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PROFINITE DUALITY GROUPS

by

Andrew Fletch

A thesis submitted to the Faculty of
Graduate Studies and Research in partial fulfilment
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Doctor of Philosophy

Department of Mathematics

Carleton University

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ABSTRACT

The pseudocompact algebra $\mathbb{Z}_p[[G]]$ over $\mathbb{Z}_p$ for some profinite group G has two naturally equivalent module categories - PC_G^p , the category of right pseudocompact G -modules, and $(C_G^{t(p)})^{opp}$, the opposite category of the category of left discrete p -torsion G -modules. This equivalence is given by $\text{Hom}_{\mathbb{Z}_p}(-, \mathbb{Z}_p)$ and it establishes the important duality

$$\text{Hom}(H_k(G, B), \mathbb{Z}_p) \cong H^k(G, \text{Hom}(B, \mathbb{Z}_p)),$$

for $B \in PC_G^p$. In this dissertation we study the consequences of this equivalence when considering profinite groups, G , having the added property that $H^k(G, A)$ is finite for all k and all $A \in C_G^{f(p)}$. These groups are the profinite analogues of discrete $\overline{FP}$ -groups.

The main work of this thesis is the definition and exposition of the profinite analogue of the discrete duality groups of Bieri and Eckmann, as first suggested by Serre in a note at the end of [6]. The main extension theorems are established as well as some theorems on profinite amalgamated products of duality groups. We also consider the relationship between discrete duality groups and their profinite completions via the concept of "goodness" defined by Serre in [23]. In the case of discrete Poincaré duality groups which are good, it is possible to say that their profinite completions are also Poincaré groups (in p) for every prime p .

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List of Notation

- $\mathbb{Z}_{=p}$ - the p -adic integers
- $\mathbb{Z}$ - the profinite completion of the integers
- $\mathbb{Z}_{=p}[[G]]$ - the pseudocompact algebra over $\mathbb{Z}_{=p}$. It is isomorphic to $\lim_{i,U} \mathbb{Z}/p^i \mathbb{Z}(G/U)$
- $\mathcal{P}_{=p}$ - the category of pseudocompact $\mathbb{Z}_{=p}$ -modules
- $\mathcal{P}_{=p}^{l,p}$ - the category of left pseudocompact $\mathbb{Z}_{=p}[[G]]$ -modules
- $\mathcal{P}_{=p}^{r,p}$ - the category of right pseudocompact $\mathbb{Z}_{=p}[[G]]$ -modules
- $\mathcal{C}_G$ - the category of discrete G -modules
- $\mathcal{C}_G^{t(p)}$ - the category of left discrete p -torsion $\mathbb{Z}_{=p}[[G]]$ -modules
- $\mathcal{C}_G^{t(p)}$ - the category of discrete $\mathbb{Z}_{=p}$ -modules
- $\mathcal{C}_G^{t(p)}$ - the category of right discrete p -torsion $\mathbb{Z}_{=p}[[G]]$ -modules
- $(-)*$ - the functor $\text{Hom}_{\mathbb{Z}_{=p}}(-, \mathbb{Z}_{=p}^\infty)$
- $\mathbb{Z}_{=p}^\infty$ - the p -torsion subgroup of $\mathbb{Q}/\mathbb{Z}$
- $\rightarrow, \rightarrow$ - monomorphism, epimorphism
- Ab - the category of abelian groups
- $H^i(G, A)$ - the i^{th} cohomology group of G with coefficients in A
- $H_i(G, B)$ - the i^{th} homology group of G with coefficients in B
- $\text{cd}_p G$ - the cohomological dimension of G with respect to discrete p -torsion G -modules.
- $\hat{\otimes}$ - the completed tensor product
- $M_G^U(A)$ - the set of continuous functions from G to A such that $\sigma(ux) = u_G(x)$
- $F(X)$ - the restricted free pseudocompact G -module on a discrete set X

(usually referred to as "free")

- $\hat{F}(X)$ - the free pseudocompact G -module on the Boolean space X
- $P(f)$ - a pseudocompact G -module M has $P(f)$ if $\text{Hom}_G(M, A)$ is finite whenever A is finite
- $\overline{\text{FP}}$ -group - a profinite group G is an $\overline{\text{FP}}$ -group if $H^k(G, A)$ is finite for all k whenever A is finite
- $A \rtimes_{\theta} G$ - the semidirect product of A by G
- $I_p[[G]]$ - the augmentation ideal of $\mathbb{Z}_p[[G]]$
- $H_{\underline{L}}^{\underline{G}} K$ - the pro- $\underline{C}$ -amalgamated product of the pro- $\underline{C}$ -groups H , L and K
- $\cap$ - the cap-product
- $\cup$ - the cup-product
- $C^i(G, B)$ - the G -linear continuous functions from $\prod_{i=1}^n G$ to B
- $\text{Tot}^*(A)$ - the total complex of A
- D_p^n -group - a duality group (in p) of dimension n
- $\hat{G}$ - the profinite completion of a discrete group G
- $\hat{G}^p$ - the pro- p -completion of a discrete group G
- G_p - a Sylow p -subgroup of a profinite group G
- G^p - the maximal p -quotient of a profinite group G
- $\bigcap$ - the intersection of all subgroups of finite index in a discrete group G
- $Z \rtimes_{\theta} Z$ - the unique non-trivial semidirect product of Z by Z
- scd_p - the strict cohomological dimension of G with respect to discrete p -torsion G -modules
- $\mathbb{Z}_p[[G/N]]$ - the free $\mathbb{Z}_p$ -module on the Boolean space of cosets of N in G

Introduction

In this study we are concerned with the existence of an analogue, in the theory of profinite groups, to discrete duality groups as studied by Bieri and Eckmann in the papers [2] through [8]. We approach profinite homology and cohomology along the lines developed by Brumer in [10].

Chapter I is devoted to the study of the two categories $C_G^{t(p)}$ and ${}_G PC^p$ - the category of discrete left p -torsion G -modules and the category of pseudocompact G -modules for any profinite group G . If $C_G^{f(p)}$ is the category of finite discrete p -torsion G -modules then $C_G^{t(p)} ({}_G PC^p)$ is the category whose objects are direct (inverse) limits of objects in $C_G^{f(p)}$.

We first review the work of Brumer; in particular we note that the two categories, $(C_G^{t(p)})^{opp}$ and ${}_G PC^p$, are naturally equivalent - the equivalence given by the functor $\text{Hom}_{\mathbb{Z}_p}(-, \mathbb{Z}_p^\infty)$. From this follows the very important duality

$$(*) \quad \text{Hom}_{\mathbb{Z}_p} (H^k(G, -), \mathbb{Z}_p^\infty) \cong H^k(G, \text{Hom}_{\mathbb{Z}_p}(-, \mathbb{Z}_p^\infty)).$$

which is used throughout this work.

We next consider projective modules in the category ${}_G PC^p$. As in the general case, these are exactly the direct summands of free modules. In fact all projectives in ${}_G PC^p$ are free since $\mathbb{Z}_p[[G]]$ is a local ring. The duality between $C_G^{t(p)}$ and ${}_G PC^p$ gives us many immediate results. In particular, projective covers exist in ${}_G PC^p$ since maximal essential extensions exist in $C_G^{t(p)}$. We introduce the property $P(f)$. A module M in ${}_G PC^p$ has $P(f)$ if and only if $\text{Hom}_G(M, A)$ is finite whenever

A belongs to $C_G^{f(p)}$. $P(f)$ is equivalent to finitely generated in ${}_G PC^P$.
 But in more general categories of pseudocompact G -modules - for example,
 the category of pseudocompact $\mathbb{Z}[[G]]$ -modules - this need not be the case.
 In any event, we establish the two natural isomorphisms of functors on
 ${}_G PC^P$ (respectively PC_G^P)

$$(i) \quad \text{Hom}_G(P, \mathbb{Z}_{=p}[[G]]) \otimes_G - \cong \text{Hom}_G(P, -), \text{ and}$$

$$(ii) \quad - \otimes_G P \cong \text{Hom}_G(\text{Hom}_G(P, \mathbb{Z}_{=p}[[G]]), -)$$

for any projective pseudocompact G -module, P , having $P(f)$.

Profinite groups, G , with the property

$$H^k(G, A) \text{ is finite for all } k \text{ whenever } A \in C_G^{f(p)}$$

are called $\overline{FP}$ -groups and we show that G is an $\overline{FP}$ -group if and only if there is a G -projective resolution of $\mathbb{Z}_{=p}$ with the property that every projective has $P(f)$, i.e., is finitely generated. Hence, our condition $\overline{FP}$, is the profinite analogue to the $\overline{FP}$ -condition in discrete group theory. In this case we have that the functor $H^*(G, -)$ commutes with the taking of inverse limits of finite p -torsion discrete G -modules; i.e.,

if $M \in {}_G C^P$, $M \cong \varprojlim M_i$ where each $M_i \in C_G^{f(p)}$; then

$$H^k(G, M) \cong \varprojlim H^k(G, M_i),$$

this isomorphism being natural and topological.

Chapter II is, in part, a rewrite of the first part of Bieri's paper, [2]. We establish the existence of cap-product morphisms between profinite cohomology and homology. The naturality of these maps is shown and we prove that modulo the duality (†) these are the same as the cup-product morphisms discussed in [21]. Hence, for $C \in PC_G^P$, and $A \in C_G^{f(p)}$, the diagram

(v)

$$H_n(G, C) \xrightarrow{(-) * \cap (-)} \text{Hom}(H_{n-q}^{n-q}(G, C \otimes A)^*, H^q(G, A)^*)$$

||

||

$$H^n(G, C^*)^* \xrightarrow{(- \cup -)^*} \text{Hom}(H^{n-q}(G, \text{Hom}(A, C^*)), H^q(G, A)^*)$$

commutes, where the two isomorphisms are due to $(\dagger)$.

Chapter III contains the main work of this thesis. We define a profinite duality group (in p) of dimension n to be a profinite group G , with the property that there is a right pseudocompact G -module C , and $e \in H_n(G, C)$ such that

$$e \cap - : H^k(G, A) \rightarrow H_{n-k}^{n-k}(G, C \otimes A)$$

is a topological isomorphism for all k and all $A \in C_G^{f(p)}$. An immediate consequence of this definition is that G is an FP-group. Profinite groups with a similar cup-product isomorphism are defined by Verdier in [26]. These are called strict Cohen-Macaulay groups (in p). Because of the relationship between cup-products and cap-products seen in the above diagram one has the following

Theorem: A profinite group G is a duality group (in p) of dimension n if and only if it is an FP-group and a strict Cohen-Macaulay group (in p) of dimension n .

In §12 we prove the profinite analogues of the main extension theorems of Bieri and Eckmann. First, that if G is a duality group (in p) so is every open subgroup of G and vice versa, that if G has an open subgroup which is a duality group (in p) then so is G . Secondly, that if

$$N \twoheadrightarrow G \twoheadrightarrow Q$$

is a short exact sequence of profinite groups such that N and Q are

duality groups (in p), then so is G . And finally, if

$$N \rightarrow G \rightarrow Q$$

is as above, G is a duality group (in p), N is an FP-group and $\text{cd}_p Q < \infty$, then both N and Q are duality groups (in p).

Next we study the relationship between discrete and profinite duality groups. This relationship is developed via the notion of "goodness" introduced by Serre in [23]. A discrete group G is good if

$$H^k(\hat{G}, A) \cong H^k(G, A)$$

for all k and all $A \in C_G^f(p)$, where $\hat{G}$ is the profinite completion of G and the isomorphism is induced by the natural map $G \rightarrow \hat{G}$.

We add to the known examples of good groups by showing that if the two diagrams

$$\begin{array}{ccc} H & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & A *_H B \end{array} \quad \text{and} \quad \begin{array}{ccc} \hat{H} & \longrightarrow & \hat{A} \\ \downarrow & & \downarrow \\ \hat{B} & \longrightarrow & \hat{A} *_H \hat{B} \end{array}$$

exist for the discrete groups H , A and B , then

$$\hat{A} \hat{H} \hat{B} \cong \hat{A} *_H \hat{B}$$

and that in this case goodness is a property closed with respect to taking amalgamated products.

As well, we show that the profinite completion of a discrete Poincaré duality group which is good, is also a Poincaré group (in p) for every prime p .

The notion of p -goodness, or goodness with respect to the pro- p completion of a discrete group, is considered and theorems similar to

those about good groups are proved..

§ 14 deals with profinite amalgamated products and their relationship to duality groups (in p). Several results similar to those proven in [8] are shown to be true for profinite groups as well.

Finally we give some examples of profinite duality groups (in p).

The idea for this dissertation came from a note added to [6] in which Serre pointed out the relationship between discrete duality groups and strict Cohen-Macaulay groups as their profinite analogues. The author is greatly indebted to Professor Serre for this observation.

CHAPTER I - PC_G^p and $C_G^t(p)$

§1 Preliminaries

Gabriel, in [12], studied locally finite categories: abelian categories with exact direct limits and a small class of generators of finite length. Examples of these categories most frequently seen in the study of profinite groups are $C_G^f(p)$ and $C_G^t(p)$: the categories of finite p -torsion, and p -torsion discrete left G -modules for some profinite group G .

The ring $\mathbb{Z}_p[[G]] \cong \varprojlim_{I,U} \mathbb{Z}/p^i \mathbb{Z}(G/U)$ is a special example of a pseudocompact ring in the sense of Gabriel and of a pseudocompact algebra over $\mathbb{Z}_p$ (the ring of p -adic integers) as studied by Brumer, [10].

Definition 1.1: A pseudocompact ring Λ is a Hausdorff, complete topological ring with a system of open neighbourhoods of (0) consisting of ideals I of Λ such that Λ/I is of finite length as a Λ -module.

Both $\mathbb{Z}_p$ and $\mathbb{Z}_p[[G]]$ are examples of pseudocompact rings.

Definition 1.2 [10]: If Ω is a commutative pseudocompact ring then a pseudocompact ring Λ is a pseudocompact algebra over Ω if

- (i) Λ is an Ω -algebra in the usual sense, and
- (ii) Λ admits a system of open neighbourhoods of (0) , consisting of ideals I such that Λ/I has finite length as an Ω -module.

Now a module M over a pseudocompact ring Λ is pseudocompact if, to begin with, it has a complete Hausdorff topology, and a system of open neighbourhoods of (0) consisting of submodules with factors of

finite length. Pseudocompact modules over Λ are therefore inverse limits of Λ -modules of finite length. For discrete $\mathbb{Z}_{-p}[[G]]$ -modules, finite length is equivalent to finite and hence pseudocompact to pro- p -abelian. We let PC_G^P (respectively ${}_G PC^P$) be the category of right (respectively left) pseudocompact G -modules.

The crucial property of PC_G^P is that it is categorically dual to $C_G^{t(p)}$. In [10], Brumer has shown that $\text{Hom}_{\mathbb{Z}_{-p}}(-, \mathbb{Z}_{-p}^\infty)$ (written $\text{Hom}(-, \mathbb{Z}_{-p}^\infty)$), where $\mathbb{Z}_{-p}^\infty$ is the p -torsion subgroup of $\mathbb{Q}/\mathbb{Z}$, defines a categorical equivalence between PC_G^P and $C_G^{t(p)}$. In this equivalence, the functorial inverse of $\text{Hom}(-, \mathbb{Z}_{-p}^\infty)$ is again $\text{Hom}(-, \mathbb{Z}_{-p}^\infty)$. In both cases the topology on $\text{Hom}(M, \mathbb{Z}_{-p}^\infty)$ is that of pointwise convergence. When $M \in PC_G^P$, this topology is discrete; when $M \in C_G^{t(p)}$, it is compact. In what follows we shall write $\text{Hom}(-, \mathbb{Z}_{-p}^\infty)$ as $(-)^*$ and $\text{Hom}(M, \mathbb{Z}_{-p}^\infty)$ as M^* , relying on context to distinguish the category with which we are dealing.

Just as $C_G^{t(p)}$ is an abelian category with exact direct limits and enough injectives, PC_G^P , by duality, is abelian, has exact inverse limits, and enough projectives. The bifunctor $\text{Hom}_G(-, -) : PC_G^P \times C_G^{t(p)} \rightarrow C^{t(p)}$ is left exact and if $P \in PC_G^P$ is projective then $\text{Hom}_G(P, -) : C_G^{t(p)} \rightarrow C^{t(p)}$ is exact. A similar statement holds for $\text{Hom}_G(-, I)$ when $I \in C_G^{t(p)}$ and I is injective. We may define the right-derived functor $\text{Ext}_G^1(A, B)$ of $\text{Hom}_G(A, B)$ by using either a projective resolution of A or an injective resolution of B and when $A \cong \mathbb{Z}_{-p}$, $\text{Ext}_G^1(\mathbb{Z}_{-p}, B) \cong H^1(G, B)$, [10].

It is now possible to define a "tensor product" for pseudocompact G -modules with the familiar universal property. Explicitly, suppose A is a right and B a left pseudocompact G -module, then $A \cong \varprojlim_{i \in I} A_i$ and $B \cong \varprojlim_{j \in J} B_j$ where each A_i (respectively B_j) is a finite right (resp. left) discrete G -module. The set $\{A_i \otimes_{G_i} B_j\}_{(i,j) \in I \times J}$ forms an inverse system of finitely discrete $\mathbb{Z}$ -modules and so $A \hat{\otimes}_G B \cong \varprojlim_{i \in I} A_i \otimes_{G_i} B_j$ is a pseudocompact $\mathbb{Z}$ -module. We call $A \hat{\otimes}_G B$ the complete tensor product of A and B . The exact sequence

$$K_{ij} \rightarrow A \otimes_{G_i} B \rightarrow A_i \otimes_{G_i} B_j$$

shows that $A \hat{\otimes}_G B$ is the completion of $A \otimes_G B$ with the topology induced by taking $\{K_{ij}\}_{(i,j) \in I \times J}$ as a fundamental system of open neighbourhoods of (0) . The bifunctor $-\hat{\otimes}_G-$ is a right exact bifunctor from the category of pseudocompact G -modules to the category of pseudocompact $\mathbb{Z}$ -modules. If either A or B is a finitely generated G -module then $A \hat{\otimes}_G B \cong A \otimes_G B$, [10]. Also, if A is projective then $A \hat{\otimes}_G-$ is an exact functor. It follows, therefore, that we may define the left-derived functor $\text{Tor}_i^G(A, B)$ of $A \hat{\otimes}_G B$ and when $B \cong \mathbb{Z}_{\mathbb{P}}$, we call $\text{Tor}_i^G(A, \mathbb{Z}_{\mathbb{P}})$ the i^{th} homology group of G with coefficients in A and write $H_i(G, A)$.

Proposition 1.3 [10]: Let $G \cong \varprojlim_i G_i$, $A \cong \varprojlim_i A_i$, $B \cong \varprojlim_i B_i$ where each A_i is a discrete G_i -module and each B_i is a pseudocompact G_i -module such that the G_i -action on A_i (resp. B_i) is compatible with the G_j -action on A_j (resp. B_j) whenever $A_i \rightarrow A_j$ (resp. $B_i \rightarrow B_j$). Then

$$(i) \ H^k(G, A) \cong \varinjlim H^k(G_i, A_i) \cong \varinjlim \text{Ext}_{G_i \mathbb{P}}^k(\mathbb{Z}_{\mathbb{P}}, A_i) \cong \text{Ext}_{G \mathbb{P}}^k(\mathbb{Z}_{\mathbb{P}}, A), \text{ and}$$

$$(ii) H_k(G, B) \cong \varprojlim H_k(G_i, B_i) \cong \varprojlim \text{Tor}_k^{G_i}(B_i, \mathbb{Z}_{\mathbb{P}}) \cong \text{Tor}_k^G(B, \mathbb{Z}_{\mathbb{P}}). //$$

We now state, in its greatest generality, the most valuable property stemming from the duality of the categories of discrete and pseudocompact modules over a pseudocompact ring.

Proposition 1.4 [10]: Let Λ and Γ be pseudocompact Ω -algebras. Let A be a left pseudocompact Λ -module, B a right (Λ, Γ) -pseudocompact module, and C be a discrete right Γ -module. Then there is a unique isomorphism:

$$\varphi : \text{Hom}_{\Lambda}(A, \text{Hom}_{\Gamma}(B, C)) \rightarrow \text{Hom}_{\Gamma}(B \otimes_{\Lambda} A, C)$$

such that $\varphi(f)(b \otimes a) = f(a)(b)$. This morphism induces a natural equivalence of functors. //

Corollary 1.5 [10]: If C is an injective Γ -module in the above proposition, then $\text{Ext}_{\Lambda}^n(A, \text{Hom}_{\Gamma}(B, C)) \cong \text{Hom}_{\Gamma}(\text{Tor}_{\Lambda}^n(B, A), C)$. //

Corollary 1.6: $H^n(G, B^*) \cong H_n(G, B)^*$. //

We end this section with a brief comment on the familiar maps "restriction" and "corestriction".

Definition 1.7: Let $H \leq G$. Then the map $H_k(H, C) \rightarrow H_k(G, C)$ induced by this inclusion is called corestriction (written cor_- to indicate it is a map between homology objects).

Proposition 1.8: If $H \leq G$ and $\text{res}^- : H^k(G, -) \rightarrow H^k(H, -)$ is the natural restriction map induced by this inclusion, then the following diagram

commutes for $C \in \text{PC}_G^{\mathbb{P}}$

$$\begin{array}{ccc} H_k(G, C)^* & \cong & H^k(G, C^*) \\ \downarrow (\text{cor}_-)^* & & \downarrow \text{res}^- \\ H_k(H, C)^* & \cong & H^k(H, C^*) \end{array}$$

Proof: $\text{cor}_- : H_k(H, C) \rightarrow H_k(G, C^*)$ is induced by the map $C \otimes_{H=p} Z[[X^k H]] \rightarrow C \otimes_{G=p} Z[[X^k G]]$ while $\text{res}_- : H^k(G, C^*) \rightarrow H^k(H, C^*)$ is induced by the map $C^k(G, C^*) \rightarrow C^k(H, C^*)$ (see §2 and [21], p.134). It is easy to check that the diagram

$$\begin{array}{ccc} (C \otimes_{H=p} Z[[X^k H]])^* \cong \text{Hom}_{H=p}(Z[[X^k H]], C^*) \cong C^k(H, C^*) & & \\ \uparrow (\text{cor}_-)^* & \uparrow & \uparrow \text{res}_- \\ (C \otimes_{G=p} Z[[X^k G]])^* \cong \text{Hom}_{G=p}(Z[[X^k G]], C^*) \cong C^k(G, C^*) & & \end{array}$$

commutes, hence taking homology yields the proposition. //

As in the non-completed case, complete tensor products are associative. As well, a projective object in PC_G^p is projective when considered as an object in PC_U^p for $U \leq G$. Hence we have the lemma of Shapiro

$$H_k(G, C \otimes_{U=p} Z[[G]]) \cong H_k(U, C)$$

for $U \leq G$, $C \in \text{PC}_U^p$. The proof of this is identical to the discrete case.

Now for $C \in \text{PC}_G^p$, U open in G , the G -morphism $C \rightarrow C \otimes_{U=p} Z[[G]]$ defined by $c \mapsto \sum x_i^{-1} \otimes x_i$, where $\{x_i\}$ is a right transversal of U in G , induces a morphism of homology groups

$$H_k(G, C) \rightarrow H_k(G, C \otimes_{U=p} Z[[G]]) \cong H_k(U, C).$$

which we call restriction (written res_-). This map also induces the map

$$\text{Hom}_U(Z[[G]], C^*) \cong (C \otimes_{U=p} Z[[G]])^* \rightarrow C^*$$

which sends $f \in \text{Hom}_U(Z[[G]], C^*)$ to $\sum x_i^{-1} f(x_i) \in C^*$ and from [10], p. 456,

we know this induces the corestriction map $\text{cor}_- : H^k(U, C^*) \cong$

$H^k(G, \text{Hom}_U(Z[[G]], C^*)) \rightarrow H^k(G, C^*)$. Hence we have the following proposition:

Proposition 1.9: For $C \in \text{PC}_G^p$, U open in G , the following diagram commutes.

$$\begin{array}{ccc}
 H_k(G, C)^* & \cong & H^k(G, C^*) \\
 \uparrow (res_-)^* & & \uparrow cor^- \\
 H_k(U, C)^* & \cong & H^k(U, C^*) \quad //
 \end{array}$$

In dealing with the morphism $cor^- : H^k(U, -) \rightarrow H^k(G, -)$ it will often be useful to view this map as one induced by certain module homomorphisms.

To this end we prove the following:

Proposition 1.10: Let U , V and G be profinite groups such that U and V are open normal subgroups of G and $U \leq V$. Then the following diagram commutes.

$$\begin{array}{ccc}
 H^k(V, Z/p^m Z) & \cong & H^k(G, Z/p^m Z(G/V)) \\
 \uparrow cor^- & & \uparrow \pi' \\
 H^k(U, Z/p^m Z) & \cong & H^k(G, Z/p^m Z(G/U))
 \end{array}$$

where π is the canonical map $Z/p^m Z(G/U) \rightarrow Z/p^m Z(G/V)$.

Proof. The diagram

$$\begin{array}{ccc}
 H^k(V, A) & \cong & H^k(G, M_G^V(A)) \\
 \uparrow cor^- & & \uparrow \alpha' \\
 H^k(U, A) & \cong & H^k(G, M_G^U(A))
 \end{array}$$

1.11 ...

$A \in C_G^{t(p)}$, commutes, where $M_G^V(A) = \{ \sigma : G \rightarrow A \text{ such that } \sigma(vg) = v\sigma(g), \\ v \in V \}$ and the map α' is induced by $\alpha : M_G^U(A) \rightarrow M_G^V(A)$ which sends σ to $\sum x_i \sigma(x_i^{-1} -)$ where $\{x_i\}$ is a transversal of U in V . This is well-known and proven by showing the commutativity at H^0 .

Moreover, the diagram of left G -modules

$$\begin{array}{ccc}
 M_G^V(Z/p^m Z) & \xrightarrow{\varphi_{m,V}} & Z/p^m Z(G/V) \\
 \uparrow \alpha & & \uparrow \pi \\
 M_G^U(Z/p^m Z) & \xrightarrow{\varphi_{m,U}} & Z/p^m Z(G/U)
 \end{array}$$

commutes, where $\varphi_{m,V}(\sigma) = \sum \sigma(z_i^{-1})z_i$ for $\sigma \in M_G^V(Z/p^m Z)$ and $\{z_i\}$ a transversal of V in G . Hence the diagram

$$\begin{array}{ccc}
 H^k(G, M_G^V(Z/p^m Z)) & \xrightarrow{\varphi'_{m,V}} & H^k(G, Z/p^m Z(G/V)) \\
 \uparrow \alpha' & & \uparrow \pi' \\
 H^k(G, M_G^U(Z/p^m Z)) & \xrightarrow{\varphi'_{m,U}} & H^k(G, Z/p^m Z(G/U))
 \end{array}$$

commutes and this, together with (1.11), yields the result. //

Notes: 1.12 It makes no difference in this proposition if $Z/p^m Z$ and $Z/p^m Z(G/V)$ are considered as left or right G -modules.

1.13 This proposition permits us to say that

$$\lim_{U, \text{cor}, m} H^k(U, Z/p^m Z) \cong \lim_{U, m} H^k(G, Z/p^m Z(G/U))$$

as either left or right G -modules. The G -action on $H^k(U, Z/p^m Z)$ is induced by the G -action on $C_U^k(G, Z/p^m Z)$ defined on page 172, [21], and the G -action on $H^k(G, Z/p^m Z(G/U))$ is the G -action on $Z/p^m Z(G/U)$.

§ 2 - Free Modules and Projective Modules

PC_G^p has enough projectives, in fact free objects exist in PC_G^p and are found as follows: Let I be any discrete index set and let $L(X)$ be the free (non-topological) $\bigoplus_{p \in \mathbb{P}} \mathbb{Z}[[G]]$ -module on a set X indexed by I . If

$$S = \{N \leq L(X) : L(X)/N \text{ is a finite } \bigoplus_{p \in \mathbb{P}} \mathbb{Z}[[G]]\text{-module and}$$

$N \text{ contains all but a finite number of elements of } X\}$

then $F(X) \cong \varprojlim_{N \in S} L(X)/N$ is a pseudocompact G -module, the restricted free pseudocompact G -module on X . It can be shown that $F(X) \cong \prod_{I \neq \emptyset} \bigoplus_{p \in \mathbb{P}} \mathbb{Z}[[G]]$, [10], hence in case X is finite, $F(X) \cong L(X)$. We write free for restricted free.

