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TITLE OF THESIS/TITRE DE LA THÈSE FINITE ELEMENT ELASTIC ANALYSIS OF
SIMPLY SUPPORTED DEEP BEAMS
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The displacement function for this element will be clearly compatible. That is, the displacements on an element edge will be linearly dependent on the end nodes for this edge only. Since the nodes are common to adjacent elements, the displacement field will be continuous. Note, however, that displacement derivatives are not continuous. This presents no problem since the displacement derivatives are not used as degrees of freedom and thus do not contribute to the strain energy of the problem.

Having determined the displacement functions, and following the steps described in the previous chapter, the following formulas are derived

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{Bmatrix} \quad (15a)$$

or

$$\{u\} = [P] \{\alpha\} \quad (15b)$$

Arising from equation (5) we also have

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix} [A]^{-1} \{\delta\} \quad (16)$$

Finite Element Elastic Analysis of
Simply Supported Deep Beams

by

Haido Papapanou, B.Eng.

A thesis submitted to the
Faculty of Graduate Studies and Research
in partial fulfillment of the requirements
for the degree of
Master of Engineering

Faculty of Engineering
Carleton University
Ottawa, Ontario

July 1980

The undersigned recommend to the Faculty of
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the thesis

"Finite Element Elastic Analysis of
Simply Supported Deep Beams"

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ABSTRACT

A finite element analysis of simply supported deep beams is presented in this thesis. The analysis refers to the elastic behaviour of deep beams considering thus that the material is isotropic and homogeneous. Constant strain triangular elements are used to represent the plane stress nature of the problem.

The parameters affecting a deep beam behaviour and thus examined here, are the ratio of the length over the height of the beam, the ratio of the length of the support over the length of the beam and the ratio of the shear span over the length (or depth) of the beam. Also three different loading conditions are considered here; uniform loading, loading at the third points and loading at the middle of the beam. All loadings are applied at the top surface of the beam.

The results of the elastic analysis of deep beams include the distribution of σ_x , σ_y , and σ_{xy} stresses along different sections as well as a visual representation of the flow and magnitude of the principal stresses within the beam, for varying loading and support conditions. The distribution of the stress zones and thus the position of the neutral axis, are determined. Maximum stresses and the regions where they occur are shown.

Along with the finite element deep beam analysis some comparisons with the classical beam theory results for long beam as well as with the finite difference method and photoelasticity experiments for deep beams are made.

The necessity for this study came out of the lack of a general theory on the elastic behaviour of deep beams.

"The struggle itself toward the heights is enough to fill a man's heart. One must imagine Sisyphus happy":

Albert Camus,
The Myth of Sisyphus

"Of course we aim for more than rolling a stone up a hill. But we are beginning to realize that for mankind there will never be a time to rest at the top of the mountain. There will be no new arcadian age. There will always be new burdens, new problems, new failures, new beginnings. And the glory of man is to respond to his harsh fate with the zest and ever-renewed effort.

This is why Sisyphus can serve as a symbol of the modern engineer. Today's engineer has lost faith in the utopia that engineers of an earlier age thought they were bringing to mankind. Yet his work, springing as it does from the most basic impulse of humanity, can fill him with existential joy".

Samuel C. Florman,
The Existential Pleasures
of Engineering

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LIST OF SYMBOLS

[A]	matrix of polynomial term coefficient multipliers in the element nodal displacement relationship
b	width of the beam (taken as unit in the present analysis)
[B]	[Δ] [P]
C	length of the support
C_i	polynomial coefficients for the distribution of the center line σ_x ($i=0,4$)
[D]	matrix containing the elastic constants of the material
E	Young's modulus
H	height of the beam
[K]	stiffness matrix of an element
[K_T]	stiffness matrix of the whole structure
L	length of the beam (middle to middle of the support)
M	bending moment
N, N_0, N_1	tensile forces at the bottom, middle and top tensile zone of a deep beam, respectively
[P]	matrix containing polynomial terms
q	uniform load intensity [$q*(L+C)=Q$]
Q_1 or $Q/2$	magnitude of one concentrated applied load
r	radius of curvature of an average isostatic
S	shear span of the beam
{U}	vector of displacements
u	displacement in the x-direction
v	displacement in the y-direction
V	shear force

z moment lever arm

α_i coefficients of displacement

$\{\delta\}$ nodes displacement vector

$\{\delta^*\}$ virtual displacement vector

$[\Delta]$ differential operator matrix

ϵ_{ii} strain at the ii -plane

$\{\epsilon\}$ vector of strains for $\{\delta\}$ displacements

$\{\epsilon_0\}$ vector of initial strains

$\{\epsilon^*\}$ vector of strains for $\{\delta^*\}$ displacements

ν Poisson's ratio

$\{\sigma\}$ stress vector

σ_x longitudinal stress along the x -axis

σ_y longitudinal stress along the y -axis

σ_{xy} shear stress

σ_I principal stress

Note: The position of the axes for all types of loading is as follows:

CHAPTER 1

Introduction1.1 General - Definition and Behaviour of Deep Beams
in the Elastic Range

The deepness or slenderness of a beam in engineering is generally expressed in the ratio of its length to its height (L/H). Thus, for values of the L/H ratio less than 6 the behaviour of a beam starts to depart from that of a slender beam. From this value of 6 to very deep beams with an L/H ratio of 1 or less there is a transition range. A precise definition of "deep beam" is thus somewhat arbitrary.

The two distinguishing characteristics of deep beam behaviour in the elastic range are:

- i) plane sections normal to the axis of the beam do not remain plane;
- ii) arching stresses as shown in Figure 1.1 become important.

Figure 1.1: Arching Stresses Developed Within a Deep Beam

In addition to the L/H ratio, the shear span over the height of the beam (S/H) ratio is often used to define deep beam behaviour. The shear span is the parameter S , shown in Figure 1.2(a), and for the doubly symmetric point loading it expresses the moment over shear ratio (M/V) for a section of the beam.

Figure 1.2(a): Shear Span S for a Beam Symmetrically Loaded

With the given loading in Figure 1.2(a) the load $Q/2$ is the shear on the beam, and considering the series of free body diagrams in Figure 1.2(b), it becomes clear that the shear span over height ratio is a measure of the relative amounts of shear and bending taking place in the beam.

Figure 1.2(b): Free Body Diagrams for Beams Symmetrically Loaded

The second beam in Figure 1.2(b) exhibits a predominantly flexural mode of behaviour, and its failure could be predicted with the flexural theory. In the first beam the predominant effect is shear, and a flexural theory will not give a good indication of the stresses.

In addition to the L/H and S/H ratios, the loading and support conditions affect the deep beam behaviour and thus are taken into consideration in the present investigation of elastic analysis of deep beams.

In practice deep beams are important structural elements. They are used in highrise buildings as "beam walls" when they are placed vertically or as "beams against the wind" when they are placed horizontally

They are also used in bridges as piers and in reservoirs as dividers, and so on. Deep beam theory can also be applied in silos where the radius of curvature of the wall is considerably larger than its thickness.

1.2 Review of the Existing Methods of Elastic Analysis of Simply Supported Deep Beams

The problem of elastic analysis in deep beams has been investigated through analytical and numerical techniques and confirmed through experimental ones.

1.2.1 Finite Difference Method

A deep beam elastic analysis using the finite difference method was first carried out by Bay in 1931 [11]. More loading cases were investigated by Chow, Conway and Winter in 1951 [15]. The above authors investigated five loading cases. These included a uniform load applied at the top or the bottom surface of the beam, a concentrated load applied at the middle of the top or bottom surface and loading at the third points of the top surface (in all cases $L/H = 1$).

