INFORMATION TO USERS

This manuscript has been reproduced from the microfilm master. UMI films the text directly from the original or copy submitted. Thus, some thesis and dissertation copies are in typewriter face, while others may be from any type of computer printer.

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleedthrough, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps.

Photographs included in the original manuscript have been reproduced xerographically in this copy. Higher quality 6" x 9" black and white photographic prints are available for any photographs or illustrations appearing in this copy for an additional charge. Contact UMI directly to order.

Bell & Howell Information and Learning
300 North Zeeb Road, Ann Arbor, MI 48106-1346 USA

UMI[®]
800-521-0600

**Investigation of the Crossed Flexure Pivot
and the Dynamics of the
Momentum Management System
Spacecraft Control Component
and the
Dynamically Tuned Gyroscope**

by

JEFFREY S. CAIN, M.Eng.

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfilment of
the requirements for the degree of
Doctor of Philosophy

Ottawa-Carleton Institute for
Mechanical and Aerospace Engineering

Department of
Mechanical and Aerospace Engineering
Carleton University
Ottawa, Ontario
February 1st, 1999

© Copyright
1999, Jeffrey S. Cain

National Library
of Canada

Acquisitions and
Bibliographic Services

395 Wellington Street
Ottawa ON K1A 0N4
Canada

Bibliothèque nationale
du Canada

Acquisitions et
services bibliographiques

395, rue Wellington
Ottawa ON K1A 0N4
Canada

Your file / Votre référence

Our file / Notre référence

The author has granted a non-exclusive licence allowing the National Library of Canada to reproduce, loan, distribute or sell copies of this thesis in microform, paper or electronic formats.

The author retains ownership of the copyright in this thesis. Neither the thesis nor substantial extracts from it may be printed or otherwise reproduced without the author's permission.

L'auteur a accordé une licence non exclusive permettant à la Bibliothèque nationale du Canada de reproduire, prêter, distribuer ou vendre des copies de cette thèse sous la forme de microfiche/film, de reproduction sur papier ou sur format électronique.

L'auteur conserve la propriété du droit d'auteur qui protège cette thèse. Ni la thèse ni des extraits substantiels de celle-ci ne doivent être imprimés ou autrement reproduits sans son autorisation.

0-612-42788-9

Canada

The undersigned recommend to
the Faculty of Graduate Studies
acceptance of the thesis.

**Investigation of the Crossed Flexure Pivot and the Dynamics of the
Momentum Management System Spacecraft Control Component
and the Dynamically Tuned Gyroscope**

submitted by

Jeffrey S. Cain, M.Eng.

in partial fulfillment of the requirements
for the degree of Doctor of Philosophy

Thesis Supervisor

Thesis Supervisor

Chair
Department of Mechanical and Aerospace Engineering

External Examiner

Carleton University

March 1, 1999

Abstract

The Momentum Management System (MMS) is a novel concept which incorporates the rate sensing capabilities of the Dynamically Tuned Gyro (DTG) and the actuation ability of a gimballed momentum wheel. The mechanical components of the MMS and DTG consist of a rotor, gimbal and shaft which are connected using crossed flexure pivots.

This document presents a static and vibration analysis of the crossed flexure pivot and a kinematic and dynamic analysis of the Dynamically Tuned Gyroscope and the Momentum Management System. Hamilton's principle of minimum energy is used to develop static and vibrational models of the crossed flexure pivot. These models were used to design the pivots used in the MMS. Comparisons are made between the static model results and those found in the literature. A discussion of the effects of large angle deflections and wide pivot strips is also included. In order to provide additional experimental validation of the static equations, a pivot was designed and tested and the results compared to the theoretical model. The static model showed that the torsional stiffness of the crossed flexure pivot is dependent on the applied loads and that the buckling load is lower than previously thought. The vibration analysis was used to find the unforced, fundamental frequencies of the crossed flexure pivots used in the MMS.

A two body stability analysis of the DTG and MMS involving the gimbal and rotor was performed using small angle equations. This included a study of the effect of using bearings instead of the crossed flexure pivots. The DTG and MMS were found to be unstable only in extreme cases. The dynamic equations were derived using Lagrange's Equations and Hamilton's Canonical Equations. The dynamic equations are used in a numerical simulation which allows the MMS to be modelled. The results show that the three body analysis involving the rotor, gimbal and shaft is more accurate than the standard two body analysis seen in the literature, especially when modelling the MMS.

Acknowledgements

I would like to express my appreciation for all the assistance that my supervisors, Professors Douglas Staley and John McPhee, have provided over the last three years. Additional thanks go to Professor Glenn Heppler who, despite not being an official supervisor, acted like one.

Finally, I would like to acknowledge my parents and Karen for all their help and support.

Contents

Acceptance Sheet	ii
Abstract	iii
Acknowledgements	v
Table of Contents	vi
List of Tables	xi
List of Figures	xiii
Nomenclature	xxvi
1 Introduction	1
1.1 Attitude Sensors	2
1.2 Momentum Exchange Devices	5
1.2.1 Three-Axis Body Stabilized Spacecraft with a Pitch Momentum Wheel	7
1.3 Concept of the Momentum Management System	11
1.4 Summary of Research	12
2 Introduction to the Dynamically Tuned Gyroscope	15
2.1 Unconventional Gyroscopes	15
2.2 Conventional Gyroscopes	17

2.3	The Dynamically Tuned Gyro	21
2.3.1	Dynamics of the DTG	22
2.3.2	Tuning the DTG	24
2.4	Summary	30
3	Introduction to the Crossed Flexure Pivot	31
3.1	History of the Crossed Flexure Pivot	35
3.1.1	Static Analysis of the Crossed Flexure Pivot	40
3.1.2	Experimental Investigations of the Crossed Flexure Pivot	42
3.2	Manufacture of the Gimbal and Crossed Flexure Pivots	44
3.3	Classical Analysis of the Crossed Flexure Pivot	51
3.3.1	Geometric Relationships of the Crossed Flexure Pivot	51
3.3.2	Static Analysis of a Single Strip	57
3.3.3	Maximum Bending Moment and Buckling Load	68
4	Static Analysis of the Crossed Flexure Pivot	70
4.1	Development of the Static Equations	72
4.2	Comparison of Analytical Results with those in the Literature	91
4.3	Discussion on the Accuracy of the Mathematical Model	94
4.3.1	Large Angle Beam Analysis	96
4.3.2	Plate and Wide Beam Analysis	101
4.4	Properties of the Crossed Flexure Pivot	104
4.5	Comparison with Experimental Results from the Literature	111
4.6	Experimental Study of the Crossed Flexure Pivot	115
4.7	Summary of Results	126
5	Crossed Flexure Pivot Sizing	129
5.1	Design Considerations	130
5.2	Material and MMS Component Properties	134
5.3	Development of the Design Equations	135
5.4	Loads Acting on the Pivot	140

5.5	Design Results and Discussion	141
5.6	Summary	148
6	Mechanical Properties of the MMS Crossed Flexure Pivots	150
6.1	Validation of the Crossed Flexure Pivot Design	151
6.2	Effect of Tolerances on Tuned Speed	155
6.3	Pivot Start-Up Loads	160
6.4	Buckling of the MMS Crossed Flexure Pivots	172
6.5	Summary	175
7	Vibration Analysis of the Crossed Flexure Pivot	177
7.1	Development of the Equations of Motion	178
7.2	Vibration of the Crossed Flexure Pivots used in the MMS Prototype	193
7.3	Summary	200
8	Small Angle Analysis of a Dynamically Tuned Gyroscope	201
8.1	Formulation of the Equations of Motion for an Idealized DTG	202
8.1.1	Complex Variable Method	205
8.1.2	State Space Method	209
8.2	Stability Analysis of the DTG	211
8.2.1	Stability of the DTG with Bearings	214
8.2.2	Numerical Study of the Litton CSG-2 DTG	216
8.3	Numerical Solution of the Equations of Motion of a Dynamically Tuned Gyro	230
8.4	Stability of the MMS	235
9	Large Angle Analysis of the MMS	239
9.1	Kinematics of the MMS	242
9.2	Derivation of the Equations of Motion	245
9.2.1	Derivation of the Equations of Motion using Lagrange's Equations	249

9.2.2	Derivation of the Equations of Motion of the MMS using Hamilton's Canonical Equations	254
9.3	Calculation of the Loads on the Crossed Flexure Pivots	258
10	Large Angle Simulation of a DTG/MMS Type System	270
10.1	Overview of the Computer Simulation	271
10.1.1	Numerical Solution of Ordinary Differential Equations	274
10.1.2	The Adams-Moulton Numerical Technique	275
10.1.3	Aliasing	277
10.2	Convergence Study	278
10.3	Validation of the Numerical Simulation	283
11	Simulation Case Studies	294
11.1	General Behaviour of the DTG	295
11.2	Comparison Between the Small Angle Analysis and the Large Angle Analysis	297
11.3	Effect of Viscous Damping on a DTG	309
11.4	Two and Three Body Models with Constant and Variable Stiff- ness	315
11.4.1	Effect of the Vibration Damper on the MMS	336
11.5	Relationship between Rotor Offset and True Tuned Speed	337
11.6	Relationship between Rotor Offset and Nutation.	344
11.7	Behaviour of an Unsymmetric Rotor	345
11.8	Response of an Open Loop MMS to a Rotation of its Case	346
11.9	Summary	353
12	Summary and Conclusions	355
13	Recommendations	365
	Bibliography	368

Appendices	376
A Kinematics of a Hooke's Joint	376
B Working Drawings and Parts List for the Test Crossed Flex- ure Pivot	385
C Derivation of the Equations of Motion using Lagrange's Equa- tions	391
C.1 Case I: $q = \theta$	391
C.2 Case II: $q = \phi$	393
C.3 Case III: $q = \Omega$	397
C.4 Case IV: $q = \beta$	402
D Derivation of the Equations of Motions using Hamilton's Canonical Equations	403
D.1 Case I: $q = \theta$	403
D.2 Case II: $q = \phi$	405
D.3 Case III: $q = \Omega$	407
D.4 Case IV: $q = \beta$	409
E Comparison of the Complex Variable and State Space Solu- tions	411
F MMS System Energy as Calculated by DTGSim	414
G Relationship between Average Shaft Speed, Rotor Offset and Precession Frequency	420

List of Tables

3.1	Sample DTG gimbal and crossed flexure pivot dimensions . . .	45
4.1	Strip dimensions used for the comparison between the non-linear beam theory, plate theory and beam theory	95
4.2	Large and small angle transverse deflections using the Euler-Bernoulli equation	98
4.3	Crossed flexure pivot dimensions used to compare theoretical results with the analytical results found in the literature. . . .	105
4.4	Properties of the crossed flexure pivots used in the experimental research by Nickols and Wunsch	112
5.1	Design criteria for the MMS gimbal and pivot assembly	133
5.2	Material properties of AerMet 100	135
5.3	Preliminary MMS component properties	135
5.4	Summary of the pivot design	149
6.1	Dimensions of the crossed flexure pivots used in the MMS . .	152
6.2	Inertial and mass properties of the MMS engineering prototype as of April 1998	152
7.1	Inertial and mass properties of the MMS engineering prototype as of April 1998	194
7.2	First four natural frequencies of the crossed flexure pivots used in the MMS prototype	195

8.1	Inertial and stiffness properties of the Litton CSG-2 Dynamically Tuned Gyro used in the stability analysis	217
8.2	Inertial and mass properties of the MMS engineering prototype as of April 1998	238
10.1	Maximum difference between the calculated angles using a specific tolerance and the angles calculated using the previous tolerance when using Lagrange's Equations	281
10.2	Maximum difference between the calculated angles using a specific tolerance and the angles calculated using the previous tolerance when using Hamilton's Canonical Equations . . .	282
11.1	Inertial and mass properties of the MMS engineering prototype as of April 1998	318
B.1	Parts list for the crossed flexure pivot used in the tensile tests	390

List of Figures

1.1	Schematic of a dynamically tuned gyroscope	4
1.2	Sketch of a satellite in orbit. In this case, the momentum wheel rotates about the y axis of the spacecraft	7
2.1	Diagram of a free rotor with an angular velocity about its spin axis	18
2.2	Freebody diagram of the DTG rotor	26
2.3	Freebody diagram of the DTG gimbal	27
3.1	Schematic of the crossed flexure pivot as used for the MMS . . .	32
3.2	Single strip hinge consisting of a notch in a bar	34
3.3	Schematic of a three strip bearing	35
3.4	Crossed flexure pivot shown in deflected and undeflected positions	37
3.5	Schematic of the crossed flexure pivot in deflected and undeflected positions used to calculate the instantaneous centre of rotation of the pivot	38
3.6	Enlarged view of the HAZ left by the EDM process	46
3.7	Schematic of the two cylinders used for the manufacture of the gimbal. In this drawing, the holes allowing access for the wire EDM process have been drilled	47

3.8	Schematic of the two cylinders used for the manufacture of the gimbal and crossed flexure pivots. One side of the strip has been machined	47
3.9	Schematic of the two cylinders used for the manufacture of the gimbal. The eight flexure strips have been cut from the surrounding gimbal material	48
3.10	Schematic of the gimbal after the strips have been cut and the inner cylinder is rotated by 180 degrees with respect to the outer cylinder. The result is four sets of crossed strips	49
3.11	Schematic of the gimbal of a DTG or the MMS	50
3.12	General loading of a crossed flexure pivot	53
3.13	Schematic of the deflection of the pivot strips	54
3.14	Deflection of a cantilever beam when transverse, F , and axial, P_a , loads are applied in addition to torque, M_O	57
3.15	Variation of normalized angular stiffness with respect to normalized horizontal and vertical loads	67
4.1	Schematic of the crossed flexure pivot showing the components of the axial and transverse forces acting on the two strips	87
4.2	Plot of the axial shortening due to bending as a function of the applied load of a cantilever beam	99
4.3	Response of a cantilever beam to a load applied to the free end using large angle Euler-Bernoulli equations	100
4.4	Shell mesh used with the FEM to solve for the deflection of a cantilever plate	102
4.5	Schematic showing the distortion of the cross-section as predicted by Euler's equation	103
4.6	Percentage change in stiffness of a crossed flexure pivot with respect to the stiffness at an angle of 0 degrees versus the rotation of the pivot	107

4.7	Non-dimensionalized moment applied to the crossed flexure pivot versus the angle of rotation of the pivot and non-dimensionalized constraint forces acting on the strips of the pivot versus the angle of rotation	108
4.8	Plots of the normalized constraint forces acting on the pivot versus normalized axial load	110
4.9	Comparison between the experimental results of Nickols and Wunsch and the predictions by the theoretical model	112
4.10	Plot of the shift of the initial centre of rotation of the crossed flexure pivot as a function of pivot rotation	113
4.11	Chart showing the ratio of yield stress (σ_y) versus Young's modulus (E) for various materials	117
4.12	Photograph of the crossed flexure pivot used in the experimental study	118
4.13	Photograph of the crossed flexure pivot mounted in the Instron Table Mounted Universal Testing Rig, model number 4465	120
4.14	Load displacement plot for ultra high molecular weight polyethylene	122
4.15	Plot of the calculated stress in the strips of a crossed flexure pivot as a function of the applied load	123
4.16	Plot of the displacement of the initial centre of rotation of a pair of crossed flexure pivots as a function of load	124
5.1	Variation of the stiffness as a function of the normalized axial load	139
5.2	Variance of the loads on the pivots with time when the shaft is operating at 1700 rpm and there is an initial rotor offset of 7 degrees about the gimbal x axis	143
5.3	Maximum tensile stress in the strips of a crossed flexure pivot under a horizontal load of 7.5 N and a vertical load of 7.9 N	144

5.4	Maximum compressive stress in the strips of a crossed flexure pivot under a horizontal load of 7.5 N and a vertical load of 7.9 N	145
5.5	Variance of the strip length and width with thickness	146
5.6	Variance of the pivot stiffness with thickness	147
6.1	Variance of the loads on the pivots with time using the MMS engineering prototype parameters	154
6.2	Plot of the normalized axial load versus the cross-sectional area of the strips at the design stress	156
6.3	Plot of the relationship between tuned speed and deviations of the strip dimensions from the design values	159
6.4	Schematic of the displacement of a pivot during the period where the rotor is accelerating to its tuned speed	162
6.5	Stress in the strips of the crossed flexure pivots as a function of the time required to go from a stationary position to a tuned speed of 1500 rpm	164
6.6	Shaft speed of the MMS engineering prototype as a function of time	165
6.7	Angular position of the MMS rotor during a 10 second acceleration to a shaft speed of 1500 rpm	167
6.8	Dynamic loads acting on the MMS pivots connecting the rotor and gimbal and the shaft and the gimbal during acceleration of the shaft from 0 to 1500 rpm	168
6.9	Gravitational loads acting on the MMS pivots during start up	170
6.10	Maximum stress acting in the strips of the pivots connecting the rotor to the gimbal and the gimbal to the shaft as a function of time	171
6.11	Plot of the maximum stress in the MMS crossed flexure pivots as a function of time required to accelerate the shaft from 0 to 1500 rpm	173

7.1	Plot of the determinate of A with respect to k	196
7.2	Plot showing the mode shape at the natural frequency of 13.53 Hz	198
8.1	Schematic of a Dynamically Tuned Gyro showing the rotor and gimbal rings, shaft and torsional elements	202
8.2	Time response of a CSG-2 Gyro with bearings in place of tor- sional springs and an initial offset of $\phi = 1$ mrad	217
8.3	Phase plane response of the CSG-2 Gyro with bearings in place of torsional springs and an initial offset of $\phi = 1$ mrad	218
8.4	Stability boundaries for the CSG-2 gyro with varying x axis stiffness	220
8.5	Characteristic curve for the CSG-2 gyro. Each hyperbola rep- resents a different x axis stiffness	221
8.6	Stability boundaries for the CSG-2 gyro with varying y axis stiffness	222
8.7	Characteristic curve for the CSG-2 gyro. Each hyperbola rep- resents a different y axis stiffness	223
8.8	Stability boundaries for the CSG-2 gyro with varying gimbal spin inertia	224
8.9	Characteristic curve for the CSG-2 gyro. Each hyperbola rep- resents a different value of I_{g_s}	225
8.10	Stability boundaries for the CSG-2 gyro with varying gimbal transverse inertia	226
8.11	Characteristic curve for the CSG-2 gyro. Each hyperbola rep- resents a different value of I_{g_t}	227
8.12	Stability boundaries for the CSG-2 gyro with varying rotor spin inertia	228
8.13	Characteristic curve for the CSG-2 gyro. Each hyperbola rep- resents a different value of the rotor spin inertia of the gyro, I_{r_s}	229

8.14	Time response of a standard CSG-2 Gyro with an initial offset of $\phi = 1$ mrad	231
8.15	Phase plane response of a standard CSG-2 Gyro with an initial offset of $\phi = 1$ mrad	232
8.16	Time response of a CSG-2 Gyro with transverse gimbal inertia, $I_{gt} = 100$ gcm ² and an initial offset of $\phi = 1$ mrad	233
8.17	Phase plane response of the CSG-2 Gyro with transverse gimbal inertia, $I_{gt} = 100$ gcm ² and an initial offset of $\phi = 1$ mrad	234
8.18	Stability boundaries for the MMS engineering prototype with varying x axis stiffness	236
8.19	Characteristic curves for the MMS. Each hyperbola represents a different torsional stiffness	237
9.1	Drawing of the MMS as of August 22, 1997	240
9.2	Schematic of the MMS showing the rotor, gimbal, shaft and vibrational damper	241
9.3	Rotor position in 'case' and 'pickoff' referenced angles	247
9.4	Freebody diagram of the MMS rotor	260
9.5	Freebody diagram of the MMS shaft	263
9.6	Freebody Diagram of the MMS vibration damper	264
9.7	Freebody diagram of the MMS gimbal	265
9.8	Rotation sequence required in order to represent the direction of the acceleration in the shaft reference frame	268
10.1	Flow chart depicting the basic organization of the Dynamically Tuned Gyroscope simulation	272
10.2	Example of aliasing due to the discrete sampling of an analog frequency	277
10.3	Orientation of the CSG-2 DTG rotor with respect to time. The tolerance specified in the integration routine was 10^{-4}	279

10.4 Orientation of the CSG-2 DTG rotor with respect to time. The tolerance specified in the integration routine was 10^{-5} . . .	280
10.5 Model of the Dynamically Tuned Gyro implemented in the Working Model 3D dynamic simulation package	284
10.6 Orientation of the CSG-2 rotor, in 'case' coordinates, with respect to time as calculated by DTGSim. The initial rotor offset was $\theta' = 1$ degree	286
10.7 Orientation of the CSG-2 rotor, in 'case' coordinates, with respect to time as calculated by RSIM. The initial rotor offset was $\theta' = 1$ degree	287
10.8 Orientation of the CSG-2 rotor, in 'case' coordinates, with respect to time as calculated by Working Model 3D. The initial rotor offset was $\theta' = 1$ degree	288
10.9 System energy in the CSG-2 DTG with an initial rotor offset of $\theta' = 1$ degree, as calculated by DTGSim	289
10.10 Orientation of the CSG-2 rotor, in 'case' coordinates, with respect to time as calculated by DTGSim. The initial rotor offset was $\phi' = 10$ degrees	290
10.11 Orientation of the CSG-2 rotor, in 'case' coordinates, with respect to time as calculated by RSIM. The initial rotor offset was $\phi' = 10$ degrees	291
10.12 Orientation of the CSG-2 rotor, in 'case' coordinates, with respect to time as calculated by Working Model 3D. The initial rotor offset was $\phi' = 10$ degrees	292
10.13 System energy in the CSG-2 Dynamically Tuned Gyroscope with an initial rotor offset of $\phi' = 10$ degrees, as calculated by DTGSim	293
11.1 Response of the Litton CSG-2 DTG to an initial offset of $\theta' = 100$ mrad	296

11.2 Response of the Litton CSG-2 to an initial rotor offset of $\theta' = 1$ mrad using small angle and large angle assumptions	298
11.3 Power spectral density analysis of the rotor position data calculated using small angle equations	299
11.4 Power spectral density analysis of the rotor position data calculated using large angle equations	300
11.5 Response of the Litton CSG-2 to an initial rotor offset of $\theta' = 10$ mrad as predicted by equations based on small angle and large angle assumptions	303
11.6 Response of the Litton CSG-2 to an initial rotor offset of $\theta' = 10$ mrad as predicted by equations based on small angle and large angle assumptions	304
11.7 Power spectral density of the rotor position data calculated using the small angle equations	305
11.8 Power spectral density of the rotor position data calculated using the large angle equations	306
11.9 Response of the Litton CSG-2 to an initial rotor offset of $\theta' = 100$ mrad as predicted by equations based on small angle and large angle assumptions	308
11.10 Power spectral density of the rotor position data calculated using small angle equations	310
11.11 Power spectral density of the rotor position data calculated using large angle equations	311
11.12 Power spectral density of the rotor position data calculated using large angle equations	312
11.13 Response of the CSG-2 gyro when axial damping is included on the two gimbal axes. In this case, the damping coefficient along an axis ($2D_x = 2D_y$) was 2×10^{-7} Nms/rad	313

11.14 Response of the CSG-2 gyro when axial damping is included on the two gimbal axes. The damping coefficient along an axis ($2D_x = 2D_y$) was set to 20×10^{-7} Nms/rad	314
11.15 Plot of rotor position, in case coordinates, versus time for the Momentum Management System. The inertia of the shaft is increased to approximate a two body analysis. The torsional stiffness of the crossed flexure pivots is assumed constant	316
11.16 Plot of shaft rate versus time for the Momentum Management System. The inertia of the shaft is increased to approximate a two body analysis. The torsional stiffness of the crossed flexure pivots is assumed constant	317
11.17 Plot of the forces acting on the crossed flexure pivots versus time for the Momentum Management System. The inertia of the shaft is increased to approximate a two body analysis. The torsional stiffness of the crossed flexure pivots is assumed constant	319
11.18 Plot of rotor position, in case coordinates, versus time for the Momentum Management System. The torsional stiffness of the crossed flexure pivots is assumed constant	320
11.19 Plot of the shaft speed versus time for the Momentum Management System. The torsional stiffness of the crossed flexure pivots is assumed constant	321
11.20 Plot of the forces acting on the crossed flexure pivots versus time for the Momentum Management System. The torsional stiffness of the crossed flexure pivots is assumed constant	323
11.21 Plot of rotor position, in case coordinates, versus time for the Momentum Management System	325
11.22 Plot of shaft rate versus time for the Momentum Management System	326

11.23 Plot of the forces acting on the crossed flexure pivots versus time	327
11.24 Plot of the variance of the axial stiffness versus time for the Momentum Management System	329
11.25 Plot of rotor position in case coordinates versus time for the Momentum Management System positioned at an angle of 45 degrees to a 1g environment	330
11.26 Plot of the shaft spin speed versus time for the Momentum Management System. The MMS is oriented at 45 degrees in a 1g gravitational field	331
11.27 Plot of the internal pivot loads versus time for the Momentum Management System when it is oriented at 45 degrees in a 1g gravitational field	333
11.28 Plot of the gravitational loads on the crossed flexure pivots versus time for the Momentum Management System when it is oriented at 45 degrees in a 1g gravitational field	334
11.29 Plot of axial stiffness versus time. The Momentum Management System has been oriented at 45 degrees to a 1g gravitational field	335
11.30 Plot of precession frequency versus average shaft speed and rotor offset for the MMS prototype	339
11.31 Lines of constant precession frequency as functions of average shaft speed and rotor offset for the MMS prototype	341
11.32 Plot of the true tuned shaft speed versus rotor offset	342
11.33 Plot of rotor position, in case coordinates, versus time for the Momentum Management System. The rotor has an initial offset of 8 degrees about the y axes in case coordinates	343
11.34 Plot of the peak to peak amplitude of the rotor nutation versus angular offset for the MMS prototype	344

11.35 Response of the Momentum Management System rotor to an unbalance of $I_{r_{y_1}} = 0.1 \text{ gcm}^2$	347
11.36 Power spectral density analysis of the response of the rotor to a rotor unbalance of $I_{r_{y_1}} = 0.1 \text{ gcm}^2$	348
11.37 Forces acting on the crossed flexure pivots as a function of time when there is a rotor imbalance of $I_{r_{y_1}} = 0.1 \text{ gcm}^2$	349
11.38 Plot of the rotor motion with respect to the case of the MMS. The case is rotating at a rate of 0.01 rad/s about its y axis	350
11.39 Plot of the motion of the rotor about the y axis in an inertial reference frame. The case of the rotor has a steady rotation of 0.1 rad/s	352
A.1 Example of gimbal motion when the rotor is offset at a particular angle	377
A.2 Rotation of a coordinate frame based on moving the shaft to the rotor position using 'gimbal' angles	379
A.3 Rotation of a coordinate frame based on moving the shaft to the rotor position using 'case' angles	381
A.4 Comparison between 'pickoff' and 'case' angles	384
B.1 Drawing of the rigid body used to attach the strips	387
B.2 Drawing of the strips of the crossed flexure pivot	388
B.3 Drawing of the blocks used to attach the strips and the rigid bodies	389
B.4 Photograph showing the assembled crossed flexure pivot	390
F.1 Energy of the MMS simulation when the mass moment of inertia of the shaft has been increased to $5.12 \times 10^{15} \text{ gcm}^2$. The stiffness of the crossed flexure pivots was held constant and the shaft speed was set to 1500 rpm	417

F.2	Energy of the MMS simulation versus time. The stiffness of the crossed flexure pivots was held constant and the shaft speed was set to 1500 rpm	418
F.3	Energy of the MMS simulation versus time. The shaft speed was set to 1500 rpm	419
G.1	Angular position of the MMS rotor versus time. The rotor was initially offset about its x axis (case coordinates) by 7 degrees and the average shaft rate was 1200 rpm	422
G.2	Angular position of the MMS rotor versus time. The rotor was initially offset about its x axis (case coordinates) by 7 degrees and the average shaft rate was 1250 rpm	423
G.3	Angular position of the MMS rotor versus time. The rotor was initially offset about its x axis (case coordinates) by 7 degrees and the average shaft rate was 1300 rpm	424
G.4	Angular position of the MMS rotor versus time. The rotor was initially offset about its x axis (case coordinates) by 7 degrees and the average shaft rate was 1350 rpm	425
G.5	Angular position of the MMS rotor versus time. The rotor was initially offset about its x axis (case coordinates) by 7 degrees and the average shaft rate was 1400 rpm	426
G.6	Angular position of the MMS rotor versus time. The rotor was initially offset about its x axis (case coordinates) by 7 degrees and the average shaft rate was 1450 rpm	427
G.7	Angular position of the MMS rotor versus time. The rotor was initially offset about its x axis (case coordinates) by 7 degrees and the average shaft rate was 1500 rpm	428
G.8	Angular position of the MMS rotor versus time. The rotor was initially offset about its x axis (case coordinates) by 7 degrees and the average shaft rate was 1550 rpm	429

G.9	Angular position of the MMS rotor versus time. The rotor was initially offset about its x axis (case coordinates) by 7 degrees and the average shaft rate was 1600 rpm	430
G.10	Angular position of the MMS rotor versus time. The rotor was initially offset about its x axis (case coordinates) by 7 degrees and the average shaft rate was 1650 rpm	431
G.11	Angular position of the MMS rotor versus time. The rotor was initially offset about its x axis (case coordinates) by 7 degrees and the average shaft rate was 1700 rpm	432
G.12	Angular position of the MMS rotor versus time. The rotor was initially offset about its x axis (case coordinates) by 7 degrees and the average shaft rate was 1750 rpm	433
G.13	Angular position of the MMS rotor versus time. The rotor was initially offset about its x axis (case coordinates) by 7 degrees and the average shaft rate was 1800 rpm	434
G.14	Angular position of the MMS rotor versus time. The rotor was initially offset about its x axis (case coordinates) by 7 degrees and the average shaft rate was 1400 rpm	435
G.15	Rotor precession frequency versus average shaft speed and rotor offset	436

Nomenclature

α	Chapters 3 - 7: Angle between a strip and the vertical axis
$d\alpha$	Chapter 2: Angle of precession of the spinning rotor
$\dot{\alpha}_1$	Chapter 2: Angular velocity of a stationary rotor about a transverse axis
$\dot{\alpha}_2$	Chapter 2: Angular velocity of a spinning rotor about a transverse axis
β	Chapter 9: Rotation of the vibration damper about its z axis with respect to the shaft
γ	Chapter 2, 9: Rotation of the rotor about its spin axis
δ	Chapter 3: Transverse deflection of a strip Chapter 6: Deflection of the strip
$\delta\mathcal{I}$	Chapter 3, 4, 7: Variance of the action integral
ΔL	Chapter 6: Difference between the desired length and the actual length
Δt	Chapter 6: Difference between the desired thickness and the actual thickness
Δw	Chapter 6: Difference between the desired width and the actual width
ϵ	Chapter 4, 6: Longitudinal strain in the strip
η	Chapter 2, 9: Y axis pickoff angle Chapter 8: Square of the shaft spin speed
θ	Chapter 1: Pitch angle of the spacecraft Chapter 2, 8, 9: Y axis gimbal angle

	Chapters 3 - 6: Rotation of one rigid body of the crossed flexure pivot with respect to the other; also the slope of the strip at the end which is allowed to deflect
θ'	Chapter 2, 9 - 11: Y axis rotor angle in case coordinates
θ_A	Chapter 7: Rotation of rigid body A with respect to an inertial frame of reference
θ_B	Chapter 7: Rotation of rigid body B with respect to an inertial frame of reference
λ_1, λ_2	Chapter 4: Lagrangian multipliers which represent the constraint forces keeping the ends of the strips apart
ν	Chapter 4: Poisson's ratio for the strip material
ρ	Chapter 5, 7: Density of the strip material
σ	Chapter 4, 7: Dummy variable representing the distance along strip '2' from the fixed end (rigid body B)
	Chapters 4 - 6: Longitudinal stress in the strip
σ_d	Chapter 5: Design stress for the strips of the crossed flexure pivots
σ_e	Chapter 5: Endurance limit of the strip material
σ_y	Chapter 3, 5: Yield stress of the strip material
τ	Chapter 4, 7: Dummy variable representing the distance along strip '1' from the fixed end (rigid body B)
Υ_1	Chapter 3: Various Coordinate frames based on the crossed flexure pivot
ϕ	Chapter 1: Roll angle of the spacecraft
	Chapter 2, 8, 9: X axis gimbal angle
	Chapter 5, 6: Angle between the crossed flexure pivot and the gravitational field
ϕ'	Chapter 2, 9 - 11: X axis rotor angle in case coordinates
ϕ_o	Chapter 2: Initial offset of the x axis gimbal angle
χ	Chapter 9: Rotation about x axis of gravitational field reference frame

ψ	Chapter 1: Yaw angle of the spacecraft Chapter 9: Rotation about y axis of gravitational field reference frame
ω	Chapter 2: Angular rate of the spinning rotor Chapter 8: Natural frequencies of the DTG
ω_{r_x}	Chapter 9: Angular rate of the MMS case about its x axis
ω_{r_y}	Chapter 9: Angular rate of the MMS case about its y axis
ω_{r_z}	Chapter 9: Angular rate of the MMS case about its z axis
ω_{g_x}	Chapter 2, 8, 9: Angular rate of the gimbal about its x axis
ω_{g_y}	Chapter 2, 8, 9: Angular rate of the gimbal about its y axis
ω_{g_z}	Chapter 2, 8, 9: Angular rate of the gimbal about its z axis
ω_n	Chapter 1: Nutation frequency of the spacecraft Chapter 7: Circular frequency of vibration of the crossed flexure pivot Chapter 8, 11: Nutation frequency of the DTG rotor in spinning coordinates
ω_n^*	Chapter 11: Nutation frequency of the DTG rotor in non spinning coordinates
ω_o	Chapter 1: Orbital rate of the spacecraft
ω_r	Chapter 2: Angular velocity of the rotor in gimbal coordinates
ω'_r	Chapter 2: Angular velocity of the rotor in case coordinates
ω_{r_x}	Chapter 2, 8, 9: Angular rate of the rotor about its x axis
ω_{r_y}	Chapter 2, 8, 9: Angular rate of the rotor about its y axis
ω_{r_z}	Chapter 2, 8, 9: Angular rate of the rotor about its z axis
ω_s	Chapter 1: Angular velocity of the momentum wheel with respect to the spacecraft Chapter 2, 5, 6, 8, 11: Angular velocity of the shaft about its spin axis
ω_{s_1}	Chapter 6: Unmodified tuned speed of the shaft
ω_{s_2}	Chapter 6: Modified tuned speed of the shaft
ω_{s_x}	Chapter 9: Angular rate of the shaft about its x axis

ω_{y_v}	Chapter 9: Angular rate of the shaft about its y axis
ω_{y_i}	Chapter 9: Angular rate of the shaft about its z axis
ω_x	Chapter 1: Angular rate of the spacecraft about its roll axis
ω_y	Chapter 1: Angular rate of the spacecraft about its pitch axis
ω_z	Chapter 1: Angular rate of the spacecraft about its yaw axis
ω_{v_i}	Chapter 9: Angular rate of the vibration damper about its z axis
Ω	Chapter 2, 9: Rotation of the input shaft with respect to the case
	Chapter 8: Square of the natural frequencies of the DTG
Ω_s	Chapter 9: Ideal 'motor rotor' position
a_1, a_2	Chapter 4: Constants of integration. Kept to simplify the static equations
A	Chapters 4 - 7: Cross-sectional area of the strips of the crossed flexure pivot
A_{g_i}	Chapter 9, 11: Load acting in the x direction on the pivots connecting the shaft and gimbal
A_{g_v}	Chapter 9, 11: Load acting in the y direction on the pivots connecting the shaft and gimbal
A_{g_i}	Chapter 9, 11: Load acting in the z direction on the pivots connecting the shaft and gimbal
A_{r_i}	Chapter 9, 11: Load acting in the x direction on the pivots connecting the gimbal and rotor
A_{r_v}	Chapter 9, 11: Load acting in the y direction on the pivots connecting the gimbal and rotor
A_{r_i}	Chapter 9, 11: Load acting in the z direction on the pivots connecting the gimbal and rotor
b_1, b_2	Chapter 4: Constants of integration. Kept to simplify the static equations
c	Chapter 9: Damping coefficient
C_i	Chapter 2, 8, 9: $\cos i$
DTG	Chapters 1 - 13: Dynamically Tuned Gyroscope

D_x	Chapter 9, 11: Damping coefficient about the gimbal x axis
D_y	Chapter 9, 11: Damping coefficient about the gimbal y axis
E	Chapters 3 - 7: Young's Modulus of the strip material
ϵ_1	Chapter 3, 4: Axial deflection of strip '1'
ϵ_2	Chapter 3, 4: Axial deflection of strip '2'
EDM	Chapter 3, 5: Electro discharge machining
f_1	Chapter 4: Normalized transverse load acting on strip '1'
f_2	Chapter 4: Normalized transverse load acting on strip '2'
f_n	Chapter 7: Frequency of vibration of the crossed flexure pivot
F	Chapter 3, 4: Transverse force acting on a strip Chapter 6: Horizontal load acting on the pivot
F_1	Chapter 3: Transverse reaction force acting on strip '1'
F_2	Chapter 3: Transverse reaction force acting on strip '2'
F_g	Chapter 6: Rotor and gimbal mass multiplied by the acceleration due to gravity
F_m	Chapter 2, 8: Figure of merit
F_x	Chapter 2, 5, 6, 9, 11: Horizontal force acting on the rotor due to the rotor/gimbal interaction
F'_x	Chapter 2, 9: Horizontal force acting on the gimbal due to the rotor/gimbal interaction
F_z	Chapter 2, 5, 6, 9, 11: Vertical force acting on the rotor due to the rotor/gimbal interaction
F'_z	Chapter 2, 9: Vertical force acting on the gimbal due to the rotor/gimbal interaction
FEM	Chapter 4: Finite element method
g	Chapter 5, 6: Gravitational constant
G_y	Chapter 5, 6, 9, 11: Horizontal force acting on the shaft due to the gimbal/shaft interaction
G'_y	Chapter 2, 9: Horizontal force acting on the gimbal due to the gimbal/shaft interaction
G_z	Chapter 5, 6, 9, 11: Vertical force acting on the shaft due to the

	gimbal/shaft interaction
G_z''	Chapter 2, 9: Vertical force acting on the gimbal due to the gimbal/shaft interaction
h	Chapter 1: Angular momentum of the momentum wheel
h	Chapter 5: Height of the gimbal ring
H	Chapter 2: Angular momentum of a spinning rotor Chapter 3, 4, 5, 6: Horizontal applied load on the pivot
HAZ	Chapter 3: Heat affected zone
H_d	Chapter 5, 6: Horizontal applied load on the pivot due to the dynamic loads
H_g	Chapter 5, 6: Horizontal applied load on the pivot due to gravity
H_{gx}	Chapter 9: Angular momentum of the gimbal about the x axis
H_{gy}	Chapter 9: Angular momentum of the gimbal about the y axis
H_{gz}	Chapter 9: Angular momentum of the gimbal about the z axis
H_{rx}	Chapter 9: Angular momentum of the rotor about the x axis
H_{ry}	Chapter 9: Angular momentum of the rotor about the y axis
H_{rz}	Chapter 9: Angular momentum of the rotor about the z axis
H_x	Chapter 1: Angular momentum of the spacecraft about its roll axis
H_y	Chapter 1: Angular momentum of the spacecraft about its pitch axis
H_z	Chapter 1: Angular momentum of the spacecraft about its yaw axis
I	Chapter 2: Mass moment of inertia of the spinning rotor Chapters 3 - 7: Area moment of inertia of the crossed flexure pivot strips Chapter 6: Sum of the spin inertia of the gimbal and rotor
$\mathcal{I}$	Chapter 3, 4: Action integral
I_{gs}	Chapter 5 - 8, 11: Mass moment of inertia of the gimbal about its spin axis
I_{gt}	Chapter 5 - 8, 11: Mass moment of inertia of the gimbal about

	its transverse axis
I_{gx}	Chapter 2, 9, 11: Mass moment of inertia of the gimbal about its x axis
I_{gy}	Chapter 2, 9, 11: Mass moment of inertia of the gimbal about its y axis
I_{gz}	Chapter 2, 9, 11: Mass moment of inertia of the gimbal about its z axis
I_{gxy}	Chapter 9: Product of inertia of the gimbal with respect to the x and y axes
I_{gxz}	Chapter 9: Product of inertia of the gimbal with respect to the x and z axes
I_{gyz}	Chapter 9: Product of inertia of the gimbal with respect to the y and z axes
I_{rx}	Chapter 5 - 8, 11: Mass moment of inertia of the rotor about its transverse axes
I_{ry}	Chapter 5 - 8, 11: Mass moment of inertia of the rotor about its spin axes
I_{rz}	Chapter 2, 9: Mass moment of inertia of the rotor about its x axis
I_{rx}	Chapter 2, 9: Mass moment of inertia of the rotor about its y axis
I_{ry}	Chapter 2, 9: Mass moment of inertia of the rotor about its z axis
I_{rxy}	Chapter 9: Product of inertia of the rotor with respect to the x and y axes
I_{rxz}	Chapter 9: Product of inertia of the rotor with respect to the x and z axes
I_{ryz}	Chapter 9, 11: Product of inertia of the rotor with respect to the y and z axes
I_{sz}	Chapter 5 - 7, 9, 11: Mass moment of inertia of the shaft about its spin axis

I_v	Chapter 9: Mass moment of inertia of the vibration damper about its z axis
I_x	Chapter 1: Mass moment of inertia of the spacecraft about its roll axis
I_y	Chapter 1: Mass moment of inertia of the spacecraft and momentum wheel about the pitch axis
I'_y	Chapter 1: Mass moment of inertia of the spacecraft about its pitch axis
I_z	Chapter 1: Mass moment of inertia of the spacecraft about its yaw axis
j	Chapter 1, 4, 8: Square root of -1
J	Chapter 1: Mass moment of inertia of the momentum wheel about its spin axis
J_A	Chapter 7: Mass moment of inertia of rigid body A
J_B	Chapter 7: Mass moment of inertia of rigid body B
k	Chapter 5, 6: Torsional stiffness of the crossed flexure pivot
k_1	Chapter 6: Design stiffness of the crossed flexure pivot
k_2	Chapter 6: Actual stiffness of the crossed flexure pivot
k_x	Chapter 2, 7 - 9, 11: Torsional stiffness of one pivot along the axis connecting the gimbal and shaft
k_y	Chapter 2, 7 - 9, 11: Torsional stiffness of one pivot along the axis connecting the rotor and gimbal
K	Chapter 3, 4: Empirical stiffness of a crossed flexure pivot
K_m	Chapter 9, 11: Stiffness in the motor
K_t	Chapter 5: Stress concentration factor
K_v	Chapter 9: Torsional stiffness connecting the shaft and the vibration damper
K_x	Chapter 8, 11: Torsional stiffness of the pair of crossed flexure pivots connecting the gimbal and shaft
K_x^*	Chapter 8: Tuned stiffness of the pair of crossed flexure pivots connecting the gimbal and shaft

K_y	Chapter 8, 11: Torsional stiffness of the pair of crossed flexure pivot connecting the rotor and gimbal
K_y^*	Chapter 8: Tuned stiffness of the pair of crossed flexure pivots connecting the rotor and gimbal
L	Chapter 3 - 7: Strip length
$\mathcal{L}$	Chapter 3, 4, 9: Lagrangian, difference between the kinetic energy and potential energy
L_o	Chapter 4: Shortened length of a bent beam
m	Chapter 4: Normalized moment applied to the crossed flexure pivot
m_A	Chapter 7: Mass of rigid body A
m_B	Chapter 7: Mass of rigid body B
M	Chapter 2: Torque applied normal to the angular momentum vector of a spinning rotor Chapter 3, 4: Moment acting on the crossed flexure pivot Chapter 6: Torque acting on the pivot
M_A	Chapter 3, 4: Reaction moment acting at the fixed end of the strip
M_B	Chapter 3: Moment acting at the free end of the strip
M_g	Chapter 5 - 7, 9, 11: Mass of the gimbal ring
M_{g_x}	Chapter 9: Moment acting about the x axis of the gimbal
M_{g_y}	Chapter 9: Moment acting about the y axis of the gimbal
M_{g_z}	Chapter 9: Moment acting about the z axis of the gimbal
M_{max}	Chapter 5: Maximum bending moment on a strip
M_o	Chapter 3: Moment applied to the strip
M_r	Chapter 5 - 7, 9, 11: Mass of the rotor
M_{r_x}	Chapter 2, 9: Moment acting about the x axis of the rotor
M_{r_y}	Chapter 2, 9: Moment acting about the y axis of the rotor
M_{r_z}	Chapter 9: Moment acting about the z axis of the rotor
M_{s_x}	Chapter 9: Moment acting about the x axis of the shaft
M_{s_y}	Chapter 9: Moment acting about the y axis of the shaft

M_{s_z}	Chapter 9: Moment acting about the z axis of the shaft
M_x	Chapter 2: Moment acting about the x axis of the rotor in a non spinning coordinate system
M_y	Chapter 2: Moment acting about the y axis of the rotor in a non spinning coordinate system
MMS	Chapters 1 - 13: Momentum Management System
n	Chapter 5: Safety factor
N	Chapter 4: Normalized axial load of a strip
N^*	Chapter 5, 6: Non-dimensionalized axial load on a strip
N_1	Chapter 4: Normalized axial load acting on strip '1'
N_2	Chapter 4: Normalized axial load acting on strip '2'
N_c	Chapter 3: Normalized compressive axial load acting on the strip
N_{cr}	Chapter 6: Normalized critical buckling load on a strip
N_t	Chapter 3: Normalized tensile axial load acting on the strip
p	Chapter 3, 4: Horizontal deflection of the instantaneous centre of rotation of the crossed flexure pivot
p_A	Chapter 7: Horizontal deflection of rigid body A
p_B	Chapter 7: Horizontal deflection of rigid body B
P	Chapter 4, 6: Axial load acting on a strip
P_1	Chapter 3, 5: Axial load acting on strip '1'
P_2	Chapter 3, 5: Axial load acting on strip '2'
P_c	Chapter 3: Compressive axial load acting on a strip
P_{cr}	Chapter 3, 6: Critical buckling load of the crossed flexure pivot
P_t	Chapter 3: Tensile axial load acting on a strip
P_q	Chapter 9: Generalized momentum with respect to the generalized coordinate q
q	Chapter 3, 4: Vertical deflection of the instantaneous centre of rotation of the crossed flexure pivot Chapter 9: Generalized coordinate
q_A	Chapter 7: Vertical deflection of rigid body A
q_B	Chapter 7: Vertical deflection of rigid body B

Q	Chapter 4: Normalized axial load acting on a strip
Q_i	Chapter 9: Generalized force
r	Chapters 3 - 7: Fraction of the length of the strip at which the strips cross
r_i	Chapter 5: Inner radius of the gimbal ring
r_o	Chapter 5: Outer radius of the gimbal ring
R	Chapter 3, 4: Radius of curvature of the deflected strip
	Chapter 5: Ratio of the minimum and maximum stresses during a fatigue test
	Chapter 6, 9: Radius at which the crossed flexure pivots are located in a DTG
S_i	Chapter 2, 8, 9: $\sin i$
SE	Chapter 4: Strain energy in a strip
t	Chapter 2: Time over which the torque is applied to the stationary rotor
	Chapters 3 - 7: Thickness of the strips of the crossed flexure pivot
	Chapter 6: Time taken for the rotor to reach the tuned speed
$\mathcal{T}$	Chapter 3, 4, 7 - 9: Kinetic energy
T_i	Chapter 2: $\tan i$
T_m	Chapter 9: Torque supplied by the motor in addition to that required to overcome bearing friction
T_x	Chapter 9: X axis torque applied to the rotor in 'pickoff' coordinates
T'_x	Chapter 2, 9: Torque acting about the rotor x axis due to the torquer coils mounted in the rotor case
T_y	Chapter 9: Y axis torque applied to the rotor in 'pickoff' coordinates
T'_y	Chapter 2, 9: Torque acting about the rotor y axis due to the torquer coils mounted in the rotor case
T'_z	Chapter 2, 9: Torque acting about the rotor z axis due to the

	torquer coils mounted in the rotor case
TW-EDM	Chapter 3: Travelling wire electro discharge machining
u_1	Chapter 3, 4, 7: Transverse deflection of strip '1'
u_2	Chapter 3, 4, 7: Transverse deflection of strip '2'
U	Chapter 6: Energy in the crossed flexure pivot
UHMWPE	Chapter 4: Ultra high molecular weight polyethylene
v_1	Chapter 4, 7: Axial displacement of strip '1' due to the axial load
v_2	Chapter 4, 7: Axial displacement of strip '2' due to the axial load
V	Chapter 3, 4 - 6: Vertical load applied to the pivot
$\mathcal{V}$	Chapter 3, 4, 7 - 9: Potential energy
V_d	Chapter 5, 6: Vertical load applied to the pivot due to the dyanmic loads
V_g	Chapter 5, 6: Vertical load applied to the pivot due to gravity
w	Chapter 4 - 7: Width of the strips in the crossed flexure pivot
W	Chapter 4: Work done by the applied loads on the crossed flexure pivot
	Chapter 9: Work done by the generalized forces
x	Chapter 3, 4: Distance along the strip from the fixed end (rigid body B)

Chapter 1

Introduction

The purpose of a spacecraft attitude control system (ACS) is to measure the spacecraft attitude and apply torques to re-orient and maintain a desired pointing direction. This is a challenging task particularly with the additional requirements of low cost and mass with maximized reliability and system lifetime.

The Momentum Management System will provide a low cost, reliable, high accuracy gimballed momentum wheel for use in the control of spacecraft attitude. This novel concept also contains rate sensing capabilities to provide the spacecraft with attitude information. The result is a low cost system incorporating both attitude sensing and attitude control capabilities in one unit. To understand the role of the Momentum Management System in a spacecraft, it is important to describe the functions of the attitude control system and the instruments involved.

Continuous monitoring and control of the spacecraft attitude is needed due to the multitude of external and internal torques acting on the vehicle. These torques may include: gravity gradient torques, solar radiation

torques, aerodynamic drag, magnetic disturbance torques, propellant slosh, etc[1]. These external and internal torques cause the attitude of the spacecraft to constantly change and continuous attitude measurement and control is needed.

The first role of the ACS is to determine the attitude of the spacecraft. Earth sensors, magnetometers, star trackers and gyroscopes are used for this purpose. With the attitude of the spacecraft determined, the ACS actuators apply torques to re-orient to the desired position, accomplished by magnetic torquers, thrusters and/or momentum exchange devices.

In a typical satellite mission it is necessary to carry all the propellant required during the mission. Recalling that minimum mass and maximum lifetime is desired, use of thrusters would be ideally limited and used as effectively as possible. Magnetic torquers use electric power to create a magnetic field which interacts with that of the Earth. While well-suited for low Earth orbits (≈ 1000 km or less) where the magnetic field is strong, they are not as effective in higher orbits. A second weakness is that magnetometers and magnetic torquers can not be used at the same time.

It follows that some type of momentum wheel should be used with thrusters and magnetic torquers applied when the wheels become saturated (or are operating at their maximum or minimum angular velocity) for continuous attitude control.

1.1 Attitude Sensors

Attitude sensors can be placed in three main categories: optical, magnetic and inertial. Optical devices include sun sensors, earth sensors and star

trackers. A typical magnetic sensor is the magnetometer and an example of an inertial instrument is the gyroscope. In a typical ACS, a combination of several sensors may be used with gyros forming the backbone of the system, while other instruments periodically update and calibrate them.

The Dynamically Tuned Gyroscope (DTG) consists of a rotor, or flywheel, which is attached to an input shaft through a gimbal, or Hooke's joint. See Figure 1.1 for a sketch of the DTG. The result is a rotor which, ideally, is isolated from the noise and motion of the attached vehicle. The rotor, acting as a freebody, remains inertially fixed while the spacecraft rotates around it. Thus, the angle between the rotor and the spacecraft can be measured and used to determine the attitude of the spacecraft. In closed loop configuration, magnetic pickoffs sense the motion of the rotor with respect to the case of the gyro then torques are applied to null the rotor¹. It is the magnitude of the torques required to null the gyro that are used to determine the attitude of the spacecraft. In reality, the rotor tends to drift from this inertially fixed position due to effects such as non-null rotor rebalancing and high and low frequency noise. The error grows over time and must be eliminated through use of an external sensor, such as a star tracker, which provides data so the gyro may be re-calibrated[2].

Finally, the gimbal joint connects the rotor to the shaft using crossed flexure pivots. These pivots replace the standard bearings or 'pin and jewels' normally used in this type of joint. The cross axis pivot provides the advantages of no friction or backlash, minimal lateral translation of the centre of rotation, high reliability and no lubrication requirement.

¹The rotor is considered to be in a null position when the rotor is aligned with the shaft.

Figure 1.1: Schematic of a dynamically tuned gyroscope. Note that the Hooke's joint in a DTG uses flexures in place of bearings.

1.2 Momentum Exchange Devices

In general, three types of devices make use of momentum for the purpose of controlling the attitude of a spacecraft². These include:

- Reaction Wheel - a flywheel with its axis fixed to the vehicle and designed to operate around a zero biased momentum.
- Momentum Wheel - similar to a reaction wheel with the exception that it is designed to operate at a biased, or non-zero, momentum. Typically, the axis of rotation is fixed with respect to the vehicle. A change in the speed of the momentum wheel causes torque on the spacecraft.
- Control Moment Gyro³ - consists of a flywheel operating at a constant rate mounted on a single or double gimbal platform. The gimbal rings allow control of the direction of the momentum vector[1][3].

Typical control systems of satellites involve the use of a momentum wheel operating about the pitch axis of the spacecraft and reaction wheels on the roll and/or yaw axes. The biased momentum of the momentum wheel provides coupling between the roll and yaw axes, adding dynamic stiffness to the system, and serving as a reaction wheel for the pitch axis[3].

An alternative to using a combination of momentum and reaction wheels is to use momentum wheels mounted on a gimballed platform. This system has been used by the Space and Communications Unit of Hughes Aircraft

²Unless otherwise stated, momentum should be taken to mean angular momentum in this document.

³Also known as a gyrotorquers[1] and sometimes as gimballed momentum wheels[8].

Company in several of their spacecraft and provides the ability for the momentum vector to be steered[4][5]. This eliminates the need for roll and yaw reaction wheels. A second alternative, patented by NASA[6] and marketed by Teledyne consists of a magnetically levitated flywheel. Through sophisticated control, it is possible to magnetically tilt the wheel, resulting in the same advantages as the system used by Hughes. Both of these systems are quite complicated, mechanically in the case of the Hughes system and, electronically in the case of the system marketed by Teledyne. The possibility of failures occurring increases with the increased complexity.

At this point, it should be noted that momentum wheels have an advantage in their length of life over reaction wheels. Never going through a zero speed situation results in less wear on the bearings. However, having said this, all types of momentum exchange devices have been used in space for many decades and, in 1987, one manufacturer quoted their devices had accumulated over 5 million hours in space applications without a failure[7].

The point of momentum and reaction wheels is to absorb external torques which are being applied to the spacecraft. A point will come where the wheels become saturated and will be operating at their maximum (or minimum) speed. When this occurs, it is necessary to 'dump' momentum using a second actuator system (such as thrusters) to restore the wheels to their nominal operating speed[1][3][8].

The final type of momentum device, the control moment gyro (CMG) stores energy using a rotor spinning at a constant rate. Because the spin axis is gimballed, a commanded gimbal rotation causes the direction of the momentum vector to change, creating a control torque parallel to the output

Figure 1.2: Sketch of a satellite in orbit. In this case, the momentum wheel rotates about the y axis of the spacecraft.

axis[1]. If a constant disturbance torque is applied to the system, the wheel axis of the CMG will gradually line up with the direction of the torque[8]. CMGs are typically used when very large torques are required[7]. A well cited example of the use of the CMG is on Skylab[1][7] and it is expected that CMGs will be used on the International Space Station[9][10].

1.2.1 Three-Axis Body Stabilized Spacecraft with a Pitch Momentum Wheel

To illustrate the effect of a momentum wheel on a body-stabilized satellite, consider a spacecraft with a momentum wheel aligned with the negative pitch axis[11]. The coordinate system used in this example has the x axis in the direction of motion (roll), the z axis pointing toward the Earth (yaw) and the y axis (pitch) completing the triad, see Figure 1.2. The momentum about

the x and z-axes can be written as

$$\begin{aligned} H_x &= I_x \omega_x \\ H_z &= I_z \omega_z \end{aligned}$$

where I_x is the inertia about the principal axis of the spacecraft, ω_x is the angular rate and H_x the momentum about the x axis. The y axis momentum is a combination of the spacecraft and wheel inertia. Taking I'_y to be the inertia of the spacecraft without the wheel and J as the spin inertia of the wheel

$$H_y = I'_y \omega_y + J(\omega_y - \omega_s)$$

which can be written as

$$H_y = (I'_y + J)\omega_y - J\omega_s$$

or

$$H_y = I_y \omega_y - h$$

where $I_y = I'_y + J$, $h = J\omega_s$ and ω_s is the angular velocity of the flywheel relative to the spacecraft.

Euler's Equations[12] can now be written as

$$I_x \dot{\omega}_x + (I_z - I_y)\omega_y \omega_z + h\omega_z = 0$$

$$I_y \dot{\omega}_y + (I_x - I_z)\omega_x \omega_z = \dot{h}$$

$$I_z \dot{\omega}_z + (I_y - I_x)\omega_x \omega_y - h\omega_x = 0$$

Assuming the angular rates are small and their products are negligible[3][11]

$$I_x \dot{\omega}_x + h\omega_z = 0 \tag{1.1}$$

$$I_y \dot{\omega}_y = \dot{h} \tag{1.2}$$

$$I_z \dot{\omega}_z - h\omega_x = 0 \tag{1.3}$$

Equation 1.2 shows that the pitch momentum wheel, represented by the h , allows direct control over the pitch rate and Equations 1.1 and 1.3 show the coupling between the roll and yaw axes.

Now, it is useful to define the spacecraft rates in terms of the yaw, roll and pitch angles[3][11]. The result is

$$\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} 0 \\ \dot{\theta} \\ 0 \end{pmatrix} + (\theta)_y \begin{pmatrix} \dot{\phi} \\ 0 \\ 0 \end{pmatrix} + (\theta)_y (\phi)_x \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix} + (\theta)_y (\phi)_x (\psi)_z \begin{pmatrix} 0 \\ -\omega_o \\ 0 \end{pmatrix} \quad (1.4)$$

where ϕ , θ and ψ are the roll, pitch and yaw angles, respectively, $(\theta)_y$ represents a rotation of the coordinate system about the y axis by an angle θ and ω_o is the orbital rate of the spacecraft. Equation 1.4 can be expanded to give

$$\omega_x = C_\theta \dot{\phi} - S_\theta C_\phi \dot{\psi} - (C_\theta S_\psi + S_\theta S_\phi C_\psi) \omega_o \quad (1.5)$$

$$\omega_y = \dot{\theta} + S_\phi \dot{\psi} - C_\phi C_\psi \omega_o \quad (1.6)$$

$$\omega_z = S_\theta \dot{\phi} + C_\theta C_\phi \dot{\psi} - (S_\theta S_\psi - C_\theta S_\phi C_\psi) \omega_o \quad (1.7)$$

where, for example, S_θ represents $\sin \theta$ and C_θ represents $\cos \theta$. If the spacecraft is in a station keeping role then small angles and rates can be assumed and Equations 1.5 to 1.7 can be simplified to give

$$\omega_x = \dot{\phi} - \psi \omega_o \quad (1.8)$$

$$\omega_y = \dot{\theta} - \omega_o \quad (1.9)$$

$$\omega_z = \dot{\psi} + \phi \omega_o \quad (1.10)$$

The roll and yaw motion can be shown to be completely uncoupled from the pitch equation[3][11]. Rewriting Equations 1.1, 1.3, 1.8 and 1.10 gives

$$I_x \dot{\omega}_x + h \omega_z = 0$$

$$I_z \dot{\omega}_z - h \omega_x = 0$$

$$\dot{\omega}_x = \dot{\phi} - \psi \omega_o$$

$$\dot{\omega}_z = \dot{\psi} + \phi \omega_o$$

Taking the Laplace transform of the above four equations yields

$$\begin{bmatrix} sI_x & h & 0 & 0 \\ -h & sI_z & 0 & 0 \\ -1 & 0 & s & -\omega_o \\ 0 & -1 & \omega_o & s \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_z \\ \phi \\ \psi \end{bmatrix} = 0 \quad (1.11)$$

The characteristic equation can be found by calculating the determinate of the coefficient matrix in Equation 1.11 and setting it to zero. After simplification, the characteristic equation is

$$(s^2 I_x I_z + h^2)(s^2 + \omega_o^2) = 0 \quad (1.12)$$

Therefore, the natural frequencies of a simple body-stabilized spacecraft with one momentum wheel aligned with the negative pitch axis are

$$\begin{aligned} s_1 &= \pm j \frac{h}{\sqrt{I_x I_z}} \\ s_2 &= \pm j \omega_o \end{aligned}$$

The first root depicts a coning motion of the spacecraft at the nutation frequency[3]

$$\omega_n = \frac{h}{\sqrt{I_x I_z}}$$

and the second root represents the effect of gyrocompassing, where an initial pure roll error becomes a pure yaw error after a quarter turn. The frequency of this mode is the orbital frequency, ω_o . This last mode shows it is possible

to control the yaw error on a spacecraft without using yaw sensors. Over time, the orbit will transform the yaw error to a roll error which can be sensed and corrected as required. The correction can be made through use of a reaction wheel or by use of a gimballed momentum wheel.

1.3 Concept of the Momentum Management System

From the above discussions, it can be seen that a typical ACS includes some combination of momentum wheels, reaction wheels, magnetic torquers and/or thrusters, gyroscopes and other sensors plus redundant components and electronics. As momentum exchange actuators can be very expensive, a full inertial ACS system can be too costly to use in small satellites. Similarly, having four to six wheels (2 or 3 wheels plus redundancy) can be costly from the point of view of mass.

These considerations have brought about the concept of a novel type of gimballed momentum wheel, which would not only provide attitude control but attitude sensing capabilities as well. This system, currently known as the Momentum Management System (MMS), is based on the structure of the tuned gyro, with the exceptions of a significantly larger rotor and tilt angle sensing. The current design specifications for small satellites allow for 1 to 4 Nms of momentum to be generated while maintaining the ability of the system to accurately sense the attitude of the spacecraft to within several arcseconds⁴. Also, the gimballed nature of the MMS will allow the rotor to

⁴Future plans call for designs which will be capable of providing momentum in the range of 50 Nms, ideal for large satellites.

tilt by 7 degrees in any direction orthogonal to the shaft axis.

To reach these goals, significant research must be done on the design of several key mechanical, electronic and magnetic systems. Also, in order to be able to predict and control the motion of the rotor, accurate analytical and numerical models of the system are required.

1.4 Summary of Research

The purpose of this research was to create accurate static and dynamic models of the crossed flexure pivot and then use these equations to design pivots for use in the MMS gimbal. A small angle, two body dynamic analysis of the DTG was performed in order to determine the stability of these instruments and the results were extended to include an analysis of the stability of the MMS engineering prototype. A large angle, four body dynamic analysis of the MMS, and DTG type systems, was performed and the results were implemented into a numerical simulation. The simulation is to be used for control systems design and testing so the closer the simulation is to running real-time, the better.

In the literature, the crossed flexure pivot has typically been analyzed by separating the strips and analyzing each separately. This is done by making the assumption that the centre of rotation of the pivot remains constant[13][14][15]. Experimental research by Young[16] and Nickols and Wunsch[17] has shown that the pivot does shift slightly. Using the calculus of variations and Hamilton's principle of minimum energy, the pivot has been analyzed as one piece and the results are found in Chapter 4.

During a literature review, it was also found that a dynamic analysis of

the crossed flexure pivot has not been performed. This implies that the pivot was always assumed to have first mode motion. Using Hamilton's principle a vibration analysis of the pivot was performed and the results can be found in Chapter 7.

Static analysis of the crossed flexure pivot show that the torsional stiffness of the pivot is affected by the applied loads. This means the stiffness along an axis varies with time and this effect will be included in the large angle analysis of the MMS, something that has not been done previously.

Dynamic analyses of the dynamically tuned gyro (DTG), on which the MMS is based, have been performed in the literature[18][19][20][21]. These analyses assume a two body system with a constant input shaft speed. This assumption is fairly accurate for a DTG type system when the figure of merit⁵ is low ($F_m < 300$) and the rotor offset is kept below half a degree; however, for the MMS which has a very high figure of merit and will have rotor offsets of up to 7 degrees, this assumption has been found to inaccurately predict the precession and nutation rates of the rotor. This can be seen in Chapter 11.

This research includes a small angle analysis which is based, in part, on the analysis by Staley[18]. This analysis is used to determine the stability of the DTG and the MMS. The stability of gyros has been discussed in the literature[22] but always with the thought that precession and nutation are indications of instability. Previous research by this author[23] and results found in Chapter 11 clearly show that the precession and nutation are cyclic in nature and represent a stable configuration. The analysis in this thesis will examine the criteria for flutter and divergence instability in DTG type

⁵The figure of merit, F_m , relates the spin inertia of the rotor to the spin and transverse inertias of the gimbal.

systems; see Chapter 8.

As mentioned earlier, the two body assumption is only valid when the DTG is operating at a small rotor offset. Note that the maximum angular offset of a rotor in a DTG is usually less than 1 mrad. The MMS is designed to operate with rotor offsets of up to 7 degrees (0.122 rad) and the models found in the literature are no longer valid. A dynamic analysis of the MMS was performed including the rotor, shaft, gimbal and a fourth body which would act as a vibration damper, see Chapter 9. Also included in the simulation was the variance of the pivot stiffness with time and the effect of a rotating reference frame. These three additions to the model and the inclusion of the effect of the torquer coils and the magnetic stiffness of the motor lead to a very accurate simulation of the Momentum Management System. Not only is this an accurate model of the MMS but it can also be used to model DTG's and further characterize the behaviour of these inertial sensors. Using this simulation when characterizing the control system and the behaviour of a DTG could result in reductions in unpredicted motion of the rotor and increase the accuracy and time between recalibration.

More time between recalibration of the gyro allows the sensors used for this recalibration, such as star trackers, to determine the orientation of the spacecraft to a higher precision. Thus, the gyro is recalibrated to a finer degree resulting in higher accuracy and longer periods between recalibration times.

Chapter 2

Introduction to the Dynamically Tuned Gyroscope

The function of a gyroscope is to measure the orientation of a spacecraft or to sense its angular rate of turn. The classical form of the gyroscope consists of a rotor, or wheel, spinning at high speeds. Examples of these mechanical, or conventional, gyros include: Dynamically Tuned Gyroscopes, flex gyroscopes and single axis gyroscopes.

In recent years, it has become acceptable to call any sensor which provides measurements of body rotation a gyroscope, even if it does not rely on the dynamic properties of a rotating body. Thus, a second class of gyroscopes, known as unconventional sensors, has arisen including sensors such as: vibratory gyroscopes, optical rate sensors and nuclear magnetic resonance gyroscopes[22][24].

2.1 Unconventional Gyroscopes

The thrust of this research is to analyze components and the behaviour of the Dynamically Tuned Gyro (DTG). However, before this type of gyro is

discussed further, the other types of gyroscopes mentioned earlier will be briefly introduced beginning with the unconventional sensors.

Vibratory gyroscopes are based on work from the 19th century[24] when Foucault demonstrated that a vibrating rod would maintain its plane of vibration even while being rotated. It is interesting to note that this principle is used in nature by some flying insects[24]. The first man-made versions began to appear in the 1950s and different forms include: vibrating wine glass sensors, tuning fork sensors and vibrating wire rate sensors.

Nuclear magnetic resonance gyroscopes have been investigated since the 1960s and have the advantages of no moving parts and no dependence on high precision manufacture. Nuclear magnetic resonance (NMR) comes from the interaction between the nuclei of specific elements, such as krypton and xenon, and an external magnetic field. In general, nuclei of these elements have an angular momentum and a magnetic dipole moment. Applying a magnetic field to the cell containing these elements causes a torque which forces the spin axis of the nuclei to precess about the direction of the magnetic field and when an angular rate is applied to the cell a change in the precession frequency occurs. This change can be measured and used to determine the angular velocity of the vehicle[24].

The final type of unconventional sensor discussed here is an optical sensor which uses electromagnetic radiation to sense rotation. A typical system uses two beams of light traveling in opposite directions (one clockwise and the other counter-clockwise). When the device is rotated, there is an apparent difference in the length of the path traveled and it is this difference which is measured. A demonstration in 1925 by Michelson and Gale[24] measured

the shift due to the Earth's rotation using a ring that had a one mile (1.6 km) circumference.

2.2 Conventional Gyroscopes

The conventional gyroscope is based on the gyroscopic inertia or rigidity[27] of a spinning rotor. If mounted in a frictionless bearing, the rotor remains fixed in inertial space while the vehicle rotates about it. Consider a simple rotor, shown in Figure 2.1, with a mass moment of inertia, I , and an angular velocity, ω . The rotor has an angular momentum of

$$H = I\omega \quad (2.1)$$

A torque, applied normal to the angular momentum vector, will have a magnitude equal to the rate of change of the angular momentum. In other words

$$M = \frac{dH}{dt} \quad (2.2)$$

Since the torque is normal to the spin axis of the rotor it can not change the magnitude of the angular velocity vector but it can change its direction, see Figure 2.1. This change is in the same direction as the applied torque and has a magnitude of

$$dH = Hd\alpha \quad (2.3)$$

where $d\alpha$ is the angle of precession. Therefore, the torque relationship becomes

$$M = H \frac{d\alpha}{dt} \quad (2.4)$$

where $d\alpha/dt$ is the precession rate of the rotor about an axis normal to the applied torque and the spin axis.

Figure 2.1: Diagram of a free rotor with an angular velocity about its spin axis. A torque is applied about an axis normal to the spin direction.

In theory, it would be possible to use a stationary rotor to measure the rotation of a vehicle; however, friction in the gimbal bearings transmit a torque to the rotor and, in time, the rotor would have an angular velocity of

$$\dot{\alpha}_1 = \frac{M}{I}t \quad (2.5)$$

If the same torque is applied to a spinning wheel, then the precession rate is

$$\dot{\alpha}_2 = \frac{M}{I\omega} \quad (2.6)$$

where ω is the spin rate of the rotor. Taking the ratio of these two expressions gives

$$\frac{\dot{\alpha}_1}{\dot{\alpha}_2} = \omega t \quad (2.7)$$

and it is clear that the effect of the torque is ωt times higher on the stationary rotor than on the rotating rotor[22].

The single axis gyroscope, or single-degree-of-freedom gyroscope (SDFG), was a common gyro used in many applications as a rate gyro or rate integrating gyro.

The rotor assembly of an SDFG is typically floated in a fluid which provides damping for the correct dynamic response, protection from shock and vibration and reduces the gimbal friction[22]. In closed loop configuration, the output from angular sensors, known as pickoffs, is amplified and then used to supply a current to a torquer. The torque generated is equal to the gyroscopic torque[22] and the orientation of the spacecraft is obtained by measuring the torquer current. The SDFG and the two-degree-of-freedom gyro (2DFG) have been well documented in the literature. Examples can be found in texts by Ishlinskii[25], Deimel[26] and Richardson[27] and articles

by Ge and Chen[28] and Boyarski and Ben-Asher[29]. A good example of the use of a SDFG is the Hubble Space Telescope[30] and an example of the use of a 2DFG is the control moment gyro (CMG).

The flex gyroscope behaves in a similar manner to the Dynamically Tuned Gyro, the main difference being that the pivot of the flex gyroscope is created by a reduction in the diameter of the drive shaft rather than using a Hooke's Joint. Compensation for the torques arising from this pivot is provided by small permanent magnets attached to the rotor. These magnets are attracted to small screws attached to a plate connected to the shaft. The resulting magnetic torques de-couple the rotor from the shaft allowing it to act freely. A good example of the design of one of these gyros is found in the paper by Shimoni[31].

The Dynamically Tuned Gyroscope was first designed in the 1940s and incorporated a spinning flywheel on an unmodified Hooke's Joint, see Appendix A. This configuration resulted in high nutation and precession rates due to the dynamic effects of the gimbal motion and was not useful¹.

Two decades later, the "Gyroflex" gyroscope was developed. The 'pins and jewels' in the Hooke's Joint had been replaced by torsional springs and magnets, attached to the gyro case, providing an attractive force on the rotor, canceling the torques due to the springs. The result was a rotor free from external torques; nevertheless several difficulties arose because the "magnetic springs" caused erratic torques making this gyro inadequate for practical use[22].

In 1963, a gyro was developed using the dynamic inertia effects of the

¹These high rates are often referred to as an unstable configuration.

gimbal to create virtual negative spring rates. This gyro is known by many names: the Dynamically Tuned Gyro (DTG), the tuned rotor gyro and the dry tuned gyro, and is still in use today.

2.3 The Dynamically Tuned Gyro

The Dynamically Tuned Gyro, see Figure 1.1, consists of a rotor attached to a shaft through a gimbal, or Hooke's Joint, where pivots, shown schematically in Figure 1.1, replace the typical 'pins and jewels' in the Hooke's Joint. The shaft is powered by a hysteresis motor and magnetic pickoffs, mounted orthogonally, sense the motion of the rotor. In closed loop configuration, the output from these sensors is amplified and returned to torquer coils which rebalance, or null, the rotor with respect to the shaft. Closed loop configuration is commonly used in strapdown systems where the gyro is attached directly to the vehicle. In this situation, the orientation of the spacecraft is obtained by applying a scale factor to the electric current used in the torquer coils.