Proposition 2.1 [10]: The morphisms of $F(X)$ into $A \in PC_G^p$ are in 1-1 correspondence with the sequences $\{a_i\}_{i \in I}$ in A which tend to 0 on the filter of complements of finite sets. //

Corollary 2.2: $F(X)$ is projective in PC_G^p .

Proof: This follows after one notices that continuous sections of epimorphisms exist in PC_G^p as they do in the category of profinite groups. //

A more general kind of "freeness" is available as well in PC_G^p .

Definition 2.3: Let X be a Boolean topological space (i.e., $X \cong \varprojlim_{i \in I} X_i$, where each map $X \rightarrow X_i$ is onto) and let $L(X)$ be the free (non-topological) G -module on X . If

$$S = \{N \leq L(X) : L(X)/N \text{ is finite and } (\ell + N) \cap X$$

$\text{is open in } X \text{ for all } \ell \in L(X)\}$

then $\hat{F}(X) \cong \varprojlim_{N \in S} L(X)/N$ is called the free pseudocompact G -module on the Boolean space X .

Proposition 2.4: Every continuous function $v : X \rightarrow A$, where $A \in PC_G^p$, extends uniquely to a continuous morphism $\hat{F}(X) \rightarrow A$.

Proof: If X is finite, then the proposition follows from Proposition 2.1.

If X is infinite, then $X = \lim_{\substack{\longrightarrow \\ i \in I}} X_i$ where each X_i is finite and $\hat{F}(X_i) = F(X_i)$.

We first show that $\hat{F}(X) \cong \lim_{\substack{\longrightarrow \\ i \in I}} \hat{F}(X_i)$.

But $\hat{F}(X) \cong \lim_{\substack{\longrightarrow \\ N \in \mathcal{N}}} L(X)/N$ and each $L(X)/N$ contains some open subset of X

so there exists X_i such that the diagram

$$\begin{array}{ccc} L(X) & \xrightarrow{\quad} & L(X)/N \\ & \searrow & \nearrow \\ & L(X_i) & \end{array}$$

commutes. Hence the finite quotients of $\hat{F}(X)$ are exactly the finite quotients of some $\hat{F}(X_i)$ for $i \in I$ and vice versa, so as inverse limits $\hat{F}(X)$ and $\lim_{\substack{\longrightarrow \\ i \in I}} \hat{F}(X_i)$ are the same.

Now X embeds in $\hat{F}(X)$ since each X_i embeds in $\hat{F}(X_i)$. If $v : X \rightarrow A$ is a continuous function and $A \in PC_G^p$ so that $A \cong \lim_{\substack{\longrightarrow \\ j \in J}} A_j$, then v corresponds to a system of maps $v_i : X_i \rightarrow A_i$ indexed by J and the diagram

$$\begin{array}{ccc} X_i & \xrightarrow{v_i} & A_i \\ \downarrow & & \downarrow \\ X_j & \xrightarrow{v_j} & A_j \end{array}$$

commutes whenever it is defined. Hence the diagram

$$\begin{array}{ccc} \hat{F}(X_i) & \xrightarrow{v'_i} & A_i \\ \downarrow & & \downarrow \\ \hat{F}(X_j) & \xrightarrow{v'_j} & A_j \end{array}$$

also commutes and v'_i and v'_j are unique with respect to the property that they extend v_i and v_j respectively. There is therefore a unique morphism $\varprojlim \hat{F}(X_i) \rightarrow A$ extending the system $\{v'_i\}$ of morphisms from $\{\hat{F}(X_i)\}$ to $\{A_i\}^\circ$. //

Notes: 2.5 $Z_{=p}[[G]]$ is the free pseudocompact $Z_{=p}$ -module on G .

2.6 $Z_{=p}[[{}^n_X G]]$ is the free pseudocompact G -module on ${}^{n-1}_X G$.

2.7 If $H \leq G$, then $Z_{=p}[[G]]$ is the free pseudocompact H -module on the Boolean space of cosets of H in G (written G/H). Notes 2.5 and 2.6 follow easily from the fact that $Z_{=p}(G)$ is the free (non-topological) $Z_{=p}$ -module on G and Definition 2.3. Note 2.7 follows from viewing $Z_{=p}[[G]]$ as $\varprojlim Z_{=p}(G/U)$ where each $Z_{=p}(G/U)$ is a free $Z_{=p}(HU/U)$ -module on the cosets of HU/U in G/U and then applying Proposition 1.7 of [15] (see Proposition 2.10).

Proposition 2.8: A free pseudocompact G -module on a Boolean space X is projective in PC_G^p .

Proof: As in Corollary 2.2. //

Now, the completed group algebra $Z_{=p}[[G]]$ is a local ring and in non-topological theory it is known that all projective modules of a local ring are free. This is also true for projective pseudocompact modules over $Z_{=p}[[G]]$.

Proposition 2.9: Let P be a projective module in PC_G^p . Then P is free.

Proof: By Corollary 1, p.393, [12], every projective $Z_{=p}[[G]]$ -module has a product decomposition into indecomposable projectives, each of which is a right ideal of $Z_{=p}[[G]]$ generated by a primitive idempotent. Hence this

is true of $\mathbb{Z}_p[[G]]$ itself. However, if e is a non-trivial idempotent in $\mathbb{Z}_p[[G]]$ so is $1-e$ and they both belong to the maximal ideal of $\mathbb{Z}_p[[G]]$ since $\mathbb{Z}_p[[G]]$ is local. Hence 1_e belongs to the maximal ideal, a contradiction. Therefore $\mathbb{Z}_p[[G]]$ has no non-trivial primitive idempotents and every indecomposable projective is isomorphic to $\mathbb{Z}_p[[G]]$ and hence is free. Therefore every projective is free. //

Next we exhibit a special projective resolution of $\mathbb{Z}_p$ which will be useful in our study. The sequence

$$\rightarrow \mathbb{Z}/p^i \mathbb{Z}(XG/U) \xrightarrow{d_{n-1}} \mathbb{Z}/p^i \mathbb{Z}(X^{n-1}G/U) \rightarrow \dots \rightarrow \mathbb{Z}/p^i \mathbb{Z}(G/U) \xrightarrow{\epsilon} \mathbb{Z}/p^i \mathbb{Z} \rightarrow 0$$

where $d_{n-1}(\bar{g}_0, \dots, \bar{g}_{n-1}) = \sum_{i=0}^{n-1} (-1)^i (\bar{g}_0, \dots, \hat{\bar{g}}_i, \dots, \bar{g}_{n-1})$ and $\epsilon(\bar{g}) = \bar{1}$

is an exact sequence of $\mathbb{Z}/p^i \mathbb{Z}(G/U)$ -modules. Moreover, the diagram

$$\rightarrow \mathbb{Z}/p^i \mathbb{Z}(XG/U) \xrightarrow{d_{n-1}} \mathbb{Z}/p^i \mathbb{Z}(X^{n-1}G/U) \rightarrow \dots \rightarrow \mathbb{Z}/p^i \mathbb{Z}(G/U) \xrightarrow{\epsilon} \mathbb{Z}/p^i \mathbb{Z} \rightarrow 0$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$\rightarrow \mathbb{Z}/p^j \mathbb{Z}(XG/V) \xrightarrow{d_{n-1}} \mathbb{Z}/p^j \mathbb{Z}(X^{n-1}G/V) \rightarrow \dots \rightarrow \mathbb{Z}/p^j \mathbb{Z}(G/V) \xrightarrow{\epsilon} \mathbb{Z}/p^j \mathbb{Z} \rightarrow 0$$

commutes when defined (i.e., when $i \geq j$ and $U \leq V$) so taking inverse limits with respect to i and U we get the exact sequence of G -modules

$$\rightarrow \mathbb{Z}_p[[XG]] \xrightarrow{d_{n-1}} \mathbb{Z}_p[[X^{n-1}G]] \rightarrow \dots \rightarrow \mathbb{Z}_p[[G]] \xrightarrow{\epsilon} \mathbb{Z}_p \rightarrow 0.$$

(since $\varprojlim$ is exact). The terms $\mathbb{Z}_p[[X^k G]]$ of this sequence are projective by Corollary 3.3, [10] or Note 2.6.

Finally we exhibit a few properties of $\mathbb{Z}_p[[G]]$ where G is an infinite profinite group. First we note that for a finite group G , $\mathbb{Z}_p[[G]]^G$ (the elements of $\mathbb{Z}_p[[G]]$ fixed by the action of G) is non-zero. However, by allowing R to represent either $\mathbb{Z}_p$ or $\mathbb{Z}/p\mathbb{Z}$ and allowing G to be infinite we have the following:

Proposition 2.10: Let $N \leq G$. If $p^\infty \nmid \#N$, then $R[[G]]^N \cong R[[G/N]]$, the free pseudocompact R -module on the Boolean space, G/N , of cosets of N in G .

Proof: $R[[G]] \cong \varprojlim R(G/U)$. If $\alpha \in R[[G]]^N$ then $\alpha = (\alpha_U)$ and $nU\alpha_U = \alpha_U$ for all $n \in N$, hence $\alpha_U \in R(G/U)^{NU/U}$. Conversely, if $(\alpha_U) \in R(G/U)^{NU/U}$ for all U in a cofinal system then $(\alpha_U) \in R[[G]]^N$ and hence $R[[G]]^N \cong \varprojlim R(G/U)^{NU/U}$.

Now each $R(G/U)^{NU/U} \cong R(G/NU)$ where G/NU is the set of cosets of NU in G , i.e., $R(G/U)^{NU/U}$ is the direct sum, as a pseudocompact R -module, of copies of $R(NU/U)^{NU/U} \cong R$ indexed by the different coset representatives of NU/U in G/U . If $U \leq V$, the map $R(G/U) \rightarrow R(G/V)$, when restricted to $R(G/U)^{NU/U}$, maps into $R(G/V)^{NV/V}$ and acts as multiplication by the index $(N \cap V : N \cap U)$, i.e., the image of an element of $R(G/U)^{NU/U}$ in $R(G/V)^{NV/V}$ is $(N \cap V : N \cap U) \cdot (\text{element of } R(G/V)^{NV/V})$. If $p^\infty \nmid \#N$, there exists U_1 such that for every $V \leq U_1$, $((N \cap V : N \cap U), p) = 1$ so this morphism is an epimorphism induced by the natural map $G/V \rightarrow G/U_1$ hence

$$\varprojlim_{U \leq U_1} R(G/U)^{NU/U} \cong \varprojlim_{U \leq U_1} R(G/NU) \cong R[[G/N]]. //$$

Corollary 2.11: $R[[G]]^G = 0$ iff every open subgroup U of G has the property that there exists V open and contained in U such that $p \mid (U:V)$.

Proof: Let $G = N$ in Proposition 2.10. From the proof of Proposition 2.10 we see that the inverse limit, $\varprojlim R(G/U)^{G/U}$, is 0 where the morphism from $R(G/U)^{G/U}$ to $R(G/V)^{G/V}$ is given by $\cdot (V:U)$. Having fixed V , if

$p \nmid (V:U)$ for every $U \leq V$, then

$$\varprojlim_{U \leq V} R(G/U)^{G/U} \cong \varprojlim_{U \leq V} R(G/U)^{G/U} \neq 0,$$

a contradiction, hence for any V , there exists $U \leq V$ such that

$p \mid (V:U)$. Conversely, if for every V there exists $U \leq V$ such that

$p \mid (V:U)$ and $R[[G]]^G \neq 0$, the proof of Proposition 2.10 shows that R

must have elements which are p^k -divisible for arbitrarily large k , and

this is not true; hence $R[[G]]^G = 0$. //

Corollary 2.12: If $R[[G]]^G \neq 0$ then $R[[G]]^G \cong R$. //

Corollary 2.13: $R[[G]]^G = 0$ iff $p^\infty \mid \#G$. //

§ 3 - $\mathbb{Z}_{=p}$ -modules

Let PC^p be the category of right pseudocompact $\mathbb{Z}_{=p}$ -modules and $C^{t(p)}$ the category of left torsion discrete $\mathbb{Z}_{=p}$ -modules. The purpose of this section is to prove the following:

Theorem 3.1: The following are equivalent for $M \in PC^p$.

- (i) M is flat as a $\mathbb{Z}_{=p}$ -module in the sense that $M \hat{\otimes}_{\mathbb{Z}_{=p}} -$ (written $M \hat{\otimes} -$) is an exact functor on PC^p .
- (ii) M^* is injective in $C^{t(p)}$
- (iii) M^* is divisible as an abelian group.
- (iv) M is torsion-free as an abelian group.
- (v) M is torsion-free as a $\mathbb{Z}_{=p}$ -module.
- (vi) M is projective in PC^p .
- (vii) M is free in PC^p .

Proof: (i) $\Rightarrow$ (v): For $\alpha \in \mathbb{Z}_{=p}$, left multiplication by α is a monomorphism $\mathbb{Z}_{=p} \xrightarrow{\cdot\alpha} \mathbb{Z}_{=p}$. Hence $M \otimes_{\mathbb{Z}_{=p}} \mathbb{Z}_{=p} \xrightarrow{\cdot\alpha} M \otimes_{\mathbb{Z}_{=p}} \mathbb{Z}_{=p}$ is monic (since M is flat) and so M is torsion-free as a $\mathbb{Z}_{=p}$ -module.

(v) $\Rightarrow$ (iv): $\mathbb{Z} \subseteq \mathbb{Z}_{=p}$ so $\mathbb{Z}$ -torsion-freeness follows from $\mathbb{Z}_{=p}$ -torsion-freeness.

(iv) $\Rightarrow$ (iii): If M is torsion-free, then multiplication by p is a monomorphism $M \xrightarrow{\cdot p} M$. Hence $M^* \xrightarrow{\cdot p} M^*$ so that M^* is p -divisible. Since M^* is p -torsion it is n -divisible for all n relatively prime to p as well.

(iii) $\Rightarrow$ (ii): $C^{t(p)}$ is exactly the p -torsion abelian groups with group homomorphisms and a divisible group is injective in this category.

(ii) $\Rightarrow$ (i): M^* injective in $C^{t(p)}$ implies $\text{Hom}(-, M^*)$ is exact. But $\text{Hom}(-, M^*) \cong \text{Hom}(M \otimes -, \mathbb{Z}_p^\infty)$ so $(M \otimes -)^*$ is exact and hence $M \otimes -$ is exact.

(ii) $\Leftrightarrow$ (vi): This is proven by the duality of PC^p and $(C^{t(p)})^{\text{opp}}$.

(vi) $\Rightarrow$ (vii): Let $G = 1$ in Proposition 2.9.

(vii) $\Rightarrow$ (vi): All free modules are projective. //

§ 4 - Covers in PC_G^P : Essential Extensions in $C_G^{t(p)}$

Definition 4.1: Let $M, N \in PC_G^P$. We say that N is small in M if $N \subseteq M$

and whenever $X \subseteq M$ is such that $N + X = M$, then $X = M$.

Definition 4.2: For $M, N \in PC_G^P$, we say that M covers N if there is an epimorphism $M \rightarrow N$ whose kernel is small in M .

Definition 4.3: Suppose $A, B \in C_G^{t(p)}$ and that $A \subseteq B$. B is said to be an essential extension of A if whenever $X \subseteq B$ such that $X \cap A = (0)$, then $X = (0)$.

Recall that maximal essential extensions are injective. By Gabriel, ([12], p. 361), $C_G^{t(p)}$ is a category with maximal essential extensions.

Duality gives us the following theorem:

Theorem 4.4: PC_G^P is a category with projective covers.

Proof: The theorem is proven by the following two lemmas.

Lemma 4.5: P is projective in PC_G^P iff P^* is injective in $C_G^{t(p)}$. //

Lemma 4.6: Given $M \rightarrow N$ in PC_G^P , then M covers N iff M^* is an essential extension of N^* in $C_G^{t(p)}$.

Proof: ($\Rightarrow$) Suppose $K \rightarrow M \rightarrow N$ is an exact sequence of pseudocompact G -modules and K is small in M . Let us suppose there exists $T \subseteq M^*$ and furthermore let us suppose that $N^* \cap T = (0)$. Then the morphism $M^{**} \rightarrow T^*$ has kernel L and the map $L \rightarrow M^{**} \rightarrow N^{**}$ is onto since the map $N^* \rightarrow M^* \rightarrow M^*/T$ is monic and $L \cong (M^*/T)^*$. Hence $L + K^{**} = M^{**}$ and since K^{**} is small in M^{**} , we have that $L = M^{**}$, hence $T = (0)$ and M^* essentially extends N^* .

($\Leftarrow$) Suppose $K \rightarrow M \rightarrow N$ is exact in PC_G^P and that M^* essentially extends N^* . If there exists $X \subseteq M$ such that $K + X = M$, then the map $K \rightarrow M \rightarrow M/X$ is epic and hence the map $(M/X)^* \rightarrow M^* \rightarrow K^*$ is monic. But this is the same as saying $N^* \cap (M/X)^* = (0)$ which, by our hypothesis, means $(M/X)^* = (0)$ or $X = M$, hence K is small in M . //.

§ 5 - Finitely Generated Pseudocompact Modules

$M \in {}_G\text{PC}^p$ is finitely generated if and only if it is finitely generated as an object of $\text{Mod}(\mathbb{Z}_p[[G]])$. It is clear that if M is finitely generated then $\text{Hom}_G(M, A)$ is finite for every $A \in C_G^{f(p)}$. We shall show in this section that the converse holds; i.e., if $M \in {}_G\text{PC}^p$ has the property that $\text{Hom}_G(M, A)$ is finite whenever $A \in C_G^{f(p)}$, then M is finitely generated. We call this property $P(f)$.

Proposition 5.1: $M \in {}_G\text{PC}^p$ has $P(f)$ iff $A \otimes M$ is finite whenever A is a finite p -torsion right discrete G -module.

Proof: $A \otimes M$ is finite iff $(A \otimes M)^*$ is finite iff $\text{Hom}_G(M, A^*)$ is finite iff M has $P(f)$. //

Theorem 5.2: Suppose $M \in {}_G\text{PC}^p$ has $P(f)$ and $P \rightarrow M$ is a projective cover of M . Then P has $P(f)$.

Proof: Suppose $A \in C_G^{f(p)}$, A simple. For any $f \in \text{Hom}_G(P, A)$ consider the diagram with exact row

$$\begin{array}{ccccc} K & \rightarrow & P & \rightarrow & M \\ & & f \downarrow & & \\ & & A & & \end{array}$$

Since A is simple and K is small in P , f can be extended to a map $f' : M \rightarrow A$ and hence $\text{Hom}_G(P, A) \cong \text{Hom}_G(M, A)$. The proof now follows easily by induction on the order of $A \in C_G^{f(p)}$. //

If $P \in {}_G\text{PC}^p$ is projective and $A \in C_G^{f(p)}$, then A is a $\mathbb{Z}/p^i\mathbb{Z}(G/U)$ -module for some pair (i, U) and there exists $s \in \mathbb{Z}^+$, $s = |A|$, such that $\bigoplus^s \mathbb{Z}/p^i\mathbb{Z}(G/U) \rightarrow A$. For every $f \in \text{Hom}_G(P, A)$ we have the diagram

$$\begin{array}{ccc}
 & & P \\
 & \swarrow f' & \downarrow f \\
 \oplus \mathbb{Z}/p^i \mathbb{Z}(G/U) & \longrightarrow & A
 \end{array}$$

and if

$$K_{i,U} = \bigcap_{h \in \text{Hom}_G(P, \mathbb{Z}/p^i \mathbb{Z}(G/U))} \text{kernel } h$$

then $K_{i,U} \subseteq \ker f' \subseteq \ker f$ and $\text{Hom}_G(P, A) \cong \text{Hom}_G(P/K_{i,U}, A)$. As well, $P/K_{i,U}$ is a projective $\mathbb{Z}/p^i \mathbb{Z}(G/U)$ -module and $P/K_{i,U}$ is finite for all pairs (i, U) if and only if P has $P(f)$.

It is well-known that if $M, A \in {}_G\text{PC}^p$, $\text{Hom}_G(M, A)$ is isomorphic to $\varprojlim \text{Hom}_G(M, A_i)$ where $A \cong \varprojlim A_i$. Hence, if M has $P(f)$, $\text{Hom}_G(M, A)$ is pseudocompact and $\text{Hom}_G(M, -)$ becomes a functor from ${}_G\text{PC}^p$ to PC^p .

We now state a theorem which is true for finitely generated projectives in ordinary module categories ([18], p. 147).

Theorem 5.3: Let $P \in {}_G\text{PC}$ be projective having $P(f)$. Then the following are true:

$$(i) \quad \text{Hom}_G(P, \mathbb{Z}_{\overline{p}}[[G]]) \hat{\otimes}_G - \cong \text{Hom}_G(P, -)$$

This isomorphism being an isomorphism of functors from ${}_G\text{PC}^p$ to PC^p .

$$(ii) \quad - \hat{\otimes}_G P \cong \text{Hom}_G(\text{Hom}_G(P, \mathbb{Z}_{\overline{p}}[[G]]), -)$$

This isomorphism being an isomorphism of functors from PC_G^p to PC^p .

Proof: We first note that by the remarks preceeding this theorem,

$\text{Hom}_G(P, \mathbb{Z}_{\overline{p}}[[G]])$ is a well-defined pseudocompact $\mathbb{Z}_{\overline{p}}$ -module. It is also a $\mathbb{Z}_{\overline{p}}[[G]]$ -module when the action is defined by the action on $\mathbb{Z}_{\overline{p}}[[G]]$.

By Proposition 2.9, P is free and from Lemma 1.2, [10], we see that

a free pseudocompact G -module has $P(f)$ if and only if it is finitely generated. Hence any projective G -module having $P(f)$ is a finitely generated free module.

The isomorphisms therefore follow immediately from Theorem 4.1 and Proposition 4.2, Chapter V, [18]. That these isomorphisms are topological isomorphisms follows directly from the fact that for any $M \in PC_G^P$, M is topologically isomorphic to $\text{Hom}_G(\mathbb{Z}_{\neq p}[[G]], M)$. //

Corollary 5.4: The functor $\text{Hom}_G(P, -)^*$, where $P \in PC_G^P$ is projective having $P(f)$, is a representable functor from PC_G^P to $C_G^{t(p)}$ although the module with respect to which $\text{Hom}_G(P, -)^*$ is representable belongs to $C_G^{t(p)}$.

Proof:

$$\text{Hom}_G(P, -)^* \cong (\text{Hom}_G(P, \mathbb{Z}_{\neq p}[[G]]) \hat{\otimes}_G -)^* \cong \text{Hom}_G(-, \text{Hom}_G(P, \mathbb{Z}_{\neq p}[[G]])^*). //$$

Note 5.5: When $M \in PC_G^P$ has $P(f)$ and $A \in C_G^{t(p)}$ then $M \otimes_G A$, the non-completed tensor product, is a discrete $\mathbb{Z}_{\neq p}$ -module as follows: $A \cong \varinjlim A_i$ where each $A_i \in C_G^{f(p)}$ and $M \otimes_G A \cong \varinjlim (M \otimes_G A_i)$ where each $(M \otimes_G A_i) \in C_G^{f(p)}$ since M has $P(f)$. Hence Corollary 5.4 can be restated to define $\text{Hom}_G(P, -)^*$ as a representable functor from $C_G^{t(p)}$ to PC^P .

Corollary 5.6: If $P \in PC_G^P$ is projective having $P(f)$ then so is

$$\text{Hom}_G(P, \mathbb{Z}_{\neq p}[[G]]).$$

Proof: From (ii) and the right exactness of $- \hat{\otimes}_G P$ one sees that

$\text{Hom}_G(\text{Hom}_G(P, \mathbb{Z}_{\neq p}[[G]]), -)$ is right exact so that $\text{Hom}_G(P, \mathbb{Z}_{\neq p}[[G]])$ is projective. It has $P(f)$ by Theorem 5.3 (i) and Proposition 5.1. //

In [10], Brumer points out that if M is a finitely generated module in PC_G^P , $M \hat{\otimes}_G - \cong M \otimes_G -$; i.e., that the tensor product of M with any

pseudocompact module is complete.

In fact the converse is also true.

Proposition 5.7: If $M \in PC_G^P$ has the property

$$M \hat{\otimes}_G - \cong M \otimes_G -$$

then M is finitely generated.

Proof: For any indexing set I ,

$$M \otimes_G \prod_I (Z_{\overline{p}}[[G]]) \cong M \hat{\otimes}_G \prod_I (Z_{\overline{p}}[[G]]) \cong \prod_I (M \otimes_G Z_{\overline{p}}[[G]]) \cong \prod_I M.$$

Hence by Lemma 1.1 (i), [7], M is finitely generated as an element of $\text{Mod}(Z_{\overline{p}}[[G]])$ and so is finitely generated as a pseudocompact $Z_{\overline{p}}[[G]]$ -module. //

To end this section we show that any module in PC_G^P having $P(f)$ is finitely generated.

Proposition 5.8: $M \in PC_G^P$ has $P(f)$ if and only if M is finitely generated.

Proof: The sufficiency is clear.

Suppose now that $M \in PC_G^P$ has $P(f)$. Then by Theorem 5.2 so does its projective cover. By Proposition 2.9 this projective is free and as we have seen in the proof of Theorem 5.3 it is also finitely generated.

Hence M is finitely generated. //

§ 6 - Profinite $\overline{\text{FP}}$ -groups

Definition 6.1: A profinite group G with the property that $H^k(G, A)$ is finite for all $k \in \mathbb{Z}^+$ whenever $A \in C_G^{f(p)}$ is said to be an $\overline{\text{FP}}$ -group. If the same statement is true for all $k \leq n$ then G is an $\overline{\text{FP}}_n$ -group. If G is an $\overline{\text{FP}}$ -group and $\text{cd}_p G = n$, then G is an $\overline{\text{FP}}$ -group of dimension n or simply an $\overline{\text{FP}}$ -group.

All finite groups are $\overline{\text{FP}}$ -groups but many infinite profinite groups fit into this class. The following two theorems characterize these groups.

Theorem 6.2: A profinite group G is an $\overline{\text{FP}}_n$ -group ($\overline{\text{FP}}$ -group) if and only if $H^k(G, \mathbb{Z}/p\mathbb{Z}(G/U))$ is finite for all U in a cofinal system of open normal subgroups of G for all $k \leq n$ (for all $k \in \mathbb{Z}^+$).

Proof: We prove the theorem for $\overline{\text{FP}}_n$ -groups, the theorem for $\overline{\text{FP}}$ -groups follows by induction. Every profinite group G is an $\overline{\text{FP}}_0$ -group. Assume then that G is an $\overline{\text{FP}}_{n-1}$ -group and that $H^k(G, \mathbb{Z}/p\mathbb{Z}(G/U))$ is finite for all $k \leq n$ and all U in a cofinal system of open normal subgroups of G . We have seen in Proposition 1.10 that

$$H^k(G, \mathbb{Z}/p\mathbb{Z}(G/U)) \cong H^k(U, \mathbb{Z}/p\mathbb{Z})$$

and hence by a simple extension argument we have that $H^k(U, X)$ is finite for all $k \leq n$ whenever X is a U -module with trivial action.

If $T \in C_G^{f(p)}$, then $T^U = T$ for some U open normal in G and so $H^k(U, T)$ is finite for $k \leq n$. But

$$H^k(U, T) \cong H^k(G, M_G^U(T))$$

and so the latter is finite for $k \leq n$. However there is a short exact sequence of G -modules

$$T \rightarrow M_G^U(T) \rightarrow Y$$

and from this comes a long exact sequence

$$\dots \rightarrow H^{n-1}(G, Y) \rightarrow H^n(G, T) \rightarrow H^n(G, M_G^U(T)) \rightarrow \dots$$

By our induction assumption $H^{n-1}(G, Y)$ is finite hence $H^n(G, T)$ is finite. and G is an FP_n -group. //

This theorem has a much simpler version if G is a pro- p -group. In this case every finite discrete p -torsion G -module has a composition series whose factors are all isomorphic to Z/pZ with trivial G -action. Hence G is an FP_n -group ($\overline{FP}$ -group) if and only if $H^k(G, Z/pZ)$ is finite for all $k \leq n$ (for all $k \in \mathbb{Z}^+$).