Generally the solution for stresses by this method is given for the points of an orthogonal grid, consequently a finer grid increases the accuracy of the solution. This is especially important in the case of the shearing stresses whose variations along the length of the beam are quite significant. In this application, the differential equations of equilibrium are solved numerically in finite difference form by

introducing the Airy stress function and solving a biharmonic equation as shown in standard textbooks in elasticity [2].

1.2.2 Analytical Energy Method

The analytical energy method was developed by Conway, Chow and Morgan in 1951 [13]. The stress distribution was given by superimposing two stress functions. The first function satisfies all but one of the boundary conditions, that of the σ_x distribution being zero along the vertical boundaries. The second function was calculated using the principle of least work and it gives the σ_x distribution. The superposition of both functions satisfied all the boundary conditions. Because of the form of the second function (polynomial parameters to be determined by the principle of least work), the accuracy of the solution is increased by increasing the number of these parameters. The cases solved with this method included beams with $L/H = 1$ and $L/H = 2$. Both cases were solved for a uniform top loading. Shear and normal stresses at certain vertical sections were found. A comparison of solutions achieved by this method with solutions achieved by the finite difference method and the linear theory (Figure 1.3) show close agreement between the analytical and finite difference methods for the σ_x distribution.

Figure 1.3: Distribution of σ_x on the Section $x = \pm a/2$
for a Beam as Shown Above [13]

1.2.3 Photoelasticity Experiments - Leonhardt's Report

Photoelasticity is the science of the measurement of stress or strain in a stressed transparent model.

Among the main advantages of photoelasticity as an experimental method are: the variety of problems that it can solve, its simplicity and the advantages of an instantaneous pictorial representation of stress.

The photoelastic experiments in deep beams are reported in a summarized form by F. Leonhardt together with the results from numerical and analytical techniques to which he refers as the theory of vertical plates [1]. According to his report, the photoelastic experiments confirm the theory of vertical plates and also permit the solution of problems involving beams of non-rectangular shapes such as an arch-shaped beam.

The elastic solutions Leonhardt reports from both theoretical and photoelastic results include normal and shearing stress distribution, the lines of principal stress direction (isostatics) the values of the tensile resultant N (Figure 3.9) and the maximum moment lever arm for different L/H ratios. The loading and support conditions examined in Leonhardt's report are i) a uniform load applied on the top or bottom surface of a beam with L/H ratio equal to 1 or 0.75 when the length C of the support varies and ii) a concentrated load applied in the middle of the top surface of the beam for L/H varying between the values 1 and 0.75.

1.3 Finite Element Analysis of Deep Beams

In this thesis the finite element method of analysis is used for the elastic analysis of simply supported deep beams. To the author's knowledge this is the only finite element elastic deep beam study that has been done. The principal objective is to investigate the state of stress within such deep beams, taking into account most of the loading or geometric parameters that may affect its stress distribution. The following two parameters were varied for the case of the uniform load: the length over height ratio (L/H) is varied over a range between 4 and 0.75 and the ratio of the length of the support over the length of the beam (C/L) is varied over a range between $1/5$ and $1/20$.

The distribution of the center line σ_x is given in the form of 3rd order polynomials for values of the L/H parameter between 4 and 1. The position of the neutral axis along the beam as well as the effect of the L/H and C/L ratios on the value of the tensile force N , are also presented.

The definition of deep beams for the case of U.D.L. is given based on the resulting non-linearity of the center line σ_x as the L/H ratio decreases.

Also the σ_y stress distribution, being quite significant for deep beams, is given when $L/H=1$ and $C/L=1/10$ for different y sections. The distribution of σ_{xy} stresses which are not of parabolic form as in long beams, is given at different x sections for different C/L ratios. Also the position and value of the maximum shear stress is presented.

Finally the flow and magnitude of the principal stresses within beams with $L/H=1$ and different C/L ratios are shown, thus giving a complete visual representation of the state of stress in a deep beam subjected to uniform loading.

The case of loads concentrated at two points equally spaced from the ends of the beam was investigated for beams with $L/H=1$ and $C/L=1/10$; the parameter varied in this case is the ratio of the shear span over the length of the beam (S/L), which expresses the position of the loads. The results presented for this case are particularly significant because they depart from the description of the stress zones presented by the classical beam theory and explain more fully than other investigations the state of stress zones within deep beams.

The σ_x distribution as well as the value of the tensile force is calculated for ratios S/L between 0.50 and 0.00. In the case of long beams or deep beams under U.D.L., the beam is divided into a top compression and a bottom tension zone. In this case a more complicated stress pattern was observed; besides the tensile force at the center section, which is usually calculated, it was possible to calculate the value of the tensile force at other regions as well.

The σ_y and σ_{xy} stresses were found for the cases of $S/L = 0.50$ and $S/L = 0.33$ (loads at the middle and at the third points respectively). Useful observations about similarities in the σ_y stresses were made for different types of loads in regions away from the loading area.

Finally, the flow and magnitude of the principal stresses is drawn for beams with $L/H=1$ and $C/L=1/10$ for three different positions of the load; 1) at the middle, 2) at the third points and 3) at the support line on the top surface of the beam.

In some of the above cases the finite element method results are compared with either the linear theory of Navier or with deep beam analysis results available through other methods (e.g. finite differences)

Thus the objectives of this thesis can be summarized in the following:

- i) to define the parameters affecting the deep beam behaviour and their values for which a beam is characterised as such,
- ii) to find the distribution of the stress zones within a deep beam and the position of the neutral axis,
- iii) to find the distribution of the σ_x , σ_y and σ_{xy} stresses and give simple equations where possible,
- iv) to give a visual representation of the flow and magnitude of the principal stresses within a deep beam under uniform or point loading,
- v) in most cases, to clarify the deviation of a deep beam behaviour from the predicted by the classical theory of Navier,
- vi) to give the maximum value of the tensile, compressive and shear stress and the position where they occur, as an aid to the designer.

The results thus presented fill a gap in the elastic analysis of beams by describing the elastic behaviour of single span deep beams more completely than has previously been possible.

CHAPTER 2

The Finite Element Method (F.E.M.)2.1 General Description of the F.E.M.

The finite element method is based on the concept of approximating a structure by subdividing it into a number of "finite" elements. For these elements a displacement field is chosen. The displacement field is continuous in each element and satisfies certain continuity conditions across element boundaries. In general, a derivative of displacement of some order and higher becomes discontinuous on element boundaries. Once the values of the displacements are known, the strain-displacement relationship permits the evaluation of strains and therefore, from the stress-strain relationships, the stresses within each of the elements. In the following, the various phases of the method are described and given in a set of algebraic equations.

First the structure is subdivided into finite elements interconnected at certain nodal points. These elements may have a variety of shapes (linear, triangular, rectangular or three dimensional shapes) and a variety of mathematical displacement fields (linear, quadratic, cubic, quintic, etc.).

Having established the displacement function $\{u\}$, the strains $\{\epsilon\}$ can be found by the relation

$$\{\epsilon\} = [\Delta] \{u\} \quad (1)$$

where $[\Delta]$ is a differential operator matrix.

From Hooke's Law, we have

$$\{\sigma\} = [D] (\{\epsilon\} - \{\epsilon_0\}) \quad (2)$$

where $[D]$ is a matrix containing the elastic constants of the material and $\{\epsilon_0\}$ is a column matrix containing an initial strain introduced to the body before loading for reasons such as temperature changes. The column matrix $\{\epsilon\}$ is the total strain so that the vector $\{\epsilon\} - \{\epsilon_0\}$ is the elastic component of strain.

