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NAME OF SUPERVISOR/NOM DU DIRECTEUR DE THÈSE Prof. J. L. HUMAR

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FINITE ELEMENT ANALYSIS
OF
DAM-RESERVOIR SYSTEM

by

MAGDY S.L. ROUFAIEL

A thesis submitted to the Faculty of Graduate Studies
and Research in partial fulfillment of the degree of

MASTER OF ENGINEERING

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OTTAWA, ONTARIO

JUNE 1980

The undersigned recommends to the Faculty of Graduate Studies
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Thesis Supervisor

Chairman, Department of
Civil Engineering

Carleton University

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ABSTRACT

A method is developed to analyze the earthquake response of a two-dimensional dam-reservoir system, using a finite element discretization of both the dam and the reservoir. The response calculations are carried out through a frequency domain analysis.

In the analysis the water is considered compressible, inviscid, and the effect of free surface waves is neglected. Representation of an infinite reservoir by one of finite length is investigated and a radiation boundary condition which adequately models the energy loss in outgoing waves is developed. In the finite element model the nodal displacements of the dam and the nodal pressures in the reservoir are considered as the basic unknowns.

A number of dam-reservoir vibration problems are solved using the frequency domain analysis and the results are compared with those obtained from a time domain method.

It is concluded that the proposed frequency domain finite element method is quite effective in the solution of the complex interaction problems of dam-reservoir vibration. It is also computationally more efficient than the time domain approach.

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CHAPTER 1

INTRODUCTION1.1 General

The safety of dams during earthquakes is extremely important because the failure of such a structure may have disastrous consequences for life and property. Dams must therefore be designed to be adequately safe against earthquake loads. Yet, the seismic analysis of dams has for long been carried out by using crude analytical methods. The traditional method is to treat earthquake loads as static horizontal forces equal to the product of mass of the system and a design ground acceleration⁽³⁷⁾. As a result of the development of high speed digital computers, during the past few years, it has been possible to conduct dynamic analyses of dams^(3,9,10,11,21,31). These analyses have shown that the dynamic behaviour of a dam is far different from that obtained by the traditional static method. The studies have also shown that the dynamic analysis of a dam-reservoir system must be carried out by a numerical procedure because of the complexities of the problem, some of which are listed below:

- 1) The complex shapes of dams require the use of a discretization procedure in analysis, not only for dynamic loads but even for static loads.
- 2) From the point of view of analysis, dams differ from other classes of structures in an important sense. Any

analysis procedure used for a dam-reservoir system must consider the effect of the reservoir on the dynamic properties and the response of the dam, a consideration which does not arise in structures that do not impound water. The ground motion and deformations of the upstream face of the dam cause hydrodynamic pressures in the reservoir, and the dam deformations in turn are affected by the hydrodynamic pressures on the upstream face.

- 3). For an inclined dam face or sloped reservoir bed and compressible water, it is not yet possible to obtain a solution by the classical methods, and it is necessary to use a discretization procedure to solve the problem.

One of the procedures which can be used to solve the dam-reservoir problem is the finite element method. In a previous investigation⁽³⁾, the method has been applied to the time domain analysis of the dam-reservoir problem. It is shown here in this thesis that the time domain finite element analysis has some serious drawbacks which can affect the accuracy of the results. In addition, the method is found to be computationally inefficient for practical problems.

The main objective of this study is to develop an accurate and efficient finite element procedure to solve the dam-reservoir problem. It is shown that the finite element method can be very effectively used for the solution of a gravity dam-reservoir system provided the analysis is carried out in the frequency domain. To

the knowledge of the author, this is the first attempt to use a frequency domain analysis in conjunction with finite element representation of both the dam and the reservoir to solve the problem of coupled vibrations of a dam-reservoir system.

1.2 Review of Past Work

The hydrodynamic dam-reservoir problem has received the attention of many investigators since 1933. Westergaard's work⁽⁶⁾ on hydrodynamic pressures on gravity dams was the first application of hydrodynamics to earthquake induced vibrations. Westergaard's solution is based on the following assumptions:

- 1) The dam is rigid.
- 2) The upstream face of the dam is vertical.
- 3) The reservoir bottom is rigid, horizontal and extends to infinity.
- 4) The ground motion is a simple harmonic horizontal excitation.
- 5) Viscosity of water is negligible.

Later studies^(7,12) have shown that Westergaard's solution is not valid when the excitation frequency of the ground is greater than the fundamental frequency of the reservoir. Several investigations have been carried out to lift some of the assumptions made by Westergaard.