Theorem 6.3: G is an FP_n -group ($\overline{FP}$ -group) if and only if there exists a $Z_p[[G]]$ -projective resolution of Z_p in $G\text{-PC}^P$

$$\dots \rightarrow P_n \rightarrow P_{n-1} \rightarrow \dots \rightarrow P_1 \rightarrow P_0 \rightarrow Z_p \rightarrow 0$$

with the property that P_k is finitely generated for each $k \leq n$ (for all $k \in \mathbb{Z}^+$).

Proof: We prove the theorem for FP_n -groups; the theorem for $\overline{FP}$ -groups follows by induction.

($\Leftarrow$) By our hypothesis, $\text{Hom}_G(P_k, A)$ is finite for all $A \in C_G^{f(p)}$ whenever $k \leq n$. Since $H^k(G, A)$ is a quotient of a subobject of $\text{Hom}_G(P_k, A)$ it too is finite; so G is an FP_n -group.

($\Rightarrow$) We proceed by induction. Since a projective resolution with $P_0 \cong Z_p[[G]]$ can be exhibited, an FP_0 -group does have a projective

resolution where P_0 is finitely generated. Suppose now that G is an FP_n -group. Then it is certainly an FP_{n-1} -group and by induction there exists a projective resolution of $\mathbb{Z}_{\overline{p}}$ where P_i is finitely generated for each $i \leq n-1$. Consider the exact sequence

$$0 \rightarrow K_{n-1} \rightarrow P_{n-1} \rightarrow \dots \rightarrow P_0 \rightarrow \mathbb{Z}_{\overline{p}} \rightarrow 0.$$

For any $A \in C_G^{f(p)}$, $\text{Hom}_G(-, A)$ applied to this sequence yields the right exact sequence

$$\text{Hom}_G(P_{n-1}, A) \rightarrow \text{Hom}_G(K_{n-1}, A) \rightarrow H^n(G, A) \rightarrow 0.$$

Since P_{n-1} is finitely generated and G is an FP_n -group, $\text{Hom}_G(K_{n-1}, A)$ is finite and so K_{n-1} has $P(f)$. Hence by Theorem 5.2, the projective cover of K_{n-1} , call it P'_n , has $P(f)$ and is finitely generated by Proposition 5.8. Now from the exact sequence

$$P'_n \rightarrow P_{n-1} \rightarrow \dots \rightarrow P_1 \rightarrow P_0 \rightarrow \mathbb{Z}_{\overline{p}} \rightarrow 0$$

it is possible to construct a projective resolution of $\mathbb{Z}_{\overline{p}}$ such that the first $n+1$ projectives are all finitely generated. This completes the proof. //

It is clear that an $\overline{FP}$ -group has the property that $\text{Ext}_G^k(\mathbb{Z}_{\overline{p}}, M)$ belongs to PC^P whenever $M \in {}_G PC^P$. However, M can be expressed as the inverse limit of a system $\{M_i\}$ of elements of $C_G^{f(p)}$ and it is natural to ask: Is $\text{Ext}_G^k(\mathbb{Z}_{\overline{p}}, M) \cong \varprojlim \text{Ext}_G^k(\mathbb{Z}_{\overline{p}}, M_i)$? This is indeed the case as is now shown.

Theorem 6.4: If G is an $\overline{FP}$ -group and $M \in {}_G PC^P$, $M \cong \varprojlim M_i$, $M_i \in C_G^{f(p)}$, then

$$\text{Ext}_G^k(\mathbb{Z}_{\overline{p}}, M) \cong \varprojlim \text{Ext}_G^k(\mathbb{Z}_{\overline{p}}, M_i).$$

Note: Hereafter we shall write $H^k(G, M)$ for $\text{Ext}_{\mathbb{Z}_p}^k(\mathbb{Z}_p, M)$ whenever $M \in \text{PC}^p$ and when we do it is understood that G is an $\overline{\text{FP}}$ -group.

Proof: Since G is an $\overline{\text{FP}}$ -group, there exists a projective resolution of $\mathbb{Z}_p$

$$\dots \rightarrow P_n \rightarrow P_{n-1} \rightarrow \dots \rightarrow P_0 \rightarrow \mathbb{Z}_p \rightarrow 0$$

where each P_i is finitely generated by Theorem 6.3.

The inverse limit of the sequence of complexes

$$0 \rightarrow \text{Hom}_G(P_0, M_1) \rightarrow \dots \rightarrow \text{Hom}_G(P_n, M_1) \rightarrow \dots$$

indexed by i , is the complex

$$0 \rightarrow \text{Hom}_G(P_0, M) \rightarrow \dots \rightarrow \text{Hom}_G(P_n, M) \rightarrow \dots$$

Since each P_k has $P(f)$, the objects $\text{Hom}_G(P_k, M_1)$ are all finite and so belong to PC^p . The functor $\varprojlim$ is exact on inverse systems of objects in PC^p (see § 1) and so the inverse limit of the k^{th} cohomology objects of the complexes indexed by i is the k^{th} cohomology object of the inverse limit sequence, i.e.

$$\varprojlim H^k(G, M_1) \cong H^k(G, M) \quad //$$

Besides finite groups, the most easily constructible examples of $\overline{\text{FP}}$ -groups are the finitely generated free pro- $\underline{C}$ -groups G where $\underline{C}$ is a subgroup-, quotient-, and extension-closed class of finite groups containing all finite p -groups. This stems from the fact that the exact sequence

$$6.5 \dots \quad \mathbb{I}_{\mathbb{Z}_p}[[G]] \twoheadrightarrow \mathbb{Z}_{\mathbb{Z}_p}[[G]] \rightarrow \mathbb{Z}_{\mathbb{Z}_p}$$

is a projective resolution of $\mathbb{Z}_{\mathbb{Z}_p}$ since $\mathbb{I}_{\mathbb{Z}_p}[[G]]$ is a free pseudocompact

G -module on the set $X-1$ where X is a finite set generating G . The proof of this closely follows the proof for the discrete case given in [18], Chapter IV; namely there is a 1-1 correspondence between $\text{Hom}_G(I_{\mathbb{P}}[[G]], A)$ and the crossed homomorphisms $G \rightarrow A$ (in the case where A is finite), and a 1-1 correspondence between these crossed homomorphisms and splittings of the exact sequence $A \rightarrow A \rtimes_{\theta} G \xrightarrow{\pi} G$, which exist if $\mathbb{P}$ is free pro- $\mathbb{C}$ on X and $\mathbb{C}$ is extension-closed so $A \rtimes_{\theta} G/U \in \mathbb{C}$ for all U open normal in G . Hence each projective in (6.5) is finitely generated so that G is an FP-group by Theorem 6.3.

Several further examples of $\overline{\text{FP}}$ -groups can be found by means of the following theorems.

Theorem 6.6: G is an $\overline{\text{FP}}$ -group if G has a single open subgroup which is an $\overline{\text{FP}}$ -group. If G is an $\overline{\text{FP}}$ -group then every open subgroup of G is an $\overline{\text{FP}}$ -group.

Proof: Suppose that $U \leq G$ is open and U is an $\overline{\text{FP}}$ -group. Let $A \in C_G^{f(p)}$, then there is a short exact sequence of G -modules

$$A \rightarrow M_G^U(A) \rightarrow K$$

and from this sequence we get the long exact cohomology sequence

$$\dots \rightarrow H^{k-1}(G, A) \rightarrow H^{k-1}(G, M_G^U(A)) \rightarrow H^{k-1}(G, K) \rightarrow H^k(G, A) \rightarrow H^k(G, M_G^U(A)) \rightarrow \dots$$

$$\begin{array}{ccc} \parallel & & \parallel \\ H^{k-1}(U, A) & & H^k(U, A) \end{array}$$

Since U is an $\overline{\text{FP}}$ -group it is clear that $H^k(G, A)$ is finite iff $H^{k-1}(G, K)$ is finite. Since A was arbitrary and K is finite the proof follows by recursion. The second part of this theorem follows easily from Shapiro's Lemma. //

Theorem 6.7: If $N \rightarrow G \rightarrow Q$ is a short exact sequence of profinite groups and

(i) N and G are $\overline{\text{FP}}$ -groups

or (ii) N and Q are $\overline{\text{FP}}$ -groups

then Q (respectively G) is an $\overline{\text{FP}}$ -group.

Proof: The proofs of (i) and (ii) are very similar, both involving spectral sequences. We show (i) to give the flavour of these proofs.

Assume, by induction, that Q is an FP_{k-1} -group (recall that all groups are FP_0 -groups) and let the diagram

represent the second "page" of the spectral sequence

$$H^p(Q, H^q(N, A)) \Rightarrow H^{p+q}(G, A),$$

where the dashed line in the diagram passes through the terms $H^p(Q, H^q(N, A))$

such that $p+q = k$. Suppose now that $A \in C_Q^{f(p)}$, then

$$E_2^{k,0} \cong H^k(Q, H^0(N, A)) \cong H^k(Q, A)$$

and $E_\infty^{k,0}$ is found by factoring $E_2^{k,0}$ a finite number of times by images of objects to the left of the dashed line on page 2 and on the succeeding pages. But our induction assumption and the fact that N is an $\overline{\text{FP}}$ -group implies that any object to the left of the dashed line on any page of the spectral sequence is finite and so $E_\infty^{k,0}$ is found by factoring $E_2^{k,0}$ a finite number of times, each time by a finite object. However

$$E_{\infty}^{k,0} \cong F^k H^k(G, A) \rightarrow H^k(G, A)$$

and so is finite. Therefore $E_2^{k,0}$ is finite; i.e., Q is an FP_k -group.

Hence, by induction, Q is an $\overline{FP}$ -group. //

In [22], Ribes states several conditions, which are expanded upon in § 13, under which a pro- $\underline{C}$ -amalgamated product of K and H with common closed subgroup L exists. In [15], under these assumptions, a Mayer-Vietoris sequence of cohomology groups is exhibited, which leads to the following theorem:

Theorem 6.8: Assuming the existence of the amalgamated product $G = H \amalg_L^{\underline{C}} K$ in the category of pro- $\underline{C}$ -groups for two pro- $\underline{C}$ -groups H and K with common closed subgroup L , and assuming that $\underline{C}$ contains all finite p -groups, then if H , K , and L are $\overline{FP}$ -groups so is G .

Proof: Under our assumptions, a Mayer-Vietoris sequence exists so that for $A \in C_G^{f(p)}$, the sequence

$$\begin{aligned} 0 \rightarrow \text{Der}(G, A) \rightarrow \text{Der}(H, A) \oplus \text{Der}(K, A) \rightarrow \text{Der}(L, A) \rightarrow H^2(G, A) \rightarrow \dots \\ \rightarrow H^n(L, A) \rightarrow H^{n+1}(G, A) \rightarrow \dots \end{aligned}$$

is exact. It is clear that for $k \geq 3$, $H^k(G, A)$ is finite since the same is true for the groups H , K and L . Since $H^1(G, A)$ is a quotient of $\text{Der}(G, A)$ we will show that $H^k(G, A)$ is finite for $k \geq 1$ if we show that $\text{Der}(H, A)$, $\text{Der}(K, A)$ and $\text{Der}(L, A)$ are finite. Recalling that for any pro-finite group N , and $A \in C_N^{f(p)}$, $\text{Der}(N, A)$ is in 1-1 correspondence with $\text{Hom}_{\mathbb{Z}_p}(\mathbb{I}_{\mathbb{Z}_p}[[N]], A)$ where $\mathbb{I}_{\mathbb{Z}_p}[[N]]$ is the augmentation ideal of $\mathbb{Z}_p[[N]]$, it remains to show that $\text{Hom}_{\mathbb{Z}_p}(\mathbb{I}_{\mathbb{Z}_p}[[N]], A)$ is finite whenever $A \in C_N^{f(p)}$; i.e., that $\mathbb{I}_{\mathbb{Z}_p}[[N]]$ has $P(f)$ when N is an $\overline{FP}$ -group.

Lemma 6.9: Let N be an FP_1 -group, then $\varprojlim_p [[N]]$ has $P(f)$.

Proof: The exact sequence

$$\varprojlim_p [[N]] \rightarrow \varprojlim_p [[N]] \rightarrow \varprojlim_p \mathbb{Z}$$

yields the exact sequence

$$\text{Hom}_{\varprojlim_p}(\varprojlim_p [[N]], A) \rightarrow \text{Hom}_{\varprojlim_p}(\varprojlim_p [[N]], A) \rightarrow H^1(N, A) \rightarrow 0$$

for any $A \in C_N^{f(p)}$. Hence $\text{Hom}_{\varprojlim_p}(\varprojlim_p [[N]], A)$ is always finite if $H^1(N, A)$ is; in other words, when N is an FP_1 -group. //

Hence $\text{Der}(H, A)$, $\text{Der}(K, A)$ and $\text{Der}(L, A)$ are all finite. Since every group is an FP_0 -group, $H^k(G, A)$ is finite for all $k \in \mathbb{Z}^+$ and all $A \in C_G^{f(p)}$ and so G is an $\overline{FP}$ -group. //

Finally, the Poincaré groups described by Serre ([23], p.I-47) are FP -groups. In particular, the Demuškin groups are FP -groups, [24]. We show in § 15 that the Poincaré groups in p of Verdier, [26], Definition 4.5, are FP -groups. In § 13, we see that the profinite completions of discrete $\overline{FP}$ -groups (discrete groups with a projective resolution of $\mathbb{Z}$ which is everywhere finitely generated, [9]) which are good ([23], p.I-16) are also $\overline{FP}$ -groups.

§ 7 - Induced Modules

Definition 7.1: A right (left) pseudocompact G -module M is called induced if $M \cong X \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[G]]$, where X is a pseudocompact $\mathbb{Z}_p$ -module and the G -action is the right (left) G -action on $\mathbb{Z}_p[[G]]$.

Proposition 7.2: $\text{Tor}_n^G(M, \mathbb{Z}_p) = 0$ for all $n > 0$ if M is induced. //

Suppose now that $C \in \text{PC}_G^p$, $C \cong \varprojlim C_i$ and $M \in \text{PC}_G^p$, $M \cong \varprojlim M_j$.

Then the complete tensor product $C \hat{\otimes} M$ can be given a G -structure as follows: $C \hat{\otimes} M \cong \varprojlim_{i,j} (C_i \otimes M_j)$ and we define a right G -action on $C_i \otimes M_j$ by $(\bar{c} \otimes \bar{m})x = \bar{c}x \otimes x^{-1}\bar{m}$, $x \in G$. Then the canonical maps $C_i \otimes M_j \rightarrow C_{i'} \otimes M_{j'}$ are G -maps and so $C \hat{\otimes} M$ becomes a right G -module via this action which we call the diagonal action of G on $C \hat{\otimes} M$.

Proposition 7.3: If $M \in \text{PC}_G^p$ is induced, then $C \hat{\otimes} M \in \text{PC}_G^p$ with diagonal action is also induced.

Proof! Suppose $M \cong X \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[G]]$, $X \cong \varprojlim X_j$, and $C \cong \varprojlim C_k$, then we may write $C \hat{\otimes} M$ as $\varprojlim (C_k \otimes (X_j \otimes \mathbb{Z}/p^i \mathbb{Z}(G/U)))$ where C_k and X_j are $\mathbb{Z}/p^i \mathbb{Z}(G/U)$ -modules. The map

$$C_k \otimes (X_j \otimes \mathbb{Z}/p^i \mathbb{Z}(G/U)) \rightarrow (C_k \otimes X_j) \otimes \mathbb{Z}/p^i \mathbb{Z}(G/U)$$

defined by $c \otimes (x \otimes \bar{g}) \mapsto (c\bar{g} \otimes x) \otimes \bar{g}^{-1}$ is a G -isomorphism by Lemma 1.3, [6]. It is straightforward to show that when the diagram

$$\begin{array}{ccc} C_k \otimes (X_j \otimes \mathbb{Z}/p^i \mathbb{Z}(G/U)) & \rightarrow & (C_k \otimes X_j) \otimes \mathbb{Z}/p^i \mathbb{Z}(G/U) \\ \downarrow & & \downarrow \\ C_{k'} \otimes (X_{j'} \otimes \mathbb{Z}/p^{i'} \mathbb{Z}(G/V)) & \rightarrow & (C_{k'} \otimes X_{j'}) \otimes \mathbb{Z}/p^{i'} \mathbb{Z}(G/V) \end{array}$$

exists and is commutative, it is G -commutative. Hence taking inverse

limits we have that

$$C \hat{\otimes} (X \hat{\otimes}_{\mathbb{Z}} \mathbb{Z}[[G]]) \cong_G (C \hat{\otimes} X) \hat{\otimes}_{\mathbb{Z}} \mathbb{Z}[[G]]$$

thus proving the proposition. //

All free pseudocompact G -modules are induced. This is true in the non-topological case and a similar proof establishes it in the pseudocompact case. For later use we prove the following.

Lemma 7.4: Let $N \rightarrow G \xrightarrow{\pi} Q$ be an exact sequence of profinite groups. If $B \in \text{PC}_G^D$, and $A \in \text{PC}_Q^D$ is induced, then $B \hat{\otimes}_N A \in \text{PC}_Q^D$, with diagonal Q -action, is induced.

Proof: The proof is essentially the same as the proof of Proposition 7.3.

The actual isomorphism is given as follows: Suppose $B \cong \varprojlim B_j$ and

$A \cong X \hat{\otimes}_{\mathbb{Z}} \mathbb{Z}[[G]] \cong \varprojlim (X_k \otimes \mathbb{Z}/p^i \mathbb{Z}(G/U))$, then

$$B_j \otimes_{N/N \cap U} (X_k \otimes \mathbb{Z}/p^i \mathbb{Z}(G/U)) \cong (B_j \otimes X_k) \otimes \mathbb{Z}/p^i \mathbb{Z}(Q/\pi(U)),$$

the isomorphism given by

$$b \otimes (x \otimes \bar{g}) \rightarrow (b\bar{g} \otimes x) \otimes \overline{\pi(\bar{g})^{-1}}$$

([6], p. 116). Hence

$$B \hat{\otimes}_N (X \hat{\otimes}_{\mathbb{Z}} \mathbb{Z}[[G]]) \cong (B \hat{\otimes} X) \hat{\otimes}_{\mathbb{Z}} \mathbb{Z}[[Q]]$$

and is Q -induced. //

Chapter II - Cap-products and Cup-products

§ 8 - Cap-products

In this chapter, the pseudocompact $\mathbb{Z}_{\overline{p}}$ -module $A \hat{\otimes} B$, where $A \in PC_G^p$, $B \in {}_G PC^p$ will always be considered as belonging to PC_G^p via the diagonal G -action described in § 7. The results of this section and § 9 follow closely the results of § 1, [2].

Definition and Theorem 8.1: Let G be a profinite group. If $A \in PC_G^p$, $B \in C_G^{f(p)}$, then there is a family of continuous $\mathbb{Z}_{\overline{p}}$ -bilinear maps, called cap-products (written $- \cap -$)

$$H_k(G, A) \times H^l(G, B) \xrightarrow{-\cap-} H_{k-l}(G, A \hat{\otimes} B)$$

denoted by $(a, b) \mapsto a \cap b$, defined for every pair of non-negative integers (k, l) , satisfying the following conditions:

(i) These maps are morphisms of functors when both are considered as covariant bifunctors from $PC_G^p \times C_G^{f(p)}$ to the category of topological $\mathbb{Z}_{\overline{p}}$ -modules.

(ii) If $A' \rightarrow A \xrightarrow{\pi} A''$ is exact in $C_G^{f(p)}$ and $C \in PC_G^p$ is such that $C \hat{\otimes} A' \rightarrow C \hat{\otimes} A \xrightarrow{\pi} C \hat{\otimes} A''$ is exact in PC_G^p , then the following diagram commutes

$$\begin{array}{ccc} H_k(G, C) \times H^l(G, A'') & \rightarrow & H_{k-l}(G, C \hat{\otimes} A'') \\ \downarrow (1, \Delta) & & \downarrow \Delta' \\ H_k(G, C) \times H^{l+1}(G, A') & \rightarrow & H_{k-l-1}(G, C \hat{\otimes} A') \end{array}$$

where Δ and Δ' are the obvious connecting homomorphisms.

Proof: For $A \in PC_G^p$, the complex of pseudocompact $\mathbb{Z}_p$ -modules

$$\dots \rightarrow A \hat{\otimes}_G \mathbb{Z}_p[[X^{\frac{n}{p}}G]] \xrightarrow{\partial^{n-1}} \dots \rightarrow A \hat{\otimes}_G \mathbb{Z}_p[[G]] \rightarrow 0$$

has as its homology objects, the homology groups of G with coefficients in A (see § 2). For $B \in C_G^{f(p)^\infty}$, the complex

$$0 \rightarrow C^0(G, B) \xrightarrow{\partial} C^1(G, B) \xrightarrow{\partial} \dots \xrightarrow{(-1)^n \partial} C^n(G, B) \rightarrow \dots$$

of discrete $\mathbb{Z}_p$ -modules

$$C^i(G, B) = \{ \sigma : X^{i+1}G \rightarrow B \mid \sigma \text{ is continuous and } \sigma(gg_0, \dots, gg_{i+1}) = g\sigma(g_0, \dots, g_{i+1}) \}$$

where $\partial(\sigma)(g_0, \dots, g_{i+1}) = \sum_{j=0}^{i+1} (-1)^j \sigma(g_0, \dots, \hat{g}_j, \dots, g_{i+1})$, has as its

cohomology objects, exactly the cohomology groups of G with coefficients in B , [21], p. 94. We establish the existence of the $\mathbb{Z}_p$ -bilinear maps

$$H_k(G, A) \times H^l(G, B) \rightarrow H_{k-l}(G, A \hat{\otimes} B)$$

by showing the existence of $\mathbb{Z}_p$ -bilinear maps

$$-\cap- : A \hat{\otimes}_G \mathbb{Z}_p[[X^{k+1}G]] \times C^l(G, B) \rightarrow (A \hat{\otimes} B) \hat{\otimes}_G \mathbb{Z}_p[[X^{k-l+1}G]]$$

which commute with the various differentials involved. In short, the map

$-\cap-$ is defined by

$$8.2 \dots -\cap-(a \otimes (g_0, \dots, g_k), \sigma) = (a \otimes \sigma(g_0, \dots, g_l)) \otimes (g_l, \dots, g_k),$$

where $\{a \otimes (g_0, \dots, g_k) : a \in A, g_i \in G\}$ generate a dense submodule of

$A \hat{\otimes}_G \mathbb{Z}_p[[X^{k+1}G]]$. We now construct this map in detail.

$$C^l(G, B) \cong \lim_{V \triangleleft \text{stabilizer}(B)} C^l(G/V, B)$$

([21], p.110), so that $\sigma \in C^l(G, B)$ may be considered as an element of $C^l(G/W, B)$ for all open normal subgroups W of G which are contained in some fixed open normal subgroup $V(\sigma) \triangleleft \text{stab}(B)$ of G . Hence for fixed

$\sigma \in C^l(G, B)$, we may write $A \hat{\otimes}_G \mathbb{Z}_p[[X^{k+1}G]]$ as $\lim_{W \triangleleft V(\sigma)} A \hat{\otimes}_{G/W} \mathbb{Z}_p[[X^{k+1}G/W]]$

where A_j is a finite G/W -module, and $(A \otimes B) \hat{\otimes}_G \mathbb{Z}_{\overline{p}}[[\frac{k-\ell+1}{X}G]]$ as

$\lim_{W \leftarrow V(\sigma)} ((A_j \otimes B) \otimes_{G/W} \mathbb{Z}_{\overline{p}}(\frac{k-\ell+1}{X}G/W))$ and the map

$$-\cap_W : A_j \otimes_{G/W} \mathbb{Z}_{\overline{p}}(\frac{k-\ell+1}{X}G/W) \rightarrow (A_j \otimes B) \otimes_{G/W} \mathbb{Z}_{\overline{p}}(\frac{k-\ell+1}{X}G/W)$$

defined by

$$-\cap_W(\bar{a} \otimes (\bar{g}_0, \dots, \bar{g}_k)) = (\bar{a} \otimes \sigma(\bar{g}_0, \dots, \bar{g}_\ell)) \otimes (\bar{g}_0, \dots, \bar{g}_k)$$

is a well-defined continuous homomorphism since the above modules are all discrete and the map $-\cap_W$ is exactly that defined in [2], § 1.1. This map commutes with the above inverse systems and so defines a continuous homomorphism

$$A \hat{\otimes}_G \mathbb{Z}_{\overline{p}}[[\frac{k+1}{X}G]] \longrightarrow (A \otimes B) \hat{\otimes}_G \mathbb{Z}_{\overline{p}}[[\frac{k-\ell+1}{X}G]]$$

which is the map (8.2) with the second variable fixed.

On the other hand, the generators $a \otimes (g_0, \dots, g_k)$ of $A \hat{\otimes}_G \mathbb{Z}_{\overline{p}}[[\frac{k+1}{X}G]]$ define morphisms

$$a \otimes (g_0, \dots, g_k) \cap : C^\ell(G, B) \rightarrow (A \otimes B) \hat{\otimes}_G \mathbb{Z}_{\overline{p}}[[\frac{k-\ell+1}{X}G]]$$

where

$$a \otimes (g_0, \dots, g_k) \cap (\sigma) = (a \otimes \sigma(g_0, \dots, g_\ell)) \otimes (g_\ell, \dots, g_k).$$

These are necessarily continuous since $C^\ell(G, B)$ is discrete. This definition extends to give morphisms

$$\alpha \cap : C^\ell(G, B) \rightarrow (A \otimes B) \hat{\otimes}_G \mathbb{Z}_{\overline{p}}[[\frac{k-\ell+1}{X}G]]$$

for arbitrary $\alpha \in A \hat{\otimes}_G \mathbb{Z}_{\overline{p}}[[\frac{k+1}{X}G]]$ in a natural way, and again these morphisms are necessarily continuous. This is the map (8.2) with the first variable fixed.