The displacement function can be written as

$$\{u\} = [P] \{\alpha\} \quad (3)$$

where $[P]$ is a matrix containing polynomial terms and $\{\alpha\}$ is a matrix vector of the polynomial coefficients.

When the coordinates of the nodes are substituted in the matrix $[P]$, then the equation (3) gives the displacements of the nodes and the equation can be written as,

$$\{\delta\} = [A] \{\alpha\} \quad (4)$$

Solving equation (4) for $\{\alpha\}$ and substituting it into (3), we have

$$\{u\} = [P] [A]^{-1} \{\delta\} \quad (5)$$

Substituting equation (5) into (1) a new expression for the strains, results as follows:

$$\{\epsilon\} = [\Delta] [P] [A]^{-1} \{\delta\} \quad (6a)$$

or

$$\{\epsilon\} = [B] [A]^{-1} \{\delta\} \quad (6)$$

where $[B] = [\Delta] [P]$

Substituting equation (6) into (2) we get the final expression for the stress which has the form,

$$\sigma = [D] [B] [A]^{-1} \{\delta\} - [D] \{\epsilon_0\} \quad (7)$$

At this stage by imposing on the element a virtual displacement $\{\delta^*\}$, an additional strain $\{\epsilon^*\}$ results within the element. This strain $\{\epsilon^*\}$ is given by the formula,

$$\{\epsilon^*\} = [B] [A]^{-1} \{\delta^*\} \quad (8)$$

The increase in strain energy within an element is given by formula (10) which is found by substituting the equations (7) and (8) into equation (9).

$$\{\delta^*\}^T \{f\} = \{\epsilon^*\}^T \{\sigma\} dV \quad (9)$$

where dV is an infinitesimal volume increment.

Thus the increase in strain energy is given by,

$$\int_{vol} [[B] [A]^{-1} \{\delta^*\}]^T [[D] [B] [A]^{-1} \{\delta\} - [D] \{\epsilon_0\}] dV \quad (10)$$

The increase in strain energy equals the external work done. This is given by the equation (11),

$$\{\delta^*\}^T \{f\} = \{\delta^*\}^T [A^{-1}]^T \int [B]^T [[D] [B] [A]^{-1} \{\delta\} - [D] \{\epsilon_0\}] dV \quad (11)$$

from which an expression for $\{f\}$ is derived in the form of the equation (12),

$$\{f\} = [K] \{\delta\} - [A^{-1}]^T \int [B]^T [D] \{\epsilon_0\} dV \quad (12)$$

where $[K]$ is the stiffness matrix of the element; it is given by equation (13)

$$[K] = [A^{-1}]^T \int [B]^T [D] [B] dV [A]^{-1} \quad (13)$$

After the stiffness matrix of elements are found, the stiffness of the whole body is assembled. Consequently, the final vector matrix $\{f\}$ includes the external forces applied to the body. From the force vector $\{f\}$, the displacements are found and then the stresses can be calculated by equation (7).

2.2 The Constant Strain Triangular Element or C.S.T. Element

As pointed out earlier, the first important step in the application of the Finite Element Method is the discretization of the structure into elements. Different forms of elements exist and the one selected to model a structure depends on the type of the problem to be solved.

For the case of a plane stress or plane strain problem, the simplest form is a constant strain triangular element. For such a triangle the displacement field is linear and the strain, which is the first derivative of the displacement function, is constant. Such a displacement field could be calculated from the equations of the following form

$$\begin{aligned} u &= \alpha_1 + \alpha_2 x + \alpha_3 y \\ v &= \alpha_4 + \alpha_5 x + \alpha_6 y \end{aligned} \tag{14}$$

The number of unknown parameters (3) in the above is the same as the number of points in the triangle to be used as nodes at the vertices of the triangle. The displacements of a constant strain triangle are shown in the Figure 2.1.

Figure 2.1: Displacements of a Constant Strain Triangular Element^[3]

The displacement function for this element will be clearly compatible. That is, the displacements on an element edge will be linearly dependent on the end nodes for this edge only. Since the nodes are common to adjacent elements, the displacement field will be continuous. Note, however, that displacement derivatives are not continuous. This presents no problem since the displacement derivatives are not used as degrees of freedom and thus do not contribute to the strain energy of the problem.

Having determined the displacement functions, and following the steps described in the previous chapter, the following formulas are derived

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{Bmatrix} \quad (15a)$$

or

$$\{u\} = [P] \{\alpha\} \quad (15b)$$

Arising from equation (5) we also have

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix} [A]^{-1} \{\delta\} \quad (16)$$

where $[A] =$

$$\begin{bmatrix} 1 & x_1 & y_1 & 0 & 0 & 0 \\ 1 & x_2 & y_2 & 0 & 0 & 0 \\ 1 & x_3 & y_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_1 & y_1 \\ 0 & 0 & 0 & 1 & x_2 & y_2 \\ 0 & 0 & 0 & 1 & x_3 & y_3 \end{bmatrix}$$

and

$$\delta = \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \end{Bmatrix}$$

The strains are calculated from displacement derivatives as

$$\epsilon_{xx} = \frac{\partial u}{\partial x} \quad \epsilon_{yy} = \frac{\partial v}{\partial y} \quad \epsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

Consequently equation (6a) takes the form

$$\epsilon = \begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix} \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix} [A]^{-1} \{\delta\} \quad (17a)$$

and equation (6) becomes

$$\{\epsilon\} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} [A]^{-1} \{\delta\} \quad (17)$$

where all matrices values are constant.

Hooke's Law for plane strain problems gives the following relationship between stresses and strains

$$\begin{aligned}\epsilon_{xx} &= \frac{1}{E} (\sigma_{xx} - \nu \sigma_{yy}) \\ \epsilon_{yy} &= \frac{1}{E} (-\nu \sigma_{xx} + \sigma_{yy}) \\ \epsilon_{xy} &= \frac{2(1+\nu)}{E} \sigma_{xy}\end{aligned}$$

which may be expressed in matrix form as

$$\begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & 0 \\ 0 & 0 & 2(1+\nu)/E \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} \quad (18)$$

Inverting equation (18) we get an equation for the stresses of the form

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{Bmatrix} \quad (19)$$

where the multiplier of the strain vector is the matrix $[D]$ of equation (7).

Then by equating the internal with the external energy of the element, the stiffness matrix $[K]$ of each element is found and thus the whole stiffness $[K_T]$ of the structure can be assembled. Consequently, the forces, displacements and stresses can be calculated.

2.3 Formulation of the Problem

In this section a description of the formulation of the problem is given, including the assumptions made for the solution, a brief description of the programs used as well as the description of the finite element grid for all the investigated cases.

The structure modelled is of uniform thickness and strains are not of interest in this investigation. Thus, the loads, and properties of the material are arbitrary, and only relative stress values, i.e. the pattern of the stress state, is of real interest. The material is considered to be homogeneous and isotropic, the solution thus being valid in practical problems, only in cases where elastic behaviour of the material is accepted (for example, concrete in its precracked state).

The program (SP23H) used to define the normal, shear and principal stresses throughout the beams, was written by Professor W. Bowes [3]. SP23H makes use of constant strain triangles (Chapter 2.2). The input data needed are, in summary, the elastic constants of the material, x and y coordinates of the nodes, numbering of the triangular elements as well as a description of the loading and support conditions. A detailed description of the program is given in reference [3].

