Spacecraft Formation Flying using $\mathcal{L}_1$ Adaptive Control Techniques

by
Jayant Sachdev

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Jayant Sachdev

This work is dedicated to my late grandfather, Brij Mohan Sehgal.
His wise jokes come to me in the most opportune of times and still brings a smile to
my face.

Abstract

Spacecraft formation flying is a fast improving technology for space missions, showcasing reduced cost, increased flexibility and efficiency and design redundancy over similar single satellite missions. Furthermore, it provides a viable solution for missions requiring high spacial and temporal resolutions.

While a lot of work has been conducted on spacecraft formation flying control, very few of these algorithms have been validated on flight hardware or through hardware experiments. The $\mathcal{L}_1$ adaptive control algorithm is a promising algorithm that can deal with various model uncertainties in real time. It's capability has been demonstrated through numerous aerospace simulations and also through real world experience on board NASA's AirSTAR aircraft. As such, this work focuses on the design, simulation, implementation and validation of the $\mathcal{L}_1$ adaptive controller for spacecraft formation flying.

More specifically, this thesis first introduces the $\mathcal{L}_1$ adaptive control architecture using examples for simple linear systems, to gain an understanding of the controller implementation and behaviour. The desired controller design decisions are discussed and the options for adaptive laws are shown.

The $\mathcal{L}_1$ adaptive controller is then designed for the spacecraft formation flying problem. For the $\mathcal{L}_1$ adaptive controller, the projection type and the piecewise-constant adaptive laws are investigated. A PD controller is also designed for performance comparisons. The on-orbit simulations are conducted for three dynamics models, ranging from a full non-linear model to the very simplified double integrator model and the results are discussed. The $\mathcal{L}_1$ adaptive controller is shown to have superior performance with significantly lower control forces, in all conditions. The differences between both adaptive laws are analysed, with the projection type adaptive laws showing better robustness and performance over the piecewise constant adaptive laws, due to high-frequency oscillations in the transient behaviour of the system utilizing the piecewise constant adaptive law.

The $\mathcal{L}_1$ adaptive and PD controllers are then experimentally validated using Carleton's Spacecraft Proximity Operations Testbed. The hardware experiments confirmed the results of the numerical simulations and demonstrated the superior capabilities of the $\mathcal{L}_1$ adaptive controller. Furthermore, it was shown that the $\mathcal{L}_1$ adaptive controller can adequately model unknown changes in mass and perturbations; even utilizing the unknown increase in mass to help dampen the effect of perturbations, through the adaptive modelling of the system.

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List of Accronyms

BIBO	Bounded-Input-Bounded-Output
BIBS	Bounded-Input-Bounded-State
DI	Double Integrator
DOF	Degree of Freedom
DSAC	Decentralized Simple Adaptive Control
ECI	Earth Centered Inertial
EKF	Extended Kalman Filter
ESA	European Space Agency
HCW	Hill-Clohessy-Wiltshire
HIL	Hardware-in-the-Loop
ISS	International Space Station
LMI	Linear Matrix Inequality
LQ	Linear Quadratic
LQR	Linear Quadratic Regulator
LTI	Linear Time Invariant
LVLH	Local-Vertical-Local-Horizon
MDA	MacDonald, Dettwiller and Associates Ltd.
MIMO	Multi-Input-Multi-Output
MRAC	Model Reference Adaptive Control
NASA	National Aeronautics and Space Administration
PCAL	Piecewise-Constant Adaptive Law
PD	Proportional-Derivative
SAC	Simple Adaptive Control
SDRE	State-Dependent Riccati Equation
SISO	Single-Input-Single-Output
SFF	Spacecraft Formation Flying
SMC	Sliding Mode Control

SPOT	Spacecraft Proximity Operations Testbed
VS-MRAC	Variable Structure Model Reference Adaptive Control

Chapter 1

Introduction

1.1 Motivation

Spacecraft formation flying (SFF) has become a critical technology for the worlds leading space agencies, ever since it was first studied at a conceptual level back in 1977 [1]. The main advantages of spacecraft formation flying lies in the fact one can obtain similar performance and reliability as a single large satellite, but with design redundancy, reduced costs and increased efficiency and flexibility. The cost reductions come from the fact that is much cheaper to launch several smaller satellites than it costs to launch one large satellite. Furthermore, since a formation of smaller satellites are replacing a single large satellite, each satellite plays a specific role within the formation mission. This distribution allows greater flexibility, including a more efficient utilization of computing and system resources, and ease of replacing or upgrading existing members of the formation [2–5]. Spacecraft formations also provide a viable solution for scientific missions requiring high spatial and temporal resolutions [5, 6].

The development of formation flying technology also required the development of various control architectures and techniques for the purpose of advanced relative motion control. In fact, there has been a growing research interest in the area of formation flying dynamics and control over the last two decades [7]. However, while there is a lot of literature on various control schemes, most of the control algorithms have not been tested or validated with hardware. Research into flight viable control strategies is the motivation behind this work.

1.2 Problem Statement

The current literature on the topic of spacecraft formation flying control mostly focuses on relative position tracking, on the basis of linear and non-linear dynamic

models [8]. In fact, due to the difficulties associated with implementing the full non-linear dynamics model for relative state's between satellites, most approached SFF control problems with linearised dynamic equations known as the Hill-Clohessy-Wiltshire (HCW) equations of motion [9–11]. The main issue with this linearised dynamic model is that it places limitation's on the desired formations and the selection of possible orbits, based on assumptions made during linearization. Furthermore, it does not accurately model perturbations, such as Earth's oblateness and aerodynamic drag, which can be highly non-linear and these inaccuracies cause drifts in the relative positions and formation centres of the spacecraft. While efforts have been made to include these perturbations with a linearised model, model uncertainties and input disturbances cannot be effectively tackled with linear control strategies [12, 13].

As such, this thesis is concerned with flight viable control laws that would be capable of dealing with the non-linear dynamic model present in spacecraft formation's, along with system uncertainties stemming from unmodelled perturbations and changes in mass.

1.3 Previous Work

Scharf et al. [1] defined spacecraft formation flying as two or more spacecraft whose dynamic states are coupled through a common control law; where at least one member of the formation, must track a desired state relative to another member, referred to as the leader; and the tracking control law must depend upon the state of the leader satellite. The system dynamics for spacecraft formation flying control uses a reference frame attached to the leader satellite. In this dissertation, the reference frame known as Local-Vertical-Local-Horizon (LVLH) will be utilized. The reference frame is defined in detail in Chapter 2.1 and shown in Fig. 2.1 on page 14.

In close-proximity formations, the spacecraft would typically be close enough to employ the well-known Clohessy–Wiltshire equations [10, 14] to model the relative dynamics. Furthermore, when the time scale is significantly less than the orbital period of the leader spacecraft, and the distance between both vehicles is limited to a few meters, a simple double-integrator model can be applied [15]. Being linear, these dynamics representations are particularly attractive for control purposes, which is

why the literature proposes numerous linear continuous control laws to regulate or track a prescribed relative trajectory.

1.3.1 Linear Control

Much of the earlier work on spacecraft formation flying control focused on simple formations based on a circular orbit. Vassar and Sherwood [16] were one of the first to tackle this problem and their work is detailed in depth in reference; they approached this problem by utilizing the HCW equations of motion and optimal control theory. As we will soon get to in Chap. 2, the HCW equations are based on some simplifying assumptions including [10, 14]:

1. the orbit of the leader satellite is circular; and,
2. the magnitude of the relative position of the follower relative to the leader satellite is negligible relative to the orbital radius of the leader satellite.

Use of the HCW allowed for the application of linear control methods, which are much simpler to implement. Vassar and Sherwood [16, 17] designed one such method, which involved using the linearized equations of motions to generate a quadratic cost function. The cost function is then optimized by solving the discrete matrix Riccati equation to obtain the steady state, in-plane and out of plane control laws.

The controller designed by Vassar and Sherwood, involving linearised dynamics and the optimization of a quadratic cost function is called a linear quadratic (LQ) controller. Many others have built upon the work of Vassar and Sherwood by using LQ controllers to tackle the formation flying problem. Ulybyshev [18] was one such individual who expanded the previous work by developing an LQ controller for a constellation of satellites. He uses the solution to the Riccati equation to obtain the control laws for satellites in the formation flying constellation.

A very common approach to the linear optimal control problem for spacecraft formation flying involves the design of a linear quadratic regulator (LQR) controller. The use of LQR controllers is attractive because it allows optimization options in the controller design and can stabilize formations up to 150 km in diameter; it does so while guaranteeing bounded relative orbits, but cannot guarantee that the steady

state tracking error will go to zero. A more in depth look into the design and stability of LQR controllers for spacecraft formation flying can be found in reference [19].

Many people who have utilized an LQR controller for spacecraft formation flying, have built upon the work done by Ulybyshev and Vassar and Sherwood. For example, Vadali et al. [13] used an LQR controller for spacecraft formation flying control using modified HCW equations that take into account J_2 perturbation; Starin [20] showed that the LQR controller can be used to reduce fuel consumption for spacecraft formation flying control by only thrusting in one axis in the orbital plane and Kapila et al. [21] showed that the LQR controller can be used to develop a linear pulse controller, to solve the multiple spacecraft formation flying problem. Yedavalli and Sparks [22] also used an LQR controller to tackle the spacecraft formation flying problem; they designed the controller based on the hybrid system stability analysis, which was applied due to the 'continuous' nature of the plant and the 'discrete' nature of the controller.

Kong and Miller [23] took a different approach to the linear optimal control problem of formation flying control. Their main objective was to design a optimal control law to determine minimum energy trajectories for the cluster of spacecraft in Techsat 21 formation flying mission. While their work involved the design of an LQ controller, the cost function was optimized and the control law was obtained by using the calculus of variation approach.

The main problem with linear control laws is that they do not accurately model highly-nonlinear perturbations, such as Earth's oblateness and aerodynamic drag. In addition, these adverse effects are known to cause drifts in the relative positions, and as such, cannot be neglected in the design of relation motion control systems. While efforts have been made to include these perturbations with linearized models, uncertainties and input disturbances cannot be effectively compensated with linear control strategies [12, 13].

1.3.2 Non-Linear Control

For these reasons, most recent research efforts focused on non-linear control systems. Similar to the linear control methods, most controller designs based on the general non-linear equation of motion for spacecraft formation flying are designed using either LQR controllers based on optimal control theory, or controllers derived from Lyapunov analysis. The advantage of using the full non-linear equations of motion are that the model more accurately describes the dynamics of spacecraft formation flying, allowing for more accurate control, along with eliminating the restrictions on the spacecraft formation due to the linearisation and simplification of the system dynamics.

Building on the work presented in the Linear Control section above, Stansbery and Cloutier [27] designed and built a variant of the LQR controller that controlled the positions and attitudes of a pair of spacecraft in formation. They utilized the The state-dependent Riccati equation (SDRE) method of non-linear regulation for control of the follower positions and attitudes. A 6-DOF simulation was used to demonstrate the performance of the controller; however, no external disturbances or perturbations were modelled into the dynamics for the simulation. Park et al. took the work of Stansbery and Cloutier [4] a step further by implementing the SDRE control law, along with an extended kalman filter (EKF) position estimation algorithm to estimate the states in the control law. They also successfully simulated the spacecraft formation flying control law in a GPS based hardware-in-the-loop (HIL) simulation test-bed.

When looking at Lyapunov controllers, the control function is determined through Lyapunov stability analysis. Examples of control architectures for spacecraft formation flying that use Lyapunov analysis include sliding mode control (SMC), variable structure model reference adaptive control, and other adaptive control algorithms. Unique control architectures, such as spacecraft formation flying control using mean orbital elements [28], also use Lyapunov stability, but this literature review will not go into these unique solutions.

Li, Pan and Kumar [29] implemented a 2nd order SMC algorithm for formation flying of multiple picosatellites; using a different sliding control surface for each of the

along-track, cross-track and radial control axis'. They went on to demonstrated efficient control using low thrust systems, through on-orbit simulations and also showed how the 2nd order SMC algorithm can be used as a fault tolerant controller by maintaining desired performance through thruster performance degradation. Due to the non-linear dynamics model, they conducted on-orbit simulations of formation flying control in circular and elliptical orbits with and without perturbations, including the J_2 perturbation.

While the aforementioned nonlinear control techniques are adequate to reducing fuel costs and/or manoeuvre times, they do not have the capability to deal with the unknown and changing parameters and uncertainties found in real world applications. Adaptive control techniques has the capability to deal with this problem by automatically adjusting the controller to achieve or maintain the desired system performance [24]. Adaptive control can be further broken down into two main classes of adaptive control: direct and indirect adaptive control. Indirect control involves the estimation of unknown parameters to update the controller; while, direct control laws directly update system parameters, thereby not requiring the use of any estimation techniques [25]. For example, Cho and Kerschen [9] utilized adaptive control techniques with the SMC algorithm. They utilized a sliding mode controller but implemented a simple adaptive law to automatically update the control gain. The function of the adaptive law in their implementation is to update the model gain to effectively suppress the effect of uncertainties. Their system was also modelled to include gravitational perturbations up to the 10th order and atmospheric drag. They conducted on-orbit simulations with a 1 km formation radius showing final positional errors on the order of mm's in each axis. Similar to Cho and Kerschen, Godard and Kumar [12] utilized the adaptive variable structure control technique to design two non-linear fault tolerant controllers using SMC architectures, to account for sensor and thruster degradation. The controller's however, were shown to be more robust as they could deal with unknown perturbations. They also showed formation control on the order of mm's, with the capability of dealing with sensor and actuator failures and degradation. While an adaptive law was effectively used by Cho and Kerschen and Godard and Kumar with the SMC architecture, adaptive control can be considered

a branch of control in itself and many solutions to the spacecraft formation flying problem utilize adaptive control techniques.

The variable structure model reference adaptive control technique (VS-MRAC), is one such adaptive control technique that augments SMC, by combining the SMC control scheme with a reference model. This allows the system to overcome a main design constraint with SMC, that requires knowledge on the upper bound of the perturbations. The VS-MRAC technique is a fairly recent technique that was first designed to solve the spacecraft formation flying problem in 2008, by Shahid and Kumar [30]. Results from their research showed that the VS-MRAC algorithm showed similar performance to an adaptive sliding mode control algorithm in formation keeping but showed major fuel savings during formation reconfigurations. Lee and Singh [31] built on the work done by Shahid and Kumar by designing an output feedback controller using the VS-MRAC scheme.

The VS-MRAC control scheme is a very specific adaptive control scheme that can be considered hybrid system, built on the work conducted on both sliding mode and adaptive control algorithms. The use of adaptive control algorithms for spacecraft formation flying goes back to 1999, when Queiroz et al. [11, 32] developed an adaptive controller for the full non-linear dynamics found in spacecraft formations. They modelled the system with unknown, constant, or slow-varying spacecraft masses, disturbance forces, and gravity forces as a constant unknown parameter vector. An estimate of this constant parameter vector was constantly updated through an adaptation law and was used to control the system. The controller guarantees global asymptotic convergence of the position tracking error. The designed controller was successfully simulated with a desired circular formation. The simulations did however show that the solution took a long time to settle, taking approximately 25 hours to reach a steady state error of 0 m in each axis.