Hence the map $-\cap-$ is a bilinear map

$$A \otimes_G \bigoplus_{\mathbb{P}} [[\overset{k+1}{X} G]] \times C^{\ell}(G, B) \rightarrow (A \otimes B) \otimes_G \bigoplus_{\mathbb{P}} [[\overset{k-\ell+1}{X} G]] .$$

It is straightforward to show that it is natural and hence $-\cap-$ is a natural transformation between the bifunctors

$-\otimes_G \bigoplus_{\mathbb{P}} [[\overset{k+1}{X} G]] \times C^{\ell}(G, -)$ and $(- \otimes -) \otimes_G \bigoplus_{\mathbb{P}} [[\overset{k-\ell+1}{X} G]]$. As in the discrete case, the equation

$$\delta(-\cap-) = (-1)^{\ell} \delta(-) \cap - + -\cap \delta(-)$$

is satisfied when applied to pairs of generators $(a \otimes (g_0, \dots, g_k), \sigma)$ and hence holds for all pairs (α, σ) in $A \otimes_G \bigoplus_{\mathbb{P}} [[\overset{k+1}{X} G]] \times C^{\ell}(G, B)$. Therefore cap-product takes the pair (boundary, coboundary) to a boundary and the pair (cycle, cocycle) to a cycle and therefore induces a well-defined natural $\bigoplus_{\mathbb{P}}$ -bilinear map

$$H_k(G, -) \times H^{\ell}(G, -) \rightarrow H_{k-\ell}(G, - \otimes -) .$$

(ii) The proof is essentially that of Lemma 1.2.2, [2]. Let

$\bar{e} \in H_k(G, C)$ have coset representative $e \in C \otimes_G \bigoplus_{\mathbb{P}} [[\overset{k+1}{X} G]]$ and $\bar{\sigma} \in H^{\ell}(G, A'')$ have coset representative $\sigma \in C^{\ell}(G, A'')$. The map

$$(1, \Delta) : H_k(G, C) \times H^{\ell}(G, A'') \rightarrow H_k(G, C) \times H^{\ell+1}(G, A')$$

has the effect of choosing $\sigma' \in C^{\ell}(G, A)$ such that $\pi \circ \sigma' = \sigma$ and then sending σ' to $\partial \sigma' \in C^{\ell+1}(G, A)$. Since $\text{Im}(\partial \sigma') \subseteq A'$, we may assume $\partial \sigma' \in C^{\ell+1}(G, A')$; so on coset representatives (e, σ) , we have that

$$(1, \Delta)((e, \sigma)) = (e, \partial \sigma') .$$

However, $(-\cap-)((e, \sigma)) = e \cap \sigma \in (C \otimes A) \otimes_G \bigoplus_{\mathbb{P}} [[\overset{k-\ell+1}{X} G]]$ and $e \cap \sigma$ is a cycle. The map Δ' replaces $e \cap \sigma$ with $e \cap \sigma' \in (C \otimes A) \otimes_G \bigoplus_{\mathbb{P}} [[\overset{k-\ell+1}{X} G]]$ as follows: whenever $\sigma(g_0, \dots, g_{\ell}) \in A''$ appears, it is replaced with

$\sigma'(g_0, \dots, g_\ell)$ since $\pi(\sigma'(g_0, \dots, g_\ell)) = \sigma(g_0, \dots, g_\ell)$. Then the differential δ is applied to $e \cap \sigma'$; but this we know yields

$$(-1)^\ell \delta e \cap \sigma' + e \cap \delta \sigma' = e \cap \delta \sigma'$$

since $\delta e = 0$. This is exactly the image of $(e, \delta \sigma')$ under $-\cap-$, hence the diagram commutes. //

Corollary 8.3: Let G be a profinite group with $\text{cd}_p G = n$ and let $C \in \text{PC}_G^p$ be torsion-free. Then if $A' \rightarrow A \rightarrow A''$ is a short exact sequence of modules in $C_G^{f(p)}$, the following diagram (with exact rows) commutes for any $\bar{e} \in H_n(G, C)$.

$$\begin{array}{ccccccc} 0 \rightarrow H^0(G, A') \rightarrow H^0(G, A) \rightarrow \dots \rightarrow H^1(G, A'') \rightarrow \dots \rightarrow H^n(G, A'') \rightarrow 0 \\ \bar{e} \cap - : \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \\ 0 \rightarrow H_n(G, C \otimes A') \rightarrow \dots \rightarrow H_{n-1}(G, C \otimes A'') \rightarrow \dots \rightarrow H_0(G, C \otimes A'') \rightarrow 0 \dots // \end{array}$$

Remark 8.4: (i) For every profinite group G , $\text{cd}_p G = \text{hd}_p G$.

(ii) When C is torsion-free and $\text{cd}_p G = n$, both $\{H^*(G, -)\}$ and

$\{H_*(G, C \otimes -)\}$ can be considered as either covariant, connected, cohomological functors or covariant, connected homological functors so that

$\bar{e} \cap -$ becomes a natural transformation of ∂ -functors.

Proposition 8.5: Let $\varphi : H \rightarrow G$ be a continuous morphism. If $C \in \text{PC}_G^p$, $A \in C_G^{f(p)}$ and both are considered as H -modules via φ , then the following diagram commutes. ($-\cap-$ is used here by abuse of notation)

$$\begin{array}{ccc} H_k(H, C) & \xrightarrow{-\cap-} & \text{Hom}(H^k(H, A), H_{k-l}(H, C \otimes A)) \\ \downarrow \varphi_- & & \downarrow \text{Hom}(\varphi^-, \varphi_-) \\ H_k(G, C) & \xrightarrow{-\cap-} & \text{Hom}(H^k(G, A), H_{k-l}(G, C \otimes A)) \end{array}$$

Proof: The diagram

$$\begin{array}{ccc}
 C \otimes_H Z_{\mathbb{P}}[[\overset{k+1}{X} H]] & \xrightarrow{-\cap-} & \text{Hom}(C^{\ell}(H, A), (C \otimes A) \otimes_H Z_{\mathbb{P}}[[\overset{k-\ell+1}{X} H]]) \\
 \downarrow \varphi_- & & \downarrow \text{Hom}(\varphi^-, \varphi_-) \\
 C \otimes_G Z_{\mathbb{P}}[[\overset{k+1}{X} G]] & \xrightarrow{-\cap-} & \text{Hom}(C^{\ell}(G, A), (C \otimes A) \otimes_G Z_{\mathbb{P}}[[\overset{k-\ell+1}{X} G]])
 \end{array}$$

commutes. To see this, let $c \otimes (h_0, \dots, h_k)$ be a generator of

$C \otimes_H Z_{\mathbb{P}}[[\overset{k+1}{X} H]]$ and $\sigma \in C^{\ell}(G, A)$, then

$$\begin{aligned}
 \text{Hom}(\varphi^-, \varphi_-) \circ -\cap- (c \otimes (h_0, \dots, h_k)) (\sigma) &= \\
 &= \text{Hom}(\varphi^-, \varphi_-) (c \otimes (h_0, \dots, h_k) \cap-) (\sigma) \\
 &= \varphi_- (c \otimes (h_0, \dots, h_k) \cap-) (\sigma \circ \varphi) \\
 &= (c \otimes \sigma(\varphi(h_0), \dots, \varphi(h_k))) \otimes (\varphi(h_0), \dots, \varphi(h_k)) \\
 &= (c \otimes (\varphi(h_0), \dots, \varphi(h_k)) \cap-) (\sigma) \\
 &= \varphi_- (c \otimes (h_0, \dots, h_k) \cap-) (\sigma) \\
 &= \varphi_- \circ -\cap- (c \otimes (h_0, \dots, h_k)) (\sigma) .
 \end{aligned}$$

Hence the diagram commutes after taking (co)homology. //

Corollary 8.6: If $H \leq G$, $C \in \text{PC}_G^{\mathbb{P}}$, $A \in C_G^{f(\mathbb{P})}$, then the following diagram commutes.

$$\begin{array}{ccc}
 H_k(H, C) & \xrightarrow{-\cap-} & \text{Hom}(H^{\ell}(H, A), H_{k-\ell}(H, C \otimes A)) \\
 \downarrow \text{cor}_- & & \downarrow \text{Hom}(\text{res}^-, \text{cor}_-) \\
 H_k(G, C) & \xrightarrow{-\cap-} & \text{Hom}(H^{\ell}(G, A), H_{k-\ell}(G, C \otimes A)) . //
 \end{array}$$

Remark 8.7: If $A \in \text{PC}_G^{\mathbb{P}}$, $A \cong \varprojlim A_i$ where each $A_i \in C_G^{f(\mathbb{P})}$, then the

functoriality expressed in Theorem 8.1 implies that the definition of cap-product can be extended to cover the case

$$-\cap- : H_k(G, C) \times \varprojlim H^{\ell}(G, A_i) \rightarrow H_{k-\ell}(G, C \otimes A) .$$

Similarly, Corollary 8.3 and Proposition 8.5 can be extended to the case $\varprojlim H^k(G, A_i)$. Hence if G is an $\overline{\text{FP}}$ -group, § 8 can be rewritten to have the cohomology coefficient modules belonging to ${}^{\text{PC}}_G$.

§ 9 - Cap-products and Spectral Sequences

Suppose $N \rightarrow G \rightarrow Q$ is an exact sequence of profinite groups. Then for $A \in C_G^{t(p)}$, $C \in PC_G^p$, the Lyndon-Hochschild-Serre spectral sequences are

$$E_2^{p,q} \cong H^p(Q, H^q(N, A)) \Rightarrow H^{p+q}(G, A)$$

$$E_{r,s}^2 \cong H_r(Q, H_s(N, C)) \Rightarrow H_{r+s}(G, C)$$

and in the case $A \in C_G^{f(p)}$, properties equivalent to Lemma 1.3.1 and Lemma 1.3.2 of [2] hold for the above sequences of profinite groups. This can be shown by essentially mimicking § 1.3 of [2] and we do so in outline form.

For $A \in C_G^{f(p)}$, $C \in PC_G^p$, let $K^{p,q}(A) = C^p(Q, C_N^q(G, A))$ and

$$K_{s,r}(C) = (C \otimes_N \mathbb{Z}[[X^{s+1}G]]) \otimes_Q \mathbb{Z}[[X^{r+1}Q]]$$

be two bicomplexes with differentials

$$\delta' : K^{p,q}(A) \rightarrow K^{p+1,q}(A) \quad \text{and} \quad \delta'' : K^{p,q}(A) \rightarrow K^{p,q+1}(A)$$

defined by

$$\delta'(\sigma) = (-1)^{p+q+1} \delta(\sigma) \quad \text{and} \quad \delta''(\sigma) = (-1)^{q+1} \sigma \circ \delta$$

where δ is the differential of the complexes $C^*(Q, -)$ and $C_N^*(G, -)$ defined in § 8; and

$$\partial' : K_{s,r}(C) \rightarrow K_{s+1,r}(C) \quad , \quad \partial'' : K_{s,r}(C) \rightarrow K_{s,r+1}(C)$$

defined by

$$\partial'((c \otimes b) \otimes b') = (c \otimes \partial b) \otimes b'$$

and

$$\partial''((c \otimes b) \otimes b') = (c \otimes b) \otimes \partial b' ,$$

where $c \in C$, $b \in Z_{-p}^{[s+1]}[[X^*G]]$, $b' \in Z_{-p}^{[r+1]}[[X^*Q]]$ and ∂ is the differential of the complexes $Z_{-p}^{[*]}[[X^*G]]$ and $Z_{-p}^{[*]}[[X^*Q]]$ defined in § 2.

The cap-product

$$-\cap- : K_{s,r}(C) \times K^{p,q}(A) \rightarrow K_{s-q, r-p}(C \otimes A)$$

can be defined as follows: let $e = (c \otimes (g_0, \dots, g_s)) \otimes (\bar{g}_0, \dots, \bar{g}_r)$ be

a generator of $K_{s,r}(C)$ and $\sigma \in C^p(Q, C_N^q(G, A))$; then

$$e \cap \sigma = (-1)^{sp} [(c \otimes (\sigma(\bar{g}_0, \dots, \bar{g}_p)) (g_0, \dots, g_q)) \otimes (g_q, \dots, g_s)] \otimes (\bar{g}_p, \dots, \bar{g}_r)$$

defines a continuous Z_{-p} -bilinear map with the following properties :

$$\partial''(e \cap \sigma) = (-1)^{p+q} \partial' e \cap \sigma + e \cap \partial'' \sigma$$

$$\partial'(e \cap \sigma) = (-1)^{p+q} \partial'' e \cap \sigma + e \cap \partial' \sigma.$$

If $\text{Tot}(K^*(A))$, $\text{Tot}(K_*(C))$ and $\text{Tot}(K_*(C \otimes A))$ are the associated simple complexes with differentials

$$\partial = \partial' + \partial'' \quad \text{and} \quad \delta = \delta' + \delta'',$$

then $-\cap-$ extends automatically to

$$9.1 \dots -\cap- : \text{Tot}(K_*(C))_n \times \text{Tot}(K^*(A))^m \rightarrow \text{Tot}(K_*(C \otimes A))_{n-m}$$

such that for $c \in \text{Tot}(K_*(C))_n$ and $\sigma \in \text{Tot}(K^*(A))^m$ we have

$$\partial(e \cap \sigma) = (-1)^m \partial e \cap \sigma + e \cap \delta \sigma.$$

The (co)homology of these total complexes is isomorphic (topologically) to the (co)homology of G . These isomorphisms are induced by the morphisms

$$\begin{aligned} \tau : C^*(G, A) &\rightarrow C_N^*(G, A) \xrightarrow{\tau} C^0(Q, C_N^*(G, A)) \\ &= K^{0,*}(A) \subseteq \text{Tot}(K^*(A)) \end{aligned}$$

$$\begin{aligned} \eta : \text{Tot}(K_*(C)) &\rightarrow K_{*,0}(C) = (C \otimes_N Z_{-p}^{[*]}[[X^*G]]) \otimes_Q Z_{-p}^{[*]}[[Q]] \\ &\xrightarrow{\eta} C \otimes_G Z_{-p}^{[*]}[[X^*G]] \end{aligned}$$

where $\tau(\bar{g}) = \bar{g}\sigma = \sigma$ while $\eta((c \otimes b) \otimes 1_Q) = c \otimes b$. Hence we have shown:

Proposition 9.2: The cap-product defined on total complexes by (9.1) induces a cap-product morphism on the (co)homology of these complexes which is exactly the cap-product defined by Theorem 8.1 so the following diagram commutes.

$$\begin{array}{ccc} H_n(\text{Tot}(K_*(C))) & \xrightarrow{-\cap-} & \text{Hom}(H^m(\text{Tot}(K^*(A))), H_{n-m}(\text{Tot}(K_*(C \otimes A)))) \\ \downarrow \eta' & & \downarrow \text{Hom}(\tau', \eta') \\ H_n(G, C) & \xrightarrow{-\cap-} & \text{Hom}(H^m(G, A), H_{n-m}(G, C \otimes A)) \quad // \end{array}$$

As well, cap-product defines a $\mathbb{Z}_p$ -bilinear map

$$E_{r,s}^n(C) \times E_n^{p,q} \xrightarrow{-\cap-} E_{r-p, s-q}^n(C \otimes A).$$

For $n = 2$ we have that

$$\begin{aligned} \zeta_* : E_{r,s}^2(C) &\cong H_r(Q, H_s(N, C)) \\ \zeta^* : E_2^{p,q}(A) &\cong H^p(Q, H^q(N, A)) \end{aligned}$$

and we can define a cap-product map

$$H_r(Q, H_s(N, C)) \times H^p(Q, H^q(N, A)) \xrightarrow{-\cap-} H_{r-p}(Q, H_{s-q}(N, C \otimes A))$$

as follows : we consider the map

$$(H_s(N, C) \otimes_{\mathbb{Q}} \mathbb{Z}_p[[\overset{r+1}{\times} Q]]) \times C^p(Q, H^q(N, A)) \xrightarrow{-\cap-} H_{s-q}(N, C \otimes A) \otimes_{\mathbb{Q}} \mathbb{Z}_p[[\overset{r-p+1}{\times} Q]]$$

given by

$$\bar{e} \otimes (\bar{g}_0, \dots, \bar{g}_p) \cap \sigma = (e \cap \sigma(\bar{g}_0, \dots, \bar{g}_p)) \otimes (\bar{g}_p, \dots, \bar{g}_r).$$

This map commutes with the necessary differentials, in particular,

$$\begin{aligned} \partial(\bar{e} \otimes (\bar{g}_0, \dots, \bar{g}_p) \cap \sigma) \\ = (-1)^p \partial(\bar{e} \otimes (\bar{g}_0, \dots, \bar{g}_p)) \cap \sigma + \bar{e} \otimes (\bar{g}_0, \dots, \bar{g}_p) \cap \partial \sigma. \end{aligned}$$

This tells us that $-\cap-$ sends (cycle, cocycle) to cycle and (boundary, coboundary) to boundary so that $-\cap-$ induces a map

$$H_r(Q, H_s(N, C)) \times H^p(Q, H^q(N, A)) \rightarrow H_{r-p}(Q, H_{s-q}(N, C \otimes A)).$$

Hence we have the following:

Proposition 9.3: The following diagram is $(-1)^{sp}$ -commutative

$$\begin{array}{ccc} E_{r,s}^2(C) \times E_2^{p,q}(A) & \xrightarrow{-\cap-} & E_{r-p,s-q}^2(C \otimes A) \\ \downarrow \zeta_* \times \zeta^* & & \downarrow \zeta_* \\ H_r(Q, H_s(N, C)) \times H^p(Q, H^q(N, A)) & \xrightarrow{-\cap-} & H_{r-p}(Q, H_{s-q}(N, C \otimes A)) \end{array}$$

i.e.

$$\zeta_*(e \cap \sigma) = (-1)^{sp} \zeta_*(e) \cap \zeta^*(\sigma) \quad //$$

§ 10 - Cap-products and Cup-products

Let G be a profinite group, $A, B \in C_G$, then $A \otimes B \in C_G$, i.e.
 $A \otimes B \cong \bigcup_U (A^U \otimes B^U)$, $g(a \otimes b) = ga \otimes gb$.

Definition and Theorem 10.1 [21] : For any profinite group G there is a unique family of $\mathbb{Z}$ -bilinear maps, called cup-products (written $-U-$)

$$H^p(G, A) \times H^q(G, B) \xrightarrow{-U-} H^{p+q}(G, A \otimes B)$$

defined for every pair (p, q) of non-negative integers and every pair of discrete G -modules (A, B) , satisfying the following conditions :

(i) These maps are morphisms of functors when we consider both as covariant bifunctors from $C_G \times C_G$ to $\underline{Ab}$,

(ii) For $p = q = 0$, the map

$$A^G \times B^G \xrightarrow{-U-} (A \otimes B)^G$$

is given by $(a, b) \mapsto (a \otimes b)$,

(iii) Let $B \in C_G$. If both

$$A' \rightarrow A \rightarrow A'' \quad \text{and} \quad A' \otimes B \rightarrow A \otimes B \rightarrow A'' \otimes B$$

are short exact sequences in C_G , then the diagram

$$\begin{array}{ccc} H^p(G, A'') \times H^q(G, B) & \xrightarrow{(\Delta, 1)} & H^{p+1}(G, A') \times H^q(G, B) \\ \downarrow -U- & & \downarrow -U- \\ H^{p+q}(G, A'' \otimes B) & \xrightarrow{\Delta'} & H^{p+q+1}(G, A' \otimes B) \end{array}$$

commutes, where Δ and Δ' are the appropriate connecting homomorphisms,

(iv) Let $A \in C_G$, if both

$$B' \rightarrow B \rightarrow B'' \quad \text{and} \quad A \otimes B' \rightarrow A \otimes B \rightarrow A \otimes B''$$

are short exact sequences in C_G , then the diagram

$$\begin{array}{ccc}
 H^p(G, A) \times H^q(G, B'') & \xrightarrow{(\Delta, 1)} & H^p(G, A) \times H^{q+1}(G, B') \\
 \downarrow -U- & & \downarrow -U- \\
 H^{p+q}(G, A \otimes B'') & \xrightarrow{(-1)^p \Delta'} & H^{p+q+1}(G, A \otimes B')
 \end{array}$$

commutes . //

If fact cup-product can be defined via a natural transformation

$$C^p(G, -) \times C^q(G, -) \xrightarrow{-U-} C^{p+q}(G, - \otimes -)$$

where each of the above functors is considered as a bifunctor

$$C_G \times C_G \rightarrow \underline{Ab}$$

and as a result, cup-product is a natural transformation

$$C^p(G, -) \xrightarrow{-U-} \text{Hom}(C^q(G, -), C^{p+q}(G, - \otimes -))$$

where $\text{Hom}(C^q(G, -), C^{p+q}(G, - \otimes B))$ is the set of natural transformations from $C^q(G, -)$ to $C^{p+q}(G, - \otimes B)$ when $B \in C_G$.

Suppose now we restrict ourselves to coefficient modules in the first variable coming from $C_G^{f(p)}$. Then one has, for $B \in C_G^{t(p)}$, the functor $\text{Hom}(-, B)$, and this functor composed with $C^p(G, -)$, yields a homomorphism

$$C^p(G, \text{Hom}(A, B)) \xrightarrow{-U-} \text{Hom}(C^q(G, A), C^{p+q}(G, A \otimes \text{Hom}(A, B)))$$

for $A \in C_G^{f(p)}$. There is a G -homomorphism

$$A \otimes \text{Hom}(A, B) \rightarrow B$$

defined by $a \otimes f \mapsto f(a)$ which defines a map

$$C^{p+q}(G, A \otimes \text{Hom}(A, B)) \rightarrow C^{p+q}(G, B)$$

and this, composed with the cup-product morphisms

$$C^q(G, A) \xrightarrow{-U-} C^{p+q}(G, A \otimes \text{Hom}(A, B))$$

yields a morphism (by abuse of notation)

$$C^p(G, \text{Hom}(A, B)) \xrightarrow{-U-} \text{Hom}(C^q(G, A), C^{p+q}(G, B)) .$$

This morphism is in fact natural ; i.e., if $A \rightarrow A'$ in $C_G^{f(p)}$, then the diagram

$$\begin{array}{ccc} C^p(G, \text{Hom}(A', B)) & \xrightarrow{-U-} & \text{Hom}(C^q(G, A'), C^{p+q}(G, B)) \\ \downarrow & & \downarrow \\ C^p(G, \text{Hom}(A, B)) & \xrightarrow{-U-} & \text{Hom}(C^q(G, A), C^{p+q}(G, B)) \end{array}$$

commutes and hence cup-product defines a natural transformation

$$C^p(G, \text{Hom}(-, B)) \rightarrow \text{Hom}(C^q(G, -), C^{p+q}(G, B)) .$$

Finally, suppose we compose this natural transformation with any fixed element f of $C^{p+q}(G, B)^*$. Then we have a natural transformation (indexed by f)

$$10.2 \dots C^p(G, \text{Hom}(-, B)) \xrightarrow{(-U-)_f} C^q(G, -)^* .$$

Since $(-)^*$ is an exact functor, and $-U-$ commutes with the differentials of the complex $\{C^*(G, -)\}$, (10.2) yields a natural transformation

$$(-U-)_f : H^p(G, \text{Hom}(-, B)) \rightarrow H^q(G, -)^* .$$

Since $B \in C_G^{t(p)}$, there exists $C \in PC_G^p$ (unique up to isomorphism) such that $B \cong C^*$. Hence, for $C \in PC_G^p$, the cup-product defines a natural morphism

$$(-U-)^* : C^{p+q}(G, C^*)^* \rightarrow \text{Hom}(C^p(G, \text{Hom}(-, C^*)), C^q(G, -)^*)$$

where $\text{Hom}(C^p(G, \text{Hom}(-, C^*)), C^q(G, -)^*)$ is the set of natural transformations from $C^p(G, \text{Hom}(-, C^*))$ to $C^q(G, -)^*$. This morphism is defined as follows:

Suppose $f \in C^{p+q}(G, C^*)^*$ and

$$-U- : C^p(G, \text{Hom}(-, C^*)) \rightarrow \text{Hom}(C^q(G, -), C^{p+q}(G, C^*)) ,$$

then $-U-$ composed with f yields a natural transformation

$$C^p(G, \text{Hom}(-, C^*)) \rightarrow C^q(G, -)^* ,$$

the images in $C^q(G, A)^*$, for any $A \in C_G^{f(p)}$, being the elements of $C^q(G, A)^*$ which factor into the cup-product maps

$$C^q(G, A) \xrightarrow{-U-} C^{p+q}(G, C^*) \xrightarrow{f} Z_p^\infty .$$

On the other hand,

$$\begin{aligned} C^{p+q}(G, C^*)^* &\cong \text{Hom}_G(Z_{\overline{p}}[[\frac{p+q+1}{x}G]], C^*)^* \\ &\cong ((C \hat{\otimes}_G Z_{\overline{p}}[[\frac{p+q+1}{x}G]])^*)^* \cong C \hat{\otimes}_G Z_{\overline{p}}[[\frac{p+q+1}{x}G]] \end{aligned}$$

(call this isomorphism α), while

$$\begin{aligned} C^p(G, \text{Hom}(-, C^*)) &\cong C^p(G, (C \otimes -)^*) \\ &\cong \text{Hom}_G(Z_{\overline{p}}[[\frac{p+1}{x}G]], (C \otimes -)^*) \\ &\cong ((C \otimes -) \hat{\otimes}_G Z_{\overline{p}}[[\frac{p+1}{x}G]])^* \end{aligned}$$

(call this isomorphism β).

We know also that cap-product defines a natural transformation

(proof of Theorem 8.1),

$$-\cap- : - \hat{\otimes}_G Z_{\overline{p}}[[\frac{p+q+1}{x}G]] \rightarrow \text{Hom}(C^q(G, -), (- \otimes -) \hat{\otimes}_G Z_{\overline{p}}[[\frac{p+1}{x}G]]) .$$

Applying $(-)^*$ to the two functors $C^q(G, -)$ and $(- \otimes -) \hat{\otimes}_G Z_{\overline{p}}[[\frac{p+1}{x}G]]$

yields a cap-product natural transformation

$$(-)^* \circ -\cap- : - \hat{\otimes}_G Z_{\overline{p}}[[\frac{p+q+1}{x}G]] \rightarrow \text{Hom}(((- \otimes -) \hat{\otimes}_G Z_{\overline{p}}[[\frac{p+1}{x}G]]^*), C^q(G, -)^*)$$

so that for $C \in PC_G^p$, cap-product induces a morphism

$$(-)^* \circ -\cap- : C \hat{\otimes}_G Z_{\overline{p}}[[\frac{p+q+1}{x}G]] \rightarrow \text{Hom}(((C \otimes -) \hat{\otimes}_G Z_{\overline{p}}[[\frac{p+1}{x}G]]^*), C^q(G, -)^*) .$$

This yields the following theorem.

Theorem 10.3: Let $C \in PC_G^p$, then

$$\begin{array}{ccc}
 C^{p+q}(G, C^*)^* & \xrightarrow{(- \cup -)^*} & \text{Hom}(C^p(G, \text{Hom}(-, C^*)), C^q(G, -)^*) \\
 \uparrow \alpha^{-1} & & \uparrow \beta' \\
 C \otimes_G Z_{\frac{p}{p}}[[\frac{p+q+1}{X}G]] & \xrightarrow{(-)^* \circ -\cap-} & \text{Hom}((C \otimes -) \otimes_G Z_{\frac{p}{p}}[[\frac{p+1}{X}G]])^*, C^q(G, -)^*
 \end{array}$$

commutes.

Proof: Let $c \otimes (g_0, \dots, g_{p+q})$ be a generator of $C \otimes_G Z_{\frac{p}{p}}[[\frac{p+q+1}{X}G]]$. Then

$(-)^* \circ -\cap- (c \otimes (g_0, \dots, g_{p+q}))$ applied to $f \in ((C \otimes A) \otimes_G Z_{\frac{p}{p}}[[\frac{p+1}{X}G]])^*$, applied to $\sigma \in C^q(G, A)$ acts as follows:

$$\begin{aligned}
 & (-)^* \circ -\cap- (c \otimes (g_0, \dots, g_{p+q}))_A (f)(\sigma) \\
 & = f(c \otimes (g_0, \dots, g_{p+q}) \cap \sigma) \in Z_p^\infty
 \end{aligned}$$

where $c \otimes (g_0, \dots, g_{p+q}) \cap \sigma = (c \otimes \sigma(g_0, \dots, g_q)) \otimes (g_q, \dots, g_{p+q})$ while

$$\begin{aligned}
 & \beta'((-)^* \circ -\cap- (c \otimes (g_0, \dots, g_{p+q})))_A (\beta_A^{-1} f)(\sigma) \\
 & = \beta_A^{-1}(f)((g_q, \dots, g_{p+q}))(\sigma(g_0, \dots, g_q))(c) \in Z_p^\infty.
 \end{aligned}$$

On the other hand, applying $(-\cup-)^* \circ \alpha^{-1}(c \otimes (g_0, \dots, g_{p+q}))$ to

$\beta_A^{-1}(f)$ and applying this to $\sigma \in C^q(G, A)$ yields

$$\begin{aligned}
 & \alpha^{-1}(c \otimes (g_0, \dots, g_{p+q})) \circ (\sigma \cup \beta_A^{-1}(f)) \\
 & = (\sigma \cup \beta_A^{-1}(f))((g_0, \dots, g_{p+q}))(c) \\
 & = \beta_A^{-1}(f)((g_q, \dots, g_{p+q}))(\sigma(g_0, \dots, g_q))(c) \in Z_p^\infty.
 \end{aligned}$$

Hence the diagram commutes. Since the diagram commutes for all $A \in C_G^{f(p)}$

the morphism $(\beta'(-)^* \circ -\cap-)_A$ is equal to $((-\cup-)^* \circ \alpha^{-1})_A$ for every

$A \in C_G^{f(p)}$. Hence $\beta' \circ (-)^* \circ -\cap-$ and $(-\cup-)^* \circ \alpha^{-1}$ are equal by definition. //

Corollary 10.4: Given $C \in PC_G^p$, $A \in C_G^{f(p)}$, the diagram

$$\begin{array}{ccc}
 H_n(G, C) & \xrightarrow{(-)* \circ -\cap-} & \text{Hom}(H_{n-q}(G, C \otimes A)^*, H^q(G, A)^*) \\
 \downarrow \alpha^{-1} & & \downarrow \beta' \\
 H^n(G, C^*)^* & \xrightarrow{(-\cup-)^*} & \text{Hom}(H^{n-q}(G, \text{Hom}(A, C^*)), H^q(G, A)^*)
 \end{array}$$

commutes, where α^{-1} and β' are exactly the isomorphisms of Corollary 1.6.