Because of the lengthy description of the input data needed by SP23H as well as their specific format, a conversational program (AUTGRID) was developed, which generates the grid, producing all the necessary data for SP23H. Due to the complete symmetry of the problem, AUTGRID

provides solution for one of the symmetrical halves, assigning zero horizontal displacements along the axis of symmetry. That way the computer time for both the preparation of the data by AUTGRID and the solution for the stresses by SP23H, is considerably reduced. AUTGRID also assigns zero vertical displacements at the nodes along the support length. The program has a choice for the number and size of the elements in both vertical and horizontal directions, and it solves for uniform or point loading.

?

A program from the APL graphics library at Carleton University was used where it was necessary to fit a stress curve between the F.E.M. solution points. The program uses the least square interpolation method in order to define the best fitting polynomial equation, which it then draws.

Another APL program (PLOT), written by F. Williams and modified by R. Mallet, was used to draw the principal stresses within the beams. Both the direction and the magnitude of the principal stresses are depicted in the form of scaled arrows.

A varying number of elements is used for the different cases with parameters L/H and C/L , the exact description of which is given in Table 2.1 for the case of uniform loading and in Table 2.2 for the case of concentrated loads. The selected grid for the analysis of results for the case of a beam with $L/H = 1$ and $C/L = 1/10$ under U.D.L. is shown in Figure 2.2 for a half symmetrical beam. The reason for selecting that particular grid is discussed in Chapter 2.4 referring to the accuracy of the solution.

Figure 2.2: Finite Element Division (360 elements) of a Beam
with $L/H=1$ and $C/L=1/10$

Table 2.1: Grid Description for a Half Symmetrical Simply

Supported Beam Under Uniform Loading				
Case (1)	Total Number of Vertical Divisions* (2)	Total Number of Horizontal Divisions (3)	Number of Elements above Support (4)	Total Number of Elements (5)=[((3)+(4))x(2)]x2
L/H=4 C/L=1/10	8	15	2	272
L/H=3 C/L=1/10	10	13	2	300
L/H=2.5 C/L=1/10	11	12	2	308
L/H=2 C/L=1/10	14	10	2	336
L/H=1 C/L=1/2	18	4	6	360
L/H=1 C/L=1/5	18	7	3	360
L/H=1 C/L=1/10	18	8	2	360
L/H=1 C/L=1/20	18	9	1	360
L/H=1 C/L=1/50	18	9	1	360
L/H=0.75 C/L=1/10	19	7	2	342

* Each division includes 2 triangular elements

Table 2.2: Grid Description for a Half Symmetrical Simply
Supported Beam Under Concentrated Loads

Case	Total Number of Vertical Divisions	Total Number of Horizontal Divisions Between C and Load	Total Number of Horizontal Divisions between Load and Support	Number of Elements Above Support	Total Number of Elements
$L/H=1$ $C/L=1/10$	(2)	(3)	(4)	(5)	(6) = [(3)+(4)+(5)] $\times$ (2) $\times$ 2
$S/L = 0.50$	18	0	8	2	360
$S/L = 0.33$	18	3	5	2	360
$S/L = 0.25$	18	3	6	2	396
$S/L = 0.00$	18	8	0	2	360

2.4 Accuracy of the Solution

The artistic aspect of analysing stresses by the finite element method consists of choosing of elements so as to produce the best stress results. The decision to use constant strain triangles for the investigation of the present problem was based on several reasons summarized as follows:

1. It is the simplest element. This has as a result an easier generation of the grid as well as need for less computer time for the solution. For example, a linear strain triangle subdivision would require a larger amount of input data (more nodes and calculation of the consistent load vector for the case of uniform loading, in order to place the appropriate load values at each one of the three nodes of an edge). Further, for the same number of degrees of freedom, the linear strain triangle usually requires more solution time owing to its larger bandwidth.
2. Practice shows that the constant strain triangle gives good results when the change in stresses is relatively smooth.
3. For stress concentrations, a C.S.T. will not reproduce the stress state except with an extremely fine grid. On the other hand, it does not go numerically unstable but continues to "rough in" average values. For example, at a point no stress concentration will be modelled, but if a base plate were present on the real structure, the C.S.T. model will provide the bearing stress values. The same cannot be said of the higher order elements. In some cases, problems can be derived for which a high

order element will model a steep strain gradient better. However, for some real engineering problems, higher order elements can easily go numerically unstable.

4. The fact that the use of C.S.T. requires a larger number of triangles than a higher order strain triangle, was helpful in generating computer graphics where more points (element stresses) are needed in order to draw a better fitting curve and more interesting plots.

After the decision on the type of elements (C.S.T.) was made, the number and dimensions of the elements had to be defined. In doing that two guidelines were followed. Almost equilateral triangles were used in most of the area of the beam; long & thin triangles were avoided because stresses should not be represented over a large distance by one element. Secondly, to evaluate the accuracy of the results, the case of the beam with $L/H=1$ and $C/L=1/10$ was solved for different grids. In analysing the beam divided into 100 elements, mentioned above, a finer grid was used every time until the desired accuracy was reached with a number of elements of 360. The center line normal stress σ_x was used as a parameter of comparison. The σ_x distribution was tested against the existing σ_x curve from the photoelasticity experiments.

In Figures 2.3 and 2.4, the σ_x curves for the different grids for the case $L/H=1$, $C/L=1/10$, are given. Also in Table 2.3, a comparison of σ_x values along the center line of the beam is shown for all the grids as well as for the photoelasticity results.

Statics were also checked at sections where stress curves were drawn.

The differences in the following values may not seem to be very large but as long as they give critical results with lower values than the Navier theory, it means that the values are not as safe as they should be.

It should also be noted that the σ_x distribution is smoother than the distribution of other stresses, even though less significant, within the beam. For example the σ_{xy} distribution varies significantly from one vertical section to the other, thus the smaller the elements the better the accuracy. It was, however, necessary to keep an upper limit on the number of elements mainly because of the problem created by the round off errors which increase with the number of the elements. Thus the case of 360 elements, which gives a good approximation of the experimental results, was selected as the one to be used for further analysis.

Figure 2.3: Distribution of the Center Line σ_x for a Beam with $L/H=1$ and $C/L=1/10$ for a Grid with 100 Elements (Top), and for a Grid with 156 Elements (Bottom)

Figure 2.4: Distribution of the Center Line σ_x for a Beam with $L/H=1$ and $C/L=1/10$ for a Grid with 240 Elements (Top) and for a Grid with 360 Elements (Bottom)

Table 2.3: Comparison of Results Along the Center
Line Section of a Beam with $L/H=1$, $C/L=1/10$

Position of neutral axis	Navier Theory	Photoelasticity	F.E.M.			
			360 Elem	240 Elem	156 Elem	100 Elem
	0.5h	0.28h	0.28h	0.27h	0.27h	0.27h
σ_x at the Bottom	$0.75 \frac{q}{b}$	$1.60 \frac{q}{b}$	$1.63 \frac{q}{b}$	$1.60 \frac{q}{b}$	$1.56 \frac{q}{b}$	$1.55 \frac{q}{b}$
Ntensile	0.19Q	0.20Q	0.19Q	0.16Q	0.16Q	0.15Q

COMPARISON RUN

Figure 2.5: σ_x Distribution for the Center Line Section of a Beam with $L/H=1$ and $C/L=1/10$

CHAPTER 3

The F.E.M. Results for Uniform Loading3.1 Transition Range: Long to Deep Beams

In the following discussion the definitions of a deep, a moderately deep or a long beam is given for the case of uniform loading. The author has chosen to define these terms based on the σ_x distribution along the center line of the beam, and its divergence from the linear theory of Navier.