Hadaegh et al. [26] implemented a very basic indirect adaptive controller for formation flying control in deep space. The dynamics used are identical to the double integrator model presented in Sec. 2.4. The controller formulation includes the dynamics of the system and unknown disturbance forces and torques, coupled into an unknown parameter vector. The unknown parameter is estimated to continuously update the

controller. The system was designed and stability was shown through Lyapunov analysis. Ulrich et al. [25] also designed an adaptive controller for the SFF problem using simple adaptive control (SAC) theory. The dynamics of the system was modelled using the HCW equations of motion and the double integrator model. Through on-orbit simulations, the SAC algorithm was shown to be vastly more effective compared to a non-adaptive PD controller, especially when dealing with unknown disturbances and uncertain mass. The SAC algorithm was also shown to be robust enough to work on a robotic platform through hardware experiments. Similarly, Ulrich [7] utilized the decentralized simple adaptive control (DSAC) technique for the SFF problem and showed its increased effectiveness versus a PD controller. The DSAC algorithm was used because it decentralizes the system and control's each axis independently. This results in better applicability to real time application on small spacecraft platforms due to the low required computational power. Lim and Bang [8] designed an adaptive back-stepping controller for SFF control under thruster misalignment, which was still effective at establishing and following a spacecraft formation.

1.3.3 $\mathcal{L}_1$ Adaptive Control

As discussed in the previous sections, many of the developed control schemes for spacecraft formation flying use linearized dynamics and control techniques, which inherently limit the types of spacecraft formations and place limitations on the leader satellites orbit. Furthermore, many of the non-linear control schemes reviewed require explicit knowledge of the perturbation models or perturbation bounds to ensure stability of the designed system and very few researchers in this area investigated flight capability of the control laws. In this context, this section introduce a recently-developed adaptive control technique, which appear to be well-suited for this particular application. This non-linear control technique is the $\mathcal{L}_1$ adaptive control approach. This particular scheme was selected based on its demonstrated robustness and performance in other aerospace applications operating under an uncertain and perturbed environment.

$\mathcal{L}_1$ adaptive control is a recent control algorithm, developed by Cao and Hovakimyan, which was first introduced in 2006 [34, 35]. The main advantage of $\mathcal{L}_1$

Figure 1.1. AirSTAR Test Aircraft [36]

adaptive control compared to other adaptive control techniques is that the architecture adapts very quickly, leading to guaranteed transient performance and robustness with a bounded away from zero, positive time-delay stability margin. Furthermore, the architecture achieves all of this without resorting to a high-gain feedback solution and uses unique techniques to reduce parameter tuning and eliminate unwanted dynamics stemming from the adaptation rates [33–37]. The architecture has already been successfully tested in numerical simulations for a variety of aeronautical and astronautical applications. The results were in fact, so promising, that the $\mathcal{L}_1$ adaptive controller was experimentally tested on NASA’s AirSTAR aircraft [36]. One of the first applications for which the $\mathcal{L}_1$ adaptive controller was used was aircraft control. Leman et al. [38] designed an $\mathcal{L}_1$ adaptive controller augmentation system for the X-48B aircraft. It was demonstrated through simulations, that the $\mathcal{L}_1$ adaptive controller ensures uniform transient and steady-state performance across the entire flight envelope. Also, the designed controller does not require tuning and does not need to be redesigned for various flight conditions. Kaminer et al. [39] utilized the $\mathcal{L}_1$ adaptive control scheme to augment conventional autopilot programs in unmanned aerial vehicles, to meet strict performance requirements with model uncertainties and environmental disturbances. Wise et al. [40] applied the $\mathcal{L}_1$ adaptive control algorithm to an unmanned combat aerial vehicle and a aerial refuelling autopilot system

and demonstrated the $\mathcal{L}_1$ adaptive control law's ability to provide guaranteed performance in the presence of non-linear uncertainties and control surface failures. All the work done on $\mathcal{L}_1$ adaptive control for aircraft control was taken a step further by Gregory et al. [36], when they utilized the $\mathcal{L}_1$ adaptive control algorithm and demonstrated its performance in a flight test, where the controller handled piloted tasks, using NASA's AirSTAR flight test vehicle.

Going beyond aircraft, the $\mathcal{L}_1$ adaptive controller has been utilized with great success in other applications in the aerospace industry. Cao and Hovakimyan [41] implemented an $\mathcal{L}_1$ adaptive output feedback control law for a missile longitudinal autopilot design. Kharisov et al. [42] applied the control law for a generic flexible crew launch vehicle, highlighting improved performance of a statically unstable vehicle. Vogel et al. [43], used the $\mathcal{L}_1$ adaptive controller to augment a magnetic momentum controller for attitude control of a satellite. The $\mathcal{L}_1$ adaptive controller was shown to improve momentum de-saturation performance and also add robustness to the system with the use of a rough geomagnetic model on-board the satellite and in the controller design. Finally, Elahidoost et al. [44] utilized the $\mathcal{L}_1$ adaptive controller for orbit stabilization. They demonstrated capability of $\mathcal{L}_1$ adaptive control to effectively deal with unknown disturbances and model uncertainties and showcased improved performance, when compared to a PD controller.

1.4 Thesis Objectives

While the $\mathcal{L}_1$ adaptive control algorithm has been implemented in numerical simulations for space applications already with great success, it has not been investigated for spacecraft formation flying yet, either in numerical simulations or through experiments. As such, this thesis is concerned with the implementation and numerical and experimental validation of the $\mathcal{L}_1$ adaptive controller for spacecraft formation flying.

To accomplish this, the dynamics for formation flying of spacecraft are derived and the $\mathcal{L}_1$ adaptive controller is designed. On-orbit simulations are conducted for a circular formation with the full non-linear dynamics, and simplified dynamics with and without perturbations. Results are compared with a PD controller for performance comparisons. Finally, to validate the on-board capability of the control algorithm,

the $\mathcal{L}_1$ adaptive control is implemented in the the Spacecraft Proximity Operations Testbed (SPOT) in Carleton University. A planar circular formation with the $\mathcal{L}_1$ adaptive controller and PD control law are compared through simulations and experiments. The effects of unknown perturbations and incorrect mass are analysed through the experiments.

1.5 Contributions

This thesis has three main contributions: the design of the $\mathcal{L}_1$ adaptive controller for spacecraft formation flying; numerical and experimental validation of the $\mathcal{L}_1$ adaptive controller and a study into the effects of unknown perturbations and incorrect mass on the performance of the $\mathcal{L}_1$ adaptive controller and the comparative benchmark PD controller. Numerical and experimental validation of the $\mathcal{L}_1$ adaptive controller includes performance comparisons of the $\mathcal{L}_1$ adaptive control to a PD controller with various perturbation and dynamic models. Furthermore, the experimental validation is one of the first hardware experiments comparing controller performance in a planar circular formation, on a ground based spacecraft testbed. Finally, individual and coupled effects of unknown perturbations and incorrect system mass have not been studied before.

1.6 Organization

This thesis is organized as follows:

The dynamics for formation flying for spacecraft is presented in Chap. 2. The chapter begins by deriving the full non-linear equations of motion for a leader-follower formation. This leads to the derivation of the linear HCW equations of motion and the assumptions required to obtain the HCW model. Finally, the double integrator model is derived.

The $\mathcal{L}_1$ adaptive control architecture is presented in Chap. 3. The control law is introduced and example problems used to gain understanding of the control law are presented. The design of the system for the unmatched SFF model is shown and the system's stability is analysed.

The on-orbit simulations of the $\mathcal{L}_1$ adaptive control are presented in Chap. 4. The $\mathcal{L}_1$ adaptive controller and PD controller parameters are presented and the various dynamic models are simulated with and without perturbations. The results are shown and performance is analysed.

Chap. 5 presents the planar circular formation experiments conducted on the SPOT facility with the $\mathcal{L}_1$ adaptive controller and the PD controller. The performance of the two controllers are compared and the effects of unknown perturbations and incorrect mass on the system performance, both coupled and decoupled, are analysed.

This thesis concludes in Chap. 6 with a summary of the results and analysis, followed by re-iterating the thesis contributions and concluding with future works and final remarks.

Chapter 2

Spacecraft Formation Flying Dynamics

In this chapter, the spacecraft formation flying dynamics will be presented. This includes an introduction to the basic on-orbit equations of motion and the on-orbit reference frame required for formation flying control. Following which, the full non-linear, HCW and Double Integrator equations of motion for spacecraft formation flying will be presented. The assumptions required to obtain the HCW and Double Integrator models will be discussed.

2.1 Relative Orbit Geometry

While there are deep space applications of spacecraft formation flying [1], many of the current applications pertain to formation flying around a celestial body, mainly Earth. As such, the formation flying dynamics presented in this section will focus on formation flying around a celestial body. It is important to note that all equations and derivations presented in this chapter have been derived from the works of Curtis [14], Vallado [45], Clohessy and Wiltshire [10] and Schaub and Junkins [46].

A common reference system that is used to describe orbital motion around Earth is called the Earth Centered Inertial (ECI) coordinate system. The ECI coordinate system is denoted by $\mathcal{F}_G$, and is placed inertially at the center of Earth and even though the Earth rotates about it's axis, the coordinate frame does not. The positive $\vec{Z}_G$ goes through the north pole and the $\vec{X}_G$, $\vec{Y}_G$ unit vectors are on the equatorial plane of Earth. More specifically, the positive $\vec{X}_G$ points toward the vernal equinox and $\vec{Y}_G$ is chosen to satisfy the right hand rule: $\vec{Y}_G = \vec{Z}_G \times \vec{X}_G$.

As established, formation flying requires the motion of the follower to be modelled as the relative motion around the leader satellite. As such, a coordinate system, originating at the leader spacecraft and rotating with the satellite as it orbits around Earth, is used. This coordinate system, known as the *Local Verticle Local Horizon*

Figure 2.1. LVLH Reference Frame

(*LVLH*) coordinate system [47], is denoted by $\mathcal{F}_L$. The $\vec{X}_L$ axis of the LVLH coordinate system is aligned with $\vec{r}_l$, which is the leader position vector with respect to the origin at $\mathcal{F}_G$; the $\vec{Z}_L$ is aligned with the angular momentum vector of the leader satellite and the $\vec{Y}_L$ axis is chosen to complete the right hand rule: $\vec{Y}_L = \vec{Z}_L \times \vec{X}_L$. Both the ECI and the LVLH coordinate systems can be seen in Fig. reffig:LVLH.

2.2 Non-Linear Relative Dynamics

In the LVLH system, the position of the follower relative to the leader is denoted by $\vec{\rho}$, which is defined as the following vectrix:

$$\vec{\rho} = \vec{X}_L \times x + \vec{Y}_L \times y + \vec{Z}_L \times z \quad (2.1)$$

and can be expressed in terms of its x , y and z components as follows:

$$\vec{\rho} = \vec{\mathcal{F}}_L^T \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad (2.2)$$

The x and y components of $\vec{\rho}$ are also called the *radial* and *along-track* components,

and represent the in-plane motion of the follower. This is because the in-plane motion is in the target orbital plane. The z component of $\vec{\rho}$ is called the *cross-track* component and represents the out of plane motion of the follower.

Using vector addition, the inertial follower position is found to be:

$$\vec{r}_f = \vec{r}_l + \vec{\rho} \quad (2.3)$$

We can also use kinematic relations to define the inertial follower velocity and acceleration:

$$\dot{\vec{r}}_f = \dot{\vec{r}}_f + \vec{\omega}_{LI} \times \vec{r}_f \quad (2.4)$$

$$\ddot{\vec{r}}_f = \ddot{\vec{r}}_f + \dot{\vec{\omega}}_{LI} \times \vec{r}_f + 2\vec{\omega}_{LI} \times \dot{\vec{r}}_f + \vec{\omega}_{LI} \times (\vec{\omega}_{LI} \times \vec{r}_f) \quad (2.5)$$

where $\vec{\omega}_{LI}$ and $\dot{\vec{\omega}}_{LI}$ are the angular velocity and angular acceleration of the LVLH frame with respect to the inertial frame. The inertial components of the follower spacecraft can be related to the inertial components of the leader spacecraft in the LVLH frame as follows:

$$\vec{r}_f = \vec{\mathcal{F}}_L^T \begin{bmatrix} r_l + x \\ y \\ z \end{bmatrix} \quad (2.6)$$

$$\dot{\vec{r}}_f = \vec{\mathcal{F}}_L^T \begin{bmatrix} \dot{r}_l + \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} \quad (2.7)$$

$$\ddot{\vec{r}}_f = \vec{\mathcal{F}}_L^T \begin{bmatrix} \ddot{r}_l + \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} \quad (2.8)$$

$$\vec{\omega}_{LI} = \vec{\mathcal{F}}_L^T \begin{bmatrix} 0 \\ 0 \\ \dot{\nu} \end{bmatrix} \quad (2.9)$$

$$\dot{\vec{\omega}}_{LI} = \vec{\mathcal{F}}_L^T \begin{bmatrix} 0 \\ 0 \\ \dot{\nu} \end{bmatrix} \quad (2.10)$$

where ν denotes the true anomaly of the leader spacecraft and r_l is the magnitude of the vector $\vec{r}_l$.

We can now use Eqs. (2.3)-(2.10) can be used to write the inertial acceleration of the follower in the LVLH frame, as follows:

$$\ddot{\vec{r}}_f = \vec{\mathcal{F}}_L^T \begin{bmatrix} \ddot{r}_l + \ddot{x} - \dot{\nu}y - 2\dot{\nu}\dot{y} - \dot{\nu}^2(r_l + x) \\ \ddot{y} + \dot{\nu}(r_l + x) + 2\dot{\nu}(\dot{r}_l + \dot{x}) - \dot{\nu}^2y \\ \ddot{z} \end{bmatrix} \quad (2.11)$$

Eq. (2.11) can be simplified by substituting in expressions for $\dot{\nu}$ and $\ddot{r}_l$. Using the fact that the angular momentum vector in a Keplerian orbit is constant and can be related to the rate of change of the true anomaly of the leader spacecraft, we can define:

$$\dot{\nu} = -2\frac{\dot{r}_l}{r_l} \quad (2.12)$$

In order to obtain the expression for $\ddot{r}_l$, let us first make use of the definition of the LVLH frame to write the leader spacecraft position vector in the LVLH reference frame:

$$\vec{r}_l = \vec{\mathcal{F}}_L^T \begin{bmatrix} r_l \\ 0 \\ 0 \end{bmatrix} \quad (2.13)$$

Differentiating this position vector twice with respect to time and noting that the motion is restricted around the $\vec{L}_x$ axis, yields the leader spacecraft acceleration vector in the LVLH frame:

$$\ddot{\vec{r}}_l = \vec{\mathcal{F}}_L^T \begin{bmatrix} \ddot{r}_l - \dot{\nu}^2 r_l \\ 0 \\ 0 \end{bmatrix} \quad (2.14)$$

We can also make use of the two body equation of motion to express the acceleration

of the leader satellite in the LVLH frame as:

$$\vec{r}_l = -\vec{\mathcal{F}}_L^T \frac{\mu}{r_l^2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad (2.15)$$

where $\mu = 3.986 \times 10^{14} \frac{\text{m}^3}{\text{s}^2}$ is the gravitational parameter for Earth.