Corollary 10.5: For $C \in PC_G^D$, there exists $e \in H_n(G, C)$ such that

$$e \cap - : H^q(G, A) \rightarrow H_{n-q}(G, C \otimes A)$$

is an isomorphism for some $A \in C_G^{f(p)}$ and $n, q \in \mathbb{Z}^+$ if and only if there exists $\hat{e} \in H^n(G, C^*)^*$ such that

$$(-\cup-)_{\hat{e}} : H^{n-q}(G, \text{Hom}(A, C^*)) \rightarrow H^q(G, A)^*$$

is an isomorphism.

Proof of Corollary 10.4: That α^{-1} and β' are the isomorphisms of Corollary 1.6 is easy to see. Also, $-\cap-$ induces a map

$$H_n(G, C) \rightarrow \text{Hom}(H^q(G, A), H_{n-q}(G, C \otimes A))$$

and hence there is another induced map

$$H_n(G, C) \rightarrow \text{Hom}(H_{n-q}(G, C \otimes A)^*, H^q(G, A)^*)$$

which must be $(-)* \circ -\cap-$ since $(-)*$ commutes with the taking of homology and cohomology. Similarly, $(-\cup-)^*$ induces a map

$$H^n(G, C^*)^* \rightarrow \text{Hom}(H^{n-q}(G, \text{Hom}(A, C^*)), H^q(G, A)^*) .$$

The diagram commutes because the diagram in Theorem 10.4 commutes. //

Proof of Corollary 10.5:

$$e \cap - : H^q(G, A) \rightarrow H_{n-q}(G, C \otimes A)$$

is an isomorphism iff

$$(-)* \circ e \cap - : H_{n-q}(G, C \otimes A)^* \rightarrow H^q(G, A)^*$$

is an isomorphism, iff

$$(-)^* \circ (\alpha^{-1}(e)) \cap - \circ \beta' : H^{n+q}(G, \text{Hom}(A, C^*)) \rightarrow H^q(G, A)^*$$

is an isomorphism, iff

$$(\alpha^{-1}(e) \cup -)^* : H^{n-q}(G, \text{Hom}(A, C^*)) \rightarrow H^q(G, A)^*$$

is an isomorphism. But

$$(\alpha^{-1}(e) \cup -)^* = (-\cup -)_{\alpha^{-1}(e)}^* //$$

Remark 10.6: Since both α^{-1} and β are topological isomorphisms, the

morphism $e \cap -$ in Corollary 10.5 will be a topological isomorphism iff

$(-\cup -)_{\alpha^{-1}(e)}^{-1}$ is likewise a topological isomorphism. In either case, the

domain is topologically discrete while the range is compact so that if

the isomorphism is topological then necessarily the compact range is

finite; i.e., $H^k(G, A)$ must be finite. Hence any group G with the pro-

perty that there exists $C \in PC_G^p$, $n \in \mathbb{Z}^+$, and $e \in H_n(G, C)$ such that

$(e \cap -)$ is a topological isomorphism for every pair (k, A) such that

$k \in \mathbb{Z}^+$ and $A \in C_G^{f(p)}$ is necessarily an FP-group. In the next chapter

we study such groups in great detail. Such groups are primary examples

of FP-groups.

CHAPTER III - Duality Groups

§ 11 - Duality Groups (in p)

Definition 11.1: A profinite group G is called a duality group (in p) of dimension n (written D_p^n -group) with respect to $C \in PC_G^p$, if there exists $e \in H_n(G, C)$ such that the cap-product map

$$11.2 \dots \quad e \cap - : H^k(G, A) \rightarrow H_{n-k}(G, C \otimes A)$$

is a topological isomorphism for all $k \in \mathbb{Z}^+$ and all $A \in C_G^{f(p)}$. We call C the dualizing module of G and e the fundamental class of G .

There are several immediate consequences of this definition. First, $cd_p G = n$. Also, $H^k(G, A)$ is finite for all k and all $A \in C_G^{f(p)}$ since $H^k(G, A)$ is discrete while $H_{n-k}(G, C \otimes A)$ is compact and these can be topologically isomorphic only when they are finite. Hence G is an FP-group. Remark 8.7 therefore allows us to extend Definition 11.1 to permit coefficient modules to be chosen from PC_G^p instead of just $C_G^{f(p)}$. Hence in particular,

$$H^n(G, \mathbb{Z}_p[[G]]) \cong H_0(G, C \hat{\otimes} \mathbb{Z}_p[[G]]) \cong (C \hat{\otimes} \mathbb{Z}_p[[G]]) \otimes_{G, \mathbb{Z}_p} \mathbb{Z}_p \cong C \otimes_{G, \mathbb{Z}_p} \mathbb{Z}_p[[G]] \cong C$$

so that the dualizing module C depends only on G , and providing it exists, it is unique up to isomorphism. In fact, this is a G -isomorphism if the G -action on $H^n(G, \mathbb{Z}_p[[G]])$ and $(C \hat{\otimes} \mathbb{Z}_p[[G]]) \otimes_{G, \mathbb{Z}_p} \mathbb{Z}_p$ is the right G -action on $\mathbb{Z}_p[[G]]$. Next,

$$H_n(G, C) \cong H_n(G, C \otimes \mathbb{Z}_p) \cong H^0(G, \mathbb{Z}_p) \cong \mathbb{Z}_p.$$

From these isomorphisms one sees that e generates $H_n(G, C)$.

In [26], Verdier describes profinite groups which he calls strict Cohen-Macaulay groups (in p) of dimension n . These are groups G , such that $\text{cd}_p G = n$ and $\lim_{\substack{\longrightarrow \\ U \subseteq G}} H^k(G, Z/pZ(G/U)) = 0$ for $k \neq n$. We give as an equivalent definition, Proposition 4.4, [26].

Definition 11.3: A profinite group G is said to be a strict Cohen-Macaulay group (in p) of dimension n if $\text{cd}_p G = n$ and there exists $I_p \in C_G^{t(p)}$ together with a canonical homomorphism:

$$\rho : H^n(G, I_p) \rightarrow Z_p^\infty$$

such that the cup-product map

$$- \cup - : H^q(G, A) \times H^{n-q}(G, \text{Hom}(A, I_p)) \rightarrow H^n(G, I_p)$$

composed with ρ yields an isomorphism (not necessarily topological)

$$H^q(G, A)^* \cong H^{n-q}(G, \text{Hom}(A, I_p))$$

for all $q \in \mathbb{Z}^+$ and all $A \in C_G^{f(p)}$. Furthermore, $I_p \cong \lim_{\substack{\longrightarrow \\ V, \text{cor}, 1}} H^n(V, Z/p^i Z)^*$.

We establish the following important theorem.

Theorem 11.4: A profinite group G is a D_p^n -group iff G is an FP-group and a strict Cohen-Macaulay group (in p) of dimension n ; in which case, $I_p \cong C^*$ where C is the dualizing module of G and ρ is the image of the fundamental class of G under the isomorphism

$$H_n(G, C) \cong H^n(G, C^*)^*.$$

Proof: This theorem is an immediate consequence of Corollary 10.5, the fact that a duality group is an FP-group, and Proposition 1.10. //

Corollary 11.5: C is torsion-free as an abelian group.

Proof: In [26], it is stated that I_p is divisible as an abelian group.

Therefore C is torsion-free by Theorem 3.1. //

Corollary 11.6: The cap-product isomorphisms

$$e \cap - : H^k(G, -) \rightarrow H_{n-k}(G, C \otimes -)$$

are δ -functor isomorphisms.

Proof: Since C is torsion-free, it satisfies the conditions of Theorem 8.1 (ii). Hence we have a δ -functor isomorphism. //

Suppose now that G is a profinite group, $\text{cd}_p G = n$, and $H^n(G, A)$ is finite whenever $A \in C_G^{f(p)}$. Then Serre ([23], p. I-29) has shown that $H^n(G, -)^* \cong \text{Hom}_G(-, I_p)$ as functors from $C_G^{t(p)}$ to Ab , where $I_p \cong \varprojlim_{i, U} H^n(G, \mathbb{Z}/p^i \mathbb{Z}(G/U))^*$, ([23], p. I-81). Applying $(-)^*$ to finite right G -modules and then the above functors, we have that

$$H^n(G, (-)^*)^* \cong \text{Hom}_G((-)^*, I_p)$$

as functors from $C_G^{f(p)}$ to Ab . However, we know

$$H^n(G, (-)^*)^* \cong H_n(G, -)$$

when applied to right pseudocompact G -modules (Corollary 1.6) so we have that

$$H_n(G, -) \cong \text{Hom}_G((-)^*, I_p)$$

on all finite right G -modules. If we call $\varprojlim_{i, U} H^n(G, \mathbb{Z}/p^i \mathbb{Z}(G/U)) = C$, then $I_p \cong C^*$ and

$$H_n(G, -) \cong \text{Hom}_G((-)^*, C^*)$$

Finally, $(-)^*$ is a full faithful functor so that

$$\text{Hom}_G((-)^*, C^*) \cong \text{Hom}_G(C, -)$$

(as abelian groups); hence we have shown that as functors on finite

right p -torsion G -modules ,

$$H_n(G, -) \cong \text{Hom}_G(C, -) .$$

Proposition 11.7: Suppose G is an FP-group , $\text{cd}_p G = n$, and C is its dualizing module, $H^n(G, \mathbb{Z}_p[[G]])$, then

$$H_n(G, -) \cong \text{Hom}_G(C, -)$$

as functors from PC_G^p to $\underline{\text{Ab}}$.

Proof: Since G is an FP-group , $H_n(G, A)$ is finite for all $A \in {}_G C^f(p)$. Hence C has $P(f)$ and the above functors can be applied to PC_G^p . //

Corollary 11.8: $\text{Hom}_G(C, C) \cong \mathbb{Z}_p$ when G is a D_p^n -group and C is its dualizing module. //

Corollary 11.9: If C is the dualizing module of a D_p^n -group G with trivial G -action, then $C \cong \mathbb{Z}_p$.

Proof: If C has trivial G -action ,

$$\text{Hom}_G(C, C) \cong \text{Hom}(C, C) \cong \text{Hom}(C^*, C^*) \cong \mathbb{Z}_p$$

by Corollary 11.8 and the fact that $(-)^*$ is a full faithful functor.

But C^* is divisible as an abelian group and is therefore isomorphic to a direct sum of copies of $\mathbb{Z}_p^\infty$. In fact it must be a single copy so that $C \cong \mathbb{Z}_p$. //

In [26] , Verdier defines Poincaré groups in p to be strict Cohen-Macaulay groups whose discrete dualizing module is isomorphic to $\mathbb{Z}_p^\infty$ as an abelian group. If, in addition, a Poincaré group in p is a pro- p -group, it is a Poincaré group in the sense of Serre [23], p. I-47 . In either case these groups are FP-groups as will be shown in § 15. Hence they are D_p^n -groups for some integer n by Theorem 11.4.

The strict cohomological dimension of a profinite group G with

$\text{cd}_p G = n$, is n or $n+1$ (see Proposition 19, p. I-30, [23]). In § 15 we show that for any D_p^n -group G , $\text{scd}_p G = n+1$ iff G has an open subgroup U such that $C_U \cong C \otimes_{U=\mathbb{Z}_p}$ is infinite, or equivalently that $(C^*)^U$ is infinite where C is the pseudocompact dualizing module.

The definition of duality groups by means of cap-product isomorphisms is rather cumbersome to use. Hence a more efficient characterization is necessary. If G is a D_p^n -group, then it is a strict Cohen-Macauley group (in p) of dimension n and so from Proposition 4.3, [26], we know that for $k \neq n$, $\lim_{V, \text{cor}} H^k(V, \mathbb{Z}/p\mathbb{Z})^* = 0$. This is equivalent, by Proposition 1.10, to having $H^k(G, \mathbb{Z}_p[[G]]) = 0$ for $k \neq n$ and $H^n(G, \mathbb{Z}_p[[G]])$ being torsion-free. In fact, these properties, in addition to being an FP-group, are sufficient to define a duality group.

Proposition 11.10. Let G be a profinite group, then G is a duality group (in p) of dimension n iff the following are true:

- (i) $\text{cd}_p G = n$
- (ii) G is an FP-group
- (iii) $H^k(G, \mathbb{Z}_p[[G]]) \leq 0$ for $k \neq n$
- (iv) $H^n(G, \mathbb{Z}_p[[G]])$ is torsion-free

Proof: Since G is an FP-group and $\text{cd}_p G = n$, there is a G -projective resolution of $\mathbb{Z}_p$

$$0 \rightarrow P_n \rightarrow P_{n-1} \rightarrow \dots \rightarrow P_1 \rightarrow P_0 \rightarrow \mathbb{Z}_p \rightarrow 0$$

such that each P_i is finitely generated. Now any projective resolution yields a resolvent functor in the sense of [13], p. 150, or Definition

1.5 [26]. Since each projective has $P(f)$ -,

$$\text{Hom}_G(P_i, -)^* \cong (\text{Hom}_G(P_i, Z_{\mathbb{P}}[[G]]) \otimes_G -)^* \cong \text{Hom}_G(-, \text{Hom}_G(P_i, Z_{\mathbb{P}}[[G]])^*),$$

by Corollary 5.4, and so the complex

$$11.11 \dots 0 \rightarrow \text{Hom}_G(P_n, Z_{\mathbb{P}}[[G]])^* \rightarrow \dots \rightarrow \text{Hom}_G(P_0, Z_{\mathbb{P}}[[G]])^* \rightarrow 0$$

is the dualizing complex (up to homotopy) for G constructed in Proposition 3.1 and Proposition 3.2 and defined by Definition 3.4 of [26].

Since $H^k(G, Z_{\mathbb{P}}[[G]]) = 0$ for $k \neq n$ and $H^n(G, Z_{\mathbb{P}}[[G]])$ is torsion-free, the complex (11.11) satisfies the conditions of Definition 4.1, [26] and hence G is a strict Cohen-Macaulay group (in p) of dimension n .

Since it is an FP-group as well, G is a duality group (in p) of dimension n by Theorem 11.4. //

There is a categorical approach to cap-product which allows us to view the cap-product isomorphism

$$e \cap - : H^k(G, A) \rightarrow H_{n-k}(G, C \otimes A)$$

in terms of a natural transformation between covariant, homological, connected, exact couniversal functors, [13], § 2.2. For any profinite group G , the functors $\{H_*(G, C \otimes -)\}$ form such a functor when $C \in \text{PC}_G^D$ is torsion-free (see Theorem 8.1 (ii)). When G is a D_p^n -group, the same is also true of the functors $\{H^*(G, -)\}$ except that in this case the method of indexing the functor is reversed. That $\{H^*(G, -)\}$, with reversed indexing, is a covariant, homological, connected, exact functor is clear. To show couniversality one must only show that for any projective $P \in \text{PC}_G^D$, $H^k(G, P) = 0$ for $k \neq n$. But $H^k(G, P) \cong$

$H_{n-k}(G, C \otimes P)$ and P is a free module by Proposition 2.9 and so $C \otimes P$ is an induced module by Proposition 7.3. Hence $H_{n-k}(G, C \otimes P) = 0$ when $k \neq n$ by Proposition 7.2, so $\{H^*(G, -)\}$ is couniversal as well. Hence cap-product, in the duality case, is a natural isomorphism

$$\{H^*(G, -)\} \rightarrow \{H_{n-k}(G, C \otimes -)\}$$

between two covariant, homological, connected, exact, couniversal functors and as such is completely determined by the cap-product transformation

$$H^n(G, -) \rightarrow H_0(G, C \otimes -).$$

We now describe fully this map.

Let G be a D_p^n -group and

$$0 \rightarrow P_n \rightarrow P_{n-1} \rightarrow \dots \rightarrow P_1 \rightarrow P_0 \rightarrow Z_p \rightarrow 0$$

be a G -projective resolution of Z_p where each P_i is a finitely generated projective left pseudocompact G -module. Writing $\text{Hom}_G(P_i, Z_p[[G]])$ as P'_i we have that the complex

$$0 \rightarrow P'_0 \rightarrow P'_1 \rightarrow \dots \rightarrow P'_{n-1} \rightarrow P'_n \rightarrow C \rightarrow 0$$

is a G -projective resolution of C by Proposition 11.10 and Corollary 5.6. Given, by Theorem 5.3, the natural isomorphisms

$$\text{Hom}_G(P_i, A) \cong P'_i \otimes_G A,$$

for $A \in {}_G\text{PC}^D$, we have natural isomorphisms

$$H^k(G, A) \cong \text{Tor}_{n-k}^G(A, C).$$

The functor $\{\text{Tor}_*^G(-, C)\}$ is a covariant, homological, connected, exact, couniversal functor with

$$\text{Tor}_0^G(A, C) \cong C \otimes_G A \cong H_0(G, C \otimes A).$$

Hence the functors $\{\text{Tor}_*^G(-, C)\}$ and $\{H_*(G, C \otimes -)\}$ are the same up to isomorphism and, therefore, so are $\{H^*(G, -)\}$ and $\{H_{n-*}(G, C \otimes -)\}$.

However, we must show that the induced isomorphisms of Theorem 5.3 are indeed the cap-product isomorphisms described in Theorem 8.1, or their inverses. This is done by adapting Theorem 4.2 of [6] to the pseudocompact case.

Theorem 11.12: Let G be an FP-group, $\text{cd}_p G = n$. Let $H^n(G, \mathbb{Z}_p[[G]]) \cong C$, then there is an element e of $H_n(G, C)$ such that

$$e \cap - : H^n(G, -) \rightarrow C \otimes_G -$$

is a natural isomorphism, inverse to the isomorphism induced by that of Theorem 5.3 (i).

Proof: If

$$0 \rightarrow P_n \rightarrow P_{n-1} \rightarrow \dots \rightarrow P_1 \rightarrow P_0 \rightarrow \mathbb{Z}_p \rightarrow 0$$

is a G -projective resolution of $\mathbb{Z}_p$ such that each P_i has $P(f)$, we have the exact sequence

$$11.13 \dots \quad P'_{n-1} \xrightarrow{\partial'} P'_n \xrightarrow{\pi} C \rightarrow 0$$

As well, the diagram

$$\begin{array}{ccc} C \otimes_G P_n & \xrightarrow{\quad \downarrow n \quad} & \text{Hom}_G(P'_n, C) \\ \downarrow \partial_1 & & \downarrow \partial'' \\ C \otimes_G P_{n-1} & \xrightarrow{\quad \downarrow n-1 \quad} & \text{Hom}_G(P'_{n-1}, C) \end{array}$$

commutes, where ∂_1 and ∂'' are induced by the map $\partial : P_n \rightarrow P_{n-1}$.

Let $e = \psi_n^{-1}(\pi)$, where $\pi \in \text{Hom}_G(P'_n, C)$ is the map in (11.13).

Then $e = (\dots, \bar{c}_1 \otimes \bar{p}_1, \dots)$, where $\bar{c}_1 \in C_{1,U} \cong H^n(G, \mathbb{Z}/p^i \mathbb{Z}(G/U))$,

and $\bar{p}_1 \in P_{1,U} \cong P_n/K_{1,U}$ and $K_{1,U}$ is defined on page 5.2. Then for

$$f \in P'_n,$$

$$\psi_n(e)(f) = (\dots, \sum \bar{c}_i \bar{f}(\bar{p}_i), \dots) = \pi(f),$$

where $\bar{f}$ is the image of f in $\text{Hom}_G(P_n, Z/p^i Z(G/U)) \cong \text{Hom}_G(P_{i,U}, Z/p^i Z(G/U))$.

Now $\partial''(\pi) = \pi \circ \partial' = 0$, (see 11.13), hence $\partial''\psi_n(e) = 0$ so that

$\psi_{n-1}\partial'(e) = 0$, and since ψ_{n-1} is monic, $e \in \ker \partial' = H_n(G, C)$.

Now for any $A \in C_G^{f(p)}$, we have the following commutative diagram:

$$\begin{array}{ccccccc} P'_{n-1} \otimes_G A & \longrightarrow & P'_n \otimes_G A & \longrightarrow & C \otimes_G A & \longrightarrow & 0 \\ \downarrow \varphi_{n-1} & & \downarrow \varphi_n & \nearrow & \downarrow \varphi'_n & & \\ \text{Hom}_G(P'_{n-1}, A) & \longrightarrow & \text{Hom}_G(P'_n, A) & \longrightarrow & H^n(G, A) & \longrightarrow & 0 \end{array}$$

with exact rows. Select $c \otimes a$ as a generator of $C \otimes_G A$ and $\gamma \in P'_n$ such that $\pi(\gamma) = c$. Then consider the map $e \cap - \circ \varphi'_n$.

$$\begin{aligned} e \cap - \circ \varphi'_n(c \otimes a) &= e \cap - \circ \varphi'_n(\gamma \otimes a) \\ &= ((\dots, \sum \bar{c}_i \otimes \bar{p}_i, \dots) \cap \overline{(\varphi_n(\gamma \otimes a))})_{j,V} \end{aligned}$$

(where we consider $\overline{\varphi_n(\gamma \otimes a)} \in \text{Hom}_G(P_{j,V}, A)$ for all pairs (j, V) such that $j \geq i$, $V \leq U$ and A is a $Z/p^i Z(G/U)$ -module.)

$$\begin{aligned} &= (\dots, \sum \bar{c}_i \otimes \overline{(\varphi_n(\gamma \otimes a)(\bar{p}_i))}, \dots); \\ &= (\dots, \sum \bar{c}_i \otimes \bar{\gamma}(\bar{p}_i)(a), \dots) \\ &= (\dots, \sum \bar{c}_i \bar{\gamma}(\bar{p}_i) \otimes a, \dots) \\ &= \pi(\gamma) \otimes a \\ &= c \otimes a. \end{aligned}$$

Hence $e \cap -$ is the inverse of φ'_n and so is itself a natural isomorphism. //

Since $e \cap -$ agrees with the inverse of φ'_n at the first dimension, it agrees with this inverse everywhere by the couniversality of

$(H^*(G, -))$.

A similar characterization for the cap-product is also possible in the discrete case once one knows that discrete duality groups are FP-groups.

Finally in this section, we consider $M \otimes_G -$ for $M \in PC_G^p$. If one considers M as a member of $\text{Mod}(\mathbb{Z}_{=p}[[G]])$, one sees that $M \otimes_G -$ commutes with $\varinjlim$. Hence for objects in $C_G^{t(p)}$, which can be expressed as direct limits of their finitely generated (finite) submodules, we have that

$$M \otimes_G \varinjlim A_i \cong \varinjlim (M \otimes_G A_i) .$$

If, in addition, M has $P(f)$ each $(M \otimes_G A_i)$ is finite and belongs to $C^{f(p)}$. Therefore $M \otimes_G \varinjlim A_i \cong \varinjlim (M \otimes_G A_i)$ belongs to $C^{t(p)}$. This allows us to define homology with coefficient modules in $C_G^{t(p)}$, provided G is an FP-group. In this case,

$$H_k(G, \varinjlim A_i) \cong \varinjlim H_k(G, A_i)$$

and $H_k(G, \varinjlim A_i)$ is a discrete torsion $\mathbb{Z}_{=p}$ -module. This yields the following theorem.

Theorem 11.13: Let G be a D_p^n -group. Then there is a natural topological isomorphism induced by the cap-product

$$H_k(G, A) \rightarrow H^{n-k}(G, \text{Hom}(C, A))$$

for all k , $A \in C_G^{t(p)}$, and $C \cong H^n(G, \mathbb{Z}_{=p}[[G]])$, the dualizing module of G .

Proof: Since G is a D_p^n -group, it is a strict Cohen-Macaulay group (in p) of dimension n and for $B \in PC_G^p$, we have the natural topolo-

gical isomorphism

$$H^k(G, B)^* \cong H^{n-k}(G, \text{Hom}(B, C^*))$$

by Proposition 4.4 of [26]. But $B \in {}_G\text{PC}^p$ implies $B \cong A^*$ where

$A \in {}_G C^{t(p)}$ so that we have

$$H^k(G, A^*)^* \cong H^{n-k}(G, \text{Hom}(A^*, C^*))$$

In the first place,

$$H^k(G, A^*)^* \cong H_k(G, A),$$

while in the second,

$$\text{Hom}(A^*, C^*) \cong_G \text{Hom}(C, A);$$

both these isomorphisms being topological and natural. Hence we have the natural topological isomorphism

$$H_k(G, A) \cong H^k(G, A^*)^* \cong H^{n-k}(G, \text{Hom}(A^*, C^*)) \cong H^{n-k}(G, \text{Hom}(C, A)). //$$

This type of inverse duality exists in the discrete case only when the dualizing module is $\mathbb{Z}$ -free. The profinite case is consistent with this fact since $H^n(G, \mathbb{Z}_{\overline{p}}[[G]])$ is $\mathbb{Z}_{\overline{p}}$ -free by Theorem 3.1.

§ 12 - Extension Theorems

Proposition 12.1 [26]: Let G be a profinite group such that $\text{cd}_p G < \infty$.

Suppose there exists an open subgroup V of G which is a strict Cohen-Macaulay group (in p), then G is a strict Cohen-Macaulay group (in p). Reciprocally, if G is a strict Cohen-Macaulay group (in p), then so is every open subgroup of G . //

This proposition, together with Theorem 6.6 and Theorem 11.4 yield the following:

Proposition 12.2: Let G be a profinite group such that $\text{cd}_p G < \infty$.

Suppose there exists an open subgroup V of G which is a D_p^n -group, then G is a D_p^n -group. Reciprocally, if G is a D_p^n -group, then so is every open subgroup of G . //

We now show that the class of duality groups (in p) is extension closed.

Theorem 12.3: If $N \rightarrow G \rightarrow Q$ is an exact sequence of profinite groups,

where N and Q are D_p^n - and D_p^m -groups respectively, then G is a D_p^{n+m} -group. Moreover,

$$H^{n+m}(G, \mathbb{Z}_p[[G]]) \cong H^n(N, \mathbb{Z}_p[[N]]) \otimes H^m(Q, \mathbb{Z}_p[[Q]]) .$$

Note: We index the various dualizing modules by their respective groups,

so that $H^n(N, \mathbb{Z}_p[[N]]) \cong C(N)$, etc.

Proof: We show that G satisfies the conditions of Proposition 11.10.

From the Lyndon-Hochschild-Serre spectral sequence, it is clear that

$\text{cd}_p G = n+m$. Furthermore, G is an FP-group by Theorem 6.7. Next we

show that $H^{n+m}(G, \mathbb{Z}_{\overline{p}}[[G]])$ is torsion-free.

$$H^{n+m}(G, \mathbb{Z}_{\overline{p}}[[G]]) \cong H^m(Q, H^n(N, \mathbb{Z}_{\overline{p}}[[G]]))$$

and we know that

$$H^n(N, \mathbb{Z}_{\overline{p}}[[G]]) \cong C(N) \otimes_N \mathbb{Z}_{\overline{p}}[[G]]$$

since N is an FP-group. By duplicating the proof of Proposition 1.8,

[6] we can show that this isomorphism is a Q -isomorphism. Hence

$$H^m(Q, H^n(N, \mathbb{Z}_{\overline{p}}[[G]])) \cong H^m(Q, C(N) \otimes_N \mathbb{Z}_{\overline{p}}[[G]]) \cong C(Q) \otimes_Q (C(N) \otimes_N \mathbb{Z}_{\overline{p}}[[G]]).$$

Finally,

$$C(Q) \otimes_Q (C(N) \otimes_N \mathbb{Z}_{\overline{p}}[[G]]) \cong (C(Q) \otimes C(N)) \otimes_G \mathbb{Z}_{\overline{p}}[[G]] \cong C(Q) \otimes C(N).$$

This isomorphism is defined by $(c_q \otimes (c_n \otimes \alpha)) \mapsto (c_q \otimes c_n) \otimes \alpha$ and is

well-defined because $C(N)$ has a two-sided G -action with the property

that for $x \in G$, and $c_n \in C(N)$, $xc_n = c_n x^{-1}$. It is clear that

$C(Q) \otimes C(N)$ is flat as a $\mathbb{Z}_{\overline{p}}$ -module and so torsion-free by Theorem 3.1.

