C_1, C_2	= constants
C_f	= $\frac{2\tau_w}{\rho U_s^2}$, skin friction coefficient
C_{fx}	= $\frac{2\tau_{x,w}}{\rho Q_{\max}^2}$ or $\frac{2\tau_{x,w}}{\rho Q_s^2}$, axial (streamwise) direction skin friction coefficient
$C_{f\theta}$	= $\frac{2\tau_{\theta,w}}{\rho Q_{\max}^2}$, tangential direction skin friction coefficient
C_{fz}	= $\frac{2\tau_{z,w}}{\rho Q_s^2}$, crossflow direction skin friction coefficient
C_u	= $\sqrt{\frac{C_{fx}}{2}} \sqrt{\frac{\tau_{x,w}}{\tau_w}}$, axial direction skin friction parameter
C_w	= $\sqrt{\frac{C_{f\theta}}{2}} \sqrt{\frac{\tau_{\theta,w}}{\tau_w}}$, tangential direction skin friction parameter
D	= cylinder diameter
D_h	= $2(r_o - r_i)$, hydraulic diameter
e_{ij}	= rate of strain tensor
f	= functional relationship

g	= functional relationship
I_1	= first invariant of a second order tensor, Eqn. (4.25)
I_2	= second invariant of a second order tensor, Eqn. (4.26) and Eqn. (E.2)
K_{ij}	= dimensionless Reynolds stress tensor, Eqn. (6.24)
k	= Von Karman constant
l	= mixing length
L_e	= dissipation length
L_o	= integral length scale of turbulence
l_x	= axial mixing length, Eqn. (3.40)
$l_{\theta r}$	= tangential mixing length, Eqn. (3.41)
l_{xy}	= streamwise mixing length, Eqn. (3.52)
l_{zy}	= crossflow mixing length, Eqn. (3.53)
P_{ij}	= Production tensor, Eqn. (6.4)
P	= static pressure
Q	= $\sqrt{U^2 + W^2}$, resultant velocity
$\bar{q}$	= $\frac{\overline{u^2} + \overline{v^2} + \overline{w^2}}{2}$, turbulent kinetic energy
Q^*	= $\sqrt{\frac{\tau_w}{\rho}}$, resultant shear velocity
r_{eff}	= effective radius of curvature of streamlines at a cylinder wall
r_i, r_o	= inner and outer radius of an annulus, respectively
r_w	= wall radius
r^*	= $\frac{(r - r_i)}{(r_o - r_i)}$, non-dimensional radial coordinate
r^+	= $\frac{r_w Q^*}{\nu}$, wall curvature parameter
Re	= Reynolds number

Re_{Dh}	$= \frac{D_h U_{ref}}{\nu}$, Reynolds number based on hydraulic diameter and reference velocity
Re_{Q_e}	$= \frac{y Q_{max}}{\nu}$, Reynolds number based on y and the maximum resultant velocity
Re_y	$= \frac{y U_e}{\nu}$, Reynolds number based on y and U_e
R_T	= turbulent Reynolds number, Eqn. (6.90)
S	= dependent variable, Eqn. (B.1)
SN	= Swirl number, Eqn. (5.3)
T	= static temperature
t_i	= independent variable, Eqn. (B.1)
U, V, W	= time-mean velocity components in the x, r, θ and x, y, z directions, respectively
u, v, w	= fluctuating velocity components in the x, r, θ and x, y, z directions, respectively
U_e	= streamwise velocity at the boundary layer edge
$-\rho \overline{u_i u_j}$	= Reynolds stress tensor, Eqn. (6.2)
U_o	= velocity at inlet to test section
U^+	$= \frac{U}{U_\tau}$ or $\frac{U}{U^{**}}$, law-of-the-wall velocity variable
U_τ	$= \sqrt{\frac{\tau_w}{\rho}}$, two-dimensional shear velocity scale
U^*	$= \sqrt{\frac{\tau_{x,y}}{\rho}}$, shear velocity scale
U^{**}	$= \sqrt{\frac{\tau_{x,y}}{\rho}} \sqrt{\frac{\tau_{x,y}}{\tau_w}}$, axial (streamwise) shear velocity scale
W^+	$= \frac{W}{W^{**}}$, tangential (crossflow) law-of-the-wall velocity variable

W^*	$= \sqrt{\frac{\tau_{\theta,w}}{\rho}}$, shear velocity scale
W^{**}	$= \sqrt{\frac{\tau_{\theta,w}}{\rho}} \sqrt{\frac{\tau_{\theta,w}}{\tau_w}}$, tangential (crossflow) shear velocity scale
W_w	$= \omega r_w$, wall velocity of a cylinder rotating about its axis
x, r, θ	$=$ axial, radial, and tangential coordinate directions, respectively
x, y, z	$=$ streamwise, normal, and crossflow coordinate directions, respectively
y	$=$ distance normal to the wall
y^+	$= \frac{y U_\tau}{\nu}$ or $\frac{y Q^*}{\nu}$, law-of-the-wall coordinate variable

GREEK SYMBOLS

β_c	$= \frac{\delta^*}{\tau_w} \frac{dP}{dx}$, Clauser equilibrium parameter
β_w	$= \tan^{-1} \left(\frac{\tau_{\theta,w}}{\tau_{x,w}} \right)$, wall shear stress angle
δ	$=$ boundary layer thickness
δ^*	$= \int_0^\delta (1 - U/U_\infty) dy$, displacement thickness
δ_i^*	$=$ displacement thickness tensor, Eqn. (4.27)
δ_{ij}	$=$ Kronecker delta, $\begin{matrix} 1, & i = j \\ 0, & i \neq j \end{matrix}$
ϵ	$=$ dissipation of turbulent kinetic energy, Eqn. (6.7)
ϵ_{11}	$=$ rate of dissipation of the Reynolds stress, $\overline{u^2}$, Eqn. (6.13)
ϵ_{22}	$=$ rate of dissipation of the Reynolds stress, $\overline{v^2}$, Eqn. (6.14)
ϵ_{33}	$=$ rate of dissipation of the Reynolds stress, $\overline{w^2}$, Eqn. (6.15)

ζ	= streamwise direction at boundary layer edge
θ_{ij}	= momentum thickness tensor, Eqn. (4.24)
μ	= dynamic viscosity
$\mu_{xr}, \mu_{\theta r}$	= eddy viscosities in the axial, x , and tangential, θ , directions, respectively
μ_{xy}, μ_{zy}	= eddy viscosities in the streamwise, x , and crossflow, z , directions, respectively
ν	= kinematic viscosity
ρ	= density
σ	= uncertainty, Eqn. (B.2)
σ_1, σ_2	= constants
τ	= shear stress
$\tau_{xr}, \tau_{\theta r}$	= shear stress in the x and θ directions, respectively
τ_{xy}, τ_{zy}	= shear stress in the x and z directions, respectively
$\tau_{x,w}$	= axial (streamwise) wall shear stress component
$\tau_{\theta,w}$	= tangential wall shear stress component
$\tau_{z,w}$	= crossflow wall shear stress component
τ_w	= $\sqrt{\tau_{x,w}^2 + \tau_{z,w}^2}$, resultant wall shear stress
Φ_{ij}	= Pressure-strain tensor, Eqn. (6.4)
Φ_{ij1}, Φ_{ij2}	= fast and slow components of pressure-strain, respectively, Eqn. (6.40)
ϕ	= $\tan^{-1}\left(\frac{W}{U}\right)$, flow angle
ψ	= coordinate system rotation angle in the X - Z plane
Ω	= $\frac{U_o}{W_w}$, non-dimensional rotation number
ω	= angular velocity

SUBSCRIPTS

<i>e</i>	= boundary layer edge
<i>i</i>	= inner wall
<i>i, j, k</i>	= tensor indices (<i>x, r, θ</i>) or (<i>x, y, z</i>)
<i>min, max</i>	= minimum and maximum values, respectively
<i>o</i>	= outer wall
<i>xr</i>	= axial component
<i>θr</i>	= tangential component
<i>w</i>	= wall value
<i>xy</i>	= streamwise component
<i>zy</i>	= crossflow component

SUPERSCRIPTS

<i>m</i>	= molecular
<i>t</i>	= turbulent

ABBREVIATIONS

BM	= Bissonnette and Mellor
2D	= two-dimensional
3D	= three-dimensional
2DBL	= two-dimensional boundary layer
2DTBL	= two-dimensional turbulent boundary layer
3DBL	= three-dimensional boundary layer
3DTBL	= three-dimensional turbulent boundary layer
NS	= Navier-Stokes
RMS	= root mean square
TKE	= turbulent kinetic energy
LRN	= low-Reynolds-number
HRN	= high-Reynolds-number

to my family

CHAPTER 1

INTRODUCTION

In the present investigation the expression "near-wall similarity" refers to the "similitude" of near-wall velocity profiles when plotted in an appropriate non-dimensionalized coordinate system. In particular, "law-of-the-wall similarity", or simply the "law of the wall", refers to a distinct class of near-wall similarity in which the non-dimensionalizing variables and scales are based on the wall shear stress, τ_w . The existence of the law of the wall for planar two-dimensional turbulent boundary layers in the form

$$\frac{U}{U_\tau} = f\left(\frac{yU_\tau}{\nu}\right), \quad (1.1a)$$

or

$$U^+ = f(y^+), \quad (1.1b)$$

is well established (eg. Coles and Hirst, 1968) for a wide range of flow conditions. Note that the two-dimensional non-dimensionalizing shear velocity scale, U_τ , is defined as

$$U_\tau = \sqrt{\frac{\tau_w}{\rho}}, \quad (1.2)$$

where ρ is the density of the fluid. Furthermore, the existence of the law-of-the-wall region, most of which is occupied by the logarithmic law (log law)

$$\frac{U}{U_\tau} = \frac{1}{k} \ln\left(\frac{yU_\tau}{\nu}\right) + B, \quad (1.3)$$

has been one of the most useful observations concerning the near-wall behavior of two-dimensional flows.¹

¹ k and B are empirical constants.

The utility of the law of the wall in either two- or three-dimensional flows is essentially threefold. Firstly, near-wall similarity is a very useful concept for the prediction of turbulent flows. Specifically, near-wall similarity can allow a significant reduction in computer storage and computational time in the solution of turbulent flows by eliminating the need for a fine computational grid near solid boundaries. For example, the two-dimensional law of the wall, Equation (1.1), presently forms the basis for wall functions in turbulence models such as the $k - \epsilon$ model.^{2,3} In addition, the law of the wall can be used to generate good initial velocity profiles for starting turbulent boundary layer predictions. Secondly, the law of the wall forms the basis for indirect techniques of measuring the wall shear stress. In particular, the Clauser plot and the Preston tube are two commonly used means for determining the wall shear stress in a two-dimensional turbulent boundary layer. Both of these methods rely on the existence of the law of the wall, Equation (1.1). Thirdly, near-wall similarity identifies the appropriate velocity and length scales for non-dimensionalizing near-wall data of all sorts, not just mean velocity data.

Attempts to identify a three-dimensional law of the wall similar to that of Equation (1.1) have only achieved limited success (Pierce et al. 1983a, and 1983b; Olcman and Simpson, 1992). However, Yowakim (1985) and more recently Kind et al. (1989) presented log-law equations for the axial and tangential velocity components, U and W , of a swirling annular flow. These equations achieved good agreement with the experiments of Yowakim (1985). A notable result of their work was the generation of component shear-velocity scales. The studies of Yowakim and Kind et al. form the starting point for the present work. In fact the primary objective of the current investigation is to test the concept of three-dimensional law-of-the-wall similarity based on component near-wall variables and log-law equations similar to those of Yowakim (1985) and Kind et al. (1989).

² The symbol k is often used in the literature to represent the turbulent kinetic energy, $k = (\overline{u^2} + \overline{v^2} + \overline{w^2}) / 2$, where u , v , and w are the fluctuating components of velocity. Furthermore, the term $k - \epsilon$ is a standard expression in the literature. However, in the present text the symbols k and $\bar{q}$ are used to represent the slope of the log-law equation, $1/k$, and the turbulent kinetic energy, respectively.

³ ϵ represents the dissipation of turbulent kinetic energy.

1.1 THESIS OUTLINE

The contents of this thesis are as follows:

Chapter 2 begins with a brief discussion of the two-dimensional form of the law of the wall and the log law, Equations (1.1) and (1.3). Following this review, some of the general features of three-dimensional turbulent boundary layers are presented. The chapter concludes with a short summary of the three-dimensional law-of-the-wall similarity models currently available in the literature.

Logarithmic laws similar to that of Equation (1.3) for use with the velocity components, U and W , of a three-dimensional boundary layer are developed in Chapter 3. Separate sets of equations are derived for the Cartesian coordinate system and the cylindrical coordinate system. These equations are applied to experimental data in Chapters 4 and 5.

Law-of-the-wall similarity plots for the experimental three-dimensional flows of McAllister (1979) are presented in Chapter 4. The appealing feature of these experiments is that the wall shear stress vector, τ_w , was measured directly by a floating element device. This is an important point if one recognizes that the use of a Clauser plot or a Preston tube to determine the wall shear stress in a three-dimensional boundary layer is somewhat problematic. Both of these techniques assume *a priori* that the two-dimensional law-of-the-wall, Equation (1.1), also applies to three-dimensional flows. Clearly, in order to avoid any such bias a direct measure of the wall shear stress is preferable. As a result, a number of important conclusions are drawn from the analysis of the McAllister data.

Comparisons with three annular flow test cases are given in Chapter 5. Firstly, the data of Yowakim (1985) is reviewed since the present study derives log-law equations which are slightly different from those of Yowakim (1985) and Kind et al. (1989). Secondly, the experimental flows of Lohmann (1976) and Bissonnette & Mellor (1974) are examined. In each of these cases a Modified Clauser plot technique is used to determine the wall shear stress vector.

In Chapter 6 a potential connection between the component log-law equations, derived in Chapter 3, and the transport equations for the Reynolds stresses $\overline{u^2}$ and $\overline{w^2}$ is considered.

Lastly, conclusions and recommendations based on the work presented in this study are presented in Chapter 7.

CHAPTER 2

LITERATURE REVIEW

2.1 THE TWO-DIMENSIONAL LAW-OF-THE-WALL

As mentioned in Chapter 1, law-of-the-wall similarity is a widely accepted experimental observation for two-dimensional turbulent boundary layers. A brief discussion of the two-dimensional case follows.

For predictive purposes, two-dimensional turbulent boundary layers are often regarded as consisting of two distinct regions, or layers: an inner, near-wall, layer and an outer layer. The inner layer is largely governed by local conditions and occupies only a small fraction of the boundary layer thickness (i.e. $y/\delta \approx 0.1$). In contrast, the outer layer is characterized by intermittency and history effects from the upstream flow. The distribution of the mean velocity in the near-wall region is considered next.

According to White (1991), Prandtl was the first to suggest that the mean velocity in the near-wall region is governed by local conditions, or variables, as

$$U = f(\tau_w, \rho, \mu, y) \quad (2.1)$$

where f represents a functional relationship, U is the mean velocity, τ_w is the wall shear stress, ρ is the density of the fluid, μ is the dynamic viscosity of the fluid, and y is the distance normal to the wall. From dimensional analysis, Equation (2.1) may be expressed as

$$\frac{U}{U_\tau} = f_1\left(\frac{yU_\tau}{\nu}\right), \quad (1.1a)$$

or

$$U^+ = f_1(y^+), \quad (1.1b)$$

where the shear velocity scale U_τ is defined as

$$U_\tau = \sqrt{\frac{\tau_w}{\rho}} \quad (1.2)$$

and the non-dimensional velocity and distance are

$$U^+ = \frac{U}{U_\tau} \quad (2.2)$$

and

$$y^+ = \frac{y U_\tau}{\nu} \quad (2.3)$$

respectively. Equation (1.1) is known as the law of the wall. This relation simply states that the near-wall mean velocity distributions for every turbulent boundary layer will collapse onto a single curve when plotted in the appropriate law-of-the-wall variables U^+ and y^+ . The existence of the law-of-the-wall for two-dimensional turbulent boundary layers (2DTBLs) is supported by experiment (Coles and Hirst, 1968). As well, a detailed and specific discussion of the range of validity of the law of the wall is given by Galbraith et al. (1977).

The functional relationship for the two-dimensional law of the wall is largely defined by two equations. The first equation is derived by recognizing that very close to the wall turbulent fluctuations are dampened almost completely such that only viscous shear is important. Accordingly, the shear stress is then given by the laminar expression

$$\tau = \mu \frac{\partial U}{\partial y} \quad (2.4)$$

The region of validity of this equation is extremely small. Consequently, the local shear stress is often assumed to be equal to the wall shear stress:

$$\tau = \tau_w = \mu \frac{\partial U}{\partial y} \quad (2.5)$$

Equation (2.5) can be non-dimensionalized by the law-of-the-wall shear velocity scale, U_τ , and integrated to produce the viscous, or linear, sublayer equation:

$$U^+ = y^+ . \quad (2.6)$$

This expression is valid for y^+ less than about 5 with this upper limit not rigidly set in the literature. The second equation, the logarithmic-law equation (log-law)

$$\frac{U}{U_\tau} = \frac{1}{k} \ln \left(\frac{y U_\tau}{\nu} \right) + B , \quad (1.3)$$

applies to the fully turbulent portion of the near-wall flow and occupies the majority of the law-of-the-wall region. The value of the empirical constants, k and B , vary with certain variables such as surface roughness and their precise values for a flow over a smooth plate are not fully agreed upon. However, the smooth plate values are typically accepted to be in the range $0.40 < k < 0.42$ and $5.0 < B < 6.0$. The reader is referred to McAllister (1979) for a listing of the values reported by various authors. For zero to moderate streamwise pressure gradients, dP/dx , the range of validity of the log-law is typically $50 < y^+ < 300$. The region between the viscous sublayer and the log-law region, in which laminar and turbulent stresses are important, is known as the buffer layer. A law-of-the-wall formula for predicting the mean velocity distribution in the buffer layer is given in Galbraith et al. (1977). One must also acknowledge that other formulas are available for predicting the near-wall mean velocity distribution (see, for instance, White, 1991). However, the present text is largely concerned with the log-law relation, Equation (1.3).

Equation (1.3) is the log-law in its simplest form and other authors have proposed modified forms to account for other factors such as the variation of the shear stress throughout the near-wall region (see, for instance, Patel, 1973). However, Galbraith & Head (1975) and Galbraith et al. (1977) have shown that Equation (1.3) is valid for all but the most strongly adverse pressure gradient flows. This point is discussed further in Chapter 3 where the log-law is derived according to the mixing-length model of Galbraith et al. (1977). In addition, a possible connection between the transport equation for the turbulent kinetic energy and the log-law is discussed in Chapter 6.

2.1.1 Terminology for the Law-of-the-Wall

The intent of this section is to briefly define the various regions of the law-of-the-wall for 2DTBLs. The present study will also use the same terminology for the near-wall region of three-dimensional boundary layers. A typical law-of-the-wall similarity plot for a two-dimensional turbulent boundary layer is given in Figure 2.1.

Viscous Sublayer: The viscous sublayer is adjacent to the wall and typically has an outer limit of $y^+ \leq 5$. Viscous shear is dominant in this region.

Buffer Layer: In this region the laminar and turbulent stresses are of the same order of magnitude. The buffer layer, or blending region, usually occupies a range of $5 < y^+ \leq 50$.

Viscous Region: This term denotes the portion of the near-wall region in which viscous shear is important. The viscous sublayer consists of both the viscous sublayer and the buffer layer.

Log-Law Region: The flow in the log-law region is fully turbulent and, hence, the influence of viscous shear is negligible. The velocity distribution in this region is given by Equation (1.3). The outer edge of the log-law region is typically at $y/\delta \approx 0.1$.

Near-Wall Region ("Inner" Region): The near-wall region includes the viscous sublayer, the buffer layer, and the log-law region.

2.2 GENERAL FEATURES OF THREE-DIMENSIONAL TURBULENT BOUNDARY LAYERS

The intent of this section is to familiarize the reader with some of the characteristics of three-dimensional turbulent boundary layers (3DTBLs). Since the turbulent structure of 3DTBLs is a very complex topic, only the time averaged mean velocity will be briefly considered. For a detailed discussion of three-dimensional turbulent boundary layers and turbulent secondary flows the reader is referred to Nash & Patel (1972) and Bradshaw (1987), respectively.

Figure 2.1 Terminology for the Near-Wall Region of a 2D Turbulent Boundary Layer. (Adapted from Bradshaw, 1978)

For simplicity, this section will deal only with 3DTBLs over planar surfaces. Therefore, a Cartesian coordinate system will be used in which the x and y axes are aligned with the streamwise, ζ , and wall normal, η , directions, respectively. In this orientation of the coordinate system, the z axis direction is referred to as the crossflow, or spanwise, direction.

This study is concerned with wall bounded three-dimensional thin shear layers in which the normal component of mean velocity, V , is small in comparison to the streamwise and crossflow velocity components, U and W :

$$V \ll U, W. \quad (2.7)$$

Thus, the magnitude and direction of the resultant mean velocity vector, Q , are given as

$$Q = \sqrt{U^2 + W^2} \quad (2.8)$$

and

$$\phi = \tan^{-1}\left(\frac{W}{U}\right), \quad (2.9)$$

respectively, where ϕ is the mean flow angle. According to Bradshaw (1987), thin or fairly thin wall bounded three-dimensional shear layers are flows in which the streamwise and the crossflow derivatives are small compared to those normal to the wall:

$$\frac{\partial}{\partial x}, \frac{\partial}{\partial z} \ll \frac{\partial}{\partial y}. \quad (2.10)$$

Thus, in such flows the streamlines essentially lie in planes which are parallel to the wall. Note that, as a matter of consistency, the term "boundary layer" will be used to include all wall bounded thin or fairly thin shear layers. The simplified mean momentum equations, according to these boundary layer assumptions, are given in Section 3.2.

In terms of the present study, a boundary layer will be regarded as three-dimensional only if the mean flow skews across the boundary layer thickness, δ . That is, if the mean flow angle, ϕ , changes considerably across the boundary layer. An isometric plot of a typical three-dimensional boundary layer is given in Figure 2.2. Although this definition may seem obvious,

Figure 2.2 Isometric Plot of the Velocity Profile for a Typical Three-Dimensional Boundary Layer.

it is interesting to note that some axisymmetric flows are often called three-dimensional even when the mean flow is largely two-dimensional in the sense that there is little skew. If one recognizes that the turbulence field is always three-dimensional, even for flows in which the mean flow is two-dimensional, then a distinction is necessary. Hereafter, the term "collateral" will be used to refer to flows in which the mean flow is two-dimensional: the velocity vectors are in a single plane.

Three-dimensional turbulent boundary layers can be identified as being either pressure or shear driven. These two classifications refer to the mechanism responsible for creating the secondary flow. In rudimentary terms, the driving force for a secondary flow may be due to a pressure gradient or a shear gradient. An example of a pressure driven three-dimensional flow is when an initially two-dimensional flow is deflected laterally by a pressure gradient $\partial P/\partial z$ to create a crossflow velocity component. Note that the majority of the three-dimensional flows reported in the literature are of this type (McAllister 1979). Shear driven 3D flows can result from the local transverse motion of bounding surfaces. In practice such flows are often produced at the junction between stationary and rotating components of machinery. Shear driven 3D flows can also occur behind swept steps. Both pressure and shear driven three-dimensional turbulent boundary layers are analyzed in this study.

2.2.1 Mean Velocity Profiles in Three-Dimensional Turbulent Boundary Layers

In the literature, the mean velocity of a 3DTBL is typically presented in separate plots for each velocity component, $U(y)$ and $W(y)$. Most often, orthogonal planes, which are normal to the wall, are used. Then, depending on the sign of the crossflow velocity profile, 3DTBLs can be identified as either unilateral or bilaterally skewed. In unilaterally skewed flows the crossflow velocity profile maintains a constant sign throughout the boundary layer thickness. In contrast, the crossflow velocity in a bilaterally skewed, or crossover, flow changes sign across the boundary layer thickness. Illustrations of these types of velocity profiles are given in Figures 2.3 and 2.4, respectively. The majority of the experimental data used in this study are unilaterally skewed flows.

Figure 2.3 Velocity Component Representation of a Unidirectional Three-Dimensional Boundary Layer.

Figure 2.4 Velocity Component Representation of a Bidirectional Three-Dimensional Boundary Layer.

Another type of mean velocity profile plot is the hodograph, or polar plot, popularized by Johnston (1960). In this case, the velocity components are projected onto a plane tangential to the wall producing a plot of W against U . A typical polar plot for a unilaterally skewed velocity profile is given in Figure 2.5. Notice that on this plot, a straight line joining the origin to the curve gives the magnitude and direction of the resultant velocity, Q .

2.3 LAW-OF-THE-WALL SIMILARITY MODELS FOR THREE-DIMENSIONAL FLOWS

The objective of this section is to provide a brief overview of the three-dimensional law-of-the-wall similarity models available in the literature. First, a short summary of the near-wall similarity review of McAllister (1979) is given. Following this, the law-of-the-wall similarity relations of Yowakim (1985) and Kind et al. (1989) are presented.

As part of a doctoral thesis, McAllister (1979) performed a comprehensive review of two- and three-dimensional near-wall similarity models. Furthermore, McAllister tested each of the 3D models with experimental data obtained from a pressure driven and two shear driven flows. His literature review and test results are summarized in Pierce et al. (1983a, 1983b, and 1983c). More recently, Olcman and Simpson (1992) applied these same 3D similarity models to nine different sets of experimental data. These reviews are quite extensive and thorough. Consequently, much of the background material for the present study was gathered from these articles. A brief summary of the 3D models presented by McAllister (1979) follows.

In this discussion, the law-of-the-wall similarity shear velocity scales, Q^* , U^{**} , W^{**} , U^* , and W^* , are defined as:

$$Q^* = \sqrt{\frac{\tau_w}{\rho}} \quad (2.11)$$

Figure 2.5 Typical Hodograph for a Unidirectional Three-Dimensional Turbulent Boundary Layer.

$$U^{**} = Q^{*} \cos(\beta_w) = \sqrt{\frac{\tau_{x,w}}{\rho}} \sqrt{\frac{\tau_{x,w}}{\tau_w}} , \quad (2.12)$$

$$W^{**} = Q^{*} \sin(\beta_w) = \sqrt{\frac{\tau_{z,w}}{\rho}} \sqrt{\frac{\tau_{z,w}}{\tau_w}} , \quad (2.13)$$

$$U^{*} = Q^{*} \sqrt{\cos(\beta_w)} = \sqrt{\frac{\tau_{x,w}}{\rho}} . \quad (2.14)$$

and

$$W^{*} = Q^{*} \sqrt{\sin(\beta_w)} = \sqrt{\frac{\tau_{z,w}}{\rho}} . \quad (2.15)$$

where, $\tau_{x,w}$ and $\tau_{z,w}$ are the wall shear stress components, τ_w is the resultant wall shear stress,

$$\tau_w = \sqrt{\tau_{x,w}^2 + \tau_{z,w}^2} . \quad (2.16)$$

and β_w is the wall shear stress angle,

$$\beta_w = \tan^{-1} \left(\frac{\tau_{z,w}}{\tau_{x,w}} \right) . \quad (2.17)$$

McAllister (1979) (see Pierce et al., 1983c) categorized 3D law-of-the-wall similarity models into two groups: scalar equations and complex equations. The scalar group consists of six equations which are essentially based on the two-dimensional log-law,

$$\frac{U}{U_{\tau}} = \frac{1}{k} \ln \left(\frac{y U_{\tau}}{\nu} \right) + B . \quad (1.3)$$

For the most part, the models in this group merely replace the velocity, U , and/or the shear velocity, U_{τ} , with an equivalent variable. Consequently, these models are considered as scalar relations since they are based on only one velocity component or near-wall velocity scale. The

general form of the scalar equations is

$$\frac{V_{eq}}{V_{eq}^*} = \frac{1}{k} \ln \left(\frac{y V_{eq}^*}{\nu} \right) + B, \quad (2.18)$$

where V_{eq} and V_{eq}^* are the equivalent velocity and near-wall scales for a three-dimensional boundary layer. The appropriate equivalent variables for each of the six scalar models are given in Table 2.1. Note that for the East and Hoxey relation, α_1 is defined as

$$\alpha_1 = \sin^{-1} \left(\left\{ U_o / K_1 Q^* \right\} \sin \beta_1 \right) - \beta_1, \quad (2.19)$$

where K_1 is a constant, U_o is a working section reference velocity, and β_1 is defined in terms of the streamwise and crossflow displacement thicknesses as

$$\beta_1 = -\tan^{-1}(\delta_x^* / \delta_z^*), \quad (2.20)$$

where

$$\delta_x^* = \int_0^{\bar{y}} \{ 1 - Q/Q_e \} dy \quad (2.21)$$

and

$$\delta_z^* = - \int_0^{\bar{y}} \frac{W}{Q_e} dy. \quad (2.22)$$

McAllister also tested four other law-of-the-wall similarity relations. These models are regarded as complex since they explicitly address the vector nature of the velocity, Q , by incorporating some or all of the following: separate consideration of the velocity components, pressure gradients, wall shear gradients, and wall shear angle gradients. A brief description of each complex model follows.

Perry and Joubert (1965) developed a 3D near-wall model based on the near-wall equilibrium arguments of Townsend (1961). This derivation includes streamwise and crossflow pressure gradients. The final form of their near-wall similarity relation is an integral similarity

<i>Model</i>	V_{eq}	V_{eq}^*
Coles (1956)	$Q \cos(\beta_w - \phi)$	Q^*
Johnston (1960)	$U / \cos(\beta_w)$	Q^*
Hornung and Joubert (1963)	U	Q^*
Pierce and Krommenhoek (1968)	U	U^*
Prahlad (1968)	Q	Q^*
East and Hoxey (1969)	$Q / \cos(\alpha_l)$	Q^*

Table 2.1 List of the Equivalent Velocities and Near-Wall Velocity Scales Used in the Six Simpler Similarity Models.

relation:

$$\frac{Q_D}{Q^*} = \frac{1}{k} \int_0^{y^*} \frac{1}{y^*} \left[1 - 2 \frac{\phi v}{\{Q^*\}^3} y^* \cos \theta + \left(\frac{\phi v}{\{Q^*\}^3} \right)^2 y^{*2} \right]^{1/4} dy^* + C, \quad (2.23)$$

where C is a constant, ϕ is the magnitude of the pressure gradient vector, θ is the angle between the pressure gradient and wall shear stress vectors, $y^* = yQ^*/v$, and the developed velocity, Q_D , is defined as

$$Q_D = \int_0^{y^*} \left[1 - \left\{ \frac{dU}{dW} \right\}^2 \right]^{1/2} dU. \quad (2.24)$$

The second of the complex models reviewed by McAllister is that of White, Lessmann, and Christoph (1975). In this model the turbulent shear stress in the streamwise direction is assumed to vary linearly away from the wall with the shear gradient being determined by the pressure gradient force. Mixing length theory is then used to relate the streamwise shear stress to the streamwise velocity gradient as

$$\tau_{xy} = \tau_{x,w} + \frac{1}{h_1} \frac{\partial P}{\partial x} = \rho \{ky\}^2 \left\{ \frac{\partial U}{\partial y} \right\}^2, \quad (2.25)$$

where h_1 is a metric coefficient. This equation can be integrated to give the streamwise profile

$$U^* = \frac{U}{U^*} = \frac{1}{k} \left[2 \{J - J_o\} + \ln \left(\frac{\{J - 1\} \{J_o + 1\}}{\{J + 1\} \{J_o - 1\}} \right) \right] \quad (2.26)$$

where

$$y^* = \frac{yU^*}{v}. \quad (2.27)$$

$$J_o = \sqrt{1 + e^{-1.8} y^*}, \quad (2.28)$$

$$J = \sqrt{1 + \alpha y^*} , \quad (2.29)$$

and

$$\alpha = \frac{\nu}{\rho \{U^*\}^3} \frac{1}{h_1} \frac{\partial P}{\partial x} . \quad (2.30)$$

White et al. mentioned attempts at developing a crossflow profile based on the same logic as that used for the streamwise profile, U , but without any success. They eventually suggested that one may use a form based on the unidirectional hodograph model of Mager (1951),

$$\frac{W}{W^*} = \frac{U}{U^*} \tan(\beta_w) \{1 - y/\delta\} . \quad (2.31)$$

Van den Berg (1975) developed a two component near-wall similarity model which takes into account inertial force and pressure gradient effects. Unlike all the other relations discussed to this point, these relations were formulated for a right-hand coordinate system in which the x coordinate direction lies in the direction of the resultant wall shear stress, τ_w . The z component is then perpendicular to this direction and the x - z coordinates form a plane parallel to the wall. Note that such a coordinate system is in general different from that of a streamwise coordinate system since the wall shear stress direction is usually different from that of the velocity at the boundary layer edge. The resulting velocity component equations are

$$\frac{Q \cos(\beta_w - \phi)}{Q^*} = \frac{1}{k} \left[\ln y^* + \frac{1}{2} \alpha_z y^* + \frac{1}{2} \gamma_z \frac{y^* \{\ln y^*\}^2}{k^2} \right] + B_z , \quad (2.32)$$

and

$$\frac{Q \sin(\beta_w - \phi)}{Q^*} = \frac{1}{k} \left[\alpha_z y^* + \gamma_z \frac{y^* \{\ln y^*\}^2}{k^2} \right] + B_z , \quad (2.33)$$

where $y^* = yQ^*/\nu$.

$$\alpha_x = \frac{v}{\rho \{Q^*\}^3} \frac{\partial P}{\partial x} . \quad (2.34)$$

$$\alpha_z = \frac{v}{\rho \{Q^*\}^3} \frac{\partial P}{\partial z} , \quad (2.35)$$

$$\gamma_x = \frac{v}{\{Q^*\}^3} \frac{\partial Q_x^*}{\partial x} , \quad (2.36)$$

and

$$\gamma_z = \frac{v}{\{Q^*\}^3} \frac{\partial Q_z^*}{\partial x} . \quad (2.37)$$

One major weak point in Van den Berg's derivation is the use of the two-dimensional log-law, Equation (1.3), to evaluate terms in the three-dimensional model. In addition, Van den Berg states that Equations (2.32) and (2.33) are restricted to flows in which $|\alpha| y^+ \ll 1$ and $|\beta| \{y^+\}^2 y^+ \ll 1$. . These restrictions limit the applicability of Equations (2.32) and (2.33) to either small pressure gradient and shear gradient flows or to relatively small y^+ values.

The last model reviewed by McAllister was that of Chandrashekhar and Swamy (1976). This model consists of separate, two-dimensional type logarithmic law equations for the streamwise and crossflow velocity components, U and W . The resulting equations are

$$\frac{U}{U^*} = A_x \ln \left(\frac{y U^*}{v} \right) + B_x , \quad (2.38)$$

and

$$\frac{W}{W^*} = A_z \ln \left(\frac{y W^*}{v} \right) + B_z , \quad (2.39)$$

where A_x, A_z, B_x , and B_z are empirical constants. From the experimental data of East and Hoxey (1969), Chandrashekhar and Swamy determined these constants to be $A_x = 5.4$, $A_z = 1.0$, $B_x = 4.9$, and $B_z = 11.8$.

McAllister (1979) (see Pierce et al., 1983a and 1983b) applied all ten models to his three-dimensional experimental data to test the validity of these similarity relations. He reported that none of the models were particularly successful. In a more recent review, Olcman and Simpson (1992) applied these equations to nine sets of three-dimensional experimental data. They found that the relation of Johnston (1960) provided a good fit to the streamwise velocity component, U . However, the empirical constants, k and B , were found to be quite different for each experiment. Also, they found that none of the models provided a good fit to the crossflow velocity, W .

As mentioned in Chapter 1, the work of Yowakim (1985) and Kind et al. (1989) form the basis for the present work. Yowakim performed three experiments on a swirling flow in a cylindrical annulus. As part of this study, Yowakim developed logarithmic law relations for use with the axial and tangential velocity components, U and W , of a swirling annular flow:

$$\frac{U}{U^{**}} = \frac{1}{k} \ln \left(\frac{y U^{**}}{\nu} \right) + B, \quad (2.40)$$

and

$$\frac{W}{W^{**}} = \{1 \pm y/r_w\} \frac{1}{k} \ln \left(\frac{y W^{**}}{\nu \{1 \pm y/r_w\}} \right) + B. \quad (2.41)$$

Note that these equations assume that the wall curvature is in the tangential direction only. The plus and minus signs in Equation (2.41) denote convex and concave wall curvature, respectively. Following the derivation of Yowakim (1985), Kind et al. (1989) obtained a slightly different form for the tangential velocity component,

$$\frac{W}{W^{**}} = \{1 \pm y/r_w\} \left[\frac{1}{k} \ln \left(\frac{y W^{**}}{\nu \{1 \pm y/r_w\}} \right) + B \right]. \quad (2.42)$$

Although the mean velocity in each of the experiments was essentially two-dimensional, these equations were considered as good candidates for establishing near-wall similarity in three-dimensional flows. In particular, these equations seem to recognize the vector nature of velocity by using a separate relation for each velocity component. Log-law relations, according to the theory of Yowakim (1985) and Kind et al. (1985), are derived in Chapter 3.

To the best of this authors' knowledge the preceding relations are the only three-dimensional law-of-the-wall similarity models available in the open literature. The author does acknowledge, however, the existence of other near-wall similarity models which do not rely on wall shear stress based similarity scales. In particular, the method of asymptotic analysis has been used to develop near-wall similarity models based on scales defined in terms of the local shear stress, τ . However, the present study is only concerned with law-of-the-wall type similarity and, as a result, did not review such models. The reader is referred to Degani et al. (1992 & 1993) and Panton (1990) for a discussion of two- and three-dimensional similarity models obtained by asymptotic matching.

2.3.1 A Note Concerning the Law-of-the-Wall Velocity Scales U^{**} and W^{**}

It is interesting to note that the log-law relations of Yowakim (1985) and Kind et al. (1989) use the same law-of-the-wall velocity scales, U^{**} and W^{**} , as Johnston (1960). However, it is worth mentioning that Yowakim and Kind et al. were not aware of Johnston's work. In fact, Yowakim deduced these law-of-the-wall velocity scales from an eddy-viscosity/mixing-length turbulence model and independently arrived at the same shear velocity scales as Johnston. Likewise, the present author only become aware of Johnston's 3D similarity equation very late in the present study. In Chapter 3, logarithmic laws are derived according to the theory of Yowakim (1985) and Kind et al. (1989). For reference, a brief outline of Johnston's 3D log-law derivation is given next.

Johnston (1960) assumed, based on experimental observations, that the mean flow in the near-wall region of a 3DTBL is collateral. Essentially, he divided the mean flow into two adjacent regions: a collateral near-wall region, *Region 1*, and an outer skewed region, *Region 2*. A hodograph indicating each region is shown in Figure 2.6. From Figure 2.6, Johnston assumed that *Region 1* lies within the viscous sublayer and, therefore, only occupies a very small portion of the boundary layer thickness, δ . Moreover, he reasoned that the mean flow angle in *Region 1* is equal to that of the wall shear stress, $\phi = \beta_w$. Johnston then assumed that the flow in the outer portion of *Region 1* to the lower portions of *Region 2* is governed by the log-law, Equation (1.3), in the form

Figure 2.6 Typical Hodograph used to Determine the Equivalent Velocity and the Law-of-the-Wall Velocity Scale, V_{eq} and V_{eq}^* , for the 3D Similarity Model of Johnston (1960).

$$\frac{Q}{Q^*} = \frac{1}{k} \ln\left(\frac{yQ^*}{\nu}\right) + B , \quad (2.43)$$

where k and B are the standard empirical values for two-dimensional flows. Consequently, Equation (2.43) can be written in terms of the velocity components as

$$\frac{U}{U^{**}} = \frac{W}{W^{**}} = \frac{1}{k} \ln\left(\frac{yQ^*}{\nu}\right) + B , \quad (2.44)$$

where

$$U = Q \cos\phi ,$$

$$U^{**} = Q^* \cos\beta_w ,$$

$$W = Q \sin\phi ,$$

and

$$W^{**} = Q^* \sin\beta_w .$$

Following these arguments, there seems to be some inconsistency in that the log-law is supposedly valid in parts of *Regions 1* and *2*. This would seem to suggest that, in 3DTBLs, the log-law applies to some portion of the viscous sublayer and the fully turbulent near-wall region. However, it is widely accepted that in the case of 2DTBLs, the log-law, Equation (1.3), only applies to the fully turbulent portion of the near-wall region.