We can equate Eqs. (2.14) and (2.15) to one another to show that the inertial acceleration of the leader spacecraft in the LVLH frame is:

$$\ddot{r}_l = \dot{\nu}^2 r_l - \frac{\mu}{r_l^2} \quad (2.16)$$

Eqs. (2.12) and (2.16) can be substituted back into (2.11) to give the simplified inertial acceleration of the follower in the LVLH frame:

$$\ddot{\vec{r}}_f = \vec{\mathcal{F}}_L^T \begin{bmatrix} \ddot{x} - \dot{\nu}^2 x - 2\dot{\nu}(\dot{y} - y\frac{\dot{r}_l}{r_l}) - \frac{\mu}{r_l^2} \\ \ddot{y} - \dot{\nu}^2 y + 2\dot{\nu}(\dot{x} - x\frac{\dot{r}_l}{r_l}) \\ \ddot{z} \end{bmatrix} \quad (2.17)$$

We can also use the two body equations of motion to express the inertial acceleration of the follower in the LVLH frame:

$$\ddot{\vec{r}}_c = -\vec{\mathcal{F}}_L^T \frac{\mu}{r_c^3} \begin{bmatrix} r_l + x \\ y \\ z \end{bmatrix} \quad (2.18)$$

The general non-linear relative equations of motion can be obtained by equating Eqs. (2.17) and (2.18) and using the components of the acceleration vector to describe the relative motion in each axis:

$$\ddot{x} - \dot{\nu}^2 x - 2\dot{\nu}(\dot{y} - y\frac{\dot{r}_l}{r_l}) - \frac{\mu}{r_l^2} = -\frac{\mu}{r_c^3}(r_l + x) \quad (2.19)$$

$$\ddot{y} - \dot{\nu}^2 y + 2\dot{\nu}(\dot{x} - x\frac{\dot{r}_l}{r_l}) = -\frac{\mu}{r_c^3}y \quad (2.20)$$

$$\ddot{z} = -\frac{\mu}{r_c^3}z \quad (2.21)$$

We can further simplify these equations for formation flying missions where the relative position components, (x, y, z) , are much smaller than the orbital radius of the leader spacecraft, by approximating:

$$r_c \approx r_l \sqrt{1 + 2\frac{x}{r_l}} \quad (2.22)$$

From here we can use of the binomial expansion theorem and neglect higher order terms to show that:

$$-\frac{\mu}{r_c^3} \begin{bmatrix} r_l + x \\ y \\ z \end{bmatrix} \approx -\frac{\mu}{r_l^3} \begin{bmatrix} r_l - 2x \\ y \\ z \end{bmatrix} \quad (2.23)$$

Making use of this result, with Eq. (2.12) we can show that the simplified general non-linear equations of relative motion can be expressed as:

$$\ddot{x} - x(\dot{\nu}^2 + 2\frac{\mu}{r_l^3}) - 2\dot{\nu}\dot{y} - \ddot{\nu}y = 0 \quad (2.24)$$

$$\ddot{y} - y(\dot{\nu}^2 - \frac{\mu}{r_l^3}) + 2\dot{\nu}\dot{x} + \ddot{\nu}x = 0 \quad (2.25)$$

$$\ddot{z} + \frac{\mu}{r_l^3}z = 0 \quad (2.26)$$

2.3 Linearized Relative Dynamics

The non linear equations presented in Eqs. (2.24) to (2.26) can be simplified further into the well known Hill-Clohessy-Wiltshire (HCW) equations of motion. As mentioned before, the non-linear equations can be simplified based on the following assumptions:

1. the orbit of the leader satellite is circular (or very near-circular); and,
2. the magnitude of the relative position of the follower relative to the leader satellite is negligible relative to the orbital radius of the leader satellite.

To begin with, it is important to note, that in a circular orbit, the orbital radius and true anomaly rate of the leader spacecraft, r_l and $\dot{\nu}$ respectively, remains constant. Furthermore, the true anomaly rate of the spacecraft, $\dot{\nu}$ is equal to the mean motion, n , of the spacecraft. We can now define the mean motion of the leader spacecraft as:

$$n = \sqrt{\frac{\mu}{r_l^3}} \quad (2.27)$$

In the linearised formation flying problem, the orbit of the leader satellite is assumed to be circular. As such, we can use the Eq. (2.27) with the constant radius and true anomaly rate, to show how Eqs. (2.24) to (2.26) can be linearized into:

$$\ddot{x} - 3n^2x - 2n\dot{y} = 0 \quad (2.28)$$

$$\ddot{y} + 2n\dot{x} = 0 \quad (2.29)$$

$$\ddot{z} + n^2z = 0 \quad (2.30)$$

Eqs. (2.24) to (2.26) are known as the *Hill's* equations or the HCW equations of motion for formation flying and proximity operations missions.

2.4 Double Integrator Relative Dynamics

The Double Integrator model simplifies the HCW equations even further by neglecting the mean motion term, n , in Eqs. (2.28)-(2.30):

$$\ddot{x} = 0 \quad (2.31)$$

$$\ddot{y} = 0 \quad (2.32)$$

$$\ddot{z} = 0 \quad (2.33)$$

As seen in Eqs. (2.24) to (2.26) and (2.28) to (2.30), the in-plane, (x, y) , and out-of-plane, (z) , motion of the satellite are decoupled. The in-plane coupling terms corresponds to Coriolis and gravity forces and the out-of-plane coupling term is an undamped oscillation resulting from the mean motion [25]. The Double Integrator

model can be used because it has been demonstrated that these terms can be dealt with as unmodelled perturbations in a feedback controller [47]. Furthermore, Saulnier et al. [15] demonstrated that for a scenario where the total relative positions are approximately 10 m or less, the coupling terms are a whole order of magnitude lower than the double integrator terms and can effectively be ignored. Saulnier et al. also explained that if the orbital manoeuvre is less than the orbital period of the leader spacecraft, then the coupling terms can also be ignored. This was backed up by Paluszek and Thomas [48], who used Bode magnitude plots to demonstrate that if the spacecraft are manoeuvring faster than the orbital mean motion, n , then the Double Integrator model can effectively be used. The bode plots of the Double Integrator model were superimposed over the linear HCW model and showed that the magnitude of the plots aligned after crossing the frequency of the mean motion.

In summary, the general non-linear relative dynamics model is the most robust dynamics model out of the three and will be used for high-fidelity numerical simulations of $\mathcal{L}_1$ adaptive control. The general non-linear dynamics model places no restrictions on orbit selection or on the formation and allows the controller to deal with all the non-linearities found in space. The HCW dynamic model, is used as an example of an application of the $\mathcal{L}_1$ adaptive controller, where the leader spacecraft is in a circular orbit with small relative distances between the leader and follower spacecraft. Finally, the double integrator dynamics model is the most simplified dynamics model that can be used in scenarios where the relative distances between the follower and leader spacecraft are very small. This simple dynamics model is ideal for use in the laboratory experiments, which will be used to validate the numerical simulations.

While all three dynamic models presented in this chapter have their advantages and disadvantages in formation flying control. A control algorithm is needed to execute any manoeuvres and keep the follower spacecraft on the desired formation. The next chapter will focus on the $\mathcal{L}_1$ adaptive controller and its application to the formation flying dynamics presented in this chapter.

Chapter 3

$\mathcal{L}_1$ Adaptive Control

In this chapter, the $\mathcal{L}_1$ adaptive control architecture is introduced. Simple Linear Time Invariant (LTI) systems are first used to gain an understanding of the architecture and basic examples of the $\mathcal{L}_1$ adaptive control are presented to demonstrate its performance. Following which, the complete architecture used for the spacecraft formation flying problem is presented and the system's performance is analysed.

3.1 Introduction

As discussed in subsection 1.3.3, $\mathcal{L}_1$ adaptive control is a relatively new control algorithm, developed by Chengyu Cao and Naira Hovakimyan [34, 35] in 2006. The main draw of $\mathcal{L}_1$ adaptive control is that the architecture adapts very quickly and has a guaranteed transient performance and robustness. The architecture does not need to resort to a high-gain feedback solution and uses unique techniques to reduce parameter tuning and eliminate unwanted dynamics stemming from the adaptation rates [33–37]. The $\mathcal{L}_1$ adaptive control algorithm has already been successfully implemented for a wide variety of aerospace applications, which have been detailed in subsection 1.3.3. The capability of the $\mathcal{L}_1$ Adaptive controller to adapt quickly, and to handle, in real time, unmodelled perturbations is what makes this control law attractive.

3.2 $\mathcal{L}_1$ Adaptive Control Architecture

To better illustrate the $\mathcal{L}_1$ adaptive control architecture, linear systems with constant unknown parameters will be utilized.

3.2.1 LTI Systems with Constant Unknown Parameters

The $\mathcal{L}_1$ adaptive control architecture builds on the following class of systems:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + b(u(t) + \theta^T x(t)), & x(0) &= x_0 \\ y(t) &= c^T x(t)\end{aligned}\tag{3.1}$$

where $x(t) \in \mathbb{R}^n$ is the measured system state vector; $u(t)$ is the control input signal; b and $c \in \mathbb{R}^n$ are known constant matrices; A is a known $n \times n$ matrix describing the dynamics of the system; θ is the unknown parameter and $y(t) \in \mathbb{R}^n$ is the regulated output.

The $\mathcal{L}_1$ adaptive controller makes use of the following control input:

$$u(t) = u_m(t) + u_{ad}(t)\tag{3.2}$$

where $u_{ad}(t)$ is the adaptive portion of the control input and u_m is defined as:

$$u_m(t) = -k_m^T x(t)\tag{3.3}$$

The static feedback gain, $k_m \in \mathbb{R}^n$, renders $A_m \triangleq A - bk_m^T$ Hurwitz and leads to the following closed-loop system:

$$\begin{aligned}\dot{x}(t) &= A_m x(t) + b(\theta^T x(t) + u_{ad}(t)), & x(0) &= x_0 \\ y(t) &= c^T x(t)\end{aligned}\tag{3.4}$$

A particularity of the $\mathcal{L}_1$ adaptive control scheme is that it incorporates a state predictor, which is given by:

$$\begin{aligned}\dot{\hat{x}}(t) &= A_m \hat{x}(t) + b(\hat{\theta}^T x(t) + u_{ad}(t)), & \hat{x}(0) &= x_0 \\ \hat{y}(t) &= c^T \hat{x}(t)\end{aligned}\tag{3.5}$$

where $\hat{x} \in \mathbb{R}^n$ is the state vector of the predictor and $\hat{\theta}(t) \in \mathbb{R}^n$ is the estimate of the unknown parameter θ , which is determined through an adaptation law.

One of the most common adaptation laws is the so-called projection-type law. Let

us first define a function $f(\theta)$ that is continuously differentiable and convex:

$$f(\theta) = \frac{(\epsilon_\theta + 1)\theta^T \theta - \theta_{max}^2}{\epsilon_\theta \theta_{max}^2} \quad (3.6)$$

where θ_{max} is the norm bound imposed on the vector θ and ϵ_θ is the projection tolerance bound defined by the user. The projection-type adaptation law can then be defined as:

$$Proj(\theta, y) \triangleq \begin{cases} y, & \text{if } f(\theta) < 0 \\ y, & \text{if } f(\theta) \geq 0 \text{ and } \nabla f^T y \leq 0 \\ y - \frac{\nabla f(\theta)(\nabla f(\theta))^T}{\|\nabla f(\theta)\|^2} y f(\theta), & \text{if } f(\theta) \geq 0 \text{ and } \nabla f^T y \geq 0 \end{cases} \quad (3.7)$$

where $\nabla f(\theta)$ is the gradient of the function $f(\theta)$.

Let us now make use of a constant $\delta > 0$ and consider a subset $\Omega_\delta = \{\theta \in R^n | f(\theta) \leq \delta\} \subset R^n$. Let's introduce $\theta^* \in \Omega_\delta$, which is not on the boundary of the subset and let us also assume $f(\theta^*) \leq \delta$. It is also important to note that $\theta \in \Omega_\delta$ lies on the boundary of the subset and let us assume $f(\theta) = \delta$. Then the following inequality takes place:

$$(\theta^* - \theta)^T \nabla f(\theta) \leq 0 \quad (3.8)$$

The projection type adaptive law can now be used to estimate the unknown parameter:

$$\dot{\hat{\theta}}(t) = \Gamma Proj(\hat{\theta}(t), -\tilde{x}^T(t) P b x(t)), \quad \hat{\theta}(0) = \hat{\theta}_0 \in \Theta \quad (3.9)$$

where $\tilde{x} \triangleq \hat{x}(t) - x(t)$ is the prediction error, $\Gamma \in \mathbb{R}^+$ is the adaptation gain, and $P = P^T > 0$ solves the Lyapunov equation $A_m^T P + P A_m = -Q$ for an arbitrary symmetric $Q = Q^T > 0$. Also, the projection is confined to the set Θ .

Finally, the adaptive control signal can be defined in the Laplace domain as:

$$u_{ad}(s) = -C(s)(\hat{\eta}(s) - k_g r(s)) \quad (3.10)$$

where $r(s)$ is the Laplace transform of the desired input vector $r(t)$, $\hat{\eta}(s)$ is the Laplace transform of $\hat{\eta} \triangleq \hat{\theta}^T(t)x(t)$, $k_g \triangleq -1/(c^T A_m^{-1} b)$, is the input tracking gain,

Figure 3.1. Block diagram of the $\mathcal{L}_1$ Adaptive Control Architecture

and $C(s)$ is a Bounded-Input-Bounded-Output (BIBO) stable and strictly proper transfer function with DC gain $C(0) = 1$, and its state-space realization assumes zero initialization. It is important to note that $r(s)$ and $\hat{\eta}(s)$ are computed in the time domain as $r(t)$ and $\hat{\eta}(t)$ and fed through the low pass filter $C(s)$ to obtain the adaptive control signal $u_{ad}(t)$ in the time domain.

The relationships defined in Eqs. (3.2),(3.4),(3.5),(3.9) and (3.10) form the basis of the $\mathcal{L}_1$ adaptive controller. The block diagram of the $\mathcal{L}_1$ adaptive control architecture can be seen in Figure 3.1.

The selection of k_m and $C(s)$ also has to ensure that the system meets the following $\mathcal{L}_1$ -norm condition:

$$\lambda \triangleq \|G(s)\|_{\mathcal{L}_1} L < 1 \quad (3.11)$$

where,

$$\begin{aligned} G(s) &= H(s)(1 - C(s)) \\ H(s) &\triangleq (s\mathbb{I} - A_m)^{-1}b \\ L &\triangleq \max_{\theta \in \Theta} \|\theta\|_1 \end{aligned} \quad (3.12)$$

The $\mathcal{L}_1$ -norm condition is important in the $\mathcal{L}_1$ adaptive control architecture because a system that meets the condition is bounded-input-bounded-state (BIBS) stable with respect to $r(t)$ and x_0 . More details on $\mathcal{L}$ -norms can be found in Appendix A.

We can also demonstrate transient and steady-state performance by demonstrating that the prediction error is uniformly bounded, independent of the adaptive control signal and by demonstrating that the adaptive control signal leads to to the

prediction error converging to 0, as time goes to infinity. To do so, let us first define the error dynamics, using Eqs. (3.4) and (3.5)

$$\dot{\tilde{x}}(t) = A_m \tilde{x}(t) + b \tilde{\theta}^T(t) x(t), \quad \tilde{x}(0) = 0 \quad (3.13)$$

where, $\tilde{\theta}(t) \triangleq \hat{\theta}(t) - \theta$.

Now, let us consider the following Lyapunov function candidate and its derivative:

$$V(\tilde{x}(t), \tilde{\theta}(t)) = \tilde{x}^T(t) P \tilde{x}(t) + \frac{1}{\Gamma} \tilde{\theta}^T(t) \tilde{\theta}(t) \quad (3.14)$$

$$\dot{V}(t) = \dot{\tilde{x}}^T(t) P \tilde{x}(t) + \tilde{x}^T(t) P \dot{\tilde{x}}(t) + \frac{1}{\Gamma} (\dot{\tilde{\theta}}^T(t) \tilde{\theta}(t) + \tilde{\theta}^T(t) \dot{\tilde{\theta}}(t)) \quad (3.15)$$

We can use the projection operator inequality shown in Eq. (3.8), along with the algebraic Lyapunov equation $A_m^T P + P A_m = -Q$ to upper bound the derivative of the Lyapunov function:

$$\dot{V}(t) \leq -\tilde{x}^T(t) Q \tilde{x}(t) \quad (3.16)$$

This implies that $\tilde{x}(t)$ and $\tilde{\theta}(t)$ are uniformly bounded. Taking this a step further, we can make use of the fact that $\tilde{x}(0) = 0$ to show:

$$\lambda_{\min}(P) \|\tilde{x}(t)\|^2 \leq V(t) \leq V(0) = \frac{\tilde{\theta}^T(0) \tilde{\theta}(0)}{\Gamma} \quad (3.17)$$

where $\lambda_{\min}(P)$ is the minimum eigenvalue of P .