DTGs are also used in open loop configuration and in these cases the gyro is mounted on a stabilized platform. When this is done, the orientation of the vehicle is determined from the pickoff output. An example of the use of a DTG on a stabilized platform can be found in the paper by Sinkiewicz et al.[32].

The case of the DTG is sealed, the air is evacuated and then refilled with hydrogen gas at a pressure of 1/3 atm. This is done to limit the development of air currents. A potential difficulty arises from the reality that the magnetics of the DTG are sensitive to temperature changes which could result in

inaccurate readings. For this reason, DTGs are typically heated to a temperature higher than ambient (for example 70°C) to minimize any temperature fluctuations.

2.3.1 Dynamics of the DTG

There has been much research into the dynamics of the DTG beginning in the late 1940s and early 1950s. Arnold and Maunder[24] performed analyses of the precursor of the DTG in order to explain the behaviour of the untuned DTG. In 1964, Howe and Savet[36] published a paper describing the dynamics of a DTG which used torsional springs rather than the ‘pins and jewels’ used by Arnold and Maunder. Although Howe and Savet’s gyro was tuned, it was found to precess when exposed to vibratory inputs at twice the spin speed ($2\omega_s$).

As this error was suspected to be due to the interaction of the gimbal and the $2\omega_s$ input, a solution was proposed which added a second gimbal at right angles to the first[21]. The intent was to give an equal and opposite $2\omega_s$ bias to the rotor. This would have a second advantage in that the spin speed of the rotor and shaft would be constant, independent of the orientation of the rotor. While there are some advantages to the use of a multi-gimbal system, there are disadvantages as well. Adding a second gimbal increases the intricacy of the design, adds mechanical complexity which increases the chance of failure and increases the size of the device.

An analysis of the dynamics of the DTG was performed by Craig[19] using Newton-Euler Equations. This analysis does contain some simplifications and the system of equations is transformed into non-spinning coordinates early

in the analysis. When performing a small angle analysis, transforming the system of equations too early can result in the loss of important terms. A second paper by Craig[20] discusses possible errors effecting the drift of the DTG rotor and includes a discussion of the sensitivity of the rotor to $2\omega_s$ vibrations, linear accelerational loads and other inputs.

Lawrence[21] performed a general analysis using a state space method and identified three periodic oscillations of the rotor orientation. These frequencies consisted of the nutation frequency of the rotor, a term which would be zero if the system was tuned, and a frequency at $2\omega_s$.

Staley[18], using Lagrange's Equations, kept the results in spinning coordinates until the last step. The result was the discovery of a fourth periodic oscillation of the rotor representing a long term precession. These findings were confirmed by a large angle analysis by Cain[23].

It should be noted that in the course of these investigations, simplifying assumptions have been made including: deflections consist of only small angles; symmetric rotor and gimbal; stiffness of the torsional springs remains constant; case of the gyro fixed to an inertial frame and the system consists of gimbal rings and rotor only, implying that the input shaft speed is constant.

A recent paper by Mansour and Lacchini[37] discusses some of the most recent developments in the design and testing of a Dynamically Tuned Gyroscope and gives insight into some of the concerns that must be considered when designing a new gyro such as the design of the pivot and drift terms.

2.3.2 Tuning the DTG

The equations required for tuning the DTG have been derived by Howe and Savet[36], Craig[19], Fogarasy[38] and Lawrence[21]. In the literature, a set of five conditions is usually listed which must be met in order for the DTG to be considered a free gyro. In reality, the tuning speed of the gyro is affected by more factors depending on the environment in which the gyro is operating. One example of this is the orientation of a DTG experiencing an accelerational load affects the tuning speed[39]. Effects such as this fluctuate and it is difficult to account for all situations; however, efforts are made to minimize them as much as possible.

The first condition to be met by the gyro is that it should operate at a tuned speed. If the rotor is offset from the shaft then the gimbal will flutter at twice the spin frequency of the shaft. As the gimbal oscillates, it generates gyroscopic torques which are applied to the rotor. The larger the torques, the larger the rotor precession and nutation rates. The magnitude of these torques is dependent on the spin speed of the gimbal and, in general, they act as negative spring rates. By adding torsional springs to the system it is possible to use the positive spring rates, of these springs, to cancel the negative spring rates, caused by the gimbal.

A simple illustration of how the tuning equation for the gyro is calculated is through the following derivation which is based on the analyses by Craig[40]. A positive rotation of the rotor about the y axis of the gimbal gives the moment

$$M_{\tau_y} = 2k_y\theta \quad (2.8)$$

where k_y represents the stiffness of one pivot along the y axis and θ represents

the angle of rotation of the rotor with respect to the gimbal.

Examining Figures 2.2 and 2.3, it can be seen that for very small angular deflections of the rotor, the moments about the x axis of the rotor can be calculated to be

$$-2k_x\phi - M_{r_x} = I_{g_x}\ddot{\phi} + (I_{g_x} - I_{g_y})\omega_{g_y}\dot{\phi}$$

This expression can be simplified to give

$$M_{r_x} = -2k_x\phi - I_{g_x}\ddot{\phi} + (I_{g_y} - I_{g_x})\omega_{g_y}\dot{\phi} \quad (2.9)$$

where the subscript 'g' refers to the gimbal, I represents the mass moment of inertia and ω_{g_x} , ω_{g_y} and ω_{g_z} represent the gimbal angular rates. Assume the torsional springs connecting the rotor and the gimbal lie along the gimbal y axis and the springs connecting the gimbal to the shaft lie along the gimbal x axis.

The gimbal angular rates can be calculated in terms of the gimbal angles, ϕ and θ , by rotating the angular rate of the shaft, ω_s , about the x axis of the gimbal and then adding the rate of the gimbal about this axis. The result is

$$\begin{aligned} \omega_g &= \begin{pmatrix} \dot{\phi} \\ 0 \\ 0 \end{pmatrix} + (\dot{\phi})_x \begin{pmatrix} 0 \\ 0 \\ \omega_s \end{pmatrix} \\ &= \begin{pmatrix} \dot{\phi} \\ S_\phi\omega_s \\ C_\phi\omega_s \end{pmatrix} \end{aligned} \quad (2.10)$$

Assuming small angles, the gimbal angular rates can be substituted into Equation 2.9 to give

$$M_{r_x} = -2k_x\phi - I_{g_x}\ddot{\phi} + (I_{g_y} - I_{g_x})\phi\omega_s \quad (2.11)$$

Figure 2.2: Freebody diagram of the DTG rotor. Contact forces F_x and F_z are the result of the connection between the rotor and the gimbal. k_y represents the torsional stiffness of the crossed flexure pivots connecting the rotor and gimbal. θ describes the orientation of the rotor with respect to the gimbal and ω_{rx} , ω_{ry} and ω_{rz} are the angular velocities of the rotor about its x, y and z axes. T'_x , T'_y and T'_z are the torques acting on the rotor due to the torquer coils mounted in the rotor case.

Figure 2.3: Freebody diagram of the DTG gimbal. Contact forces F'_x and F'_z are the result of the connection between the rotor and the gimbal; and, G'_y and G'_z are the forces due to the interaction of the gimbal and the shaft. k_y and k_x are the torsional stiffness of the crossed flexure pivots connected to the rotor and the shaft, respectively. θ is the angle between the gimbal and the rotor; and, ϕ is the angle between the gimbal and the shaft. ω_{gx} , ω_{gy} and ω_{gz} are the angular rates about its x, y and z axes.

Defining ϕ in terms of an initial offset, ϕ_o , gives

$$\phi = \phi_o \cos \omega_s t$$

The derivative, and second derivative, of ϕ with respect to time are then

$$\dot{\phi} = -\omega_s \phi_o \sin \omega_s t$$

and

$$\ddot{\phi} = -\omega_s^2 \phi_o \cos \omega_s t$$

Similarly, the angle θ can be defined by realizing it has a 90 degree phase shift from ϕ

$$\theta = \phi_o \sin \omega_s t$$

Substituting these relationships into Equations 2.11 and 2.8 and simplifying gives

$$M_{r_x} = \left[(I_{g_x} + I_{g_y} - I_{g_z}) \omega_s^2 - 2k_x \right] \phi_o \cos \omega_s t \quad (2.12)$$

$$M_{r_y} = 2k_y \phi_o \sin \omega_s t \quad (2.13)$$

Equations 2.12 and 2.13 are written in the local coordinate system of the rotor. It is possible to transform these expressions to a non-spinning coordinate system by a rotation of $-\omega_s t$ about the z axis of the rotor. The resulting expressions for the torques on the rotor are

$$M_x = \left[(I_{g_x} + I_{g_y} - I_{g_z}) \omega_s^2 - 2k_x \right] \phi_o \cos^2 \omega_s t - 2k_y \phi_o \sin^2 \omega_s t \quad (2.14)$$

$$M_y = \left[(I_{g_x} + I_{g_y} - I_{g_z}) \omega_s^2 - 2k_x \right] \phi_o \cos \omega_s t \sin \omega_s t + 2k_y \phi_o \cos \omega_s t \sin \omega_s t \quad (2.15)$$

The above expressions can be simplified to give

$$M_x = \frac{1}{2} \left[(I_{g_x} + I_{g_y} - I_{g_z}) \omega_s^2 - (2k_x + 2k_y) \right] \phi_o \\ + \frac{1}{2} \left[(I_{g_x} + I_{g_y} - I_{g_z}) \omega_s^2 - (2k_x - 2k_y) \right] \phi_o \cos 2\omega_s t \quad (2.16)$$

$$M_y = \frac{1}{2} \left[(I_{g_x} + I_{g_y} - I_{g_z}) \omega_s^2 - (2k_x - 2k_y) \right] \phi_o \sin 2\omega_s t \quad (2.17)$$

Ideally, the moments on the rotor would be zero; however, it can be seen that it would be difficult to make all the terms in the above expressions zero. The tuning condition can be found by forcing the first term of Equation 2.16 to be zero. The tuning condition is expressed as[19]

$$\omega_s = \sqrt{\frac{2k_x + 2k_y}{I_{g_x} + I_{g_y} - I_{g_z}}} \quad (2.18)$$

The remaining terms show the influence of the gimbal flutter on the rotor. Using a second gimbal would cause torques in the opposite direction, with a net moment of zero. A small angle analysis by Fogarasy[38] has shown that the actual tuning condition is more complicated than this. It has been shown that the second term cannot vanish but can be minimized by using the remaining four tuning conditions. The analyses by Staley[18] and Lawrence[22] agree with this conclusion, as do the expressions found in Chapter 8.

In addition to satisfying the tuning condition, damping torques may not act on the rotor. The stiffness about the y axis of the gimbal, k_y , should also be zero. If this was true it would be possible to eliminate the remaining terms in the above expressions. This condition is typically ignored because of other error terms caused by the non-symmetry of the springs. The fourth condition requires that

$$I_{g_y} + I_{g_x} - I_{g_z} = 0$$

which would be satisfied if the gimbal was a thin disk but would also result in an infinite tuning speed. The remaining condition states that the rotor should also be a thin disk or

$$I_{r_y} + I_{r_x} - I_{r_z} = 0$$

where ‘ r ’ represents the rotor. Again, this is not practical and the result is that the rotor nutates and precesses[22].

In general, the quality of a DTG is represented by the Figure of Merit, as introduced by Craig[19]. The Figure of Merit, F_m , relates the spin inertia of the rotor to the inertias of the gimbal. Mathematically,

$$F_m = \frac{I_{r_s}}{I_{g_y} + I_{g_x} - I_{g_z}} \quad (2.19)$$

If the fourth tuning condition was met, the Figure of Merit would be infinite. In practice, the gimbal is made small with respect to the spin inertia of the rotor to give the largest Figure of Merit possible. The F_m of a typical high quality gyro is in the range of 200 to 300.

2.4 Summary

Despite the array of alternatives for the measurement of the rotation of a spacecraft, the Dynamically Tuned Gyro is still considered an attractive option because of its low cost per axis, its accuracy and reliability. Several small angle analyses of the behaviour of the Dynamically Tuned Gyro have been performed in the literature. While useful for the design of DTGs, these approximations are not always appropriate for the accurate determination of the behaviour of the DTG.

Chapter 3

Introduction to the Crossed Flexure Pivot

The concept of the Momentum Management System (MMS) is based on the design of the Dynamically Tuned Gyro (DTG). This was done to allow accurate positioning of the rotor to facilitate station keeping and rotational maneuvers of the spacecraft. The original prototypes of the DTG included bearings or 'pins and jewels' as the method of connecting the rotor to the gimbal and the gimbal to the shaft. The resulting gyro was found to be very unstable due to the dynamic inertia effect of the gimbal. The implementation of crossed flexure pivots¹, see Figure 3.1, allowed the DTG to operate properly and, to date, this joint is still used for this purpose. The crossed flexure pivot has proven a reliable method of allowing limited rotational motion about one axis while remaining rigid about the remaining axes.

Using thin strips of metal to support the moving parts of sensitive instruments has been in practice for many years. An example of the use of

¹Crossed flexure pivots are known by a variety of names, including cross-spring pivot and leaf-spring flexible joints.

Figure 3.1: Schematic of the crossed flexure pivot as used for the MMS. A second pair of pivots orthogonal to the first is not shown in this diagram for the sake of clarity.

this type of hinge is the torsional balance used in experimental physics. Invented independently by Michell in 1750, Coulomb in 1785 and Cavendish in 1798, it has been in constant use, especially for the measurement of small torques[41]. Originally the torsional members were wires or threads; however, in 1826, Ohm, realizing that a strip would be weaker in torsion than a wire for the same cross section, modified the device. The precision afforded by these joints was aptly demonstrated by Holbourn's (1936) use of the torsional balance to measure a torque of the magnitude 2×10^{-11} dyn-cm[42]². Flexural bearings rely on the stretching of atomic bonds during elastic motion, and, in the typical flexural bearing, there are millions of planes of atoms.

²This experiment involved the measurement of the angular momentum of circularly polarized light.

This provides an averaging effect allowing atomically smooth motion. It is for this reason that flexural bearings are used in scanning electron microscopes, allowing the tip of the microscope to scan the surface of a sample with subatomic resolution[41].

There are many different types of flexure pivots and these include: angle spring hinges and single, two and three strip pivots. The choice of which type of pivot to use is dependent on the task at hand. All flexure pivots have the advantages of no friction, low damping (as it is limited to material damping), the ability to withstand sudden changes in load which could unseat a knife edge³, lubrication is not necessary and no wear occurs.

Angle-spring hinges are used in situations where the axis of rotation can be placed to one side of the component, much like the hinges of a door. In their simplest form, the angle-spring hinge is compact and easily manufactured; however, there is a tendency for the axis of rotation to shift significantly[43]. It is possible to develop compensated hinges[43][44] which have very good performance, having less than a $5\text{ }\mu\text{m}$ shift of the axis of rotation with a $1/2$ radian twist. Compensated angle-spring hinges are more complex and less compact.

Even simpler in form are the single strip hinges. These pivots are made by machining a notch in a bar or rod, see Figure 3.2. This notch allows rotary motion about a single axis while remaining relatively stiff in the remaining axes. If two degrees of freedom are required then machining two notches at right angles to each other allows for a type of universal motion. These hinges are simple to manufacture but allow very limited motion without translation

³These joints are sometimes advertised as shock proof torsional suspensions[42].

Figure 3.2: Single strip hinge consisting of a notch in a bar.

of the axis of rotation. They are also susceptible to buckling under compressive loading. Eastman[45], Weinstein[46], Paros and Weisbord[47] have performed small angle deflection analyses of the single strip pivot including, in the case of Paros and Weisbord, calculating the stiffness of the pivot about the non-sensitive axes.

Three strip bearings, see Figure 3.3, are the most complicated of the flexure pivots and they have two important advantages. First, the centre of rotation remains constant for large angles of rotation. The possibility to design the bearing so it is self-compensating or unaffected by loads applied to the joint is the second advantage. An analysis of the three strip pivot has been performed by Weinstein[15].

The two strip pivot, or crossed flexure pivot, consists of two, flat strips that cross each other without touching, see Figure 3.1. These strips attach two rigid bodies and allow rotational motion of one with respect to the other. The crossed flexure pivot offers the advantages of a significant increase in the resistance to buckling and has greater stiffness in the non-sensitive axes than the single strip pivot. Finally, the motion of the centre of rotation is negligible for small angular deflections.

The angle-spring pivot, the three strip pivot and the crossed flexure pivot are all viable alternatives for the gimbal joint in the MMS. However, the

Figure 3.3: Schematic of a three strip bearing. The jagged lines represent springs whose stiffness can be selected such that loads acting on the pivot do not affect its stiffness. The three thin strips connect the inner and outer cylinders.

crossed flexure pivot, used in DTG's for decades, has been found to be a reliable and precise hinge and is straightforward to manufacture. For these reasons, it was chosen to be used in the gimbal joint for the MMS.

3.1 History of the Crossed Flexure Pivot

The crossed flexure pivot consists of two thin flexures that cross without touching. The point at which these strips cross can be taken as the centre of rotation for the pivot for small angles of deflection. This fact can be demonstrated by finding the location of the instantaneous centre of rotation for the crossed flexure pivot.

Figure 3.4 shows a drawing of a crossed flexure pivot where the top body is assumed fixed with respect to an inertial frame and the lower body is allowed

to rotate. The diagram shows both deflected and undeflected positions. As the lower body moves, the ends of strips '1' and '2' undergo translations from A_1 to A'_1 and A_2 to A'_2 , respectively.

The motion of the ends of the strips can be broken down into two components consisting of axial and transverse deflections, e_i and u_i , where i represents strips '1' and '2'. Note that a positive axial deflection has been defined as an elongation of the strip, hence in Figure 3.4 the axial deflection is negative. A simple schematic of the motion is shown in Figure 3.5. The triangle A_2OA_1 represents the strips of the pivot in its undeflected position where O is the point where the strips cross, L is the length of each strip and r represents the fraction of the strip between the crossing point, O , and the lower body, A_1A_2 . The angle between the strips is 2α . In this drawing, $A'_1A'_2$ represents the deflected position of the lower body. The lines $A_1A'_1$ and $A_2A'_2$ are linear approximations of the motion of the ends of the strips.

The instantaneous centre of rotation is found by projecting two lines perpendicular to the motion of the points A_1 and A_2 . Where these lines meet is the instantaneous centre of rotation and is represented by the point IC' in the diagram.

The vector $A_2A'_2$ can be expressed as

$$A_2A'_2 = \begin{pmatrix} u_2C_\alpha - e_2S_\alpha \\ -u_2S_\alpha - e_2C_\alpha \\ 0 \end{pmatrix}$$

where C_α and S_α represent $\cos \alpha$ and $\sin \alpha$, respectively. Similarly

$$A_1A'_1 = \begin{pmatrix} u_1C_\alpha + e_1S_\alpha \\ u_1S_\alpha - e_1C_\alpha \\ 0 \end{pmatrix}$$

Figure 3.4: Crossed flexure pivot shown in deflected and undeflected positions. One end of the pivot is assumed fixed while the other is allowed to move. The points A_1 and A_2 are at the ends of strips '1' and '2'. A'_1 and A'_2 represent the displaced positions of the two strips. The motion of the ends of the strips from their undeflected to deflected positions can be expressed in terms of a transverse, represented by u , and axial, represented by e , components. A positive axial deflection is defined as an elongation of the strip. The initial centre of rotation occurs at O and moves to O' after the pivot has deflected. This motion can be broken into horizontal, p , and vertical, q , components.

Figure 3.5: Schematic of the crossed flexure pivot in deflected and undeflected positions used to calculate the instantaneous centre of rotation of the pivot. Only the lower portion of the pivot is shown. A_1A_2 and $A'_1A'_2$ represent the lower rigid body in its undeflected and deflected positions, respectively. rL is the fraction of each strip below the point where the strips cross at O . The lines $A_1A'_1$ and $A_2A'_2$ are linear approximations of the motion of the ends of the strips and u_1 and e_1 are the transverse and axial components of this motion, respectively. The lower portions of strips '1' and '2' are given by OA_1 and OA_2 , respectively.

Perpendiculars to these vectors can be constructed by rotating the coordinate system by 90 degrees. In other words

$$(A_2A'_2)^\perp = (90)_z (A_2A'_2)$$

where $(A_2A'_2)^\perp$ denotes a perpendicular vector to $(A_2A'_2)$ and $(90)_z$ represents a rotation of 90 degrees about the z axis. Therefore

$$(A_2A'_2)^\perp = \begin{pmatrix} -u_2S_\alpha - e_2C'_\alpha \\ -u_2C'_\alpha + e_2S_\alpha \\ 0 \end{pmatrix}$$

$$(A_1A'_1)^\perp = \begin{pmatrix} u_1S_\alpha - e_1C'_\alpha \\ -u_1C'_\alpha - e_1S_\alpha \\ 0 \end{pmatrix}$$

Using these two relationships it is possible to construct equations to represent the lines A_2IC' and A_1IC' . With respect to a coordinate system fixed to A_2 the equations can be written as

$$y_1 = \frac{u_2C'_\alpha - e_2S_\alpha}{u_2S_\alpha + e_2C'_\alpha}x$$

and

$$y_2 = -\left(\frac{u_1C'_\alpha + e_1S_\alpha}{u_1S_\alpha - e_1C'_\alpha}\right)(x - 2rLS_\alpha)$$

where y_1 and y_2 represents A_1IC and A_2IC' , respectively. The point of intersection of these two curves will yield the location of the instantaneous centre of rotation. After simplification, the location of the instantaneous centre of rotation with respect to A_2 is

$$x = \frac{2rLS_\alpha(u_1C_\alpha + e_1S_\alpha)(u_2S_\alpha + e_2C_\alpha)}{(u_1S_\alpha - e_1C_\alpha)(u_2C_\alpha - e_2S_\alpha) + (u_1C_\alpha + e_1S_\alpha)(u_2S_\alpha + e_2C_\alpha)} \quad (3.1)$$

$$y = \frac{2rLS_\alpha(u_1C_\alpha + e_1S_\alpha)(u_2C_\alpha + e_2S_\alpha)}{(u_1S_\alpha - e_1C_\alpha)(u_2C_\alpha - e_2S_\alpha) + (u_1C_\alpha + e_1S_\alpha)(u_2S_\alpha + e_2C_\alpha)} \quad (3.2)$$

For small angles of rotation $e_1 \ll u_1$ and $e_2 \ll u_2$. This means that Equations 3.1 and 3.2 can be simplified to give

$$x = rL \sin \alpha$$

$$y = rL \cos \alpha$$

which also describes the position of point O , the spot where the two strips cross. Therefore, for small angles of rotation the centre of rotation can be assumed to be the point where the strips cross.

If the angle between the strips is zero ($2\alpha = 0$) then the strips are parallel and the location of the centre of rotation of the joint approaches infinity. In this special case, the two strip flexure is a parallel or rectilinear spring. This spring is the subject of several analyses including those by Jones[48][49], Jones and Young[50] and Plainevaux[51][52]. The analyses performed by these authors follow similar methods to those performed for the case when the angle between the pivots is non-zero.

The MMS requires rotary rather than rectilinear movement so the crossed axis pivots used in the MMS will have a positive angle between the strips ($2\alpha > 0$). There have been several attempts to model the crossed flexure pivot analytically and hence determine its behaviour. It should be noted that the majority of the work on the crossed axis pivot was performed during the 1940s and 1950s and that no significant research on the crossed axis pivot has been performed since.

3.1.1 Static Analysis of the Crossed Flexure Pivot

Weinstein[15] and Wittrick[13] have taken the approach of analyzing the strips independently and then combining the results through the use of force

balance equations. This approach is based on the assumption that the centre of rotation is nearly stationary during angular deflection. The angle at which the two strips cross has been kept as a variable in the solutions by both authors. Weinstein, and a later paper by Wittrick[14], also include the generality of moving the point where the strips cross; however, only Wittrick actually investigated the effect of changing the point of crossing. The paper by Wittrick also contains an analysis of the axial displacement of the strips due to bending.

The earlier work by Wittrick[13] describes the construction of a crossed flexure pivot with strip lengths of 6 inches and then mentions experiments performed to compare the stiffness of the actual pivot with that predicted by the theory. The results were in close agreement. It was found by both Weinstein and Wittrick that the stiffness of the pivot is sensitive to applied loads and if these loads are non-zero then the angle of rotation is not exactly proportional to the applied couple. It was also found that there is a limited range of direct loads where the pivot remains stable (i.e. has a positive stiffness). Outside this range, the pivot has a negative stiffness. Finally, Wittrick[14] found that if the strips crossed at a point $\frac{\sqrt{5}}{6}$ times the strip length from the midpoint of a strip towards the fixed end, the axis of rotation and pivot stiffness were the most stable.

Haringx[53] extended the work done by Wittrick to include large angle deflections where only a moment is applied. From this work it was found that the centre of rotation moved only slightly for small angles of deflection but this motion increased rapidly with angle and also that the stiffness of the hinge was linear with the applied moment for small angles and began

to deviate slightly as the moment and angle grew larger. In the small angle analysis by Haringx, it was shown that the stiffness of the pivot changed as horizontal or vertical loads were applied. The possibility of the pivot being unstable, or having a negative stiffness, was also found to exist under some loading conditions. The consequence of a pivot with a negative stiffness is that the pivot would actually want to rotate and resistance would have to be applied to stop this motion.

Scieszko and Mansour[54][55] took a slightly different approach. Forces acting in a direction parallel to a strip were assumed to cause an axial deflection but not affect the bending of the strips. This is opposite to the approach taken by the previous authors. Unfortunately, the results found from this study are in a form which makes it difficult to compare with other studies and experimental research.

All in all, analysis of the crossed flexure pivot has been limited to small angular deflections where movement of the centre of rotation can be neglected and the strips can then be analyzed separately. Only Wittrick[14] and Haringx[53] have attempted to determine the motion of the centre of rotation analytically but even then only after having separated the strips and analysing them separately.

3.1.2 Experimental Investigations of the Crossed Flexure Pivot

Realizing that the analytical solution for the crossed flexure pivot was difficult to determine for large angle deflections and for the motion of the centre of rotation, two researchers performed experiments on the crossed flexure pivot.

Young[16] focused attention on pivots with strip lengths over one inch while Nickols and Wunsch[17] focused on pivots with strip lengths under 3/4 of an inch. Both of these papers deal with pivots that have no external forces being applied. Comparing the stiffness of the pivot versus angle found by Haringx to that found by Nickols and Wunsch indicates only a 4 percent deviation and a similar comparison between the work of Nickols and Wunsch and that of Young finds they are within $7\frac{1}{2}\%$ for small angles. Slight discrepancies can be seen when comparing the measurements and predictions of the movement of the centre of rotation. The results of the two experimental researchers compare for rotations of up to 5 degrees. However, the predictions made by Haringx seem to be less than the measured values by both Young and Nickols and Wunsch.

The experiments show that as strip length decreases, stiffness and the amount the initial centre of rotation moves increase. The calculated pivot stiffness seems slightly larger than the measured pivot stiffness. This was attributed to the fact that the effective strip length is not the same as the actual strip length. Compensating for this by using the following empirical formula gives results which agree with the experimental data to within 1.5%[17]

$$K = 4 \frac{EI}{L + X}$$

where X is 0.089 cm or 0.035 inches, E is the Young's modulus, I is the area moment of inertia of a strip, L is the length of the strip and K is its stiffness.

Some of the conclusions reached in the experimental work by Young include:

- Stress in the strips can be reduced by crossing the pivots at their middle.

- The intersection angle between the pivots should be 90 degrees to produce a more rigid pivot as a whole. However angles as low as 30 degrees could be used if it is necessary to reduce pivot motion. This conclusion agrees with that by Haringx.
- Displacement of the initial axis of rotation can be neglected for deflection angles of up to 15 degrees (conditional on less than a 6 lb-in torque being applied to the pivot[17]).

Note, these results are only for cases where there is a pure torque and no forces acting on the pivot.

Finally, it should be noted that the literature survey includes only papers openly available to the scientific community. The crossed axis pivot is a popular joint and is manufactured by several companies (for example Lucas[56][41]). This implies the possibility of proprietary documents not being accessible to the public.

3.2 Manufacture of the Gimbal and Crossed Flexure Pivots

One benefit of using crossed flexure pivots in the MMS is that they can be incorporated directly into the design of the gimbal. The dimensions (and tolerances) of a DTG gimbal and pivots (see Table 3.1) require the manufacturing process be done with the utmost care.

Where possible the material is drilled and reamed; however, fine work, such as machining the strips of the pivot, is performed using traveling wire-electro discharge machining, TW-EDM or EDM. As seen in Table 3.1, the

Dimension	Value
Gimbal Outer Radius	9.0 mm
Gimbal Inner Radius	7.0 mm
Gimbal Height	7.0 mm
Strip Length	2.000 ± 0.100 mm
Strip Width	1.000 ± 0.100 mm
Strip Thickness	0.0500 ± 0.0025 mm

Table 3.1: Sample DTG gimbal and crossed flexure pivot dimensions.

pivot strips are very thin ($50.0 \mu\text{m}$) which means the thermally influenced layers must be minimized during machining in order to reduce the number of crack initiation sites in the strips and also ensure that the original properties of the material are maintained.

The thermally influenced layers or heat-affected zone (HAZ), see Figure 3.6, are a recast layer, consisting of small amounts of spherical electrode material that cooled too quickly to be washed away, a white zone, which has undergone heating to high temperatures and then quickly cooled causing brittleness and an annealed layer. The white zone also, typically, has a higher carbon content due to absorption of carbon while the strip is at elevated temperatures. The annealed layer is an area in which the material was heated but not above its melting temperature. This layer commonly has lower hardness than the bulk material. In addition, the residual stress distribution in this layer can affect material strength and fatigue properties[57][58].

In general, the depth of the HAZ is proportional to the amount of current used in the EDM process. Rough cuts using high amperage can cause HAZs more than 0.2 mm deep. Finishing operations, in which low current is used, normally have HAZs of less than $4 \mu\text{m}$. The thickness of the recast layer and the white zone is usually equal to the thickness of the annealed layer[57].

Figure 3.6: Enlarged view of the heat affected zone left by the EDM process.

There are many ways to reduce the thickness of the HAZ including: chemical polishing, electrochemical machining and vibratory finishing. The method used typically depends on the application. In the case of DTG's, attempts have been made to chemically polish the flexure strips without affecting the rest of the gimbal with little success and the entire assembly is usually submersed in the chemical polish. One recent improvement in the EDM process resulting in higher surface quality has come about by limiting the amount of stray voltage around the work piece[57], known as anti-electrolysis. Wire EDM creates a certain amount of stray current in the dielectric, or flushing fluid, and this current can cause surface degradation, such as rusting of iron based materials and micro cracking. When an anti-electrolysis process is used, the depth of the white and recast layers is minimized reducing the need for polishing the material after machining[57].

The gimbal consists of two concentric cylinders and the first step in manufacturing the pivots is to drill two pairs of holes, at 90 degrees to each other, into these cylinders to allow the wire used in the EDM process access (see Figure 3.7). A single strip is then machined out of the cylinders at the four locations, two at a time, as shown in Figures 3.8 and 3.9.

The inner cylinder is rotated 180 degrees with respect to the outer cylin-

Figure 3.7: Schematic of the two cylinders used for the manufacture of the gimbal. In this drawing, the holes allowing access for the wire EDM process have been drilled.

Figure 3.8: Schematic of the two cylinders used for the manufacture of the gimbal and crossed flexure pivots. One side of the strip has been machined.

Figure 3.9: Schematic of the two cylinders used for the manufacture of the gimbal. The eight flexure strips have been cut from the surrounding gimbal material.

der, resulting in the crossed strip configuration shown in Figure 3.10. The cylinders are welded together along their ends and one end of the pivot is freed from the cylinders by cutting vertical slots at the four pivot locations. Flanges are attached to these ends and will be used to connect the rotor to the gimbal and the gimbal to the shaft, see Figure 3.11.

It should be noted that the resulting strips are curved along their length. This curvature is typically ignored in DTG's since the pivots are only a small portion of the circumference of the cylinders. Should it be a problem, it is possible to flatten the cylinders where the pivots are located removing any curvature. Several gimbals were manufactured for the engineering prototype of the MMS, some with flat pivots and others with curved pivots, in order to determine the best option.

Figure 3.10: Schematic of the gimbal after the strips have been cut and the inner cylinder is rotated by 180 degrees with respect to the outer cylinder. The result is four sets of crossed strips.

Tolerances on the dimensions are usually quite strict because slight deviations can change the properties of the gyro. This will be discussed further in Chapter 5. It is always necessary to specify tolerances which are physically possible. In order to determine the best possible tolerances and to help choose between prospective machining companies, the pivots for the MMS prototype were ordered with the specification "Manufacture to the highest tolerance possible!"

Figure 3.11: Schematic of the gimbal of a DTG or the MMS. This drawing is drawn to scale using the dimensions listed in Table 3.1. Vertical cuts have been made from the original drill holes in order to free one end of the pivots from the gimbal ring. Flanges, connecting pivots on the same axis, have been added in order to attach the rotor (the outside ring) and the shaft (the inside flange).

3.3 Classical Analysis of the Crossed Flexure Pivot

3.3.1 Geometric Relationships of the Crossed Flexure Pivot

When examining crossed flexure pivots it is important to derive the relationships between the deflections of the two strips. The following geometric analysis was performed in order to validate those found in the literature and also develop additional relationships required in this research. In Figure 3.4 both deflected and undeflected positions of a crossed flexure pivot are shown. Figure 3.4 also contains an enlarged view of the strip end deflections and shows the two components of the deflections. As discussed earlier, the first component is a deflection perpendicular to the beams, depicted by u_1 and u_2 for strips '1' and '2', respectively. This lateral deflection is due to the bending of the thin strips. The second deflection is a 'shortening', 'e', parallel to each of the strips. This deflection is, in part, due to the bending of the beam and also due to axial compression (tension) because of axial loading.

The linear distance A_1A_2 is constant since the strips are attached to a rigid body. This fact can be used to find a relationship between the axial and transverse deflections of the strips and the angle of rotation of the rigid body. When the pivot is loaded as shown in Figure 3.12, it undergoes a rotation of θ as shown in Figure 3.4 and schematically in Figure 3.13.

The vertical deflections of the strips must be equivalent to the vertical projected height of $A'_1A'_2$. In this case, vertical implies a direction perpen-

dicular to A_1A_2 . The result is the following relationship

$$2rL \sin \alpha \sin \theta = (u_1 + u_2) \sin \alpha - (e_1 - e_2) \cos \alpha \quad (3.3)$$

where 2α is the angle between the undeflected strips, r is the position along the length at which the strips originally cross, θ is the angle of rotation of one rigid body with respect to the other, and u_i and e_i are the lateral and parallel deflections of the i th beam, respectively. The above equation can be rearranged to give

$$u_1 + u_2 - 2rL \sin \theta - (e_1 - e_2) \cot \alpha = 0 \quad (3.4)$$

Similar reasoning can be used in determining the horizontal component of $A'_1A'_2$, parallel to A_1A_2 , and the horizontal deflection of the two strips. The result is

$$2rL \sin \alpha \cos \theta = 2rL \sin \alpha + (u_1 - u_2) \cos \alpha + (e_1 + e_2) \sin \alpha \quad (3.5)$$

This can be rewritten in the form

$$u_1 - u_2 + (e_1 + e_2) \tan \alpha + 2rL \tan \alpha (1 - \cos \theta) = 0 \quad (3.6)$$

Using geometry, it can be seen that the slope of a strip at $x_i = L$ and the rotation of the rigid body are equivalent. From this fact, the following relationships can be derived

$$u_{1x} = \tan \theta \quad (3.7)$$

$$u_{2x} = \tan \theta \quad (3.8)$$

where u_{ix} is the derivative of u_i with respect to x_i . This relationship is valid for both large and small angle deflections. For small angle deflections $\tan \theta \rightarrow \theta$.

Figure 3.12: General loading of a crossed flexure pivot. V and H are the applied loads and M is the applied torque. P_1 and P_2 are axial reaction forces and F_1 and F_2 are transverse reaction loads. The subscripts indicate which strip the reaction loads are acting upon, specifically strip '1' or '2'.

Figure 3.13: Schematic of the deflection of the pivot strips. The triangle A_2OA_1 represents the lower portion of the crossed flexure pivot where the line A_2A_1 represents the rigid body which is allowed to rotate. The point O is the initial centre of rotation and coincides with the point where the strips cross. The line $A_2'A_1'$ represents the deflected position of the rigid body and the point O' represents the location of the initial centre of rotation after deflection. It should be noted that this point does not exist in a physical manner in the deflected position. The points A_1 and A_2 also represent the ends of strips '1' and '2', respectively, in their undeflected positions. A_1' and A_2' represent the deflected positions of the ends of the strips. u and e represent orthogonal components of the motion of the strips and are the transverse and axial deflections, respectively. Note that an elongation of a strip is considered positive as denoted by the coordinate systems γ_4 and γ_5 . Finally, θ represents the rotation of the rigid body. Note that A_2OA_1 and $A_2'O_1A_1'$ are similar triangles.

Assuming the rotation of the rigid body is small, then $\cos \theta \approx 1$ and $\sin \theta \approx \theta$ and Equations 3.4 and 3.6 can be simplified to the form

$$u_1 + u_2 = 2rL\theta \sin \alpha + (c_1 - c_2) \cot \alpha$$

$$u_1 - u_2 = -(c_1 + c_2) \tan \alpha$$

These two equations can be combined to give the following relationships

$$u_1 = rL\theta + (c_1 - c_2) \frac{\cot \alpha}{2} - (c_1 + c_2) \frac{\tan \alpha}{2} \quad (3.9)$$

and

$$u_2 = rL\theta + (c_1 - c_2) \frac{\cot \alpha}{2} + (c_1 + c_2) \frac{\tan \alpha}{2} \quad (3.10)$$

If it is assumed that, since c_1 is a function of the square of the angular rotation, c_1 is small at small angular deflections then Equations 3.9 and 3.10 simplify to become

$$u_1 = u_2 = rL\theta \quad (3.11)$$

Equation 3.11 indicates that the transverse deflections of the two strips are equivalent. This implies that the location of the centre of rotation remains constant. This result agrees with those found in the literature[14].

If the axial displacement of the ends of the strips is not neglected then an expression for the displacement of the initial centre of rotation can be found. Using the coordinate frame Υ_2 , fixed to the point A_2 , the position of the initial centre of rotation is given as

$$O = \begin{pmatrix} rL \sin \alpha \\ rL \cos \alpha \\ 0 \end{pmatrix} \quad (3.12)$$

The position of O' , in the coordinate frame Υ_2 , can be found by defining the position of O' in the coordinate frame Υ_3 and then moving the frame to

match Υ_2 . This can be accomplished by a rotation of $-\theta$ about the z axis and then a translation from A'_2 to A_2 . This is expressed mathematically as

$$O' = \begin{pmatrix} u_2 \cos \alpha - e_2 \sin \alpha \\ -u_2 \sin \alpha - e_2 \cos \alpha \\ 0 \end{pmatrix} + (-\theta)_z \begin{pmatrix} rL \sin \alpha \\ rL \cos \alpha \\ 0 \end{pmatrix}$$

Expanding the above and simplifying gives

$$O' = \begin{pmatrix} u_2 \cos \alpha - e_2 \sin \alpha + rL \sin \alpha \cos \theta - rL \cos \alpha \sin \theta \\ -u_2 \sin \alpha - e_2 \cos \alpha + rL \sin \alpha \sin \theta + rL \cos \alpha \cos \theta \\ 0 \end{pmatrix}$$

Now define the displacement of the initial centre of rotation as

$$\Delta O = O' - O = \begin{pmatrix} p \\ q \\ 0 \end{pmatrix}$$

Therefore, the following relationships have been derived

$$p = u_2 \cos \alpha - e_2 \sin \alpha + rL \sin \alpha (\cos \theta - 1) - rL \cos \alpha \sin \theta \quad (3.13)$$

$$q = -u_2 \sin \alpha - e_2 \cos \alpha + rL \sin \alpha \sin \theta + rL \cos \alpha (\cos \theta - 1) \quad (3.14)$$

Combining Equations 3.5 and 3.13 and Equations 3.3 and 3.14 give the following equations

$$p = \frac{1}{2} (u_1 + u_2) \cos \alpha + \frac{1}{2} (e_1 - e_2) \sin \alpha - rL \cos \alpha \sin \theta \quad (3.15)$$

$$q = \frac{1}{2} (u_1 - u_2) \sin \alpha - \frac{1}{2} (e_1 + e_2) \cos \alpha + rL \cos \alpha (\cos \theta - 1) \quad (3.16)$$

Finally, Equations 3.15 and 3.16 can be simplified further using Equations 3.4 and 3.6, respectively, to get

$$p = \frac{1}{2 \sin \alpha} (e_1 - e_2) \quad (3.17)$$

Figure 3.14: Deflection of a cantilever beam when transverse, F , and axial, P_c , loads are applied in addition to torque, M_O .

$$q = \frac{1}{2 \cos \alpha} [2rL (\cos \theta - 1) - (c_1 + c_2)] \quad (3.18)$$

In this section several constraints on the motion of the ends of the strips have been derived. It should be noticed that Equations 3.4 and 3.6 are components of the constraint on the system that $A_1 A_2 = A'_1 A'_2$. Should it be necessary, it is possible to sum the square of the two equations to get a single constraint which eliminates the term involving the angular rotation of the rigid body. Similarly, Equations 3.7 and 3.8 give the system constraint that the slope of the two strips must be equal to the rotation of the body and each other. Lastly, it should be noted that the equations of constraint derived in this section and the relationships between p , q and θ with u_1 , u_2 , c_1 , c_2 are only valid at the ends of the strips attached to the moving body.

3.3.2 Static Analysis of a Single Strip

Based on the result from Equation 3.11 that the lateral deflections of the strips are the same, it is valid to analyze each strip separately and then

combine the results through use of a force and moment balance on the pivot as a whole. Taken individually, each strip acts like a cantilever beam and is analyzed accordingly. A sketch of a single strip under general loading conditions is shown in Figure 3.14. Assuming a uniform cross-section, the potential energy for a beam in bending is

$$\mathcal{V} = \frac{1}{2} \int_0^L EI u_{xx}^2 dx \quad (3.19)$$

where E is the Young's modulus of the beam, I is the area moment of inertia of the strip about its neutral axis, u is the transverse deflection and u_{xx} represents the second derivative of u with respect to x , x being the distance along the strip from the fixed edge.

Before proceeding, it is important to point out that several assumptions about the system have been made to arrive at this expression for the strain energy in the beam. This equation is for an Euler-Bernoulli beam where it has been assumed that [59][60]:

- Plane transverse cross-sections of the beam remain plane during bending.
- Shear has been neglected (i.e. there is no lateral stresses in the beam).
- The radius of curvature of the beam during bending is large compared to the thickness of the beam.
- The Modulus of Elasticity of the material is the same in compression and tension.

The first assumption in this list is valid when the length of the beam is much greater than the thickness. This is true for the case of the strips in a crossed

flexure pivot because low bending stiffness is usually considered an asset. The second assumption is similar to the first and depends on the ratio of the length to thickness and width. In this case, the pivot has a large width when compared to the thickness; however the loads in the plane perpendicular to the plane in which the bending occurs are usually small. The last assumption is made to simplify the analysis of the problem and is based on the fact that an isotropic material is used and that these properties are valid whether the beam is in tension or compression.

The third assumption comes about from the requirement that the elastic limit of the material is not exceeded during bending. In general, if the following equation is obeyed then this assumption is valid[60]

$$\frac{R}{t} = \frac{E}{2\sigma_y}$$

where R is the radius of curvature, t is the thickness of the beam and σ_y is the yield stress of the material.

It is possible to include the forces and moment in Equation 3.19 using virtual work. The result is

$$\mathcal{V} = \frac{1}{2} \int_0^L (EIu_{xx}^2 - P_c u_x^2) dx - (Fu + M_B u_x)_{x=L} \quad (3.20)$$

where P_c is the axial compressive load, F is the transverse force on the strip and M_B is the moment acting on the end of the strip. The notation $(\cdot)_{x=L}$ refers to the fact that the terms within the brackets are evaluated at $x = L$.

Hamilton's Principle can then be used to determine the beam equations. If the forces acting on the system are conservative, then Hamilton's Principle states that the first variation of the action integral is zero, where the action

integral is the integral of the Lagrangian. Mathematically, the action integral is expressed as

$$\mathcal{I} = \int_{t_1}^{t_2} \mathcal{L} dt$$

and $\mathcal{L} = \mathcal{T} - \mathcal{V}$ is the Lagrangian, $\mathcal{T}$ and $\mathcal{V}$ are the kinetic and potential energy of the system and t_1 and t_2 are arbitrary points in time.

For a static case, the kinetic energy is zero and the action integral can be written as

$$\mathcal{I} = \int_{t_1}^{t_2} \left[\frac{1}{2} \int_0^L (EI u_{xx}^2 - P_c u_x^2) dx - (F u + M_B u_x)_{x=L} \right] dt \quad (3.21)$$

Taking the variation of the action integral yields

$$\begin{aligned} \delta \mathcal{I} = \int_{t_1}^{t_2} & \left[\int_0^L (EI u_{xx} \delta u_{xx} - P_c u_x \delta u_x) dx \right. \\ & \left. - (F \delta u + M_B \delta u_x)_{x=L} \right] dt = 0 \end{aligned} \quad (3.22)$$

Equation 3.22 can be simplified by using integration by parts. For example

$$\begin{aligned} \int_0^L EI u_{xx} \delta u_{xx} dx &= [EI u_{xx} \delta u_x - EI u_{xxx} \delta u]_0^L \\ &+ \int_0^L EI u_{xxxx} \delta u dx \end{aligned} \quad (3.23)$$

and

$$\int_0^L P_c u_x \delta u_x dx = [P_c u_x \delta u]_0^L - \int_0^L P_c u_{xx} \delta u dx \quad (3.24)$$

Using Equations 3.23 and 3.24, Equation 3.22 can be rewritten as

$$\begin{aligned} \delta \mathcal{I} = \int_{t_1}^{t_2} & \left[\int_0^L (EI u_{xxxx} + P_c u_{xx}) \delta u dx \right. \\ & + [EI u_{xx} \delta u_x - EI u_{xxx} \delta u - P_c u_x \delta u]_0^L \\ & \left. - [F \delta u + M_B \delta u_x]_{x=L} \right] dt = 0 \end{aligned} \quad (3.25)$$

In order for the previous expression to be true each term in the integral must equal zero, since the variations of δu and δu_x are independent of each other. This fact gives the following relationship from the double integral

$$EIu_{xxxx} + P_c u_{xx} = 0 \quad (3.26)$$

In addition to the previous relationship, the natural boundary conditions of the system are found to be

$$(EIu_{xx} + P_c u_x) \delta u = 0 \quad (3.27)$$

$$EIu_{xx} \delta u_x = 0 \quad (3.28)$$

when $x = 0$ and

$$(-EIu_{xx} - P_c u_x - F) \delta u = 0 \quad (3.29)$$

$$(EIu_{xx} - M_B) \delta u_x = 0 \quad (3.30)$$

when $x = L$. Equation 3.26 can be rewritten as

$$u_{xxxx} + N_c^2 u_{xx} = 0 \quad (3.31)$$

where

$$N_c^2 = \frac{P_c}{EI}$$

This differential equation can be solved to give

$$u = C_1 \sin N_c x + C_2 \cos N_c x + C_3 x + C_4 \quad (3.32)$$

One end of the beam has been assumed to be fixed. For this to be true, the requirement that the transverse displacement and the slope of the beam at

the fixed end be zero is necessary. This leads to the two forced boundary conditions $\delta u = \delta u_x = 0$ when $x = 0$. Therefore

$$u(0) = u_x(0) = 0 \quad (3.33)$$

It should also be noted that at $x = L$, $\delta u, \delta u_x \neq 0$. From $u(0) = 0$

$$C_4 = -C_2 \quad (3.34)$$

and from $u_x(0) = 0$

$$C_3 = -N_c C_1 \quad (3.35)$$

Therefore, Equation 3.32 can be rewritten as

$$u = C_1 (\sin N_c x - N_c x) + C_2 (\cos N_c x - 1) \quad (3.36)$$

The first, second and third derivatives of Equation 3.36 are

$$u_x = -N_c C_1 (\cos N_c x - 1) - N_c C_2 \sin N_c x \quad (3.37)$$

$$u_{xx} = -N_c^2 C_1 \sin N_c x - N_c^2 C_2 \cos N_c x \quad (3.38)$$

$$u_{xxx} = -N_c^3 C_1 \cos N_c x + N_c^3 C_2 \sin N_c x \quad (3.39)$$

It is possible to combine Equations 3.29, 3.37 and 3.39 to give

$$\begin{aligned} -N_c^3 C_1 \cos N_c L + N_c^3 C_2 \sin N_c L + N_c^3 C_1 (\cos N_c L - 1) \\ - N_c^3 C_2 \sin N_c L = -\frac{F}{EI} \end{aligned} \quad (3.40)$$

The above equation can be reduced to the form

$$C_1 = \frac{F}{EIN_c^3} \quad (3.41)$$

Substituting Equation 3.38 into Equation 3.30, replacing C_1 with the expression from Equation 3.41 and solving for C_2 gives

$$C_2 = -\frac{1}{N_c^2 EI \cos N_c L} \left(M_B + \frac{F}{N_c} \sin N_c L \right) \quad (3.42)$$

Now Equation 3.36 can be written as

$$u = \frac{F}{N_c^3 EI} (\sin N_c x - N_c x) + \frac{1}{N_c^2 EI \cos N_c L} \left(M_B + \frac{F}{N_c} \sin N_c L \right) (1 - \cos N_c x) \quad (3.43)$$

In general, the moment at any point along an Euler-Bernoulli beam can be expressed as[61]

$$EI u_{xx} = M(x) \quad (3.44)$$

or

$$-\frac{F}{N_c} \sin N_c x + \frac{\cos N_c x}{\cos N_c L} \left(M_B + \frac{F}{N_c} \sin N_c L \right) = M(x) \quad (3.45)$$

At $x = 0$, the equation above simplifies to

$$\frac{M_B}{\cos N_c L} + \frac{F}{N_c \cos N_c L} \sin N_c L = M(0) = M_A \quad (3.46)$$

This means that C_2 can be simplified to obtain

$$C_2 = -\frac{M_A}{N_c^2 EI} \quad (3.47)$$

Equation 3.43 can now be written as

$$u = \frac{M_A}{P_c} (1 - \cos N_c x) + \frac{F}{N_c P_c} (\sin N_c x - N_c x) \quad (3.48)$$

where $P_c = N_c^2 EI$.

When $x = L$, the deflection of the strip is δ and, for small angles, the slope is θ . Using these relationships with Equation 3.48, the following equations result

$$\delta = \frac{F}{N_c P_c} (\sin N_c L - N_c L) + \frac{M_A}{P_c} (1 - \cos N_c L) \quad (3.49)$$

$$\theta = \frac{F}{P_c} (\cos N_c L - 1) + \frac{N_c M_A}{P_c} \sin N_c L \quad (3.50)$$

At $x = L$, $u = \delta = rL\theta$ (from Equation 3.11 for small angles). Using this fact and Equations 3.49 and 3.50, it is possible to solve for F and M_A

$$F = \frac{P_c \theta (\cos N_c L + r N_c L \sin N_c L - 1)}{2 \cos N_c L + N_c L \sin N_c L - 2} \quad (3.51)$$

$$M_A = \frac{P_c \theta (r N_c L \cos N_c L - \sin N_c L + N_c L(1 - r))}{N_c (2 \cos N_c L + N_c L \sin N_c L - 2)} \quad (3.52)$$

which can be simplified to give

$$F = P_c \theta \left(\frac{r N_c L \cot \frac{1}{2} N_c L - 1}{2 - N_c L \cot \frac{1}{2} N_c L} \right) \quad (3.53)$$

$$M_A = EI N_c \theta \left(\frac{r N_c L - \frac{1}{2} N_c L}{2 - N_c L \cot \frac{1}{2} N_c L} + \frac{1}{2} \cot \frac{1}{2} N_c L \right) \quad (3.54)$$

In general, the moment at the centre of rotation is

$$M_O = M_A - FL(1 - r) \quad (3.55)$$

Substituting Equations 3.51 and 3.52 into Equation 3.55, non-dimensionalizing and simplifying gives

$$\frac{M_O L}{\theta EI} = N_c L \left(\frac{\frac{1}{2} N_c L - r N_c^2 L^2 (1 - r) \cot \frac{1}{2} N_c L}{2 - N_c L \cot \frac{1}{2} N_c L} + \frac{1}{2} \cot \frac{1}{2} N_c L \right) \quad (3.56)$$

Up to this point only a compressive force has been considered. Setting $P_t = -P_c$, the equations for the case where the axial load is tensile can be

calculated. This can be done by setting $N_c = iN_t$ and using the following relationships

$$\sin N_c = i \sinh N_t$$

$$\cos N_c = \cosh N_t$$

Equations 3.53, 3.54 and 3.56 now become

$$F = P_t \theta \left(\frac{1 - r N_t L \coth \frac{1}{2} N_t L}{2 - N_t L \coth \frac{1}{2} N_t L} \right) \quad (3.57)$$

$$M_A = EI N_t \theta \left(\frac{r N_t L - \frac{1}{2} N_t L}{N_t L \coth \frac{1}{2} N_t L - 2} + \frac{1}{2} \coth \frac{1}{2} N_t L \right) \quad (3.58)$$

$$\begin{aligned} \frac{M_O L}{\theta EI} = N_t L \left(\frac{\frac{1}{2} N_t L - r N_t^2 L^2 (1 - r) \coth \frac{1}{2} N_t L}{N_t L \coth \frac{1}{2} N_t L - 2} \right. \\ \left. + \frac{1}{2} \coth \frac{1}{2} N_t L \right) \end{aligned} \quad (3.59)$$

where P_t is a tensile force.

For the special case where $r = \frac{1}{2}$, Equations 3.56 and 3.59 simplify to give

$$\frac{M_O L}{\theta EI} = \beta \quad (3.60)$$

$$\frac{M_O L}{\theta EI} = \gamma \quad (3.61)$$

where

$$\text{When } P_c \text{ is compressive, } \beta = \frac{N_c L}{2} \left(\cot \frac{1}{2} N_c L + \frac{1}{2} N_c L \right)$$

$$\text{When } P_t \text{ is tensile, } \gamma = \frac{N_t L}{2} \left(\coth \frac{1}{2} N_t L - \frac{1}{2} N_t L \right)$$

The stiffness of the pivot can be found by summing the moments about the pivot centre

$$\frac{KL}{EI} = (\beta_1 \text{ or } \gamma_1) + (\beta_2 \text{ or } \gamma_2) \quad (3.62)$$

where K is the pivot stiffness and the subscript refers to the individual strips. It should be noted at this point that as $N_i \rightarrow 0$, $\frac{KL}{EI} \rightarrow 2$. This is the nondimensionalized stiffness of the pivot when there are no applied loads.

Examining Figure 3.12, it is possible to sum the forces in the horizontal direction to obtain

$$H = (F_1 + F_2) \cos \alpha - (P_1 - P_2) \sin \alpha \quad (3.63)$$

Similarly, summing the forces in the vertical direction gives

$$V = (F_1 - F_2) \sin \alpha + (P_1 + P_2) \cos \alpha \quad (3.64)$$

If the strips cross at their centre, that is, $r = \frac{1}{2}$, then Equations 3.53 and 3.57 simplify to give

$$F = -\frac{1}{2} P_c \theta \quad (3.65)$$

$$F = \frac{1}{2} P_t \theta \quad (3.66)$$

Solving for P_1 and P_2 using Equation 3.66 in conjunction with Equations 3.63 and 3.64 gives

$$P_1 = \frac{V(2 \sin \alpha + \theta \cos \alpha) - H(2 \cos \alpha - \theta \sin \alpha)}{(\theta^2 + 4) \cos \alpha \sin \alpha} \quad (3.67)$$

$$P_2 = \frac{V(2 \sin \alpha - \theta \cos \alpha) + H(2 \cos \alpha + \theta \sin \alpha)}{(\theta^2 + 4) \cos \alpha \sin \alpha} \quad (3.68)$$

Typically, θ is small and can be neglected when compared to the other terms. If this assumption is made, then Equations 3.67 and 3.68 simplify to give

$$P_1 = \frac{1}{2} (V \sec \alpha - H \csc \alpha) \quad (3.69)$$

$$P_2 = \frac{1}{2} (V \sec \alpha + H \csc \alpha) \quad (3.70)$$

Figure 3.15: Variation of normalized angular stiffness with respect to normalized horizontal and vertical loads. Negative values of V imply a compressive force on the pivot while positive values depict a tensile force.

It can be shown that Equations 3.69 and 3.70 are also valid when the axial force is compressive. From these equations it can be seen that it is possible to have a combination of loads which causes one strip to be in tension while the other is in compression. Determining the relationship between stiffness and the applied loads is now possible. Figure 3.15 gives this relationship. Notice that the magnitude of the forces on the pivot affects its stiffness significantly. It is also apparent that there is only a small range of normalized forces where the stiffness of the pivot is positive.

3.3.3 Maximum Bending Moment and Buckling Load

Having already calculated the loads on the pivots, the maximum stress in the strips can now be calculated. Recall from Equation 3.44

$$EIu_{xx} = M$$

Taking the second derivative of Equation 3.48 with respect to x , Equation 3.44 becomes

$$M = EI N_c \left(\frac{N_c M_A}{P_c} \cos N_c x - \frac{F}{P_c} \sin N_c x \right) \quad (3.71)$$

Substituting Equations 3.53 and 3.54 into 3.71 and simplifying gives

$$M = EI \theta N_c \left[\left(\frac{r N_c L - \frac{1}{2} N_c L}{2 - N_c L \cot \frac{1}{2} N_c L} + \frac{1}{2} \cot \frac{1}{2} N_c L \right) \cos N_c x - \left(\frac{r N_c L \cot \frac{1}{2} N_c L - 1}{2 - N_c L \cot \frac{1}{2} N_c L} \right) \sin N_c x \right] \quad (3.72)$$

For the special case where the strips cross at the middle ($r = \frac{1}{2}$), Equation 3.72 can be simplified further to give

$$M = \frac{EI \theta N_c}{2} \frac{\cos N_c (x - \frac{1}{2} L)}{\sin \frac{1}{2} N_c L} \quad (3.73)$$

This is a maximum at the centre of the strips, hence at $x = \frac{1}{2} L$

$$M = \frac{EI \theta N_c}{2} \csc \frac{1}{2} N_c L \quad (3.74)$$

A similar analysis can be performed to calculate the maximum bending moment for a tensile axial load. For the case where $r = \frac{1}{2}$, the maximum moment occurs at the ends of the strips and has a magnitude of

$$M = \frac{EI N_t \theta}{2} \coth \frac{1}{2} N_t L \quad (3.75)$$

From Equation 3.74 it can be seen that as

$$\frac{1}{2}N_c L \rightarrow \pi$$

the maximum bending moment approaches infinity and buckling occurs. It is possible to calculate the buckling load for the pivot as follows

$$\frac{1}{2}N_c L = \pi$$

which can be rewritten as

$$\frac{1}{4} \frac{P_c L^2}{EI} = \pi^2$$

Solving for P_c gives

$$P_{cr} = \frac{4\pi^2 EI}{L^2} \quad (3.76)$$

where P_{cr} is the critical buckling load for the pivot.

It has been suggested by Young[16] that it is possible to use the equations for lateral buckling found in the book “Theory of Elastic Stability” by Timoshenko[62] (or in any good mechanics text) but, according to Young, using these formulas would be definitely overly conservative as the strips of the pivot tend to exert restraining forces on each other which increases the buckling load. This restraining action seems to increase as the deflection angle increases[16] and hence leads to even greater stability. In fact, the buckling load for the crossed flexure pivot can be calculated by using the equations for a beam with both ends held fixed (known as a fixed-fixed beam)[15][14][53] resulting in a possible axial load 16 times greater than that which would be allowed for a simple cantilevered beam (fixed-free). This increase in the resistance to buckling is one of the advantages of the crossed flexure pivot over the single strip pivot.

Chapter 4

Static Analysis of the Crossed Flexure Pivot

In the existing literature the crossed flexure pivot has been analyzed assuming that the initial centre of rotation does not move, as indicated in Chapter 3. In other words, the axial deflection of the ends of the strips was ignored and it followed that the transverse deflection of each strip would be equal. It has been shown through the experiments of Young[16] and Nickols and Wunsch[17], that the initial centre of rotation does move. It has been shown in Chapter 3, Section 3.3.1 that the motion of the initial centre of rotation is related to the axial deflection of the strips. Therefore, in order for the analysis to be complete, axial deflection should be taken into account.

An excellent method of including the axial effects in the analysis of the crossed flexure pivot, without making too many assumptions, is to use Hamilton's principle of least action. The potential energy of the system can be formulated, and then, by including the constraint equations and relationships derived in the previous chapter, equations that describe the static behaviour of the pivot can be formulated.

In this analysis, only bending and motion in the plane perpendicular to its sensitive axis will be considered (see Figure 3.4). In other words, only a two-dimensional analysis of the pivot will be performed.

In general, the crossed flexure pivot used in the MMS or DFG gimbals experiences forces acting in each of the three orthogonal directions. The moments acting on the pivot are dominated by one acting about the sensitive axis of the pivot. Due to its width, forces acting on the pivot act as distributed loads and cause slight torques about the remaining two axes; however, these are small in comparison to the dominant moment and can be neglected. In general, the two rigid bodies that are connected to the pivot restrict the loads on the pivot to one moment and two in-plane loads and one out-of-plane load. Since superposition of the out-of-plane and in-plane displacements is valid in a case such as this, the out-of-plane load will not affect the performance of the pivot as a torsional spring and will be ignored in the analysis. It will be assumed that the pivot strips behave like beams and not plates. Specifically, the strips will be assumed to behave as Euler-Bernoulli beams, using the same reasoning as presented in the previous chapter.

Finally, since this study is for a static case only, determining the motion of one rigid body with respect to the other is all that is necessary. One body will be fixed in an inertial frame while the second is allowed to translate and rotate.

4.1 Development of the Static Equations

Hamilton's Principle states that the variation of the action integral is zero where the action integral is defined as

$$\mathcal{I} = \int_{t_1}^{t_2} \mathcal{L} dt$$

and $\mathcal{L}$ is known as the Lagrangian and is given by $\mathcal{L} = \mathcal{T} - \mathcal{V}$, $\mathcal{T}$ and $\mathcal{V}$ are the kinetic and potential energy of the system and t_1 and t_2 are arbitrary points in time. For the static case, the kinetic energy is zero and it is only necessary to express the potential energy of the system. The strain energy of each strip in the crossed flexure pivot is

$$\mathcal{SE} = \int_0^L (EI u_{ixx}^2 + EA v_{ix}^2) dx$$

where the indice i represents strips '1' or '2', E is the Young's Modulus of the material, I and A are the area moment of inertia and the cross-sectional area of the strip, u_i is the transverse deflection of the strip due to bending, v_i is the axial deflection of the strip due to an axial load and u_{ixx} represents the second derivative of u_i with respect to x where x is the distance along the strip from the fixed end.

Since the forces acting on the pivot are conservative and not influenced by the position of the pivot they can be expressed in the form of potential energy. These forces act at the initial center of rotation of the pivot, which, ideally, is the centre of mass of the two rigid bodies (the rotor and gimbal, for example). The work done by these forces is a function of the displacement of the initial centre of rotation. Mathematically,

$$W = Hp + Vq + M\theta$$

where W represents the work done by the loads, H and V are the horizontal and vertical loads, p and q are the horizontal and vertical displacements of the initial centre of rotation, M is the moment acting on the pivot and θ is the angle of rotation of one rigid body with respect to the other, see Figures 3.12 and 3.13.

Recall in Section 3.3.1, Equations 3.17 and 3.18 gave the relationships between p and q and the axial displacement of the strips. These were

$$p = \frac{1}{2 \sin \alpha} (\epsilon_1 - \epsilon_2)$$

$$q = \frac{1}{2 \cos \alpha} [2rL (\cos \theta - 1) - (\epsilon_1 + \epsilon_2)]$$

where ϵ_i represents the axial displacement of the strip and is given by

$$\epsilon_i = v_i - \frac{1}{2} \int_0^L u_{ix}^2 dx$$

The first term in the above equation represents the elongation of the strip due to the axial load. The second term is the fore-shortening of the strip due to bending.

Strictly speaking, the displacement of the strip due to bending is found using [63]

$$L = \int_0^{L_0} \sqrt{1 + u_{ix}^2} dx$$

where L_0 is the length of the 'shortened' beam. Assuming that u_{ix} is small, this can be expanded using a Taylor's series expansion. Truncating after the second term yields

$$L = L_0 + \frac{1}{2} \int_0^{L_0} u_{ix}^2 dx$$

Therefore, the change in length is given by

$$L - L_0 = \Delta = \frac{1}{2} \int_0^{L_0} u_{ix}^2 dx$$

Assuming that $L \approx L_0$ for small deflections, in accordance to linear beam theory, the axial shortening of the strip due to bending is expressed as

$$\Delta = \frac{1}{2} \int_0^L u_{1\tau}^2 d\tau$$

which is the second term in c_1 .

Including the strain energy from both strips and the work energy due to the loads, the potential energy of the system is

$$\begin{aligned} \mathcal{V} = & \frac{1}{2} \int_0^L (EI u_{1\tau\tau}^2 + EA v_{1\tau}^2) d\tau + \frac{1}{2} \int_0^L (EI u_{2\sigma\sigma}^2 + EA v_{2\sigma}^2) d\sigma \\ & - \left\{ \frac{1}{2} H \csc \alpha (v_1 + v_2) \right. \\ & \quad \left. + \frac{1}{2} V \sec \alpha (2rL(\cos \theta - 1) - (v_1 + v_2)) + M\theta \right\}_{\tau, \sigma = L} \\ & - \frac{1}{4} (V \sec \alpha - H \csc \alpha) \int_0^L u_{1\tau}^2 d\tau \\ & - \frac{1}{4} (V \sec \alpha + H \csc \alpha) \int_0^L u_{2\sigma}^2 d\sigma \end{aligned} \quad (4.1)$$

where τ and σ are variables used for integration and represent the distance along strips '1' and '2', respectively.

The fact that the two strips are connected to a rigid body has not been taken into account in this equation. To take this into account, it is necessary to include the relationships between the displacements and the slopes of the ends of the two strips. The distance between the ends of the strips is constant since the strips are connected to a rigid body. This relationship is given by Equations 3.4 and 3.6 which are perpendicular components of this constraint. The slopes of the two strips are the same and equal to the angle of rotation of the body. This relationship is given in Equations 3.7 and 3.8. The constraints are then

$$u_1 + u_2 - 2rL \sin \theta - (e_1 - e_2) \cot \alpha = 0 \quad (4.2)$$

$$u_1 - u_2 + (c_1 + c_2) \tan \alpha + 2rL \tan \alpha (1 - \cos \theta) = 0 \quad (4.3)$$

$$u_{1x} = \tan \theta \quad (4.4)$$

$$u_{2x} = \tan \theta \quad (4.5)$$

At this point seven coordinates have been defined in order to describe the behaviour of the crossed flexure pivot. Of these coordinates, p , q and θ are not independent and can be related to the remaining coordinates using Equations 3.17 and 3.18 and one of Equations 4.4 or 4.5. Therefore there are four independent coordinates (u_1 , u_2 , v_1 and v_2) and two constraints. Note that equations 4.2 and 4.3 are components of one constraint.

The degrees of freedom of the system can be written as the difference between the number of independent coordinates and the number of constraints. Therefore the current system has two degrees of freedom. At first glance it seems that there should be three degrees of freedom, two translations and a rotation; however, due to the geometry of the problem, the rotation is actually a function of the translation and is not an independent motion.

