Software for combinatorial power series

by

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Abstract

Generating functions (i.e. power series) have applications throughout enumerative and analytic combinatorics. In this thesis we present Genfunlib, a new Mathematica package containing a selection of implementations of symbolic methods related to generating functions. With Genfunlib one can find the generating functions for regular languages, compute the initial terms of a generating function, convert between generating function equations and recurrences, and find asymptotics. This thesis gives mathematical background, extensive documentation for Genfunlib, and tutorials.

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1 Introduction

According to the NSF, a scientist's worth is not dependent solely on publications. ... In the next five years, I believe that it will become routine to track—and to value—citations to an online lab notebook, contributions to a software library, bookmarks to data sets from content-sharing sites such as Pinterest and Delicious.

– Heather Piwowar, 2013 [27]

This thesis is on Genfunlib, a package for the Mathematica computer algebra system, which was written by the author of this document. The goal of Genfunlib is to provide commands for obtaining and using generating functions to solve combinatorial problems. The full source code of Genfunlib can be found in the Appendix; it is also available on the web at this address:

<https://github.com/amacfie/Genfunlib>

Genfunlib consists of implementations of a number of well-known procedures not currently available in Mathematica or any third-party Mathematica package. Genfunlib includes comprehensive finite automata functionality that allows the user to convert between automata and regular expressions, and perform standard operations such as union, intersection, and reversal. Generating functions that count the number of words in a given regular language can be computed. These generating functions can be useful for counting problems that involve regular languages such as pattern avoidance in words. Genfunlib uses known methods to produce asymptotic information about a counting sequence given its generating function. From a power series equation, Genfunlib can compute the initial terms of the solution series. Genfunlib improves on Mathematica's built-in capabilities for converting between recurrence relations and power series equations. The symbolic method for obtaining generating functions

from constructible combinatorial classes is also implemented in Genfunlib.

The contents of this thesis are as follows. In Section 2, we give an overview of enumerative combinatorics, generating functions, and their applications to analysis of algorithms. Thanks to this section, the mathematical content of this thesis is fairly self-contained. Section 3 is a brief overview of Mathematica programming to allow readers to better understand the code in this thesis. Section 4 goes through each of Genfunlib’s commands and explains how to use them, what they do and, where educative, how they do it. It is divided into subsections that cover all the commands related to one mathematical topic. This section can be considered to be both user and developer documentation. Boxes labeled “Mathematica note” can be ignored by readers unfamiliar with Mathematica. In Section 5 we present a number of relatively elementary combinatorial problems, and solutions that make use of Genfunlib and Mathematica. In the solutions, we attempt to give a sense of the range of Genfunlib’s functionality and try to minimize the number of operations performed “by hand”. The problems in Section 5.1 come from high school mathematics contests, or contest preparation material. Finally, in Section 6 we conclude by offering possible paths for further work.

Naturally, Genfunlib does not implement all known algorithms that one may want to use with generating functions. Areas for expansion and improvement are discussed as they are relevant at various points in the body of the thesis. Further, there are tests and operations on power series for which no algorithm has been found, e.g. whether a given univariate rational function has only positive coefficients [16].

2 Mathematics for the analysis of algorithms

In theoretical computer science, the greatest innovation is the realisation that algorithms are mathematical objects, and can be rigorously analysed in terms of their consumption of scarce resources, including space, time, and randomness.

– Jeffrey Shallit, 2006 [33]

Given a computer program, one may desire to know how long it will take to run in absolute terms, that is, in physical units of time. Indeed ultimately, such concrete predictions are always the end goal, and engineering methods involving statistics, simulation, and detailed knowledge of particular computing environments are certainly used (when rigorous this is known as the *experimental analysis of algorithms*). However, given the complexity of actual computers, a more abstract problem is taken to be the typical one in the field of computer science called *analysis of algorithms*.

First, instead of executable computer programs, the objects of study are *algorithms* [13], generally given at a level of abstraction where pseudocode is sufficient. The physical running time of an algorithm is not defined, but the number of steps it takes is. It is assumed that the number of steps taken by an algorithm gives information related to actual running time of computer implementations. From now on, the “running time” of an algorithm means the number of steps it takes.

Second, because algorithms are abstract objects and do not run on computers, the goal is not to predict absolute timings, but to compare algorithms with each other. Let $T(A, i)$ be the running time of an algorithm A on an input i . Let $I(n)$ be the set of inputs with size n . We can represent the running time of

A with respect to a probability mass function $f_n : I(n) \rightarrow [0, 1]$ by the sequence

$$(T_A(n)), \text{ where } T_A(n) = \sum_{i \in I(n)} T(A, i) f_n(i).$$

That is, $T_A(n)$ is the expected running time of A according to the probability distribution defined by f_n . The most common distribution is “worst-case”, which assigns probability 1 to an input in $I(n)$ on which the running time of A is maximal. Another common distribution used is “average-case”, where

$$f_n(i) = 1/|I(n)|,$$

for all i . More nuanced analysis has been done beyond these two distributions, notably smoothed analysis and analysis using the Solomonoff–Levin measure [19].

Once the distribution has been chosen, for two algorithms A and B computing the same function, we have the two sequences $(T_A(n))$ and $(T_B(n))$. Obviously, if $T_A(n) \geq T_B(n)$ for all n , it is easy to compare A and B , but this ordering is not always enough. Instead, analysis of algorithms uses *asymptotics*: relations are established between $T_A(n)$ and $T_B(n)$ as n approaches infinity. Now, instead of directly comparing $T_A(n)$ to $T_X(n)$ for each X , we can, analogously to the “interlingua” approach to machine translation [15], find a relation between $T_A(n)$ and a simple standard sequence, after which, to compare algorithms we need only compare simple standard sequences, which is easy.

Definition 1 (Asymptotic notation). *As $n \rightarrow \infty$,*

- $f(n) = O(g(n))$ means $\exists c > 0, N > 0 : \forall n, n > N \implies f(n) < cg(n)$,
- $f(n) = \Omega(g(n))$ means $g(n) = O(f(n))$,
- $f(n) = \Theta(g(n))$ means $f(n) = O(g(n))$ and $g(n) = O(f(n))$,

- $f(n) = o(g(n))$ means $\lim_{n \rightarrow \infty} \frac{g(n)}{f(n)} = \infty$,
- $f(n) \sim g(n)$ means $\lim_{n \rightarrow \infty} \frac{g(n)}{f(n)} = 1$, and
- $f(n) = \omega(g(n))$ means $\lim_{n \rightarrow \infty} \frac{g(n)}{f(n)} = 0$.

Example 1. Suppose $T_A(n) = \sum_{k=1}^n \frac{n}{k}$, $T_B(n) = 2^n$, and $T_C(n) = \frac{1}{n+1} \binom{2n}{n}$. Then we have, as $n \rightarrow \infty$,

$$T_A(n) = \Theta(n \log(n)),$$

$$T_B(n) = \Theta(2^n),$$

$$T_C(n) = \Theta\left(\frac{4^n}{n^{3/2}}\right).$$

By comparing $n \log(n)$, 2^n , and $\frac{4^n}{n^{3/2}}$, we see $T_A(n) = o(T_B(n))$ and $T_B(n) = o(T_C(n))$, as $n \rightarrow \infty$.

Asymptotics in this setting are perhaps surprisingly useful. The obvious objection to their use is that, for example, even though $(1 + \exp(-10^{10}))^n$ is in every asymptotic sense greater than n^{100} , the former is in every practical sense less than the latter. It is simply an observed fact that asymptotic statements are usually practically relevant, even though so much information is lost.

Thus, we see that the prototypical methodology in analysis of algorithms is the (incomputable in general) passage from algorithms to running time sequences to asymptotic expressions. (As always in mathematics, more abstract problems may be solved in analysis of algorithms which support this methodology indirectly.) This progression can be broken down into its two stages.

Obtaining exact running time sequences is closely related to the field of mathematics called enumerative combinatorics, a.k.a. counting. For example, let S_n be the highest-cardinality set of states that an algorithm A enters on an input of size n . Then $|S_n|$ is the worst-case running time of A on inputs

of size n . Counting sets such as S_n is precisely the field of enumerative combinatorics. According to Wilf [39], when an algorithm is found that takes n and returns $|S_n|$ and runs in at most exponential time, the combinatorial counting problem associated with the sets S_n has been solved. To obtain these cardinality algorithms, a set of counting strategies/techniques is used, including the principle of inclusion–exclusion, recurrences, refinements, decompositions, and Pólya theory.

Instead of passing directly from a set to its cardinality, in enumerative combinatorics, establishing a bijection between two sets is a very common goal. If something has been proved about the cardinality of a set S_1 , which is shown to be in bijection with S_2 , the same result applies to S_2 .

Problems in enumerative combinatorics may come from places other than analysis of algorithms, such as combinatorial number theory, algebra, and graph theory.

Given a sequence (f_n) , we define the *ordinary generating function* of (f_n) to be

$$F(z) = \sum_{n \geq 0} f_n z^n,$$

and we define the *exponential generating function* of (f_n) to be

$$\hat{F}(z) = \sum_{n \geq 0} f_n \frac{z^n}{n!}.$$

It turns out that these power series representations can be very useful for solving combinatorial problems, and especially useful for obtaining an asymptotic estimate of a sequence.

We define two common classes of power series. A power series $f(z) \in \mathbb{Q}[[z]]$

is *algebraic* iff there exist $p_0(z), \dots, p_j(z) \in \mathbb{Q}[z]$ such that

$$\sum_{k=0}^j p_k(z) f(z)^k = 0.$$

A power series $f(z) \in \mathbb{Q}[[z]]$ is *holonomic* iff there exist $p_0(z), \dots, p_j(z) \in \mathbb{Q}[z]$ such that

$$\sum_{k=0}^j p_k(z) f(z)^{(k)} = 0,$$

where $f(z)^{(k)}$ is the k th formal derivative of $f(z)$. Algebraic power series are holonomic [36].

3 A few notes on Mathematica

In order to fully understand all of the Mathematica code used in this thesis, the reader would have to be quite familiar with the Mathematica programming language. Ruskeepää's book [32] is ample preparation, for example. However, readers who are not familiar with Mathematica should still be able to follow the important ideas, and this section provides some essential Mathematica information to help in this regard.

The Mathematica code in this thesis is shown as it appears in an interactive session: There are input lines, which are typed in by the user after the prompt `In[k] :=`, and output lines, which Mathematica returns. Input lines that end in a semicolon do not produce an output line. Here is a basic example:

```
In[1] := 2 + 2
```

```
Out[1] = 4
```

```
In[2] := 3 + 3
```

```
Out[2]= 6
```

Every input line is numbered and its output line has the same number.

The character % stands for the expression in the previous output line.

```
In[3]:= 4 - 2
```

```
Out[3]= 2
```

```
In[4]:= % - 2
```

```
Out[4]= 0
```

In Mathematica, functions take their arguments in square brackets.

```
In[5]:= Sin[Pi]
```

```
Out[5]= 0
```

To write an anonymous function, write the value in terms of the argument #, and place an & at the end. In the next snippet, we define a function `f` that squares its argument.

```
In[6]:= f = #^2 &;
```

```
In[7]:= f[4]
```

```
Out[7]= 16
```

Some of Mathematica's built-in functions use assumptions to control their behavior. They work like this:

```
In[8]:= FullSimplify[(-1)^(2a)]
```

```
Out[8]= (-1)^(2a)
```

```
In[9]:= FullSimplify[(-1)^(2a), Assumptions ->
      Element[a, Integers]]
```

```
Out[9]= 1
```

In In[9] we pass an `Assumptions` option to the algebraic simplifier function `FullSimplify` that says `a` can be assumed to be an integer. Alternatively, assumptions can be specified by setting the global `$Assumptions` variable:

```
In[10]:= $Assumptions = Element[a, Integers];
```

```
In[11]:= FullSimplify[(-1)^(2a)]
```

```
Out[11]:= 1
```

Lists in Mathematica are made with braces and commas:

```
In[12]:= list = {5, 4, 3, 2, 1, 0};
```

```
In[13]:= Length[list]
```

```
Out[13]= 6
```

Mathematica lists can contain anything:

```
In[14]:= l = {5, "this is a string", {"string in a nested list!",
      True, symbol}};
```

It is perfectly fine for a function to not evaluate:

```
In[15] := Sin[2]
```

```
Out[15]= Sin[2]
```

If we want the numerical value of $\sin(2)$, for example, we pass it to the `N` function:

```
In[16] := N[Sin[2]]
```

```
Out[16]= 0.909297
```

Mathematica has a huge number of built-in functions. To learn about a built-in function, simply search the online documentation [29].

4 Guide to Genfunlib

4.1 Power series and their coefficients

Converting between power series and their sequences of coefficients is a basic operation in computer algebra. In general, power series are represented by equations that they satisfy, which may be differential equations. This includes equations implied by “closed-form” expressions. Analogously, sequences may be given by recurrence relations or closed-form expressions.

It is tempting to say that when performing the conversion between power series and coefficient sequences, closed-form power series equations should become closed-form expressions for coefficients, and nontrivial power series equations should become recurrence relations. However, while it generally can, why *should* this aspect of the structure of an equation be preserved? Why not transform

$$F(z) = \frac{1}{1 - 2z}$$

to

$$f_n = 2f_{n-1}, \quad f_0 = 1 \quad \text{instead of} \quad f_n = 2^n?$$

Indeed, if one wants a recurrence relation for computing initial terms, for computational efficiency reasons it is preferable to have a recurrence relation that includes a finite set of lower-index terms, as opposed to one that uses the full-history of the sequence. Thus, unless the form that the result should be in is explicitly stated, there are many ways the translation can be performed [23].

Mathematica 9 includes some functionality for converting between power series and their coefficient sequences; Genfunlib extends this functionality by increasing the set of power series equations/recurrence relations on which the transformation can be applied.

Mathematica note: In computer algebra, representing a power series by an equation it satisfies is useful because of the level of generality it affords. Mathematica has support for two kinds of power series equation objects that can be used with `Series` and `SeriesCoefficient`.

Solutions to linear differential equations are represented as `DifferentialRoot` objects:

```
In[17]:= FullSimplify[
  SeriesCoefficient[
    DifferentialRoot[Function[{y, x}, {y'[x] == y[x], y[0] == 1}
    ]][x], {x, 0, n}],
  Assumptions :> Element[n, Integers] && n >= 0]

Out[17]= 1/n!
```

Solutions to algebraic equations are represented as `Root` objects:

```
In[18]:= Series[Root[#^5+a #+1&,1],{a,0,5}]
```

```
Out[18]= -1+a/5+a^2/25+a^3/125-(21 a^5)/15625+0[a]^6
```

Genfunlib overrides the `SeriesCoefficient` command for extracting the coefficients of a power series.

Using Mathematica's built-in `SeriesCoefficient` we can compute the coefficients of explicit power series like $\exp(z)$.

```
In[19]:= $Assumptions=(n>=0);
```

```
In[20]:= SeriesCoefficient[Exp[z],{z,0,n}]
```

```
Out[20]= 1/n!
```

Genfunlib adds the ability to work with *symbolic* power series, as in the following example.

```
In[21]:= $FullAnalytic=True;
```

```
In[22]:= SeriesCoefficient[f[z]*g[z],{z,0,n}]
```

```
Out[22]:= Sum[SeriesCoefficient[f[z], {z, 0, $5}, Assumptions ->
  n >= 0 && Element[$5, Integers] && $5 >= 0]*
  SeriesCoefficient[g[z], {z, 0, n - $5},
  Assumptions -> n >= 0 && Element[$5, Integers] && $5 >= 0],
  {$5, 0, n}]
```

```
In[23]:= Replace[%,Verbatim[Rule][Assumptions,_]:>
```



```
Sequence[], Infinity]
```

```
Out[23]= Sum[SeriesCoefficient[f[z], {z, 0, $5}]*
SeriesCoefficient[g[z], {z, 0, n - $5}], {$5, 0, n}]
```

In this snippet, we expand $[z^n](f(z)g(z))$. We first set the global option `$FullAnalytic=True`. The `$FullAnalytic` option, introduced in Genfunlib, determines whether or not expressions such as `f[z]` should be assumed to be power series. In general, they could be Puiseux series

$$\sum_{n \geq n_0} c_n z^{n/k}, \quad k \in \mathbb{Z}_{>0}, n_0 \in \mathbb{Z},$$

with negative and/or fractional exponents such as $z^{-1/2}$. Assuming that we are working with power series (or analytic functions) allows us to unconditionally use the Cauchy product formula, for example.

Mathematica note: The Mathematica command `Series` has the option `Analytic` which is used for the same purpose as `$FullAnalytic`.

Next, we extract coefficients from `f[z]*g[z]` which returns a large expression containing `Assumptions` expressions. Using `Replace` to remove the assumptions, we get a result corresponding to $\sum_{j=0}^n ([z^j]f(z))([z^{n-j}]g(z))$.

As well as products of symbolic power series, Genfunlib also adds support for sums, shifts, derivatives, integrals, and powers. The general form of the expansion of $[z^n]f(g(z))$ is an expression with multiple iterated sums that cannot be nicely represented in Mathematica notation, but special cases of power series composition such as $f(kz)$, for constant k , are handled by Genfunlib. Because Genfunlib enhances `SeriesCoefficient` with the ability to extract the coefficients of symbolic power series, we could use `SeriesCoefficient` on both sides

of a power series equation and obtain a recurrence relation.

Mathematica's `GeneratingFunction` command is used for converting in the other direction: from coefficients to power series. In this case, the built-in version of `GeneratingFunction` comes with the ability to handle symbolic input.

```
In[24]:= GeneratingFunction[Sum[f[j]*g[n-j],{j,0,n}],n,z]
```

```
Out[24]= GeneratingFunction[f[n],n,z] GeneratingFunction[g[n],n,z]
```

In this example, Mathematica's `GeneratingFunction` recognizes that the coefficient `Sum[f[j]*g[n-j],{j,0,n}]` is in the form of a Cauchy product, and thus is the coefficient of the product of the generating functions of `f[n]` and `g[n]`.

Genfunlib adds functionality to `GeneratingFunction` in a few areas. First, it handles coefficients of the form

$$\frac{p(n)}{q(n)}f(n),$$

where $p(n)$ and $q(n)$ are polynomials in the index symbol n , and $f(n)$ is symbolic. The generating function $\sum_{n \geq 0} \frac{p(n)}{q(n)} f(n) z^n$ can be expressed in terms of integrals and derivatives of $\sum_{n \geq 0} f(n) z^n$. Problems arise if $q(n)$ has positive integer roots, so we ignore that case. (Specifically, some coefficients would be undefined so it is not clear what the generating function should be.)

Mathematica note: Unfortunately, Mathematica 9 prevents the common case of $q(n) = 1$ from working as expected. The `GeneratingFunctions` package of Mallinger [21] can be used for this, however.

```
In[25]:= GeneratingFunction[f[n]/(n+1),n,z]
```

```
Out[25]= Integrate[GeneratingFunction[f[n], n, $3], {$3, 0, z}]/z
```

This example shows that

$$\sum_{n \geq 0} \frac{1}{n+1} f(n) z^n$$

is automatically computed by Genfunlib to be

$$z^{-1} \int_0^z \left(\sum_{n \geq 0} f(n) s^n \right) ds.$$

Genfunlib also implements the “multi-section” formula for summing only on indices with a given residue modulo an integer.

```
In[26]:= GeneratingFunction[Boole[Divisible[n,2]]f[n],n,z]
```

```
Out[26]= 1/2 (GeneratingFunction[f[n],n,-z]+
GeneratingFunction[f[n],n,z])
```

From the point of view of Genfunlib’s source code, adding functionality to `GeneratingFunction` and `SeriesCoefficient` is straightforward using pattern matching and replacement rules. For example, the snippet

```
SeriesCoefficient[expr1_ + expr2_, {x_, 0, n_},
  opts:OptionsPattern[]] := SeriesCoefficient[expr1, {x, 0, n},
  opts] + SeriesCoefficient[expr2, {x, 0, n}, opts];
```

from Genfunlib (adapted slightly for simplicity) says that expressions that look like

```
SeriesCoefficient[expr1 + expr2, {x, 0, n}]
```

are to be replaced by

```
SeriesCoefficient[expr1, {x, 0, n}] + SeriesCoefficient[
    expr2, {x, 0, n}]
```

4.2 Numerical coefficients

Given a power series equation, it is a problem of computer algebra to evaluate the coefficients of the solutions. Obviously, computation speed is relevant, and for various sequence classes, there is active research around the problem of efficiently computing sequences in the class. [2, 3, 5, 17] One must note that there is a difference between computing *all* of the first n terms of a sequence, and computing just the n th. Computing 2^n , for some integer n , is quickly done by using the method of repeated squaring. However, to compute $2^0, 2^1, \dots, 2^n$, it is faster to simply perform one multiplication for each term. Genfunlib focuses on the problem of computing all initial terms.

Mathematica note:

One special type of power series equation is the inverse equation, which has the form

$$z = \phi(f(z)),$$

where $f(z)$ is the unknown power series. Various algorithms can be used to compute the expansion of $f(z)$, including the Lagrange inversion formula and a more recent method due to Dominici [7]. Mathematica includes the function `InverseSeries` which can be used to solve this problem. In the following example the equation is $z = \sin(f(z))$.

```
In[27]:= Series[Sin[f], {f, 0, 10}]
```

```
Out[27]= f - f^3/6 + f^5/120 - f^7/5040 + f^9/362880 + O[f]^11
```

```
In[28]:= InverseSeries[%, z]
```

```
Out[28]= z + z^3/6 + (3*z^5)/40 + (5*z^7)/112 + (35*z^9)/1152 +  
0[z]^11
```

A very general method for computing coefficients of power series uses the Taylor coefficient formula

$$[z^n]f(z) = \frac{f^{(n)}(0)}{n!}.$$

If an equation $F(f(z), z) = 0$ is given in a power series $f(z)$, the value $[z^0]f(z) = f(0)$ can be computed by setting $z = 0$ and solving $F(f(0), 0) = 0$ for $f(0)$. Then, for $n > 0$, assuming we know $[z^0]f(z), [z^1]f(z), \dots, [z^{n-1}]f(z)$, we compute and solve

$$\left. \frac{d^n}{dz^n} F(f(z), z) \right|_{z=0} = 0$$

to find $[z^n]f(z)$. This can be generalized to differential equations, integral equations, etc.

Genfunlib implements this algorithm as the `CoefsByDerivs` command.

```
In[29]:= ?CoefsByDerivs
```

```
CoefsByDerivs[{eqn1, eqn2,...}, {f,g, ...}, {x, 0, nx},  
  {y, 0, ny}, ...] gives the Taylor expansions at (0,0,...)  
of the power series f,g,... satisfying the equations.
```

```
In[30]:= CoefsByDerivs[{z*f[z]^2 + 1 - f[z] == 0}, {f[z]},  
  {z, 0, 10}]
```

```
Out[30]= {f[z] == 1 + z + 2*z^2 + 5*z^3 + 14*z^4 + 42*z^5 +
132*z^6 + 429*z^7 + 1430*z^8 + 4862*z^9 + 16796*z^10
+ 0[z]^11}
```

In this example, we pass a system of equations consisting of just one equation,

$$zf(z)^2 + 1 - f(z) = 0,$$

then a list of power series which we want to expand, and a triple $\{z, 0, 10\}$ indicating that the power series is in the indeterminate z , the series is to be expanded around 0, and should have order 10. Genfunlib only supports expansions around 0, but multivariate problems are possible.

Mathematica note: Genfunlib calls `Series` to take the derivatives and `Solve` to solve the equations.

The next example shows the expansion of $f(u, v)$ and $g(v)$ given by

$$f(u, v) = ug(v)$$

$$g(2v) = \frac{1}{2} \exp(v),$$

up to $[u^4v^3]f(u, v)$ and $[v^3]g(v)$.

```
In[31]:= CoefsByDerivs[{f[u, v] == u*g[v],
g[2 v] == 1/2*Exp[v]}, {f[u, v], g[v]}, {u, 0, 4}, {v, 0, 3}]
```

```
Out[31]= {f[u, v] == 0[v]^4 + (1/2 + v/4 + v^2/16 + v^3/96)*u +
0[v]^4*u^2 + 0[v]^4*u^3 + 0[v]^4*u^4 + 0[u]^5,
g[v] == 1/2 + v/4 + v^2/16 + v^3/96}
```

Newton iteration is a classical algorithm for numerically finding a zero of a continuously differentiable real function f . Starting at an arbitrary real number x_0 , Newton iteration involves computing the sequence defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Using a Taylor–Lagrange expansion of f around x^* , we get

$$0 = f(x^*) = f(x_n) + f'(x_n)(x^* - x_n) + \frac{1}{2}f''(\xi_n)(x^* - x_n)^2,$$

where ξ_n is between x_n and x^* . This equation implies

$$x^* - x_{n+1} = \frac{-f''(\xi_n)}{2f'(x_n)}(x^* - x_n)^2. \quad (1)$$

Thus, under some conditions on f' and f'' , we have that the sequence x_n in Newton iteration converges to x^* quadratically.