Since the projection operator ensures that $\hat{\theta}(t) \in \Theta$, we can show that the prediction error is uniformly bounded, for every $\tau \geq 0$:

$$\|\tilde{x}_\tau\|_{\mathcal{L}_\infty} \leq \sqrt{\frac{\theta_{\max}}{\lambda_{\min}(P)\Gamma}}, \quad \theta_{\max} \triangleq 4 \max_{\theta \in \Theta} \|\theta\|^2 \quad (3.18)$$

Furthermore, since the bound on $\|\tilde{x}\|_{\mathcal{L}_\infty}$ was derived independently of $u_{ad}(t)$, $x(t)$ and $\hat{x}(t)$ can diverge at the same rate to maintain the uniformly bounded error. As long as the $\mathcal{L}_1$ -norm condition shown in Eq. (3.11) is met, we can also show that the estimated state vector $\hat{x}$ with the effect of the adaptive input signal $u_{ad}(t)$ included, is uniformly bounded. We can then use of Barbalat's lemma to show the asymptotic

Figure 3.2. Example 1: System performance of an LTI system performance with no external disturbances and $C(s) = \frac{160}{s+160}$

convergence of the prediction error to zero [49], that is:

$$\lim_{t \rightarrow \infty} \tilde{x}(t) = 0 \quad (3.19)$$

More information on LTI Systems with Constant Unknown Parameters, including more detailed information on the $\mathcal{L}_1$ architecture and detailed performance and stability analysis can be found in references [33, 35, 50].

To gain some practical experience with the $\mathcal{L}_1$ adaptive controller, an example problem was implemented from the references above. The $\mathcal{L}_1$ adaptive control architecture was used with the following parameters:

$$A_m = \begin{bmatrix} 0 & 1 \\ -1 & -1.4 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \theta = \begin{bmatrix} 4 \\ -4.5 \end{bmatrix}$$

Furthermore, $\hat{\theta} = [\hat{\theta}_1, \hat{\theta}_2]^T \in \mathbb{R}^2 \in [-10, 10]$ for $i = 1, 2$, which leads to $L = 20$. We can then implement the $\mathcal{L}_1$ adaptive controller with $k_m = 0$, $\Gamma = 10000$, and $C(s) = \frac{160}{s+160}$. The simulation results are shown in Fig. 3.2 with the initial state vector $x(0) = \hat{x}(0) = 0$ and a time-varying reference signal $r(t) = 100 \cos(0.2t)$.

Fig. 3.2a shows that the $\mathcal{L}_1$ adaptive controller tracks the desired reference input with a small time delay and Fig. 3.2b shows the control signal input $u(t)$, that led to the tracking results in Fig. 3.2a. Fig. 3.3 shows the prediction error converge to zero

Figure 3.3. Example 1: Analysing the bounds on the prediction error and the associated adaptive estimates

when θ is constant and also provides insight to the adaptation parameter, which does not converge to θ but varies as a function of time to continuously adapt the system and allow the prediction error to remain uniformly bounded.

3.2.2 Systems with Uncertain System Input Gain

While the $\mathcal{L}_1$ adaptive control architecture introduced in subsection 3.2.1 provides a good introduction to the system and its stability and performance bounds, the system introduced cannot effectively deal with real-world control problems due to the incapability of the system to tackle input disturbances and time-varying parameters. For this reason, the $\mathcal{L}_1$ adaptive control architecture was then extended to handle

unknown system input gain [33]. The closed-loop system introduced in Eq. (3.4) can be modified to reflect this desired capability:

$$\begin{aligned}\dot{x}(t) &= A_m x(t) + b(\omega u(t) + \theta^T x(t) + \sigma(t)), & x(0) &= x_0 \\ y(t) &= c^T x(t)\end{aligned}\tag{3.20}$$

where $\omega \in \mathbb{R}$ is an unknown constant with known sign and $\sigma(t)$ models the input disturbances. It is also important to note that $u_m(t)$ is modelled through the Hurwitz A_m matrix, and, as such, the control input $u(t)$ is simply given by u_{ad} . The input disturbance bound is assumed to be known, along with the sign and upper and lower limit of the unknown constant.

The state predictor must also be modified to reflect this updated architecture:

$$\begin{aligned}\dot{\hat{x}}(t) &= A_m \hat{x}(t) + b(\hat{\omega} u(t) + \hat{\theta}^T x(t) + \hat{\sigma}(t)), & \hat{x}(0) &= x_0 \\ \hat{y}(t) &= c^T \hat{x}(t)\end{aligned}\tag{3.21}$$

where, $\hat{\omega}$, $\hat{\sigma}(t)$ and $\hat{\theta}(t)$ are adaptive estimates of their respective parameters.

Furthermore, the following projection-based adaptive laws can be used to model the adaptation mechanism:

$$\begin{aligned}\dot{\hat{\theta}}(t) &= \Gamma Proj(\hat{\theta}(t), -\tilde{x}^T(t) P b x(t)), & \hat{\theta}(0) &= \hat{\theta}_0 \\ \dot{\hat{\sigma}}(t) &= \Gamma Proj(\hat{\sigma}(t), -\tilde{x}^T(t) P b), & \hat{\sigma}(0) &= \hat{\sigma}_0 \\ \dot{\hat{\omega}}(t) &= \Gamma Proj(\hat{\omega}(t), -\tilde{x}^T(t) P b u(t)), & \hat{\omega}(0) &= \hat{\omega}_0\end{aligned}\tag{3.22}$$

where the following relationships can be used to bound the projection estimates: $|\sigma(t)| \leq \Delta$; $\theta(t) \in \Theta$, and $\omega \in \Omega \triangleq [\omega_l, \omega_u]$, $0 < \omega_l < \omega_u$.

The control signal can now be defined as:

$$u(s) = -kD(s)(\hat{\eta}(s) - k_g r(s))\tag{3.23}$$

where $r(s)$ is the Laplace transform of the input signal; $\hat{\eta}(s)$ is the Laplace transform of $\hat{\eta}(t) \triangleq \hat{\omega}(t)u(t) + \hat{\theta}^T(t)x(t) + \hat{\sigma}(t)$; $k_g \triangleq \frac{-1}{c^T A_m^{-1} b}$ is the feed-forward gain; $k > 0$ is the feedback gain and $D(s)$ is a strictly proper transfer function leading to a strictly

proper and stable transfer function $C(s)$, which is given by:

$$C(s) = \frac{\omega k D(s)}{1 + \omega k D(s)} \quad \forall \omega \in \Omega \quad (3.24)$$

The system still must meet the $\mathcal{L}_1$ -norm condition:

$$\lambda \triangleq \|G(s)\|_{\mathcal{L}_1} L < 1 \quad (3.25)$$

where,

$$\begin{aligned} G(s) &= H(s)(1 - C(s)) \\ H(s) &\triangleq (s\mathbb{I} - A_m)^{-1}b \\ L &\triangleq \max_{\theta \in \Theta} \|\theta\|_1 \end{aligned} \quad (3.26)$$

Similar to basic system introduced in subsection 3.2.1, we can use Lyapunov analysis to show that the prediction error $\tilde{x}(t)$ is uniformly bounded at any time instant, $\tau \geq 0$:

$$\|\tilde{x}_\tau\|_{\mathcal{L}_\infty} \leq \sqrt{\frac{\theta_m}{\lambda_{\min}(P)\Gamma}} \quad (3.27)$$

where,

$$\begin{aligned} \theta_m &\triangleq 4 \max_{\theta \in \Theta} \|\theta\|^2 + 4\Delta^2 + (\omega_u - \omega_l)^2 + 4 \frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)} (d_\theta \max_{\theta \in \Theta} \|\theta\| + d_\sigma \Delta) \\ \|\dot{\theta}(t)\| &< d_\theta < \infty \quad \forall t \geq 0 \\ \|\dot{\sigma}(t)\| &< d_\sigma < \infty \quad \forall t \geq 0 \end{aligned} \quad (3.28)$$

It is important to note that Eqs. (3.27) and (3.28) only work with the assumption that the unknown parameter $\theta(t)$ and the input disturbances parameter $\sigma(t)$ are continuously differentiable with uniformly bounded derivatives. Furthermore, as shown earlier, P and Q are determined from the solution to the Lyapunov equation $A_m^T P + P A_m = -Q$ for an arbitrary $Q = Q^T > 0$.

Furthermore, if the unknown system parameters are constant, we can again show that the prediction error converges to zero as time goes to infinity. This cannot be proven if the parameters are not constant, as changes in the parameters causes

Figure 3.4. Example 2: System performance of an LTI system performance with external disturbances and $D(s) = \frac{1}{s}$

changes to the prediction error. The desired performance is demonstrated through comparisons with a non-adaptive closed-loop reference system to show that the difference between the reference state variable and $\mathcal{L}_1$ state variable, along with the difference between the reference control input and $\mathcal{L}_1$ control input, are bounded. More detailed derivations and analysis on the $\mathcal{L}_1$ adaptive controller for systems with uncertain input gain can be found in references [33] and [51].

To gain a better understanding of the $\mathcal{L}_1$ architecture with uncertain system input gain, an example from reference [33] was also implemented. The $\mathcal{L}_1$ adaptive control architecture introduced in this section was used with the following parameters:

$$A_m = \begin{bmatrix} 0 & 1 \\ -1 & -1.4 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\theta(t) = [2 + \cos(\pi t), 2 + 0.3 \sin(\pi t) + 0.2 \cos(2t)]^T$$

$$\sigma(t) = \sin\left(\frac{\pi}{2}t\right)$$

$$\omega = 1$$

Furthermore the projection bounds were set to, $\Theta = \{v = [v_1, v_2]^T \in \mathbb{R}^2 : v_i \in [-5, 5], i = 1, 2\}$; $\omega = [0.1, 2]$ and $\Delta = 50$.

In order to satisfy the $\mathcal{L}_1$ norm condition, ωk must be greater than 30, as such, with the knowledge that the conservative lower bound of ω is 0.5, we can set $k = 60$. We

Figure 3.5. Example 2: Analysing the bounds on the prediction error

can then implement the $\mathcal{L}_1$ adaptive controller with $\Gamma = 10000$, and $D(s) = \frac{1}{s}$. The simulation results are shown in Fig. 3.4 with the initial state vector $x(0) = \hat{x}(0) = 0$ and a time-varying reference signal $r(t) = \cos(2t/\pi)$.

Fig. 3.4a shows that the $\mathcal{L}_1$ adaptive controller tracks the desired reference input with a time delay and Fig. 3.4b shows the control signal input $u(t)$ that led to the tracking results in Fig. 3.4a. Fig. 3.5 shows the bounded prediction error for the system and Fig. 3.6 shows the adaptive parameters that modelled $\hat{x}(t)$ and lead to the bounded prediction error.

In the architecture introduced in Eq. (3.21), it can be seen that the control input, $u(t)$, and the input disturbances, $\sigma(t)$, have the same dimensions as the state vector, $x(t)$. As such, the control input can directly account for the effect of the input disturbances; this is referred to as a matched system. However, in the formation flying problem, the dimensions of the control input do not match the dimensions of the state vector. This is because the derivative of the state vector, $\dot{x}(t)$, consists of both velocity and acceleration, whereas the control input is representative of a control force that would effect the dynamics only through the acceleration portion of the system dynamics. As such, the dimensions of the control input do not match the dimensions of the state vector; these systems are referred to as unmatched systems. The $\mathcal{L}_1$ adaptive control architecture for formation flying, which is introduced below, is structured to deal with unmatched systems.

Figure 3.6. Example 2: Analysing the behaviour of the adaptive estimates

3.3 $\mathcal{L}_1$ Adaptive Control Architecture for the SFF Problem

The $\mathcal{L}_1$ adaptive control architecture required for the spacecraft formation flying problem is slightly different than the architectures introduced in the previous sections. As we have already discussed, the architecture must account for the unmatched dynamics found in the formation flying problem. Furthermore, the spacecraft formation flying problem is a non-linear problem and requires the architecture to be capable of dealing with non-linear systems. Also, the architectures introduced previously are for single input-single-output (SISO) systems, whereas the formation flying problem is a multi-input-multi-output (MIMO) system. As such, while the $\mathcal{L}_1$ adaptive control

architectures introduced so far are excellent to gain an understanding on implementation of the architecture; they do need to be expanded to be able to deal with the formation flying problem. The $\mathcal{L}_1$ adaptive control architecture for non-linear systems in the presence of unmatched uncertainties has the capability to be used for the formation flying problem and will be introduced below.

3.3.1 $\mathcal{L}_1$ Adaptive Controller for Non-Linear Systems in the Presence of Unmatched Uncertainties

Let's begin by considering the following system dynamics:

$$\begin{aligned}
 \dot{x}(t) &= A_m x(t) + B_m(\omega u(t) + f_1(t, x(t), z(t))) + B_{um} f_2(t, x(t), z(t)), \\
 x(0) &= x_0 \\
 y(t) &= Cx(t) \\
 \dot{x}_z(t) &= g(t, x_z(t), x(t)), \quad x_z(0) = x_{z0} \\
 z(t) &= g_0(t, x_z(t))
 \end{aligned} \tag{3.29}$$

where $x(t) \in \mathbb{R}^n$ is the system state vector and $y(t) \in \mathbb{R}^m$ is the regulated output. The matrix A_m is a known Hurwitz $n \times n$ matrix that is observable and controllable and describes the desired dynamics of the closed-loop system; $u(t) \in \mathbb{R}^m$ is the control signal ($m \leq n$); $B_m \in \mathbb{R}^{n \times m}$ is a known, controllable, full-rank constant matrix; $B_{um} \in \mathbb{R}^{n \times (n-m)}$ is a constant matrix such that $B_m^T B_{um} = 0$ and $\text{rank}([B_m, B_{um}]) = n$; $C \in \mathbb{R}^{m \times n}$ is a known, observable, full-rank constant matrix; $\omega \in \mathbb{R}^{m \times m}$ is the unknown system input gain matrix; $z(t) \in \mathbb{R}^p$ and $x_z(t) \in \mathbb{R}^l$ are the output and state vector of the unmodelled dynamics; and $f_1 : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^m$ and $f_2 : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^{(n-m)}$ are functions representing the matched and unmatched portions of the non-linear uncertainties, respectively, which also satisfy:

$$\begin{bmatrix} f_1(t, x(t), z(t)) \\ f_2(t, x(t), z(t)) \end{bmatrix} = [B_m \ B_{um}]^{-1} f(t, x(t), z(t)) \tag{3.30}$$

The corresponding state predictor is given by:

$$\begin{aligned}
\dot{\hat{x}}(t) &= A_m \hat{x}(t) + B_m(\hat{\omega}u(t) + \hat{\theta}_1^T \|x(t)\|_{\mathcal{L}^\infty} + \hat{\sigma}_1(t)) + B_{um}(\hat{\theta}_2^T \|x(t)\|_{\mathcal{L}^\infty} + \hat{\sigma}_2(t)) \\
\hat{x}(0) &= x_0 \\
\hat{y}(t) &= C\hat{x}(t)
\end{aligned} \tag{3.31}$$

where, $\hat{\omega} \in \mathbb{R}^m \times m$, $\hat{\sigma}_1(t) \in \mathbb{R}^m$, $\hat{\theta}_1(t) \in \mathbb{R}^m$, $\hat{\sigma}_2(t) \in \mathbb{R}^{n-m}$ and $\hat{\theta}_2(t) \in \mathbb{R}^{n-m}$ are adaptive estimates.