3.1.1 Effect of the L/H and C/L Ratios on the Distribution of the Center Line σ_x and its Resultant Tensile Force N.

In this section, a fundamental characteristic of deep beam behaviour is investigated by observing the distribution of stresses σ_x on a center line cross section for differing L/H (length over depth) beam ratios and for a uniformly distributed loading pattern. Figures 3.1 to 3.6 illustrate these differences well, as the L/H ratio is varied between 0.75 and 4.0. It is noted here that the individual points on these curves represent stress results of element centroids not along a perfectly straight normal line, but rather along a zig-zag line joining element centroids at midspan. This accounts for some of the zig-zag in the results. The fitted curves are of 3rd order and were obtained using an available package resident on the Carleton University Sigma 9 computer.)

The nonlinear variation in σ_x is once again confirmed in this study for deep beams. In order to facilitate the use of an algebraic expression for σ_x , for any L/H ratio less than 4, a set of curves for the polynomial coefficients C_0, C_1, C_2, C_3 is given in Figures 3.7 to 3.8 where the general expression for σ_x is given by the equation

$$\sigma_x = C_0 + C_1 y + C_2 y^2 + C_3 y^3$$

For the particular case of L/H ratio = 0.75, the fitted σ_x curve is a 4th order polynomial with a C_4 coefficient equal to .0064. The reason for not selecting a 3rd order curve for this particular case, is that a 4th order oscillation was taking place at the end of the σ_x curve near the top surface of the beam (Figure 3.1). For L/H ratio less than 0.75 the σ_x distribution remains practically the same. Also the σ_x curve for the cases of L/H = 0.75 and L/H = 1 remains almost the same for a height of the L/H = 0.75 beam equal to 0.75 H. This indicates that the upper part of the beam practically does not participate in the resistance in flexure.

It should also be noted that for a beam with L/H = 4, although a 3rd order σ_x curve was selected, a 1st order curve also gave a very good approximation for the resulting points. For this reason the L/H case was considered as the limiting case between deep and long beams, belonging most properly in the category of long beams for the uniform loading pattern. This is also evident from the steeper slope of the coefficient curves when the L/H ratio, reaches the value of 4.

$$C/L = 1/10 \quad L/H = 0.75$$

Figure 3.1: Distribution of the Center Line σ_x for a Beam with $L/H=0.75$ under U.D.L.

$$C/L = 1/10 \quad L/H = 1$$

Figure 3.2: Distribution of the Center Line σ_x for a Beam with $L/H=1$ under U.D.L.

$$C/L = 1/10 \quad L/H = 2$$

Figure 3.3: Distribution of the Center Line σ_x for a Beam with $L/H=2$ under U.D.L.

Figure 3.4: Distribution of the Center Line σ_x for a Beam with $L/H=2.5$ under U.D.L.

$C/L=1/10$ $L/H=3$

Figure 3.5: Distribution of the Center Line σ_x for a Beam with $L/H=3$ under U.D.L.

Figure 3.6: Distribution of the Center Line σ_x for a Beam with $L/H=4$ under U.D.L.

Figure 3.7: Coefficients C_0 , C_1 of the Center Line
 σ_x Equation for Different L/H Ratios

Figure 3.8: Coefficients C_2 , C_3 of the Center Line σ_x Equation for Different L/H Ratios

The ratio L/H also has an effect on the value of the tensile force N at the center line. Increasing the ratio L/H , higher values of N are obtained.

In Figure 3.9, the relationship between the ratio of tensile force N over the total load Q , and the ratio L/H is indicated for $C/L=1/10$. Along with the finite element results, the plate theory (finite difference method) [1] as well as the Navier theory results are presented. From this graph it becomes evident that the tensile force N , for L/H ratios less than 1, diverges significantly from the Navier theory, which in that case gives rather unreliable results.

The C/L ratio also affects the distribution of the center line σ_x at the bottom part of the beam; the greater the C/L ratio, the smaller the σ_x at the bottom edge of the beam. Consequently, the tensile force N at the center line section will have lower values. In the Figure 3.10 the relation between the ratio N/Q and C/L is shown for a beam with $L/H = 1$.

Figure 3.9: Values of the Tensile Force Over the Total Applied Load Ratio, for Different L/H Values, for a Beam Under U.D.L.

Figure 3.10: Values of the Tensile Force Over the Applied Load Ratio for Different L/C Values for a Beam Under U.D.L.

3.1.2 Effect of the L/H Ratios on the Position of the Neutral Axis

According to the Navier theory, the neutral axis of a vertically symmetrical beam coincides with its centroidal axis, thus dividing a top-loaded beam into a compression zone (top) equal to a tension zone (bottom) all along its length. This theory does not prove satisfactory in determining the position of the neutral axis for beams with ratio L/H less than 4.

As L/H becomes less than 4, the neutral axis moves downwards; consequently, the compression zone area increases at the expense of the tension zone. Further, the neutral axis is not a straight line anymore, but a curve of parabolic form. This is shown in Figure 3.11 for the case of $L/H=1$. For ratios L/H less than one there is a slight parallel movement of the axis downwards.

The effect of the C/L ratio on the position of the neutral axis is negligible.

NEUTRAL AXIS

Figure 3.11: Position of the Neutral Axis for a Deep Beam
($L/H=1$) Under U.D.L.

3.2 Deep Beams with $L/H = 1$

In this section, the stress distribution for a typical deep beam ($L/H \leq 1$) is analysed. The results are mainly for a beam with ratio $L/H=1$, the reason being that below that ratio the changes in the stresses within the beam are rather small.

3.2.1 Distribution of σ_x in Vertical Sections

The σ_x distribution along the center line of a beam with $L/H=1$ is given in section 3.1.1. In Figures 3.12 to 3.14 the distribution of σ_x at different sections along the beam is shown. From these it becomes evident that the tension zone decreases slightly for sections closer to the support, while the maximum σ_x at the bottom edge remains practically the same, equal to $1.6q/b$.

Figure 3.12: Distribution of the Center Line σ_x for a Beam with $L/H=1$ Under U.D.L.

Figure 3.13: Distribution of σ_x Along the Section $X=0.2L$ (AA')
for a Beam Under U.D.L.

Figure 3.14: Distribution of σ_x Along the Section $X=0.3L$ (BB')
for a Beam Under U.D.L.

3.2.2 Distribution of the Shear Stress σ_{xy} in Vertical Sections

The classical beam theory for the case of a beam under U.D.L., gives a σ_{xy} distribution of a parabolic form with the maximum value at the neutral axis, depending on the value of the total shear at that section.

In the case of a deep beam under U.D.L. the classical beam theory does not apply. The finite element analysis of a beam with $L/H=1$, gives a σ_{xy} distribution dependent on both the x and y directions. Its form along the y -axis is not parabolic but of higher order and its maximum value, as it follows the position of the neutral axis, appears lower than the middle line, the exact position depending on the x -coordinate of the cross section. In more detail, in the particular case, for example, of a deep beam having as parameters $L/H=1$ and $C/L=1/10$, the finite element method shows the following results: At a section $x=0.2*L$, the maximum value of the shear stress σ_{xy} appears at a height from the bottom of the beam $y=0.2*H$ and its value equals $0.34q/b$, while at a section $x=0.4*L$ its maximum value is $0.70q/b$ and appears at the section $y=0.1*H$.

For beams with $L/H=1$, three cases of support length were studied ($C/L=1/5$, $C/L=1/10$ and $C/L=1/20$). The effect of the support length in the σ_{xy} distribution becomes evident from the Figures 3.15 to 3.20 as well as from the Table 3.1.