If f is not a real function but a mapping $\mathbb{Q}[[x]] \rightarrow \mathbb{Q}[[x]]$ defined by $y(x) \mapsto F(x, y(x))$, where $F(x, y) \in \mathbb{Q}[[x, y]]$, Newton iteration can be applied to compute initial terms of the solution $y(x)^*$ of the power series equation

$$F(x, y^*(x)) = 0,$$

provided

$$\left. \frac{\partial}{\partial y^*(x)} F(x, y^*(x)) \right|_{x=y^*(x)=0} \neq 0.$$

If we set $y_0(x) = [x^0]y^*(x)$, the key observation to make is that instead of Equation (1) referring to the distance between real numbers, we have

$$y(x)^* - y_{n+1}(x) = c_n(x) (y(x)^* - y_n(x))^2,$$

which implies that if $y(x)^* - y_n(x) \equiv 0 \pmod{x^{2^n}}$, then $y(x)^* - y_{n+1}(x) \equiv 0 \pmod{x^{2^{n+1}}}$. In other words, the n th iteration step generates 2^n correct initial coefficients [38].

Genfunlib implements the Newton iteration method for computing expansions of solutions to power series equations with the `CoefsByNewton` command.

```
In[32] := ?CoefsByNewton
CoefsByNewton[eqn, f, {x, 0, nx}] gives the Maclaurin
expansion in x of f satisfying eqn.
```

The code for `CoefsByNewton` comes partly from the implementation by Koepf [18] (which contains a slight bug on line 4 – compare with Genfunlib’s source code for details). The expression

```
CoefsByNewton[lhs==rhs, f[x,y,...], {x,0,n}]
```

where `lhs == rhs` is an equation in `f[x,y,...]`, evaluates to the series expansion of `f[x,y,...]` around 0 in `x` of order `n`. The following example shows the expansion of $f(z)$ given $zf(z)^2 + 1 - f(z) = 0$.

```
In[33] := CoefsByNewton[z*f[z]^2 + 1 - f[z] == 0, f[z], {z, 0, 5}]
```

```
Out[33]= 1 + z + 2*z^2 + 5*z^3 + 14*z^4 + 42*z^5 + 0[z]^6
```

Genfunlib’s `CoefsByNewton` cannot handle systems of equations, however if a system of equations can be broken up into independent equations by other methods, such as algebraic elimination [6], `CoefsByNewton` can be used.

4.3 Regular languages

A right regular grammar is a formal grammar (N, Σ, P, S) , where N is a set of non-terminal symbols, Σ is an alphabet, P is a set of production rules, and

$S \in N$ is the start symbol, such that all the production rules in P are of one of the following forms:

- $B \rightarrow a$, where B is a non-terminal symbol in N and a is a terminal symbol in Σ ,
- $B \rightarrow aC$, where B and C are in N and a is in Σ ,
- $B \rightarrow \epsilon$, where B is in N and ϵ is the empty string.

A word w is *generated* by a right regular grammar if and only if, starting from S we can obtain w by applying a finite sequence of the production rules. A regular language is a set of words generated by a right regular grammar, or satisfying any of a number of alternative specifications. These include deterministic finite automata (DFA), nondeterministic finite automata (NFA), and regular expressions [35]. Any one of these specifications can be called a *regular language representation* (RLR).

Let L be a regular language, let $|w|$ be the length of a word w in L , and let

$$c_L(n) = |\{w \in L : |w| = n\}|.$$

In enumerative combinatorics, a key property of regular languages is that the ordinary generating function of $c_L(n)$, that is $\sum_{n \geq 0} c_L(n)z^n$, is a rational power series. This fact is hinted at by the linear structure of right regular grammars.

Indeed, this rational power series is computable in closed form, given an instance of an RLR. Let r be a regular expression, and let $L(r)$ be the language it successfully parses. We say that r is *ambiguous* iff there exists a word $w \in L(r)$ which can be parsed by r in more than one way. Examples of ambiguous regular expressions include

$$\epsilon^*, a^{**}, b|b, a^*|b^*,$$

while examples of unambiguous regular expressions include

$$a^*, a|aa|aaa, a|ab|\epsilon.$$

It can be seen that if r is an unambiguous regular expression, applying the following transformations to r yields its generating function [11]:

- concatenation becomes multiplication,
- union becomes addition, and
- Kleene star becomes the quasi-inverse operation $1/(1 - \cdot)$.

Any RLR instance can be converted to a regular expression [35], so we can count the words in a regular language if we have an algorithm that takes a regular expression and returns an equivalent unambiguous one. It turns out that this can be done by simply converting from regular expression to NFA to DFA and back to regular expression according to the algorithms in [35]. Essentially, the “determinism” of a DFA carries over to a regular expression in the form of unambiguity. We note that Brabrand and Thomsen [4] give a useful algorithm for finding the particular subexpressions of a regular expression that gives rise to ambiguity.

Genfunlib implements a number of operations on RLR instances corresponding to some of their abundant decidable properties, and in particular, Genfunlib allows regular expressions to be disambiguated. Genfunlib allows regular languages to be represented by any of the following RLRs: NFA, DFA, regular expression, right regular grammar, or directed graph with labeled vertices. Directed graphs in which the vertices are labeled with letters are a nontraditional RLR which have the benefit that they may be visualized using Mathematica’s built-in graph plotting. Such a graph $G = (V, E, l, \Sigma, S, E, e)$ is a set of vertices V , a set of edges E , a labeling function $l : V \rightarrow \Sigma$, where Σ is an alphabet, a

set $S \subseteq V$ of starting vertices, a set $E \subseteq V$ of ending vertices, and an integer e which is 1 if ϵ is accepted and 0 otherwise. We say a labeled digraph G accepts a nontrivial word w if and only if there exists a path $v_1, v_2, \dots, v_k \in V$ in G with $v_1 \in S$, and $v_k \in E$, such that $l(v_1)l(v_2) \cdots l(v_k) = w$.

Genfunlib allows an instance any RLR to be converted to an instance of any other. The following operations on RLRs are supported: union, intersection, complement, reverse, concatenation, and Kleene star. Generating functions for regular expressions can be computed by specifying a weight/marker for each letter in the alphabet.

Mathematica note: Since Mathematica does not use a type system, expressions must be inspected dynamically in order to determine whether they satisfy certain properties. The author of a Mathematica function must decide how to handle input expressions that the function is not supposed to take. One example of this is a function on the integers being passed a symbol rather than an integer — there are many reasons why a user might want to do this. According to the Guidebook for programming [37],

As a rule of thumb, messages are not generated for “symbolic” input if the function they appear in is used in classical mathematics.

Thus `Table[1, {n}]` generates a message, while `Sin[z]` does not. Genfunlib attempts to follow this convention, so all functions related to regular languages do not accept symbolic input or input that cannot be directly used for computation.

In Mathematica, input validation checks can be complete or shallow. A shallow check would only make sure the input is in roughly the right form, but would not check whether the input would cause an error later on. Generally, openly available Mathematica packages do not perform as much validation as

functions in the Mathematica kernel.

The FiniteFields package largely does not do input validation. It sometimes performs shallow syntactic validity checks, sometimes performs total semantic validity checks, and sometimes sends error messages. (On failure, all checks result in an expression returning unevaluated). The Splines package does only shallow syntactic input validation. Mathematica built-in functions often validate *any* sequence of arguments and send messages on errors, and generally do not produce results that will cause errors “later on”.

Since Mathematica expressions must be inspected dynamically, one may seek a method of caching properties of expressions explicitly to avoid overcomputation. For example, a function may always return a prime number, so the values of this function do not need to be tested later on for primality. In Genfunlib, the result of a successful `RegUnion` command, for example, is guaranteed to be a valid RLR, but when it is passed to another function, it is checked for validity anyway. One possible strategy is for all functions to store the validity of their results right before they return them, by setting a value of the "validate" function, which would look something like this:

```
validate[ret] = True; Return[ret]
```

This system could be altered by making the validate symbol only remember the last n such expressions to save memory.

If a public function calls another public function, it presumably always passes valid input. One way to avoid unnecessary computation of the validity is to pass an option saying “validation not required”; another is for public functions never to call public functions.

The validation scheme of Genfunlib is as follows. Public functions call validation directly (right in their definition) unless told not to by the

`validationRequired` option; private functions do not do validation. Using `validationRequired` saves some computation at the expense of more complicated code. Data representation symbols like `DFA` do not do validation themselves; indeed they are not defined as functions. This follows the convention exhibited in Mathematica symbols like `RegularExpression` and `Graph`.

The authors of the Combinatorica package for Mathematica [25] say, “Our aim in introducing permutation groups into Combinatorica is primarily for solving combinatorial enumeration problems. We make no attempt to efficiently represent permutation groups or to solve many of the standard computational problems in group theory.” The situation for this package and automata/grammar algorithms is similar – performance is not the highest priority.

In Genfunlib, letters are represented by nonempty `Strings` and words are represented by `Lists` of letters.

Mathematica note: An alternative approach to providing regular language functionality in Mathematica would have been to make a J/Link interface to the dk.brics.automaton Java package available at <http://www.brics.dk/automaton/>.

The expression formats used by Genfunlib for each RLR are as follows.

An **NFA** expression is of the form

```
NFA[numStates, alphabet, transitionMatrix,  
    acceptStates, initialState]
```

The number of states is a nonnegative integer; any NFA with zero states accepts no words. The alphabet is a sorted list of distinct strings, not containing `"`. A value of `{}` (the empty list) means the NFA accepts either the empty language

or $\{\epsilon\}$. The transition matrix is a `numStates` by `Length[alphabet]+1` matrix where entry i, j is a list of integers corresponding to states accessible from state i and, if $j \leq \text{Length}[\text{alphabet}]$, on the letter `alphabet[[j]]`, otherwise, on an ϵ -move. If the number of states is zero, then we must have `transitionMatrix == {}`. If the alphabet is empty, then `transitionMatrix` has one column (if it has any rows). The accept states are a list of integers between 1 and the number of states. The initial state is an integer between 1 and number of states, or `Null` iff `numStates == 0`.

As an example, we can represent the NFA in Figure 1 by the following expression.

```
NFA[6, {"0", "1"}, {
  {{2}, {2}, {3}},
  {{4}, {2}, {}},
  {{4}, {2}, {}},
  {{6}, {3}, {}},
  {{}, {}, {}},
  {{4}, {5}, {}},
}, {3, 4}, 1]
```

A regular expression that is equivalent to the NFA in Figure 1 is

$$\begin{aligned} & ((\epsilon|1(1)^*0|1(1)^*00(00)^*0|0(0)^*|(1(1)^*01|1(1)^*00(00) \\ & *01|0(0)^*1)((\epsilon|1(1)^*0)1|(0|1(1)^*0)0(00)^*01))^*(\epsilon|0 \\ & |1(1)^*0|(0|1(1)^*0)0(00)^*0)). \end{aligned}$$

A **DFA** expression is of the form

```
DFA[numStates, alphabet, transitionMatrix,
  acceptStates, initialState]
```

Figure 1: A six-state NFA.

The number of states is a nonnegative integer; any DFA with zero states accepts no words. The alphabet is a sorted list of distinct strings, not containing "". A value of {} means the DFA accepts either the empty language or $\{\epsilon\}$. The transition matrix is a `numStates` by `Length[alphabet]` matrix where entry i, j is the state accessible from state i and letter `alphabet[[j]]`. If the number of states is zero, then `transitionMatrix == {}`. If the alphabet is empty, then `transitionMatrix == {{}, {}, ...}`. The accept states are a list of integers between 1 and the number of states. The initial state is an integer between 1 and the number of states, or `Null` if `numStates == 0`.

As an example, we can represent the DFA shown in Figure 2 with the following expression.

```

DFA[5, {"0", "1"}, {
  {2, 3},
  {2, 4},
  {4, 3},
  {5, 5},
  {5, 5}
}, {4}, 1]

```

Figure 2: A five-state DFA.

A regular expression that is equivalent to the DFA in Figure 2 is

$$(0(0)^*1|1(1)^*0).$$

In Genfunlib, a **regular expression** is represented by an expression with head `Regex`, and an argument built up from nonempty strings, `EmptyWord`, `RegexStar`, `RegexConcat`, and `RegexOr`. The expression `Regex[Null]` matches no words (the empty language).

As an example, the regular expression $(a|bb)^*bbb$ is represented as

```
Regex[RegexConcat[RegexStar[RegexOr["a", RegexConcat["b", "b"]]],
      "b", "b", "b"]]
```

Applying Mathematica's `TreeForm` command to this expression produces Figure 3.

Right regular grammars are represented in Genfunlib by expressions with head `RRGrammar`, and one argument which is a list of rules in the form `sym -> RHS` or `sym[n] -> RHS`, where `sym` is a symbol, `n` is an integer, and `RHS` is either

- `EmptyWord`,

Figure 3: The parse tree of $(a|bb)^*bbb$.

- a string,
- an expression on the left hand side of another rule,
- `RRGrammarConcat[str, sym]` (where `str` is a string),
- `RRGrammarConcat[str, sym[n]]`, or
- `RRGrammarOr[args]`, where `args` is a sequence of any of the things in this list.

Strings cannot be empty. An empty list corresponds to the empty language.

For example, let G be the right regular grammar such that $N = \{S, A\}$, $\Sigma = \{a, b, c\}$, and the production rules P are

$$S \rightarrow aS$$

$$S \rightarrow bA$$

$$A \rightarrow \epsilon$$

$$A \rightarrow cA.$$

It can be seen that G is equivalent to the regular expression a^*bc^* . In Genfunlib, G becomes

```
RRGrammar[{
  s -> RRGrammarConcat["a", s],
  s -> RRGrammarConcat["b", a],
  a -> EmptyWord,
  a -> RRGrammarConcat["c", a]
}]
```

A **directed graph with labeled vertices** is represented by an expression of the form `Digraph[graph, startVertices, endVertices, eAccepted]`. The

graph is a directed Mathematica `Graph`, whose vertices are all labeled with nonempty strings. The starting vertices are a list of vertices of the graph; if this list is empty, the language associated with the graph contains no words except possibly ϵ . The ending vertices are a list of vertices of the graph; if this list is empty, the language associated with the graph contains no words except possibly ϵ . Finally, `eAccepted` is `True` if ϵ is accepted and `False` otherwise.

Any of the five RLRs can be converted to any other using the commands `ToNFA`, `ToDFA`, `ToRegex`, `ToRRGrammar`, and `ToDigraph`.

```
In[34]:= ?ToDFA
ToDFA[NFA[...]] is a DFA defined from a NFA.
ToDFA[Regex[...]] is a DFA defined from a symbolic regular
expression.
ToDFA[RRGrammar[...]] is a DFA defined from a right
regular grammar.
ToDFA[Digraph[...]] is a DFA defined from a digraph
with labeled vertices.

In[35]:= ToDFA[Regex[RegexOr["a", RegexStar["b"]]]]

Out[35]= DFA[4, {"a", "b"}, {{2, 3}, {4, 4}, {4, 3}, {4, 4}}, {1, 2,
3}, 1]
```

The algorithms for performing these conversions come from the textbook of Sipser [35] with two exceptions. First, given an NFA M with state set Q , the standard algorithm for obtaining a DFA that is equivalent to M involves computing the powerset 2^Q , which is used as the state set for the DFA. In some cases, there is no DFA equivalent to M with a state set with fewer elements than

2^Q [34]. However, the algorithm used in Genfunlib produces an automaton with many fewer states, when this is possible. The first step is to remove states from the NFA that are inaccessible from the start state or from every accepting state. Each state removed halves the maximum size of the resulting DFA. Next, when computing the set of states for the DFA, Genfunlib only includes those sets in 2^Q that are accessible from the initial state by performing an incremental search starting at the initial state. Finally, the DFA is minimized using Hopcroft’s algorithm [35]. Experimental tests showed a huge speed increase when using these steps to avoid creating a DFA with $2^{|Q|}$ states. In testing, Genfunlib found the minimal DFA of an NFA with 379 states in under half an hour on a computer with an AMD Phenom II X4 995 CPU and 8 gigabytes of RAM. However, the resulting DFA had only 8 states, so in terms of the length of the output, Genfunlib’s performance still has room for improvement. The dk.brics.automaton Java package [22] uses a similar determinization algorithm, but the performance of that library is much higher than that of Genfunlib, presumably because Java is a lower-level language than Mathematica.

The other somewhat unusual conversion algorithm used in Genfunlib is the one that maps labeled digraphs to automata and vice versa, since the digraph RLR is not included in textbooks on automata theory. Because in Genfunlib’s digraphs, vertices are labeled, and in the graphs derived from automata, edges are labeled, something like the line digraph construction can be used. The *line digraph* of a digraph G has one vertex for each edge of G , and has an edge between two of its vertices if and only if the corresponding edges in G form a length-two directed path in G . “Graphs” derived from automata are actually hypergraphs, but the line digraph definition can be adapted to this case. When Genfunlib converts something to a DFA, the output is a minimal DFA, but when converting to labeled digraphs or NFAs, the output may not be minimal.

Figure 4: A DFA with eight transitions.

Figure 5: The digraph constructed from the DFA in Figure 4. Vertices are labeled by the transition they correspond to in the DFA.

Minimizing NFAs is a PSPACE-complete problem [14].

As an illustration, the DFA in Figure 4 has each of its “edges” shown with an integer label, in brackets. Genfunlib converts this DFA to the digraph shown in Figure 5. The vertices of this digraph are shown with extra labels in brackets indicating the edge in the original DFA to which they correspond.

In addition to conversions between the five RLRs, Genfunlib includes functions for converting between Genfunlib regular expressions and Mathematica built-in regular expressions, which have head `RegularExpression`. To be converted, the Mathematica regular expressions must have a restricted syntax:

Only letter characters, parentheses, asterisks, and bars (|) are allowed.

The following example converts the regular expression $(c|a)^*$ from Mathematica syntax to Genfunlib syntax and back, and demonstrates how built-in Mathematica functions such as `StringMatchQ` can be used with Mathematica regular expressions.

```
In[36]:= ToRegex[RegularExpression["(c|a)*"]]
```

```
Out[36]= Regex[RegexStar[RegexOr["c", "a"]]]
```

```
In[37]:= ToRegularExpression[%]
```

```
Out[37]= RegularExpression["((c|a))*"]
```

```
In[38]:= StringMatchQ["accaccccaaa", %]
```

```
Out[38]= True
```

Since, in Genfunlib, letters are represented by strings which may be longer than one character (and words are represented by lists of strings), some issues may arise when converting to Mathematica regular expressions. In Genfunlib, `{"a", "aa"}` and `{"aa", "a"}` are different words. However, since in Mathematica represents letters as length-one strings and represented words as strings,

```
Regex[RegexOr[RegexConcat["a", "aa"], RegexConcat["aa", "a"]]]
```

gets converted to

```
RegularExpression["(aaa|aaa)"]
```

Regular languages have a rich set of closure properties [35], which Genfunlib implements in the form of six operations: Kleene star (`RegStar`), comple-

ment (`RegComplement`), reverse (`RegReverse`), union (`RegUnion`), concatenation (`RegConcat`), and intersection (`RegIntersection`).

The algorithms for performing these operations all come from Sipser’s textbook [35]. For each operation, Genfunlib implements such an algorithm that works for one type of RLR. Then the operation is implemented for all other RLR types by converting them to the type for which an algorithm is implemented, running the algorithm, then converting back again. As an example, the concatenation operation is directly implemented only for right regular grammars, but users can call `RegConcat` on two DFAs because they will get automatically converted to and from grammars.

The unary operations `RegStar` and `RegReverse` simply take one RLR as an argument. The operation `RegComplement` takes one RLR r and an alphabet Σ and returns an RLR r' of the same type as r such that $L(r') = \Sigma^* \setminus L(r)$, where $L(r)$ is the language of r .

```
In[39] := ?RegComplement
```

```
RegComplement[NFA[...], {"str1", "str2", ...}] performs the  
complement operation on an NFA with respect to a given  
alphabet.
```

```
RegComplement[DFA[...], {"str1", "str2", ...}] performs the  
complement operation on a DFA with respect to a given  
alphabet.
```

```
RegComplement[Regex[...], {"str1", "str2", ...}] performs the  
complement operation on a symbolic regular expression  
with respect to a given alphabet.
```

```
RegComplement[RRGrammar[...], {"str1", "str2", ...}] performs the  
complement operation on a right regular grammar with respect  
to a given alphabet.
```

`RegComplement[Digraph[...], {"str1", "str2",...}]` performs the complement operation on a directed graph with respect to a given alphabet.

```
In[40]:= RegComplement[Regex[RegexStar["a"]], {"a", "b"}]
```

```
Out[40]= Regex[ RegexConcat[RegexStar["a"], "b",
    RegexStar[RegexOr["a", "b"]]]]
```

The binary operations `RegUnion`, `RegConcat`, and `RegIntersection` take two RLRs of the same type.

Genfunlib overloads `GeneratingFunction` to take a regular expression and a list of rules mapping letters to weights. The regular expression is automatically disambiguated, and the letters are replaced with their weights to obtain a rational power series. Here we compute the generating function for the ambiguous regular expression $a|a|b|b^*$ that marks a with u and b with v :

```
In[41]:= GeneratingFunction[Regex[RegexOr["a","a","b",
    RegexStar["b"]]]], {"a"->u,"b"->v}]
Out[41]= 1+u+v/(1-v)
```

4.4 Symbolic method

Genfunlib contains a basic implementation of the symbolic method. An appropriate introduction to the symbolic method may be found in the author's honours project [20] or Flajolet and Sedgewick's book [11].

Genfunlib allows combinatorial classes to be defined by specifications, and can convert these specifications to corresponding systems of equations in generating functions.

A specification is represented by an expression of the form

Labeled		Unlabeled	
Sum	SMPlus	Sum	SMPlus
Product	SMTimes	Product	SMTimes
Sequence	SMSeq	Sequence	SMSeq
Cycle	SMCyc	Cycle	SMCyc
Set	SMSet	Powerset	SMSet
Pointing	SMPointing	Multiset	SMMultiset
Substitution	SMSub	Pointing	SMPointing
		Substitution	SMSub

Table 1: Admissible constructions supported by Genfunlib.

`Spec[{lhs1==rhs1,...},labeled?]`

In the list of equations, the left hand sides are symbols representing classes, and the right hand sides are expressions built with combinatorial constructions, symbols found in a left hand side, restrictions (**Restricted**[...]), and atomic and neutral classes. The second argument is **True** if the classes are labeled, and **False** if they are unlabeled.

Genfunlib supports multiple atomic classes, each of which is indexed by a positive integer and is marked separately in generating functions. They have the form **ZClass**[1], **ZClass**[2], The neutral class is represented by the symbol **EClass**.

The combinatorial constructions in Genfunlib are shown in Table 1, along with their syntax. Sum and product are commutative binary constructions; sequence, cycle, set, multiset, and pointing are unary constructions. Substitution is a binary construction, but it is not commutative. The expression **SMSub**[**A**,**B**] corresponds to $\mathcal{A}(\mathcal{B})$, if $\mathcal{A}$ is represented by the symbol **A** and $\mathcal{B}$ is represented by the symbol **B**.

The set, multiset, sequence, and cycle constructions take an optional second argument in the form **Cardinality**->**f**, where **f** is a predicate on the integers which gives the allowed number of components. For example,

```
SMSeg[class, Cardinality->(#<=k&)]
```

represents the class of finite sequences of elements of `class` with at most `k` terms. It is fine to use a symbol like `k` rather than an integer.

The other kind of restriction that can be done is with the `Restricted` symbol. The expression `Restricted[class,f]`, where `f` is a predicate on the integers, represents the class of objects from `class` with sizes that `f` maps to `True`. For example,

```
Restricted[SMSeg[SMTimes[ZClass[1],ZClass[2]]], #>2&]
```

represents the class containing all the objects in $\text{SEQ}(\mathcal{Z}_1 \times \mathcal{Z}_2)$ except the singleton sequence.

Mathematica note: Predicates on the integers must return unevaluated when given symbolic input. The predicate `Mod[#, 2]&` satisfies this requirement, while `PrimeQ[#]&` does not.