Equations 4.2 to 4.5 can be included in the action integral, so that the system is properly represented, through the use of Lagrange multipliers. After simplification, the action integral is given as

$$\begin{aligned} \mathcal{I} = & \int_{t_1}^{t_2} \left[\frac{1}{2} \int_0^L \left(EI u_{1\tau\tau}^2 + EA v_{1\tau}^2 \right. \right. \\ & \left. \left. - \frac{1}{2} (V \sec \alpha - H \csc \alpha + 2\lambda_1 \cot \alpha - 2\lambda_2 \tan \alpha) u_{1\tau}^2 \right) d\tau \right. \\ & + \frac{1}{2} \int_0^L \left(EI u_{2\sigma\sigma}^2 + EA v_{2\sigma}^2 \right. \\ & \left. \left. - \frac{1}{2} (V \sec \alpha + H \csc \alpha - 2\lambda_1 \cot \alpha - 2\lambda_2 \tan \alpha) u_{2\sigma}^2 \right) d\sigma \right. \\ & \left. - \left\{ \frac{1}{2} (H \csc \alpha - V \sec \alpha - 2\lambda_1 \cot \alpha + 2\lambda_2 \tan \alpha) v_1 \right. \right. \end{aligned}$$

$$\begin{aligned}
 & - \frac{1}{2} (H \csc \alpha + V \sec \alpha - 2\lambda_1 \cot \alpha - 2\lambda_2 \tan \alpha) v_2 \\
 & + (\lambda_1 + \lambda_2) u_1 + (\lambda_1 - \lambda_2) u_2 + \lambda_3 u_{1\tau} + \lambda_4 u_{2\sigma} \\
 & + M\theta + (VrL \sec \alpha - 2rL\lambda_2 \tan \alpha) (\cos \theta - 1) \\
 & - (\lambda_3 + \lambda_4) \tan \theta - 2rL\lambda_1 \sin \theta \Big\}_{\tau, \sigma = L} dt \quad (4.6)
 \end{aligned}$$

Taking the variation of the action integral gives the following result

$$\begin{aligned}
 \delta \mathcal{I} = & \int_{t_1}^{t_2} \left[\int_0^L (EI u_{1\tau\tau} \delta u_{1\tau\tau} + EA v_{1\tau} \delta v_{1\tau} \right. \\
 & \left. - \frac{1}{2} (V \sec \alpha - H \csc \alpha + 2\lambda_1 \cot \alpha - 2\lambda_2 \tan \alpha) u_{1\tau} \delta u_{1\tau} \right) d\tau \\
 & + \int_0^L (EI u_{2\sigma\sigma} \delta u_{2\sigma\sigma} + EA v_{2\sigma} \delta v_{2\sigma} \\
 & \left. - \frac{1}{2} (V \sec \alpha + H \csc \alpha - 2\lambda_1 \cot \alpha - 2\lambda_2 \tan \alpha) u_{2\sigma} \delta u_{2\sigma} \right) d\sigma \\
 & - \left\{ \frac{1}{2} (H \csc \alpha - V \sec \alpha - 2\lambda_1 \cot \alpha + 2\lambda_2 \tan \alpha) \delta v_1 \right. \\
 & \left. - \frac{1}{2} (H \csc \alpha + V \sec \alpha - 2\lambda_1 \cot \alpha - 2\lambda_2 \tan \alpha) \delta v_2 \right. \\
 & + (\lambda_1 + \lambda_2) \delta u_1 + (\lambda_1 - \lambda_2) \delta u_2 + \lambda_3 \delta u_{1\tau} + \lambda_4 \delta u_{2\sigma} \\
 & + (M - (VrL \sec \alpha - 2rL\lambda_2 \tan \alpha) \sin \theta \\
 & \left. - (\lambda_3 + \lambda_4) \sec^2 \theta - 2rL\lambda_1 \cos \theta) \delta \theta \right\}_{\tau, \sigma = L} \\
 & - (u_1 + u_2 - 2rL \sin \theta - (v_1 - v_2) \cot \alpha \\
 & \quad + \frac{\cot \alpha}{2} \int_0^L u_{1\tau}^2 d\tau - \frac{\cot \alpha}{2} \int_0^L u_{2\sigma}^2 d\sigma) \delta \lambda_1 \\
 & - (u_1 - u_2 + 2rL \tan \alpha (1 - \cos \theta) + (v_1 + v_2) \tan \alpha \\
 & \quad - \frac{\tan \alpha}{2} \int_0^L u_{1\tau}^2 d\tau - \frac{\tan \alpha}{2} \int_0^L u_{2\sigma}^2 d\sigma) \delta \lambda_2 \\
 & - (u_{1\tau} - \tan \theta) \delta \lambda_3 - (u_{2\sigma} - \tan \theta) \delta \lambda_4 \Big] dt = 0 \quad (4.7)
 \end{aligned}$$

It is possible to simplify Equation 4.7 using integration by parts. For

example

$$\begin{aligned} \int_{t_1}^{t_2} \int_0^L u_{1\tau\tau} \delta u_{1\tau\tau} d\tau dt &= \int_{t_1}^{t_2} (u_{1\tau\tau} \delta u_{1\tau} - u_{1\tau\tau\tau} \delta u_1)_{\tau=0}^L dt \\ &+ \int_{t_1}^{t_2} \int_0^L u_{1\tau\tau\tau\tau} \delta u_1 d\tau dt \end{aligned}$$

and

$$\int_{t_1}^{t_2} \int_0^L v_{1\tau} \delta v_{1\tau} d\tau dt = \int_{t_1}^{t_2} (v_{1\tau} \delta v_1)_{\tau=0}^L dt - \int_{t_1}^{t_2} \int_0^L v_{1\tau\tau} \delta v_1 d\tau dt$$

Taking advantage of the relationships above, Equation 4.7 can be rewritten to obtain

$$\begin{aligned} \delta \mathcal{I} &= \int_{t_1}^{t_2} \left[\int_0^L (EI u_{1\tau\tau\tau\tau} \delta u_1 - EA v_{1\tau\tau} \delta v_1 \right. \\ &\quad \left. - \frac{1}{2} (V \sec \alpha - H \csc \alpha + 2\lambda_1 \cot \alpha - 2\lambda_2 \tan \alpha) u_{1\tau\tau} \delta u_1) d\tau \right. \\ &\quad \left. + \int_0^L (EI u_{2\sigma\sigma\sigma\sigma} \delta u_2 - EA v_{2\sigma\sigma} \delta v_2 \right. \\ &\quad \left. - \frac{1}{2} (V \sec \alpha + H \csc \alpha - 2\lambda_1 \cot \alpha - 2\lambda_2 \tan \alpha) u_{2\sigma\sigma} \delta u_2) d\sigma \right. \\ &\quad \left. + \{ EI u_{1\tau\tau} \delta u_{1\tau} - EI u_{1\tau\tau\tau} \delta u_1 + EI u_{2\sigma\sigma} \delta u_{2\sigma} - EI u_{2\sigma\sigma\sigma} \delta u_2 \right. \\ &\quad \left. + EA v_{1\tau} \delta v_1 + EA v_{2\sigma} \delta v_2 \right. \\ &\quad \left. - \frac{1}{2} (V \sec \alpha - H \csc \alpha + 2\lambda_1 \cot \alpha - 2\lambda_2 \tan \alpha) u_{1\tau} \delta u_1 \right. \\ &\quad \left. - \frac{1}{2} (V \sec \alpha + H \csc \alpha \right. \\ &\quad \left. - 2\lambda_1 \cot \alpha - 2\lambda_2 \tan \alpha) u_{2\sigma} \delta u_2 \}_{\tau,\sigma=0}^{\tau,\sigma=L} \right. \\ &\quad \left. - \left\{ \frac{1}{2} (H \csc \alpha - v \sec \alpha - 2\lambda_1 \cot \alpha + 2\lambda_2 \tan \alpha) \delta v_1 \right. \right. \\ &\quad \left. - \frac{1}{2} (H \csc \alpha + V \sec \alpha - 2\lambda_1 \cot \alpha - 2\lambda_2 \tan \alpha) \delta v_2 \right. \\ &\quad \left. + (\lambda_1 + \lambda_2) \delta u_1 + (\lambda_1 - \lambda_2) \delta u_2 + \lambda_3 \delta u_{1\tau} + \lambda_4 \delta u_{2\sigma} \right. \\ &\quad \left. + (M - (VrL \sec \alpha - 2rL\lambda_2 \tan \alpha) \sin \theta \right. \end{aligned}$$

$$\begin{aligned}
 & -(\lambda_3 + \lambda_4) \sec^2 \theta - 2rL\lambda_1 \cos \theta) \delta \theta \\
 & - (u_1 + u_2 - 2rL \sin \theta - (v_1 - v_2) \cot \alpha \\
 & \quad + \frac{\cot \alpha}{2} \int_0^L u_{1\tau}^2 d\tau - \frac{\cot \alpha}{2} \int_0^L u_{2\sigma}^2 d\sigma) \delta \lambda_1 \\
 & - (u_1 - u_2 + 2rL \tan \alpha (1 - \cos \theta) + (v_1 + v_2) \tan \alpha \\
 & \quad - \frac{\tan \alpha}{2} \int_0^L u_{1\tau}^2 d\tau - \frac{\tan \alpha}{2} \int_0^L u_{2\sigma}^2 d\sigma) \delta \lambda_2 \\
 & - (u_{1\tau} - \tan \theta) \delta \lambda_3 - (u_{2\sigma} - \tan \theta) \delta \lambda_4 \} \Big|_{\tau, \sigma=0}^L dt = 0 \quad (4.8)
 \end{aligned}$$

For a non-trivial solution, δu_1 , δu_2 , δv_1 , δv_2 , $\delta \theta$, $\delta \lambda_1$, $\delta \lambda_2$, $\delta \lambda_3$ and $\delta \lambda_4$ are arbitrary for all of τ and σ . This means that the coefficients of each of the terms in Equation 4.8 must equal zero. Four differential equations describing the behaviour of the crossed flexure pivot are obtained from Equation 4.8

$$EIu_{1\tau\tau\tau\tau} + \frac{1}{2}(V \sec \alpha - H \csc \alpha + 2\lambda_1 \cot \alpha - 2\lambda_2 \tan \alpha) u_{1\tau\tau} = 0 \quad (4.9)$$

$$EIu_{2\sigma\sigma\sigma\sigma} + \frac{1}{2}(V \sec \alpha + H \csc \alpha - 2\lambda_1 \cot \alpha - 2\lambda_2 \tan \alpha) u_{2\sigma\sigma} = 0 \quad (4.10)$$

$$EA v_{1\tau\tau} = 0 \quad (4.11)$$

$$EA v_{2\sigma\sigma} = 0 \quad (4.12)$$

At the beginning of this analysis, it was stated that the strips were fixed at one end and governed by the constraint conditions at the other. This means that $\delta u_1 = \delta u_2 = \delta v_1 = \delta v_2 = \delta u_{1\tau} = \delta u_{2\sigma} = 0$ when $\tau, \sigma = 0$ and that the following forced boundary conditions exist

$$u_1(0) = u_{1\tau}(0) = v_1(0) = 0 \quad (4.13)$$

$$u_2(0) = u_{2\sigma}(0) = v_2(0) = 0 \quad (4.14)$$

For a non-trivial solution when $\tau = \sigma = L$, δu_1 , δu_2 , δv_1 , δv_2 , $\delta u_{1\tau}$, $\delta u_{2\sigma} \neq 0$. This gives the following natural boundary conditions

$$EIu_{1\tau\tau\tau} + \frac{1}{2}(V \sec \alpha - H \csc \alpha + 2\lambda_1 \cot \alpha - 2\lambda_2 \tan \alpha)u_{1\tau} + (\lambda_1 + \lambda_2) = 0 \quad (4.15)$$

$$EIu_{2\sigma\sigma\sigma} + \frac{1}{2}(V \sec \alpha + H \csc \alpha - 2\lambda_1 \cot \alpha - 2\lambda_2 \tan \alpha)u_{2\sigma} + (\lambda_1 - \lambda_2) = 0 \quad (4.16)$$

$$EIu_{1\tau\tau} - \lambda_3 = 0 \quad (4.17)$$

$$EIu_{2\sigma\sigma} - \lambda_4 = 0 \quad (4.18)$$

$$EAv_{1\tau} + \frac{1}{2}(V \sec \alpha - H \csc \alpha + 2\lambda_3 \cot \alpha - 2\lambda_4 \tan \alpha) = 0 \quad (4.19)$$

$$EAv_{2\tau} + \frac{1}{2}(V \sec \alpha + H \csc \alpha - 2\lambda_3 \cot \alpha - 2\lambda_4 \tan \alpha) = 0 \quad (4.20)$$

$$M = (VrL \sec \alpha - 2rL\lambda_2 \tan \alpha) \sin \theta - (\lambda_3 + \lambda_4) \sec^2 \theta - 2rL\lambda_1 \cos \theta = 0 \quad (4.21)$$

Finally, the four constraint equations are also realized since $\delta \lambda_1$, $\delta \lambda_2$, $\delta \lambda_3$, $\delta \lambda_4 \neq 0$.

Before continuing, it would be wise to compare the number of unknowns to the total number of equations. The four differential equations, Equations 4.9 to 4.12, result in a total of 12 integration constants. There are also the four Lagrange multipliers, λ_1 to λ_4 , and the dependent coordinate θ . Therefore, there are a total of 17 unknowns. At the same time there are 6 forced boundary conditions, Equations 4.13 and 4.14, 7 natural boundary equations, Equations 4.15 to 4.21, and four constraint equations, Equations 4.2 to 4.5. Therefore, the total number of equations is 17 which can be used to find the 17 unknowns.

It is now useful to define the following constants

$$N_1^2 = \frac{1}{2EI} (V \sec \alpha - H \csc \alpha + 2\lambda_1 \cot \alpha - 2\lambda_2 \tan \alpha)$$

$$N_2^2 = \frac{1}{2EI} (V \sec \alpha + H \csc \alpha - 2\lambda_1 \cot \alpha - 2\lambda_2 \tan \alpha)$$

$$f_1 = \frac{1}{EI} (\lambda_1 + \lambda_2)$$

$$f_2 = \frac{1}{EI} (\lambda_1 - \lambda_2)$$

Note that these are all functions of the unknown Lagrange multipliers, λ_1 and λ_2 . From Equations 4.17 and 4.18, it is possible to solve for two of the Lagrange multipliers which are

$$\lambda_3 = EI u_{1\tau\tau}$$

$$\lambda_4 = EI u_{2\sigma\sigma}$$

It is then possible to rewrite Equations 4.9 and 4.10 to obtain

$$u_{1\tau\tau\tau\tau} + N_1^2 u_{1\tau\tau} = 0 \quad (4.22)$$

$$u_{2\sigma\sigma\sigma\sigma} + N_2^2 u_{2\sigma\sigma} = 0 \quad (4.23)$$

The remaining natural boundary conditions, which exist when $\tau = \sigma = L$, are rewritten to obtain

$$u_{1\tau\tau\tau} + N_1^2 u_{1\tau} + f_1 = 0 \quad (4.24)$$

$$u_{2\sigma\sigma\sigma} + N_2^2 u_{2\sigma} + f_2 = 0 \quad (4.25)$$

$$EA v_{1\tau} + E I N_1^2 = 0 \quad (4.26)$$

$$EA v_{2\sigma} + E I N_2^2 = 0 \quad (4.27)$$

$$\begin{aligned} m = & (u_{1\tau\tau} + u_{2\sigma\sigma}) \sec^2 \theta - rL \cos \theta (f_1 + f_2) \\ & - rL \sin \theta (N_1^2 + N_2^2) = 0 \end{aligned} \quad (4.28)$$

where $m = \frac{M}{EI}$.

The differential equations, Equations 4.9 and 4.10 are now in a form which can be easily solved. Assuming that Equation 4.22 has the solution $e^{r\tau}$, it has the characteristic equation

$$r^4 + N_1^2 r^2 = 0$$

which has the following roots, assuming that N_1^2 is positive,

$$r = \pm jN_1$$

$$r = 0$$

$$r = 0$$

Therefore, Equation 4.22 has the solution

$$u_1 = a_1 \cos N_1 \tau + a_2 \sin N_1 \tau + a_3 \tau + a_4 \quad (4.29)$$

where a_1 , a_2 , a_3 and a_4 are constants of integration. Similarly, Equation 4.23 has the solution

$$u_2 = b_1 \cos N_2 \sigma + b_2 \sin N_2 \sigma + b_3 \sigma + b_4 \quad (4.30)$$

where b_1 , b_2 , b_3 and b_4 are constants of integration.

Equations 4.11 and 4.12 can be integrated using the same method to obtain

$$v_1 = c_1 \tau + c_2 \quad (4.31)$$

and

$$v_2 = d_1 \sigma + d_2 \quad (4.32)$$

where c_1 , c_2 , d_1 and d_2 are constants of integration.

Using the forced boundary conditions, Equations 4.13 and 4.14, it is possible to solve for six of the twelve integration constants as follows

$$a_1 = -a_4$$

$$b_1 = -b_4$$

$$a_2 = -N_1 a_3$$

$$b_2 = -N_2 b_3$$

$$c_2 = 0$$

$$d_2 = 0$$

Equations 4.29 to 4.32 can be rewritten as

$$u_1 = a_1 (\cos N_1 \tau - 1) + a_2 (\sin N_1 \tau - N_1 \tau) \quad (4.33)$$

$$u_2 = b_1 (\cos N_2 \sigma - 1) + b_2 (\sin N_2 \sigma - N_2 \sigma) \quad (4.34)$$

$$v_1 = c_1 \tau \quad (4.35)$$

$$v_2 = d_1 \sigma \quad (4.36)$$

It is now possible to take the first derivative of Equation 4.35 and use it in Equation 4.26 in order to solve for c_1 . This gives

$$c_1 = -\frac{I}{A} N_1^2$$

The elongation of strip '1' can be expressed as

$$v_1 = -\frac{I}{A} N_1^2 \tau \quad (4.37)$$

Similarly, taking the first derivative of Equation 4.36 with respect to σ and combining it with Equation 4.27 allows the constant d_1 to be found and thus the elongation of strip '2' is given by

$$v_2 = -\frac{I}{A} N_2^2 \sigma \quad (4.38)$$

It is now necessary to find the values of the four remaining integration constants. The first, second and third derivatives of Equation 4.33 with respect to τ are

$$\begin{aligned} u_{1\tau} &= -N_1 a_1 \sin N_1 \tau + N_1 a_2 (\cos N_1 \tau - 1) \\ u_{1\tau\tau} &= -N_1^2 a_1 \cos N_1 \tau - N_1^2 a_2 \sin N_1 \tau \\ u_{1\tau\tau\tau} &= N_1^3 a_1 \sin N_1 \tau - N_1^3 a_2 \cos N_1 \tau \end{aligned}$$

Substituting the relevant equations into Equation 4.24 gives the equation

$$N_1^3 a_1 \sin N_1 L - N_1^3 a_2 \cos N_1 L - N_1^3 a_1 \sin N_1 L + N_1^3 a_2 (\cos N_1 L - 1) + f_1 = 0$$

which means that

$$a_2 = \frac{f_1}{N_1^3}$$

Similarly, using Equation 4.25

$$b_2 = \frac{f_2}{N_2^3}$$

Equations 4.33 and 4.34 can now be written as

$$u_1 = a_1 (\cos N_1 \tau - 1) + \frac{f_1}{N_1^3} (\sin N_1 \tau - N_1 \tau) \quad (4.39)$$

$$u_2 = b_1 (\cos N_2 \sigma - 1) + \frac{f_2}{N_2^3} (\sin N_2 \sigma - N_2 \tau) \quad (4.40)$$

There are still two integration constants left to find and yet there is only one boundary condition, Equation 4.28, remaining which introduces a third unknown, θ . Therefore, it is necessary to use two of the constraint equations in addition to the remaining forced boundary condition in order to solve the problem. Equations 3.7 and 3.8 relate the slope of the ends of the strips to θ and will be used for this task. Substituting the first derivatives of Equations 4.39 and 4.40 into Equations 3.7 and 3.8 yields the equations

$$-N_1 a_1 \sin N_1 L + \frac{f_1}{N_1^2} (\cos N_1 L - 1) = \theta \quad (4.41)$$

$$-N_2 b_1 \sin N_2 L + \frac{f_2}{N_2^2} (\cos N_2 L - 1) = \theta \quad (4.42)$$

Substituting the second derivatives of Equations 4.39 and 4.40 into Equation 4.28 and linearizing with respect to θ gives the result

$$\begin{aligned} m &= -N_1^2 a_1 \cos N_1 L + \frac{f_1}{N_1} \sin N_1 L \\ &+ -N_2^2 b_1 \cos N_2 L + \frac{f_2}{N_2} \sin N_2 L \\ &- rL (f_1 + f_2) - rL\theta (N_1^2 + N_2^2) = 0 \end{aligned} \quad (4.43)$$

Using Equations 4.41 to 4.43, it is possible to solve for θ , a_1 and b_1 and obtain

$$\theta = \frac{m + \frac{f_1}{N_1} \tan \frac{1}{2} N_1 L + \frac{f_2}{N_2} \tan \frac{1}{2} N_2 L - rL (f_1 + f_2)}{N_1 \cot N_1 L + N_2 \cot N_2 L + rL (N_1^2 + N_2^2)} \quad (4.44)$$

$$\begin{aligned} a_1 &= -\frac{f_1}{N_1^3} \tan \frac{1}{2} N_1 L \\ &- \frac{m + \frac{f_1}{N_1} \tan \frac{1}{2} N_1 L + \frac{f_2}{N_2} \tan \frac{1}{2} N_2 L - rL (f_1 + f_2)}{N_1 \sin N_1 L (N_1 \cot N_1 L + N_2 \cot N_2 L + rL (N_1^2 + N_2^2))} \end{aligned} \quad (4.45)$$

$$b_1 = -\frac{f_2}{N_2^3} \tan \frac{1}{2} N_2 L$$

$$= \frac{m + \frac{f_1}{N_1} \tan \frac{1}{2} N_1 L + \frac{f_2}{N_2} \tan \frac{1}{2} N_2 L - rL(f_1 + f_2)}{N_2 \sin N_2 L (N_1 \cot N_1 L + N_2 \cot N_2 L + rL(N_1^2 + N_2^2))} \quad (4.46)$$

The values of the integration constants have been found in terms of λ_1 and λ_2 . The two Lagrange multipliers, λ_1 and λ_2 , still remain, however; the two remaining constraint equations can be used to find the values of these Lagrange multipliers. The problem lies in the fact that the equations are transcendental in nature making it difficult to complete the analytical solution of the differential equations. In order to determine the values of λ_1 and λ_2 , a non-linear root finding technique will have to be implemented after which, N_1 , N_2 and all previous expressions can be evaluated.

At this point, it is useful to examine the constants N_1^2 , N_2^2 , f_1 and f_2 and their relationship with the loads acting on the system. Recall that in Chapter 3 a simple cantilever beam was examined, see Figure 3.14, and the transverse deflection of this beam was expressed as

$$u = \frac{M_A}{P} (1 - \cos Qx) + \frac{F}{QP} (\sin Qx - Qx)$$

where $Q^2 = \frac{P}{EI}$, F is a transverse load and P a compressive load applied to the beam, M_A is the moment acting at the fixed end and x is the distance along the beam from the fixed end.

Comparison of the above equation and Equation 4.39, derived earlier, shows that

$$Q = N_1$$

and

$$\frac{F}{QP} = \frac{f_1}{N_1^3}$$

Earlier, N_1^2 and f_1 were defined as

$$N_1^2 = \frac{1}{2EI} (V \sec \alpha - H \csc \alpha + 2\lambda_1 \cot \alpha - 2\lambda_2 \tan \alpha)$$

$$f_1 = \frac{1}{EI} (\lambda_1 + \lambda_2)$$

Therefore the following relationships can be found

$$P = \frac{1}{2} (V \sec \alpha - H \csc \alpha + 2\lambda_1 \cot \alpha - 2\lambda_2 \tan \alpha) \quad (4.47)$$

$$F = \lambda_1 + \lambda_2$$

This implies that $N_1^2 L^2$ and $f_1 L^2$ represent the non-dimensionalized axial and transverse loads acting on each strip. The conclusion can now be drawn that the Lagrange multipliers, λ_1 and λ_2 , represent forces acting on the ends of the two strips and that they enforce the constraints placed on the system, specifically the constraint requiring the distance between the strips remain constant. In fact, it can be shown that the two multipliers act in directions perpendicular to the strips, see Figure 4.1.

It should also be noted that the directions of λ_1 and λ_2 on each strip are such that

$$(\lambda_1 \csc \alpha)_{\text{strip 1}} + (\lambda_1 \csc \alpha)_{\text{strip 2}} = 0$$

and

$$(\lambda_2 \sec \alpha)_{\text{strip 1}} + (\lambda_2 \sec \alpha)_{\text{strip 2}} = 0$$

This result is expected since λ_1 and λ_2 are due to the constraints and, therefore, should not add energy to the system, which they would have if the sum was not zero.

Figure 4.1: This figure shows a schematic of the crossed flexure pivot and the components of the axial and transverse forces acting on the two strips. V and H are the loads that have been applied to the pivot and α is the angle between each strip and the vertical. λ_1 and λ_2 are the Lagrange multipliers which are present in the relationships for the normalized axial and transverse loads, N_1^2 , N_2^2 , f_1 and f_2 .

A final comparison between the equations for a simple cantilever beam and the ones derived earlier in this section gives

$$a_1 = -\frac{M_A}{P}$$

Equation 4.45 gives the relationship between a_1 , the applied moment and the horizontal and vertical loads applied to the pivot. It is observed that coupling between the beams occurs in this torque, with the remainder of the coupling occurring in the two unsolved Lagrange multipliers. It is the moment acting on the ends of the strips that is most affected by the coupling of the two strips.

Until this point, it has been assumed that the axial loads are compressive in nature, that is, N_1^2 and $N_2^2 > 0$. If the axial load is tensile, N_1^2 or $N_2^2 < 0$, the static equations can be derived by realizing that

$$\sin(jN) = j \sinh N$$

$$\cos(jN) = \cosh N$$

The equations describing the deflection of the pivot are now given by

$$v_1 = \frac{I}{A} N_1^2 \tau \quad (4.48)$$

$$v_2 = \frac{I}{A} N_2^2 \sigma \quad (4.49)$$

$$u_1 = a_1 (\cosh N_1 \tau - 1) - \frac{f_1}{N_1^3} (\sinh N_1 \tau - N_1 \tau) \quad (4.50)$$

$$u_2 = b_1 (\cosh N_2 \sigma - 1) - \frac{f_2}{N_2^3} (\sinh N_2 \sigma - N_2 \sigma) \quad (4.51)$$

where

$$\theta = \frac{m + \frac{f_1}{N_1} \tanh \frac{1}{2} N_1 L + \frac{f_2}{N_2} \tanh \frac{1}{2} N_2 L - rL (f_1 + f_2)}{N_1 \coth N_1 L + N_2 \coth N_2 L - rL (N_1^2 + N_2^2)}$$

$$\begin{aligned}
 a_1 &= -\frac{f_1}{N_1^3} \tanh \frac{1}{2} N_1 L \\
 &+ \frac{m + \frac{f_1}{N_1} \tanh \frac{1}{2} N_1 L + \frac{f_2}{N_2} \tanh \frac{1}{2} N_2 L - rL(f_1 + f_2)}{N_1 \sinh N_1 L (N_1 \coth N_1 L + N_2 \coth N_2 L - rL(N_1^2 + N_2^2))} \\
 b_1 &= -\frac{f_2}{N_2^3} \tanh \frac{1}{2} N_2 L \\
 &+ \frac{m + \frac{f_1}{N_1} \tanh \frac{1}{2} N_1 L + \frac{f_2}{N_2} \tanh \frac{1}{2} N_2 L - rL(f_1 + f_2)}{N_2 \sinh N_2 L (N_1 \coth N_1 L + N_2 \coth N_2 L - rL(N_1^2 + N_2^2))}
 \end{aligned}$$

Due to the complexity of the solution, using L'Hopital's rule to find the solution when N_1^2 or $N_2^2 \rightarrow 0$ would be more difficult than simply re-deriving the static equations. If $N_1 = N_2 = 0$ then

$$v_1 = 0$$

$$v_2 = 0$$

and Equations 4.9 and 4.10 take the form

$$u_{1\tau\tau\tau\tau} = 0$$

$$u_{2\sigma\sigma\sigma\sigma} = 0$$

Similarly, the natural boundary conditions at $\tau = \sigma = L$ become

$$u_{1\tau\tau\tau} + f_1 = 0$$

$$u_{2\sigma\sigma\sigma} + f_2 = 0$$

$$m - u_{1\tau\tau} - u_{2\sigma\sigma} - rL(f_1 + f_2) = 0$$

when θ is assumed small. The characteristic equations of the differential equations have four roots at $r = 0$. The homogeneous solutions of the differential equations are then

$$u_1 = a_1 \tau^3 + a_2 \tau^2 + a_3 \tau + a_4$$

$$u_2 = b_1 \sigma^3 + b_2 \sigma^2 + b_3 \sigma + a_4$$

Using the forced boundary conditions at $\tau = \sigma = 0$ and the first two natural boundary conditions at $\tau = \sigma = L$, the above equations can be simplified to give

$$u_1 = a_2 \tau^2 + \frac{f_1}{6} \tau^3$$

$$u_2 = b_2 \sigma^2 + \frac{f_2}{6} \sigma^3$$

Finally, Equations 3.7 and 3.8, which relate the angle θ to the slope of the strips, and the remaining natural boundary condition can be used to find the values of the two remaining constants of integration and θ . These expressions are

$$a_2 = \frac{1}{4} \left(m + f_1 L \left(\frac{3}{2} - r \right) + f_2 L \left(\frac{1}{2} - r \right) \right)$$

$$b_2 = \frac{1}{4} \left(m + f_1 L \left(\frac{1}{2} - r \right) + f_2 L \left(\frac{3}{2} - r \right) \right)$$

$$\theta = \frac{1}{2} \left(mL + f_1 L^2 \left(\frac{1}{2} - r \right) + f_2 L^2 \left(\frac{1}{2} - r \right) \right)$$

It is noticed that, when the strips cross at their centre, the expression for θ reduces to give the normalized stiffness of the pivot as

$$\frac{mL}{\theta} = \frac{KL}{EI} = 2$$

This result is independent of the transverse load on the strips and only dependent on the fact that $r = \frac{1}{2}$ and the axial loads on both strips are zero. If $N_1^2 = 0$ and $N_2^2 > 0$ the static equations have the form

$$\begin{aligned} v_1 &= 0 \\ v_2 &= -\frac{I}{A} N_2^2 \sigma \\ u_1 &= a_2 r^2 - \frac{f_1}{6} r^3 \\ u_2 &= b_1 (\cos N_2 \sigma - 1) + \frac{f_2}{N_2^3} (\sin N_2 \sigma - N_2 \sigma) \\ a_2 &= \frac{f_1 L}{4} + \frac{\theta}{2L} \\ b_1 &= -\frac{f_2}{N_2^3} \tan \frac{1}{2} N_2 L - \frac{\theta}{N_2} \csc N_2 L \\ \theta &= \frac{mL + f_1 L^2 \left(\frac{1}{2} - r \right) - r f_2 L^2 + \frac{f_2 L}{N_2} \tan \frac{1}{2} N_2 L}{1 + N_2 L \cot N_2 L + r N_2^2 L^2} \end{aligned}$$

The solutions for other combinations of N_1^2 and N_2^2 can be derived using similar methods.

4.2 Comparison of Analytical Results with those in the Literature

In general, four equations have been derived that, together, describe the deflection of the crossed flexure pivot under a given set of loads. In the literature, papers by Wittrick[13][14] and Weinstein[15][46], for example, only two equations were found and this was because the elongation of the strips due to the axial load was ignored.

At this point, it is of interest to compare the equations from Section 4.1 with those found in the literature. This will help to determine the effect of assuming the axial deflection, e_1 , to be zero, as done in the literature.

Based on this assumption, the elongation of the strips, e_1 , must be small enough to be negligible when compared to the transverse deflections. Similarly, the shortening due to bending must also be small. A second assumption must also be made in order for the relationships in Section 4.1 to be comparable to those in the literature. It must be assumed that λ_1 and λ_2 are small when compared with the applied loads V and H . If this is done, it can be shown that the expressions for the axial loads in Section 4.1 are the same as those in the literature.

When the axial displacements, e_1 , are assumed to be negligible and θ is assumed small, the constraint equations, Equations 3.4 and 3.6, reduce to the form

$$u_1 - u_2 = rL\theta$$

as shown in Section 3.3.1. From Equations 3.7 and 3.8, the slope of each strip at $\tau = \sigma = L$ is

$$u_{1\tau} = u_{2\sigma} = \theta$$

Taking the above relationships and combining them with Equation 4.39, which gives the transverse deflection of strip '1', yields the expressions

$$rL\theta = a_1 (\cos N_1 L - 1) + \frac{f_1}{N_1^3} (\sin N_1 L - N_1 L)$$

and

$$\theta = -N_1 a_1 \sin N_1 L + \frac{f_1}{N_1^2} (\cos N_1 L - 1)$$

Solving for a_1 and f_1 in terms of θ gives

$$f_1 = -\theta N_1^2 \left(\frac{\cos N_1 L + r L N_1 \sin N_1 L - 1}{2 \cos N_1 L + N_1 L \sin N_1 L - 2} \right)$$

$$a_1 = \frac{\theta}{N_1} \left(\frac{N_1(1-r) - \sin N_1 L + r L N_1 \cos N_1 L}{2 - 2 \cos N_1 L - N_1 L \sin N_1 L} \right)$$

These two relationships are seen to be equivalent to those derived using the method found in the literature, Equations 3.51 and 3.52, when it is recalled that

$$N_1^2 = \frac{P_1}{EI}$$

$$f_1 = \frac{F_1}{EI}$$

$$a_1 = -\frac{M_{A_1}}{P_1}$$

as shown in the previous section.

If it is assumed that the pivots cross at their centre, that is, $r = \frac{1}{2}$, then

$$f_1 = -\frac{\theta N_1^2}{2}$$

and, similarly,

$$f_2 = -\frac{\theta N_2^2}{2}$$

A relationship between the applied moment, m , and the angular deflections of the pivot is given by Equation 4.44. Substituting the above values of f_1 and f_2 into this equation gives

$$\theta = \frac{m + \frac{N_1^2 \theta}{2 N_1 \sin N_1 L} (\cos N_1 L - 1)}{N_1 \cot N_1 L + N_2 \cot N_2 L + \frac{1}{2} L (N_1^2 + N_2^2)}$$

$$+ \frac{\frac{N_2^2 \theta}{2 N_2 \sin N_2 L} (\cos N_2 L - 1) + \frac{L \theta}{4} (N_1^2 + N_2^2)}{N_1 \cot N_1 L + N_2 \cot N_2 L + \frac{1}{2} L (N_1^2 + N_2^2)}$$

Solving for m/θ and simplifying yields

$$\frac{KL}{EI} - \frac{mL}{\theta} = \frac{1}{2}N_1L \cot \frac{1}{2}N_1L + \frac{1}{2}N_2L \cot \frac{1}{2}N_2L + \frac{L^2}{4}(N_1^2 + N_2^2)$$

This expression is identical to the one derived using the method in the literature when both strips are loaded in compression. Similar derivations can be performed in order to show that the expressions are identical, independent of the direction of the loads on the pivot.

It has now been shown that the equations derived in the previous section give the same results as those from the literature, when the same assumptions are made, that is, negligible axial deflection.

4.3 Discussion on the Accuracy of the Mathematical Model

Several assumptions about the nature of the crossed flexure pivot strips were made before the static analysis of the crossed flexure pivot was performed. While any assumption that was made has some effect on the accuracy of the theoretical model, two of them are worth discussing in more detail. The first of these was the use of the linearized Euler-Bernoulli equation. It is possible for the pivot to undergo large rotations where the value of u_x , the slope of the strip, is not small.

The second assumption is that the strips behave as beams and not plates. As pivots are designed to be rigid about two axes and have a low stiffness about the third, it is often necessary to increase the width of the strips quite significantly. In fact, it will be shown in Chapter 5 that the ratios of length to width to thickness for the strips of the MMS crossed flexure pivots

<i>Property</i>	<i>Value</i>
Length, L	60.0 mm
Width, w	25.4 mm
Thickness, t	1.5 mm
Young's Modulus, E	194.4 GPa
Poisson's Ratio, ν	0.33

Table 4.1: Strip dimensions used for the comparison between the non-linear beam theory, plate theory and beam theory. The dimensions are scaled up from the strip dimensions of the crossed flexure pivots used in the MMS engineering prototype.

is approximately 40:17:1. A similarly proportioned material would be the common $8' \times 4' \times \frac{1}{4}''$ sheet of plywood available at lumber yards. With these dimensions, an investigation into plate versus beam behaviour is warranted.

In depth analysis into the effects of these two assumptions is beyond the scope of this work and will be left for future researchers; however brief discussions on non-linear bending and plate analysis will be made in the following sections.

For this study, the simple case of a cantilever beam will be discussed. Ideally, these investigations would be done symbolically in order to maintain some generality; however, in this case, both the non-linear analysis and the plate analysis require iterative processes and so actual strip dimensions must be used. To tie these results in with later discussions scaled dimensions of the MMS pivot strips will be used; see Table 4.1. At the same time, the material properties of AerMet 100 will be used for the analysis.

In order to study the effects of these assumptions, comparison to the results from a linear Euler-Bernoulli analysis must be made. The linear version of the Euler-Bernoulli equation is based on the assumption that u_x

is small. The result is the well known equation

$$M = EIu_{xx}$$

In the analysis of the crossed flexure pivot, this expression was used to determine the strain energy in the strips due to bending.

Applying a transverse load, F , to the free end of the cantilever beam makes the moment acting at a distance x from the fixed end of the strip as

$$M = F(x - L) = EIu_{xx}$$

Integrating this equation once yields the slope of the strip at any point x and integrating a second time gives the transverse displacement. In other words,

$$u_x = \frac{Fx}{EI} \left(\frac{1}{2}x - L \right) \quad (4.52)$$

and

$$u = \frac{Fx^2}{2EI} \left(\frac{1}{3}x - L \right) \quad (4.53)$$

Assuming that the load of $F = 50N$ is applied to the end of the strip gives the deflection of the strip as

$$u = 2.592 \text{ mm} \quad (\text{downwards})$$

This analysis can now be compared to a plate analysis and a large angle analysis.

4.3.1 Large Angle Beam Analysis

In general, the Euler-Bernoulli equation is written as

$$\frac{M}{EI} = \frac{1}{R}$$

where R is the radius of curvature of the strip. It can be shown that the radius of curvature is equal to

$$R = \frac{(1 + u_x^2)^{\frac{3}{2}}}{u_{xx}}$$

and, therefore, the Euler-Bernoulli equation becomes

$$\frac{M}{EI} = \frac{u_{xx}}{(1 + u_x^2)^{\frac{3}{2}}}$$

Usually it can be assumed that the slope of the beam is small and u_x^2 can be neglected. The result is the well known form of the Euler-Bernoulli equation. However, when dealing with large angles, this term can not be neglected resulting in a much more complicated relationship.

As stated earlier, a cantilever beam, with a transverse load applied to the free end, is to be examined. Including this in the above equation gives

$$\frac{M}{EI} = \frac{F}{EI} (x - L_0) = \frac{u_{xx}}{(1 + u_x^2)^{\frac{3}{2}}}$$

where L_0 is the axial length of the strip in its deflected position. Integrating once yields

$$\frac{u_x}{\sqrt{1 + u_x^2}} = \frac{Fx}{2EI} (x - 2L_0) + C$$

where C is a constant of integration. It is known that $u_x(0) = 0$ and so it is possible to solve for the integration constant to get $C = 0$.

Letting $f = \frac{F}{EI}$ and rewriting the above in terms of u_x gives

$$u_x = \frac{fx(x - 2L_0)}{\sqrt{4 - (fx(x - 2L_0))^2}} \quad (4.54)$$

Integrating this gives the deflection of the strip at any point. The value of L_0 can be found by a 'trial and error procedure'[57] from the equation

$$L = \int_0^{L_0} \sqrt{1 + u_x^2} dx \quad (4.55)$$

<i>Load (N)</i>	<i>Transverse Deflection (mm)</i>			<i>Slope (degrees)</i>		
	<i>Linear</i>	<i>Large Angle</i>	<i>% Diff.</i>	<i>Linear</i>	<i>Large Angle</i>	<i>% Diff.</i>
50	2.592	2.587	0.2	3.71	3.71	0.0
100	5.185	5.145	0.8	7.43	7.42	0.1
150	7.777	7.647	1.7	11.14	11.13	0.1
200	10.369	10.068	3.0	14.85	14.82	0.2

Table 4.2: Differences between the deflections of a cantilever beam 60 mm long, 25.4 mm long and 1.5 mm thick as calculated by small and large angle Euler-Bernoulli equations. The Young's Modulus of AerMet 100 is 194.4 GPa.

In other words, a value for L_0 must be assumed. The length L can be calculated using Equation 4.55 and compared to the actual length. This procedure must be repeated until the proper value of L is obtained.

Using the values for the strips from Table 4.1 and Simpson's rule for the integration routine, the transverse and axial deflection of the cantilever strip was found, see Figures 4.2 and 4.3. Figure 4.2 shows the shortening of the beam due to the bending. At a applied load of 100 N, which corresponds roughly to a slope at the free end of 7 degrees, it is seen that the axial deflection is just over 0.265 mm or 0.45% of the undeflected length. The upper plot in Figure 4.3 shows a plot of transverse deflection of the strip as a function of the applied load. Table 4.2 contains some points from this curve and compares them to the values obtained using the linear Euler-Bernoulli equation, Equation 4.53. The lower plot in Figure 4.3 shows the relationship between the slope of the beam and the applied load. The negative sign indicates that the beam is convex upwards.

It can be seen that as the load increases the difference between the large and small angle deflections increases. In fact, it is seen that the large angle

Figure 4.2: Plot of the axial shortening due to bending as a function of the applied load of a cantilever beam. The initial strip length was 60 mm. The width of the beam is 25.4 mm and the thickness is 1.5 mm. The material used is AerMet 100 which has a Young's modulus of 194.4 GPa.

Figure 4.3: Response of a cantilever beam to a load applied to the free end using large angle Euler-Bernoulli equations. The top plot shows transverse deflection of the free end as a function of the applied load. The negative sign refers to the fact that the beam is bending downwards. The bottom plot shows the slope of the free end as a function of the applied load. the negative slope indicates that the beam is convex upwards. The undeflected axial length of the strip was 60 mm and the strip thickness and width were 1.5 mm and 25.4 mm, respectively. The Young's Modulus of the material was 194.4 GPa.

equations predict the beam to be stiffer. At 50 N, this error is approximately 0.2% and by the time the load equals 100 N the error has grown to 0.8%. Since the pivots in the MMS will deflect by a maximum of 7 degrees, the error due to using the linear Euler-Bernoulli equation can be neglected.

4.3.2 Plate and Wide Beam Analysis

The fact that the width of the strips is just over a factor of 2 less than the length and that the thickness of the strip is much less than the width means that a look at plate analysis and the errors incurred by not using it is warranted.

In general, plate analysis is more difficult than beam analysis and only a select group of boundary conditions have analytical solutions. Since an analytical solution for a fixed-free-free-free plate, or a cantilever plate, has not been found in the literature, the Finite Element Method (FEM) will be used to find the strip deflection.

The FEM package[93] employed for this analysis is based on cubic elements with 16 nodes per shell element, see Figure 4.4. Convergence studies were performed and a 2 by 4 mesh was found to be ideal. Then a uniformly distributed tip load was applied so that a consistent force vector was generated. The result is equivalent to the 50 N load applied in the beam analysis.

Even though the FEM software used has been well tested, a case study was performed to verify that it was working properly. In order to do this, Poisson's ratio was reduced to 0.001 and the problem was solved. It was not reduced to zero due to limitations in the FEM package which required a non zero Poisson's ratio. The end deflection of the strip was compared to

Figure 4.4: Shell mesh used with the FEM to solve for the defection of a cantilever plate with length 60 mm, width 25.4 mm and a thickness of 1.5 mm.

the results from the beam equation and they were found to be equivalent to within 0.04%.

Raising the Poisson's ratio back to 0.33, the problem of the bending of a cantilever plate was solved. The deflection was found to be symmetric about the centreline of the strip and varied from 2.473 mm at the edges to 2.482 mm on the centreline.

When comparing the mean of the deflection, 2.48 mm, to that calculated by the beam equation it is seen that the difference is 4.4%. Again, the beam analysis is predicting a lower strip stiffness.

This discrepancy is due to the fact that this is a thin cross-section in which distortion must be considered¹. As the strip bends, the convex side of the neutral surface² is subject to a contraction governed by Poisson's ratio. Conversely, the concave side of the beam undergoes a dilation[60].

¹For a more complete explanation see the correspondence by Timoshenko[60].

²The neutral surface is the locus of the neutral axes along the length of the beam[61].

Figure 4.5: Schematic showing the distortion of the cross-section as predicted by Euler's equation. N.A. stands for neutral axis.

The neutral axis does not remain straight either and its curvature is given by [62] ν/R ; see Figure 4.5. When the width of the strip is similar to the thickness and ν/R is a small quantity, this distortion can be ignored and the bending of the neutral axis doesn't interfere with the assumption that cross-sections remain plane.

When the width of the strip is large, the distortion predicted by the theory can no longer be ignored. However, experiments, and plate theory, show that this distortion does not occur. In fact, in the bending of thin wide strips, the cross-section remains basically the same, with the distortion being slightly more pronounced at the edges.

The distortion of the cross-section can be prevented by applying forces along the edges. These forces affect the longitudinal strain and superimpose a contraction of $\nu^2\sigma/E$ acting in the plane of the cross-section. The elongation of a strip along the convex side of the beam is now

$$\epsilon = \frac{\sigma}{E} - \frac{\sigma\nu^2}{E}$$

and therefore

$$\sigma = \frac{E}{1 - \nu^2}\epsilon$$

In essence, a factor of $1/(1 - \nu^2)$ has been applied to the Young's Modulus. It can be shown that Euler's equation now becomes

$$u_{\tau\tau} = \frac{M}{EI}(1 - \nu^2)$$

Using this modified equations, a strip deflection of 2.310 mm is calculated for the strip. This result is much stiffer, by 11%, then the result from the unmodified equation. In fact, it is seen that the deflection is less than that predicted by the FEM analysis as well.

The above equation is actually used in cases where the beam is considered to be wide. It is seen from the results that the strips used in the MMS crossed flexure pivots fall somewhere in between an ideal beam and a wide beam. It has been shown that by assuming the strips to be beams the strips will be more flexible than if a plate analysis was used.

4.4 Properties of the Crossed Flexure Pivot

Earlier it was shown, if the same assumptions are made, that the equations derived in Section 4.1 can be used to arrive at the same solution as in the literature. To determine the effect that these assumptions have on the results has not yet been demonstrated. To do this, it will be necessary to solve the equations presented in Section 4.1 and then compare the results with those from the literature.

When the equations describing the static deflection of the crossed flexure pivot were developed it was found that the values of two Lagrange multipliers, λ_1 and λ_2 , could not be found analytically. In order to complete the solution, these two constants were found by finding the roots of transcendental

Dimension	Value
Strip Length, L	2.00 mm
Strip Width, w	1.00 mm
Strip Thickness, t	0.05 mm
Initial Centre of Rotation, r	$\frac{1}{2}$
Angle at which the Strips Cross, α	45 degrees
Young's Modulus, E	194.4×10^3 MPa

Table 4.3: Crossed flexure pivot dimensions used to compare theoretical results with the analytical results found in the literature.

Equations 3.4 and 3.6 using a non-linear root finding technique.

For this purpose, a Fortran program was developed which uses a modified Powell hybrid algorithm and a finite difference approximation of the Jacobian[64]. This algorithm is a variation of Newton's method. The documentation of the subroutine warns that with single precision calculations, the Jacobian could be incorrect and that the program may terminate far from the roots. In order to avoid this pitfall, double precision calculations were performed. But even with this precaution it was found that the root-finding technique would occasionally have difficulty converging to the proper roots.

Ideally, the static equations would be non-dimensionalized so that they could be solved without having to use specific pivot dimensions. The fact that the elongation of the strip, v_1 , is included in the solution makes this impossible and specific strip dimensions had to be used. The dimensions used are shown in Table 4.3 and are based on the approximate dimensions listed in Table 3.1. The results have been non-dimensionalized to attempt to eliminate any influence that the choice of pivot dimensions would have on the results.

It has been shown that the non-dimensionalized stiffness of the crossed

flexure pivot is 2 using both methods. It can also be shown that, when θ is zero, the variation of the normalized stiffness of the pivot with the applied loads is approximately the same.

The results in the literature predict that this is the stiffness of the pivot for all values of θ . This is not the case for the solution found in this research. Figure 4.6 shows the percentage change in stiffness versus the angular rotation of the pivot, θ , when the applied loads are zero. This plot demonstrates that the stiffness of the pivot, calculated by taking the ratio of the current normalized moment and the current angle of rotation, steadily increases and reaches a value slightly larger than 1.8% greater than the stiffness at $\theta = 0$ when the angle is 20 degrees. At angles of 7 degrees, the difference in the stiffness is closer to 0.3%.

Figure 4.7 contains two plots obtained by setting the applied loads to zero and then increasing the moment applied to the pivot. The top plot shows the normalized moment on the pivot versus the angle of rotation of the pivot. This is the plot from which the pivot stiffness was calculated. The plot is not quite linear, with a slight curve upwards. The bottom plot has the non-dimensionalized values of the Lagrange multipliers, λ_1 and λ_2 , with respect to the angle of rotation. Since the two multipliers represent the constraint forces acting on the strips, it is expected that these forces should be zero when there is no rotation, and no applied loads, and would increase as the angle increases. The negative sign just refers to the fact that the direction of the multipliers is opposite to that which was chosen.

Figure 4.8 shows the response of the constraint forces when a vertical load is applied to the pivot. The direction of the vertical load was such

Figure 4.6: Percentage change in stiffness of a crossed flexure pivot with respect to the stiffness at an angle of 0 degrees versus the rotation of the pivot. The properties of the pivot used for this plot are shown in Table 4.3.

Figure 4.7: The top plot shows the non-dimensionalized moment applied to the crossed flexure pivot versus the angle of rotation of the pivot. The lower plot contains the non-dimensionalized constraint forces acting on the strips of the pivot versus the angle of rotation. The properties of the pivot used for this plot are shown in Table 4.3.

that the pivot is in compression and, therefore, the plots give the buckling load of the pivots. The top plot is for a pivot where a pure vertical load is applied and the magnitude of the normalized constraint forces is plotted against the normalized axial load. It can be seen that buckling occurs at a normalized axial load of 9 or 2.86π . In theory, the buckling load of the pivot, as calculated in the literature, only applies when there is a slight rotation of the pivot. If there is no rotation, then the pivot would not buckle. This result shows otherwise.

As with the top plot, the bottom plot shows the response of the constraint forces as a vertical load is applied to the pivot versus normalized axial load. In this case a very small moment was applied to the pivot in order for it to have a small rotation. Buckling now occurs at a normalized axial load of 5.6 or 1.78π . In other words, buckling occurs when

$$N_1 L = 1.78\pi$$

In comparison, the buckling load predicted in the literature occurs at

$$N_1 L = 2\pi$$

Therefore, the buckling load predicted by the equations derived in this research is only 89% of that predicted in the literature. This discrepancy is due to the fact that the constraint forces have, effectively, been neglected in the literature. It appears that they play an important part in this calculation. It should also be pointed out that this is the first buckling mode. The second buckling mode occurs at a normalized load of 2.86π , the same value as for the undeflected pivot.

Figure 4.8: Plots of the normalized constraint forces acting on the pivot versus normalized axial load. Figure (a) shows the case where a pure vertical load is applied to an undeflected pivot. Figure (b) shows the normalized constraint forces versus normalized axial load when the pivot has a slight rotation. In this case, a moment of 1×10^{-2} Nmm was applied which resulted in a rotation of 0.28 degrees (5×10^{-3} rad). The properties of the pivot used for this plot are shown in Table 4.3.

4.5 Comparison with Experimental Results from the Literature

A general examination of the properties of the crossed flexure pivot has been made. Still of interest is a comparison between the measurements made by Nickols and Wunsch[17] and Young[16] and those made from the theory developed in this chapter.

As has been mentioned previously, Nickols and Wunsch and Young performed experiments to determine the motion of the pivot centre as a function of the angle of rotation. Nickols and Wunsch went even further to include strip length in this relationship.

The theory allows the transverse and axial deflections of the strips to be calculated when the applied loads are known. Using this information the displacement of the initial centre of rotation can be calculated using Equations 3.17 and 3.18.

$$p = \frac{1}{2 \sin \alpha} (e_1 - e_2)$$

$$q = \frac{1}{2 \cos \alpha} [2rL (\cos \theta - 1) - (e_1 + e_2)]$$

The pivots used by Nickols and Wunsch consisted of two pivots connected in parallel and had the properties listed in Table 4.4.

These dimensions were used to calculate the displacement of the initial centre. Plots of the angle of rotation of the pivot versus applied moment and the shift of the initial centre of rotation as a function of the angular rotation are shown in Figures 4.9 and 4.10.

Figure 4.9 contains a plot of the pivot rotation as a function of the applied moment. In this experiment, a moment was applied to the pivot while the

	<i>Pivot 1</i>	<i>Pivot 2</i>
Strip Length, L (inches)	0.66	0.41
Strip Width, w (inches)	0.25	0.25
Strip Thickness, t (inches)	0.01	0.01
Young's Modulus, E (psi)	27.5×10^6	27.5×10^6
α (degrees)	45	45
r	0.5	0.5

Table 4.4: Properties of the crossed flexure pivots used in the experimental research by Nickols and Wunsch[17].

Figure 4.9: Comparison between the experimental results of Nickols and Wunsch[17] and the predictions by the theoretical model developed in this chapter. The dimensions of the crossed flexure pivots were $t = 0.01$ inches, $w = 0.25$ inches and $E = 27.5 \times 10^6$ psi.

Figure 4.10: Plot of the shift of the initial centre of rotation of the crossed flexure pivot as a function of pivot rotation. Data shown includes; empirical results from Nickols and Wunsch[17], the empirical formula developed by Young[16], the theoretical prediction by Haringx[53] and the theoretical model developed in this chapter. The dimensions of the crossed flexure pivots were $t = 0.01$ inches, $w = 0.25$ inches and $E = 27.5 \times 10^6$ psi.

vertical and horizontal loads were kept fixed at zero. The plot shows data from the experiments by Nickols and Wunsch and the results calculated by the mathematical model of the pivot. It can be seen that the mathematical model is predicting a stiffer pivot than what the experimental results show.

In fact, Nickols and Wunsch found that using an effective length 0.035 inches longer than the actual strip length, their results compared with those from the literature. Using this effective length in the mathematical model results in a solution which is closer to the results of Nickols of Wunsch yet there is still a slight difference. Note that this also demonstrates the difference between the mathematical model from this chapter and those from the literature.

Figure 4.10 shows the shift of the initial centre of rotation as a function of the angle of rotation of the pivot. The data has been normalized with respect to the strip length. It can be seen that the results of the mathematical model have the same form as the experimental data yet it predicts a stiffer pivot than is seen from the experimental data. It should also be noted that the models by Haringx and Young, as well as the model developed in this research, predict that the normalized shift of the initial centre of rotation is independent of the length of the strips. The results from Nickols and Wunsch show that it is dependent on the length. Using the effective length rather than the strip length causes only a slight shift of the results towards that of Nickols and Wunsch.

It is at this point that the findings from the previous section should be discussed. Recall that the estimated errors caused by using a beam rather than plate analysis was close to 5%. However, the error was such that a plate

analysis results in a stiffer pivot. The results here show that the beam theory is already predicting the pivot to be too stiff.

It is obvious that while the beam versus plate discussion is valid, it is not the significant cause for the differences between the theoretical model and the experimental results. It is possible that the reduction in the stiffness is due to the fact that the free end of the strip is not actually free. In fact, the cross-section is not able to distort. It was seen in the previous section that the stiffness of a plate is higher than that for a beam of the same dimensions due to the constraints placed on it by its own width. It may be possible that the restriction of the cross-section distortion by the rigid body actually makes the strip more flexible.

4.6 Experimental Study of the Crossed Flexure Pivot

The previous section dealt with comparisons between the theoretical model and experimental results from the literature. Since the details of the experimental procedure used by the researchers is not known it may be possible that this may not be a proper comparison.

For this reason, as well as to do an independent check on the accuracy of the mathematical model, a crossed flexure pivot was designed and manufactured. A test consisting of applying a vertical load on the pivot was then performed and the displacement of the initial centre of rotation was measured.

In order to maintain some consistency, enable proper comparison between the earlier results and the test results and to gain some knowledge of the

crossed flexure pivots used in the MMS prototype, the test pivot was designed as a 10:1 reproduction of the pivot used in the MMS, see Table 4.1.

The choice of material used for the strips was based on the desire to have a large motion of the centre with respect to the applied load. Since the goal was large displacements, it was also necessary that the yield stress of the material was high enough to allow large loads to be applied. In other words, the higher the ratio of the yield stress to the Young's Modulus, the better suited the material was for this application.

A chart showing the ratios of yield stress to Young's modulus is found in Figure 4.11 which shows that the ideal material is Teflon. However, it was found that ultra high molecular weight polyethylene (UHMWPE) was readily available making this the material used for the strips. The yield stress of UHMWPE is 48.3 MPa and the Young's modulus is 690 MPa[91].

The pieces to which the strips attach, the rigid bodies, were made out of 6061 T651 aluminum. This choice was made in order to ensure the material was much stiffer than that of the strips. A second consideration was the availability of the material.

Examination of a single crossed flexure pivot shows that if the rigid bodies are not held firmly, they will have a tendency to 'corkscrew' or twist as a vertical load is applied. This is due to the fact that the two strips are offset about the centreline of the pivot. The reaction forces, while counteracting each other, are not on the same line of action causing a couple. Therefore, in order to avoid this problem, a double pivot was designed. Detailed drawings can be found in Appendix B. A photograph of the double crossed flexure pivot is seen in Figure 4.12.

Figure 4.11: Chart showing the ratio of yield stress (σ_y) versus Young's modulus (E) for various materials. The abbreviations are as follows: LDPE - low density polyethylene, HDPE - high density polyethylene, PVC - polyvinyl chloride, UHMWPE - ultra high molecular weight polyethylene[91].

Figure 4.12: Photograph of the crossed flexure pivot used in the experimental study.

Two steel eye hooks were attached to the centre of the top and bottom aluminum pieces and were used to attach the pivot to the test rig. To perform the tensile test, the pivot was mounted in an Instron Table Mounted Universal Testing Rig³ and aligned, by eye, with the vertical axis of the rig, see Figure 4.13. A load was slowly applied and the distance traveled by the grip was measured. This distance corresponds to the motion of the initial centre of rotation.

The system is displacement controlled and the rate of travel of the upper grip was set to 1 mm/minute. The load acting on the pivot was monitored during the tests and when it approached a value of 60 to 70 kg_f the test was stopped, the grip returned to its initial position and the pivot removed from the test rig. This was done to prevent creep and stress relaxation from occurring in the strips.

Creep occurs when a constant load is applied to the polymer. This load causes the material to stretch over time. Stress relaxation occurs when a constant strain is applied to the polymer and the stress in the material reduces over time. Polymers are viscoelastic materials and, given enough time, the polymer chains slide over one another by viscous flow. This motion is the cause of both creep and stress relaxation.

The maximum load of 60 to 70 kg_f was based on limiting the stress level in the strips to low percentages of the yield stress. If the polymer chains in UHMWPE simply stretch the material would behave in a linear elastic manner; however polyethylene is a viscoelastic material and some distortion

³The standard frame Instron test rig had a capacity of 5 kN (1125 lb). Its model number was 4465 and the frame serial number was C3143. The test rig was controlled using an Instron 4400 Console, serial number C2525 and a personal computer with a 486 processor.

Figure 4.13: Photograph of the crossed flexure pivot mounted in the Instron Table Mounted Universal Testing Rig, model number 4465. The grips are attached to the eye hooks. Two nuts, one on each eye hook, were used to lock the eye hooks in position. The bottom grip in the Instron was fixed, while the top one was free to rotate about the vertical axis and translate in one direction. It could not move in a side to side motion. In order to achieve proper alignment the pivot was mounted in the position shown above so that the position of the eye within the grip could be adjusted.

of the material will occur causing a slight non-linear response. This non-linear behaviour becomes more pronounced at higher stress levels. A scrap of material left over when the strips were machined was used in a tensile test to determine the response of UHMWPE to an applied load. The results are shown in Figure 4.14. The non-linear behaviour of the material is clearly seen.

Before the tests were conducted, a plot of the stress in the strips versus applied load was made, see Figure 4.15, using the mathematical model and Euler's equation

$$M = EIu_{xx}$$

It can be seen that when the load reaches 100 kg_f (981 Newtons) the stress in the strips is just under 11 MPa. This is around a quarter of the yield stress of ultra high molecular weight polyethylene.

Three tests were performed and after each the pivot was removed from the test rig. The results of the three tests are shown in Figure 4.16. Data was taken continuously during the tests and was output in the form of a graph. Since these graphs were not portable to different software packages, discrete points were taken from these plots at intervals of 0.05 mm and are seen in Figure 4.16.

The experimental data is tightly grouped with very little scatter, indicating that there was very good repeatability in the tests. The solid line represents the mathematical model of the double pivot. It can be seen that for loads of up to 20 to 30 kg_f there is very good correlation between the theoretical model and the experimental data. As the load increases, the results diverge, with the theoretical model predicting a linear response and

Figure 4.14: Load displacement plot for ultra high molecular weight polyethylene. The horizontal axis represents displacement in millimetres and the vertical axis shows the load in kN. This plot also gives an example of the output from the Instron test rig. The material was strained at a rate of 2 mm/min.

Figure 4.15: Plot of the calculated stress in the strips of a crossed flexure pivot as a function of the applied load. The applied load was along the vertical axis of two pivots, thus halving the load experienced by each pivot. The length, width and thickness of the pivot strips were 60 mm, 25.4 mm and 1.54 mm, respectively. The strips were oriented at an angle of 90 degrees to each other.

Figure 4.16: Plot of the displacement of the initial centre of rotation of a pair of crossed flexure pivots as a function of load. The discrete points represent experimental data and the solid line represents the predictions by the mathematical model. The dashed line represents the predicted motion of the initial centre of rotation when one of the rigid bodies has an initial rotation of 4 degrees with respect to the other. The applied load was along the vertical axis of two pivots, thus halving the load experienced by each pivot. The length, width and thickness of the pivot strips were 60 mm, 25.4 mm and 1.54 mm, respectively. The strips were oriented at an angle of 90 degrees to each other.

the experimental data indicating that the pivot becomes more flexible as the load increases. The form of the curve followed by the experimental data is that of a quadratic. The difference between the theoretical model and the experimental results is just under 15% at a load of 55 kg_f. It should be pointed out that had a plate analysis been used, the prediction would have been even further from the experimental results.

There are several sources of error in this test. One is the fact that the tapped holes used to attach the strips to the aluminum blocks were not precisely located. The result was that some of the strips were initially in compression and others were in tension. In order to avoid having any large stresses in the strips due to transportation of the pivot, the pivot was assembled immediately before testing. This means there was little chance stress relaxation had occurred in the strips before the tests were conducted.

A second possible source of error is that the pivot may have been mounted so that there was a slight rotation of one rigid body with respect to the other. The result would be a shift of the load/displacement curve, as seen by the dashed curve in Figure 4.16 which represents the case where the pivot had an initial rotation 4 degrees. The angle of 4 degrees was chosen as the curve fits the experimental data quite nicely. It is expected that any initial rotation of the pivot would have been closer to 1 degree than 4 degrees. A rotation of 1 degree would mean only a small shift in the curve making the error only a minor factor.

A related source of error is that the pivot could have been mounted slightly off axis. This would mean that the loading on the pivot would have actually been some combination of vertical and horizontal loads. It can be

shown that the error caused by this would be small. For example, in order for the vertical load to be reduced to 60 kg_f from 65 kg_f the misalignment would have to be in the range of 20 degrees. When the pivot was mounted in the grips, misalignments of this magnitude would have been noticed and corrected.

A final source of error is the fact that the stress strain curve for ultra high molecular weight polyethylene is non-linear. In fact, it is seen that as the load increases, the material becomes more flexible. This could account for the behaviour observed in the experimental results. The load on the pivot was kept low in hopes that this behaviour would be avoided; however, it is possible that this tactic was not entirely successful.

With this said, it can be seen that the model predicts the behaviour of the crossed flexure pivot fairly closely. It appears that the mathematical model of the crossed flexure pivot is a valid one.

4.7 Summary of Results

This chapter contains a derivation of equations that describe the static deflection of the crossed flexure pivot. The equations were derived by taking the system as a whole and determining its potential energy. Four equations were found that describe the deflected position of the pivot in response to the applied forces and moments. Unfortunately, these equations were dependent on two additional constants, namely, λ_1 and λ_2 , which turn out to represent the constraint forces acting on the ends of the strips and keep them a constant distance apart. In order to find the value of these constants, a system of two transcendental equations must be solved. This was done using

a non-linear root finding technique based on a modified Newton's method, known as modified Powell hybrid algorithm.

The equations were found to be the same as those from the literature when the same assumptions are made. An investigation of the new equations was performed and, in most respects, the results were found to be similar to those from the literature. The largest difference was seen to occur when buckling of the pivot was examined. The normalized buckling load of the pivot was found to be 2.81π when the pivot was undeflected and 1.763π when deflected. These can be compared to the buckling load of 2π predicted in the literature.

Comparisons of the mathematical model were made with experimental results from the literature. The trends of the data were modeled almost perfectly, however the magnitudes differed. It was found that theoretical model tended to show that the crossed flexure pivot was stiffer than what was measured in the literature. An independent test was performed which used two pivots connected in parallel and loaded along their vertical axes. The displacement of the initial centre of rotation was measured and compared to the predictions by the mathematical model. The results were found to be very close at low loads but as the load on the pivots increased, the results diverged. This divergence indicated again that the model predicts a stiffer pivot than the experimental data shows.

The fact that a linear beam analysis was used rather than large angle analysis or plate analysis were studied in detail and it was found that the current analysis results in a more flexible system. This flexible system models the experimental results more closely than a plate model would. A discussion of whether the pivot strips should be treated as wide beams showed that the

strips fall somewhere in between ideal beams and a wide beams. Again, using a wide beam would result in a stiffer model.

Since the theoretical model and the model developed in the literature are very similar and the theoretical model can not be solved analytically, equations derived using the literature method will be used to develop a design procedure to be used in order to properly size MMS crossed flexure pivots.

Chapter 5

Crossed Flexure Pivot Sizing

The crossed flexure pivots in the Momentum Management System (MMS) and Dynamically Tuned Gyro (DTG) allow the rotor to spin about an axis that is not parallel to the spin axis of the shaft. Since a detailed analysis of the crossed flexure pivot has been reported in Chapter 4, developing the procedures and design equations necessary to determine the dimensions of the pivots used in the MMS engineering prototype is now possible. In order for the MMS to behave as a momentum exchange device for a micro-satellite, it will provide a range of angular momentum spanning approximately 0.8 Nms to 1.2 Nms with rotor spin speeds of 1500 ± 200 rpm. In the future, to meet the requirements of larger satellites, this speed may be increased to 5200 ± 800 rpm resulting in an increase in the maximum angular momentum of the MMS to 4 Nms[66]. Therefore, it is desirable that the design of the crossed flexure pivots be easily changed in order to simplify the transition from one speed range to another.

The function of the crossed flexure pivots is to allow rotary motion of the gimbal and rotor about the gimbal x and y axes while remaining stiff about

the nonsensitive axis. The pivot must be able to withstand the constraint forces and moments resulting from the connection of the rotor to the gimbal and the gimbal to the shaft. When the rotor is offset from the null position the gimbal flutters at twice the spin frequency. Consequently, it is essential for the pivots to withstand a high number of stress reversals. If a mission length of 10 years and a shaft speed of 1500 rpm is assumed, then the number of cycles experienced by the pivots is calculated to be over 1.5×10^{10} .

There are two pairs of pivots in the MMS, along perpendicular axes. In order to simplify the design and manufacture of these pivots, one design is used for all of them. The result was that the torsional stiffness about each of the transverse axes of the gimbal is the same.

5.1 Design Considerations

Prior to the development of design equations, a discussion of the ideal parameters for the MMS pivots is necessary. The failure of one pivot during operation means the MMS would cease to function properly. It is for this reason that the pivots are designed as 'fail-safe'.

The first priority of the design is to ensure that the MMS does not fail statically under the maximum applied loads. These loads will be discussed in detail later; although, it should be noted that the MMS pivots will experience constraint forces, torques and loads due to gravity and acceleration. The crossed flexure pivot is made up of two strips and it is these strips which must withstand the applied loads.

The high number of cycles to be experienced by the pivots is of concern considering the endurance limit for most materials is specified at 10^7 cycles

and very little testing extends beyond this point. While this limit is reasonable from a testing point of view, tests of the required length would take years (perhaps even decades), it leaves doubt as to the properties of the material at a very high number of cycles. For this reason, the design of the pivots is based on ensuring the strips do not fail in fatigue. For the design, it is assumed the MMS rotor will operate at its maximum offset of 7 degrees for its entire life (a remote, but real, possibility) and at the high end of its speed range. It will also be assumed that the MMS is oriented in a 1g environment in such a way that the stress in the strips is at its highest. Note that while in service in a space environment, the accelerational loads on the pivot will be quite low, however, the prototype will be tested on Earth and therefore will experience a 1g acceleration.

Normally, when determining the modified endurance limit, factors are applied to the laboratory endurance limit in order to model the practical situation. For this design no modifying factors were applied. This is taken into account by being conservative in the choice of a safety factor. However, with this said, a brief discussion on the magnitudes of the modifying functions should be made. It can be assumed the surface of the strip is polished, due to the care taken during the EDM process. Other factors include: effective diameter, stress concentration due to the radii at the end of the strips, residual stress in the strip from the EDM process, temperature factors and load factors[67]. Standard design procedure would warrant the calculation of the modified endurance limit using these factors; however, in this situation there is an uncertainty about the stress level for infinite life of a specimen undergoing this many cycles. Therefore, a fairly large safety factor was applied to

the unmodified endurance limit and used for the design.

Buckling of the pivot during operation of the MMS must also be avoided. The design is such that under normal loading conditions buckling will not occur. When the MMS experiences high accelerations, the pivot will be allowed to buckle but will be physically restrained by a stop so that the stress in the strips does not exceed 50 to 75% of the yield stress, thus avoiding plastic damage to the strips.

Several factors required to tune an ideal DTG system were discussed in Chapter 2. If these conditions are not met then the result is precession of the rotor, which must be compensated for using torques from the torquer coils. One of the critical factors is the torsional stiffness of the two sets of pivots. The higher the stiffness, the more power required to control the rotor. Another consideration is that the MMS is only able to be tuned for one specific speed. Since the shaft speed can vary by 400 rpm and the stiffness of the pivots is dependent on the applied loads, the tuning of the MMS for all situations is difficult.

As mentioned earlier, the stiffness of the pivots is dependent on the applied loads. Therefore, it should be possible to determine the required stiffness of each set of pivots at specific rotor offsets and spin speeds and then apply a load to the pivots resulting in a tuned system at any point in time. The forces, which would act in a vertical direction in this case, would be generated through use of magnetic coils 'pushing' and 'pulling' on the rotor and gimbal. However, a 'back of the envelope' calculation shows that the power required to do this is greater than the power needed to compensate for the larger precession rates that result from the non-tuned system[69]. Power is a

<i>Design Criteria</i>	<i>Value at a Tuned Speed of 1500 rpm</i>
Failure by Fatigue ($n\sigma < \sigma_e$)	$n > 2.15$
Failure by Yield ($n\sigma < \sigma_y$)	$n > 4$
Buckling	N/A
Strip Thickness	$t > 0.127$ mm
Pivot Stiffness (k)	$k < 12.2$ Nmm/rad

Table 5.1: Design criteria for the MMS gimbal and pivot assembly. The design stress to be used against fatigue was chosen as 413.7 MPa (60 ksi) which corresponds to a safety factor of 2.15 against fatigue. The ideal pivot stiffness was calculated by Staley and Ower[65].

critical design point which must be kept as low as possible. This means that this solution is impractical for a space-bound MMS. The second solution is for the MMS to be designed so it is tuned when operating at 1500 rpm with no loads on the pivots.

Finally, in order to facilitate the manufacture of the strips, a minimum strip thickness has been specified. Table 5.1 contains a summary of the criteria discussed and the ideal safety factors and parameters.

This chapter also includes an estimate of the gimbal properties and dimensions, which are dependent on the pivot design. This was done by assuming that the gimbal consists of a cylinder with no cut outs. More detailed calculations of the gimbal dimensions, including the criteria used for the gimbal design, can be found in the Technical Note by Chesser[66].

It should be pointed out that the design of the pivot was an iterative process with the design of the gimbal and many of the restrictions on the gimbal translating to restrictions on the pivots. This includes maximum pivot widths, as the gimbal must fit within an envelope provided by the rotor and

shaft and a minimum gimbal height to pivot length ratio to ensure the gimbal remains stiff relative to the pivot, as well as other considerations. Details of the gimbal design process have not been included as this chapter is meant to discuss the procedure and rationale used in order to size the crossed flexure pivots.

5.2 Material and MMS Component Properties

In earlier chapters it was shown that the stiffness of a pivot with no applied load is given by the expression

$$k = \frac{2EI}{L}$$

Therefore, for a low stiffness, a low Young's modulus, E , is preferred. At the same time, resistance to fatigue and static failure is required. This translates to a high yield stress. Ideally, this means that the ratio σ_y/E for the material used in the MMS pivots should be high. AerMet 100, an alloy steel with a high yield stress, good σ_y/E ratio and excellent fatigue properties, was chosen in order to meet these criteria. Its properties are listed in Table 5.2.

Other known quantities required for the design of the pivots include the mass moment of inertia of the rotor about its transverse and spin axes, the rotor mass and the spin inertia of the shaft. These quantities are found in Table 5.3. It should be noted that the inertias of the gimbal are also required; however, these properties depend on the pivot design and cannot be calculated until after the dimensions of the pivot have been chosen.

Property	Value
Young's Modulus, E	194.4 GPa
Density, ρ	7892 kg/m ³
Ultimate Tensile Strength	1965 MPa
Yield Stress, σ_y	1724 MPa
Endurance Limit, σ_e	890 MPa
$(R = -1, K_t = 1, 10^7 \text{ cycles})$	

Table 5.2: Material properties of AerMet 100. The composition of AerMet 100 is as follows: 0.23% carbon, 3.10% chromium, 11.10% nickel, 1.20% molybdenum, 13.4% cobalt and the balance is iron[68].

Property	Value
Rotor Mass	2.05 kg
Rotor Spin Inertia, I_{r_s}	6283 kg mm ²
Rotor Transverse Inertia, I_{r_t}	3418 kg mm ²
Shaft Spin Inertia, I_{s_s}	1.8 kg mm ²

Table 5.3: Preliminary MMS component properties.

5.3 Development of the Design Equations

It was shown that the equations derived in Chapter 4 are similar to those from the literature. Therefore, for simplicity, the expressions used for the design of the crossed flexure pivot are based on the results found in the literature and in Chapter 3.

The relationship between the axial load on the strips and the loads applied to the pivot was given in Chapter 3, Equations 3.69 and 3.70, as

$$P_1 = \frac{1}{2} (V \sec \alpha - H \csc \alpha) \quad (5.1)$$

$$P_2 = \frac{1}{2} (V \sec \alpha + H \csc \alpha) \quad (5.2)$$

where the subscripts 1 and 2 represent the two strips, V and H are the vertical and horizontal loads applied to the pivot respectively, and 2α is

the angle between the strips. The angle between the strips was set to 90 degrees for simplicity and ease of manufacture. Setting the angle between the strips to a value other than 90 degrees can reduce the anisoelectricity of the system[70]. The anisoelectricity is a drift term dependent on the square of the acceleration. This drift is of concern if the MMS undergoes high acceleration while operating. Since the MMS will not be operating during periods of high acceleration this term is not as critical and a 90 degree angle can be used. The strips are also assumed to cross at their mid-points. This reduces the stress in the strips and gives a more rigid pivot[16] with respect to the applied loads.

As always, the worst case situation is desired for the analysis. This will occur if the applied loads are both positive and Equation 5.2 is used. The maximum bending moment in a strip, Equations 3.74 and 3.75, is expressed as

$$M_{max} = \begin{cases} (EI\theta/L) N^{\bullet} \csc N^{\bullet} & \text{if } P \text{ is compressive} \\ (EI\theta/L) N^{\bullet} \coth N^{\bullet} & \text{if } P \text{ is tensile} \end{cases} \quad (5.3)$$

where L is the length of the strip, E is the Young's modulus of the material, I is the area moment of inertia of the strip, θ is the angle of rotation of the pivot about its sensitive axis and $N^{\bullet 2}$ is the non-dimensionalized axial load, defined as

$$N^{\bullet 2} = \frac{PL^2}{4EI} \quad (5.4)$$

Note that $N^{\bullet 2}$ is not quite the same N^2 , found in Chapters 3 and 4. The relationship between them is given by $N^{\bullet 2} = N^2 L^2 / 4$. Combining the maximum axial and bending stresses gives the total stress in a strip as

$$\sigma = \frac{P}{wt} + \frac{M_{max} t}{2I} \quad (5.5)$$

where w is the width and t is the thickness of a strip. The above expression can be rewritten to give

$$\sigma = \begin{cases} P/A + (\theta/2) \sqrt{3PE/A} \csc N^* & \text{if } P \text{ is compressive} \\ P/A + (\theta/2) \sqrt{3PE/A} \coth N^* & \text{if } P \text{ is tensile} \end{cases} \quad (5.6)$$

where A is the cross-sectional area of a strip.

The stress has now been expressed as a function of the cross-sectional area of the strips and the normalized axial load. The result is a series of curves dependent on N^* and A at the assumed angle of 7 degrees. It is possible to find a relationship describing the intersection of those curves and the design stress, σ_d , which is defined as the endurance limit divided by the safety factor. Solving for the cross-sectional area in terms of the normalized axial load gives

$$A = \frac{1}{2\sigma_d^2} \left(2\sigma_d P + C'^2 + C' \sqrt{C'^2 + 4\sigma_d P} \right) \quad (5.7)$$

where

$$C' = \begin{cases} (\theta/2) \sqrt{3PE} \csc N^* & \text{if } P \text{ is compressive} \\ (\theta/2) \sqrt{3PE} \coth N^* & \text{if } P \text{ is tensile} \end{cases}$$

The second solution can be ignored since only positive values for the cross-sectional area have any meaning. In this equation θ is considered to be positive. Examining the equation for the maximum bending moment when a compressive axial load is applied shows that as long as N^* remains below π , buckling will not occur. This is not quite true, as shown in Section 4.4, but it will be used in this chapter as a quick check for failure by buckling.

The relationship between the strip length and the area is found using the relationship for the normalized axial load, that is

$$N^{*2} = \frac{PL^2}{4EI}$$

Isolating the ratio L/t yields

$$\frac{L}{t} = \sqrt{\frac{EAN^{\bullet 2}}{3P}}$$

This is valid whether P is compressive or tensile.