However, there is no use of an unknown parameter in the method used to model the formation flying problem. As such, the state predictor can be simplified to:

$$\begin{aligned}
\dot{\hat{x}}(t) &= A_m \hat{x}(t) + B_m(\hat{\omega}u(t) + \hat{\sigma}_1(t)) + B_{um}\hat{\sigma}_2(t) \\
\hat{x}(0) &= x_0 \\
\hat{y}(t) &= C\hat{x}(t)
\end{aligned} \tag{3.32}$$

Furthermore, the projection-based adaptive laws, introduced earlier can be used as the adaptation mechanism:

$$\begin{aligned}
\dot{\hat{\omega}}(t) &= \Gamma Proj(\hat{\omega}(t), -(\tilde{x}^T(t)PB_m)^T u(t)), & \hat{\omega}(0) &= \hat{\omega}_0 \\
\dot{\hat{\sigma}}_1(t) &= \Gamma Proj(\hat{\sigma}_1(t), -(\tilde{x}^T(t)PB_m))^T, & \hat{\sigma}_1(0) &= \hat{\sigma}_{10} \\
\dot{\hat{\sigma}}_2(t) &= \Gamma Proj(\hat{\sigma}_2(t), -(\tilde{x}^T(t)PB_{um}))^T, & \hat{\sigma}_2(0) &= \hat{\sigma}_{20}
\end{aligned} \tag{3.33}$$

where $\tilde{x} \triangleq \hat{x}(t) - x(t)$ is the prediction error, $\Gamma \in \mathbb{R}^+$ is the adaptation gain, and $P = P^T > 0$ solves the Lyapunov equation $A_m^T P + PA_m = -Q$ for an arbitrary symmetric $Q = Q^T > 0$. Also, the adaptive estimates are bounded their respective sets.

Before proceeding any further, it is important to introduce some key components

of the $\mathcal{L}_1$ adaptive controller:

$$\begin{aligned}
H_{xm} &\triangleq (s\mathbb{I}_n - A_m)^{-1}B_m \\
H_{xum} &\triangleq (s\mathbb{I}_n - A_m)^{-1}B_{um} \\
H_m &= C(s\mathbb{I}_n - A_m)^{-1}B_m \\
H_{um} &= C(s\mathbb{I}_n - A_m)^{-1}B_{um}
\end{aligned} \tag{3.34}$$

Furthermore, the $\mathcal{L}_1$ adaptive controller involves the use of a feedback gain matrix $K \in \mathbb{R}^{m \times m}$ and an $m \times m$ strictly proper transfer matrix $D(s)$ which are combined, for all $\omega \in \Omega$, to form a strictly proper and stable transfer function $C(s)$, which is given by:

$$C(s) \triangleq wKD(s)(\mathbb{I}_m + wKD(s))^{-1}, \quad C(0) = \mathbb{I}_m \tag{3.35}$$

The choice of $D(s)$ also needs to ensure that $C(s)H_m^{-1}(s)$ is a proper stable transfer matrix. Furthermore, the choice of K needs to ensure that $-\omega K$ is Hurwitz.

For proof of stability, the system must meet the following $\mathcal{L}_1$ norm condition:

$$\|G_m(s)\|_{\mathcal{L}_1} L_1 + \|G_{um}(s)\|_{\mathcal{L}_1} L_2 < 1 \tag{3.36}$$

where,

$$\begin{aligned}
G_m(s) &\triangleq H_{xm}(s)(\mathbb{I}_m - C(s)) \\
G_{um}(s) &\triangleq (\mathbb{I}_n - H_{xm}(s)C(s)H_m^{-1}(s)C(s))H_{xum}(s)
\end{aligned} \tag{3.37}$$

and L_1 and L_2 are the uniform bounds on the derivatives of the matched and unmatched non-linear uncertainties.

The control signal can now be defined as:

$$u(s) = -KD(s)\hat{\eta}(s) \tag{3.38}$$

where $\hat{\eta}(s)$ is the Laplace transform of the signal:

$$\hat{\eta}(t) \triangleq \hat{\omega}(t)u(t) + \hat{\eta}_1(t) + \hat{\eta}_{2m}(t) - r_g(t)$$

with $r_g(s) \triangleq K_g(s)r(s)$, $\hat{\eta}_{2m}(s) \triangleq H_m^{-1}(s)H_{um}(s)\hat{\eta}_2(s)$, and $\hat{\eta}_1(t)$ and $\hat{\eta}_2(t)$ defined

as: $\hat{\eta}_i(t) \triangleq \hat{\sigma}_i(t)$ for $i = 1, 2$. It is important to note that $K_g(t)$ is a BIBO-stable feedforward pre-filter that will be used as $K_g(t) = -(CA_m^{-1}B_m)^{-1}$. Just as is the case with previous architectures, the adaptive parameters and tracking signal are calculated in the time domain and passed through the low pass filter, $D(s)$, to obtain the control signal $u(t)$ in the time domain.

For this specific control architecture to be used, a few criteria must be met. First of all, the non-linear uncertainties must be uniformly bounded for all $t \geq 0$. Furthermore, the $\mathcal{L}_1$ norm condition requires that the partial derivatives of the non-linear uncertainties be uniformly bounded with respect to x for all $x \in \mathbb{R}^n$. Finally, the transmission zeros of the transfer matrix $H_m(s)$ must lie in the open left half plane. There is also an assumption that the stability of unmodelled dynamics, x_z , is BIBO stable with respect to both initial conditions, x_{z0} , and the state vector $x(t)$. However, the spacecraft formation flying problem does not have any unmodelled dynamics and as such, this criteria is a non-factor.

We can also show that the prediction error is bounded, at any time instant, τ , same as the $\mathcal{L}_1$ control architecture for the linear systems introduced before:

$$\|\tilde{x}_\tau\|_{\mathcal{L}_\infty} \leq \sqrt{\frac{\theta_m(\rho r)}{\lambda_{\min}(P)\Gamma}} \quad (3.39)$$

where,

$$\begin{aligned} \theta_m(\rho r) &\triangleq 4(\max_{\omega \in \Omega} \text{tr}(\omega^T \omega)^2 + \sigma_{b1}^2)m + \sigma_{b2}(n - m) + 4\frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)}((\sigma_{b1}d_{\sigma_1})m + (\sigma_{b2}d_{\sigma_2})(n - m)) \\ \|\hat{\sigma}_i\|_\infty &< \sigma_{bi} \quad \text{for } i = 1, 2 \\ \|\dot{\sigma}_1(t)\| &< d_{\sigma_1} < \infty \quad \forall t \geq 0 \\ \|\dot{\sigma}_2(t)\| &< d_{\sigma_2} < \infty \quad \forall t \geq 0 \end{aligned} \quad (3.40)$$

3.3.2 Piecewise Constant Adaptive Laws

The piecewise constant adaptive law (PCAL) uses a fast estimation scheme and the adaptation rate can be directly associated with the sampling rate of an available CPU, which is beneficial in real-time operations [33, 52]. For the formation flying

problem, it can be paired with the same system dynamics introduced in Eq. (3.29) and it has been shown to have similar performance bounds to the projection-based adaptive laws introduced earlier [33].

Let's now modify the state predictor model for the PCAL:

$$\begin{aligned}\dot{\hat{x}}(t) &= A_m \hat{x}(t) + B_m(\omega_0 u(t) + \hat{\sigma}_1(t)) + B_{um} \hat{\sigma}_2(t) \\ \hat{x}(0) &= x_0 \\ \hat{y}(t) &= C \hat{x}(t)\end{aligned}\tag{3.41}$$

where everything is the same as before except ω_0 , which is the best initial guess for ω .

For this system, let $T_s > 0$ be the adaptation sampling time, which can be associated with the sampling rate of the available CPU. We can now introduce:

$$\Phi(T_s) \triangleq A_m^{-1}(e^{A_m T_s} - \mathbb{I}_n)\tag{3.42}$$

which can be used to define the piecewise constant adaptive laws for $\hat{\sigma}_1(t)$ and $\hat{\sigma}_2(t)$:

$$\begin{aligned}\begin{bmatrix} \hat{\sigma}_1(t) \\ \hat{\sigma}_2(t) \end{bmatrix} &= \begin{bmatrix} \hat{\sigma}_1(iT_s) \\ \hat{\sigma}_2(iT_s) \end{bmatrix}, \quad t \in [iT_s, (i+1)T_s], \\ \begin{bmatrix} \hat{\sigma}_1(iT_s) \\ \hat{\sigma}_2(iT_s) \end{bmatrix} &= - \begin{bmatrix} \mathbb{I}_m & 0 \\ 0 & \mathbb{I}_{n-m} \end{bmatrix} B^{-1} \Phi^{-1}(T_s) \mu(iT_s)\end{aligned}\tag{3.43}$$

with,

$$\mu(iT_s) = e^{A_m T_s} \tilde{x}(iT_s)\tag{3.44}$$

Following which the control signal can be defined as:

$$u(s) = -KD(s)\hat{\eta}(s)\tag{3.45}$$

where $\hat{\eta}(s)$ is the Laplace transform of the signal:

$$\hat{\eta}(t) \triangleq \omega_0(t)u(t) + \hat{\eta}_1(t) + \hat{\eta}_{2m}(t) - r_g(t)$$

with $r_g(s) \triangleq K_g(s)r(s)$, $\hat{\eta}_{2m}(s) \triangleq H_m^{-1}(s)H_{um}(s)\hat{\eta}_2(s)$; $\hat{\eta}_1(t)$ and $\hat{\eta}_2(t)$ defined as: $\hat{\eta}_i(t) \triangleq \hat{\sigma}_i(t)$ for $i = 1, 2$ and $K_g(t) = -(CA_m^{-1}B_m)^{-1}$ being defined same as before.

The PCAL also requires the non-linear uncertainties to meet the semiblobal Lipschitz condition, which implies that, for an arbitrary $\delta > 0$, there exist positive $K_{1\delta}$ and $K_{2\delta}$, such that:

$$\|f_i(t, X_1) - f_i(t, X_2)\|_\infty \leq K_{i\delta}\|X_1 - X_2\|_\infty, \quad \forall i = 1, 2 \quad (3.46)$$

Furthermore, similar to the $\mathcal{L}_1$ architecture with the projection type laws, it can be shown that the prediction error is bounded, at any time instant τ , if the state vector and control input are also bounded [33].

More information on the $\mathcal{L}_1$ adaptive control architecture for non-linear systems in the presence of unmatched uncertainties along with the projection and piecewise constant adaptive laws, including performance and stability derivations can be found in references [33, 52].

With the $\mathcal{L}_1$ architectures now introduced, the next chapter presents the design and numerical simulation validation of the $\mathcal{L}_1$ adaptive controller for the spacecraft formation flying problem.

Chapter 4

Simulation Results

In this chapter, the spacecraft formation flying dynamics will be converted to state-space form and integrated into the $\mathcal{L}_1$ adaptive controller. The controllers design will be highlighted, especially pertaining to the system requirements introduced in Chapter 3. A proportional-derivative (PD) controller will be introduced as a benchmark to test and compare the performance of the $\mathcal{L}_1$ adaptive controller. Following which, the on-orbit simulations will be presented for a desired circular formation with radius of 100 km using the HCW, full non-linear and double integrator formation flying dynamic models.

4.1 Introduction

With the dynamic model and the $\mathcal{L}_1$ adaptive control architectures now introduced, the main thesis objective of designing an $\mathcal{L}_1$ adaptive controller for the formation flying problem can be addressed. This involves converting the dynamics introduced in Chapter 2 into a state-space formulation, as required for the $\mathcal{L}_1$ adaptive control architecture introduced in Chapter 3, as well as selecting the various design parameters.

A decision was made to design both the $\mathcal{L}_1$ adaptive controller and the PD controller around the HCW dynamic model and then keep the same controller with the different dynamic models to demonstrate the controller's ability to handle various dynamics and uncertainties. The HCW dynamic model was chosen as the baseline model because it has been shown to be an accurate linearisation of the general non-linear model in a circular orbit with small relative distances, which results in a good trade off between model accuracy and computational requirements for the initial controller design [47].

All of the on-orbit simulations will be conducted for a formation where the leader

satellite is orbiting Earth with a semi-major axis of $a = 7200$ km. Furthermore, the chaser satellite mass was selected to be $m = 9.7$ kg.

The HCW model introduced in equations (2.28) to (2.30) can be written in state-space form as:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 3n^2 & 0 & 0 & 0 & 2n & 0 \\ 0 & 0 & 0 & -2n & 0 & 0 \\ 0 & 0 & -n^2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} \quad (4.1)$$

Using the fact that we wish for the system to be controlled through a control force in each axis (i.e $u \in \mathbb{R}^3$), it can be seen that the dynamics fits into the following control architecture:

$$\begin{aligned} \dot{x} &= Ax + Bu; \\ y &= Cx + Du; \end{aligned} \quad (4.2)$$

where,

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} \quad A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 3n^2 & 0 & 0 & 0 & 2n & 0 \\ 0 & 0 & 0 & -2n & 0 & 0 \\ 0 & 0 & -n^2 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \frac{1}{m} \begin{bmatrix} 0_{3 \times 3} \\ \mathbb{I}_{3 \times 3} \end{bmatrix} \quad C = \begin{bmatrix} \alpha \mathbb{I}_{3 \times 3} & \mathbb{I}_{3 \times 3} \end{bmatrix} \quad D = 0;$$

where $\alpha = 1$, denotes the positive-definite ratio of position to velocity. This ensures that the relative position can still be controlled through the position portion of the system output; however, the addition of the velocity component introduces beneficial damping affects into the control signal [25].

This architecture can now be used in the design of the PD controller.

uncertainties introduced in Sec. 3.3, with the uncertainties modelled as:

$$f_1(t) = \begin{bmatrix} 2 + 2 \sin(0.1t) \\ 2 + 2 \sin(0.1t) \\ 2 + 2 \sin(0.1t) \end{bmatrix}$$

$$f_2(t) = 0;$$

The unmatched uncertainties was chosen to be 0 because in the formation flying problem, all of the uncertainties will present itself as external perturbation forces, which fall under the matched uncertainties. Furthermore, the design involves:

1. Selecting the static feedback gain k_m , such that the state matrix A_m is Hurwitz and specifying the desired closed-loop dynamics.
2. Insuring the transmission zeros of the transfer matrix H_m lie in the left-hand side of the complex plane.
3. Selecting the bandwidth of the low-pass filter $C(s)$, along with the input gain, ω , and feedback gain, K , so the system can track the reference signals and compensate for the uncertainties within a pre-specified range of frequencies.
4. Selecting the bounds for the adaptive control parameters, including the uncertainties and the uncertain input gain.