The command for passing from a combinatorial specification to a system of generating function equations is `ToGFEqns`. It takes two arguments, the first is a `Spec` expression and the second is a symbol to be used as the indeterminate. If the symbol `x` is supplied, `ZClass[1]` is marked with `x[1]`, `ZClass[2]` is marked with `x[2]`, and so on. In the output, the generating function of a class is written as the symbol for the class, with arguments.

```
In[42] := ?ToGFEqns
```

`ToGFEqns[spec, indeterminate]` gives the system of power series equations corresponding to `spec`, using the given indeterminate.

```
In[43] := ToGFEqns[
```

```
Spec[{c == SMTimes[ZClass[1], SMSeq[ZClass[1]]]}, True], z]
```

```
Out[43]= {c[z[1]] == z[1]/(1 - z[1])}
```

The code that performs the translation to generating functions uses pattern matching and replacement rules. The following example snippet is a simplified segment of Genfunlib's code that takes a specification called `spec` and replaces, e.g. `ZClass[2]` and `ZClass[3]` with `z[2]` and `z[3]`, and replaces `EClass` with 1.

```
Replace[spec,
{
  ZClass[n_] :> z[n],
  EClass :> 1,
},
{1, Infinity}, Heads -> True
];
```

Genfunlib includes one more command related to the symbolic method: `ToGenfunlibSpec`. This command takes two arguments. The first is a string containing the Maple code for a Combstruct grammar in the format used on the Encyclopedia of Combinatorial Structures (ECS) website at

<http://algo.inria.fr/ecs/>

The second argument is a Boolean value indicating whether the classes are labeled. The output is a `Spec` expression representing the same specification as the Combstruct grammar. Since Genfunlib and Combstruct specifications are similar in structure, the conversion is not complex and mostly consists of term-for-term translation. Class symbols are converted from upper case to lower case (and reduplicated) to conform to Mathematica conventions.

As an example, we can take entry #9 from the ECS: words on the alphabet $\{a, b\}$ with at most one consecutive b -letters. The ECS gives the following unlabeled Combstruct specification:

```
{S = Prod(X,Sequence(Prod(a,X))), X = Sequence(b,card <= 1),
  a = Atom, b = Atom }
```

In Genfunlib, the class S becomes ss .

```
In[44] := ?ToGenfunlibSpec
ToGenfunlibSpec[mapleString, labeled] converts from Combstruct to
  Genfunlib combinatorial specification syntax.
```

```
In[45] := ToGenfunlibSpec["{S = Prod(X,Sequence(Prod(a,X))), X =
  Sequence(b,card <= 1), a = Atom, b = Atom }", False]
```

```
Out[45]= Spec[{ss == SMTimes[xx, SMSeq[SMTimes[a, xx]]],
  xx == SMSeq[b, Cardinality -> (#1 <= 1 &)], a == ZClass[1],
  b == ZClass[1]}, False]
```

We can convert to generating functions using indeterminate z and solve for $ss[z[1]]$.

```
In[46] := ToGFEqns[%, z]
```

```
Out[46]= {ss[z[1]] == xx[z[1]]/(1 - a[z[1]] xx[z[1]]),
  xx[z[1]] == 1 + b[z[1]], a[z[1]] == z[1], b[z[1]] == z[1]}
```

```
In[47] := Solve[Eliminate[%, {a[z[1]], b[z[1]], xx[z[1]]}],
  ss[z[1]]]
```

```
Out[47]= {{ss[z[1]] -> (-1 - z[1])/(-1 + z[1] + z[1]^2)}}
```

There is room for expansion of Genfunlib's symbolic method capabilities. Symbolic method specifications are useful because tractable operations can be performed on them. This is of note since in general, combinatorial objects are too complex for such simple notation. Assuming the user is able to provide a combinatorial specification, one valuable question may be whether the specification is well-formed. The class $\mathcal{I}$ defined by $\mathcal{I} = \text{SEQ}(\mathcal{E})$ has an infinite set of objects of size 0, so the specification should be edited before anything is done with it. Work on checking specifications for well-definedness has been published by Salvy and coauthors [10, 26]. Genfunlib does not implement these complete validity checks on specifications; however, errors are likely to be found at a later state when working with power series. For example, $\mathcal{I} = \text{SEQ}(\mathcal{E})$ would generate a division by 0 error.

Lastly, there exists a notation system and framework for combinatorial objects similar to symbolic combinatorics called species [20]. Currently, Genfunlib provides support for the symbolic method only, although the possibility of implementing species instead and including a converter for specifications from the symbolic method to species was considered. It does appear as though a natural mapping exists between the sequence, set, and cycle constructions on the one hand, and the species of linear orders, sets, and cyclic permutations on the other. However, problems arise eventually. Whereas the pointing construction in the symbolic method corresponds, in unlabeled structures, to the power series operator xD , for a species F the type generating series $\widetilde{F}^\circ(x)$ cannot be expressed in terms of $\widetilde{F}(x)$, but is given by

$$\widetilde{F}^\circ(x) = x \left(\frac{\partial}{\partial x_1} Z_F \right) (x, x^2, x^3, \dots).$$

Thus, in species, the pointing operation cannot correspond to any admissible construction in the symbolic method. This kind of subtle difference also exists between the substitution “construction” in the symbolic method and “operation” in species. Salvy et al. discuss perhaps less serious issues with converting the unlabeled multiset construction to the language of species [26].

4.5 Asymptotics

Flajolet et al. [10] define the class $\mathcal{E}$ of *elementary functions* as the set of functions containing the monomials 1 and z , and closed under the operations of $\{+, \times, Q, L, E\}$, where

$$Q(f) = \frac{1}{1-f}, \quad L(f) = \log \frac{1}{1-f}, \quad E(f) = \exp(f).$$

They proceed to give algorithms that can be used to automatically extract Maclaurin coefficient asymptotics for functions in a subset of $\mathcal{E}$. Before going further here, we note that at this point some confusion can appear related to the terminology and notation used. First, even though when talking about asymptotics we need to think of generating functions as complex functions, for the purpose of computer algebra we really still compute with power series. That is, “ $\exp(z)$ ” cannot, strictly speaking, be called a function. Also, we actually want to talk about *expressions* built with the operations $+, \times, Q, L, E$, etc. — we are talking about syntax, not semantics. Deciding whether a given expression semantically belongs in $\mathcal{E}$ is in general impossible, and not our main goal.

The class $\mathcal{E}$ is subdivided into three disjoint sets

$$\mathcal{E} = \mathcal{E}_{\text{AL}} \dot{\cup} \mathcal{E}_{\text{entire}} \dot{\cup} \mathcal{E}_{\text{other}},$$

where $\mathcal{E}_{\text{entire}}$ is the set of functions in $\mathcal{E}$ that are entire, and $\mathcal{E}_{\text{AL}}$ is the functions

with *algebraic-logarithmic* growth, meaning that $f \in \mathcal{E}_{\text{AL}}$ if and only if f has radius of convergence $0 < r < \infty$ then there exist $\alpha \in \mathbb{R}, k \in \mathbb{Z}_{\geq 0}$ such that

$$f(z) \sim \frac{1}{(1 - z/r)^\alpha} \log^k \frac{1}{1 - z/r}, \quad \text{as } z \rightarrow r^-.$$

The algorithm for computing asymptotic expansions of the Maclaurin coefficients of functions in $\mathcal{E}_{\text{AL}}$ described by Flajolet et al. [10] was implemented in Combstruct for Maple and is now partly implemented in Genfunlib for Mathematica as the `CoefLimit` command. This command takes an aperiodic function $f \in \mathcal{E}_{\text{AL}}$ and returns something asymptotically equivalent to the general term of the Maclaurin coefficient sequence of f . By saying that f is aperiodic, we mean that $f(z)$ cannot be expressed as $z^a g(z^b)$ where $b > 1$ and a are integers, and $g \in \mathcal{E}_{\text{AL}}$.

```
In[48]:= ?CoefLimit
```

```
CoefLimit[f,z,n] gives an expression that is asymptotically
equivalent to SeriesCoefficient[f, {z,0,n}] as n goes to
infinity.
```

```
In[49]:= CoefLimit[1/(1 - z*Exp[z]), z, n]
```

```
Out[49]= ProductLog[1]^(-1 - n)/(1 + E^ProductLog[1])
```

```
In[50]:= CoefLimit[1/(1 - Log[1/(1 - z*Exp[z])])], z, n]
```

```
Out[50]= -(((1 - (-1 + E)/E) ProductLog[(-1 + E)/E]^(-1 -
n))/(-((-1 + E)/E) - E^ProductLog[(-1 + E)/E]))
```

In the case where $f \in \mathcal{E}_{\text{AL}}$ has positive radius of convergence r and

$$f(z) \sim \frac{1}{(1 - z/r)^\alpha} \log^k \frac{1}{1 - z/r} \quad \text{as } z \rightarrow r^-,$$

where $\alpha \leq 0$, `CoefLimit` merely returns the exact expression for the coefficients, using Mathematica's `SeriesCoefficient`.

```
In[51]:= CoefLimit[Log[1/(1 - z*Exp[z])], z, n]
```

```
Out[51]= SeriesCoefficient[Log[1/(1 - E^z z)], {z, 0, n}]
```

```
In[52]:= $Assumptions = n > 1;
```

```
In[53]:= CoefLimit[1/(1 - z) Log[1/(1 - 2 z)], z, n]
```

```
Out[53]= -I*Pi - 2^(1 + n) LerchPhi[2, 1, 1 + n]
```

Here `LerchPhi[2, 1, 1 + n]` represents $\sum_{k \geq 0} 2^k / (k + n + 1)$.

Mathematica note: The original algorithm [10] works for periodic generating functions and can obtain an asymptotic expansions of the form

$$C \rho^{-n} n^s \log^k n \left(1 + O\left(\frac{1}{\log n}\right) \right).$$

However, in Mathematica asymptotic series are represented with `SeriesData` objects, and these only support big-Oh terms of the form $O(n^k)$. Thus, `CoefLimit` does not give big-Oh terms, merely the dominant term. Since periodic generating functions have zeroes in their coefficients, the coefficients are not asymptotically equivalent to anything, so Genfunlib cannot give an asymptotic

expression in this case.

A general way to specify a generating function is to define it implicitly by a power series equation. Genfunlib provides the `TreeAsymptot` command for computing the coefficient asymptotics of generating functions $f(z)$ satisfying equations of the form

$$f(z) = z\phi(f(z)),$$

where $\phi \in \mathcal{E}_{\text{AL}}$ satisfies $\phi(z) \neq c_0 + c_1 z$ and $\phi(0) \neq 0$. In this case, we can apply Theorem VI.6 [11] to derive the asymptotics of $[z^n]f(z)$ provided we can compute the minimum solution τ to the characteristic equation

$$\phi(\tau) - \tau\phi'(\tau) = 0, \quad \tau > 0,$$

and the periodicity of ϕ . The value of τ can be computed by Mathematica's built-in equation solver, and we can obtain the necessary information about the period of ϕ via the “Reduction” algorithm from Flajolet et. al [10].

The `TreeAsymptot` command uses this method to give an asymptotic expression consisting of a main estimate and a big O term.

```
In[54]:= ?TreeAsymptot
TreeAsymptot[y[z]==z*phi,n] gives an asymptotic series for
SeriesCoefficient[y[z], {z,0,n}] as n goes to infinity,
where y[z] is the solution to y[z]==z*phi that is
analytic at 0.
```

```
In[55]:= TreeAsymptot[f[z] == z*Exp[f[z]], n]
```

```
Out[55]= E^n ( (n^(-1))^(3/2)/Sqrt[2*Pi] + O[1/n]^(5/2))
```

The case where $\phi(z)$ is periodic is allowed.

```
In[56]:= TreeAsymptot[f[z] == z*Exp[f[z]^2], n]
```

```
Out[56]= (2 E)^(n/2) Boole[Mod[n, 2] == 1] ((n^(-1))^(3/2)/
  Sqrt[2*Pi] + O[1/n]^(5/2))
```

Flajolet and Sedgewick [11] give an algorithm for obtaining coefficient asymptotics of generating functions defined by algebraic equations. This surely useful algorithm has so far not been implemented in any computer algebra system.

5 Genfunlib in action

5.1 Mathematics contest problems

Elementary problems from combinatorics are often chosen for high school math contests. This subsection contains a selection of contest-style problems and shows how Genfunlib can help to solve them; more contest problems solved with the aid of a computer algebra system can be found in an article by Dumas and Salvy [8].

Problem 1. How many arrangements are there of the letters in the word MATHEMATICS? [30]

Solution: As Zeilberger says [12], “Enumerating specific finite sets is no longer considered mathematics. A genuine mathematical fact has to incorporate infinitely many facts, and the generic enumeration problem is to enumerate not just one set but all the sets in an infinite family.” In particular, this problem can be solved easily “by computer” simply by performing a brute-force enumeration (in a math contest, this method is too slow). However, Genfunlib could also be used.

The word MATHEMATICS contains 11 letters. If we create sequences of 11 atoms $\textcircled{1}$, we can associate the label of the i th atom with the position of the i th letter of MATHEMATICS in an arrangement of the letters. For example,

$$\left(\textcircled{8}, \textcircled{4}, \textcircled{5}, \textcircled{2}, \textcircled{7}, \textcircled{1}, \textcircled{3}, \textcircled{10}, \textcircled{9}, \textcircled{11}, \textcircled{6}\right)$$

corresponds to the arrangement MHAATSEMITC. The problem is that not all letters in MATHEMATICS are distinct, so not every different labeled sequence corresponds to a different arrangement. We fix this by creating sequences of sets of atoms, where there is one set for each different letter in MATHEMATICS. This way, each set of positions of a letter is counted, not each sequence of positions.

Our specification of the class $\mathcal{M}$ of rearrangements of MATHEMATICS is

$$\begin{aligned}\mathcal{M} &= \text{SET}_2 \left(\textcircled{1}\right)^3 \times \text{SET}_1 \left(\textcircled{1}\right)^5 \\ &= \text{SET}_2 \left(\textcircled{1}\right)^3 \times \textcircled{1}^5.\end{aligned}$$

With Genfunlib, we convert this to an exponential generating function $m[z]$ where the coefficient of z^{11} multiplied by $11!$ is the solution to the problem.

```
In[57]:= ToGFEqns[Spec[{
  m == SMTimes[
    SMSet[ZClass[1], Cardinality -> (# == 2 &)],
    SMSet[ZClass[1], Cardinality -> (# == 2 &)],
    SMSet[ZClass[1], Cardinality -> (# == 2 &)],
    ZClass[1],
    ZClass[1],
    ZClass[1],
    ZClass[1],
```

Figure 6: An NFA N for the allowed sequences of coin tosses in Problem 2.

```

ZClass[1]
]], True], z] /. z[1] -> z

```

```

Out[57]= {m[z] == z^11/8}

```

From here we observe that the solution is

```

In[58]:= 11!/8

```

```

Out[58]= 4989600

```

Problem 2. A fair coin is to be tossed ten times. Find the probability that heads never occur on three consecutive tosses. (This is a modified version of AIME 1990 #9, a typical Fibonacci problem [31].)

Solution: Say the coin is tossed n times. Then we can represent the sequence of tosses as an n -word over the alphabet $\{H, T\}$. Let N be the NFA in Figure 6. Then the language accepted by N is the set of words representing coin tosses in which heads never occur on three consecutive tosses. Using Genfunlib, we can convert N to an equivalent regular expression.

```

In[59]:= ToRegex[NFA[3, {"H", "T"}, {
  {{2}, {1}, {}},

```

```

{{3},{1},{}},
{{},{3},{}}
},{1,2,3},1]]

```

```

Out[59]= Regex[RegexOr[RegexStar[T],RegexConcat[
  RegexStar[T],H,RegexStar[RegexConcat[T,RegexStar[T],H]],
  RegexOr[EmptyWord,RegexConcat[T,RegexStar[T]]]],
  RegexConcat[RegexStar[T],H,RegexStar[RegexConcat[T,
  RegexStar[T],H]],H,RegexStar[T]]]]

```

Now, we obtain the generating function of this regular expression to find the number of words of length 10.

```

In[60]:= GeneratingFunction[%, {"H" -> z, "T" -> z}]

```

```

Out[60]= (1 + z + z^2/(1 - z))/(1 - z - z^2)

```

```

In[61]:= SeriesCoefficient[%, {z, 0, 10}]

```

```

Out[61]= 232

```

Thus, the answer is $232/2^{10} = 29/128$.

Problem 3. Find a recursion for the number of partitions of an n -set [9].

First we can note that the combinatorial class $\mathcal{P}$ of set partitions is defined

$$\mathcal{P} = \text{SMSET}(\text{SMSET}_{\geq 1}(\textcircled{1})).$$

That is, a set partition is simply a set of nonempty sets. With Genfunlib we can get the exponential generating function $P(z)$ for the class $\mathcal{P}$, represented by `p[z]`.

```
In[62]:= ToGFEqns[
  Spec[{p == SMSet[SMSet[ZClass[1], Cardinality ->
    (# >= 1 &)]}], True], z] /. z[1] -> z
```

```
Out[62]= {p[z] == E^(-1 + E^z)}
```

This is actually an explicit equation for the generating function $p[z]$. To arrive at a recurrence relation, we logarithmically differentiate both sides and solve for $p[z]$.

```
In[63]:= Solve[D[Log[p[z]], z] == D[Log[E^(-1 + E^z)], z], p[z]]
```

```
Out[63]= {{p[z] -> E^-z p'[z]}}
```

Using Genfunlib's enhancement to `SeriesCoefficient`, we extract coefficients of the right hand side.

```
In[64]:= SeriesCoefficient[E^-z p'[z], {z, 0, n}]
```

```
Out[64]= Sum[((-1)^$6*(1 + n - $6)*SeriesCoefficient[p[z],
  {z, 0, 1 + n - $6}, Assumptions -> Element[$6, Integers] &&
  $6 >= 0)]/$6!, {$6, 0, n}]
```

If we let $p_n = n![z^n]P(z)$, we can use the above result to arrive at the relation

$$p_0 = 1, \quad p_n = p_{n-1} - \sum_{j=1}^{n-1} \binom{n-1}{j} (-1)^j p_{n-j}.$$

Problem 4. Prove that the number of binary n -words with exactly m 01-blocks is $\binom{n+1}{2m+1}$ [9].

Solution: The way we attack this problem is defining the NFA N_b in Figure 7. The language that N_b accepts is all words over the alphabet $\{0, 1, b\}$ where

Figure 7: An NFA N_b that marks 01 blocks with b .

the letter b only appears as part of the subword $0b1$, and the subword 01 never appears. Counting the number of words with n letters 0 and 1 and m occurrences of b will solve the problem.

We define `nfab` to be N_b from Figure 7 with Genfunlib.

```

In[65]:= nfab = NFA[4, {"0", "1", "b"}, {
  {{2}, {1}, {}, {}},
  {{2}, {}, {3}, {}},
  {{}, {4}, {}, {}},
  {{2}, {1}, {}, {}},
  {}, {1, 2, 4}, 1];

```

We obtain a generating function from `nfab`, with u marking 0 and 1, and v marking b .

```

In[66]:= GeneratingFunction[ToRegex[nfab],
  {"0" -> u, "1" -> u, "b" -> v}]

```

```

Out[66]= (1 + u/(1 - u))/(1 - u - (u^2 v)/(1 - u))

```

Now, we extract the coefficient of v^m .

```
In[67]:= $Assumptions = m >= 0;
```

```
In[68]:= SeriesCoefficient[1/(1 + u (-2 + u - u v)), {v, 0, m}]
```

```
Out[68]= (u^2/(-1 + u)^2)^(1 + m)/u^2
```

Lastly we compare this to $\sum_{n \geq 2m} \binom{n+1}{2m+1} u^n$.

```
In[69]:= Out[68] == Sum[Binomial[n + 1, 2 m + 1] u^n,
  {n, 2 m, Infinity}] // FullSimplify
```

```
Out[69]= True
```

This may seem like an indirect route to the result. The explanation is that, as long as we want to use Mathematica and Genfunlib as much as possible, the most direct route sometimes does not work but something more creative does.

Problem 5. How many n -words from the alphabet $\{0, 1, 2\}$ are such that the neighbors differ by at most 1? [9]

Solution: One can see that this is a regular language accepted by the NFA N_d shown in Figure 8. We define N_d in Genfunlib as `nfadif`.

```
nfadif = NFA[4, {"0", "1", "2"}, {
  {{2}, {3}, {4}, {}},
  {{2}, {3}, {}, {}},
  {{2}, {3}, {4}, {}},
  {{}, {3}, {4}, {}},
  {}, {1, 2, 3, 4}, 1];
```

Figure 8: The NFA N_d that accepts words from the alphabet $\{0, 1, 2\}$ such that neighboring letters differ by at most 1.

All that remains in order to count the n -words is to obtain the generating function and extract coefficients.

```

In[70]:= GeneratingFunction[
  ToRegex[nfadif], {"0" -> z, "1" -> z, "2" -> z}] //
  FullSimplify

Out[70]= -((1 + z)/(-1 + z (2 + z)))

In[71]:= $Assumptions = n >= 1;

In[72]:= SeriesCoefficient[-((1 + z)/(-1 + z (2 + z))), {z, 0, n}]

Out[72]= (-2 (1 - Sqrt[2])^n + Sqrt[2] (1 - Sqrt[2])^n +
  2 (1 + Sqrt[2])^n + Sqrt[2] (1 + Sqrt[2])^n)/(2 Sqrt[2])

```

Simplified and in standard mathematical notation, this says that there are

$$\frac{1}{2} \left(\left(1 - \sqrt{2}\right)^{n+1} + \left(1 + \sqrt{2}\right)^{n+1} \right)$$

n -words from the alphabet $\{0, 1, 2\}$ such that neighboring letters differ by at most 1.

Problem 6. If n people sit around a circular table, how many of the $n!$ arrangements are distinct, i.e., do not have the same neighboring relations?

Solution: The first thing to note about this problem is that it appears to be ambiguous. Is the neighboring relation who is sitting next to whom, or who is sitting to the left of whom? Let us solve the problem for the latter case. It turns out that the final answers simply differ by a factor of 2.

We represent the seating arrangements by the combinatorial class $\mathcal{C} = \text{Cyc}(\textcircled{1})$. That is, the objects in $\mathcal{C}$ are labeled cycles. With Genfunlib, we can find $c[z]$, the generating function of $\mathcal{C}$, and extract coefficients.

```
In[73]:= ToGFEqns[Spec[{c == SMCyc[ZClass[1]]}, True], z]
/. z[1] -> z
```

```
Out[73]= {c[z] == Log[1/(1 - z)]}
```

```
In[74]:= SeriesCoefficient[Log[1/(1 - z)], {z, 0, n}]
```

```
Out[74]= 1/n
```

Since $c[z]$ is an exponential generating function, the number of distinct arrangements is $n!/n = (n-1)!$.

Problem 7. A permutation p of the set $\{1, \dots, n\}$ is called an *involution* if $p \circ p = \text{Id}$. Find a recursion for the number t_n of involutions of $\{1, \dots, n\}$. Also find a closed formula in the form of a sum [9].

Solution: The functional graph of an involution is a set of 1- and 2-cycles.

So we define the class $\mathcal{G}$ by

$$\mathcal{G} = \text{SET}(\text{CYC}_{\leq 2}(\textcircled{1})).$$

With Genfunlib, we find the generating function $g[z]$ for $\mathcal{G}$.

```
In[75]:= ToGFEqns[Spec[{g == SMSet[SMCyc[ZClass[1]],
    Cardinality -> (# <= 2 &)]}, True], z] /. z[1] -> z
```

```
Out[75]= {g[z] == 1 + Log[1/(1 - z)] + 1/2 Log[1/(1 - z)]^2}
```

From this expression for $g[z]$, Mathematica can produce a recurrence relation for t_n , which we call `tn`.

```
In[76]:= tn = FullSimplify[ n! SeriesCoefficient[
    1 + Log[1/(1 - z)] + 1/2 Log[1/(1 - z)]^2, {z, 0, n}]];
```

```
In[77]:= DifferenceRootReduce[tn]
```

```
Out[77]= DifferenceRoot[
    Function[{y, n}, {n^2 y[n] + (-1 - 2 n) y[1 + n] +
    y[2 + n] == 0, y[1] == 1, y[2] == 2}]] [n]
```

That is, $t_1 = 1, t_2 = 2$, and, for $n \geq 3$,

$$n^2 t_n - (2n + 1)t_{n+1} + t_{n+2} = 0.$$

Now, finding a sum for t_n should be easy since we have the generating function $g[z]$. There is a small problem, though, which is that Mathematica tries to “simplify” the sum into something else as soon as it is obtained. To avoid this, we

replace $\log(1/(1-z))$ in $g[z]$ with the placeholder $f(z)$, then extract coefficients from $g[z]$ in terms of the coefficients of $f(z)$, using Genfunlib's enhancements to Mathematica's `SeriesCoefficient`. Then we replace coefficients of $f(z)$ with the coefficients of $\log(1/(1-z))$, and we are done.

```
In[78]:= $FullAnalytic = True;
```

```
In[79]:= n! SeriesCoefficient[
  1 + Log[1/(1 - z)] + 1/2 Log[1/(1 - z)]^2 /.
  Log[1/(1 - z)] -> f[z], {z, 0, n}]
```

```
Out[79]= n!*(SeriesCoefficient[f[z], {z, 0, n}] +
  Sum[SeriesCoefficient[f[z], {z, 0, n - $14}, Assumptions ->
  n >= 1 && Element[n, Integers] && Element[$14, Integers] &&
  $14 >= 0]*SeriesCoefficient[f[z], {z, 0, $14},
  Assumptions -> n >= 1 && Element[n, Integers] &&
  Element[$14, Integers] && $14 >= 0], {$14, 0, n}]/2)
```

```
In[80]:= % /. SeriesCoefficient[f[z], {z, 0, j_}, ___] :>
  Boole[j > 0]/j
```

```
Out[80]= n!*(Boole[n > 0]/n + Sum[(Boole[n - $14 > 0]*
  Boole[$14 > 0])/((n - $14)*$14), {$14, 0, n}]/2)
```

The result is a bit cluttered, but after a little inspection, it says that, for $n \geq 1$,

$$t_n = n! \left(\frac{1}{n} + \frac{1}{2} \sum_{j=1}^{n-1} \frac{1}{j(n-j)} \right).$$

Problem 8. Let $f(n)$ be the number of n -words without neighboring zeros from the alphabet $\{0, 1, 2\}$. Find a recursion and a formula for $f(n)$.