Figure 3.15: Distribution of the Shear Stress Along the Section $x=0.2 \cdot L$ for a Beam with $L/H=1$ and $C/L=1/5$ Under U.D.L.

Figure 3.16: Distribution of the Shear Stress Along the Section $x=0.4*L$ for a Beam with $L/H=1$ and $C/L=1/5$ Under U.D.L.

Figure 3.17: Distribution of the Shear Stress Along the Section $x=0.2 \cdot L$ for a Beam with $L/H=1$ and $C/L=1/10$ Under U.D.L.

Figure 3.18: Distribution of the Shear Stress Along the Section $x=0.4*L$ for a Beam with $L/H=1$ and $C/L=1/10$ Under U.D.L.

Figure 3.19: Distribution of the Shear Stress Along the Section $x=0.2 \cdot L$ for a Beam with $L/H=1$ and $C/L=1/20$ Under U.D.L.

Figure 3.20: Distribution of the Shear Stress Along the Section $x=0.4*L$ for a Beam with $L/H=1$ and $C/L=1/20$ Under U.D.L.

Table 3.1: Maximum Values for the σ_{xy} Stress
for a Beam with $L/H=1$ under U.D.L.

C/L	Maximum σ_{xy} at		Beam with $L/H=1$
	section A-A' $x=0.2L$	section C-C' $x=0.4L$	
1/20	$0.46q/b$	$0.82q/b$
1/10	$0.35q/b$	$0.70q/b$	
1/5	$0.35q/b$	$0.62q/b$	

3.2.3 Distribution of σ_y in Horizontal Sections

In long beams, changes in the σ_y stress along the depth of the beam are not considered.

In deep beams however, the variation of σ_y along such sections is rather important. It was mentioned earlier that a distinguishing feature of deep beams is the presence of arching stresses occurring between top loads and bottom reactions. These stresses are mainly σ_y at the reaction and σ_x at the top center of the arch. In Figure 3.21 the distribution of σ_y is shown for three different sections: $y_1=0.25H$, $y_2=0.50H$, $y_3=0.75H$.

The ratio C/L has an effect on the magnitude of σ_y ; as C/L increases, the σ_y values decrease. It is worthwhile mentioning here that while the self weight of a deep beam has negligible effect on the distribution of other stresses like σ_x or σ_{xy} , it does affect the σ_y ; taking into account the self weight of such a beam as a uniformly distributed load all over its surface, the negative values of σ_y start to increase, giving a distribution that starts from lower negative values at a section near the support and positive values as the section gets closer to the center line [1].

One could say that the distribution of σ_y along horizontal sections, taking into account the self-weight of the beam, is an intermediate distribution between that caused by a uniform loading on top of such a beam and a uniform loading at its bottom [1].

Figure 3.21: Distribution of the σ_y Stress Along Different Horizontal Sections for a Beam with $L/H=1$ and $C/L=1/10$ Under U.D.L.

3.2.4 Flow of the Principal Stresses for Beams with $L/H=1$ and Varying C/L Ratios

The distribution of σ_x , σ_y and σ_{xy} stresses becomes important for the calculation of the direction and magnitude of the principal stresses. The two families of lines intersecting at right angles, which represent the flow of the principal stresses within a structural element, are called principal stress lines. Their usefulness consists in helping to visualize structural action, and the exact means by which a structure carries its loads.

In the Figures 3.22 to 3.24 the flow of the principal stresses for a beam with $L/H=1$ under uniform loading, for three difference values of C/L are shown; the principal stresses are drawn at the centroid of each triangular element as a set of arrows in the appropriate directions. The arrowheads are directed inward for compression and outward for tension.

From these graphs the effect of the ratio C/L on the principal stresses becomes more evident. By decreasing the C/L , the values of the principal stresses increase and the lines bend downwards.

From the same graphs it can also be seen that the maximum stress appears immediately next to the support and increases as the length of the support decreases.

Figure 3.22: Flow of the Principal Stresses for a Half
Symmetrical Deep Beam with $L/H=1$ and $C/L=1/20$
Under U.D.L.

Figure 3.23: Flow of the Principal Stresses Within a Half Symmetrical Deep Beam with $L/H=1$ and $C/L=1/10$ Under U.D.L.

Figure 3.24: Flow of the Principal Stresses Within a
Half Symmetrical Deep Beam with $L/H=1$
and $C/L=1/5$ Under U.D.L.

CHAPTER 4

The F.E.M. Results for Concentrated Loads4.1 General

A different pattern of analysis than in the case of U.D.L. is followed in this chapter. The parameter of this analysis is the shear span (Figure 4.1) over length ratio (S/L) (or S/H ratio, since in the presently investigated case $L=H$). Considering that the L/H would have the same effect on the stress distribution as in the case of the U.D.L. and that the load position would be the most important parameter affecting the stresses, only a beam with $L/H=1$ and $C/L=1/10$ was thus studied for S/L ratios varying over a range between 0.00 and 0.50

Figure 4.1: Shear Span for a Symmetrically Loaded
Deep Beam ($L/H=1$)

4.2 Effect of the S/L Ratio on the Distribution of the σ_x Stress for a Beam with L/H=1 and C/L=1/10

In the following, the distribution of σ_x is to be examined for two different positions of the load. These are loads applied at the middle of the top surface of the beam (S/L=0.50); and loads applied at the third point (S/L=0.33).

4.2.1 Distribution of σ_x Along Vertical Sections for a Beam with S/L=0.50

In Figures 4.2 to 4.4 the distribution of σ_x along different vertical sections is shown. What is noteworthy about these curves is that they indicate a different stress zone distribution than predicted by classical beam theory or by finite element investigation of a similar deep beam under U.D.L. Along the center line section the presence of four zones is evident (Figure 4.2). Two intermediate zones, one a tension and the other compression zone, are shown. Along a section further away from the center line, (Figure 4.3) the stress zones are a top compression and a bottom tension. The σ_x distribution shown for the third section, which lies closer to the support (Figure 4.4), shows the zones distributed as follows: a small tension zone at the top, a compression zone at the middle and a tension zone at the bottom part of the beam.

The σ_x distribution at the above sections, as well as at other vertical sections, makes possible an approximate visual representation of the stress zones within a beam with L/H=1 when it is subjected to a concentrated load at the middle of its top surface, (Figure 4.5).

Figure 4.2: Distribution of the Center Line (AA') σ_x for a

Beam with $L/H=1$ and $S/L=0.5$

SHEAR SPAN = 0.5 L

Figure 4.3: Distribution of σ_x Along the $x=0.17L$ Section (BB') for a Beam with $L/H=1$ and $S/L=0.5$

SHEAR SPAN = 0.5 L

Figure 4.4: Distribution of σ_x Along the $x=0.4L$ Section (CC') for a Beam with $L/H=1$ and $S/L=0.5$

Figure 4.5: Approximate Distribution of the Stress

Zones Within a Deep Beam ($L/H=1$) Under a

Point Load at the Middle of its Top Surface

The equation corresponding to the center line σ_x distribution is not given here because, being a higher order polynomial, it is of little practical use.