As introduced in Eq. (3.29), the $\mathcal{L}_1$ adaptive controller for spacecraft formation flying can be represented by the following system:

$$\begin{aligned} \dot{x}(t) &= A_m x(t) + B_m(\omega u(t) + f_1(t, x(t), z(t))) + B_{um} f_2(t, x(t), z(t)), \\ x(0) &= x_0 \\ y(t) &= Cx(t) \end{aligned}$$

where,

$$A_m = A - B_m k_m^T$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 3n^2 & 0 & 0 & 0 & 2n & 0 \\ 0 & 0 & 0 & -2n & 0 & 0 \\ 0 & 0 & -n^2 & 0 & 0 & 0 \end{bmatrix}$$

$$B_m = \frac{1}{m} \begin{bmatrix} 0_{3 \times 3} \\ \mathbb{I}_{3 \times 3} \end{bmatrix} \quad B_{um} = \frac{1}{m} \begin{bmatrix} \mathbb{I}_{3 \times 3} \\ 0_{3 \times 3} \end{bmatrix} \quad C = \begin{bmatrix} \alpha \mathbb{I}_{3 \times 3} & \mathbb{I}_{3 \times 3} \end{bmatrix}, \quad \alpha = 1$$

and where k_m was determined through trial and error to be:

$$k_m = 30 \begin{bmatrix} \mathbb{I}_{3 \times 3} \\ \mathbb{I}_{3 \times 3} \end{bmatrix}$$

which, as shown in Table 4.1, leads to A_m having eigenvalues with strictly negative real parts, thus proving that A_m is Hurwitz [49].

Table 4.1. Eigenvalues of A_m using HCW dynamics for the simulation orbit

Eigenvalues
$\lambda_1 = -1.5483 + 0.8386i$
$\lambda_2 = -1.5483 - 0.8386i$
$\lambda_3 = -1.5445 + 0.8365i$
$\lambda_4 = -1.5445 - 0.8365i$
$\lambda_5 = -1.5464 + 0.8375i$
$\lambda_6 = -1.5464 - 0.8375i$

This was also confirmed by checking the transmission zeros of the transfer matrix H_m , all of which, as seen in Table 4.2, have negative real numbers. One of the most important design criteria for the $\mathcal{L}_1$ adaptive controller is the selection of $D(s)$, ω and K , which leads to a proper and stable low-pass filter $C(s)$.

Table 4.2. Transmission zeros of H_m using HCW dynamics for the simulation orbit

Transmission Zeros
$z_1 = z_2 = z_3 = -1.0000 + 0.0000i$
$z_4 = -1.5464 + 0.8375i$
$z_5 = -1.5464 - 0.8375i$
$z_6 = z_7 = z_8 = -1.5483 + 0.8386i$
$z_9 = z_{10} = z_{11} = -1.5483 - 0.8386i$
$z_{12} = z_{13} = z_{14} = -1.5445 + 0.8365i$
$z_{15} = z_{16} = z_{17} = -1.5445 - 0.8365i$

The term ω is the uncertain system input gain and for the purposes of the $\mathcal{L}_1$ adaptive control architecture, one must only know the sign and initial estimate of the parameter. For the spacecraft formation flying problem, ω was estimated to be 1, as this does not alter the input control signal. The selection of K must insure that $-\omega K$ is Hurwitz. Furthermore, the transfer function representing the system can be represented by a cascade of a high-pass filter $(\mathbb{I}_m - C(s))$ and low-pass filters H_m and H_{um} . It can be shown that as the maximum negative eigenvalue of $-\omega K$ goes to infinity, the $\mathcal{L}_1$ norm of the high pass filter goes to 0:

$$\lim_{\lambda_{max}(-\omega K) \rightarrow \infty} \|\mathbb{I}_m - C(s)\|_{\mathcal{L}_1} = 0$$

Linear matrix inequalities were used to demonstrate that a reasonable upper bound on the $\mathcal{L}_1$ norm of a system (see Appendix A), defined by the HCW spacecraft formation flying dynamics, is $\|G(s)\|_{\mathcal{L}_1} = 0.023$. As such, K does not need to be very high and was selected as:

$$K = \mathbb{I}_m$$

and, $-\omega K$ was found to be Hurwitz since all of its eigenvalues have strictly negative real parts (ie. $\lambda_1 = \lambda_2 = \lambda_3 = -1$).

The selection of $D(s)$ is more challenging as it determines the bandwidth of the low-pass filter $C(s)$. Increasing the bandwidth of $C(s)$, also reduces the $\mathcal{L}_1$ norm of the system $\|G(s)\|_{\mathcal{L}_1}$, which increases the performance of the system; however this results in reducing the time-delay margin of the system which reduces the robustness

of the system. An appropriate $D(s)$ was determined through trial and error to be:

$$D(s) = \frac{1}{s(\frac{s}{140} + 1)(\frac{s}{175} + 1)(\frac{s}{500} + 1)(\frac{s}{115} + 1)(\frac{s}{225} + 1)(\frac{s}{5} + 1)(\frac{s}{200} + 1)} \mathbb{I}_3$$

A comparison between a low bandwidth filter with $D(s)_1 = \frac{1}{s} \mathbb{I}_3$ and $D(s)_2 = D(s)$ can be seen in Fig. 4.2. Beyond just improving the performance of the system, it can be seen that the higher bandwidth filter also reduces noise in the control signal and furthermore results in a $3\times$ reduction in the control signal magnitude, which is a significant advantage.

Finally, the last design decisions are related to the adaptive laws. For the projection type adaptive laws, the bounds need to be selected for the matched and unmatched uncertainties and also the uncertain input gain. Since the maximum value for the matched uncertainties, f_1 , is 4 and the initial estimate of ω was chosen to be 1, the bounds for the uncertainties and uncertain input gain were chosen conservatively as:

$$|\hat{\sigma}_1| \leq 10, |\hat{\sigma}_2| \leq 10, 0.5 \leq \hat{\omega} \leq 1.3$$

Also, the adaptation gain was selected to be:

$$\Gamma = 800000$$

For the PCAL, the adaptive estimate of the uncertain input gain is replaced by the estimate of the uncertain input gain, $\hat{\omega} = \omega_0$, and the uncertainties are modelled according to Eqs. (3.42) - (3.44). The sampling time was determined by the fixed time step of the simulations and was set to: $T_s = 0.0001$ s.

4.4 On-orbit Simulations

The $\mathcal{L}_1$ and PD controllers designed above can now be used for the on-orbit simulations. Each dynamic model will be simulated under nominal and perturbed conditions, with the general non-linear model also simulated with an orbital eccentricity of $e = 0.3$. In nominal conditions, the uncertainties are set to $f_1 = f_2 = 0$, to represent

(a) Control Signal with the low bandwidth $D(s)_1$

(b) Control Signal with the higher bandwidth $D(s)_2$

Figure 4.2. Effect of the low-pass filter bandwidth on the control input

an ideal, unperturbed, case; this is extremely helpful in establishing the baseline performance of the controllers. With the perturbations included, the uncertainties are set to $f_1(t) = 2 + 2\sin(0.1t)$ and $f_2 = 0$, along each axis. Furthermore, all simulations are initialized with the following relative initial conditions:

$$r = \begin{bmatrix} 0.0 \\ -0.2 \\ 0.0 \end{bmatrix} \text{ m}, \quad v = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \frac{\text{m}}{\text{s}}$$

The simulations for the HCW dynamic model are presented first; this is followed by the general non-linear model and finally the DI model simulations will be shown. The DI integrator model simulations will also demonstrate the capability of the $\mathcal{L}_1$ adaptive controller to adequately control the robotic spacecraft platforms in the hardware experiments conducted in Chap. 5, which are also modelled by the DI model.

4.4.1 HCW Model: Nominal Conditions

As can be seen in Figs. 4.3 - 4.6, both the $\mathcal{L}_1$ and PD control algorithms yield good tracking of the desired position. Furthermore, in the case of the $\mathcal{L}_1$ adaptive controller, the PCALs and the projection type adaptive laws results in almost identical performance. Where the projection type adaptation law has some noise in its control signal, it can be seen that this does not effect the PCALs. The PCALs resulted in quicker simulation times with average simulation times taking between 7 s and 9 s for the PCAL scheme and 13 s to 15 s for the projection type adaptive laws, when simulating the HCW dynamic model. Also, it can be seen in Fig. 4.4 that prediction error for the $\mathcal{L}_1$ adaptive controller is bounded, which results in the tracking performance of the controller. Between the $\mathcal{L}_1$ adaptive controller and the PD controller, it is evident through looking at the in-plane motion, that the $\mathcal{L}_1$ adaptive controller has a much tighter performance bound. The maximum transient separation in the in-plane motion of the $\mathcal{L}_1$ adaptive controller is similar in magnitude to the maximum steady-state separation of the PD controller. Furthermore, the $3\times$ reduction in control magnitude, demonstrated by the careful selection of the low-pass filter, can be seen directly in the control force comparison between the PD and $\mathcal{L}_1$ controllers.

In the transient portion of the simulation, the $\mathcal{L}_1$ controller requires just under $3\times$ the control force as the PD controller and this difference is even greater in steady-state conditions where, the $\mathcal{L}_1$ adaptive controller requires almost $10\times$ less control force than the PD controller. The $\mathcal{L}_1$ adaptive controller also reaches steady-state conditions much sooner than the PD controller, with a settling time of 6 s, while the PD controller takes almost $4\times$ longer at 23 s to reach steady steady-state conditions. This is especially evident when looking at the absolute tracking error, which shows the total deviation from the desired circular formation and highlights the performance of the controllers.

Figure 4.3. On-orbit simulations: $\mathcal{L}_1$ adaptive controller, projection-based adaptive law and the HCW dynamic model in nominal conditions

Figure 4.4. On-orbit simulations: $\mathcal{L}_1$ adaptive controller, projection-based adaptive law and the HCW dynamic model in nominal conditions - prediction Error Bounds

(a) Out-of-plane motion

(b) In-plane motion

(c) Control signal $u(t)$ (d) Steady state control signal $u(t)$

(e) Deviation from the desired position in each axis

(f) Deviation from the circle

Figure 4.5. On-orbit simulations: $\mathcal{L}_1$ adaptive controller, PCAL scheme and the HCW model in nominal conditions

Figure 4.6. On-orbit simulations: PD controller with the HCW model in nominal conditions

4.4.2 HCW Model: Perturbations Conditions

The results from the perturbation conditions can be seen in Figs. 4.7 - 4.9. The $\mathcal{L}_1$ adaptive controller with both the adaptive laws achieve satisfactory tracking results for both the in-plane and out-of-plane motion. However, the PCAL scheme shows some high frequency oscillations in the transient portion of the control signal which results in a high frequency oscillation in the in-plane motion of the satellite. Furthermore, one can see that although the control signals resulting from the two adaptive laws are very similar, the maximum deviation seen in the in-plane motion is less when using the projection type adaptive law. Since the PCAL and projection type adaptive laws have similar performance, the trade off come down to the minor gains in performance bounds and a slightly more robust performance with the projection type adaptive laws, compared to faster computation times for the PCAL scheme. For the spacecraft formation flying problem, the computational load was demonstrated to be a non-issue for either adaptive law and, as such, the projection type adaptive law will be used as the main adaptive law in the $\mathcal{L}_1$ adaptive controller moving forward. The benefit of which is a tighter performance bounds and no high frequency oscillations in the control signal resulting from the adaptive laws.

When comparing to the results obtained in nominal conditions, the $\mathcal{L}_1$ adaptive controller in perturbed conditions exhibits good tracking performance regardless of the perturbations. The performance of the $\mathcal{L}_1$ adaptive controller is highlighted when compared to the PD controller. One can see that the in-plane deviation of the PD controller reaches up to 2.5 m in the x-axis, where by comparison, the $\mathcal{L}_1$ adaptive controller with the PCAL scheme has maximum in-plane deviation of 0.175 m in the x-axis. The PD controller also cannot adapt to the perturbations and the oscillations resulting from the perturbations can be seen in the out-of plane motion. The control force follows a similar trend to the nominal case, with the $\mathcal{L}_1$ adaptive controller having a maximum transient control force just under $3\times$ less than the equivalent PD controller and a maximum steady-state control force approximately $10\times$ less than the PD controller, showcasing both the transient and steady-state performance of the $\mathcal{L}_1$ adaptive controller.

Figure 4.7. On-orbit simulations: $\mathcal{L}_1$ adaptive controller, projection-based adaptive law - HCW dynamic model with perturbations

(a) Out-of-plane motion

(b) In-plane motion

(c) Control signal $u(t)$ (d) Steady state control signal $u(t)$

(e) Deviation from the desired position in each axis

(f) Deviation from the circle

Figure 4.8. On-orbit simulations: $\mathcal{L}_1$ adaptive controller, PCAL scheme - HCW model with perturbations

Figure 4.9. On-orbit simulations: PD controller - HCW model with perturbations

4.4.3 Non-Linear Model

As mentioned earlier, the HCW dynamic model is an accurate model for the relative dynamics of spacecraft, where the leader is in a circular orbit and the relative distances between the spacecraft are much smaller than the orbital radius of the leader spacecraft. The biggest advantage of the HCW dynamic model is the computational simplicity of the model; this however, results in a lot of restrictions in the mission design and the orbit selection for the formation flying mission. To alleviate these restrictions, the general non-linear dynamic model should be used.

In terms of controller implementation, the general non-linear dynamics will be utilized. As a result, the first change is related to the definition of the matrix A , which is now time-varying:

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ (\dot{\nu} - \frac{\mu}{r_f^3}) & \ddot{\nu} & 0 & 0 & 2\dot{\nu} & 0 \\ -\ddot{\nu} & (\dot{\nu} - \frac{\mu}{r_f^3}) & 0 & -2\dot{\nu} & 0 & 0 \\ 0 & 0 & -\frac{\mu}{r_f^3} & 0 & 0 & 0 \end{bmatrix}$$

where $r_f = |\vec{r}_f|$, is the inertial position of the follower satellite, which can be found using its relationship to the leader satellite, r_l , derived in Eq. (2.3). The true anomaly, ν and radius, r_l , of the leader satellite can be determined using the relationships derived in Eqs. (2.12) and (2.16).

The second change is related to the addition of the non-linear matrix, $G(x)$, in the equation of motion [7]. This can be reflected through the following change in the state-space representation of the system:

$$\dot{x}(t) = A_m x(t) + B_m(\omega u(t) + f_1(t, x(t), z(t))) + B_{um} f_2(t, x(t), z(t)) + G(x),$$

$$x(0) = x_0$$

$$y(t) = Cx(t)$$

where $G(x)$ is defined as,

$$G(x) = \begin{bmatrix} 0 & 0 & 0 & \mu(\frac{1}{r_l^2} - \frac{r_l}{r_f^3}) & 0 & 0 \end{bmatrix}^T$$

All other parameters of the $\mathcal{L}_1$ adaptive controller and the PD controller remain the same as the controller for the HCW dynamic model.

Similar to the HCW dynamic model simulations, the first simulation will take place in the same circular orbit as the HCW dynamic model in nominal conditions. Following this, the perturbations simulations will be presented and finally the case with an elliptical orbit and perturbations will be presented.

4.4.4 Non-Linear Model: Nominal

One can see that the results presented in Figs. 4.10 and 4.11, from the non-linear model in nominal conditions, are nearly identical to the results obtained from using the HCW dynamic model. This first of all, validates the use of the HCW model in circular orbits with small relative distances, as the results show that the simplifications made during the linearisation hold under the specific dynamic conditions. The relative trajectories resulting from both controllers are nearly identical and the control forces also look very similar to the HCW simulation results, with the $\mathcal{L}_1$ controller requiring a lot less control input than the PD controller, while maintaining tighter bounds. The one small difference, when comparing the HCW simulation to the non-linear nominal simulation would be the noise in the control signal of the $\mathcal{L}_1$ adaptive controller. It can be seen that there is less visible noise in the non-linear control signal of the $\mathcal{L}_1$ adaptive controller.