Finally, we must show that $H^k(G, \mathbb{Z}_{\overline{p}}[[G]]) = 0$ when $k \neq n+m$.

Since $\mathbb{Z}_{\overline{p}}[[G]]$ is an induced pseudocompact N -module (see Note 2.7),

$H^k(N, \mathbb{Z}_{\overline{p}}[[G]]) = 0$ for $k \neq n$ because N is a D_p^n -group. Hence the

spectral sequence

$$H^p(Q, H^q(N, \mathbb{Z}_{\overline{p}}[[G]])) \rightarrow H^{p+q}(G, \mathbb{Z}_{\overline{p}}[[G]])$$

collapses to a single line

$$(H^i(Q, H^n(N, \mathbb{Z}_{\overline{p}}[[G]])), i = 0, 1, \dots, m)$$

and from this it is clear that $H^k(G, \mathbb{Z}_{\overline{p}}[[G]]) = 0$ for $k < n$. Now each

$$H^i(Q, H^n(N, \mathbb{Z}_{\overline{p}}[[G]])) \cong H^{i+n}(G, \mathbb{Z}_{\overline{p}}[[G]]),$$

while

$$H^n(N, \mathbb{Z}_{\underline{p}}[[G]]) \cong C(N) \otimes_N \mathbb{Z}_{\underline{p}}[[G]]$$

is an induced Q -module by Lemma 7.4. Hence

$$H^i(Q, H^n(H, \mathbb{Z}_{\underline{p}}[[G]])) \cong H_{m-1}(Q, C(Q) \otimes (Q\text{-induced})) \cong 0$$

for $i \neq m$ by Proposition 7.3. Hence $H^k(G, \mathbb{Z}_{\underline{p}}[[G]]) = 0$ for $k \neq n+m$.

This completes the proof. //

If G is a D_p^n -group, it is a reasonable question to ask when G^p , the maximal p -quotient of G , is a duality group (necessarily in p) of some finite dimension. Sufficient conditions for this to happen are:

(i) that the spectral sequence

$$H^p(G^p, H^q(N, -)) \Rightarrow H^{p+q}(G, -)$$

defined for the exact sequence $N \rightarrow G \rightarrow G^p$, collapses to yield only base terms

$$H^*(G^p, H^0(N, -)) \cong H^*(G^p, -) \cong H^*(G, -)$$

when applied to elements of $C_{G^p}^{f(p)}$,

(ii) $H^k(G^p, \mathbb{Z}_{\underline{p}}[[G^p]]) = 0$ for $k \neq \text{cd}_p G^p$ and,

(iii) $H^m(G^p, \mathbb{Z}_{\underline{p}}[[G^p]])$ be torsion-free for $m = \text{cd}_p G^p$.

As we shall see at the end of § 13, this always occurs when G is the profinite completion of an orientable Poincaré duality group which is good and p -good.

We now show that the duality (in p) of any profinite group can be characterized by considering the pseudocompact $\mathbb{Z}/p\mathbb{Z}$ -algebra, $\mathbb{Z}/p\mathbb{Z}[[G]]$, and the cohomology functor $\text{Ext}_{\mathbb{Z}/p\mathbb{Z}[[G]]}^*(\mathbb{Z}/p\mathbb{Z}, -)$, (hereafter written $\text{Ext}_G^*(\mathbb{Z}/p\mathbb{Z}, -)$, applied to the subcategory $\mathcal{C}_G^{t(p)}$ of $\mathcal{C}_G^{t(p)}$ consisting

of all p -torsion discrete left G -modules annihilated by p . In fact

$$\text{Ext}_G^*(\mathbb{Z}/p\mathbb{Z}, -) \cong H^*(G, -)$$

when both are applied to $\bar{C}_G^{t(p)}$. This can be shown by noting that every injective in $\bar{C}_G^{t(p)}$ is a direct summand of $M_G(\bigoplus^s \mathbb{Z}/p\mathbb{Z})$ for some cardinal s and since $H^*(G, M_G(\bigoplus^s \mathbb{Z}/p\mathbb{Z})) = 0$ in dimensions other than 0, the same is true when $H^*(G, -)$ is applied to injectives in $\bar{C}_G^{t(p)}$. The two cohomological functors are therefore isomorphic because they are isomorphic at dimension 0. An immediate consequence of this isomorphism is the following:

Theorem 12.5: A profinite group G is an FP-group with respect to the group algebra $\mathbb{Z}_p[[G]]$ iff it is an FP-group with respect to the group algebra $\mathbb{Z}/p\mathbb{Z}[[G]]$.

Proof: With respect to $\mathbb{Z}/p\mathbb{Z}[[G]]$, FP-groups are defined analogously to Definition 6.1. As well, analogues exist for Theorems 6.2, 6.3, and 6.4. Hence the necessity of the theorem follows immediately and the sufficiency by a simple extension argument. //

Note: In the remainder of this section, the analogues of Definition 6.1, and Theorems 6.2, 6.3 and 6.4 will be denoted as Definition 6.1', and Theorems 6.2', 6.3', and 6.4' although they will not be stated explicitly.

Definition 12.6: A duality group of dimension n over the ring $\mathbb{Z}/p\mathbb{Z}$ is a profinite group G which is an FP-group over $\mathbb{Z}/p\mathbb{Z}$, having $\text{cd}_{\mathbb{Z}/p\mathbb{Z}}(G) = n$, and $\text{Ext}_G^k(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}[[G]]) \cong H^k(G, \mathbb{Z}/p\mathbb{Z}[[G]]) = 0$ for $k \neq n$.

Theorem 12.7: G is a D_p^n -group iff G is a duality group over Z/pZ of dimension n .

Proof: $cd_p G = cd_{Z/pZ}(G)$ (see § 3.1 [23]). Moreover, G is an FP-group iff it is an FP-group over Z/pZ by Theorem 12.5. If G is a D_p^n -group, then $H^k(G, Z/pZ[[G]]) = 0$ for $k \neq n$ since $Z/pZ[[G]]$ is G -induced. Conversely, if $H^k(G, Z/pZ[[G]]) = 0$ for $k \neq n$, then $H^k(G, \underline{Z}_p[[G]]) = 0$ for $k \neq n$ by extension, and from the short exact sequence of G -modules

$$\underline{Z}_p[[G]] \xrightarrow{\cdot p} \underline{Z}_p[[G]] \longrightarrow Z/pZ[[G]]$$

one sees immediately that $H^{n-1}(G, Z/pZ[[G]]) = 0$ implies

$$H^n(G, \underline{Z}_p[[G]]) \xrightarrow{\cdot p} H^n(G, \underline{Z}_p[[G]])$$

is monic so that $H^n(G, \underline{Z}_p[[G]])$ is torsion-free. //

We now use this relationship between D_p^n -groups and duality groups over Z/pZ to prove the following general extension theorem about profinite groups.

Theorem 12.8: Let $N \twoheadrightarrow G \rightarrow Q$ be a short exact sequence of profinite groups. Suppose N is an FP-group, then if G is a D_p^n -group, and $cd_p Q < \infty$, both N and Q are duality groups (in p) and $n = cd_p N + cd_p Q$.

Proof: First we show that $n = cd_p N + cd_p Q$. Let $cd_p N = \ell$, and $cd_p Q = k$, then we know $n \leq k + \ell$. It is easy to see that both $C(N)$ and $C(Q)$ are non-zero.

Note: The term $H^k(Q, \underline{Z}_p[[Q]])$ is well-defined since by Theorem 6.7, Q is an FP-group.

From the Lyndon-Hochschild-Serre spectral sequence we have that

$$H^k(Q, H^l(N, \mathbb{Z}_{\underline{p}}[[G]])) \cong H^{k+l}(G, \mathbb{Z}_{\underline{p}}[[G]]) .$$

By Theorem 11.12

$$H^l(N, \mathbb{Z}_{\underline{p}}[[G]]) \cong C(N) \otimes_{\mathbb{Z}_{\underline{p}}} \mathbb{Z}_{\underline{p}}[[G]] ,$$

as in the proof of Theorem 12.3, this is a $\mathbb{Q}$ -isomorphism. Therefore

$$H^k(Q, H^l(N, \mathbb{Z}_{\underline{p}}[[G]])) \cong C(Q) \otimes_Q (C(N) \otimes_{\mathbb{Z}_{\underline{p}}} \mathbb{Z}_{\underline{p}}[[G]]) \cong (C(Q) \otimes C(N)) \otimes_G \mathbb{Z}_{\underline{p}}[[G]] ;$$

the latter isomorphism defined as in the proof of Theorem 12.3. Hence

$$H^{k+l}(G, \mathbb{Z}_{\underline{p}}[[G]]) \cong C(Q) \otimes C(N)$$

and in order to show $\text{cd}_p G = k+l$, it is enough to show $C(Q) \otimes C(N)$

is non-zero. Certainly

$$C(Q) \otimes C(N) \rightarrow H^k(Q, \mathbb{Z}/p\mathbb{Z}[[Q]]) \otimes H^l(N, \mathbb{Z}/p\mathbb{Z}[[N]])$$

and neither $H^k(Q, \mathbb{Z}/p\mathbb{Z}[[Q]])$ nor $H^l(N, \mathbb{Z}/p\mathbb{Z}[[N]])$ is zero so that as a vector space over $\mathbb{Z}/p\mathbb{Z}$,

$$H^k(Q, \mathbb{Z}/p\mathbb{Z}[[Q]]) \otimes H^l(N, \mathbb{Z}/p\mathbb{Z}[[N]])$$

has dimension greater than zero and hence $C(Q) \otimes C(N)$ is non-zero so that $\text{cd}_p G = k+l$.

In order now to show that N and Q are duality groups it is sufficient, by Theorem 12.7, to show that $H^i(Q, \mathbb{Z}/p\mathbb{Z}[[Q]]) = 0$ when $i \neq k$ and $H^j(N, \mathbb{Z}/p\mathbb{Z}[[N]]) = 0$ when $j \neq l$.

Since N is an FP-group, there exists a commutative diagram

$$\begin{array}{ccccccc}
0 & \rightarrow & \text{Hom}_N(P_0, Z/pZ[[G]]) & \rightarrow & \dots & & \\
& & \downarrow & & & & \\
0 & \rightarrow & \text{Hom}_N(P_0, Z/pZ[[N]]) \otimes_N Z/pZ[[G]] & \rightarrow & \dots & & \\
& & & & \dots & \rightarrow & \text{Hom}_N(P_\ell, Z/pZ[[G]]) \rightarrow H^\ell(N, Z/pZ[[G]]) \rightarrow 0 \\
& & & & \downarrow & & \downarrow \\
& & \dots & \rightarrow & \text{Hom}_N(P_\ell, Z/pZ[[N]]) \otimes_N Z/pZ[[G]] & \rightarrow & C(N) \otimes_N Z/pZ[[G]] \rightarrow 0
\end{array}$$

This follows from Theorem 6.3' and a statement analogous to Theorem 5.3 for pseudocompact $Z/pZ[[N]]$ -modules. Taking cohomology of both the above complexes gives $H^1(N, Z/pZ[[G]])$ on the one hand, and $H^1(N, Z/pZ[[N]]) \otimes_N Z/pZ[[G]]$ on the other, because $Z/pZ[[G]]$ is $Z/pZ[[N]]$ -projective. Hence

$$H^1(N, Z/pZ[[G]]) \cong H^1(N, Z/pZ[[N]]) \otimes_N Z/pZ[[G]]$$

and as in the case $i = k$, this isomorphism is a Q -isomorphism.

The spectral sequence

$$H^p(Q, H^q(N, Z/pZ[[G]])) \Rightarrow H^{p+q}(G, Z/pZ[[G]])$$

can therefore be rewritten

$$H^p(Q, (H^q(N, Z/pZ[[N]]) \otimes_N Z/pZ[[G]])) \Rightarrow H^{p+q}(G, Z/pZ[[G]]).$$

Now $H^q(N, Z/pZ[[N]]) \otimes_N Z/pZ[[G]]$, with diagonal Q -action, is Q -isomorphic to $H^q(N, Z/pZ[[N]]) \otimes Z/pZ[[Q]]$ where the Q -action is the left Q -action on $Z/pZ[[Q]]$. In turn, $H^q(N, Z/pZ[[N]]) \cong \prod_\alpha (Z/pZ)$ for some cardinal α since it is a pro- p -abelian group annihilated by p , so that

$$H^q(N, Z/pZ[[N]]) \otimes Z/pZ[[Q]] \cong \prod_\alpha (Z/pZ) \otimes Z/pZ[[Q]]$$

$$\cong \prod_{\alpha} (Z/pZ \otimes Z/pZ[[Q]]) \cong \prod_{\alpha} Z/pZ[[Q]]$$

and this isomorphism is as Q -modules. Hence

$$\begin{aligned} H^p(Q, (H^q(N, Z/pZ[[N]]) \otimes Z/pZ[[Q]])) &\cong H^p(Q, \prod_{\alpha} Z/pZ[[Q]]) \\ &\cong \prod_{\alpha} H^p(Q, Z/pZ[[Q]]) \\ &\cong H^q(N, Z/pZ[[N]]) \otimes H^p(Q, Z/pZ[[Q]]) . \end{aligned}$$

The second isomorphism in this sequence exists because Q is an FP-group and so $H^p(Q, -)$ commutes with inverse limits.

We therefore have a spectral sequence converging to $H^*(G, Z/pZ[[G]])$ where $E_2^{p,q} \cong H^q(N, Z/pZ[[N]]) \otimes H^p(Q, Z/pZ[[Q]])$. If q' is the smallest integer such that $H^{q'}(N, Z/pZ[[N]]) \neq 0$ and p' is the smallest integer such that $H^{p'}(Q, Z/pZ[[Q]]) \neq 0$ then

$$E_2^{p',q'} \cong H^{q'}(N, Z/pZ[[N]]) \otimes H^{p'}(Q, Z/pZ[[Q]]) \neq 0 .$$

It is easy to see from the spectral sequence that

$$E_2^{p',q'} \cong H^{p'+q'}(G, Z/pZ[[G]])$$

as well, and since G is a duality group of dimension $k+l$, $p' = k$ and $q' = l$ necessarily. Hence $H^q(N, Z/pZ[[N]]) = 0$ for $q \neq l$ and $H^p(Q, Z/pZ[[Q]]) = 0$ for $p \neq k$ so that both N and Q are duality groups by Theorem 12.5. //

As corollaries to Theorem 12.8 we have the following.

Corollary 12.9: Let G be a D_p^n -group and suppose that G_p , a Sylow p -subgroup of G , has a normal complement in G ; then G_p is a D_p^n -group.

Proof: If G_p has a normal complement in G there is a short exact

sequence of profinite groups

$$N \twoheadrightarrow G \twoheadrightarrow G_p$$

where $\text{cd}_p N = 0$. Therefore N is an FP-group with respect to p and

since $\text{cd}_p G = \text{cd}_p G_p$, we apply the theorem to say G_p is a D_p^n -group. //

Corollary 12.10: Let G be a D_p^n -group and suppose G_p is normal in

G . If, in addition, G_p is an FP-group then G_p is a D_p^n -group.

Proof: Since $\text{cd}_p G/G_p = 0$, Theorem 12.8 can be applied to give the result. //

The converse to the above corollaries is true with no conditions on G_p .

Theorem 12.11: Let G be a profinite group and G_p be a Sylow- p -subgroup. If G_p is a D_p^n -group, then so is G .

Proof: The restriction map

$$H^k(G, -) \rightarrow H^k(G_p, -)$$

is a monomorphism, ([21], p.139), and so G is clearly an FP-group

if G_p is. As well, $\text{cd}_p G = \text{cd}_p G_p$, so by Proposition 11.20 it remains

to show $H^k(G, \mathbb{Z}_p[[G]]) = 0$ for $k \neq n = \text{cd}_p G$ and that $H^n(G, \mathbb{Z}_p[[G]])$

is torsion-free.

By Proposition 2.10, $\mathbb{Z}_p[[G]]$ is an induced G_p -module and by

Proposition 7.3, $C(G_p) \otimes \mathbb{Z}_p[[G]]$ is again induced as a G_p -module.

Hence

$$H^k(G_p, \mathbb{Z}_p[[G]]) \cong H_{n-k}(G_p, C(G_p) \otimes \mathbb{Z}_p[[G]]) = 0$$

for $k \neq n$ since homology vanishes on induced modules in dimensions

other than 0. But

$$H^k(G, Z_{-p}[[G]]) \rightarrow H^k(G_p, Z_{-p}[[G]])$$

and so $H^k(G, Z_{-p}[[G]]) = 0$ for $k \neq n$. Similarly $H^{n-1}(G, Z/pZ[[G]]) = 0$

and this, together with the short exact sequence

$$Z_{-p}[[G]] \xrightarrow{\cdot p} Z_{-p}[[G]] \longrightarrow Z/pZ[[G]]$$

shows that

$$H^n(G, Z_{-p}[[G]]) \xrightarrow{\cdot p} H^n(G, Z_{-p}[[G]])$$

is a monomorphism and so $H^n(G, Z_{-p}[[G]])$ is torsion-free. //

Note 12.12: In both Corollary 12.9 and Corollary 12.10 the dualizing

module of G , $H^n(G, Z_{-p}[[G]])$, is isomorphic to

$$H^n(G_p, Z_{-p}[[G_p]]) \otimes H^0(N, Z_{-p}[[N]])$$

where N is some profinite group with the property that $cd_p N = 0$;

i.e., that $p \nmid \#N$. Hence $H^0(N, Z_{-p}[[N]]) \cong Z_{-p}$ by Corollaries 2.12 and

2.13 so

$$H^n(G, Z_{-p}[[G]]) \cong H^n(G_p, Z_{-p}[[G_p]])$$

§ 13 - Discrete Duality Groups and Good Groups

Let us recall a few facts about the profinite completions of discrete groups. If G is a discrete group, then

$$\hat{G} \cong \varprojlim_U G/U$$

(where the inverse limit is taken over all normal subgroups of G having finite index in G) is called the profinite completion of G . For every U and V normal of finite index in G such that $U \leq V$ there is a commutative diagram

and so by the universal property of inverse limits there is a morphism $G \xrightarrow{h} \hat{G}$ such that the diagram

commutes for all U normal of finite index in G . If the topology on G has as its open neighbourhoods of (1) , precisely the normal subgroups of finite index, then h is a continuous morphism. Let $\underline{G}$ be the intersection of all subgroups of finite index in G . If $G/\underline{G}$ has the quotient topology induced by the topology of G , then $h(G)$, with the relative topology, is topologically isomorphic to $G/\underline{G}$ since the kernel of h is $\underline{G}$.

Moreover, if U is a subgroup of finite index in G then $\hat{U}$ is

naturally topologically isomorphic to the closure of $h(U)$ in G , while $\overline{h(U)} \cap h(G) = h(U)$, or equivalently $h^{-1}(\overline{h(U)}) = U$.

Definition 13.1: A discrete group G is good if $H^k(\hat{G}, A) \cong H^k(G, A)$ for all k and for all $A \in C_G^f$ where $\hat{G}$ is the profinite completion of G and the isomorphism is induced by the natural morphism $G \xrightarrow{h} \hat{G}$.

Good groups were introduced by Serre in [23], Chapter 1. He gives several characterizations which are equivalent to the following.

Proposition 13.2: A discrete group G is good (i) iff $\lim_{\substack{\longrightarrow \\ U, \text{res}^-}} H^k(U, M) = 0$

for all $k > 0$ and all $M \in C_G^f$ where the direct limit is taken over all subgroups of finite index in G and (ii) iff $\lim_{\substack{\longrightarrow \\ U, \text{res}^-}} H^k(U, \mathbb{Z}/p\mathbb{Z}) = 0$ for $k > 0$.

Proof: (i): One characterization given by Serre is that G is good if and only if for every pair $(k, M) \in \mathbb{N} \times C_G^f$ and every $x \in H^k(G, M)$, there is a subgroup G_0 of G , the inverse image under h of an open subgroup of $\hat{G}$, such that $\text{res}^-(x) = 0$ in $H^k(G_0, M)$. It is clear that if $\lim_{\substack{\longrightarrow \\ U, \text{res}^-}} H^k(U, M) = 0$ for all pairs $(k, M) \in \mathbb{N} \times C_G^f$ then G is good.

An exercise in the same section of [23] shows that if G is good and G_0 is as above then G_0 is good as well. But every subgroup of finite index in G is the inverse image under h of some open subgroup of $\hat{G}$, namely the closure of its image in $\hat{G}$, and so the converse is also true by an induction argument.

(ii): Proposition 4.5, [19]. //

In this section we wish to study the connections between good groups

and discrete duality groups studied by Bieri and Eckmann, [6]. We would like to know what duality groups are good groups; and in the event one does have a good duality group, whether its profinite completion is a D_p^n -group for some p and some n .

Brown, [9], has shown that all discrete duality groups are FP-groups - groups with a projective resolution of $\mathbb{Z}$ where each projective is finitely generated. The class of discrete FP-groups is closed with respect to taking subgroups of finite index (and as we have seen, so is the class of good groups). As well, the class of FP-groups is extension-closed. If $N \rightarrow G \rightarrow Q$ is a short exact sequence of discrete groups such that N and Q are good groups and N is an FP-group then we may apply Exercise 2(c), p.I-16, [23], to say that G is good as well. Hence we have shown:

Theorem 13.3: The class of discrete good FP-groups is closed under the taking of subgroups of finite index and under the taking of extensions. //

Corollary 13.4: The class of discrete good duality groups is closed under the taking of subgroups of finite index and under the taking of extensions. //

We now wish to show that under certain conditions, amalgamated products of good groups are good and then to show that these conditions can be applied to the case of duality groups. To do this we must first look at profinite amalgamated products.

Let H and K be two profinite groups with common closed subgroup L .

In [22], the profinite group

$$H \amalg_L K \cong \varprojlim_N (H *_L K / N)$$

(where $H *_L K$ is the ordinary amalgamated product of H and K over L and N is any subgroup of $H *_L K$ of finite index such that $N \cap H$ and $N \cap K$ are open in H and K respectively) is defined to be the profinite amalgamated product of H and K over L provided the canonical maps

$$H \rightarrow H \amalg_L K \quad \text{and} \quad K \rightarrow H \amalg_L K$$

are both monomorphisms. A corollary to Theorem 1.2, [22], is the following, which we include for completeness.

Proposition 13.5 [22]: If H and K are two profinite groups with common closed subgroup L then the profinite amalgamated product $H \amalg_L K$ exists iff there is a set $u(H)$ of open normal subgroups of H and a similar set $u(K)$ such that

$$(i) \quad \bigcap_{U \in u(H)} U = 1 \quad \text{and} \quad \bigcap_{U \in u(K)} U = 1$$

$$(ii) \quad \text{For each } U \in u(H) \text{ there is a } U' \in u(K) \text{ such that}$$

$U \cap L = U' \cap L$ and reciprocally for each $U \in u(K)$ there is a $U' \in u(H)$ such that $U \cap L = U' \cap L$. //

Theorem 13.6: Let A and B be two discrete groups with common subgroup

H . If $\hat{H} \rightarrow \hat{A}$ and $\hat{H} \rightarrow \hat{B}$ then $\hat{A} \hat{*}_H \hat{B} \cong \hat{A} \amalg_{\hat{H}} \hat{B}$.

Proof: The diagram

is a pushout in the category of groups. It is well-known that the profinite completion functor has a right adjoint and hence preserves pushouts. Hence

$$\begin{array}{ccc} \hat{H} & \xrightarrow{\quad} & \hat{A} \\ \downarrow & & \downarrow \\ \hat{B} & \xrightarrow{\quad} & A \hat{*}_H B \end{array}$$

is a pushout in the category of profinite groups.

Now the canonical maps $\hat{A} \rightarrow \hat{A} \amalg_{\hat{H}} \hat{B}$ and $\hat{B} \rightarrow \hat{A} \amalg_{\hat{H}} \hat{B}$ agree on $\hat{H}$ and so there is a map $A \hat{*}_H B \xrightarrow{\alpha} \hat{A} \amalg_{\hat{H}} \hat{B}$ extending these maps (since $\dagger$ is a pushout). If N is any open subgroup of $\hat{A} \amalg_{\hat{H}} \hat{B}$, the sequence

$$A \hat{*}_H B \rightarrow A \hat{*}_H B \xrightarrow{\alpha} \hat{A} \amalg_{\hat{H}} \hat{B} \rightarrow \hat{A} \amalg_{\hat{H}} \hat{B} / N$$

is an onto morphism from $A \hat{*}_H B$ to $\hat{A} \amalg_{\hat{H}} \hat{B} / N$ and hence the image of

$A \hat{*}_H B$ is dense in $\hat{A} \amalg_{\hat{H}} \hat{B}$ so that α is an onto morphism as well. On the

other hand the diagram

$$\begin{array}{ccc} \hat{H} & \xrightarrow{\quad} & \hat{A} \\ \downarrow & & \downarrow \\ \hat{B} & \xrightarrow{\quad} & \hat{A} \hat{*}_H \hat{B} \end{array}$$

is a pushout in the category of discrete groups so there exists a map

$\hat{A} \hat{*}_H \hat{B}$ to $A \hat{*}_H B$ which extends to a map

$$\beta : \hat{A} \amalg_{\hat{H}} \hat{B} \rightarrow A \hat{*}_H B.$$

In order to show that α is monic it is enough to show that $\beta\alpha$ is the identity on $A \hat{*}_H B$. But to do this it is enough to show that $\beta\alpha$ is the identity on the image of $A \hat{*}_H B$ in $A \hat{*}_H B$. This is done by a simple diagram chase of the appropriate diagram. //

Corollary 13.7: If A and B are two discrete groups with common subgroup

H , such that $\hat{H} \twoheadrightarrow \hat{A}$ and $\hat{H} \twoheadrightarrow \hat{B}$, then $\hat{A} \amalg_{\hat{H}} \hat{B}$ exists if and only if

$$\hat{A} \twoheadrightarrow A \hat{*}_H B \quad \text{and} \quad \hat{B} \twoheadrightarrow A \hat{*}_H B.$$

Proof: By Theorem 13.6, $\hat{A} \amalg_{\hat{H}} \hat{B} \cong A \hat{*}_H B$ and the corollary follows immediately from the definition of existence of profinite amalgamated products given in [22] or on page 13.4. //

We now make some statements about the "goodness" of amalgamated products.

Theorem 13.8: Let A , B and H be discrete groups such that H is a common subgroup of A and B .

$$\hat{H} \twoheadrightarrow \hat{A}, \hat{H} \twoheadrightarrow \hat{B}, \hat{A} \twoheadrightarrow A \hat{*}_H B \text{ and } \hat{B} \twoheadrightarrow A \hat{*}_H B$$

then (i), if A and B are good then H is good iff $A \hat{*}_H B$ is good; and

(ii), if H and $A \hat{*}_H B$ are good then so are A and B .

Proof: (i): By our hypothesis $\hat{A} \amalg_{\hat{H}} \hat{B}$ exists and is isomorphic to $A \hat{*}_H B$.