4.2.2 Distribution of σ_x Along Vertical Sections for a Beam with $S/L=0.33$

A different type of σ_x distribution results in the case of a deep beam loaded at the third points. The σ_x curve drawn for the center line section (Figure 4.6) as well as the σ_x curve drawn for a section closer to the support (Figure 4.8), indicate the presence of a small tension zone at the top, a compression zone at the middle and a tension zone at the bottom of the beam; the σ_x curve along the section where the load is applied indicates the presence of a top compression and a bottom tension zone as well as two intermediate ones, one tension and one compression.

$$\text{SHEAR SPAN} = 0.33 \cdot L$$

VERTICAL DIRECTION

Figure 4.6: Distribution of the σ_x Stress Along the Center Line Section (AA')

for a Beam with $L/H=1$ and $S/L=0.33$

Figure 4.7: Distribution of the σ_x Stress Along the Section $x=0.17L$ (BB') for a Beam with $L/H=1$ and $S/L=0.33$

$$\text{SHEAR SPAN} = 0.33 \cdot L$$

Figure 4.8: Distribution of the σ_x Stress Along the Section $x=0.4L$ (CC') for a Beam with $L/H=1$ and $S/L=0.33$

The approximate stress distribution of the zones for such a beam as well as the sections along which the σ_x curves of the Figures 4.4 to 4.6 are drawn, is shown in Figure 4.9.

Figure 4.9: Approximate Distribution of the Stress Zones Within a Deep Beam ($l/H \geq 1$) Under Loads at its Third Points

4.3 Effect of the S/L Ratio on the Value of the Tensile Force

In the following, a set of σ_x curves corresponding to S/L ratios equal to 0.00, 0.25, 0.33, 0.50 are presented in Figures 4.10 and 4.11. Also, the value of the tensile force N, which results from the Navier theory and the F.E.M. solution, is shown in Figure 4.11. The F.E.M. values for N are calculated as the integrals of the σ_x curves in the tensile region. According to the linear theory of flexure, the tensile force in a beam with $L/H=1$, having maximum bending moment M and corresponding lever arm z, is given by the formula

$$N = \frac{M}{z} = \frac{SQ_1}{z} = \frac{SQ_1}{\frac{4}{3} \frac{H}{2}} = \frac{3}{2} \frac{SQ_1}{L}$$

Since Q_1 is one half of the total load Q, the above equation becomes

$$N = \frac{3}{4} Q \left(\frac{S}{L} \right) = 0.75Q \left(\frac{S}{L} \right)$$

The value for the tensile force resulting from the F.E.M. analysis for the σ_x distribution is given by the formula $N = .109Q + .301 Q \left(\frac{S}{L} \right)$. From the graph presented in Figure 4.12 it becomes evident that for values of the S/L ratio less than 0.25, the Navier theory underestimates N, while for values over 0.50 it overestimates the values given by the F.E.M. solution.

In view of the presence of other tensile zones besides the one at the bottom part of the beam, the maximum values of the tensile forces at other regions are also calculated for the cases of $S/L=0.50$ and $S/L=0.33$. These are compared, wherever possible, with values given by the Navier theory and with those presented by F. Leonhardt [1] (Tables 4.1 and 4.2).

Figure 4.10: Distribution of the Center Line σ_x for
S/L Ratios Equal to 0.00 and 0.25

Figure 4.11: Distribution of the Center Line σ_x for
S/L Ratios Equal to 0.33 and 0.50

Figure 4.12: Diagram Showing the Variations of the Ratio of Tensile Force Q/N Over the Total Applied Load Q in Terms of the Shear Span Over Length Ratio for a Beam with $L/H=1$ and $C/L=1/10$

Table 4.1: Values of the Maximum Tensile Forces N , N_1 , N_0
for the Case of a Beam with $L/H=1$ and $S/L=0.50$

	F.E.M. Solution	Navier Theory	Leonhardt's Report	Case
N	0.265 Q	0.37 Q	0.280 Q
N_1	0.090 Q	-	0.080 Q	
N_0	0.025 Q	-	0.020 Q	

Table 4.2: Values of the Maximum Tensile Forces N , N_1 , N_0
for the Case of a Beam with $L/H=1$ and $S/L=0.33$

	F.E.M. Solution	Navier Theory	Leonhardt's Report	Case
N	0.208 Q	0.247 Q	-
N_1	0.080 Q	-	-	
N_0	0.025 Q	-	-	

4.4 Distribution of σ_{xy} along Vertical Sections for Beams with $S/L=0.50$ or $S/L=0.33$

One of the major problems in finding an accurate σ_{xy} distribution in the case of a beam loaded with two point loads is that the vertical section in which such a distribution is checked should not be close to regions where changes in the shear force occur. Such regions are either close to the point of load application or near the support. Nevertheless, the σ_{xy} curve which results from the F.E.M. solution is drawn for a vertical section at a distance $x=0.4*L$ from the center line of the beam, for the case of $S/L=0.5$ and for the case of $S/L=0.33$ (Figure 4.13). What is apparent, especially in the case of $S/L=0.33$, is the strong effect of the load in the σ_{xy} distribution at the upper half of the beam while at the lower half the support section gives rise to higher σ_{xy} stresses. This is shown in Figure 4.14 where the upper part of the line 1-1' is closer to the loading region, while the lower part of the line 2-2' is closer to the support area. Consequently, the difference in the σ_{xy} values between the two rows of elements is due to their closer position either to the loading or support area.

Taking into account that the distance between the two lines along which the centroids of the triangles lie is less than $.0018*L$, it becomes clear that a decrease in the size of elements would not offer much improvement in results unless it was quite significant in which case round off errors might be accumulated.

Figure 4.13: Shear Stress Distribution Along the Section
 $x=0.4L$ for Beams with $L/H=1$ and $S/L=0.50$
 and for Beams with $L/H=1$ and $S/L=0.33$

Figure 4.14: Finite Element Division of a Vertical Section of a Deep Beam

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4.5 Distribution of σ_y in Horizontal Sections for Beams with $S/L=0.50$ and $S/L=0.33$

In the following Figures 4.15 and 4.16, the σ_y distribution is shown for three different y-sections ($y=0.25 \cdot H$, $0.50 \cdot H$ and $0.75 \cdot H$).

What is interesting in comparing these figures with the U.D.L. case in Figure 3.21, is the application of St. Venant's Principle. Thus, the σ_y distribution at the section F-F' is exactly the same for all three cases while the values of the σ_y -stress in the region of point G' is the same for both $S/L=0.50$ and $S/L=0.33$ cases. Further in the case of $S/L=0.50$, the σ_y values for points I and F' are the same. This could be expected because of the same loading conditions in these regions (compressive load $Q/2$ applied in a vertical distance $0.25H$ above the point I and equal compressive load $Q/2$ applied at the support which lies at a vertical distance $0.25H$ below the point F').

SHEAR SPAN = $0.50 \times L$

Figure 4.15: Distribution of σ_y Along Different Horizontal Sections for a Beam with $L/H=1$ and $S/L=0.50$

SHEAR SPAN=0.33×L

Figure 4.16: Distribution of σ_y Along Different Horizontal Sections for a Beam with $L/H=1$ and $S/L=0.33$

4.6 Flow of the Principal Stresses for Beams with Varying

S/L Ratios

In Figures 4.18 to 4.20 the principal stresses are drawn as scaled arrows with their arrowheads directed inwards for compression and outwards for tension. Their magnitude is found by dividing the arrows length by 0.4. The resulting values are in terms of the $\frac{Q}{1.1xLxb}$ ratio.

Also, the stress zones are shown thus giving a complete image of the elastic behaviour of such a beam for different positions of the point loads. Considering that the flow of the principal tensile stresses coincides with the direction of the isostatics, the shape of the latter makes clear the type of the stress zones. Thus, if r is the radius of an average isostatic curve (Figure 4.17), the magnitude of the σ_I stress at that point is given by the formula

$$\sigma_I = \frac{Q}{br} \quad [10]$$

where

Q is the total applied load

and b is the width of the beam.