Solution: To specify the language of interest, we define the grammar G with the following productions.

$$A \rightarrow 1 \cdot A$$

$$A \rightarrow 2 \cdot A$$

$$A \rightarrow 0 \cdot B$$

$$B \rightarrow 1 \cdot A$$

$$B \rightarrow 2 \cdot A$$

$$A \rightarrow \epsilon$$

$$B \rightarrow \epsilon$$

In Genfunlib, we can define the corresponding grammar `oneZero` and compute the generating function.

```
In[81]:= oneZero = RRGrammar[{
  a -> RRGrammarConcat["1", a],
  a -> RRGrammarConcat["2", a],
  a -> RRGrammarConcat["0", b],
  b -> RRGrammarConcat["1", a],
  b -> RRGrammarConcat["2", a],
  a -> EmptyWord,
  b -> EmptyWord}];
```

```
In[82]:= GeneratingFunction[ ToRegex[oneZero],
```

```
{"0" -> z, "1" -> z, "2" -> z}]
```

```
Out[82]= (1 + z)/(1 - 2 z - 2 z^2)
```

Figure 9 shows a digraph with labeled vertices corresponding to `oneZero`, which was created with Genfunlib's `ToDigraph` command and edited slightly for clarity.

Finding a formula for $f(n)$ is now as simple as extracting coefficients using Mathematica's `SeriesCoefficient` command.

```
In[83]:= SeriesCoefficient[%, {z, 0, n}]
```

```
Out[83]= (-2 (1 - Sqrt[3])^n + Sqrt[3] (1 - Sqrt[3])^n +
          2 (1 + Sqrt[3])^n + Sqrt[3] (1 + Sqrt[3])^n)/(2 Sqrt[3])
```

After some simplification, we have that

$$f(n) = \frac{1}{6} \left(3 + 2\sqrt{3} \right) \left(1 + \sqrt{3} \right)^n - \frac{1}{6} \left(2\sqrt{3} - 3 \right) \left(1 - \sqrt{3} \right)^n.$$

Obtaining a recurrence relation is as easy as calling a Mathematica command called `DifferenceRootReduce`.

```
In[84]:= DifferenceRootReduce[%, n] // FullSimplify
```

```
Out[84]= DifferenceRoot[ Function[{y, n}, {-2 y[n] -
          2 y[1 + n] + y[2 + n] == 0, y[0] == 1, y[1] == 3}]] [n]
```

That is, $f(0) = 1$, $f(1) = 3$, and, for $n \geq 2$,

$$f(n+2) = 2f(n) + 2f(n+1).$$

Figure 10: The 12 binary trees with 3 labeled leaves.

Problem 9. How many binary trees with n labeled leaves are there? [9]

Solution:

We assume that “binary trees” are rooted plane trees such that every node has either 0 or 2 children. We assume that these leaf-labeled binary trees are binary trees whose leaves are uniquely labeled. The task is to count how many leaf-labeled binary trees there are with n leaves. Figure 10 shows all leaf-labeled binary trees with 3 leaves.

First we define a combinatorial specification for the classes of leaves, $\mathcal{L}$, inner nodes $\mathcal{I}$, and trees $\mathcal{T}$.

$$\begin{aligned}\mathcal{L} &= \textcircled{1} \\ \mathcal{I} &= \mathcal{E} \\ \mathcal{T} &= \mathcal{L} + \mathcal{T} \times \mathcal{I} \times \mathcal{T}\end{aligned}$$

In Genfunlib, we can convert this specification to equations in the exponential generating functions $l[z]$ for $\mathcal{L}$, $i[z]$ for $\mathcal{I}$, and $t[z]$ for $\mathcal{T}$.

```
In[85]:= ToGFEqns[Spec[{
  l == ZClass[1],
  i == EClass,
  t == SMPlus[l, SMTimes[t, i, t]]
}, True], z] /. z[1] -> z
```

```
Out[85]= {l[z] == z, i[z] == 1, t[z] == l[z] + i[z] t[z]^2}
```

We use Mathematica's `Eliminate` to provide us with an equation in just our generating function of interest, $t[z]$.

```
In[86]:= eqn = Eliminate[%, {l[z], i[z]}]
```

```
Out[86]= (1 - t[z]) t[z] == z
```

The power series equation $(1 - T(z))T(z) = z$ has two solutions in general, but only one with nonnegative coefficients: $(1/2)(1 - \sqrt{1 - 4z})$. Extracting coefficients with Mathematica's `SeriesCoefficient` and multiplying by $n!$ gives the final answer for the number of leaf-labeled binary trees with n leaves.

```
In[87]:= $Assumptions = n >= 1;
```

```
In[88]:= n! SeriesCoefficient[1/2 (1 - Sqrt[1 - 4 z]),
    {z, 0, n}]
```

```
Out[88]= -(-1)^n 2^(-1 + 2 n) Binomial[1/2, n] n!
```

Mathematica reports here that the formula is

$$-(-1)^n 2^{2n-1} n! \binom{\frac{1}{2}}{n},$$

which is just a more cumbersome form of the expected result

$$\frac{(2(n-1))!}{(n-1)!}.$$

We note that one way to come up with the specification in this solution is to find the counting sequence for leaf-labeled binary trees in the Encyclopedia of Combinatorial Objects at <http://algo.inria.fr/ecs>. It is entry number 51, “Labelled Plane Binary Trees”. The Encyclopedia shows the unlabeled Maple-syntax specification $\{S = \text{Union}(Z, \text{Prod}(S, S))\}$, which we can use to get a specification in Genfunlib via the `ToGenfunlibSpec` command.

```
In[89]:= ToGenfunlibSpec["{S = Union(Z,Prod(S,S))}", False]
```

```
Out[89]= Spec[{ss == SMPlus[ZClass[1], SMTimes[ss, ss]]}, False]
```

Problem 10. Let k and n be integers such that $1 \leq k \leq n$. Consider all finite sequences of positive integers with sum n . Suppose that the term k occurs $T(n, k)$ times in all these sequences. Find $T(n, k)$. [9]

Solution: We start with a specification of the class $\mathcal{S}$ of integer compositions where $\mathcal{Z}_1$ marks the number of parts equal to k , and $\mathcal{Z}_2$ marks the sum of all parts.

$$\mathcal{P}_1 = \text{MSET}_{\geq 1, < k}(\mathcal{Z}_2)$$

$$\mathcal{P}_2 = \text{MSET}_k(\mathcal{Z}_2) \times \mathcal{Z}_1$$

$$\mathcal{P}_3 = \text{MSET}_{> k}(\mathcal{Z}_2)$$

$$\mathcal{S} = \text{SEQ}(\mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3).$$

Here, $\mathcal{P}_1$ is the set of parts smaller than k but no smaller than 1; $\mathcal{P}_2$ is the singleton set containing the part of size k , marked with $\mathcal{Z}_1$; and $\mathcal{P}_3$ is the set of parts larger than k . Now, we ultimately wish to count the number of occurrences of k in all compositions of n . We do this using the symbolic method by defining a class $\mathcal{F}$ each of whose elements corresponds to an occurrence of k in a composition of n . It suffices to define $\mathcal{F}$ as the class obtained by taking $\mathcal{S}$ and *pointing* separately at each atom $\mathcal{Z}_1$, then *substituting* $\mathcal{E}$ for $\mathcal{Z}_1$ so that the objects in $\mathcal{F}$ are only weighted by $\mathcal{Z}_2$, which marks the total size of the composition.

$$\mathcal{F} = (\Theta\mathcal{S}) \circ \mathcal{E}$$

We give the whole specification to Genfunlib and compute the corresponding system of ordinary generating function equations. (We assume $k \geq 2$ for simplicity; the case $k = 1$ can easily be handled separately.)

```
In[90]:= $Assumptions = Element[k, Integers] && k >= 2;
```

```
In[91]:= spec = Spec[{
  p1 == SMMultiset[ZClass[2], Cardinality -> (1 <= # < k &)],
  p2 == SMTimes[ZClass[1],
```

```

SMMultiset[ZClass[2], Cardinality -> (# == k &)]],
p3 == SMMultiset[ZClass[2], Cardinality -> (# > k &)],
s == SMSeq[SMPplus[p1, p2, p3]],
f == SMSub[SMPointing[s], EClass]
}, False];

```

```
In[92]:= eqns = ToGFEqns[spec, z];
```

The full system of equations stored in `eqns` is long, so we do not show it in full. Let $S(z_1, z_2)$ be the ordinary generating function for $\mathcal{S}$ and let $F(z_2)$ be the ordinary generating function for $\mathcal{F}$. We solve the system to obtain $S(z_1, z_2)$, represented in Genfunlib by `s[z[1], z[2]]`.

```

In[93]:= s[z[1], z[2]] = s[z[1], z[2]] /. Solve[Eliminate[
eqns, {p1[z[1], z[2]], p2[z[1], z[2]], p3[z[1], z[2]]}],
s[z[1], z[2]]] // First

```

```

Out[93]= -((-1 + z[2])/(1 - 2 z[2] + z[2]^k - z[1] z[2]^k -
z[2]^(1 + k) + z[1] z[2]^(1 + k)))

```

We manually compute $F(z_2)$, represented by `f[z[2]]`, using `s[z[1], z[2]]` and the equation

$$F(z_2) = (\partial/\partial z_1)S(z_1, z_2)|_{z_1=1}.$$

```
In[94]:= f[z[2]] = D[%, z[1]] /. z[1] -> 1
```

```

Out[94]= ((-1 + z[2]) (-z[2]^k + z[2]^(1 + k)))/(1 - 2 z[2])^2

```

Extracting coefficients from $F(z_2)$ with a command built into Mathematica, we get the final result:

$$T(n, k) = \begin{cases} 2^{-k+n-2}(-k+n+3) & \text{if } k \leq n-2, \\ 2^{n-k} & \text{if } k = n-1, \\ 2^{n-k}(-k+n+1) & \text{if } k = n. \end{cases}$$

Problem 11. An n -term sequence $(x_1, x_2, \dots, x_n)$ in which each term is either 0 or 1 is called a binary sequence of length n . Let a_n be the number of binary sequences of length n containing no three consecutive terms equal to 0, 1, 0 in that order. Let b_n be the number of binary sequences of length n that contain no four consecutive terms equal to 0, 0, 1, 1 or 1, 1, 0, 0 in that order. Prove that $b_{n+1} = 2a_n$ for all positive integers n . (USA Mathematical Olympiad 1996 #4)

Solution: We use the letters a and b to represent 0 and 1, respectively, so n -term binary sequences become n -words over the alphabet $\{a, b\}$.

In an actual math contest, it would be most efficient to solve this problem using a bijection. However, a brute-force solution is feasible if it puts Genfunlib to work.

We define a regular expression r_1 that matches words that avoid aba . In Genfunlib we represent this regular expression by `r1`, which is set to the complement of the regular expression $(a|b)^*aba(a|b)^*$ with respect to the alphabet $\{a, b\}$.

```
In[95]:= r1 = RegComplement[
  ToRegex[RegularExpression["(a|b)*(aba)(a|b)*"]], {"a", "b"}]
```

```
Out[95]= Regex[RegexConcat[
  RegexStar[RegexOr["b", RegexConcat["a", RegexStar["a"]],
```

```
"b", "b"]]], RegexOr[EmptyWord, RegexConcat["a",
RegexStar["a"], RegexOr[EmptyWord, "b"]]]]]]
```

We represent the generating function $\sum_{n \geq 0} a_n z^n$ by `f1`.

```
In[96]:= f1 = FullSimplify[GeneratingFunction[r1,
{"a" -> z, "b" -> z}]]
```

```
Out[96]:= (1 + z^2)/(1 + z (-2 + z - z^2))
```

We define `r2` as the complement of $(a|b)^*aabb(a|b)^*$, and we define `r3` as the complement of $(a|b)^*bbaa(a|b)^*$.

```
In[97]:= r2 = RegComplement[
ToRegex[RegularExpression["(a|b)*(aabb)(a|b)*"]],
{"a", "b"}];
```

```
In[98]:= r3 = RegComplement[
ToRegex[RegularExpression["(a|b)*(bbaa)(a|b)*"]],
{"a", "b"}];
```

We compute the generating function $\sum_{n \geq 0} b_n z^n$, represented by `f2`, by taking the intersection of `r2` and `r3`.

```
In[99]:= f2 = GeneratingFunction[
RegIntersection[r2, r3], {"a" -> z, "b" -> z}]
// FullSimplify
```

```
Out[99]= (1 + z^2 + z^3)/(1 + z (-2 + z - z^2))
```

Now, we can conclude by showing that `f2 - 2*z*f1` is a constant.

```
In[100]:= f2 - 2 z*f1 // FullSimplify
```

```
Out[100]= 1
```

5.2 Miscellaneous problems

Problem 12. A word w avoids a subword pattern p if and only if no contiguous subword (a.k.a factor) of w is equal to p . Let $f_k(n)$ be the number of n -words over the alphabet $\{0,1\}$ that avoid the subword pattern $1^k = 11 \cdots 1$. Find $F_k(z) = \sum_{n \geq 0} f_k(n) z^n$.

Solution: This is a generalization of Problem 2, which is the case $k = 3$. We cannot define automata with parameters such as k , but we can use a recurrence relation to solve this problem instead. An n -word $w \in \{0,1\}^*$ avoids 1^k if and only if $w = 1^j 0 w'$, where $0 \leq j < k$ and w' is a binary $(n - j - 1)$ -word that avoids 1^k . If we define $f(n) = 0$ for all $n < 0$, this leads to a recurrence relation for $f_k(n)$:

$$f_k(n) = \sum_{i=1}^k f_k(n-i) + [n < k], \quad \text{for all } n \in \mathbb{Z}.$$

The $[n < k]$ term accounts for the words 1^n , where $n < k$.

Now we write this equation out in Mathematica using `f[n]` to represent $f(n)$, and apply the Genfunlib-enhanced `GeneratingFunction` command to perform the generating function transform to both sides.

```
In[101]:= $Assumptions =
```

```
Element[i, Integers] && i >= 1 && Element[k, Integers]
&& k >= 1;
```

```
In[102]:= eqn = f[n] == Sum[f[n - i], {i, 1, k}] + Boole[n < k];
```

```
In[103]:= GeneratingFunction[eqn, n, z]
```

```
Out[103]= GeneratingFunction[f[n], n, z] == (-1 + z^k)/(-1 + z) +
Sum[GeneratingFunction[f[-i + n], n, z], {i, 1, k}]
```

For some reason, this output must be evaluated again to get the desired result.

```
In[104]:= GeneratingFunction[f[n], n, z] == (-1 + z^k)/(-1 + z) +
Sum[GeneratingFunction[f[-i + n], n, z], {i, 1, k}]
```

```
Out[104]= GeneratingFunction[f[n], n, z] == (-1 + z^k)/(-1 + z) +
Sum[z^i*GeneratingFunction[f[n], n, z] + Sum[z^(i + n)*f[n],
{n, -i, -1}], {i, 1, k}]
```

We can replace `GeneratingFunction[f[n], n, z]`, which represents $F(z)$, with the shorter expression `ff[z]`; and we can remove remaining `f[n]` terms with zero, because they are all equal to zero.

```
In[105]:= % /. {GeneratingFunction[f[n], n, z] -> ff[z], f[n] -> 0}
```

```
Out[105]= ff[z] == (-1 + z^k)/(-1 + z) + (z (-1 + z^k) ff[z])/(-1 + z)
```

In mathematical notation, this is

$$F(z) = \frac{zF(z)(z^k - 1)}{z - 1} + \frac{z^k - 1}{z - 1}.$$

All that remains is to solve this linear equation.

```
In[106]:= ff[z] /. Solve[%, ff[z]] // First
```

```
Out[106]= (1 - z^k)/(1 - 2 z + z^(1 + k))
```


So the final answer is

$$F(z) = \frac{1 - z^k}{z^{k+1} - 2z + 1}.$$

Problem 13. Let $f(n)$ be the number of n -words without neighboring zeros from the alphabet $\{0, 1, 2\}$. Find an asymptotic expression for $f(n)$ as n goes to infinity.

Solution: This $f(n)$ is the same as the one from Problem 8, where we discovered that the ordinary generating function for $f(n)$ is

$$\sum_{n \geq 0} f(n)z^n = \frac{1 + z}{1 - 2z - 2z^2}.$$

The required asymptotic expression is extracted by simply applying Genfunlib's `CoefLimit` command to the generating function.

```
In[107]:= CoefLimit[(1 + z)/(1 - 2 z - 2 z^2), z, n]
```

```
Out[107]= ((1/2 (-1 + Sqrt[3]))^(-1 -
n) (1 + 1/2 (-1 + Sqrt[3])))/(2 + 2 (-1 + Sqrt[3]))
```

After automatic simplification, this says

$$f(n) \sim \frac{1}{3} \left(3 + \sqrt{3}\right) 2^{n-1} \left(\sqrt{3} - 1\right)^{-n-1}, \text{ as } n \rightarrow \infty.$$

Problem 14. Define the class of plane trees, where $\mathcal{Z}_1$ marks nodes and $\mathcal{Z}_2$ marks path length, by

$$\mathcal{T} = \mathcal{Z}_1 \times \text{SEQ}(\mathcal{T} \circ (\mathcal{Z}_1 \times \mathcal{Z}_2)).$$

The path length of a tree is the sum of the distances of all nodes to the root.

1. Find an asymptotic expression for the number of plane trees with n nodes, as n goes to infinity.
2. Find the number of plane trees with 7 nodes and pathlength 8.
3. Find an asymptotic expression for the average path length of a plane tree with n nodes, as n goes to infinity.

Solution: To solve part 1, we remove $\mathcal{Z}_2$ from the specification and use Genfunlib's `ToGFEqns` command to get an equation in $T(z_1)$, which is represented by `t[z[1]]`.

```
In[108]:= spec = Spec[{
    t == SMTimes[ZClass[1], SMSeq[t]]
}, False];
```

```
In[109]:= treeEqn = First[ToGFEqns[spec, z]]
```

```
Out[109]= t[z[1]] == z[1]/(1 - t[z[1]])
```

The returned equation (`treeEqn`), which is $t(z_1) = z_1/(1 - t(z_1))$, can be given to Genfunlib's `TreeAsymptot` command to compute an asymptotic expression for $[z_1^n]t(z_1)$.

```
In[110]:= TreeAsymptot[treeEqn, n]
```

```
Out[110]= 4^n ((1/n)^(3/2)/(4 Sqrt[Pi])+O[1/n]^(5/2))
```

Converted to L^AT_EX, this says that the number of plane trees with n nodes is

$$[z_1^n]T(z_1) = 4^n \left(\frac{\left(\frac{1}{n}\right)^{3/2}}{4\sqrt{\pi}} + O(n^{-5/2}) \right).$$

Now, we assign the full specification, including Z_2 , to the `augSpec` variable, and save the corresponding bivariate ordinary generating function equation in the `eqn` variable.

```
In[111]:= augSpec = Spec[{
  t == SMTimes[ZClass[1],
    SMSeq[SMSub[t, SMTimes[ZClass[1], ZClass[2]]]]
  ] }, False];
```

```
In[112]:= eqn = First[ToGFEqns[augSpec, z]]
```

```
Out[112]= t[z[1], z[2]] == z[1]/(1 - t[z[1] z[2], z[2]])
```

That is, we have

$$T(z_1, z_2) = \frac{z_1}{1 - T(z_1 z_2, z_2)}. \quad (2)$$

Equation (2) is all we need to answer part 2, since we can pass it to Genfunlib's `CoefsByDerivs` command to compute the initial terms of $t[z[1], z[2]]$.

```
In[113]:= CoefsByDerivs[eqn, t[z[1], z[2]], {z[1], 0, 7},
  {z[2], 0, 8}]
```

```
Out[113]= {t[z[1], z[2]] == 0[z[2]]^9 + (1 + 0[z[2]]^9) z[1] +
  (z[2] + 0[z[2]]^9) z[1]^2 + (z[2]^2 + z[2]^3 + 0[z[2]]^9) z[1]^3 +
  (z[2]^3 + 2 z[2]^4 + z[2]^5 + z[2]^6 + 0[z[2]]^9) z[1]^4 +
  (z[2]^4 + 3 z[2]^5 + 3 z[2]^6 + 3 z[2]^7 + 2 z[2]^8 +
  0[z[2]]^9) z[1]^5 + (z[2]^5 + 4 z[2]^6 + 6 z[2]^7 +
  7 z[2]^8 + 0[z[2]]^9) z[1]^6 + (z[2]^6 + 5 z[2]^7 + 10 z[2]^8 +
  0[z[2]]^9) z[1]^7 + 0[z[1]]^8}
```

This says $[z_1^7 z_2^8]T(z_1, z_2) = 10$.

For part 3, we need to compute (an asymptotic estimate of)

$$\frac{[z_1^n]T(z_1, 1)}{[z_1^n] \frac{\partial}{\partial z_2} T(z_1, z_2) \Big|_{z_2=1}}.$$

We set z_2 to 1 on both sides of Equation (2), and save the result in `eqn1`. Then we solve that equation for $T(z_1, 1)$ and store the solution as `t00`.

```
In[114]:= eqn1 = eqn /. z[2] -> 1;
```

```
In[115]:= t00 = First[t[z[1], 1] /. Solve[eqn1, t[z[1], 1]]]
```

```
Out[115]= 1/2 (1 - Sqrt[1 - 4 z[1]])
```

We define `eqn2` as the equation obtained by differentiating both sides of Equation (2) with respect to z_2 , and setting z_2 to 1.

```
In[116]:= eqn2 = D[eqn, z[2]] /. z[2] -> 1;
```

We define `eqn3` as the equation obtained by differentiating both sides of Equation 2 with respect to z_1 .

```
In[117]:= eqn3 = D[eqn1, z[1]];
```

Finally, we define `eqn4` as the equation obtained by eliminating $T(z_1, 1)$ and $\frac{\partial}{\partial z_1} T(z_1, 1)$ from the system `eqn1 && eqn2 && eqn3`.

```
In[118]:= eqn4 = Eliminate[
    eqn1 && eqn2 && eqn3, {t[z[1], 1],
    Derivative[1, 0][t][z[1], 1]};
```

As `eqn1` is for $T(z_1, 1)$, `eqn4` is a single equation in which the only unknown is $\frac{\partial}{\partial z_2} T(z_1, z_2) \Big|_{z_2=1}$. We solve `eqn4` for $\frac{\partial}{\partial z_2} T(z_1, z_2) \Big|_{z_2=1}$ and save the solution in `t01`.

```

In[119]:= t01 = PowerExpand[ Simplify[First[(t^(0, 1))[z[1], 1] /.
      Solve[eqnt4, Derivative[0, 1][t][z[1], 1]]]]];
Out[119]= (z[1] - 4 z[1]^2 - I z[1] (-1 + 4 z[1])^(3/2))/
      (2 (1 - 4 z[1])^2)

```

Now, Genfunlib does not have the ability to obtain an asymptotic expression for the coefficients of `t00` or `t01`, but Mathematica does. We define the function `coefAsympt` which uses Mathematica's `Series` and `SeriesCoefficient` commands to extract an expression for coefficient `n` of a power series, and convert to an asymptotic form.

```

In[120]:= $Assumptions = n > 1 && Element[n, Integers];

```

```

In[121]:= coefAsympt[series_, var_] := FullSimplify[
      Series[SeriesCoefficient[series, {var, 0, n}],
      {n, Infinity, 1}]];

```

All that remains is to apply `coefAsympt` to `t00` and `t01` and divide, obtaining the answer to part 3.

```

In[122]:= coefAsympt[t01,z[1]]/coefAsympt[t00,z[1]]
Out[122]= 1/2 Sqrt[Pi] n^(3/2)-n/2+1/Sqrt[0[1/n]]

```

That is, as n goes to infinity,

$$\frac{[z_1^n]T(z_1, 1)}{[z_1^n] \left. \frac{\partial}{\partial z_2} T(z_1, z_2) \right|_{z_2=1}} = \frac{1}{2} \sqrt{\pi} n^{3/2} - \frac{n}{2} + O(\sqrt{n}).$$

6 Further work

In terms of further work, there is a number of possible directions. As mentioned previously, Mathematica itself causes difficulties when one wants to work with general big-O expressions, iterated sums with a symbolic number of dummy variables, and recurrence relations with polynomial coefficients. To get around these issues, perhaps a package could be written to provide alternatives or enhancements to the core Mathematica commands involved. We also mentioned that performance in Genfunlib should be improved across the board, and specifically in the area of regular languages. Multi-dimensional Newton iteration for computing coefficients from transcendental power series equations could be added.