CHAPTER 3

MATHEMATICAL FORMULATION

3.1 COORDINATE SYSTEMS

The objective of this section is to specify the orthogonal coordinate systems used in Chapters 4 and 5. Cartesian coordinates are used to analyze the planar flows presented in Chapter 4. Similarly, flows in annular ducts are examined in Chapter 5. As a result, cylindrical coordinates are the natural choice for these cases. Particular issues, such as the treatment of convex or concave wall curvature, are briefly considered in this section.

3.1.1 Cylindrical System

The three-dimensional orthogonal cylindrical coordinate system consists of three components: radial, tangential, and axial. These components, along with the corresponding velocity components, are illustrated in Figure 3.1. The radial and tangential components for convex and concave surfaces are shown in Figure 3.2. For the annular flows presented in Chapter 5, the axial axis, x coordinate direction, is regarded as positive when oriented in the direction of the flow. Consequently, referring to Figure 3.2, the positive x coordinate axis is directed out of the page for both the convex and the concave walls. The distance normal to the surface of the cylinder is denoted by y . Hence, the radial component of a particular point is given by

$$r = r_w \pm y, \quad (3.1)$$

where r_w is the cylinder radius. The plus and minus signs in Equation (3.1) indicate convex and concave wall curvature, respectively. The distance normal to the convex side of a cylinder is

$$y = r - r_w, \quad (3.2)$$

while the distance normal to the concave side of a cylinder is

Figure 3.1 Coordinate System for Flows Along the Axis of a Cylinder.

Figure 3.2 Coordinate System for Annular Flows.

$$y = r_o - r . \quad (3.3)$$

From Figure 3.2, the unit normals for y and r coincide for the convex wall and are in opposite directions for the concave wall.

The gradients in the direction normal to the cylinder surface are obtained by applying the chain rule. Neglecting gradients in the tangential and axial directions, the chain rule produces

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial r_i} \frac{\partial r_i}{\partial y} + \frac{\partial}{\partial r} \frac{\partial r}{\partial y} \quad (3.4)$$

for the convex wall, and

$$\frac{\partial}{\partial y} = - \left[\frac{\partial}{\partial r_o} \frac{\partial r_o}{\partial y} + \frac{\partial}{\partial r} \frac{\partial r}{\partial y} \right] \quad (3.5)$$

for the concave wall. Since the wall radii are constant their partial derivatives with respect to y are zero.

$$\frac{\partial r_i}{\partial y} = \frac{\partial r_o}{\partial y} = 0 .$$

From Equation (3.2), the partial derivative of r with respect to y for the convex wall is

$$\frac{\partial r}{\partial y} = 1 .$$

Substituting the above results into Equation (3.4) shows that gradients in the direction of the outward normal and the radial direction are the same for the convex wall,

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial r} . \quad (3.6)$$

The same result is obtained for the concave wall by differentiating Equation (3.3) with respect to y and substituting into Equation (3.5).

Although the preceding is elementary calculus, it is important that the sign of the gradients be correct. This will become evident in the derivation of the logarithmic laws for an axisymmetric

flow presented in Section 3.7. Furthermore, while the results are trivial in this case such issues are more complicated if one chooses to employ general orthogonal curvilinear coordinates for use with more complicated body geometries.

3.1.2 Cartesian System

The Cartesian coordinate system consists of a component normal to the wall, y , and two mutually orthogonal components x and z . Unlike the cylindrical coordinate system, the choice of orientation of the coordinate system in the plane parallel to the wall is arbitrary. However, the experimental data for the planar flows in Chapter 4 are presented in relation to the streamwise direction at each measuring station. Consequently, the x coordinate direction is taken as the streamwise direction. The z coordinate direction is then regarded as the crossflow direction. This coordinate system, along with the corresponding velocity components, is shown in Figure 3.3.

3.2 GOVERNING EQUATIONS

This section presents the governing equations for three-dimensional planar and axisymmetric turbulent boundary layers. Although the equations of motion are not needed in this study, they are included here for completeness. Note that the current investigation regards a boundary layer to be three-dimensional only if the mean velocity vectors skew, vary in magnitude and direction, across the boundary layer thickness, δ .

It is generally accepted that the time-dependent three-dimensional Navier-Stokes (NS) equations and the continuity equation describe turbulent flows completely. These equations apply to the instantaneous flow field. However, engineers are primarily concerned with the mean motion of the flow. As a result, the equations of motion are often time-averaged to produce a solvable form. These equations then become an unclosed system due to the introduction of six turbulent, or Reynolds, stresses.

The Navier-Stokes equations, in their complete form, are nonlinear. For wall bounded

Figure 3.3 Cartesian Coordinate System for Planar Flows.

shear flows, the NS equations can be reduced to a less complicated parabolic form by applying boundary layer, or thin shear layer, approximations. Note that in such flows, the boundary layer thickness, δ , is small in relation to the length, L , over which the flow develops. The parabolized momentum, or boundary layer, equations are valid as long as no flow reversal occurs. The boundary layer equations for a three-dimensional flow in cylindrical and Cartesian coordinates are given next.

3.2.1 Cylindrical Equations

Consider the cylindrical coordinate system presented in Figure 3.1. Let U , V , and W be the time-mean velocities in the x , r , and θ directions respectively and u , v , and w be the corresponding turbulent fluctuating velocities. The time-mean continuity and parabolized Navier-Stokes equations can be written in cylindrical notation by applying the following assumptions.

- 1) The flow is assumed to be steady, incompressible, and axisymmetric.
- 2) Axial derivatives of viscous and turbulent stresses are assumed to be negligible.

Continuity Equation

$$\frac{\partial(rU)}{\partial x} + \frac{\partial(rV)}{\partial r} = 0 \quad (3.7)$$

Momentum Equations

Axial Momentum

$$U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial r} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \frac{1}{\rho r} \frac{\partial(r\tau_{rx})}{\partial r} \quad (3.8)$$

Tangential Momentum

$$U \frac{\partial W}{\partial x} + \frac{V}{r} \frac{\partial(rW)}{\partial r} = \frac{1}{\rho r^2} \frac{\partial(r^2\tau_{r\theta})}{\partial r} \quad (3.9)$$

Radial Momentum

$$\frac{\rho W^2}{r} = \frac{\partial P}{\partial r} - \frac{1}{r} \frac{\partial(r\tau_{rr})}{\partial r} + \frac{\tau_{\theta\theta}}{r} \quad (3.10)$$

The stress terms in these equations are given as the sum of the viscous and turbulent stresses. Thus, the tensor form of the fluid stress is represented as the sum of two second order symmetrical tensors. One tensor consists of the viscous, or molecular, stresses while the other consists of the turbulent, or Reynolds, stresses where

$$\tau_{ij} = \tau_{ij}^m + \tau_{ij}^t \quad (i, j = x, r, \theta) ,$$

$$\tau_{rr} = \mu \frac{\partial U}{\partial r} - \rho \overline{uv} , \quad (3.11)$$

$$\tau_{rr} = -\rho \overline{v^2} , \quad (3.12)$$

$$\tau_{\theta r} = \mu \left[r \frac{\partial(W/r)}{\partial r} \right] - \rho \overline{vw} , \quad (3.13)$$

and

$$\tau_{\theta\theta} = -\rho \overline{w^2} . \quad (3.14)$$

3.2.2 Cartesian Equations

From Section 3.1.2, the Cartesian coordinate system used in the present study is shown in Figure 3.3. U , V , and W represent the mean velocity components in the x , y , and z directions, respectively, and u , v , and w represent the corresponding turbulent velocity fluctuations. The time-mean continuity and parabolized Navier-Stokes equations for the Cartesian coordinate system are produced in the same manner as for the preceding cylindrical coordinate system. However, one additional condition is that the crossflow, z , as well as the streamwise, x , direction derivatives

of viscous and turbulent stresses be negligible.

Continuity Equation

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = 0 \quad (3.15)$$

Momentum Equations

Streamwise Momentum

$$U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} + W \frac{\partial U}{\partial z} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \frac{1}{\rho} \frac{\partial (\tau_{xy})}{\partial y} \quad (3.16)$$

Crossflow Momentum

$$U \frac{\partial W}{\partial x} + V \frac{\partial W}{\partial y} + W \frac{\partial W}{\partial z} = -\frac{1}{\rho} \frac{\partial P}{\partial z} + \frac{1}{\rho} \frac{\partial (\tau_{zy})}{\partial y} \quad (3.17)$$

Normal Momentum

$$\frac{\partial P}{\partial y} = \rho \frac{\partial (\tau_{yy})}{\partial y} \quad (3.18)$$

The stress terms in these equations again consist of a molecular portion and a turbulent portion, where

$$\tau_{xy} = \mu \frac{\partial U}{\partial y} - \rho \overline{uv} \quad (3.19)$$

$$\tau_{yy} = -\rho \overline{v^2} \quad (3.20)$$

and

$$\tau_{zy} = \mu \frac{\partial W}{\partial y} - \rho \overline{vw} \quad (3.21)$$

Clearly, the presence of the Reynolds stresses in the momentum equations introduces a closure problem in that the number of unknowns is greater than the number of equations.

Accordingly, turbulence models are required to provide additional relations to be able to solve the time-averaged equations of motion, Equations (3.7-10) and Equations (3.15-18). Several models for calculating turbulence stresses for incompressible flows are reviewed by White (1991). However, in the following sections one of the simplest turbulence models, the eddy-viscosity/mixing-length model, is used to derive logarithmic laws for two- and three-dimensional turbulent boundary layers. A description of the eddy viscosity hypothesis is given next.

3.3 THE EDDY-VISCOSITY CONCEPT

For engineering analysis, the eddy-viscosity concept has long been regarded as a good method for evaluating the turbulent shear stresses in boundary layer flows. In the eddy-viscosity concept, the turbulent shear stresses are related to the rate of strain of the mean flow field in the same manner as the laminar shear stresses. The shear stress, $-\rho \overline{uv}$, in a two-dimensional planar flow is then given as

$$-\rho \overline{uv} = \mu' \frac{\partial U}{\partial y} . \quad (3.22)$$

where μ' is the eddy viscosity. Similarly, in cylindrical coordinates, the axial and tangential components of the turbulent shear stress, $-\rho \overline{uv}$ and $-\rho \overline{vw}$, in Equations (3.11) and (3.13) are then expressed as

$$-\rho \overline{uv} = \mu_{x,r} \frac{\partial U}{\partial r} \quad (3.23)$$

and

$$-\rho \overline{vw} = \mu_{\theta,r} \left[r \frac{\partial (w/r)}{\partial r} \right] . \quad (3.24)$$

where $\mu_{x,r}$ and $\mu_{\theta,r}$ are the eddy viscosities in the axial and tangential directions, respectively. Substituting Equations (3.23) and (3.24) into (3.11) and (3.13), respectively, the shear stresses $\tau_{x,r}$ and $\tau_{\theta,r}$ can be written as

$$\tau_{sr} = (\mu + \mu_{sr}) \frac{\partial U}{\partial r} \quad (3.25)$$

and

$$\tau_{\theta r} = (\mu + \mu_{\theta r}) \left[r \frac{\partial (W/r)}{\partial r} \right]. \quad (3.26)$$

Applying the eddy-viscosity concept to Equations (3.19) and (3.21), similar relationships are obtained for the streamwise and crossflow directions of a Cartesian coordinate system:

$$-\rho \overline{uv} = \mu_{xy} \frac{\partial U}{\partial y}, \quad (3.27)$$

$$-\rho \overline{vw} = \mu_{xy} \frac{\partial W}{\partial y}, \quad (3.28)$$

$$\tau_{xy} = (\mu + \mu_{xy}) \frac{\partial U}{\partial y}, \quad (3.29)$$

and

$$\tau_{xy} = (\mu + \mu_{xy}) \frac{\partial W}{\partial y}. \quad (3.30)$$

Obviously, the eddy viscosities should be easily obtained if these equations are to be of any practical use in the prediction of turbulent flows. One of the simplest methods for calculating the eddy viscosity, the Prandtl mixing-length definition, is described in the following section.

3.4 THE PRANDTL MIXING-LENGTH DEFINITION OF EDDY VISCOSITY

As one of the first attempts to model two-dimensional turbulent boundary layers, Prandtl developed an eddy-viscosity model based on an analogy to the kinetic theory of gases. He simply assumed that a lump of fluid, or fluid element, conserves its linear momentum when displaced a distance, l , normal to the mean direction of the flow. The magnitudes of the fluctuating velocities,

u and v , are then assumed to be proportional to the mean velocity gradient. Consequently, the two-dimensional representation of the Reynolds shear stress is given as

$$-\rho \overline{uv} = \mu' \frac{\partial U}{\partial y} = \rho l^2 \left\{ \frac{\partial U}{\partial y} \right\}^2, \quad (3.31)$$

where l is the mixing length and μ' is the eddy viscosity. From Equation (3.31), the corresponding two-dimensional eddy viscosity is

$$\mu' = \rho l^2 \frac{\partial U}{\partial y}. \quad (3.32)$$

The difficulty now is that one must be able to specify the mixing length, l . Prandtl deduced that in the near-wall region the mixing length should be proportional to the distance from the wall, $l = ky$, with k being an empirical constant.

According to Whitaker (1968), the general scalar form of Equation (3.32), for use with three-dimensional boundary layers, is

$$\mu' = \rho l^2 \sqrt{4e_{ij}:e_{ij}}, \quad (3.33)$$

where e_{ij} is the rate of strain tensor for the mean flow and $e_{ij}:e_{ij}$ is a reduced form of I_2 , the second invariant of the rate of strain tensor. This reduced form of I_2 is obtained by applying the boundary layer assumptions to the second invariant of the rate of strain tensor, as described in Appendix E. Although Equation (3.33) seems to be a reasonable model, one must acknowledge that there is a considerable amount of experimental evidence which suggests that the eddy viscosity is not a scalar in three-dimensional boundary layers. A discussion regarding the anisotropic nature of three-dimensional boundary layers follows.

3.4.1 Anisotropic Nature of Turbulent Shear Stress in Three-Dimensional Boundary Layers

The intent of this section is to briefly explain why the general scalar form of the eddy-viscosity/mixing-length model, Equation (3.33), is not appropriate in a three-dimensional turbulent boundary layer. Following this, a modified version of Equation (3.33) is presented.

It is important to recognize that the dynamic viscosity, μ , is a scalar fluid property which only depends on the local thermodynamic state of the fluid. However, unlike the dynamic viscosity, μ , the eddy viscosity, μ^t , is not a property of the fluid but is a model of the dynamics of the flow field. For two-dimensional turbulent boundary layers, the eddy viscosity, μ^t , can be interpreted as a scalar since, in general, only the Reynolds shear stress component $-\rho \overline{u'v'}$ is significant. In contrast, the flow field in a steady three-dimensional turbulent boundary layer is generally non-homogeneous and anisotropic. Thus, for three-dimensional turbulent boundary layers, the eddy viscosity should be defined such that this anisotropic behavior is accounted for. Justification for interpreting the eddy viscosity as a pseudo vector follows.

For thin shear layers, or boundary layers, the turbulent shear stress in a three-dimensional turbulent boundary layer is often regarded as a pseudo vector of the form $(\overline{uv}, 0, -\overline{wv})$ (Nash and Patel, 1972; and Bradshaw, 1987). Moreover, there is considerable experimental evidence that in a three-dimensional turbulent boundary layer the direction of this Reynolds shear stress vector,

$$\phi_r = \tan^{-1} \left(\frac{\overline{wv}}{\overline{uv}} \right), \quad (3.34)$$

may vary substantially from that of the mean rate of strain vector,

$$\phi_{rr} = \tan^{-1} \left(\frac{\partial W / \partial y}{\partial U / \partial y} \right), \quad (3.35)$$

(Johnston, 1970; Lilley and Chigier, 1971; Van den Berg and Elsenaar, 1972; Anderson and Eaton, 1987; Driver and Hebbar, 1987; and Van den Berg, 1989). Consequently, one must acknowledge that utilizing a scalar form of the eddy viscosity in three-dimensional flows is inconsistent with experimental findings. This in turn implies that the general scalar eddy-viscosity model, Equation (3.33), should be modified in order to accurately predict the dynamics of the flow. In the present study the eddy-viscosity is regarded as a pseudo vector based on the definitions of Lilley and Chigier (1971), as described next.

Lilley and Chigier (1971) proposed an anisotropic eddy-viscosity model for annular swirling flows by utilizing anisotropic mixing lengths in Equation (3.33):

$$\mu_{xr} = \rho l_{xr}^2 \left[\left\{ \frac{\partial U}{\partial r} \right\}^2 + \left\{ r \frac{\partial}{\partial r} \left(\frac{W}{r} \right) \right\}^2 \right]^{1/2} \quad (3.36)$$

and

$$\mu_{\theta r} = \rho l_{\theta r}^2 \left[\left\{ \frac{\partial U}{\partial r} \right\}^2 + \left\{ r \frac{\partial}{\partial r} \left(\frac{W}{r} \right) \right\}^2 \right]^{1/2} . \quad (3.37)$$

Note that Equations (3.34) and (3.35) can accommodate a difference between the direction of the shear stress vector, ϕ_r , and the mean rate of strain vector, ϕ_{rs} , by using anisotropic mixing-length definitions. Expressions for l_{xr} and $l_{\theta r}$ in the near-wall region of a three-dimensional turbulent boundary layer are given in Section 3.6.

In the strict mathematical sense, there are difficulties associated with assuming a zero, first, second, or fourth order eddy-viscosity tensor (Hinze, 1975 and Stanisic, 1988). Thus, from a mathematical perspective, it seems that no simple general form of the eddy viscosity can be derived. In spite of this, eddy-viscosity/mixing-length models have achieved a good degree of success in two- and three-dimensional turbulent boundary layer calculations (see, for example, Baldwin and Lomax, 1978). As a result, the shear stress relations of Equations (3.24-25) and (3.28-29) have been very useful from an engineering perspective. A model for the Reynolds shear stress vector, according to the eddy-viscosity definitions of Equations (3.36) and (3.37), is given in Section 3.6.

3.5 THE TWO-DIMENSIONAL LOG LAW AND THE EDDY-VISCOSITY/MIXING-LENGTH HYPOTHESIS

This section derives the two-dimensional (2D) logarithmic law, Equation (1.3), from the classical eddy-viscosity/mixing-length representation of the Reynolds shear stress $-\rho \overline{uv}$ presented in Section 3.4. Particular attention is given to the variation of $-\rho \overline{uv}$.

Recall from Section 3.4 that the Prandtl eddy-viscosity/mixing-length representation of

the Reynolds shear stress, $-\rho \overline{uv}$, in a two-dimensional planar boundary layer is

$$-\rho \overline{uv} = \mu' \frac{\partial U}{\partial y} = \rho l^2 \left\{ \frac{\partial U}{\partial y} \right\}^2. \quad (3.31)$$

Substituting the Prandtl mixing length expression, $l = ky$, into Equation (3.31) gives

$$-\overline{uv} = (ky)^2 \left\{ \frac{\partial U}{\partial y} \right\}^2. \quad (3.38)$$

Note that, in general, the turbulent shear stress, $-\rho \overline{uv}$, is a function of the distance normal to the wall, y . Therefore, it is apparent that one must specify the functional form of $-\overline{uv}(y)$ in order to integrate Equation (3.38). However, in the open literature the shear stress, τ , is often assumed to be constant and equal to the wall shear stress value throughout the near-wall region. Consequently, the turbulent shear stress in the log-law region is given as

$$\tau' = -\rho \overline{uv} = \tau_w.$$

Substituting this expression in Equation (3.8) and integrating, one obtains the 2D log law

$$\frac{U}{U_\tau} = \frac{1}{k} \ln \left(\frac{y U_\tau}{\nu} \right) + B, \quad (1.3)$$

where k and B are empirically determined constants, and

$$U_\tau = \sqrt{\frac{\tau_w}{\rho}} \quad (1.2)$$

is the shear velocity scale. From the streamwise momentum equation, Equation (3.16), the assumption of constant shear stress in the near-wall region implies that the log law, Equation (1.3), can only exist in zero pressure gradient flows, $dP/dx = 0$. However, there is considerable experimental evidence that Equation (1.3) applies in flows which experience significant streamwise pressure gradients.

Galbraith et al. (1977) demonstrated that the two-dimensional log law, Equation (1.3), existed for a number of flows which were subjected to strong streamwise pressure gradients. This is considered to be a severe test of the validity of the log law since the shear stress, $-\rho \overline{uv}$,

varies significantly through the near-wall region. They proposed that the log law, Equation (1.3), has a broader range of validity than the Prandtl mixing length hypothesis, $l = \kappa y$. If one accepts this argument, it is then apparent that the mixing length should be modified to accommodate the variation of the shear stress in the log-law region. Galbraith et al. suggest that the mixing length should be

$$l = \kappa y \sqrt{\frac{\tau'}{\tau_w}} \quad (3.39)$$

where

$$\tau' = -\rho \overline{uv}.$$

Substituting Equation (3.39) into the shear stress representation,

$$-\overline{uv} = l^2 \left\{ \frac{\partial U}{\partial y} \right\}^2 \quad (3.31)$$

and integrating the result produces the log-law equation, Equation (1.3). As a result, the two-dimensional planar log law is then obtained without assuming constant shear stress throughout the near-wall region. Consequently, the rather convincing argument of Galbraith et al. is indirect evidence for the near-wall mixing-length relation of Equation (3.39). Modified forms of Equation (3.39) for use with axisymmetric and three-dimensional boundary layers are given next.

3.6 THE EDDY-VISCOSITY/MIXING-LENGTH MODEL IN THREE-DIMENSIONAL BOUNDARY LAYERS

This section gives expressions for the eddy viscosity components of a three-dimensional turbulent boundary layer. These expressions will later be combined with the Reynolds shear stress definitions of Section 3.3 to produce log-law relations for the velocity components, U and W . Note that this section and Section 3.7 largely reiterate the works of Yowakim (1985) and Kind et al. (1989).

Consider an axisymmetric flow as described in Section 3.2.1. In the fully turbulent

portion of the near-wall region, the laminar contribution to the total shear stress is negligible. As a result, the laminar viscosity in Equations (3.25) and (3.26) is neglected. Now a key assumption is that mixing lengths, l_{rx} and $l_{r\theta}$, in the log-law region are given by the relations similar to that proposed by Galbraith et al. (1977), where

$$l_{rx} = ky \sqrt{\frac{\tau_{rx}}{\tau_{x,w}}} \quad (3.40)$$

and

$$l_{r\theta} = ky \sqrt{\frac{\tau_{\theta r}}{\tau_{\theta,w}}} \quad (3.41)$$

Substitution of these mixing length expressions into Equations (3.36) and (3.37) produces the anisotropic eddy-viscosities

$$\mu_{rx} = \rho (ky)^2 \frac{\tau_{rx}}{\tau_{x,w}} \left[\left\{ \frac{\partial U}{\partial r} \right\}^2 + \left\{ r \frac{\partial}{\partial r} \left(\frac{W}{r} \right) \right\}^2 \right]^{1/2} \quad (3.42)$$

and

$$\mu_{r\theta} = \rho (ky)^2 \frac{\tau_{\theta r}}{\tau_{\theta,w}} \left[\left\{ \frac{\partial U}{\partial r} \right\}^2 + \left\{ r \frac{\partial}{\partial r} \left(\frac{W}{r} \right) \right\}^2 \right]^{1/2} \quad (3.43)$$

Substituting Equations (3.42) and (3.43) into the shear stress definitions of Equations (3.24) and (3.25), the turbulent shear stresses in the log-law region are given as

$$\tau_{rx} = \rho (ky)^2 \frac{\tau_{rx}}{\tau_{x,w}} \left[\left\{ \frac{\partial U}{\partial r} \right\}^2 + \left\{ r \frac{\partial}{\partial r} \left(\frac{W}{r} \right) \right\}^2 \right]^{1/2} \frac{\partial U}{\partial r} \quad (3.44)$$

and

$$\tau_{\theta r} = \rho (ky)^2 \frac{\tau_{\theta r}}{\tau_{\theta,w}} \left[\left\{ \frac{\partial U}{\partial r} \right\}^2 + \left\{ r \frac{\partial}{\partial r} \left(\frac{W}{r} \right) \right\}^2 \right]^{1/2} \left\{ r \frac{\partial}{\partial r} \left(\frac{W}{r} \right) \right\} \quad (3.45)$$

From Equations (3.44) and (3.45), the wall shear stress components are related to the rates of

strain in the log-law region by

$$\tau_{x,w} = \rho (ky)^2 \left[\left\{ \frac{\partial U}{\partial r} \right\}^2 + \left\{ r \frac{\partial}{\partial r} \left(\frac{W}{r} \right) \right\}^2 \right]^{1/2} \frac{\partial U}{\partial r} \quad (3.46)$$

and

$$\tau_{\theta,w} = \rho (ky)^2 \left[\left\{ \frac{\partial U}{\partial r} \right\}^2 + \left\{ r \frac{\partial}{\partial r} \left(\frac{W}{r} \right) \right\}^2 \right]^{1/2} \left\{ r \frac{\partial}{\partial r} \left(\frac{W}{r} \right) \right\}. \quad (3.47)$$

Note that the resultant wall shear stress is the sum of the component wall shear stresses,

$$\tau_w = \sqrt{\tau_{x,w}^2 + \tau_{\theta,w}^2}. \quad (3.48)$$

Substituting Equations (3.46) and (3.47) into Equation (3.48), one obtains an expression for the resultant rate of strain in the log-law region:

$$\left[\left\{ \frac{\partial U}{\partial r} \right\}^2 + \left\{ r \frac{\partial}{\partial r} \left(\frac{W}{r} \right) \right\}^2 \right]^{1/2} = \frac{1}{ky} \sqrt{\frac{\tau_w}{\rho}}. \quad (3.49)$$

Simpler expressions for $\tau_{x,w}$ and $\tau_{\theta,w}$ are achieved by substituting Equation (3.49) into Equations (3.46) and (3.47):

$$\tau_{x,w} = \rho ky \sqrt{\frac{\tau_w}{\rho}} \frac{\partial U}{\partial r} \quad (3.50)$$

and

$$\tau_{\theta,w} = \rho ky \sqrt{\frac{\tau_w}{\rho}} \left\{ r \frac{\partial}{\partial r} \left(\frac{W}{r} \right) \right\}. \quad (3.51)$$

Integration of Equations (3.50) and (3.51) will produce log-law relations for the axial and tangential directions of an axisymmetric flow. The derivation of such equations is given in Section 3.7.

Similar expressions can be obtained for a three-dimensional boundary layer on a planar

surface. In the planar case, the mixing lengths, l_{xy} and l_{zy} , in the log-law region are defined as

$$l_{xy} = ky \sqrt{\frac{\tau_{xy}}{\tau_{x,w}}} \quad (3.52)$$

and

$$l_{zy} = ky \sqrt{\frac{\tau_{zy}}{\tau_{z,w}}} . \quad (3.53)$$

The wall shear stress component relations are then obtained by following the same procedure as for the preceding axisymmetric case:

$$\tau_{x,w} = \rho ky \sqrt{\frac{\tau_w}{\rho}} \frac{\partial U}{\partial y} \quad (3.54)$$

and

$$\tau_{z,w} = \rho ky \sqrt{\frac{\tau_w}{\rho}} \frac{\partial W}{\partial y} . \quad (3.55)$$

The log-law relations for the streamwise and the crossflow directions of a three-dimensional planar flow are produced by integrating Equations (3.54) and (3.55), as given in Section 3.8.

Two notable results come from the preceding derivation. First, the eddy viscosity definitions of Equations (3.42) and (3.43) are different. As discussed in Section 3.4.1, such definitions are consistent with the anisotropic nature of a three-dimensional boundary layer. Secondly, Yowakim (1985) and Kind et al. (1989) recognized that suitable law-of-the-wall velocity scales are obtained directly from Equations (3.50) and (3.51):

$$U^{**} = \sqrt{\frac{\tau_{x,w}}{\rho}} \sqrt{\frac{\tau_{x,w}}{\tau_w}} \quad (3.56)$$

and

$$W^{**} = \sqrt{\frac{\tau_{\theta,w}}{\rho}} \sqrt{\frac{\tau_{\theta,w}}{\tau_w}} \quad (3.57)$$

Log-law relations based on these near-wall velocity scales are presented in the following sections.

3.7 COMPONENT LOGARITHMIC LAWS FOR A CYLINDRICAL COORDINATE SYSTEM

This section presents the log-law relations for the axial and tangential velocity components of an axisymmetric flow. In Chapter 5, these equations will be applied to two- and three-dimensional flows.

3.7.1 A Logarithmic Law for the Axial Velocity Component

It was shown in Section 3.1.1, that the gradients in the wall normal and radial directions are equal:

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial r} \quad (3.6)$$

This result allows Equation (3.50),

$$\tau_{x,w} = \rho k y \sqrt{\frac{\tau_w}{\rho}} \frac{\partial U}{\partial r} \quad (3.50)$$

to be expressed as a linear, first order differential equation in y ,

$$\frac{\partial U}{\partial r} = \frac{dU}{dy} = \rho k y \sqrt{\frac{\tau_{x,w}}{\rho}} \sqrt{\frac{\tau_{x,w}}{\tau_w}} \quad (3.58)$$

This equation may be converted to a non-dimensional form by using the near-wall similarity scales U^{**} and y/Q^* . Equation (3.58) is then solved by separating variables and integrating. The

solution of this equation yields the relationship for the distribution of the axial velocity, U , in the log-law region,

$$\frac{U}{U^{**}} = \frac{1}{k} \ln \left(\frac{y Q^*}{\nu} \right) + B, \quad (3.59)$$

where,

$$U^{**} = \sqrt{\frac{\tau_{x,w}}{\rho}} \sqrt{\frac{\tau_{x,w}}{\tau_w}}$$

and

$$Q^* = \sqrt{\frac{\tau_w}{\rho}}.$$

As mentioned previously in Section 2.1, the precise values of the empirical constants, k and B , are not generally agreed upon. However, the two-dimensional constants of Patel (1965), $k = 0.42$ and $B = 5.0$, are commonly used values and will be used in the present study unless stated otherwise. Note that Equation (3.59) is similar to that of the two-dimensional log law,

$$\frac{U}{U_\tau} = \frac{1}{k} \ln \left(\frac{y U_\tau}{\nu} \right) + B. \quad (1.3)$$

The difference between the two equations is attributed to the definition of the shear velocity scales U^{**} and U_τ . Equation (3.59) is used to analyze the cylindrical flows presented in Chapter 5.

3.7.2 Logarithmic Laws for the Tangential Velocity Component

Two forms of the tangential log law are obtained. One form applies to convex walls while the other applies to concave walls. A linear, first order, differential equation is obtained by substituting the relationship,

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial r} \quad (3.6)$$

into Equation (3.51),

$$\tau_{s,w} = \rho k y \sqrt{\frac{\tau_w}{\rho}} \left\{ r \frac{\partial}{\partial r} \left(\frac{W}{r} \right) \right\} \quad (3.51)$$

The resulting differential equation is

$$\frac{\partial}{\partial y} \left(\frac{W}{r} \right) = \frac{d}{dy} \left(\frac{W}{r} \right) = \frac{1}{k y r} \sqrt{\frac{\tau_{s,w}}{\rho}} \sqrt{\frac{\tau_{s,w}}{\tau_w}} \quad (3.60)$$

This equation may be converted to a non-dimensional form by using the near-wall similarity scales W^{**} and v/Q^* as well as the radial coordinate definition,

$$r = r_w \pm y \quad (3.1)$$

The non-dimensional form of Equation (3.60) then is

$$\frac{\partial}{\partial y^*} \left(\frac{W^*}{r_w^* \pm y^*} \right) = \frac{1}{k y^* \{ r_w^* \pm y^* \}} \quad (3.61)$$

where the dimensionless variables W^* , y^* , r_w^* and are defined as

$$W^* = \frac{W}{W^{**}} \quad ,$$

$$y^* = \frac{y Q^*}{v} \quad ,$$

and

$$r_w^* = \frac{r_w Q^*}{v} \quad .$$

The plus sign in Equation (3.61) denotes a convex wall while the minus sign indicates a concave wall. Equation (3.61) can be solved by integration of partial fractions to obtain

$$\frac{W^+}{r_w^+ \pm y^+} = \frac{1}{k r_w^+} \ln \left(\frac{y^+}{r_w^+ \pm y^+} \right) + C, \quad (3.62)$$

where C is a constant of integration.

Clearly the form of the integration constant, C , should be such that Equation (3.62) reduces to the planar form,

$$\frac{W}{W^{**}} = \frac{1}{k} \ln \left(\frac{y Q^+}{v} \right) + B_0, \quad (6.63)$$

as the wall radius becomes large. However, it appears that this constant may be defined in a number of ways. For instance, the primary difference between the tangential log-law equations of Yowakim (1985),

$$\frac{W}{W^{**}} = \frac{\{1 \pm y/r_w\}}{k} \ln \left(\frac{y W^{**}}{v \{1 \pm y/r_w\}} \right) + B, \quad (2.41)$$

and Kind et al. (1989),

$$\frac{W}{W^{**}} = \{1 \pm y/r_w\} \left[\frac{1}{k} \ln \left(\frac{y W^{**}}{v \{1 \pm y/r_w\}} \right) + B \right], \quad (2.42)$$

stem from how the integration constant, C , of Equation (3.62) is incorporated into the log law constant, B . The present study utilizes the definition of Kind et al. (1989),

$$B_0 = C' - \frac{1}{k} \ln(r_w^+), \quad (3.64)$$

where $C' = r_w^+ C$ is a constant. The tangential logarithmic law for a convex wall and a concave wall are then

$$\frac{W}{W^{**}} = \{1 \pm y/r_w\} \left[\frac{1}{k} \ln \left(\frac{y Q^+}{v \{1 \pm y/r_w\}} \right) + B_0 \right] \quad (3.65a)$$

and

$$\frac{W}{W^{**}} = \{1 - y/r_w\} \left[\frac{1}{k} \ln \left(\frac{yQ^*}{v\{1 - y/r_w\}} \right) + B_0 \right], \quad (3.65b)$$

respectively. From Equations (3.65a) and (3.65b), it is apparent that as the wall radius becomes large these relations reduce to the three-dimensional planar form, Equation (3.63). In addition, if the axial velocity then becomes zero, Equations (3.65a) and (3.65b) reduce to the two-dimensional log-law relation, Equation (1.3).

The tangential log-law relations, Equations (3.65a) and (3.65b), include the radius of wall curvature as a parameter. Note that the type of wall curvature, either convex or concave, has a considerable effect on the tangential velocity distribution. For example, Equations (3.65a) and (3.65b) will produce different tangential velocity distributions even if the wall radius and the shear velocity scales are the same. Furthermore, the influence of curvature in these equations is seen to increase as the radius of wall curvature decreases. Equations (3.65a) and (3.65b) are used to analyze the cylindrical flows presented in Chapter 5.

3.8 COMPONENT LOGARITHMIC LAWS FOR A CARTESIAN COORDINATE SYSTEM

The component logarithmic laws for a three-dimensional planar flow are obtained from the differential equations given in Section 3.5:

$$\tau_{x,w} = \rho k y \sqrt{\frac{\tau_w}{\rho}} \frac{\partial U}{\partial y} \quad (3.54)$$

and

$$\tau_{z,w} = \rho k y \sqrt{\frac{\tau_w}{\rho}} \frac{\partial W}{\partial y}. \quad (3.55)$$

Equations (3.54) and (3.55) may be expressed in non-dimensional form by using the near-wall similarity scales U^{**} , W^{**} , and v/Q^* . Separating variables and integrating, the component log-law equations are

$$\frac{U}{U^{**}} = \frac{1}{k} \ln \left(\frac{\gamma Q^*}{v} \right) + B_s \quad (3.66)$$

and

$$\frac{W}{W^{**}} = \frac{1}{k} \ln \left(\frac{\gamma Q^*}{v} \right) + B_s \quad (3.67)$$

where,

$$U^{**} = \sqrt{\frac{\tau_{x,w}}{\rho}} \sqrt{\frac{\tau_{x,w}}{\tau_w}} \quad .$$

$$W^{**} = \sqrt{\frac{\tau_{z,w}}{\rho}} \sqrt{\frac{\tau_{z,w}}{\tau_w}} \quad .$$

and

$$Q^* = \sqrt{\frac{\tau_w}{\rho}} \quad .$$

As noted in Section 2.3.1, Johnston (1960) was the first to propose Equations (3.66) and (3.67) for use in three-dimensional turbulent boundary layers. However, during the early stages of this thesis, the present author first derived these equations without prior knowledge of Johnston's work. It will be shown in the following section that Equations (3.66) and (3.67) can be applied to any direction other than that of the resultant wall shear stress direction. In other words, the orientation of the x -axis for three-dimensional flows is largely arbitrary, but some restrictions apply. In the analysis of the three-dimensional planar flows presented in Chapter 4, Equations (3.66) and (3.67) are applied to the streamwise and crossflow coordinate directions, respectively.

3.9 A LOGARITHMIC LAW FOR THE RESULTANT VELOCITY

Two means of deriving a log-law equation for the resultant velocity, Q , are discussed. Both methods are based on the eddy-viscosity/mixing-length hypothesis discussed in Section 3.6.

The first approach uses the turbulent shear stress model presented in Section 3.6. The other method is to combine the component log-laws.

Consider an axisymmetric flow to demonstrate the first approach. Recall from Section 3.6 that Equation (3.49),

$$\left[\left\{ \frac{\partial U}{\partial r} \right\}^2 + \left\{ r \frac{\partial}{\partial r} \left(\frac{W}{r} \right) \right\}^2 \right]^{1/2} = \frac{1}{ky} \sqrt{\frac{\tau_w}{\rho}} = \frac{Q^*}{ky} \quad (3.49)$$

relates the resultant wall shear stress to the local resultant rate of strain for an axisymmetric flow. To obtain a relationship for the resultant velocity, the velocity components in Equation (3.49) must be expressed in terms of the resultant velocity, Q . For a given coordinate system orientation, the mean flow angle is defined as

$$\phi = \tan^{-1} \left(\frac{W}{U} \right). \quad (2.9)$$

The velocity components are then written in terms of the resultant velocity and the flow angle as

$$U = Q \cos \phi$$

and

$$W = Q \sin \phi.$$

Substitution of these velocity component definitions into Equation (3.49) produces

$$\left[\left\{ \frac{\partial (Q \cos \phi)}{\partial r} \right\}^2 + \left\{ r \frac{\partial}{\partial r} \left(\frac{Q \sin \phi}{r} \right) \right\}^2 \right]^{1/2} = \frac{Q^*}{ky}. \quad (3.68)$$

This relation does not lend itself to a simple analytical solution, since, in general both the resultant velocity and the flow angle are functions of the radial coordinate, r . Equation (3.68) demonstrates that it is difficult to obtain a resultant velocity log-law relationship for a three-dimensional skewed flow. As a result, the present study focused only on producing simple logarithmic relations for the velocity components.