Recall that the stiffness of the pivot can be expressed by, Equation 3.62,

$$\frac{kL}{EI} = (\beta_1 \text{ or } \gamma_1) + (\beta_2 \text{ or } \gamma_2)$$

where the indices represent the two strips and

$$\begin{aligned} \beta &= N^{\bullet}(\cot N^{\bullet} + N^{\bullet}) && \text{when } P \text{ is compressive} \\ \gamma &= N^{\bullet}(\coth N^{\bullet} - N^{\bullet}) && \text{when } P \text{ is tensile} \end{aligned}$$

This expression shows that the stiffness is a function of the applied loads as seen in Figure 5.1. In order to tune the system, the axial load is set to zero and the above expression simplifies to give

$$\frac{kL}{EI} = 2$$

or

$$k = \frac{Ewt^3}{6L}$$

The relationship between the gimbal inertias, pivot stiffness and the spin speed of the shaft, as given by the tuning condition, is

$$\omega_s^2 = \frac{4k}{2I_{g_t} - I_{g_s}} \quad (5.8)$$

For a first approximation, the gimbal is assumed to be a perfect cylinder with height, h , and inner and outer radii, r_i and r_o , respectively. Therefore

$$I_{g_s} = \frac{1}{2}\pi h\rho(r_o^4 - r_i^4)$$

Figure 5.1: Variation of the stiffness as a function of the normalized axial load.

and

$$I_{gt} = \frac{\pi h \rho}{4} (r_o^4 - r_i^4) + \frac{h^3 \pi \rho}{12} (r_o^2 - r_i^2)$$

Equation 5.8 becomes

$$\frac{4k}{\omega_s^2} = \frac{\pi h \rho}{2} (r_o^4 - r_i^4) + \frac{h^3 \pi \rho}{6} (r_o^2 - r_i^2) - \frac{1}{2} \pi h \rho (r_o^4 - r_i^4) \quad (5.9)$$

which can be simplified to give

$$h^3 = \frac{24k}{\omega_s^2 \pi \rho (r_o^2 - r_i^2)} \quad (5.10)$$

The height calculated in this equation can be used to ensure the design is practical and that the gimbal is stiff relative to the pivots. This information can be used by the gimbal designer to develop a design to fit the pivot. Note that the actual height will be slightly larger than the height calculated using Equation 5.10 because the above calculation neglects the pivot cut-outs and the gap between the inner and outer cylinder.

5.4 Loads Acting on the Pivot

In general, the crossed flexure pivots in the MMS will experience loads and torques in the three orthogonal directions. Because the pivot is stiffer along the width of the strips, only a two dimensional problem will be considered and the pivot will be assumed to be rigid out-of-plane.

Each pivot will experience vertical and horizontal loads and a torque. The torque can be taken into account by deflecting the pivot by 7 degrees, the maximum deflection of the MMS rotor. The forces on the pivots can be divided into two components; the load due to gravity and the load caused by

the dynamics of the rotor, gimbal and shaft. The components due to gravity are

$$V_g = \frac{1}{2}g(M_r + M_g)\sin\phi$$

$$H_g = \frac{1}{2}g(M_r + M_g)\cos\phi$$

where g is the acceleration due to gravity, M_r and M_g are the masses of the rotor and gimbal, respectively, and ϕ is the orientation of the pivot in the gravitational field. It can be shown that the ‘worst case scenario’ occurs when $\phi = 45^\circ$; therefore

$$V_g = H_g = 3.47(M_r + M_g)$$

Assuming that a typical mass of the gimbal is 50 grams, the vertical and horizontal gravitational loads are 7.3 N.

The dynamic loads, V_d and H_d , are calculated using a simplified version of the numerical simulation of the MMS found in Chapter 10 and the rotor properties listed in Table 5.3¹. The initial iteration assumes that the loads are zero and the simulation is used for calculating the dynamic loads for subsequent iterations.

5.5 Design Results and Discussion

The design process consists of determining the stresses in the strips, choosing dimensions for the strips that are able to withstand these stresses and then submitting these numbers for the design of the gimbal[66]. The model of the gimbal and pivot can then be updated and the new design used to calculate

¹The simplified simulation contains the shaft, gimbal and rotor but does not account for the effect of the variation of the pivot stiffness and the stiffness inherent in the motor.

the dynamic loads. As this process is quite long, only the last iteration is shown.

Figure 5.2 contains a plot of the dynamic loads on the pivot. Summing the peak data from this plot and the gravitational loads gives the horizontal and vertical loads on the pivot

$$H = 7.5 \text{ N}$$

$$V = 7.9 \text{ N}$$

Given these loads, it is possible to calculate the tensile and compressive stresses in the strips. Figure 5.3(a) gives the tensile stress in the strip as a function of the cross-sectional area and the square root of the normalized axial load. Figure 5.3(b) gives the relationship between the area and the axial load at the design stress. Figure 5.4(a) shows a plot of the compressive stress in a strip and Figure 5.4(b) is the intersection of the curves and the design stress.

Comparing Figures 5.3 and 5.4 it can be seen that the magnitude of the tensile stress is larger than the compressive stress and was therefore used for the design. The horizontal line, in the plots, represents the design stress. As stated earlier, buckling is avoided by ensuring that the square root of the normalized axial load, N^* , never reaches π .

Choosing several different values for the cross-sectional area, corresponding values for N^* can be calculated using the design stress. These numbers can then be used to determine the variance of the length and width with thickness, see Figure 5.5. It can be seen that as the thickness of the strip increases, the length increases and the width decreases. Figure 5.6 shows the variance of the pivot stiffness with thickness. From this plot it is apparent

Figure 5.2: Variance of the loads on the pivots with time when the shaft is operating at 1700 rpm and there is an initial rotor offset of 7 degrees about the gimbal x axis. The rotor and gimbal specifications listed in this chapter were used for the simulation. F_x and F_z represent the loads on the pivots connecting the rotor and the gimbal and are in the local coordinate frame of the rotor. G_y and G_z are the loads on the pivots connecting the gimbal and the shaft and are in the shaft coordinate frame.

Figure 5.3: (a) Maximum tensile stress in the strips of a crossed flexure pivot under a horizontal load of 7.5 N and a vertical load of 7.9 N. The design stress is depicted by the horizontal line. (b) Plot of the change of the square root of the normalized axial load with cross-sectional area at the design stress. Any design falling on the right side of the curve and below $N^* = \pi$ can be considered safe.

Figure 5.4: (a) Maximum compressive stress in the strips of a crossed flexure pivot under a horizontal load of 7.5 N and a vertical load of 7.9 N. The design stress is depicted by the horizontal line. Note that the curve for a value of $N^* = 3$ is off the plot (to the right). (b) Plot of the variance of the square root of the normalized axial load with cross-sectional area at the design stress. Any design falling in the pocket on the right can be considered safe.

Figure 5.5: Variance of the strip length and width with thickness. The length of the strip is given by the curves with the positive slope.

that the stiffness of the pivot increases with thickness. Examining Figure 5.6, it is clear that the stiffness specified in Table 5.1 cannot be met using these values of cross-sectional area.

The difficulty now lies in choosing a thickness and area that meets the criteria as closely as possible while still permitting a good gimbal design. This compromise was made by choosing a strip thickness of 0.15 mm and a cross-sectional area of 0.38 mm². The length, width and stiffness are as follows:

- $L = 6.00$ mm

Figure 5.6: Variance of the pivot stiffness with thickness.

- $w = 2.53 \text{ mm}$
- $k = 46.11 \text{ Nmm/rad}$

With this design, the safety factor against the endurance limit is just over 2.25. It should be noted that the desired stiffness has been exceeded by almost a factor of four. It was found that this was the lowest obtainable stiffness while maintaining a good gimbal design. A second factor which should be examined is the variance of the stiffness of a pivot during operation. As the cross-sectional area of the strip increases, the normalized axial load decreases and therefore the variance of the stiffness decreases. The variance that occurs with this design is approximately $\pm 50\%$.

Using Equation 5.10, an estimate of the height of the gimbal can be found. For this design, the gimbal must have a height of 23.08 mm to tune the MMS for an operational speed of 1500 rpm. This number is much larger than the length of the strip, ensuring a stiff gimbal. Note the actual height of the gimbal is 23.9 mm showing good correlation between the estimate and the calculation which takes all the details of the gimbal design into account.

5.6 Summary

This chapter introduced the procedure used to determine the best dimensions for the MMS engineering prototype pivots. This procedure can also be used for the sizing of crossed flexure pivots intended for any application.

Several iterations were required to arrive at the final design for the MMS, summarized in Table 5.4. The desired specifications of the pivot were listed before the design process began and it was found that these specifications

<i>Dimensions</i>	<i>Value</i>
Strip Length (mm)	6.00
Strip Width (mm)	2.53
Strip Thickness (mm)	0.15
Pivot Torsional Stiffness (Nmm/rad)	46.11
Gimbal Outer Radius (mm)	17.85
Gimbal Inner Radius (mm)	12.00
Gimbal Height (mm)	23.90
Gimbal Spin Inertia (kg mm^2)	18.0
Gimbal Transverse Inertia (kg mm^2)	12.74

Table 5.4: Summary of the pivot design. Also included are some working dimensions of the MMS gimbal[71].

were difficult to meet and it was necessary to compromise somewhat to achieve a well-rounded design. Chapter 6 contains calculations of additional properties of the design. It was found that the gimbal design dictated the pivot dimensions rather than the other way around.

With the completion of the design of the pivots and gimbal, additional properties of the crossed flexure pivots can be found, such as the sensitivity of the design to errors in manufacture and the stress in the strips due to start up loads can be calculated. This will be shown in the next chapter.

Chapter 6

Mechanical Properties of the MMS Crossed Flexure Pivots

The crossed flexure pivots used in the MMS are designed with the maximum stress in the strips and the torsional stiffness of the pivot in mind. The pivot is also designed using equations based on the assumption that the axial deflection of the strips is small. There are additional properties of the pivot that have not yet been discussed. The sensitivity of the design to errors in manufacture and the effect on the tuned speed of the system must be analyzed, as should the response of the pivot to start-up and dynamic loads.

The pivots used in the MMS are designed as though they will operate in a 1g environment at a seven degree offset. Large safety factors are also used for the design so it will operate safely within this type of environment; however, if the MMS was to experience high accelerations, such as those experienced during launch or the shock from an accidental drop, the flexures could buckle and suffer permanent damage. Careful handling of the MMS and proper protection during shipping will prevent damage from a drop but

launch loads are inevitable. It is thus important to know what loads an unprotected pivot could withstand.

Finally, the pivots were sized early in the MMS design process and the pivot dimensions were ‘fixed’ early because of the dependency of the entire design on the pivots. As the design of the MMS evolved, some critical properties, such as the rotor mass, changed significantly. Therefore it is prudent to verify that the pivot still has acceptable safety factors against failure.

The following chapter deals with the above questions and delves deeper into the characteristics of the crossed flexure pivots designed in Chapter 5.

6.1 Validation of the Crossed Flexure Pivot Design

The design arrived at in the previous chapter hasn’t changed since an early stage in the design of the MMS; therefore, before any properties of the pivots can be discussed, it is necessary to confirm the design is still valid considering the final parameters of the engineering prototype.

The dimensions of the crossed flexure pivots used in the MMS are listed in Table 6.1 where L is the length, w is the width, and t is the thickness of the strips, 2α is the angle between the two strips, r indicates the point at which the strips cross and k is the stiffness of the pivot when the applied loads are zero. These parameters are also shown in Figure 3.13.

The final design of the gimbal, rotor and shaft are found in Table 6.2. It should be noticed that the inertias of the gimbal and rotor have only changed by about 0.5% from those used while designing the pivots, see Tables 5.3 and 5.4; however, the mass of the rotor is nearly 1.5 times higher. This

<i>Dimensions</i>	<i>Value</i>
Strip Length (L)	6.00 mm
Strip Width (w)	2.53 mm
Strip Thickness (t)	0.15 mm
Pivot Torsional Stiffness (k)	46.11 Nmm/rad
Angle between the strips, 2α	90 degrees
Crossing point of the strips, r	0.5

Table 6.1: Dimensions of the crossed flexure pivots used in the MMS.

<i>Component</i>	<i>Parameter</i>	<i>Value</i>
Rotor	Spin Inertia (I_{r_s})	6324 kg mm ²
	Transverse Inertia (I_{r_t})	3432 kg mm ²
	Mass (M_r)	2.9 kg
Gimbal	Spin Inertia (I_{g_s})	18.05 kg mm ²
	Transverse Inertia (I_{g_t})	12.76 kg mm ²
	Mass (M_g)	0.077 kg
Shaft	Spin Inertia (I_{s_i})	5.12 kg mm ²

Table 6.2: Inertial and mass properties of the MMS engineering prototype as of April 1998. The values for the rotor and gimbal were obtained from the *Final Report: Smallsat Critical Technology Development - Phase 2B*[72]. The shaft inertia was obtained by personnel communication[73].

increases the loads on the pivots proportionately. Originally, the speed range of the MMS rotor was to be 1300 to 1700 rpm. This has been increased to 1200 to 1800 rpm. The increase in the maximum speed are results in an increase in the loads acting on the pivots.

In Section 5.4, the load on a pivot due to gravity was calculated to be

$$V_g = \frac{1}{2}g(M_r + M_g) \sin \phi$$

$$H_g = \frac{1}{2}g(M_r + M_g) \cos \phi$$

Using the rotor and gimbal masses listed in Table 6.2 and assuming the MMS is oriented at an angle $\phi = 45$ degrees to the gravitational field, these loads are

$$V_g = H_g = 10.33\text{N}$$

as compared to 7.3 N which was used in Chapter 5.

Using the parameters of the gimbal, rotor and shaft listed in Table 6.2 in DTGSim, a non-linear simulation of the MMS discussed in Chapter 10, the dynamic loads acting on the pivots can be found. Figure 6.1 shows the loads occurring in 0.05 second time span. It should be noted that the loads are periodic and this pattern does not change for an open loop system with no damping. It can also be assumed that the torque from the motor just matches the friction in the bearings. The maximum dynamic loads, found by examining the peaks of the curves, are $V_d = 0.71$ N and $H_d = 0.21$ N.

Summing the dynamic and gravitational loads, the maximum loads acting on the pivots are calculated to be $V = 11.04$ N and $H = 10.54$ N. These loads are the highest possible loads on the pivot when it is situated at the

Figure 6.1: Variance of the loads on the pivots with time using the MMS engineering prototype parameters, see Table 6.2. F_x and F_z represent the loads on the pivots connecting the rotor and the gimbal and are in the local frame of the rotor. G_y and G_z are the loads on the pivots connecting the gimbal and the shaft and are in the shaft coordinate frame.

worst orientation. The maximum tensile and compressive stress in the strips can be calculated using Equation 5.6 which is listed below

$$\sigma = \begin{cases} P/A + (\theta/2) \sqrt{3PE/A} \csc N^* & \text{if } P \text{ is compressive} \\ P/A + (\theta/2) \sqrt{3PE/A} \coth N^* & \text{if } P \text{ is tensile} \end{cases}$$

where A is the cross-sectional area of a strip, θ is the rotation of the pivot, N^{*2} is the normalized axial load given by

$$N^{*2} = \frac{PL^2}{4EI} = \frac{N^2 L^2}{4}$$

and P is calculated using the relationship

$$P = \frac{1}{2} (V \sec \alpha + H \csc \alpha)$$

Plots of the change of the normalized axial load versus the cross-sectional area of the strips, A , are shown in Figure 6.2. The '*' represents the current pivot design and the solid line is calculated by substituting the design stress of 60 ksi (413.7 MPa) into Equation 5.6 which corresponds to a safety factor of 2.15 applied against the endurance limit of AerMet 100. The plot on the right gives the stress in the strip when a tensile load is applied and the plot on the left is the stress occurring when a compressive load is applied to the strip. Since anything lying to the right of the curves represents a safe design, it is seen that the pivot no longer meets the design criteria when the strips are in tension. In fact, the safety factor has only decreased to a value slightly larger than 2 and therefore this is still an acceptable pivot design.

6.2 Effect of Tolerances on Tuned Speed

Machining a part to meet rigid specifications is virtually impossible and it is necessary to specify the maximum departure from the ideal dimensions

Figure 6.2: Plot of the normalized axial load versus the cross-sectional area of the strips at the design stress of 60 ksi (413.7 MPa). The plot on the right is for a tensile load and the plot on the left is for a compressive load. The '*' represent the position of the current pivot design.

that is allowable. These tolerances are partly dependent on the accuracy obtainable by the machining process. In this case, the parts are so small the tolerances must be quite tight or of the order of ± 0.01 mm. In this section, the effect of a change in the pivot dimensions on the tuned speed of the system will be discussed.

The tuned speed of the MMS can be calculated using the equation

$$\omega_s^2 = \frac{4k}{2I_{g1} - I_{g2}} \quad (6.1)$$

where k is the torsional stiffness of one pivot and is given by

$$k = \frac{Ewt^3}{6l} \quad (6.2)$$

Since the inertias of the gimbal remain constant the following relationship can be derived

$$\frac{\omega_{s2}^2}{\omega_{s1}^2} = \frac{k_2}{k_1} \quad (6.3)$$

where k_2 is the actual stiffness and k_1 is the desired stiffness of the pivot. It should be noted, in practice, when $k_2 \neq k_1$ that a change in the gimbal inertias would be made in order to obtain the tuned speed of 1500 rpm. This change can be measured by examining the spin speed of the shaft, hence the use of the tuned speed as the variable.

The following analysis will assume that the error in the dimension is constant. In other words, the strips have no curvature due to manufacture. The effect of dimensioning errors can be found by breaking the analysis into three parts. First, assume a change in the thickness of Δt . In this case, k_2 can be expressed as

$$k_2 = \frac{Ew(t + \Delta t)^3}{6L}$$

Equation 6.3 can be written as

$$\frac{\omega_{s2}^2}{\omega_{s1}^2} = \frac{6L}{Ewt^3} \frac{Ew(t + \Delta t)^3}{6L} \quad (6.4)$$

If the error in the thickness of the strip is 10%, a common specification, then the error in tuning speed when the system is operating at 1500 rpm is 15%, or 230 rpm. This error is quite high meaning that the variance of the thickness must be kept small.

A similar analysis can be performed by varying the width. In this case, Equation 6.3 takes the form

$$\frac{\omega_{s2}}{\omega_{s1}} = \sqrt{1 + \frac{\Delta w}{w}} \quad (6.5)$$

In this case, a change of 10% in the width of the strip results in 5% change in the tuned speed. The final calculation considers the effect of an error in the length of the strip. This time Equation 6.3 takes on the simple form of

$$\frac{\omega_{s2}}{\omega_{s1}} = \sqrt{\frac{L}{L + \Delta L}} \quad (6.6)$$

and a change of 10% in the length results in a decrease in the tuned speed of 5%. Plots of the change in tuning speed due to changes in the strip dimensions are found in Figure 6.3.

It must be considered that there will be errors on every dimension and the result could compound to the point where the difference between the tuned speed and the desired speed is too large. A good example is to consider the scenario where there is a 10% error in the length, width and thickness.

Figure 6.3: Plot of the relationship between tuned speed and deviations of the strip dimensions from the design values.

Equation 6.3 would take the form

$$\frac{\omega_{s2}}{\omega_{s1}} = \frac{(1 + \Delta w/w)(t + \Delta t/t)^3}{(1 + \Delta L/L)} \quad (6.7)$$

The result is a change of 27.5% in the tuned speed. Thus, the higher the tolerance achieved, the closer the tuned speed will be to the desired one and less modifications to the gimbal will be required in order for the MMS to operate at the desired tuned speed.

The gimbal used in the MMS prototype has been tested by Bristol Aerospace in order to determine the torsional stiffness of the pivots along each axis. The measured values are 94 Nmm/rad and 96.5 Nmm/rad about the x and y axes[72], respectively. This corresponds to deviations of 2% and 5% which indicate that fairly good tolerances were maintained. The result is a change of just over 1.5% (25 rpm) in the tuned speed.

6.3 Pivot Start-Up Loads

The rotor of the MMS has a large mass moment of inertia. This has to be considered since the pivots must be able to transmit enough torque for the rotor to accelerate to its tuned speed of 1500 rpm. This section deals with the evaluation of the loads on the pivot occurring during this start-up time. First, assume it takes t seconds for the rotor to reach 1500 rpm and this occurs when the rotor is in a null position. This last assumption can be made with a fair amount of confidence since the torquer coils can ensure the rotor is in the null position.

Assuming it is constant, the acceleration of the rotor speed is given by

$$\dot{\omega}_s = \frac{\omega_s}{t} \quad (6.8)$$

Now consider the inner pivots which connect the shaft to the gimbal. These pivots must provide enough torque to counteract the inertia of both the rotor and the gimbal. Letting $I = I_{r_s} + I_{g_s}$ means the torque that must be transmitted by the two pivots is

$$M = I\dot{\omega}_s \quad (6.9)$$

There are two pivots mounted along the shaft at a distance R from the centre of rotation. The horizontal force on each of these pivots is given by

$$F = \frac{M}{2R} = \frac{I\dot{\omega}_s}{2R} \quad (6.10)$$

The forces on the pivots due to start-up will act in such a way as to cancel any rotation of the gimbal and rotor. Therefore, there will be a horizontal deflection of the gimbal with respect to the shaft as seen in Figure 6.4. Recall that this analysis is simply meant to estimate the stress on the strips during the start-up process and is not a detailed analysis. In order to determine any problem locations, an FEM analysis of the MMS[71] will be conducted.

Assuming the deflection of the strip is small and that bending is negligible, the strain, ϵ , in the strip is given by

$$\epsilon = \frac{\delta \sin \alpha}{L} \quad (6.11)$$

Therefore, the stress, σ , in the strip is

$$\sigma = \frac{E\delta \sin \alpha}{L} \quad (6.12)$$

where E is the Young's modulus and α is the angle between the strips. The energy, U , in the system can be expressed as

$$U = 2 \left(\frac{1}{2} \sigma \epsilon AL \right)$$

Figure 6.4: Schematic of the displacement of a pivot during the period where the rotor is accelerating to its tuned speed of 1500 rpm. δ represents the deflection of one of the rigid bodies, represented by the rectangles, with respect to the other.

$$\frac{EA\delta^2 \sin^2 \alpha}{L} \quad (6.13)$$

where A is the cross-sectional area of the strip. The energy in the pivot can also be written as

$$U = \frac{1}{2} F \delta$$

Therefore, the displacement of the strip is calculated to be

$$\delta = \frac{FL}{2EA \sin^2 \alpha} \quad (6.14)$$

The stress in the strip is then

$$\sigma = \frac{I \omega_s}{4tRA \sin \alpha} \quad (6.15)$$

This relationship is shown in Figure 6.5. If the rotor takes 10 seconds to reach the tuned speed, the stress in the strips would be 6.2 MPa if the load due to gravity is ignored. This result indicates that the stress in the strips during start-up is quite small.

If the rotor of the MMS is off at some angle, say 7 degrees about the gimbal x axis, when the rotor begins to accelerate then the analysis is much more complicated and the program DTGSim is required. Assume that the MMS is undergoing an uncontrolled start-up such that there is a constant torque applied to the shaft but no control torques acting on the rotor. In this example, the torque on the shaft accelerates the shaft to the tuned speed in 10 seconds, see Figure 6.6.

It will also be assumed that the MMS is oriented 45 degree to the Earth's gravitational field. Implementing the parameters listed in Table 6.2 in DTGSim¹, a simulation of the response of the MMS can be performed.

¹See Chapter 10.

Figure 6.5: Stress in the strips of the crossed flexure pivots as a function of the time required to go from a stationary position to a tuned speed of 1500 rpm. For this calculation the rotor and gimbal spin inertias were 63240 gcm^2 and 180.5 gcm^2 , respectively, the radius was 1.4925 cm , the cross-sectional area of the strips was 0.0038 cm^2 and the strips were oriented at 90° to each other.

Figure 6.6: Shaft speed of the MMS engineering prototype as a function of time. The torque applied to the system is 99.6 Nmm.

Figure 6.7 shows the angular position of the rotor in gimbal coordinates. The initial offset of the rotor about the gimbal x axis (top diagram) decreases rapidly and approaches zero. The bottom diagram indicates that the offset about the y axis increases to an angle of approximately 4.5 degrees before returning to zero again. Thus it can be seen that even an uncontrolled rotor will return to null position during the start-up process. It should be noted that damping along the gimbal x and y axes was neglected for this analysis and, if it was taken into account, would ensure that the rotor does go to a null position.

Figure 6.8 shows the dynamic loads acting on the pivots. Figures 6.8(a) and 6.8(b) show the horizontal and vertical forces, respectively, acting on the pivots connecting the rotor and gimbal. The directions of the forces are indicated in Figure 2.2. Figures 6.8(c) and 6.8(d) show the horizontal and vertical loads acting on the pivots connecting the shaft and gimbal. It is apparent that the vertical loads on the pivots are small compared to the horizontal loads. The fact that F_x is positive indicates that the rotor is being pushed by the gimbal and the negative values for G_y show that the gimbal is being pulled by the shaft. Both of these results agree with intuition.

The forces acting on the pivots due to gravity are shown in Figure 6.9. It was calculated earlier that the load due to gravity when the MMS was oriented at 45 degrees was 10.33 N which is confirmed by examining Figures 6.9(c) and 6.9(d). These two plots show the horizontal, A_{gy} , and vertical, A_{gz} , components of the gravitational load acting on the pivots connecting the shaft and the gimbal. The vertical component is independent of the angular rotation of the shaft about its z axis, as seen by the straight line in

Figure 6.7: Angular position of the MMS rotor during a 10 second acceleration to a shaft speed of 1500 rpm. The rotor was initially offset about the gimbal x axis by 7 degrees. The top diagram shows the position of the rotor about the gimbal x axis and the bottom diagram shows the angle of rotation of the rotor about the gimbal y axis.

Figure 6.8: Figures (a) and (b): Dynamic loads acting on the MMS pivots connecting the rotor and gimbal during acceleration of the shaft from 0 to 1500 rpm. Figures (c) and (d): Dynamic loads action on the pivots connecting the gimbal to the shaft during acceleration of the shaft from 0 to 1500 rpm. The right column represents the vertical pivot loads and the left column represents the horizontal loads. The rotor was initially offset about the gimbal x axis by 7 degrees.

Figure 6.9(d). As the shaft rotates, the horizontal component moves into and out of the plane of the pivot. This is indicated by the oscillation of the curve in Figure 6.9(c). The faster the shaft rotates, the faster the oscillations occur.

Figures 6.9(a) and 6.9(b) show the gravitational loads acting on the pivots connecting the gimbal and the rotor. The rotor is not necessarily oriented at 45 degrees to the gravitational field; in fact, it starts at an orientation of 52 degrees². Therefore, the components of the force do not always have the values of 10.33 N. It can be seen that as the rotor approaches a null position, the loads approach those of the shaft. Note that the results are 90 degrees out of phase due to the orthogonality of the two pairs of pivots.

Combining the dynamic and gravitational loads acting on the pivots, the maximum stress in the strips at any point in time can be calculated, see Figure 6.10. The maximum stress, at any point in time, in the strips of the gimbal/shaft pivots and the rotor/gimbal pivots are shown in Figures 6.10(a) and 6.10(b), respectively. It should be noted that a negative stress indicates tension and a positive stress indicates compression; however, this does not show the behaviour of the strips, just the direction of the normalized axial load.

During a 10 second start-up, the largest stress in the strips of the crossed flexure pivots is 315.2 MPa. Comparing this to the yield stress of AerMet 100 gives a safety factor of 5.5 against yield. It should also be noted that taking 10 seconds for the rotor to get up to the operating speed is quite quick and the actual time is most likely much longer giving a slightly larger safety

²A 45 degree offset of the shaft with respect to the gravitational field plus a 7 degrees offset of the rotor with respect to the shaft.

Figure 6.9: Gravitational loads acting on the MMS pivots during start up. Figures (a) and (b) represent the horizontal and vertical components of the loads on the rotor/gimbal pivots, respectively, and Figures (c) and (d) represent the horizontal and vertical components acting on the gimbal/shaft pivots, respectively. The shaft was accelerated from 0 to 1500 rpm in 10 seconds and the rotor had an initial offset of 7 degrees about the x axis of the gimbal.

Figure 6.10: Maximum stress acting in the strips of the pivots connecting the rotor to the gimbal [Figure (a)] and the gimbal to the shaft [Figure (b)] as a function of time. The shaft was accelerated from 0 to 1500 rpm in 10 seconds and the rotor had an initial offset of 7 degrees about the x axis of the gimbal.

factor. See Figure 6.11 for a plot of the maximum stress versus time required to reach the tuned speed.

6.4 Buckling of the MMS Crossed Flexure Pivots

It was mentioned in Chapter 5 that a translational stop would be used to limit the relative deflection of the rotor with respect to the shaft. This stop is used to prevent damage from occurring to the strips when the MMS is experiencing high accelerations. It is of interest to determine just how high the acceleration must be to be considered 'high'. This can be done by examining the buckling load of the pivots and comparing it to the load caused by a 1g acceleration.

In Section 3.3.3, the normalized critical buckling load was found to be

$$\frac{N_{cr}L}{2} = \pi$$

In Chapter 4, it was found that this term should be modified by a factor of 0.89 in order to take into account the effects of the constraint forces acting on the strips. The result is that

$$\frac{N_{cr}L}{2} = 0.89\pi$$

where $N_{cr}^2 = P_{cr}/EI$. Therefore, the critical buckling load of the pivot is given by

$$P_{cr} = \frac{3.17\pi^2 EI}{L^2}$$

where E is the Young's modulus, I is the area moment of inertia of the strip and L is the length of the strip.

Figure 6.11: Plot of the maximum stress in the MMS crossed flexure pivots as a function of time required to accelerate the shaft from 0 to 1500 rpm. The rotor initially has an offset of 7 degrees about the gimbal x axis. The circles represent points obtained through the simulation DTGSim and the line is fit to the data using a least squares method. The relationship between the stress and the time is given by $\sigma = 315.5 + \frac{1.7}{t} + \frac{63.4}{t^2}$.

Again, assuming that the MMS is oriented 45 degrees to the direction of acceleration, the largest load on a strip is given by Equation 4.47 as

$$P = \frac{1}{2} (V \sec \alpha + H \csc \alpha)$$

where the constraint forces, λ_1 and λ_2 , have been neglected since they are small compared to the applied loads and $\alpha = 45$ degrees. The accelerational load is given by

$$V = H = \frac{1}{2} F_g \sin 45 \\ 0.354 F_g$$

where F_g is the sum of the rotor and gimbal mass multiplied by the acceleration due to gravity. Therefore, the axial load is

$$P = 0.5 (M_r + M_g) 9.81 = 14.6 \text{ N}$$

For this calculation, the shaft speed of the MMS will be assumed to be zero.

Substituting the dimensions of the MMS pivots from Table 6.1, the critical buckling load is calculated to be

$$P_{cr} = 120.2 \text{ N}$$

The acceleration the MMS can withstand without the pivots buckling is

$$\#G = \frac{P_{cr}}{P} \\ \approx 8.2g$$

Therefore, the pivots can withstand accelerations up to 8 times that of gravity. This value is large enough to withstand any normal accelerations

that the MMS would experience once the satellite is in its proper orbit but it is not enough for the pivots to withstand launch loads which can be 20g or higher. The necessity of some mechanism to protect the pivots from damage during these periods of high acceleration has been justified. Discussion of this mechanism is outside the scope of this research.

6.5 Summary

The design of the crossed flexure pivot was based on ensuring that the pivot can withstand the operational loads acting on it when on Earth. Since the pivots are a critical component in the MMS they were designed early in the overall design process. The pivot design was then fixed so that other components could be properly sized. The result was that the final rotor and gimbal design were different than what was used to design the pivots. Once the MMS design was complete it was necessary to revisit the pivot design and ensure that it was still valid. A check was performed and the design was found to still be suitable for its purpose.

The design process also ignored the effects of start-up and changes in the dimensions of the strips. Calculations were performed and it was found that the start-up loads could be withstood by the pivots. It was found that changes in the dimensions of the strips had a large effect on the tuned speed of the system and so it is important to ensure that the machining process is very accurate.

The pivot design is based on a static analysis of the crossed flexure pivot. The system will be operating at rotor speeds of 1200 to 1800 rpm which means that the gimbal will be fluttering at 40 to 60 Hz. This is also the

frequency of oscillation of the pivots. A vibrational analysis of the crossed flexure pivot will be made in order to ensure that the MMS does not operate at a resonance frequency of the pivots and to determine the mostly likely form of the pivot deflection. This is done in Chapter 7.

Chapter 7

Vibration Analysis of the Crossed Flexure Pivot

The static behaviour of the crossed flexure pivot was examined in Chapter 4 and it is now necessary to determine the dynamic behaviour of the system. This study will consist of examining the unforced response of the pivot. Hamilton's principle of least action served well for the static case and it will be used again in this analysis.

Only small vibrations about the undeflected position will be examined which means that conventional linear vibration analysis techniques can be utilized. The results will give an indication of the modes of vibration of the pivot and the frequencies at which they occur. The equations for the stiffness of the pivot, found in Chapter 3, Section 3.3.2 and Chapter 4, Section 4.1, have been based on the assumption that the deflection of the strips will be equivalent to the first mode shape of a cantilever beam and this analysis should determine the validity of that assumption.

In the previous chapter, the strips of the pivot were assumed to be held fixed at one end and governed by the constraint equations at the other. When

examining the pivot statically this assumption is valid since the reference frame can always be fixed to one of the two rigid bodies. For a dynamic analysis this could be done but any motion of the reference frame would have to be taken into account as well.

Derivation of the equations of motion for a crossed flexure pivot are more straightforward if the reference frame is fixed inertially which means that motion of both rigid bodies must be considered. Therefore, for this dynamic analysis both ends of the strips will be bound by the constraint equations set out in Chapter 3, Section 3.3.1.

It is assumed that the strips will behave as Euler-Bernoulli beams and that both strips will have constant density ρ , cross-sectional area A , Young's modulus E and area moment of inertia I . The rigid bodies at the two ends of the strips have mass moments of inertia J_B and J_A , respectively. In the static analysis, θ defined the angular rotation of the freebody with respect to its original position. In this analysis θ_A and θ_B will represent the rotations of the bodies A and B. This analysis is not concerned with any rigid body translation or rotation of the pivot so these terms will be neglected.

7.1 Development of the Equations of Motion

The kinetic energy, $\mathcal{T}$, of the crossed flexure pivot is given by

$$\begin{aligned} \mathcal{T} = & \frac{1}{2} \int_0^L (\rho A \dot{u}_1^2 + \rho A \dot{v}_1^2) d\tau + \frac{1}{2} \int_0^L (\rho A \dot{u}_2^2 + \rho A \dot{v}_2^2) d\sigma \\ & + \left[J_B \dot{\theta}_B^2 + m_B \dot{p}_B^2 + m_B \dot{q}_B^2 \right]_{\tau=\sigma=0} \\ & + \left[J_A \dot{\theta}_A^2 + m_A \dot{p}_A^2 + m_A \dot{q}_A^2 \right]_{\tau=\sigma=L} \end{aligned} \quad (7.1)$$

where $\dot{u}_1$ is the time derivative of u_1 , u_1 and u_2 are the transverse displacements of strips '1' and '2' and v_1 and v_2 are the axial displacements of the strips. The variables τ and σ represent the distance along strips '1' and '2' from the rigid body with inertia J_B . The masses of the two rigid bodies are represented by m_A and m_B .

Finally, note that

$$p = \frac{1}{2} \csc \alpha (v_1 - v_2) \quad (7.2)$$

$$q = \frac{1}{2} \sec \alpha [2rL (\cos \theta - 1) - (v_1 + v_2)] \quad (7.3)$$

These relationships were derived in Chapter 3, Section 3.3.1 and the values of p and q represent the horizontal and vertical displacement of each of the two rigid bodies. The following constraint equations governing the motion of the ends of the strips were also derived

$$u_1 + u_2 - 2rL \sin \theta - (v_1 - v_2) \cot \alpha = 0 \quad (7.4)$$

$$u_1 - u_2 + 2rL \tan \alpha (1 - \cos \theta) + (v_1 + v_2) \tan \alpha = 0 \quad (7.5)$$

$$u_{1\tau} - \tan \theta = 0 \quad (7.6)$$

$$u_{2\sigma} - \tan \theta = 0 \quad (7.7)$$

Rewriting Equation 7.5 in terms of the ' $\cos \theta - 1$ ' term and substituting this into Equation 7.3 gives the relationship

$$q = \frac{1}{2} \sec \alpha \left[\frac{u_1 - u_2}{\tan \alpha} + (v_1 + v_2) - (v_1 + v_2) \right]$$

which simplifies to give

$$q = \frac{1}{2} \csc \alpha (u_1 - u_2) \quad (7.8)$$

This substitution eliminates the ' $1 - \cos \theta$ ' term which disappears if a linearization with respect to θ is made. Taking the derivative of Equations 7.2 and 7.8 with respect to time yields

$$\dot{p} = \frac{1}{2} \csc \alpha (\dot{v}_1 - \dot{v}_2) \quad (7.9)$$

$$\dot{q} = \frac{1}{2} \csc \alpha (\dot{u}_1 - \dot{u}_2) \quad (7.10)$$

The potential energy, $\mathcal{V}$, of the system is given by

$$\begin{aligned} \mathcal{V} = & \frac{1}{2} \int_0^L (EA v_{1\tau}^2 + El u_{1\tau\tau}^2) d\tau \\ & + \frac{1}{2} \int_0^L (EA v_{2\sigma}^2 + El u_{2\sigma\sigma}^2) d\sigma \end{aligned} \quad (7.11)$$

where, for example, $v_{1\tau}$ represents the first derivative of v_1 with respect to τ . It should be noted that the motion of the two rigid bodies is with respect to the initial centre of rotation of the pivot.

The variables u_i , v_i and θ_i are not independent at the ends of the strips but are related through constraint Equations 7.4 to 7.7. These equations are valid when $\tau = \sigma = 0$ and $\tau = \sigma = L$.

If a comparison is made between the equations listed here and those shown in the Chapters 3 and 4, it will be noticed that e_i has been replaced with v_i . Earlier, Section 4.1, it was stated that

$$e_i = v_i - \frac{1}{2} \int_0^L u_{ix}^2 dx$$

where v_i represents the axial deflection of the strip and the second term represents the shortening of the strip due to bending.

In this analysis, the second term will be considered negligible. It was shown in Section 4.3.1 that this deflection is small even when the slope at

the ends of the strips is approximately 7 degrees. In fact, the shortening due to bending is a function of the square of the slope of the strips, that is $u_{1\tau}^2$. Therefore neglecting this term is the equivalent to neglecting terms involving θ^2 in c_1 .

Including the constraint equations, the action integral can be expressed as

$$\begin{aligned}
 \mathcal{I} = & \int_{t_1}^{t_2} \left[\frac{1}{2} \int_0^L (\rho A \dot{u}_1^2 + \rho A \dot{v}_1^2 - E A v_{1\tau}^2 - E I u_{1\tau\tau}^2) d\tau \right. \\
 & + \frac{1}{2} \int_0^L (\rho A \dot{u}_2^2 + \rho A \dot{v}_2^2 - E A v_{2\sigma}^2 - E I u_{2\sigma\sigma}^2) d\sigma \\
 & + \left[\frac{1}{2} J_B \dot{\theta}_B^2 + \frac{1}{2} m_B \left(\frac{1}{2} \csc \alpha (\dot{v}_1 - \dot{v}_2) \right)^2 + \frac{1}{2} m_B \left(\frac{1}{2} \csc \alpha (\dot{u}_1 - \dot{u}_2) \right)^2 \right. \\
 & + \lambda_1 (u_1 + u_2 - 2rL \sin \theta_B - (v_1 - v_2) \cot \alpha) \\
 & + \lambda_2 (u_1 - u_2 + (v_1 + v_2) \tan \alpha + 2rL \tan \alpha (1 - \cos \theta_B)) \\
 & + \lambda_3 (u_{1\tau} - \tan \theta_B) + \lambda_4 (u_{2\sigma} - \tan \theta_B) \Big|_{\tau=\sigma=0} \\
 & + \left[\frac{1}{2} J_A \dot{\theta}_A^2 + \frac{1}{2} m_A \left(\frac{1}{2} \csc \alpha (\dot{v}_1 - \dot{v}_2) \right)^2 + \frac{1}{2} m_A \left(\frac{1}{2} \csc \alpha (\dot{u}_1 - \dot{u}_2) \right)^2 \right. \\
 & + \lambda_5 (u_1 + u_2 - 2rL \sin \theta_A - (v_1 - v_2) \cot \alpha) \\
 & + \lambda_6 (u_1 - u_2 + (v_1 + v_2) \tan \alpha + 2rL \tan \alpha (1 - \cos \theta_A)) \\
 & \left. \left. + \lambda_7 (u_{1\tau} - \tan \theta_A) + \lambda_8 (u_{2\sigma} - \tan \theta_A) \Big|_{\tau=\sigma=L} \right] dt \right] \quad (7.12)
 \end{aligned}$$

Hamilton's principle of least action states that the first variance of the action integral is zero; therefore

$$\begin{aligned}
 \delta \mathcal{I} = & \int_{t_1}^{t_2} \left[\frac{1}{2} \int_0^L (\rho A \dot{u}_1 \delta \dot{u}_1 + \rho A \dot{v}_1 \delta \dot{v}_1 - E A v_{1\tau} \delta v_{1\tau} - E I u_{1\tau\tau} \delta u_{1\tau\tau}) d\tau \right. \\
 & + \frac{1}{2} \int_0^L (\rho A \dot{u}_2 \delta \dot{u}_2 + \rho A \dot{v}_2 \delta \dot{v}_2 - E A v_{2\sigma} \delta v_{2\sigma} - E I u_{2\sigma\sigma} \delta u_{2\sigma\sigma}) d\sigma \\
 & \left. + \left[J_B \dot{\theta}_B \delta \dot{\theta}_B + \frac{1}{4} m_B \csc^2 \alpha (\dot{v}_1 - \dot{v}_2) (\delta \dot{v}_1 - \delta \dot{v}_2) \right. \right.
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{4} m_B \csc^2 \alpha (\dot{u}_1 - \dot{u}_2) (\delta \dot{u}_1 - \delta \dot{u}_2) \\
 & + \lambda_1 (\delta u_1 + \delta u_2 - 2rL \cos \theta_B \delta \theta_B - (\delta v_1 - \delta v_2) \cot \alpha) \\
 & + \lambda_2 (\delta u_1 - \delta u_2 + (\delta v_1 + \delta v_2) \tan \alpha + 2rL \tan \alpha \sin \theta_B \delta \theta_B) \\
 & + \lambda_3 (\delta u_{1\tau} - \sec^2 \theta_B \delta \theta_B) + \lambda_4 (\delta u_{2\sigma} - \sec^2 \theta_B \delta \theta_B) \\
 & + \delta \lambda_1 (u_1 + u_2 - 2rL \sin \theta_B - (v_1 - v_2) \cot \alpha) \\
 & + \delta \lambda_2 (u_1 - u_2 + (v_1 + v_2) \tan \alpha + 2rL \tan \alpha (1 - \cos \theta_B)) \\
 & + \delta \lambda_3 (u_{1\tau} - \tan \theta_B) + \delta \lambda_4 (u_{2\sigma} - \tan \theta_B) \Big|_{\tau = \sigma = 0} \\
 & + \left[J_A \dot{\theta}_A \delta \dot{\theta}_A + \frac{1}{4} m_A \csc^2 \alpha (\dot{v}_1 - \dot{v}_2) (\delta \dot{v}_1 - \delta \dot{v}_2) \right. \\
 & \quad + \frac{1}{4} m_A \csc^2 \alpha (\dot{u}_1 - \dot{u}_2) (\delta \dot{u}_1 - \delta \dot{u}_2) \\
 & \quad + \lambda_5 (\delta u_1 + \delta u_2 - 2rL \cos \theta_A \delta \theta_A - (\delta v_1 - \delta v_2) \cot \alpha) \\
 & \quad + \lambda_6 (\delta u_1 - \delta u_2 + (\delta v_1 + \delta v_2) \tan \alpha + 2rL \tan \alpha \sin \theta_A \delta \theta_A) \\
 & \quad + \lambda_7 (\delta u_{1\tau} - \sec^2 \theta_A \delta \theta_A) + \lambda_8 (\delta u_{2\sigma} - \sec^2 \theta_A \delta \theta_A) \\
 & \quad + \delta \lambda_5 (u_1 + u_2 - 2rL \sin \theta_A - (v_1 - v_2) \cot \alpha) \\
 & \quad + \delta \lambda_6 (u_1 - u_2 + (v_1 + v_2) \tan \alpha + 2rL \tan \alpha (1 - \cos \theta_A)) \\
 & \quad \left. + \delta \lambda_7 (u_{1\tau} - \tan \theta_A) + \delta \lambda_8 (u_{2\sigma} - \tan \theta_A) \Big|_{\tau = \sigma = L} \right] dt = 0
 \end{aligned}$$

Integration by parts can now be used to simplify the integrand. For example

$$\begin{aligned}
 \int_{t_1}^{t_2} \int_0^L \dot{u}_1 \delta \dot{u}_1 d\tau dt &= \int_0^L [\dot{u}_1 \delta u_1]_{t_1}^{t_2} d\tau - \int_{t_1}^{t_2} \int_0^L \ddot{u}_1 \delta u_1 d\tau dt \\
 \int_{t_1}^{t_2} \int_0^L v_{1\tau} \delta v_{1\tau} d\tau dt &= \int_{t_1}^{t_2} [v_{1\tau} \delta v_1]_0^L d\tau - \int_{t_1}^{t_2} \int_0^L v_{1\tau\tau} \delta v_1 d\tau dt \\
 \int_{t_1}^{t_2} \int_0^L u_{1\tau\tau} \delta u_{1\tau\tau} d\tau dt &= \int_{t_1}^{t_2} [u_{1\tau\tau} \delta u_{1\tau} - u_{1\tau\tau\tau} \delta u_1]_0^L d\tau \\
 &\quad + \int_{t_1}^{t_2} \int_0^L u_{1\tau\tau\tau\tau} \delta u_1 d\tau dt
 \end{aligned}$$

$$\int_{t_1}^{t_2} \dot{\theta}_B \delta \dot{\theta}_B dt = [\dot{\theta}_B \delta \theta_B]_{t_1}^{t_2} - \int_{t_1}^{t_2} \ddot{\theta}_B \delta \theta_B dt$$

After simplification and collection of like terms, the variance of the action integral can be written as

$$\begin{aligned} \delta \mathcal{I} = & \int_{t_1}^{t_2} \int_0^L [(EA v_{1\tau\tau} - \rho A \ddot{v}_1) \delta v_1 - (EI u_{1\tau\tau\tau\tau} + \rho A \ddot{u}_1) \delta u_1] d\tau dt \\ & + \int_{t_1}^{t_2} \int_0^L [(EA v_{2\sigma\sigma} - \rho A \ddot{v}_2) \delta v_2 - (EI u_{2\sigma\sigma\sigma\sigma} + \rho A \ddot{u}_2) \delta u_2] d\sigma dt \\ & + \int_{t_1}^{t_2} \left[\left(-EI u_{1\tau\tau\tau} - \frac{1}{4} m_B \csc^2 \alpha (\ddot{u}_1 - \ddot{u}_2) + \lambda_2 + \lambda_1 \right) \delta u_1 \right. \\ & \quad + (EI u_{1\tau\tau} + \lambda_3) \delta u_{1\tau} \\ & \quad + \left(EA v_{1\tau} - \frac{1}{4} m_B \csc^2 \alpha (\ddot{v}_1 - \ddot{v}_2) + \lambda_2 \tan \alpha - \lambda_1 \cot \alpha \right) \delta v_1 \\ & \quad + \left(-EI u_{2\sigma\sigma\sigma} + \frac{1}{4} m_B \csc^2 \alpha (\ddot{u}_1 - \ddot{u}_2) - \lambda_2 + \lambda_1 \right) \delta u_2 \\ & \quad + (EI u_{2\sigma\sigma} + \lambda_4) \delta u_{2\sigma} \\ & \quad + \left(EA v_{2\sigma} + \frac{1}{4} m_B \csc^2 \alpha (\ddot{v}_1 - \ddot{v}_2) \right. \\ & \quad \quad \left. + \lambda_2 \tan \alpha + \lambda_1 \cot \alpha \right) \delta v_2 \Big|_{\tau=\sigma=0} dt \\ & + \int_{t_1}^{t_2} \left[\left(EI u_{1\tau\tau\tau} - \frac{1}{4} m_A \csc^2 \alpha (\ddot{u}_1 - \ddot{u}_2) + \lambda_6 + \lambda_5 \right) \delta u_1 \right. \\ & \quad + (-EI u_{1\tau\tau} + \lambda_7) \delta u_{1\tau} \\ & \quad + \left(-EA v_{1\tau} - \frac{1}{4} m_A \csc^2 \alpha (\ddot{v}_1 - \ddot{v}_2) + \lambda_6 \tan \alpha - \lambda_5 \cot \alpha \right) \delta v_1 \\ & \quad + \left(EI u_{2\sigma\sigma\sigma} + \frac{1}{4} m_A \csc^2 \alpha (\ddot{u}_1 - \ddot{u}_2) - \lambda_6 + \lambda_5 \right) \delta u_2 \\ & \quad + (-EI u_{2\sigma\sigma} + \lambda_8) \delta u_{2\sigma} \\ & \quad + \left(-EA v_{2\sigma} + \frac{1}{4} m_A \csc^2 \alpha (\ddot{v}_1 - \ddot{v}_2) \right. \\ & \quad \quad \left. + \lambda_6 \tan \alpha + \lambda_5 \cot \alpha \right) \delta v_2 \Big|_{\tau=\sigma=L} dt \\ & + \int_{t_1}^{t_2} \left[-J_B \ddot{\theta}_B + 2\lambda_2 r L \tan \alpha \sin \theta_B - 2\lambda_1 r L \cos \theta_B \right. \\ & \quad \left. - \lambda_3 \sec^2 \theta_B - \lambda_4 \sec^2 \theta_B \right]_{\tau=\sigma=0} \delta \theta_B dt \end{aligned}$$

$$\begin{aligned}
 & + \int_{t_1}^{t_2} \left[-J_A \ddot{\theta}_A + 2\lambda_6 r L \tan \alpha \sin \theta_A - 2\lambda_5 r L \cos \theta_A \right. \\
 & \quad \left. - \lambda_7 \sec^2 \theta_A - \lambda_8 \sec^2 \theta_A \right]_{\tau=\sigma=L} \delta \theta_A dt \\
 & + \int_{t_1}^{t_2} \left[(\delta \lambda_2 (u_1 - u_2 + (v_1 + v_2) \tan \alpha + 2r L \tan \alpha (1 - \cos \theta_B)) \right. \\
 & \quad + \delta \lambda_1 (u_1 + u_2 - 2r L \sin \theta_B - (v_1 - v_2) \cot \alpha) \\
 & \quad + \delta \lambda_3 (u_{1\tau} - \tan \theta_B) + \delta \lambda_4 (u_{2\sigma} - \tan \theta_B))_{\tau=\sigma=0} \\
 & \quad + (\delta \lambda_6 (u_1 - u_2 + (v_1 + v_2) \tan \alpha + 2r L \tan \alpha (1 - \cos \theta_A)) \\
 & \quad + \delta \lambda_5 (u_1 + u_2 - 2r L \sin \theta_A - (v_1 - v_2) \cot \alpha) \\
 & \quad \left. + \delta \lambda_7 (u_{1\tau} - \tan \theta_A) + \delta \lambda_8 (u_{2\sigma} - \tan \theta_A))_{\tau=\sigma=L} \right] dt \\
 & + \left[(J_B \dot{\theta}_B \delta \theta_B)_{\tau=\sigma=0} + (J_A \dot{\theta}_A \delta \theta_A)_{\tau=\sigma=L} \right. \\
 & \quad \left. + \rho A \int_0^L (\dot{u}_1 \delta u_1 + \dot{v}_1 \delta v_1) d\tau + \rho A \int_0^L (\dot{u}_2 \delta u_2 + \dot{v}_2 \delta v_2) d\sigma \right]_{t_1}^{t_2} = 0
 \end{aligned}$$

Since δu_1 , δu_2 , δv_1 , δv_2 , $\delta u_{1\tau}$, $\delta u_{2\sigma}$, $\delta \theta_A$ and $\delta \theta_B$ are arbitrarily assigned, each term in the above expression must equal zero in order to avoid a trivial solution. Further, the variants have been defined such that they equal zero when evaluated at t_1 and t_2 and those terms can be ignored. With this in mind, the following four differential equations are obtained

$$\rho A \ddot{u}_1 + E I u_{1\tau\tau\tau\tau} = 0 \quad (7.13)$$

$$\rho A \ddot{u}_2 + E I u_{2\sigma\sigma\sigma\sigma} = 0 \quad (7.14)$$

$$\rho A \ddot{v}_1 - E A v_{1\tau\tau} = 0 \quad (7.15)$$

$$\rho A \ddot{v}_2 - E A v_{2\sigma\sigma} = 0 \quad (7.16)$$

as well as the following 14 natural boundary conditions. The first 7 are valid when $\tau = \sigma = 0$ and are

$$E I u_{1\tau\tau\tau} + \frac{1}{4} m_B \csc^2 \alpha (\ddot{u}_1 - \ddot{u}_2) - \lambda_2 - \lambda_1 = 0 \quad (7.17)$$

$$EIu_{1\tau\tau} + \lambda_3 = 0 \quad (7.18)$$

$$EA v_{1\tau} - \frac{1}{4}m_B \csc^2 \alpha (\ddot{v}_1 - \ddot{v}_2) + \lambda_2 \tan \alpha - \lambda_1 \cot \alpha = 0 \quad (7.19)$$

$$EIu_{2\sigma\sigma\sigma} - \frac{1}{4}m_B \csc^2 \alpha (\ddot{u}_1 - \ddot{u}_2) + \lambda_2 - \lambda_1 = 0 \quad (7.20)$$

$$EIu_{2\sigma\sigma} + \lambda_4 = 0 \quad (7.21)$$

$$EA v_{2\sigma} + \frac{1}{4}m_B \csc^2 \alpha (\ddot{v}_1 - \ddot{v}_2) + \lambda_2 \tan \alpha + \lambda_1 \cot \alpha = 0 \quad (7.22)$$

$$\begin{aligned} & -J_B \ddot{\theta}_B + 2\lambda_2 r L \tan \alpha \sin \theta_B - 2\lambda_1 r L \cos \theta_B \\ & - \lambda_3 \sec^2 \theta_B - \lambda_4 \sec^2 \theta_B = 0 \end{aligned} \quad (7.23)$$

The remaining 7 natural boundary conditions are valid when $\tau = \sigma = L$

$$EIu_{1\tau\tau\tau} - \frac{1}{4}m_A \csc^2 \alpha (\ddot{u}_1 - \ddot{u}_2) + \lambda_6 + \lambda_5 = 0 \quad (7.24)$$

$$EIu_{1\tau\tau} - \lambda_7 = 0 \quad (7.25)$$

$$EA v_{1\tau} + \frac{1}{4}m_A \csc^2 \alpha (\ddot{v}_1 - \ddot{v}_2) - \lambda_6 \tan \alpha + \lambda_5 \cot \alpha = 0 \quad (7.26)$$

$$EIu_{2\sigma\sigma\sigma} + \frac{1}{4}m_A \csc^2 \alpha (\ddot{u}_1 - \ddot{u}_2) - \lambda_6 + \lambda_5 = 0 \quad (7.27)$$

$$EIu_{2\sigma\sigma} - \lambda_8 = 0 \quad (7.28)$$

$$EA v_{2\sigma} - \frac{1}{4}m_A \csc^2 \alpha (\ddot{v}_1 - \ddot{v}_2) - \lambda_6 \tan \alpha - \lambda_5 \cot \alpha = 0 \quad (7.29)$$

$$\begin{aligned} & -J_A \ddot{\theta}_A + 2\lambda_6 r L \tan \alpha \sin \theta_A - 2\lambda_5 r L \cos \theta_A \\ & - \lambda_7 \sec^2 \theta_A - \lambda_8 \sec^2 \theta_A = 0 \end{aligned} \quad (7.30)$$

The remaining terms in the variance of the action integral make up the constraint equations at both ends of the strips. In other words, when $\tau = \sigma = 0$

$$u_1 + u_2 - 2rL \sin \theta_B - (v_1 - v_2) \cot \alpha = 0 \quad (7.31)$$

$$u_1 - u_2 + (v_1 + v_2) \tan \alpha + 2rL \tan \alpha (1 - \cos \theta_B) = 0 \quad (7.32)$$

$$u_{1\tau} - \tan \theta_B = 0 \quad (7.33)$$

$$u_{2\sigma} - \tan \theta_B = 0 \quad (7.34)$$

and when $\tau = \sigma = L$,

$$u_1 + u_2 - 2rL \sin \theta_A - (v_1 + v_2) \cot \alpha = 0 \quad (7.35)$$

$$u_1 - u_2 + (v_1 + v_2) \tan \alpha + 2rL \tan \alpha (1 - \cos \theta_A) = 0 \quad (7.36)$$

$$u_{1\tau} - \tan \theta_A = 0 \quad (7.37)$$

$$u_{2\sigma} - \tan \theta_A = 0 \quad (7.38)$$

It is possible to isolate the Lagrange multipliers using Equations 7.17, 7.18, 7.20, 7.21, 7.24, 7.25, 7.27 and 7.28 and obtain

$$\begin{aligned} \lambda_1 &= \frac{1}{2}EI(u_{1\tau\tau\tau} + u_{2\sigma\sigma\sigma}) \\ \lambda_2 &= \frac{1}{2}\left(EIu_{1\tau\tau\tau} - EIu_{2\sigma\sigma\sigma} + \frac{1}{2}m_B \csc^2 \alpha (\ddot{u}_1 - \ddot{u}_2)\right) \\ \lambda_3 &= -EIu_{1\tau\tau} \\ \lambda_4 &= -EIu_{2\sigma\sigma} \\ \lambda_5 &= -\frac{1}{2}EI(u_{1\tau\tau\tau} + u_{2\sigma\sigma\sigma}) \\ \lambda_6 &= -\frac{1}{2}\left(EIu_{1\tau\tau\tau} - EIu_{2\sigma\sigma\sigma} - \frac{1}{2}m_A \csc^2 \alpha (\ddot{u}_1 - \ddot{u}_2)\right) \\ \lambda_7 &= EIu_{1\tau\tau} \\ \lambda_8 &= EIu_{2\sigma\sigma} \end{aligned}$$

The 6 remaining natural boundary conditions can be rewritten as

$$\begin{aligned} EA v_{1\tau} &= \frac{1}{4}m_B \csc^2 \alpha (\ddot{v}_1 - \ddot{v}_2) \\ &+ \frac{1}{2} \tan \alpha \left(EIu_{1\tau\tau\tau} - EIu_{2\sigma\sigma\sigma} + \frac{1}{2}m_B \csc^2 \alpha (\ddot{u}_1 - \ddot{u}_2) \right) \\ &- \frac{1}{2}EI \cot \alpha (u_{1\tau\tau\tau} + u_{2\sigma\sigma\sigma}) = 0 \end{aligned} \quad (7.39)$$

$$\begin{aligned}
 EA v_{2\sigma} &+ \frac{1}{4} m_B \csc^2 \alpha (\ddot{v}_1 - \ddot{v}_2) \\
 &+ \frac{1}{2} \tan \alpha \left(EI u_{1\tau\tau\tau} - EI u_{2\sigma\sigma\sigma} + \frac{1}{2} m_B \csc^2 \alpha (\ddot{u}_1 - \ddot{u}_2) \right) \\
 &+ \frac{1}{2} EI \cot \alpha (u_{1\tau\tau\tau} + u_{2\sigma\sigma\sigma}) = 0
 \end{aligned} \tag{7.10}$$

$$\begin{aligned}
 -J_B \ddot{\theta}_B &+ rL \tan \alpha \sin \theta_B \left(EI u_{1\tau\tau\tau} - EI u_{2\sigma\sigma\sigma} + \frac{1}{2} m_B \csc^2 \alpha (\ddot{u}_1 - \ddot{u}_2) \right) \\
 &- EIrL \cos \theta_B (u_{1\tau\tau\tau} + u_{2\sigma\sigma\sigma}) \\
 &+ EI \sec^2 \theta_B (u_{1\tau\tau} + u_{2\tau\tau}) = 0
 \end{aligned} \tag{7.11}$$

when $\tau = \sigma = 0$ and

$$\begin{aligned}
 EA v_{1\tau} &+ \frac{1}{4} m_A \csc^2 \alpha (\ddot{v}_1 - \ddot{v}_2) \\
 &+ \frac{1}{2} \tan \alpha \left(EI u_{1\tau\tau\tau} - EI u_{2\sigma\sigma\sigma} - \frac{1}{2} m_A \csc^2 \alpha (\ddot{u}_1 - \ddot{u}_2) \right) \\
 &- \frac{1}{2} EI \cot \alpha (u_{1\tau\tau\tau} + u_{2\sigma\sigma\sigma}) = 0
 \end{aligned} \tag{7.12}$$

$$\begin{aligned}
 EA v_{2\sigma} &- \frac{1}{2} m_A \csc^2 \alpha (\ddot{v}_1 - \ddot{v}_2) \\
 &+ \frac{1}{2} \tan \alpha \left(EI u_{1\tau\tau\tau} - EI u_{2\sigma\sigma\sigma} - \frac{1}{2} m_A \csc^2 \alpha (\ddot{u}_2 - \ddot{u}_1) \right) \\
 &+ \frac{1}{2} EI \cot \alpha (u_{1\tau\tau\tau} + u_{2\sigma\sigma\sigma}) = 0
 \end{aligned} \tag{7.13}$$

$$\begin{aligned}
 -J_A \ddot{\theta}_A &- rL \tan \alpha \sin \theta_A \left(EI u_{1\tau\tau\tau} - EI u_{2\sigma\sigma\sigma} - \frac{1}{2} m_A \csc^2 \alpha (\ddot{u}_1 - \ddot{u}_2) \right) \\
 &+ EIrL \cos \theta_A (u_{1\tau\tau\tau} + u_{2\sigma\sigma\sigma}) \\
 &- EI \sec^2 \theta_A (u_{1\tau\tau} + u_{2\tau\tau}) = 0
 \end{aligned} \tag{7.14}$$

when $\tau = \sigma = L$.

Assuming separation of variables, that is

$$u_1(\tau, t) = U_1(\tau)e^{j\omega t}$$

$$u_2(\sigma, t) = U_2(\sigma)e^{j\omega t}$$

$$v_1(\tau, t) = V_1(\tau)e^{j\omega t}$$

$$v_2(\sigma, t) = V_2(\sigma)e^{j\omega t}$$

then Equation 7.13 can be rewritten as

$$-\omega^2 \rho A U_1 + EI U_1'''' = 0$$

where U_1'''' is the fourth derivative of U_1 with respect to τ . This has the solution

$$U_1 = c_1 \cos k\tau + c_2 \sin k\tau + c_3 \cosh k\tau + c_4 \sinh k\tau \quad (7.45)$$

where c_i are integration constants and

$$k^4 = \frac{\omega^2 \rho A}{EI}$$

Similar solutions can be found for Equations 7.14 to 7.16 and are expressed as

$$U_2 = c_5 \cos k\sigma + c_6 \sin k\sigma + c_7 \cosh k\sigma + c_8 \sinh k\sigma \quad (7.46)$$

$$V_1 = c_9 \cos gk^2\tau + c_{10} \sin gk^2\tau \quad (7.47)$$

$$V_2 = c_{11} \cos gk^2\tau + c_{12} \sin gk^2\tau \quad (7.48)$$

where $g^2 = I/A$.

Constraint Equations 7.33 and 7.37 can be used to find the relationship between the angles θ_A and θ_B and the slope of the strips. Linearizing with respect to θ_i gives

$$\theta_B = U_1'(0)$$

$$\theta_A = U_1'(L)$$

Using these relationships, neglecting terms involving θ_1^2 or higher and realizing that, for example, $\ddot{v}_1 = -\omega^2 V_1$, allows the six boundary conditions and six remaining constraint equations to be simplified and rewritten as

$$\begin{aligned} V_1' &+ cg^2k^4(V_1 - V_2) - ag^2(U_1''' + U_2''') \\ &+ bg^2(U_1''' - U_2''') - dg^2k^4(U_1 - U_2) = 0 \end{aligned} \quad (7.49)$$

$$\begin{aligned} V_2' &- cg^2k^4(V_1 - V_2) + ag^2(U_1''' + U_2''') \\ &+ bg^2(U_1''' - U_2''') - dg^2k^4(U_1 - U_2) = 0 \end{aligned} \quad (7.50)$$

$$2hk^4U_1' + U_1'' + U_2'' - l(U_1''' + U_2''') = 0 \quad (7.51)$$

$$U_1 + U_2 - 2lU_1' - 2a(V_1 - V_2) = 0 \quad (7.52)$$

$$U_1 - U_2 + 2b(V_1 + V_2) = 0 \quad (7.53)$$

$$U_2' - U_1' = 0 \quad (7.54)$$

$$\begin{aligned} V_1' &- cg^2k^4(V_1 - V_2) - ag^2(U_1''' + U_2''') \\ &+ bg^2(U_1''' - U_2''') + fg^2k^4(U_1 - U_2) = 0 \end{aligned} \quad (7.55)$$

$$\begin{aligned} V_2' &+ cg^2k^4(V_1 - V_2) + ag^2(U_1''' + U_2''') \\ &+ bg^2(U_1''' - U_2''') + fg^2k^4(U_1 - U_2) = 0 \end{aligned} \quad (7.56)$$

$$2ik^4U_1' + l(U_1''' + U_2''') - U_1'' - U_2'' = 0 \quad (7.57)$$

$$U_1 + U_2 - 2lU_1' - 2a(V_1 - V_2) = 0 \quad (7.58)$$

$$U_1 - U_2 + 2b(V_1 + V_2) = 0 \quad (7.59)$$

$$U_2' - U_1' = 0 \quad (7.60)$$

where Equations 7.49 to 7.54 are evaluated at $\tau = \sigma = 0$ and Equations 7.55 to 7.60 are evaluated at $\tau = \sigma = L$. The constants introduced in these equations are defined as

$$\begin{aligned} a &= \frac{1}{2} \cot \alpha \\ b &= \frac{1}{2} \tan \alpha \\ c &= \frac{1}{4} \frac{m_B \csc^2 \alpha}{\rho A} \\ d &= \frac{m_B \csc 2\alpha}{2\rho A} \\ e &= \frac{1}{4} \frac{m_a \csc^2 \alpha}{\rho A} \\ f &= \frac{m_A \csc 2\alpha}{2\rho A} \\ h &= \frac{J_B}{2\rho A} \\ i &= \frac{J_A}{2\rho A} \\ l &= rL \end{aligned}$$

One term in Equations 7.41 and 7.44 was neglected and further discussion is required at this point. The term that is being discussed is

$$\frac{rL}{2} \tan \alpha \left(EIU_1''' - EIU_2''' - \frac{1}{2} m_B \omega^2 \csc^2 \alpha (U_1 - U_2) \right) \theta_B$$

from Equation 7.41 and the similar term in Equation 7.44. When examining this term it is natural to break it into two parts; one involving the transverse deflection terms, $U_1 - U_2$, and the other involving the shear terms, $U_1''' - U_2'''$. It was shown in Chapter 3, Section 3.3.1 that for small angles of rotation

the transverse deflection of the two strips can be assumed equal. Therefore $U_1 - U_2 \approx 0$ and this removes the first of the two parts. The shear terms are the result of a transverse force acting on the strips (see Chapter 4) which is dependent on the angle of rotation of the rigid bodies. In this study only small angular deflections are being examined so this term can be neglected since it is multiplied by θ_B which makes it dependent on θ_B^2 . Similar arguments can be made for the term in Equation 7.44.

Neglecting this term when simplifying Equations 7.41 and 7.44 is the same as neglecting the ' $\cos \theta - 1$ ' term in Equation 7.5. It is uncertain how omitting this term effects the calculated properties of the crossed flexure pivot and this would be a good area in which future work could be performed. For the purposes of this research it is enough to say that the term will be small when compared to the rest and can be ignored.

It is at this point in the analysis that the boundary and constraint equations must be used to solve for the 13 unknowns, c_1 through c_{12} and ω . Substituting the solutions to the differential equations, Equations 7.45 to 7.48, into Equations 7.49 to 7.60 yields twelve equations which are functions of the integration constants and ω . Putting the equations into matrix form yields

$$A\mathbf{C} = \mathbf{0}$$

where $\mathbf{C}$ represents a column of the integration constants and A contains the coefficients of these constants. In order to obtain a non trivial solution, the determinate of A must equal zero, the result is known as the characteristic equation. Solving for the roots of the characteristic equation gives the natural frequencies of vibration of the system.

In this case, the characteristic equation is formed by calculating the de-

terminate of a 12 by 12 matrix. This is too large and cumbersome to do manually and therefore a combination of Maple V[92] and Matlab[76] was used to assist in the calculations. Maple was used to do the symbolic manipulation and Matlab was used to solve the determinate and calculate the mode shapes.

The resulting characteristic equation is too large to examine directly. It can be simplified and written in terms of k and trigonometric terms but it would be very difficult to find the roots of this transcendental equation without the use of a numerical technique.

For this case, the roots can be found using the Matlab 'fzero' command and verified by plotting determinate of A as a function of k . Recall that k was defined as

$$k^4 = \frac{\omega^2 \rho A}{EI}$$

which can be rewritten to give

$$\omega_n = \sqrt{\frac{EI}{\rho A}} k_n^2$$

where ω_n is a circular frequency of the system and is related to the natural frequency by $f_n = \frac{1}{2\pi} \omega_n$.

Once the natural frequencies have been found, it is necessary to determine the eigenfunctions of the system. In this case, this is done by determining the relative magnitudes of the integration constants. Recall that there were 13 unknowns but only 12 equations. This means that it is not possible to solve for all the unknowns. However, by finding the values of the constants with respect to c_{12} relative magnitudes can be calculated. This allows the mode shapes to be determined.

It was shown earlier that the 6 boundary conditions and 6 constraint equations could be written in the form

$$AC = 0$$

Dividing A and C into sub matrices, the equation can be rewritten to give

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = 0$$

where A_{11} is an 11×11 matrix, A_{12} is an 11×1 vector, A_{21} is a 1×11 row vector and A_{22} is a scalar. C_1 is a 11×1 column vector and C_2 is a scalar containing c_{12} . Using the top two elements, the following equation can be obtained

$$A_{11}C_1 + A_{12}C_2 = 0$$

Solving for C_1 yields

$$C_1 = -A_{11}^{-1}A_{12}C_2$$

where A_{11}^{-1} is the inverse of A_{11} . The integration constants have now been written in terms of c_{12} since $C_2 = c_{12}$.

7.2 Vibration of the Crossed Flexure Pivots used in the MMS Prototype

Determining the vibrational behaviour of the crossed flexure pivot strips is very case dependent since an analytic solution could not be found. The goal of this detailed investigation of the pivot is to understand their behaviour in terms of the DTG and MMS. Therefore, it is useful to investigate the vibrational behaviour of the crossed flexure pivots used in the MMS prototype.

<i>Component</i>	<i>Parameter</i>	<i>Value</i>
Rotor	Spin Inertia (I_{r_s})	63240 g cm ²
	Transverse Inertia (I_{r_t})	34320 g cm ²
Gimbal	Mass (M_r)	2900 g
	Spin Inertia (I_{g_s})	180.5 g cm ²
	Transverse Inertia (I_{g_t})	127.6 g cm ²
	Mass (M_g)	77.0 g
Shaft and Motor	Spin Inertia (I_{s_t})	51.2 g cm ²
Crossed Flexure Pivots	Gimbal X Axis (k_x)	46.11×10^4 dyn cm/rad
	Gimbal Y Axis (k_y)	46.11×10^4 dyn cm/rad

Table 7.1: Inertial and mass properties of the MMS engineering prototype as of April 1998[72]. The stiffness listed are for a single pivot located along the referenced axis.

The relevant properties of the crossed flexure pivots and the rotor, gimbal and shaft are found in Table 7.1. A pair of pivots connect the rotor and the gimbal and another connect the gimbal to the shaft. Since the gimbal is the most active component it will be used as rigid body A in the analysis and the shaft, or rotor, will be rigid body B. For example

$$J_A = \frac{I_{g_t}}{2}$$

$$m_A = \frac{M_g}{2}$$

The division by two is performed because there are two pivots acting along each axis and only one is used in the analysis. It is assumed that the pivots distribute the load evenly.

Looking at Table 7.1 it is noticed that the transverse inertia of the shaft has been neglected. This is because the shaft is only allowed to rotate about its spin axis. It can then be assumed that the pivot is fixed on this end. In fact, it can be noticed that the transverse inertia of the rotor is over 250

	Value			
k	0.1996	60.25	62.78	63.39
f (Hz)	13.53	1.233×10^6	1.339×10^6	1.365×10^6

Table 7.2: First four natural frequencies of the crossed flexure pivots used in the MMS prototype.

times that of the gimbal. Similarly, the mass of the rotor is nearly 40 times that of the gimbal. It is safe to assume that the rotor is fixed with respect to the gimbal as well.

Therefore, it seems that the constraint equations and natural boundary conditions acting at $\tau = \sigma = 0$ can be replaced with the following forced boundary conditions

$$\begin{aligned} U_1(0) &= 0 & U_2(0) &= 0 \\ U'_1(0) &= 0 & U'_2(0) &= 0 \\ V_1(0) &= 0 & V_2(0) &= 0 \end{aligned}$$

The result is a simpler analysis which is consistent for both cases regarding the rotor and the shaft.

Using the 'fzero' command from Matlab and plotting the determinate of A against k , the first four natural frequencies of vibration are found, see Table 7.2. Figure 7.1 shows a plot of the determinate versus k for the first root and clearly shows the point at which the curve intersects the x axis.