Since in the present analysis b was taken as of unit length the above formula becomes

$$\sigma_I = \frac{Q}{r}$$

This equation shows that for negative values of the r , the σ_I is compressive (point A) for positive values of the r , the σ_I is tensile

(point B) and when r becomes infinity, σ_I is zero (point C).

Figure 4.17: Approximate Average Isostatic for a
Beam with $L/H=1$ and $S/L=0.50$

It should also be noted that the high values of stresses in the loading area are due to load concentration. The solution presented here represents only good "engineering values" for the bearing stresses in those locations. Only a solution with a finer finite element grid would give more accurate results for such regions. However, the direction of the stresses is still valid.

The application of the St. Venant's Law of proximity is evident here, too. It can be seen that in regions further from the loading area, the stress flow and magnitude remains unchanged giving this the same maximum stresses in the support area.

Figure 4.18: Flow of the Principal Stresses for a Half Symmetrical Beam
with $L/H=1$ and $S/L=0.50$

Figure 4.19: Flow of the Principal Stresses Half Symmetrical Beam
with $L/H=1$ and $S/L=0.33$

Figure 4.20: Flow of the Principal Stresses for a Half Symmetrical Beam
with $L/H=1$ and $S/L=0.00$

CHAPTER 5

Summary and Discussion

The purpose of this chapter is to present, in summary, the results of the finite element analysis of deep beams. The chapter is divided into four sections. The first contains the definition of a deep beam under uniform or point loading; the second summarizes the stress distribution within such a beam; the third presents in a visual form the magnitude and direction of maximum stresses as an aid to the designer; and the fourth includes general recommendations for further elastic analysis of deep beams.

5.1 Definition of a Deep Beam

The deepness of a beam is most clearly expressed by its length-to-depth ratio, using as the basic criterion the divergence of its σ_x stress from a straight line. Using this as a basis, the author concludes that beams could be categorized as shown in the following table:

Table 5.1: Beam Designation According to the L/H or S/H Ratio

<u>L/H(S/H) Ratio</u>	<u>Beam Designation</u>
≤ 1 (0.5)	Very deep beam
≤ 2 (1)	Deep Beam
$\leq 3, 4$ (1.5, 2)	Moderately deep beam
≤ 5 and up (2.5 and up)	Long Beam

These designations apply for both point loaded beams and uniformly loaded beams. For the uniformly loaded beam, the full beam length-to-

depth is considered, while for a point loaded beam, the shear span to depth ratios is used in the above table.

Although the shear stress distribution is not considered in this definition, it continues to be inherently tied to the variation of σ_x .

5.2 Stress Distribution Within a Deep Beam

For the cases of both uniform and point loading, the stress distribution in deep beams is quite different from that predicted by the theory of strength of materials. The most significant results of the finite element analysis can be summarized as follows:

For the case of uniform loading, the σ_x distribution at the center line, as well as at other vertical sections, is given by 3rd order polynomials. The neutral axis is of parabolic form and at the center line section lies at a height $y=0.28 \cdot H$. The σ_y distribution for a deep beam under U.D.L. changes significantly depending on both the vertical and horizontal position of the "finite" element at which the stress occurs. Generally, in regions away from the loading or support areas, the σ_y decreases significantly. The shear stress σ_{xy} curve is not of parabolic but of a higher-order distribution. One could postulate that since shear stress in the Navier theory of beams is obtained from integrals of σ_x stresses giving rise to a mathematical curve one order higher than the straight line, therefore shear stress curves will be approximately one order higher in the case of deep beams. The author is prepared to postulate, but a proof is beyond the scope of this thesis. The shear stress

maximum value along every vertical section follows the position of the neutral axis at that section. Its absolute maximum value is presented at the section $x=0.4*H$.

For the case of point loading, the σ_x distribution differs considerably from vertical section to vertical section due to the differences in the distribution of stress zones from those presented by classical beam theory. The order of the polynomials which gives the σ_x distribution also changes for different vertical sections, in all cases being higher than 3rd order. The curves presented earlier in this thesis indicate a transition to a fourth order polynomial. The σ_y distribution for a beam under point loads is similar to the σ_y distribution for a beam under U.D.L. for regions away from the points where the load is applied as suggested by St. Venant's Principle. The maximum value of the shear stress (σ_{xy}) along a vertical section appears closer to the top of the beam for sections close to the point of the load application, while for sections close to the support region it appears near the bottom edge of the beam. The shear stress curves presented in this study are not highly accurate because the curves do change significantly between vertical sections fairly close together. The results obtained are considered adequate for the purposes of the present study; however, if one wishes to obtain accurate stress values over short shear spans, a much finer grid is recommended.

For both the cases of U.D.L. and concentrated loading, the maximum principal stress is tensile and appears near the bottom edge of the beam, next to the support.

5.3 Guidelines for Design of Deep Beam Structures

For the structural designer, maximum stresses are of significance in selecting a final design. In this section these stresses, obtained from the finite element results, are summarized. In Figures 5.1 to 5.3 the location and magnitude of the maximum σ_x , σ_{xy} and maximum principal stresses for deep beam structures are shown. The magnitudes are given in terms of the load magnitudes (q for uniform load intensity and Q for the point load).

5.4 Recommendations for Future Work on the Elastic Analysis of Deep Beams

The scope of this thesis was to give a complete elastic analysis of single span simply supported deep beams and to emphasize differences in behaviour with that of long beams. However, in order to keep an upper limit on the complexity of the problem, a choice had to be made among possible parameters. More complete investigations of parameters not studied here will increase the sensitivity and scope of this type of analysis.

For instance, uniform and point loading cases were selected because they most often occur in practice. However, in order to obtain a more complete representation of loading conditions, other types of loads such as triangular or trapezoidal might be examined. Further the case of loading applied at the bottom part of the beam could be analyzed as it gives a different stress distribution than the case of loading applied at the top of the beam.

The self weight of the beam was not taken into account in this investigation. This could have a serious effect on the stress distribution if the magnitude of the self weight is comparable to the magnitude of the applied load. In particular, the self weight of the beam would have an effect on the distribution of the τ_y stress and, consequently, on the magnitude and distribution of the principal stresses. Thus, the self weight of the beam, over the total applied load could be taken into account in the analysis, as it generally affects stress distribution within a deep beam.

A possible extension to this study which could be of practical value would be the investigation of prestressed concrete deep beams. The application of prestressed forces to the finite element grid given here presents no increased difficulty, and the tools developed for this work could be implemented directly. One would have to first investigate the extent of the need for such a study.

Finally, in order to have more accurate results for the shear stress distribution, a different type of grid in the region where the shear stress is measured, is suggested.

SYMBOLS

used for

Figures 5.1 to 5.3

finite area at the centroid of which
the stress occurs.

max. σ_x^+ or σ_{xy}

max. σ_y^- occurring
at an angle θ
with the y-axis.

arrow scale: $1\text{cm} = 1q/b$ or $1Q/11Lb$

Figure 5.1: Maximum σ_x , σ_{xy} , and Principal Stresses for a Beam with $L/H=1$, $C/L=1/10$ Under Uniform Loading

Figure 5.3: Maximum σ_x , σ_{xy} and Principal Stresses for a Beam with $L/H=1$, $C/L=1/10$ and $S/L=0.33$

Figure 5.2: Maximum σ_x , σ_{xy} and Principal Stresses for a Beam with $L/H=1$, $C/L=1/10$ and $S/L=0.50$

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