It is interesting to note that Kind et al. (1989) derived a resultant velocity log law for an

annular swirling flow. This relation assumes that the mean flow is two-dimensional, in that the resultant wall shear stress and velocity both lie in the same plane throughout the boundary layer thickness. In order to obtain a simple first order differential equation similar to that of Equation (3.60),

$$\frac{\partial}{\partial y} \left(\frac{W}{r} \right) = \frac{1}{k y r} \sqrt{\frac{\tau_{\theta, w}}{\rho}} \sqrt{\frac{\tau_{\theta, w}}{\tau_w}} \quad (3.60)$$

an effective radius of curvature for the wall was determined. The effective radius of curvature, r_{eff} , was defined as the radius of curvature of the limiting streamline at the wall. The resulting differential equation then is

$$\frac{\partial}{\partial y} \left(\frac{Q}{r} \right) = \frac{1}{k y r} \sqrt{\frac{\tau_w}{\rho}} \quad (3.69)$$

where the radial coordinate, r , is

$$r = r_{eff} \pm y \quad (3.70)$$

Integrating Equation (3.69), Kind et al. (1989) obtained a resultant velocity log law,

$$\frac{Q}{Q^*} = \{1 \pm y/r_{eff}\} \left[\frac{1}{k} \ln \left(\frac{y Q^*}{v \{1 \pm y/r_{eff}\}} \right) + B \right] \quad (3.71)$$

where the plus and minus signs denote convex and concave walls, respectively. This relation provides a good correlation to the experimental data of Yowakim (1985) except for the 15 degree swirl flow (Kind et al. 1989).

The second approach to deriving a log-law for the resultant velocity utilizes the vector nature of the velocity. The magnitude of the resultant velocity vector can be expressed in terms of the velocity components as

$$Q = \sqrt{U^2 + W^2} \quad (2.8)$$

Similarly, the magnitude of the resultant wall shear stress may be expressed in terms of the wall shear stress components as

$$Q^* = \sqrt{U^{*2} + W^{*2}} . \quad (3.72)$$

Equations (2.8) and (3.72) can be combined to produce an expression for the resultant log-law in terms of the component log-laws,

$$\left\{ \frac{Q}{Q^*} \right\}^2 = \left\{ \frac{U}{U^{**}} \right\}^2 \left(\frac{U^{**}}{Q^*} \right)^2 + \left\{ \frac{W}{W^{**}} \right\}^2 \left(\frac{W^{**}}{Q^*} \right)^2 . \quad (3.73)$$

In order to use Equation (3.73) the component log-law laws and the wall shear stress components must be known. Note that Equation (3.73) may become complicated even if the component log-law equations are relatively simple. Thus, there seems to be no apparent advantage to developing a log-law equation for the resultant velocity. In fact, the component log-law equations appear to be more versatile. For instance, given a particular wall geometry, component log-laws can be developed according to the theory of Section 3.4. It seems that relatively simple component log-laws may be derived with an appropriate choice of coordinate system. This suggests that component log-law relations for complicated geometries may be possible if one uses general orthogonal curvilinear coordinates. Therefore, it appears that using component log-laws is more convenient and practical.

Equation (3.73) also provides some useful information about the component log-laws. Consider a three-dimensional planar flow in which the x -axis is aligned with the streamwise velocity direction. The component shear velocity scales may then be defined in terms of the resultant shear velocity and the wall shear stress angle as,

$$U^{**} = Q^* \cos \beta_w , \quad (2.12)$$

and

$$W^{**} = Q^* \sin \beta_w , \quad (2.13)$$

where

$$\beta_w = \tan^{-1} \left(\frac{\tau_{1,w}}{\tau_{2,w}} \right) \quad (2.17)$$

is the wall shear stress angle. From Equations (2.12) and (2.13) it is seen that one of the component shear velocity scales will become zero when its corresponding coordinate axis is aligned with the wall shear stress vector. This presents a problem in three-dimensional flows because the component wall shear stress may become zero while the corresponding velocity component is not zero. Thus, to avoid a singularity one must choose an alignment for the coordinate system such that neither of the axes is aligned with the wall shear stress.

CHAPTER 4

TURBULENT BOUNDARY LAYERS ON PLANAR SURFACES

This Chapter presents law-of-the-wall similarity plots based on the similarity scales U^{**} , W^{**} , and v/Q^* , introduced in Section 3.6. First, a sample of two-dimensional turbulent boundary layer data are considered. Following this, the three-dimensional turbulent boundary layer data of McAllister (1979) are analyzed. Particular issues, such as the validity of the component log-law equations, derived in Section 3.8, are addressed.

4.1 LAW-OF-THE-WALL SIMILARITY FOR TWO-DIMENSIONAL FLOWS

In general, three-dimensional flows are much more difficult to predict than two-dimensional flows. Therefore, it seems reasonable to assume that a group of law-of-the-wall variables which do not successfully establish near-wall similarity in two-dimensional turbulent boundary layers (2DTBLs) will not correlate the near-wall region of three-dimensional turbulent boundary layers (3DTBLs). One can then eliminate certain three-dimensional near-wall variables by simply applying them to a sample of two-dimensional flows. As a result, this section generates law-of-the-wall plots, based on the scales U^{**} , W^{**} , and v/Q^* , for a few two-dimensional flows. The sample of two-dimensional flows used in this study is given next.

In the present study, four two-dimensional turbulent boundary layer experiments were chosen from the literature: those by Bradshaw & Ferriss (1965), Schubauer & Spangenberg (1960), Weighardt & Tillman (1944), and Herring & Norbury (1967). These experiments were selected in order to cover a range of streamwise pressure gradients, dP/dx , ranging from adverse to favorable. Note that only one streamwise data station, x , was analyzed for each experiment. Moreover, the mean velocity profile, $U(y)$, for each experiment was obtained from the first *Stanford Conference on Turbulent Flows* (Coles and Hirst, 1968).

Based on the definition of Clauser (1954 and 1956), the experimental flows of Bradshaw & Ferriss (1965), Weighardt & Tillman (1944), and Herring & Norbury (1967) are considered to be equilibrium boundary layers (Coles and Hirst, 1968). In this case, the term "equilibrium boundary layer" denotes a boundary layer which is in turbulent equilibrium. That is, the flow is dynamically similar at each streamwise station, x . Clauser demonstrated that a two-dimensional turbulent boundary layer is in turbulent equilibrium if the pressure gradient parameter, β_c , remains constant for each streamwise station, x . The Clauser equilibrium parameter, β_c , is defined as

$$\beta_c = \frac{\delta^*}{\tau_w} \frac{dP}{dx}, \quad (4.1)$$

where,

$$\delta^* = \int_0^{\delta} \{1 - U/U_e\} dy \quad (4.2)$$

is the displacement thickness, τ_w is the wall shear stress, dP/dx is the streamwise pressure gradient, and U_e is the velocity at the boundary layer edge. From a computational perspective, equilibrium flows are generally regarded as being relatively easy to predict, since they contain no history effects.

As mentioned in Chapter 1 and Section 2.1, law-of-the-wall similarity in the form

$$\frac{U}{U_\tau} = f\left(\frac{yU_\tau}{\nu}\right) \quad (1.1)$$

is rather well established for two-dimensional turbulent boundary layers. Clearly, one must first determine the value of the wall shear stress in order to calculate the shear velocity scale, U_τ . For the sample of 2D flows considered in this study, the two-dimensional shear velocity scale, U_τ , was calculated from the skin friction coefficient as

$$U_\tau = \sqrt{\frac{\tau_w}{\rho}} = U_e \sqrt{\frac{C_f}{2}}, \quad (4.3)$$

where C_f is defined as

$$C_f = \frac{2\tau_w}{\rho U_\tau^2} . \quad (4.4)$$

Note that these skin friction values were obtained by the Modified Clauser plot technique outlined in Appendix C. Modified Clauser plots for the sample of 2DTBLs are provided in Figures 4.1 and 4.2. In addition, the skin friction values are listed in Table 4.1. A traditional law-of-the-wall plot, according to Equation (1.1), for the sample 2D data is shown in Figure (4.3). From Figure (4.3) it is apparent that data collapse onto a single curve throughout the near-wall region. Moreover, the log-law equation,

$$\frac{U}{U_\tau} = \frac{1}{k} \ln \left(\frac{\gamma U_\tau}{\nu} \right) + B , \quad (1.3)$$

is seen to correlate the data rather well in the fully turbulent portion of the near-wall layer. The component law-of-the-wall velocity scales, U^{**} and W^{**} , were calculated from U_τ , as described next.

A simple test for component near-wall similarity is to divide a two-dimensional flow into two mean velocity components. In the present investigation, the coordinate system for each of the sample of 2D flows was rotated in the X - Z plane to produce two mean velocity components, U and W . Clearly, a rotation of the coordinate system does not produce a three-dimensional boundary layer. The sign convention for the coordinate system rotation angle, ψ , is shown in Figure 4.4. A clockwise rotation is considered positive. Accordingly, the component shear velocity scales, U^{**} and W^{**} , in the rotated coordinate system are

$$U^{**} = Q^* \cos \psi , \quad (4.5)$$

and

$$W^{**} = Q^* \sin \psi , \quad (4.6)$$

where,

Figure 4.1 Modified Clauser Plot for the Determination of the Skin Friction Coefficient, C_f .

Figure 4.2 Modified Clauser Plot for the Determination of the Skin Friction Coefficient, C_f .

<i>Experiment</i>	<i>Streamwise Distance x</i>	<i>U_e</i>	<i>C_f</i>
<i>Bradshaw and Ferriss (1965) $\alpha = -0.255$ {# 2600}</i>	6.917 ft	98.3 ft/s	0.00132
<i>Schubauer and Spangenberg (1960) Flow B {# 4500}</i>	6.667 ft	61.04 ft/s	0.00135
<i>Weighardt and Tillman (1944) Flat Plate {# 1400}</i>	1.987 m	33.00 m/s	0.00275
<i>Herring and Norbury (1967) $\beta = -0.35$ {# 2700}</i>	4.000 ft	97.10 ft/s	0.00340

1968 Stanford Conference Identification Number

Table 4.1 Calculated Skin Friction Values for a Sample of Two-Dimensional Flows.

Figure 4.3 Semi-Logarithmic Plot for the Sample of Two-Dimensional U^+ velocity Profiles.

Figure 4.4 Nomenclature for a Coordinate System Rotation in the X - Z Plane.

$$Q^* = U_* . \quad (4.7)$$

Component law-of-the-wall plots, using the similarity scales U^{**} , W^{**} , and v/Q^* , for the flow of Bradshaw and Ferriss (1965) are given in Figures 4.5 and 4.6. These figures illustrate that near-wall similarity, in both velocity components, is maintained for an X - Z coordinate system rotation. Based on these results, the similarity scales, U^{**} , W^{**} , and v/Q^* , appear to be good candidates for establishing law-of-the-wall similarity in three-dimensional flows. For comparison, component law-of-the-wall plots based on the near-wall similarity scales of Yowakim (1985) and Kind et al. (1989), U^{**} , W^{**} , v/U^{**} , and v/W^{**} , are given in Figures 4.7 and 4.8. It is seen that, for a rotated coordinate system, the near-wall data for each velocity component does not collapse onto one curve. This suggests that v/U^{**} and v/W^{**} may not be the appropriate near-wall similarity component length scales for two- and three-dimensional flows. One may also arrive at these conclusions by realizing that the two-dimensional log-law, Equation (1.3), is still valid in the rotated coordinate system, as described next.

The more appropriate form of the near-wall similarity scales, and hence the component log-law equations, can be deduced by simply considering the vector nature of the resultant velocity. The resultant velocity, neglecting the small component of velocity normal to the wall, may be expressed as

$$Q^2 = U^2 + W^2 . \quad (2.8)$$

Using the shear velocity scales Q^* , U^{**} , and W^{**} , the non-dimensional form of Equation (2.8) is

$$\left\{ \frac{Q}{Q^*} \right\}^2 = \left\{ \frac{U}{U^{**}} \right\}^2 \left(\frac{U^{**}}{Q^*} \right)^2 + \left\{ \frac{W}{W^{**}} \right\}^2 \left(\frac{W^{**}}{Q^*} \right)^2 \quad (3.73)$$

or

$$\left\{ \frac{Q}{Q^*} \right\}^2 = \left\{ \frac{U}{U^{**}} \right\}^2 \cos^2 \psi + \left\{ \frac{W}{W^{**}} \right\}^2 \sin^2 \psi . \quad (4.8)$$

Figure 4.5 Effect of Coordinate System Rotation on the X-Direction Velocity Profiles for the Streamwise Station $x = 6.917$ ft.: Experiment of Bradshaw and Ferriss (1965).

Figure 4.6 Effect of Coordinate System Rotation on the Z-Direction Velocity Profiles for the Streamwise Station $x = 6.917$ ft.: Experiment of Bradshaw and Ferriss (1965).

Figure 4.7 Effect of Coordinate System Rotation on the X-Direction Velocity Profiles Using the Law-of-the-Wall Length Scale of Yowakim (1985) and Kind et al. (1989).

Figure 4.8 Effect of Coordinate System Rotation on the X-Direction Velocity Profiles Using the Law-of-the-Wall Length Scale of Yowakim (1985) and Kind et al. (1989).

From Equation (4.8), the component log-law equations must combine to reproduce the two-dimensional log-law,

$$\frac{Q}{Q^*} = \frac{1}{k} \ln \left(\frac{yQ^*}{\nu} \right) + B. \quad (4.9)$$

Also, if the two-dimensional log-law constants, k and B , have universal values then the component log-law equations must be of the form

$$\frac{U}{U^{**}} = \frac{W}{W^{**}} = \frac{1}{k} \ln \left(\frac{yQ^*}{\nu} \right) + B \quad (4.10)$$

in order to satisfy Equation (4.9).

The preceding analysis demonstrates that the component shear velocity scales, U^{**} and W^{**} , must be used in conjunction with the length scale, ν/Q^* , to yield near-wall similarity and to reproduce the logarithmic law in two-dimensional flows. The significance of these results is that the log-law relations of Yowakim (1985) and Kind et al. (1989), for swirling annular flows, should use ν/Q^* as the near-wall similarity length scale. Law-of-the-wall plots for the three-dimensional flows of McAllister (1979) are presented in the following sections.

4.2 THREE-DIMENSIONAL TURBULENT BOUNDARY LAYER DATA

The objective of this section is to familiarize the reader with the three-dimensional turbulent boundary layer experiments of McAllister (1979). Law-of-the-wall plots based on the near-wall scales, U^{**} , W^{**} , and ν/Q^* , will be presented in Sections 4.3 and 4.4. Note, however, that much of the information provided in this section will later be used to interpret the apparent range of validity of the component log-law relations, Equations (3.66) and (3.67). As a result, a detailed description of the experiments is provided.

4.2.1 Description of Experiments

McAllister (1979) performed three separate three-dimensional turbulent boundary layer experiments: for one pressure driven and two shear driven flows. The wind tunnel configuration used by McAllister is shown in Figure 4.9. A pressure driven three-dimensional flow was generated by a cylindrical teardrop body mounted perpendicular to a flat plate. Two shear driven flows were created by a swept step mounted on a flat plate. Sweep angles, α_0 , for the two shear driven flows were 45 degrees and 30 degrees. The teardrop body and swept step configurations are shown in Figures 4.10 and 4.11. Tunnel unit Reynolds numbers, $Re_{unit} = Q_e/\nu$, for the teardrop, 45° step, and 30° step flows were $1.322 \times 10^6(1/m)$, $1.333 \times 10^6(1/m)$, and $1.322 \times 10^6(1/m)$, respectively.

The mean flow quantities measured in each experiment included static pressure, velocity, and wall shear stress. No turbulence measurements were made. Most notably, direct wall shear stress measurements were obtained by a floating element wall shear stress meter (McAllister, 1979). This device measured the magnitude and the direction of the wall shear stress vector, τ_w , simultaneously. Pressure gradient, wall shear stress, and limiting wall velocity vector maps for each of the experiments are presented in Figures 4.12 - 4.17. Referring to Figure 4.12, the magnitude of the pressure gradient increases as the teardrop body is approached. In the case of the step flows, Figures 4.14 and 4.16 indicate that the magnitude of the pressure gradient decreases with the downstream distance from the step. Note that the pressure gradient and wall shear stress scales are different for the teardrop and the swept step figures.

In order to assist the reader in identifying the velocity profile for each data station, a brief description of the numbering system is provided. As a matter of consistency, the present study uses the velocity profile identification notation of McAllister (1979), as follows.

Teardrop Flow: ZX 0A

- 1) ZX denotes the grid location. Referring to Figure 4.18, Z indicates the column while X specifies the row.

Figure 4.9 Wind Tunnel Configuration Used by McAllister (1979).
(Obtained from McAllister 1979)

Figure 4.10 Teardrop Body Used to Create the Pressure Driven Flow.
(Obtained from McAllister 1979)

Figure 4.11 Swept Step Used to Create the Shear Driven Flows, $\alpha_0 = 45^\circ$ and 30° .
(Obtained from McAllister 1979)

Figure 4.12 Pressure Gradient Map for One Quadrant of the Teardrop Body. (Obtained from McAllister 1979)

Figure 4.13 Wall Shear and Velocity Vector Map for the Teardrop Body. (Obtained from McAllister 1979)

Figure 4.15 Wall Shear and Velocity Vector Map Behind the 45° Step. (Obtained from McAllister 1979)

Figure 4.14 Pressure Gradient Map Behind the 45° Step. (Obtained from McAllister 1979)

Figure 4.16 Pressure Gradient Map Behind the 30° Step.
(Obtained from McAllister 1979)

Figure 4.17 Wall Shear and Velocity Vector Map Behind the
30° Step. (Obtained from McAllister 1979)

2

MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS
STANDARD REFERENCE MATERIAL 1963-Oa
(ANSI and ISO TEST CHART No. 2)

Swept Step Flows: SNN Z X A

- 1) S45 and S30 designate the 45° and 30° swept steps, respectively.
- 2) Z and X, given in inches, denote the lateral distance from the tunnel centerline and the distance from the back of the step, respectively.

In each of these definitions, A is the velocity profile number. Since the velocity profiles were measured twice at each location, A may be either 1 or 2. A few velocity profiles for each experiment are given next.

Two-dimensional flow existed well upstream of the teardrop body and the swept steps. The flow became increasingly three-dimensional as the teardrop body was approached due to pressure generated spanwise vorticity. Typical streamwise and crossflow velocity profiles for the teardrop flow are shown in Figures 4.18 and 4.19. The distribution of the flow angle, ϕ , given in Figure 4.20, indicates that the flow is skewed throughout the boundary layer thickness for the teardrop flow.

For the two swept step experiments, the flow travels over the step and reattaches on the lee side of the plate. Typical streamwise and crossflow velocity profiles for the 45° and 30° steps are shown in Figures 4.21 - 4.22 and 4.24 - 4.25, respectively. The corresponding flow angle, ϕ , distribution for these data stations is given in Figures 4.23 and 4.26. Recall from Section 2.2, that the term "collateral" refers to a two-dimensional flow. That is, the velocity vectors lie in a plane. In contrast to the teardrop flow, Figures 4.23 and 4.26 indicate that a collateral region exists for some portion of the near-wall region, but the collateral region does not necessarily extend completely to the wall. The existence of a collateral near-wall region in these flows will be discussed further in Section 4.4. In general, it seems that only data stations in areas of relatively small pressure gradient exhibited a collateral near-wall region.

In conclusion, the primary motive for McAllister's work was to establish the validity of the law-of-the-wall similarity models available in the literature. Accordingly, McAllister recognized that the most definitive means of accomplishing this is to measure the wall shear stress vector, τ_w , directly. This then allows one to calculate law-of-the-wall velocity scales without any

Figure 4.18 Typical Streamwise Velocity Profiles for the Teardrop Flow of McAllister (1979).

Figure 4.19 Typical Crossflow Velocity Profiles for the Teardrop Flow of McAllister (1979).

Figure 4.20 Distribution of Flow Angle, ϕ , for a Few Data Stations. Teardrop Flow of McAllister (1979).

Figure 4.21 Typical Streamwise Velocity Profiles for the 45° Step Flow of McAllister (1979).

Figure 4.22 Typical Crossflow Velocity Profiles for the 45° Step Flow of McAllister (1979).

Figure 4.23 Distribution of Flow Angle, ϕ , for a Few Data Stations. 45° Step Flow of McAllister (1979).

Figure 4.24 Typical Streamwise Velocity Profiles for the 30° Step Flow of McAllister (1979).

Figure 4.25 Typical Crossflow Velocity Profiles for the 30° Step Flow of McAllister (1979).

Figure 4.26 Distribution of Flow Angle, ϕ , for a Few Data Stations. 30° Step Flow of McAllister (1979).

bias towards a particular model or equation. One must acknowledge, however, that a direct measure of both the magnitude and direction of the wall shear stress vector is a particularly difficult task. Specifically, one should note that McAllister did not determine the effect of pressure gradients on the accuracy of his floating element wall shear stress meter. As a result, it is this author's opinion that the accuracy of the wall shear stress vector measurements in regions of significant pressure gradients may be suspect. For a discussion of the problems associated with floating element devices, the reader is referred to Winter (1978). The uncertainty in the wall shear stress measurements is an important topic, as discussed in the following section.

4.3 LAW-OF-THE-WALL SIMILARITY PLOTS FOR THREE-DIMENSIONAL FLOWS

This section presents law-of-the-wall plots based on the similarity scales U^{**} , W^{**} , and v/Q^* , developed in Section 3.6. One must recognize that establishing the validity of similarity variables and models is largely a graphical approach. Therefore, law-of-the-wall plots are used extensively in this section and throughout the remainder of this study.

Component law-of-the-wall plots, using the similarity scales U^{**} , W^{**} , and v/Q^* , for each of the 3DTBL experiments of McAllister (1979) are provided in Figures 4.27 - 4.32. These figures contain a sample of data stations and represent the worst and typical good near-wall similarity agreement. Each figure contains both the streamwise and crossflow velocity components. The velocity components are calculated with respect to the streamwise coordinate system shown in Figure 3.3. As a result, the crossflow velocity profile is the one that becomes zero at the boundary layer edge. The viscous sublayer equations,

$$U^+ = y^+ \quad (A.10)$$

and

$$W^+ = y^+ , \quad (A.11)$$

are also included in each law-of-the-wall plot. The derivation of these equations is given in Appendix A. A discussion concerning these similarity plots follows.

Figure 4.27 Worst Near-Wall Similarity Agreement for the Streamwise and Crossflow Velocity Profiles: Teardrop Flow Experiment of McAllister (1979).

Figure 4.28 Typical Good Near-Wall Similarity Agreement for the Streamwise and Crossflow Velocity Profiles: Teardrop Flow Experiment of McAllister (1979).

Figure 4.29 Worst Near-Wall Similarity Agreement for the Streamwise and Crossflow Velocity Profiles: 45° Step Flow Experiment of McAllister (1979).

Figure 4.30 Typical Good Near-Wall Similarity Agreement for the Streamwise and Crossflow Velocity Profiles: 45° Step Flow Experiment of McAllister (1979).

Figure 4.31 Worst Near-Wall Similarity Agreement for the Streamwise and Crossflow Velocity Profiles: 30° Step Flow Experiment of McAllister (1979).

Figure 4.32 Typical Good Near-Wall Similarity Agreement for the Streamwise and Crossflow Velocity Profiles: 30° Step Flow Experiment of McAllister (1979).

The pressure gradient maps shown in Figures 4.12, 4.14, and 4.16 indicate that some of the stations shown in Figures 4.27, 4.29, and 4.31 experience significant pressure gradients. In general, only data stations in regions of small pressure gradient exhibited near-wall similarity. Uncertainty estimates in the law-of-the-wall variables, U/U^{**} , W/W^{**} , and yQ^*/ν , for a few data stations are given in Figures 4.33 - 4.35. The calculation of the uncertainty is outlined in Appendix B. These figures show that the data stations which experience large pressure gradients do not lay completely within the uncertainty limits of those which experience relatively small pressure gradients. Furthermore, the uncertainty analysis indicates that the uncertainty in the wall shear stress vector, τ_w , is by far the largest contributor to the uncertainty in the law-of-the-wall similarity variables. According to Winter (1978), pressure gradient effects are a potential source of error in floating element wall shear stress measuring devices. Consequently, it is this author's opinion that the measured wall shear stress data in the regions of substantial pressure gradient are not reliable. Hence, in terms of the present investigation, only the near zero pressure gradient data stations were used to draw conclusions about the success of the near-wall similarity scales.

Figures 4.28, 4.30, and 4.32 are component law-of-the-wall plots for a sample of data stations in regions of relatively small pressure gradient. Near-wall similarity is seen to exist for both velocity components in the teardrop and the swept step flows. However, in the case of the teardrop flow, only the streamwise velocity component, U , contains a log-linear y^+ region. In contrast, the swept step flows exhibit log-linear y^+ regions in both the streamwise and crossflow velocity components, U and W .

In the strict mathematical sense, similarity implies that all the velocity profiles must collapse onto one curve when non-dimensionalized by an appropriate set of similarity scales. Figures 4.28, 4.30, and 4.32 show that there is a distinct difference in the near-wall similarity for the crossflow profiles between the teardrop and the swept step flows. In particular, only the crossflow velocity profile of the swept step flows contain a log-linear y^+ near-wall region. Since the similarity curves in the near-wall region are different for the two types of flows, one may assume that secondary effects are important. Consequently, it seems that other variables must be included in the list of independent parameters used to describe the near-wall region in order to develop a general near-wall similarity model for three-dimensional turbulent boundary layers.

Figure 4.33 Typical Uncertainty in the Near-Wall Similarity Variables: Teardrop Flow of McAllister (1979).

Figure 4.34 Typical Uncertainty in the Near-Wall Similarity Variables: 45° Step Flow of McAllister (1979).

Figure 4.35 Typical Uncertainty in the Near-Wall Similarity Variables: 30° Step Flow of McAllister (1979).

The findings of this section indicate that the law of the wall for three-dimensional flows does not appear to be as robust as that for two-dimensional flows. It seems that the extent of the law-of-the-wall region may be considerably reduced in three-dimensional flows. A possible explanation for a lack of 3D near-wall similarity, based on turbulence equilibrium arguments, is given in Section 6.6. Nevertheless, the swept step flow data is encouraging in that the law-of-the-wall plots display a log-linear y^+ region. Presumably, this log-linear region lies in the fully turbulent portion of the near-wall region. One then expects that the velocity distribution can be predicted by the component log-law equations, Equations (3.66) and (3.67), derived in Section 3.8.

4.4 COMPONENT LOG-LAW EQUATIONS IN THREE-DIMENSIONAL FLOWS

This section applies the component log-law equations, developed in Section 3.8, to the swept step data. In particular, the slope and intercept of the log-law lines and the skew of the flow are considered.

Law-of-the-wall plots for the 45° and 30° step flows are shown in Figures 4.36 and 4.37, respectively. A comparison of log-law lines using the two-dimensional constants of Patel (1965), $k = 0.42$ and $B = 5.5$, with those determined from the step flows is given in Figure 4.38. The slope of the log-law lines seems to be quite different from that of the two-dimensional value of Patel (1965). Moreover, the log-law intercepts are very different for the streamwise and the crossflow directions. Comparing Figures 4.36 and 4.37, it appears that empirical values of $k = 0.7$, $B_x = 10.0$, and $B_z = 6.0$, fit the 45° and 30° step data equally well. One should be aware, however, that these values were essentially obtained by visual inspection. In fact, a linear least-squares regression analysis was only conducted for one data station, S45 0 14 2. The resulting best fit lines, over a range of $y^+ = 33.8$ to 169.2, are shown in Figure 4.39. A linear least-squares regression analysis was not performed for the other data stations since the uncertainty in the law-of-the-wall variables for each velocity profile make it difficult to determine the exact limits of the log-law region. As a result, the slope, $1/k$, of the log-law lines was estimated as the average value of those obtained for the velocity components, U^+ and W^+ , of data station S45 0 14 2. The intercepts, B_x and B_z , of the log-law lines were then obtained by visual inspection.

Figure 4.36 Semi-Logarithmic Plot of the Streamwise and Crossflow Velocity Profiles for the 45° Step Flow Experiment of McAllister (1979).

Figure 4.37 Semi-Logarithmic Plot of the Streamwise and Crossflow Velocity Profiles for the 30° Step Flow Experiment of McAllister (1979).

Figure 4.38 Comparison of the Logarithmic-Law Lines Using Standard Two-Dimensional Constants With Those Determined from the Step Flows of McAllister (1979).

Figure 4.39 Calculated Best Fit Lines for the Data Station S45 0 14 2.

The flow angle, ϕ , as calculated from the component log-law equations,

$$U^+ = \frac{1}{k} \ln y^+ + B_x \quad (3.66)$$

and

$$W^+ = \frac{1}{k} \ln y^+ + B_z, \quad (3.67)$$

is

$$\phi(y) = \tan^{-1}\left(\frac{W}{U}\right) = \tan^{-1}\left(\frac{W^{**}}{U^{**}} \frac{\left\{\frac{1}{k} \ln(yQ^+/v) + B_z\right\}}{\left\{\frac{1}{k} \ln(yQ^+/v) + B_x\right\}}\right). \quad (4.11)$$

From Equation (4.11), it is apparent that the calculated flow angle, ϕ , is only equal to the wall shear stress angle, β_w , when the log-law intercepts, B_x and B_z , and slopes, $1/k$, are the same. That is, Equation (4.11) then would simply reduce to the two-dimensional case

$$\phi(y) = \tan^{-1}\left(\frac{\tau_w \sin(\beta_w)}{\tau_w \cos(\beta_w)}\right) = \beta_w. \quad (4.12)$$

Thus, based on the empirical values $k = 0.72$, $B_x = 10.0$, and $B_z = 6.0$, Equation (4.11) indicates that the flow angle, ϕ , may be quite different from that of the wall shear stress angle, β_w , in the log-law region. However, it is also apparent that the log-law equations do not predict skewing of the mean flow. To illustrate this point, consider the variation of the ratio,

$$\frac{\left\{\frac{1}{0.72} \ln(yQ^+/v) + 6.0\right\}}{\left\{\frac{1}{0.72} \ln(yQ^+/v) + 10.0\right\}} \quad (4.13)$$

across the range $y^+ = 70$ to 300. Equation (4.13) evaluated at $y^+ = 70$ and $y^+ = 300$ is 0.748 and 0.777, respectively. Therefore, Equations (3.66) and (3.67) can only be applied to the portion of the near-wall region in which the flow angle does not change significantly. The present study will use the term "nearly-collateral" to describe such a region.

It should be acknowledged that there is no consensus among researchers as to the range or even the existence of a collateral, or nearly-collateral, region in a three-dimensional turbulent boundary layer (Degani et al., 1992). Nevertheless, the flow angle figures for the swept step flows of McAllister (1979), Figures 4.23 and 4.26, indicate that a nearly-collateral region exists in these flows, but that the flow is not collateral throughout the near-wall region. In fact, Equation (4.11), demonstrates that different values for the log-law intercept, in the streamwise and crossflow directions, will result in a nearly-collateral region in which the flow angle is different from that of the wall shear stress angle. Figures 4.23 and 4.26 seemingly support the hypothesis that a collateral region can exist solely in the fully turbulent portion of the near-wall region. The flow could then conceivably continue to skew through the viscous sublayer. In general, the flow in a 3DTBL will skew continuously to the wall. As a result, the existence of a collateral near-wall region must be viewed as an exception rather than a rule. However, this does not mean that research into developing correlations for the log-law slope and intercept should not be pursued. Suggestions for correlating the slope of the log-law and intercept of the log-law lines are given in Section 4.5.

4.4.1 Interpreting k as a Vector

From Figures 4.36 and 4.37, the slope of the streamwise and crossflow log-law lines appear to have the same value, $1/k = 1.39$. This observation is consistent with the eddy-viscosity/mixing-length model presented in Section 3.6. However, if the slopes are determined to be different for the streamwise and crossflow directions, the mixing length model presented in Section 3.6 should be modified to accommodate k as a vector. The component mixing lengths and eddy viscosities might then be of the form,

$$l_{xy} = k_x y \sqrt{\frac{\tau_w}{\tau_{x,w}}} \sqrt{\frac{\tau'_{xy}}{\tau_{x,w}}} , \quad (4.14)$$

$$l_{xy} = k_s y \sqrt{\frac{\tau_w}{\tau_{x,w}}} \sqrt{\frac{\tau'_{xy}}{\tau_{x,w}}} \quad (4.15)$$

$$\mu_{xy} = \rho l_{xy}^2 \left| \frac{\partial U}{\partial y} \right| \quad (4.16)$$

and

$$v_{xy} = l_{xy}^2 \left| \frac{\partial W}{\partial y} \right| \quad (4.17)$$

respectively. The component log-law equations,

$$\frac{U}{U^*} = \frac{1}{k_s} \ln \left(\frac{y Q^*}{\nu} \right) + B_s \quad (4.18)$$

and

$$\frac{W}{W^*} = \frac{1}{k_s} \ln \left(\frac{y Q^*}{\nu} \right) + B_s \quad (4.19)$$

are then obtained by substituting the eddy viscosities definitions of Equations (4.16) and (4.17) into the shear stress models,

$$-\rho \overline{uv} = \mu_{xy} \frac{\partial U}{\partial y} \quad (4.20)$$

and

$$-\rho \overline{wv} = \mu_{xy} \frac{\partial W}{\partial y} \quad (4.21)$$

and integrating the result. Note that these mixing length, eddy viscosity, and turbulent shear stress definitions are different from those presented in Section 3.6.

4.5 DETERMINATION OF k , B_x , AND B_z

From Figures 4.36 and 4.37, the slope and intercept of the component log-law lines for the swept step flows are much different from those of the standard two-dimensional constants, $k = 0.42$ and $B = 5.5$. Moreover, it seems that the slope and intercept of the log-law lines in three-dimensional flows can not be regarded as universal constants but must be viewed as variables. For instance, it is unlikely that the slope and intercept are step functions in which they take on different constant values in three-dimensional and two-dimensional flows. The two-dimensional constants should be limiting values for the more general three-dimensional case as the degree of three-dimensionality becomes small. Thus, in an effort to extend the usefulness of the component log-law equations to 3DTBLs, a few variables for correlating the constant, k , and the intercept, B , were considered.

One data station, S30 0 14 2, from the 30° step flow experiment was subjected to an X - Z coordinate system rotation to determine the effect on the slope of the log-laws. The nomenclature for the coordinate system rotation is shown in Figure (4.4). A clockwise rotation is considered positive. Component law-of-the-wall plots for the rotated coordinate system are given in Figures 4.40 and 4.41. These figures indicate that, for the step flow experiment, the slope of the log-law lines, k , remains relatively constant. Thus, in this case, the slope can be considered to be invariant with respect to a coordinate system rotation. This may prove to be a useful concept in future research, particularly when dealing with general orthogonal curvilinear coordinates. For example, if one can correlate the log-law slope and intercept in a streamline coordinate system then perhaps these values can be transformed to another coordinate system which is not fixed to the streamwise direction.

Since the slope of the component log-law equations seemed to be the same for both velocity components and invariant with respect to an X - Z coordinate system rotation, k was assumed to be a scalar. This assumption is consistent with the derivation of the log-law lines presented in Chapter 3. Two attempts were then made to correlate k based on a single scalar variable. As a matter of computational convenience, the present study chose variables which could be easily calculated from mean flow velocity data. The variables used were

Figure 4.40 Effect of an X-Z Coordinate System Rotation on the Streamwise Velocity Profiles for the Data Station S30 0 14 2: Experiment of McAllister (1979).

Figure 4.41 Effect of an X-Z Coordinate System Rotation on the Crossflow Velocity Profiles for the Data Station S30 0 14 2: Experiment of McAllister (1979).

$$(\beta_w - \bar{\phi}) \quad (4.22)$$

and

$$\frac{2I_{20}}{I_{10}^2} \quad (4.23)$$

The first variable, Equation (4.22), is the difference between the wall shear stress angle, β_w , and the average value of the flow angle, ϕ , in the log-law region. The variation of the flow angle in the apparent log-law region, for the data stations considered, was typically less than one degree. The second variable, Equation (4.23), is calculated from the invariants of the momentum thickness tensor. The momentum thickness tensor in Cartesian coordinates is (Nash and Patel, 1972)

$$\theta_{ij} = \frac{1}{Q_e^2} \int_0^{\delta} \{U_{e,i} - U_i\} U_j dy \quad (i, j = x, z, y) \quad (4.24)$$

where Q_e is the resultant velocity at the boundary layer edge, and U_i is the velocity component. The first and second invariants of the cartesian momentum thickness tensor for a three-dimensional boundary layer then are

$$I_{10} = \theta_{xx} + \theta_{zz} \quad (4.25)$$

and

$$I_{20} = \frac{1}{2} (\theta_{xx}^2 + \theta_{zz}^2 - 2\theta_{xx}\theta_{zz}) \quad (4.26)$$

Similarly, an effort was made to correlate the intercept of the log-law lines, B_x and B_z , as vector components based on the shape factors $\delta_x^*/(\theta_{xx} + \theta_{zz})$ and $\delta_z^*/(\theta_{xx} + \theta_{zz})$ where

$$\delta_i^* = \frac{1}{Q_e} \int_0^{\delta} \{U_{e,i} - U_i\} dy \quad (i = x, z, y) \quad (4.27)$$

is the displacement thickness vector. The results of these correlation attempts are given in Figures 4.42 - 4.45. Note that these plots use the empirical values, $k = 0.72$, $B_x = 10.0$, and $B_z = 6.0$, determined from Figures 4.36 and 4.37. From Figures 4.42 - 4.45, there is a substantial amount of scatter in the data and, as a result, no definitive correlation was obtained by these variables.

Figure 4.42 Plot to Correlate the Scalar Slope, k , of the Logarithmic-Law Line Based on the Scalar Variable $\beta_w - \bar{\phi}$.

Figure 4.43 Plot to Correlate the Scalar Slope, k , of the Logarithmic-Law Line Based on the Scalar Variable $2I_{2\theta}/I_{1\theta}^2$.

Figure 4.44 Plot to Correlate the Intercept, B_x , of the Streamwise Logarithmic-Law Line.

Figure 4.45 Plot to Correlate the Intercept, B_z , of the Crossflow Logarithmic-Law Line.

Consequently, the present study did not produce correlating functions for the slope and intercept of the component log-law lines. Clearly, such trial and error methods are not very efficient.

An approach similar to that of Spalart (1988) appears to be a more promising method of accurately determining the slope, k , and intercept, B , of the component log-law lines. Spalart (1988) suggested that the slope and range of the logarithmic law region, for two-dimensional turbulent boundary layers, can be determined from a $y\partial U/\partial y$ versus y plot. This approach has much in common with the Clauser plot method for determining C_f in that the value of k is determined graphically. An adapted version of this method, for use with the velocity components of a three-dimensional boundary layer follows.

According to Section 3.5, the mean velocity gradients in the log-law region of a 3DTBL may be expressed as

$$\frac{\partial U}{\partial y} = \frac{U^{**}}{ky} \quad (3.54)$$

and

$$\frac{\partial W}{\partial y} = \frac{W^{**}}{ky} . \quad (3.55)$$

In non-dimensional form these equations are

$$\frac{\partial U^*}{\partial y^*} = \frac{1}{ky^*} \quad (4.28)$$

and

$$\frac{\partial W^*}{\partial y^*} = \frac{1}{ky^*} . \quad (4.29)$$

If the velocity gradients and the wall shear stress vector are known, figures for each velocity component may be generated from the relations

$$y^* \frac{\partial U^*}{\partial y^*} = f(y^*) \quad (4.30)$$

and

$$y^+ \frac{\partial W^+}{\partial y^+} = g(y^+) . \quad (4.31)$$

where f and g are functions which must be determined from experimental data. Referring to Equations (4.30) and (4.31), the log-law intercept, k , can then be determined from the flat near-wall region of these $y^+(\partial U^+/\partial y^+)$ versus y^+ and $y^+(\partial W^+/\partial y^+)$ versus y^+ plots. Typical figures that one may expect for the streamwise and crossflow directions of a streamline coordinate system are shown in Figures 4.46 and 4.47. The log-law intercepts, B_x and B_z , are then calculated from the component log-law equations by selecting a y_0^+ value which lies in the log-law region:

$$B_x = U_0^+ - \frac{1}{k} \ln(y_0^+) \quad (4.32)$$

and

$$B_z = W_0^+ - \frac{1}{k} \ln(y_0^+) . \quad (4.33)$$

where U_0^+ and W_0^+ are the corresponding values at y_0^+ . The primary advantage of this technique is that the value of k is determined from the flat portion of a plot rather than from the slope of a law-of-the-wall plot. In addition, this technique may help determine if the limits of the log-law region and the slope of the log-law lines are the same for both velocity components.