The first mode occurs at a frequency of 13.53 Hz. The MMS operates at shaft speeds of 20 to 30 Hz and the gimbal flutters at twice the frequency of the shaft or 40 to 60 Hz. This means that during normal operation, resonance of this mode would not occur. During start up and shut down care should be taken in order to avoid the gimbal flutter at the frequency of 13.53 Hz. This means that a constant shaft speed of 405.9 rpm (6.77 Hz) should be avoided.

The relative magnitudes of the integration constants at this mode are

Figure 7.1: Plot of the determinate of A with respect to k. The point where the curve intersects the x axis indicates a solution of the characteristic equation.

given as

$$C = \begin{bmatrix} 0.50 \\ 0.0 \\ -0.50 \\ 0.0 \\ 0.50 \\ 0.03 \\ -0.50 \\ -0.03 \\ 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \end{bmatrix}$$

Recall that c_1 to c_8 apply to the transverse deflection of the strips while c_9 to c_{12} occur in the equations describing the axial deflection of the strips. It is seen that the mode shape is governed by the bending of the strips with no contribution from axial deformation. The deformation of the pivot at this frequency is shown in Figure 7.2. The solid lines indicate the deflected position while the dashed lines refer to the undeflected position. It is seen that this mode agrees with the deflection predicted in the static analysis. The proximity of the operating frequency to this mode and the distance to the next mode indicates that this is how the deformation of the strips will appear.

The second, third and fourth modes occur at 1.23 MHz, 1.34 MHz and 1.37 MHz. The magnitude of these frequencies make any calculation of the eigenfunctions very difficult and the results are suspect. Therefore, the form of the deflection can not be examined.

This analysis was performed assuming that the rotor, like the shaft, was fixed. If this is not done then a fifth frequency is found which occurs at

Figure 7.2: Plot showing the mode shape at the natural frequency of 13.53 Hz. The dimensions are in centimetres. The plot shows only the two strips of the pivot. The top of the strips connect to the rotor (rigid body B) and the lower ends of the strips are attached to the gimbal (rigid body A). The solid and dashed lines indicate the deflected and undeflected forms of the pivot.

2.85 Hz. The mode shape is based mostly on bending as shown by the values of the integration constants shown below.

$$C = \begin{bmatrix} -0.014 \\ 0.500 \\ 0.014 \\ -0.500 \\ -0.014 \\ 0.502 \\ 0.014 \\ -0.502 \\ 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \end{bmatrix}$$

In fact, this mode is due to the motion of the rotor and the gimbal. The other modes remain at frequencies which are within 1.3% of the values calculated previously.

This mode will occur at 2.85 Hz when the MMS is at rest. However, when the rotor is spinning, its effective transverse inertia increases, as shown in Chapter 2, Section 2.2. This means that this mode will occur at a much lower frequency and may not even be noticeable. The error on the frequencies of the other modes will decrease and the system will behave as if the rotor was fixed.

A comparison between the first mode at 13.53 Hz can now be made with the calculation of the fundamental frequency of a simple mass and torsional spring. Assuming that the pivot is an ideal torsional spring and that the rotor is held fixed, the frequency of vibration is given by

$$f_1 = \frac{1}{2\pi} \sqrt{\frac{K}{I}}$$

where K is the stiffness of the pivot and I is half the inertia of the gimbal. Therefore

$$f_1 = \frac{1}{2\pi} \sqrt{\frac{4.611 \times 10^5}{63.8}} \quad (7.61)$$

$$= 13.53 \text{ Hz} \quad (7.62)$$

This result is identical to that of the first mode and shows that the crossed flexure pivot does indeed act like an ideal torsional spring for small displacements.

7.3 Summary

A small angle dynamic analysis of the crossed flexure pivot was performed. Using numerical methods, the characteristic equation describing the system was analyzed and the frequencies of motion of the pivot were found.

The first natural frequency was found to be very low (< 3 Hz) and would actual be much lower in an operating MMS due to the 'inertial rigidity' of the rotor and shaft. The second mode occurred at 13.53 Hz and is dominated by bending motion. Due to the fact that the next frequency of response is at 1.23 MHz, which is significantly higher than the flutter frequency of the gimbal during operation of the MMS, it is felt that the mode shape at 13.53 Hz will be the governing one.

Chapter 8

Small Angle Analysis of a Dynamically Tuned Gyroscope

The Dynamically Tuned Gyroscope (DTG) consists of a spinning rotor attached to a shaft through a gimbal joint. Ideally, the rotor is symmetric with transverse inertia, I_{rt} and spin inertia, I_{rs} . Similarly, it is desired that the gimbal be symmetric having transverse inertia, I_{gt} and spin inertia, I_{gs} .

The gimbal and shaft are connected along the gimbal x axis by crossed flexure pivots with torsional stiffness, K_x . The rotor is attached to the gimbal along the gimbal y axis through a set of torsional springs with stiffness, K_y . Figure 8.1 is a schematic of the rotor, gimbal and shaft assembly. In the case of an ideal DTG, damping can be ignored. The result is the rotor can be offset with respect to the shaft.

For this analysis the crossed flexure pivots have been replaced by torsional springs. It was shown in Chapter 7 that the pivots behave like torsional springs for small angles of deflection. Restricting the stability study to small rotor deflections and ignoring the effect of gravity allows this simplifying assumption to be made.

Figure 8.1: Schematic of a Dynamically Tuned Gyro showing the rotor and gimbal rings, shaft and torsional elements. Note that one of the two outer torsional springs is hidden behind the shaft.

In the following analysis, it will be assumed that the gyro case is fixed in inertial space. It will also be assumed the motor and rotor inertias and motor stiffness are such that the shaft spins at a constant rate, ω_s .

8.1 Formulation of the Equations of Motion for an Idealized DTG

To derive the equations of motion, it is necessary to define the generalized coordinates to be used. For the DTG, these coordinates are ϕ , which represents the rotation of the gimbal about its x axis with respect to the shaft, and θ , which represents the rotation of the rotor about its y axis with respect to the gimbal. Based on these coordinates, the angular velocity of the gimbal can be calculated by rotating the shaft speed, ω_s , into the gimbal coordinate

frame and adding the rate about the x axis. In other words

$$\omega_g = \begin{pmatrix} \dot{\phi} \\ 0 \\ 0 \end{pmatrix} + (\phi)_x \begin{pmatrix} 0 \\ 0 \\ \omega_s \end{pmatrix}$$

where $(\phi)_x$ represents a rotation by ϕ about its x axis. Expanding, the angular rates of the gimbal can be found

$$\omega_{gx} = \dot{\phi} \quad (8.1)$$

$$\omega_{gy} = S_\phi \omega_s \approx \phi \omega_s \quad (8.2)$$

$$\omega_{gz} = C_\phi \omega_s \approx \omega_s \quad (8.3)$$

where S_ϕ represents $\sin \phi$ and C_ϕ represents $\cos \phi$. The approximations were made by realizing that $C_\phi \approx 1$ and $S_\phi \approx \phi$ when ϕ is small. Rotating the gimbal angular velocities into the rotor coordinate frame and then adding the rate about the y axis gives the angular velocities of the rotor. That is

$$\omega_r = \begin{pmatrix} 0 \\ \dot{\theta} \\ 0 \end{pmatrix} + (\theta)_y \begin{pmatrix} \omega_{gx} \\ \omega_{gy} \\ \omega_{gz} \end{pmatrix}$$

which expands to give

$$\omega_{rx} = C_\theta \dot{\phi} - S_\theta C_\phi \omega_s \approx \dot{\phi} - \theta \omega_s \quad (8.4)$$

$$\omega_{ry} = \dot{\theta} + S_\phi \omega_s \approx \dot{\theta} + \phi \omega_s \quad (8.5)$$

$$\omega_{rz} = S_\theta \dot{\phi} + C_\theta C_\phi \omega_s \approx \omega_s \quad (8.6)$$

Again, the approximations were made by realizing that θ is small. It was also assumed that the angular rates, at these small angles, were small. Now that the angular rates of each of the bodies is known, the kinetic energy, $\mathcal{T}$, of the system can be expressed as

$$\mathcal{T} = \frac{1}{2} \left[I_{rx} \omega_{rx}^2 + I_{ry} (\omega_{rx}^2 + \omega_{ry}^2) + I_{gz} \omega_{gz}^2 + I_{gy} (\omega_{gz}^2 + \omega_{gy}^2) \right] \quad (8.7)$$

Also, the potential energy, $\mathcal{V}$, is given as,

$$\mathcal{V} = \frac{1}{2}K_x\phi^2 + \frac{1}{2}K_y\theta^2 \quad (8.8)$$

Lagrange's equations can now be used to obtain the equations of motion. For this system, Lagrange's equations have the form

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) - \frac{\partial \mathcal{L}}{\partial \phi} &= 0 \\ \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) - \frac{\partial \mathcal{L}}{\partial \theta} &= 0 \end{aligned}$$

where $\mathcal{L}$ is known as the Lagrangian and is defined as

$$\mathcal{L} = \mathcal{T} - \mathcal{V}$$

After manipulation, Lagrange's equations become

$$\begin{aligned} C_\theta(I_{r_t} + I_{g_t})\ddot{\omega}_{r_z} + S_\theta(I_{r_s} + I_{g_t})\ddot{\omega}_{r_z} + (C_\theta I_{r_s}\omega_{r_z} - S_\theta I_{r_t}\omega_{r_z} + I_{g_t}\omega_{g_z})\omega_{r_y} \\ - (2I_{g_t} - I_{g_s})\omega_{g_y}\omega_{g_z} - I_{r_t}\omega_{g_z}\omega_{r_y} + K_x\phi = 0 \end{aligned} \quad (8.9)$$

and

$$I_{r_t}\ddot{\omega}_{r_y} - (I_{r_s} - I_{r_t})\omega_s\omega_{r_z} + K_y\theta = 0 \quad (8.10)$$

These equations were obtained by utilizing the following relationships

$$\begin{aligned} \frac{\partial \omega_{g_z}}{\partial \phi} &= 0 & \frac{\partial \omega_{g_y}}{\partial \phi} &= \omega_{g_z} & \frac{\partial \omega_{g_z}}{\partial \phi} &= -\omega_{g_y} \\ \frac{\partial \omega_{r_z}}{\partial \phi} &= S_\theta \omega_{g_y} & \frac{\partial \omega_{r_y}}{\partial \phi} &= \omega_{g_z} & \frac{\partial \omega_{r_z}}{\partial \phi} &= -C_\theta \omega_{g_y} \\ \frac{\partial \omega_{g_z}}{\partial \phi} &= 1 & \frac{\partial \omega_{g_y}}{\partial \phi} &= 0 & \frac{\partial \omega_{g_z}}{\partial \phi} &= 0 \\ \frac{\partial \omega_{r_z}}{\partial \phi} &= C_\theta & \frac{\partial \omega_{r_y}}{\partial \phi} &= 0 & \frac{\partial \omega_{r_z}}{\partial \phi} &= S_\theta \end{aligned} \quad (8.11)$$

and

$$\begin{array}{ccccc}
 \frac{\partial \omega_{g_x}}{\partial \theta} & 0 & \frac{\partial \omega_{g_y}}{\partial \theta} & 0 & \frac{\partial \omega_{g_z}}{\partial \theta} & 0 \\
 \frac{\partial \omega_{r_x}}{\partial \theta} & -\omega_{r_z} & \frac{\partial \omega_{r_y}}{\partial \theta} & 0 & \frac{\partial \omega_{r_z}}{\partial \theta} & \omega_{r_x} \\
 \frac{\partial \omega_{g_x}}{\partial \theta} & 0 & \frac{\partial \omega_{g_y}}{\partial \theta} & 0 & \frac{\partial \omega_{g_z}}{\partial \theta} & 0 \\
 \frac{\partial \omega_{r_x}}{\partial \theta} & 0 & \frac{\partial \omega_{r_y}}{\partial \theta} & 1 & \frac{\partial \omega_{r_z}}{\partial \theta} & 0
 \end{array} \quad (8.12)$$

Equations 8.9 and 8.10 can be simplified further using small angle approximations. The results are

$$(I_{r_t} + I_{g_t})\dot{\omega}_{r_x} + (I_{g_t} + I_{r_s} - I_{r_t})\omega_s\omega_{r_y} + [K_x - (2I_{g_t} - I_{g_s})\omega_s^2]\phi = 0 \quad (8.13)$$

$$I_{r_t}\dot{\omega}_{r_y} - (I_{r_s} - I_{r_t})\omega_s\omega_{r_x} + K_y\theta = 0 \quad (8.14)$$

Equations 8.13 and 8.14 are the dynamic equations of motion of the Dynamically Tuned Gyro and the characteristic equation for this system can be derived using one of two methods; either by using complex numbers or a state space approach. Both of these methods will be outlined in the following sections.

8.1.1 Complex Variable Method

In this section complex variables will be used to determine the characteristic equation. It is possible to combine Equations 8.13 and 8.14 by multiplying Equation 8.14 by j and then adding it to Equation 8.13. After doing this, the equations take the form

$$\begin{aligned}
 I_{r_t}(\dot{\omega}_{r_x} + j\dot{\omega}_{r_y}) + I_{g_t}\dot{\omega}_{r_x} + (I_{r_s} - I_{r_t})\omega_s(\omega_{r_y} - j\omega_{r_x}) \\
 + I_{g_t}\omega_s\omega_{r_y} + (K_x - 2J\omega_s^2)\phi + jK_y\theta = 0 \quad (8.15)
 \end{aligned}$$

where

$$J = I_{gt} - \frac{I_{gs}}{2}$$

and $j = \sqrt{-1}$. Using the following relationships, Equation 8.15 can be simplified even further

$$\omega = \omega_{rx} + j\omega_{ry}$$

$$\omega = \omega_{rx} - j\omega_{ry}$$

$$\psi = \phi + j\theta$$

$$\psi = \phi - j\theta$$

and be rewritten as

$$\begin{aligned} I_{rt}\dot{\omega} + I_{gt}\left(\frac{\dot{\omega} + \dot{\omega}}{2}\right) - j(I_{rs} - I_{rt})\omega_s\omega - jI_{gt}\omega_s\left(\frac{\omega - \omega}{2}\right) \\ + (K_x - 2J\omega_s^2)\left(\frac{\psi + \psi}{2}\right) + K_y\left(\frac{\psi - \psi}{2}\right) = 0 \end{aligned} \quad (8.16)$$

Collecting terms gives

$$\begin{aligned} \left(I_{rt} + \frac{I_{gt}}{2}\right)\dot{\omega} + \frac{I_{gt}}{2}(\dot{\omega} + j\omega\omega_s) - j\left(I_{rs} - \left(I_{rt} + \frac{I_{gt}}{2}\right) + I_{gt}\right)\omega_s\omega \\ + (K_x + K_y - 2J\omega_s^2)\frac{\psi}{2} \\ + (K_x - K_y - 2J\omega_s^2)\frac{\psi}{2} = 0 \end{aligned} \quad (8.17)$$

Now define

$$\begin{aligned} \omega_n &= \frac{I_{rs}\omega_s}{I_{rt} + \frac{I_{gt}}{2}} \\ \alpha &= \frac{K_x + K_y - 2J\omega_s^2}{2I_{rt} + I_{gt}} \\ \beta &= \frac{K_x - K_y - 2J\omega_s^2}{2I_{rt} + I_{gt}} \\ \gamma &= \frac{I_{gt}}{2I_{rt} + I_{gt}} \end{aligned}$$

where ω_n is the nutation frequency of the system. It should also be noted that the numerator of α is the tuning condition for a DTG and that in a typical gyro, γ is small. Equation 8.17 can be rewritten as

$$\dot{\omega} + \gamma(\dot{\omega} + j\omega_s\omega) - j(\omega_n - \omega_s + 2\gamma\omega_s)\omega + \alpha\psi + \beta\dot{\psi} = 0 \quad (8.18)$$

Finally, using Equations 8.4 and 8.5, it is possible to show that

$$\begin{aligned} \omega &= \dot{\psi} + j\omega_s\psi \\ \dot{\omega} &= \ddot{\psi} - j\omega_s\dot{\psi} \end{aligned}$$

Equation 8.18 becomes

$$\begin{aligned} \ddot{\psi} + j(2\omega_s - \omega_n - 2\gamma\omega_s)\dot{\psi} + (\alpha + \omega_s(\omega_n - \omega_s) + 2\gamma\omega_s^2)\psi \\ + \gamma\dot{\psi} + (\beta + \gamma\omega_s^2)\psi = 0 \end{aligned} \quad (8.19)$$

The Laplace transform of this equation can now be taken and the expression becomes

$$\begin{aligned} [s^2 + j(2\omega_s - \omega_n - 2\gamma\omega_s)s + (\alpha + \omega_s(\omega_n - \omega_s) + 2\gamma\omega_s^2)]\psi \\ + [\gamma s^2 + (\beta + \gamma\omega_s^2)]\psi = 0 \end{aligned} \quad (8.20)$$

The complex conjugate of Equation 8.20 is

$$\begin{aligned} [s^2 - j(2\omega_s - \omega_n - 2\gamma\omega_s)s + (\alpha + \omega_s(\omega_n - \omega_s) + 2\gamma\omega_s^2)]\bar{\psi} \\ + [\gamma s^2 + (\beta + \gamma\omega_s^2)]\bar{\psi} = 0 \end{aligned} \quad (8.21)$$

Equations 8.20 and 8.21 can only be solved if the determinate of the coefficients (or in other words, the characteristic equation) is zero. The characteristic equation is

$$\begin{aligned} [s^2 + \omega_s(\omega_n - \omega_s) + \alpha + 2\gamma\omega_s^2]^2 + (\omega_n - 2\omega_s + 2\gamma\omega_s)^2 s^2 \\ - [\gamma(s^2 + \omega_s^2) + \beta]^2 = 0 \end{aligned} \quad (8.22)$$

and can be reduced to

$$\begin{aligned} & (s^2 + \omega_s^2)^2 \\ & + \frac{\omega_n(\omega_n - 2\omega_s + 4\gamma\omega_s) + 2(\alpha - \gamma\beta) + 4\gamma\omega_s^2(\gamma - 1)}{(1 - \gamma^2)} (s^2 + \omega_s^2) \\ & + \frac{2\alpha\omega_s(\omega_n - 2\omega_s + 2\gamma\omega_s) + \alpha^2 - \beta^2}{(1 - \gamma^2)} = 0 \end{aligned} \quad (8.23)$$

The above expression can be solved in order to determine the frequency response of the system. The characteristic equation resulting from the state space method is better suited for the stability analysis, see Section 8.1.2; however, it is informative to find the roots of Equation 8.23. The result is

$$\begin{aligned} s^2 + \omega_s^2 = & -\frac{\omega_n(\omega_n - 2\omega_s) + 2(\alpha - \gamma\beta) + 4\gamma\omega_s(\omega_s(\gamma - 1) + \omega_n)}{2(1 - \gamma^2)} \\ & \pm \frac{1}{2(1 - \gamma^2)} \left(\left| \omega_n(\omega_n - 2\omega_s) + 2(\alpha - \gamma\beta) \right. \right. \\ & \quad \left. \left. + 4\gamma\omega_s(\omega_s(\gamma - 1) + \omega_n) \right|^2 \right. \\ & \quad \left. - 4(1 - \gamma^2) [2\alpha\omega_s(\omega_n - 2\omega_s + 2\gamma\omega_s) \right. \\ & \quad \left. + \alpha^2 - \beta^2] \right)^{1/2} \end{aligned} \quad (8.24)$$

By taking the square root of the above expression two more solutions are found. Thus, this system has four natural frequencies. It should be noted that α , β and γ depend on the ratio of the gimbal to rotor inertias and are small. Using this fact, it is possible to expand the roots to the second order using the binomial series to give

$$\begin{aligned} s^2 + \omega_s^2 = & \frac{1}{1 - \gamma^2} \left[-\frac{\omega_n}{2}(\omega_n - 2\omega_s) - (\alpha - \gamma\beta) - 2\gamma\omega_s(\omega_s(\gamma - 1) + \omega_n) \right. \\ & \pm \left(\frac{\omega_n}{2}(\omega_n - 2\omega_s) + (\alpha - \gamma\beta) + 2\gamma\omega_s(\omega_s(\gamma - 1) + \omega_n) \right. \\ & \quad \left. \left. - \frac{(1 - \gamma^2) [2\alpha\omega_s(\omega_n - 2\omega_s + 2\gamma\omega_s) + \alpha^2 - \beta^2]}{\omega_n(\omega_n - 2\omega_s) + 2(\alpha - \gamma\beta) + 4\gamma\omega_s(\omega_s(\gamma - 1) + \omega_n)} \right) \right] \end{aligned} \quad (8.25)$$

If the system is tuned, that is

$$K_x + K_y - 2J\omega_s^2 = 0 \quad (8.26)$$

then the above expression can be simplified to give

$$\begin{aligned} s^2 + \omega_s^2 &= (1 + \gamma^2) \left[-\frac{\omega_n}{2} (\omega_n - 2\omega_s) + \gamma\beta - 2\gamma\omega_s (\omega_s (\gamma - 1) + \omega_n) \right. \\ &\quad \pm \left(\frac{\omega_n}{2} (\omega_n - 2\omega_s) - \gamma\beta + 2\gamma\omega_s (\omega_n (\gamma - 1) + \omega_n) \right. \\ &\quad \left. \left. + \frac{(1 - \gamma^2)\beta^2}{\omega_n (\omega_n - 2\omega_s) - 2\gamma\beta + 4\gamma\omega_s (\omega_s (\gamma - 1) + \omega_n)} \right) \right] \end{aligned} \quad (8.27)$$

Rewriting gives the two equations as

$$s^2 + \omega_s^2 = \frac{(1 - \gamma^2)\beta^2}{\omega_n (\omega_n - 2\omega_s) - 2\gamma\beta + 4\gamma\omega_s (\omega_s (\gamma - 1) + \omega_n)} \quad (8.28)$$

$$\begin{aligned} s^2 + \omega_s^2 &= (1 + \gamma^2) \left[-\omega_n (\omega_n - 2\omega_s) + 2\gamma\beta - 4\gamma\omega_s (\omega_s (\gamma - 1) + \omega_n) \right. \\ &\quad \left. - \frac{(1 - \gamma^2)\beta^2}{\omega_n (\omega_n - 2\omega_s) - 2\gamma\beta + 4\gamma\omega_s (\omega_s (\gamma - 1) + \omega_n)} \right] \end{aligned} \quad (8.29)$$

Therefore the effect of tuning the system was to eliminate the majority of the first order terms. It should be noticed that the roots of the characteristic equation are dependent on the system parameters only and not on the orientation of the rotor.

8.1.2 State Space Method

An alternative to using complex variables to find the characteristic equations is to use the state space method. Instead of summing the two equations of motion, it is possible to replace the rotor and gimbal angular rates in

Equations 8.13 and 8.14 using the relationships found in Equations 8.1 to 8.3 and 8.4 to 8.6. This gives

$$(I_{r_t} + I_{g_t})(\ddot{\phi} - \dot{\theta}\omega_s) + (I_{g_s} + I_{r_s} - I_{r_t})\omega_s(\dot{\theta} + \phi\omega_s) + [K_x - (2I_{g_t} - I_{g_s})\omega_s^2]\phi = 0 \quad (8.30)$$

and

$$I_{r_t}(\ddot{\theta} + \dot{\phi}\omega_s) - (I_{r_s} - I_{r_t})\omega_s(\dot{\phi} - \theta\omega_s) + K_y\theta = 0 \quad (8.31)$$

These equations can be rewritten in matrix form as

$$\begin{aligned} & \begin{bmatrix} I_{r_t} + I_{g_t} & 0 \\ 0 & I_{r_t} \end{bmatrix} \begin{bmatrix} \ddot{\phi} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} 0 & -(2I_{r_t} - I_{r_s})\omega_s \\ (2I_{r_t} - I_{r_s})\omega_s & 0 \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \end{bmatrix} \\ & + \begin{bmatrix} K_x + (I_{g_s} - I_{g_t} + I_{r_s} - I_{r_t})\omega_s^2 & 0 \\ 0 & K_y + (I_{r_s} - I_{r_t})\omega_s^2 \end{bmatrix} \begin{bmatrix} \phi \\ \theta \end{bmatrix} \\ & = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned} \quad (8.32)$$

Assuming the solution has the form

$$\begin{bmatrix} \phi \\ \theta \end{bmatrix} = \mathbf{A}e^{j\omega t}$$

Equation 8.32 becomes

$$\begin{aligned} & \left\{ -\omega^2 \begin{bmatrix} I_{r_t} + I_{g_t} & 0 \\ 0 & I_{r_t} \end{bmatrix} + j\omega \begin{bmatrix} 0 & -(2I_{r_t} - I_{r_s})\omega_s \\ (2I_{r_t} - I_{r_s})\omega_s & 0 \end{bmatrix} \right. \\ & \left. + \begin{bmatrix} K_x + (I_{g_s} - I_{g_t} + I_{r_s} - I_{r_t})\omega_s^2 & 0 \\ 0 & K_y + (I_{r_s} - I_{r_t})\omega_s^2 \end{bmatrix} \right\} \mathbf{A}e^{j\omega t} \\ & = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned} \quad (8.33)$$

Note that this is not the same as taking the Laplace transform. By taking the Laplace transform, the solution is assumed to have the form, $\mathbf{A}e^{st}$. In order to compare the two cases it is necessary to replace s with $j\omega$. The

characteristic equation of the system is

$$\begin{aligned} \Delta &= \left[-\omega^2 (I_{r_t} + I_{g_t}) + K_x + (I_{g_s} + I_{r_s} - I_{r_t} - I_{g_t}) \omega_s^2 \right] \\ &\quad \times \left[-\omega^2 I_{r_t} + K_y + (I_{r_s} - I_{r_t}) \omega_s^2 \right] \\ &\quad - \omega^2 (2I_{r_t} - I_{r_s}^2) \omega_s^2 = 0 \end{aligned} \quad (8.34)$$

This can be rearranged to give

$$\begin{aligned} \Delta &= I_{r_t} (I_{r_t} + I_{g_t}) \omega^4 \\ &\quad - \omega^2 \{ I_{r_t} (K_x + K_y) + I_{g_t} K_y \\ &\quad + \omega_s^2 [I_{r_t} (2I_{r_t} - 2I_{r_s} + I_{g_s} - I_{g_t}) \\ &\quad + I_{g_t} (I_{r_s} - I_{r_t}) + I_{r_s}^2] \} \\ &\quad + \left[K_x + (I_{g_s} + I_{r_s} - I_{r_t} - I_{g_t}) \omega_s^2 \right] \\ &\quad \times \left[K_y + (I_{r_s} - I_{r_t}) \omega_s^2 \right] = 0 \end{aligned} \quad (8.35)$$

The characteristic equation is now in a form which can be used for a stability analysis. It is shown in Appendix E that Equations 8.23 and 8.35 are equivalent.

8.2 Stability Analysis of the DTG

The following stability analysis is based on the results found in Section 8.1.2. In that section, the solution of Equations 8.13 and 8.14 were assumed to have the form $\mathbf{A}e^{j\omega t}$. Because of this, it is necessary that the square of the roots of the characteristic equation (Equation 8.35) be positive and real for the system to be stable.

Given that ω represents the natural (or fundamental) frequencies of the system and its square belongs to the real domain, that is $\omega^2 \in \mathfrak{R}$, then if

$\omega^2 < 0$ at least one value of $j\omega$ will be positive and real. This results in an instability due to divergence. If ω^2 is positive, it is still possible the system is unstable. This occurs when the roots of ω^2 have imaginary parts resulting in a pair of complex, positive roots which cause the system to oscillate with an increasingly larger magnitude. This instability is known as flutter.

At this point, the square of the natural frequency (ω^2) will be defined as Ω and will be referred to as the frequency of the system, and the square of the shaft spin speed (ω_s^2) will be defined as η and will be referred to as the shaft speed. Normalizing with respect to the rotor transverse inertia, Equation 8.35 can be rewritten as

$$\Delta(\Omega) = \Lambda\Omega^2 - (\Xi + \Upsilon\eta)\Omega + \left(\frac{K_x}{I_{r_t}} + \Psi\eta\right)\left(\frac{K_y}{I_{r_t}} + \Gamma\eta\right) = 0 \quad (8.36)$$

where

$$\begin{aligned} \Lambda &= 1 + \frac{I_{g_t}}{I_{r_t}} \\ \Xi &= -\frac{K_x + K_y}{I_{r_t}} + \frac{I_{g_t}K_y}{I_{r_t}^2} \\ \Upsilon &= \left(\frac{I_{r_s}}{I_{r_t}} - 1\right)^2 + 1 + \frac{I_{g_s} - 2I_{g_t}}{I_{r_t}} + \frac{I_{g_t}I_{r_s}}{I_{r_t}^2} \\ \Psi &= \frac{I_{g_s} + I_{r_s} - I_{g_t}}{I_{r_t}} - 1 \\ \Gamma &= \frac{I_{r_s}}{I_{r_t}} - 1 \end{aligned}$$

As stated earlier, the system will experience a divergent instability if one, or both, of the roots of the characteristic equation (Equation 8.36) are less than zero ($\Omega_{1,2} < 0$). The boundary between the stable regions and the unstable regions, known as the divergence boundary, is found using the

following relationships[74][75]

$$\Delta(\Omega) = 0 \quad \text{when} \quad \Omega_i = 0 \quad i = 1 \text{ or } 2 \quad (8.37)$$

For at least one root of the characteristic equation to be zero, it is necessary that

$$\left(\frac{K_x}{I_{r_i}} + \Psi\eta \right) \left(\frac{K_y}{I_{r_i}} + \Gamma\eta \right) = 0 \quad (8.38)$$

This equation for the divergence boundary can be expanded into a second order polynomial involving the shaft speed, η

$$\Gamma\Psi\eta^2 + \frac{K_x\Gamma + K_y\Psi}{I_{r_i}}\eta^2 + \frac{K_xK_y}{I_{r_i}^2} = 0 \quad (8.39)$$

It should be noted that if both of the roots of the characteristic equation are zero, then the following relationship must be true as well

$$\Xi + \Upsilon\eta = 0 \quad (8.40)$$

If the roots of the characteristic equation are complex then the system will loose stability because of flutter. The flutter boundaries can be determined by satisfying the following equations

$$\Delta(\Omega) = 0 \quad (8.41)$$

$$\frac{\partial\Delta(\Omega)}{\partial\Omega} = 0 \quad (8.42)$$

For the characteristic equation given above

$$\frac{\partial\Delta}{\partial\Omega} = 2\Lambda\Omega - (\Xi + \Upsilon\eta) \quad (8.43)$$

The roots of the characteristic equation are

$$\Omega_{1,2} = \frac{(\Xi + \Upsilon\eta) \pm \sqrt{(\Xi + \Upsilon\eta)^2 - 4\Lambda \left(\frac{K_x}{I_{r_i}} + \Psi\eta \right) \left(\frac{K_y}{I_{r_i}} + \Gamma\eta \right)}}{2\Lambda} \quad (8.44)$$

The roots from Equation 8.44 can now be substituted into Equation 8.43. This is then squared and the result is

$$(\Xi + \Upsilon\eta)^2 - 4\Lambda \left(\frac{K_x}{I_{rt}} + \Psi\eta \right) \left(\frac{K_y}{I_{rt}} + \Gamma\eta \right) = 0 \quad (8.45)$$

Equation 8.45 can be expanded to give

$$\begin{aligned} (\Upsilon^2 - 4\Lambda\Gamma\Psi)\eta^2 + \left[2\Xi\Upsilon - 4\Lambda \frac{K_x\Gamma + K_y\Psi}{I_{rt}} \right] \eta \\ + \left(\Xi^2 - 4\Lambda \frac{K_x K_y}{I_{rt}^2} \right) = 0 \end{aligned} \quad (8.46)$$

8.2.1 Stability of the DTG with Bearings

A special case worth examining, occurs when the pivots are replaced with bearings. In other words, the stiffness of the pivots is zero. With $K_x = K_y = 0$, the characteristic equation becomes

$$\Lambda\Omega^2 - \Upsilon\eta\Omega + \Gamma\Psi\eta^2 = 0$$

The roots can be found by using the quadratic equation, and are given as

$$\Omega = \frac{\Upsilon\eta \pm \sqrt{\Upsilon^2\eta^2 - 4\Lambda\Gamma\Psi\eta^2}}{2\Lambda}$$

which can be simplified to give

$$\Omega = \frac{\eta}{2\Lambda} \left(\Upsilon \pm \sqrt{\Upsilon^2 - 4\Lambda\Gamma\Psi} \right)$$

In order for the system to be unstable by divergence, it is necessary that one of the roots of the characteristic equation be less than zero. By definition, $\Lambda > 1$, which means that for this to occur

$$\sqrt{\Upsilon^2 - 4\Lambda\Gamma\Psi} > \Upsilon$$

This can be rewritten in the form

$$\Lambda\Gamma\Psi < 0$$

Given the restriction on Λ stated earlier, this equation is only true if either, but not both, Γ or Ψ is negative. Therefore, using the definitions of Γ and Ψ , the region of instability of the gyro is

$$I_{r_s} < I_{r_t} < I_{r_s} + I_{g_s} - I_{g_t}$$

which is a very limited and unusual distribution of inertias. Thus, it has been shown there is only a small region where a gyro with bearings would be unstable by divergence. There is also the possibility that the gyro could become unstable through flutter. This can be checked using Equation 8.45, which becomes

$$(\Upsilon^2 - 4\Lambda\Gamma\Psi)\eta^2 < 0$$

Since the square of the shaft spin speed can never be negative, flutter occurs only when

$$\Upsilon^2 < 4\Lambda\Gamma\Psi$$

At this time, it is useful to re-examine the definitions of Λ , Υ , Γ and Ψ . In the typical gyro, $I_{r_t} \gg I_{g_s}$ and I_{g_t} , which means that the following approximations can be found

$$\begin{aligned}\Lambda &\approx 1 \\ \Upsilon &\approx 1 + \left(\frac{I_{r_s}}{I_{r_t}} - 1\right)^2 \\ \Psi &\approx \frac{I_{r_s}}{I_{r_t}} - 1 \\ \Gamma &\approx \frac{I_{r_s}}{I_{r_t}} - 1\end{aligned}$$

Therefore, the above relationship can be expressed as

$$\left(1 + \left(\frac{I_{r_s}}{I_{r_t}} - 1\right)^2\right)^2 < 4 \left(\frac{I_{r_s}}{I_{r_t}} - 1\right)^2$$

and after simplification, this can be rewritten as

$$\left(1 - \left(\frac{I_{r_s}}{I_{r_t}} - 1\right)^2\right)^2 < 0$$

Since the square of a term can never be less than zero, the above equation implies that the system will not experience flutter instability. However, it should be noted that there is a flutter boundary at $\eta = 0$ and so the system is on the verge of instability when the shaft speed is zero. See Figures 8.2 and 8.3 for time and phase plots of the response of a Litton CSG-2 gyro whose properties are shown in Table 8.1. Note that the axial stiffness has been reduced to zero for this study. These plots were obtained by numerically solving the equations of motion, Equations 8.13 and 8.14.

8.2.2 Numerical Study of the Litton CSG-2 DTG

At this point it is possible to perform a few numerical studies to observe the effect of changing various parameters on the stability of a DTG. This study is based on the Litton CSG-2 gyro, the properties of which are given in Table 8.1. The study is performed by changing the value of one parameter while keeping the remaining ones constant. While, in most of the proceeding plots, negative values of the η are plotted, these values are physically impossible to achieve and are only shown in order to give a clear understanding of the trends of the curves.

The first study involves changing the value of the inner torsional stiffness, K_x . Figure 8.4 gives the stability boundaries of the system. It can be seen

Figure 8.2: Time response of a CSG-2 Gyro with bearings in place of torsional springs (i.e. $K_x = K_y = 0$) and an initial offset of $\phi = 1$ mrad. The response of the system was obtained by numerically integrating the equations of motion using the Matlab function ODE15s.

Parameter	Value
I_{r_s}	70.0 gcm ²
I_{r_t}	40.5 gcm ²
I_{g_s}	1.37 gcm ²
I_{g_t}	0.8267 gcm ²
K_x	6.226×10^4 dyncm/rad
K_y	6.226×10^4 dyncm/rad

Table 8.1: Inertial and stiffness properties of the Litton CSG-2 Dynamically Tuned Gyro used in the stability analysis.

Figure 8.3: Phase plane response of the CSG-2 Gyro with bearings in place of torsional springs (i.e. $K_x = K_y = 0$) and an initial offset of $\phi = 1$ mrad.

that it is possible for the system to go unstable by divergence when the torsional stiffness is negative. However, the system can be made stable by increasing the spin speed of the input shaft. A point of interest occurs when the shaft speed is zero and the system is in its tuned condition. At this point, the system is touching the tip of the flutter boundary indicating a system on the verge of flutter instability. It should be noted that in an actual system the stiffnesses of the flexures would not be exactly the same nor would the system be perfectly tuned so the actual system would be sitting to either side of this point. From this analysis, it can be seen that the system will remain stable for all reasonable operating conditions. Figure 8.5 shows the characteristic curves for the system and the results agree with those from the stability plot.

Figures 8.6 and 8.7 give the stability boundaries and characteristic curve for the CSG-2 gyro when the stiffnesses of the outer torsional springs are varied. The results are similar to the previous case.

Varying the spin inertia of the gimbal generates the curves shown in Figures 8.8 and 8.9. It can be seen from Figure 8.8 that the gyro is stable for all practical values of η . When the shaft speed is zero, the system is positioned on the flutter boundary for all values of I_{θ} . The characteristic curves for this case are shown in Figure 8.9.

For large gimbal transverse inertias, it is possible for the gyro to be unstable, as seen in Figure 8.10. This occurs for inertias over 40.5 gm^2 or at a transverse inertia almost 50 times that used in the actual Litton gyro. From Figures 8.10 and 8.11 it is observed that the value of η at which flutter occurs decreases as the transverse inertia of the gimbal increases from

Figure 8.4: Stability boundaries for the CSG-2 gyro with varying x axis stiffness. K_x^* is twice the ideal, tuned stiffness of the pivot. D represents divergence, S represents a stable region and F represents an unstable system due to flutter. Note that a negative value for the square of the shaft spin speed (η) is physically impossible and is only shown to give an understanding of the trends of the curves.

Figure 8.5: Characteristic curve for the CSG-2 gyro. Each hyperbola represents a different x axis stiffness. Note that a negative value for the square of the shaft spin speed (η) is physically impossible and is only shown to give an understanding of the trends of the curves.

Figure 8.6: Stability boundaries for the CSG-2 gyro with varying y axis stiffness. K_y^* is twice the ideal, tuned stiffness of the pivot. D represents divergence, S represents a stable region and F represents an unstable system due to flutter. Note that a negative value for the square of the shaft spin speed (η) is physically impossible and is only shown to give an understanding of the trends of the curves.

Figure 8.7: Characteristic curve for the CSG-2 gyro. Each hyperbola represents a different y axis stiffness. Note that a negative value for the square of the shaft spin speed (η) is physically impossible and is only shown to give an understanding of the trends of the curves.

Figure 8.8: Stability boundaries for the CSG-2 gyro with varying gimbal spin inertia. D represents divergence, S represents a stable region and F represents an unstable system due to flutter. Note that a negative value for the square of the shaft spin speed (η) is physically impossible and is only shown to give an understanding of the trends of the curves.

Figure 8.9: Characteristic curve for the CSG-2 gyro. Each hyperbola represents a different value of I_{gs} . Note that a negative value for the square of the shaft spin speed (η) is physically impossible and is only shown to give an understanding of the trends of the curves.

Figure 8.10: Stability boundaries for the CSG-2 gyro with varying gimbal transverse inertia. D represents divergence, S represents a stable region and F represents an unstable system due to flutter. Note that a negative value for the square of the shaft spin speed (η) is physically impossible and is only shown to give an understanding of the trends of the curves.

the tuned value. The flutter boundary remains fairly constant for values of transverse inertia smaller than that shown in Table 8.1.

Figure 8.12 shows that for low values of I_{r_s}/I_{r_t} , the DTG can be unstable by divergence at specific shaft speeds. This area is small (it lies between the stable areas above $\eta = 1700$). It is important to ensure that I_{r_s} does not decrease to low values and thus avoid a possible problem. The system encounters the flutter boundary for all values of I_{r_s} when $\eta = 0$, with the

Figure 8.11: Characteristic curve for the CSG-2 gyro. Each hyperbola represents a different value of I_g/I_h . Note that a negative value for the square of the shaft spin speed (η) is physically impossible and is only shown to give an understanding of the trends of the curves.

Figure 8.12: Stability boundaries for the CSG-2 gyro with varying rotor spin inertia. D represents divergence, S represents a stable region and F represents an unstable system due to flutter. Note that a negative value for the square of the shaft spin speed (η) is physically impossible and is only shown to give an understanding of the trends of the curves.

exception of the area near $I_{rs}/I_t = 2$. At this point the flutter boundary decreases rapidly. Figure 8.13 gives a plot of the characteristic curves for different values of I_{rs}/I_t .

It should be noted in Figures 8.4 to 8.13 that the system does not enter a region of flutter instability for a positive shaft spin speed. It is also observed that the gyro only enters regions of divergent instability in extreme cases, where, for example, the stiffness of the torsional springs have a very large

Figure 8.13: Characteristic curve for the CSG-2 gyro. Each hyperbola represents a different value of the rotor spin inertia of the gyro, I_{rs} . Note that a negative value for the square of the shaft spin speed (η) is physically impossible and is only shown to give an understanding of the trends of the curves.

negative value or the transverse inertia of the gimbal is quite large. In fact, it has been shown that for a practical gyro design, the system would be stable. The stability of the gyro is insensitive to practical changes in the physical properties of the system.

8.3 Numerical Solution of the Equations of Motion of a Dynamically Tuned Gyro

The stability of the gyro has been discussed and it is useful to determine the response of the system and confirm these findings. The equations that describe the system, Equations 8.30 and 8.31, are a pair of ordinary differential equations and can be solved using one of many techniques, from Taylor series methods to multi-step methods. It should be noted that these equations are considered stiff and this fact must be taken into consideration when choosing a numerical integration technique. A system is considered stiff if the response of the system has relative high and low frequency terms.

In order to numerically simulate the gyro, a Matlab[76] variable order solver, 'ode15s', was implemented and the results are shown in Figures 8.14 to 8.17.

In order to demonstrate the stable and unstable response of the gyro, two cases were evaluated. The first, seen in Figures 8.14 and 8.15, shows the response of an unmodified CSG-2 gyro with the physical parameters listed in Table 8.1. Figure 8.14 gives the time response of the system with an initial offset of the gimbal about the x axis of $\phi = 1$ mrad. Note that the results are still in spinning coordinates; and the response is periodic and stable, as predicted by the stability diagrams discussed earlier. Figure 8.15 gives the

Figure 8.14: Time response of a standard CSG-2 Gyro with an initial offset of $\phi = 1$ mrad.

phase plane plots and shows the response is periodic, indicating stability.

In order to observe the response of an unstable system, the transverse inertia of the gimbal (I_{gt}) was increased to 100 gcm^2 . The time response of the system is shown in Figure 8.16. It can be seen that the gimbal angles grow exponentially and quickly go outside the practical range of the pivots. (Note the different time scale in Figure 8.16 versus that of Figure 8.14.) Phase plane plots, Figure 8.17, show that the response is linear, starting at the initial conditions and increasing as the angle increases. This is a good example of divergent instability in a gyroscope.

Figure 8.15: Phase plane response of a standard CSG-2 Gyro with an initial offset of $\phi = 1$ mrad.

Figure 8.16: Time response of a CSG-2 Gyro with transverse gimbal inertia, $I_{gt} = 100 \text{ gcm}^2$ and an initial offset of $\phi = 1 \text{ mrad}$.

Figure 8.17: Phase plane response of the CSG-2 Gyro with transverse gimbal inertia, $I_{g_t} = 100 \text{ gcm}^2$ and an initial offset of $\phi = 1 \text{ mrad}$.

8.4 Stability of the MMS

The stability of a typical Dynamically Tuned Gyroscope has now been discussed and it has been shown that the DTG is inherently stable except under extreme conditions, such as a very high gimbal transverse inertia. A similar analysis must now be performed to ensure that the Momentum Management System (MMS) is stable.

Figure 8.18 gives the stability boundaries for the MMS when the torsional stiffness about the gimbal x axis is varied. The inertial properties of the MMS engineering prototype are shown in Table 8.2. Figure 8.19 contains a plot of the characteristic curves for various stiffnesses. Notice that similar trends to that of the DTG are observed. However, due to the larger rotor, the system will become stable once the rotor begins to spin. A comparison can be made at a value of $K_x/K_x^* = -1$ where the DTG is unstable at rotor spin speeds of less than 44.7 rpm and the MMS is only unstable at speeds less than 6 rpm.

It is at this point that a small discussion of the relative Figures of Merit should be made. The Figure of Merit can be calculated using Equation 2.19,

$$F_m = \frac{I_{r_z}}{I_{g_y} + I_{g_x} - I_{g_z}} \quad (8.47)$$

It is found that the F_m of the Litton gyro used for this analysis is approximately 250 while the Figure of Merit of the MMS is 847, over three times larger. In Chapter 2 it was mentioned that the Figure of Merit was a measure of the quality of a gyro. It gives an indication of how isolated the rotor is from the rest of the gyro assembly. The higher the number, the more the rotor acts as a freely spinning body and the more stable the system.

Figure 8.18: Stability boundaries for the MMS engineering prototype with varying x axis stiffness. K_x^* is twice the ideal, tuned stiffness of the pivots. D indicates a region where instability due to divergence will occur, S represents a stable region and F represents an area of instability due to flutter. Note that a negative value for the square of the shaft spin speed (η) is physically impossible and is only shown to give an understanding of the trends of the curves.

Figure 8.19: Characteristic curves for the MMS engineering prototype. Each hyperbola represents a different torsional stiffness. Note that a negative value for the square of the shaft spin speed (η) is physically impossible and is only shown to give an understanding of the trends of the curves.

<i>Component</i>	<i>Parameter</i>	<i>Value</i>
Rotor	Spin Inertia (I_{r_s})	6324 kg mm ²
	Transverse Inertia (I_{r_t})	3432 kg mm ²
	Mass (M_r)	2.9 kg
Gimbal	Spin Inertia (I_{g_s})	18.05 kg mm ²
	Transverse Inertia (I_{g_t})	12.76 kg mm ²
	Mass (M_g)	0.077 kg
Shaft and Motor	Spin Inertia (I_{s_s})	5.12 kg mm ²
Crossed Flexure Pivots	Gimbal X Axis (k_x)	46.105 Nmm/rad
	Gimbal Y Axis (k_y)	46.105 Nmm/rad

Table 8.2: Inertial and mass properties of the MMS engineering prototype as of April 1998[72]. The stiffnesses listed are for a single pivot located along the referenced axis.

Chapter 9

Large Angle Analysis of the MMS

The Momentum Management System (MMS), shown in detail in Figure 9.1 and schematically in Figure 9.2, is based on the Dynamically Tuned Gyroscope (DTG) and consists of a rotor, gimbal and shaft, all connected by crossed flexure pivots. The schematic in Figure 9.2 also includes a proposed fourth body, intended to act as a vibrational damper, in order to decrease the amount of shaft oscillation.

Several methods for deriving the equations of motion of the MMS were studied in order to find the most efficient derivation. These included: Kane's equations, Lagrange's Equations, Hamilton's Canonical Equations and Newton-Euler's Equations¹. Also, two different sets of rotational coordinates, namely, quaternions and Euler angles were examined.

Use of quaternions[11][77], or one of their many variations, would eliminate the need for trigonometry in most of the analysis but for this problem

¹Kane[78] has produced a paper which discusses the differences between several potential methods of deriving the equations of motion of a spacecraft.

Figure 9.1: Drawing of the MMS as of August 22, 1997.

Figure 9.2: Schematic of the MMS showing the rotor, gimbal, shaft and vibrational damper.

they were found to be difficult to implement and so Euler angles were used instead.

Kane's Equations[79] are well suited for many types of problems, especially those with non-holonomic constraints (see paper by Kane[80] for a good example). For the DTG, the same solution resulted as derived using Newton-Euler's Equations but was more difficult to implement.

For this type of problem, the algebra involved using Newton-Euler Equations can be tedious and prone to error when compared to using Lagrange's Equations[2]. While this eliminates the use of Newton-Euler Equations to determine the equations of motion, a partial solution by this method will be required in order to find the forces acting on the crossed flexure pivots.

The two remaining methods, Lagrange's Equations[11][81] and Hamil-

ton's Canonical Equations[81][78] are well suited for this type of problem and both will be used to derive the equations of motion for the MMS.

9.1 Kinematics of the MMS

To develop the equations of motion of the MMS, it is necessary to select the generalized coordinates to be used. Define Ω as the angle of rotation of the shaft with respect to the case, about the z axis, β as the angle of rotation of the vibration damper with respect to the shaft about the z axis, ϕ as the angle of rotation of the gimbal with respect to the shaft about the x axis and θ as the angle of rotation of the rotor with respect to the gimbal about the y axis. The last two angles are the 'gimbal' angles.

The MMS, and DTGs, are designed to measure the angular rate of the vehicle to which they are attached. This means the case of the MMS, which is attached to the spacecraft, will have some angular velocity and, in general, this angular velocity will take the form

$$\boldsymbol{\omega}_c = \begin{pmatrix} \omega_{c_x} \\ \omega_{c_y} \\ \omega_{c_z} \end{pmatrix} \quad (9.1)$$

where ω_{c_i} represents the angular velocities of the case about the Cartesian reference frame. Because this investigation is meant to be as general as possible it is assumed that the angular rates of the case are constant, in other words

$$\dot{\boldsymbol{\omega}}_c = 0$$

This eliminates the need to take the inertial properties of the spacecraft into consideration. The angular velocity of the shaft is found by rotating the

angular velocity of the case about the z axis and then adding the spin speed of the shaft with respect to the case, $\dot{\Omega}$,

$$\omega_s = \begin{pmatrix} 0 \\ 0 \\ \dot{\Omega} \end{pmatrix} + (\dot{\Omega})_z \begin{pmatrix} \omega_{c_x} \\ \omega_{c_y} \\ \omega_{c_z} \end{pmatrix} \quad (9.2)$$

which can be simplified to obtain

$$\omega_s = \begin{pmatrix} C_{\dot{\Omega}}\omega_{c_x} + S_{\dot{\Omega}}\omega_{c_y} \\ C_{\dot{\Omega}}\omega_{c_y} - S_{\dot{\Omega}}\omega_{c_x} \\ \dot{\Omega} + \omega_{c_z} \end{pmatrix} \quad (9.3)$$

It is possible to find the angular acceleration of the shaft by taking the time derivative of Equation 9.3. The result is

$$\dot{\omega}_s = \begin{pmatrix} \dot{\Omega}\omega_{s_y} \\ -\dot{\Omega}\omega_{s_x} \\ \ddot{\Omega} \end{pmatrix} \quad (9.4)$$

The angular velocity of the vibrational damper is derived by a rotation of the shaft velocity about the z axis and adding the result to the angular rate of the damper with respect to the shaft

$$\begin{aligned} \omega_v &= \begin{pmatrix} 0 \\ 0 \\ \dot{\beta} \end{pmatrix} + (\dot{\beta})_z \begin{pmatrix} \omega_{s_x} \\ \omega_{s_y} \\ \omega_{s_z} \end{pmatrix} \\ &= \begin{pmatrix} C_{\dot{\beta}}\omega_{s_x} + S_{\dot{\beta}}\omega_{s_y} \\ C_{\dot{\beta}}\omega_{s_y} - S_{\dot{\beta}}\omega_{s_x} \\ \dot{\beta} + \omega_{s_z} \end{pmatrix} \end{aligned} \quad (9.5)$$

The angular accelerations are calculated to be

$$\dot{\omega}_v = \begin{pmatrix} (\dot{\beta} + \dot{\Omega})\omega_{v_y} \\ -(\dot{\beta} + \dot{\Omega})\omega_{v_x} \\ \ddot{\beta} + \ddot{\Omega} \end{pmatrix} \quad (9.6)$$

The angular velocity of the gimbal is the sum of the gimbal rate with respect to the shaft and the shaft velocity expressed in the gimbal coordinate frame

$$\begin{aligned}\omega_g &= \begin{pmatrix} \dot{\phi} \\ 0 \\ 0 \end{pmatrix} + (\phi)_x \begin{pmatrix} \omega_{s_x} \\ \omega_{s_y} \\ \omega_{s_z} \end{pmatrix} \\ &= \begin{pmatrix} \dot{\phi} + \omega_{s_x} \\ C_\phi \omega_{s_y} + S_\phi \omega_{s_z} \\ C_\phi \omega_{s_z} - S_\phi \omega_{s_y} \end{pmatrix} \end{aligned} \quad (9.7)$$

and the corresponding angular accelerations are

$$\dot{\omega}_g = \begin{pmatrix} \ddot{\phi} + \dot{\omega}_{s_x} \\ \dot{\phi}\omega_{s_z} - C_\phi \dot{\omega}_{s_y} + S_\phi \dot{\omega}_{s_z} \\ -\dot{\phi}\omega_{s_y} + C_\phi \dot{\omega}_{s_z} + S_\phi \dot{\omega}_{s_x} \end{pmatrix} \quad (9.8)$$

Finally, rotating the gimbal angular velocities into the rotor coordinate frame and adding the rate about the y axis gives the angular velocities of the rotor as

$$\begin{aligned}\omega_r &= \begin{pmatrix} 0 \\ \dot{\theta} \\ 0 \end{pmatrix} + (\theta)_y \begin{pmatrix} \omega_{g_x} \\ \omega_{g_y} \\ \omega_{g_z} \end{pmatrix} \\ &= \begin{pmatrix} C_\theta \omega_{g_x} - S_\theta \omega_{g_z} \\ \dot{\theta} + \omega_{g_y} \\ S_\theta \omega_{g_x} + C_\theta \omega_{g_z} \end{pmatrix} \end{aligned} \quad (9.9)$$

and angular accelerations of the rotor as

$$\dot{\omega}_r = \begin{pmatrix} -\dot{\theta}\omega_{r_x} + C_\theta \ddot{\phi} - S_\theta C_\phi \ddot{\omega}_{s_z} + S_\theta \dot{\phi}\omega_{g_y} + (C_\theta \omega_{s_y} - S_\theta S_\phi \omega_{s_z}) \dot{\omega}_{s_z} \\ \ddot{\theta} + S_\phi \dot{\omega}_{s_z} + \dot{\phi}\omega_{g_z} - C_\phi \dot{\omega}_{s_x} \\ \dot{\theta}\omega_{r_z} + S_\theta \ddot{\phi} + C_\theta C_\phi \ddot{\omega}_{s_z} - C_\theta \dot{\phi}\omega_{g_y} + (S_\theta \omega_{s_y} + C_\theta S_\phi \omega_{s_z}) \dot{\omega}_{s_z} \end{pmatrix} \quad (9.10)$$

This completes the required kinematic description of the system.

9.2 Derivation of the Equations of Motion

Now that the angular rates of the bodies have been derived, it is possible to calculate the equations of motion of a DTG-type system. Recall from the small angle analysis of a DTG that, when the generalized forces are derivable from a potential, Lagrange's equations can be expressed as

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}} \right) - \frac{\partial \mathcal{L}}{\partial q} = 0 \quad (9.11)$$

where q represents a generalized coordinate and $\mathcal{L}$ is known as the Lagrangian and is given by

$$\mathcal{L} = \mathcal{T} - \mathcal{V}$$

Neglecting any translation in the system, the kinetic energy, $\mathcal{T}$, of the MMS can be written as

$$\begin{aligned} \mathcal{T} = & \frac{1}{2} \left(I_{r_i} \omega_{r_i}^2 + I_{r_v} \omega_{r_v}^2 + I_{r_i} \omega_{r_i}^2 + I_{g_i} \omega_{g_i}^2 \right. \\ & \left. + I_{g_v} \omega_{g_v}^2 + I_{g_i} \omega_{g_i}^2 + I_{s_i} \omega_{s_i}^2 + I_{v_i} \omega_{v_i}^2 \right) \\ & - \left(I_{r_i v} \omega_{r_i} \omega_{r_v} + I_{r_i i} \omega_{r_i} \omega_{r_i} + I_{r_v i} \omega_{r_v} \omega_{r_i} \right. \\ & \left. + I_{g_i v} \omega_{g_i} \omega_{g_v} + I_{g_i i} \omega_{g_i} \omega_{g_i} + I_{g_v i} \omega_{g_v} \omega_{g_i} \right) \end{aligned} \quad (9.12)$$

where I_{i_r} , I_{i_v} and I_{i_i} are the mass moments of inertia of the i^{th} body where i could represent the rotor, r; the gimbal, g; the shaft, s; or the vibrational damper, v; and are defined, in general, by

$$I_x = \int (z^2 + y^2) dm$$

$$I_y = \int (z^2 + x^2) dm$$

$$I_z = \int (x^2 + y^2) dm$$

and I_{rxv} , I_{rxz} , I_{rvz} and I_{gzv} , I_{gzx} , I_{gvz} are the products of inertia of the rotor and gimbal, respectively, and are defined as

$$I_{xv} = \int xy \, dm$$

$$I_{xz} = \int xz \, dm$$

$$I_{vz} = \int yz \, dm$$

The potential energy of the system, $\mathcal{V}$, is

$$\begin{aligned} \mathcal{V} = & k_x \phi^2 + k_v \theta^2 + \frac{1}{2} K_v \beta^2 + \frac{1}{2} K_m (\Omega - \Omega_s)^2 \\ & - T_m \Omega - T_x \phi' - T_v \eta \end{aligned} \quad (9.13)$$

where k_x and k_v represent the stiffness of the crossed flexure pivots along the gimbal x and y axes, respectively. K_v is the torsional stiffness between the shaft and the vibrational damper. K_m represents the stiffness in the motor and $(\Omega - \Omega_s)$ is the angle describing the 'motor hunt', or the angle between the ideal 'motor rotor' position, Ω_s , and the actual position, Ω . β is the angle about the z axis between the shaft and the vibration damper. T_x and T_v represent the torques being applied to the rotor by the torquing coils and T_m represents any torque provided by the motor in addition to that required to overcome bearing friction. The torques provided by the torquer coils are in the 'case' coordinate frame and act over the angles ϕ' and η , see Figure 9.3.

The relationships between these angles and ϕ and θ are

$$\begin{aligned} \phi' &= \arctan \left(\frac{C_\theta S_\phi C_\Omega - S_\theta S_\Omega}{C_\theta C_\phi} \right) \\ \eta &= \arctan \left(\frac{S_\theta C_\Omega + C_\theta S_\phi S_\Omega}{C_\theta C_\phi} \right) \end{aligned}$$

Figure 9.3: Rotor position in 'case' and 'pickoff' referenced angles. The two rotation sequences are given by: $(\gamma)_z(\theta')_y(\phi')_x$ and $(\gamma')_z(\phi'')_x(\eta)_y$. η and ϕ' are the angles that are acted upon by the torquer coils.

where, for example C_θ represents $\cos \theta$ and S_θ represents $\sin \theta$.

The potential and kinetic energy in the system have been defined. The open loop response of the MMS could now be derived if damping along the x and y gimbal axes was ignored. In practice, damping along these axes will exist, although it will be quite small. Because of this fact, the MMS can not be treated like a conservative system. Lagrange's equation for a non-conservative system is

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) - \frac{\partial \mathcal{L}}{\partial q_i} = Q_i$$

where the quantities Q_i are known as generalized forces and are associated with the generalized coordinates q_i . The work done by these forces when the system is subjected to an arbitrary infinitesimal displacement is given by[82]

$$\delta W = \sum_{i=1}^n Q_i \delta q_i$$

It will be assumed that the damping in the MMS is viscous and occurs due to the fact that the case of the MMS is filled with a gas which impedes the motion of the rotor and gimbal, there is also damping in the material of the pivots, etc. The damping force is expressed as[82]

$$Q_i = -c_i \dot{q}_i$$

where c is the damping coefficient. Therefore, the following relationships can be expressed

$$\begin{aligned} Q_\phi &= -2D_x \dot{\phi} \\ Q_\theta &= -2D_y \dot{\theta} \end{aligned}$$

where $2D_x$ and $2D_y$ are the damping coefficients relating the x and y axes of the gimbal, respectively. Note that any damping occurring about the spin axis of the shaft is assumed to be countered by torques from the motor.

9.2.1 Derivation of the Equations of Motion using Lagrange's Equations

It is possible to use Lagrange's Equations directly to derive the equations of motion. Substituting Equations 9.12 and 9.13 into the Lagrangian give the form of Lagrange's Equations as

$$\begin{aligned}
& \frac{d}{dt} \left(I_{r_z} \omega_{r_z} \frac{\partial \omega_{r_z}}{\partial \dot{q}} + I_{r_y} \omega_{r_y} \frac{\partial \omega_{r_y}}{\partial \dot{q}} + I_{r_x} \omega_{r_x} \frac{\partial \omega_{r_x}}{\partial \dot{q}} + I_{g_z} \omega_{g_z} \frac{\partial \omega_{g_z}}{\partial \dot{q}} \right. \\
& \quad \left. + I_{g_y} \omega_{g_y} \frac{\partial \omega_{g_y}}{\partial \dot{q}} + I_{g_x} \omega_{g_x} \frac{\partial \omega_{g_x}}{\partial \dot{q}} + I_{s_z} \omega_{s_z} \frac{\partial \omega_{s_z}}{\partial \dot{q}} + I_{v_z} \omega_{v_z} \frac{\partial \omega_{v_z}}{\partial \dot{q}} \right. \\
& - \left[I_{r_{zy}} \left(\frac{\partial \omega_{r_z}}{\partial \dot{q}} \omega_{r_y} + \omega_{r_z} \frac{\partial \omega_{r_y}}{\partial \dot{q}} \right) + I_{r_{xz}} \left(\frac{\partial \omega_{r_z}}{\partial \dot{q}} \omega_{r_x} + \omega_{r_z} \frac{\partial \omega_{r_x}}{\partial \dot{q}} \right) \right. \\
& \quad \left. + I_{r_{yz}} \left(\frac{\partial \omega_{r_y}}{\partial \dot{q}} \omega_{r_z} + \omega_{r_y} \frac{\partial \omega_{r_z}}{\partial \dot{q}} \right) + I_{g_{zy}} \left(\frac{\partial \omega_{g_z}}{\partial \dot{q}} \omega_{g_y} + \omega_{g_z} \frac{\partial \omega_{g_y}}{\partial \dot{q}} \right) \right. \\
& \quad \left. + I_{g_{xz}} \left(\frac{\partial \omega_{g_z}}{\partial \dot{q}} \omega_{g_x} + \omega_{g_z} \frac{\partial \omega_{g_x}}{\partial \dot{q}} \right) + I_{g_{yz}} \left(\frac{\partial \omega_{g_y}}{\partial \dot{q}} \omega_{g_z} + \omega_{g_y} \frac{\partial \omega_{g_z}}{\partial \dot{q}} \right) \right] \Bigg) \\
& \quad I_{r_z} \omega_{r_z} \frac{\partial \omega_{r_z}}{\partial q} + I_{r_y} \omega_{r_y} \frac{\partial \omega_{r_y}}{\partial q} + I_{r_x} \omega_{r_x} \frac{\partial \omega_{r_x}}{\partial q} + I_{g_z} \omega_{g_z} \frac{\partial \omega_{g_z}}{\partial q} \\
& \quad + I_{g_y} \omega_{g_y} \frac{\partial \omega_{g_y}}{\partial q} + I_{g_x} \omega_{g_x} \frac{\partial \omega_{g_x}}{\partial q} + I_{s_z} \omega_{s_z} \frac{\partial \omega_{s_z}}{\partial q} + I_{v_z} \omega_{v_z} \frac{\partial \omega_{v_z}}{\partial q} \\
& - \left[I_{r_{zy}} \left(\frac{\partial \omega_{r_z}}{\partial q} \omega_{r_y} + \omega_{r_z} \frac{\partial \omega_{r_y}}{\partial q} \right) + I_{r_{xz}} \left(\frac{\partial \omega_{r_z}}{\partial q} \omega_{r_x} + \omega_{r_z} \frac{\partial \omega_{r_x}}{\partial q} \right) \right. \\
& \quad \left. + I_{r_{yz}} \left(\frac{\partial \omega_{r_y}}{\partial q} \omega_{r_z} + \omega_{r_y} \frac{\partial \omega_{r_z}}{\partial q} \right) + I_{g_{zy}} \left(\frac{\partial \omega_{g_z}}{\partial q} \omega_{g_y} + \omega_{g_z} \frac{\partial \omega_{g_y}}{\partial q} \right) \right. \\
& \quad \left. + I_{g_{xz}} \left(\frac{\partial \omega_{g_z}}{\partial q} \omega_{g_x} + \omega_{g_z} \frac{\partial \omega_{g_x}}{\partial q} \right) + I_{g_{yz}} \left(\frac{\partial \omega_{g_y}}{\partial q} \omega_{g_z} + \omega_{g_y} \frac{\partial \omega_{g_z}}{\partial q} \right) \right] \\
& - 2k_x \phi \frac{\partial \phi}{\partial q} - 2k_y \theta \frac{\partial \theta}{\partial q} - K_v \beta \frac{\partial \beta}{\partial q} - K_m (\Omega - \Omega_s) \frac{\partial \Omega}{\partial q} + T_m \frac{\partial \Omega}{\partial q}
\end{aligned}$$

$$\begin{aligned}
& + \frac{T_x}{C_\theta^2 C_\phi^2 + (C_\theta S_\phi C_\Omega - S_\theta S_\Omega)^2} \left[C_\theta C_\phi \frac{\partial}{\partial q} (C_\theta S_\phi C_\Omega - S_\theta S_\Omega) \right. \\
& \quad \left. - (C_\theta S_\phi C_\Omega - S_\theta S_\Omega) \frac{\partial}{\partial q} (C_\theta C_\phi) \right] \\
& + \frac{T_y}{C_\theta^2 C_\phi^2 + (S_\theta C_\Omega + C_\theta S_\phi S_\Omega)^2} \left[C_\theta C_\phi \frac{\partial}{\partial q} (S_\theta C_\Omega + C_\theta S_\phi S_\Omega) \right. \\
& \quad \left. - (S_\theta C_\Omega + C_\theta S_\phi S_\Omega) \frac{\partial}{\partial q} (C_\theta C_\phi) \right] + Q
\end{aligned} \tag{9.14}$$

where q represents one of the four generalized coordinates, θ , ϕ , Ω and β , introduced earlier and Q represents the appropriate generalized force due to damping. The equations of motion for the MMS can now be calculated using Equation 9.14 and the appropriate partial derivatives. Four cases must be considered, each one using a different generalized coordinate.

The result of this analysis, see Appendix C, is four second order equations that describe the motion of the MMS. Writing the equations in matrix form gives

$$\begin{bmatrix} A & -B & C & 0 \\ -B & E & F & 0 \\ C & F & H & I \\ 0 & 0 & I & I \end{bmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{\phi} \\ \ddot{\Omega} \\ \ddot{\beta} \end{pmatrix} = \begin{pmatrix} D \\ G \\ J \\ K \end{pmatrix} \tag{9.15}$$

Where

$$A = I_{r_v}$$

$$B = C_\theta I_{r_{zv}} + S_\theta I_{r_{vz}}$$

$$C = S_\phi I_{r_v} + C_\phi (S_\theta I_{r_{zv}} - C_\theta I_{r_{vz}})$$

$$\begin{aligned}
D &= H_{r_z} \omega_{r_z} - H_{r_z} \omega_{r_z} + \dot{\theta} (I_{r_{vz}} \omega_{r_z} - I_{r_{zv}} \omega_{r_z}) \\
&+ \dot{\phi} [(S_\theta I_{r_{zv}} - C_\theta I_{r_{vz}}) \omega_{g_v} - I_{r_v} \omega_{g_z}] \\
&+ \dot{\Omega} [(C_\phi I_{r_v} - S_\phi (S_\theta I_{r_{zv}} - C_\theta I_{r_{vz}})) \omega_{g_z} + B \omega_{g_v}]
\end{aligned}$$

$$\begin{aligned}
& - T_x \frac{C_\phi S_\Omega}{C_\theta^2 C_\phi^2 + (C_\theta S_\phi C_\Omega - S_\theta S_\Omega)^2} \\
& + T_y \frac{C_\phi C_\Omega}{C_\theta^2 C_\phi^2 + (S_\theta C_\Omega + C_\theta S_\phi S_\Omega)^2} \\
& - 2k_y \theta - 2D_y \theta
\end{aligned}$$

$$E = I_{g_x} + C_\theta^2 I_{r_x} + S_\theta^2 I_{r_z} - S_{2\theta} I_{r_{xz}}$$

$$F = C_\phi \left(\frac{1}{2} S_{2\theta} (I_{r_x} - I_{r_z}) - C_{2\theta} I_{r_{xz}} - I_{g_{xz}} \right) - S_\phi \left(C_\theta I_{r_{xy}} + S_\theta I_{r_{yz}} + I_{g_{xy}} \right)$$

$$\begin{aligned}
G &= \dot{\theta} [S_\theta H_{r_x} - C_\theta H_{r_z} + (C_\theta I_{r_x} - S_\theta I_{r_z}) \omega_{r_x} - (S_\theta I_{r_x} - C_\theta I_{r_z}) \omega_{r_z}] \\
&+ \dot{\phi} \left[(C_\theta I_{r_{xy}} + S_\theta I_{r_{yz}} + I_{g_{xy}}) \omega_{g_x} \right. \\
&\quad \left. - \left(\frac{1}{2} S_{2\theta} (I_{r_x} - I_{r_z}) + C_{2\theta} I_{r_{xz}} + I_{g_{xz}} \right) \omega_{g_y} \right] \\
&- \Omega \left[E \omega_{s_y} + \omega_{s_x} \left(S_\phi \left(\frac{1}{2} S_{2\theta} (I_{r_x} - I_{r_z}) - C_{2\theta} I_{r_{xz}} - I_{g_{xz}} \right) \right. \right. \\
&\quad \left. \left. + C_\phi (C_\theta I_{r_{xy}} + S_\theta I_{r_{yz}} + I_{g_{xy}}) \right) \right] \\
&+ (S_\theta H_{r_x} - C_\theta H_{r_z} - H_{g_x}) \omega_{g_y} + (H_{r_y} + H_{g_y}) \omega_{g_x} - 2k_x \phi - 2D_x \dot{\phi} \\
&+ T_x \frac{C_\theta^2 C_\Omega - \frac{1}{2} S_{2\theta} S_\phi S_\Omega}{C_\theta^2 C_\phi^2 + (C_\theta S_\phi C_\Omega - S_\theta S_\Omega)^2} + T_y \frac{C_\theta^2 S_\Omega + \frac{1}{2} S_{2\theta} S_\phi C_\Omega}{C_\theta^2 C_\phi^2 + (S_\theta C_\Omega + C_\theta S_\phi S_\Omega)^2}
\end{aligned}$$

$$\begin{aligned}
H &= S_{2\phi} (S_\theta I_{r_{xy}} - C_\theta I_{r_{yz}} - I_{g_{yz}}) + I_{s_x} + I_{v_x} \\
&+ C_\phi^2 (S_\theta^2 I_{r_x} + C_\theta^2 I_{r_z} + S_{2\theta} I_{r_{xz}} + I_{g_x}) + S_\phi^2 (I_{r_y} + I_{g_y})
\end{aligned}$$

$$I = I_{v_x}$$

$$\begin{aligned}
J &= \dot{\theta} [C_\phi (C_\theta H_{r_x} + S_\theta H_{r_z}) - \omega_{r_x} (S_\phi I_{r_{xy}} + C_\phi (S_\theta I_{r_x} + C_\theta I_{r_z})) \\
&\quad + \omega_{r_z} (S_\phi I_{r_{yz}} - C_\phi (C_\theta I_{r_x} + S_\theta I_{r_z}))] \\
&+ \dot{\phi} [S_\phi (C_\theta H_{r_x} - S_\theta H_{r_z} + H_{g_x}) - C_\phi (H_{r_y} + H_{g_y}) \\
&\quad + \omega_{g_y} (S_\phi (S_\theta I_{r_{xy}} - C_\theta I_{r_{yz}} - I_{g_{yz}})
\end{aligned}$$

$$\begin{aligned}
& + C_\phi \left(S_\theta^2 I_{r_x} + C_\theta^2 I_{r_z} + S_{2\theta} I_{r_{xz}} + I_{g_z} \right) \\
& - \omega_{g_z} \left(S_\phi \left(I_{r_v} + I_{g_v} \right) + C_\phi \left(S_\theta I_{r_{zv}} - C_\theta I_{r_{vz}} - I_{g_{vz}} \right) \right) \\
& - \dot{\Omega} \left[\omega_{s_x} \left(C_{2\phi} \left(I_{g_{vz}} - S_\theta I_{r_{zv}} + C_\theta I_{r_{vz}} \right) \right. \right. \\
& \quad \left. \left. + \frac{1}{2} S_{2\phi} \left(S_\theta^2 I_{r_x} + C_\theta^2 I_{r_z} + S_{2\theta} I_{r_{xz}} - I_{r_v} - I_{g_v} + I_{g_z} \right) \right) \right. \\
& \quad \left. - \omega_{s_v} \left(S_\phi \left(C_\theta I_{r_{zv}} + S_\theta I_{r_{vz}} + I_{g_{zv}} \right) \right. \right. \\
& \quad \left. \left. + C_\phi \left(C_{2\theta} I_{r_{xz}} + \frac{1}{2} S_{2\theta} \left(I_{r_x} - I_{r_z} \right) + I_{g_{xz}} \right) \right) \right] \\
& + \omega_{s_x} \left[S_\phi \left(H_{g_z} + C_\theta H_{r_z} - S_\theta H_{r_x} \right) - C_\phi \left(H_{r_v} - H_{g_v} \right) \right] \\
& + \omega_{s_v} \left(C_\theta H_{r_x} + S_\theta H_{r_z} + H_{g_z} \right) \\
& - \frac{1}{2} T_x \frac{C_\theta^2 S_{2\phi} S_\Omega + S_{2\theta} C_\phi C_\Omega}{C_\theta^2 C_\phi^2 + (C_\theta S_\phi C_\Omega - S_\theta S_\Omega)^2} - \frac{1}{2} T_v \frac{S_{2\theta} C_\phi S_\Omega - C_\theta^2 S_{2\phi} C_\Omega}{C_\theta^2 C_\phi^2 + (S_\theta C_\Omega + C_\theta S_\phi S_\Omega)^2} \\
& - K_m (\Omega - \Omega_s) + T_m
\end{aligned}$$

$$K = -K_v \beta$$

and where, for example, H_{r_x} and H_{r_z} represent the angular momentum of the rotor about the x and z axes, respectively.