Many counting problems involve not regular languages but context-free languages. Context-free languages do not have all the decidable properties that regular languages do, e.g. testing whether a context-free grammar is not decidable, and neither is testing whether a context-free grammar has an equivalent deterministic context-free grammar. However, if grammars are simply assumed to be unambiguous, something could be built, similar to Genfunlib's symbolic method functionality, to obtain generating functions that count the number of words of length n in a context-free language.

Genfunlib focuses on *counting* combinatorial objects, but there are also the problems of enumeration and random generation. For families of objects represented by combinatorial species, a recent work by Pivoteau et. al [26] presents algorithms based on combinatorial Newton iteration [1] that are relevant to both of these problems. So far, the algorithm for random generation with species using Boltzmann sampling has not been implemented.

Extracting an asymptotic estimate for the coefficients $[z^n]F(z)$ of a power series is an interesting problem that Genfunlib approaches only on a relatively

basic level. Algorithms that handle a larger set of power series $F(z)$ are needed. In addition, algorithms and implementations for multivariate asymptotics would be useful. Some results in this area have been found [24, 28], and Raichev’s amgf package for Sage implements a method for computing the asymptotic expansion of

$$[z_1^{r\alpha_1} z_2^{r\alpha_2} \dots] F(z_1, z_2, \dots),$$

as $r \rightarrow \infty$, for a certain class of multivariate power series $F(z_1, z_2, \dots)$.

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Appendix

Asymptotics.m

```
(* Mathematica Package *)
```

```
BeginPackage["Genfunlib`Asymptotics`"]
```

```
TreeAsymptot::usage = "TreeAsymptot[y[z]==z*phi,n] gives an asymptotic " <>
"series for SeriesCoefficient[y[z], {z,0,n}] as " <>
"n goes to infinity, where y[z] is the solution to y[z]==z*phi that is " <>
"analytic at 0.";
```

```
CoefLimit::usage = "CoefLimit[f,z,n] gives an expression that is " <>
"asymptotically equivalent to SeriesCoefficient[f, {z,0,n}] as n goes to " <>
"infinity.";
```

```
Begin["`Private`"]
```

```

(* define GCD and Mod when Infinity is an argument *)
protected = Unprotect[GCD, Mod];

GCD[Infinity, rest__] := GCD[rest];

GCD[0] := Infinity;

Mod[n_, Infinity] := n;

Protect[Evaluate[protected]];

(* validation of equations given as input to asymptotic functions *)

TreeAsymptot::invalid = "Not an elementary functional equation.";
CoefLimit::invalid = "Not an elementary function.";
TreeAsymptot::invalidArgumentSyntax = "Invalid argument syntax.";
CoefLimit::invalidArgumentSyntax = "Invalid argument syntax.";

(* what can usually be used as a variable, according to
   ref/message/General/ivar *)
variablePattern = Except[_String | _?NumberQ | _Plus | _Times |
    _Sum | _Product | _^_Integer];

validatePhi[expr_, y_[z_]] := Module[
    {
        ok = True
    },
    ],

```

```

(* must not be deg 1 polynomial *)
ok = ok && !(PolynomialQ[expr, y[z]] && Exponent[expr, y[z]] <= 1);
ok = ok && SeriesCoefficient[expr, {y[z], 0, 0}] != 0;
(* and elementary function *)
ok = ok && ea[expr, y[z]];

If[ ok != True, Message[TreeAsymptot::invalid]];
ok
];

validateExpr[expr_, x_] := Module[
{
    ok = True
},
ok = ok && ea[expr, x];

If[ ok != True, Message[CoefLimit::invalid]];
ok
];

(* ea tests whether an expression is an "elementary function" *)

ea[expr_, x_] /; PolynomialQ[expr, x] := Module[
{
    ok = True
},
ok = ok &&
    And @@ ((Element[#, Rationals] && # >= 0) & /@ CoefficientList[expr, x]);
ok = ok && CoefficientList[expr, x] != {0};
ok
];

```

```

];

ea[1/(1 + arg_), x_] := Module[
{
  ok = True
},
ok = ok && ea[-arg, x];
ok = ok && (Limit[arg, x -> 0] === 0);
ok
];

ea[Log[1/(1 + arg_)], x_] := Module[
{
  ok = True
},
ok = ok && ea[-arg, x];
ok = ok && (Limit[arg, x -> 0] === 0);
ok
];

ea[Exp[arg_], x_] := Module[
{
  ok = True
},
ok = ok && ea[arg, x];
ok = ok && (Limit[arg, x -> 0] === 0);
ok
];

ea[arg1_ + arg2_, x_] := ea[arg1, x] && ea[arg2, x];

```

```

ea[arg1_ * arg2_, x_] := ea[arg1, x] && ea[arg2, x];

ea[Power[expr_, j_Integer?Positive], x_] := ea[expr, x];
ea[Power[expr_, j_Integer?(#<-1&)], x_] := ea[1/expr, x];

ea[___] := False;

exponents[y_, x_] :=
  Union[Flatten[CoefficientRules[y, x][[All, 1]]]];

(* reduction implements the algorithm in Flajolet et al. (Automatic *)
(* average-case analysis of algorithms) on p78 *)
reduction[y_, x_] := Module[
  {exps = exponents[y, x]},
  (
    If[Length[exps] == 1,
      {First[exps], Infinity},
      {Mod[First[exps], GCD @@ Differences[exps]],
       GCD @@ Differences[exps]}
    ]
  ) /; PolynomialQ[y, x]
];

reduction[y : 1/(1 + arg_), x_] := Module[
  {
    arga, argp
  },
  {arga, argp} = reduction[-arg, x];
  {0, GCD[arga, argp]}
];

```

```

];

reduction[y : Log[1/(1 + arg_)], x_] := Module[
{
  arga, argp
},
{arga, argp} = reduction[-arg, x];
{0, GCD[arga, argp]}
];

reduction[y : Exp[arg_], x_] := Module[
{
  arga, argp
},
{arga, argp} = reduction[arg, x];
{0, GCD[arga, argp]}
];

(* some details of this algorithm in Flajolet et al. were left out. to prove *)
(* that the following is correct, *)
(* give equivalent definitions of a and p in terms of GCD and Mod, *)
(* show that the following supplies that; note: term1a=term2a (mod pp) *)

reduction[term1_ + term2_, x_] := Module[
{
  term1a, term1p, term2a, term2p,
  pp
},
{term1a, term1p} = reduction[term1, x];
{term2a, term2p} = reduction[term2, x];

```



```

pp = GCD[term1p, term2p, term1a - term2a];
{Mod[term1a, pp], pp}
];

reduction[term1_ * term2_, x_] := Module[
{
term1a, term1p, term2a, term2p,
pp
},
{term1a, term1p} = reduction[term1, x];
{term2a, term2p} = reduction[term2, x];
pp = GCD[term1p, term2p];
{Mod[term1a + term2a, pp], pp}
];

reduction[Power[expr_, j_Integer?Positive], x_] := reduction[Unevaluated[
expr * Power[expr, j - 1]], x];
reduction[Power[expr_, j_Integer?(#<-1&)], x_] := reduction[Unevaluated[
Power[expr, -1] * Power[expr, j + 1]], x];

(* assumes expr is from elementary iterative class, so characteristic \
eqn is satisfied and period is computed *)
TreeAsymptot[(y:variablePattern)[z:variablePattern] ==
(z:variablePattern) * expr_, n:variablePattern] := Module[
{
period = reduction[expr, y[z]], p,
t, res, nres,
d1, r
},
(

```

```

p = If[period[[2]] == Infinity,
  period[[1]], period[[2]]];
(* solve characteristic equation *)
res = Solve[expr - y[z] * D[expr, y[z]] == 0 && y[z] > 0, y[z]];
If[MatchQ[res, {__List}],
  t = Min[y[z] /. res],
  nres =
    NSolve[expr - y[z]*D[expr, y[z]] == 0 && y[z] > 0, y[z]];
  t = Min[y[z] /. nres]
];
r = t / (expr /. y[z] -> t);
d1 = Sqrt[(2 * (expr /. y[z] -> t))/(
  D[expr, {y[z], 2}] /. y[z] -> t)];
If[p >= 2,
  Boole[Mod[n, p] == 1] * r ^ -n *
    SeriesData[n, DirectedInfinity[1], {(p*d1)/(2*Sqrt[Pi]), 0},
      3, 5, 2],
  r^-n*
    SeriesData[n, DirectedInfinity[1], {d1/(2*Sqrt[Pi]), 0}, 3,
      5, 2]
]
) /; validatePhi[expr, y[z]]
];
TreeAsymptot[___] /; (Message[TreeAsymptot::invalidArgumentSyntax]; False) :=
  Null;

(* radius implements the algorithm from Flajolet et al. on p77 *)

radius[y_, x_] /; PolynomialQ[y, x] := Infinity;

```

```
radius[Exp[y_], x_] := radius[y, x];
```

```
radius[1/(1 + y_), x_] := Module[
{
  res, r, nres
},
res = Solve[-y == 1 && x > 0, x];
r = If[MatchQ[res, {__List}],
  Min[x /. res],
  nres = NSolve[-y == 1 && x > 0, x];
  Min[x /. nres]
];
r
];
```

```
radius[Log[1/(1 + y_)], x_] := Module[
{
  res, r, nres
},
res = Solve[-y == 1 && x > 0, x];
r = If[MatchQ[res, {__List}],
  Min[x /. res],
  nres = NSolve[-y == 1 && x > 0, x];
  Min[x /. nres]
];
r
];
```

```
radius[term1_ + term2_, x_] := Min[radius[term1, x], radius[term2, x]];
```

```
radius[factor1_*factor2_, x_] := Min[radius[factor1, x], radius[factor2, x]];
```

```
radius[Power[expr_, j_Integer?Positive], x_] := radius[Unevaluated[
    expr * Power[expr, j - 1]], x];
```

```
radius[Power[expr_, j_Integer?(#<-1&)], x_] := radius[Unevaluated[
    Power[expr, -1] * Power[expr, j + 1]], x];
```

```
(* directions implements the algorithm from Flajolet et al. on p79 *)
```

```
directions[Exp[expr_], x_] := directions[expr, x];
```

```
directions[1/(1 + expr_), x_] := Module[
```

```
{
    a, p, q
},
{a, p} = reduction[-expr, x];
q = GCD[a, p];
2*Range[1, q]*Pi/q
];
```

```
directions[Log[1/(1 + expr_)], x_] := Module[
```

```
{
    a, p, q
},
{a, p} = reduction[-expr, x];
q = GCD[a, p];
2*Range[1, q]*Pi/q
];
```

```
directions[Plus[term1_, term2_], x_] := Module[
```

```

{
  r1 = radius[term1, x],
  r2 = radius[term2, x]
},
If[r1 < r2,
  Return[directions[term1, x]]
];
If[r1 > r2,
  Return[directions[term2, x]]
];
Return[Union[directions[term1, x], directions[term2, x]]];
];

directions[Times[term1_, term2_], x_] := Module[
{
  r1 = radius[term1, x],
  r2 = radius[term2, x]
},
If[r1 < r2,
  Return[directions[term1, x]]
];
If[r1 > r2,
  Return[directions[term2, x]]
];
Return[Union[directions[term1, x], directions[term2, x]]];
];

directions[Power[expr_, j_Integer?Positive], x_] := directions[Unevaluated[
  expr * Power[expr, j - 1]], x];
directions[Power[expr_, j_Integer?(#<-1&)], x_] := directions[Unevaluated[

```

```

Power[expr, -1] * Power[expr, j + 1]], x];

(* allimit implements the Expansion algorithm from Flajolet et al. on p81. *)
(* it returns a 4-tuple {a, b, c, d}, where the expansion is *)
(* a*(1-z/d)^b*Log[1-z/d]^c *)

allimit[1/(1 + expr_), x_] := Module[
{
  r = radius[1/(1 + expr), x]
},
{1/(r*D[-expr, x] /. x -> r), -1, 0, r}
];

allimit[Log[1/(1 + expr_)], x_] := Module[
{
  r = radius[Log[1/(1 + expr)], x]
},
{1, 0, 1, r}
];

allimit[Exp[expr_], x_] := Module[
{
  expra, exprb, exprc, exprr,
  r = radius[Exp[expr], x]
},
{exprc, expra, exprb, exprr} = allimit[expr, x];
If[MatchQ[{exprc, expra, exprb, exprr}, {_, 0, 1, r}],
{Exp[exprb], -expra, 0, r},
Throw["experr"]
]

```

```

];

alllimit[term1_ + term2_, x_] := Module[
{
    term1limit, term1a, term1b, term1c, term1r,
    term2limit, term2a, term2b, term2c, term2r
},

term1limit = Catch[allimit[term1, x]];
term2limit = Catch[allimit[term2, x]];
If[term1limit == term2limit == "experr",
    Throw["experr"]
];

If[term1limit == "experr",
    If[term2limit[[4]] < radius[term1, x],
        Return[term2limit],
        Throw["experr"]
    ]
];

If[term2limit == "experr",
    If[term1limit[[4]] < radius[term2, x],
        Return[term1limit],
        Throw["experr"]
    ]
];

{term1c, term1a, term1b, term1r} = term1limit;
{term2c, term2a, term2b, term2r} = term2limit;

```

```

If[term1r < term2r,
  Return[term1limit];
];
If[term2r > term1r,
  Return[term1limit];
];
If[term1a < term2a,
  Return[term1limit];
];
If[term2a < term1a,
  Return[term2limit];
];
If[term1b > term2b,
  Return[term1limit];
];
If[term2b > term1b,
  Return[term2limit];
];
{term1c + term2c, term1a, term1b, term1r}
];

allimit[factor1_ * factor2_, x_] := Module[
{
  factor1limit, factor1a, factor1b, factor1c, factor1r,
  factor2limit, factor2a, factor2b, factor2c, factor2r
},
factor1limit = Catch[allimit[factor1, x]];
factor2limit = Catch[allimit[factor2, x]];

If[factor1limit == factor2limit == "experr",

```



```

        Throw["experr"]
];

If[factor1limit == "experr",
    If[factor2limit[[4]] < radius[factor1, x],
        {factor2c, factor2a, factor2b, factor2r} = factor2limit;
        Return[{factor2c * Limit[factor1, x -> factor2r], factor2a, factor2b,
            factor2r}],
        Throw["experr"]
    ]
];

If[factor2limit == "experr",
    If[factor1limit[[4]] < radius[factor2, x],
        {factor1c, factor1a, factor1b, factor1r} = factor1limit;
        Return[{factor1c * Limit[factor2, x -> factor1r], factor1a, factor1b,
            factor1r}],
        Throw["experr"]
    ]
];

{factor1c, factor1a, factor1b, factor1r} = factor1limit;
{factor2c, factor2a, factor2b, factor2r} = factor2limit;

If[factor1r < factor2r,
    Return[{factor1c*(factor2 /. x -> factor1r), factor1a, factor1b,
        factor1r}];
];

If[factor2r < factor1r,
    Return[{(factor1 /. x -> factor2r)*factor2c, factor2a, factor2b,
        factor2r}];
];

```

```

];

{factor1c*factor2c, factor1a + factor2a, factor1b + factor2b, factor1r}

];

allimit[Power[expr_, j_Integer?Positive], x_] := allimit[Unevaluated[
    expr * Power[expr, j - 1]], x];
allimit[Power[expr_, j_Integer?(#<-1&)], x_] := allimit[Unevaluated[
    Power[expr, -1] * Power[expr, j + 1]], x];

allimit[poly_, x_] /; PolynomialQ[poly,x] := {poly /. x-> Infinity,
    0, 0, Infinity};

CoeftLimit[expr_, x:variablePattern, n:variablePattern] := Module[
{
    r = radius[expr, x],
    dirs = directions[expr, x],
    expra, exprb, exprc, exprr, exprlimit
},
(
    If[dirs != {2 * Pi} || r == Infinity,
        Return[SeriesCoefficient[expr, {x, 0, n}]];
    ];
    exprlimit = Catch[allimit[expr, x]];
    If[exprlimit == "experr",
        Return[SeriesCoefficient[expr, {x, 0, n}]]
    ];
    {exprc, expra, exprb, exprr} = exprlimit;
    If[expra < 0,
        exprc*exprr^-n (n^-expra Log[n]^exprb)/(n*Gamma[-expra]),
        Return[SeriesCoefficient[expr, {x, 0, n}]];
    ]
];

```

```

]
) /; validateExpr[expr, x]
];
CoefLimit[___] /; (Message[CoefLimit::invalidArgumentSyntax]; False) := Null;

End[]
EndPackage[]

```

GFeqn2rec.m

```

(* Mathematica Package *)

BeginPackage["Genfunlib`GFeqn2rec`"]

(* this file defines no public functions, however it does *)
(* override SeriesCoefficient *)

$FullAnalytic::usage = "$FullAnalytic is True iff SeriesCoefficient should " <>
    "assume that " <>
    "all subexpressions of the expression to be expanded are " <>
    "analytic, when the " <> "expansion is around 0.";

Begin["`Private`"]

protected = Unprotect[SeriesCoefficient];

$FullAnalytic = False;

(* what can usually be used as a variable, according to
    ref/message/General/ivar *)
variablePattern = Except[_String | _?NumberQ | _Plus | _Times |

```

```

    _Sum | _Product | _^_Integer];

arg2 = {x:variablePattern, 0, n_};

(* coefficients of negative or noninteger powers are 0 *)
SeriesCoefficient[_, arg2, opts:OptionsPattern[]] := Module[
    {},
    0 /; NumericQ[n] && (n < 0 || !IntegerQ[n]) && $FullAnalytic
];

(* addition *)
SeriesCoefficient[expr1_ + expr2_, arg2, opts:OptionsPattern[]] :=
    SeriesCoefficient[expr1, {x, 0, n}, opts] + SeriesCoefficient[expr2,
    {x, 0, n}, opts];

(* scaling (can produce simpler expressions than general multiplication) *)
SeriesCoefficient[fac_ * expr_, arg2, opts:OptionsPattern[]] /; FreeQ[fac, x] &&
    !FreeQ[expr, x] :=
    fac * SeriesCoefficient[expr, {x, 0, n}, opts];

(* shifting (can produce simpler expressions than multiplication) *)
SeriesCoefficient[x_^(k_) * expr_, arg2,
    opts:OptionsPattern[]] := Module[
    {},
    (
    Boole[n - k >= 0] * SeriesCoefficient[expr,
        {x, 0, n - k}, opts]
    ) /; !FreeQ[expr, x] && $FullAnalytic && FreeQ[k, x]
];

```

```

(* multiplication *)

(* doesn't seem to interfere with things like E^x*1/(1-x) *)
SeriesCoefficient[expr1_ * expr2_, arg2, opts:OptionsPattern[]] := With[
{
  iterator = Unique[]
},
Module[{
  newOpts = Sequence@@(DeleteCases[{opts}, Verbatim[Rule][Assumptions, _]] ~
  Append ~ (Assumptions ->
  OptionValue[Assumptions] && Element[iterator, Integers] && iterator >= 0))
},
  Sum[SeriesCoefficient[expr1, {x, 0, iterator}, newOpts]
    * SeriesCoefficient[expr2, {x, 0, n - iterator}, newOpts],
    {iterator, 0, n}]
] /; !FreeQ[expr1, x] && !FreeQ[expr2, x] && $FullAnalytic
];

```

```

(* derivatives and integrals *)

SeriesCoefficient[Derivative[k_][a_Symbol][(j_:1) * x_], arg2,
opts:OptionsPattern[]] :=
  Pochhammer[n + 1, k] j^n SeriesCoefficient[a[x], {x, 0, n + k}, opts] /;
  Simplify[Implies[OptionValue[Assumptions], k > 0]] && FreeQ[j, x];

SeriesCoefficient[Derivative[0, k2_][a_Symbol][j_ * x_, expr_], arg2,
opts:OptionsPattern[]] :=
  j^n SeriesCoefficient[Derivative[0, k2][a][x, expr], {x, 0, n}, opts] /;
  Simplify[Implies[OptionValue[Assumptions], k2 > 0]] && FreeQ[expr, x] &&
  FreeQ[j, x];

```

```

SeriesCoefficient[Derivative[k1_, 0][a_Symbol][expr_, j_ * x_], arg2,
opts:OptionsPattern[]] :=
  j^n SeriesCoefficient[Derivative[k1, 0][a][expr, x], {x, 0, n}, opts] /;
  Simplify[Implies[OptionValue[Assumptions], k1 > 0]] && FreeQ[expr, x] &&
  FreeQ[j, x];

SeriesCoefficient[Derivative[k1_, k2_][a_Symbol][(j_:1) * x_, expr_], arg2,
opts:OptionsPattern[]] :=
  Pochhammer[n + 1, k1] * j^n * SeriesCoefficient[
    Derivative[0, k2][a][x, expr], {x, 0, n + k1}, opts] /;
  Simplify[Implies[OptionValue[Assumptions], k1 > 0 && k2 >= 0]] &&
  FreeQ[expr, x] && FreeQ[j, x];

SeriesCoefficient[Derivative[k2_, k1_][a_Symbol][expr_, (j_:1) * x_], arg2,
opts:OptionsPattern[]] :=
  Pochhammer[n + 1, k1] * j^n * SeriesCoefficient[
    Derivative[k2, 0][a][expr, x], {x, 0, n + k1}, opts] /;
  Simplify[Implies[OptionValue[Assumptions], k1 > 0 && k2 >= 0]] &&
  FreeQ[expr, x] && FreeQ[j, x];

SeriesCoefficient[Derivative[k1_, k2_][a_Symbol][(j_:1) * x_, (k_:1) * x_],
arg2, opts:OptionsPattern[]] :=
  With[
    {iterator = Unique[], first = Unique[], second = Unique[]},
    (
      Sum[k^iterator * j^(n - iterator) *
        Pochhammer[iterator + 1, k1] * Pochhammer[n - iterator + 1, k2]
        * SeriesCoefficient[a[first, second], {first, 0, iterator + k1},
          {second, 0, n - iterator + k2}, opts], {iterator, 0, n}]
    ) /; $FullAnalytic &&

```

```

Simplify[Implies[OptionValue[Assumptions], k1 >= 0 && k2 >= 0]] &&
  FreeQ[j, x] && FreeQ[k, x];
];

SeriesCoefficient[Integrate[expr_,
  {t_, 0, x_}], arg2, opts:OptionsPattern[]] := Module[
  {},
  (
  Piecewise[{
    {
      (1/n) * SeriesCoefficient[expr /. t -> x, {x, 0, n - 1}, opts],
      n >= 1
    }
  }]
  ) /; $FullAnalytic
];

(* powers of functions *)
SeriesCoefficient[a_Symbol[args_]^(k_Integer?Positive), arg2,
  opts:OptionsPattern[]] /; !FreeQ[{args}, x] :=
  SeriesCoefficient[
    Unevaluated[a[args] * a[args]^(k - 1)],
    {x, 0, n}, opts
  ];

(* compositions *)
SeriesCoefficient[a_Symbol[k_ * x_], arg2, opts:OptionsPattern[]] /;
  FreeQ[k, x] := k^n * SeriesCoefficient[a[x], {x, 0, n}, opts];

SeriesCoefficient[a_Symbol[(k_:1) * x_, (j_:1) * x_], arg2,

```

```

opts:OptionsPattern[] := With[
{iterator = Unique[], first = Unique[], second = Unique[]},
(
Sum[k^iterator * j^(n - iterator) *
SeriesCoefficient[a[first, second], {first, 0, iterator},
{second, 0, n - iterator}, opts], {iterator, 0, n}]
) /; $FullAnalytic && FreeQ[k, x] && FreeQ[j, x]
];

SeriesCoefficient[a_Symbol[k_*x_, j_], arg2, opts:OptionsPattern[]] /;
FreeQ[j, x] && FreeQ[k, x] :=
k^n SeriesCoefficient[a[x, j], {x, 0, n}, opts];
SeriesCoefficient[a_Symbol[j_, k_*x_], arg2, opts:OptionsPattern[]] /;
FreeQ[j, x] && FreeQ[k, x] :=
k^n SeriesCoefficient[a[j, x], {x, 0, n}, opts];

(* misc *)
SeriesCoefficient[Sum[expr_, {i_, lb_, ub_}], arg2, opts:OptionsPattern[]] /;
FreeQ[lb, x] && FreeQ[ub, x] :=
Sum[SeriesCoefficient[expr, {x, 0, n}, opts], {i, lb, ub}];

SeriesCoefficient[expr_, {x:variablePattern, 0, n_}, iters__List,
opts:OptionsPattern[]] :=
SeriesCoefficient[SeriesCoefficient[expr, {x, 0, n}, opts], iters,
opts];

Protect[Evaluate[protected]];

End[]
EndPackage[]