Figure 4.10. On-orbit simulations: $\mathcal{L}_1$ adaptive controller - Non-Linear model in nominal conditions

Figure 4.11. On-orbit simulations: PD controller - Non-Linear model in nominal conditions

4.4.5 Non-Linear Model: Perturbed

Similarly, the results from the perturbed conditions on the circular orbit with the non-linear model, shown in Figs. 4.12 and 4.13, are very similar to the results from the perturbed HCW model simulations. The $\mathcal{L}_1$ adaptive controller performs better in the in-plane motion when compared to the HCW model, when utilizing the non-linear model, with improved steady-state tracking and smoother trajectories. The PD controller with the non-linear model behaves extremely similarly to the PD controller in the HCW model. The control forces and the error bounds are all very similar to the perturbation simulations utilizing the HCW model and both demonstrate that the $\mathcal{L}_1$ adaptive controller outperforms the PD controller.

Considering a perturbed elliptical orbit adds complexity to the problem and tests the control algorithms even further. The results, shown in Figs. 4.14 and 4.15, show that an elliptical orbit has minimal effect on the controller performance with both the $\mathcal{L}_1$ and PD performing very similarly to the to the perturbed circular orbit. The effect of the elliptical orbit can be seen more clearly through the coupled in-plane motion on the $\mathcal{L}_1$ adaptive controller, with its smaller control forces and tighter performance bounds. The elliptical orbit does result in a slightly worse error bound on the $\mathcal{L}_1$ adaptive controller when compared to the circular orbit, but it is still noticeably better than the PD controller.

Figure 4.12. On-orbit simulations: $\mathcal{L}_1$ adaptive controller - Non-Linear model and perturbations

Figure 4.13. On-orbit simulations: PD controller - Non-Linear model and perturbations

Figure 4.14. On-orbit simulations: $\mathcal{L}_1$ adaptive controller - Non-Linear model with perturbations and elliptical orbit

Figure 4.15. On-orbit simulations: PD controller - Non-Linear model with perturbations and elliptical orbit

4.4.6 Double Integrator Model

The final set of simulations are conducted using the Double Integrator Model. The model is further simplified than the HCW dynamic model by decoupling the in-plane motion from the out-of-plane motion. As demonstrated in the HCW and non-linear nominal simulations presented above, the in-plane separation is less than 0.2 m on a 100 m projected circular relative trajectory; under such circumstances, the dynamics coupling can be neglected. The hardware experiments that will be shown in Chapter 5, are based on planar circular formation conducted on a granite table. This experimental facility effectively emulates the double integrator dynamics. The purpose of using the double integrator model in the simulations is to demonstrate that the same $\mathcal{L}_1$ adaptive controller can adequately deal with the simplified dynamics introduced in with this model.

In order to implement the Double Integrator model, all components of the $\mathcal{L}_1$ and PD controller are kept the same, except of course, the dynamic state matrix A , which is changed to represent the Double Integrator dynamics:

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

As shown in Figs. 4.16 and 4.17, perfect tracking of the in-plane motion is obtained under nominal conditions, with similar tracking as other dynamic models in the out-of-plane motion. The perturbed case, shown in Figs. 4.18 and 4.19, exhibits very similar results to what has been obtained with the other dynamics models.

To summarize, the simulation results demonstrated that the $\mathcal{L}_1$ adaptive controller outperforms the benchmark PD controller, in terms of handling unknown perturbations. This improved performance will also be demonstrated through hardware experiments in the next chapter.

Figure 4.16. On-orbit simulations: $\mathcal{L}_1$ adaptive controller - Double Integrator model in nominal conditions

Figure 4.17. On-orbit simulations: PD controller - Double Integrator model in nominal conditions

Figure 4.18. On-orbit simulations: $\mathcal{L}_1$ adaptive controller - Double Integrator model with perturbations

Figure 4.19. On-orbit simulations: PD controller - Double Integrator model with perturbations

Chapter 5

Hardware Experiments

In this chapter, the $\mathcal{L}_1$ adaptive controller developed and demonstrated in Chap. 4 is implemented onto the floating spacecraft platforms Carleton University’s Spacecraft Proximity Operations Testbed (SPOT) facility and used to test and compare the controller’s performance in a real world hardware experiment setup. The effects of unknown mass and unmodelled perturbations will be analysed on the $\mathcal{L}_1$ adaptive controller’s performance. All the results obtained with the $\mathcal{L}_1$ adaptive controller will be compared against a PD benchmark controller.

5.1 Introduction

One of the main thesis objectives was to validate the on-board capability of the $\mathcal{L}_1$ adaptive controller. The capability of the controller to deal with the SFF dynamics and various unmodelled perturbations have already been demonstrated in Chap. 4. As such, hardware experiments using the SPOT facility at Carleton University will not only validate the performance and usability of the $\mathcal{L}_1$ adaptive controller under various environmental conditions, but also provide insight on the ease of implementation and computational requirements associated with the controller. The hardware experiments also allow a unique opportunity to test the controller with an incorrectly modelled mass and unmodelled perturbations, and determine the effect of each on the performance of the controller.

5.2 SPOT Facility

The hardware experiments were conducted on the SPOT facility, which is the largest of its kind in Canada. The SPOT facility consists of a smooth, large and flat granite table, that is used as the centrepiece of the facility. Other components of the SPOT

facility are the air bearing floating spacecraft platforms; the air refill station; the ground truth tracking system and the ground station.

There are two spacecraft platforms that were designed and built to take advantage of the granite table's smooth surface, by using air-bearings to create a thin film of air between the spacecraft platforms and the granite table. This allows the platforms to float and creates a near friction-free environment for the floating spacecraft platforms, which allows the emulation of the double-integrator dynamics, found in deep-space or in very close-proximity operations. Both platforms were built with the same structure and basic systems; however, one of the spacecraft platforms was designed as the chaser platform, while the other was designed as the target. The chaser platform houses additional systems, including vision systems, and the capability for attachments, such as robotic arms. Both platforms use four air bearings to float and eight thrusters to rotate and translate around the table [54]. For the $\mathcal{L}_1$ adaptive control experiments only the chaser platform was utilized, as the purpose of formation flying is to control the relative position of the chaser relative to the leader, the leader position was set to the center of the table. At the heart of the spacecraft platform, is a Raspberry Pi 3 that is used to control the thrusters and implement the desired control algorithms. The control algorithms themselves are compiled on the ground station and the Raspberry Pi 3 executes it on board, in a fully autonomous way.

On the ground station, there is a pre-built MATLAB/Simulink program simulates the complete hardware experiment. There are different stages of the pre-built Simulink blocks, including, floating after receiving ground truth data; manoeuvring to the initial x-y location on the table; executing the experiment, and then manoeuvring to the end location before turning off the flotation system. The simulation can be used to properly tune control laws to ensure the desired performance is achieved. Once satisfied with the controller performance, the Simulink embedder coder is used to generate C++ code of the experiment diagram, which is then compiled into a format that can be executed by the on-board compiler on the Raspberry Pi 3.

In order for the control algorithm to work effectively, the true position of the spacecraft on the granite table must be fed into the chaser platform in real time. This is accomplished using the ground truth system and the ground station computer.

Figure 5.1. SPOT Facility at Carleton University

The ground truth system uses eight cameras that lock onto LEDs at each of the corners of the chaser platform to determine its position and attitude. The data is then transmitted to the ground station at 240 Hz. The data is stored on the ground station and the latest data is then transmitted wirelessly to the chaser platform at 20 Hz [54].

The data transfer frequencies are important because it means that the feedback control law can only run at 20 Hz frequency at best, which results in a time delay of up to 50 ms in the system, just from data acquisition. There are also other time delays in the system resulting from transmission delays and computations; however, these the time delays are extremely small and can be ignored. To account for all the time-delays in the system, the control algorithms are run at a frequency of 10 Hz.

The combination of all these components allow the SPOT facility to experimentally validate innovative spacecraft control technologies. This provides a unique opportunity to test and validate the $\mathcal{L}_1$ adaptive controller, simulated on-orbit in Chap. 4. The SPOT facility can be seen in Fig. 5.1.

5.3 Hardware Experiment Design

The main objective of the hardware experiments is to validate the on-orbit simulation performance of the $\mathcal{L}_1$ adaptive controller and conduct real world tests that expands the simulations by studying the coupled and decoupled effects of incorrect system mass and perturbations on the controller performance. This involves modifying the PD and $\mathcal{L}_1$ adaptive controllers for use with the spacecraft platforms; designing the mechanism of introducing perturbations and establishing comparison criteria to compare the performance of each of the controllers. Similar to the on-orbit simulations, the $\mathcal{L}_1$ and PD controllers aimed at tracking the platform along a 0.7 m circular trajectory on the table.

Modifying the controllers involve selecting the dynamic model that best represents the dynamics of the spacecraft platform and re-tuning the various gains, as well as the low-pass filter in the case of the $\mathcal{L}_1$ adaptive controller. This process is outlined in Sec. 5.4.

A decision was made to use a large fan to blow air to perturb the motion of the spacecraft platform. A large fan was installed on a rotating stand, and its location was marked for experiment repeatability. The perturbation fan and its location can be seen in Fig. 5.2. The platforms were made to float and the effects of the fan were studied on the motion of the platforms. The fan speeds were then selected for different segments of the platforms desired trajectories, based on the distance of the platforms from the fan and the magnitude of the perturbations at that distance.

In the incorrect mass experiments, 9 kg of mass was added to the center of mass (COM) of the spacecraft platform. The controllers were unaltered from the nominal case, and as such are validated in terms of their robustness to handle parametric uncertainty in the form of additional mass.

Furthermore, a major difference between the on-orbit simulations and the hardware experiments is the limited available thrust on-board the floating spacecraft platforms. As such, the calculated control force is fed through an on-board control mixer that actively reduces the thrust, proportionally along each axis. The control mixer works by using an experimentally-determined maximum thrust force for each thruster, and using the maximum thrust axis to proportionally scale all the thrust forces. This

Figure 5.2. Perturbation Device Acting on the Spacecraft Platform

experimentally-determined maximum thrust force adds some error in the $\mathcal{L}_1$ adaptive controller due to the fact that the maximum available thrust is not a singular maximum value, but a function of the number of active thrusters and air-pressure in the tank. For example, if only one of the eight thrusters is activated, the maximum thrust could be almost double the set maximum thrust, compared to when all eight thrusters are active. As such, this leads to discrepancies between the calculation of the estimated states, which uses the nominal thrust, and the actual states, as measured by the ground truth system.

In order to add some additional real-world limitations, a decision was made to use a PD controller to control the attitude of the satellite and have it always face toward the leader satellite at the center of the circle. This has the effect of reducing the maximum available thrust and again stresses a controller's ability to efficiently handle the various perturbations while sharing thrusting resources with the attitude controller.

With all this in place, the $\mathcal{L}_1$ adaptive controller performance will be evaluated through comparisons to a non-adaptive PD controller. Based on the results shown in Chap. 4, the $\mathcal{L}_1$ adaptive controller is expected to yield better performance, even in the presence of perturbations and parametric uncertainties. The coupled and uncoupled

effect of incorrect system mass and perturbations were not studied through on-orbit simulations, so this will provide additional insight on the effect of each on the overall performance of the controller. In order to demonstrate repeatability, each experiment was conducted five times, and the controller performance will be evaluated in terms of: (1) tracking performance, (2) capability to handle incorrect mass and external perturbations, and (3) thruster efficiency. The tracking performance and capability to cope with plant uncertainties and external perturbations are analysed through the ability to track the desired circular trajectory, and evaluating the overall deviation from the circle with the 3σ errors over the 5 experiments. A discrete digital thruster, on-off, graph for each of the 8 thrusters are used to compare the thrusters' activity for both controllers under the various operating scenarios.

5.4 Controller Design for Hardware Experiments

To implement the controllers on the platforms, it is essential to tune the control parameters for on-board use. The dynamic model used previously in simulations will be replaced by the real-time position received from the SPOT ground truth system. However, for the $\mathcal{L}_1$ adaptive controller, a dynamic model for the state predictor and low-pass filter for the control input is still required. The dynamics of the spacecraft platforms can be accurately modelled through the double integrator dynamic model introduced in Chap. 4 with the state matrix, A , defined as:

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

As shown previously, both the PD and $\mathcal{L}_1$ adaptive controllers can adequately control a system modelled by the double integrator dynamic model in the on-orbit simulations. For the laboratory experimentation, the complexity is increased due to various factors, such as actuator limitations.

The PD controller does not depend on a reference model and a state observer, and, as such, its design is simpler. The gains were tuned through simulations with a nominal chaser platform mass of 17.24 kg, and were selected as:

$$K_p = 0.023$$

$$K_d = 0.065$$

The design of the $\mathcal{L}_1$ adaptive controller is more complicated, as there is the design choice between the various adaptive laws and also the selection of the low-pass filter and the static feedback gain k_m , which makes the system matrix, A_m , Hurwitz. The first decision is related to the choice of the adaptive laws. Between the PCAL and the projection type adaptive laws, the PCAL looks promising for the hardware experiments due to its inherently low computational requirement. However, due to issues with finding a suitable sampling time, along with the presence of high frequency oscillations demonstrated in the on-orbit simulations, when dealing with perturbations, a decision was made not to use PCALs. As such, the projection type adaptive laws were chosen to update the adaptive parameters with the same state estimator used by the PCALs, introduced in Eqs. (3.41) and (3.45):

$$\dot{\hat{x}}(t) = A_m \hat{x}(t) + B_m(\hat{\omega}_0 u(t) + \hat{\sigma}_1(t)) + B_{um} \hat{\sigma}_2(t)$$

$$\hat{x}(0) = x_0$$

$$\hat{y}(t) = C^T \hat{x}(t)$$

The advantage being that the estimated dynamics model is controlled entirely through the matched and unmatched system uncertainties, which provides computational savings since the uncertain input gain does not need to be computed. As seen with the PCAL on-orbit simulations, this can result in bounds that are slightly less than the model including the uncertain system input gains. However, the use of the project laws eliminates the high frequency behaviour found in the on-orbit simulations.

The estimate of the uncertain system input gain and feedback gain matrix K were

selected to remain the same as in the simulations:

$$\begin{aligned} K &= \mathbb{I}_m \\ \hat{\omega}_0 &= 1 \end{aligned}$$

The static feedback gain k_m , however, was changed slightly to improve the controller performance:

$$k_m = 40 \begin{bmatrix} \mathbb{I}_{3 \times 3} \\ \mathbb{I}_{3 \times 3} \end{bmatrix}$$

Furthermore, to reduce the computational load, make the system less susceptible to measurement noise and improve the robustness of the system, the bandwidth of the low-pass filter was reduced with the selection of the transfer function:

$$D(s) = \frac{1}{s(\frac{s}{15} + 1)(\frac{s}{5} + 1)(\frac{s}{10} + 1)} \mathbb{I}_3$$

This was at the cost of increased control input forces required by the controller, but even with the reduced filter bandwidth, the $\mathcal{L}_1$ adaptive controller was still expected to be more efficient than the PD controller and the overall benefits far exceeded the downfalls in this case. Finally, the adaptation gain and the projection bounds were chosen as:

$$\begin{aligned} |\hat{\sigma}_1| &\leq 5, \quad |\hat{\sigma}_2| \leq 1, \\ \Gamma &= 80000 \end{aligned}$$

5.5 SPOT Simulations

With both the controllers designed, the next phase was to use the SPOT simulation tool to validate the platform behaviour prior to the experimental campaign. The trajectories of both the PD and $\mathcal{L}_1$ adaptive controllers were simulated for all 4 experiments: nominal; with unknown mass; with unknown perturbations and with unknown mass and perturbations. This provides a good insight as to what to expect in terms of experimental performance. The nominal simulation results from the $\mathcal{L}_1$ adaptive controller and PD controller are provided in Figs. 5.3 and 5.4.