This matrix can also be rewritten in the familiar form

$$M\ddot{\mathbf{q}} = \mathbf{Q} \quad (9.16)$$

Multiplying both sides of the above expression by the inverse of the coefficient matrix and solving allows $\ddot{\theta}$, $\ddot{\phi}$, $\ddot{\Omega}$ and $\ddot{\beta}$ to be isolated. Therefore

$$\begin{aligned}
\ddot{\theta} &= I [B [F (K - J) + G (H - I)] - C [E (J - K) - FG]] \\
&- D [F^2 - E (H - I)] \div \det M
\end{aligned} \quad (9.17)$$

$$\begin{aligned}
\ddot{\phi} &= I [A [F (K - J) + G (H - I)] - B [C (J - K) + D (I - H)] \\
&- C (CG - DF)] \div \det M
\end{aligned} \quad (9.18)$$

$$\ddot{\Omega} = \frac{I[A(E(J-K) - FG) + B^2(K-J) - B(CG + DF) - CDE]}{\det M} \quad (9.19)$$

$$\begin{aligned} \ddot{\beta} = & \left[A\{E(HK - IJ) - F(FK - GI)\} + B^2(IJ - HK) \right. \\ & \left. + B[DFI - C(2FK - GI)] - CE(CK - DI) \right] \div \det M \quad (9.20) \end{aligned}$$

where

$$\det M = I \left[A \left[E(H - I) - F^2 \right] + B^2(I - H) - C(2BF + CE) \right] \quad (9.21)$$

In order for the above equations to be true it is necessary that the determinate of M does not equal zero. This is usually true since the mass matrix, M , is symmetric positive definite. There is one exception; the equations above include the effects of the vibration damper and, if not present, Equation 9.20 would approach infinity. Calculating the limit of Equations 9.17 through 9.20 as $I \rightarrow 0$ and setting $K = 0$ gives the following results

$$\ddot{\theta} \rightarrow - \frac{B(FJ - GH) + C(EJ - FG) + D(F^2 - EH)}{A(EH - F^2) - B^2H - C(2BF + CE)} \quad (9.22)$$

$$\ddot{\phi} \rightarrow - \frac{A(FJ - GH) + B(CJ - DH) + C(CG - DF)}{A(EH - F^2) - B^2H - C(2BF + CE)} \quad (9.23)$$

$$\ddot{\Omega} \rightarrow \frac{A(EJ - FG) - B^2J - B(CG + DF) - CDE}{A(EH - F^2) - B^2H - C(2BF + CE)} \quad (9.24)$$

$$\ddot{\beta} \rightarrow 0 \quad (9.25)$$

Four second order equations have been derived which describe the motion of the MMS or a DTG.

9.2.2 Derivation of the Equations of Motion of the MMS using Hamilton's Canonical Equations

A second method of deriving the equations of motion is to use Hamilton's Canonical Equations. By defining the generalized momentum, P_q , as [81]

$$P_q = \frac{\partial \mathcal{L}}{\partial \dot{q}}$$

Lagrange's Equations can now be rewritten in terms of this generalized momentum to obtain

$$\frac{d}{dt}(P_q) = \frac{\partial \mathcal{L}}{\partial q} + Q$$

Therefore, the rate of change of the momentum is equal to the partial derivative of the Lagrangian with respect to q . The equations of motion can be rewritten as

$$\dot{P}_q = \frac{\partial \mathcal{L}}{\partial q} + Q \quad (9.26)$$

$$P_q = \frac{\partial \mathcal{L}}{\partial \dot{q}} \quad (9.27)$$

and are known as Hamilton's Canonical Equations. Substituting Equations 9.12 and 9.13 into Equations 9.26 and 9.27 gives

$$\begin{aligned} \dot{P}_q &= I_{r_x} \omega_{r_x} \frac{\partial \omega_{r_x}}{\partial q} + I_{r_y} \omega_{r_y} \frac{\partial \omega_{r_y}}{\partial q} + I_{r_z} \omega_{r_z} \frac{\partial \omega_{r_z}}{\partial q} + I_{g_x} \omega_{g_x} \frac{\partial \omega_{g_x}}{\partial q} \\ &\quad + I_{g_y} \omega_{g_y} \frac{\partial \omega_{g_y}}{\partial q} + I_{g_z} \omega_{g_z} \frac{\partial \omega_{g_z}}{\partial q} + I_{s_x} \omega_{s_x} \frac{\partial \omega_{s_x}}{\partial q} + I_{v_x} \omega_{v_x} \frac{\partial \omega_{v_x}}{\partial q} \\ &= \left[I_{r_{xy}} \left(\frac{\partial \omega_{r_x}}{\partial q} \omega_{r_y} + \omega_{r_x} \frac{\partial \omega_{r_y}}{\partial q} \right) + I_{r_{xz}} \left(\frac{\partial \omega_{r_x}}{\partial q} \omega_{r_z} + \omega_{r_x} \frac{\partial \omega_{r_z}}{\partial q} \right) \right. \\ &\quad + I_{r_{yz}} \left(\frac{\partial \omega_{r_y}}{\partial q} \omega_{r_z} + \omega_{r_y} \frac{\partial \omega_{r_z}}{\partial q} \right) + I_{g_{xy}} \left(\frac{\partial \omega_{g_x}}{\partial q} \omega_{g_y} + \omega_{g_x} \frac{\partial \omega_{g_y}}{\partial q} \right) \\ &\quad \left. + I_{g_{xz}} \left(\frac{\partial \omega_{g_x}}{\partial q} \omega_{g_z} + \omega_{g_x} \frac{\partial \omega_{g_z}}{\partial q} \right) + I_{g_{yz}} \left(\frac{\partial \omega_{g_y}}{\partial q} \omega_{g_z} + \omega_{g_y} \frac{\partial \omega_{g_z}}{\partial q} \right) \right] \end{aligned}$$

$$\begin{aligned}
& - 2k_x \phi \frac{\partial \phi}{\partial q} - 2k_y \theta \frac{\partial \theta}{\partial q} - K_v \beta \frac{\partial \beta}{\partial q} - K_m (\Omega - \Omega_s) \frac{\partial \Omega}{\partial q} + T_m \frac{\partial \Omega}{\partial q} \\
& + \frac{T_x}{C_\theta^2 C_\phi^2 + (C_\theta S_\phi C_\Omega - S_\theta S_\Omega)^2} \left[C_\theta C_\phi \frac{\partial}{\partial q} (C_\theta S_\phi C_\Omega - S_\theta S_\Omega) \right. \\
& \quad \left. - (C_\theta S_\phi C_\Omega - S_\theta S_\Omega) \frac{\partial}{\partial q} (C_\theta C_\phi) \right] \\
& + \frac{T_y}{C_\theta^2 C_\phi^2 + (S_\theta C_\Omega + C_\theta S_\phi S_\Omega)^2} \left[C_\theta C_\phi \frac{\partial}{\partial q} (S_\theta C_\Omega + C_\theta S_\phi S_\Omega) \right. \\
& \quad \left. - (S_\theta C_\Omega + C_\theta S_\phi S_\Omega) \frac{\partial}{\partial q} (C_\theta C_\phi) \right] + Q \tag{9.28}
\end{aligned}$$

and

$$\begin{aligned}
P_q &= I_{r_x} \omega_{r_x} \frac{\partial \omega_{r_x}}{\partial \dot{q}} + I_{r_y} \omega_{r_y} \frac{\partial \omega_{r_y}}{\partial \dot{q}} + I_{r_z} \omega_{r_z} \frac{\partial \omega_{r_z}}{\partial \dot{q}} + I_{g_x} \omega_{g_x} \frac{\partial \omega_{g_x}}{\partial \dot{q}} \\
&+ I_{g_y} \omega_{g_y} \frac{\partial \omega_{g_y}}{\partial \dot{q}} + I_{g_z} \omega_{g_z} \frac{\partial \omega_{g_z}}{\partial \dot{q}} + I_{s_x} \omega_{s_x} \frac{\partial \omega_{s_x}}{\partial \dot{q}} + I_{v_x} \omega_{v_x} \frac{\partial \omega_{v_x}}{\partial \dot{q}} \\
&- \left[I_{r_{xy}} \left(\frac{\partial \omega_{r_x}}{\partial \dot{q}} \omega_{r_y} + \omega_{r_x} \frac{\partial \omega_{r_y}}{\partial \dot{q}} \right) + I_{r_{xz}} \left(\frac{\partial \omega_{r_x}}{\partial \dot{q}} \omega_{r_z} + \omega_{r_x} \frac{\partial \omega_{r_z}}{\partial \dot{q}} \right) \right. \\
&+ I_{r_{yz}} \left(\frac{\partial \omega_{r_y}}{\partial \dot{q}} \omega_{r_z} + \omega_{r_y} \frac{\partial \omega_{r_z}}{\partial \dot{q}} \right) + I_{g_{xy}} \left(\frac{\partial \omega_{g_x}}{\partial \dot{q}} \omega_{g_y} + \omega_{g_x} \frac{\partial \omega_{g_y}}{\partial \dot{q}} \right) \\
&+ I_{g_{xz}} \left(\frac{\partial \omega_{g_x}}{\partial \dot{q}} \omega_{g_z} + \omega_{g_x} \frac{\partial \omega_{g_z}}{\partial \dot{q}} \right) \\
&\quad \left. + I_{g_{yz}} \left(\frac{\partial \omega_{g_y}}{\partial \dot{q}} \omega_{g_z} + \omega_{g_y} \frac{\partial \omega_{g_z}}{\partial \dot{q}} \right) \right] \tag{9.29}
\end{aligned}$$

Equations 9.28 and 9.29 can be used to find eight first order equations which describe the motion of the MMS or a DTG, see Appendix D. As with the solution found using Lagrange's Equations, four of these equations must be solved simultaneously in order to isolate $\dot{\theta}$, $\dot{\phi}$, $\dot{\Omega}$ and $\dot{\beta}$. Writing the

equations in matrix form gives

$$\begin{bmatrix} A & -B & C & 0 \\ E & -B & F & 0 \\ C & F & H & I \\ 0 & 0 & I & I \end{bmatrix} \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \\ \dot{\Omega} \\ \dot{\beta} \end{pmatrix} = \begin{pmatrix} D_2 \\ G_2 \\ J_2 \\ K_2 \end{pmatrix} \quad (9.30)$$

where

$$D_2 = P_\theta + B\omega_{s_x} - \left[C'_\phi I_{r_v} + S_\phi (S_\theta I_{r_{xv}} - C'_\theta I_{r_{vz}}) \right] \omega_{s_v} - C'\omega_{c_x}$$

$$\begin{aligned} G_2 = P_\phi + & \left[C'_\phi (C'_\theta I_{r_{xv}} + S_\theta I_{r_{vz}} + I_{g_{xv}}) \right. \\ & \left. + S_\phi \left(\frac{1}{2} S_{2\theta} (I_{r_x} - I_{r_z}) - C'_{2\theta} I_{r_{xz}} - I_{g_{xz}} \right) \right] \omega_{s_v} \\ & - E\omega_{s_x} - F\omega_{c_x} \end{aligned}$$

$$\begin{aligned} J_2 = P_\Omega - H\omega_{c_x} - F\omega_{s_x} \\ - \left[C'_{2\phi} (S_\theta I_{r_{xv}} - C'_\theta I_{r_{vz}} - I_{g_{vz}}) \right. \\ \left. + \frac{1}{2} S_{2\phi} (I_{r_v} + I_{g_v} - S_\theta^2 I_{r_x} - C_\theta'^2 I_{r_z} - S_{2\theta} I_{r_{xz}} - I_{g_x}) \right] \omega_{s_v} \end{aligned}$$

$$K_2 = P_\beta - I_{v_x} \omega_{c_x}$$

Note that the above expression is identical to that obtained using Lagrange's Equations with the exception of the right hand side of the equation. Multiplying both sides by the inverse of the coefficient matrix and solving gives the following solutions

$$\begin{aligned} \dot{\theta} = & -[B[F(J_2 - K_2) + G_2(I - H)] + C[E(J_2 - K_2) - FG_2] \\ & + D_2[F^2 - E(H - I)]] \div [A[E(H - I) - F^2] \\ & + B^2(I - H) - C(2BF + CE)] \end{aligned} \quad (9.31)$$

$$\begin{aligned}\dot{\phi} = & -[A[F(J_2 - K_2) + G_2(I - H)] + B[C(J_2 - K_2) + D_2(I - H)] \\ & + C(CG_2 - D_2F)] \div [A[E(H - I) - F^2] + B^2(I - H) \\ & - C(2BF + CE)]\end{aligned}\quad (9.32)$$

$$\dot{\Omega} = \frac{A[E(J_2 - K_2) - FG_2] + B^2(K_2 - J_2) - B(CG_2 + D_2F) - CD_2E}{A[E(H - I) - F^2] + B^2(I - H) - C(2BF + CE)}\quad (9.33)$$

$$\begin{aligned}\dot{\beta} = & [A[E(HK_2 - IJ_2) - F(FK_2 - G_2I)] + B^2(IJ_2 - HK_2) \\ & + B[D_2FI - C(2FK_2 - G_2I)] - CE(CK_2 - D_2I)] \\ & \div [A[E(H - I) - F^2] + B^2(I - H) - C(2BF + CE)]\end{aligned}\quad (9.34)$$

If the inertia and stiffness of the damper are set to zero then the solutions have the form

$$\theta = \frac{B(FJ_2 - G_2H) + C(EJ_2 - FG_2) + D_2(F^2 - EH)}{A(EH - F^2) - B^2H - C(2BF + CE)}\quad (9.35)$$

$$\phi = \frac{A(FJ_2 - G_2H) + B(CJ_2 - D_2H) + C(CG_2 - D_2F)}{A(EH - F^2) - B^2H - C(2BF + CE)}\quad (9.36)$$

$$\Omega = \frac{A(EJ_2 - FG_2) - B^2J_2 - B(CG_2 + D_2F) - CD_2E}{A(EH - F^2) - B^2H - C(2BF + CE)}\quad (9.37)$$

$$\dot{\beta} = 0\quad (9.38)$$

Thus, a set of eight first order differential equations have also been derived to express the equations of motion of the MMS using Hamilton's Canonical Equations. The first four are expressed above and the remaining four are:

$$\begin{aligned}\dot{P}_\theta = & H_{r_i}\omega_{r_i} - H_{r_z}\omega_{r_i} - 2k_y\theta - 2D_y\dot{\theta} \\ & - T_x \frac{C_\phi S_\Omega}{C_\theta^2 C_\phi^2 + (C_\theta S_\phi C_\Omega - S_\theta S_\Omega)^2} \\ & + T_y \frac{C_\phi C_\Omega}{C_\theta^2 C_\phi^2 + (S_\theta C_\Omega + C_\theta S_\phi S_\Omega)^2}\end{aligned}$$

$$\begin{aligned}
\dot{P}_\phi &= (S_\theta H_{r_x} + C_\theta H_{r_z} + H_{g_y}) \omega_{g_y} + (H_{r_y} + H_{g_y}) \omega_{g_z} - 2k_x \phi - 2D_x \dot{\phi} \\
&\quad + T_x \frac{C_\theta^2 C_\Omega + \frac{1}{2} S_{2\theta} S_\phi S_\Omega}{C_\theta^2 C_\phi^2 + (C_\theta S_\phi C_\Omega + S_\theta S_\Omega)^2} \\
&\quad + T_y \frac{C_\theta^2 S_\Omega + \frac{1}{2} S_{2\theta} S_\phi C_\Omega}{C_\theta^2 C_\phi^2 + (S_\theta C_\Omega + C_\theta S_\phi S_\Omega)^2}
\end{aligned}$$

$$\begin{aligned}
\dot{P}_\Omega &= \omega_{s_x} [S_\phi (H_{g_x} + C_\theta H_{r_z} + S_\theta H_{r_x}) + C_\phi (H_{r_y} + H_{g_y})] \\
&\quad + \omega_{s_y} (C_\theta H_{r_x} + S_\theta H_{r_z} + H_{g_x}) - \frac{1}{2} T_x \frac{C_\theta^2 S_{2\phi} S_\Omega + S_{2\theta} C_\phi C_\Omega}{C_\theta^2 C_\phi^2 + (C_\theta S_\phi C_\Omega + S_\theta S_\Omega)^2} \\
&\quad - \frac{1}{2} T_y \frac{S_{2\theta} C_\phi S_\Omega + C_\theta^2 S_{2\phi} C_\Omega}{C_\theta^2 C_\phi^2 + (S_\theta C_\Omega + C_\theta S_\phi S_\Omega)^2} - K_m (\Omega - \Omega_s) + T_m
\end{aligned}$$

$$\dot{P}_\beta = -K_v \beta$$

The procedure used to derive these equations is found in Appendix D.

9.3 Calculation of the Loads on the Crossed Flexure Pivots

The constraint forces in the system have been avoided by using energy methods to derive the equations of motion. However, the stiffness of the crossed flexure pivots, which act as torsional springs connecting the rotor to the gimbal and the gimbal to the shaft, are affected by these constraint forces. To complete the model it is now necessary to find these forces.

The static analysis of the crossed flexure pivot involves the assumption that one of the rigid bodies remains fixed while the other rotates. In the MMS, and DTGs, the rotor and the shaft remain relatively fixed (except about the spin axis) when compared to the gimbal. It is for this reason

that the forces on the pivots will be calculated in the rotor and the shaft coordinate frames.

Figure 9.4 is a freebody diagram of the rotor. The rotor is connected to the gimbal through pivots mounted on the y axis and at these connection points, two sets of perpendicular forces are present. Torques are applied about each of the axes and are due to: i) the interaction of the magnetic field of the torquer coils and the permanent magnet attached to the rotor and, ii) the stiffness of the crossed flexure pivots.

The applied torques are about the x and y axes of the case and must be rotated to the rotor coordinate system in order to determine their effect on the rotor. This is done by rotating about the z axis of the shaft and then about the x and y axes of the gimbal, in other words

$$\begin{pmatrix} T'_x \\ T'_y \\ T'_z \end{pmatrix} = (\theta)_y (\phi)_x (\Omega)_z \begin{pmatrix} T_x \\ T_y \\ 0 \end{pmatrix} = \begin{pmatrix} (C_\theta C_\Omega - S_\theta S_\phi S_\Omega) T_x + (C_\theta S_\Omega + S_\theta S_\phi C_\Omega) T_y \\ C_\phi C_\Omega T_y - C_\phi S_\Omega T_x \\ (S_\theta C_\Omega + C_\theta S_\phi S_\Omega) T_x + (S_\theta S_\Omega - C_\theta S_\phi C_\Omega) T_y \end{pmatrix} \quad (9.39)$$

where T_x and T_y are the applied torques about the case x and y axes and T'_x , T'_y and T'_z are the same torques acting about the rotor x, y and z axes.

It should be noted at this point that it has been assumed that the centre of mass of the rotor is not undergoing an acceleration. With this in mind, it is clear that the forces on the rotor are equal and act in opposite directions, resulting in a net force of zero; however, they do create force couples about the x and z axes and act directly on the crossed axis pivots. Finally, it should be noted that the rotor is free to rotate about all three axes, as represented by the angular velocity terms. From the freebody diagram, the net moment

Figure 9.4: Freebody diagram of the MMS rotor. F_z and F_x are the forces acting on the rotor and are due to the interaction with the gimbal, T'_x , T'_y and T'_z are the torques acting on the rotor caused by the magnetic torquers. k_y represents the individual stiffness of the crossed flexure pivots connecting the rotor and the gimbal and θ is the rotation of the rotor with respect to the gimbal. The damping torque is assumed to act about the y axis and in the same direction as the torque due to the stiffness. The magnitude of the damping torque is $2D_y\dot{\theta}$. It has been assumed that the centre of mass of the rotor is not undergoing any acceleration.

acting on the rotor is

$$\begin{pmatrix} M_{r_x} \\ M_{r_y} \\ M_{r_z} \end{pmatrix} = \begin{pmatrix} 2F_z R + (C_\theta C_\Omega - S_\theta S_\phi S_\Omega) T_x + (C_\theta S_\Omega + S_\theta S_\phi C_\Omega) T_y \\ -2k_y \theta - 2D_y \dot{\theta} + C_\phi C_\Omega T_y - C_\phi S_\Omega T_x \\ 2F_r R + (S_\theta C_\Omega + C_\theta S_\phi S_\Omega) T_r + (S_\theta S_\Omega - C_\theta S_\phi C_\Omega) T_y \end{pmatrix} \quad (9.40)$$

where R is the radius at which the load acts. The moments can also be expressed using Euler's equations[12] as

$$\begin{aligned} \mathbf{M}_r &= \begin{pmatrix} \dot{H}_{r_x} - H_{r_y} \omega_{r_z} + H_{r_z} \omega_{r_y} \\ \dot{H}_{r_y} - H_{r_z} \omega_{r_x} + H_{r_x} \omega_{r_z} \\ \dot{H}_{r_z} - H_{r_x} \omega_{r_y} + H_{r_y} \omega_{r_x} \end{pmatrix} \\ &\quad \begin{pmatrix} I_{r_x} \dot{\omega}_{r_x} - I_{r_{xy}} \dot{\omega}_{r_y} - I_{r_{xz}} \dot{\omega}_{r_z} - H_{r_y} \omega_{r_z} + H_{r_z} \omega_{r_y} \\ I_{r_y} \dot{\omega}_{r_y} - I_{r_{xy}} \dot{\omega}_{r_x} - I_{r_{yz}} \dot{\omega}_{r_z} - H_{r_z} \omega_{r_x} + H_{r_x} \omega_{r_z} \\ I_{r_z} \dot{\omega}_{r_z} - I_{r_{xz}} \dot{\omega}_{r_x} - I_{r_{yz}} \dot{\omega}_{r_y} - H_{r_x} \omega_{r_y} + H_{r_y} \omega_{r_x} \end{pmatrix} \end{aligned} \quad (9.41)$$

Each of the terms in Equation 9.41 can be calculated using the equations of motion derived in the previous sections. The above relationships and the ones expressed in Equation 9.40 can be used to find the forces acting on the pivots connecting the rotor and the gimbal. The forces acting on the pivots are

$$\begin{aligned} F_x &= \frac{1}{2R} \left[I_{r_x} \dot{\omega}_{r_x} - I_{r_{xz}} \dot{\omega}_{r_z} - I_{r_{xy}} \dot{\omega}_{r_y} - H_{r_z} \omega_{r_y} + H_{r_y} \omega_{r_z} \right. \\ &\quad \left. - (S_\theta C_\Omega - C_\theta S_\phi S_\Omega) T_x - (S_\theta S_\Omega + C_\theta S_\phi C_\Omega) T_y \right] \end{aligned} \quad (9.42)$$

and

$$\begin{aligned} F_z &= \frac{1}{2R} \left[I_{r_z} \dot{\omega}_{r_z} - I_{r_{zy}} \dot{\omega}_{r_y} - I_{r_{zx}} \dot{\omega}_{r_x} - H_{r_y} \omega_{r_x} + H_{r_x} \omega_{r_y} \right. \\ &\quad \left. - (C_\theta C_\Omega + S_\theta S_\phi S_\Omega) T_x - (C_\theta S_\Omega - S_\theta S_\phi C_\Omega) T_y \right] \end{aligned} \quad (9.43)$$

A freebody diagram of the shaft is shown in Figure 9.5. The pivots connect the shaft and the gimbal along the x axis and this is where the

forces and torques act. Applied torques act about the z axis and represent the torque due to the motor, T_m . Two additional torques act about the spin axis and are due to the connection with the vibrational damper, see Figure 9.6, and the stiffness associated with the magnetic fields in the motor. The shaft is free to rotate about only the z axis and, similar to the rotor, the forces sum to zero. The bearings provide reaction torques which cancel all torques acting on the body except those acting about the z axis. With this in mind, the moments acting on the shaft are expressed as

$$\begin{pmatrix} M_{s_x} \\ M_{s_y} \\ M_{s_z} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2RG_y + K_v\beta - K_m(\Omega - \Omega_s) + T_m \end{pmatrix} \quad (9.44)$$

where $M_{s_i} = I_{s_i}\dot{\omega}_{s_i}$. The force acting parallel to the y axis can be calculated as

$$G_y = \frac{1}{2R} |I_{s_z}\dot{\omega}_{s_z} - K_v\beta + K_m(\Omega - \Omega_s) - T_m| \quad (9.45)$$

The remaining constraint force can be calculated by examining the free-body diagram of the gimbal, Figure 9.7. Like the rotor, the gimbal is free to rotate about any axis. The moments acting on the gimbal are

$$\begin{pmatrix} M_{g_x} \\ M_{g_y} \\ M_{g_z} \end{pmatrix} = \begin{pmatrix} -2F'_z R - 2k_x\phi - 2D_x\dot{\phi} \\ -2G'_z R + 2k_y\theta + 2D_y\dot{\theta} \\ -2G'_y R - 2F'_x R \end{pmatrix} \quad (9.46)$$

where F'_x , F'_z are the rotor forces, F_x and F_z in the gimbal coordinate frame and G'_y , G'_z are the shaft forces in the gimbal coordinate frame. Solving for G'_y and G'_z yields

$$G'_y = -F'_x - \frac{M_{g_x}}{2R} \quad (9.47)$$

$$G'_z = \frac{1}{2R} (2k_y\theta + 2D_y\dot{\theta} - M_{g_y}) \quad (9.48)$$

Figure 9.5: Freebody diagram of the MMS shaft. R_x and R_y represent the bearing reactions and G_y and G_z are the forces acting on the shaft and are due to the interaction with the gimbal. k_x represents the torsional stiffness of the individual crossed flexure pivots connecting the gimbal and shaft, K_m represents the resistance provided by the motor when the actual shaft position is offset from the ideal position and K_v represents the stiffness of the torsional spring connecting the vibration damper and the shaft. T_m is the torque supplied by the motor in addition to that needed to overcome bearing friction. The damping torque is assumed to act about the x axis and in the same direction as the torque due to the stiffness. The magnitude of the damping torque is $2D_x\phi$.

Figure 9.6: Freebody Diagram of the MMS vibration damper. K_v represents the stiffness of the torsional spring connecting the shaft and the damper and β is the angle between the shaft and the damper. Note that this component is not shown in the drawing of the MMS.

Figure 9.7: Freebody diagram of the MMS gimbal. The forces F'_x , F'_z , G'_y and G'_z are the forces acting on the gimbal and are caused by its interaction with the rotor and shaft, respectively. The prime indicates that the forces are in the local coordinate system of the gimbal. k_x and k_y represent the individual stiffness of the crossed flexure pivots connecting the gimbal to the shaft and rotor, respectively. The damping torque is assumed to act about the x axis and in the same direction as the torque due to the stiffness. The magnitude of the damping torque is $2D_x\phi$. It is assumed that the shaft is not undergoing any acceleration.

F'_x is also calculated by rotating the rotor forces into the gimbal coordinate frame, giving

$$\begin{pmatrix} F'_x \\ 0 \\ -F'_z \end{pmatrix} = (\theta)_y \begin{pmatrix} -F_x \\ 0 \\ F_z \end{pmatrix}$$

$$\begin{pmatrix} F'_x \\ 0 \\ F'_z \end{pmatrix} = \begin{pmatrix} C_\theta F_x - S_\theta F_z \\ 0 \\ C_\theta F_z + S_\theta F_x \end{pmatrix} \quad (9.49)$$

Therefore

$$G'_y = -C_\theta F_x + S_\theta F_z - \frac{M_{g_z}}{2R} \quad (9.50)$$

and

$$G'_z = \frac{1}{2R} (2k_y \theta + 2D_y \dot{\theta} - M_{g_y}) \quad (9.51)$$

These forces must now be equated to the shaft forces by rotating them through an angle of $-\phi$. That is

$$\begin{pmatrix} 0 \\ G_y \\ -G_z \end{pmatrix} = (-\phi)_x \begin{pmatrix} 0 \\ -G'_y \\ G'_z \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ G_y \\ G_z \end{pmatrix} = \begin{pmatrix} 0 \\ C_\phi G'_y + S_\phi G'_z \\ -S_\phi G'_y + C_\phi G'_z \end{pmatrix} \quad (9.52)$$

Therefore

$$\begin{aligned} G_z &= \frac{C_\phi}{2R} (2k_y \theta + 2D_y \dot{\theta} - M_{g_y}) + S_\phi \left(C_\theta F_x - S_\theta F_z + \frac{M_{g_z}}{2R} \right) \\ &= \frac{C_\phi}{2R} (2k_y \theta + 2D_y \dot{\theta} - I_{g_y} \dot{\omega}_{g_y} + I_{g_z y} \dot{\omega}_{g_z} + I_{g_y z} \dot{\omega}_{g_z} + H_{g_z} \omega_{g_z} - H_{g_z} \omega_{g_z}) \\ &\quad + \frac{S_\phi}{2R} \left[C_\theta (I_{r_x} \dot{\omega}_{r_x} - I_{r_{xz}} \dot{\omega}_{r_z} - I_{r_{yz}} \dot{\omega}_{r_y} - H_{r_x} \omega_{r_y} + H_{r_y} \omega_{r_x} \right. \\ &\quad \left. - (S_\theta C_\Omega - C_\theta S_\phi S_\Omega) T_x - (S_\theta S_\Omega + C_\theta S_\phi C_\Omega) T_y \right. \\ &\quad \left. - S_\theta (I_{r_x} \dot{\omega}_{r_x} - I_{r_{zy}} \dot{\omega}_{r_y} - I_{r_{xz}} \dot{\omega}_{r_z} - H_{r_y} \omega_{r_z} + H_{r_z} \omega_{r_y} \right] \end{aligned}$$

$$\begin{aligned}
& - (C_\theta C_\Omega + S_\theta S_\phi S_\Omega) T_x - (C_\theta S_\Omega - S_\theta S_\phi C_\Omega) T_y \\
& + I_{g_x} \dot{\omega}_{g_x} - I_{g_{xz}} \dot{\omega}_{g_z} - I_{g_{yz}} \dot{\omega}_{g_y} - H_{g_x} \omega_{g_y} + H_{g_y} \omega_{g_x} \Big] \quad (9.53)
\end{aligned}$$

The magnitude of the loads on the pivots due to the constraint forces have now been calculated and are found from Equations 9.42, 9.43, 9.45 and 9.53.

A final consideration is that the MMS may be operating while in an accelerational field, such as that due to gravity. If the MMS is in such a field then the loads acting on the pivots will be a combination of the dynamic loads and the accelerational loads. In order to include the effect of this load on the pivots it is first necessary to describe the orientation of the shaft in this field. Assume that the shaft is oriented in the field so rotations of the angles χ and ψ about the x and y axes are required to move the coordinate system from one based on the accelerational load to that of the shaft, see Figure 9.8. The loads on the pivots attached to the shaft are then

$$\begin{aligned}
\begin{pmatrix} A_{g_x} \\ A_{g_y} \\ A_{g_z} \end{pmatrix} &= \frac{1}{2} (\Omega)_z (\psi)_y (\chi)_x \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} (M_r + M_g) g \\
&= \frac{1}{2} \begin{pmatrix} C_\Omega S_\psi C_\chi - S_\Omega S_\chi \\ -S_\Omega S_\psi C_\chi - C_\Omega S_\chi \\ -C_\psi C_\chi \end{pmatrix} (M_r + M_g) g \quad (9.54)
\end{aligned}$$

where M_r and M_g represent the mass of the rotor and gimbal, respectively, and g is the accelerational load.

The loads on the pivots connecting the gimbal and rotor can be calculated by rotating the field from the shaft coordinate system into the rotor frame of reference. In other words

$$\begin{pmatrix} A_{r_x} \\ A_{r_y} \\ A_{r_z} \end{pmatrix} = \frac{1}{2} (\theta)_y (\phi)_x \begin{pmatrix} A_{r_x} \\ A_{r_y} \\ A_{r_z} \end{pmatrix} \left(\frac{M_r}{M_r + M_g} \right) g$$

Figure 9.8: Rotation sequence required in order to represent the direction of the acceleration in the shaft reference frame.

$$= \frac{1}{2} \begin{pmatrix} C_\theta A_{g_z} + S_\theta (S_\phi A_{g_y} - C_\phi A_{g_x}) \\ C_\phi A_{g_y} + S_\phi A_{g_x} \\ S_\theta A_{g_z} + C_\theta (C_\phi A_{g_x} - S_\phi A_{g_y}) \end{pmatrix} \left(\frac{M_r}{M_r + M_g} \right) g$$

In order to include the above loads it is necessary to sum the constraint load and the appropriate load due to the acceleration while keeping the direction of the constraint load in mind. For example, the loads acting on a crossed flexure pivot located along the positive x axis (see Figure 9.5) connecting the gimbal and the shaft are

$$G_{total_x} = -G_z + A_{g_x}$$

$$G_{total_y} = G_y + A_{g_y}$$

and the loads acting on the pivot located along the negative x axis are

$$G_{total_x} = G_z + A_{g_x}$$

$$G_{total_y} = -G_y + A_{g_y}$$

Similar calculations can be performed in order to find the loads on the pivots connecting the rotor and the gimbal.

Relationships for the large angle equations of motion and the forces acting on the crossed flexure pivots have now been derived. These results will now be implemented in a numerical simulation in order to simulate the motion of the MMS in the next chapter.

Chapter 10

Large Angle Simulation of a DTG/MMS Type System

In Chapter 9, the equations of motion of a Dynamically Tuned Gyroscope and similar systems were derived. These equations can now be solved to determine the response of a DTG and the Momentum Management System to a set of initial conditions describing the orientation and rate of the rotor and shaft.

Ideally, an analytical solution of the equations of motion which describe the motion of the system would be found; however, the complexity of the equations means that such a solution would be extraordinarily difficult to obtain. An analytical solution for this system is only practical when simplifying assumptions are made, such as those used in Chapter 8.

In order to solve the dynamic equations without the use of simplifying assumptions, a computer simulation was developed to numerically integrate the equations of motion.

10.1 Overview of the Computer Simulation

The goals of the computer simulation are to accurately predict the motion of a DTG and to perform these calculations as quickly as possible. For the purpose of this research, a user interface will be included to facilitate interaction with the simulation; however, the program will be structured so it can easily be modified to form a subroutine in a larger program. It is in this second form that the speed of the simulation is critical as it will be used in 'hardware-in-the-loop' simulations of the MMS and its components.

The structure of the program is determined by the form of input from a 'hardware-in-the-loop' simulation. In this situation, only the orientation and rates of the rotor, as measured by the pickoffs, and the torques demanded of the motor and torquer coils will be available to the simulation. This information will be in 'pickoff' coordinates and must be transformed to 'gimbal' coordinates. It is also assumed that any information used internally by the program will be lost when it is exited. Therefore all variables must be re-initialized each time the program is called. This translates to a slower simulation because of the additional calculations required.

A flow chart depicting the organization of the simulation is shown in Figure 10.1. Additional calculations are required if Hamilton's Canonical Equations are used and these additions are denoted by the dashed lines. The procedure is straight forward and involves obtaining the initial conditions and desired time step, converting the angles from pickoff to gimbal coordinates then initializing any remaining variables (such as the pivot properties and the inertial properties of the DTG). The equations of motion are integrated and the results are transformed back to pickoff coordinates.

Figure 10.1: Flow chart depicting the basic organization of the Dynamically Tuned Gyroscope simulation. t_c represents the current simulation time and t_f represents the desired end time for the simulation.

Since the input is in the form of angles and rates, calculation of the generalized momenta and the conversion of the momenta back to rates is required if Hamilton's Canonical Equations are used.

During each iteration, the numerical integration routine calculates new rates, accelerations and, when using Hamilton's Canonical Equations, new rates of change of the momenta. Using these new rates and accelerations, the forces acting on the pivots are calculated and the new stiffnesses of the pivots found. This is repeated as required.

To evaluate and conduct tests of the simulation, a user interface was developed. The interface allows the user to specify the inertial properties of the system, the crossed flexure pivot dimensions and the torsional stiffness of the remaining springs. Other properties that can be specified by the user are the sampling rate and the form of the equations of motion (that is, the user can choose to use Hamilton's Canonical Equations, Lagrange's Equations or the small angle equations derived in Chapter 8).

The resulting program provides a flexible, yet detailed simulation of the motion of Dynamically Tuned Gyroscopes that is easily modified for implementation into a 'hardware-in-the-loop' simulation of the MMS. The simulation was coded using Fortran 90 standard and the numerical integration routine came from the International Mathematical and Statistical Libraries (IMSL). The user interface requires a Microsoft Windows 95[83] environment to function.

10.1.1 Numerical Solution of Ordinary Differential Equations

In order to solve the equations of motion a numerical integration routine must be used. There are many different techniques to numerically integrate a set of ordinary differential equations, ranging from Taylor series methods to multi-step methods and, to choose the best method, the equations must first be examined.

One crucial consideration is to decide whether the equations are stiff or not. In general, an equation is defined as stiff if its solution has a small time constant in comparison to the time interval over which it is being solved[84]. When examining a set of ordinary differential equations, this situation arises if the natural frequencies of the system are widely separated, whereby some are very low frequencies and some very high ones.

Analyses in the literature, see Staley[18] and in Chapter 8, have calculated four natural frequencies for the DTG. Two of these frequencies are at nearly twice the spin frequency of the shaft while the remaining two are much lower (one is of the order of 10^{-3} Hz.) Hence, the equations of motion of the Dynamically Tuned Gyroscope must be considered stiff. This fact must be taken into consideration when choosing a numerical integration technique. For stiff equations, a numerical method with a large region of absolute stability must be used otherwise the step size has to be very small[85]. A small step size is typically considered only a temporary solution since this only delays the point at which the integration method becomes unstable[84].

Three numerical methods were considered for this problem; Runge-Kutta-Vernier method, Adams-Moulton method and Gear's backward differentia-

tion formulas (BDF) method. Runge-Kutta methods are popular for problems involving rotating bodies and are well tested; however, they are most efficient when used for non-stiff problems and require a very small step size when integrating stiff equations. The Adams-Moulton method, a multi-step method, has a large region of stability and is generally considered to be a very efficient method for solving stiff differential equations[84][85]. Gear's BDF method is another multistep method and is recommended for very stiff problems.

In order to choose the best method, all three were implemented in the simulation. A 5th-6th order Runge-Kutta-Vernier method, a 5th order Gear's BDF method and a 12th order Adams-Moulton method were evaluated. All three methods were obtained from the IMSL libraries. The Adams-Moulton method was found to be the most efficient, resulting in faster simulations with accuracies similar to both of the other methods. This result agrees with the recommendations made in the literature.

10.1.2 The Adams-Moulton Numerical Technique

The Adams-Moulton method is an example of a predictor-corrector, multi-step method. A fourth order Adams-Moulton method is given by[85]

$$y_{n+1} = y_n + \frac{h}{24} [9f(x_{n+1}, y_{n+1}) + 19f(x_n, y_n) - 5f(x_{n-1}, y_{n-1}) + f(x_{n-2}, y_{n-2})]$$

where h is the step size and f is the function that is being integrated (in other words, $Y'(x) = f(x, Y(x))$). This equation is typically referred to as the corrector. Note that y_{n+1} appears on both sides of the equation which

means that calculating y_{n+1} is a non-linear root-finding problem. Because of this, the Adams-Moulton numerical technique is known as an implicit method. In contrast, methods where y_{n+1} is found directly are known as explicit methods.

Standard root-finding techniques could be used to solve for y_{n+1} ; however this can be a time consuming process. For this reason, an iterative approach is usually used. A predictor is used to obtain an initial estimate of the value of y_{n+1} . The usual choice of the predictor is based on the Adams-Bashforth method which provides a good guess for the initial iteration. An example of a fourth order predictor is given by[85]

$$y_{n+1} = y_n + \frac{h}{24} [55f(x_n, y_n) - 59f(x_{n-1}, y_{n-1}) + 37f(x_{n-2}, y_{n-2}) - 9f(x_{n-3}, y_{n-3})]$$

The combination of an accurate initial estimate and small step size means the iteration converges rapidly and only one or two iterations are required.

It is possible to use the predictor alone to numerically integrate the equations of motion and it has been found that there is no significant gain in accuracy by using an Adams-Moulton method over an Adams-Bashford method of the same order. The computational cost appears to be about equal as well. The desirability of the Adams-Moulton method over the Adams-Bashford method stems from the fact that the Adams-Moulton method has a larger region of absolute stability[85]. This large region of stability is required when integrating stiff ordinary differential equations. Should a region of instability be encountered, oscillation of the solution may result, adding false frequencies to the calculated system response.

Figure 10.2: Example of aliasing due to the discrete sampling of an analog frequency. The solid curve is the actual frequency. In this case, the sampling rate is 5 Hz and the result is that the measured frequency is 1 Hz, as shown by the dashed line.

10.1.3 Aliasing

Whenever discrete sampling of an analog frequency is performed, a phenomenon known as aliasing can develop where high frequencies appear as low frequencies due to the sampling rate. A good example of this is shown in Figure 10.2.

In general, passing the data through a low pass filter before analysis would remove any high frequency components and reduce the possibility of aliasing. In this case, a low pass filter was not used since spurious high frequencies should not be present in the results and any high frequencies present are true components of the response of the MMS.

In order to avoid aliasing, a sampling rate of at least a factor of 2 faster than the highest frequency is recommended[86]. In some situations, such as long term tests, this is not always possible and it is important to keep aliasing in mind when analyzing the data.

10.2 Convergence Study

The speed and accuracy of a numerical integration process depends on the required precision and, in the Adams-Moulton method, this precision is dictated by specifying a tolerance. The tolerance is used to control the norm of the local error so that the global error is proportional to the tolerance[64]. Making the tolerance more stringent results in an increase of the accuracy of the solution; however, the length of time required to perform the integration increases as well.

In order to determine the sensitivity of the simulation to a change in the tolerance a convergence study was performed. The study involved simulating the motion of a CSG-2 DTG, see Table 8.1¹, while specifying various tolerances for the integration routine. The inertia of the shaft was chosen to be 700 gm² so that the simulation would approximate a two body solution but yet still allow the use of Hamilton's Canonical Equations. The reasons behind this choice are discussed later in this section. The shaft speed was set to 105.5 Hz, the tuned speed of the CSG-2 gyro.

Figures 10.3 and 10.4 show an extreme example of the differences that can result due to a change in the specified tolerance. Figure 10.3 shows the precession of the rotor which was initially offset by 7 degrees about the x axis of the case. The tolerance for the integration routine was set to 10^{-4} . It can be seen that the frequency of the rotor precession, the oscillatory motion, is changing slightly over time. This is especially apparent in the plot of θ' versus time. Figure 10.4 represents the solution of the rotor motion using

¹Note that the torsional stiffness specified in this table is the axial stiffness. The simulation requires a pivot stiffness, which is half the axial stiffness. Damping has been assumed to be zero.

Figure 10.3: Orientation of the CSG-2 DTG rotor with respect to time. The tolerance specified in the integration routine was 10^{-4} . The rotor had an initial offset about the x axis of 7 degrees. The system was operating at a tuned speed.

the same initial conditions except with the tolerance increased to 10^{-5} . The precession rate is seen to be much more constant in this case.

Tables 10.1 and 10.2 contain the results from the convergence study. These results can vary as they are dependent on the amount of time simulated as well as the sampling rate and are presented only to give an impression of the relationship between an increase in the specified tolerance and the increase in accuracy. The length of time taken to simulate the 10 seconds of rotor motion is also listed and demonstrates that the amount of time required

Figure 10.4: Orientation of the CSG-2 DTG rotor with respect to time. The tolerance specified in the integration routine was 10^{-5} . The rotor had an initial offset about the x axis of 7 degrees. The system was operating at a tuned speed.

<i>Tolerance</i>	<i>Change in Angle when Tolerance is Increased</i>		<i>Simulation Time (sec)</i>	<i>Actual Time (sec)</i>
	<i>X Axis (arcsec)</i>	<i>Y Axis (arcsec)</i>		
10^{-4}	-	-	10	57.1
10^{-5}	25270.41 (100.3%)	22157.95 (87.9%)	10	77.6
10^{-6}	1282.00 (5.09%)	1513.87 (6.01%)	10	89.9
10^{-7}	100.13 (0.40%)	117.87 (0.47%)	10	115.1
10^{-8}	22.02 (0.09%)	26.02 (0.10%)	10	125.5
10^{-9}	2.43 (0.010%)	2.88 (0.011%)	10	158.9
10^{-10}	0.57 (0.002%)	0.68 (0.003%)	10	189.6
10^{-11}	0.02 (0.00008%)	0.02 (0.0001%)	10	209.7

Table 10.1: Maximum difference between the calculated angles using a specific tolerance and the angles calculated using the previous tolerance. Lagrange's Equations were used for this study. The rotor had an initial offset about the x axis of 7 degrees in 'case' coordinates. Percentages were calculated by taking the ratio of the change in angle and the initial rotor offset of 7 degrees.

for the integration process increases as the required tolerance increases. Note that the sampling rate for the simulation was 1000 Hz.

The numerical iteration routine was not able to meet the specified tolerance of 10^{-11} when Hamilton's Canonical Equations were used. This is due to the fact that the generalized momentum of the shaft is large compared to the generalized momenta of the rotor and gimbal because of the high spin inertia of the shaft. This means that the generalized shaft momentum (P_{Ω}) becomes the critical term when trying to match the tolerance specification. When lower values of shaft inertia are used, this problem is not encountered, hence the reasoning behind the choice of the relatively low moment of inertia for the shaft used in these tests. Comparing the results in Tables 10.1 and 10.2 shows that the results from Hamilton's Canonical Equations converge faster than those obtained using Lagrange's Equations. However, the

<i>Tolerance</i>	<i>Change in Angle when Tolerance is Increased</i>		<i>Simulation Time (sec)</i>	<i>Actual Time (sec)</i>
	<i>X Axis (arcsec)</i>	<i>Y Axis (arcsec)</i>		
10^{-4}	-	-	10	115.8
10^{-5}	465.99 (1.85%)	547.88 (2.17%)	10	143.3
10^{-6}	26.29 (0.10%)	30.96 (0.12%)	10	155.0
10^{-7}	6.63 (0.026%)	7.82 (0.031%)	10	193.3
10^{-8}	0.62 (0.002%)	0.73 (0.003%)	10	236.0
10^{-9}	0.08 (0.0003%)	0.09 (0.0004%)	10	276.4
10^{-10}	0.005 (0.00002%)	0.006 (0.00002%)	10	294.8
10^{-11}	N/A	N/A	N/A	N/A

Table 10.2: Maximum difference between the calculated angles using a specific tolerance and the angles calculated using the previous tolerance. Hamilton's Canonical Equations were used for this study. The rotor had an initial offset about the x axis of 7 degrees in 'case' coordinates. Percentages were calculated by taking the ratio of the change in angle and the initial rotor offset of 7 degrees.

length of time required to solve the equations at a specific tolerance is much less (up to a minute and a half) using the equations developed through Lagrange's Equations.

Unless otherwise stated, the tolerance used for the integration of the equations of motion was 10^{-11} and 10^{-10} for Lagrange's and Hamilton Canonical Equations, respectively. Also, with the exception of the length of time required to integrate the equations of motion, Hamilton's Canonical Equations and Lagrange's Equations were found to give the same results, as expected. Therefore, either method could be used for numerical experimentation. In most cases, Lagrange's Equations were used for this research.

10.3 Validation of the Numerical Simulation

Several methods were employed in order to evaluate the validity of the numerical simulation. These included monitoring the energy of the system and comparing the results from the simulation to the results obtained by other methods. For this purpose, the commercial packages ADAMS[87] and Working Model 3D[88], as well as earlier code created by this author[23], were used to simulate the motion of a Dynamically Tuned Gyro.

For the purposes of this research, damping along the x and y axes of the gimbal has been assumed to be negligible and therefore the energy in the system should remain constant over time. Thus, calculating the energy of the system at each time step is an excellent method of ensuring that the simulation is working properly. However, in a numerical simulation, it is possible for the energy to deviate slightly over time due to the truncation and rounding that occurs. Using double precision variables (using variables that are accurate to 14 or 15 digits) and increasing the tolerance are two ways of minimizing round off errors.

ADAMS² and Working Model 3D³ are two popular commercial packages used to derive and solve the equations of motion of rigid bodies. ADAMS uses Gear's BDF method to do this and Working Model uses a Runge-Kutta routine. Models of the Dynamically Tuned Gyroscope were implemented in both packages, omitting the vibration damper, and a graphical representation of the model used in Working Model is shown in Figure 10.5. The model implemented in ADAMS was very similar. It should be noted that the con-

²Version 9.0.9 of ADAMS was used.

³Version 3.0 of Working Model 3D was used for these tests.

Figure 10.5: Model of the Dynamically Tuned Gyro implemented in the Working Model 3D[88] dynamic simulation package.

nections between the shaft, gimbal and rotor can not be seen in the figures due to the nature of the output from the programs.

Finally, the model was also implemented in a program known as RSIM, which was written previously by this author. This code uses equations of motion derived by assuming a constant shaft speed. This program has been validated by comparison with experimental results[23].

The model used for the comparison of the four programs is based on the CSG-2 DTG, see Table 8.1, with a shaft inertia of 1×10^{10} gcm². Two cases were used: the first involved an initial rotor offset of 1 degree about the y axis (in 'case' coordinates) and, the second, an initial rotor offset of 10 degrees about the x axis (in 'case' coordinates). The simulations were run for

1 second (simulation time) and the output compared. Note that the stiffness of the pivots was assumed constant and equal so that proper comparisons between the simulations could be made.

Figures 10.6 to 10.8 show the calculated response of the DTG rotor to the initial rotor offset of $\theta' = 1$ degree as calculated by the simulation described in earlier sections (for brevity, this simulation will be referred to as DTGSim), RSIM and Working Model, respectively. Comparison between the plots reveals that the solutions are clearly the same. Figure 10.9 shows the system energy as a function of time as calculated by DTGSim which was found to remain constant throughout the simulation.

Figures 10.10 to 10.12 show the calculated response of the DTG rotor to an initial rotor offset of $\phi' = 10$ degrees as calculated by DTGSim, RSIM and Working Model, respectively. Again, the plots can be compared and the solutions are in agreement. The energy of the system, see Figure 10.13, remained constant.

The simulation has now been verified and can be used to examine the behaviour of the Dynamically Tuned Gyroscope and the Momentum Management System. It can also be used in 'hardware-in-the-loop' simulations for control system and electronics testing. The following chapter contains several case studies used to develop an understanding of the behaviour of a DTG and the MMS.

Figure 10.6: Orientation of the CSG-2 rotor, in 'case' coordinates, with respect to time as calculated by DTGSim. The initial rotor offset was $\theta' = 1$ degree and the mass moment of inertia of the shaft is $1 \times 10^{10} \text{ gcm}^2$.

Figure 10.7: Orientation of the CSG-2 rotor, in 'case' coordinates, with respect to time as calculated by RSIM[23]. The initial rotor offset was $\theta' = 1$ degree and the mass moment of inertia of the shaft is $1 \times 10^{10} \text{ gcm}^2$.

Figure 10.8: Orientation of the CSG-2 rotor, in 'case' coordinates, with respect to time as calculated by Working Model 3D[88]. The initial rotor offset was $\theta' = 1$ degree and the mass moment of inertia of the shaft is 1×10^{10} gcm².

Figure 10.9: System energy in the CSG-2 Dynamically Tuned Gyro with an initial rotor offset of $\theta' = 1$ degree, as calculated by DTGSim.

Figure 10.10: Orientation of the CSG-2 rotor, in 'case' coordinates, with respect to time as calculated by DTGSim. The initial rotor offset was $\phi' = 10$ degrees and the mass moment of inertia of the shaft is 1×10^{10} gcm².

Figure 10.11: Orientation of the CSG-2 rotor, in 'case' coordinates, with respect to time as calculated by RSIM[23]. The initial rotor offset was $\phi' = 10$ degrees and the mass moment of inertia of the shaft is $1 \times 10^{10} \text{ gcm}^2$.

Figure 10.12: Orientation of the CSG-2 rotor, in 'case' coordinates, with respect to time as calculated by Working Model 3D[88]. The initial rotor offset was $\phi' = 10$ degrees and the mass moment of inertia of the shaft is $1 \times 10^{10} \text{ gcm}^2$.

Figure 10.13: System energy in the CSG-2 Dynamically Tuned Gyro with an initial rotor offset of $\phi' = 10$ degrees, as calculated by DTGSim.

Chapter 11

Simulation Case Studies

The computer simulation DTGSim has been validated in the previous chapter and it is now possible to perform a series of numerical tests in order to gain a better understanding of the behaviour of the Dynamically Tuned Gyroscope (DTG) and similar systems. For these studies, the Litton CSG-2 DTG and the Momentum Management System (MMS) are modeled. The CSG-2 is used to illustrate the general response of a DTG to a rotor offset since it has a higher tuned speed. With a tuned speed of 105.5 Hz the precession frequency is much higher than that for the MMS which operates in the range of 20 to 30 Hz. Studies of the effects of using a four body model rather than a two body model (as used in the literature) are performed using the design parameters for the MMS in order to determine its characteristics.

After the general response of the DTG to a rotor offset is discussed, a comparison is made between the small angle model, developed in Chapter 8, and the large angle model, developed in Chapter 9. The effect of including shaft oscillation and variable pivot stiffness on the behaviour of the rotor and the forces acting on the pivots is also studied. Finally, by varying both the

initial offset of the rotor and the average shaft speed, a relationship between the rotor offset and the tuned speed is developed.

It should be noted that a large number of tests could be performed each at many different gimbal angles. This means that tests could be performed *ad infinitum*. To limit the time involved in this study, only a few select configurations are used.

11.1 General Behaviour of the DTG

In this section, the response of the Dynamically Tuned Gyroscope to an initial rotor offset is discussed. As mentioned earlier, the Litton CSG-2 DTG is used as a model and the results are shown in Figure 11.1.

The rotor motion shown in this plot is in response to an initial rotor offset of $\theta' = 100 \text{ mrad}^1$. Figures 11.1(a) and 11.1(b) show the response of the rotor in case coordinates. In these plots, a low frequency precession term is observed. It is also apparent that the two 'pickoff' angles are shifted by 90 degrees with respect to each other. At this angular offset, the period of precession is approximately 4 seconds.

Recall that the gyro is ideally operating at the tuned speed. In theory, this means the rotor is isolated from the rest of the system and, therefore, the precession should be zero. Clearly it is not, which confirms the predictions of the theoretical analysis and small angle simulation performed earlier in Chapter 8.

Figure 11.1(c) contains a plot of θ' versus ϕ' . In essence, this plot shows the tracing of a normal to the rotor on the inside of a sphere. It can be

¹Recall that θ' and ϕ' are case angles as defined in Chapter 2

Figure 11.1: Response of the Litton CSG-2 DTG to an initial offset of $\theta' = 100$ mrad. The system was operating at its tuned speed of 105.5 Hz. The inertial properties of the gyro are found in Table 8.1. Figures (a) and (b) show the angular position of the rotor about the x and y case coordinates, respectively. Figure (c) shows the motion of a vector normal to the rotor. The width of the line is caused by the nutation of the rotor and the circular shape is due to the precession of the rotor. In this diagram, the precession is in a counter clockwise direction, indicating direct precession. Figure (d) shows the y axis component of the nutation. Note the different time scales.

seen that, after experiencing two complete cycles of precession, the motion is periodic and stable.

Figure 11.1(d) gives the motion of the rotor at a much smaller time scale. In this plot it is possible to see a component of the nutation of the rotor. A beat frequency confirms the existence of a second frequency located near the nutation frequency.

Therefore, precession, nutation and the second frequency near the nutation frequency are all easily identifiable. The fourth frequency, predicted by the small angle equations, is not as easy to identify and would be observed as a slight oscillation of the precession. The motion shown in Figure 11.1 is in case coordinates, which is a non-spinning reference frame. This reference frame is very similar to the 'pickoff' coordinates, see Chapter 2. It will be shown in later sections that the non-linear transformation from spinning to non-spinning coordinates results in the introduction of super and sub harmonics.

11.2 Comparison Between the Small Angle Analysis and the Large Angle Analysis

A comparison between the small angle and large angle analysis is performed in this section. Again, the parameters of the CSG-2 are used and in order to ensure that the various numerical experiments are compatible, the rotor will be offset about a single axis only.

The results in the previous section show that the response of the DTG is periodic. Therefore, the results from any analysis dealing with an initial rotor offset about one axis will be shifted by 90 degrees from the results of

Figure 11.2: Response of the Litton CSG-2 to an initial rotor offset of $\theta' = 1$ mrad using (a) a small angle assumption, and (b) a large angle assumption.

an analysis involving an offset about the other axis.

Figure 11.2(a) shows the response of the CSG-2 with a rotor offset, about the y axis, of 1 mrad (0.057 degrees) using small angle equations. Figure 11.2(b) shows the response given by the large angle equations. It should be noted that the pivot stiffness is held constant during these simulations.

From these plots it can be seen that the response is identical and, therefore, the simplified equations derived for small angle motion are valid. Power spectral density analyses of the response are found in Figures 11.3 and 11.4. Comparison of these two plots confirms that they are identical.

Figure 11.3: Power spectral density analysis of the rotor position data calculated using small angle equations. The top plot is a PSD analysis of the motion of the rotor about the x axis in case coordinates and the bottom plot is a PSD analysis of the motion of the rotor about the y axis in case coordinates.

Figure 11.4: Power spectral density analysis of the rotor position data calculated using large angle equations. The top plot is a PSD analysis of the motion of the rotor about the x axis in case coordinates and the bottom plot is a PSD analysis of the motion of the rotor about the y axis in case coordinates.

The power spectral density (PSD) analysis reveals there are three frequencies, 211 Hz, 182.6 Hz and 28.4 Hz, in the response. The frequency at 182.6 Hz is the nutation frequency, ω_n^* , and is close to the theoretical nutation frequency calculated in spinning coordinates. Recall that this was calculated in Chapter 8 using the following equation

$$\omega_n = \frac{I_{r_s} \omega_s}{I_{r_t} + \frac{I_{g_t}}{2}}$$

where ω_n is defined as the nutation frequency, I_{r_s} and I_{r_t} are the spin and transverse inertia of the rotor (assuming it is symmetric about the transverse axes), I_{g_t} is the transverse inertia of the gimbal and ω_s is the spin speed of the shaft. For the CSG-2, operating at 105.5 Hz, the value of ω_n is 180.5 Hz. While ω_n does not give the nutation frequency in non-spinning coordinates, it does seem to provide an excellent prediction². The frequency at 211.0 Hz is exactly twice the spin speed of the shaft and this response is due to the Hooke's joint. It is the interaction between the $2\omega_s$ and ω_n^* terms that creates the beat frequency. The frequency response at 28.4 Hz is not easily identifiable in the time-domain plots of the response of the system.

The precession frequency is not shown on the plot due to the fact that the length of the simulation was not long enough to capture it; nor was the resolution, which is governed by the step size of the simulation, fine enough to accurately portray it. For this test, data is generated for 10 seconds of simulation time, of which only the first second is shown in Figure 11.2. Using all the data it is possible to estimate the precession frequency using

²In order to maintain the nomenclature that has been used up to this point, ω_n^* will be used to indicate the nutation frequency in non-spinning coordinates and ω_n will be used to describe the nutation frequency in spinning coordinates.

the following equation

$$\phi' = A \sin(2\pi t f) \quad (11.1)$$

by solving for the frequency, f . In this equation, A represents the maximum angular offset of the rotor when nutation is neglected and ϕ' is the angular position of the rotor about its x axis in case coordinates at the current time, t . Using Equation 11.1, the precession frequency is estimated to be -1.35×10^{-3} Hz. The negative sign represents the fact that the rotor is undergoing retrograde precession. In other words, the rotor precession is in the opposite direction of the shaft spin.

Using either Equation 8.26 or 8.35 it is possible to calculate the natural frequencies of the system as predicted by the small angle equations. These are 211.0 Hz, 182.6 Hz, 28.4 Hz and -1.36×10^{-3} Hz. It should be noted that the frequencies obtained directly from Equations 8.26 and 8.35 are circular frequencies and must be multiplied by $1/2\pi$ in order to give the results listed above. These frequencies are the same as those obtained from the large and small angle numerical simulations³.

Increasing the initial rotor offset to $\theta' = 10$ mrad (≈ 0.57 degrees) yields the response shown in Figure 11.5, where Figure 11.5(a) is the calculated response using the small angle equations and Figure 11.5(b) gives the response as dictated by the large angle equations. Comparing the two ϕ' versus time plots reveals that the precession calculated by the large angle equations has reversed direction with respect to the results from the small angle equations. Comparing Figures 11.2(a) and 11.5(a) shows that the precession frequency is the same when using the small angle equations for the two different rotor

³Note that this result also confirms the validity of the large angle simulation.

Figure 11.5: Response of the Litton CSG-2 to an initial rotor offset of $\theta' = 10$ mrad as predicted by equations based on (a) a small angle assumption, and (b) a large angle assumption.

offsets; however, it should be noted that the amplitude of the precession and nutation has changed.

Figure 11.6 contains the same information as Figure 11.5 but over a shorter time frame. This was done in order to show the nutation of the rotor as calculated by the small and large angle equations.

Figures 11.7 and 11.8 contain PSD analyses of the data and it can be seen that the curves are very similar, with the magnitude of the plots in Figure 11.7 being slightly higher. Again, the natural frequencies shown in

Figure 11.6: Response of the Litton CSG-2 to an initial rotor offset of $\theta' = 10$ mrad as predicted by equations based on (a) a small angle assumption, and (b) a large angle assumption.

Figure 11.7: Power spectral density of the rotor position data calculated using the small angle equations, see Figure 11.5(a). The top plot is a PSD analysis of the motion of the rotor about the x axis in case coordinates and the bottom plot is a PSD analysis of the motion of the rotor about the y axis in case coordinates.

these plots are 211 Hz, 182.6 Hz and 28.4 Hz. There is also a peak at 422 Hz. This is not a fundamental frequency of the system but is introduced through the non-linear transformation of gimbal coordinates to case coordinates. In fact, this is a super harmonic of the $2\omega_s$ frequency. It should be noted that this super harmonic will be seen by the 'pickoffs' and must be dealt with, or at least taken into account, when designing a control system.

Again, the precession frequency is too low to be measured in this analysis;

Figure 11.8: Power spectral density of the rotor position data calculated using the large angle equations, see Figure 11.5(b). The top plot is a PSD analysis of the motion of the rotor about the x axis in case coordinates and the bottom plot is a PSD analysis of the motion of the rotor about the y axis in case coordinates.

however, an estimate can be found using Equation 11.1. Again, the precession frequency, as calculated by the small angle equations, is -1.35×10^{-3} Hz. The precession frequency calculated by the large angle equations is 1.19×10^{-3} Hz, where the difference in the sign of the two numbers denotes the change in direction of the precession. Recall that a negative precession indicates a retrograde precession and a positive value represents a direct (or prograde) precession.

There is now a large difference in the precession frequencies calculated by the small and large angle equations. The difference between Figures 11.5(a) and 11.5(b) demonstrates that the small angle assumption is not valid at this angle even though it is much less than the generally accepted limit for small angle assumptions of 5 to 10 degrees.

The response of the DTG to an initial rotor offset of 100 mrad (≈ 5.7 degrees) is shown in Figure 11.9. Figures 11.9(a) and 11.9(b) show the response calculated by the small angle and large angle equations, respectively. The rotor motion in Figure 11.9(a) is identical to that found in Figures 11.2(a) and 11.5(a) with the exception that the magnitude of the nutation and precession has changed. This clearly demonstrates the fact that, according to the small angle equations, the precession frequency is independent of the rotor offset. Figure 11.9(b) shows otherwise. It is seen that when the small angles aren't assumed, the precession depends on the rotor position. It is clearly important to use a model which does not assume small angles when attempting to predict the precession of the MMS, which will operate at rotor angles as large as 7 degrees.

Figures 11.10 and 11.11 contain plots of the PSD versus frequency for the

Figure 11.9: Response of the Litton CSG-2 to an initial rotor offset of $\theta' = 100$ mrad as predicted by equations based on (a) a small angle assumption, and (b) a large angle assumption.

data in Figure 11.9. The fundamental frequencies are the same as before; however, this time the super, sub and supersubharmonics⁴ are more evident. Figure 11.12 shows the frequency response in the range of 0 to 15 Hz which allows the identification of the precession frequency at 0.25 Hz which corresponds to a period of four seconds. This agrees with Figure 11.9(b) which shows a quarter cycle in the one second plot. The peak at 10.75 Hz is a super or sub harmonic of one of the fundamental frequencies. To determine which it is can be an extremely difficult task and is beyond the scope of this research. The discovery of the existence of these super and sub harmonics is enough.

From this study it is seen that the frequency of precession of the rotor is dependent on the orientation of the rotor. As the angular offset increases, so does the period of precession. The remaining three fundamental frequencies appear to be independent of offset and are accurately predicted using the equations derived in Chapter 8. It has also been shown that any system designed to control the response of a DTG rotor must not only take the fundamental frequencies into account but also a multitude of super and sub harmonics.

11.3 Effect of Viscous Damping on a DTG

Before moving on to study the behaviour of the MMS prototype, a brief look at the effect of viscous damping can be made. It has already been assumed in Chapter 9 that any damping in the system acts about the two gimbal axes and this will have some effect on the response of a DTG.

⁴Supersubharmonics are defined as $\frac{n}{N}f$, where n and N are positive integers and $n \neq N$.

Figure 11.10: Power spectral density of the rotor position data calculated using small angle equations, see Figure 11.9(a). The top plot is a PSD analysis of the motion of the rotor about the x axis in case coordinates and the bottom plot is a PSD analysis of the motion of the rotor about the y axis in case coordinates.

Figure 11.11: Power spectral density of the rotor position data calculated using large angle equations, see Figure 11.9(b). The top plot is a PSD analysis of the motion of the rotor about the x axis in case coordinates and the bottom plot is a PSD analysis of the motion of the rotor about the y axis in case coordinates.

Figure 11.12: Power spectral density of the rotor position data calculated using large angle equations, see Figure 11.9(b). The top plot is a PSD analysis of the motion of the rotor about the x axis in case coordinates and the bottom plot is a PSD analysis of the motion of the rotor about the y axis in case coordinates.

Figure 11.13: Response of the CSG-2 gyro when axial damping is included on the two gimbal axes. In this case, the damping coefficient along an axis ($2D_x = 2D_y$) was 2×10^{-7} Nms/rad. The length of the simulation is 10 seconds and the rotor was initially offset about the y axis by 0.1 radians in case coordinates.

As before, the CSG-2 is used for this study due to the fact that it has a high operating speed and, thus, a fairly high precession rate when compared to that of the MMS. Using a value of 2×10^{-4} Nmms for the axial damping coefficient gives the results shown in Figure 11.13. It is seen that axial damping causes the rotor to slowly spiral inward toward the null position. A change in the magnitude of the damping coefficient causes a change in the time to reach the null position.

Figure 11.14: Response of the CSG-2 gyro when axial damping is included on the two gimbal axes. The damping coefficient along an axis ($2D_x = 2D_y$) was set to 20×10^{-7} Nms/rad. The length of the simulation is 10 seconds and the rotor is initially offset about the y axis by 0.1 radians in case coordinates.

If the damping coefficient is changed to 20×10^{-4} Nmms the rate increases significantly, see Figure 11.14. In this figure, it is seen that the rotor does not complete even a single period of precession before moving to the null position.

It is clear that the behaviour of a DTG is affected by the magnitude of the damping acting along the gimbal axes. In general, the damping coefficients are quite low and this, in combination with control system update intervals that are measured in milliseconds, means that damping only adds a small

error in any position calculations.

11.4 Two and Three Body Models with Constant and Variable Stiffness

Up to this point, only the CSG-2 has been modeled to demonstrate the behaviour of the DTG in a reasonable time scale. The errors introduced by assuming that the MMS operates at small angles has been demonstrated in the previous section, justifying a large angle approach in analyzing the system. It is now useful to begin a study of the behaviour of the MMS.

Figure 11.15 depicts the response of the MMS to an initial rotor offset of 7 degrees ($\phi' = 7$ degrees). The moment of inertia of the shaft is increased by a factor of 10^{13} in order to approximate a two body model, or to keep ω , approximately constant, and the remaining inertial properties of the MMS can be found in Table 11.1. The system is operating at its tuned speed of 25 Hz.

For this case the rotor is experiencing direct precession with a period of precession of approximately 11.7 seconds. This high rate of precession would require large torquer coils to keep the rotor in the desired position when the MMS is operating in closed loop configuration.

Figure 11.16 contains a plot of the shaft rate versus time. It can be seen that increasing the shaft inertia has the desired effect of keeping the shaft rate constant, which is the assumption made in the two body analysis.

The pivot design depends on the loads experienced by the pivots. For comparison, the loads for this model are plotted in Figure 11.17, where F_x and F_z are the loads on the pivots connecting the rotor and gimbal and

Figure 11.15: Plot of rotor position, in case coordinates, versus time for the Momentum Management System. The rotor has an initial offset of 7 degrees about the x axis in case coordinates. The inertia of the shaft is increased to approximate a two body analysis. The torsional stiffness of the crossed flexure pivots is assumed constant.

Figure 11.16: Plot of shaft rate versus time for the Momentum Management System. The rotor has an initial offset of 7 degrees about the x axis in case coordinates. The inertia of the shaft is increased to approximate a two body analysis. The torsional stiffness of the crossed flexure pivots is assumed constant.

<i>Component</i>	<i>Parameter</i>	<i>Value</i>
Rotor	Spin Inertia (I_{r_s})	63240 g cm ²
	Transverse Inertia (I_{r_t})	34320 g cm ²
	Mass (M_r)	2900 g
Gimbal	Spin Inertia (I_{g_s})	180.5 g cm ²
	Transverse Inertia (I_{g_t})	127.6 g cm ²
	Mass (M_g)	77.0 g
Shaft and Motor	Spin Inertia (I_{s_s})	51.2 g cm ²
Motor	Stiffness (K_m)	3.361×10^6 dyn cm/rad
Crossed Flexure	Gimbal X Axis (k_x)	46.11×10^4 dyn cm/rad
Pivots	Gimbal Y Axis (k_y)	46.11×10^4 dyn cm/rad

Table 11.1: Inertial and mass properties of the MMS engineering prototype as of April 1998[72]. The stiffness listed is for a single pivot located along the referenced axis. The motor stiffness is estimated by assuming the peak torque occurs when the shaft position is out of phase with the stator magnetic field by 45 degrees and then assuming the torque curve is linear.

G_y and G_z are the loads acting on the pivots connecting the gimbal and the shaft⁵. In this model, the horizontal loads on the pivots are nearly 7.5 Mdynes (75 N). Loads of this magnitude would result in very high stresses in the pivots and would force the pivots to be redesigned to be much larger and stiffer.

Figure 11.18 shows the response of the rotor if the actual moments of inertia of the MMS shaft and motor are used. The precession frequency has decreased dramatically to approximately 1.108×10^{-4} Hz, which translates to a period of 9023.7 seconds. Rather than the shaft rate remaining fixed, it oscillates at a frequency twice the average shaft speed, see Figure 11.19. The range of the oscillation is approximately 1.5% of the mean shaft spin rate.

⁵For a more thorough definition of the loads and the directions in which they act see Chapter 9.

Figure 11.17: Plot of the forces acting on the crossed flexure pivots versus time for the Momentum Management System. The rotor has an initial offset of 7 degrees about the x axis in case coordinates. The inertia of the shaft is increased to approximate a two body analysis. The torsional stiffness of the crossed flexure pivots is assumed constant.

Figure 11.18: Plot of rotor position, in case coordinates, versus time for the Momentum Management System. The rotor has an initial offset of 7 degrees about the x axis in case coordinates. The torsional stiffness of the crossed flexure pivots is assumed constant.

Figure 11.19: Plot of the shaft speed versus time for the Momentum Management System. The rotor has an initial offset of 7 degrees about the x axis in case coordinates. The torsional stiffness of the crossed flexure pivots is assumed constant.

At this point, the procedure used to simulate the system with a variable shaft speed should be discussed. To solve the equations of motion, the initial values of the generalized coordinates and their derivatives are required. This means that, for the MMS, or a DTG, the gimbal angles and rates must be specified along with the initial position and rate of the shaft; however, these are the instantaneous rates and positions rather than average values. In order to properly tune the system, the average shaft rate is required; this is difficult to predict. A good estimation of the average speed is obtained by running each simulation twice. The ratio between the average speed and the specified instantaneous speed is obtained during the first test run. The initial conditions can then be adjusted to obtain the desired average shaft rate.

In order to simulate a motor, an ideal motor speed was input into the simulation. Ideally, this motor speed corresponds to the average shaft speed. Small differences between this ideal motor speed and the actual average shaft speed can cause small errors in the simulation. This is demonstrated in a plot of the system energy as a function of the simulation time, see Appendix F. A portion of the oscillation of the energy is a result of the difference in the estimated ideal speed and the actual ideal speed.