```


RegularLanguages.m

(* Mathematica Package *)

BeginPackage["Genfunlib`RegularLanguages`"]

NFA::usage = "NFA[n, alphabet, transitions, acceptStates, initialState] is " <>

"an NFA " <> "with states [1..n], and transitions given by a matrix.";

DFA::usage = "DFA[n, alphabet, transitions, acceptStates, initialState] is " <>

"a DFA " <> "with states [1..n], and transitions given by a matrix.";

Regex::usage = "Regex is a wrapper for expressions built with strings, " <>

"RegexOr, " <> "RegexConcat, RegexStar, and EmptyWord.";

RRGrammar::usage = "RRGrammar[{lhs -> rhs, ...}] is a right regular " <>

"grammar given " <> "by a list of productions.";

Digraph::usage = "Digraph[graph, startVertices, endVertices, " <>

"\[Epsilon]Accepted] " <>

"is a directed graph with labeled vertices for counting " <>

"walks " <> "from start vertices to end vertices.";

ToNFA::usage = "ToNFA[DFA[...]] is an NFA defined from a DFA.\n" <>

"ToNFA[Regex[...]] is an NFA defined from a symbolic " <>

"regular expression.\n" <>

"ToNFA[RRGrammar[...]] is an NFA defined from a right " <>

"regular grammar.\n" <>

"ToNFA[Digraph[...]] is an NFA defined from a digraph with " <>

"labeled vertices.";

ToDFA::usage = "ToDFA[NFA[...]] is a DFA defined from a NFA.\n" <>

"ToDFA[Regex[...]] is a DFA defined from a symbolic " <>

"regular expression.\n" <>

"ToDFA[RRGrammar[...]] is a DFA defined from a right " <>

"regular grammar.\n" <>

```

    "ToDFA[Digraph[...]] is a DFA defined from a digraph with " <>
    "labeled vertices.";
ToRegex::usage = "ToRegex[NFA[...]] is a symbolic regular expression " <>
    "defined from an NFA.\n" <>
    "ToRegex[DFA[...]] is a symbolic regular expression defined from a DFA.\n" <>
    "ToRegex[RegularExpression[\"regex\"] is a symbolic regular expression " <>
    "defined from a restricted Mathematica regular expression.\n" <>
    "ToRegex[RRGrammar[...]] is a symbolic regular expression defined " <>
    "from a right regular grammar.\n" <>
    "ToRegex[Digraph[...]] is a symbolic regular expression defined from a " <>
    "digraph with labeled vertices.";
ToRRGrammar::usage = "ToRRGrammar[NFA[...]] is a right regular grammar " <>
    "defined from a NFA.\n" <>
    "ToRRGrammar[DFA[...]] is a right regular grammar defined from a DFA.\n" <>
    "ToRRGrammar[Regex[...]] is a right regular grammar defined from a " <>
    "symbolic regular expression.\n" <>
    "ToRRGrammar[Digraph[...]] is a right regular grammar defined from a " <>
    "digraph with labeled vertices.";
ToDigraph::usage = "ToDigraph[NFA[...]] is a directed graph defined " <>
    "from an NFA.\n" <>
    "ToDigraph[DFA[...]] is a directed graph defined from a DFA.\n" <>
    "ToDigraph[Regex[...]] is a directed graph defined from a " <>
    "symbolic regular expression.\n" <>
    "ToDigraph[RRGrammar[...]] is a directed graph defined " <>
    "from a right regular grammar.";

RegStar::usage = "RegStar[NFA[...]] performs the star operation on an NFA.\n" <>
    "RegStar[DFA[...]] performs the star operation on a DFA.\n" <>
    "RegStar[Regex[...]] performs the star operation on a symbolic " <>
    "regular expression.\n" <>

```

```

"RegStar[RRGrammar[...]] performs the star operation on a right " <>
    "regular grammar.\n" <>
"RegStar[Digraph[...]] performs the star operation on a directed graph.";

RegComplement::usage = "RegComplement[NFA[...], {\\"str1\\", \\"str2\\",...}] " <>
    "performs the " <>
    "complement operation on an NFA with respect to a given alphabet.\n" <>
"RegComplement[DFA[...], {\\"str1\\", \\"str2\\",...}] performs the complement " <>
    "operation on a DFA with respect to a given alphabet.\n" <>
"RegComplement[Regex[...], {\\"str1\\", \\"str2\\",...}] performs the " <>
    "complement operation on a symbolic regular expression with respect " <>
    "to a given alphabet.\n" <>
"RegComplement[RRGrammar[...], {\\"str1\\", \\"str2\\",...}] performs the " <>
    "complement operation on a right regular grammar with respect to a " <>
    "given " <> "alphabet.\n" <>
"RegComplement[Digraph[...], {\\"str1\\", \\"str2\\",...}] performs the " <>
    "complement " <>
    "operation on a directed graph with respect to a given alphabet.";

RegReverse::usage = "RegReverse[NFA[...]] performs the reversal " <>
    "operation on an NFA.\n" <>
"RegReverse[DFA[...]] performs the reversal operation on a DFA.\n" <>
"RegReverse[Regex[...]] performs the reversal operation on a symbolic " <>
    "regular expression.\n" <>
"RegReverse[RRGrammar[...]] performs the reversal operation on a right " <>
    "regular grammar.\n" <>
"RegReverse[Digraph[...]] performs the reversal operation on a directed " <>
    "graph.";

RegUnion::usage = "RegUnion[NFA[...], NFA[...]] performs the reversal " <>

```

```

    "operation on two NFAs.\n" <>
"RegUnion[DFA[...], DFA[...]] performs the reversal operation on two " <>
    "DFAs.\n" <>
"RRegUnionRegex[...], Regex[...]] performs the reversal operation on two " <>
    "symbolic " <> "regular expressions.\n" <>
"RegUnion[RRGrammar[...], RRGrammar[...]] performs the reversal operation " <>
    "on two " <> "right regular grammars.\n" <>
"RegUnion[Digraph[...], Digraph[...]] performs the reversal operation on " <>
    "two directed " <> "graphs.";

RegConcat::usage = "RegConcat[NFA[...], NFA[...]] performs the " <>
    "concatenation operation on two NFAs.\n" <>
"RegConcat[DFA[...], DFA[...]] performs the concatenation operation on " <>
    "two DFAs.\n" <>
"ReRegConcatRegex[...], Regex[...]] performs the concatenation operation on " <>
    "two symbolic " <> "regular expressions.\n" <>
"RegConcat[RRGrammar[...], RRGrammar[...]] performs the concatenation " <>
    "operation on two " <> "right regular grammars.\n" <>
"RegConcat[Digraph[...], Digraph[...]] performs the concatenation " <>
    "operation on two directed " <> "graphs.";

RegIntersection::usage = "RegIntersection[NFA[...], NFA[...]] performs the " <>
    "intersection operation on two NFAs.\n" <>
"RegIntersection[DFA[...], DFA[...]] performs the intersection operation " <>
    "on two DFAs.\n" <>
"RegIntersection[Regex[...], Regex[...]] performs the intersection " <>
    "operation on two symbolic " <>
    "regular expressions.\n" <>
"RegIntersection[RRGrammar[...], RRGrammar[...]] performs the intersection " <>
    "operation on two " <> "right regular grammars.\n" <>

```

```

"RegIntersection[Digraph[...], Digraph[...]] performs the intersection " <>
  "operation on two directed " <> "graphs.";

ToRegularExpression::usage = "ToRegularExpression[Regex[...]] converts a " <>
  "symbolic regular " <> "expression to a Mathematica regular expression.";

RegexOr::usage = "RegexOr is a head used to define symbolic regular " <>
  "expressions. It " <>
  "represents the regular expression equal to the union of its arguments.";

RegexConcat::usage = "RegexConcat is a head used to define symbolic " <>
  "regular expressions. It " <>
  "represents the regular expression equal to the concatenation of its " <>
  "arguments.";

RegexStar::usage = "RegexStar is a head used to define symbolic regular " <>
  "expressions. It " <>
  "represents the regular expression equal to the star of its argument.";

RRGrammarOr::usage = "RRGrammarOr is a head used to define right regular " <>
  "grammars. It " <> "represents a possible choice between its arguments.";

RRGrammarConcat::usage = "RRGrammarConcat is a head used to define right " <>
  "regular grammars. It " <>
  "is used to concatenate a letter with a nonterminal symbol.";

EmptyWord::usage = "EmptyWord represents the empty word. It may be used in " <>
  "symbolic regular expressions.";

GeneratingFunction::usage = "GeneratingFunction[Regex[...], {"letter1\" -> " <>

```

```

"expr1, \"letter2\" -> expr2,...}] " <>

    "gives the generating function for the regular language specified by a " <>

    "symbolic " <>

    "regular expression. It takes as its second argument a list of rules " <>

    "giving a marker/weight " <> "for each letter in the alphabet.\n" <>

    GeneratingFunction::usage;

Begin["`Private`"] (* Begin Private Context *)

(* all public symbols in this package *)
publicSymbols = {NFA, DFA, Regex, RRGrammar, Digraph,
    ToNFA, ToDFA, ToRegex, ToRRGrammar, ToDigraph, ToRegularExpression,
    RegStar, RegComplement, RegReverse, RegUnion, RegConcat, RegIntersection,
    Disambiguate, AmbiguousQ,
    RegexOr, RegexConcat, RegexStar, EmptyWord,
    RRGrammarConcat, RRGrammarOr};

(* ::Section:: *)
(* Input validation *)

(* returns a list of the unique nonterminal symbols in a right regular *)
(* grammar *)
nonTerminals[RRGrammar[grammar_]] := Module[
    {
        extendedGrammar
    },
    (* expand or[...] 's into separate rules in extendedGrammar *)
    expandedGrammar = grammar /. (rhs_ -> RRGrammarOr[args_]) :>
        Sequence@@((rhs -> # &) /@ {args});
    Union@Flatten[{

```

```

grammar[[All, 1]],
Cases[grammar, Verbatim[Rule][lhs_, sym_Symbol /; sym != EmptyWord
    ] -> {lhs, sym}],
Cases[grammar, Verbatim[Rule][lhs_, sym_Symbol[n_]] -> {lhs, sym[n]}],
Cases[grammar, Verbatim[Rule][lhs_, RRGrammarConcat[_ , expr_] -> expr]
    ]
];

```

```

NFA::invalid = "Invalid NFA.";
DFA::invalid = "Invalid DFA.";
RegularExpression::invalid = "Invalid Mathematica regular expression.";
Regex::invalid = "Invalid symbolic regular expression.";
RRGrammar::invalid = "Invalid right regular grammar.";
Digraph::invalid = "Invalid directed graph.";
GeneratingFunction::invalidRules = "Invalid rules.";
RegComplement::invalidAlphabet = "Invalid alphabet.";

```

```

(* validate NFAs *)
validateRLR[NFA[numStates_Integer?NonNegative, alphabet:{___String},
    transitionMatrix_, acceptStates:{___Integer},
    initialState:_Integer|Null]] := Module[ {
    ok = True
},
    (* alphabet is valid *)
    ok = ok && !MemberQ[alphabet, ""];

    (* matrix entries are valid lists *)
    ok = ok && ((numStates == 0 && transitionMatrix == {}) ||
        MatrixQ[transitionMatrix, MatchQ[#, {___Integer}] &&
            (Max[#] <= numStates && Min[#] >= 1) &]);

```

```

(* matrix dimensions are correct *)
ok = ok && ((numStates == 0 && transitionMatrix == {}) ||
  Dimensions[transitionMatrix, 2] ==
  {numStates, Length[alphabet] + 1});

(* accept states are valid *)
ok = ok && ((numStates == 0 && acceptStates == {}) ||
  (numStates > 0 && Max[acceptStates] <= numStates &&
  Min[acceptStates] >= 1));

(* initial state is valid *)
ok = ok && ((numStates == 0 && initialState == Null) ||
  (numStates > 0 && 1 <= initialState <= numStates));

If[ !ok, Message[NFA::invalid]];
ok

];

validateRLR[NFA[___]] := (Message[NFA::invalid];False);

(* validate DFAs *)
validateRLR[DFA[numStates_Integer?NonNegative, alphabet:{___String},
  transitionMatrix_, acceptStates:{___Integer},
  initialState:_Integer|Null]] := Module[ {
  ok = True
},
  (* alphabet is valid *)
  ok = ok && !MemberQ[alphabet, ""];

  (* matrix entries are valid numbers *)

```



```

ok = ok && ((numStates == 0 && transitionMatrix == {}) ||
    MatrixQ[transitionMatrix, MatchQ[#, _Integer] &&
        (# <= numStates && # >= 1) &]);

(* matrix dimensions are correct *)
ok = ok && ((numStates == 0 && transitionMatrix == {}) ||
    Dimensions[transitionMatrix, 2] ==
        {numStates, Length[alphabet]});

(* accept states are valid *)
ok = ok && ((numStates == 0 && acceptStates == {}) ||
    (numStates > 0 && Max[acceptStates] <= numStates &&
        Min[acceptStates] >= 1));

(* initial state is valid *)
ok = ok && ((numStates == 0 && initialState == Null) ||
    (numStates > 0 && 1 <= initialState <= numStates));

If[ !ok, Message[DFA::invalid]];
ok

];
validateRLR[DFA[___]] := (Message[DFA::invalid];False);

(* validate Mathematica regular expressions *)
validateRLR[RegularExpression[regex_String]] := Module[
{
    ok = True
},
ok = ok && StringMatchQ[regex, (LetterCharacter | DigitCharacter | "\\*" |
    "(" |

```

```

    ")" | "(" ...];

    (* test validity as a regex according to Mathematica *)
    ok = ok && Check[StringMatchQ["", RegularExpression[regex]], $Failed] !=
        $Failed;

    If[!ok, Message[RegularExpression::invalid]];
    ok
];

validateRLR[RegularExpression[Null]] := True;
validateRLR[RegularExpression[___]] := (Message[RegularExpression::invalid];
    False);

(* validate symbolic regex without Regex head *)
validateRawRegex[Null] := True;
validateRawRegex[EmptyWord] := True;
validateRawRegex[str_String /; str != ""] := True;
validateRawRegex[RegexStar[regex_]] := validateRawRegex[regex];
validateRawRegex[RegexOr[regexes__]] := And @@ validateRawRegex /@ {regexes};
validateRawRegex[RegexConcat[regexes__]] := And @@ validateRawRegex /@
    {regexes};
validateRawRegex[___] := False;

(* validate symbolic regex *)
validateRLR[Regex[regex_]] := Module[
    {
        ok = True
    },
    ok = ok && validateRawRegex[regex];

    If[!ok, Message[Regex::invalid]];

```

```

    ok
];

validateRLR[Regex[___]] := (Message[Regex::invalid];False);

(* validate right regular grammars *)
validateRLR[g:RRGrammar[grammar:{(_ -> _) ...}]] := Module[
{
    ok = True,
    nonTerms = nonTerminals[g],
    validateTerm
},
(* validate left-hand sides *)
ok = ok && MatchQ[nonTerms, {(_Symbol | _Symbol[_Integer]) ...}];

(* validate right-hand sides *)
validateTerm[EmptyWord] := True;
validateTerm[sym_Symbol] := True;
validateTerm[sym_Symbol[n_Integer]] := True;
validateTerm[str_String /; str != ""] := True;
validateTerm[RRGrammarConcat[str_String /; str != "", sym_Symbol]] := True;
validateTerm[RRGrammarConcat[str_String /; str != "", sym_Symbol[
    n_Integer]]] := True;
validateTerm[_] := False;

ok = ok && MatchQ[grammar[[All,2]], {( _?validateTerm | RRGrammarOr[
    _?validateTerm .. ]) ...}];

If[!ok, Message[RRGrammar::invalid]];
ok
];

```

```

validateRLR[RRGrammar[___]] := (Message[RRGrammar::invalid];False);

(* validate digraphs *)
validateRLR[Digraph[graph_Graph, startVertices_List, endVertices_List, True |
False]] := Module[
{
    ok = True,
    vertices

},
(* validate graph *)
ok = ok && (EmptyGraphQ[graph] || DirectedGraphQ[graph]);
vertices = VertexList[graph];
ok = ok && MatchQ[vertices, {__Integer}];

(* make sure vertex labels are nonempty strings *)
ok = ok && And @@ ( (Head[PropertyValue[{graph, #}, VertexLabels]] ===
String &&
PropertyValue[{graph, #}, VertexLabels] != "") & /@ vertices );
(* validate vertex lists *)
ok = ok && Union[startVertices, endVertices, vertices] == Union[vertices];

If[!ok, Message[Digraph::invalid]];
ok
];
validateRLR[Digraph[___]] := (Message[Digraph::invalid];False);

(* validate rules passed to GeneratingFunction for making letters *)
validateRules[rules_, regex_] := Module[
{
    strings = Cases[regex, _String, {0, Infinity}],

```

```

        ok = True
    },

    ok = ok && MatchQ[rules, {(_String -> _) ...}];

    ok = ok && Complement[strings, rules[[All, 1]]] == {};

    If[!ok, Message[GeneratingFunction::invalidRules]];
    ok
];

(* validate alphabet passed to RegComplement *)
validateAlphabet[alphabet_] := Module[
    {
        ok = True
    },

    ok = ok && MatchQ[alphabet, {_String ...}];

    ok = ok && DeleteDuplicates[alphabet] == alphabet;

    If[!ok, Message[RegComplement::invalidAlphabet]];
    ok
];

(* ::Section:: *)
(* Regex to NFA conversion *)

(* This algorithm is simply described in Flajolet and Sedgewick on p 735 *)

```

```

regex2nfa[Null] := NFA[0, {}, {}, {}, Null];

regex2nfa[str_String] := NFA[
    2,
    {str},
    {
        {{2}, {}},
        {}, {}
    },
    {2}, 1
];

regex2nfa[EmptyWord] := NFA[
    1, {},
    {
        {}
    },
    {1}, 1
];

regex2nfa[RegexOr[first_, rest_]] :=
    RegUnion[regex2nfa[first], regex2nfa[RegexOr[rest]]];

regex2nfa[RegexOr[regex_]] := regex2nfa[regex];

regex2nfa[RegexStar[regex_]] :=
    RegStar[regex2nfa[regex]];

regex2nfa[RegexConcat[first_, rest_]] :=
    RegConcat[regex2nfa[first], regex2nfa[RegexConcat[rest]]];

```

```

regex2nfa[RegexConcat[regex_]] := regex2nfa[regex];

ToNFA[Regex[regex_], OptionsPattern[]] := Module[
  {},
  (
    regex2nfa[regex]
  )/; !OptionValue[validationRequired] || validateRLR[Regex[regex]]
];

(* ::Section:: *)
(* RRGrammar to NFA conversion *)
(* in this straightforward algorithm, the nonterminal symbols become states *)
(* in the NFA *)

ToNFA[RRGrammar[{}], OptionsPattern[]] := NFA[0, {}, {}, {}, Null];

ToNFA[g: RRGrammar[grammar_], OptionsPattern[]] :=
  Module[
    {
      (* added for rhs -> string rules *) acceptState = Unique[],
      stateSet,
      alphabet = Cases[grammar, _String, Infinity]//Union,
      (* lookup functions *) stateNumber, alphabetNumber,
      expandedGrammar,
      transitionMatrix, acceptStates
    },
    (
      stateSet = Flatten[{nonTerminals[g], acceptState}];

```

```

stateNumber[state_ /; MemberQ[stateSet, state]] := Position[stateSet,
    state, {1}, Heads -> False][[1,1]];
alphabetNumber[str_] := Position[alphabet,
    str, {1}, Heads -> False][[1,1]];

(* initialize acceptStates *)
acceptStates = {stateNumber[acceptState]};

(* expand or[...]s into separate rules in extendedGrammar *)
expandedGrammar = grammar /. (rhs_ -> RRGrammarOr[args_]) :>
    Sequence@@((rhs -> # &) /@ {args});
transitionMatrix = ConstantArray[{}, {Length[stateSet],
    Length[alphabet] + 1}];

(* incorporate each rule into transitionMatrix and acceptStates *)
Function[rule,
    rule /. {
        (lhs_ -> EmptyWord) :> (acceptStates = Union[acceptStates, {
            stateNumber[lhs]}]),
        (lhs_ -> st:(_Symbol|_Symbol[_])) :> (transitionMatrix[[
            stateNumber[lhs], -1]] =
            Union[transitionMatrix[[stateNumber[lhs], -1]], {
            stateNumber[st]}]),
        (lhs_ -> RRGrammarConcat[str_, st:(_Symbol|_Symbol[_]))] :> (transitionMatrix[[stateNumber[lhs],
            alphabetNumber[str]]] =
            Union[transitionMatrix[[stateNumber[lhs],
            alphabetNumber[str]]], {stateNumber[st]}]),
        (lhs_ -> str_String) :> (transitionMatrix[[stateNumber[lhs],

```



```

        alphabetNumber[str]]] =
        Union[transitionMatrix[[stateNumber[lhs],
        alphabetNumber[str]]], {stateNumber[acceptState]}];)
    }
] /@ expandedGrammar;

NFA[Length[stateSet], alphabet, transitionMatrix, acceptStates,
    grammar[[1,1]]//stateNumber]

) /; (!OptionValue[validationRequired] || validateRLR[g])
];

(* ::Section:: *)
(* Digraph to NFA conversion *)

ToNFA[dg: Digraph[graph_, startVertices_, endVertices_, eAccepted_],
    OptionsPattern[]] :=
Module[
    {
        edges = EdgeList[graph],
        alphabet =
            Union[PropertyValue[{graph, #}, VertexLabels] & /@
                VertexList[graph]],
        numStates, initialState, acceptState,
        transitionMatrix, vertex2LetterPos, edge2StatePos
    },
    (
        (* the states are all edges, plus a start state and an accept state *)

        (* lookup alphabet number for vertex *)

```

```

vertex2LetterPos[vertex_] :=
  Position[alphabet,
    PropertyValue[{graph, vertex}, VertexLabels], {1},
    Heads -> False][[1,1]];
(* lookup state number for edge *)
edge2StatePos[edge_] :=
  Position[edges, edge, {1}, Heads -> False][[1,1]];

{initialState, acceptState, numStates} = Length[edges] + {1, 2, 2};

(* initialize transitionMatrix *)
transitionMatrix =
  ConstantArray[{}, {numStates, Length[alphabet] + 1}];

(* add emove from initial state to accept state if eAccepted *)
If[eAccepted,
  transitionMatrix[[initialState, -1]] = {acceptState}];

(* if a vertex is a start vertex and an end vertex, add an
   edge labeled with its label from initialState to acceptState *)
Map[Function[vertex,
  transitionMatrix[[initialState, vertex2LetterPos[vertex]]] =
    Union[transitionMatrix[[initialState,
      vertex2LetterPos[vertex]]], {acceptState}];],
  Intersection[endVertices, startVertices]];

(* if an edge begins at a start vertex, add an edge labeled with
   that start vertex's label between initialState and the state
   corresponding to the edge in the digraph *)
Map[Function[edge,

```

```

transitionMatrix[[initialState, vertex2LetterPos[edge[[1]]]]] =
  Union[
    transitionMatrix[[initialState,
      vertex2LetterPos[edge[[1]]]], {edge2StatePos[edge]}];],
EdgeList[graph, DirectedEdge[Alternatives@@startVertices, _]]
];

(* if an edge ends at an end state, add a transition from that edge
to acceptState labeled by the end state's label *)
Map[Function[edge,
  transitionMatrix[[edge2StatePos[edge],
    vertex2LetterPos[edge[[2]]]]] =
  Union[transitionMatrix[[edge2StatePos[edge],
    vertex2LetterPos[edge[[2]]]], {acceptState}];],
EdgeList[graph, DirectedEdge[_ , Alternatives@@endVertices]]
];

(* if one edge goes into a vertex and one edge comes out,
add a transition between the edges labeled with the label
of the common vertex *)
Map[Function[edgePair,
  transitionMatrix[[edge2StatePos[edgePair[[1]]],
    vertex2LetterPos[edgePair[[1, 2]]]]] =
  Union[transitionMatrix[[edge2StatePos[edgePair[[1]]],
    vertex2LetterPos[edgePair[[1, 2]]]], {edge2StatePos[
    edgePair[[2]]]}];],
Union@Flatten[
  Outer[List, EdgeList[graph, DirectedEdge[_ , #]],
    EdgeList[graph, DirectedEdge[#, _]]] & /@ VertexList[graph],
  2]