From these figures, it is evident that both controllers achieve excellent trajectory

Figure 5.3. Simulation of the $\mathcal{L}_1$ adaptive controller on the chaser platform in nominal conditions

tracking performance under nominal conditions; however, the $\mathcal{L}_1$ adaptive controller yields greater overshoot. Going back to the results from the on-orbit simulations, the $\mathcal{L}_1$ adaptive controller is expected to reach steady-state tracking faster than the PD controller due to a quicker settling time, resulting in steeper transient trajectory. This overshoot is due to a combination of the steep transient trajectory coupled with the low actuating thrust provided by the air nozzles. In fact, it was determined through various experiments, that each thruster is only capable of delivering 0.25 N of force. Increasing the maximum thrust force of 0.25 N by 30%, completely eliminates the overshoot and the results become much closer to those obtained with the on-orbit simulations. In other words, the overshoot would not be present if the thrusters were providing slightly higher forces. Another way to reduce this overshoot would be to alter the desired transient performance by using path-planning techniques to plan an optimal transient trajectory to, say, minimize control input forces.

Figure 5.4. Simulation of the PD controller on the chaser platform in nominal conditions

5.6 Planar Experiments

5.6.1 Nominal Conditions

With the simulations completed, the hardware experiments were conducted. To begin with, the controllers were tested under nominal conditions, the results of which, can be seen in Figs. 5.5 and 5.6. As expected, the overshoot related to the use of the $\mathcal{L}_1$ adaptive controller is present. Also, one can see the performance bounds of the $\mathcal{L}_1$ adaptive controller with the maximum error cycling between -3 mm and 1 mm. The similarities between all five trials show that the overall performance bound trend is due to the state estimator of the $\mathcal{L}_1$ architecture. This is backed up by the 3σ error for the $\mathcal{L}_1$ adaptive controller in nominal conditions of -0.6 ± 5.3 mm. The PD controller seems to have a more random error between the five trials but the overall tracking error also has an offset. The error bounds of the $\mathcal{L}_1$ adaptive controller can

Figure 5.5. Planar Experiments: $\mathcal{L}_1$ and PD Controllers in nominal conditions

be reduced by improving the thrust estimation, including the uncertain input gain projection law into the adaptive model and also including the table slope into the state estimator. The overall error bounds of the PD controller seems to be marginally less when compared to the $\mathcal{L}_1$ adaptive controller, with a 3σ error of -0.8 ± 1.3 mm. It is also important that improving the accuracy of estimating the thrust output of the platforms should improve the error bounds of the $\mathcal{L}_1$ adaptive controller.

The other comparison point between the two control algorithms is thruster efficiency, which can be seen through time histories of all 8 thrusters for one trial, as shown in Fig. 5.6. At first glance, both the $\mathcal{L}_1$ adaptive and PD controllers seem fairly similar in-terms of thruster activity, but on closer inspection, it is clear that the $\mathcal{L}_1$ adaptive controller is marginally more efficient than the PD controller in nominal

Figure 5.6. Thrust Input for all 8 thrusters with the $\mathcal{L}_1$ and PD Controllers in nominal conditions

conditions.

5.6.2 Unknown Mass Conditions

In the unknown mass condition, 9 kg of mass was added to the COM of the chaser platform and the experiment was performed again, without re-tuning the controllers. The results in Figs. 5.7 and 5.8 showcase the capability of the $\mathcal{L}_1$ adaptive controller in less than ideal situations. The $\mathcal{L}_1$ adaptive controller modelled the higher required forces resulting from the increased mass as perturbations, overcompensating initially before adjusting mid-way through the experiment, whereas the PD controller has a consistent offset due to the unadjusted controller for the increased mass. The 3σ errors of 1.1 ± 4.1 mm and -1.4 ± 1.4 mm for the $\mathcal{L}_1$ and PD controllers, respectively reflect this. The overall error range is comparable to the nominal case for both.

The thruster time histories and the control efficiency, are a little harder to compare, as the PD controller does not compensate for the extra mass while the $\mathcal{L}_1$ adaptive controller does, and this reflects in the thruster comparisons. While there are periods where each seems to utilize less fuel, it seems that the lack of mass compensation in the PD controller makes it marginally more efficient in this situation.

Figure 5.7. Planar Experiments: $\mathcal{L}_1$ and PD Controllers with added mass

Figure 5.8. Thrust Input for all 8 thrusters with the $\mathcal{L}_1$ and PD Controllers with added mass

Figure 5.9. Planar Experiments: $\mathcal{L}_1$ and PD Controllers with added perturbations

5.6.3 Unknown Perturbation Conditions

In the perturbation experiment, the fan was only turned on after the spacecraft platform reached steady state tracking. At the beginning of the steady state trajectory, the platform was at its closest point to the fan and as such, the fan speed was set to its lowest settings. As shown in Figs. 5.9 and 5.10, even at this lowest setting, the perturbations affect the actual trajectory, especially for the PD controller. As half of the circle is completed, the fan speed is increased to the medium setting to account for the increased distance between the fan and the spacecraft platform. It is important to note that the mechanism of blowing air through a fan resulted in some inconsistencies in the magnitude of perturbation applied. This is shown in the

Figure 5.10. Thrust Input for all 8 thrusters with the $\mathcal{L}_1$ and PD Controllers with added perturbations

slightly inconsistent results obtained from all trials.

The discrepancy results highlight the improved performance of the $\mathcal{L}_1$ adaptive controller under perturbations, with a 3σ error of 0.4 ± 25.1 mm compared to the PD controller, which had a 3σ error of 0.2 ± 56.3 mm. The ability of the $\mathcal{L}_1$ adaptive controller to model and adjust for the perturbations resulted in the ± 31.2 mm 3σ error improvement of the $\mathcal{L}_1$ adaptive controller over the PD controller. The overall offset of the two controllers are very similar and can be considered negligible.

The $\mathcal{L}_1$ thruster efficiency is more evident in Fig. 5.10, where the $\mathcal{L}_1$ adaptive controller thrusters are saturated less than the PD controller, which appears to be saturated for most of the duration of the experiment, from 70 s when the fan turns on, until 220 s, when the platform completes the circular trajectory.

5.6.4 Unknown mass and Perturbations Conditions

The final experiment was the ultimate performance test for both controllers, combining the effects of unknown mass and unmodelled perturbations. The results shown in Figs. 5.11 and 5.12 really outline the capabilities of the $\mathcal{L}_1$ adaptive controller, which appears to track almost as well as in nominal conditions. On the other hand, the PD controller yielded its worst tracking performance of all experiments.

The 3σ error for the $\mathcal{L}_1$ adaptive controller and the PD controller was found to be 0.2 ± 12.3 mm and 1.5 ± 89.4 mm respectively. The $\mathcal{L}_1$ adaptive controller

Figure 5.11. Planar Experiments: $\mathcal{L}_1$ and PD Controllers with added mass and perturbations

performance resulted in almost no offset and a very small 3σ error. This is in stark contrast to the PD controller, which resulted in a larger offset and a very high 3σ error, which is ± 77.1 mm more than the $\mathcal{L}_1$ adaptive controller. The final 3σ errors for all experiments are summarized in Table 5.1.

Overall, the hardware experiments re-enforced the results obtained in the on-orbit simulations, showcasing that similar performance will be obtained in nominal conditions for both the $\mathcal{L}_1$ adaptive and PD controllers. However, the $\mathcal{L}_1$ adaptive controller better handles system uncertainties, due to the controller actively counteracting the estimated effect of these uncertainties. Use of a higher bandwidth low-pass filter would further improve the controller efficiency and tracking results. In terms of computational capability, the controllers ran on a Raspberry pi 3 microprocessor

Figure 5.12. Thrust Input for all 8 thrusters with the $\mathcal{L}_1$ and PD Controllers with added mass and perturbations

with no difficulties, meaning real-time use of the $\mathcal{L}_1$ adaptive controller on space flight hardware should not be an issue, regardless of the adaptive laws utilized.

Table 5.1. 3σ steady state tracking error for the $\mathcal{L}_1$ and PD controller's under various conditions

Experiment	3σ Error, E
L1 Nominal	$E = -0.6 \pm 5.3$ mm
PD Nominal	$E = -0.8 \pm 1.3$ mm
L1 Mass only	$E = 1.1 \pm 4.1$ mm
PD Mass only	$E = -1.4 \pm 1.4$ mm
L1 Pert. Only	$E = 0.4 \pm 25.1$ mm
PD Pert. Only	$E = 0.2 \pm 56.3$ mm
L1 Mass and Pert.	$E = 0.2 \pm 12.3$ mm
PD Mass and Pert.	$E = 1.5 \pm 89.4$ mm

Chapter 6

Conclusions

In this chapter, the motivation and objective of this thesis will be revisited. Following which, a brief review of the $\mathcal{L}_1$ adaptive controller design decisions, on-orbit simulation results and SPOT circular planar results will be presented. Future works will also be presented.

6.1 Thesis Summary

Spacecraft formation flying is a critical technology for the worlds leading space agencies, promoting benefits over conventional single satellite missions, such as reduced costs, increased flexibility and efficiency, design redundancy and capability for missions requiring high spacial and temporal resolutions. While a lot of work has been done on developing control algorithms for spacecraft formation flying, few of these algorithms have been tested or validated with hardware experiments. The $\mathcal{L}_1$ adaptive control algorithm was identified as a promising algorithm to cope with various model uncertainties in real time. It's capability was demonstrated through numerous aerospace simulations and also real world experience on board NASA's AirSTAR aircraft. In this context, this thesis considered flight viable control strategies for the spacecraft formation flying problem by designing, simulating, implementing, testing and validating the $\mathcal{L}_1$ adaptive controller.

This was accomplished by first introducing the various spacecraft formation flying dynamics model available for use. The complete non-linear formation flying model was presented first. This model was simplified for a circular orbit, with small relative distances between the target and chaser spacecraft, which is represented by the linear CW dynamic model. The CW dynamic model was then simplified further for missions with very small relative distances distances and short time scales, as well as deep space missions. This dynamic model was paramount in this study because it was the

dynamics model emulated by Carleton’s SPOT hardware experiments.

The $\mathcal{L}_1$ adaptive control architecture was introduced next, using examples for simple linear systems to gain an understanding of the controller implementation and behaviour. The $\mathcal{L}_1$ adaptive control algorithm for the full non-linear spacecraft formation flying problem was shown, along with the desired controller design decisions and the projection type and piecewise constant adaptive laws.

The $\mathcal{L}_1$ adaptive controller was designed for the spacecraft formation flying problem, with the main design decisions of the low-pass filter and various gains discussed. A PD controller was also designed for performance comparison purposes. The on-orbit simulations were then conducted for all three dynamic models introduced in Chap. 2 and the results were discussed. The $\mathcal{L}_1$ adaptive controller was shown to have superior performance with significantly lower control forces, in all conditions. The differences between the PCAL and projection type adaptive laws were also investigated and the projection type adaptive laws showed better robustness and performance over the PCAL, due to high-frequency oscillations in the transient behaviour of the system utilizing PCAL’s.

The $\mathcal{L}_1$ adaptive and PD controllers were then modified for use in Carleton’s SPOT facility, where hardware experiments were conducted to validate the results from the numerical simulations. The hardware experiments confirmed the results of the simulations and showcased the superior capabilities of the $\mathcal{L}_1$ adaptive controller. However, the reduction of the bandwidth of the $\mathcal{L}_1$ adaptive controller resulted in less of a control input difference between the $\mathcal{L}_1$ adaptive and PD controllers than observed in the simulations.

6.2 Future Work

Future work can focus on several areas. To begin with, the less than ideal thruster force highlighted the issue of overshoot with the $\mathcal{L}_1$ adaptive controller and its transient behaviour. Work can be done to incorporate path-planning and trajectory optimization with the $\mathcal{L}_1$ adaptive controller to further improve the performance of the controller while reducing any unwanted overshoots.

Another area of focus would be to incorporate the on-orbit simulations within

a GPS hardware-in-the-loop simulator, to validate on-orbit performance of the $\mathcal{L}_1$ adaptive controller with hardware GPS receivers. Several authors [4, 55–57] have conducted GPS hardware-in-the-loop experiments with various controllers for spacecraft formation flying, and this would add additional credibility to the performance of the $\mathcal{L}_1$ adaptive controller. Such a facility is currently being established at Carleton University.

Work can also be done on the perturbation mechanisms on the SPOT facility in terms of perturbation actuation and also perturbation measurements. The fan used to introduce perturbations in the hardware experiments was slightly inconsistent between experiments and was reflected in the slightly varied results. Designing a system that can more accurately introduce perturbations will result in experiments that will allow better performance comparisons and improved simulations of the hardware experiments.

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Appendix A

Accompanying Definitions and Derivations

This Appendix provides additional definitions and derivations to mathematical methods utilized in the derivation of the $\mathcal{L}_1$ adaptive control architecture.

A.1 $\mathcal{L}$ -norms

Similar to vector norms, we can define norms of functions, $f : [0, \infty) \rightarrow \mathbb{R}^n$. These function norms are referred to as $\mathcal{L}$ -norms, and the functions themselves belong in $\mathcal{L}$ -spaces.

The bounded $\mathcal{L}_p$ norm of a piecewise-continuous integrable function, can be defined as [33]:

$$\|f\|_{\mathcal{L}_p} \triangleq \left(\int_0^\infty \|f(\tau)\|^p d\tau \right)^{1/p} \leq \infty$$

in $\mathcal{L}_p^n$ space.

The $\mathcal{L}_\infty$ norm of a piecewise-continuous bounded function, can be defined as [33]:

$$\|f\|_{\mathcal{L}_\infty} \triangleq \max_{1 \leq i \leq n} \left\{ \sup_{\tau \geq 0} |f_i(\tau)| \right\} \leq \infty$$

Since $\mathcal{L}$ -norm needs to be finite to belong in $\mathcal{L}$ -space, we can consider a function that is truncated as to always have a finite $\mathcal{L}$ -norm:

$$f_\tau(t) = \begin{cases} f(t), & 0 \leq t \leq \tau, \\ 0, & t > \tau. \end{cases}$$

where $f_\tau(t)$ is the truncation of the function $f(t)$ such that the only restriction is that the function f does not have a finite escape time less than τ . These truncated functions belong in the extended space $\mathcal{L}_e^n$ [33].

A.2 Maximum $\mathcal{L}_1$ -norm of LTI system's using LMI's

Lets consider the following LTI system:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t), & x(0) &= x_0 \\ y(t) &= c^T x(t)\end{aligned}$$

Where $x(t) \in \mathbb{R}^n$ is the measured system state vector; $u(t)$ is the control input signal; B and $C \in \mathbb{R}^n$ are known constant matrices; A is a known $n \times n$ matrix describing the dynamics of the system and $y(t) \in \mathbb{R}^n$ is the regulated output.

Now, if there exists a positive definite symmetric matrix $P_\alpha \in \mathbb{R}^{n \times n}$ solving the LMI [33]:

$$AP_\alpha + P_\alpha A^T + \alpha P_\alpha + \frac{1}{\alpha} BB^T \leq 0$$

for some $\alpha > 0$, then there exists an upper bound on the $\mathcal{L}_1$ -norm of the system given by:

$$\|g\|_{\mathcal{L}_1} \leq \|CP_\alpha C^T\|_2.$$

We can also define a conservative bound on the feasible solution of α given by [33]:

$$\alpha \in (0, -2\mathcal{R}(\lambda_{\max}(A))],$$

where $\mathcal{R}(\lambda_{\max}(A))$ is the maximum real part of the eigenvalue of the matrix A .

A tight approximation on the $\mathcal{L}_1$ -norm of the system over all possible α is called the $\star$ -norm:

$$\|g\|_\star \triangleq \inf_{\alpha} \|CP_\alpha C^T\|_2.$$

Please see reference [58] by Nazin, Polak and Topunov for the implementation to this LMI solution.