It is possible to use this method with experimental data. However, determining smooth first derivatives from experimental data is a difficult task. For instance, in order to produce smooth first derivatives, the experimental velocity profiles would have to be fitted with piecewise smooth functions. Discontinuities in the first derivative, at the junction of the piecewise functions, may then be a problem. Thus, it seems that this method is best suited to a computational investigation so that smooth functions for the mean velocity gradients, $\partial U/\partial y$ and $\partial W/\partial y$, can be calculated. Clearly, this would involve a 3D Navier-Stokes calculation. A turbulence model that one might use in such a study is discussed in Section 6.8. Such studies may eventually produce functions for the slope and intercept of the log-law lines in three-dimensional flows.

Figure 4.46 Hypothetical Plot to Determine the Slope, k , of the Logarithmic-Law Line for the Streamwise Direction.

Figure 4.47 Hypothetical Plot to Determine the Slope, k , of the Logarithmic-Law Line for the Crossflow Direction.

CHAPTER 5

ANNULAR FLOWS

This Chapter presents law-of-the-wall similarity plots for three annular flow experiments: Yowakim (1985), Lohmann (1976), and Bissonnette & Mellor (1974). These plots are based on the near-wall similarity scales, U^{**} , W^{**} , and v/Q^* , introduced in Chapter 3. A brief description of these experiments is provided.

5.1 DESCRIPTION OF EXPERIMENTS

The intent of this section is to familiarize the reader with the experimental flows of Yowakim (1985), Lohmann (1976), and Bissonnette & Mellor (1974). Law-of-the-wall similarity plots for these flows will be presented in the next section. The discussion concerning these law-of-the-wall plots will refer to much of the information provided in this section.

Schematic figures of the experimental arrangements for the flows of Yowakim (1985), Lohmann (1974), and Bissonnette & Mellor (1974) (hereafter referred to as BM) are given in Figures 5.1, 5.2, and 5.3, respectively. Yowakim (1985) investigated the development of swirling flows in an annular duct of constant inner and outer radii, r_i and r_o , respectively. The experiments of Lohmann (1976) and BM (1974) were concerned with the development of a turbulent boundary layer along the axis of a spinning cylinder. Although only data for the inner spinning cylinder were reported for the Lohmann and the BM experiments, the present study regards these experiments as annular flows since they were performed in a cylindrical annulus. A partial list of some of the measurements taken in each experiment is given in Table 5.1. Note that the Reynolds number given in Table 5.1 is defined as

$$Re_{D_h} = \frac{D_h U_{ref}}{\nu} , \quad (5.1)$$

where

Figure 5.1 Schematic Figure of the Experimental Arrangement of Yowakim (1985). (Obtained from Yowakim, 1985)

Figure 5.2 Schematic Figure of the Experimental Apparatus of Lohmann (1976). (Obtained from Lohmann, 1974)

Figure 5.3 Schematic Figure of the Experimental Apparatus of Bissonnette and Mellor (1974).
(Obtained from Bissonnette and Mellor, 1974)

<i>Experiment</i>	<i>Description</i>	<i>Mean Flow Velocity</i> U, W	<i>Reynolds Stress Tensor Components</i> $\overline{u^2}, \overline{v^2}, \overline{w^2}$ $-\overline{uv}, -\overline{wx}, -\overline{uw}$	<i>Annulus Radii</i> r_i, r_o (m)	<i>Reference Velocity, U_{ref}</i> (m/s)	<i>Reynolds Number, Re_{D_h}</i>
Bissonnette and Mellor (1974)	Flow along the axis of a spinning cylinder	All stations	All axial stations	0.0635, 0.203	19.48	3.5×10^5
Loimann (1976)	Flow along the axis of a spinning cylinder	All stations	Axial stations 3, 5, and 8 only	0.134, 0.254	17.53	2.7×10^5
Yowakim (1985)	Swirling flow in a cylindrical annulus	All stations	Axial stations 1, 3, and 5 only	0.127, 0.203	32.0	3.1×10^5

Table 5.1 Brief Summary of Annular Flow Experiments Indicating Some of the Measurements Taken.

$$D_h = 2(r_o - r_i) \quad (5.2)$$

is the hydraulic diameter and U_{ref} is a reference velocity. For the experiment of Yowakim, U_{ref} is the mean axial velocity. Whereas for the experiments of Lohmann and BM, U_{ref} is the velocity at the inlet to the test section. The experimental data of Lohmann (1976) and BM (1974) were obtained from the 1980-81 *Stanford Conference on Complex Turbulent Flows* (Kline et al., 1982).

Yowakim performed three separate swirling flow experiments with the arrangement shown in Figure 5.1. Each experiment was based on a different inlet flow angle: $\phi = 15^\circ$, 30° , and 45° . Swirl was imparted to the flow at the start of the test section by a row of inlet guide vanes. A row of outlet guide vanes was used to remove the swirl from the flow at the end of the test section. For swirling flows in a cylindrical annulus, the swirl number, SN , is defined as the ratio of the axial flux of tangential momentum to axial momentum divided by the outer wall radius, r_o ,

$$SN = \frac{\int_{r_i}^{r_o} U W r^2 dr}{\int_{r_i}^{r_o} r_o U^2 r dr} \quad (5.3)$$

Nominal initial swirl numbers, SN , for the three experiments of Yowakim were 0.23, 0.47, and 0.87 corresponding to the inlet flow angles of 15° , 30° and 45° , respectively. Clearly, the swirl number and flow angle decrease as the flow progresses downstream due to the retarding torque of the wall shear stress. Axial and tangential direction velocity profiles for the 45° swirl, $SN = 0.87$, flow are shown in Figures 5.4 and 5.5, respectively, and corresponding flow angle plots are given in Figures 5.6 and 5.7. Note that U_{max} and W_{max} are the respective maximum values of the velocity components in the annulus for a particular axial data station, x . In addition, the flow angle, ϕ , in Figures 5.6 and 5.7 is defined as

$$\phi = \tan^{-1} \left(\frac{W}{U} \right) \quad (5.4)$$

The velocity component figures indicate that only a few near-wall velocity measurements were made in these experiments. In fact, Yowakim (1985) states that the closest near-wall mean velocity measurement is at $y = 0.37 \text{ mm}$, which corresponds to an approximate law-of-the-wall

Figure 5.4 Radial Distribution of the Axial Velocity for $SN = 0.87$: Experiment of Yowakim (1985).

Figure 5.5 Radial Distribution of the Tangential Velocity for $SN = 0.87$: Experiment of Yowakim (1985).

Figure 5.6 Radial Distribution of Flow Angle, ϕ , at Different Axial Stations for $SN = 0.87$: Experiment of Yowakim (1985).

Figure 5.7 Radial Distribution of Flow Angle, ϕ , at Different Axial Stations for $SN = 0.87$: Experiment of Yowakim (1985).

coordinate value of $y^+ = 70$. Thus, it appears that no mean flow velocity measurements were taken in the viscous sublayer. The flow angle figures show that the degree of skew is very small in these flows. Consequently, the mean flow is essentially two-dimensional. Figures 5.4 - 5.7 are representative of the other swirl numbers, and the present study will only present the analysis results for the 45° swirl flow. Therefore, in the following section the observations and conclusions regarding law-of-the-wall similarity for the 45° swirl flow also apply to the 15° and 30° swirl flows unless otherwise stated.

The flows of Lohmann (1976) and BM (1974) will be described together since their experimental arrangements are similar. In these experiments, the test boundary layer develops along the axis of a cylinder. The cylinder is segmented into two components of the same diameter: a stationary section followed immediately by a section which rotates about its axis. On the stationary section, the mean velocity is two-dimensional and consists of only the axial velocity component, U . A transverse velocity, W , is produced when the flow encounters the rotating section. Thus, a three-dimensional boundary layer is created on the rotating section due to the sudden imposition of transverse shear. Since the present study is mainly concerned with three-dimensional boundary layers, only the data on the rotating portion of these experiments was used. The reader should note that both Lohmann (1976) and BM (1974) performed two experiments based on different cylinder rotation rates, ω . Each of these experiments is identified by a different rotation number,

$$\Omega = \frac{U_o}{W_w} , \quad (5.5)$$

where

$$W_w = \omega r_w \quad (5.6)$$

is the tangential velocity of the rotating cylinder at the wall and U_o is the velocity at inlet of the test section. However, in the present investigation only one rotation number was analyzed for each experiment since this was the only data provided in Kline et al. (1982). Consequently, the rotation number, Ω , and the tangential velocity at the wall, W_w , for the Lohmann experiment are 0.936 and 24.74 m/s, respectively. Similarly, for the BM experiment, the rotation number and tangential velocity are 1.45 and 18.14 m/s, respectively.

In the present investigation, the flows of Lohmann (1976) and BM (1974) are analyzed from a rotating frame of reference. The rationale for this decision, is to enforce the standard no-slip boundary layer condition, $W = 0$ at $y = 0$, at the wall. Furthermore, in terms of the mixing length theory presented in Chapter 3, a negative velocity gradient implies that the log-law slope, $1/k$, must also be negative. This is inconsistent with the physical interpretation of the mixing length, l . As a result, a relative frame of reference, rotating at a constant rotational velocity of ω , is used such that the tangential velocity at the wall, W_w , is zero. The tangential velocity in this rotating reference frame is then written as

$$W_{r'} = |W - \omega r| \quad (5.7)$$

where

$$r = r_w + y \quad (3.1)$$

is the radial coordinate, W is the tangential velocity measured in the inertial reference frame, and ω is the angular velocity of the rotating cylinder. Strictly speaking, the momentum equations for a cylindrical coordinate system presented in Section 3.2.1, should be modified for use in a relative frame of reference. However, in the present study, the form of these equations will not change since the body force is merely changed from a centrifugal to a Coriolis force. Axial and tangential velocity profiles for the experimental flows of Lohmann (1976) and BM (1974) are given in Figures 5.8 - 5.9 and 5.12 - 5.13, respectively. Note that the tangential velocity component is non-dimensionalized by ωr_{max} , where r_{max} is simply the last radial measurement position and should not be interpreted as the boundary layer edge. These figures indicate that the tangential velocity gradients in the near-wall region are quite large for the first few data stations due to the sudden imposition of transverse shear. Corresponding flow angle plots are given in Figures 5.10 - 5.11 and 5.14 - 5.15, respectively. These figures indicate that the flow is initially three-dimensional since the flow angle changes considerably across the boundary layer thickness, δ . In the rotating frame of reference, the flow is then seen to approach a two-dimensional state as it progresses along the axis of the rotating cylinder. Furthermore, for most of the data stations, a nearly-collateral region is seen to exist in what is probably the fully turbulent portion of the near-wall region. However, for some of these data stations, this nearly-collateral region does not extend completely to the wall. It is unclear at this time if this observation is result of the measurement uncertainties associated with the viscous sublayer data

Figure 5.8 Experimental Axial Velocity Profiles from Lohmann (1976).

Figure 5.9 Experimental Tangential Velocity Profiles from Lohmann (1976).

Figure 5.10 Distribution of the Flow Angle, ϕ , at Different Axial Stations for the Experiment of Lohmann (1976).

Figure 5.11 Distribution of the Flow Angle, ϕ , at Different Axial Stations for the Experiment of Lohmann (1976).

Figure 5.12 Experimental Axial Velocity Profiles from Bissonnette and Mellor (1974).

Figure 5.13 Experimental Tangential Velocity Profiles from Bissonnette and Mellor (1974).

Figure 5.14 Distribution of the Flow Angle, ϕ , at Different Axial Stations for the Experiment of Bissonnette and Mellor (1974).

Figure 5.15 Distribution of the Flow Angle, ϕ , at Different Axial Stations for the Experiment of Bissonnette and Mellor (1974).

or to actual skewing of the mean velocity in the viscous sublayer.

The following section will present component law-of-the-wall plots for these annular flow experiments based on the near-wall similarity scales U^{**} , W^{**} , and v/Q^* . One expects that such plots will correlate the experimental data of Yowakim (1985) since the mean flow is two-dimensional in this case. However, the experiments of Lohmann (1976) and BM (1974) are expected to be stronger tests for these near-wall similarity scales since the mean flow at some of the axial stations is substantially three-dimensional.

5.2 LAW-OF-THE-WALL SIMILARITY FOR ANNULAR FLOWS

This section applies the logarithmic laws derived in Section 3.7 to establish near-wall similarity in the three annular flow experiments. Particular issues, such as the calculation of the wall shear stress vector, τ_w , and the values of the slope, $1/k$, and intercept, B , of the log-law lines are discussed.

In Section 3.7 component logarithmic laws were developed for a cylindrical coordinate system. Recall from Section 3.7, that the axial and tangential direction log-law equations are

$$\frac{U}{U^{**}} = \frac{1}{k} \ln \left(\frac{yQ^*}{v} \right) + B_x \quad (3.59)$$

and

$$\frac{W}{W^{**}} = \left\{ i \pm y/r_w \right\} \left[\frac{1}{k} \ln \left(\frac{yQ^*}{v \{1 \pm y/r_w\}} \right) + B_s \right] \quad (3.65)$$

respectively, where

$$U^{**} = \sqrt{\frac{\tau_{x,w}}{\rho}} \sqrt{\frac{\tau_{x,w}}{\tau_w}} .$$

$$W^{**} = \sqrt{\frac{\tau_{\theta,w}}{\rho}} \sqrt{\frac{\tau_{\theta,w}}{\tau_w}} ,$$

and

$$Q^* = \sqrt{\frac{\tau_w}{\rho}} .$$

The plus and minus signs in Equations (3.59) and (3.65) denote convex and concave wall curvature, respectively. Clearly, in order to use these equations one must know the magnitude and direction of the wall shear stress vector, τ_w , such that the law-of-the-wall scales, U^{**} , W^{**} and v/Q^* , can be calculated. In the present study, the wall shear stress components, $\tau_{x,w}$ and $\tau_{\theta,w}$, were calculated from the experimental data, as follows.

The skin friction coefficient values,

$$C_{fx} = \frac{2\tau_{x,w}}{\rho Q_{max}^2} \quad (C.4)$$

and

$$C_{f\theta} = \frac{2\tau_{\theta,w}}{\rho Q_{max}^2} , \quad (C.5)$$

for each annular flow experiment were calculated by the component Modified Clauser plot technique outlined in Appendix C. Note that for the annular swirling flow experiment of Yowakim (1985), Q_{max} is defined as the maximum resultant velocity in the annulus for a given axial data station, x ,

$$Q_{max} = \sqrt{U^2 + W^2} \Big|_{max} . \quad (5.8)$$

Similarly, for the spinning cylinder experiments of Lohmann (1976) and BM (1974), the resultant velocity Q_{max} is defined as

$$Q_{max} = \sqrt{U_z^2 + (\omega r_{max} - W|_{r=r_{max}})^2} \quad (5.9)$$

for a given axial data station, x . These definitions of Q_{max} were chosen for computational convenience and should not be interpreted as the resultant velocity at the edge of the boundary layer. A sample of typical Modified Clauser plots for determining the component skin friction coefficients, C_{fx} and $C_{f\theta}$, are shown in Figures 5.16 - 5.21. Within the Modified Clauser plot method the region of constant C_f indicates data from the log-law region and the indicated C_f is expected to be the true one. These figures are representative of the agreement obtained for each experiment except for the first axial data station of both the Lohmann and the BM experiments. Modified Clauser plots for determining the skin friction parameter $C_{f\theta}$ at the first data station on the rotating section for both of these experiments are shown in Figures 5.22 and 5.23. Note that a clearly defined flat near-wall portion is not present in these plots, and as a result a value of $C_{f\theta}$ was not obtained for these data stations. An explanation for this is given in Section 6.5.1. Once the skin friction coefficients were determined a number of useful variables were calculated, as discussed next.

Wall shear stress data, calculated from the skin friction coefficients, C_{fx} and $C_{f\theta}$, for each of the annular flow experiments is given in Tables 5.2 - 5.5. Note that the shear scales U^* and W^* are defined as

$$U^* = \sqrt{\frac{\tau_{x,w}}{\rho}} \quad (5.10)$$

and

$$W^* = \sqrt{\frac{\tau_{\theta,w}}{\rho}} \quad (5.11)$$

As discussed later, these scales are used to non-dimensionalize the Reynolds shear stress terms $-\rho \overline{uv}$ and $-\rho \overline{wv}$. From Tables 5.2 - 5.5 it is seen that the calculated value of the wall shear stress angle, β_w , generally decreases with the downstream axial distance, x . This result seems to be consistent with all three of the annular flow experiments. For instance, in the case of the Yowakim's swirling flow experiment, one expects that the retarding torque of the wall shear stress

Figure 5.16 Modified Clauser Plot for Determining the Skin Friction Coefficient C_{fx} at the Axial Station $x = 22.0 \text{ in.}$ for $SN = 0.87$: Experiment of Yowakim (1985).

Figure 5.17 Modified Clauser Plot for Determining the Skin Friction Coefficient $C_{f\theta}$ at the Axial Station $x = 22.0 \text{ in.}$ for $SN = 0.87$: Experiment of Yowakim (1985).

Figure 5.18 Modified Clauser Plot for Determining the Skin Friction Coefficient C_{fx} at the Axial Station $x = 10.0 \text{ in.}$: Experiment of Lohmann (1976).

Figure 5.19 Modified Clauser Plot for Determining the Skin Friction Coefficient $C_{f\theta}$ at the Axial Station $x = 10.0 \text{ in.}$: Experiment of Lohmann (1976).

Figure 5.20 Modified Clauser Plot for Determining the Skin Friction Coefficient C_{fx} at the Axial Station $x = 4.0$ in.: Experiment of Bissonnette and Mellor (1974).

Figure 5.21 Modified Clauser Plot for Determining the Skin Friction Coefficient $C_{f\theta}$ at the Axial Station $x = 4.0$ in.: Experiment of Bissonnette and Mellor (1974).

Figure 5.22 Modified Clauser Plot for Determining the Skin Friction Coefficient $C_{f\theta}$ at the Axial Station $x = 1.0$ in.: Experiment of Lohmann (1976).

Figure 5.23 Modified Clauser Plot for Determining the Skin Friction Coefficient $C_{f\theta}$ at the Axial Station $x = 1.0$ in.: Experiment of Bissonnette and Mellor (1974).

<i>Station</i>	<i>1</i>	<i>2</i>	<i>3</i>	<i>4</i>	<i>5</i>
<i>x</i> (inches)	10.0	22.0	34.0	46.0	62.0
$Q_{\max}$ (m/s)	52.6	50.8	49.4	48.6	48.3
$C_{fx} \times 10^3$	2.13	2.06	2.05	2.04	2.08
$C_{f\theta} \times 10^3$	2.07	1.93	1.92	1.87	1.88
U^{**} (m/s)	1.452	1.394	1.351	1.335	1.341
W^{**} (m/s)	1.415	1.304	1.265	1.220	1.211
r_w^+	16415	15452	14981	14638	14625
U^* (m/s)	1.716	1.631	1.581	1.554	1.556
W^* (m/s)	1.694	1.578	1.530	1.485	1.479
$\tau_{x,w}/\tau_w$	0.716	0.730	0.730	0.738	0.742
$\tau_{\theta,w}/\tau_w$	0.698	0.683	0.683	0.674	0.670
β_w (deg.)	44.3	43.1	43.1	42.4	42.1

Table 5.2 Calculated Data for the Experiment of Yowakim (1985) $SN = 0.87$ Convex Wall.

Station	1	2	3	4	5
x (inches)	10.0	22.0	34.0	46.0	62.0
$Q_{\max}$ (m/s)	52.6	50.8	49.4	48.6	48.3
$C_{fx} \times 10^3$	2.13	2.22	2.28	2.43	2.47
$C_{f\theta} \times 10^3$	2.18	2.25	2.26	2.36	2.41
U^{**} (m/s)	1.434	1.419	1.403	1.435	1.434
W^{**} (m/s)	1.470	1.438	1.395	1.394	1.400
r_w^+	26569	26140	25598	25889	25927
U^* (m/s)	1.716	1.693	1.666	1.695	1.695
W^* (m/s)	1.737	1.704	1.661	1.670	1.675
$\tau_{x,w}/\tau_w$	0.698	0.703	0.709	0.717	0.715
$\tau_{\theta,w}/\tau_w$	0.716	0.712	0.705	0.697	0.699
β_w (deg.)	45.7	45.4	44.8	44.2	44.3

Table 5.3 Calculated Data for the Experiment of Yowakim (1985) $SN = 0.87$ Concave Wall.

Station	3	4	5	6	7	8
x (inches)	3.0	5.0	8.0	10.0	13.0	16.0
Q_{max} (m/s)	33.7	33.8	33.8	33.8	34.6	34.6
$C_{fx} \times 10^3$	1.20	1.20	1.27	1.24	1.29	1.30
$C_{f\theta} \times 10^3$	2.50	2.50	2.40	2.28	2.20	2.20
U^{**} (m/s)	0.544	0.545	0.581	0.581	0.625	0.629
W^{**} (m/s)	1.132	1.135	1.100	1.069	1.066	1.064
r_w^+	10817	10846	10710	10477	10642	10647
U^* (m/s)	0.826	0.828	0.850	0.841	0.879	0.882
W^* (m/s)	1.193	1.196	1.169	1.140	1.148	1.147
$\tau_{x,w}/\tau_w$	0.433	0.433	0.467	0.478	0.506	0.509
$\tau_{\theta,w}/\tau_w$	0.902	0.902	0.885	0.879	0.863	0.861
β_w (deg.)	64.4	64.4	62.2	61.5	59.6	59.4

Table 5.4 Calculated Data for the Experiment of Lohmann (1976).

<i>Station</i>	<i>2</i>	<i>3</i>	<i>4</i>	<i>5</i>	<i>6</i>
<i>x</i> (inches)	4.0	8.0	13.0	18.0	23.0
Q_{max} (m/s)	31.7	31.9	32.7	32.8	33.5
$C_{fx} \times 10^3$	1.85	1.79	1.68	1.63	1.58
$C_{f\theta} \times 10^3$	2.34	2.07	1.91	1.74	1.66
U^{**} (m/s)	0.757	0.772	0.768	0.722	0.781
W^{**} (m/s)	0.960	0.893	0.875	0.827	0.822
r_w^+	4989	4819	4750	4617	4629
U^* (m/s)	0.962	0.955	0.945	0.935	0.941
W^* (m/s)	1.084	1.027	1.009	0.967	0.966
$\tau_{x,w}/\tau_w$	0.619	0.654	0.660	0.682	0.689
$\tau_{\theta,w}/\tau_w$	0.785	0.756	0.751	0.731	0.725
β_w (deg.)	51.8	49.1	48.7	47.0	46.5

Table 5.5 Calculated Data for the Experiment of Bissonnette and Mellor (1974).

will reduce the flow angle as the flow progresses downstream. Similarly, in the spinning cylinder experiments of Lohmann and BM, the near-wall velocity gradients decrease with downstream distance as the flow adjusts to the spinning wall. Thus, it is expected that the wall shear stress angle will decrease with downstream distance due to a reduction in the tangential wall shear stress component. Having determined the wall shear stress components, law-of-the-wall plots are generated by using the near-wall similarity scales U^{**} , W^{**} , and v/Q^* .

Law-of-the-wall similarity plots for the experiment of Yowakim (1985) are shown in Figures 5.24 - 5.27. These figures show that the log-law relations, Equations (3.59) and (3.65), correlate the data in the fully turbulent portion of the near-wall region. As noted in Section 5.1 only a few near-wall velocity measurements were taken in this experiment. Moreover, these velocity measurements did not extend deep into the near-wall region. Thus, it is expected that the near-wall data begin in the log-law region, and this view seems to be supported by the lack of any viscous sublayer data in the law-of-the-wall plots. This is an important point if one considers using v/U^{**} and v/W^{**} as the near-wall similarity length scales. As discussed previously in Section 4.1, the length scale v/Q^* seems to be a more appropriate near-wall similarity length scale than v/U^{**} and v/W^{**} . A sample of law-of-the-wall similarity plots using the tangential log-law equations of Yowakim (1985) and Kind et al. (1989),

$$\frac{W}{W^{**}} = \frac{\{1 \pm y/r_w\}}{k} \ln \left(\frac{yW^{**}}{v\{1 \pm y/r_w\}} \right) + B_0 \quad (2.41)$$

and

$$\frac{W}{W^{**}} = \{1 \pm y/r_w\} \left[\frac{1}{k} \ln \left(\frac{yW^{**}}{v\{1 \pm y/r_w\}} \right) + B_0 \right]. \quad (2.42)$$

are given in Figures 5.28 and 5.29. Agreement with the experimental data in these figures appears to be good. However, this is partly due to the fact that the measured velocity data do not extend into the viscous sublayer. As demonstrated in Section 4.1, the effect of using v/U^{**} and v/W^{**} as the near-wall length scales is to shift the near-wall similarity data along the y^+ axis. This contradicts the concept of near-wall similarity since the viscous sublayer data then will not collapse onto a single curve. In terms of the Yowakim's experimental data, this distinction is not

Figure 5.24 Semi-Logarithmic Plot of the Convex Wall Axial Velocity Profiles for $SN = 0.87$: Experiment of Yowakim (1985).

Figure 5.25 Semi-Logarithmic Plot of the Convex Wall Tangential Velocity Profiles for $SN = 0.87$: Experiment of Yowakim (1985).

Figure 5.26 Semi-Logarithmic Plot of the Concave Wall Axial Velocity Profiles for $SN = 0.87$: Experiment of Yowakim (1985).

Figure 5.27 Semi-Logarithmic Plot of the Concave Wall Tangential Velocity Profiles for $SN = 0.87$: Experiment of Yowakim (1985).

Figure 5.28 Semi-Logarithmic Plot of the Convex Wall Tangential Velocity Profile at the Axial Station $x = 10.0 \text{ in.}$ for $SN = 0.23$: Experiment of Yowakim (1985).

Figure 5.29 Semi-Logarithmic Plot of the Convex Wall Tangential Velocity Profile at the Axial Station $x = 10.0 \text{ in.}$ for $SN = 0.87$: Experiment of Yowakim (1985).

easily detected since no velocity measurements were taken in the viscous sublayer. From the flow angle figures, Figures 5.6 and 5.7, it is apparent that the mean flow in Yowakim's experiment is essentially two-dimensional since the change in the flow angle, ϕ , across the annulus height is small. Recognizing that a two-dimensional flow is a relatively weak test of component near-wall similarity, one then expects that the component log-law relations, Equations (3.59) and (3.65), will be successful in this case. However, the mean flow in the experiments of Lohmann and of BM is three-dimensional and these experiments may be considered as more demanding tests for the component log-law equations.

Law-of-the-wall plots for the experiments of Lohmann (1976) and of BM (1974) are shown in Figures 5.30 - 5.35. Note that in each of these figures the component log-laws, Equations (3.59) and (3.65), use the standard two dimensional constants $k = 0.42$ and $B = 5.5$. The viscous sublayer equations,

$$U^+ = y^+ \quad (\text{A.14})$$

and

$$W^+ = \{y^+ + r_w^+\} \ln(1 + y^+/r_w^+) , \quad (\text{A.20})$$

are included in each figure. The derivation of these equations is given in Appendix A. The component log-law equations, Equations (3.59) and (3.65), appear to provide a good fit to the Lohmann data. The agreement with the BM data is less satisfactory and is progressively worse for each downstream data station. Law-of-the-wall plots for the last two data stations of the BM experiment are shown in Figures 5.34 and 5.35. The poorest log-law agreement is observed at these two data stations. Referring to Figures 5.34 and 5.35, the tangential direction log-law line seems to correlate the data at a y^+ range which is much larger than that of the axial direction. However, the author regards this correlation in the outer layer as a coincidence since the y^+ range is well beyond that of the expected law-of-the-wall range. One possible explanation for this poor agreement with the tangential velocity data of the BM experiment is that perhaps the slope of the tangential log-law line, $1/k$, is too large at these data stations. It seems that a better fit to the data would be obtained if the slope of the tangential log-law line was reduced. Another possibility is that perhaps the destabilizing effects of the transverse rotation become more important as the flow

Figure 5.30 Semi-Logarithmic Plot of the Axial Velocity Profiles for the Experiment of Lohmann (1976).

Figure 5.31 Semi-Logarithmic Plot of the Tangential Velocity Profiles for the Experiment of Lohmann (1976).

Figure 5.32 Semi-Logarithmic Plot of the Axial Velocity Profiles for the Experiment of Bissonnette and Mellor (1974).

Figure 5.33 Semi-Logarithmic Plot of the Tangential Velocity Profiles for the Experiment of Bissonnette and Mellor (1974).

Figure 5.34 Semi-Logarithmic Plot of the Axial Velocity Profiles for the Experiment of Bissonnette and Mellor (1974).

Figure 5.35 Semi-Logarithmic Plot of the Tangential Velocity Profiles for the Experiment of Bissonnette and Mellor (1974).

progresses downstream so as to substantially influence the near-wall region. The effect of curvature and rotation on the limits of validity of the tangential direction log-law equation, Equation (3.65), is addressed in Section 5.3. At the present time it is difficult to determine which, if either, of these explanations is valid. A direct measurement of the wall shear stress would perhaps resolve these issues by allowing one to determine the slope of the log-law lines from the data. However, such measurements were not taken in these experiments. Therefore, it is necessary resort to other means, such as plots of the turbulent shear stress measurements, to assess the accuracy of the wall shear stress components.

For their experiment, Bissonnette and Mellor (1974) calculated the magnitude and direction of the wall shear stress vector by extrapolating the measured Reynolds shear stress data, $-\rho \overline{uv}$ and $-\rho \overline{wv}$, to the wall. Although this seems to be a reasonable approach, the accuracy of this method is questionable, especially since the uncertainties in $-\overline{uv}$ and $-\overline{wv}$ are typically in the range of 10% to 20%. Nevertheless, in terms of the present study a similar approach may be used to assess the accuracy of the calculated wall shear stress components, as discussed next.

Consider that one may plot the non-dimensional variables

$$-\frac{\rho \overline{uv}}{\tau_{x,w}} = \frac{-\overline{uv}}{\{U^*\}^2} \quad (5.12)$$

and

$$-\frac{\rho \overline{wv}}{\tau_{\theta,w}} = \frac{-\overline{wv}}{\{W^*\}^2} \quad (5.13)$$

as functions of the distance normal to the wall, y . Obviously, the molecular contribution to the total shear stress dominates in the very near-wall region. As a result, both $-\rho \overline{uv}$ and $-\rho \overline{wv}$ rapidly decrease in the viscous sublayer and are zero at the wall. Accordingly, plots based on Equations (5.12) and (5.13) are then only accurate representations of the total shear stress outside of the viscous near-wall region. However, one may extrapolate the data from the fully turbulent near-wall region to the wall. Thus, such plots indicate if the wall shear stress components, $\tau_{x,w}$ and $\tau_{\theta,w}$, are consistent with the trend of the shear stress in the fully turbulent

portion of the near-wall region.

Component turbulent shear stress plots for the experiments of Yowakim (1985), Lohmann (1976), and BM (1974) are shown in Figures 5.36 - 5.43. Note that the curves in these figures are intended to facilitate reading of the graphs and are not correlations of the data. Also, the uncertainties given in these figures are those of the measurement uncertainty in the Reynolds shear stresses. In each figure, the closest near-wall turbulent shear stress measurement is typically at $y/\delta = 0.1$. Consequently, one expects that the first near-wall data point is in the log-law region. The shear stress plots for Yowakim's experiment are seen to approach the value of 1.0 at the wall. This suggests the values of the wall shear stress vector, obtained from the Modified Clauser plots, are reliable in this case. In contrast, the shear stress plots for Lohmann's experiment imply that the calculated wall shear stress components are overestimated at all the data stations since the near-wall data is below 1.0. The trend of the Reynolds shear stress data for BM's experimental data indicate that the calculated tangential wall shear stress component, $\tau_{\theta,w}$, is overestimated at some stations while underestimated at others. Based on the Lohmann as well as the BM figures, it appears that some of the wall shear stress components may not be accurate for these cases. This would seem to suggest that the slope and intercept of the component log-law lines, Equations (3.59) and (3.65), may have to be changed. Further justification for altering these values in order to predict the correct value of the wall shear stress components, $\tau_{x,w}$ and $\tau_{\theta,w}$, will be considered next.

In terms of the flow angle figures, similarities exist between the shear generated flows of McAllister (1979), presented in Chapter 4, and those of Lohmann (1976) and BM (1974). Recall from Section 4.5 that the planar component log-law lines, Equations (3.66) and (3.67), were shown to apply to the nearly-collateral region of a three-dimensional planar boundary layer. However, the slope and intercept of these equations were found to be very different from those of the standard two-dimensional values, $1/k = 2.38$ and $B = 5.5$. This then led to the hypothesis that slope and intercept of the log-law lines in a 3DTBL may be considered as vectors rather than scalars. The flow angle figures for the experiments of Lohmann and BM, Figures 5.12 - 5.13 and 5.14 - 5.15, also indicate that a nearly-collateral region exists in the fully turbulent portion of the near-wall region for these flows. Furthermore, the nearly-collateral regions do not appear to extend completely to the wall. This was also found to be the case for the swept step data of

Figure 5.36 Distribution of Axial Turbulent Shear Stress at Different Axial Stations: Experiment of Yowakim (1985) $SN = 0.87$ Convex Wall.

Figure 5.37 Distribution of Tangential Turbulent Shear Stress at Different Axial Stations: Experiment of Yowakim (1985) $SN = 0.87$ Convex Wall.

Figure 5.38 Distribution of Axial Turbulent Shear Stress at Different Axial Stations: Experiment of Yowakim (1985) $SN = 0.87$ Concave Wall.

Figure 5.39 Distribution of Tangential Turbulent Shear Stress at Different Axial Stations: Experiment of Yowakim (1985) $SN = 0.87$ Concave Wall.

Figure 5.40 Distribution of Axial Turbulent Shear Stress at Different Axial Stations: Experiment of Lohmann (1976).

Figure 5.41 Distribution of Tangential Turbulent Shear Stress at Different Axial Stations: Experiment of Lohmann (1976).

Figure 5.42 Distribution of Axial Turbulent Shear Stress at Different Axial Stations: Experiment of Bissonnette and Mellor (1974).

Figure 5.43 Distribution of Tangential Turbulent Shear Stress at Different Axial Stations: Experiment of Bissonnette and Mellor (1974).

McAllister (1979). These similarities suggest that the slope and intercepts in the cylindrical component log-law equations, Equations (3.59) and (3.65), may also have to be changed in order to correlate the three-dimensional experimental flows of Lohmann and of BM. However, unlike the experiment of McAllister (1979), the wall shear stress vector was not measured directly in the annular flow experiments of Lohmann (1976) and BM (1974). Thus, in these cases one must resort to other evidence such as the law-of-the-wall and the shear stress plots to determine if one should alter the values of k and B .

Note that in this chapter the component log-law lines, Equations (3.59) and (3.65), both use the standard two-dimensional constants $k = 0.42$ and $B = 5.5$. Referring to the law-of-the-wall plots, Figures 5.12 - 5.13 and 5.17 - 5.18, these values appear to correlate the experimental data of Yowakim (1985) and Lohmann (1976) rather well. However, for Lohmann's data, the axial and tangential direction turbulent shear stress plots, Figures 5.40 and 5.41, indicate that the wall shear stress components, $\tau_{x,w}$ and $\tau_{\theta,w}$, may be too large. Thus, it appears that in this case a reduction in the values of log-law intercepts, B_x and B_θ , used in the Modified Clauser plots may provide more accurate wall shear stress values. To see how this would be accomplished consider the axial component log-law equation. If U^{**} has been overestimated then a reduction in its value will cause the U/U^{**} data to shift upwards on a law-of-the-wall plot. Hence, an increase in the value of the axial direction log-law intercept, B_x , is then required in order to correlate the data. Similar recommendations can be made for improving the correlation with the BM data. From the law-of-the-wall plots for the BM data, Figures 5.32 - 5.35, it appears that the value of $k = 0.42$ provides a good fit to the axial velocity component. As noted earlier, it appears that a better fit to the tangential data would be obtained if the slope of the tangential log-law were reduced. In addition, from the tangential direction shear stress plot, Figure 5.43, it appears that the tangential direction wall shear stress, $\tau_{\theta,w}$, has been overestimated at the last data station. Therefore, in this case, one may have to change both the slope and the intercept of the tangential log-law equation. Again, these results suggest that k and B may both be vectors. Interpreting k and B as vectors in 3DTBLs was discussed in Section 4.6. The effect of altering k and B on the predicted flow angle, ϕ , in a cylindrical annulus follows.

The flow angle, ϕ , as calculated from the component log-law equations, Equations (3.59) and (3.65), is

$$\phi(y) = \tan^{-1} \left(\frac{W^{\infty}}{U^{\infty}} \frac{\{1 \pm y/r_w\} \left[\frac{1}{k_0} \ln \left(\frac{y^*}{1 \pm y/r_w} \right) + B_0 \right]}{1/k_x \ln y^* + B_x} \right) \quad (5.14)$$

If $k = k_x = k_0 = 0.42$ and $B_x = B_0 = 5.5$, then the flow angle, ϕ , predicted by Equation (5.14) will not differ significantly from that of the wall shear stress angle, β_w . In order to demonstrate this point, consider the variation of the ratio

$$\frac{\{1 + y/r_w\} \left[\frac{1}{0.42} \ln \left(\frac{y^*}{1 + y/r_w} \right) + 5.5 \right]}{\frac{1}{0.42} \ln y^* + 5.5} \quad (5.15)$$

across the range $y^* = 70$ to 300 for the axial data station $x = 13.0$ in. of Lohmann (1976). Equation (5.15) evaluated at $y^* = 70$ and $y^* = 300$ is 1.01 and 1.03, respectively. Thus, if one uses the same values of k and B for both of the component log-law equations, then the ratio of the component log-law equations, Equation (5.15), is essentially 1.0. Moreover, with regard to Equation (5.14), this then ensures that the flow angle, ϕ , in the log-law region will be equal to that of the wall shear stress angle, β_w . However, if the flow angle in the log-law region is quite different from that of the wall shear stress angle then it follows that the ratio of the component log-laws, Equation (5.15), must not be one. As discussed in Section 4.4, this in turn implies that the slope and intercept of the log-law lines may then be different for the axial and tangential directions. Recall that the flow angle figures of Lohmann and BM, Figures 5.9 - 5.10 and 5.13 - 5.14, indicate that a nearly-collateral region exists in the near-wall region for some of the data stations. In addition, these figures show that the flow may skew substantially in the viscous sublayer. Thus, the wall shear stress angle may be very different from that of the flow angle in the log-law region. Consequently, a better fit to the experimental data of Lohmann and BM may be obtained if one changes the values of k and B . Approaches to developing correlations for k and B , based on mean flow quantities, have been discussed in Section 4.5.

5.3 INFLUENCE OF CURVATURE AND ROTATION ON THE TANGENTIAL DIRECTION LOG-LAW EQUATION

As noted in Section 5.2, the tangential direction log-law equation, Equation (3.65), did not provide a good fit to some of the experimental data of Bissonnette and Mellor (1974). It was suggested that the tangential log-law may not be valid when transverse wall curvature and rotation significantly influence the near-wall region. Therefore, the intent of this section is to address such issues as the range of applicability and the appropriate functional form of the tangential direction log-law equation.