```

```

];

NFA[numStates, alphabet, transitionMatrix, {acceptState},
  initialState]
  ); !OptionValue[validationRequired] || validateRLR[dg]
];

(* ::Section:: *)
(* conversion from NFA to DFA (determinization) *)

(* hopcroft implements the algorithm at http://bit.ly/1kT6P2n directly *)
hopcroft[DFA[numStates:(0|1), alphabet_, transitionMatrix_, acceptStates_,
  initialState_]] :=
DFA[numStates, alphabet, transitionMatrix, acceptStates, initialState];

hopcroft[DFA[numStates_, alphabet_, transitionMatrix_, acceptStates_,
  initialState_]] := Module[
{
  p = {Sort@acceptStates,
    Complement[Range[numStates], acceptStates]},
  w = {Sort@acceptStates}, a, c, x
},
While[Length[w] > 0,
  a = First[w]; w = Delete[w, 1];
For[c = 1, c <= Length[alphabet], c++,
  x = Flatten@
    Position[transitionMatrix[[All, c]],
      _?(MemberQ[a, #] &), {1}, Heads -> False];
Map[
  (

```

```

p =
  Union[p /. # :>
    Sequence[Intersection[x, #], Complement[#, x]]];
If[MemberQ[w, #],
  w =
    Union[w /. # :>
      Sequence[Intersection[x, #], Complement[#, x]]],
  (* else *)
  If[Length[Intersection[x, #]] <= Length[Complement[#, x]],
    w = Union[w, {Intersection[x, #]}],
    (* else *)
    w = Union[w, {Complement[#, x]}]
  (* end else *)
]
(* end else *)
]
) &,
Select[p, Intersection[#, x] != {} &]
];
];
];
p = DeleteCases[p, {}];
DFA[
  Length[p],
  alphabet,
  Table[
    First@Flatten@
      Position[
        p, _?(MemberQ[#, transitionMatrix[[First@p[[s]], c]]] &), {1},
        Heads -> False], {s, 1, Length[p]}, {c, 1, Length[alphabet]}],

```

```

    Flatten@
      Position[p, _?(Intersection[#, acceptStates] != {} &), {1},
        Heads -> False],
  First@
    Flatten@Position[p, _?(MemberQ[#, initialState] &), {1},
      Heads -> False]
]
];

SetAttributes[floydWarshall, HoldFirst];

(* this computes transitive closure of a graph via the Floyd-Warshall *)
(* algorithm *)
(* "pass by reference": assigns to the symbol passed *)
floydWarshall[m_] := Module[
  {n = Length[m], i, j, k},
  For[k = 1, k <= n, ++k,
    For[i = 1, i <= n, ++i,
      For[j = 1, j <= n, ++j,
        m[[i, j]] = m[[i, j]] || (m[[i, k]] && m[[k, j]]);
      ]
    ]
  ];
];

(* deadStates finds states with no path to an accept state in an NFA *)
deadStates[NFA[numStates_, _, transitionMatrix_, acceptStates_, _]] :=
  Module[
    {adj = Flatten /@ transitionMatrix},
    graph = Graph[
      Range[numStates],

```

```

    MapIndexed[DirectedEdge[First[#2], #1] &, adj, {2}] // Flatten // Union];
Complement[Range[numStates],
  VertexInComponent[graph, acceptStates]
]
];

```

```

ToDFA[nfa:NFA[0, alphabet_, _, _, _], OptionsPattern[]] := Module[
  {},
  (
    DFA[0, alphabet, {}, {}, Null]
  ) /; !OptionValue[validationRequired] || validateRLR[nfa]
];

```

```

ToDFA[nfa :
  NFA[numStates_, alphabet_, transitionMatrix_, acceptStates_,
    initialState_], OptionsPattern[]] := Module[
{
  states = {}, queue,
  adjacency =
    Table[ReplacePart[ConstantArray[False, numStates],
      List /@ (transitionMatrix[[k, -1]]) -> True], {k, 1,
        numStates}],
  nfaDeadStates = deadStates[nfa],
  (* function that stores transitions in the DFA *) tr,
  state, neighbor,
  dfaInitialState, dfaAcceptStates, stateNumber,
  dfaTransitionMatrix
},
  (
    If[MemberQ[nfaDeadStates, initialState],

```

```

Return[DFA[0, {}, {}, {}, Null]]];
floydWarshall[adjacency];
(* incrementally search for possible states in the DFA *)
queue = {Complement[
  Flatten[{initialState,
    Position[adjacency[[initialState]], True]}], nfaDeadStates]};
While[Length[queue] > 0,
  (* process "state": the next state that was found *)
  {state, queue} = {First[queue], Rest[queue]};
  If[! MemberQ[states, state],
    For[i = 1, i <= Length[alphabet], ++i,
      (* assign to "neighbor" the set of accessible states from "state" *)
      neighbor =
        Flatten[Map[
          Function[
            st, {transitionMatrix[[st, i]],
              Map[Position[adjacency[[#]], True] &,
                transitionMatrix[[st, i]]}], state
          ]~Complement~nfaDeadStates; (* remove dead states from list *)
        (* store neighbor in tr *)
        tr[state, i] = neighbor;
        queue = Append[queue, neighbor];
      ];
    states = Append[states, state];
  ];
];
tr[_, _] = {};
stateNumber[set_] :=
  Position[states, set, {1}, Heads -> False][[1, 1]];
dfaAcceptStates =

```



```

stateNumber /@
Select[states, (Intersection[#, acceptStates] != {}) &];
dfaTransitionMatrix =
Table[stateNumber@tr[states[[s]], a], {s, 1, Length[states]}, {a, 1,
Length[alphabet]}}];
DFA[Length[states], alphabet, dfaTransitionMatrix,
dfaAcceptStates, 1] // hopcroft (* minimize just before returning *)
) /; ! OptionValue[validationRequired] || validateRLR[nfa]
];

(* ::Section:: *)
(* conversion from DFA to Regex *)

(* simplifyRawRegex takes a symbolic regex, without the head Regex, and *)
(* simplifies it if possible by removing redundancies *)
simplifyRawRegex[regex_] := FixedPoint[
Replace[#, {
RegexStar[RegexStar[args_]] :> RegexStar[args],
RegexStar[Null] -> EmptyWord,
RegexStar[EmptyWord] -> EmptyWord,
RegexOr[x_] :> x,
RegexOr[i____, Null, f____] :> RegexOr[i, f],
RegexOr[i____, x_, m____, x_, f____] :> RegexOr[i, x, m, f],
RegexOr[i____, RegexOr[args_], f____] :> RegexOr[i, args, f],
RegexConcat[i____, RegexConcat[args_], f____] :> RegexConcat[i, args, f],
RegexConcat[x_] :> x,
RegexConcat[____, Null, ____] -> Null,
RegexConcat[i____, EmptyWord, f____] :> RegexConcat[i, f]
}, {0, Infinity}] &,
regex]

```

```

ToRegex[DFA[0, _, _, _, _], OptionsPattern[]] := Regex[Null];

ToRegex[dfa: DFA[numStates_, alphabet_, transitionMatrix_, acceptStates_,
  initialState_], OptionsPattern[]] :=
Module[
{
(* edgeRawRegex is a function that stores the current regex corresponding *)
(* to each transition during the conversion process *)
  edgeRawRegex,
  k, i, j},
(
  edgeRawRegex[_ , _] = Null;
MapIndexed[
  Function[{endState, pos},
    (*pos[[1]]=startState, pos[[2]]=letter*)
    edgeRawRegex[pos[[1]], endState] =
      RegexOr[edgeRawRegex[pos[[1]], endState], alphabet[[pos[[2]]]]];],
  transitionMatrix, {2}];
(* setup beginning and end states *)
edgeRawRegex[0, initialState] = EmptyWord;
edgeRawRegex[x_?(MemberQ[acceptStates, #] &), numStates + 1] =
  EmptyWord;
For[k = 1, k <= numStates, ++k,
  Do[edgeRawRegex[i, j] =
    RegexOr[edgeRawRegex[i, j],
      RegexConcat[edgeRawRegex[i, k], RegexStar[edgeRawRegex[k, k]],
        edgeRawRegex[k, j]] // simplifyRawRegex, {i,
    Append[Range[k + 1, numStates + 1], 0]}, {j,
    Append[Range[k + 1, numStates + 1], 0]}

```

```

];
];
Regex[edgeRawRegex[0, numStates + 1]]
)/; !OptionValue[validationRequired] || validateRLR[dfa]
];

(* ::Section:: *)
(* conversion from NFA to RRGrammar *)

ToRRGrammar[NFA[0, _, _, _, _], OptionsPattern[]] := RRGrammar[{}];

(* simply convert states to nonterminal symbols *)
ToRRGrammar[nfa: NFA[numStates_, alphabet_, transitionMatrix_, acceptStates_,
  initialState_], OptionsPattern[]] :=
Module[
  {
    nonTermHead = Unique[],
    nonTerms,
    augmentedAlphabet = {alphabet, EmptyWord} // Flatten,
    grammar,
    posOfInitial
  },
  (
    (* create nonterminal for each state *)
    nonTerms = nonTermHead /@ Range[numStates];
    (* convert transitions to grammar productions *)
    grammar = Flatten@MapIndexed[
      (* #1 == list of stateNums *)
      (* #2[[1]] == stateNum, #2[[2]] == letterNum *)
      (nonTermHead[#2[[1]]] -> RRGrammarOr @@ (Function[st,

```

```

RRGrammarConcat[augmentedAlphabet[[#2[[2]]]],
nonTermHead[st]]] /@ #1)) &,
transitionMatrix, {2}
];
(* simplify resulting grammar *)
grammar = grammar /. RRGrammarConcat[EmptyWord, x_] :> x;
grammar = grammar /. (_ -> RRGrammarOr[]) :> Sequence[];
grammar = grammar /. (lhs_ -> RRGrammarOr[arg_]) :> (lhs -> arg);
(* handle acceptStates *)
grammar = {grammar, (nonTermHead[#] -> EmptyWord) & /@ acceptStates} //
Flatten;

(* if the initial state has no outgoing edges,
and is not an accepting state, return {} *)
If[FreeQ[grammar[[All, 1]], nonTermHead[initialState]], Return[
RRGrammar[{}]]];
(* if start state appears, make one of its rules the first one *)
posOfInitial = Position[grammar, nonTermHead[initialState]][[1, 1]];

If[posOfInitial != 1,
grammar = Permute[grammar, Cycles[{{1, posOfInitial}}]]
];

RRGrammar[grammar]
); !OptionValue[validationRequired] || validateRLR[nfa]
];

(* ::Section:: *)
(* conversion from DFA to Digraph *)

```

```

cartesian[lists___List] := Flatten[Outer[List, lists, 1], Length[{lists}] - 1];

ToDigraph[DFA[0, _, _, _, _], OptionsPattern[]] := Digraph[Graph[{}], {}, {},
False];

ToDigraph[dfa: DFA[numStates_, alphabet_, transitionMatrix_, acceptStates_,
initialState_], OptionsPattern[]] := Module[
{
(* each transition corresponds to a vertex *)
vertexSet = Flatten[MapIndexed[
(* #1 = to, #2[[1]] = from, #2[[2]] = letterNum *)
{#2[[1]], #1, #2[[2]]}&, transitionMatrix, {2}], 1],
vertexNum,
edgeSet,
graph,
startVertices, endVertices, eAccepted
},
(
(* ranking function *)
vertexNum[vertex_] := Position[vertexSet, vertex,
{1}, Heads -> False][[1, 1]];

startVertices = vertexNum /@ Cases[vertexSet, {initialState, _, _}];
endVertices = vertexNum /@ Cases[vertexSet, {_, st_, _} /;
MemberQ[acceptStates, st]];
eAccepted = MemberQ[acceptStates, initialState];

(* create the edge set *)
edgeSet = Cases[cartesian[vertexSet, vertexSet],
{_, c_, _}, {c_, _, _}]];

```

```

edgeSet = Map[vertexNum, edgeSet, {2}];
edgeSet = edgeSet /. {i_, f_} :> DirectedEdge[i, f];

graph = Graph[Range[Length[vertexSet]], edgeSet];

(* start vertices are colored green, end vertices have dotted edges *)
(PropertyValue[{graph, #}, VertexStyle] = Green;)& /@ startVertices;
(PropertyValue[{graph, #}, VertexStyle] = EdgeForm[{Thick, Dotted}];
  )& /@ endVertices;
(PropertyValue[{graph, #}, VertexStyle] =
  Directive[Green, EdgeForm[{Thick, Dotted}]];& /@ Intersection[
    startVertices, endVertices];

(* add String labels to the vertices *)
(# /. {from_, to_, letterNum_} :> (PropertyValue[{graph,
  vertexNum[{from, to, letterNum}], VertexLabels] =
    alphabet[[letterNum]];&))& /@ vertexSet;

Digraph[graph, startVertices, endVertices, eAccepted]
  //; !OptionValue[validationRequired] || validateRLR[dfa]
];

(* ::Section:: *)
(* Regular expression conversions *)

(* conversion from Mathematica regular expression to symbolic regular *)
(* expression *)

(* "," not allowed in the regex *)
(* pars takes a Mathematica regular expression and comes up with a symbolic *)

```

```

(* regex (except for the head Regex) *)
pars[""] := EmptyWord;
pars[regex_] :=
Module[{temp=regex},
  (*insert commas between things that are concatenated*)
  temp = FixedPoint[
    StringReplace[#,
      a_ ~~ b_ /;
        a != "(" && b != ")" && b != "*" && a != "," && b != "," &&
        a != "|" && b != "|" :> a <> "," <> b] &, temp];
  (*insert EmptyWord*)
  temp = FixedPoint[
    StringReplace[#,
      {
        "||" -> "|EmptyWord|",
        "(|" -> "(EmptyWord|",
        "|)" -> "|EmptyWord)",
        "()" -> "(EmptyWord)"
      }
    ] &, temp];
  (*make the star operation the squaring operation *)
  temp = StringReplace[temp, "*" -> "^2"];
  (*make the concatenation operation the ** operation *)
  temp = StringReplace[temp, "," -> "**"];
  (* use ToExpression to obtain a Mathematica expression from the string *)
  temp = ToExpression[temp, InputForm, Hold];
  (* replace the operations present in the expressions with the ones *)
  (* found in Regex expressions *)
  temp =
    Replace[temp,
      sym_Symbol?(

```

```

!MemberQ[{Hold, RegexOr, RegexConcat, RegexStar,
EmptyWord}, #] &)>
Module[{chars = Characters@ToString[Unevaluated[sym]]},
If[Length[chars] == 1, chars[[1]],
RegexConcat @@ chars
]
], Infinity];
temp =
FixedPoint[
Replace[#, {Power[expr_, _] :> RegexStar[expr],
NonCommutativeMultiply[args__] :> RegexConcat[args],
Verbatim[Alternatives][args__] :> RegexOr[args]}, {0,
Infinity}] &, temp];
ReleaseHold[temp]
];

ToRegex[RegularExpression[Null], OptionsPattern[]] := Regex[Null];

ToRegex[r:RegularExpression[regex_], OptionsPattern[]] := Module[
{ },
Regex@pars["(" <> regex <> ")"] /; !OptionValue[validationRequired] ||
validateRLR[r]
];

(* symbolic regular expression to Mathematica regular expression conversion *)
regex2regularexpression[Null] := Null;
regex2regularexpression[str_String] := str;
regex2regularexpression[EmptyWord] := "()";
regex2regularexpression[RegexStar[expr_]] := "(" <>
regex2regularexpression[expr] <> ")*";

```



```

regex2regularexpression[RegexConcat[args_]] := StringJoin[
    regex2regularexpression /@ {args}];
regex2regularexpression[RegexOr[args_]] := "(" <>
    StringJoin[Riffle[regex2regularexpression /@ {args}, "|"] <> ")";

ToRegularExpression[r:Regex[regex_], OptionsPattern[]] := Module[
    {simp = simplifyRawRegex[regex]},
    RegularExpression[regex2regularexpression[simp]] /;
    !OptionValue[validationRequired] || validateRLR[r]
];

(* ::Section:: *)
(* Closure properties *)

(* Kleene star for right regular grammars *)
RegStar[RRGrammar[{}], OptionsPattern[]] := RRGrammar[{Unique[] -> EmptyWord}];

RegStar[grammar : RRGrammar[_], OptionsPattern[]] := Module[
    {
        initial1 = grammar[[1, 1, 1]]
    },
    (
        RRGrammar[Flatten[{ initial1 -> EmptyWord,
            Replace[First[grammar],
                {
                    Verbatim[Rule][nonTerm_, EmptyWord] :> (nonTerm ->
                        RRGrammarOr[initial1, EmptyWord]),
                    RRGrammarOr[pre___, EmptyWord, post___] :>
                        RRGrammarOr[pre, EmptyWord, initial1, post]
                }
            ]
        }]]
    )
]

```

```

    },
    Infinity
  ]}]
  )/; !OptionValue[validationRequired] || validateRLR[grammar]
];

(* complement for DFAs *)
RegComplement[dfa:DFA[numStates_, alphabet_, transitionMatrix_, acceptStates_,
  initialState_], compAlphabet_, OptionsPattern[]] := Module[
{
  compDFA,
  intersectionAlphabet = Intersection[alphabet, compAlphabet],
  intersectionAlphabetRegex,
  compNotAlphabetLetters = Complement[compAlphabet, alphabet],
  compNotAlphabetRegex
},
(
  (* compute the DFA that accepts the words, over the alphabet "alphabet", *)
  (* that dfa rejects *)
  compDFA = If[numStates == 0,
    (* no states *)
    If[alphabet == {},
      ToDFA[Regex[EmptyWord]],
      ToDFA[Regex[RegexStar[RegexOr@@alphabet]]]
    ],
    (* states *)
    DFA[numStates, alphabet, transitionMatrix,
      Complement[Range[numStates], acceptStates], initialState]
  ];

```

```

(* define a regex that parses all words with letters found in alphabet *)
(* and compAlphabet *)
intersectionAlphabetRegex = If[intersectionAlphabet == {},
  Regex[EmptyWord],
  Regex[RegexStar[RegexOr@@intersectionAlphabet]]
];

(* define a regex that parses all words containg a letter in compAlphabet *)
(* but not alphabet *)
compNotAlphabetRegex = If[compNotAlphabetLetters == {},
  Regex[Null],
  Regex[RegexConcat[
    RegexStar[RegexOr@@compAlphabet],
    RegexOr@@compNotAlphabetLetters,
    RegexStar[RegexOr@@compAlphabet]]
  ]
];

(* return the DFA that accepts any words with a letter in compAlphabet *)
(* that is not in alphabet AND any words over compAlphabet that dfa *)
(* rejects *)
RegUnion[RegIntersection[ToRegex[compDFA], intersectionAlphabetRegex],
  compNotAlphabetRegex]//ToDFA
)/; !OptionValue[validationRequired] ||
  (validateRLR[dfa] && validateAlphabet[compAlphabet])
];

(* reverse for regexes *)

```

```

(* regexReverse reverses a symbolic regular expression lacking the head *)
(* Regex *)
regexReverse[str_String] := str;

regexReverse[Null] := Null;

regexReverse[RegexOr[args_]] := RegexOr @@ regexReverse /@ {args};

regexReverse[EmptyWord] := EmptyWord;

regexReverse[RegexConcat[args_]] :=
  RegexConcat @@ regexReverse /@ Reverse[{args}];

regexReverse[RegexStar[arg_]] := RegexStar[regexReverse[arg]];

RegReverse[Regex[regex_], OptionsPattern[]] := Module[
  {},
  (
    Regex[regexReverse[regex]]
  )/; !OptionValue[validationRequired] || validateRLR[Regex[regex]]
];

(* union for right regular grammars *)
RegUnion[grammar1 : RRGrammar[_], RRGrammar[{}], OptionsPattern[]] := Module[
  {},
  (
    grammar1
  )/; !OptionValue[validationRequired] || validateRLR[grammar1]
];

RegUnion[RRGrammar[{}], grammar2 : RRGrammar[_], OptionsPattern[]] := Module[

```

```

    {}},
    (
      grammar2
    )/; !OptionValue[validationRequired] || validateRLR[grammar2]
  ];

RegUnion[grammar1 : RRGrammar[_], grammar2 : RRGrammar[_], OptionsPattern[]] :=
Module[
  {
    nonTerms1 = nonTerminals[grammar1],
    nonTerms2 = nonTerminals[grammar2],
    initial1, initial2,
    commonNonTerms,
    replacements, rules
  },
  (
    (* do some stuff to make sure the two grammars don't share any *)
    (* nonterminals *)
    commonNonTerms = Intersection[nonTerms1, nonTerms2];
    replacements = Table[Unique[], {Length[commonNonTerms]}];
    rules = MapThread[Rule, {commonNonTerms, replacements}];
    initial1 = grammar1[[1, 1, 1]];
    initial2 = grammar2[[1, 1, 1]] /. rules;

    (* combine the grammars *)
    RRGrammar[{Unique[] -> RRGrammarOr[initial1, initial2],
      grammar1//First, grammar2 /. rules //First} // Flatten]

  )/; !OptionValue[validationRequired] ||
  (validateRLR[grammar1] && validateRLR[grammar2])

```

```

];

(* concatenation for right regular grammars *)
RegConcat[grammar1 : RRGrammar[_], RRGrammar[{}], OptionsPattern[]] := Module[
  {},
  RRGrammar[{}] /; !OptionValue[validationRequired] ||
    validateRLR[grammar1]
];

RegConcat[grammar1 : RRGrammar[_], grammar2 : RRGrammar[_], OptionsPattern[]] :=
Module[
  {
    nonTerms1 = nonTerminals[grammar1],
    nonTerms2 = nonTerminals[grammar2],
    initial2,
    commonNonTerms,
    replacements, rules
  },
  (
    (* do some stuff to make sure the two grammars don't share any *)
    (* nonterminals *)
    commonNonTerms = Intersection[nonTerms1, nonTerms2];
    replacements = Table[Unique[], {Length[commonNonTerms]}];
    rules = MapThread[Rule, {commonNonTerms, replacements}];
    initial2 = grammar2[[1, 1, 1]] /. rules;

    RRGrammar[Flatten[{
      First[grammar1] /. {
        Verbatim[Rule][nonTerm_, EmptyWord] :>
          (nonTerm -> initial2),

```

```

RRGrammarOr[pre____, EmptyWord, post____] :=>
    RRGrammarOr[pre, initial2, post]
    }, First[grammar2] /. rules
    }]]
);

!OptionValue[validationRequired] ||
(validateRLR[grammar1] && validateRLR[grammar2])

];

(* intersection for DFAs *)
RegIntersection[dfa1:DFA[numStates1_, alphabet1_, transitionMatrix1_,
    acceptStates1_, initialState1_], dfa2:DFA[numStates2_, alphabet2_,
    transitionMatrix2_,
    acceptStates2_, initialState2_], OptionsPattern[]] := Module[
{
    newAlphabet = Intersection[alphabet1, alphabet2],
    (* new state set is cartesian product of state sets *)
    stateSet = cartesian[Range[numStates1], Range[numStates2]],
    stateNum,
    newTransitionMatrix, newAcceptStates, newInitialState,
    alphabet1Num, alphabet2Num
},
(
If[numStates1 == 0 || numStates2 == 0,
    Return[DFA[0, newAlphabet, {}, {}, Null]];
];

(* the alphabet1Num function looks up the position of a letter in *)
(* alphabet1 *)
alphabet1Num[alphabetNum_] := Position[alphabet1,
    newAlphabet[[alphabetNum]], {1}, Heads -> False][[1, 1]];

```

```

(* the alphabet2Num function looks up the position of a letter in *)
(* alphabet2 *)
alphabet2Num[alphabetNum_] := Position[alphabet2,
    newAlphabet[[alphabetNum]], {1}, Heads -> False][[1, 1]];

(* the stateNum function looks up the position of a state in stateSet *)
stateNum[state_] := Position[stateSet, state, {1}, Heads-> False][[1, 1]];

newInitialState = stateNum[{initialState1, initialState2}];

newAcceptStates = stateNum /@ Select[stateSet,
    MemberQ[acceptStates1, #][[1]]] && MemberQ[acceptStates2, #][[2]]]&];

(* compute new transition matrix *)
newTransitionMatrix = Table[
    stateNum[
        {
            transitionMatrix1[[stateSet[[state]][[1]], alphabet1Num[c]]],
            transitionMatrix2[[stateSet[[state]][[2]], alphabet2Num[c]]]
        }
    ],
    {state, 1, Length[stateSet]},
    {c, 1, Length[newAlphabet]}
];

DFA[Length[stateSet], newAlphabet, newTransitionMatrix,
    newAcceptStates, newInitialState]
)/; !OptionValue[validationRequired] ||
    (validateRLR[dfa1] && validateRLR[dfa2])
];