Along with a decrease in the precession frequency, the results in Figure 11.20 show a corresponding decrease in the forces acting on the pivot. The maximum horizontal force acting on the pivots has decreased from 7.5 Mdynes to approximately 40 kdynes. According to the two body analysis, the pivots would be too large and too stiff to be practical. When the more accurate three body model is used, the forces decrease to manageable values.

Figure 11.20: Plot of the forces acting on the crossed flexure pivots versus time for the Momentum Management System. The rotor has an initial offset of 7 degrees about the x axis in case coordinates. The torsional stiffness of the crossed flexure pivots is assumed constant.

It should be noted that these forces are not the maximum experienced by the pivots; they would occur when the MMS is operating at its maximum speed of 1800 rpm (188.5 rad/s).

Finally, note that the loads are oscillatory and cyclic in nature. The loads on the pivot connecting the gimbal and the rotor are shifted by 90 degrees from those acting on the pivot connecting the gimbal and the shaft. This phase shift agrees with intuition as the sets of pivots are located at right angles to each other.

In the previous two cases, the crossed flexure pivots were replaced with torsional springs having constant stiffness. A final study involves replacing the torsional springs with a stiffness that is a function of the applied loads. The translational motion of the pivot, shown to be negligible in Chapters 3 and 4, is ignored and only the change in torsional stiffness is included. Figure 11.21 shows the motion of the rotor in case coordinates. It can be seen that the magnitude of the nutation is the same as the case with constant stiffness. The precession frequency is also approximately the same. Very careful analysis shows that the precession rate is 0.026% faster when the variance in the stiffness is included. This translates to a decrease in the precession period of approximately 2 seconds over the original 9024 second period. Therefore, the effect of the varying pivot stiffness is very small in low gravity situations such as those encountered in space. Figure 11.22 shows the shaft rate as a function of time. The response is the same as for the constant stiffness case.

Plots of the loads on the pivots as a function of time are found in Figure 11.23. It can be seen that the forces are approximately the same as for the case with the constant pivot stiffness. A detailed comparison between the two is made difficult due to the small difference in the precession rates. However, taking the maximum forces from the two cases shows a difference of $7.5 \times 10^{-8}\%$ which can be ignored. It should be noted that the horizontal loads acting on the pivots connecting the rotor and the gimbal are higher than those on the pivots connecting the gimbal and shaft. This agrees with intuition since both shaft and gimbal are acting on the pivots connecting the rotor and gimbal. Only the shaft acts on the pivots connecting the gimbal and shaft, resulting in lower loads.

Figure 11.21: Plot of rotor position, in case coordinates, versus time for the Momentum Management System. The rotor has an initial offset of 7 degrees about the x axis in case coordinates.

Figure 11.22: Plot of shaft rate versus time for the Momentum Management System. The rotor has an initial offset of 7 degrees about the x axis in case coordinates.

Figure 11.23: Plot of the forces acting on the crossed flexure pivots versus time. The rotor has an initial offset of 7 degrees.

Figure 11.24 contains a plot of the axial stiffness with respect to time. Recall that K_x is the axial stiffness between the gimbal and the shaft and that K_y represents the axial stiffness between the gimbal and the rotor. The larger loads on the pivots along the gimbal y axis translates to a large variation of the axial stiffness about the y axis. The difference in the two curves is due to the difference in the magnitudes of the vertical loads on the pivots. The characteristics of the oscillation are also governed by the behaviour of the loads. It should be noted that the mean of the axial stiffness of both pivots is 92.22 Nmm/rad (9.222×10^5 dyne cm/rad), which is twice the design stiffness of one pivot.

The oscillation of the axial stiffness connecting the gimbal and the shaft, the top plot in Figure 11.24, is approximately $6.4 \times 10^{-4}\%$ of the design value and the range of the stiffness for the pivots connecting the rotor and the gimbal is approximately $6.77 \times 10^{-4}\%$ of the design value.

The final situation that needs to be examined is when gravity is applied to the system. Normally, gravity is ignored when analyzing a DTG since it acts through the centre of mass of the rotor which is also the centre of rotation. Assuming the pivots have constant stiffness gravity would have no effect on a symmetric Dynamically Tuned Gyro.

It has been shown that the axial stiffness does vary due to the internal loads even though these loads are quite small. It is clear that including loads due to gravity, which are a magnitude larger than the internal loads, will have a significant effect on the variance of the pivot stiffness and on the behaviour of the pivot in general.

In order to determine the magnitude of the influence of gravity on the

Figure 11.24: Plot of the variance of the axial stiffness versus time for the Momentum Management System. The rotor has an initial offset of 7 degrees about the x axis in case coordinates.

Figure 11.25: Plot of rotor position in case coordinates versus time for the Momentum Management System positioned at an angle of 45 degrees to a 1g environment. The rotor has an initial offset of 7 degrees about the x axis in case coordinates.

system, a simulation can be performed in which the MMS is placed at 45 degrees to a 1g field. The response of the rotor can then be simulated and this is shown in Figure 11.25. Note that the scale along the y axis for both plots are much different than in Figures 11.18 and 11.21. Adding the loads due to gravity has caused the precession frequency to increase from 1.108×10^{-4} Hz to 4.18×10^{-3} Hz, or from having a period of over 9000 seconds to one just less than 240 seconds. It should also be noted that the direction of precession has changed as well.

Figure 11.26: Plot of the shaft spin speed versus time for the Momentum Management System. The rotor has an initial offset about the x axis of 7 degrees in case coordinates. The MMS is oriented at 45 degrees in a $1g$ gravitational field.

The shaft rate, shown in Figure 11.26, is the same as that obtained from the case with no gravity. In fact, the variance in the shaft rate is based on the ratios of the inertia of the rotor and shaft and the angle between them. Note that the motor stiffness is also a factor since it acts to increase the effective inertia of the shaft.

Figure 11.27 shows the forces acting on the pivot due to the internal loads. It is apparent that the vertical loads on the pivots has decreased slightly. This change is due to the system now operating at an average shaft

speed that is less than the tuned speed. The retrograde precession lowers the forces acting on the pivots. This can be verified using Equation 9.43

$$F_z = \frac{1}{2R} \left[I_{rx} \dot{\omega}_{r_x} - I_{r_{xy}} \dot{\omega}_{r_y} - I_{r_{xz}} \dot{\omega}_{r_z} - H_{r_y} \omega_{r_x} + H_{r_x} \omega_{r_y} \right. \\ \left. - (C_\theta C_\Omega + S_\theta S_\phi S_\Omega) T_x - (C_\theta S_\Omega - S_\theta S_\phi C_\Omega) T_y \right]$$

The critical term is $H_{r_y} \omega_{r_x}$ because of the large value of ω_{r_x} . As the rate of precession increases, the angular momentum of the rotor about the y axis (H_{r_y}) grows larger. When the precession is direct, H_{r_y} is negative which means higher pivot loads. But, if the rotor is experiencing retrograde precession, H_{r_y} is positive which means that the forces decrease in magnitude.

The loads due to gravity are shown in Figure 11.28. The reason why they have not been combined with the internal pivot loads is that the forces affect each pivot differently. It is possible to use the Equations derived in Section 9.3 to find the loads on each pivot. The top two diagrams represent the horizontal and vertical loads due to gravity on the rotor/gimbal pivots, respectively. Again, reference to Section 9.3 should be made as to the directions of the loads.

As the rotor precesses, its orientation in the gravitational field changes and so the loads acting on this axis change as well. This effect is seen in Figure 11.28(b). The vertical loads on the gimbal x axis are constant due to the fact that they are in the coordinate frame of the shaft which is fixed in the gravitational field (except about the spin axis). Figures 11.28(a) and 11.28(c) show the horizontal loads, which oscillate about zero at the spin frequency of the shaft.

The effect of all these forces acting on the pivots is a significant change in the mean axial stiffness, see Figure 11.29. The top plot shows the axial

Figure 11.27: Plot of the internal pivot loads versus time for the Momentum Management System when it is oriented at 45 degrees in a 1g gravitational field. The rotor has an initial offset about the x axis of 7 degrees in case coordinates.

Figure 11.28: Plot of the gravitational loads on the crossed flexure pivots versus time for the Momentum Management System when it is oriented at 45 degrees in a 1g gravitational field. The rotor has an initial offset about the x axis of 7 degrees in case coordinates. Note the different scales along the y axes.

Figure 11.29: Plot of axial stiffness versus time. The top diagram is the axial stiffness of the gimbal x axis and the bottom plot is the stiffness of the gimbal y axis. The Momentum Management System has been oriented at 45 degrees to a 1g gravitational field. The rotor has an initial offset about the x axis of 7 degrees in case coordinates.

stiffness along the gimbal x axis. The mean stiffness is now 120.67 Nmm/rad. The axial stiffness along the gimbal y axis is shown in the bottom plot and, again, the mean stiffness is 120.67 Nmm/rad.

Therefore, one effect of including gravity in the system is a change in the average axial stiffness. This means that the tuned speed has changed and

the new one can be calculated using

$$\omega_s = \sqrt{\frac{K_x + K_y}{I_{gx} + I_{gy} - I_{gz}}} \times \frac{60}{2\pi}$$

1715.3 rpm

This is a 14.4% change in the tuned speed. The further the system is from its tuned speed, the faster the precession and this explains the significant increase in precession frequency that results when gravity is included.

The fact that operating in a gravitational field increases the frequency of precession is an important finding since the MMS will be tested on Earth. Had an analysis from the literature been used, this behaviour would have been missed.

11.4.1 Effect of the Vibration Damper on the MMS

The oscillation of the shaft reduces the forces on the crossed flexure pivots. This fact makes it inadvisable to use a damper to limit this oscillation; however, since the damper was included in the derivation of the equations of motion, a brief discussion of its effect on the system is warranted.

In order to limit the vibration of a body, a small mass can be attached by a weak spring. The result is that the bulk of the vibration is experienced by the small mass and not the main body. In the current model, this was the intended role of the damper.

Closer examination of the system reveals that the shaft oscillation is a result of the kinematic constraints rather than due to dynamic loading. This implies that the rotor and/or shaft must oscillate about their respective spin axes. The result is that the damper will not work in the intended manner.

In fact, in order to dampen the oscillation of the shaft, the damper should have a high inertia and the spring should be stiff, exactly the opposite of the traditional requirements. By doing this, the effective inertia of the shaft is increased since the damper is connected to it, and the ratio of the rotor to shaft inertia is reduced. This results in a decrease in the oscillation of the shaft and an increase in the oscillation of the rotor. This would also result in higher forces acting on the pivots which is an undesirable scenario. However, if this was the goal, then the same effect could be achieved more economically by increasing the inertia of the shaft and motor, or by increasing the stiffness of the motor (K_m).

11.5 Relationship between Rotor Offset and True Tuned Speed

It has been shown in Chapter 2 that the equation used to calculate the tuned speed of a Dynamically Tuned Gyroscope is only an approximation. In order to tune the system properly, the moments acting on the rotor must be zero. Analytically, it is difficult to see how such a solution would be obtained. Comparing Figures 11.2(b) and 11.5(b) shows that the direction of the precession has changed as a result of a change in the initial angular offset of the rotor. In order for this to occur, the precession frequency must have gone through zero, indicating a point where the system is truly tuned.

To examine this phenomenon, a series of numerical experiments are conducted which involves changing the average shaft spin rate and the angular offset of the rotor. Again, these simulations are conducted with an offset about a single axis; however, the symmetry of the response of the rotor in-

dicates that the solution would be valid for any initial rotor offset.

The result of this sequence of numerical experiments is the generation of 6 sets of 13 plots of rotor position versus time. An example showing one set of plots that were obtained can be found in Appendix G. Compiling the results, a 3-dimensional plot of precession frequency versus average shaft speed and rotor offset is obtained, see Figure 11.30. In this plot, a negative sign indicates retrograde precession and a positive value indicates direct precession of the rotor.

The surface is derived using a least squares fit to find a polynomial approximation of the data in the form

$$f = c_1 + c_2r + c_3r^2 + c_4a + c_5a^2$$

where r represents the average shaft speed in rpm, a is the rotor offset in degrees and c_i are constants determined by the data. It should be noted that by completing the square this equation describes a hyperboloid of one sheet[90], if the equation is written in terms of the square root of the frequency. The result after finding the values of c_1 to c_5 is the equation

$$\frac{f}{1.470 \times 10^{-2}} + \frac{(r - 2.974 \times 10^3)^2}{2.163 \times 10^6} - \frac{(a - 4.050 \times 10^{-1})^2}{4.251 \times 10^3} = 1 \quad (11.2)$$

which describes the relationship between the precession frequency, rotor offset and average shaft speed, in Hertz, degrees and rpm, respectively. The frequency of precession is estimated using Equation 11.1 from the plots of rotor position versus time.

A contour plot is shown in Figure 11.31 and contains lines of constant precession frequency. One of the curves indicates the values of rotor offset and average shaft speed which result in the rotor not precessing at all. This

Figure 11.30: Plot of precession frequency versus average shaft speed and rotor offset for the MMS prototype. Negative and positive values indicate retrograde and direct precession, respectively.

curve represents the true tuning relationship for the MMS, see Figure 11.32. It can be calculated that the MMS is tuned at 1500 rpm when the rotor is offset by 4.45 degrees. It is also seen that the tuned speed predicted by the tuning equation

$$\omega_s = \sqrt{\frac{2k_x + 2k_y}{I_{gx} + I_{gy} - I_{gz}}}$$

has an error of 2.8 rpm (0.293 rad/s) when the rotor offset is zero. It should be noted that, in general, no precession is observed when a symmetric rotor is at a null position, independent of the shaft speed. However, there is usually some error in the system, which results in an electronic null that does not agree with the physical null position and so the rotor usually has some offset.

Figure 11.32 also verifies that the precession frequency is a function of the rotor offset and not independent as was calculated by the small angle analysis. To determine the accuracy of this plot, a simulation was performed where the rotor was offset about the y axis by 8 degrees and the shaft speed was set to 1492.89 rpm, the true tuned speed as predicted by Figure 11.32; note that Figure 11.32 was created using offsets about the x axis. The result is shown in Figure 11.33 where the plot shows 100 seconds of simulation time. Using Equation 11.1, the frequency of precession is estimated to be 3.42×10^{-6} Hz which means the precession has a period of just over 3 days 9 hours and 14 minutes.

If the MMS is operating at a shaft speed of 1500 rpm, it can be shown that the period of precession of the rotor would be just under 2 hours. Comparing this with the result obtained earlier demonstrates a significant increase in the period of the precession and that the actual tuning speed has been closely approached by using Equation 11.2.

Figure 11.31: Lines of constant precession frequency as functions of average shaft speed and rotor offset for the MMS prototype. Negative and positive values indicate retrograde and direct precession, respectively.

Figure 11.32: Plot of the true tuned shaft speed versus rotor offset.

Figure 11.33: Plot of rotor position, in case coordinates, versus time for the Momentum Management System. The rotor has an initial offset of 8 degrees about the y axes in case coordinates.

Figure 11.34: Plot of the peak to peak amplitude of the rotor nutation versus angular offset for the MMS prototype. The curve is a second order polynomial that has been fit to the simulation (shown here as the diamonds) using a least squares method.

11.6 Relationship between Rotor Offset and Nutation.

The results of the simulations used in the last section also allow for a brief discussion on the relationship between the magnitude of the nutation and the angular offset of the rotor. Since this is only a quick look at the relationship, the average shaft speed will be fixed at 1500 rpm allowing the initial offset of the rotor to be varied.

The results are found in Figure 11.34 where the diamonds represent points obtained from the simulation data. It is seen that the magnitude of the nutation increases with rotor angle. A second order polynomial can be fit to the data giving the relationship between rotor offset and the magnitude of the rotor nutation as

$$Magnitude_N = 7.76 \times 10^{-2}a^2 + 24.4a$$

where a is the angular offset of the rotor in degrees and $Magnitude_N$ is the magnitude of the nutation in arcseconds. Note that ‘magnitude of nutation’ means the diameter of the nutation coning action of the rotor.

The coefficient of the second order term is very small in comparison to the first order term, which means that the relationship is very nearly linear but that the curve is slightly concave upwards. It has been found that the non-linear effect of the angular offset of the rotor is only very slight at any reasonable angle (i.e. angles less than 10 degrees).

11.7 Behaviour of an Unsymmetric Rotor

This case study involves determining the effects of a rotor imbalance on the open loop behaviour of the system. For this numerical experiment, the MMS is operating at an average shaft rate of 25 Hz (1500 rpm) and the rotor is initially in the null position. It is in this manner that the effect of a rotor imbalance is most easily viewed.

The design specifications state that the MMS rotor is balanced such that the largest imbalance is less than 0.1 gcm²[89]. Therefore, based on this specification, a rotor imbalance of $I_{r_{\nu_1}} = 0.1 \text{ gcm}^2$ will be introduced into

the system. The resulting response is shown in Figure 11.35. It should be noted that while the motion about the y case axis has a mean of zero, the motion about the x axis is shifted upwards slightly. A power spectral density (PSD) analysis, see Figure 11.36, reveals the response of the rotor is governed by at least three frequencies at 4 Hz, 25 Hz and 46 Hz. The first of these is observed in Figure 11.35 as the 'carrier' frequency. The middle frequency is at the shaft spin rate and the high frequency is slightly less than twice the spin rate. Close examination of the PSD plot shows there may also be one very low frequency.

The fact that there is a rotor imbalance also results in loads on the crossed flexure pivots, as shown in Figure 11.37. It can be seen that the effect of the rotor imbalance is fairly small with 3 dynes (3×10^{-5} N) being the largest load on the pivots.

11.8 Response of an Open Loop MMS to a Rotation of its Case

The response of the Momentum Management System to various initial conditions has been discussed in the previous sections. This final section involves the situation where the case of the MMS is rotating. This simulates the reaction of the MMS to a rotation of the spacecraft. As always, the system is in an open loop configuration; that is, there is no feedback or control loop.

In this simulation, the MMS rotor begins in a null position and the case is given a constant rate of 0.01 rad/s about its positive y axis. The length of the simulation is 10 seconds, which gives a final case rotation of 0.1 radians (5.73 degrees). The shaft speed of the MMS is 25 Hz.

Figure 11.35: Response of the MMS rotor to an unbalance of $I_{r_{vz}} = 0.1 \text{ gcm}^2$. The initial rotor position is at null and the system is operating at an average shaft speed of 25 Hz.

Figure 11.36: Power spectral density analysis of the response of the rotor to a rotor unbalance of $I_{r_{y_1}} = 0.1 \text{ gcm}^2$. The rotor is initially at null and the system is operating at an average shaft speed of 25 Hz. The sampling rate is 1000 Hz.

Figure 11.37: Plot of the forces acting on the crossed flexure pivots versus time. The loads are due to a rotor imbalance of $I_{r_{yz}} = 0.1 \text{ gcm}^2$. The MMS is operating at 25 Hz and there is no initial offset of the rotor.

Figure 11.38: Plot of the rotor motion with respect to the case of the MMS. The case is rotating at a rate of 0.01 rad/s about its y axis. The MMS is operating in a open loop configuration and the average shaft speed is 25 Hz.

The results of this study are found in Figure 11.38. The plot of θ' shows the relative motion of the rotor with respect to the case. Since the case has rotated by 0.1 radians in the 10 seconds, this plots shows that the rotor has remained relatively motionless in an inertial frame. By measuring the motion of the rotor with respect to the case, a measurement of the case, or spacecraft motion, can be made.

Subtracting the case motion from the data shown in Figure 11.38 allows the accuracy of the measurement to be checked, see Figure 11.39. The top plot gives the rotor motion with respect to an inertial frame and the bottom plot gives a linear approximation of the motion, neglecting the nutation. It can now be shown that the precession of the rotor about the y axis is slightly greater than 12 arcseconds and this would be the error in the measurement of the case position. This corresponds to an error of 0.06% on the reading if the precession is not taken into account. While this error appears to be insignificant, over several minutes this error may accumulate to become a significant factor.

In earlier sections it has been observed that most of the rotor precession is seen about the axis orthogonal to that about which the rotor is offset. In this case, this means the precession should be easily observed about the x axis, see the top plot in Figure 11.38. It is observed that the rotor experiences direct precession at small angles but this changes to retrograde precession at larger angles. This agrees with the results found in Section 11.5. It was shown in Figure 11.32 that the tuned frequency of the MMS was 1500 rpm when the offset of the rotor was 4.45 degrees. At this point there is a reversal of the direction of precession, see Figure 11.30, which is what is observed in

Figure 11.39: Plot of the motion of the rotor about the y axis in an inertial reference frame. The case of the rotor has a steady rotation of 0.1 rad/s.

Figure 11.38.

The effect of precession on the open loop MMS has been observed to be a 12 arcsecond error over the 10 seconds of simulation. When the MMS is operating in a closed loop configuration, the interval between control system updates may be as little as 1 millisecond. Using the linear approximation of the error, the total error after 10 seconds would also be 12 arcseconds.

11.9 Summary

Using the computer simulation DTGSim, which is based on the dynamic equations derived in Chapter 9, the dynamics of the Dynamically Tuned Gyroscope and the Momentum Management System were examined. To demonstrate the generic properties of the DTG, the Litton CSG-2 DTG was used since it operates at high shaft speeds. A comparison between the small angle analysis found in the literature and the large angle analysis conducted in Chapter 9 showed that the small angle analysis inaccurately predicted the precession rate of the rotor except at very small rotor angles because it failed to include the relationship between rotor offset and precession frequency. The remaining three frequencies, one of which is rotor nutation, were accurately predicted.

The effect of viscous damping acting on the x and y gimbal axes was to cause the rotor to spiral to a null position. It took less time to reach the null position as the damping coefficient increased.

Comparisons between the two body analysis used in the literature and the three body analysis used in this work were made using the properties of the MMS. It was seen that the assumption of constant shaft speed was not

valid. When the rotor was offset at 7 degrees the shaft oscillated with a range that was 1.5% of the average shaft speed. Including the variable stiffness of the pivots made only a slight impact on the system dynamics when gravity was neglected. When gravity was included, the tuned speed changed by over 200 rpm and the period of precession for a rotor which was offset by 7 degrees decreased from 9000 to 240 seconds. This effect must be considered when conducting tests of the MMS on Earth.

A fourth body had been added to act as a vibrational damper for the shaft. It was shown that the shaft oscillation actually decreased the precession rate of the rotor and that less oscillation was not desirable. It was also determined that the damper would perform its job but only very inefficiently and the same effect could be made by increasing the spin inertia of the shaft, should it be desired.

In the literature, the tuned speed was calculated using Equation 2.18. Using numerical experimentation, a relationship between the tuned speed and rotor offset was determined. Using this new equation it is possible to obtain the proper tuned speed for any offset of the rotor. It was also shown that the tuned speed calculated in the literature is off by 2.8 rpm where the rotor offset is zero for the MMS.

The dynamics of the MMS and DTG has now been characterized. The effect of the crossed flexure pivot is also included in this analysis and the result was the discovery that the tuned speed of the MMS is different when the system is operating in a gravitational field.

Chapter 12

Summary and Conclusions

The Dynamically Tuned Gyroscope (DTG) has been used for angular rate sensing in spacecraft for decades. In combination with momentum and reaction wheels, it forms the backbone of a spacecraft attitude control system (ACS). The demand on the accuracy of these systems is continually increasing due to scientific, military and commercial missions that have stringent pointing requirements. At the same time demands for low cost, minimal mass and maximum reliability have to be met.

The Momentum Management System (MMS) will provide a low cost, reliable and high precision alternative to the DTG/momentum wheel/reaction wheel package. Based on DTG technology, the MMS uses a gimballed momentum wheel, providing attitude control about three orthogonal axes and rate sensing capabilities about two axes.

In order for the MMS to be a high precision instrument as well as an actuator, an excellent understanding of its dynamics is required. Since the MMS is based on the DTG, a detailed analysis of these devices was performed. The Dynamically Tuned Gyro has been in existence for over three decades and

during this time many analyses have been performed. These studies have all made several simplifying assumptions which result in overlooked characteristics of the DTG.

Mechanically, the DTG and MMS are made up of three main components, the rotor, gimbal and shaft. The relative motion of the rotor with respect to the case is used to measure the angular rate and position of the vehicle. In the situation of the MMS, the rotor is also used to remove or add angular momentum to the spacecraft.

The shaft is connected to the gimbal ring through the use of a pair of crossed flexure pivots. The gimbal and the rotor are also attached in the same manner. These pivots act as torsional springs and make the gimbal a modified universal or Hooke's joint. This enables the rotor to rotate about its transverse axes. The torsional stiffness of the pivots enables the system to be dynamically tuned. If tuning was not possible then high precession rates would result, a situation which, in the past, has led to the belief that the predecessors of the DTG were unstable.

Crossed flexure pivots are made up of two thin strips which cross each other without touching. The ends of the strips are connected to rigid bodies, the shaft and gimbal or the rotor and gimbal. The result is a joint which allows rotational motion about one axis while remaining stiff about the other two. Translational motion about all three axes is also kept to a minimum.

Quite a few analytical studies of the static behaviour of this joint have been performed; however, the majority of these studies assume that the translational motion of the pivot can be neglected. This means that the strips could be analyzed separately and then combined through the use of force

balance equations. Haringx[53] treated the joint as a whole in his analysis but only for the case where only a torque was applied. Wittrick[14] tried to account for the translational motion by including it after the shape equations had been formed.

At the same time, Young[16] and Nickols and Wunsch[17] performed empirical studies which measured the motion of the pivot centre and its stiffness when a pure moment was applied. The experimental results demonstrated that the initial centre of rotation did shift as the pivot deflected although this motion was small.

The fact that there was motion of the initial centre of rotation raises questions as to the validity of the assumption that it is negligible. Therefore, an analysis was performed using Hamilton's principle of minimum energy which treated the system as a whole. In addition to including the transverse displacement of the strips, the axial displacement due to axial loads and the bending of the strips were also included. Wittrick[14] had included shortening due to bending in his paper but did not include deformation due to the axial load. Scieszko and Mansour[54], on the other hand, included the deformation due to the axial load but not due to the bending of the strips. The current analysis included both.

The result of this analysis was four equations describing the axial and transverse deflection of the strips. Using these equations, the stiffness of a pivot can be calculated as can the displacement of the initial centre of rotation. These can be done under any two-dimensional loading conditions where previously any analytic calculations of the motion were limited.

One difficulty arose in the fact that two Lagrange multipliers, representing

the constraint loads acting on the strips, remained and had to be found by solving a set of transcendental equations. Fortran code, implementing a hybrid Powell root-finding method, was used to calculate the magnitude of the multipliers.

Comparison between the stiffness calculated by the theoretical model and the models from the literature shows that they were very similar. In fact, it was found that the difference between the stiffness calculations was less than 2% for rotations up to 20 degrees.

A major discrepancy was found when the buckling load was calculated. In the literature, the buckling load of the pivot is calculated to occur at a normalized axial load of $NL = 2\pi$. The current analysis found that $NL = 1.78\pi$, or a value 89% of that from the literature. It is assumed that this difference comes from the fact that the constraint loads are neglected in the analyses in the literature. Any pivot design must take this lower buckling load into account.

Comparisons between the theoretical model and the empirical results from the literature show that the theoretical model predicts the proper trend but it predicts a slightly stiffer pivot compared to the results from the literature. Investigation into two of the assumptions made during this analysis, 1) small angles of deflection and 2) the strips are beams rather than plates, made it apparent that errors introduced by these assumptions were approximately 5% and would have made the pivot even stiffer.

A test pivot was designed and constructed in order to provide an independent check on the theoretical model. Three experimental tests were performed in which a tensile vertical load was applied to the pivot during

which the motion of the initial centre of rotation was measured. The theoretical model predicted the response of the pivot quite closely. It should be noted that the model predicted a linear response while the actual motion was slightly non-linear. With this said, the error only grew to 15% and the non-linearity may have been due to the material properties of the strips as well as tolerancing errors in the test pivot. The validity of the model has been demonstrated.

A procedure was then developed and used to size the pivots used in the MMS. The procedure outlined can be used to design crossed flexure pivots for any application. Then several studies were performed in order to ensure that the design would function in all operational modes, including start-up and shut-down.

Using Hamilton's principle of least action, a vibrational analysis of the crossed flexure pivot was performed. In the literature only static analyses have been conducted. Finding the determinant of the resulting 12 by 12 matrix was too complex to arrive at a general characteristic equation for the crossed flexure pivot. However, should the dimensions of the strips and the inertial properties of the rigid bodies be known, the frequencies of vibration and the corresponding mode shapes can be calculated for an unforced system.

Using the properties of the MMS pivots and the mass and inertial properties of the gimbal ring, the vibrational characteristics of the crossed flexure pivots in the MMS were calculated. It was found that the dominant frequency occurred at 13.53 Hz and that the corresponding mode shape agreed with the predicted shape used for the static analysis. This frequency also corresponds to the natural frequency which is calculated when it is assumed

that the pivot is a perfect torsional spring. The next higher frequency occurred at 1.23 MHz which is significantly higher than the operational speeds of the MMS.

With the properties and behaviour of the crossed flexure pivots understood, the research then moved on to look at the dynamics of the MMS and DTG. A stability analysis of the DTG and the MMS was performed. In the literature, it had been stated that the DTG is unstable when using bearings instead of crossed flexure pivots. This statement was made after empirical observations of the response of a gyro without pivots (or some type of torsional stiffness). It was felt that an analysis of the DTG was required in order to determine the regions and types of instability for the gyro.

To accomplish this task a two body, small angle analysis of the DTG was performed. Once the dynamic equations had been determined, the characteristic equation was calculated using the complex variable and state space methods. The characteristic equation was used in a stability analysis and it was found that a DTG with bearings rather than pivots would be unstable by divergence only in the region

$$I_{r_s} < I_{r_t} < I_{r_s} + I_{g_s} - I_{g_t}$$

The transverse and spin inertia of the gimbal, I_{g_t} and I_{g_s} , are small when compared to I_{r_s} and this results in a small region of instability. Also note that the transverse inertia of the rotor, I_{r_t} , is always less than the spin inertia.

It was found that a DTG with bearings will never experience flutter instability, although when the shaft speed is zero, it will be on the verge of flutter instability. From these facts, it can be implied that what was observed in the literature was actually just high precession rates and not instability

after all. This is not surprising since high precession rates would make the response appear as though it is unstable.

A stability study of the DTG showed that under specific circumstances it was possible for the system to be unstable. The response at low shaft speeds and negative pivot stiffness and the response at high ratios of the transverse inertia of the gimbal and rotor are two excellent examples. These situations were extreme cases and, in general, the DTG is a very stable device. An analysis of the stability of the MMS demonstrated that it was even more stable than the DTG.

At this point a four body, large angle analysis of the MMS and DTG was performed using Lagrange's Equations and Hamilton's Canonical Equations. Other methods were evaluated but it was felt that these two were the most suitable for the problem at hand. In the literature, the most complicated analysis involved a two body, large angle analysis of a DTG. This results in the assumption that the shaft speed is constant which is not true for all cases and especially not for the MMS. This is due to the spin inertia of the rotor being extremely high. For this reason, an analysis involving the rotor, gimbal and shaft was required for the MMS. A fourth body, intended to act as a vibrational damper, was added as well. Once the equations of motion were derived, Newton-Euler Equations were used to determine the forces acting on the pivots. These results were then implemented in a computer simulation called DTGSim of the MMS and DTG. The variance of the pivot stiffness with applied load was also included in the simulation. In this simulation, an Adams-Moulton method was used to integrate the equations of motion.

Validation of the simulation consisted of modeling the Litton CSG-2 DTG

in DTGSim, two commercial packages (Adams and Working Model) and a program (RSIM) developed by this author at an earlier date. The code was made so that it could be used on its own to examine the open-loop characteristics of the MMS and DTG or it could be used as a sub-component in a larger program which evaluates the control system design. It could also be implemented in a 'hardware-in-the-loop' simulation of the MMS electronics.

The computer simulation was then used to analyze different aspects of the dynamic behaviour of the MMS and DTG. In general, it was shown that the rotor of the DTG and MMS undergoes precession and nutation when it is not at a null position. If damping is assumed to be negligible then the precession is periodic. The frequency and direction of the precession is dependent on the angle of offset of the rotor and the shaft spin speed.

Analysis of the Litton CSG-2 DTG was used to demonstrate the various properties of the DTG. Comparisons of the power spectral density analyses of the DTGSim results with the natural frequencies calculated by the small angle equations showed one large difference. The small angle analysis indicated that the precession frequency is independent of the rotor offset while the large angle analysis showed otherwise. When the dynamics were viewed from case coordinates several super and sub harmonics were observed. These harmonics must be taken into account by any control system intended for the MMS.

Viscous damping was seen to cause the rotor to spiral in toward a null position. The higher the damping, the quicker the rotor reaches null. Since precession frequency is dependent on rotor offset, it can be seen that it will be continuously changing due to the damping under open loop conditions.

Using the properties of the MMS, a comparison between the two body and three body model was made. The rotor in the three body model was found to precess at a much slower rate than the rotor in the two body model. It was also seen that the shaft speed oscillated with a range that was 1.5% of the mean shaft rate. A two body analysis involving just the rotor and gimbal is clearly unsuitable for use when analyzing the MMS.

Including the fact that the pivot stiffness is a function of the applied loads did not have a serious effect on the dynamics of the system when gravity was assumed to be zero. This demonstrates that a constant stiffness model will accurately predict the response of the rotor in a micro-gravity environment. However, any experimentation on a prototype will be performed on the Earth and so gravity must be taken into account. Studies including the effect of a 1g environment demonstrated that gravity has a large effect on the system and effectively changed the tuned speed by over 200 rpm. Therefore, including the variance in the pivot stiffness is essential when modeling the MMS on the Earth. To date, the variable pivot stiffness has never been included in a model of a DTG.

The vibrational damper, the fourth body in the analysis, was found to be an ineffective method of limiting shaft oscillation. In fact, the oscillation of the shaft reduces the precession frequency of the rotor and any reduction in the shaft oscillation would be inadvisable. If it was desired, increasing the inertia of the shaft would accomplish the task more efficiently.

Using the parameters of the MMS, a relationship between precession frequency, rotor offset and average shaft speed was found. From this, the proper tuned speed at any rotor offset can be determined. Since the rotor of the

MMS is meant to operate at non-null positions, it is now possible to determine the ideal average shaft speed in order to reduce the precession frequency. Previously, the tuned speed was calculated using Equation 2.18 and the result was assumed to be correct. The tuned speed can now be accurately calculated for any gyro, by determining the precession rate at various shaft speeds.

In the literature, the analysis of the Dynamically Tuned Gyroscope has been limited to small angles and two bodies, the rotor and gimbal. This research shows that the small angle analysis does not accurately predict the fact that precession is related to the rotor offset. In addition, the rotor of the MMS has a very large inertia and it can not be assumed that the shaft rate is constant. The analysis in the literature also assumes that the stiffness of the torsional springs is constant. However, the torsional springs are actually crossed flexure pivots whose stiffness is a function of the loads acting on it. Including this fact in the analysis of the DTG and MMS shows that in a 1g environment there is a significant effect on the behaviour of the system. By including the effect of gravity on the pivots it is possible to properly test and calibrate the MMS while on Earth.

This result can be extended to a DTG under high accelerations where the rotor may not be as heavy as the MMS rotor but the loads may increase enough to affect the performance of the device. Examples where this would be of concern is gyros operating in rockets or high performance aircraft.

Chapter 13

Recommendations

This research has added to the understanding of the crossed flexure pivot and the dynamics of the Dynamically Tuned Gyroscope (DTG). Through this work, behavioral characteristics of the Momentum Management System (MMS) have been identified that would not have shown up using analyses from the literature. In fact, the MMS may not have been recognized as practical using these previous analyses. However, this work is not definitive and there are several areas in which further study could be conducted.

The crossed flexure pivot is a deceptively simple mechanism. By crossing the two strips the pivot gains some interesting qualities such as high buckling load and variable stiffness. In Chapter 4 it was found that the buckling load is lower than that calculated in the literature. This warrants further investigation and empirical data should be collected.

Actually, much more experimental data on the crossed flexure pivot is required. Tests from the literature involve determining the behaviour of the pivot when a torsional load is applied. This is only a small subset of the actual conditions that may arise and tests which also include horizontal and

vertical applied loads should be conducted.

The research by Nickols and Wunsch[17] indicated that using an effective length make the experimental and theoretical results converge. It has yet to be determined whether this effective length is a function of the method by which the strips are joined to the rigid body. Specifically, by making the pivots out of one solid piece of material, do the theory and experimental results converge without using an effective length? Similarly, does the effective length also apply to vibration and non-linear dynamic analyses?

The static analysis of the crossed flexure pivots should be extended to include large angle deflections and wider strips. To do this, an analytical solution may not be possible and an FEM package may be required or some other alternative method.

The vibration analysis of the crossed flexure pivot developed in Chapter 7 appears to be the first, and further work can definitely be performed on this front. A non-linear analysis could be performed so that all the terms would be kept. This way, the effect of neglecting terms can be determined.

In general, the unforced natural frequencies still hold when the system is under loading conditions. However, the properties of the crossed flexure pivot change when loads are applied which may add new frequencies of response. It has been shown, statically, that the stiffness of the pivot changes under applied loads. Therefore, even if a constant force is applied to the system, a shift in the natural frequencies should occur. This behaviour needs to be studied.

The model of the MMS and DTG developed in Chapter 9 is quite detailed; however, translational motion of the rotor and gimbal has been neglected. It

has been shown that the initial centre of rotation of the crossed flexure pivots moves and this means that the rotor and gimbal move as well. Although this motion is very small, it would be interesting to determine its effect on the system dynamics.

As with the crossed flexure pivots, comparisons between the theoretical and experimental behaviour of a DTG or the MMS would be useful. This would allow any deviation, if there are any, to be found.

Finally, adding a control system to DTGSim would allow both open and closed loop situations to be studied. The application of various gains could be studied and the control system could be refined without endangering the actual prototype.

Bibliography

- [1] Wertz, James R. *Spacecraft Attitude Determination and Control, Volume 73*. D. Reidel Publishing Company, Boston, 1978.
- [2] Cain, J.S., Staley, D.A. and Hartl, D. "Inertial Sensors in a Spacecraft Application", *Canadian Aeronautics and Space Journal*, Vol. 43, No. 2, 1997, pp. 101-105.
- [3] Hughes, Peter C. *Spacecraft Attitude Dynamics*. John Wiley & Sons, Inc. Toronto, 1986.
- [4] Yuan, Mark S. and Wittmann, Alois. May 12, 1992. *Tilting Momentum Wheel for Spacecraft*. U.S. Patent 5,112,012.
- [5] Bender, Douglas J. and Brehove, James D. Assignee: Hughes Aircraft Company. Nov. 16, 1993. *Momentum Wheel Platform Steering System and Method*. U.S. Patent 5,261,631.
- [6] Studer, Philip A. Assignee: National Aeronautics and Space Administration. Mar. 22, 1988. *Three Axis Attitude Control System*. U.S. Patent 4,732,353.
- [7] Lademann, Ernest E. "Gyro Technology", *Advances in the Astronautical Sciences*, Vol. 63, 1987, pp. 207-235.
- [8] Bryson, Arthur E. *Control of Spacecraft and Aircraft*. Princeton University Press, Princeton New Jersey, 1994.
- [9] Vadali, S.R. and Oh, H.S. "Attitude Control and Momentum Management of the Space Station", *Advances in the Astronautical Sciences*, Vol. 75, 1991, pp 609-620.

- [10] Chipman, Richard, Hu, Tsay-Hsin G., Barber, Mark and Holmes, Eric. "Impact of Suntracking on Space Station Controllability", *Advances in the Astronautical Sciences*, Vol. 84, 1993, pp. 371-389.
- [11] Staley, D.A. *Orbital Mechanics and Spacecraft Control*, Carleton University, Ottawa, 1997.
- [12] Meriam, J.L. and Kraige, L.G. *Engineering Mechanics Volume 2: Dynamics*, John Wiley & Sons, Inc. Toronto, 1986.
- [13] Wittrick, W.H. "The Theory of Symmetrical Crossed Flexure Pivots", *Australian Journal of Scientific Research*, Vol. 1, 1948, pp. 121-134.
- [14] Wittrick, W.H. "The Properties of Crossed Flexure Pivots and the Influence of the Point at which the Strips Cross", *The Aeronautical Quarterly*, Vol II, February 1951, pp. 272-292.
- [15] Weinstein, Warren D. "Flexure-Pivot Bearings, Part II", *Machine Design*, Vol. 37, July 8, 1965, pp. 136-145.
- [16] Young, W.E. "An Investigation of the Cross-Spring Pivot", *Journal of Applied Mechanics* June 1944, pp. A-114-A-120.
- [17] Nickols, L.W. and Wunsch, H.L. "The Design Characteristics of Cross Spring Pivots", *Machinery*, Vol. 79, October 1951, pp. 645-651.
- [18] Staley, D. and Cain, J.S., "Dynamics of a Tuned Rotor Gyroscope", *Canadian Aeronautics and Space Journal*, Vol. 43, No. 2, 1997, pp. 106-110.
- [19] Craig R.J.G., "Theory of Operation of an Elastically Supported, Tuned Gyroscope", *IEEE Transactions on Aerospace and Electronic Systems*, Vol. AES-8, No. 3, 1972, pp. 280-288.
- [20] Craig R.J.G., "Theory of Errors of a Multigimbal, Elastically Supported, Tuned Gyroscope", *IEEE Transactions on Aerospace and Electronic Systems*, Vol. AES-8, No. 3, 1972, pp. 289-297.
- [21] Lawrence, A.W., "A Simulation of a Dynamically Tuned Gyroscope", *Gyrodynamics: Euromech 38 Colloquim*, University of Louvain, 1974, pp. 56-62.

- [22] Lawrence, A., *Modern Inertial Technology*, Springer-Verlag, 1993, pp. 127-147.
- [23] Cain, J.S., "Non-Linear Analysis of a Gyroscopic Rotor", M.Eng. Thesis, Department of Mechanical and Aerospace Engineering, Carleton University, Ottawa, April, 1996.
- [24] Titterton, D.H. and Weston, J.L. *Strapdown Inertial Navigation Technology*, Peter Peregrinus Ltd., 1997.
- [25] Ishlinskii, A.Yu. *Mechanics of Gyroscopic Systems*. Translated by A. Barouch. Israel Program for Scientific Translations, Jerusalem, 1965.
- [26] Deimel, Richard F. *Mechanics of the Gyroscope: The Dynamics of Rotation*. Dover Publications, Inc. USA, 1950.
- [27] Richardson, K.I.T. *The Gyroscope Applied*. Hutchinson's Scientific and Technical Publications, Toronto, 1955.
- [28] Ge, Zheng-Ming and Chen, Heng-Hui. "Stability of Rate Gyro in a Vehicle with Uncertain Angular Velocity and Acceleration", *Transactions of the CSME*, Vol. 19, No. 4, 1995, pp. 417-432.
- [29] Boyarski, Shmuel and Ben-Asher, Joseph Z. "Minimum-Time Reorientation of a Two-Degree-of-Freedom Gyroscope", *Journal of Guidance, Control and Dynamics*, Vol. 18, No. 4, July-August, 1995, pp. 782-791.
- [30] Dougherty, H., Rodden, J., Reschke, L.F., Tompetrini, K., Weinstein, S.P. and Slater, D. "Performance Characterization of the Hubble Space Telescope Rate Gyro Assembly", *IFAC Automatic Control in Space Symposium*. Toulouse, France, 1985, pp. 219-222.
- [31] Shimoni, U. "Tamam Minitune Gyro", *DGON Symposium Gyro Technology*, Stuttgart, 1983.
- [32] Sinkiewicz, J.S., Hartl, D.A., Staley, D.A. and Vinnins, M. "A Gyro Stabilized Airborne Gravimetry Platform", *Canadian Aeronautics and Space Journal*, Vol. 43, No. 2, June 1997, pp 123-125.
- [33] Aucktor, E., Schmelz, F., and Seherr-Thoss, H.-Chr. *Universal Joints and Driveshafts*, Springer-Verlag, 1992, pp. 1-8.

- [34] Martin, G.H., *Kinematics and Dynamics of Machines*, McGraw-Hill Book Company, Toronto, 1969, pp. 62-64.
- [35] Wittenburg, J. *Dynamics of Systems of Rigid Bodies*. B.G. Teuber, Stuttgart, 1977.
- [36] Howe, E.W. and Savet, P.H. "The Dynamically Tuned Free Rotor Gyro", *Control Engineering*, June 1964, pp. 67-72.
- [37] Mansour, W.M. and Lacchini, C., "Two Axis Tuned-Rotor Gyroscopes: Design and Technology", *Journal of Guidance, Control and Dynamics*, June 1993, pp. 417-425.
- [38] Fogarasy, A.A., "A Contribution to the Dynamics of a Spring Restrained Hooke's Joint Gyroscope", *Gyrodynamics: Euromech 38 Colloquium*, University of Louvain, 1974, pp. 49-55.
- [39] Albrecht, W.G. "Errors of DTG's on Acceleration Induced Mistuning and Rotor Axis Tilt", *DGON Symposium Gyro Technology*, Stuttgart, 1983.
- [40] Craig, R.J.G. "Dynamically Tuned Gyros in Strapdown Systems", *AGARD Conference on Inertial Navigation Computers and Systems*. Florence, Italy. October, 1972. pp. 12-1 to 12-17.
- [41] Slocum, Alexander H., *Precision Machine Design*. Prentice Hall, Inc. New Jersey, 1992.
- [42] Jones, R.V. "Some Uses of Elasticity in Instrument Design", *Instruments and Experiences: Papers on Measurement and Instrument Design*. John Wiley & Sons, Toronto. 1988. pp. 122-144.
- [43] Jones, R.V. "Angle-Spring Hinges", *Journal of Scientific Instruments*. Vol. 32, September 1955, pp. 337-338.
- [44] Jones, R.V. "Angle-Spring Hinges", *Journal of Scientific Instruments*. Vol. 33, June 1956, p. 245.
- [45] Eastman, Fred S. "The Design of Flexure Pivots", *Journal of the Aeronautical Sciences*. Vol. 5, November 1937, pp. 16-21.

- [46] Weinstein, Warren D. "Flexure-Pivot Bearings, Part I", *Machine Design*. Vol. 37, June 10, 1965, pp. 150-157.
- [47] Paros, J.M. and Weisbord, L. "How to Design Flexure Hinges", *Machine Design*. Vol. 37, No. 27, November 25, 1965, pp. 151-156.
- [48] Jones, R.V. "Parallel and Rectilinear Spring Movements", *Journal of Scientific Instruments*. Vol. 28, February 1951, pp. 38-41.
- [49] Jones, R.V. "A Parallel-Spring Cross-Movement For an Optical Bench", *Journal of Scientific Instruments*. Vol. 33, July 1956, pp. 279-280.
- [50] Jones, R.V. and Young, I.R. "Some Parasitic Deflexions in Parallel Spring Movements", *Journal of Scientific Instruments*. Vol. 33, January 1956, pp. 11-15.
- [51] Plainevaux, J.E. "Etude des Déformations d'une Lame de Suspension Élastique", *Nuovo Cimento*. Vol. 4, October 1956, pp. 922-928.
- [52] Plainevaux, J.E. "Mouvement de Tangage d'une Suspension Élémentaire sur Lames Élastiques", *Nuovo Cimento*. Vol. 4, November 1956, pp. 1133-1141.
- [53] Haringx, J.A. "The Cross-Spring Pivot as a Constructional Element", *Applied Science Research*. Vol A1, 1949, pp. 313-332.
- [54] Scieszko, Jan L. and Mansour, William M. "Analysis and Design of Leaf-Spring Flexible Joints for Driving Gyroscopic Rotors", *Proceedings of the 10th IMACS World Congress on System Simulation and Scientific Computation*. Montreal, Quebec. August 1982, pp. 155-157.
- [55] Scieszko, J.L. and Mansour, W.M. "Analysis and Design of Composite Universal Flexure Joints", *Proceedings of the Second International Computer Engineering Conference*. San Diego, California. August 1982, pp. 238-241.
- [56] Lucas Free-Flex Flexural Pivot, Promotional Pamphlet. Obtained February, 1997.
- [57] Ferris, Kristina "Electrical Discharge Machining" Industrial Research and Development Institute. Ref. No. 702. 1996.

- [58] "Controlling EDM Surface Integrity", *American Machinist and Automated Manufacturing*, November 1987, pp. 80-83.
- [59] Benham, P.P and Crawford, R.J. *Mechanics of Engineering Materials*. John Wiley and Sons, Inc. New York, 1991.
- [60] Timoshenko, S. "Mathematical Determination of the Modulus of Elasticity", *Mechanical Engineering*, Vol. 45, No. 4, April 1923. pp. 259-260.
- [61] Popov, Edgar P., *Engineering Mechanics of Solids*. Prentice Hall, Inc. New Jersey, 1990.
- [62] Timoshenko, Stephen P., *Theory of Elasticity*. McGraw-Hill. New York, 1961.
- [63] Fertis, Demeter G. *Nonlinear Mechanics*. CRC Press, Inc., United States, 1993.
- [64] "IMSL Math Library", *Fortran Powerstation 4.0 Books Online*. Microsoft Corporation, 1995.
- [65] Discussion with Doug Staley and Cameron Ower. Autumn 1997.
- [66] Chesser, H. *MMS Gimbal Sizing*. Technical Memo TN-006, File 613-03-2.
- [67] Shigley, J.E. and Mischke, C.R. *Mechanical Engineering Design*. Fifth edition. McGraw-Hill, Inc. Toronto. 1989.
- [68] "AerMet 100 Alloy" *Alloy Data*. Carpenter Technology Corporation, Carpenter Steel Division, received by fax May 9, 1997.
- [69] Tyc, George. "Comparison of the Power Requirements for a Dynamically Tuned and an Untuned Flexure." Received by Fax August 2, 1997.
- [70] Pearson, Brian. Email message dated April 10, 1997.
- [71] Chesser, H. *Preliminary FEM Results of Prototype Gimbal Design*. Internal memorandum dated December 12, 1997.
- [72] Chesser, H. et al. *Final Report: Smallsat Critical Technology Development - Phase 2B*. CSA Report #CSA-ST-CR-1998-0011. April 1998.
- [73] Taylor, Bruce. Email message dated December 5, 1997.

- [74] Yamanaka, Keiji, *On the Stability Analysis of Gyroelastic Systems*. Ph.D Thesis, University of Waterloo, 1993.
- [75] Huseyin, K., *Vibrations and Stability of Multiple Parameter Systems*. Noordhoff International Publishing, 1978.
- [76] *Matlab: The Language of Technical Computing, Using Matlab Version 5*, The MathWorks, Inc. January, 1997.
- [77] Armanovitch, L., "Spacecraft Orientation Based on Space Object Observations by Means of Quaternion Algebra", *Journal of Guidance, Control and Dynamics*, Vol. 18, No. 4, July-August, 1995, pp. 859-866.
- [78] Kane, Thomas R. and Levinson, David A., "Formulation of Equations of Motion for Complex Spacecraft", *Journal of Guidance and Control*, Vol. 3, No. 2, March-April, 1980, pp. 99-112.
- [79] Kane, Thomas R. and Levinson, David A., *Dynamics: Theory and Applications*. McGraw-Hill Book Company, Toronto, 1985.
- [80] Kane, Thomas R., "Formulation of Dynamical Equations of Motion", *American Journal of Physics*, Vol. 51, No. 11, November 1983, pp. 974-977.
- [81] Hildebrand, Francis B., *Methods of Applied Mathematics*. Dover Publications Inc., Toronto Ontario. 1992.
- [82] Thomson, William T. *Theory of Vibration with Applications*, Prentice Hall, 1988.
- [83] *Microsoft Windows 95*, Microsoft Corporation, USA, 1995.
- [84] Forsythe, George E., Malcolm, Michael A. and Moler, Cleve B., *Computer Methods for Mathematical Computations*, Prentice-Hall, 1977.
- [85] Atkinson, K., *Elementary Numerical Analysis*, John Wiley & Sons, 1985.
- [86] Blackman, R.B. and Tukey, J.W., *The Measurement of Power Spectra*, Dover Publications, Inc., 1959.
- [87] *ADAMS: Using ADAMS/View*. Mechanical Dynamics, Inc., Limited Edition, USA, 1997.

- [88] *Working Model 3D: User's Manual*, Knowledge Revolution, USA, 1997.
- [89] Staley, D.A. Interoffice memo, dated March 2, 1997. Received by email.
- [90] *CRC Handbook of Mathematical Sciences*. Ed. William H. Beyer. CRC Press, Inc. 5th Edition, 1978, pp. 362-376.
- [91] Askeland, Donald R. *The Science and Engineering of Materials*. PWS Publishing Company. Boston, USA, 1994.
- [92] *Maple V Release 4* Waterloo Maple Inc. 1996.
- [93] Discussion with Glenn Heppler. Spring 1998.

Additional References on Electrical Discharge Machining:

Bromann, Randall L. "Cut-Surface Quality: The Pluses and Minuses of EDM" *Tooling & Production*. Huebner Publications Inc. January 1985. pp. 37-41.

Electrical Discharge Machining Industrial Research and Development Institute (IRDI) Seminar and Presentation Notes. November 21, 1996.

Appendix A

Kinematics of a Hooke's Joint

The gimbal, which behaves like a Hooke's Joint, allows the spin axis of the rotor to be oriented in a different direction than that of the shaft. In other words, it allows the rotor to tilt about both x and y axes.

A Hooke's Joint, more commonly known as a universal joint, allows rotary movement through an angled joint. It does not transmit rotary movement steadily and the result is the sinusoidal motion of one shaft with respect to the rotor. A second aspect of the Hooke's Joint is that the gimbal must 'flutter' in order to allow transmission of the rotary motion. This 'flutter', shown in Figure A.1, occurs at twice the spin frequency of the shaft and causes the crossed flexure pivots to exert torsional forces on the rotor, affecting its response. In fact, 'tuning' the DTG (and the MMS) involves eliminating the first order effects of these torsional forces (see Section 2.3.2).

Two classical analyses of the kinematics of the Hooke's Joint have been performed by Aucklor et al.[33] and Martin[34]. These discussions find the relationship between input and output shaft rotations when the shafts are offset to each other by a particular angle. The results are useful when the

Figure A.1: Example of gimbal motion when the rotor is offset at a particular angle. Diagram (a) shows the system at the start of rotation. At this point the inner crossed flexure pivots are twisted and the outer ones are not. One quarter turn later (Diagram (b)), the outer pivots are twisted while the inner ones are not.

coordinate system can be placed in the plane of the two shafts. In the case of a DTG this is not possible and a more general analysis is required[23]. In general, two angles are needed to describe the position of the rotor with respect to the shaft. It is possible to relate the shaft and the rotor by rotating the coordinate system, using rotation matrices, from the shaft to the rotor. This involves rotations about the x, y, and z axes and it is the order of these rotations that is important.

The first rotation sequence is based on the gimbal axes and the angles involved are known as 'gimbal' angles. For this sequence, the first rotation is about the spin axis of the shaft (or z axis) followed by rotations about the x and y axes of the gimbal. Shown in Figure A.2, this sequence, A , can be expressed as

$$A = (\theta)_y(\phi)_x(\Omega)_z \quad (A.1)$$

where

$(\theta)_y$ = rotation by θ about its y axis, etc.

Ω = input shaft rotation

ϕ = x axis gimbal angle

θ = y axis gimbal angle

expanding Equation A.1 gives

$$A = \begin{pmatrix} C_\theta C_\Omega - S_\theta S_\phi S_\Omega & C_\theta S_\Omega + S_\theta S_\phi C_\Omega & -S_\theta C_\phi \\ -C_\phi S_\Omega & C_\phi C_\Omega & S_\phi \\ S_\theta C_\Omega + C_\theta S_\phi S_\Omega & S_\theta S_\Omega - C_\theta S_\phi C_\Omega & C_\theta C_\phi \end{pmatrix} \quad (A.2)$$

where, for example

C_ϕ represents $\cos \phi$

S_ϕ represents $\sin \phi$

Figure A.2: Rotation of a coordinate frame based on moving the shaft (XYZ) to the rotor position ($X''Y''Z''$) using 'gimbal' angles. The order of rotation in this instance is zxy .

A second possible rotation sequence involves rotating the coordinate system about the x and y axes first and then rotating about the rotor spin axis¹. In this case, the x and y angles are known as 'case' angles since they are essentially fixed to the case. This sequence, defined as A' and shown in Figure A.3, is expressed as

$$A' = (\gamma)_z(\theta')_y(\phi')_x \quad (A.3)$$

¹The angles involved in a rotation using this sequence are sometimes referred to as Bryant or Cardan angles[35].

where

γ = the rotation of the rotor

ϕ' = the x axis rotor angle

θ' = the y axis rotor angle

Equation A.3 can be expanded to give

$$A' = \begin{pmatrix} C_\gamma C_{\theta'} & C_\gamma S_{\theta'} S_{\phi'} + S_\gamma C_{\phi'} & -C_\gamma S_{\theta'} C_{\phi'} + S_\gamma S_{\phi'} \\ -S_\gamma C_{\theta'} & -S_\gamma S_{\theta'} S_{\phi'} + C_\gamma C_{\phi'} & S_\gamma S_{\theta'} C_{\phi'} + C_\gamma S_{\phi'} \\ S_{\theta'} & -C_{\theta'} S_{\phi'} & C_{\theta'} C_{\phi'} \end{pmatrix} \quad (A.4)$$

Equations A.2 and A.4 describe the same orientation so they must be equal. From this, the relationship between the shaft and rotor can be derived. To find this relationship take

$$\frac{A_{21}}{A_{22}} = \frac{A'_{21}}{A'_{22}}$$

which can be simplified to give

$$T_\gamma = \frac{T_\Omega C_{\theta'}}{C_{\theta'} + T_\Omega S_{\theta'} S_{\phi'}} \quad (A.5)$$

where, for example

T_γ represents $\tan \gamma$

Using similar procedures, it is possible to find the relationships between the case and gimbal angles. These relationships are

$$S_\theta = S_{\theta'} C_\Omega - C_{\theta'} S_{\phi'} S_\Omega \quad (A.6)$$

$$S_\phi = S_{\theta'} S_\gamma + S_{\phi'} C_\gamma \quad (A.7)$$

$$S_{\theta'} = S_\theta C_\gamma + C_\theta S_\phi S_\gamma \quad (A.8)$$

$$S_{\phi'} = C_\gamma S_\phi - S_\theta C_\phi S_\gamma \quad (A.9)$$

$$T_\gamma = \frac{C_\phi T_\Omega}{C_\theta - S_\theta S_\phi T_\Omega} \quad (A.10)$$

Figure A.3: Rotation of a coordinate frame based on moving the shaft (XYZ) to the rotor position ($X''Y''Z''$) using 'case' angles. The order of rotation is xyz .

The case and gimbal angular velocities can also be related by transforming them into the final orientation of the rotor. To find the angular velocity of the rotor, rotate the shaft velocity about the x and y axes and rotate $\dot{\phi}$ about the x axis. The transform becomes

$$\omega_r = \begin{pmatrix} 0 \\ \dot{\theta} \\ 0 \end{pmatrix} + (\theta)_y \begin{pmatrix} \dot{\phi} \\ 0 \\ 0 \end{pmatrix} + (\theta)_y(\phi)_x \begin{pmatrix} 0 \\ 0 \\ \dot{\Omega} \end{pmatrix}$$

which can be simplified to give

$$\omega_r = \begin{pmatrix} C_\theta \dot{\phi} - S_\theta C_\phi \dot{\Omega} \\ \dot{\theta} + S_\phi \dot{\Omega} \\ S_\theta \dot{\phi} + C_\theta C_\phi \dot{\Omega} \end{pmatrix} \quad (\text{A.11})$$

A similar transformation can be made to the case angular velocities to give

$$\omega'_r = \begin{pmatrix} C_\gamma C_{\theta'} \dot{\phi}' + S_\gamma \dot{\theta}' \\ -S_\gamma C_{\theta'} \dot{\phi}' + C_\gamma \dot{\theta}' \\ \dot{\gamma} + S_{\theta'} \dot{\phi}' \end{pmatrix} \quad (\text{A.12})$$

Equations A.11 and A.12 can be equated to give the following relationships

$$\dot{\phi} = \frac{1}{C_\theta} (C_\gamma C_{\theta'} \dot{\phi}' + S_\gamma \dot{\theta}' + S_\theta C_\phi \dot{\Omega}) \quad (\text{A.13})$$

$$\dot{\theta} = C_\gamma \dot{\theta}' - S_\gamma C_{\theta'} \dot{\phi}' - S_\phi \dot{\Omega} \quad (\text{A.14})$$

$$\dot{\phi}' = \frac{1}{C_{\theta'}} (C_\gamma C_\theta \dot{\phi} - S_\gamma \dot{\theta} - (C_\gamma S_\theta C_\phi + S_\gamma S_\phi) \dot{\Omega}) \quad (\text{A.15})$$

$$\dot{\theta}' = S_\gamma C_\theta \dot{\phi} + C_\gamma \dot{\theta} - (S_\gamma S_\theta C_\phi - C_\gamma S_\phi) \dot{\Omega} \quad (\text{A.16})$$

$$\dot{\gamma} = S_\theta \dot{\phi} + C_\theta C_\phi \dot{\Omega} - S_{\theta'} \dot{\phi}' \quad (\text{A.17})$$

Finally, it should be noted that while ϕ' describes the rotation about the x axis of the case, θ' is actually about an intermediate axis and so it is necessary to relate the angle that a pickoff would measure to θ' . To do this, a transformation of the coordinate frame fixed to the case to the position of

the rotor is required with a rotation about the y axis first, then about the x and z axes. This sequence, shown in Figure A.4, is given by

$$A'' = (\gamma''), (\phi'')_r(\eta)_y \quad (\text{A.18})$$

where η is the y axis pickoff angle. Equation A.18 can be expanded to give

$$A'' = \begin{pmatrix} C_{\gamma''}C_{\eta} + S_{\gamma''}S_{\phi''}S_{\eta} & S_{\gamma''}C_{\phi''} & -C_{\gamma''}S_{\eta} + S_{\gamma''}S_{\phi''}C_{\eta} \\ -S_{\gamma''}C_{\eta} + C_{\gamma''}S_{\phi''}S_{\eta} & C_{\gamma''}C_{\phi''} & S_{\gamma''}S_{\eta} + C_{\gamma''}S_{\eta} + C_{\gamma''}S_{\phi''}C_{\eta} \\ C_{\phi''}S_{\eta} & -S_{\phi''} & C_{\phi''}C_{\eta} \end{pmatrix} \quad (\text{A.19})$$

Equating this to Equation A.3 gives

$$T_{\eta} = \frac{T_{\theta'}}{C_{\phi'}} \quad (\text{A.20})$$

Taking the derivative of Equation A.20 and solving for $\dot{\theta}'$:

$$\dot{\theta}' = C_{\theta'}^2 \left(\frac{C_{\phi'}}{C_{\eta}^2} \dot{\eta} - S_{\phi'} T_{\eta} \dot{\phi}' \right) \quad (\text{A.21})$$

Therefore, it is possible to relate the gimbal angles and velocities with the measurements from the pickoffs.

Figure A.4: Comparison between 'pickoff' and 'case' angles. The angles ϕ' and θ' are 'case' angles while η and ϕ' are pickoff angles.

Appendix B

Working Drawings and Parts List for the Test Crossed Flexure Pivot

During the course of this research, questions were raised as to the validity of the beam assumption when strips as wide as those on the MMS crossed flexure pivots are used. It was also observed that while some test results exist in the literature, only certain cases were evaluated and those involved a pure torque applied to the system. These two facts raise the question as to the accuracy of the theoretical model developed in Chapter 4. For this reason, a crossed flexure pivot was designed, manufactured and tested. This appendix will focus on the first of these.

It was desired that the crossed flexure pivot used in the tests be proportionally similar to the pivots used in the Momentum Management System (MMS). It should be noticed, however, that the scale of the MMS pivots is such that very accurate machining techniques are required for its manufacture. In order to allow parts to be milled rather than EDM'ed all the dimensions of the MMS pivot were multiplied by a factor of 10. Similarly, in

order to facilitate the machining of the parts, the dimensions were converted from metric to imperial. This was due to the fact that the milling machine worked in decimal inches.

During the conversion process it was realized that a slight change in the width of the strips from 2.53 mm to 2.54 mm would mean that the width would be 1 inch instead of some fraction thereof. When the ultra high molecular weight polyethylene (UHMWPE) was measured it was found that the 'as is' thickness was 1.54 mm (as compared to the ideal thickness of 1.5 mm). Therefore, in order to simplify the machining of the parts, the width and thickness of the strips were changed to 2.54 mm and 1.54 mm, respectively.

The test pivot is made of three different parts; the strips, the rigid bodies which connect the ends of the strips and small blocks used to attach the strips to the rigid bodies. The working drawings for each of these parts are found in Figures B.1 to B.3. A parts list is found in Table B.1. The strips were attached to the rigid bodies using common robertson round head screws. Eye hooks attached to the two rigid bodies provided connection points for the Instron test rig. Each part was milled to the proper dimensions, including the UHMWPE. See Figure B.4 for a picture showing the assembled pivot.

Figure B.1: Drawing of the rigid body used to attach the ends of the strips.
The part is make out of 6061 T651 Aluminum.

PART B
DRAWING TO SCALE
ALL DIMENSIONS IN INCHES
MAY 21, 1998 J. CAIN

Figure B.2: Drawing of the strips of the crossed flexure pivot. The part is make out of ultra high molecular weight polyethylene.

PART C
DRAWING TO SCALE
ALL DIMENSIONS IN INCHES
MAY 21, 1998 J.CAIN

Figure B.3: Drawing of the blocks used to attach the strips and the rigid bodies. The part is make out of 6061 T651 Aluminum.

<i>Part</i>	<i>Quantity</i>	<i>Material</i>
≈ 10 32UNF $\times \frac{3}{4}$ round head bolt	8	1020 Steel [†]
$\frac{1}{4}$ 28UNF $\times 2\frac{1}{2}$ eye bolt	2	1020 Steel [†]
A	2	6061 T651 Aluminum
B	1	UHMWPE [‡]
C	8	6061 T651 Aluminum

Table B.1: Parts list for the crossed flexure pivot used in the tensile tests.

[†]This is a minimum requirement. Standard “off the shelf” bolts can be used.

[‡]UHMWPE stands for ultra high molecular weight polyethylene.

Figure B.4: Photograph showing the assembled crossed axis pivot.

Appendix C

Derivation of the Equations of Motion using Lagrange's Equations

The equations of motion for the MMS can be calculated using Lagrange's Equation, Equation 9.14, and the appropriate partial derivatives. Four cases must be considered, one for each of the generalized coordinates.