In order to provide a rudimentary description of the effects of streamline curvature on the structure of turbulent boundary layers consider a two-dimensional flow in which the wall curvature is in the streamwise direction. In such flows, numerous experiments have consistently demonstrated that convex wall curvature tends to suppress turbulent mixing while concave wall curvature enhances turbulent mixing (Cheng and Farokhi, 1992). Therefore, in general the stabilizing effect of convex wall curvature attenuates the Reynolds stresses whereas the destabilizing effect of concave wall curvature intensifies the Reynolds stresses. Similar arguments can be extended to include two-dimensional rotating flows in which the rotation is in the transverse direction. According to the Rayleigh stability criterion, such flows are unstable if the angular momentum, Ur , decreases with increasing radius of curvature, r . Although this criterion strictly applies to inviscid flows, it is often used to describe the behavior of laminar and turbulent boundary layers on rotating surfaces. Thus, one then expects that turbulence mixing will be augmented by rotation if the angular momentum decreases with the radial distance. In terms of the annular flows analyzed in Section 5.2, these observations of two-dimensional curved boundary layers only serve as a simplified view to understanding the effects of streamline curvature on the turbulence structure. One must realize that the issues of transverse wall curvature and rotation in the spinning cylinder experiments of Lohmann (1976) and of BM (1974) are more complicated since these flows are three-dimensional.

It must be acknowledged that the effects of curvature and rotation on the structure of turbulent flows is a complicated phenomenon and is currently an active area of research. Furthermore, it is not within the scope of the present work to provide a detailed discussion

regarding the effects of streamline curvature on turbulent boundary layers. The reader is directed to Bradshaw (1973) for a thorough discussion of this topic. At the same time, streamline curvature may be an important factor in the annular flow experiments discussed in Section 5.2. Consequently, how streamline curvature may effect the range of validity of the tangential direction log-law equation, Equation (3.65), will now be considered.

Referring to Figures 5.34 and 5.36, it appears that the tangential log-law equation, Equation (3.65), does not provide a good fit to the last two axial stations of the Bissonnette and Mellor data. As previously discussed in Section 5.2, this agreement may be improved by reducing the value of the slope, $1/k_0$, of the tangential log-law line. However, another explanation for this poor agreement is that perhaps the effects of transverse wall curvature and rotation are important in this experiment. Consider that as the flow progresses along the axis of the rotating cylinder these effects may substantially alter the flow structure in the near-wall region. For instance, history effects could be introduced into the near-wall region via enhanced turbulence mixing. If such is the case, one expects that the outer limit of the law-of-the-wall region will be reduced since the outer portion of the near-wall region can be considerably influenced by the outer region of the flow. The influence of such history effects on the near-wall region is discussed further in Section 6.6. However, in terms of the present study, this issue can not be resolved since the correct values of the slope and intercept of the component log-law lines are also unresolved issues. Thus, it seems that the best approach would be to refer to a few strongly curved two-dimensional boundary layers in order to determine if the outer limit of the tangential direction log-law, Equation (3.65), is significantly reduced in such strongly curved flows.

A recommendation for future research is to determine the influence of wall curvature and rotation on the range of validity of the two-dimensional log-law equation. For a two-dimensional boundary layer on a curved wall the log-law can be expressed in the same form as Equation (3.65),

$$\frac{U}{U_\tau} = \{1 \pm y/r_w\} \left[\frac{1}{k} \ln \left(\frac{y U_\tau}{\nu} \frac{1}{\{1 \pm y/r_w\}} \right) + B \right]. \quad (5.16)$$

A number of experiments which deal with two-dimensional boundary layers on curved walls are available in the literature. For example, a list of such wall bounded shear flows is available in

Holloway and Tavoularis (1992). It appears that it would be worthwhile to apply Equation (5.16) to a few strongly curved flows to determine if the range of the log-law is reduced by streamline curvature. Such investigations may help determine if the poor agreement with the tangential velocity data of Bissonnette and Mellor is due to curvature and rotation effects or to incorrect values of the log-law slope, $1/k_0$.

Another issue concerning the flow in a cylindrical annulus is the appropriate functional form of the tangential direction log-law equation. It seems reasonable to question whether the tangential log-law equation,

$$\frac{W}{W^{**}} = \{1 \pm y/r_w\} \left[\frac{1}{k} \ln \left(\frac{yQ^*}{\nu \{1 \pm y/r_w\}} \right) + B_0 \right] \quad (3.65)$$

is really much different than that of the planar form,

$$\frac{W}{W^{**}} = \frac{1}{k} \ln \left(\frac{yQ^*}{\nu} \right) + B_0. \quad (3.67)$$

in the log-law region. A comparison between these two log-law equations is given in Figures 5.44 and 5.45. These figures indicate that there is a distinct difference between the two equations for small values of wall radius, r_w . However, these variations are largely confined to the outer region of the boundary layer where the log-law equations are not expected to apply. Thus, it appears that the tangential log-law equation, Equation (3.65), is numerically not very different from that of the planar equation, Equation (3.67), in the log-law portion of the near-wall region. If such is the case, then as a matter of consistency one expects that cylindrical annulus flows should use Equation (3.65) rather than Equation (3.67) as the tangential direction log-law equation. Again, it seems the best way to resolve this issue is to refer to a number of strongly curved two-dimensional boundary layer experiments to determine if the tangential log-law equation, Equation (3.65), provides a better fit to the near-wall data than the planar log-law equation, Equation (3.67).

Figure 5.44 Comparison of the Tangential Log-Law Line, Equation (3.65), and the Planar Log-Law Line, Equation (3.67), for a Convex Wall.

Figure 5.45 Comparison of the Tangential Log-Law Line, Equation (3.65), and the Planar Log-Law Line, Equation (3.67), for a Concave Wall.

CHAPTER 6

THE LOGARITHMIC LAWS AND THE TURBULENT TRANSPORT EQUATIONS

The objective of this chapter is to provide an alternative derivation of the eddy-viscosity/mixing-length models presented in Chapters 3 and 4. The derivation of the two-dimensional eddy-viscosity/mixing-length model from the turbulent kinetic energy equation, according to Townsend (1961), is presented. This method is then modified for use with three-dimensional turbulent boundary layers. The turbulent transport equations, the structural-similarity hypothesis of Townsend, and the turbulent kinetic energy dissipation model of Batchelor (1953) are briefly discussed.

6.1 THE TURBULENT TRANSPORT EQUATIONS

This section presents the transport equations for the turbulent kinetic energy (TKE), $\bar{q}$, and the components of the Reynolds stress tensor, $-\rho \overline{u_i u_j}$. These equations will be expressed in Cartesian tensor notation. The eddy-viscosity/mixing length model and the logarithmic laws will later be derived from these equations based on certain assumptions and models of the turbulence structure.

The turbulent kinetic energy is defined as the sum of the mean squares of the fluctuating velocities,

$$\bar{q} = \frac{1}{2} \overline{u_i u_i} = \frac{\overline{u^2} + \overline{v^2} + \overline{w^2}}{2} . \quad (6.1)$$

where u , v , and w are the fluctuating components of velocity in the x , y , and z directions, respectively. The Reynolds stress tensor is given as

$$-\rho \overline{u_i u_k} = -\rho \begin{bmatrix} \overline{u^2} & \overline{uv} & \overline{uw} \\ \overline{vu} & \overline{v^2} & \overline{vw} \\ \overline{wu} & \overline{wv} & \overline{w^2} \end{bmatrix}. \quad (6.2)$$

Note that the Reynolds stress tensor is a second order tensor while the turbulent kinetic energy is a scalar. Transport equations for the turbulent kinetic energy and the Reynolds stress tensor (hereafter denoted as $-\overline{u_i u_k}$ rather than $-\rho \overline{u_i u_k}$) are obtained from the Navier-Stokes equations. The derivation of these equations is a fundamental topic and is described in most fluid dynamic text books which deal with turbulent flows; see for instance, Hinze (1975).

Neglecting body forces, the complete transport equation for the turbulent kinetic energy, in Cartesian tensor notation is (Bradshaw, 1978)

$$\frac{D\left(\frac{1}{2} \overline{u_i u_i}\right)}{Dt} = \underbrace{-\overline{u_i u_j} \frac{\partial U_i}{\partial x_j}}_i - \underbrace{\frac{\partial}{\partial x_j} \left(\overline{u_j \left\{ \frac{p}{\rho} + \frac{1}{2} u_i u_i \right\}} \right)}_{ii} + \underbrace{\overline{v u_i} \frac{\partial^2 u_i}{\partial x_j \partial x_j}}_{iii}. \quad (6.3)$$

Similarly, the complete transport equation for $\overline{u_i u_k}$, in Cartesian tensor notation is (Launder et al., 1975)

$$\begin{aligned} \frac{D(\overline{u_i u_k})}{Dt} = & \underbrace{\left(-\overline{u_j u_k} \frac{\partial U_i}{\partial x_j} - \overline{u_i u_j} \frac{\partial U_k}{\partial x_j} \right)}_I + \underbrace{\frac{p}{\rho} \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right)}_{II} - \underbrace{2\nu \frac{\partial u_i}{\partial x_j} \frac{\partial u_k}{\partial x_j}}_{III} \\ & + \underbrace{\left\{ -\frac{1}{\rho} \left(\frac{\partial \overline{p u_k}}{\partial x_i} + \frac{\partial \overline{p u_i}}{\partial x_k} \right) + \nu \frac{\partial \overline{u_i u_k}}{\partial x_j \partial x_j} - \frac{\partial \overline{u_i u_j u_k}}{\partial x_j} \right\}}_{IV}. \end{aligned} \quad (6.4)$$

The indices, i , j , and k , in Equations (6.3) and (6.4) represent the Cartesian directions:

$$i, j, k = x, y, z.$$

The substantive derivative in Equations (6.3) and (6.4) is defined as

$$\frac{D(\quad)}{Dt} = \frac{\partial(\quad)}{\partial t} + U_j \frac{\partial(\quad)}{\partial x_j}, \quad (6.5)$$

and represents the rate of change of TKE and $\overline{u_i u_i}$, respectively. The first two terms on the right hand side of Equation (6.3), *i* and *ii*, represent the production and diffusion of TKE, respectively. The last term on the right hand side of Equation (6.3), *iii*,

$$\overline{v u_i \frac{\partial^2 u_i}{\partial x_j \partial x_j}} = \text{viscous diffusion} + \text{viscous dissipation}, \quad (6.6)$$

represents the combined effects of viscous diffusion and viscous dissipation of TKE. In general, for turbulent boundary layers, the contribution of viscous diffusion to Equation (6.6) is negligible outside the viscous sublayer. Moreover, the present study is largely concerned with the fully turbulent portion of the near-wall region. As a result, throughout the remainder of this study the viscous diffusion portion of Equation (6.6) will be neglected unless stated otherwise. Accordingly, Equation (6.6) reduces to the viscous dissipation term, denoted as ϵ . The viscous dissipation of TKE, or simply the dissipation, in Cartesian tensor notation is

$$\text{viscous dissipation} = -\epsilon = -\nu \overline{\frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j}}, \quad (6.7)$$

The physical interpretation of the terms on the right side of the Reynolds transport equations, Equation (6.4), is not as obvious as those of the TKE transport equation, Equation (6.3). However, the standard practice is to draw an analogy between the TKE transport equation and the Reynolds stress transport equations. Therefore, the terms on the right hand side of Equation (6.4) are often called the production (*I*), pressure-strain (*II*), dissipation (*III*), and diffusion (*IV*) of Reynolds stress. The reader is referred to Bradshaw (1978) for a discussion of the physical interpretation of the terms in the Reynolds stress transport equations.

6.2 STRUCTURAL SIMILARITY

This section presents some of the results of the structural-similarity hypothesis of Townsend (see Townsend, 1976). These results will later be used to model the Reynolds shear

stress terms, $-\overline{uv}$ and $-\overline{wv}$, in the TKE transport equation and the transport equations for $\overline{u^2}$ and $\overline{w^2}$.

Townsend proposed a structural-similarity hypothesis for the turbulence field of a two-dimensional flow. A number of assumptions were made by Townsend in developing this hypothesis. However, it is not within the scope of the present work to critically evaluate his work. Hence, only the results which are pertinent to the present study will be stated here. The reader should consult the book by Townsend (1976) for a more detailed discussion of his structural-similarity hypothesis.

The structural-similarity hypothesis of Townsend states that the turbulence field rapidly adjusts itself to an equilibrium state in which the degree of anisotropy and the ratio of the shear stress to the TKE are independent of the rate of strain that produced them. Under such conditions the Reynolds stress tensor is determined by a single velocity scale, $(\overline{q})^{1/2}$. Thus, according to this hypothesis the Reynolds shear stress term, $-\overline{uv}$, for a two-dimensional flow may be written as

$$-\overline{uv} = a \overline{q}, \quad (6.8)$$

where a is a constant. Although this hypothesis was formulated for two-dimensional flows, the present study will assume that it can be applied to a three-dimensional boundary layer in which the Reynolds shear stress term $-\overline{uw}$ is small. In such a 3DTBL, the shear stress in the fully turbulent region may then be regarded as a pseudo vector of the form $(-\rho \overline{uv}, 0, -\rho \overline{wv})$. This assumes that the x -axis is aligned with the freestream velocity and the y -axis is normal to the wall. Referring to the structural-similarity hypothesis, the shear stress terms, $-\overline{uv}$ and $-\overline{wv}$, for a three-dimensional boundary layer may then be written as

$$-\overline{uv} = a_1 \overline{q} \quad (6.9)$$

and

$$-\overline{wv} = a_3 \overline{q}, \quad (6.10)$$

where a_1 and a_3 are constants. One must recognize that the value of the shear stresses, $-\rho \overline{uv}$ and $-\rho \overline{wv}$, depend on the orientation of the α coordinate system. As a result, a_1 and a_3 are

functions of the orientation of the coordinate axes.

Equations (6.8), (6.9), and (6.10) will be used in the turbulent transport equations, Equations (6.3) and (6.4), to derive the eddy-viscosity/mixing-length models presented in Chapters 3 and 4. In order to accomplish this, models for the dissipation terms in the turbulence transport equations are required, as described next.

6.3 MODELS FOR VISCOUS DISSIPATION

A model for the dissipation term, ϵ , in the TKE transport equation as given in Batchelor (1953) is presented. This expression will be used in the next section to derive the two-dimensional form of the eddy-viscosity/mixing length model from the TKE transport equation. In addition, this TKE dissipation model will be modified for use in three-dimensional boundary layers.

To assist in understanding the origin of the TKE dissipation model of Batchelor (1953), a simplified interpretation of the dynamics of the turbulence field is provided. This simplified explanation generalizes a number of key assumptions and theories. It is not within the scope of the present work to provide details on such theories. Therefore, the reader is referred to Townsend (1976) or Hinze (1975) for detailed discussions of the various theories available for analyzing turbulent flows.

One characteristic of turbulence is its dissipative nature. Thus, a turbulent flow field is not self-sustaining in that an energy input is required in order to maintain turbulence. For turbulent boundary layers and other shear layers, the turbulence flow field extracts energy from the mean flow by interacting with the mean flow shear. With these observations in mind, one may regard the turbulence field as a collection of eddies which constitute an energy cascade. A hierarchy exists in this cascade such that turbulent kinetic energy is passed from larger energy containing eddies to smaller eddies. The loss of TKE to the surroundings during the transfer process can be considered to be negligible. This transfer process continues until the eddy size is so small that the energy transferred to these eddies from the larger eddies is then dissipated into

internal energy through viscous interactions. At large Reynolds numbers the small eddy structure adjusts itself so as to accept energy from the largest eddies. Under such circumstances, the small eddy structure can be viewed as an energy sink in that these eddies will only dissipate the amount of energy which is transferred to them. The large energy containing eddies then dictate the dissipation of TKE. Hence, this implies the dissipation of turbulent kinetic energy becomes independent of the viscosity of the fluid. According to Batchelor (1953), from dimensional reasoning the dissipation term in the TKE equation may then be written as

$$\epsilon = A \frac{\{u_o\}^3}{L_o} , \quad (6.11)$$

where A is a proportional constant, u_o is a characteristic velocity of the turbulence, and L_o is an integral length scale of the turbulence. Equation (6.11) is often written as (Townsend, 1976)

$$\epsilon = \frac{\{\bar{q}\}^{3/2}}{L_\epsilon} , \quad (6.12)$$

where L_ϵ is the dissipation length. The dissipation length, L_ϵ , is proportional to the integral length scale of the turbulence, L_o , and may be viewed as a measure of the size of the dominant energy containing eddies.

In order to derive component logarithmic laws from the Reynolds stress transport equations, Equation (6.12) will be modified to represent the viscous dissipation terms in the transport equations for the Reynolds stresses, $\overline{u^2}$, $\overline{v^2}$, and $\overline{w^2}$. The viscous dissipation term, ϵ , in the TKE transport equation is the contraction of the second order dissipation tensor ϵ_{ij} . Consequently, ϵ is the trace of ϵ_{ij} .

$$\epsilon = tr(\epsilon_{ij}) = \epsilon_{ii} = \epsilon_{11} + \epsilon_{22} + \epsilon_{33} ,$$

where ϵ_{11} , ϵ_{22} , and ϵ_{33} are the viscous dissipation terms in the transport equations for $\overline{u^2}$, $\overline{v^2}$, and $\overline{w^2}$, respectively. The present study will assume that these component dissipation terms are then directly proportional to the dissipation term ϵ . The component dissipation terms can then be written as,

$$\epsilon_{11} = \nu \left[\overline{\left\{ \frac{\partial u}{\partial x} \right\}^2} + \overline{\left\{ \frac{\partial u}{\partial y} \right\}^2} + \overline{\left\{ \frac{\partial u}{\partial z} \right\}^2} \right] = b_{11} \frac{\{\bar{q}\}^{3/2}}{L_t} , \quad (6.13)$$

$$\epsilon_{22} = \nu \left[\overline{\left\{ \frac{\partial v}{\partial x} \right\}^2} + \overline{\left\{ \frac{\partial v}{\partial y} \right\}^2} + \overline{\left\{ \frac{\partial v}{\partial z} \right\}^2} \right] = b_{22} \frac{\{\bar{q}\}^{3/2}}{L_t} , \quad (6.14)$$

and

$$\epsilon_{33} = \nu \left[\overline{\left\{ \frac{\partial w}{\partial x} \right\}^2} + \overline{\left\{ \frac{\partial w}{\partial y} \right\}^2} + \overline{\left\{ \frac{\partial w}{\partial z} \right\}^2} \right] = b_{33} \frac{\{\bar{q}\}^{3/2}}{L_t} , \quad (6.15)$$

where b_{11} , b_{22} , and b_{33} are constants which obey the relationship

$$b_{11} + b_{22} + b_{33} = 1 . \quad (6.16)$$

6.4 THE TWO-DIMENSIONAL LOGARITHMIC LAW AND THE TURBULENT KINETIC ENERGY TRANSPORT EQUATION

The objective of this section is to derive the two-dimensional eddy-viscosity/mixing-length model from the TKE transport equation. The two-dimensional logarithmic law is then obtained from this eddy-viscosity/mixing-length model according to the derivation described in Section 3.6.

For two-dimensional boundary layers, the TKE Transport equation, Equation (6.3), can be reduced to a simpler form by applying the boundary layer assumptions

$$W = 0 , \quad V \ll U ,$$

and

$$\frac{\partial}{\partial x} \ll \frac{\partial}{\partial y} .$$

Note that since the only significant Reynolds shear stress term is $-\bar{u}\bar{v}$, the other production

terms in Equation (6.3) are assumed to be of second order. The resulting TKE transport equation for a two-dimensional boundary layer then is

$$\begin{array}{ccccccc}
 U \frac{\partial \bar{q}}{\partial x} + V \frac{\partial \bar{q}}{\partial y} & = & - \overline{uv} \frac{\partial U}{\partial y} & - & \frac{\partial}{\partial y} \left(\overline{v \left\{ \frac{p}{\rho} + q \right\}} \right) & - & \epsilon . \\
 \text{Convection} & & \text{Production} & & \text{Diffusion} & & \text{Dissipation}
 \end{array} \quad (6.17)$$

A typical plot of the terms in Equation (6.17), across the boundary layer thickness, δ , for a two-dimensional flat plate boundary layer, is shown in Figure 6.1. From Figure 6.1 it is seen that the TKE production and dissipation terms become large and are of the same order of magnitude near the wall. Furthermore, these terms dominate the TKE transport equation, Equation (6.17), in the fully turbulent portion of the near-wall region. For the remainder of this study, the condition in which TKE production is balanced by dissipation will be called "TKE equilibrium".

The two-dimensional log law can be derived from the TKE transport equation, Equation (6.17), by converting the TKE transport equation to a form expressed in terms of the eddy-viscosity/mixing-length model. This is primarily achieved by using the concept of near-wall TKE equilibrium and by modelling the Reynolds shear stress and TKE dissipation terms in the TKE transport equation. This has been done by a number of authors including Townsend (1961) and Rotta (1962). For illustrative purposes the method of Townsend (1961) will be presented.

Townsend (1961) interprets the momentum transport theory of Prandtl as an expression of the equilibrium between the production and dissipation of turbulent kinetic energy. Thus, local equilibrium is implied when convection and diffusion are small compared with the production and dissipation terms. Equation (6.17) then becomes

$$- \overline{uv} \frac{\partial U}{\partial y} = \epsilon . \quad (6.18)$$

The structural-similarity hypothesis of Townsend, discussed in Section 6.2, is then used to express the shear stress, $-\overline{uv}$, in terms of the TKE, $\bar{q}$, as

$$-\overline{uv} = a \bar{q} . \quad (6.8)$$

The TKE dissipation model of Batchelor (1953), presented in Section 6.3, is then used to express

Figure 6.1 Turbulent Kinetic Energy Balance in a Two-Dimensional Boundary Layer Along a Smooth Wall With Zero Pressure Gradient. (Adapted from Hinze 1975)

ϵ in terms of the TKE, $\bar{q}$, and the dissipation length, L_ϵ , as

$$\epsilon = \frac{\{\bar{q}\}^{3/2}}{L_\epsilon} . \quad (6.12)$$

Combining Equations (6.18), (6.8), and (6.12), a direct relationship between the mean rate of strain and the turbulent shear stress is obtained,

$$-\overline{uv} = a^3 L_\epsilon^2 \left\{ \frac{\partial U}{\partial y} \right\}^2 . \quad (6.19)$$

Substitution of Equations (6.19), (6.8), and (6.12) into the definition of the eddy viscosity,

$$-\overline{uv} = \nu' \frac{\partial U}{\partial y} ,$$

yields the result that the eddy viscosity is given by

$$\nu' = a^3 L_\epsilon^2 \frac{\partial U}{\partial y} . \quad (6.20)$$

Thus, Equation (6.20) is equivalent to the Prandtl eddy-viscosity/mixing-length model,

$$\nu' = l^2 \frac{\partial U}{\partial y} , \quad (3.31)$$

if

$$l = a^{3/2} L_\epsilon . \quad (6.21)$$

The significance of Equation (6.21) is that the mixing length, l , is proportional to the integral length scale of the turbulence. Thus, the mixing length can then be regarded as an appropriate length scale to model the Reynolds shear stress. Obviously, one must be able to specify either the mixing length or the dissipation length if Equation (6.21) is to be of any use.

Consider the mixing length definition of Galbraith et al. (1977), presented in Section 3.5:

$$l = ky \sqrt{\frac{\tau'}{\tau_w}} . \quad (3.39)$$

where $\tau' = -\rho \overline{u'v'}$. Substituting Equation (3.39) into Equations (6.21) and (6.19), one obtains the shear stress representation

$$-\overline{u'v'} = (ky)^2 \frac{\tau'}{\tau_w} \left\{ \frac{\partial U}{\partial y} \right\}^2. \quad (6.22)$$

The two-dimensional log-law,

$$\frac{U}{U_\tau} = \frac{1}{k} \ln \left(\frac{y U_\tau}{\nu} \right) + B, \quad (1.3)$$

is then obtained by integrating Equation (6.22). The implication of the preceding derivation is that the momentum mixing-length theory, and hence the logarithmic-law equation, is the result of an equilibrium between the production and dissipation of turbulent kinetic energy.

If one accepts that the log law and near-wall TKE equilibrium occur simultaneously, it is then conceivable that no log-law region will exist if the near-wall TKE balance between production and dissipation is significantly disturbed. Simpson et al. (1974) have shown that the log-law, Equation (1.3), is valid up to the beginning of intermittent separation. In addition, Simpson et al. (1977 and 1981a) demonstrate that the normal stress contribution to the production of TKE becomes significant as separation is approached. As a result, some secondary terms must be retained in the TKE equation (Simpson et al., 1981b), as follows

$$U \frac{\partial \overline{q}}{\partial x} + V \frac{\partial \overline{q}}{\partial y} = -\overline{u'v'} \frac{\partial U}{\partial y} - (\overline{u'^2} - \overline{v'^2}) \frac{\partial U}{\partial x} - \frac{\partial}{\partial y} \left(\overline{v \left\{ \frac{p}{\rho} + q \right\}} \right) - \epsilon. \quad (6.23)$$

The second term on the right hand side of Equation (6.23) represents the production of TKE due to differences in the normal Reynolds stresses. In addition, Simpson et al. (1981b) show that TKE diffusion becomes increasingly significant as separation is approached. Hence, from Equation (6.23) it is seen that the effect of a strong adverse pressure gradient is to upset the balance of TKE equilibrium in the near-wall region by introducing new production terms and by increasing the importance of the diffusion terms in the TKE. Consequently, this lack of near-wall TKE equilibrium might explain why the log-law, Equation (1.3), ceases to exist in very strong adverse pressure gradient flows.

6.5 THE TWO-DIMENSIONAL LOGARITHMIC LAW AND THE $\overline{u^2}$ TRANSPORT EQUATION

This section derives the two-dimensional eddy-viscosity/mixing-length model, and hence the log-law, from the $\overline{u^2}$ transport equation. A hypothesis similar to that of the near-wall TKE balance between production and dissipation, presented in Section 6.4, is given.

When dealing with the Reynolds stress transport equations and the turbulence field in general, it is convenient to express the Reynolds stress tensor in dimensionless form as

$$K_{ij} = \frac{\overline{u_i u_j}}{\overline{u_i u_i}} = \frac{\overline{u_i u_j}}{2\bar{q}}. \quad (6.24)$$

This tensor is a measure of the anisotropy of the turbulence field for a given flow. For example, in the case of an isotropic flow K_{ij} reduces to

$$K_{ij} = \frac{1}{3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (6.25)$$

However, it is well known that the turbulence field for two- and three-dimensional boundary layers is generally anisotropic. Based on a survey of data available in the literature, Tavoularis and Kamik (1989) determined the K_{ij} tensor, averaged over a number of uniform shear flow experiments, to be

$$K_{ij} = \begin{bmatrix} 0.51 \pm 0.04 & -0.16 \pm 0.01 & 0 \\ -0.16 \pm 0.01 & 0.22 \pm 0.02 & 0 \\ 0 & 0 & 0.27 \pm 0.03 \end{bmatrix}. \quad (6.26)$$

Since a two-dimensional uniform shear flow is similar to that of a two-dimensional boundary layer, it is expected that the K_{ij} tensor for a two-dimensional boundary layer will be numerically similar to Equation (6.26). Note that in terms of the nomenclature used in Sections 6.2 and 6.4, the constants a , a_1 , and a_3 may be written as

$$a = -2K_{12} = -2K_{21}, \quad (6.27)$$

$$a_1 = -2K_{12} = -2K_{21}, \quad (6.28)$$

and

$$a_3 = -2K_{13} = -2K_{31}. \quad (6.29)$$

In the following discussion, K_{ij} will be used to model the pressure-strain term in the Reynolds stress transport equations.

It is important to recognize that the sum of the transport equations for the Reynolds stresses, $\overline{u^2}$, $\overline{v^2}$, and $\overline{w^2}$, produces the TKE transport equation. The question then arises as to respective contribution of each transport equation to the terms in the TKE equation. In Section 6.4 it was shown that the two-dimensional TKE equation, Equation (6.17), may be reduced to a balance between production and dissipation in the log-law region,

$$-\overline{uv} \frac{\partial U}{\partial y} = \varepsilon. \quad (6.18)$$

Production Dissipation

In this case, the only significant production term,

$$-\overline{uv} \frac{\partial U}{\partial y}, \quad (6.20)$$

comes from the $\overline{u^2}$ transport equation. Moreover, according to Tavoularis and Karnik (1989), the only available measurements of turbulence velocity derivatives for a uniform high shear flow indicate a strong streamwise dissipation anisotropy:

$$\frac{\varepsilon_{11}}{\varepsilon} = 0.6, \quad (6.31)$$

which implies that

$$\frac{\epsilon_{22}}{\epsilon} \approx \frac{\epsilon_{33}}{\epsilon} \approx 0.2 . \quad (6.32)$$

If these uniform high shear flow measurements are indicative of a two-dimensional boundary layer, then it appears that the transport equation for $\overline{u^2}$ is the major contributor to the TKE equation in the log-law region. This, then leads one to believe that the two-dimensional eddy-viscosity/mixing length model, and hence the log-law, may be derived from the transport equation for $\overline{u^2}$. However, one important point is that one may not assume a near-wall balance exists solely between production and dissipation in the $\overline{u^2}$ transport equation. Another term, namely the pressure-strain, Φ_{ij} , is also of importance. The pressure-strain term represents the inter-component energy transfer between Reynolds stress components by means of pressure fluctuations. Note that the TKE equation does not contain a pressure-strain term, since the sum of the pressure-strain terms from the $\overline{u^2}$, $\overline{v^2}$, and $\overline{w^2}$ transport equations is zero.

In general, the pressure-strain term tends to move the turbulence toward isotropy by reducing the magnitude of the shear stress and by increasing the weaker normal stresses at the expense of the stronger (Bradshaw, 1978). Based on an analogy with the TKE equation, the present study proposes that the $\overline{u^2}$ transport equation reduces to a balance between production, pressure-strain, and dissipation in the log-law region:

$$0 = P_{11} + \Phi_{11} - \epsilon_{11} \quad (6.33)$$

where

$$P_{11} = -\overline{u_1 u_j} \frac{\partial U_1}{\partial x_j} = -\overline{uv} \frac{\partial U}{\partial y} . \quad (6.34)$$

$$\epsilon_{11} = \overline{u_1 \frac{\partial^2 u_1}{\partial x_j \partial x_j}} = \overline{u \frac{\partial^2 u}{\partial x^2}} + \overline{u \frac{\partial^2 u}{\partial y^2}} + \overline{u \frac{\partial^2 u}{\partial z^2}} \quad (6.35)$$

and

$$\Phi_{11} = \overline{p \left(\frac{\partial u_1}{\partial x_j} + \frac{\partial u_j}{\partial x_1} \right)} . \quad (6.36)$$

Hence, it is assumed that the two-dimensional eddy-viscosity/mixing-length model and the log-law may be derived from Equation (6.33). Implicit in this assumption is that the pressure-strain and dissipation terms can be modelled correctly. For the remainder of this study, the condition in which a balance exists between production, pressure-strain, and dissipation in the $\overline{u^2}$ transport equation will be called " $\overline{u^2}$ equilibrium". Supporting evidence for this hypothesis follows.

The hypothesis of a near-wall $\overline{u^2}$ equilibrium for a two-dimensional turbulent boundary layer certainly requires some experimental verification. However, to the best of this authors knowledge, experimental measurements of the complete $\overline{u^2}$ transport equation are not currently available. Obviously, the difficulty lies in measuring the fluctuating pressure and velocities with sufficient spatial accuracy. Thus, it appears that one must resort to numerical simulation data in order to test the hypothesis of a near-wall $\overline{u^2}$ equilibrium.

Kim et al. (1987) performed a direct numerical simulation of a low Reynolds number channel flow by solving the unsteady Navier-Stokes equations. Mansour et al. (1988) later used this simulation data to compute the terms in the Reynolds stress transport equations. A typical plot of the terms in the $\overline{u^2}$ transport equation, at the channel centerline, is shown in Figure 6.2. Note that the velocity pressure-gradient term, Π_{11} , given in Figure 6.2 is defined as

$$\Pi_{11} = \frac{\overline{u^2}}{\overline{q}} \Pi_{11} + \Phi_{11} . \quad (6.37)$$

where

$$\frac{\overline{u^2}}{\overline{q}} \Pi_{11} . \quad (6.38)$$

is the pressure-diffusion term and

$$\Pi_{11} - \frac{\overline{u^2}}{\bar{q}} \Pi_{11} = \Phi_{11} \quad (6.39)$$

is the pressure-strain term. The contribution of the pressure-strain and pressure-diffusion terms to the velocity pressure-gradient term is shown in Figure 6.3. Note that in the apparent log-law region, $50 < y^+ < 150$, the pressure-diffusion term is negligible compared to that of the pressure-strain term. As a result, outside the viscous sublayer the velocity pressure-gradient term, Π_{11} , may simply be regarded as the pressure-strain term, Φ_{11} . Figure 6.2 shows a balance between production, pressure-strain, and dissipation in the log-law region. The corresponding terms of the TKE equation are given in Figure 6.4. Note that TKE equilibrium also exists in the apparent log-law region, $50 < y^+ < 150$. Consequently, this simulation data seems to support the hypothesis of near-wall $\overline{u^2}$ equilibrium. A model for the pressure-strain term, Φ_{ij} , follows.

One must acknowledge that the modelling of the Reynolds stress transport equations, and the pressure-strain term in particular, is an active area of research. However, for illustrative purposes the present study will only consider the "standard" model of Launder et al. (1975). As well, since it is not within the scope of the present study to discuss the derivation of this model, only the final result is presented. Outside of the viscous sublayer, the pressure-strain term is usually modelled as two components, ϕ_{ij1} and ϕ_{ij2} , in order to reflect the dual nature of the pressure fluctuations:

$$\Phi_{ij} = \phi_{ij1} + \phi_{ij2} \quad (6.40)$$

The first component, ϕ_{ij1} , is regarded as the "fast" part since it responds immediately to changes in the mean strain rate, $\partial U_i / \partial x_j$. Conversely, the second component, ϕ_{ij2} , is called the "slow" part due to its slow response to changes in the mean strain rate. In the simplified model of Launder et al. (1975), the fast component of the pressure strain is taken to be linearly proportional to the production, while the slow response is assumed to be linearly proportional to the anisotropy of the turbulence:

$$\phi_{ij1} = -C_1 \left[P_{ij} - \frac{1}{3} \delta_{ij} P_{kk} \right] \quad (6.41)$$

and

Figure 6.2 Terms in the Budget of $\overline{u_1 u_1}$ in Wall Coordinates. P_{11} = Production; T_{11} = Turbulent Transport; D_{11} = Viscous Diffusion; ϵ_{11} = Dissipation Rate; Π_{11} = Velocity Pressure-Gradient Term. (Obtained from Mansour et al., 1988)

Figure 6.3 Split of the Velocity Pressure-Gradient Term, Π_{11} , into a Pressure Transport Term, $(\overline{u_1 u_1} / 2k) \Pi_{11}$, and a Pressure-Strain Term, $\Pi_{11} - (\overline{u_1 u_1} / 2k) \Pi_{11}$. (Obtained from Mansour et al., 1988)

Figure 6.4 Terms in the Budget of Turbulent Kinetic Energy in Wall Coordinates.
 P_k = Production; T_k = Turbulent Transport; D_k = Viscous Diffusion;
 ϵ_k = Dissipation Rate; Π_k = Velocity Pressure-Gradient Term.
 (Obtained from Mansour et al., 1988)

$$\phi_{ij2} = -C_2 \frac{\epsilon}{\bar{q}} \left[\overline{u_i u_j} - \frac{1}{3} \delta_{ij} \overline{u_k u_k} \right] \quad (6.42)$$

where C_1 and C_2 are empirical constants and the Kronecker delta, δ_{ij} , is defined as

$$\delta_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (6.43)$$

Accordingly, the pressure-strain term in the $\overline{u^2}$ transport equation is then modeled as

$$\Phi_{11} = \phi_{11,1} + \phi_{11,2} \quad (6.44)$$

where

$$\phi_{11,1} = -C_1 \left[P_{11} - \frac{1}{3} \{P_{11} + P_{22} + P_{33}\} \right] \quad (6.45)$$

and

$$\phi_{11,2} = -C_2 \frac{\epsilon}{\bar{q}} \left[\overline{u^2} - \frac{2\bar{q}}{3} \right]. \quad (6.46)$$

From Equation (6.13), the $\overline{u^2}$ dissipation term, ϵ_{11} , may be written in terms of the TKE dissipation, ϵ , as

$$\frac{\epsilon_{11}}{\epsilon} = b_{11} \quad (6.47)$$

Substituting Equation (6.47) into Equation (6.46), and recognizing that

$$\frac{\overline{u^2}}{\bar{q}} = 2K_{11} \quad (6.48)$$

the slow pressure-strain term may be written as

$$\phi_{11,2} = \frac{-C_2 \epsilon_{11}}{b_{11}} \left[2K_{11} - \frac{2}{3} \right]. \quad (6.49)$$

If K_{11} and b_{11} are constant over the log-law range, then Equation (6.49) reduces to

$$\phi_{11,2} = -C'_2 \epsilon_{11} , \quad (6.50)$$

where

$$C'_2 = \frac{C_2}{b_{11}} \left[2K_{11} - \frac{2}{3} \right] \quad (6.51)$$

is a constant. If the production terms in the crossflow and normal directions, z and y , are negligible compared to that of the streamwise, x , direction,

$$P_{33}, P_{22} \ll P_{11} , \quad (6.52)$$

then the fast portion of the pressure-strain term reduces to

$$\phi_{11,1} = -C'_1 P_{11} . \quad (6.53)$$

where

$$C'_1 = \frac{2C_1}{3} \quad (6.54)$$

is a constant. Substituting Equations (6.50) and (6.54) into the near-wall $\overline{u^2}$ transport equation, Equation (6.33), gives

$$P_{11} = A_1 \epsilon_{11} , \quad (6.55)$$

where

$$A_1 = \frac{(1 + C'_2)}{(1 - C'_1)} \quad (6.56)$$

is a constant. Hence, in this case, the pressure-strain term has been absorbed into the production and dissipation terms. Moreover, the near-wall equilibrium form of the $\overline{u^2}$ transport equation, Equation (6.55), is similar to the TKE equilibrium equation, Equation (6.18). Consequently, one may obtain the two-dimensional form of the eddy-viscosity/mixing-length model and the log-law from Equation (6.55) in the same manner as that outlined in Section 6.4. The resulting expression for the Reynolds shear stress is

$$-\overline{uv} = \frac{1}{A_1^2} \left[\frac{-2K_{12}}{b_{11}} \right]^3 L_t^2 \left\{ \frac{\partial U}{\partial y} \right\}^2. \quad (6.57)$$

Equation (6.57) is equivalent to the eddy-viscosity/mixing-length model representation of the turbulent shear stress,

$$-\overline{uv} = l^2 \left\{ \frac{\partial U}{\partial y} \right\}^2, \quad (3.31)$$

if

$$l = \frac{1}{A_1} \left[\frac{-2K_{12}}{b_{11}} \right]^{3/2} L_t. \quad (6.58)$$

Substituting the mixing length definition of Galbraith et al. (1977),

$$l = ky \sqrt{\frac{-\rho \overline{uv}}{\tau_w}},$$

into Equations (6.59) and (6.58) and integrating the result produces two-dimensional log-law,

$$\frac{U}{U_\tau} = \frac{1}{k} \ln \left(\frac{y U_\tau}{\nu} \right) + B. \quad (1.3)$$

The preceding derivation has demonstrated that one may obtain the two-dimensional eddy-viscosity/mixing-length model and the log-law equation, Equation (1.3), directly from the $\overline{u^2}$ transport equation. However, a number of key assumptions and simplifications were made in arriving at this result. Firstly, a balance between production, pressure-strain, and dissipation was assumed to exist in the log-law region. Secondly, the pressure-strain term, Φ_{11} , was absorbed into the production, P_{11} , and dissipation terms, ϵ_{11} , by means of the standard pressure-strain model of Launder et al. (1975). Thirdly, the non-dimensional Reynolds stress tensor, K_{ij} , and the ratio of $\overline{u^2}$ dissipation to TKE dissipation, b_{11} , were considered to be constants throughout the log-law portion of the near-wall region. Lastly, the dissipation model of Batchelor, presented in Section 6.3, was assumed to accurately model the TKE dissipation in the log-law region.