```



```

(* ::Section:: *)

(* To generating functions *)

(* takes an unambiguous regex lacking the head Regex and returns the *)
(* corresponding generating function *)
regex2GF[regex_, rules_] :=
FixedPoint[Replace[#,
  {
    str_String -> (str /. rules),
    Null -> 0,
    EmptyWord -> 1,
    RegexOr[args_] -> Plus[args],
    RegexConcat[args_] -> Times[args],
    RegexStar[arg_] -> 1/(1 - arg)
  },
  {0, Infinity}]]&,
  regex
];

previousOptions = Options[GeneratingFunction]

Unprotect[GeneratingFunction];
Options[GeneratingFunction] = Union[
  previousOptions,
  {validationRequired -> True}
];
Protect[GeneratingFunction];

```

```

Unprotect[GeneratingFunction];

GeneratingFunction[regex:Regex[_], rules_, OptionsPattern[]] :=

Module[
  {},
  (
    (* disambiguate the regex and pass to regex2GF *)
    regex2GF[First@ToRegex@ToDFA@regex, rules]
  )/; !OptionValue[validationRequired] || validateRules[rules, regex]
];

Protect[GeneratingFunction];

(* ::Section:: *)

(* Code generation for repetitive bits and pieces *)

(* in this section we do dynamic code generation to set up the automatic *)
(* conversion necessary before and/or after the operations and conversions *)
(* that are implemented above *)
(* for example, we implemented the intersection operation for DFAs only; this *)
(* section defines how to convert each RLR to and from DFAs in order to *)
(* use that implementation *)

viaAlgorithms = {
  (* {to, from, via} *)
  {ToNFA, DFA, ToRegex},
  {ToDFA, Regex, ToNFA},
  {ToDFA, RRGrammar, ToNFA},
  {ToDFA, Digraph, ToNFA},
  {ToRegex, NFA, ToDFA},
  {ToRegex, RRGrammar, ToDFA},
  {ToRegex, Digraph, ToDFA},

```

```

{ToRRGrammar, DFA, ToNFA},
{ToRRGrammar, Regex, ToNFA},
{ToRRGrammar, Digraph, ToNFA},
{ToDigraph, NFA, ToDFA},
{ToDigraph, Regex, ToDFA},
{ToDigraph, RRGrammar, ToDFA}
};

(* define conversion algorithms that go via
   another conversion *)
(# /. {to_, from_, via_} :>
  Hold[
    to[name : from[___], opts:OptionsPattern[]] :=
      to[via[name, opts], validationRequired -> False];
  ]) & /@
viaAlgorithms // ReleaseHold;

(* complement *)
complementVias = {NFA, Regex, RRGrammar, Digraph};

(Hold[
  RegComplement[name : #[___], alpha_, opts:OptionsPattern[]] :=
    ToExpression["To" <> ToString[#]] [
      RegComplement[ToDFA[name, opts], alpha, validationRequired -> False]
    ];
  ]) & /@
complementVias // ReleaseHold;

(* star and reverse *)
starReverseVias = {

```

```

    {RegStar, DFA, ToRRGrammar},
    {RegStar, Regex, ToRRGrammar},
    {RegStar, NFA, ToRRGrammar},
    {RegStar, Digraph, ToRRGrammar},
    {RegReverse, NFA, ToRegex},
    {RegReverse, DFA, ToRegex},
    {RegReverse, RRGrammar, ToRegex},
    {RegReverse, Digraph, ToRegex}
};

(# /. {op_, rlr_, via_} :> Hold[
    op[name : rlr[___], opts:OptionsPattern[]] :=
        ToExpression["To" <> ToString[rlr]][
            op[via[name, opts], validationRequired -> False]
        ];
    ]) & /@

starReverseVias // ReleaseHold;

(* union, concat, intersection *)
uciVias = {
    {RegUnion, DFA, ToRRGrammar},
    {RegUnion, Regex, ToRRGrammar},
    {RegUnion, NFA, ToRRGrammar},
    {RegUnion, Digraph, ToRRGrammar},
    {RegConcat, DFA, ToRRGrammar},
    {RegConcat, Regex, ToRRGrammar},
    {RegConcat, NFA, ToRRGrammar},
    {RegConcat, Digraph, ToRRGrammar},
    {RegIntersection, NFA, ToDFA},
    {RegIntersection, Regex, ToDFA},

```

```

    {RegIntersection, RRGrammar, ToDFA},
    {RegIntersection, Digraph, ToDFA}
};

(# /. {op_, rlr_, via_} :> Hold[
    op[name1 : rlr[___], name2 : rlr[___], opts:OptionsPattern[]] :=
    ToExpression["To" <> ToString[rlr]] [
        op[via[name1, opts], via[name2, opts], validationRequired -> False]
    ];
]) & /@
uciVias // ReleaseHold;

(* add options for downvalued symbols except GeneratingFunction *)
(* also the symbols with catch-all definitions *)
symbolsWithOptions = {
    ToNFA, ToDFA, ToRegex, ToRRGrammar, ToDigraph, ToRegularExpression,
    RegStar, RegComplement, RegReverse, RegUnion, RegConcat, RegIntersection
};

(MessageName[#, "invalidArgumentSyntax"] = "Invalid argument syntax.")& /@
symbolsWithOptions;

Hold[#[___] /; (Message[#::invalidArgumentSyntax]; False) := Null;]& /@
symbolsWithOptions //ReleaseHold;

(Options[#] = {validationRequired -> True})& /@ symbolsWithOptions;

End[] (* End Private Context *)

```

```
EndPackage[]
```

GFeqn2coefs.m

```
(* Mathematica Package *)
```

```
BeginPackage["Genfunlib`GFeqn2coefs`"]
```

```
CoefsByDerivs::usage = "CoefsByDerivs[{eqn1, eqn2,...}, {f,g, ...}, " <>
  "{x, 0, nx}, {y, 0, ny}, ...] gives the Taylor expansions at (0,0,...) " <>
  "of the power series f,g,... satisfying the equations.";
CoefsByNewton::usage = "CoefsByNewton[eqn, f, {x, 0, nx}] gives the " <>
  "Maclaurin expansion in x of f satisfying eqn.";
```

```
Begin["`Private`"] (* Begin Private Context *)
```

```
(* what can usually be used as a variable, according to
   ref/message/General/ivar *)
variablePattern = Except[_String | _?NumberQ | _Plus | _Times |
  _Sum | _Product | _^_Integer];
```

```
CoefsByDerivs::invalid = "Invalid input.";
CoefsByNewton::invalid = "Invalid input.";
CoefsByDerivs::invalidArgumentSyntax = "Invalid argument syntax.";
CoefsByNewton::invalidArgumentSyntax = "Invalid argument syntax.";
```

```
(* validate the equation or system of equations that the user supplies *)
```

```
validateSystem[system : {HoldPattern[_ == _]..}, series: { variablePattern.. },
  iters:({variablePattern, 0, _Integer?NonNegative}..)] := Module[
{
```

```

        ok = True
    },
    If[ !ok, Message[CoefsByDerivs::invalid]];
    ok
];

validateSystem[___] := (Message[CoefsByDerivs::invalid];False);

validateEquation[lhs_ == rhs_, series:variablePattern,
    {var:variablePattern, 0, _Integer?NonNegative}] := Module[
    {ok = True},

    ok = ok && !FreeQ[series, var] && !FreeQ[lhs-rhs, series];
    ok = ok && (D[lhs - rhs, series] /. {var->0, series->0}) != 0;

    If[ !ok, Message[CoefsByNewton::invalid]];
    ok
];

validateEquation[___] := (Message[CoefsByNewton::invalid];False);

CoefsByDerivs[x:Except[_List], y__] := CoefsByDerivs[{x}, y];
CoefsByDerivs[x_, y:Except[_List], z__] := CoefsByDerivs[x, {y}, z];

CoefsByDerivs[system_, series_, iters__] := Module[
    {
        indets = {iters} /. {x_, 0, _} :> x,
        systemCanon = system /. (l_ == r_ :> l - r),
        expansion,
        eqns, subs

```

```

    },
    (
        expansion = Normal[Series[systemCanon, iters]];
        eqns = Thread[Flatten[CoefficientList[expansion, indets]] == 0];
        subs = Flatten[Solve[eqns]];
        Thread[series == (Series[series, iters] /. subs)]
    ) /; validateSystem[system, series, iters]
];

CoefsByDerivs[____] /; (Message[CoefsByDerivs::invalidArgumentSyntax]; False) :=
    Null;

CoefsByNewton[lhs_ == rhs_, series_, {var_, 0, n_}] := Module[
    {
        f = lhs - rhs, ff, m = 2^Ceiling[Log[2, n]],
        approx = First[
            (series/.var->0) /. Solve[(lhs==rhs)/.var->0, series/.var->0]]
    },
    (
        ff[z_] := (f /. series :> z);
        Do[
            approx = Normal[# - ff[#]/ff'[#] &[approx] + 0[var]^(2^k)],
            {k, 1, Log[2, m] + 1}
        ];
        approx + 0[var]^(n + 1)
    ) /; validateEquation[lhs==rhs, series, {var, 0, n}]
];

CoefsByNewton[____] /; (Message[CoefsByNewton::invalidArgumentSyntax]; False) :=
    Null;

```



```

End[] (* End Private Context *)

EndPackage[]

rec2GFeqn.m

(* Mathematica Package *)

BeginPackage["Genfunlib`rec2GFeqn`"]

(* this file defines no public functions, however it does *)
(* override GeneratingFunction*)

Begin["`Private`"]

(* http://bit.ly/1lgtAt1 *)
GeneratingFunction;
protected = Unprotect[GeneratingFunction];

(* what can usually be used as a variable, according to
   ref/message/General/ivar *)
variablePattern = Except[_String | _?NumberQ | _Plus | _Times |
    _Sum | _Product | _^_Integer];

arg2 = Sequence[n:variablePattern, x:variablePattern];

(* rational function (over syntactic integers) coefficients *)
(* http://mathematica.stackexchange.com/a/32611/208 *)
(* partial fraction expansion *)
GeneratingFunction[expr_, arg2,

```

```

opts:OptionsPattern[]] /; expr != Apart[expr, n] :=
GeneratingFunction[Apart[expr, n], n, x, opts];

GeneratingFunction[expr_ * (n_ + k_Integer?Positive)^(j_Integer?Negative), arg2,
opts:OptionsPattern[]] := With[
{t = Unique[]},
x^(-k) * Integrate[t^(k-1) * GeneratingFunction[expr * (n + k)^(j + 1),
n, t, opts], {t, 0, x}]
];

(* symbolic sums *)
GeneratingFunction[Sum[expr_, {i_, lb_, ub_}], arg2, opts:OptionsPattern[]] /;
FreeQ[lb, n] && FreeQ[ub, n] := Sum[GeneratingFunction[expr, n, x, opts],
{i, lb, ub}];

(* different ways to set terms to 0 *)
GeneratingFunction[Boole[expr_] * fac_, arg2, opts:OptionsPattern[]] :=
GeneratingFunction[Piecewise[{{fac, expr}}], n, x, opts];

GeneratingFunction[UnitStep[expr_] * fac_, arg2, opts:OptionsPattern[]] :=
GeneratingFunction[Piecewise[{{fac, expr >= 0}}], n, x, opts];

GeneratingFunction[Piecewise[{{expr_, Divisible[n_, k_]}}], arg2,
opts:OptionsPattern[]] :=
GeneratingFunction[Piecewise[{{expr, Mod[n, k] == 0}}], n, x, opts];

(* "multisection formula" *)
GeneratingFunction[Piecewise[{{expr_, Mod[n_, k_] == j_}}], arg2,
opts:OptionsPattern[]] /; FreeQ[k, n] :=
With[{var = Unique[]},

```

```

      k^(-1) * Sum[Exp[-var*j*2*Pi*I/k] * GeneratingFunction[
        expr, n, Exp[var * 2 * Pi *I /k] * x, opts],
        {var, 0, k-1}]
];

Protect[Evaluate[protected]];

End[]
EndPackage[]

SymbolicMethod.m

(* Mathematica Package *)

BeginPackage["Genfunlib`SymbolicMethod`"]

Spec::usage = "Spec[{lhs1 == rhs1, ...}, labeled] is a combinatorial " <>
  "specification.";
SMPlus::usage = "SMPlus is the addition construction from the symbolic method.";
SMTimes::usage = "SMTimes is the multiplication construction from the " <>
  "symbolic method.";
SMSeq::usage = "SMSeq is the sequence construction from the symbolic method.";
SMCyc::usage = "SMCyc is the cycle construction from the symbolic method.";
SMSet::usage = "SMSet is the set construction from the symbolic method.";
SMPointing::usage = "SMPointing is the pointing construction from the " <>
  "symbolic method.";
SMSub::usage = "SMSub is the substitution construction from the symbolic " <>
  "method.";
SMMultiset::usage = "SMMultiset is the multiset construction from the " <>
  "symbolic method.";
Cardinality::usage = "Cardinality is an option for symbolic method " <>

```

```

    "constructions which takes a Function on the integers with which to " <>
    "restrict the construction.";
Restricted::usage = "Restricted[class, pred] is the class of objects from " <>
    "class with sizes in pred(-1)(True).";
ZClass::usage = "ZClass[n] is the nth atomic class.";
EClass::usage = "EClass is the epsilon class.";
ToGenfunlibSpec::usage = "ToGenfunlibSpec[mapleString, labeled] converts " <>
    "from Combstruct to Genfunlib combinatorial specification syntax.";
ToGFEqns::usage = "ToGFEqns[spec, indeterminate] gives the system of power " <>
    "series equations corresponding to spec, using the given indeterminate."

Begin["`Private`"] (* Begin Private Context *)

(* ::Section:: *)
(* Syntactic input validation *)

Spec::invalid = "Invalid specification.";

validatePred[_Function] := True;
validatePred[___] := False;

validateRHS[sym_, nonTerms_] /; MemberQ[nonTerms, sym] := True;
validateRHS[EClass, _] := True;
validateRHS[ZClass[n_Integer?Positive], _] := True;

validateRHS[SMPlus[args__], nonTerms_] := And @@ (validateRHS[#, nonTerms]& /@
    {args});
validateRHS[SMTimes[args__], nonTerms_] := And @@ (validateRHS[#, nonTerms]& /@
    {args});

```

```

validateRHS[SMSeq[arg_], nonTerms_] := validateRHS[arg, nonTerms];
validateRHS[SMSeq[arg_, Cardinality -> pred_], nonTerms_] := validateRHS[arg,
    nonTerms] && validatePred[pred];

validateRHS[SMCyc[arg_], nonTerms_] := validateRHS[arg, nonTerms];
validateRHS[SMCyc[arg_, Cardinality -> pred_], nonTerms_] := validateRHS[arg,
    nonTerms] && validatePred[pred];

validateRHS[SMSet[arg_], nonTerms_] := validateRHS[arg, nonTerms];
validateRHS[SMSet[arg_, Cardinality -> pred_], nonTerms_] := validateRHS[arg,
    nonTerms] && validatePred[pred];

validateRHS[SMMultiset[arg_], nonTerms_] := validateRHS[arg, nonTerms];
validateRHS[SMMultiset[arg_, Cardinality -> pred_], nonTerms_] := validateRHS[
    arg, nonTerms] && validatePred[pred];

validateRHS[SMSub[first_, second_], nonTerms_] := validateRHS[first,
    nonTerms] && validateRHS[second, nonTerms];
validateRHS[SMPointing[arg_], nonTerms_] := validateRHS[arg, nonTerms];

validateRHS[Restricted[arg_, pred_], nonTerms_] := validatePred[pred] &&
    validateRHS[arg, nonTerms];

validateRHS[___] := False;

validateSpecSyntax[spec:Spec[list:{HoldPattern[_ == _]..},
    labeled:True|False]] := Module[
{
    ok = True,
    lhss = list[[All, 1]],

```

```

        rhss = list[[All, 2]]
    },
    ok = ok && MatchQ[lhss, {(_Symbol|(_Symbol)[_Integer])...}];
    ok = ok && Length@lhss == Length@Union@lhss;
    If[labeled && MemberQ[list, SMMultiset, {0, Infinity}, Heads -> True],
        ok = False];
    ok = ok && And @@ (validateRHS[#, lhss]& /@ rhss);

    If[!ok, Message[Spec::invalid]];
    ok
];
validateSpecSyntax[___] := False;

(* ::Section:: *)
(* To GF eqns *)

(* the function restrictedSum returns a simplified form of *)
(* Sum[generalTerm*Boole[pred[n]], {n, slb, sub}] *)
restrictedSum[generalTerm_, Function[GreaterEqual[ub_: Infinity, Slot[1],
    lb_: 0]], {n_, slb_, sub_}] :=
    Sum[generalTerm, {n, Max[slb, lb], Min[sub, ub]}];

restrictedSum[generalTerm_, Function[LessEqual[lb_: 0, Slot[1], ub_: Infinity]],
    {n_, slb_, sub_}] :=
    Sum[generalTerm, {n, Max[slb, lb], Min[sub, ub]}];

restrictedSum[generalTerm_, pred_, {n_, slb_, sub_}] :=
    Sum[Boole[pred[n]] * generalTerm, {n, slb, sub}];

(* labeled *)

```

```

ToGFEqns[ spec:Spec[list_List, True], indet_Symbol ] := Module[
{
    numAtomicClasses = Max @@ First /@ Cases[list, ZClass[n_], Infinity] //
    Max[0, #]&,
    nonTerms = list[[All, 1]],
    indets,
    ret = Hold[list]
},
(
    indets = Sequence @@ indet /@ Range[numAtomicClasses];

    ret = Replace[ret,
    {
        ZClass[n_] :> indet[n],
        EClass :> 1,
        sym_Symbol?(MemberQ[nonTerms, #1] &) :> sym[indets]
    },
    {1, Infinity}, Heads -> True
];

    ret = FixedPoint[Function[iter, Replace[iter,
    {
        SMPlus[args__] :> Plus[args],
        SMTimes[args__] :> Times[args],

        SMSeq[arg_] :> 1 / (1 - arg),
        SMSeq[arg_, Cardinality -> pred_] :> With[{unique = Unique[]},
            restrictedSum[arg^unique, pred, {unique, 0, Infinity}]],

        SMCyc[arg_] :> Log[ 1/(1 - arg) ],

```

```

SMCyc[arg_, Cardinality -> pred_] := With[{unique = Unique[]},
  Boole[pred[0]] + restrictedSum[arg^unique / unique, pred, {
    unique, 1, Infinity}]],

SMSet[arg_] := Exp[arg],
SMSet[arg_, Cardinality -> pred_] := With[{unique = Unique[]},
  restrictedSum[arg^unique / (unique!), pred, {unique, 0,
    Infinity}]],

Restricted[expr_, {}] := expr,
Restricted[expr_, pred_] := With[{unique = Unique[]},
  restrictedSum[
    SeriesCoefficient[expr, {indet[1], 0, unique}] * indet[1]^
    unique,
    pred, {unique, 0, Infinity}
  ]],

SMPointing[expr_] := indet[1] * D[expr, indet[1]],

SMSub[func_, arg_] := (func /. indet[1] := arg)

}, {1, Infinity}, Heads -> True
]], ret];

ReleaseHold[ret]

) /; validateSpecSyntax[spec]
];

(* unlabeled *)

```



```

ToGFEqns[spec:Spec[list_List, False], indet_Symbol] := Module[
{
    numAtomicClasses = Max @@ First /@ Cases[list, ZClass[n_], Infinity] //
    Max[0, #]&,
    nonTerms = list[[All, 1]],
    indets,
    unique, uniqueAux,
    ret = Hold[list]
},
(
    indets = Sequence @@ indet /@ Range[numAtomicClasses];

    ret = Replace[ret,
    {
        ZClass[n_] :> indet[n],
        EClass :> 1,
        sym_Symbol?(MemberQ[nonTerms, #1] &) :> sym[indets]
    },
    {1, Infinity}, Heads -> True
];

    ret = FixedPoint[Function[iter, Replace[iter,
    {
        SMPlus[args_] :> Plus[args],
        SMTimes[args_] :> Times[args],

        SMSeq[arg_] :> 1 / (1 - arg),
        SMSeq[arg_, Cardinality -> pred_] :> With[{unique = Unique[]},
            restrictedSum[arg^unique, pred, {unique, 0, Infinity}]],

```

```

SMCyc[arg_] := With[{unique = Unique[]},
  Sum[EulerPhi[unique]/unique * Log[1/(1 - (arg /. indet[n_] :=>
    indet[n]^unique))], {unique, 1, Infinity}]],
(* p. 730 of Flajolet and Sedgewick *)
SMCyc[arg_, Cardinality -> pred_] := With[{unique = Unique[],
  uniqueAux = Unique[]},
  restrictedSum[
    SeriesCoefficient[
      Sum[EulerPhi[unique]/unique * Log[1/(1 - uniqueAux^
        unique * (arg /. indet[n_] :=> indet[n]^unique))],
        {unique, 1, Infinity}
      ],
      {uniqueAux, 0, unique}
    ],
    pred,
    {unique, 0, Infinity}
  ]
],

SMSet[arg_] := With[{unique = Unique[]},
  Exp@Sum[(-1)^(unique - 1)/unique * (arg /. indet[n_] :=>
    indet[n]^unique), {unique, 1, Infinity}]],
SMSet[arg_, Cardinality -> pred_] := With[{unique = Unique[],
  uniqueAux = Unique[]},
  restrictedSum[
    SeriesCoefficient[
      Exp@Sum[(-1)^(unique - 1)/unique * uniqueAux^unique *
        (arg /. indet[n_] :=> indet[n]^unique),
        {unique, 1, Infinity}
      ],
      {uniqueAux, 0, unique}
    ],
    pred,
    {unique, 0, Infinity}
  ]
],

```

```

        {uniqueAux, 0, unique}
    ],
    pred,
    {unique, 0, Infinity}
]
],

SMMultiset[arg_] := With[{unique = Unique[]},
    Exp@Sum[1/unique * (arg /. indet[n_] := indet[n]^unique),
        {unique, 1, Infinity}]],
SMMultiset[arg_, Cardinality -> pred_] := With[{unique = Unique[],
    uniqueAux = Unique[]},
    restrictedSum[
        SeriesCoefficient[
            Exp@Sum[1/unique * uniqueAux^unique * (arg /.
                indet[n_] := indet[n]^unique),
                {unique, 1, Infinity}
            ],
            {uniqueAux, 0, unique}
        ],
        pred,
        {unique, 0, Infinity}
    ]
],

Restricted[expr_, {}] := expr,
Restricted[expr_, param_] := With[{unique = Unique[]},
    restrictedSum[
        SeriesCoefficient[expr, {indet[1], 0, unique}] *
        indet[1]^unique,

```

```

        pred, {unique, 0, Infinity}
    ]],

    SMPointing[expr_] := indet[1] * D[expr, indet[1]],

    SMSub[func_, arg_] := (func /. indet[1] :=> arg)

    }, {1, Infinity}, Heads -> True
  ]], ret];

ReleaseHold[ret]

) /; validateSpecSyntax[spec]
];

(* ::Section:: *)
(* Construct grammar to Genfunlib spec *)

(* this function assumes there are no parentheses except to group function *)
(* arguments, assumes nonterminals are only one character, and *)
(* assumes the grammar is explicit *)
ToGenfunlibSpec[str_String, labeled:(True|False)] := Module[
{
    ret = str
},
ret = StringReplace[ret, c_?UpperCaseQ :> ToLowerCase[c] <> ToLowerCase[c]];
ret = StringReplace[ret, {
    " = " -> " == ",
    "(" -> "[", ")" -> "]",
    "pprod" -> "SMTimes",

```

```

    "uunion" -> "SMPlus",
    "sset" -> "SMSSet",
    "ppowersset" -> "SMSSet",
    "ccycle" -> "SMCyc",
    "ssubst" -> "SMSSub",
    "ssequence" -> "SMSeq",
    "zz" -> "ZClass[1]",
    "aatom" -> "ZClass[1]",
    "eepsilon" -> "EClass"
  }];

ret = ToExpression[ret, InputForm, Hold];
ret = Replace[ret,
  SMSSub[first_, second_] :> SMSSub[second, first], {0, Infinity}];
ret = Replace[ret, {
  (head : (SMSSet | SMSeq | SMCyc))[arg_, rel_[Global`card, k_]] :>
    head[arg, Cardinality -> (rel[#, k] &)],
  (head : (SMSSet | SMSeq | SMCyc))[arg_, rel_[k_, Global`card]] :>
    head[arg, Cardinality -> (rel[k, #] &)]
},
{0, Infinity}];

Spec[ret//ReleaseHold, labeled]
];

ToGFEqns::invalidArgumentSyntax = "Invalid argument syntax.";
ToGenfunlibSpec::invalidArgumentSyntax = "Invalid argument syntax.";
ToGFEqns[____] /; (Message[ToGFEqns::invalidArgumentSyntax]; False) := Null;
ToGenfunlibSpec[____] /; (Message[ToGenfunlibSpec::invalidArgumentSyntax];
  False) := Null;

```

```
End[] (* End Private Context *)
```

```
EndPackage[]
```