Hence we have two amalgamated products - $A \hat{*}_H B$ and $\hat{A} \amalg_{\hat{H}} \hat{B}$ - and associated with these, two Mayer-Vietoris sequences in cohomology (for the former see, for example, Exercise 14.3, Chapter IV, [17], and for the latter see Theorem 1.13, [16] and Corollary 14.2) so that for any finite discrete

$A \hat{*}_H B$ -module M we have the following commutative diagram.

$$\begin{array}{ccccccc} \dots \rightarrow H^{k-1}(H, M) & \rightarrow & H^k(A \hat{*}_H B, M) & \rightarrow & H^k(A, M) \oplus H^k(B, M) & \rightarrow & H^k(H, M) \rightarrow \dots \\ & \uparrow h' & \uparrow (a \hat{*}_h b)' & & \uparrow a' \oplus b' & & \uparrow h' \\ \dots \rightarrow H^{k-1}(\hat{H}, M) & \rightarrow & H^k(A \hat{*}_H B, M) & \rightarrow & H^k(\hat{A}, M) \oplus H^k(\hat{B}, M) & \rightarrow & H^k(\hat{H}, M) \rightarrow \dots \end{array}$$

If H , A and B are good then h' and $a' \oplus b'$ are isomorphisms for all $k \in \mathbb{Z}^+$ and so $(a \hat{*}_h b)'$ is an isomorphism for all $k \in \mathbb{Z}^+$ and $A \hat{*}_H B$ is good.

Suppose now that $A \hat{*}_H B$, A and B are good. Clearly, from the above diagram, for every discrete $A \hat{*}_H B$ -module M , h' is an

isomorphism for every $k \in \mathbb{Z}^+$. Let X be any finite discrete $\hat{H}$ -module. Then X is a trivial U -module for some open normal subgroup U of $\hat{H}$ and since $\hat{H} \rightarrow A \hat{*}_H B$, there is an open normal subgroup V of $A \hat{*}_H B$ such that $V \cap \hat{H} = \hat{H}_V \subseteq U$ and of course, X is a trivial $\hat{H}_V$ -module. Consider the abelian group

$$\mathbb{Z}(A \hat{*}_H B / V) \otimes_{\hat{H}_V / V} X,$$

call it X_V . Then X_V is a finite discrete $A \hat{*}_H B$ -module and there is an $\hat{H}$ -embedding of X into X_V ; namely, $x \mapsto 1 \otimes x$. Hence we have a short exact sequence of finite discrete $\hat{H}$ -modules

$$X \rightarrow X_V \rightarrow K$$

where X_V is an $A \hat{*}_H B$ -module. From this sequence we have the following commutative diagram

$$\begin{array}{ccccccc} \dots & \rightarrow & H^k(H, X_V) & \rightarrow & H^k(H, K) & \rightarrow & H^{k+1}(H, X) \rightarrow H^{k+1}(H, X_V) \rightarrow \dots \\ 13.9 \dots & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \dots & \rightarrow & H^k(\hat{H}, X_V) & \rightarrow & H^k(\hat{H}, K) & \rightarrow & H^{k+1}(\hat{H}, X) \rightarrow H^{k+1}(\hat{H}, X_V) \rightarrow \dots \end{array}$$

where the vertical maps are induced by the map $H \rightarrow \hat{H}$. We argue by induction on k . H is good in dimension 0 so assume that for all finite discrete $\hat{H}$ -modules T , $H^i(\hat{H}, T) \cong H^i(H, T)$ whenever $i \leq k$. Then

$H^k(\hat{H}, K) \cong H^k(H, K)$ and from (13.9) one sees therefore that

$H^{k+1}(\hat{H}, X) \rightarrow H^{k+1}(H, X)$ since: $H^{k+1}(\hat{H}, X_V) \cong H^{k+1}(H, X_V)$. Now one can show that because $H^{k+1}(\hat{H}, X_V) \cong H^{k+1}(H, X_V)$, given any $x \in H^{k+1}(H, X_V)$,

there exists $T \in C_{\hat{H}}^f$ such that $X_V \rightarrow T$ and the image of x in

$H^{k+1}(H, T)$ is zero. Since $X \rightarrow X_V$, the same property therefore holds

for each $x \in H^{k+1}(H, X)$ by the functoriality of $H^*(H, -)$ and so H satisfies Condition C_{k+1} , p.I-15, [23]. Hence the map

$$H^{k+1}(\hat{H}, X) \rightarrow H^{k+1}(H, X)$$

is onto and H is good, by induction.

(ii): This can be shown by an argument analogous to that used to show H is good. //

The argument used in the proof of Theorem 13.8 can be repeated to prove the following general property of discrete groups.

Proposition 13.10: If $H \subseteq G$ are discrete groups such that $\hat{H} \rightarrow \hat{G}$ and $H^k(\hat{H}, X) \cong H^k(H, X)$ for all k and all finite discrete G -modules X , then H is good. //

In [8], Bieri and Eckmann prove the following theorem.

Theorem 13.11 [8]: Let $A *_H B$ be an amalgamated product.

(i) If A and B are FP-groups, then $A *_H B$ is an FP-group iff H is an FP-group; and,

(ii) If H and $A *_H B$ are FP-groups, then so are A and B . //

This, together with Theorem 13.8, yields the following:

Corollary 13.12: Let A , B and H be discrete groups satisfying the hypothesis of Theorem 13.6.

(i) If A and B are good FP-groups, then $A *_H B$ is a good FP-group iff H is a good FP-group; and,

(ii) If H and $A *_H B$ are good FP-groups, then so are A and B . //

Hence we see that under certain conditions, the class of good

FP-groups is closed under the taking of amalgamated products. The following proposition exhibits some examples of groups which satisfy the hypothesis of Theorem 13.6.

Proposition 13.13: Let A , B and H be discrete groups.

- (i) $\hat{A} \amalg \hat{B}$ exists
- (ii) If $H \subseteq A$ and $\hat{H} \subseteq \hat{A}$, then $\hat{A} \amalg_{\hat{H}} \hat{A}$ exists
- (iii) If H is finitely generated and normal in both A and B , $\hat{H} \rightarrow \hat{A}$, $\hat{H} \rightarrow \hat{B}$, then $\hat{A} \amalg_{\hat{H}} \hat{B}$ exists.

Proof: (i) follows from Corollary 1.3, [22].

(ii) follows from Theorem 2.1, [22].

(iii) follows from Theorem 2.4, [22]. //

Let G be a discrete Poincaré duality group (for the definition see [2]). Then the dualizing module of G is $\mathbb{Z}$; either with trivial G -action, in which case G is called orientable; or with the only possible non-trivial G -action, in which case G is called non-orientable. If G is a non-orientable Poincaré duality group it has an orientable Poincaré duality group as a subgroup of index 2 and so without loss of generality we may consider only orientable Poincaré duality groups when dealing with questions of "goodness" since we know that if a discrete group contains a subgroup of finite index which is good, then the group itself is good and vice versa.

Theorem 13.14: Let G be a discrete orientable Poincaré duality group of dimension n which is good. Then G is a D_p^n -group for every prime p .

Proof: Since G is a duality group it is an FP-group, [9], and so $H^k(G, X)$ is finite for all k and all finite G -modules, X . Since $H^k(\hat{G}, X) \cong H^k(G, X)$ whenever $X \in C_G^f$ the same holds for $\hat{G}$ and so $\hat{G}$ is a profinite FP-group for every prime p . Since $H^n(G, Z/pZ) \neq 0$, $cd_p \hat{G} = n$, and so by Theorem 12.7 it only remains to show $H^k(\hat{G}, Z/pZ[[\hat{G}]]) = 0$ for $k \neq n$.

Now since both G and $\hat{G}$ are FP-groups, the functors $\{H^*(G, -)\}$ and $\{H^*(\hat{G}, -)\}$ both commute with inverse limits of systems of finite discrete $\hat{G}$ -modules. This is shown in Theorem 6.4 for $\{H^*(\hat{G}, -)\}$ and an analogous proof works in the case of $\{H^*(G, -)\}$ precisely because Z has a G -projective resolution of finitely generated projectives. Hence $H^k(\hat{G}, Z/pZ[[\hat{G}]]) = 0$ for $k \neq n$ iff $H^k(G, Z/pZ[[\hat{G}]]) = 0$ for $k \neq n$. By the fact that G is an orientable Poincaré duality group this is equivalent to $H_k(G, Z/pZ[[\hat{G}]]) = 0$ for $k \neq 0$. In this case the right G -action on $Z/pZ[[\hat{G}]]$ is defined by $(\lambda)_g = g^{-1}\lambda$ where $\lambda \in Z/pZ[[\hat{G}]]$. Hence this module is isomorphic to $Z/pZ[[\hat{G}]]$ with the normal right G -action. But

$$\begin{aligned} H_k(G, Z/pZ[[\hat{G}]]) &\cong H_k(G, \varprojlim Z/pZ(G/U)) \\ &\cong \varprojlim H_k(G, Z/pZ \otimes_U Z(G)) \cong \varprojlim_{U, \text{cor}} H_k(U, Z/pZ). \end{aligned}$$

By Proposition 5.2, p.28, [17], there is an isomorphism

$$H_k(U, Z/pZ)^* \cong H^k(U, Z/pZ^*) \cong H^k(U, Z/pZ)$$

and since G (and hence U) is an FP-group we have that $H_k(U, Z/pZ)$ is finite for all U and

$$H_k(U, Z/pZ) \cong H_k(U, Z/pZ)^{**} \cong H^k(U, Z/pZ)^* .$$

Hence for $U \subseteq V$, the diagram

$$\begin{array}{ccc} H_k(U, Z/pZ) & \cong & H^k(U, Z/pZ)^* \\ \downarrow \text{cor} & & \downarrow (\text{res}^-)^* \\ H_k(V, Z/pZ) & \cong & H^k(V, Z/pZ)^* \end{array}$$

commutes and so

$$\varprojlim_{\text{cor}} H_k(U, Z/pZ) \cong \varprojlim_{(\text{res}^-)^*} H^k(U, Z/pZ)^* \cong (\varprojlim_{\text{res}} H^k(U, Z/pZ))^*$$

and hence $\varprojlim_{\text{cor}} H_k(U, Z/pZ) = 0$ iff $\varprojlim_{\text{res}^-} H^k(U, Z/pZ) = 0$. Since G is

good, $\varprojlim_{\text{res}} H^k(U, Z/pZ) = 0$ for $k \neq 0$ by Proposition 13.2 so

$$\begin{aligned} 0 &= \varprojlim_{\text{cor}} H_k(U, Z/pZ) \cong \varprojlim_{\text{cor}} H_k(G, Z/pZ \otimes_U Z(G)) \\ &\cong \varprojlim_{\text{cor}} H_k(G, Z/pZ(G/U)) \cong H_k(G, Z/pZ[[G]]) \end{aligned}$$

for $k \neq 0$. Hence the theorem is proved. //

Corollary 13.15: Let G be a discrete Poincaré duality group of dimension n which is good. Then $\hat{G}$ is a D_p^n -group for every prime p .

Proof: As we have seen G has a subgroup U of index 2 which is an orientable Poincaré duality group of dimension n . If G is good, then U is good by Corollary 13.4. Hence $\hat{U}$ is a D_p^n -group for every prime p by the theorem. Since $\hat{U} \rightarrow \hat{G}$ and is of finite index in $\hat{G}$, $\hat{G}$ is a D_p^n -group for every prime p by Proposition 12.2. //

Proposition 13.16: Let G be a discrete Poincaré duality group, $\text{cd} G = 2$.

Then G is a good group iff $p \nmid \#G$ for every prime p .

Proof: We consider the case of orientable Poincaré duality groups, the proposition is then immediately true for non-orientable Poincaré duality groups. G is good iff $\lim_{U, \text{res}} H^2(U, \mathbb{Z}/p\mathbb{Z}) = 0$ for every prime p , since $\lim_{\rightarrow} H^1(U, \mathbb{Z}/p\mathbb{Z}) = 0$ for all groups G . As we saw in the proof of Theorem 13.14, this is equivalent to $H^0(G, \mathbb{Z}/p\mathbb{Z}[[\hat{G}]])$ being 0 for every prime p . Since

$$H^0(G, \mathbb{Z}/p\mathbb{Z}[[\hat{G}]]) \cong H^0(\hat{G}, \mathbb{Z}/p\mathbb{Z}[[\hat{G}]])$$

the corollary follows from Corollary 2.13. //

Note 13.17: All known discrete orientable Poincaré duality groups of dimension 2 except $\mathbb{Z} \times \mathbb{Z}$ (which is good) are subgroups of finite index of the group $F(x, y) *_{[x, y]} F(x, y)$. This group is good since its first integral homology group is non-trivial free abelian hence the abelianization of its profinite completion is non-trivial free pro-abelian and so the completion satisfies the hypothesis of Proposition 13.16.

At this point it is worthwhile to discuss another kind of goodness.

Let G be a discrete group and $\hat{G}^p$ be its pro- p -completion. Then

$$G \xrightarrow{h} \hat{G}^p \cong \lim_{(G:U)=p^i} G/U.$$

We call G , p -good, if $H^k(\hat{G}^p, A) \cong H^k(G, A)$ for all k and all $A \in C_{G^p}^{f(p)}$.

Proposition 13.18: Let G be a discrete group. If G is p -good so is

every normal subgroup of G of index a power of p . If, on the otherhand,

G contains a normal subgroup of index a power of p which is p -good,

then G is p -good.

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Proof: Suppose G is p -good and $U \triangleleft G$, $(G:U) = p^i$. For any $A \in C_{0p}^{f(p)}$, $\text{Hom}_{0p}(Z[[G^p]], A) \in C_{G^p}^{f(p)}$. In fact, it is clear that

$$\text{Hom}_{0p}(Z[[G^p]], A) \cong \text{Hom}_U(Z(G), A).$$

On the one hand

$$\text{Hom}_{0p}(Z[[G^p]], A) \cong M_{G^p}^{U^p}(A)$$

$$\cong \{ \text{all } U^p\text{-linear continuous functions from } G^p \text{ to } A \}.$$

However, $A^{\bar{V}} = A$ for some open normal subgroup $\bar{V}$ of U^p and without

loss of generality we may assume that $\bar{V}$ is normal in G^p . Hence

$$M_{G^p}^{U^p}(A) \cong M_{G^p/\bar{V}}^{U^p/\bar{V}}(A) \cong \{ \text{all } U^p/\bar{V}\text{-linear functions from } G^p/\bar{V} \text{ to } A \}. \text{ If}$$

$V = h^{-1}(\bar{V}) \subseteq G$, then $A^V = A$, $U^p/\bar{V} \cong U/V$, $G^p/\bar{V} \cong G/V$ and so

$$M_{G^p/\bar{V}}^{U^p/\bar{V}}(A) \cong M_{G/V}^{U/V}(A) \cong \{ \text{all } U/V\text{-linear functions from } G/V \text{ to } A \}$$

and this is clearly isomorphic to $\text{Hom}_U(Z(G), A)$. Hence the diagram

$$H^k(G, \text{Hom}_{0p}(Z[[G^p]], A)) \xleftarrow{h'_G} H^k(G^p, \text{Hom}_{0p}(Z[[G^p]], A))$$

13.9...

$$H^k(U, A) \xleftarrow{h'_U} H^k(U^p, A)$$

commutes where the vertical isomorphisms are those of Shapiro's Lemma

because of the isomorphism $\text{Hom}_{0p}(Z[[G^p]], A) \cong \text{Hom}_U(Z(G), A)$. Since h'_G is an isomorphism, so is h'_U ; i.e., U is p -good.

Now suppose G is a discrete group and U is a normal subgroup of index a power of p . Furthermore, assume U is p -good. Then from the diagram 13.19, for any $A \in C_{G^p}^{f(p)}$, we have that h'_U and hence h'_G is an isomorphism. We show G is p -good by showing that G satisfies condition C_n , p.I-15, [23]; i.e., by showing that for any $x \in H^k(G, A)$ there is $x \in C_{G^p}^{f(p)}$ such that $A \mapsto x$ and $x \mapsto 0 \in H^k(G, X)$.

Now $A \rightarrow \text{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p[[\hat{G}^p]], A)$ and so $x \in H^k(G, A)$ maps to

$\bar{x} \in H^k(G, \text{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p[[\hat{G}^p]], A))$. From diagram 13.19 we see that

$$H^k(G, \text{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p[[\hat{G}^p]], A)) \cong H^k(\hat{G}^p, \text{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p[[\hat{G}^p]], A))$$

and so $\bar{x}$ is the image of some $\bar{y} \in H^k(\hat{G}^p, \text{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p[[\hat{G}^p]], A))$. Since $\hat{G}^p$

is a profinite group there exists $x \in C_{\hat{G}^p}^{f(p)}$ such that

$$\text{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p[[\hat{G}^p]], A) \rightarrow X$$

and $\bar{y} \mapsto 0 \in H^k(\hat{G}^p, X)$. Now

$$A \rightarrow \text{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p[[\hat{G}^p]], A) \rightarrow X$$

and the diagram

$$H^k(G, A) \rightarrow H^k(G, \text{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p[[\hat{G}^p]], A)) \rightarrow H^k(G, X)$$

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$$H^k(\hat{G}^p, \text{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p[[\hat{G}^p]], A)) \rightarrow H^k(\hat{G}^p, X)$$

commutes. Hence we see that $x \in H^k(G, A)$ is mapped to $0 \in H^k(G, X)$.

Therefore G satisfies condition C_n of Serre and so G is p -good. //

As in the case of Proposition 13.2, it is now true, using Proposition 13.18, that if G is a discrete group then G is p -good iff

$$\lim_{(G:U)=p^i} H^k(U, A) = 0 \text{ for } k > 0 \text{ and } A \in C_G^{f(p)}.$$

$$\lim_{\rightarrow} H^k(U, \mathbb{Z}/p\mathbb{Z}) = 0 \text{ (see Proposition 13.2 (ii))}$$

Theorem 13.20: Let G be an orientable Poincaré duality group of dimension n and suppose G is p -good. Then $\hat{G}^p$ is a Poincaré group of dimension n in the sense of Serre, [23].

Proof: The proof that $\hat{G}^p$ is a $D_{\hat{G}^p}^n$ -group is identical to that of Theorem

13.14. It remains therefore to show that $H^n(G, \mathbb{Z}_p[[\hat{G}^p]]) \cong \mathbb{Z}_p$.

However,

$$\begin{aligned} H^n(G^p, \mathbb{Z}_p[[G^p]]) &\cong H^n(G, \mathbb{Z}_p[[G^p]]) \cong \lim_{i, (G:U)=p} H^n(G, \mathbb{Z}/p^i \mathbb{Z} \otimes_{\mathbb{Z}} (G/U)) \\ &\cong \lim_{i, \bar{U}; \text{cor}} H^n(U, \mathbb{Z}/p^i \mathbb{Z}) \cong \lim_{i, \bar{U}; \text{res}} \mathbb{Z} \otimes_U \mathbb{Z}/p^i \mathbb{Z} \end{aligned}$$

(Theorem 4.5, [5]) and in this case, res_- is clearly an isomorphism so

$$\lim_{i, \bar{U}; \text{res}} \mathbb{Z} \otimes_U \mathbb{Z}/p^i \mathbb{Z} \cong \lim_{i, \bar{U}; \text{res}} \mathbb{Z}/p^i \mathbb{Z} \cong \mathbb{Z}_p. //$$

It is not necessarily true that Poincaré duality groups need be p -good even in dimension 2. Let $G = \mathbb{Z} \rtimes_{\theta} \mathbb{Z}$ be the non-orientable Poincaré duality group, $\text{cd} G = 2$. If $p \neq 2$, and $V \trianglelefteq G$ is of index a power of $p \neq 2$ it is easy to see that V contains the normal copy of $\mathbb{Z}$ in G and so $G/V \cong \mathbb{Z}/p^i \mathbb{Z}$ for some i . Hence $\hat{G}^p \cong \mathbb{Z}_p$ and although this is a Poincaré group in the sense of Serre it has cohomological dimension 1 and not 2. However, for discrete orientable Poincaré duality groups of dimension 2 we have the following:

Proposition 13.21: Let G be a discrete orientable Poincaré duality group of dimension 2. The following are equivalent:

- (i) G is p -good
- (ii) $\# \hat{G}^p = p^{\infty}$
- (iii) G has a normal subgroup of index a power of p for arbitrarily high powers of p
- (iv) G is not perfect.

Proof: (i) $\Leftrightarrow$ (ii) The proof is analogous to that of Proposition 13.16 and so is omitted.

(ii) $\Leftrightarrow$ (iii) Since G is finitely generated there are only finitely many subgroups of a fixed index. But if $\#G^p = p^\infty$ then the inverse limit is taken over infinitely many quotients of G of arbitrarily high order, and conversely.

(iii) = (iv) By the Universal Coefficients Theorem for Cohomology (see [14], p.49) and the fact that $H^2(G, A) \cong A$ for all trivial G -modules A , we have that $H_1(G, \mathbb{Z}) \cong G_{ab}$ is free abelian of finite rank. It is zero if and only if G is perfect. If it is non-zero then (iii) is satisfied and if (iii) is satisfied then G has a non-zero abelian quotient so is not perfect. //

Finally, if G is a discrete group and $\hat{G}$ is its profinite completion it is well-known that the maximal p -quotient of $\hat{G}$ is the pro- p -completion of G . We write both as G^p . Hence we have the following proposition.

Proposition 13.23: Let G be a discrete orientable Poincaré duality group of dimension n . If G is good and p -good then $\hat{G}$ is a D_p^n -group, $H^n(\hat{G}, \mathbb{Z}_p[[\hat{G}]]) \cong \mathbb{Z}_p$, and G^p is a Poincaré group in the sense of Serre.

Proof: By Corollary 13.15, Theorem 13.20 and the fact that the pro- p -completion of G is the maximal p -quotient of $\hat{G}$ the theorem is proven except for the isomorphism $H^n(\hat{G}, \mathbb{Z}_p[[\hat{G}]]) \cong \mathbb{Z}_p$. This follows by the same argument used in Theorem 13.20 to show $H^n(G^p, \mathbb{Z}_p[[G^p]]) \cong \mathbb{Z}_p$. //

§ 14 - Amalgamated products of Profinite Groups and Duality

In this section we follow closely the work of Bieri and Eckmann in [8]. In [22], Ribes states conditions under which amalgamated products of profinite groups over a common closed subgroup exist. In what follows we beg the question of existence and simply assume that where need be, the necessary amalgamated products do exist.

The purpose of this section is to consider what conditions are necessary for amalgamated products of duality groups to be duality groups. To do this we use very strongly the Mayer-Vietoris sequence for profinite amalgamated products established in [16].

Theorem 14.1 [16]: Assume the existence of the amalgamated product, $G = G_1 \amalg_L G_2$, of the two profinite groups G_1 and G_2 over a common closed subgroup L and suppose $A \in C_G$, the category of discrete G -modules. Then the following sequence is exact.

$$0 \rightarrow \text{Der}(G, A) \rightarrow \text{Der}(G_1, A) \oplus \text{Der}(G_2, A) \rightarrow \text{Der}(L, A) \rightarrow H^2(G, A) \rightarrow \dots$$

$$\dots \rightarrow H^{n-1}(L, A) \rightarrow H^n(G, A) \rightarrow H^n(G_1, A) \oplus H^n(G_2, A) \rightarrow H^n(L, A) \rightarrow \dots$$

where the map

$$H^k(G, A) \rightarrow H^k(G_1, A) \oplus H^k(G_2, A)$$

is the pair $(\text{res}^-, \text{res}^-)$ and the map

$$H^k(G_1, A) \oplus H^k(G_2, A) \rightarrow H^k(L, A)$$

is the pair $(\text{res}^-, -\text{res}^-)$. $\square$

Corollary 14.2: With the above assumptions, the following sequence is exact.

$$0 \rightarrow A^G \rightarrow A^{G_1} \oplus A^{G_2} \rightarrow A^L \rightarrow H^1(G, A) \rightarrow H^1(G_1, A) \oplus H^1(G_2, A) \rightarrow \dots$$

$$H^1(L, A) \rightarrow H^2(G, A) \rightarrow \dots$$

Proof: The proof is straightforward and identical to the discrete case. //

An immediate consequence of this long exact sequence is the following:

Theorem 14.2: Let $G = G_1 \amalg_{L_2} G_2$. If L , G_1 , and G_2 are $\overline{\text{FP}}$ -groups,

then so is G . Furthermore, if G_1 , G_2 and G are $\overline{\text{FP}}$ -groups so is

L , while if G and L are $\overline{\text{FP}}$ -groups so are G_1 and G_2 .

Proof: From the Mayer-Vietoris sequence of Corollary 14.2, it is clear

that if L , G_1 and G_2 are $\overline{\text{FP}}$ -groups, $H^k(G, A)$ is finite for all $k \geq 0$ and all $A \in C_G^{f(p)}$ and so G is an $\overline{\text{FP}}$ -group.

Suppose now that G , G_1 and G_2 are $\overline{\text{FP}}$ -groups. By Theorem 6.2 it is sufficient to show $H^k(L, Z/pZ(L/V))$ is finite for all $k \leq n$ and V open normal in L , to show that L is an FP_n -group. We show that L is an FP_n -group for every $n \in \mathbb{Z}^+$ and hence that L is an $\overline{\text{FP}}$ -group.

L is an FP_0 -group so assume, by induction, that L is an FP_{n-1} -group. $Z/pZ(L/V) \cong M_L^V(Z/pZ)$ and this isomorphism is well-known. Furthermore, since $L \leq G$, there is an open subgroup U of G such that $U \cap L = V$. Hence $M_G^U(Z/pZ)$ is a finite G -module and by the Mayer-Vietoris sequence $H^n(L, M_G^U(Z/pZ))$ is finite.

We claim that as L -modules,

$$M_L^V(Z/pZ) \xrightarrow{\sim} M_G^U(Z/pZ).$$

For $\sigma \in M_L^V(Z/pZ) \cong M_L^{U \cap L}(Z/pZ)$, define $\iota(\sigma)$ as follows:

$\iota(\sigma)(Ux) = \sigma(Vx)$ if $Ux \cap L \neq \emptyset$, since in this case

we may assume $x \in L$

$= 0$ if $Ux \cap L = \emptyset$.

Then $\iota(\iota\sigma)(Ux) = (\iota\sigma)(Vx)$

$= \sigma(Vx\ell)$ if $Ux \cap L \neq \emptyset$ and since we choose

$x \in L$ we have $x\ell \in L$

$= 0$ if $Ux \cap L = \emptyset$;

while $\iota(\iota\sigma)(Ux) = \iota(\sigma)(Ux\ell) = \sigma(Vx\ell)$

if $Ux\ell \cap L \neq \emptyset$ and clearly this

happens only if $Ux \cap L \neq \emptyset$

$= 0$ if $Ux\ell \cap L = \emptyset$ and clearly this

happens only when $Ux \cap L = \emptyset$.

Hence ι is a well-defined L -module monomorphism and since L is an FP_{n-1} -group and $H^n(L, M_G^U(Z/pZ))$ is finite, $H^n(L, M_L^{U \cap L}(Z/pZ))$ is finite as well so that L is an FP_n -group. Hence, by induction, L is an $\overline{FP}$ -group. Similarly, if G and L are $\overline{FP}$ -groups then G_1 and G_2 are $\overline{FP}$ -groups. //

Before proceeding any further with the discussion of duality properties of G , it is possible to establish some conditions on the cohomological dimension of the subgroups G_1 , G_2 and L of G .

Proposition 14.4: If $G = G_1 \amalg_L G_2$ is a duality group (in p) of dimension n for every prime p and G_1 and G_2 are FP -groups such that $p \nmid (G:G_i)$ for $i = 1, 2$ then

$$n-1 \leq cd_p L \leq cd_p G_i \leq cd_p G = n \text{ for } i = 1, 2.$$

Proof: Using Corollary 1.7, Proposition 1.8, and the Mayer-Vietoris sequence or Corollary 14.2 we have for any $B \in PC_G^p$, the following long exact sequence

$$\begin{aligned} \dots \rightarrow H_k(L, B) &\xrightarrow{(\text{cor}_-, -\text{cor}_-)} H_k(G_1, B) \oplus H_k(G_2, B) \xrightarrow{(\text{cor}_-, \text{cor}_-)} H_k(G, B) \rightarrow \dots \\ \dots \rightarrow H_1(G, B) &\rightarrow H_0(L, B) \rightarrow H_0(G_1, B) \oplus H_0(G_2, B) \rightarrow H_0(G, B) \rightarrow 0 \end{aligned}$$

and for $k = n$, we have

$$0 \rightarrow H_n(L, B) \rightarrow H_n(G_1, B) \oplus H_n(G_2, B) \rightarrow H_{n-1}(L, B)$$

Suppose now that G is a duality group, $C = H^n(G, \mathbb{Z}_p[[G]])$, and $\text{cd}_p L \leq n-1$, then the above sequence shows that

$$\mathbb{Z}_p = H_n(G, C) = H_n(G_1, C) \oplus H_n(G_2, C).$$

Hence, without loss of generality, we may assume $H_n(G_2, C) \neq 0$ (and hence, necessarily, $H_n(G_1, C) = 0$).