Admittedly, this is a formidable list of restrictions. In fact, an obvious recommendation for future research is to provide further justification for these assumptions by conducting a thorough review of the experimental data, simulation data, and Reynolds stress turbulence transport models available in the literature.

If one accepts the near-wall $\overline{u^2}$ equilibrium hypothesis for a 2DTBL, then it seems plausible that similar results may be obtained for the velocity components, U and W , of a 3DTBL. In this case, one must consider the $\overline{w^2}$ transport equation in addition to the $\overline{u^2}$ transport equation.

6.6 THE NEAR-WALL EQUILIBRIUM CONDITION IN THREE-DIMENSIONAL TURBULENT BOUNDARY LAYERS

In the preceding derivation of the two-dimensional log-law, the condition of near-wall equilibrium in the transport equation for $\overline{u^2}$ is a key assumption. If one accepts that the component log-laws and the condition of near-wall equilibrium in the $\overline{u^2}$ and $\overline{w^2}$ transport equations occur simultaneously, then this may explain why the component log-law equations are successful in correlating the experimental data presented in Chapters 4 and 5.

Recall from Section 4.4, that log-linear y^+ regions existed for both velocity components in the 45° and 30° swept step experiments of McAllister (1979). Conversely, in the case of the teardrop body experiment, only the streamwise velocity component exhibited a log-linear y^+ region. One possible explanation for this behavior is that perhaps the condition of near-wall equilibrium exists in both the $\overline{u^2}$ and $\overline{w^2}$ transport equations for the step flows but only in the $\overline{u^2}$ transport equation for the teardrop flow. Thus, history effects, such as diffusion, may be important in the $\overline{w^2}$ transport equation of the teardrop flow. Similarly, in Chapter 5 the component log-law equations,

$$\frac{U}{U^*} = \frac{1}{k} \ln \left(\frac{yQ^*}{\nu} \right) + B, \quad (3.59)$$

and

$$\frac{W}{W^*} = \{1 \pm y/r_w\} \left[\frac{1}{k} \ln \left(\frac{yQ^*}{v \{1 \pm y/r_w\}} \right) + B_0 \right]. \quad (3.65)$$

seem to provide a good fit to the near-wall data for the three annular flow experiments except for the last two axial data stations of the Bissonnette and Mellor (1974) experiment. It is well known that the effects of curvature and rotation can significantly change the turbulence structure, and that these effects are more pronounced as the radius of curvature decreases or the rate of rotation increases (Bradshaw, 1973). However, these effects are generally confined to the outer regions of the boundary layer. In the case of the annular swirling flow experiment of Yowakim (1985), it is then conceivable that the success of the component log-laws can be largely attributed to the existence of near-wall equilibrium in the $\overline{u^2}$ and $\overline{w^2}$ transport equations. Likewise, in the experiment of Lohmann (1976), the effects of curvature and rotation may not have influenced the near-wall region so that near-wall equilibrium may exist in both of the transport equations. However, in the case of the Bissonnette and Mellor (1974) experiment, it appears that the tangential log-law equation, Equation (3.65), does not provide a good fit to the near-wall data at the last two axial data stations. It may be that at these stations the effects of curvature and rotation have penetrated the near-wall layer upsetting the balance between production, pressure-strain, and dissipation in the $\overline{w^2}$ transport equation. It is clear that the assumption of near-wall equilibrium in the $\overline{u^2}$ and $\overline{w^2}$ transport equations of a three-dimensional boundary layer is a topic that warrants future experimental verification.

It is interesting to note that, based on the concept of component near-wall equilibrium, one may derive a tangential log-law equation from the $\overline{w^2}$ transport equation. The derivation according to the material presented in Sections 6.2, 6.3, and 6.5 is as follows. Firstly, it is assumed that the only significant production term in the $\overline{w^2}$ transport equation is

$$-\overline{wv} \frac{1}{r} \frac{\partial(Wr)}{\partial r}. \quad (6.59)$$

Secondly, a balance between production, pressure-strain, and dissipation is assumed to exist in the fully turbulent portion of the near-wall region. Furthermore, it is assumed that the pressure-strain term may be modelled such that it can be absorbed into the production and dissipation terms. The resulting reduced $\overline{w^2}$ transport equation then is

$$-\frac{\overline{wv}}{r} \frac{\partial(Wr)}{\partial r} = A_2 \epsilon_{33} . \quad (6.60)$$

where

$$r = r_w \pm y \quad (3.1)$$

and A_1 is a constant. From Sections 6.2, 6.3, and 6.5, the tangential direction shear stress may be written as

$$-\overline{wv} = \frac{1}{rA_2^2} \left(\frac{-2K_{32}}{b_{33}} \right)^3 L_t^2 \left[\frac{\partial(Wr)}{\partial r} \right]^2 = \frac{l_{tr}^2}{r} \left[\frac{\partial(Wr)}{\partial r} \right]^2 . \quad (6.61)$$

Substituting the mixing length definition

$$l_{tr} = ky \sqrt{\frac{-\rho \overline{wv}}{\tau_{\theta,w}}} \sqrt{\frac{\tau_w}{\tau_{\theta,w}}} \quad (6.62)$$

into Equation (6.61) and non-dimensionalizing the result one obtains

$$\frac{1}{r_w^* \pm y^*} \frac{\partial(W^* \{r_w^* \pm y^*\})}{\partial y^*} = \frac{1}{ky^*} . \quad (6.63)$$

where

$$W^* = \frac{W}{W^*} ,$$

$$y^* = \frac{yQ^*}{v} ,$$

and

$$r_w^* = \frac{r_w Q^*}{v} .$$

Integration of Equation (6.63) gives

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$$W^* \{r_w^* \pm y^*\} = \frac{r_w^*}{k} \ln(y^*) \pm \frac{y^*}{k} + C, \quad (6.64)$$

where C is a constant of integration. Equation (6.64) may be expressed as

$$W^* = \frac{1}{\{1 \pm y/r_w\}} \left[\frac{1}{k} \ln(y^*) \pm \frac{y^*}{k r_w^*} + B \right], \quad (6.65)$$

where $B = Cr_w^*$. Note that the top and bottom signs in Equation (6.65) denote convex and concave wall curvature, respectively. A law-of-the-wall plot for the last two data stations of the Bissonnette and Mellor (1974) experiment is given in Figure 6.5. From Figure 6.5, Equation (6.65) is seen to correlate the data rather well. Recall from Section 5.2, that the tangential log-law

$$\frac{W}{W^*} = \{1 + y/r_w\} \left[\frac{1}{k} \ln \left(\frac{y Q^*}{\nu \{1 + y/r_w\}} \right) + B_0 \right] \quad (3.65a)$$

derived in Section 3.7, did not provide a good fit to the last two data stations of the BM (1974) experiment. For the purposes of comparison, Figure 5.35 accompanies Figure 6.5. As noted in Section 5.3, it appears that one should apply Equations (6.65) and (3.65) to a number of two-dimensional boundary layers on curved surfaces to determine which equation provides a better correlation to curved wall flow data. This is a topic for future research.

6.7 OTHER ISSUES CONCERNING THE LAW-OF-THE-WALL AND THE TURBULENT TRANSPORT EQUATIONS

The objective of this section is to provide more detail on some of the points mentioned in Sections 6.5 and 6.6. Although an emphasis is placed on three-dimensional boundary layers, the following discussion also applies to two-dimensional boundary layers.

The success of the log-law, Equation (1.3), in two-dimensional boundary layers provides indirect support for the use of the structural-similarity hypothesis of Townsend and the TKE dissipation model of Batchelor (1953) in the near-wall region of two-dimensional boundary layers.

Figure 6.2 Semi-Logarithmic Plot of the Tangential Velocity Profiles for the Last Three Data Stations of Bissonnette and Mellor (1974).

Figure 5.35 Semi-Logarithmic Plot of the Tangential Velocity Profiles for the Experiment of Bissonnette and Mellor (1974).

However, the modifications to these postulates for use in three-dimensional boundary layers, proposed in Sections 6.2, 6.3 and 6.5, needs to be verified. In a three-dimensional boundary layer experiment, Fernholz and Vagt (1981) observed that, for a given data station, the ratios $-\overline{uv}/\bar{q}$ and $\sqrt{(\overline{uv})^2 + (\overline{wv})^2} / \bar{q}$ can increase significantly across the boundary layer thickness, δ . They also found that this variation increased as the mean flow became more skewed. However, their figures also indicate that the ratios, $-\overline{uv}/\bar{q}$ and $\sqrt{(\overline{uv})^2 + (\overline{wv})^2} / \bar{q}$, may be constant in the near-wall region. The difficulty with determining if such constant ratios exist in the near-wall region of their experiment is that only a few data points are shown in the near-wall region. Thus, it appears that a number of near-wall data points are necessary if one is to resolve the issue of structural similarity in the near-wall region of a three-dimensional boundary layer.

In two-dimensional boundary layers the outer limit of validity of the log-law, Equation (1.3), is generally regarded to be approximately $y/\delta = 0.10$. It then seems reasonable to assume that the component log-law equations, in a three-dimensional boundary layer, will have similar outer limits of validity. One recommendation for future research is to perform a three-dimensional boundary layer experiment that includes detailed Reynolds stress measurements in the very near-wall region. Such an experiment should have a number of data points in the near-wall region, say in the range $0.02 < y/\delta < 0.15$. Another recommendation for future research is to calculate the ratios of shear stress to TKE, in the near-wall region, for other three-dimensional boundary layer experiments. A number of three-dimensional experiments are available in the literature. Lists of experiments which include measurements of all the components of the Reynolds stress tensor are given in Fernholz and Vagt (1981), Van den Berg (1989), and Olcmen & Simpson (1992). Such investigations may determine if the modified structural-similarity hypothesis, presented in Section 6.2, is valid in the near-wall region of three dimensional boundary layers.

It is important to recognize that more than one production term may be significant in two- and three-dimensional boundary layers. Consider the case of a three-dimensional boundary layer to demonstrate this point. It seems that production terms associated with $-\overline{uw}$ may be significant in the $\overline{u^2}$ and $\overline{w^2}$ transport equations. Several authors have found that, in three-dimensional boundary layers, the direction of the streamlines in the plane parallel to the wall, the X-Z plane, can be quite different from those of the mean velocity gradient (Van den Berg, 1989).

Moreover, the Reynolds shear stress $-\overline{uw}$ may be large in comparison to $-\overline{uv}$ and $-\overline{vw}$ (Bradshaw, 1987) which implies that the boundary layer, or thin shear layer, approximations are no longer valid. Consequently, the near-wall production terms in transport equations for $\overline{u^2}$ and $\overline{w^2}$ may then be written as

$$P_{11} = -\overline{uv} \frac{\partial U}{\partial y} + -\overline{uw} \frac{\partial U}{\partial z} \quad (6.66)$$

and

$$P_{33} = -\overline{vw} \frac{\partial W}{\partial y} + -\overline{wu} \frac{\partial W}{\partial x} , \quad (6.67)$$

respectively. In general, one expects that $\partial U/\partial z$ and $\partial W/\partial x$ will be functions of the distance normal to the wall, y , in a three-dimensional boundary layer. Note the reduced transport equations will then be partial differential equations rather than ordinary differential equations. How one might solve these equations is a topic for further research.

As a final note, it must be realized that the assumption of near-wall equilibrium in the $\overline{u^2}$ and $\overline{w^2}$ transport equations requires verification. If other terms such as convection or diffusion are important in the $\overline{u^2}$ and $\overline{w^2}$ transport equations, then perhaps the development of near-wall similarity equations for three-dimensional turbulent boundary layers may not be possible.

6.8 $\bar{q} - \epsilon$ TURBULENCE MODELS

As mentioned in Chapters 4 and 5, the issue of different slopes, $1/k$, for the component log-law equations might be resolved by a computational study. Clearly, a turbulence model is required for such an investigation. Since much of this Chapter has dealt with the TKE transport equation, a brief description of the $\bar{q} - \epsilon$ turbulence model is given.¹ First, two $\bar{q} - \epsilon$ models for planar two-dimensional turbulent boundary layers are presented. Following this, a modified $\bar{q} - \epsilon$ model for use with planar three-dimensional turbulent boundary layers is considered.

The $\bar{q} - \epsilon$ is a two-equation turbulence model in that two transport equations, one for the TKE, $\bar{q}$, and one for the rate of TKE dissipation, ϵ , must be solved in addition to the Navier-Stokes equations and the continuity equation. In the hierarchy of turbulence models, $\bar{q} - \epsilon$ models are regarded as moderately complex. Zero equation models, or mixing length models, represent the simplest form of turbulence models while Reynolds stress models and large-eddy simulations are the most complex means of predicting turbulent flows. The reader is referred to White (1991) for an introduction to turbulence models. In addition, a more detailed discussion of turbulence models is given in Lakshminarayana (1986).

$\bar{q} - \epsilon$ turbulence models can be generally classified into two groups: high-Reynolds-number and low-Reynolds-number models. The high-Reynolds-number (HRN) models use wall functions to bridge the near-wall layer. As a result, the first computational point lies within the fully turbulent log-law region. In contrast, the low-Reynolds-number (LRN) models solve the $\bar{q}$ and ϵ transport equations all the way to the wall. One must acknowledge that many forms of both types of these models are available in the literature. However, the present study only considered the HRN model of Launder and Spalding (1974) (hereafter referred to as the standard $\bar{q} - \epsilon$ model) and the LRN model of Chen and Patel (1988). A description of the standard

¹ In the literature the letter k is often used to represent the turbulent kinetic energy of the velocity fluctuations (TKE) while the letter κ is used to denote the inverse slope of the log-law, Equation (1.3). However, the present study uses the symbol $\bar{q}$ and the letter k to identify the TKE and inverse slope of the log-law, respectively.

$\bar{q} - \epsilon$ model for two-dimensional turbulent boundary layers follows.

6.8.1 The Standard $\bar{q} - \epsilon$ Turbulence Model

For two-dimensional turbulent boundary layers, the time-averaged high-Reynolds-number $\bar{q}$ and ϵ transport equations are

$$U \frac{\partial \bar{q}}{\partial x} + V \frac{\partial \bar{q}}{\partial y} = \underbrace{v' \left\{ \frac{\partial U}{\partial y} \right\}^2}_{\text{Production}} - \underbrace{\frac{\partial}{\partial y} \left(\frac{v'}{\sigma_1} \frac{\partial \bar{q}}{\partial y} \right)}_{\text{Diffusion}} - \underbrace{\epsilon}_{\text{Dissipation}} \quad (6.68)$$

and

$$U \frac{\partial \epsilon}{\partial x} + V \frac{\partial \epsilon}{\partial y} = -C_1 v' \frac{\epsilon}{\{\bar{q}\}} \left\{ \frac{\partial U}{\partial y} \right\}^2 - \frac{\partial}{\partial y} \left(\frac{v'}{\sigma_2} \frac{\partial \bar{q}}{\partial y} \right) + C_2 \frac{\epsilon^2}{\{\bar{q}\}} \quad (6.69)$$

where the eddy-viscosity is defined as

$$v' = C_\mu \frac{\{\bar{q}\}^2}{\epsilon} \quad (6.70)$$

and C_μ , C_1 , C_2 , σ_1 , and σ_2 are empirical constants. Details on the derivation of these model equations is given in Launder and Spalding (1974). Equations (6.68) and (6.69) only apply to the fully turbulent portion of a two-dimensional boundary layer. Near the wall the log-law, Equation (1.3), is assumed to be valid. As a result, the computations begin at a data point, y_p , in the log-law region, typically $y_p^+ = 50$. At the point y_p the mean flow velocity, TKE, and rate of TKE dissipation are given by the wall functions

$$\frac{U_p}{U_\tau} = \frac{1}{k} \ln \left(\frac{y_p U_\tau}{\nu} \right) + B \quad (1.3)$$

$$\bar{q}_p = \frac{U_\tau^2}{C_\mu^{1/2}} \quad (6.71)$$

and

$$\epsilon_p = \frac{U_\tau^3}{k y} . \quad (6.72)$$

respectively, where k and B are the standard two-dimensional log-law constants and

$$U_\tau = \sqrt{\frac{\tau_w}{\rho}} .$$

The advantage of the wall function approach is that one may start the computations at a point well away from the wall, therefore avoiding the need for a very fine computational mesh near the wall. Although the wall function approach has achieved a good degree of success, it seems that this method assumes that the turbulent shear stress in the log-law region is constant and equal to the wall shear stress. Furthermore, it appears that Equations (6.71) and (6.72) are derived based on the assumption of near-wall TKE equilibrium in the log-law region. To illustrate these points the derivation of the wall functions $\bar{q}_p$ and ϵ_p , based on Sections 6.2, 6.3, and 6.4, follows.

According to the near-wall TKE equilibrium condition, the two-dimensional turbulent boundary layer form of the TKE equation, Equation (6.17), reduces to a balance between production and dissipation:

$$- \overline{uv} \frac{\partial U}{\partial y} = \epsilon . \quad (6.18)$$

From Sections 6.2 and 6.3, the Reynolds shear stress and the TKE dissipation may be expressed as

$$- \overline{uv} = a \bar{q} \quad (6.8)$$

and

$$\epsilon = \frac{\{\bar{q}\}^{3/2}}{L_t} . \quad (6.12)$$

In Section 6.4, it was shown that by substituting Equations (6.8) and (6.12) into (6.18), one obtains a relation for the turbulent shear stress in the log-law region:

$$-\overline{uv} = a^3 L_t^2 \left\{ \frac{\partial U}{\partial y} \right\}^2. \quad (6.19)$$

Moreover, Equation (6.19) is then equivalent to the Prandtl mixing length model,

$$-\overline{uv} = l^2 \left\{ \frac{\partial U}{\partial y} \right\}^2, \quad (3.31)$$

if

$$l = a^{3/2} L_t. \quad (6.21)$$

As a result, from Equation (6.19), the TKE may be expressed in terms of the mixing length as

$$\overline{q} = a^2 L_t^2 \left\{ \frac{\partial U}{\partial y} \right\}^2 = \frac{l^2}{a} \left\{ \frac{\partial U}{\partial y} \right\}^2. \quad (6.73)$$

Substituting the mean velocity gradient,

$$\frac{\partial U}{\partial y} = \frac{U_\tau}{ky}, \quad (6.74)$$

and the mixing length relation of Galbraith et al. (1977),

$$l = ky \sqrt{\frac{\tau'}{\tau_w}}, \quad (3.39)$$

into Equation (6.73), one obtains the value of the TKE at a point y_p in the log-law region:

$$\overline{q}_p = \frac{\tau'_p}{\tau_w} \frac{U_\tau^2}{C_\mu^{1/2}}, \quad (6.75)$$

where

$$\tau'_p = -\rho \overline{uv} |_{y=y_p} \quad (6.76)$$

and

$$C_\mu^{1/2} = a. \quad (6.77)$$

Substituting Equation (6.75) into Equation (6.12), the TKE dissipation at y_p is

$$\epsilon_p = \frac{U_\tau^3}{k y_p} \frac{\tau_p'}{\tau_w} . \quad (6.78)$$

Thus, the wall functions relations of Equations (6.71) and (6.72) are only equivalent to Equations (6.75) and (6.78) if the turbulent shear stress in the log-law region is equal to that of the wall shear stress:

$$-\rho \overline{uv} = \tau_w . \quad (6.79)$$

As discussed previously in Section 3.5, Galbraith et al. (1977) have demonstrated that the log-law, Equation (1.3), applies to 2DTBLs in which the shear stress varies substantially throughout the near-wall region. If one accepts the validity of the log-law Galbraith et al. (1977) deduced that the mixing length must then be of the form given by Equation (3.39). This implies that the standard $\overline{q} - \epsilon$ turbulence model should use Equations (6.75) and (6.78) as the wall functions rather than Equations (6.71) and (6.72).

From the preceding derivation, it seems that the wall function approach is based on the validity of the log-law and near-wall TKE equilibrium. A two-layer LRN model which does not assume near-wall TKE equilibrium is given next.

6.8.2 A Two-Layer Low-Reynolds-Number Turbulence Model

This section presents the LRN model of Chen and Patel (1988). This model is an alternative to the standard HRN model described in Section 6.8.2.

In the two-layer model of Chen and Patel (1988), the flow domain is divided into two regions: a near-wall region, *Region 1*, and an outer region, *Region 2*. *Region 1* includes the linear sublayer, the buffer layer, and part of the fully turbulent layer. A one equation model is used in *Region 1*, whereas the standard $\overline{q} - \epsilon$ model is used in *Region 2*. Note that this model differs from LRN models which require the solution of transport equations for $\overline{q}$ and ϵ all the way to the wall. That is, the one equation model for *Region 1* only requires the solution to the TKE equation in addition to the Navier-Stokes equations and the continuity equation. The TKE

dissipation, ϵ , in *Region 1* is expressed as an algebraic relation. The model equations for *Region 1* are given next.

One must recognize that the high-Reynolds-number TKE equation, Equation (6.68), only applies to the fully turbulent portion of the flow domain. Consequently, Equation (6.68) neglects the molecular diffusion of TKE. However, LRN models solve the TKE and TKE diffusion transport equations all the way to the wall. For the two-layer model of Chen and Patel (1988), molecular diffusion is accounted for by simply including the molecular viscosity in the TKE equation, Equation (6.68). The resulting TKE equation for the near-wall region, *Region 1*, then is

$$\underbrace{U \frac{\partial \bar{q}}{\partial x} + v \frac{\partial \bar{q}}{\partial y}}_{\text{Convection}} = \underbrace{v' \left\{ \frac{\partial U}{\partial y} \right\}^2}_{\text{Production}} - \underbrace{\frac{\partial}{\partial y} \left(\left\{ v + v' / \sigma_1 \right\} \frac{\partial \bar{q}}{\partial y} \right)}_{\text{Diffusion}} - \underbrace{\epsilon}_{\text{Dissipation}} \quad (6.80)$$

where the eddy viscosity, v' , is defined as

$$v' = C_\mu \sqrt{\bar{q}} L_t^* \quad (6.81)$$

The corresponding rate of TKE energy dissipation in *Region 1* is given by

$$\epsilon = \frac{\{\bar{q}\}^{3/2}}{L_t} \quad (6.12)$$

In these equations, Chen and Patel (1988) define L_t and L_t^* as

$$L_t = C_\mu^{-3/4} k y \left(1 - \exp\{-R_T/A_t\} \right) \quad (6.82)$$

and

$$L_t^* = f_\mu L_t \quad (6.83)$$

where

$$f_{\mu} = \frac{(1 - \exp\{-R_T/A_{\mu}\})}{(1 - \exp\{-R_T/A_{\epsilon}\})} \quad (6.84)$$

R_T is the turbulence Reynolds number,

$$R_T = \frac{y\sqrt{\bar{q}}}{\nu} \quad (6.85)$$

and C_{μ} , A_{μ} , and A_{ϵ} are empirical constants. The advantage of this model is that it does not assume TKE equilibrium in the near-wall region. As well, the use of a one equation model in *Region 1* makes the modelling of the near-wall simpler than that of two equation LRN models. Chen and Patel (1988) have demonstrated their two-layer LRN model accurately predicts a number of experimental two-dimensional boundary layers.

As with the wall function approach of the standard $\bar{q} - \epsilon$ model presented in Section 6.8.1, the present study recommends that the dissipation length model of Equation (6.82) be altered to reflect the findings of Galbraith et al. (1977). In agreement with the mixing length model of Galbraith et al. (1977), Equation (3.39), the dissipation length model of Chen and Patel (1988), Equation (6.82), may be written as

$$L_{\epsilon} = C_{\mu}^{-3/4} k y \sqrt{\frac{\tau'}{\tau_w}} (1 - \exp\{-R_T/A_{\epsilon}\}) \quad (6.86)$$

where

$$\tau' = -\rho \overline{uv} \quad (6.87)$$

These modifications might improve the predictive capabilities of the LRN model of Chen and Patel. Clearly, the testing of this modified turbulence model is a topic for future research.

6.8.3 A Near-Wall Turbulence Model for Three-Dimensional Turbulent Boundary Layers

The intent of this section is to propose a two-layer LRN $\bar{q} - \epsilon$ model for use with 3DTBLs. A model similar to that of the 2D model of Chen and Patel (1988), presented in Section 6.8.2, is given. The present author recognizes that the turbulence modelling is a complex subject. Consequently, the two-layer model presented in this section is largely intended as a basis for future investigations.

For simplicity, this section will only consider planar three-dimensional turbulent boundary layers. Accordingly, the x -axis is assumed to be aligned with the freestream direction and the y -axis is normal to the wall. The near-wall one equation model for a 3DTBL is as follows.

In accordance with the two-layer model of Chen and Patel (1988), the present study will assume that the near-wall TKE transport equation for a planar 3DTBL is similar to that of Equation (6.80):

$$\begin{aligned}
 U \frac{\partial \bar{q}}{\partial x} + v \frac{\partial \bar{q}}{\partial y} + w \frac{\partial \bar{q}}{\partial z} = & v_{xy} \left\{ \frac{\partial U}{\partial y} \right\}^2 + v_{zy} \left\{ \frac{\partial W}{\partial y} \right\}^2 \\
 & - \frac{\partial}{\partial y} \left(\{v + v'/\sigma_1\} \frac{\partial \bar{q}}{\partial y} \right) - \epsilon
 \end{aligned}
 \tag{6.88}$$

where

$$v' = \sqrt{\{v_{xy}\}^2 + \{v_{zy}\}^2} \tag{6.89}$$

Note that the TKE production terms in Equation (6.88) are written in terms of anisotropic eddy viscosities as

$$- \bar{u}v \frac{\partial U}{\partial y} = v_{xy} \left\{ \frac{\partial U}{\partial y} \right\}^2 \tag{6.90}$$

and

$$-\overline{wv} \frac{\partial W}{\partial y} = v_{xy} \left\{ \frac{\partial W}{\partial y} \right\}^2 . \quad (6.91)$$

In the inner region, the anisotropic eddy viscosities may be expressed in forms similar to that of Equation (6.81):

$$v_{xy} = C_{\mu x} \sqrt{\overline{q}} L_{xx} \quad (6.92)$$

and

$$v_{yy} = C_{\mu z} \sqrt{\overline{q}} L_{zz} . \quad (6.93)$$

where

$$L_{xx} = C_{\mu x}^{-3/4} k y \sqrt{\frac{\tau_{xy}}{\tau_{x,w}}} \sqrt{\frac{\tau_w}{\tau_{x,w}}} (1 - \exp\{-R_T/A_{xx}\}) . \quad (6.99)$$

$$L_{zz} = C_{\mu z}^{-3/4} k y \sqrt{\frac{\tau_{yy}}{\tau_{z,w}}} \sqrt{\frac{\tau_w}{\tau_{z,w}}} (1 - \exp\{-R_T/A_{zz}\}) . \quad (6.95)$$

A_{xx} and A_{zz} are empirical constants and the turbulence Reynolds number, R_T , is defined as

$$R_T = \frac{y \sqrt{\overline{q}}}{\nu} . \quad (6.85)$$

The corresponding TKE diffusion relation in the near-wall region then is

$$\varepsilon = \frac{\{\overline{q}\}^{3/2}}{L_\epsilon} , \quad (6.12)$$

where the dissipation length is

$$L_e = C_\mu^{-3/4} k y \sqrt{\frac{\tau'}{\tau_w}} \left(1 - \exp\{-R_T/A_e\} \right), \quad (6.96)$$

and A_e is an empirical constant. Note that in the three-dimensional case $C_{\mu x}$, $C_{\mu z}$, and C_μ may be constants or functions (Lakshminarayana, 1986). Lakshminarayana (1986) notes that some turbulence modelers have attempted to modify C_μ by means of algebraic Reynolds stress equations. In the outer region, the component, or anisotropic, eddy viscosities might be expressed as

$$\nu_{xy} = C_{\mu x} \frac{\{\bar{q}\}^2}{\epsilon} \quad (6.97)$$

and

$$\nu_{zy} = C_{\mu z} \frac{\{\bar{q}\}^2}{\epsilon}. \quad (6.98)$$

One should note that these relations only form the starting point for developing a complete 3D turbulence model. Clearly, other issues such as the appropriate form of the TKE and TKE dissipation transport equations in the inner and outer regions, as well as the correct form of the variables and constants in these equations are topics for further research.

CHAPTER 7

SUMMARY, CONCLUSIONS, AND RECOMMENDATIONS

7.1 SUMMARY

The objective of this investigation was to establish near-wall similarity in three-dimensional turbulent boundary layers based on the law-of-the-wall similarity scales U^{**} , W^{**} , and y/Q^* . These scales were used to generate the law-of-the-wall variables,

$$\frac{U}{U^{**}},$$

$$\frac{W}{W^{**}},$$

and

$$\frac{yQ^*}{\nu},$$

where U and W are the components of velocity and y is the distance normal to the wall. Note that the comments provided in this chapter refer only to these law-of-the-wall similarity scales and variables unless stated otherwise. Clearly, the verification of near-wall similarity is largely a graphical approach. As a result, extensive law-of-the-wall plots based on these near-wall variables were included in this study. The main body of work presented in this thesis is summarized as follows.

- 1) Logarithmic law equations were developed in Chapter 3. These equations are used to predict the distribution of the velocity components, U and W , in the fully turbulent portion of the near-wall region. The component log-law equations for a Cartesian coordinate system are

$$\frac{U}{U^{**}} = \frac{1}{k} \ln \left(\frac{yQ^*}{\nu} \right) + B_x \quad (3.66)$$

and

$$\frac{W}{W^{**}} = \frac{1}{k} \ln \left(\frac{yQ^*}{\nu} \right) + B_s, \quad (3.67)$$

respectively. Similarly, the component log-law equations for the axial and tangential velocity components of a cylindrical coordinate system are

$$\frac{U}{U^{**}} = \frac{1}{k} \ln \left(\frac{yQ^*}{\nu} \right) + B_s, \quad (3.59)$$

and

$$\frac{W}{W^{**}} = \{1 \pm y/r_w\} \left[\frac{1}{k} \ln \left(\frac{yQ^*}{\nu \{1 \pm y/r_w\}} \right) + B_s \right], \quad (3.65)$$

respectively. Note that Equations (3.59) and (3.65) are not completely general in that wall the curvature, r_w , is assumed to be in the tangential, θ , direction only.

- 2) Law-of-the-wall plots for the three-dimensional planar flows of McAllister (1979) were presented in Chapter 4. In these experiments a floating element device was used to measure the wall shear stress vector, τ_w , directly. As a result, the law-of-the-wall similarity scales were calculated from these experimentally obtained values. The log-law equations, Equations (3.66) and (3.67), were then applied to the streamwise and crossflow velocity components.
- 3) Law-of-the-wall plots for the axisymmetric annular flow experiments of Yowakim (1985), Lohmann (1976), and BM (1974) were presented in Chapter 5. Note that the wall shear stress vector, τ_w , was not measured directly in these experiments. Consequently, in the present study the value of the wall shear stress vector was determined by the Modified Clauser plot method. This technique is described in Appendix C. The log-law equations, Equations (3.59) and (3.65), were applied to the axial and tangential velocity components.
- 4) A possible connection between the component log-law equations and the transport equations for the turbulent stresses, $\overline{u^2}$ and $\overline{w^2}$, was discussed in Chapter 6.

7.2 CONCLUSIONS

The main conclusions drawn from the present work are as follows:

- 1) In Section 4.1 it was shown that the scales ν/U^{**} and ν/W^{**} are not the appropriate length scales for establishing law-of-the-wall similarity in two-dimensional boundary layers. As a result, these length scales were assumed not to be good candidates for establishing law-of-the-wall similarity in three-dimensional boundary layers. Instead, the present study used ν/Q^* as the near-wall similarity length scale for both velocity components, U and W .
- 2) The law-of-the-wall plots for the teardrop, pressure driven flow of McAllister (1979), presented in Section 4.3, exhibited a log-linear y^+ region only for the streamwise velocity, U .
- 3) The law-of-the-wall plots for the two swept step, shear driven flows of McAllister (1979), presented in Section 4.3, revealed a log-linear y^+ region for both velocity components, U and W . The log-law relations, Equations (3.66) and (3.67), were shown to apply to the streamwise and the crossflow velocity components. However, the slope, $1/k$, and intercepts, B_x and B_z , of these log-law lines were found to be very different from the standard two-dimensional values of $1/k = 2.38$ and $B = 5.5$. For instance, in these two experiments the slopes of the streamwise and crossflow log-law lines were the same for both velocity components: $1/k = 1.39$. In addition, the streamwise and crossflow log-law intercepts, B_x and B_z , were determined to be 10.0 and 6.0, respectively. Based on these findings it was then suggested that the slope, $1/k$, may be a scalar while the intercept may be a vector with the components B_x and B_z .
- 4) According to the discussion given in Sections 4.4 and 5.2, it is apparent that the component log-law equations, Equations (3.66 & 3.67) and (3.59 & 3.65), only apply to the nearly-collateral region of a three-dimensional turbulent boundary layer. That is to say these equations do not predict skewing of the mean velocity profile. However, depending on the value of the constants, k and B , the flow angle, ϕ , in such a nearly-collateral region may differ substantially from that of the wall shear stress angle, β_w . As mentioned in Section 4.4, the existence of such a collateral region is an exception rather than a rule in three-dimensional turbulent boundary layers.

- 5) The Modified Clauser plot technique, outlined in Appendix C, appeared to be a good method for determining the skin friction coefficients, C_{fx} and $C_{f\theta}$, for all of the annular flows examined in Chapter 5. However, this method was unable to give the tangential skin friction component, $C_{f\theta}$, at the first data station for the experiments of Lohmann (1976) and Bissonnette & Mellor (1974). It was suggested in Section 6.5 that this may be due to an imbalance between production and dissipation in the transport equations for $\overline{u^2}$ and $\overline{w^2}$ at the first data station.

- 6) As shown in Section 5.2, the axial and tangential direction log-law equations, Equation (3.59) and (3.65), provided a good fit to the experimental data of Yowakim (1985) for all three inlet swirl numbers: $SN = 0.23, 0.47, \text{ and } 0.87$. Furthermore, the calculated wall shear stress data were shown to be consistent with the trend of the turbulent shear stresses, $-\overline{uv}$ and $-\overline{wv}$. Hence, the axial and tangential direction log-law equations, Equations (3.59) and (3.65), seem to be valid for a swirling flow in which the mean flow is two-dimensional.

- 7) The component log-law, Equations (3.59) and (3.65), also provided a good fit to the experimental data of Lohmann (1976). However, the agreement with the tangential velocity data of Bissonnette and Mellor (1974) was less satisfactory, particularly at the last two axial data stations. Two possible explanations for this poor agreement were given. First, the slope of the tangential log-law line, $1/k$, may be too large. This then implies that the constant k may be a vector consisting of the components k_x and k_θ . Second, the effects of transverse curvature and rotation may have become significant so as to upset the balance between production and dissipation in the log-law region. If this is the case, history effects would be important in the near-wall region. Note that the experiments of Lohmann and BM were considered to be more severe tests of the validity of Equations (3.59) and (3.65) since the mean flow was significantly three-dimensional at some of the data stations.

- 8) The shear stress plots for the experiments of Lohmann (1976) and BM (1974), given in Section 5.2, indicated that the calculated value of the wall shear stress vector, τ_w , may not be accurate. It was suggested that different values of k and B , other than those of $k = 0.42$ and $B = 5.5$ might improve this agreement.

- 9) Based on the law-of-the-wall plots for the swept step experiments of McAllister and the spinning cylinder experiments of Lohmann and BM it appears that one may regard the slope and intercept of log-law equations, $1/k$ and B , as variables. In particular, k and B may be viewed as vectors in three-dimensional boundary layers. Clearly, the usefulness of the component log-law equations in 3DTBLs would then be greatly enhanced if one was to develop correlations for k and B . In Section 4.5 it was suggested that the best approach to determining k and B would be through a computational investigation.
- 10) In Chapter 6, it was proposed that a successful application of the component log-law equations is accompanied by a balance between production, pressure-strain, and dissipation in the transport equations for $\overline{u^2}$ and $\overline{w^2}$. If such is the case, this may explain why log-linear y^+ regions do not exist in some instances and why the value of k may be very different from the standard two-dimensional value, $k = 0.42$.

In summary, based on the results of the present study, law-of-the-wall similarity does not seem to be as extensive in 3DTBLs as in 2DTBLs. In particular, the law-of-the-wall plots based on the near-wall scales, U^{**} , W^{**} , and v/Q^* , did not collapse all the near-wall data in the fully turbulent region. In spite of this, the log-law equations derived in Chapter 3 were shown to provide a good fit to some of the data discussed in this investigation. Specifically, these log-law equations are confined to the special case of 3DTBLs which have a collateral region in the fully turbulent portion of the near-wall region. It was suggested that the usefulness of these equations could be extended if k and B were interpreted as variable constants rather than universal constants.

7.3 RECOMMENDATIONS

The present study has not proven that law-of-the-wall similarity exists in three-dimensional turbulent boundary layers. In fact a number of important issues remain unresolved. However, some of the results of this study are encouraging and warrant further investigation. Consequently, the main recommendations for additional work are as follows:

- 1) According to the suggestions provided in Chapter 6, more detailed three-dimensional boundary layer experimental data are required in order to test the validity of law-of-the-wall relations. In particular, such experiments should include a number of turbulence measurements in the near-wall region. Also, a direct measure of the wall shear stress vector, presumably obtained by a floating element device, is desirable although admittedly very difficult to obtain. Perhaps the best approach would be to perform an experiment in which a 3DTBL is generated by shear forces rather than pressure forces. This would conceivably minimize the adverse effects of pressure gradients on the accuracy of a floating element device.
- 2) Perform a 3D Navier-Stokes computational study in conjunction with a 3DTBL experimental study. This may allow one to correlate, or at least accurately determine the slope and intercept, $1/k$ and B , of the log-law equations. Such studies may also help identify the variables which govern these constants. Also, the choice of a turbulence closure model is a topic for further investigation. However, the equilibrium concepts discussed in Chapter 6 suggest that a variant of the two-layer $k - \epsilon$ turbulence model described in Chen and Patel (1988) may be more appropriate.^{1,2}
- 3) Refer to a few two-dimensional flows over curved surfaces to determine how wall

¹ The symbol k is often used in the open literature to represent the turbulent kinetic energy, $k = (\overline{u^2} + \overline{v^2} + \overline{w^2}) / 2$, where u , v , and w are the fluctuating components of velocity. Furthermore, the term $k - \epsilon$ is a standard expression in the open literature. However, in this thesis the symbols k and $\bar{q}$ are used to represent the slope of the log-law equation, $1/k$, and the turbulent kinetic energy, respectively.