C.1 Case I: $q = \theta$

When $q = \theta$, Lagrange's Equation, Equation 9.14, becomes

$$\begin{aligned}
 I_{r_y} \dot{\omega}_{r_y} - I_{r_{xy}} \dot{\omega}_{r_x} - I_{r_{yx}} \dot{\omega}_{r_x} = & \\
 I_{r_x} \omega_{r_x} \omega_{r_x} - I_{r_x} \omega_{r_x} \omega_{r_x} + I_{r_{xy}} \omega_{r_y} \omega_{r_x} + I_{r_{xz}} \omega_{r_x}^2 & \\
 - I_{r_{xz}} \omega_{r_x}^2 - I_{r_{yz}} \omega_{r_x} \omega_{r_y} - 2k_y \theta - 2D_y \dot{\theta} & \\
 + T_x \frac{[C_\theta C_\phi (-S_\theta S_\phi C_\Omega - C_\theta S_\Omega) + S_\theta C_\phi (C_\theta S_\phi C_\Omega - S_\theta S_\Omega)]}{C_\theta^2 C_\phi^2 + (C_\theta S_\phi C_\Omega - S_\theta S_\Omega)^2} & \\
 + T_y \frac{[C_\theta C_\phi (C_\theta C_\Omega - S_\theta S_\phi S_\Omega) + S_\theta C_\phi (S_\theta C_\Omega + C_\theta S_\phi S_\Omega)]}{C_\theta^2 C_\phi^2 + (S_\theta C_\Omega + C_\theta S_\phi S_\Omega)} & \quad (C.1)
 \end{aligned}$$

where Equation C.1 is based on the following partial derivatives

$$\frac{\partial \omega_{r_x}}{\partial \theta} = -\omega_{r_z} \quad \frac{\partial \omega_{r_z}}{\partial \theta} = \omega_{r_x} \quad (\text{C.2})$$

$$\frac{\partial \omega_{r_y}}{\partial \theta} = 1 \quad (\text{C.3})$$

and the remaining partial derivatives are zero. Substituting the angular accelerations of the rotor, Equation 9.10, into Equation C.1 gives

$$\begin{aligned} & I_{r_y} \left(\ddot{\theta} + S_\phi \ddot{\Omega} + \dot{\phi} \omega_{g_z} - C_\phi \dot{\Omega} \omega_{g_z} \right) \\ & - I_{r_{xy}} \left(-\dot{\theta} \omega_{r_z} + C_\theta \ddot{\phi} - S_\theta C_\phi \ddot{\Omega} + S_\theta \dot{\phi} \omega_{g_y} + (C_\theta \omega_{g_y} - S_\theta S_\phi \omega_{g_z}) \dot{\Omega} \right) \\ & - I_{r_{yz}} \left(\dot{\theta} \omega_{r_x} + S_\theta \ddot{\phi} + C_\theta C_\phi \ddot{\Omega} - C_\theta \dot{\phi} \omega_{g_y} + (S_\theta \omega_{g_y} + C_\theta S_\phi \omega_{g_z}) \dot{\Omega} \right) \\ & \quad \left(I_{r_x} \omega_{r_z} - I_{r_{xz}} \omega_{r_z} - I_{r_{yz}} \omega_{r_y} \right) \omega_{r_x} - \left(I_{r_z} \omega_{r_x} - I_{r_{xz}} \omega_{r_x} - I_{r_{yz}} \omega_{r_z} \right) \omega_{r_z} \\ & - T_x \frac{C_\phi S_\Omega}{C_\theta^2 C_\phi^2 + (C_\theta S_\phi C_\Omega - S_\theta S_\Omega)^2} + T_y \frac{C_\phi C_\Omega}{C_\theta^2 C_\phi^2 + (S_\theta C_\Omega + C_\theta S_\phi S_\Omega)^2} \\ & - 2k_y \theta - 2D_y \dot{\theta} \end{aligned} \quad (\text{C.4})$$

The above expression can be rewritten and simplified to give

$$\begin{aligned} & \ddot{\theta} I_{r_y} - \ddot{\phi} \left(C_\theta I_{r_{xy}} + S_\theta I_{r_{yz}} \right) + \ddot{\Omega} \left[S_\phi I_{r_y} + C_\phi \left(S_\theta I_{r_{xy}} - C_\theta I_{r_{yz}} \right) \right] \\ & - H_{r_x} \omega_{r_x} - H_{r_z} \omega_{r_z} + \dot{\theta} \left(I_{r_{yz}} \omega_{r_x} - I_{r_{xz}} \omega_{r_z} \right) \\ & + \dot{\phi} \left[\left(S_\theta I_{r_{xy}} - C_\theta I_{r_{yz}} \right) \omega_{g_y} - I_{r_y} \omega_{g_z} \right] \\ & + \dot{\Omega} \left[\left(C_\phi I_{r_y} - S_\phi \left(S_\theta I_{r_{xy}} - C_\theta I_{r_{yz}} \right) \right) \omega_{g_z} + \left(C_\theta I_{r_{xy}} + S_\theta I_{r_{yz}} \right) \omega_{g_y} \right] \\ & - T_x \frac{C_\phi S_\Omega}{C_\theta^2 C_\phi^2 + (C_\theta S_\phi C_\Omega - S_\theta S_\Omega)^2} + T_y \frac{C_\phi C_\Omega}{C_\theta^2 C_\phi^2 + (S_\theta C_\Omega + C_\theta S_\phi S_\Omega)^2} \\ & - 2k_y \theta - 2D_y \dot{\theta} \end{aligned} \quad (\text{C.5})$$

where H_{r_x} and H_{r_z} represent the angular momentum of the rotor about the x and y axes, respectively. Let

$$A = I_{r_y}$$

$$\begin{aligned}
 B &= C_\theta I_{r_{zy}} + S_\theta I_{r_{yz}} \\
 C &= S_\phi I_{r_y} + C_\phi (S_\theta I_{r_{zy}} - C_\theta I_{r_{yz}}) \\
 D &= H_{r_x} \omega_{r_x} - H_{r_z} \omega_{r_z} + \dot{\theta} (I_{r_{yz}} \omega_{r_z} - I_{r_{zy}} \omega_{r_z}) \\
 &+ \dot{\phi} [(S_\theta I_{r_{zy}} - C_\theta I_{r_{yz}}) \omega_{g_y} - I_{r_y} \omega_{g_z}] \\
 &+ \dot{\Omega} [(C_\phi I_{r_y} - S_\phi (S_\theta I_{r_{zy}} - C_\theta I_{r_{yz}})) \omega_{g_x} + B \omega_{g_y}] \\
 &- T_x \frac{C_\phi S_\Omega}{C_\theta^2 C_\phi^2 + (C_\theta S_\phi C_\Omega - S_\theta S_\Omega)^2} \\
 &+ T_y \frac{C_\phi C_\Omega}{C_\theta^2 C_\phi^2 + (S_\theta C_\Omega + C_\theta S_\phi S_\Omega)^2} \\
 &- 2k_y \theta - 2D_y \dot{\theta}
 \end{aligned}$$

It is possible to rewrite Equation C.5 as

$$A\ddot{\theta} - B\ddot{\phi} + C\ddot{\Omega} = D \quad (C.6)$$

C.2 Case II: $q = \phi$

When $q = \phi$, the partial derivatives of the body angular rates become

$$\begin{aligned}
 \frac{\partial \omega_{g_y}}{\partial \phi} &= \omega_{g_z} & \frac{\partial \omega_{g_z}}{\partial \phi} &= -\omega_{g_y} \\
 \frac{\partial \omega_{r_x}}{\partial \phi} &= S_\theta \omega_{g_y} & \frac{\partial \omega_{r_y}}{\partial \phi} &= \omega_{g_z} & \frac{\partial \omega_{r_z}}{\partial \phi} &= -C_\theta \omega_{g_y}
 \end{aligned} \quad (C.7)$$

$$\begin{aligned}
 \frac{\partial \omega_{g_x}}{\partial \phi} &= 1 \\
 \frac{\partial \omega_{r_x}}{\partial \phi} &= C_\theta & \frac{\partial \omega_{r_z}}{\partial \phi} &= S_\theta
 \end{aligned} \quad (C.8)$$

and the remaining partial derivatives are zero. Substituting the angular accelerations of the rotor, Equation C.7, and the gimbal, Equation C.8, into

Lagrange's Equation, Equation 9.14, yields

$$\begin{aligned}
 & \frac{d}{dt} \left(C_\theta I_{r_x} \dot{\omega}_{r_x} + S_\theta I_{r_x} \dot{\omega}_{r_z} + I_{g_x} \dot{\omega}_{g_x} - C_\theta I_{r_{xy}} \dot{\omega}_{r_y} - C_\theta I_{r_{xz}} \dot{\omega}_{r_z} \right. \\
 & \quad \left. - S_\theta I_{r_{xz}} \dot{\omega}_{r_x} - S_\theta I_{r_{yz}} \dot{\omega}_{r_y} - I_{g_{xy}} \dot{\omega}_{g_y} - I_{g_{xz}} \dot{\omega}_{g_z} \right) \\
 & \quad S_\theta I_{r_x} \dot{\omega}_{r_x} \omega_{g_y} + I_{r_y} \dot{\omega}_{r_y} \omega_{g_z} - C_\theta I_{r_x} \dot{\omega}_{r_x} \omega_{g_y} + I_{g_y} \dot{\omega}_{g_y} \omega_{g_z} \\
 & \quad - I_{g_x} \dot{\omega}_{g_x} \omega_{g_y} - S_\theta I_{r_{xy}} \dot{\omega}_{r_y} \omega_{g_y} - I_{r_{xy}} \dot{\omega}_{r_x} \omega_{g_z} - S_\theta I_{r_{xz}} \dot{\omega}_{r_x} \omega_{g_y} \\
 & \quad + C_\theta I_{r_{xz}} \dot{\omega}_{r_x} \omega_{g_y} - I_{r_{yz}} \dot{\omega}_{r_z} \omega_{g_z} + C_\theta I_{r_{yz}} \dot{\omega}_{r_y} \omega_{g_y} - I_{g_{xy}} \dot{\omega}_{g_x} \omega_{g_z} \\
 & \quad + I_{g_{xz}} \dot{\omega}_{g_y} \omega_{g_z} - I_{g_y} \dot{\omega}_{g_x}^2 + I_{g_y} \dot{\omega}_{g_y}^2 - 2k_x \phi - 2D_x \dot{\phi} \\
 & \quad + T_x \frac{C_\theta^2 C_\Omega - \frac{1}{2} S_{2\theta} S_\phi S_\Omega}{C_\theta^2 C_\phi^2 + (C_\theta S_\phi C_\Omega - S_\theta S_\Omega)^2} \\
 & \quad + T_y \frac{C_\theta^2 S_\Omega + \frac{1}{2} S_{2\theta} S_\phi C_\Omega}{C_\theta^2 C_\phi^2 + (S_\theta C_\Omega + C_\theta S_\phi S_\Omega)^2} \quad (C.9)
 \end{aligned}$$

The above expression can be rearranged to obtain

$$\begin{aligned}
 & \dot{\theta} S_\theta (I_{r_{xy}} \dot{\omega}_{r_y} + I_{r_{xz}} \dot{\omega}_{r_z} - I_{r_x} \dot{\omega}_{r_x}) - \dot{\theta} C_\theta (I_{r_{xz}} \dot{\omega}_{r_x} + I_{r_{yz}} \dot{\omega}_{r_y} - I_{r_x} \dot{\omega}_{r_x}) \\
 & \quad + (C_\theta I_{r_x} - S_\theta I_{r_{xz}}) \dot{\omega}_{r_x} - (C_\theta I_{r_{xy}} + S_\theta I_{r_{yz}}) \dot{\omega}_{r_y} + (S_\theta I_{r_x} - C_\theta I_{r_{xz}}) \dot{\omega}_{r_z} \\
 & \quad + I_{g_x} \dot{\omega}_{g_x} - I_{g_{xy}} \dot{\omega}_{g_y} - I_{g_{xz}} \dot{\omega}_{g_z} \\
 & \quad - S_\theta \omega_{g_y} (I_{r_x} \dot{\omega}_{r_x} - I_{r_{xy}} \dot{\omega}_{r_y} - I_{r_{xz}} \dot{\omega}_{r_z}) + \omega_{g_x} (I_{r_y} \dot{\omega}_{r_y} - I_{r_{xy}} \dot{\omega}_{r_x} - I_{r_{yz}} \dot{\omega}_{r_z}) \\
 & \quad - C_\theta \omega_{g_y} (I_{r_x} \dot{\omega}_{r_x} - I_{r_{xz}} \dot{\omega}_{r_z} - I_{r_{yz}} \dot{\omega}_{r_y}) + \omega_{g_x} (I_{g_y} \dot{\omega}_{g_y} - I_{g_{xy}} \dot{\omega}_{g_x} - I_{g_{yz}} \dot{\omega}_{g_z}) \\
 & \quad - \omega_{g_y} (I_{g_x} \dot{\omega}_{g_x} - I_{g_{xz}} \dot{\omega}_{g_z} - I_{g_{xy}} \dot{\omega}_{g_y}) + T_x \frac{C_\theta^2 C_\Omega - \frac{1}{2} S_{2\theta} S_\phi S_\Omega}{C_\theta^2 C_\phi^2 + (C_\theta S_\phi C_\Omega - S_\theta S_\Omega)^2} \\
 & \quad + T_y \frac{C_\theta^2 S_\Omega + \frac{1}{2} S_{2\theta} S_\phi C_\Omega}{C_\theta^2 C_\phi^2 + (S_\theta C_\Omega + C_\theta S_\phi S_\Omega)^2} - 2k_x \phi - 2D_x \dot{\phi}
 \end{aligned}$$

Substituting Equations 9.10 and 9.8 into the above equation gives the following result

$$\dot{\theta} (C_\theta H_{r_x} - S_\theta H_{r_z}) + I_{g_x} (\ddot{\phi} + \omega_{s_y} \dot{\Omega})$$

$$\begin{aligned}
 & + (C_\theta I_{r_x} - S_\theta I_{r_{xz}}) \left(-\dot{\theta} \omega_{r_x} + C_\theta \ddot{\phi} - S_\theta C_\phi \ddot{\Omega} \right. \\
 & \quad \left. + S_\theta \dot{\phi} \omega_{g_v} + (C_\theta \omega_{s_v} - S_\theta S_\phi \omega_{s_x}) \dot{\Omega} \right) \\
 & - (C_\theta I_{r_{xv}} + S_\theta I_{r_{vz}}) \left(\ddot{\theta} + S_\phi \ddot{\Omega} + \dot{\phi} \omega_{g_z} - C_\phi \dot{\Omega} \omega_{s_x} \right) \\
 & + (S_\theta I_{r_x} - C_\theta I_{r_{xz}}) \left(\dot{\theta} \omega_{r_x} + S_\theta \ddot{\phi} + C_\theta C_\phi \ddot{\Omega} \right. \\
 & \quad \left. - C_\theta \dot{\phi} \omega_{g_v} + (S_\theta \omega_{s_v} + C_\theta S_\phi \omega_{s_x}) \dot{\Omega} \right) \\
 & - I_{g_{xv}} \left(\dot{\phi} \omega_{g_z} - C_\phi \dot{\Omega} \omega_{s_x} + S_\phi \dot{\Omega} \right) - I_{g_{xz}} \left(-\dot{\phi} \omega_{g_v} + C_\phi \dot{\Omega} + S_\phi \dot{\Omega} \omega_{s_x} \right) \\
 & \quad (S_\theta H_{r_x} - C_\theta H_{r_z} - H_{g_z}) \omega_{g_v} + (H_{r_v} + H_{g_v}) \omega_{g_z} - 2k_x \dot{\phi} - 2D_x \dot{\phi} \\
 & + T_x \frac{C_\theta^2 C_\phi^2 \dot{\Omega} - \frac{1}{2} S_{2\theta} S_\phi S_\Omega}{C_\theta^2 C_\phi^2 + (C_\theta S_\phi C_\phi \dot{\Omega} - S_\theta S_\Omega)^2} + T_y \frac{C_\theta^2 S_\Omega + \frac{1}{2} S_{2\theta} S_\phi C_\phi \dot{\Omega}}{C_\theta^2 C_\phi^2 + (S_\theta C_\phi \dot{\Omega} + C_\theta S_\phi S_\Omega)^2}
 \end{aligned}$$

where, for example, H_{r_v} and H_{g_v} represent the angular momenta of the rotor and gimbal, respectively, about their y axes. This equation can be simplified to give

$$\begin{aligned}
 & \ddot{\phi} \left(I_{g_x} + C_\theta^2 I_{r_x} + S_\theta^2 I_{r_z} - S_{2\theta} I_{r_{xz}} \right) - \ddot{\theta} \left(C_\theta I_{r_{xv}} + S_\theta I_{r_{vz}} \right) \\
 & + \ddot{\Omega} \left[\frac{1}{2} S_{2\theta} C_\phi (I_{r_x} - I_{r_z}) - C_{2\theta} C_\phi I_{r_{xz}} \right. \\
 & \quad \left. - S_\phi (C_\theta I_{r_{xv}} + S_\theta I_{r_{vz}} + I_{g_{xv}}) - C_\phi I_{g_{xz}} \right] \\
 & - \dot{\theta} [S_\theta H_{r_x} - C_\theta H_{r_z} + (C_\theta I_{r_x} - S_\theta I_{r_{xz}}) \omega_{r_x} - (S_\theta I_{r_x} - C_\theta I_{r_{xz}}) \omega_{r_z}] \\
 & - \dot{\phi} \left[(C_\theta I_{r_{xv}} + S_\theta I_{r_{vz}} + I_{g_{xv}}) \omega_{g_z} \right. \\
 & \quad \left. - \left(\frac{1}{2} S_{2\theta} (I_{r_x} - I_{r_z}) + C_{2\theta} I_{r_{xz}} + I_{g_{xz}} \right) \omega_{g_v} \right] \\
 & + \dot{\Omega} \left[\omega_{s_v} \left(I_{g_x} + C_\theta^2 I_{r_x} + S_\theta^2 I_{r_z} - S_{2\theta} I_{r_{xz}} \right) \right. \\
 & \quad \left. + \omega_{s_x} \left(S_\phi \left(\frac{1}{2} S_{2\theta} (I_{r_x} - I_{r_z}) - C_{2\theta} I_{r_{xz}} - I_{g_{xz}} \right) \right. \right. \\
 & \quad \left. \left. + C_\phi (C_\theta I_{r_{xv}} + S_\theta I_{r_{vz}} + I_{g_{xv}}) \right) \right] \\
 & - (S_\theta H_{r_x} - C_\theta H_{r_z} - H_{g_z}) \omega_{g_v} + (H_{r_v} + H_{g_v}) \omega_{g_z} - 2k_x \dot{\phi} - 2D_x \dot{\phi}
 \end{aligned}$$

$$+ T_x \frac{C_\theta^2 C_\Omega - \frac{1}{2} S_{2\theta} S_\phi S_\Omega}{C_\theta^2 C_\phi^2 + (C_\theta S_\phi C_\Omega - S_\theta S_\Omega)^2} T_y \frac{C_\theta^2 S_\Omega + \frac{1}{2} S_{2\theta} S_\phi C_\Omega}{C_\theta^2 C_\phi^2 + (S_\theta C_\Omega + C_\theta S_\phi S_\Omega)^2}$$

Isolating $\ddot{\theta}$, $\ddot{\phi}$ and $\ddot{\Omega}$ on the left hand side of the equation yields

$$\begin{aligned} & \ddot{\phi} \left(I_{g_x} + C_\theta^2 I_{r_x} + S_\theta^2 I_{r_z} - S_{2\theta} I_{r_{xz}} \right) - \ddot{\theta} \left(C_\theta I_{r_{xy}} + S_\theta I_{r_{yz}} \right) \\ & + \ddot{\Omega} \left[C_\phi \left(\frac{1}{2} S_{2\theta} (I_{r_z} - I_{r_x}) - C_{2\theta} I_{r_{xz}} - I_{g_{xz}} \right) \right. \\ & \quad \left. - S_\phi (C_\theta I_{r_{xy}} + S_\theta I_{r_{yz}} + I_{g_{xy}}) \right] \\ & - \dot{\theta} [S_\theta H_{r_x} - C_\theta H_{r_z} + (C_\theta I_{r_x} - S_\theta I_{r_{xz}}) \omega_{r_x} - (S_\theta I_{r_z} - C_\theta I_{r_{xz}}) \omega_{r_z}] \\ & + \dot{\phi} \left[(C_\theta I_{r_{xy}} + S_\theta I_{r_{yz}} + I_{g_{xy}}) \omega_{g_x} \right. \\ & \quad \left. - \left(\frac{1}{2} S_{2\theta} (I_{r_z} - I_{r_x}) + C_{2\theta} I_{r_{xz}} + I_{g_{xz}} \right) \omega_{g_y} \right] \\ & - \dot{\Omega} \left[\omega_{s_y} (I_{g_x} + C_\theta^2 I_{r_x} + S_\theta^2 I_{r_z} - S_{2\theta} I_{r_{xz}}) \right. \\ & \quad + \omega_{s_z} \left(S_\phi \left(\frac{1}{2} S_{2\theta} (I_{r_z} - I_{r_x}) - C_{2\theta} I_{r_{xz}} - I_{g_{xz}} \right) \right. \\ & \quad \left. \left. + C_\phi (C_\theta I_{r_{xy}} + S_\theta I_{r_{yz}} + I_{g_{xy}}) \right) \right] \\ & + T_x \frac{C_\theta^2 C_\Omega - \frac{1}{2} S_{2\theta} S_\phi S_\Omega}{C_\theta^2 C_\phi^2 + (C_\theta S_\phi C_\Omega - S_\theta S_\Omega)^2} \\ & + T_y \frac{C_\theta^2 S_\Omega + \frac{1}{2} S_{2\theta} S_\phi C_\Omega}{C_\theta^2 C_\phi^2 + (S_\theta C_\Omega + C_\theta S_\phi S_\Omega)^2} \\ & + (S_\theta H_{r_x} - C_\theta H_{r_z} - H_{g_x}) \omega_{g_y} \\ & + (H_{r_y} + H_{g_y}) \omega_{g_z} - 2k_x \phi - 2D_x \dot{\phi} \end{aligned} \tag{C.10}$$

By setting

$$\begin{aligned} E &= I_{g_x} + C_\theta^2 I_{r_x} + S_\theta^2 I_{r_z} - S_{2\theta} I_{r_{xz}} \\ F &= C_\phi \left(\frac{1}{2} S_{2\theta} (I_{r_z} - I_{r_x}) - C_{2\theta} I_{r_{xz}} - I_{g_{xz}} \right) - S_\phi (C_\theta I_{r_{xy}} + S_\theta I_{r_{yz}} + I_{g_{xy}}) \\ G &= \dot{\theta} [S_\theta H_{r_x} - C_\theta H_{r_z} + (C_\theta I_{r_x} - S_\theta I_{r_{xz}}) \omega_{r_x} - (S_\theta I_{r_z} - C_\theta I_{r_{xz}}) \omega_{r_z}] \\ &+ \dot{\phi} \left[(C_\theta I_{r_{xy}} + S_\theta I_{r_{yz}} + I_{g_{xy}}) \omega_{g_x} \right. \end{aligned}$$

$$\begin{aligned}
 & - \left(\frac{1}{2} S_{2\theta} (I_{r_z} - I_{r_x}) + C_{2\theta} I_{r_{xz}} + I_{g_{xz}} \right) \omega_{g_y} \Big] \\
 & - \Omega \left[E \omega_{g_y} + \omega_{s_z} \left(S_\phi \left(\frac{1}{2} S_{2\theta} (I_{r_z} - I_{r_x}) - C_{2\theta} I_{r_{xz}} - I_{g_{xz}} \right) \right. \right. \\
 & \quad \left. \left. + C_\phi \left(C_\theta I_{r_{xy}} + S_\theta I_{r_{yz}} + I_{g_{xy}} \right) \right) \right] \\
 & + (S_\theta H_{r_x} - C_\theta H_{r_z} - H_{g_x}) \omega_{g_y} + (H_{r_y} + H_{g_y}) \omega_{g_z} - 2k_x \phi - 2D_x \dot{\phi} \\
 & + T_x \frac{C_\theta^2 C_\Omega - \frac{1}{2} S_{2\theta} S_\phi S_\Omega}{C_\theta^2 C_\phi^2 + (C_\theta S_\phi C_\Omega - S_\theta S_\Omega)^2} + T_y \frac{C_\theta^2 S_\Omega + \frac{1}{2} S_{2\theta} S_\phi C_\Omega}{C_\theta^2 C_\phi^2 + (S_\theta C_\Omega + C_\theta S_\phi S_\Omega)^2}
 \end{aligned}$$

Equation C.10 can be rewritten as

$$-B\ddot{\theta} + E\ddot{\phi} + F\ddot{\Omega} = G \quad (\text{C.11})$$

C.3 Case III: $q = \Omega$

When $q = \Omega$, taking the partial derivatives of the shaft, vibration damper, gimbal and rotor motion, Equations 9.3, 9.5, 9.7 and 9.9, with respect to the shaft angle, Ω , gives the following equations

$$\begin{aligned}
 \frac{\partial \omega_{s_x}}{\partial \Omega} &= \omega_{s_y} & \frac{\partial \omega_{s_y}}{\partial \Omega} &= -\omega_{s_z} & \frac{\partial \omega_{s_z}}{\partial \Omega} &= 0 \\
 \frac{\partial \omega_{v_x}}{\partial \Omega} &= \omega_{v_y} & \frac{\partial \omega_{v_y}}{\partial \Omega} &= -\omega_{v_z} & \frac{\partial \omega_{v_z}}{\partial \Omega} &= 0 \\
 \frac{\partial \omega_{g_x}}{\partial \Omega} &= \omega_{s_y} & \frac{\partial \omega_{g_y}}{\partial \Omega} &= -C_\phi \omega_{s_z} & \frac{\partial \omega_{g_z}}{\partial \Omega} &= S_\phi \omega_{s_z} \\
 \frac{\partial \omega_{r_x}}{\partial \Omega} &= C_\theta \omega_{s_y} - S_\theta S_\phi \omega_{s_z} & \frac{\partial \omega_{r_y}}{\partial \Omega} &= -C_\phi \omega_{s_z} & \frac{\partial \omega_{r_z}}{\partial \Omega} &= S_\theta \omega_{s_y} + C_\theta S_\phi \omega_{s_z}
 \end{aligned} \quad (\text{C.12})$$

$$\begin{aligned}
 \frac{\partial \omega_{s_x}}{\partial \Omega} &= 0 & \frac{\partial \omega_{s_y}}{\partial \Omega} &= 0 & \frac{\partial \omega_{s_z}}{\partial \Omega} &= 1 \\
 \frac{\partial \omega_{v_x}}{\partial \Omega} &= 0 & \frac{\partial \omega_{v_y}}{\partial \Omega} &= 0 & \frac{\partial \omega_{v_z}}{\partial \Omega} &= 1 \\
 \frac{\partial \omega_{g_x}}{\partial \Omega} &= 0 & \frac{\partial \omega_{g_y}}{\partial \Omega} &= S_\phi & \frac{\partial \omega_{g_z}}{\partial \Omega} &= C_\phi \\
 \frac{\partial \omega_{r_x}}{\partial \Omega} &= -S_\theta C_\phi & \frac{\partial \omega_{r_y}}{\partial \Omega} &= S_\phi & \frac{\partial \omega_{r_z}}{\partial \Omega} &= C_\theta C_\phi
 \end{aligned} \quad (\text{C.13})$$

Substituting the above partial differentials into Equation 9.14 yields

$$\begin{aligned}
 \frac{d}{dt} & \left(-S_\theta C_\phi I_{r_x} \omega_{r_x} + S_\phi I_{r_y} \omega_{r_y} + C_\theta C_\phi I_{r_z} \omega_{r_z} + S_\phi I_{g_y} \omega_{g_y} + C_\phi I_{g_z} \omega_{g_z} \right. \\
 & \quad + I_{s_x} \omega_{s_x} + I_{v_z} \omega_{v_z} + S_\theta C_\phi I_{r_{xy}} \omega_{r_{xy}} - S_\phi I_{r_{yz}} \omega_{r_{yz}} + S_\theta C_\phi I_{r_{xz}} \omega_{r_{xz}} \\
 & \quad - C_\theta C_\phi I_{r_{xz}} \omega_{r_{xz}} - S_\phi I_{r_{yz}} \omega_{r_{yz}} - C_\theta C_\phi I_{r_{yz}} \omega_{r_{yz}} - S_\phi I_{g_{xy}} \omega_{g_{xy}} \\
 & \quad \left. - C_\phi I_{g_{xz}} \omega_{g_{xz}} - S_\phi I_{g_{yz}} \omega_{g_{yz}} - C_\phi I_{g_{yz}} \omega_{g_{yz}} \right) \\
 & \quad I_{r_x} \omega_{r_x} (C_\theta \omega_{s_y} - S_\theta S_\phi \omega_{s_x}) - C_\phi I_{r_y} \omega_{r_y} \omega_{s_x} + I_{r_z} \omega_{r_z} (S_\theta \omega_{s_y} + C_\theta S_\phi \omega_{s_x}) \\
 & \quad + I_{g_x} \omega_{g_x} \omega_{s_y} - C_\phi I_{g_y} \omega_{g_y} \omega_{s_x} + S_\phi I_{g_z} \omega_{g_z} \omega_{s_x} - (C_\theta \omega_{s_y} - S_\theta S_\phi \omega_{s_x}) I_{r_{xy}} \omega_{r_{xy}} \\
 & \quad + C_\phi I_{r_{xy}} \omega_{r_{xy}} \omega_{s_x} + C_\phi I_{r_{yz}} \omega_{r_{yz}} \omega_{s_x} - (S_\theta \omega_{s_y} + C_\theta S_\phi \omega_{s_x}) I_{r_{yz}} \omega_{r_{yz}} \\
 & \quad - (C_\theta \omega_{s_y} - S_\theta S_\phi \omega_{s_x}) I_{r_{xz}} \omega_{r_{xz}} - (S_\theta \omega_{s_y} + C_\theta S_\phi \omega_{s_x}) I_{r_{xz}} \omega_{r_{xz}} \\
 & \quad - I_{g_{xy}} \omega_{g_{xy}} \omega_{s_y} + C_\phi I_{g_{xy}} \omega_{g_{xy}} \omega_{s_x} - I_{g_{xz}} \omega_{g_{xz}} \omega_{s_y} \\
 & \quad - S_\phi I_{g_{xz}} \omega_{g_{xz}} \omega_{s_x} + C_\phi I_{g_{yz}} \omega_{g_{yz}} \omega_{s_x} - S_\phi I_{g_{yz}} \omega_{g_{yz}} \omega_{s_x} \\
 & \quad - T_x \frac{C_\theta^2 S_\phi C_\phi S_\Omega + S_\theta C_\theta C_\phi C_\Omega}{C_\theta^2 C_\phi^2 + (C_\theta S_\phi C_\Omega - S_\theta S_\Omega)^2} - T_y \frac{C_\theta S_\theta C_\phi S_\Omega - C_\theta^2 S_\phi C_\phi C_\Omega}{C_\theta^2 C_\phi^2 + (S_\theta C_\Omega + C_\theta S_\phi S_\Omega)^2} \\
 & \quad - K_m (\Omega - \Omega_s) + T_m
 \end{aligned} \tag{C.14}$$

Rearranging gives

$$\begin{aligned}
 & \dot{\phi} C_\phi (I_{r_y} \omega_{r_y} - I_{r_{xy}} \omega_{r_{xy}} - I_{r_{yz}} \omega_{r_{yz}}) - (S_\theta C_\phi I_{r_x} + C_\theta C_\phi I_{r_{xz}} + S_\phi I_{r_{xy}}) \dot{\omega}_{r_x} \\
 & \quad - (\dot{\theta} C_\theta C_\phi - \dot{\phi} S_\theta S_\phi) (I_{r_x} \omega_{r_x} - I_{r_{xy}} \omega_{r_{xy}} - I_{r_{xz}} \omega_{r_{xz}}) \\
 & \quad - (\dot{\theta} S_\theta C_\phi + \dot{\phi} C_\theta S_\phi) (I_{r_x} \omega_{r_x} - I_{r_{xz}} \omega_{r_{xz}} - I_{r_{yz}} \omega_{r_{yz}}) \\
 & \quad + \dot{\phi} C_\phi (I_{g_y} \omega_{g_y} - I_{g_{xy}} \omega_{g_{xy}} - I_{g_{yz}} \omega_{g_{yz}}) - \dot{\phi} S_\phi (I_{g_x} \omega_{g_x} - I_{g_{xz}} \omega_{g_{xz}} - I_{g_{yz}} \omega_{g_{yz}}) \\
 & \quad + (S_\phi I_{r_y} + S_\theta C_\phi I_{r_{xy}} - C_\theta C_\phi I_{r_{yz}}) \dot{\omega}_{r_y} \\
 & \quad + (C_\theta C_\phi I_{r_x} + S_\theta C_\phi I_{r_{xz}} - S_\phi I_{r_{yz}}) \dot{\omega}_{r_z} + (C_\phi I_{g_x} - S_\phi I_{g_{yz}}) \dot{\omega}_{g_x} \\
 & \quad - (S_\phi I_{g_{xy}} + C_\phi I_{g_{xz}}) \dot{\omega}_{g_z} + (S_\phi I_{g_y} - C_\phi I_{g_{yz}}) \dot{\omega}_{g_y} + I_{s_x} \dot{\omega}_{s_x} + I_{v_z} \dot{\omega}_{v_z} \\
 & \quad = (C_\theta \omega_{s_y} - S_\theta S_\phi \omega_{s_x}) (I_{r_x} \omega_{r_x} - I_{r_{xy}} \omega_{r_{xy}} - I_{r_{xz}} \omega_{r_{xz}})
 \end{aligned}$$

$$\begin{aligned}
 & -C_\phi \omega_{s_x} (I_{r_v} \omega_{r_v} - I_{r_{zv}} \omega_{r_z} - I_{r_{vz}} \omega_{r_z}) + \omega_{s_v} (I_{g_z} \omega_{g_z} - I_{g_{zv}} \omega_{g_v} - I_{g_{zv}} \omega_{g_z}) \\
 & + (S_\theta \omega_{s_v} + C_\theta S_\phi \omega_{s_x}) (I_{r_z} \omega_{r_z} - I_{r_{xz}} \omega_{r_x} - I_{r_{vz}} \omega_{r_v}) \\
 & - C_\phi \omega_{s_x} (I_{g_v} \omega_{g_v} - I_{g_{zv}} \omega_{g_z} - I_{g_{vz}} \omega_{g_z}) \\
 & + S_\phi \omega_{s_x} (I_{g_z} \omega_{g_z} - I_{g_{xz}} \omega_{g_x} - I_{g_{vz}} \omega_{g_v}) \\
 & - T_x \frac{C_\theta^2 S_\phi C_\phi S_\Omega + S_\theta C_\theta C_\phi C_\Omega}{C_\theta^2 C_\phi^2 + (C_\theta S_\phi C_\Omega - S_\theta S_\Omega)^2} - T_y \frac{C_\theta S_\theta C_\phi S_\Omega - C_\theta^2 S_\phi C_\phi C_\Omega}{C_\theta^2 C_\phi^2 + (S_\theta C_\Omega + C_\theta S_\phi S_\Omega)^2} \\
 & - K_m (\Omega - \Omega_s) + T_m
 \end{aligned} \tag{C.15}$$

Substituting the appropriate angular accelerations, obtained from Equations 9.4, 9.6, 9.8 and 9.10, into the above yields

$$\begin{aligned}
 & \dot{\phi} C_\phi H_{r_v} - (\dot{\theta} C_\theta C_\phi - \dot{\phi} S_\theta S_\phi) H_{r_z} \\
 & - (\dot{\theta} S_\theta C_\phi + \dot{\phi} C_\theta S_\phi) H_{r_x} + \dot{\phi} C_\phi H_{g_v} - \dot{\phi} S_\phi H_{g_z} \\
 & - (S_\theta C_\phi I_{r_x} + C_\theta C_\phi I_{r_{xz}} + S_\phi I_{r_{zv}}) (-\dot{\theta} \omega_{r_x} + C_\theta \ddot{\phi} \\
 & \quad - S_\theta C_\phi \ddot{\Omega} + S_\theta \dot{\phi} \omega_{g_v} + (C_\theta \omega_{s_v} - S_\theta S_\phi \omega_{s_x}) \dot{\Omega}) \\
 & + (S_\phi I_{r_v} + S_\theta C_\phi I_{r_{zv}} - C_\theta C_\phi I_{r_{vz}}) (\ddot{\theta} + S_\phi \ddot{\Omega} + \dot{\phi} \omega_{g_z} - C_\phi \dot{\Omega} \omega_{s_x}) \\
 & + (C_\theta C_\phi I_{r_x} + S_\theta C_\phi I_{r_{xz}} - S_\phi I_{r_{vz}}) (\dot{\theta} \omega_{r_x} + S_\theta \ddot{\phi} \\
 & \quad + C_\theta C_\phi \ddot{\Omega} - C_\theta \dot{\phi} \omega_{g_v} + (S_\theta \omega_{s_v} + C_\theta S_\phi \omega_{s_x}) \dot{\Omega}) \\
 & - (S_\phi I_{g_{zv}} + C_\phi I_{g_{zx}}) (\ddot{\phi} + \dot{\Omega} \omega_{s_v}) + I_{s_z} \ddot{\Omega} + I_{v_z} (\ddot{\Omega} + \ddot{\beta}) \\
 & + (S_\phi I_{g_v} - C_\phi I_{g_{vz}}) (\dot{\phi} \omega_{g_z} - C_\phi \dot{\Omega} \omega_{s_x} + S_\phi \ddot{\Omega}) \\
 & + (C_\phi I_{g_x} - S_\phi I_{g_{vz}}) (-\dot{\phi} \omega_{g_v} + C_\phi \ddot{\Omega} + S_\phi \dot{\Omega} \omega_{s_x}) \\
 & - (C_\theta \omega_{s_v} - S_\theta S_\phi \omega_{s_x}) H_{r_x} - C_\phi \omega_{s_x} H_{r_v} + (S_\theta \omega_{s_v} + C_\theta S_\phi \omega_{s_x}) H_{r_z} \\
 & + \omega_{s_v} H_{g_z} - C_\phi \omega_{s_x} H_{g_v} + S_\phi \omega_{s_x} H_{g_z} \\
 & - T_x \frac{C_\theta^2 S_\phi C_\phi S_\Omega + S_\theta C_\theta C_\phi C_\Omega}{C_\theta^2 C_\phi^2 + (C_\theta S_\phi C_\Omega - S_\theta S_\Omega)^2} - T_y \frac{C_\theta S_\theta C_\phi S_\Omega - C_\theta^2 S_\phi C_\phi C_\Omega}{C_\theta^2 C_\phi^2 + (S_\theta C_\Omega + C_\theta S_\phi S_\Omega)^2} \\
 & - K_m (\Omega - \Omega_s) + T_m
 \end{aligned} \tag{C.16}$$

It is now possible to solve for $\ddot{\theta}$, $\ddot{\phi}$, $\ddot{\Omega}$ and $\ddot{\beta}$ as follows

$$\begin{aligned}
 & \ddot{\theta} \left[S_{\phi} I_{r_v} + C_{\phi} (S_{\theta} I_{r_{zv}} - C_{\theta} I_{r_{vz}}) \right] \\
 & + \ddot{\phi} \left[C_{\phi} \left(\frac{1}{2} S_{2\theta} (I_{r_z} - I_{r_x}) - C_{2\theta} I_{r_{xz}} - I_{g_{xz}} \right) \right. \\
 & \quad \left. - S_{\phi} (C_{\theta} I_{r_{zv}} + S_{\theta} I_{r_{vz}} + I_{g_{zv}}) \right] \\
 & + \ddot{\Omega} \left[S_{2\phi} (S_{\theta} I_{r_{zv}} - C_{\theta} I_{r_{vz}}) + I_{s_z} + I_{v_z} - S_{2\phi} I_{g_{vz}} \right. \\
 & \quad \left. + C_{\phi}^2 (S_{\theta}^2 I_{r_z} + C_{\theta}^2 I_{r_x} + S_{2\theta} I_{r_{xz}} + I_{g_z}) + S_{\phi}^2 (I_{r_v} + I_{g_v}) \right] \\
 & + \ddot{\beta} I_{v_z} \\
 & \ddot{\theta} \left[C_{\phi} (C_{\theta} H_{r_z} + S_{\theta} H_{r_x}) - \omega_{r_z} (S_{\theta} I_{r_{zv}} + C_{\phi} (S_{\theta} I_{r_z} + C_{\theta} I_{r_{xz}})) \right. \\
 & \quad \left. + \omega_{r_x} (S_{\phi} I_{r_{vz}} - C_{\phi} (C_{\theta} I_{r_z} + S_{\theta} I_{r_{xz}})) \right] \\
 & + \ddot{\phi} \left[S_{\phi} (C_{\theta} H_{r_z} - S_{\theta} H_{r_x} + H_{g_z}) - C_{\phi} (H_{r_v} + H_{g_v}) \right. \\
 & \quad \left. + \omega_{g_v} (S_{\phi} (S_{\theta} I_{r_{zv}} - C_{\theta} I_{r_{vz}} - I_{g_{vz}}) \right. \\
 & \quad \left. + C_{\phi} (S_{\theta}^2 I_{r_z} + C_{\theta}^2 I_{r_x} + S_{2\theta} I_{r_{xz}} + I_{g_z})) \right. \\
 & \quad \left. - \omega_{g_z} (S_{\phi} (I_{r_v} + I_{g_v}) + C_{\phi} (S_{\theta} I_{r_{zv}} - C_{\theta} I_{r_{vz}} - I_{g_{vz}})) \right] \\
 & - \ddot{\Omega} \left[\omega_{s_z} \left((I_{g_{vz}} - S_{\theta} I_{r_{zv}} + C_{\theta} I_{r_{vz}}) C_{2\phi} \right. \right. \\
 & \quad \left. \left. + \frac{1}{2} S_{2\phi} (S_{\theta}^2 I_{r_z} + C_{\theta}^2 I_{r_x} + S_{2\theta} I_{r_{xz}} - I_{r_v} - I_{g_v} + I_{g_z}) \right) \right. \\
 & \quad \left. - \omega_{s_v} (S_{\phi} (C_{\theta} I_{r_{zv}} + S_{\theta} I_{r_{vz}} + I_{g_{zv}}) \right. \\
 & \quad \left. + C_{\phi} (C_{2\theta} I_{r_{xz}} + \frac{1}{2} S_{2\theta} (I_{r_z} - I_{r_x}) + I_{g_{xz}})) \right] \\
 & + \omega_{s_x} [S_{\phi} (H_{g_z} + C_{\theta} H_{r_z} - S_{\theta} H_{r_x}) - C_{\phi} (H_{r_v} - H_{g_v})] \\
 & + \omega_{s_v} (C_{\theta} H_{r_z} + S_{\theta} H_{r_x} + H_{g_z}) \\
 & - \frac{1}{2} T_x \frac{C_{\theta}^2 S_{2\phi} S_{\Omega} + S_{2\theta} C_{\phi} C_{\Omega}}{C_{\theta}^2 C_{\phi}^2 + (C_{\theta} S_{\phi} C_{\Omega} - S_{\theta} S_{\Omega})^2} - \frac{1}{2} T_y \frac{S_{2\theta} C_{\phi} S_{\Omega} - C_{\theta}^2 S_{2\phi} C_{\Omega}}{C_{\theta}^2 C_{\phi}^2 + (S_{\theta} C_{\Omega} + C_{\theta} S_{\phi} S_{\Omega})^2} \\
 & - K_m (\Omega - \Omega_s) + T_m
 \end{aligned} \tag{C.17}$$

Let

$$I = I_{v_i}$$

$$\begin{aligned}
 H &= S_{2\phi} (S_{\theta} I_{r_{zv}} - C_{\theta} I_{r_{vz}} - I_{g_{vz}}) + I_{s_z} + I_{v_z} \\
 &+ C_{\phi}^2 (S_{\theta}^2 I_{r_z} + C_{\theta}^2 I_{r_z} + S_{2\theta} I_{r_{zz}} + I_{g_z}) + S_{\phi}^2 (I_{r_v} + I_{g_v}) \\
 J &= \theta \left[C_{\phi} (C_{\theta} H_{r_z} + S_{\theta} H_{r_z}) - \omega_{r_z} (S_{\phi} I_{r_{zv}} + C_{\phi} (S_{\theta} I_{r_z} + C_{\theta} I_{r_{zz}})) \right. \\
 &\quad \left. + \omega_{r_z} (S_{\phi} I_{r_{vz}} - C_{\phi} (C_{\theta} I_{r_z} + S_{\theta} I_{r_{zz}})) \right] \\
 &+ \phi \left[S_{\phi} (C_{\theta} H_{r_z} - S_{\theta} H_{r_z} + H_{g_z}) - C_{\phi} (H_{r_v} + H_{g_v}) \right. \\
 &\quad \left. + \omega_{g_v} (S_{\phi} (S_{\theta} I_{r_{zv}} - C_{\theta} I_{r_{vz}} - I_{g_{vz}})) \right. \\
 &\quad \left. + C_{\phi} (S_{\theta}^2 I_{r_z} + C_{\theta}^2 I_{r_z} + S_{2\theta} I_{r_{zz}} + I_{g_z}) \right. \\
 &\quad \left. - \omega_{g_z} (S_{\phi} (I_{r_v} + I_{g_v}) + C_{\phi} (S_{\theta} I_{r_{zv}} - C_{\theta} I_{r_{vz}} - I_{g_{vz}})) \right] \\
 &- \Omega \left[\omega_{s_z} (C_{2\phi} (I_{g_{vz}} - S_{\theta} I_{r_{zv}} + C_{\theta} I_{r_{vz}}) \right. \\
 &\quad \left. + \frac{1}{2} S_{2\phi} (S_{\theta}^2 I_{r_z} + C_{\theta}^2 I_{r_z} + S_{2\theta} I_{r_{zz}} - I_{r_v} - I_{g_v} + I_{g_z})) \right. \\
 &\quad \left. - \omega_{s_v} (S_{\phi} (C_{\theta} I_{r_{zv}} + S_{\theta} I_{r_{vz}} + I_{g_{zv}}) \right. \\
 &\quad \left. + C_{\phi} (C_{2\theta} I_{r_{zz}} + \frac{1}{2} S_{2\theta} (I_{r_z} - I_{r_z}) + I_{g_{zz}})) \right] \\
 &+ \omega_{s_z} [S_{\phi} (H_{g_z} + C_{\theta} H_{r_z} - S_{\theta} H_{r_z}) - C_{\phi} (H_{r_v} - H_{g_v})] \\
 &+ \omega_{s_v} (C_{\theta} H_{r_z} + S_{\theta} H_{r_z} + H_{g_z}) \\
 &- \frac{1}{2} T_x \frac{C_{\theta}^2 S_{2\phi} S_{\Omega} + S_{2\theta} C_{\phi} C_{\Omega}}{C_{\theta}^2 C_{\phi}^2 + (C_{\theta} S_{\phi} C_{\Omega} - S_{\theta} S_{\Omega})^2} - \frac{1}{2} T_y \frac{S_{2\theta} C_{\phi} S_{\Omega} - C_{\theta}^2 S_{2\phi} C_{\Omega}}{C_{\theta}^2 C_{\phi}^2 + (S_{\theta} C_{\Omega} + C_{\theta} S_{\phi} S_{\Omega})^2} \\
 &- K_m (\Omega - \Omega_s) + T_m
 \end{aligned}$$

Therefore, Equation C.17 can be rewritten as

$$C\ddot{\theta} + F\ddot{\phi} + H\ddot{\Omega} + I\ddot{\beta} = J \quad (\text{C.18})$$

C.4 Case IV: $q = \beta$

When $q = \beta$, the partial derivatives of the body rates which are not zero are

$$\frac{\partial \omega_{v_x}}{\partial \beta} = \omega_{v_y} \quad \frac{\partial \omega_{v_y}}{\partial \beta} = -\omega_{v_x} \quad (\text{C.19})$$

$$\frac{\partial \omega_{v_z}}{\partial \beta} = 1 \quad (\text{C.20})$$

The above partial derivatives can be substituted into Lagrange's Equation (Equation 9.14) to obtain the following equation

$$I_{v_x} \dot{\omega}_{v_x} = -K_v \beta \quad (\text{C.21})$$

After simplification, this becomes

$$I_{v_x} \ddot{\beta} + I_{v_x} \ddot{\Omega} = -K_v \beta \quad (\text{C.22})$$

Let

$$K = -K_v \beta$$

Equation C.22 can be rewritten as

$$I \ddot{\beta} + I \ddot{\Omega} = K \quad (\text{C.23})$$

The result is four equations that can be rewritten in matrix form, as done in Chapter 9, Section 9.2.1.

Appendix D

Derivation of the Equations of Motions using Hamilton's Canonical Equations

It is possible to find the equations of motion for the MMS by substituting the appropriate partial derivatives into Hamilton's Canonical Equations, Equations 9.28 and 9.29. This process can be broken into four cases, one for each of the generalized coordinates.

D.1 Case I: $q = \theta$

If $q = \theta$ then the set of canonical equations have the form

$$\dot{P}_\theta = \frac{\partial \mathcal{L}}{\partial \theta} + Q \quad (\text{D.1})$$

and

$$P_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} \quad (\text{D.2})$$

In the Chapter 9, the partial derivatives of the angular rates with respect

to θ were found. Substituting these into Equation 9.29 gives

$$P_{\theta} = I_{r_y}\omega_{r_y} - I_{r_{zy}}\omega_{r_z} - I_{r_{yz}}\omega_{r_z} \quad (D.3)$$

Using Equations 9.7 and 9.9, the above expression can be written in terms of the generalized coordinates and their derivatives with respect to time

$$\begin{aligned} P_{\theta} = & I_{r_y}\dot{\theta} - (C_{\theta}I_{r_{zy}} + S_{\theta}I_{r_{yz}})(\dot{\phi} + \omega_{sz}) + I_{r_y}(C_{\phi}\omega_{sy} + S_{\phi}\omega_{sz}) \\ & + (S_{\theta}I_{r_{zy}} - C_{\theta}I_{r_{yz}})(C_{\phi}\omega_{sz} - S_{\phi}\omega_{sy}) \end{aligned} \quad (D.4)$$

After simplification, the final form of this expression is

$$\begin{aligned} I_{r_y}\dot{\theta} = & (C_{\theta}I_{r_{zy}} + S_{\theta}I_{r_{yz}})\dot{\phi} + [S_{\phi}I_{r_y} + C_{\phi}(S_{\theta}I_{r_{zy}} - C_{\theta}I_{r_{yz}})]\dot{\Omega} \\ P_{\theta} + & (C_{\theta}I_{r_{zy}} + S_{\theta}I_{r_{yz}})\omega_{sz} - [C_{\phi}I_{r_y} + S_{\phi}(S_{\theta}I_{r_{zy}} - C_{\theta}I_{r_{yz}})]\omega_{sy} \\ & - [S_{\phi}I_{r_y} + C_{\phi}(S_{\theta}I_{r_{zy}} - C_{\theta}I_{r_{yz}})]\omega_{sz} \end{aligned} \quad (D.5)$$

Let

$$A = I_{r_y}$$

$$B = C_{\theta}I_{r_{zy}} + S_{\theta}I_{r_{yz}}$$

$$C = S_{\phi}I_{r_y} + C_{\phi}(S_{\theta}I_{r_{zy}} - C_{\theta}I_{r_{yz}})$$

$$D_2 = P_{\theta} + B\omega_{sz} - [C_{\phi}I_{r_y} + S_{\phi}(S_{\theta}I_{r_{zy}} - C_{\theta}I_{r_{yz}})]\omega_{sy} - C\omega_{sz}$$

Equation D.5 can be rewritten as

$$A\dot{\theta} - B\dot{\phi} + C\dot{\Omega} = D_2 \quad (D.6)$$

The second canonical equation can be found using Equations C.2 and 9.28 and simplifying

$$\dot{P}_{\theta} = H_{r_z}\omega_{r_z} - H_{r_x}\omega_{r_x} - 2k_y\theta - 2D_y\dot{\theta}$$

$$\begin{aligned} & -T_x \frac{C_\phi S_\Omega}{C_\theta^2 C_\phi^2 + (C_\theta S_\phi C_\Omega - S_\theta S_\Omega)^2} \\ & + T_y \frac{C_\phi C_\Omega}{C_\theta^2 C_\phi^2 + (S_\theta C_\Omega + C_\theta S_\phi S_\Omega)^2} \end{aligned} \quad (D.7)$$

where, for example, H_{r_x} represents the angular momentum of the rotor about its x axis.

D.2 Case II: $q = \phi$

When $q = \phi$, the two canonical equations have the following forms

$$\dot{P}_\phi = \frac{\partial \mathcal{L}}{\partial \phi} + Q \quad (D.8)$$

$$P_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \quad (D.9)$$

The partial derivatives from Equation C.8 can be combined with Equation 9.29 to give

$$\begin{aligned} P_\phi &= (C_\theta I_{r_x} - S_\theta I_{r_{xx}}) \omega_{r_x} - (C_\theta I_{r_{xy}} + S_\theta I_{r_{yz}}) \omega_{r_y} \\ &+ (S_\theta I_{r_x} - C_\theta I_{r_{xx}}) \omega_{r_z} + I_{g_x} \omega_{g_x} - I_{g_{xy}} \omega_{g_y} - I_{g_{xz}} \omega_{g_z} \end{aligned} \quad (D.10)$$

Substituting for the body angular velocities, the above expression can be rewritten as

$$\begin{aligned} P_\phi &= - (C_\theta I_{r_{xy}} + S_\theta I_{r_{yz}}) \dot{\theta} + (C_\theta^2 I_{r_x} + S_\theta^2 I_{r_z} - S_{2\theta} I_{r_{xx}} + I_{g_x}) (\dot{\phi} + \omega_{s_x}) \\ &- (C_\theta I_{r_{xy}} + S_\theta I_{r_{yz}} + I_{g_{xy}}) (C_\phi \omega_{s_y} + S_\phi \omega_{s_z}) \\ &+ \left[\frac{1}{2} S_{2\theta} (I_{r_x} - I_{r_z}) - C_{2\theta} I_{r_{xx}} - I_{g_{xz}} \right] (C_\phi \omega_{s_z} - S_\phi \omega_{s_y}) \end{aligned} \quad (D.11)$$

Solving for $\dot{\theta}$, $\dot{\phi}$ and $\dot{\Omega}$, Equation D.11 can be rewritten as

$$(C_\theta^2 I_{r_x} + S_\theta^2 I_{r_z} - S_{2\theta} I_{r_{xx}} + I_{g_x}) \dot{\phi} - (C_\theta I_{r_{xy}} + S_\theta I_{r_{yz}}) \dot{\theta}$$

$$\begin{aligned}
 & + \left[C_\phi \left(\frac{1}{2} S_{2\theta} (I_{r_z} - I_{r_x}) - C_{2\theta} I_{r_{xz}} - I_{g_{xz}} \right) \right. \\
 & \quad \left. - S_\phi (C_\theta I_{r_{zv}} + S_\theta I_{r_{vz}} + I_{g_{zv}}) \right] \dot{\Omega} \\
 & - P_\phi + \left[C_\phi (C_\theta I_{r_{rv}} + S_\theta I_{r_{rz}} + I_{g_{rv}}) \right. \\
 & \quad \left. + S_\phi \left(\frac{1}{2} S_{2\theta} (I_{r_z} - I_{r_x}) - C_{2\theta} I_{r_{xz}} - I_{g_{xz}} \right) \right] \omega_{g_v} \\
 & - (C_\theta^2 I_{r_x} + S_\theta^2 I_{r_z} - S_{2\theta} I_{r_{xz}} + I_{g_x}) \omega_{s_x} \\
 & - \left[C_\phi \left(\frac{1}{2} S_{2\theta} (I_{r_z} - I_{r_x}) - C_{2\theta} I_{r_{xz}} - I_{g_{xz}} \right) \right. \\
 & \quad \left. - S_\phi (C_\theta I_{r_{zv}} + S_\theta I_{r_{vz}} + I_{g_{zv}}) \right] \omega_{c_z} \tag{D.12}
 \end{aligned}$$

Recall that

$$\begin{aligned}
 E &= C_\theta^2 I_{r_x} + S_\theta^2 I_{r_z} - S_{2\theta} I_{r_{xz}} + I_{g_x} \\
 F &= C_\phi \left(\frac{1}{2} S_{2\theta} (I_{r_z} - I_{r_x}) - C_{2\theta} I_{r_{xz}} - I_{g_{xz}} \right) \\
 & \quad - S_\phi (C_\theta I_{r_{zv}} + S_\theta I_{r_{vz}} + I_{g_{zv}})
 \end{aligned}$$

and let

$$\begin{aligned}
 G_2 &= P_\phi + \left[C_\phi (C_\theta I_{r_{rv}} + S_\theta I_{r_{rz}} + I_{g_{rv}}) \right. \\
 & \quad \left. + S_\phi \left(\frac{1}{2} S_{2\theta} (I_{r_z} - I_{r_x}) - C_{2\theta} I_{r_{xz}} - I_{g_{xz}} \right) \right] \omega_{g_v} \\
 & \quad - E \omega_{s_x} - F \omega_{c_z}
 \end{aligned}$$

Equation D.12 can be rewritten as

$$E\dot{\phi} - B\dot{\theta} + F\dot{\Omega} = G_2 \tag{D.13}$$

The second canonical equation can be found by using the partial differential equations calculated in Equation C.8 and substituting them into Equation 9.28 to give

$$\dot{P}_\phi = (S_\theta H_{r_x} - C_\theta H_{r_z} - H_{g_x}) \omega_{g_v} + (H_{r_v} + H_{g_v}) \omega_{g_x} - 2k_x \phi - 2D_x \dot{\phi}$$

$$\begin{aligned} & + T_x \frac{C_\theta^2 C_\Omega - \frac{1}{2} S_{2\theta} S_\phi S_\Omega}{C_\theta^2 C_\phi^2 + (C_\theta S_\phi C_\Omega - S_\theta S_\Omega)^2} \\ & + T_y \frac{C_\theta^2 S_\Omega + \frac{1}{2} S_{2\theta} S_\phi C_\Omega}{C_\theta^2 C_\phi^2 + (S_\theta C_\Omega + C_\theta S_\phi S_\Omega)^2} \end{aligned} \quad (D.14)$$

D.3 Case III: $q = \Omega$

When $q = \Omega$, Hamilton's Canonical Equations are of the form

$$\dot{P}_\Omega = \frac{\partial \mathcal{L}}{\partial \Omega} \quad (D.15)$$

and

$$P_\Omega = \frac{\partial \mathcal{L}}{\partial \dot{\Omega}} \quad (D.16)$$

Substituting the partial derivatives calculated in Equation C.13 into Equation 9.29 gives

$$\begin{aligned} P_\Omega &= - \left(S_\theta C_\phi I_{r_x} + S_\phi I_{r_{xy}} + C_\theta C_\phi I_{r_{xz}} \right) \omega_{r_x} \\ &+ \left(S_\phi I_{r_y} + S_\theta C_\phi I_{r_{xy}} - C_\theta C_\phi I_{r_{yz}} \right) \omega_{r_y} \\ &+ \left(C_\theta C_\phi I_{r_x} + S_\theta C_\phi I_{r_{xz}} - S_\phi I_{r_{yz}} \right) \omega_{r_z} - \left(S_\phi I_{g_{xy}} + C_\phi I_{g_{xz}} \right) \omega_{g_x} \\ &+ \left(S_\phi I_{g_y} - C_\phi I_{g_{yz}} \right) \omega_{g_y} + \left(C_\phi I_{g_z} - S_\phi I_{g_{yz}} \right) \omega_{g_z} \\ &+ I_{s_x} \omega_{s_x} + I_{v_x} \omega_{v_x} \end{aligned} \quad (D.17)$$

Substituting Equations 9.3, 9.5, 9.7 and 9.9 into Equation D.17 results in the following equation

$$\begin{aligned} P_\Omega &= \left[S_\phi I_{r_y} + C_\phi \left(S_\theta I_{r_{xy}} - C_\theta I_{r_{yz}} \right) \right] \dot{\theta} \\ &+ \left[C_\phi \left(S_\theta C_\theta I_{r_x} + S_\theta^2 I_{r_{xz}} \right) - S_\theta S_\phi I_{r_{yz}} - S_\phi C_\theta I_{r_{xy}} \right. \\ &\quad \left. - C_\phi \left(S_\theta C_\theta I_{r_x} + C_\theta^2 I_{r_{xz}} \right) - S_\phi I_{g_{xy}} - C_\phi I_{g_{xz}} \right] \left(\dot{\phi} + \omega_{s_x} \right) \end{aligned}$$

$$\begin{aligned}
 & + \left[S_\phi I_{r_v} + C_\phi (S_\theta I_{r_{zv}} - C_\theta I_{r_{vz}}) + S_\phi I_{g_v} - C_\phi I_{g_{vz}} \right] (C_\phi \omega_{s_v} + S_\phi \omega_{s_z}) \\
 & + \left[S_\phi S_\theta I_{r_{zv}} + C_\phi (S_\theta^2 I_{r_z} + S_\theta C_\theta I_{r_{zz}}) + C_\phi (C_\theta^2 I_{r_z} + S_\theta C_\theta I_{r_{zz}}) \right. \\
 & \quad \left. - S_\phi C_\theta I_{r_{vz}} + C_\phi I_{g_z} - S_\phi I_{g_{vz}} \right] (C_\phi \omega_{s_z} - S_\phi \omega_{s_v}) \\
 & + I_{v_z} (\dot{\beta} + \omega_{s_z}) + I_{s_z} \omega_{s_z}
 \end{aligned} \tag{D.18}$$

This can be simplified to give the following expression

$$\begin{aligned}
 & \left[S_\phi I_{r_v} + C_\phi (S_\theta I_{r_{zv}} - C_\theta I_{r_{vz}}) \right] \dot{\theta} \\
 & + \left[C_\phi \left(\frac{1}{2} S_{2\theta} (I_{r_z} - I_{r_x}) - C_{2\theta} I_{r_{zz}} - I_{g_{zz}} \right) \right. \\
 & \quad \left. - S_\phi (S_\theta I_{r_{vz}} + C_\theta I_{r_{zv}} + I_{g_{zv}}) \right] \dot{\phi} \\
 & + \left[S_\phi^2 (I_{r_v} + I_{g_v}) + S_{2\phi} (S_\theta I_{r_{zv}} - C_\theta I_{r_{vz}} - I_{g_{vz}}) \right. \\
 & \quad \left. + C_\phi^2 (S_\theta^2 I_{r_z} + C_\theta^2 I_{r_z} + S_{2\theta} I_{r_{zz}} + I_{g_z}) + I_{v_z} + I_{s_z} \right] \dot{\Omega} \\
 & + I_{v_z} \dot{\beta} \\
 & P_\Omega - \left[S_\phi^2 (I_{r_v} + I_{g_v}) + S_{2\phi} (S_\theta I_{r_{zv}} - C_\theta I_{r_{vz}} - I_{g_{vz}}) \right. \\
 & \quad \left. + C_\phi^2 (S_\theta^2 I_{r_z} + C_\theta^2 I_{r_z} + S_{2\theta} I_{r_{zz}} + I_{g_z}) + I_{v_z} + I_{s_z} \right] \omega_{c_z} \\
 & - \left[C_\phi \left(\frac{1}{2} S_{2\theta} (I_{r_z} - I_{r_x}) - C_{2\theta} I_{r_{zz}} - I_{g_{zz}} \right) \right. \\
 & \quad \left. - S_\phi (S_\theta I_{r_{vz}} + C_\theta I_{r_{zv}} + I_{g_{zv}}) \right] \omega_{s_z} \\
 & - \left[C_{2\phi} (S_\theta I_{r_{zv}} - C_\theta I_{r_{vz}} - I_{g_{vz}}) \right. \\
 & \quad \left. + \frac{1}{2} S_{2\phi} (I_{r_v} + I_{g_v} - S_\theta^2 I_{r_z} - C_\theta^2 I_{r_z} - S_{2\theta} I_{r_{zz}} - I_{g_z}) \right] \omega_{s_v} \tag{D.19}
 \end{aligned}$$

Let

$$I = I_{v_z}$$

$$\begin{aligned}
 H &= S_\phi^2 (I_{r_v} + I_{g_v}) + S_{2\phi} (S_\theta I_{r_{zv}} - C_\theta I_{r_{vz}} - I_{g_{vz}}) \\
 & \quad + C_\phi^2 (S_\theta^2 I_{r_z} + C_\theta^2 I_{r_z} + S_{2\theta} I_{r_{zz}} + I_{g_z}) + I_{v_z} + I_{s_z}
 \end{aligned}$$

$$\begin{aligned}
 J_2 = & P_\Omega - H\omega_{c_i} - F\omega_{s_x} \\
 & - \left[C_{2\phi} (S_\theta I_{r_{xy}} - C_\theta I_{r_{yz}} - I_{g_{yz}}) \right. \\
 & \left. + \frac{1}{2} S_{2\phi} (I_{r_v} + I_{g_v} - S_\theta^2 I_{r_x} - C_\theta^2 I_{r_z} - S_{2\theta} I_{r_{xz}} - I_{g_x}) \right] \omega_{s_v}
 \end{aligned}$$

Therefore, Equation D.19 can be rewritten as

$$C\dot{\theta} + F\dot{\phi} + H\dot{\Omega} + I\dot{\beta} = J_2 \quad (\text{D.20})$$

The second canonical equation can be derived by substituting the partial derivatives in Equation C.12 into Equation 9.28 to give

$$\begin{aligned}
 \dot{P}_\Omega = & \omega_{s_x} [S_\phi (H_{g_x} + C_\theta H_{r_x} - S_\theta H_{r_z}) - C_\phi (H_{r_v} - H_{g_v})] \\
 & + \omega_{s_v} (C_\theta H_{r_x} + S_\theta H_{r_z} + H_{g_x}) - \frac{1}{2} T_x \frac{C_\theta^2 S_{2\phi} S_\Omega + S_{2\theta} C_\phi C_\Omega}{C_\theta^2 C_\phi^2 + (C_\theta S_\phi C_\Omega - S_\theta S_\Omega)^2} \\
 & - \frac{1}{2} T_v \frac{S_{2\theta} C_\phi S_\Omega - C_\theta^2 S_{2\phi} C_\Omega}{C_\theta^2 C_\phi^2 + (S_\theta C_\Omega + C_\theta S_\phi S_\Omega)^2} - K_m (\Omega - \Omega_s) + T_m \quad (\text{D.21})
 \end{aligned}$$

D.4 Case IV: $q = \beta$

If $q = \beta$, Hamilton's Canonical Equations take the form

$$\dot{P}_\beta = \frac{\partial \mathcal{L}}{\partial \beta} \quad (\text{D.22})$$

and

$$P_\beta = \frac{\partial \mathcal{L}}{\partial \dot{\beta}} \quad (\text{D.23})$$

Substituting the results from Equation C.20 into Equation D.23 yields

$$P_\beta = I_{v_i} \omega_{c_i} \quad (\text{D.24})$$

which simplifies to give

$$I_{v_i} \dot{\Omega} + I_{v_i} \dot{\beta} = P_\beta - I_{v_i} \omega_{c_i} \quad (\text{D.25})$$

Let

$$K_2 = P_\beta - I_{v_i} \omega_{c_i}$$

and Equation D.25 can be rewritten as

$$I\dot{\Omega} + I\dot{\beta} = K_2 \tag{D.26}$$

The rate of change of the generalized momentum can be written as

$$\dot{P}_\beta = -K_v \beta \tag{D.27}$$

The result is eight equations which describe the motion of the MMS.

Appendix E

Comparison of the Complex Variable and State Space Solutions

The characteristic equation for the Dry-Tuned Gyroscope has been derived using complex variable and state space approaches. The result should be the same; however, at first glance, this does not appear to be true. A comparison of the resulting equations must be made to ensure that the characteristic equations derived by the two methods are the same.

Recall that the characteristic equation derived using the complex variable method, Equation 8.22, is

$$\begin{aligned} \left[s^2 + \omega_s(\omega_n - \omega_s) + \alpha + 2\gamma\omega_s^2 \right]^2 + (\omega_n - 2\omega_s + 2\gamma\omega_s)^2 s^2 \\ - \left[\gamma(s^2 + \omega_s^2) + \beta \right]^2 = 0 \quad (\text{E.1}) \end{aligned}$$

and the characteristic equation derived using the state space method, Equation 8.35

$$I_{rt} (I_{rt} + I_{gt}) \omega^4 - \omega^2 \{ I_{rt} (K_x + K_y) + I_{gt} K_y$$

$$+ \omega_s^2 \left[I_{r_t} (2I_{r_t} - 2I_{r_s} + I_{g_s} - I_{g_t}) + I_{g_t} (I_{r_s} - I_{r_t}) + I_{r_s}^2 \right] \} \\ + \left[K_x + (I_{g_s} + I_{r_s} - I_{r_t} - I_{g_t}) \omega_s^2 \right] \left[K_y + (I_{r_s} - I_{r_t}) \omega_s^2 \right] = 0 \quad (\text{E.2})$$

The first step is to substitute the values of α , β , γ and J into Equation E.1 to get

$$\left[s^2 + \frac{K_x + K_y - (2I_{g_t} - I_{g_s}) \omega_s^2}{2I_{r_t} + I_{g_t}} + \left(\frac{2I_{r_s}}{2I_{r_t} + I_{g_t}} - 1 \right) \omega_s^2 + \frac{2I_{g_t}}{2I_{r_t} + I_{g_t}} \omega_s^2 \right]^2 \\ + \left[\left(\frac{2I_{r_s} + 2I_{g_t}}{2I_{r_t} + I_{g_t}} \right) \omega_s - 2\omega_s \right]^2 s^2 \\ - \left[\frac{I_{g_t} (s^2 + \omega_s^2) + K_x - K_y - (2I_{g_t} - I_{g_s}) \omega_s^2}{2I_{r_t} + I_{g_t}} \right]^2 = 0 \quad (\text{E.3})$$

Multiplying Equation E.3 by $(2I_{r_t} + I_{g_t})^2$ and simplifying gives

$$[(2I_{r_t} + I_{g_t})s^2 + K_x + K_y + (2I_{r_s} + I_{g_s} - 2I_{r_t} - I_{g_t})\omega_s^2]^2 \\ + [(2I_{r_s} - 4I_{r_t})]^2 \omega_s s^2 - [K_x - K_y + I_{g_t} s^2 + (I_{g_s} - I_{g_t})\omega_s^2]^2 = 0 \quad (\text{E.4})$$

It is possible to expand Equation E.4 in terms of s and simplify to get

$$4I_{r_t}(I_{r_t} + I_{g_t})s^4 \\ + s^2 [4I_{r_t}K_x + 2I_{g_t}K_y - 2I_{g_t}K_x + 2I_{g_t}K_y + 2I_{g_t}K_x + 4I_{r_t}K_y \\ + \omega_s^2 [(2I_{r_s} - 4I_{r_t})^2 + (4I_{r_t} + 2I_{g_t})(2I_{r_s} + I_{g_s} - 2I_{r_t} - I_{g_t}) \\ - 2I_{g_t}(I_{g_s} - I_{g_t})]] \\ + [K_x + K_y + (2I_{r_s} + I_{g_s} - 2I_{r_t} - I_{g_t})\omega_s^2]^2 \\ - [K_x - K_y + (I_{g_s} - I_{g_t})\omega_s^2]^2 = 0 \quad (\text{E.5})$$

The s^2 component of Equation E.5 can be simplified by collecting like terms and the s^0 component can be simplified by factoring. The result is

$$4I_{r_t}(I_{r_t} + I_{g_t})s^4$$

$$\begin{aligned}
 & + s^2 [4I_{r_t}(K_x + K_y) + 4I_{g_t}K_y \\
 & \quad + \omega_s^2 [4I_{r_s}^2 + 4I_{r_t}(2I_{r_t} - 2I_{r_s} + I_{g_s} - I_{g_t}) + 4I_{g_t}(I_{r_s} - I_{r_t})]] \\
 & + (2K_x + (2I_{r_s} - 2I_{r_t} + 2I_{g_s} - 2I_{g_t})\omega_s^2) \\
 & \quad (2K_y + (2I_{r_s} - 2I_{r_t})\omega_s^2) = 0
 \end{aligned} \tag{E.6}$$

At this point it should be noted that in order to solve the system of ordinary differential equations using the complex variable method a solution of the form $\mathbf{A}e^{st}$ had been assumed, while a solution of the form $\mathbf{A}e^{j\omega t}$ had been assumed when using the state space method. This must now be taken into account. Dividing Equation E.6 by 4 and replacing s with $j\omega$ gives

$$\begin{aligned}
 & I_{r_t}(I_{r_t} + I_{g_t})\omega^4 - \omega^2 \{ I_{r_t}(K_x + K_y) + I_{g_t}K_y \\
 & \quad + \omega_s^2 [I_{r_t}(2I_{r_t} - 2I_{r_s} + I_{g_s} - I_{g_t}) + I_{g_t}(I_{r_s} - I_{r_t}) + I_{r_s}^2] \} \\
 & + [K_x + (I_{g_s} + I_{r_s} - I_{r_t} - I_{g_t})\omega_s^2] [K_y + (I_{r_s} - I_{r_t})\omega_s^2] = 0
 \end{aligned} \tag{E.7}$$

Equations E.2 and E.7 are equivalent and so the characteristic equations derived by the state space and complex variable methods are the same.

Appendix F

MMS System Energy as Calculated by DTGSim

In order to determine the validity of DTGSim, comparison between similar two body models were made. However, when the three body model with variable crossed flexure pivot stiffness is used, it is difficult to ensure that the simulation is correct. One method available is to measure the energy in the system and verify that it remains steady. For simple cases, the energy in the system will remain constant. In more complicated systems, such as the three body problem mentioned above, numerical inaccuracies and the procedure used to integrate the equations of motion can combine to result in a system energy that changes over time.

In an accurate simulation, the change in the energy will be a small percentage of the total energy. To determine the accuracy of DTGSim, numerical experiments were run with a rotor offset of 7 degrees about the x axis, in case coordinates. The MMS was operated at an average shaft rate of 1500 rpm. The first simulation involved a system where the shaft inertia was very high, in order to determine the energy when the shaft speed and stiffness of

the crossed flexure pivots are constant. The results are shown in Figure F.1. It can be seen that the energy remains constant throughout the 10 second simulation.

The second simulation involved keeping the stiffness of the pivots constant while returning the moment of inertia of the shaft to its proper value. Note that the inertial properties of the MMS used in this simulation can be found in Table 11.1. The energy in the system is shown in Figure F.2. The energy is now changing over time. The first thing to note is this change is oscillatory with a mean value of 78.3 Joules. The fact that this is less than the value from Figure F.1 is due to the reduction of the moment of inertia of the shaft. The range of the oscillation is less than 0.004% of the mean value.

Allowing the stiffness of the pivots to change, the energy of the system takes on the form shown in Figure F.3. The mean of the energy is still 78.3 Joules and the amplitude of the oscillation remained the same as well. In fact, the maximum difference between the two plots is 1×10^{-5} Joules or $1.3 \times 10^{-5}\%$ of the mean energy.

The oscillation of the system energy is due to the numerical accuracy of the simulation and the method used to determine the motor and shaft speeds. By increasing the tolerances used in the integration routine, round-off errors could be lessened and the simulation would be made more accurate; however, there is always a trade off between accuracy and speed with the machine tolerance thrown in as well. The tolerances used with DTGSim were as high as the machine tolerance would allow.

The second reason for the oscillation was the method used to determine the influence of the motor stiffness on the system. Each simulation was run

twice with the mean shaft rate of the first run used to determine the initial shaft speed for the second run. A difference between the average shaft speed and the motor speed would result in the system oscillating slightly about the shaft, gimbal and rotor spin axes. This would result in oscillation of the system energy since the motor speed remains constant and would act as a torque on the system.

With this said, the change in energy is so slight that the energy can be considered constant and thus the simulation is validated, at least in this one respect.

Figure F.1: Energy of the MMS simulation when the mass moment of inertia of the shaft has been increased to $5.12 \times 10^{15} \text{ gcm}^2$. The stiffness of the crossed flexure pivots was held constant and the shaft speed was set to 1500 rpm. The initial rotor offset was 7 degrees about the x axis (in case coordinates).

Figure F.2: Energy of the MMS simulation versus time. The stiffness of the crossed flexure pivots was held constant and the shaft speed was set to 1500 rpm. The initial rotor offset was 7 degrees about the x axis (in case coordinates).

Figure F.3: Energy of the MMS simulation versus time. The shaft speed was set to 1500 rpm and the initial rotor offset was 7 degrees about the x axis (in case coordinates).

Appendix G

Relationship between Average Shaft Speed, Rotor Offset and Precession Frequency

The 3 dimensional surface shown in Figure 11.30, which describes the precession of the rotor as a function of the average shaft speed and rotor offset, was generated by setting the initial rotor offset to a specific angle and then running DTGSIM with different average shaft speeds. The curve was derived through a series of 78 different simulations which resulted in 78 points of data. The following is an example to demonstrate the generation of 13 of these points.

Each simulation was run using the specifications listed in Table 11.1. For this case, the rotor was given an initial offset about the x axis (in case coordinates) of 7 degrees. The average shaft speed was set to 1200 rpm for the first simulation and the shaft speed was incremented by 50 rpm for each subsequent run until the maximum shaft speed of 1800 rpm was reached. The results are shown in Figures G.1 to G.13.

A quarter period of the precession can be measured directly from Fig-

ures G.1 to G.4 and G.10 to G.13. Equation 11.1, shown below, was used to determine the frequency of precession for the remainder of the cases. Recall that A is the amplitude of the precession, f is the precession frequency and ϕ' is the offset of the rotor about the x axis in case coordinates at the current time, t .

$$\phi' = A \sin(2\pi t f)$$

By changing the scale of the plots, Figure G.14, an accurate estimate of the position of the rotor at the 100 second point can be made. It is then possible to estimate the amplitude of the precession and solve for the frequency using Equation 11.1. Note the nutation is neglected for these calculations and the amplitude of the precession is taken to be the initial offset of the rotor.

Each subsequent set of points were obtained by changing the initial rotor offset to a new value and varying the shaft speed. These values are plotted to obtain Figure 11.30. Figure G.15 shows the points derived from this series of curves.

Figure G.1: Angular position of the MMS rotor versus time. The rotor was initially offset about its x axis (case coordinates) by 7 degrees and the average shaft rate was 1200 rpm.

Figure G.2: Angular position of the MMS rotor versus time. The rotor was initially offset about its x axis (case coordinates) by 7 degrees and the average shaft rate was 1250 rpm.

Figure G.3: Angular position of the MMS rotor versus time. The rotor was initially offset about its x axis (case coordinates) by 7 degrees and the average shaft rate was 1300 rpm.

Figure G.4: Angular position of the MMS rotor versus time. The rotor was initially offset about its x axis (case coordinates) by 7 degrees and the average shaft rate was 1350 rpm.

Figure G.5: Angular position of the MMS rotor versus time. The rotor was initially offset about its x axis (case coordinates) by 7 degrees and the average shaft rate was 1400 rpm.

Figure G.6: Angular position of the MMS rotor versus time. The rotor was initially offset about its x axis (case coordinates) by 7 degrees and the average shaft rate was 1450 rpm.

Figure G.7: Angular position of the MMS rotor versus time. The rotor was initially offset about its x axis (case coordinates) by 7 degrees and the average shaft rate was 1500 rpm.

Figure G.8: Angular position of the MMS rotor versus time. The rotor was initially offset about its x axis (case coordinates) by 7 degrees and the average shaft rate was 1550 rpm.

Figure G.9: Angular position of the MMS rotor versus time. The rotor was initially offset about its x axis (case coordinates) by 7 degrees and the average shaft rate was 1600 rpm.

Figure G.10: Angular position of the MMS rotor versus time. The rotor was initially offset about its x axis (case coordinates) by 7 degrees and the average shaft rate was 1650 rpm.

Figure G.11: Angular position of the MMS rotor versus time. The rotor was initially offset about its x axis (case coordinates) by 7 degrees and the average shaft rate was 1700 rpm.

Figure G.12: Angular position of the MMS rotor versus time. The rotor was initially offset about its x axis (case coordinates) by 7 degrees and the average shaft rate was 1750 rpm.

Figure G.13: Angular position of the MMS rotor versus time. The rotor was initially offset about its x axis (case coordinates) by 7 degrees and the average shaft rate was 1800 rpm.

Figure G.14: Angular position of the MMS rotor versus time. The rotor was initially offset about its x axis (case coordinates) by 7 degrees and the average shaft rate was 1400 rpm.

Figure G.15: Rotor precession frequency versus average shaft speed and rotor offset. The data for the case when the rotor offset is 7 degrees about the x axis (case coordinates) is highlighted when compared to the rest of the data.