But $H_n(G_2, C) = H_n(G, C \otimes_{\mathbb{Z}_p[[G]]} \mathbb{Z}_p[[G]])$, and the module $C \otimes_{\mathbb{Z}_p[[G]]} \mathbb{Z}_p[[G]]$ is G -isomorphic to the module $C \otimes_{\mathbb{Z}_p[[G/G_2]]} \mathbb{Z}_p[[G/G_2]]$ where $\mathbb{Z}_p[[G/G_2]]$ is the free pseudocompact $\mathbb{Z}_p[[G]]$ -module on the Boolean space G/G_2 , of cosets of G_2 in G and the G -action on $C \otimes_{\mathbb{Z}_p[[G]]} \mathbb{Z}_p[[G/G_2]]$ is diagonal. This isomorphism is given by $c \otimes g \rightarrow cg \otimes g^{-1}g_2$. Hence

$$H_n(G, C \otimes_{\mathbb{Z}_p[[G]]} \mathbb{Z}_p[[G]]) \cong H_n(G, C \otimes_{\mathbb{Z}_p[[G/G_2]]} \mathbb{Z}_p[[G/G_2]])$$

and since G is a duality group,

$$H_n(G, C \otimes_{\mathbb{Z}_p[[G/G_2]]} \mathbb{Z}_p[[G/G_2]]) \cong H_n^0(G, \mathbb{Z}_p[[G/G_2]]) \otimes_{\mathbb{Z}_p[[G/G_2]]} \mathbb{Z}_p[[G/G_2]]^G.$$

Now since $p^i \mid (G:G_1)$ for $i = 1, 2$, $\mathbb{Z}_p[[G/G_2]]^G$ must be 0. This follows by an argument analogous to that which proves Proposition 2.10.

Namely, if $\mathbb{Z}_p[[G/G_2]]^G$ is nontrivial it is isomorphic to $\varprojlim_{U \leq G} \mathbb{Z}_p$.

where if $i_2 \leq U \leq V$, the morphism between Z_p (indexed by U) and Z_p (indexed by V) is multiplication by $(V:U)$. If $p^\infty \mid (G:G_2)$, $Z_p[[G/G_2]]^G$ is non-zero if and only if Z_p has an element which is divisible by p^k for arbitrarily high values of k and this does not happen. Therefore $Z_p[[G/G_2]]^G = 0$, a contradiction, and so $\text{cd}_p L \geq n-1$. //

Remark 14.5: Since both G_1 and L are FP-groups, it is straightforward to show, using the Mayer-Vietoris sequence, that if $\text{cd}_{p_1} G_1 < n = \text{cd}_p G$, then $\text{cd}_{p_1} L = \text{cd}_p L$. In fact, in all of the propositions that follow, $\text{cd}_p L$ is necessarily greater than or equal to $n-1$, where $n = \text{cd}_p G$.

We note that if we restrict ourselves to pro-p-groups, and pro-p-amalgamated products (assuming, of course, that they exist) then Proposition 14.4 has a particularly strong form.

Corollary 14.6: Let G_1 , G_2 and L be pro-p-groups which are also FP-groups and suppose $G = G_1 \amalg_L G_2$, the pro-p-amalgamated product of G_1 and G_2 over L exists. Then if G is a D_p^n -group, $\text{cd}_p L \geq n-1$.

Proof: Using the Mayer-Vietoris sequence one sees, as in Proposition 14.4, that assuming $\text{cd}_p L < n-1$ yields $H^0(G, Z_p[[G/G_i]]) = 0$ for at least one $i = 1$ or 2 . If the amalgamated product is non-trivial (i.e., $L \neq G_i$ for either $i = 1$ or 2) then $p^\infty \mid (G:G_i)$ and so $H^0(G, Z_p[[G/G_i]]) = 0$, a contradiction. Hence $\text{cd}_p L \geq n-1$. //

Theorem 14.7: Let $G = G_1 \amalg_L G_2$, G_1 , G_2 and L be FP-groups. If G_1 and G_2 are D_p^n -groups and L is a D_p^{n-1} -group, then G is a D_p^n -group.

Proof: It is clear from the Mayer-Vietoris sequence that $\text{cd}_p G = n$ and

$H^k(G, \mathbb{Z}_p[[G]]) = 0$ if $k \neq n$. We also have the short exact sequence

$$0 \rightarrow H^{n-2}(L, \mathbb{Z}/p\mathbb{Z}[[G]]) \rightarrow H^{n-1}(G, \mathbb{Z}/p\mathbb{Z}[[G]]) \\ \rightarrow H^{n-1}(G_1, \mathbb{Z}/p\mathbb{Z}[[G]]) \oplus H^{n-1}(G_2, \mathbb{Z}/p\mathbb{Z}[[G]]) \rightarrow 0$$

and since $\mathbb{Z}/p\mathbb{Z}[[G]]$ is an induced G -module it is also an induced

L -, G_1 -, and G_2 -module so the above sequence collapses since L ,

G_1 and G_2 are duality groups. Therefore, $H^{n-1}(G, \mathbb{Z}/p\mathbb{Z}[[G]]) = 0$ and

as in the proof of Theorem 12.7, this implies $H^n(G, \mathbb{Z}_p[[G]])$ is torsion-

free. Hence, by Proposition 11.10, G is a D_p^n -group. //

Theorem 14.8: Let $G = G_1 \amalg_L G_2$, G_1 , G_2 and L be $\overline{\text{FP}}$ -groups. If G_2 is a D_p^n -group and G_1 and L are D_p^{n-1} -groups such that

$$\text{res}^- : H^{n-1}(G_1, A) \rightarrow H^{n-1}(L, A)$$

is a monomorphism for all induced G_1 -modules, A , then G is a D_p^n -group.

Note: The condition that the restriction morphism is monic is fulfilled

when, for example, $p \nmid (G_1 : L)$.

Proof: Clearly, from the Mayer-Vietoris sequence, $\text{cd}_p G = n$, and

$H^k(G, \mathbb{Z}_p[[G]]) = 0$ for $k < n-1$. In addition, both $\mathbb{Z}_p[[G]]$ and

$\mathbb{Z}/p\mathbb{Z}[[G]]$ are induced G_1 -modules and so the exact sequences

$$0 \rightarrow H^{n-1}(G, \mathbb{Z}_p[[G]]) \rightarrow H^{n-1}(G_1, \mathbb{Z}_p[[G]]) \xrightarrow{\text{res}^-} H^{n-1}(L, \mathbb{Z}_p[[G]])$$

and

$$0 \rightarrow H^{n-1}(G, \mathbb{Z}/p\mathbb{Z}[[G]]) \rightarrow H^{n-1}(G_1, \mathbb{Z}/p\mathbb{Z}[[G]]) \xrightarrow{\text{res}^-} H^{n-1}(L, \mathbb{Z}/p\mathbb{Z}[[G]])$$

show that both $H^{n-1}(G, \mathbb{Z}_p[[G]])$ and $H^{n-1}(G, \mathbb{Z}/p\mathbb{Z}[[G]])$ are zero since

res^- is monic in both cases. Hence, G is an $\overline{\text{FP}}$ -group, $\text{cd}_p G = n$,

$H^k(G, \mathbb{Z}_p[[G]]) = 0$ for $k \neq n$ and $H^n(G, \mathbb{Z}_p[[G]])$ is torsion-free as in the previous theorem, so that G is a D_p^n -group by Proposition 11.10. //

Finally we have

Theorem 14.9: Let $G = G_1 \amalg_L G_2$, G_1 , G_2 and hence L be $\overline{FP}$ -groups and suppose G_1 , G_2 and L are D_p^n -groups. Then if $cd_p G = n$, G is a D_p^n -group as well.

Proof: Follows directly from the Mayer-Vietoris sequence. //

§ 15 - Examples

We first make some comments on the condition FP. Verdier defines a Poincaré group (in p), G , to be a strict Cohen-Macaulay group (in p) with discrete dualizing module isomorphic to $\mathbb{Z}_p^\infty$ as an abelian group. This condition on the discrete dualizing module forces G to be an FP-group as we now show.

Proposition 15.1: Let G be a Poincaré group (in p) of dimension n .

Then G is an FP-group.

Proof: By Proposition 4.4, [26], there is an isomorphism

$$\text{Hom}_Z(H^q(G, \mathbb{Z}/p\mathbb{Z}), \mathbb{Q}/\mathbb{Z}) \cong H^{n-q}(G, \text{Hom}(\mathbb{Z}/p\mathbb{Z}, I))$$

where I , being the discrete dualizing module, is isomorphic to $\mathbb{Z}_p^\infty$ as an abelian group. Hence $\text{Hom}(\mathbb{Z}/p\mathbb{Z}, I) \cong \mathbb{Z}/p\mathbb{Z}$ and there exists U , open normal in G , such that the U -action is trivial on both these modules. But U is also a Poincaré group by Proposition 4.7, [26] and so

$$\text{Hom}_Z(H^q(U, \mathbb{Z}/p\mathbb{Z}), \mathbb{Q}/\mathbb{Z}) \cong H^{n-q}(U, \mathbb{Z}/p\mathbb{Z})$$

and the same isomorphism holds true for any V open normal in U . In this context the functor $\text{Hom}_Z(-, \mathbb{Q}/\mathbb{Z})$ is not the equivalence between $C_G^t(p)$ and PC_G^p but simply the functor from $C_G^t(p)$ to $\underline{\text{Ab}}$. Applying this functor again to the above isomorphism we get

$$\begin{aligned} \text{Hom}_Z(\text{Hom}_Z(H^q(U, \mathbb{Z}/p\mathbb{Z}), \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z}) &\cong \text{Hom}_Z(H^{n-q}(U, \mathbb{Z}/p\mathbb{Z}), \mathbb{Q}/\mathbb{Z}) \\ &\cong H^q(U, \mathbb{Z}/p\mathbb{Z}) \end{aligned}$$

and so $H^q(U, \mathbb{Z}/p\mathbb{Z})$ is isomorphic to its double dual, and is therefore finitely generated and hence finite. This is also true for every open

normal subgroup V of U and so by Theorem 6.2, U is an FP-group.

Hence G is also an FP-group. //

We now turn to § 13.

Theorem 15.2: Let G be a finitely generated discrete one-relator group of cohomological dimension 2. If $H^2(G, Z/pZ) = 0$, then G is p -good.

If $H^2(G, Z/pZ) \neq 0$ then G is p -good, iff $\text{cd}_p G^p = 2$.

Proof: From the remark following Proposition 13.18 we see that G is p -good iff $\lim_{(G:\bar{U})=p^1} H^k(G, M) = 0$ for $k > 0$, $M \in C_{G^p}^{f(p)}$. As in the good group case, it is clear that this is true for $k = 1$ and so by Exercise 1, p.I-15, [23], $H^1(G^p, M) \cong H^1(G, M)$ and $H^2(G^p, M) \cong H^2(G, M)$ for all $M \in C_{G^p}^{f(p)}$. If $H^2(G, Z/pZ) = 0$ then $H^2(G^p, Z/pZ) = 0$ and so $\text{cd}_p G^p = 1$. However, every $M \in C_{G^p}^{f(p)}$ has a composition series with factors isomorphic to Z/pZ by Corollary, p.I-32, [23], and so if $H^2(G, Z/pZ) = 0$, then $H^2(G, M) = 0$ for all $M \in C_{G^p}^{f(p)}$. Hence G is p -good.

If $H^2(G, Z/pZ) \neq 0$ then it is straightforward to show that $H^2(G, Z/pZ) \cong Z/pZ$. If G is p -good then

$$H^2(G^p, Z/pZ) \cong H^2(G, Z/pZ) \cong Z/pZ$$

and

$$H^3(G^p, Z/pZ) \cong H^3(G, Z/pZ) \cong 0$$

so $\text{cd}_p G^p = 2$.

Conversely, if $\text{cd}_p G^p = 2$, then

$$0 \neq H^2(G^p, Z/pZ) \cong H^2(G, Z/pZ) \cong Z/pZ$$

and so $H^2(G^p, Z/pZ) \cong H^2(G, Z/pZ)$. If $M \in C_{G^p}^{f(p)}$, let

$$X \rightarrow M \rightarrow Z/pZ$$

be an exact sequence of G^p -modules. Assume, by induction, that

$$H^2(\hat{G}^p, X) \cong H^2(G, X) \quad \text{and of course we have } H^2(\hat{G}^p, Z/pZ) \cong H^2(G, Z/pZ).$$

Then we have the diagram

$$\begin{array}{ccccccc} \dots & \rightarrow & H^2(G, X) & \rightarrow & H^2(G, M) & \rightarrow & H^2(G, Z/pZ) \rightarrow 0 \\ & & \parallel & & \parallel & & \parallel \\ \dots & \rightarrow & H^2(\hat{G}^p, X) & \rightarrow & H^2(\hat{G}^p, M) & \rightarrow & H^2(\hat{G}^p, Z/pZ) \rightarrow 0 \end{array}$$

where the map $H^2(\hat{G}^p, M) \rightarrow H^2(\hat{G}^p, Z/pZ)$ is onto since $\text{cd}_p G^p = 2$. It is now easy to show that $H^2(\hat{G}^p, M) \cong H^2(G, M)$; i.e., that G is p -good at H^0 , H^1 , and H^2 and so G is p -good. //

Note 15.: (i) The usefulness of this theorem is found in the fact that

$\hat{G}^p$ has the same relator as G and relators for which $\hat{G}^p$ has cohomological dimension 2. have been studied by Labute, Gildenhuys and others.

(ii) If G is a discrete one-relator group of dimension 2 and

$\text{cd}_p \hat{G}^p = 2$, then since $H^0(\hat{G}^p, Z/pZ[[\hat{G}^p]]) = 0$, $\hat{G}^p$ is a D_p^n -group by Theorem 12.7 iff $H^1(\hat{G}^p, Z/pZ[[\hat{G}^p]]) = 0$. However, G is p -good and so

$$H^1(\hat{G}^p; Z/pZ[[\hat{G}^p]]) \cong H^1(G, Z/pZ[[\hat{G}^p]]) \cong \lim_{(G:U)=p^i, \text{cor}} H^1(U, Z/pZ)$$

and so $\hat{G}^p$ is a D_p^n -group iff this inverse limit is zero. For FP-

groups this inverse limit being zero is equivalent to the direct limit

$$\lim_{(G:U)=p^i, \text{res}} H_1(U, Z/pZ)$$

being zero.

For profinite FP-groups, the conditions (iii) and (iv) of

Proposition 11.10 have a characterization which involves only finite coefficient modules and not inverse limits.

Proposition 15.4: Let G be a profinite FP-group. Then the following are equivalent:

- (i) $H^k(G, \mathbb{Z}/p\mathbb{Z}[[G]]) = 0$
- (ii) For every $\mathbb{Z}/p^i\mathbb{Z}(G/U)$, and every $x \in H^k(G, \mathbb{Z}/p^i\mathbb{Z}(G/U))$, $x \neq 0$, there exists a pair (j, V) such that $\mathbb{Z}/p^j\mathbb{Z}(G/V) \rightarrow \mathbb{Z}/p^i\mathbb{Z}(G/U)$ and $x \notin \text{Im}(H^k(G, \mathbb{Z}/p^j\mathbb{Z}(G/V))) \subseteq H^k(G, \mathbb{Z}/p^i\mathbb{Z}(G/U))$
- (iii) For every $M \in C_G^{f(p)}$, and $0 \neq x \in H^k(G, M)$, there exists $M' \in C_G^{f(p)}$ such that $M' \rightarrow M$ and $x \notin \text{Im}(H^k(G, M')) \subseteq H^k(G, M)$
- (iv) For every $M \in C_G^{f(p)}$, there exists $M' \in C_G^{f(p)}$ such that $M' \rightarrow M$ and $0 = \text{Im}(H^k(G, M')) \subseteq H^k(G, M)$.

Proof: (i) $\Leftrightarrow$ (ii) — This is clear.

(ii) $\Rightarrow$ (iii) Let $M \in C_G^{f(p)}$. Then there exists i and U such that M is an $\mathbb{Z}/p^i\mathbb{Z}(G/U)$ -module and for some $s \in \mathbb{Z}^+$,

$$\bigoplus^s \mathbb{Z}/p^i\mathbb{Z}(G/U) \rightarrow M$$

is a G -morphism. Now suppose $0 \neq x \in H^k(G, M)$ and

$x \in \text{Im}(H^k(G, \bigoplus^s \mathbb{Z}/p^i\mathbb{Z}(G/U)))$. If $x \notin \text{Im}(H^k(G, \bigoplus^s \mathbb{Z}/p^i\mathbb{Z}(G/U)))$ we are done, otherwise the preimage of x in $H^k(G, \bigoplus^s \mathbb{Z}/p^i\mathbb{Z}(G/U))$ is finite and so there exists j and V such that

$$\bigoplus^s \mathbb{Z}/p^j\mathbb{Z}(G/V) \rightarrow \bigoplus^s \mathbb{Z}/p^i\mathbb{Z}(G/U)$$

and

$$\text{Im}(H^k(G, \bigoplus^s \mathbb{Z}/p^j\mathbb{Z}(G/V))) \cap \{\text{preimage of } x\} = \emptyset,$$

i.e., $\bigoplus^s \mathbb{Z}/p^j\mathbb{Z}(G/V) \rightarrow M$ and $x \notin \text{Im}(H^k(G, \bigoplus^s \mathbb{Z}/p^j\mathbb{Z}(G/V))) \subseteq H^k(G, M)$

(iii) $\Rightarrow$ (iv) For every $M \in C_G^{f(p)}$ there is a pair (i, U) and a natural number s such that $\bigoplus^s \mathbb{Z}/p^i\mathbb{Z}(G/U) \rightarrow M$. It is clear that since

$H^k(G, \bigoplus \mathbb{Z}/p^i \mathbb{Z}(G/U))$ is finite we can find, for every $x \in H^k(G, \bigoplus \mathbb{Z}/p^i \mathbb{Z}(G/U))$, a pair (j, V) such that $x \notin \text{Im}(H^k(G, \bigoplus \mathbb{Z}/p^j \mathbb{Z}(G/V))) \subset H^k(G, \bigoplus \mathbb{Z}/p^i \mathbb{Z}(G/U))$ and so taking $\max(j_x) = n$ and $W = \bigcap V_x$ then $\bigoplus \mathbb{Z}/p^n \mathbb{Z}(G/W) \rightarrow M$ and satisfies the hypothesis of (iv).

(iv) $\Rightarrow$ (ii) If $M \rightarrow \mathbb{Z}/p^i \mathbb{Z}(G/U)$ satisfies the hypothesis there exists $m \in M$ such that $m \rightarrow \bar{1}$ and a pair (j, V) such that M is a $\mathbb{Z}/p^j \mathbb{Z}(G/V)$ -module and there is a map $\mathbb{Z}/p^j \mathbb{Z}(G/V) \rightarrow M$ which sends $\bar{1}$ to m . The composition

$$\mathbb{Z}/p^j \mathbb{Z}(G/V) \rightarrow M \rightarrow \mathbb{Z}/p^i \mathbb{Z}(G/U)$$

satisfies condition (ii).

This proves the proposition. //

It has been pointed out to the author that a theorem more general than Theorem 13.6 is true, by the same proof. In particular, if A and B are discrete groups and $\hat{A}^p$ and $\hat{B}^p$ are their respective pro- p -completions, then the pro- p -amalgamated product, $\hat{A}^p \amalg \hat{B}^p$ is isomorphic to $(\hat{A * B})^p$. We use this fact to consider the structure of certain pro- p -groups. In what follows we let G be a pro- p -group satisfying the following conditions:

- (i) G is finitely presented
- (ii) $\text{cd}_p G = 2$
- (iii) the relations of G are finite words in the generators of G
- (iv) The number of generators in the given presentation is minimal.

Now let H be the discrete group with the same presentation as that of G . This is possible since the relators of G are finite words. Then

it is well-known that G is the pro- p -completion, $\hat{H}^p$, of H . Suppose that H is not a duality group of dimension 2 but that $\text{cd } H = 2$.

Then H is a non-trivial free product of groups

$$H \cong H_1 * H_2 * \dots * H_n$$

by the Stallings' Structure Theorem. Moreover,

$$G \cong \hat{H}^p \cong \hat{H}_1^{pp} \amalg \hat{H}_2^{pp} \dots \amalg \hat{H}_n^{pp}$$

(the pro- p -free product) by the generalization of Theorem 13.6 and since the number of generators of each H_i is strictly less than the number of generators of H (Gruško's Theorem) which, by condition (iii), is the minimal number of generators of G we have that G is a non-trivial pro- p -free product as well; i.e., at least two of the groups, $\hat{H}_i^p$, are non-trivial. Now, by Corollary 14.6, a pro- p -free product (which always exists) can not be a duality group (in p) unless it has at most, p -cohomological dimension 1. Hence we have shown the following:

Theorem 15.5: Let G be a finitely presented pro- p -group satisfying the following conditions:

- (i) $\text{cd}_p G = 2$
- (ii) the relators of G are all finite words in the generators of G
- (iii) the number of generators in the given presentation is minimal.

If H is the discrete group with the same presentation as G , $\text{cd } H = 2$ and G is a D_p^n -group, then H is a duality group of dimension 2. //

We note that the situation is somewhat simplified in case G is a one-relator group satisfying the conditions of the theorem. The theorem applies if H is torsion-free. In this case, H is p -good by Theorem

15.2 . Since G is given to be a D_p^2 -group, we have

$$\lim_{(G:U)=p^i} H^1(U, \mathbb{Z}/p\mathbb{Z}) = 0$$

and since H is p -good this same condition also applies to H ; i.e.,

$$15.0 \dots \lim_{(H:U)=p^i} H^1(U, \mathbb{Z}/p\mathbb{Z}) = 0.$$

It is a conjecture of the author that the converse to Theorem 15.5 is

true in this special case where G is a one-relator pro- p -group. What

needs to be proven is that if H is a duality group of dimension 2

then H satisfies the equality (15.0). A consequence of this conjecture

being true would be a Stallings' Structure Theorem for one-relator pro- p -

groups satisfying the hypothesis of Theorem 15.5; i.e., if G is a one-

relator pro- p -group satisfying the hypothesis of Theorem 15.5 then either

G is a D_p^2 -group or G is a pro- p -free product of D_p^i -groups for

$i = 1, 2$ (see [6]).

Before we give some examples in the theory of profinite duality

groups we make a brief comment about strict cohomological dimension.

From Corollary 5.5, [10], we see that if a profinite group G has

$\text{cd}_p G = n$, then $\text{scd}_p G = n$ if and only if $H^n(U, \mathbb{Z}_p^\infty) = 0$ for every open

subgroup U of G . Now $\mathbb{Z}_p^\infty \cong \varinjlim \mathbb{Z}/p^i\mathbb{Z}$ where, if $i \leq j$, the map

$\mathbb{Z}/p^i\mathbb{Z} \rightarrow \mathbb{Z}/p^j\mathbb{Z}$ found in this direct limit has, as its cokernel, the nat-

ural map $\mathbb{Z}/p^j\mathbb{Z} \rightarrow \mathbb{Z}/p^{j-1}\mathbb{Z}$. Hence if G is an FP-group and

$H^n(G, \mathbb{Z}_p^\infty) = 0$, there exists $j \geq 1$ such that the map $\mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}/p^j\mathbb{Z}$

induces the 0-map from $H^n(G, \mathbb{Z}/p\mathbb{Z})$ to $H^n(G, \mathbb{Z}/p^j\mathbb{Z})$. Therefore the map

$\mathbb{Z}/p^j\mathbb{Z} \rightarrow \mathbb{Z}/p^{j-1}\mathbb{Z}$ induces an isomorphism

$$H^n(G, \mathbb{Z}/p^j \mathbb{Z}) = H^n(G, \mathbb{Z}/p^{j-1} \mathbb{Z}) .$$

Moreover, this is true if j is replaced with any $k \geq j$; i.e., the canonical map $\mathbb{Z}/p^k \mathbb{Z} \rightarrow \mathbb{Z}/p^{j-1} \mathbb{Z}$ induces an isomorphism

$$H^n(G, \mathbb{Z}/p^k \mathbb{Z}) \cong H^n(G, \mathbb{Z}/p^{j-1} \mathbb{Z}) .$$

Hence we have that

$$H^n(G, \mathbb{Z}_p) \cong \varprojlim_i H^n(G, \mathbb{Z}/p^i \mathbb{Z}) \cong H^n(G, \mathbb{Z}/p^{j-1} \mathbb{Z})$$

and so is finite. But if G is an FP-group, so is every open subgroup and so we have shown:

Proposition 15.7: Let G be an FP-group, $\text{cd}_p G = n$. Then $\text{scd}_p G = n$ iff $H^n(U, \mathbb{Z}_p)$ is finite for every open subgroup U of G . //

Corollary 15.8: Let G be an FP-group, $\text{cd}_p G = n$. Let I_p be the discrete dualizing module of G in the sense of Serre, Proposition 17, p.1-27, [23], $I_p \cong \varprojlim_{i, U} H^n(U, \mathbb{Z}/p^i \mathbb{Z})^*$, then $\text{scd}_p G = n+1$ iff there exists an open subgroup U of G such that I_p^U is infinite.

Note: Compare this corollary with Proposition 19, p.1-30, [23].

Proof: For every U open in G ,

$$H^n(U, \mathbb{Z}_p) \cong \varprojlim_i H^n(U, \mathbb{Z}/p^i \mathbb{Z}) .$$

Hence $\text{scd}_p G = n+1$ iff $\varprojlim_i H^n(U, \mathbb{Z}/p^i \mathbb{Z})$ is infinite for some open U in G . However $\varprojlim_i H^n(U, \mathbb{Z}/p^i \mathbb{Z})$ is infinite iff

$$(\varprojlim_i H^n(U, \mathbb{Z}/p^i \mathbb{Z}))^* \cong \varprojlim_i H^n(U, \mathbb{Z}/p^i \mathbb{Z})^*$$

is infinite and the latter is exactly $(\varprojlim_{i, U} H^n(U, \mathbb{Z}/p^i \mathbb{Z})^*)^U \cong I_p^U$. //

As examples we have seen that all known discrete orientable Poincaré

groups G , $\text{cd } G = 2$, are p -good and so their pro- p -completions are profinite Poincaré groups whose dualizing module is isomorphic to $\mathbb{Z}_p$ with trivial G^p -action. The strict cohomological dimension of these groups is therefore 3.

We now give some examples of profinite duality groups.

Example 1: All finitely generated profinite groups G with $\text{cd}_p G \leq 1$ are D_p^1 -groups (see p.v-24, [26]) or D_p^0 -groups.

Example 2: All Poincaré groups (which we have shown to be FP-groups) are D_p^* -groups.

Example 3: The Demuškin groups, [24], are D_p^2 -groups.

Example 4: As was seen in Theorem 12.11, any profinite group whose Sylow p -subgroup is a D_p^n -group is also a D_p^n -group. The converse is manifestly not true since a finitely generated free profinite group is a D_p^1 -group but it has an infinitely generated Sylow p -subgroup.

Example 5: It is not true that if G is a discrete duality group of dimension n and G is good, that $\hat{G}$ is a D_p^n -group. The discrete group

$$G = \langle x, y, z, w \mid x^y = x^2, y^z = y^2, z^w = z^2, w^x = w^2 \rangle$$

is known as the Higman group and in Example 5, p.171, [5], G is shown to be a duality group of dimension 2. Now G has no subgroups of finite index and so $\hat{G} = 1$. However, $\text{tcd } G = 0$; i.e., $H^k(G, A) = 0$ for all $k \neq 0$ whenever A is a trivial G -module. From this it is clear that G is good and its profinite completion is not a D_p^2 -group.

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