² ϵ represents the dissipation of turbulent kinetic energy.

curvature affects the limits of validity of the two-dimensional log-law equation. At the same time, one could determine which log-law relation provides the best fit to such data:

$$\frac{U}{U_\tau} = \frac{1}{k} \ln \left(\frac{y U_\tau}{\nu} \right) + B, \quad (1.3)$$

$$\frac{U}{U_\tau} = \{1 \pm y/r_w\} \left[\frac{1}{k} \ln \left(\frac{y U_\tau}{\nu \{1 \pm y/r_w\}} \right) + B \right], \quad (5.20)$$

or

$$\frac{U}{U_\tau} = \frac{1}{\{1 \pm y/r_w\}} \left[\frac{1}{k} \ln \left(\frac{y U_\tau}{\nu} \right) \mp \frac{y}{k r_w} + B \right]. \quad (6.65)$$

- 4) Refer to other three-dimensional experimental data to determine if the modifications to the structural similarity hypothesis of Townsend (1961) and the dissipation model of Batchelor (1953), presented in Sections 6.3 and 6.4, are appropriate for the near-wall region of a 3DTBL.³
- 5) Examine experimental and numerical simulation data to determine if a near-wall balance exists between production, pressure-strain, and dissipation in the transport equations for the turbulent stresses, $\overline{u^2}$ and $\overline{w^2}$, in 3DTBLs.³ At the same time one could identify the significant production terms in these equations. This may explain why the slope of the log-law equations may be different from that of the standard two-dimensional value, $1/k = 2.38$. Moreover, such investigations may also serve as a guide to selecting appropriate near-wall models for the dissipation and pressure-strain terms.

³ It is suggested that one first consult all the 3DTBL data available in the literature. Otherwise new experiments may be necessary.

- 6) Investigate other means of deriving near-wall similarity relations for two- and three-dimensional turbulent boundary layers. In particular, the method of asymptotic analysis has been used to derive log-law equations for 2DTBLs. However, the application of this technique to 3DTBLs is currently an active area of research: see for instance Degani et al. (1992 & 1993). Hence, it seems worthwhile to determine the merits of such approaches.

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APPENDIX A

VISCOUS SUBLAYER EQUATIONS

The purpose of this appendix is to derive the viscous sublayer equations that are used in the figures of Chapters 4 and 5. These equations merit some discussion since they are based on the near-wall similarity scales, U^{**} , W^{**} , and v/Q^* , introduced in Chapter 3.

Consider that, in the very near-wall region of a boundary layer flow, the Reynolds stress contribution to the total shear stress is negligible. Thus, in the very near-wall region, the total shear stress consists primarily of the molecular, or viscous, portion. In the case of a three-dimensional planar flow, the shear stress Equations (3.19) and (3.21) then reduce to

$$\tau_{xy} = \tau_{xy}^m = \mu \frac{\partial U}{\partial y} \quad (\text{A.1})$$

and

$$\tau_{xy} = \tau_{xy}^m = \mu \frac{\partial W}{\partial y} . \quad (\text{A.2})$$

If the variation of the shear stress components across the sublayer is considered small, the viscous shear stress components may then be assumed to be equal to their respective wall shear stress values. This is a reasonable assumption since the viscous sublayer thickness is extremely thin. Equations (A.1) and (A.2) can then be written as

$$\tau_{xy}^m = \mu \frac{\partial U}{\partial y} = \tau_{x,w} \quad (\text{A.3})$$

and

$$\tau_{xy}^m = \mu \frac{\partial W}{\partial y} = \tau_{x,w} . \quad (\text{A.4})$$

respectively. Equations (A.3) and (A.4) may be converted into non-dimensional form by using the component velocity scales

$$U^{**} = \sqrt{\frac{\tau_{x,w}}{\rho}} \sqrt{\frac{\tau_{x,w}}{\tau_w}}$$

and

$$W^{**} = \sqrt{\frac{\tau_{x,w}}{\rho}} \sqrt{\frac{\tau_{x,w}}{\tau_w}}$$

and the length scale

$$\frac{Q^*}{v} = \frac{1}{v} \sqrt{\frac{\tau_w}{\rho}} .$$

The resulting non-dimensional viscous sublayer equations are then given as

$$\frac{\partial U^*}{\partial y^*} = 1 \quad (A.5)$$

and

$$\frac{\partial W^*}{\partial y^*} = 1 , \quad (A.6)$$

where

$$U^* = \frac{U}{U^{**}} ,$$

$$W^* = \frac{W}{W^{**}} ,$$

and

$$y^* = \frac{yQ^*}{v} .$$

Integration of Equations (A.5) and (A.6) yields expressions for the mean velocity distribution in the viscous sublayer:

$$U^* = y^* + C_1 \quad (\text{A.7})$$

and

$$W^* = y^* + C_2, \quad (\text{A.8})$$

where C_1 and C_2 are constants of integration. Application of the no-slip boundary condition at the wall,

$$U^* = W^* = 0 \quad \text{at } y^* = 0, \quad (\text{A.9})$$

then gives

$$U^* = y^* \quad (\text{A.10})$$

and

$$W^* = y^*. \quad (\text{A.11})$$

The important feature of Equations (A.10) and (A.11) is that the velocity components are non-dimensionalized by their respective shear velocity scales. These two equations produce the same curve in the law-of-the-wall plots of Chapter 4.

The viscous sublayer equations for the annular flows, presented in Chapter 5, are derived in the same manner as for the preceding planar case. The significant difference between the two cases is in the interpretation of the integration constant for the tangential direction. The non-dimensional sublayer equations for the axial and the tangential directions are

$$\frac{\partial U^*}{\partial y^*} = 1 \quad (\text{A.12})$$

and

$$r^* \frac{\partial}{\partial y^*} \left(\frac{W^*}{r^*} \right) = 1. \quad (\text{A.13})$$

respectively. Integration of Equation (A.12) and the application of the no-slip boundary condition, Equation (A.9), produces the axial direction viscous sublayer equation

$$U^* = y^* . \quad (\text{A.14})$$

Integration of Equation (A.13) gives

$$\frac{W^*}{r^*} = \pm \ln(r_w^* \pm y^*) + C , \quad (\text{A.15})$$

where C is a constant of integration. The plus and minus signs in Equation (A.15) denote convex and concave walls, respectively. The form of the integration constant, C , is determined by the fact that Equation (A.15) should approach the planar form,

$$W^* = y^* + C_2 , \quad (\text{A.16})$$

as r_w approaches infinity. This may be accomplished by rewriting Equation (A.15) as

$$W^* = \{r_w^* \pm y^*\} \left[\pm \ln(1 \pm y^*/r_w^*) + C \pm \ln r_w^* \right] . \quad (\text{A.17})$$

The integration constant of Equation (A.15) can then be defined in terms of the planar integration constant, C_2 , as

$$C = \frac{C_2}{r_w^*} \mp \ln r_w^* . \quad (\text{A.18})$$

Substitution of Equation (A.18) into Equation (A.17) gives

$$W^* = \{y^* \pm r_w^*\} \ln(1 \pm y^*/r_w^*) + C_2 \{1 \pm y^*/r_w^*\} . \quad (\text{A.19})$$

The first term in Equation (A.19) reduces to y^* as r_w approaches infinity. Hence, Equation (A.19) reduces to the planar form, Equation (A.16), as r_w tends to infinity. Applying the no-slip boundary condition, Equation (A.9), the viscous sublayer equation for the tangential direction becomes

$$W^* = \{y^* \pm r_w^*\} \ln(1 \pm y^*/r_w^*) , \quad (\text{A.20})$$

where

$$r_w^+ = \frac{r_w Q^+}{\nu}.$$

Note that the functional forms of Equations (A.20) and (A.11) are quite different. However, one must recognize that the upper limit of validity of the viscous sublayer equations is typically $y^+ = 5$ and that the smallest value of r_w^+ given in Chapter 5 is $r_w^+ = 4530.0$. Consequently, for all practical purposes Equation (A.11) can be used in place of Equation (A.20) since they are not numerically different in the viscous sublayer region. Nevertheless, as a matter of consistency the law-of-the-wall plots for the annular flows presented in Chapter 5 use Equation (A.20) rather than Equation (A.11).

APPENDIX B

UNCERTAINTY ANALYSIS

The objective of this appendix is to estimate the uncertainty in the law-of-the-wall variables, U^+ , W^+ , and y^+ , for the experimental data of McAllister (1979). These uncertainty estimates are included in a few of the law-of-the-wall plots presented in Chapter 4.

In the present investigation, uncertainty estimates are calculated based on a root mean square (RMS) formula. One must acknowledge that such an analysis is an external estimate and can not detect systematic errors such as unsuspected sources of noise, time and temperature drift, zero offset, and human error. Hence, the measured data is assumed to be free of any such bias. The reader is referred to Kline (1985) for a complete discussion of uncertainty analysis. A brief summary concerning the calculation of RMS uncertainty follows.

Consider a dependent variable, S , which is calculated from a number of independently measured variables, $t_1, t_2, \dots, t_n$, such that

$$S = f(t_1, t_2, t_3, \dots, t_n) \quad (\text{B.1})$$

In dimensional form, the RMS uncertainty in S is then given as

$$\sigma_s = \sqrt{\sum_{i=1}^n \left(\frac{\partial S}{\partial t_i} \right)^2 \sigma_{t_i}^2}, \quad (i = 1, \dots, n) \quad (\text{B.2})$$

where the subscript i denotes an independent measure and σ_{t_i} is the uncertainty in the measure. Equation (B.2) can be written in fractional, or non-dimensional, form as

$$\frac{\sigma_s}{S} = \sqrt{\sum_{i=1}^n C_i \left(\frac{\sigma_{t_i}}{t_i} \right)^2}. \quad (\text{B.3})$$

where

$$C_i = \left(\frac{\partial \ln S}{\partial \ln t_i} \right) \quad (\text{B.4})$$

are the influence coefficients. The influence coefficient, C_i , connects a one-percent uncertainty in a given measureand, t_i , to the percent uncertainty in the result, S . Therefore, the appealing feature of Equation (B.3) is that the influence coefficients are explicitly stated. This then allows one to easily identify which measureands contribute the most to the uncertainty in the computed result. In the present investigation, Equation (B.3) was used to calculate the uncertainty in the law-of-the-wall variables for the experimental data of McAllister (1979).

From Chapter 3, the component law-of-the-wall variables for a Cartesian coordinate system are defined as

$$U^+ = \frac{U}{U^{**}} , \quad (\text{B.5})$$

$$W^+ = \frac{W}{W^{**}} , \quad (\text{B.6})$$

and

$$y^+ = \frac{yQ^+}{\nu} . \quad (\text{B.7})$$

The shear velocity scales in these equations are given as

$$Q^+ = \sqrt{\frac{\tau_w}{\rho}} ,$$

$$U^{**} = \sqrt{\frac{\tau_{x,w}}{\rho}} \sqrt{\frac{\tau_{x,w}}{\tau_w}} = \sqrt{\frac{\tau_w}{\rho}} \cos \beta_w = Q^+ \cos \beta_w ,$$

and

$$W^{**} = \sqrt{\frac{\tau_{x,w}}{\rho}} \sqrt{\frac{\tau_{z,w}}{\tau_w}} = \sqrt{\frac{\tau_w}{\rho}} \sin \beta_w = Q^* \sin \beta_w ,$$

where β_w is the wall shear stress angle. In Chapter 4, Equations (B.5) and (B.6) are applied to the streamwise and crossflow velocity components, respectively, of a three-dimensional turbulent boundary layer. Clearly, one must first determine the uncertainties in τ_w , β_w , ϕ , μ , γ , Q , and ρ , in order to estimate the uncertainty in the law-of-the-wall variables.

A partial list of the measurement uncertainties for the experimental data of McAllister (1979) is given in Table B.1. Note that uncertainty estimates for the air density, ρ , and the magnitude of the resultant velocity, Q , were not stated by McAllister. Therefore, for the purpose of an uncertainty analysis, the present study assumed that these values were calculated from simple relations. For instance, the air was assumed to follow the ideal gas law,

$$\rho = \frac{P}{RT} ,$$

where $R = 287 \text{ J/(kg-K)}$ is the gas constant of air. The uncertainty in the gas constant R was considered to be negligible. Furthermore, since the Mach number of the air flow was small, $M = 0.07$, the flow was regarded as incompressible. The resultant mean velocity, Q , was then assumed to be calculated from the Bernoulli equation as

$$Q = \sqrt{\frac{2 \Delta P}{\rho}} ,$$

where ΔP is the experimentally measured stagnation-static pressure difference. Consequently, using the ideal gas law and the Bernoulli equation, one may express the uncertainty in the law-of-the-wall variables in terms of the fundamental measureands: τ_w , β_w , ϕ , μ , γ , P , T , and ΔP . The uncertainty equations for the law-of-the-wall variables are then given by Equation (B.3).

Using Equation (B.3), the squared fractional uncertainties in the law-of-the-wall variables, y^+ , U^+ , and W^+ , are

<i>Variable</i>	<i>Uncertainty</i>
Barometric Pressure, P (mmHg)	± 0.762 mmHg
Temperature, T ($^{\circ}\text{C}$)	± 0.56 $^{\circ}\text{C}$
Pitot-Static Pressure Difference, ΔP (mmH ₂ O)	± 0.076 mmH ₂ O, $\Delta P < 2.5$ mmH ₂ O ± 0.127 mmH ₂ O, $\Delta P \geq 2.5$ mmH ₂ O
Vertical Position, y (mm)	± 0.0254 mm
Mean Flow Angle, ϕ (degrees)	$\pm 1.0^{\circ}$
Dynamic Viscosity, μ (kg/[m-s])	Change in μ corresponding to ± 0.56 $^{\circ}\text{C}$

Table B.1 A Partial List of Measurement Uncertainties from McAllister (1979).

$$\left(\frac{\sigma_{y^*}}{y^*}\right)^2 = \frac{1}{4}\left(\frac{\sigma_{\tau_w}}{\tau_w}\right)^2 + \left(\frac{\sigma_y}{y}\right)^2 + \left(\frac{\sigma_\mu}{\mu}\right)^2 + \frac{5}{4}\left(\frac{\sigma_P}{P}\right)^2 + \frac{5}{4}\left(\frac{\sigma_T}{T}\right)^2, \quad (\text{B.8})$$

$$\begin{aligned} \left(\frac{\sigma_{U^*}}{U^*}\right)^2 &= \frac{1}{4}\left(\frac{\sigma_{\tau_w}}{\tau_w}\right)^2 + \frac{1}{4}\left(\frac{\sigma_{\Delta P}}{\Delta P}\right)^2 + \frac{1}{2}\left(\frac{\sigma_P}{P}\right)^2 + \frac{1}{2}\left(\frac{\sigma_T}{T}\right)^2 \\ &+ [\phi \tan \phi]^2 \left(\frac{\sigma_\phi}{\phi}\right)^2 + [\beta_w \tan \beta_w]^2 \left(\frac{\sigma_{\beta_w}}{\beta_w}\right)^2, \end{aligned} \quad (\text{B.9})$$

and

$$\begin{aligned} \left(\frac{\sigma_{W^*}}{W^*}\right)^2 &= \frac{1}{4}\left(\frac{\sigma_{\tau_w}}{\tau_w}\right)^2 + \frac{1}{4}\left(\frac{\sigma_{\Delta P}}{\Delta P}\right)^2 + \frac{1}{2}\left(\frac{\sigma_P}{P}\right)^2 + \frac{1}{2}\left(\frac{\sigma_T}{T}\right)^2 \\ &+ [\phi \cot \phi]^2 \left(\frac{\sigma_\phi}{\phi}\right)^2 + [\beta_w \cot \beta_w]^2 \left(\frac{\sigma_{\beta_w}}{\beta_w}\right)^2, \end{aligned} \quad (\text{B.10})$$

respectively. Note that these equations only apply to a Cartesian coordinate system in which the velocity components are defined as

$$U = Q \cos \phi$$

and

$$W = Q \sin \phi,$$

where

$$\phi = \tan^{-1}\left(\frac{W}{U}\right)$$

is the mean flow angle. The RMS fractional uncertainties are then obtained by taking the square root of Equations (B.8), (B.9), and (B.10). For the present study, a computer program was written to calculate these equations for each data point. A sample calculation for one data point is provided in Section B.1.

A notable feature of the three-dimensional flow experiments of McAllister (1979) is that the wall shear stress vector, τ_w , was measured directly by a floating element device. However, as mentioned in Chapter 4, such measurements are particularly difficult. One then expects that the fractional uncertainty in the wall shear stress vector will be the largest contributor to the fractional uncertainty in the law-of-the-wall variables. In particular, from Equations (B.9) and (B.10), it is seen that the influence coefficients for the wall shear stress angle, β_w , may become very large. This is primarily due to the trigonometric terms, $\tan(\beta_w)$ and $\cot(\beta_w)$, which are infinite at 0° and 90° , respectively. Thus, the influence coefficients for the wall shear stress angle, β_w , indicate that precise wall shear stress measurements are essential for reducing the fractional uncertainty in law-of-the-wall variables.

B.1 SAMPLE UNCERTAINTY CALCULATION FOR DATA STATION S45 0 6 1

A sample calculation to determine the fractional uncertainty in the law-of-the-wall variables, U^+ , W^+ , and y^+ , for the data station S45 0 6 1 is given. For illustrative purposes, only one data point, $y = 0.81 \text{ mm}$, will be considered.

From McAllister (1979), the resultant mean velocity and flow angle at the vertical position $y = 0.81 \text{ mm}$ of S45 0 6 1 are

$$Q = 10.01 \frac{m}{s}$$

and

$$\phi = 42.6^\circ ,$$

respectively. Measured values of the wall shear stress vector, the barometric pressure, and the temperature at the wall are given in Table B.2. Furthermore, calculated values of the air density, the pitot-static pressure difference, the law-of-the-wall shear velocity scales, and the law-of-the-wall variables are listed in Table B.3.

The uncertainty estimates for the law-of-the-wall variables, U^+ , W^+ , and y^+ , are calculated

<i>Measured Quantity</i>	<i>Value</i>
Wall Shear Stress, τ_w	$0.61 \text{ Pa} \pm 0.033 \text{ Pa}$
Wall Shear Stress Angle, β_w	$60.9^\circ \pm 3.1^\circ$
Barometric Pressure, P	714.2 mmHg
Temperature, T	26.7°C

Table B.2 A Partial List of the Measured Data at Station S45 0 6 1.

<i>Quantity</i>	<i>Equation</i>	<i>Value</i>
Density of Air, ρ	$\rho = \frac{P}{RT}$	1.104 kg/m ³
Pressure Difference, ΔP	$\Delta P = \rho \frac{Q^2}{2}$	5.657 mmH ₂ O
Shear Velocity Scale, Q^*	$Q^* = \sqrt{\frac{\tau_w}{\rho}}$	0.743 m/s
Streamwise Shear Velocity, U^{**}	$U^{**} = Q^* \cos \beta_w$	0.361 m/s
Crossflow Shear Velocity, W^{**}	$W^{**} = Q^* \sin \beta_w$	0.649 m/s
y^+	$y^+ = \frac{y Q^*}{\nu}$	35.8
U^+	$U^+ = \frac{Q \cos \phi}{U^{**}}$	20.4
W^+	$W^+ = \frac{Q \sin \phi}{W^{**}}$	10.4

Table B.3 Calculated Values for the Data Station S45 0 6 1 at the Position $y = 0.81$ mm.

by substituting the measured values and corresponding uncertainties, given in Tables B.1, B.2, and B.3, into Equations (B.5), (B.6), and (B.7). From Equation (B.5), the fractional uncertainty in y^+ is

$$\begin{aligned}\frac{\sigma_{y^+}}{y^+} &= \sqrt{\frac{1}{4} \left(\frac{0.033}{0.61} \right)^2 + \left(\frac{0.0256}{0.81} \right)^2 + \left(\frac{0.005}{1.855} \right)^2 + \frac{5}{4} \left(\frac{0.762}{714.2} \right)^2 + \frac{5}{4} \left(\frac{0.56}{26.7} \right)^2} \\ &= \underline{0.0478 = 4.8\%}.\end{aligned}$$

Similarly, the fractional uncertainties in U^+ and W^+ , as calculated from Equations (B.6) and (B.7), are

$$\begin{aligned}\left(\frac{\sigma_{U^+}}{U^+} \right)^2 &= \frac{1}{4} \left(\frac{0.033}{0.61} \right)^2 + \frac{1}{4} \left(\frac{0.127}{5.657} \right)^2 + \frac{1}{2} \left(\frac{0.762}{714.2} \right)^2 + \frac{1}{2} \left(\frac{0.56}{26.7} \right)^2 \\ &\quad + \left[\frac{42.6^\circ \pi}{180} \tan 42.6^\circ \right]^2 \left(\frac{1.0}{42.6} \right)^2 + \left[\frac{60.9^\circ \pi}{180} \tan 60.9^\circ \right]^2 \left(\frac{3.1}{60.9} \right)^2 \\ &= 0.011\end{aligned}$$

$$\therefore \frac{\sigma_{U^+}}{U^+} = \underline{0.105 = 10.5\%}$$

and

$$\begin{aligned}\left(\frac{\sigma_{W^+}}{W^+} \right)^2 &= \frac{1}{4} \left(\frac{0.033}{0.61} \right)^2 + \frac{1}{4} \left(\frac{0.127}{5.657} \right)^2 + \frac{1}{2} \left(\frac{0.762}{714.2} \right)^2 + \frac{1}{2} \left(\frac{0.56}{26.7} \right)^2 \\ &\quad + \left[\frac{42.6^\circ \pi}{180} \cot 42.6^\circ \right]^2 \left(\frac{1.0}{42.6} \right)^2 + \left[\frac{60.9^\circ \pi}{180} \cot 60.9^\circ \right]^2 \left(\frac{3.1}{60.9} \right)^2 \\ &= 0.0023\end{aligned}$$

$$\therefore \frac{\sigma_{W^+}}{W^+} = \underline{0.048 = 4.8\%},$$

respectively.

APPENDIX C

MODIFIED CLAUSER PLOT METHOD TO DETERMINE THE SKIN FRICTION COEFFICIENTS C_f , C_{fx} , AND $C_{f\theta}$

The objective of this appendix is to present the Modified Clauser plot method of determining the wall shear stress, τ_w . These type of plots are used in Chapter 4 and 5.

C.1 MODIFIED CLAUSER PLOTS FOR TWO-DIMENSIONAL TURBULENT BOUNDARY LAYERS

The Clauser plot method for determining the wall shear stress, τ_w , has been one of the most useful applications of the two-dimensional law of the wall. Specifically, the two-dimensional log-law forms the basis of the Clauser plot method and a brief description of this procedure follows. In addition, a modified version of this method, according to Sjolander (1980), is presented.

The skin friction coefficient, for two-dimensional turbulent boundary layers, is often defined as

$$C_f = \frac{2 \tau_w}{\rho U_e^2} . \quad (C.1)$$

where ρ is the density of the fluid and U_e is the velocity at the boundary layer edge. Using this definition, the shear velocity scale, U_τ , may be written as

$$U_\tau = \sqrt{\frac{\tau_w}{\rho}} . \quad (C.2)$$

Substituting Equation (C.2) into the two-dimensional log-law equation,

$$\frac{U}{U_\tau} = \frac{1}{k} \ln \left(\frac{y U_\tau}{\nu} \right) + B, \quad (1.3)$$

one obtains the Clauser plot equation

$$\frac{U}{U_\tau} \sqrt{\frac{2}{C_f}} = \frac{1}{k} \ln \left(Re_\tau \sqrt{\frac{C_f}{2}} \right) + B, \quad (C.3)$$

where

$$Re_\tau = \frac{y U_\tau}{\nu}.$$

From Equation (C.3) one can generate a chart of U/U_τ as a function of Re_τ for different values of C_f . Experimental data in the log-law region will then fall on one of these lines of constant C_f . A sample Clauser plot is given in Figure C.1. Although such plots can be easily generated, this is somewhat of an iterative approach in that one must first assume a C_f value and then generate a corresponding C_f line on the Clauser plot. The procedure is complete when one is satisfied that a particular C_f line agrees with the experimental data. As a matter of convenience, the present study employed a modified version of the Clauser plot technique, as described next.

For a given experimental data point, $U(y)$, Equation (C.3) is an implicit equation in a single unknown, C_f . Thus, for each point Equation (C.3) may be solved iteratively for a corresponding value of C_f . However, this will be the true C_f only if the data point were measured in the log-law region; other data points will give different values of C_f . Such data points from the log-law region are easily recognized by plotting the calculated C_f values against y or Re_τ , as shown in Figure C.2. The true value of C_f can then be read from the axis. This approach is considerably more convenient than using the traditional Clauser plot.

In the present study, a computer program was written to generate Modified Clauser plots for the sample of two-dimensional flows analyzed in Section 4.1. This program used the Newton-Raphson technique to solve Equation (C.3). A component form of the Modified Clauser plot method, for use with the log-law equations developed in Chapter 3, is described next.

Figure C.1 Traditional Clauser Plot for Determining C_f . Figure C.2 Modified Clauser Plot for Determining C_f .

C.2 MODIFIED CLAUSER PLOTS FOR COMPONENTS OF VELOCITY

A means for determining the component skin friction coefficients based on the log-law equations introduced in Chapter 3 is given. This method is a modified version of those outlined in Sjolander (1980) and Yowakim (1985).

Consider the cylindrical annular flow experiments of Yowakim (1985), Lohmann (1976), and Bissonnette & Mellor (1974), presented in Chapter 5. The axial and tangential skin friction coefficients for an annular flow may be written as

$$C_{fx} = \frac{2\tau_{x,w}}{\rho Q_{max}^2} \quad (C.4)$$

$$C_{f\theta} = \frac{2\tau_{\theta,w}}{\rho Q_{max}^2} \quad (C.5)$$

and

$$C_f = \sqrt{C_{fx}^2 + C_{f\theta}^2} \quad (C.6)$$

Note that the choice of the velocity scale used to define these skin friction coefficients is somewhat arbitrary. For instance, a mass averaged or momentum averaged velocity might have been used. However, as a matter of computational convenience, the present study used the maximum resultant velocity, Q_{max} , in the definition of the skin friction coefficients. For the experiment of Yowakim, Q_{max} is the maximum resultant velocity in the annulus at a given axial station, x . Similarly, for the experiments of Lohmann (1976) and Bissonnette & Mellor (1974), Q_{max} is the resultant velocity at the furthest radial measurement coordinate for a particular axial station, x . Using Equations (C.4) and (C.5), the axial and tangential shear velocity scales, U^{**} and W^{**} , are given as

$$U^{**} = \sqrt{\frac{C_{fz}}{2}} \sqrt{\frac{\tau_{z,w}}{\tau_w}} Q_{max} = \sqrt{\frac{C_{fz}}{2}} \sqrt{\frac{C_{fz}}{C_f}} Q_{max} \quad (C.7)$$

and

$$W^{**} = \sqrt{\frac{C_{f\theta}}{2}} \sqrt{\frac{\tau_{\theta,w}}{\tau_w}} Q_{max} = \sqrt{\frac{C_{f\theta}}{2}} \sqrt{\frac{C_{f\theta}}{C_f}} Q_{max} . \quad (C.8)$$

Substituting these shear velocity definitions into the axial and tangential log-law relations,

$$\frac{U}{U^{**}} = \frac{1}{k} \ln \left(\frac{y Q^*}{\nu} \right) + B_s \quad (3.59)$$

and

$$\frac{W}{W^{**}} = \{1 \pm y/r_w\} \left[\frac{1}{k} \ln \left(\frac{y Q^*}{\nu \{1 \pm y/r_w\}} \right) + B_\theta \right], \quad (3.65)$$

one produces the component Clauser plot equations

$$\frac{U}{Q_{max}} \sqrt{\frac{2 C_f}{C_{fz}^2}} = \frac{1}{k} \ln \left(Re_\theta \sqrt{\frac{2 C_f}{C_{fz}^2}} \right) + B_s \quad (C.9)$$

and

$$\frac{W}{Q_{max}} \sqrt{\frac{2 C_f}{C_{f\theta}^2}} = \{1 \pm y/r_w\} \left[\frac{1}{k} \ln \left(\frac{Re_\theta}{1 \pm y/r_w} \sqrt{\frac{2 C_f}{C_{f\theta}^2}} \right) + B_\theta \right], \quad (C.10)$$

where

$$Re_\theta = \frac{y Q_{max}}{\nu} .$$

Note that the top sign in Equation (C.10) refers to the inner, convex, wall and the bottom sign denotes the outer, concave, wall. Furthermore, these equations assume that the wall curvature, r_w , is in the tangential direction only.

Given the experimental axial and tangential velocity profiles, $U(y)$ and $W(y)$, Equations (C.9) and (C.10) can be solved for C_{fx} and $C_{f\theta}$ by iteration. However, the presence of the resultant skin friction coefficient, C_f , in both of these equations indicates that they must be solved simultaneously. A computer program which solves these two non-linear equations is given in Appendix D. Having solved the equations, axial and tangential Modified Clauser plots are generated by plotting C_{fx} and $C_{f\theta}$ as functions of Re_{Q_e} . For each velocity component, U and W , the data points which are actually in the log-law region are then easily identified on the Modified Clauser plots as flat lines of constant C_{fx} and $C_{f\theta}$, respectively. Typical Modified Clauser plots for the experiments of Yowakim (1985), Lohmann (1976), and Bissonnette & Mellor (1974) are shown in Figures C.3 to C.8.

Similar figures can be generated for the three-dimensional planar data of McAllister (1979), presented in Chapter 4. In this case, the skin friction coefficients may be defined as

$$C_{fx} = \frac{2\tau_{x,w}}{\rho Q_e^2} \quad (C.11)$$

and

$$C_{fz} = \frac{2\tau_{z,w}}{\rho Q_e^2} \quad (C.12)$$

where C_{fx} and C_{fz} correspond to the streamwise and crossflow directions, respectively, and Q_e is the velocity at the boundary layer edge. The resulting Clauser plot equations are then

$$\frac{U}{Q_e} \sqrt{\frac{2C_f}{C_{fx}^2}} = \frac{1}{k} \ln \left(Re_{Q_e} \sqrt{\frac{2C_f}{C_{fx}^2}} \right) + B_x \quad (C.13)$$

and

Figure C.3 Modified Clauser Plot for Determining the Skin Friction Coefficient C_{fx} at the Axial Station $x = 46.0 \text{ in.}$ for $SN = 0.87$: Experiment of Yowakim (1985).

Figure C.4 Modified Clauser Plot for Determining the Skin Friction Coefficient $C_{f\theta}$ at the Axial Station $x = 46.0 \text{ in.}$ for $SN = 0.87$: Experiment of Yowakim (1985).

Figure C.5 Modified Clauser Plot for Determining the Skin Friction Coefficient C_{fx} at the Axial Station $x = 8.0 \text{ in.}$: Experiment of Lohmann (1976).

Figure C.6 Modified Clauser Plot for Determining the Skin Friction Coefficient C_{f0} at the Axial Station $x = 8.0 \text{ in.}$: Experiment of Lohmann (1976).

Figure C.7 Modified Clauser Plot for Determining the Skin Friction Coefficient C_{fx} at the Axial Station $x = 8.0 \text{ in.}$: Experiment of Bissonnette and Mellor (1974).

Figure C.8 Modified Clauser Plot for Determining the Skin Friction Coefficient $C_{f\theta}$ at the Axial Station $x = 8.0 \text{ in.}$: Experiment of Bissonnette and Mellor (1974).

$$\frac{W}{Q_s} \sqrt{\frac{2C_f}{C_{fs}^2}} = \frac{1}{k} \ln \left(Re_{Q_s} \sqrt{\frac{2C_f}{C_{fs}^2}} \right) + B_s, \quad (C.14)$$

where

$$Re_{Q_s} = \frac{yQ_s}{\nu}.$$

Typical streamwise and crossflow Modified Clauser plots for the swept step data of McAllister (1979) are shown in Figures C.9 and C.10.

Figure C.9 Modified Clauser Plot for Determining the Streamwise Direction Skin Friction Coefficient, C_{fx} , at the Data Station S30 0 14 2: Experiment of McAllister (1979).

Figure C.10 Modified Clauser Plot for Determining the Crossflow Direction Skin Friction Coefficient, C_{fx} , at the Data Station S30 0 14 2: Experiment of McAllister (1979).

APPENDIX D

PROGRAM CODE FOR THE CALCULATION OF COMPONENT MODIFIED CLAUSER PLOTS

This appendix gives the computer program code used to generate the Modified Clauser plots for the annular flows presented in Chapter 5. A brief description of the program structure is provided.

From Appendix C, the Clauser plot equations for the axial and tangential velocity components, U and W , of a cylindrical annular flow are

$$\frac{U}{Q_{\max}} \sqrt{\frac{2C_f}{C_{fx}^2}} = \frac{1}{k} \ln \left(Re_Q \sqrt{\frac{2C_f}{C_{fx}^2}} \right) + B, \quad (C.9)$$

and

$$\frac{W}{Q_{\max}} \sqrt{\frac{2C_f}{C_{f\theta}^2}} = \{1 \pm y/r_w\} \left[\frac{1}{k} \ln \left(\frac{Re_Q}{1 \pm y/r_w} \sqrt{\frac{2C_f}{C_{f\theta}^2}} \right) + B_0 \right] \quad (C.10)$$

respectively, where

$$C_{fx} = \frac{2\tau_{x,w}}{\rho Q_{\max}^2},$$

$$C_{f\theta} = \frac{2\tau_{\theta,w}}{\rho Q_{\max}^2},$$

$$C_f = \sqrt{C_{fx}^2 + C_{f\theta}^2}.$$

and

$$Re_Q = \frac{yQ_{\max}}{\nu}.$$

Note that Equations (C.9) and (C.10) assume that the wall curvature, r_w , is in the tangential direction. For a given set of experimental mean velocity data, $U(y)$ and $W(y)$, Equations (C.9) and (C.10) must be solved simultaneously in order to determine the component skin friction coefficients, C_{fx} and $C_{f\theta}$. The difficulty, however, is that these equations are non-linear and require an iterative solution method. For the present investigation, a computer program was developed which uses a two-equation Newton-Raphson method to solve the Clauser plot equations. The Newton-Raphson method is a common means of solving non-linear equations and the details of this technique can be found in text books which deal with numerical methods: see for instance Chapra and Canale (1988). Maintaining a stable, converging calculation is one of the main difficulties with iterative solution techniques. Stability problems were encountered in the solution to Equations (C.9) and (C.10), as described next.

Initially, the program was formulated to solve the Clauser plot equations directly for the component skin friction coefficients, C_{fx} and $C_{f\theta}$. However, the solution was found to be unstable and usually diverged on the first data point. To overcome this problem, Equations (C.9) and (C.10) were rewritten in terms of two new skin friction parameters,

$$C_u = \sqrt{\frac{C_{fx}^2}{2C_f}} = \sqrt{\frac{C_{fx}}{2}} \sqrt{\frac{\tau_{x,w}}{\tau_w}} \quad (D.1)$$

and

$$C_v = \sqrt{\frac{C_{f\theta}^2}{2C_f}} = \sqrt{\frac{C_{f\theta}}{2}} \sqrt{\frac{\tau_{\theta,w}}{\tau_w}}. \quad (D.2)$$

Incorporating these parameters into the Clauser plot equations gives

$$\frac{U}{C_u Q_{\max}} = \frac{1}{k} \ln \left(Re_Q \sqrt{C_u^2 + C_v^2} \right) + B, \quad (D.3)$$

and

$$\frac{W}{C_w Q_{\text{max}}} = \left\{ 1 \pm y/r_w \right\} \left[\frac{1}{k} \ln \left(\frac{Re_0}{1 \pm y/r_w} \sqrt{C_x^2 + C_w^2} \right) + B_0 \right]. \quad (\text{D.4})$$

The program was then modified to solve these equations for C_u and C_w . The skin friction coefficients, C_{fx} and C_{f0} , were then calculated from the skin friction parameters, C_u and C_w , as

$$C_{fx} = \frac{2}{Q_{\text{max}}} C_u \sqrt{C_x^2 + C_w^2} \quad (\text{D.5})$$

and

$$C_{f0} = \frac{2}{Q_{\text{max}}} C_w \sqrt{C_x^2 + C_w^2}. \quad (\text{D.6})$$

Equations (D.3) and (D.4) converged to a solution rather quickly for the experimental data of Yowakim (1985), Lohmann (1976), and Bissonnette & Mellor (1974). A short summary of the program structure is given next.

The computer program code consists of a main program and five subroutines: CLRSR, FLUID, UWPLLOT, REPLLOT, and MODCL. Comment statements throughout the program listing describe the procedures of each subroutine in detail. Consequently, only a brief statement of the function of each subroutine will be provided here. The user is required to supply the main program with a data file containing the mean velocity components $U(y)$ and $W(y)$. The subroutine CLRSR removes any text or graphics from the screen. In the subroutine FLUID, the user is prompted for the log-law constants, k , B_x and B_0 , the wall radius, r_w , the type of wall curvature, either convex or concave, and the kinematic viscosity of the fluid, ν . The Clauser plot equations, Equations (D.3) and (D.4), are solved in the subroutine MODCL. Lastly, velocity and Modified Clauser plots are plotted on screen in the subroutines UWPLLOT and REPLLOT, respectively. A complete program listing is provided next.

The programming language of the following code is standard FORTRAN 77. Note that the code uses the WATCOM Graphical Kernel System (GKS) environment of WATFOR-77 to plot graphs on screen. The reader is referred to Coshi & Schueler (1989) and Yach (1989), respectively, for details regarding WATFOR-77 and WATCOM GKS.

```

*****
** COMPONENT MODIFIED CLAUSER PLOTS **
(ANGULAR FLOWS)

WRITTEN BY: DALE LADON

THIS PROGRAM SOLVES THE MODIFIED CLAUSER EQUATIONS FOR FLOWS OVER
CURVED SURFACES. CURVATURE IS ASSUMED TO BE IN THE TANGENTIAL
DIRECTION (S-DIR'N) ONLY. THE COMPONENT CLAUSER PLOT EQUATIONS
ARE SOLVED SIMULTANEOUSLY FOR THE SKIN FRICTION PARAMETERS Cf AND
Cw. A MULTIPLE-EQUATION NEWTON-RAPHSON METHOD IS USED TO SOLVE
THESE NONLINEAR COMPONENT EQUATIONS. THE SKIN FRICTION
PARAMETERS, Cf AND Cw, ARE THEN USED TO CALCULATE THE SKIN FRICTION
COEFFICIENTS Cf,x AND Cf,z. CONSULT THE MASTERS THESIS OF D.W. LADON
(1993) FOR THE EXACT FORM OF THE MODIFIED CLAUSER PLOT EQUATIONS

THE PROGRAM ACCEPTS USER SUPPLIED MEAN FLOW VELOCITY DATA. CONSULT
THE PROGRAM CODE FOR THE REQUIRED FORMAT OF THE INPUT DATA FILE.

MODIFIED CLAUSER PLOTS CAN BE VIEWED ON SCREEN. THIS IS
ACCOMPLISHED BY USING THE WATCON Graphics Kernel System
(WATCON GKS). THE PROGRAM IS CONFIGURED FOR A HERCULES Monochrome
Graphics Adapter. IF THE USER WISHES TO CHANGE THE TYPE OF ADAPTER
THE FOLLOWING COMMAND MUST BE CHANGED IN THE SUBROUTINES CLAUSER,
UNPLOT, AND REPLOT:

CALL GINERC( WTYPE )

CONSULT Yach (1989) FOR THE APPROPRIATE COMMANDS. AS WELL, IN ORDER
TO RUN THE PROGRAM IN THE WATFOR-77 ENVIRONMENT THE USER MUST LOAD
WATCON GKS INTO MEMORY. THIS PROCEDURE IS ILLUSTRATED IN Coechl and
Schueler (1989).

THE PROGRAM STRUCTURE CONSISTS OF A MAIN PROGRAM AND FIVE
SUBROUTINES: CLAUSER, FLUID, UNPLOT, REPLOT, AND MODCL. A BRIEF
DESCRIPTION OF THE FUNCTION OF EACH SUBROUTINE IS PROVIDED BELOW.

THE MAIN PURPOSE FOR DEVELOPING THIS PROGRAM WAS TO PROVIDE AN
ANALYTICAL TOOL FOR THE MASTERS THESIS WORK OF D.W. LADON.

SUBROUTINES
*****
CLAUSER - CLEARS THE SCREEN
FLUID - PROMPTS THE USER FOR THE KINEMATIC VISCOSITY AND THE LOG-
LAW CONSTANTS
UNPLOT - PLOTS MEAN VELOCITY PROFILES
REPLOT - PLOTS SEMI-LOGARITHMIC GRAPHS (LOG BASE 10)
MODCL - SOLVES THE COMPONENT MODIFIED CLAUSER PLOT EQUATIONS
SIMULTANEOUSLY BY THE TWO EQUATION NEWTON-RAPHSON METHOD

REFERENCES
*****
Coechl, G. and Schueler, J.B., 1989, 'WATFOR-77 User's Guide
IBM PC with DOS' 5th Edition, WATCON Publications Ltd.: Waterloo,
Ontario, CANADA.

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* Yach, D., 1989, 'WATCON GKS Graphics Tutorial and Reference,'
* WATCON Publications Ltd.: Waterloo, Ontario, CANADA.
*****
REAL Y(100),U(100),W(100),Q(100),UMAX,QMAX,VIS,K,B1,B2,BW
REAL YW(100)
INTEGER I,N,START,VPILOT,CALC,W
CHARACTER*20 DFILE

5 START=0
I=0
N=0

C DECLARE CONSTANTS, FLUID PROPERTIES, AND WALL CURVATURE TYPE.
CALL FLUID(K,B1,B2,VIS,BW,W)

C READ MEAN VELOCITY DATA FROM A USER SUPPLIED DATA FILE. THE INPUT
C DATA FILE IS ASSUMED TO BE OF THE FORM Y, U, W, AND Q. THE FIRST
C LINE OF THE DATA FILE MUST ONLY CONTAIN UMAX, QMAX AND N (THE
C NUMBER OF DATA POINTS). Y MUST BE IN METERS WHILE U, W, AND Q ARE
C NORMALIZED BY THEIR RESPECTIVE MAXIMUM VALUES. UMAX, WMAX, AND QMAX
C ARE ASSUMED TO BE IN m/s.
PRINT*
PRINT* 'INPUT DATA'
PRINT* '*****'
PRINT*
PRINT* 'ENTER DATA FILE NAME (INCLUDING THE *.DAT * EXTENSION)'
READ(*,*) (A) DFILE
CALL CLAUSER
OPEN(UNIT=8,FILE=DFILE,STATUS='OLD')
READ(8,10) UMAX,WMAX,QMAX,N
FORMAT(2X,F6.3,2X,F6.3,2X,F6.3,2X,I3)
DO 30 I=1,N
READ(8,20) Y(I),U(I),W(I),Q(I)
FORMAT(1X,F8.5,1X,F7.5,1X,F7.5,1X,F7.5)
30 CONTINUE

C ELIMINATE ANY 0.0 OR NEGATIVE VALUES FROM THE DATA. THIS AVOIDS
C PROBLEMS WITH TRYING TO CALCULATE THE LOGARITHMIC VALUE OF NEGATIVE
C OR ZERO VARIABLES. THE DATA IS ASSUMED TO BE IN ASCENDING ORDER.
DO 70 I=1,N
IF(U(I).LE.0.0) THEN
DO 40 J=I,(N-1)
U(J)=U(J+1)
W(J)=W(J+1)
Q(J)=Q(J+1)
Y(J)=Y(J+1)
CONTINUE
N=N-1
ENDIF
ENDIF
IF(W(1).LE.0.0) THEN
DO 50 J=1,(N-1)
U(J)=U(J+1)
W(J)=W(J+1)
Q(J)=Q(J+1)
Y(J)=Y(J+1)
CONTINUE
N=N-1
ENDIF
50

```

```

        IF(Y(I)-LE,0,0) THEN
          DO 60 J=1,(N-1)
            U(J)=0/(J+1)
            W(J)=0/(J+1)
            Q(J)=0/(J+1)
            X(J)=Y(J+1)
          CONTINUE
        N=N-1
      ENDDIF
    60 CONTINUE
    70 CONTINUE

    C CALCULATE (1 +/- Y/TW) TERMS.
    DO 80 I=1,N
      IF(VC.EQ.1) THEN
        YRM(I)=1.0-Y(I)/RW
      ELSE
        YRM(I)=1.0-Y(I)/RW
      ENDDIF
    80 CONTINUE

    C USE THE SUBROUTINE UNPLOT TO VIEW THE NORMALIZED VELOCITY PROFILES.
    VPLOT=0
    WRITE(*,*) 'A' ENTER 1 TO VIEW THE RESULTANT VELOCITY PROFILE',
    * , OR ZERO TO CONTINUE'
    WRITE(*,*) ' (1 OR 0 MUST BE AN INTEGER VALUE): '
    READ(5,*) VPLOT
    IF (VPLOT.EQ.1) THEN
      CALL UNPLOT(Y,Q,N)
    ELSE
      CALL CLPSCR
    ENDDIF
    VPLOT=0
    WRITE(*,*) 'B' ENTER 1 TO VIEW THE X-DIRECTION VELOCITY PROFILE',
    * , OR ZERO TO CONTINUE'
    WRITE(*,*) ' (1 OR 0 MUST BE AN INTEGER VALUE): '
    READ(5,*) VPLOT
    IF (VPLOT.EQ.1) THEN
      CALL UNPLOT(Y,U,N)
    ELSE
      CALL CLPSCR
    ENDDIF
    VPLOT=0
    WRITE(*,*) 'C' ENTER 1 TO VIEW THE Z-DIRECTION VELOCITY PROFILE',
    * , OR ZERO TO CONTINUE'
    WRITE(*,*) ' (1 OR 0 MUST BE AN INTEGER VALUE): '
    READ(5,*) VPLOT
    IF (VPLOT.EQ.1) THEN
      CALL UNPLOT(Y,W,N)
    ELSE
      CALL CLPSCR
    ENDDIF

    C PROVIDE THE OPTION TO CALCULATE MODIFIED CLAUSER PLOTS.
    CLCALC=0
    WRITE(*,*) 'D' ENTER 1 TO PRODUCE COMPONENT MODIFIED CLAUSER',
    * , PLOTS OR ZERO TO CONTINUE'
    WRITE(*,*) ' (1 OR 0 MUST BE AN INTEGER VALUE): '
    READ(5,*) CLCALC
    IF (CLCALC.EQ.1) THEN
      CALL MODCL(QMAX,UHMAX,WMAX,YRM,VIS,K,B1,B2,X,U,W,N,DFILE)
    ELSE
      PRINT*
    ENDIF
  ENDIF

  CALL CLPSCR
  ENDDIF

  C PROVIDE THE OPTION TO RERUN THE PROGRAM.
  CALL CLPSCR
  CLOSE(6,STATUS='KEEP')
  CLOSE(9,STATUS='KEEP')
  WRITE(*,*) 'E' ENTER 1 TO RERUN PROGRAM FROM THE BEGINNING OR',
  * , ZERO TO QUIT'
  WRITE(*,*) ' (1 OR 0 MUST BE AN INTEGER): '
  READ(5,*) START
  IF (START.EQ.1) THEN
    GO TO 5
  ENDDIF
  ENDDIF
END

.....
SUBROUTINE CLPSCR
.....
* THIS SUBROUTINE CLEARS THE SCREEN.
  INTEGER ERRFID,WKTYPE,WKID,CONVID,CTRL
  PARAMETER (ERRFID=0,WKTYPE=3,WKID=14,CONVID=0,CTRL=1)

  C INITIALIZE GKS
  CALL GHERC(WKTYPE)
  CALL GOKPS(ERRFID)
  CALL GOKP(WKID,CONVID,WKTYPE)
  CALL GACWK(WKID)

  C CLEAR CURRENT SCREEN
  CALL GCLWRK(WKID,CTRL)

  C UNDO GKS INITIALIZATION
  CALL GDAWK(WKID)
  CALL GCLWK(WKID)
  CALL GCLKS()

  RETURN
END

.....
SUBROUTINE FLUID(K,B1,B2,VIS,RW,WC)
.....
* THIS SUBROUTINE PROMPTS THE USER FOR THE KINEMATIC VISCOSITY, AND THE
* LOG-LAW CONSTANTS K, B1, AND B2. THESE VALUES ARE PASSED TO THE MAIN
* PROGRAM.
  REAL K,B1,B2,VIS,RW
  INTEGER LOOP,WC

  WC=1
  CALL CLPSCR
  WRITE(*,6)
  6 FORMAT(23X,'MODIFIED CLAUSER PLOT DATA ANALYSIS')
  WRITE(*,7)
  7 FORMAT(23X,' (ANGULAR FLOWS) ')
  WRITE(*,8)
  8 FORMAT(23X,'.....')
  PRINT*

```

```

PRINT*, 'ENTER THE FOLLOWING VALUES:'
PRINT*, '-----'
PRINT*
READ(5,*) VIS
VIS=0.00001556
PRINT*
PRINT*, '2) LOG-LAW SLOPE, k: '
READ(5,*) K
PRINT*
PRINT*, '3) X-DIRECTION LOG-LAW CONSTANT, B1: '
READ(5,*) B1
B1=5.5
PRINT*
PRINT*, '4) 2-DIRECTION LOG-LAW CONSTANT, B2: '
READ(5,*) B2
PRINT*
WRITE(*,*) '5) WALL RADIUS, rw (meters): '
READ(*,*) RW
PRINT*
WRITE(*,*) 'ENTER 1 FOR CONVEX WALL CURVATURE OR ZERO FOR CONCAVE',
*
WRITE(*,*) '1 OR ZERO MUST BE AN INTEGER: '
READ(*,*) WC
CALL CLSCR

PRINT*, 'THE FOLLOWING VALUES HAVE  N ENTERED:'
PRINT*, '-----'
PRINT*
WRITE(6,10) K
FORMAT(1X,'K = ',F5.3)
WRITE(6,20) B1
FORMAT(1X,'B1 = ',F5.2)
WRITE(6,30) B2
FORMAT(1X,'B2 = ',F5.2)
WRITE(6,40) VIS
FORMAT(1X,'VISCOSITY = ',E11.4,' m2/s')
WRITE(6,50) RW
FORMAT(1X,'WALL RADIUS = ',F7.4,' m')
IF(WC.EQ.1) THEN
  WRITE(*,*) 'WALL CURVATURE TYPE: CONVEX'
ELSE
  WRITE(*,*) 'WALL CURVATURE TYPE: CONCAVE'
ENDIF
PRINT*
PRINT*, 'ENTER 1 TO CHANGE VALUES OR 0 TO CONTINUE'
READ*, LOOP
IF(LOOP.EQ.1) THEN
  GO TO 5
ENDIF
RETURN
END

.....
SUBROUTINE UNPLOT(Y,VEL,N)
* THIS SUBROUTINE PLOTS NORMALIZED VELOCITY PROFILES. A VELOCITY

```

* COMPONENT, Q, U or W, AND ITS CORRESPONDING DISTANCE, Y, ARE PASSED
 * FROM THE MAIN PROGRAM. THE ABSCISSA IS THE NORMALIZED VELOCITY,
 * Q/Qmax, U/Umax OR W/Wmax, WHILE THE ORDINATE IS THE DISTANCE
 * NORMAL TO THE WALL, Y.

```

REAL VWXL,VWXR,VWYT,VWYB,XLEFT,XRIGHT,YBOTM,YTOP,SCALE
REAL XBORD(5),YBORD(5),VEL(100),Y(100)
INTEGER ERNFID,MKTYPE,MKID,CONMID,TRANS,CIRCLE,CTRL
PARAMETER (ERNFID=0,MKTYPE=3,MKID=14,CONMID=0)
PARAMETER (TRANS=1,CIRCLE=4,SCALE=0.1,CTRL=1)
PARAMETER (VWXL=0.2,VWXR=0.8,VWYB=0.02,VWYT=0.82)
PARAMETER (XLEFT=0.0,YBOTM=0.0)

C SET TOP Y-COORDINATE OF VIEWPORT TO THE LARGEST Y VALUE. (ASSUMING
C THE DATA IS ASCENDING ORDER)
YTOP=Y(N)
XRIGHT=1.1
XBORD(1)=XLEFT
YBORD(1)=YBOTM
XBORD(2)=XLEFT
YBORD(2)=YTOP
XBORD(3)=XRIGHT
YBORD(3)=YTOP
XBORD(4)=XRIGHT
YBORD(4)=YBOTM
XBORD(5)=XLEFT
YBORD(5)=YBOTM

C INITIALIZE GKS
CALL GHERC( MKTYPE )
CALL GCPES( ERNFID )
CALL GCPMK( MKID, CONMID, MKTYPE )
CALL GACWK( MKID )
CALL GCLWKR( WKID, CTRL )

C CLEAR CURRENT SCREEN
CALL GSVF( TRANS,VWXL,VWXR,VWYB,VWYT )

C DEFINE VIEWPORT BASED ON NORMALIZED DEVICE COORDINATES.
CALL GSVF( TRANS,VWXL,VWXR,VWYB,VWYT )

C SELECT NORMALIZATION TRANSFORMATION NUMBER AS THE CURRENT ONE.
CALL GSELAT( TRANS )

C DEFINE GLOBAL WINDOW.
CALL GSWN( TRANS,XLEFT,XRIGHT,YBOTM,YTOP )

C SELECT NORMALIZATION TRANSFORMATION NUMBER 1 AS THE CURRENT ONE.
CALL GSELAT( TRANS )

C DRAW A GRAPH BORDER.
CALL GPL( 5, XBORD, YBORD )

C SET MARKER TO TYPE 2. THIS WILL DRAW CIRCLES FOR THE DATA POINTS.
C CIRCLES ARE SCALED TO NEAREST MARKER SIZE.
CALL GSWK( CIRCLE )
CALL GSWKSC( SCALE )

C PLOT THE DATA POINTS WITH CIRCLES.
CALL GPM( N,VEL,Y )

C UNDO GKS INITIALIZATION

```

```

CALL GDAWK( WKID )
CALL GCLWK( WKID )
CALL GCLLS( )
RETURN
END

*****
* THIS SUBROUTINE CREATES SEMI-LOGARITHMIC PLOTS. THE ORIGINATE IS
* LINEAR. THE ABSCESSA IS AN APPROXIMATE LOGARITHMIC SCALE (BASE 10).
*****

SUBROUTINE REPLOT(Y, YCALC, X, N, YTOP)
*****
REAL VMXL, VMXR, VMYT, VMYB, XLEFT, XRIGHT, YBOTH, YTOP, SCALE
REAL XBORD(5), YBORD(5), X(100), Y(100), Y1(100), Y2(100), Y3(100)
REAL Y4(100), X1LOG(100), X2LOG(100), X3LOG(100), X4LOG(100)
REAL YCALC(100), YC1(100), YC2(100), YC3(100), YC4(100), YC5(100)
REAL Y5(100), X5LOG(100),
INTEGER J, K, L, M, P
PARAMETER (ERRFID=0, WKTYPE=3, WKID=14, CONNID=0)
PARAMETER (TRANS=1, CIRCLE=4, SCALE=0.1, CTRL=1)
PARAMETER (VMTB=0.02, VMYT=0.02, YBOTH=0.0)

C INITIALIZE COUNTERS
J=0
K=0
L=0
M=0
P=0

C SORT X-AXIS DATA
DO 10 I=1,N
  IF (X(I).GE.10.0.AND.X(I).LE.100.0) THEN
    J=J+1
    X1LOG(J)=ALOG10(X(I))
    Y1(J)=Y(I)
    YC1(J)=YCALC(I)
  ENDIF
  IF (X(I).GE.100.0.AND.X(I).LE.1000.0) THEN
    K=K+1
    X2LOG(K)=ALOG10(X(I))
    Y2(K)=Y(I)
    YC2(K)=YCALC(I)
  ENDIF
  IF (X(I).GE.1000.0.AND.X(I).LE.10000.0) THEN
    L=L+1
    X3LOG(L)=ALOG10(X(I))
    Y3(L)=Y(I)
    YC3(L)=YCALC(I)
  ENDIF
  IF (X(I).GE.10000.0.AND.X(I).LE.100000.0) THEN
    M=M+1
    X4LOG(M)=ALOG10(X(I))
    Y4(M)=Y(I)
    YC4(M)=YCALC(I)
  ENDIF
  IF (X(I).GE.100000.0.AND.X(I).LE.1000000.0) THEN
    P=P+1
    X5LOG(P)=ALOG10(X(I))
    Y5(P)=Y(I)
  ENDIF

*****
CALL GDAWK( WKID )
CALL GCLWK( WKID )
CALL GCLLS( )
RETURN
END

*****
* PLOT LOG(100) TO LOG(1000) DATA *
*****

YC5(P)=YCALC(I)
ENDIF
10 CONTINUE

C INITIALIZE GKS
CALL GHERC( WKTYPE )
CALL GOKS( ERRFID )
CALL GOMK( WKID, CONNID, WKTYPE )
CALL GACWK( WKID )

C CLEAR CURRENT SCREEN
CALL GCLRWK( WKID, CTRL )

*****
* PLOT LOG(10) TO LOG(100) DATA *
*****
C DEFINE VIEWPORT BASED ON NORMALIZED DEVICE COORDINATES.
VMXL=0.0
VMXR=0.2
CALL GSWP( TRANS, VMXL, VMXR, VMYT, VMYT )

C SELECT NORMALIZATION TRANSFORMATION NUMBER AS THE CURRENT ONE.
CALL GSELNT( TRANS )

C DEFINE GLOBAL WINDOW.
XLEFT=1.0
XRIGHT=2.0
CALL GSWN( TRANS, XLEFT, XRIGHT, YBOTH, YTOP )

C SELECT NORMALIZATION TRANSFORMATION NUMBER AS THE CURRENT ONE.
CALL GSELNT( TRANS )

C DRAW A GRAPH BORDER.
XBORD(1)=XRIGHT
YBORD(1)=YBOTH
XBORD(2)=XLEFT
YBORD(2)=YBOTH
XBORD(3)=XLEFT
YBORD(3)=YTOP
XBORD(4)=XRIGHT
YBORD(4)=YTOP
CALL GPL( 4, XBORD, YBORD )

C SET MARKER TO TYPE 2. THIS WILL DRAW CIRCLES FOR THE DATA POINTS.
C CIRCLES ARE SCALED TO NEAREST MARKER SIZE.
CALL GSKK( CIRCLE )
CALL GSKSC( SCALE )

C THE EXPERIMENTAL DATA WILL BE PLOTTED AS CIRCLES WHILE THE CALCULATED
C DATA WILL BE JOINED BY LINES.
IF (J.GT.0) THEN
  CALL GPH( J, X1LOG, Y1 )
ENDIF
IF (J.GE.2) THEN
  CALL GPL( J, X1LOG, YC1 )
ENDIF

*****
* PLOT LOG(100) TO LOG(1000) DATA *
*****
VMXL=0.2

```

```

VWXR=0.4
CALL GSP( TRANS,VNXL,VWXR,VWYB,VWYT )

CALL GSELNT( TRANS )

XLEFT=2.0
XRIGHT=3.0
CALL GSN( TRANS,XLEFT,XRIGHT,YBOTM,YTOP )
CALL GSELNT( TRANS )

XBORD(1)=XRIGHT
YBORD(1)=YBOTM
XBORD(2)=XLEFT
YBORD(2)=YBOTM
CALL GPL( 2, XBORD, YBORD )
XBORD(1)=XLEFT
YBORD(1)=YTOP
XBORD(2)=XRIGHT
YBORD(2)=YTOP
CALL GPL( 2, XBORD, YBORD )

CALL GSNK( CIRCLE )
CALL GSKSC( SCALE )

IF (K.GT.0) THEN
  CALL GPM( K,X2LOG,Y2 )
ENDIF
IF (K.GE.2) THEN
  CALL GPL( K,X2LOG,YC2 )
ENDIF

*****
* PLOT LOG(1000) TO LOG(10000) DATA *
*****
VWXL=0.4
VWXR=0.6
CALL GSP( TRANS,VNXL,VWXR,VWYB,VWYT )
CALL GSELNT( TRANS )

XLEFT=3.0
XRIGHT=4.0
CALL GSN( TRANS,XLEFT,XRIGHT,YBOTM,YTOP )
CALL GSELNT( TRANS )

XBORD(1)=XRIGHT
YBORD(1)=YBOTM
XBORD(2)=XLEFT
YBORD(2)=YBOTM
CALL GPL( 2, XBORD, YBORD )
XBORD(1)=XLEFT
YBORD(1)=YTOP
XBORD(2)=XRIGHT
YBORD(2)=YTOP
CALL GPL( 2, XBORD, YBORD )

CALL GSNK( CIRCLE )
CALL GSKSC( SCALE )

IF (L.GT.0) THEN
  CALL GPM( L,X3LOG,YC3 )
ENDIF
IF (L.GE.2) THEN
  CALL GPL( L,X3LOG,YC3 )
ENDIF

*****
* PLOT LOG(10000) TO LOG(100000) DATA *
*****
VWXL=0.6
VWXR=1.0
CALL GSP( TRANS,VNXL,VWXR,VWYB,VWYT )
CALL GSELNT( TRANS )

XLEFT=5.0
XRIGHT=6.0
CALL GSN( TRANS,XLEFT,XRIGHT,YBOTM,YTOP )
CALL GSELNT( TRANS )

XBORD(1)=XLEFT
YBORD(1)=YBOTM
XBORD(2)=XRIGHT
YBORD(2)=YBOTM
XBORD(3)=XRIGHT
YBORD(3)=YTOP
XBORD(4)=XLEFT
YBORD(4)=YTOP
CALL GPL( 4, XBORD, YBORD )

CALL GSNK( CIRCLE )
CALL GSKSC( SCALE )

```



```

IF (P.GT.0) THEN
  CALL GPM( P,XSLOG,Y5 )
ENDIF
IF (P.GE.2) THEN
  CALL GPL( P,XSLOG,YCS )
ENDIF

C UNDO GRS INITIALIZATION
CALL GDANK( MRID )
CALL GCLMK( MRID )
CALL GCLKS( )
RETURN
END

.....
SUBROUTINE MODCL(QMAX,UNAX,WMAX,YRW,VIS,K,B1,B2,Y,U,W,N,DFILE)
.....
* THIS SUBROUTINE SOLVES THE MODIFIED CLAUSER PLOT EQUATIONS
* SIMULTANEOUSLY FOR THE SKIN FRICTION PARAMETERS CU AND CV. THE SKIN
* FRICTION COEFFICIENTS, CFX AND CFZ, ARE THEN CALCULATED FROM CU AND
* CV.
REAL QMAX,UNAX,WMAX,VIS,K,B1,B2,Y(100),U(100),W(100),DFC1,DFC2
REAL REQ(100),C0,C1,C2,C3,C4,C5,CFX(100),CFZ(100),FM,GN,FMS,GNS
REAL CU(100),CW(100),CUT,CWT,CT,CTSR,FCU,GCU,DFCU,DFCW,DOC1,DOC2
REAL DDCU,DGCU,DET,CDO,CWO,F(100),G(100),DIF1,DIF2,YTOP,YRW(100)
REAL CFPMAX,CFZMAX
INTEGER I,N,ITER,LVITEM,PFILE,ITERN(100),GO,CONTIN,TERM,USER
CHARACTER*20 DFILE,DFILE

C CALCULATE CONSTANTS AND REYNOLDS NUMBERS.
C0=QMAX/VIS
C1=UNAX/QMAX
C2=WN1/QMAX
C3=1.0/K
DO 10 I=1,N
  REQ(I)=Y(I)*C0
10 CONTINUE

C ITERATE TO DETERMINE CU AND CV. THE USER IS PROVIDED WITH TWO
C OPTIONS:
* 1) INPUT INITIAL GUESSES FOR CU AND CV
* 2) ALLOW THE PROGRAM TO CALCULATE INITIAL GUESSES FOR CU
  AND CV.

C AUTOMATED INITIAL GUESSES FOR THE FIRST DATA POINT ARE CALCULATED BY
C NEGLECTING THE LOGARITHMIC TERM IN THE MODIFIED CLAUSER PLOT
C EQUATIONS. ALL SUCCEEDING DATA POINTS USE THE CONVERGED SOLUTIONS OF
C THE PREVIOUS DATA POINT AS THE INITIAL GUESSES.
USER=0
CALL CLSCSR
WRITE(*,*)'SKIN FRICTION ITERATION PROCEDURE'
WRITE(*,*)'.....'
PRINT*
PRINT*
WRITE(*,*)'1) TYPE 1 TO ENTER STARTING CU AND CV VALUES OR ZERO',
* 'FOR AUTOMATED VALUES'
WRITE(*,*)' (1 OR 0 MUST BE AN INTEGER)'
READ*,USER

IF (USER.EQ.1) THEN
  PRINT*
  PRINT*
  PRINT*
  PRINT*
  WRITE(*,*)'USER SPECIFIED STARTING VALUES FOR CU AND CV'
  WRITE(*,*)'-----'
  PRINT*
  WRITE(*,*)'1) ENTER A CU VALUE:'
  READ*,CUT
  PRINT*
  WRITE(*,*)'2) ENTER A CV VALUE:'
  READ*,CWT
  ELSE
    CUT=U(1)*C1/(YRW(1)*B1)
    CWT=W(1)*C2/B2
  ENDIF
  ST1=0.00001
  ST2=0.0001
  ITER=0
  TERM=0
  DO 50 I=1,N
    ITER=ITER+1
    CT=CUT+CWT*CWT
    CTSR=SQR(CT)
    C CALCULATE THE MODIFIED CLAUSER PLOT FUNCTIONS, F AND G, AND THEIR
    C FIRST DERIVATIVES WITH RESPECT TO CU AND CV.
    FCU=C3*ALOG(REQ(I)*CTSR)+B1-U(I)*C1/CUT
    GCU=W(1)*(C3*ALOG(REQ(I)*CTSR)+B2)-W(I)*C2/CWT
    DFCU=C3/CUT*CT*U(I)*C1/(CUT*CWT)
    DFCW=C3*CWT/CT
    DDCU=YRW(I)*C3*CWT/CT*W(I)*C2/(CWT*CWT)
    DDCW=YRW(I)*C3/CUT/CT
    DOC1=ABS(DFCU)
    DOC2=ABS(DFCW)
    DOC1=ABS(DDCU)
    DOC2=ABS(DDCW)
    IF(DFC1.LE.0.001.OR.DFC2.LE.0.001) TERM=1
    IF(DOC1.LE.0.001.OR.DOC2.LE.0.001) TERM=1
    CONTIN=0
    IF(TERM.EQ.1) THEN
      CALL CLSCSR
      PRINT*
      PRINT*
      PRINT*
      WRITE(*,*)'.... ITERATION PROCEDURE TERMINATED ....'
      FORMAT(21X,'*** ITERATION PROCEDURE TERMINATED ***')
      PRINT*
      WRITE(*,*)'FIRST DERIVATIVES ARE BECOMING SMALL'
      PRINT*
      N=N+1
      IF(N.LT.1) THEN
        PRINT*
        PRINT*
        WRITE(*,*)'*** SOLUTION DIVERGED ON THE FIRST DATA',
        * 'POINT ***'
      ENDIF
    ENDIF
  END DO

```

```

      PRINT*
      WRITE(*,24)
      FORMAT(16X,'REUN PROGRAM WITH NEW FIRST GUESSES FOR CU',
      * AND CW')
      ENDIF
      PRINT*
      PRINT*
      WRITE(6,25)
      FORMAT(31X,'ENTER 1 TO CONTINUE')
      READ*,CONTIN
      IF(CONTIN.EQ.1) GOTO 55
      ENDIF
      PRINT*,I,ITER
      PRINT*,CUT,FCU,DFCU,DFCW
      PRINT*,CWT,GCM,DGCU,DFCW
      *
      *
      C CALCULATE THE DETERMINATE OF THE JACOBIAN.
      DET=DFCU*DGCM-DFCW*DGCU
      *
      C CALCULATE THE NEW VALUES OF CU AND CW.
      CU=CUT
      CW=CWT
      CUT=CUO-(FCU*DGCM-GCW*DFCW)/DET
      CWT=CWO-(FCU*DGCU-GCW*DFCU)/DET
      PRINT*,CUT,CWT,REQ(1)
      READ*,CONTIN
      *
      *
      C THE ITERATION PROCEDURE WILL BE STOPPED IF THE VALUES OF CU AND CW
      C BECOME LARGE (I.E. IF THE SOLUTION BEGINS TO DIVERGE). THIS AVOIDS
      C FLOATING POINT OVERFLOW ERRORS. ONLY DATA VALUES (I.E. CONVERGED
      C SOLUTIONS) PRIOR TO HALTING THE PROGRAM ARE STORED.
      CONTIN=0
      IF(CUT.GE.1.0.OR.CWT.GE.1.0) THEN
        CALL CLRSCR
        PRINT*
        PRINT*
        PRINT*
        WRITE(6,26)
        FORMAT(21X,'..... ITERATION PROCEDURE TERMINATED .....')
        PRINT*
        PRINT*
        WRITE(6,30)
        FORMAT(21X,'      CU AND/OR CW ARE > 1.0')
        PRINT*
        N=1-1
        IF(N.LT.1) THEN
          PRINT*
          PRINT*
          WRITE(*,31)
          FORMAT(16X,'..... SOLUTION DIVERGED ON THE FIRST DATA ',
          * , POINT .....')
          PRINT*
          PRINT*
          WRITE(*,32)
          FORMAT(16X,'REUN PROGRAM WITH NEW FIRST GUESSES FOR CU',
          * AND CW')
          ENDIF
          PRINT*
          PRINT*
          WRITE(6,48)
          FORMAT(31X,'ENTER 1 TO CONTINUE')
          READ*,CONTIN
          IF(CONTIN.EQ.1) GOTO 55
        ENDIF
        IF(DIF1.LE.ST1.AND.DIF2.LE.ST1.AND.FNS.LE.ST2.AND.
        * GNS.LE.ST2) THEN
          CU(I)=CUT
          CW(I)=CWT
          F(I)=FN
          G(I)=GN
          *
          *
          PRINT*,I,CU(I),CW(I)
          PRINT*,F(I),G(I)
          READ(*,*) CONTIN
          *
          *
          ITER=ITER+1
          IF(ITER=0)
            ELSE
              C NEGATIVE VALUES OF CU AND CW ARE CONVERTED TO POSITIVE VALUES DURING

```

```

C THE ITERATION PROCEDURE.
  IF(CUT.LT.0.0) CUT=-1.0*CUT
  IF(CWT.LT.0.0) CWT=-1.0*CWT
  GOTO 20
ENDIF

50 CONTINUE

55 CALL CLSCR

C CALCULATE THE COMPONENT SKIN FRICTION COEFFICIENTS, CFx AND CFz.
C THESE VALUES ARE MULTIPLIED BY 1000 FOR PLOTTING PURPOSES (I.E.
C CFx X 1000 AND CFz X 1000).
  IF(N.GT.0) THEN
    DO 60 I=1,N
      C5=SQRT(CU(I)*CU(I)+CW(I)*CW(I))
      CFX(I)=2000.0*C5
      CFZ(I)=2000.0*C5
    CONTINUE
  C PRINT THE CONVERGED RESULTS ON SCREEN.
  WRITE(6,70)
  70 FORMAT(8X,'N',3X,'ITER',4X,'Cu',7X,'Cw',8X,'f',9X,'g',
    + 6X,'CFx*1000',2X,'CFz*1000')
  75 FORMAT(7X,'---',2X,'-----',2(2X,'-----'),2(2X,'-----'),
    + 2(2X,'-----'))
    PRINT*
    DO 85 I=1,N
      WRITE(6,80) I,ITERN(I),CU(I),CW(I),F(I),G(I),CFX(I),CFZ(I)
      80 FORMAT(7X,12,2X,14,4X,F7.5,2X,F7.5,2X,F8.5,2X,F8.5,2X,F8.5,
        + 2X,F8.5)
    CONTINUE
    PRINT*
  90 WRITE(6,90)
    FORMAT(30X,'ENTER 1 TO CONTINUE')
    READ*,QO

C DETERMINE THE MAXIMUM CFx AND CFz VALUES. THESE WILL BE USED FOR
C SCALING THE PLOTS.
    CFMAX=CFX(1)
    CFZMAX=CFZ(1)
    DO 95 I=1,N
      IF(CFX(I).GT.CFMAX) CFMAX=CFX(I)
      IF(CFZ(I).GT.CFZMAX) CFZMAX=CFZ(I)
    CONTINUE

95 CONTINUE

C USE THE SUBROUTINE RENPLOT TO VIEW THE MODIFIED CLAUSER PLOTS.
  LVIEW=0
  CALL CLSCR
  WRITE(*,*)'11) ENTER 1 TO VIEW THE X-DIRECTION CLAUSER PLOT',
    + 'OR ZERO TO CONTINUE'
  READ(5,*) LVIEW
  IF(LVIEW.EQ.1) THEN
    YTOP=1.3*CFZMAX
    CALL RENPLOT(CFZ,CFZ,REQ,N,YTOP)
  ELSE
    CALL CLSCR
    ENDIF
  LVIEW=0

  WRITE(*,*)'111) ENTER 1 TO VIEW THE Z-DIRECTION CLAUSER PLOT',
    + 'OR ZERO TO CONTINUE'
  WRITE(*,*) LVIEW
  IF(LVIEW.EQ.1) THEN
    YTOP=1.3*CFZMAX
    CALL RENPLOT(CFZ,CFZ,REQ,N,YTOP)
  ELSE
    CALL CLSCR
    ENDIF
  ENDIF
  RETURN
  END

C SAVE X-DIR,N AND Z-DIR,N CLAUSER PLOT DATA IN A FILE. THIS FILE
C CAN THEN BE IMPORTED INTO A GRAPHING PROGRAM.
  PFIL=0
  WRITE(*,*)'1V) ENTER 1 TO CREATE A DATA FILE CONTAINING THE ',
    + 'COMPONENT CLAUSER PLOT'
  WRITE(*,*)' DATA OR ZERO TO CONTINUE (1 OR 0 MUST BE AN',
    + 'INTEGER VALUE): '
  READ(5,*) PFIL
  IF(PFIL.EQ.1) THEN
    CALL CLSCR
    PRINT*,'COMPONENT MODIFIED CLAUSER PLOT DATA FILE'
    PRINT*,'*****'
    PRINT*
    WRITE(6,100) DFILE
    PRINT*
    FORMAT('INPUT DATA FILE: ',A)
    WRITE(*,*)'V) ENTER DATA FILE NAME (INCLUDING THE *.DAT',
      + 'EXTENSION): '
    READ(*,*) (A) DOFILE
    OPEN(UNIT=9,FILE=DOFILE,STATUS='NEW')
    DO 120 I=1,N
      WRITE(9,110) REQ(I),CFX(I),CFZ(I)
      110 FORMAT(F10.1,1X,F8.5,1X,F8.5)
    CONTINUE
    CALL CLSCR
    ELSE
      CALL CLSCR
    ENDIF
  ENDIF
  RETURN
  END

```

APPENDIX E

DERIVATION OF THE SECOND INVARIANT, I_2

This appendix gives a detailed derivation of the reduced second invariant, I_2 , of the rate of strain tensor for the mean flow of a three-dimensional boundary layer. In Chapter 3, this reduced form is used to define the component eddy viscosities for Cartesian and cylindrical coordinate systems.

The local rate of strain tensor for Cartesian coordinates is given as (Malvern, 1969)

$$e_{ij} = \frac{1}{2} \left[\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right] = \frac{1}{2} \begin{bmatrix} 2 \frac{\partial U}{\partial x} & \left\{ \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \right\} & \left\{ \frac{\partial U}{\partial z} + \frac{\partial W}{\partial x} \right\} \\ \left\{ \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \right\} & 2 \frac{\partial V}{\partial y} & \left\{ \frac{\partial V}{\partial z} + \frac{\partial W}{\partial y} \right\} \\ \left\{ \frac{\partial W}{\partial x} + \frac{\partial U}{\partial z} \right\} & \left\{ \frac{\partial W}{\partial y} + \frac{\partial V}{\partial z} \right\} & 2 \frac{\partial W}{\partial z} \end{bmatrix}. \quad (\text{E.1})$$

Consider this tensor to illustrate the derivation of the term $e_{ij}:e_{ij}$ for a three-dimensional boundary layer. The second invariant, I_2 , of the rate of strain tensor is (Malvern 1969)

$$I_2 = \frac{1}{2} \left(e_{ij} e_{ij} - \{ tr(e_{ij}) \}^2 \right), \quad (\text{E.2})$$

where $tr(e_{ij})$ is the trace of the tensor e_{ij} . For incompressible fluids the trace of the rate of strain tensor results in the left hand side of the continuity equation, Equation (3.15):

$$tr(e_{ij}) = 2 \left\{ \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} \right\}. \quad (\text{E.3})$$

Thus, the trace is zero and the second invariant reduces to

$$I_2 = e_{ij}:e_{ij} = \frac{1}{2} (e_{ij} e_{ij}). \quad (\text{E.4})$$

Expanding Equation (E.4) one obtains

$$I_2 = \frac{1}{2} (e_{11}e_{11} + e_{22}e_{22} + e_{33}e_{33}) + e_{12}^2 + e_{13}^2 + e_{23}^2. \quad (\text{E.5})$$

Applying the three-dimensional boundary layer assumptions,

$$V \ll U, W$$

and

$$\frac{\partial}{\partial x}, \frac{\partial}{\partial z} \ll \frac{\partial}{\partial y},$$

to the components of the rate of strain tensor, e_{ij} , Equation (E.5) reduces to

$$I_2 = \frac{1}{4} \left[\left\{ \frac{\partial U}{\partial y} \right\}^2 + \left\{ \frac{\partial W}{\partial y} \right\}^2 \right]. \quad (\text{E.6})$$

As a result, for Cartesian coordinates the term $\sqrt{4e_{ij}:e_{ij}}$ becomes

$$\sqrt{4e_{ij}:e_{ij}} = \left[\left\{ \frac{\partial U}{\partial y} \right\}^2 + \left\{ \frac{\partial W}{\partial y} \right\}^2 \right]^{1/2}. \quad (\text{E.7})$$

Similar results are obtained for an axisymmetric flow. The rate of strain tensor in cylindrical coordinates is (White, 1991)

$$e_{ij} = \frac{1}{2} \begin{bmatrix} \frac{\partial V}{\partial r} & \left\{ \frac{1}{r} \frac{\partial V}{\partial \theta} + r \frac{\partial}{\partial r} \left(\frac{W}{r} \right) \right\} & \left\{ \frac{\partial U}{\partial r} + \frac{\partial V}{\partial x} \right\} \\ \left\{ \frac{1}{r} \frac{\partial V}{\partial \theta} + r \frac{\partial}{\partial r} \left(\frac{W}{r} \right) \right\} & \left\{ \frac{1}{r} \frac{\partial W}{\partial \theta} + \frac{V}{r} \right\} & \left\{ \frac{1}{r} \frac{\partial U}{\partial \theta} + \frac{\partial W}{\partial x} \right\} \\ \left\{ \frac{\partial U}{\partial r} + \frac{\partial V}{\partial x} \right\} & \left\{ \frac{1}{r} \frac{\partial U}{\partial \theta} + \frac{\partial W}{\partial x} \right\} & \frac{\partial U}{\partial x} \end{bmatrix}. \quad (\text{E.8})$$

The boundary layer and axisymmetric conditions for cylindrical flows are

$$V \ll U, W,$$

$$\frac{\partial}{\partial x} \ll \frac{\partial}{\partial r} .$$

and

$$\frac{\partial}{\partial \theta} = 0 .$$

The term $\sqrt{4e_{ij}:e_{ij}}$ for an axisymmetric flow in cylindrical coordinates then is

$$\sqrt{4e_{ij}:e_{ij}} = \left[\left\{ \frac{\partial U}{\partial r} \right\}^2 + \left\{ r \frac{\partial}{\partial r} \left(\frac{w}{r} \right) \right\}^2 \right]^{1/2} . \quad (\text{E.9})$$

END

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FIN