Zangar and Chwang have studied the effect of an inclined upstream face of the dam. Zangar⁽²²⁾ obtained the hydrodynamic pressure distribution on the inclined face of a dam by using an electric

analogue and constructing flow nets. Chwang⁽¹³⁾ developed an exact solution for the hydrodynamic pressures on the inclined face using an integral solution based on the two-dimensional potential-flow theory. Both Zangar and Chwang assumed that the water was incompressible. The other assumptions made by them were similar to those of Westergaard. A mathematical solution for the hydrodynamic pressure on the inclined dam face has not yet been obtained for the case of compressible water.

Chopra^(7,8) used a frequency domain analysis to find the hydrodynamic pressures on a rigid gravity dam with vertical upstream face under both horizontal and vertical earthquake ground motion. He found that typical strong motion earthquakes cause hydrodynamic pressures of appreciable magnitude, that vertical components of earthquakes have a very significant effect on the response, and that the compressibility of water has a significant influence on the pressures.

The effect of flexibility of the dam on hydrodynamic pressure was first studied by Brahtz and Heilbron⁽²⁷⁾ in 1933. They assumed a linear deformation of the dam and used iterative procedures to obtain the response of the coupled system. Bustamante⁽²⁸⁾ assumed a parabolic deformation shape for the dam and concluded that the effect of the flexibility of the dam on hydrodynamic pressure was very significant. The effects of dam-reservoir interaction were further investigated by Chopra^(9,29) assuming that the dam vibrated only in its fundamental mode. Later, Chakrabarti and Chopra^(10,11,30,31,32) developed a more general method in which any number of modes could be considered. In their analysis, they assumed that the deformation of

the dam could be represented as a linear combination of the first few modes of the dam without the reservoir. The major conclusion of their analysis was that consideration of the vertical component of excitation and compressibility of water could lead to an increase in stresses in the dam by more than 100%. Finn and Varoglu^(34,35) first solved the problem of vibrations of a rectangular dam analytically and later extended their solution to a finite element model of a gravity dam. They did not place any restriction on the deformation of the dam.

All investigations of the dam-reservoir systems reported above treated the reservoir as a continuum for which the governing differential equations of motion could be solved directly. This approach requires the reservoir boundary to be greatly idealized. It is necessary to assume that the upstream face of the dam is vertical and the reservoir bed is horizontal, whereas in a practical situation, such may not be the case.

When arbitrary geometries of the reservoir and an arbitrary excitation need to be considered, a discretization process is the only feasible method of analysis. Nath⁽²⁶⁾ used a finite element discretization to study the effect of vertical harmonic ground motion on the hydrodynamic pressures on an inclined rigid dam face. This work was limited to finite reservoirs and could not account for the energy dissipating mechanism of an infinite reservoir. A more general approach has been proposed by Zienkiewicz and Newton⁽²⁾ to solve the problem of structure-fluid interaction. They treat the coupled system as an assemblage of finite elements having nodal displacements as the

basic unknowns for the structure and nodal pressure as the basic unknown for the fluid. The procedure has been applied by Sharan⁽³⁾ to the present problem. He obtained the transient response by direct numerical integration of the coupled equations of motion using the constant acceleration method, and concluded that the existing methods of analyzing a dam-reservoir system, which treated the reservoir as a continuum and obtained a closed form solution for an idealized reservoir boundary, were inadequate for predicting the response of gravity dams, when the reservoir boundaries were not simple. He also confirmed the conclusions of the past analyses that the compressibility of water and the vertical components of an earthquake could not be ignored in the design of dams.

1.3 Objectives and Scope of Work

Although the time domain analysis provides a direct solution of the problem, it sometimes is more convenient to perform the analysis in the frequency domain⁽¹⁸⁾. Also, as will be discussed later, the modelling of energy loss in the outgoing waves may present some problems when a finite element discretization is used for the reservoir. To get an adequate representation of the energy loss, the boundary condition at the far end of the reservoir must involve an expression which depends on the exciting frequency. When such is the case, a frequency domain analysis is the only feasible method of obtaining the solution. The main objective of this work was to combine the finite element discretization with a frequency domain analysis to solve the problem of vibration of the dam-reservoir system.

The work was approached in the following stages:

- 1) Review of the mathematical solutions of the complex frequency response of an idealized reservoir to establish a framework for comparing the results of the finite element formulation of the reservoir problem.
- 2) Review and investigation of the finite element formulation of the reservoir problem.
- 3) Investigation of the finite element modelling of long or infinite reservoirs.
- 4) Development of a computer program to obtain the complex frequency response for the reservoir problem using the finite element method.
- 5) Review of the finite element formulation of the gravity dams.
- 6) Development of a computer program to obtain the earthquake response of the dam-reservoir problem, using a time domain analysis.
- 7) Development of a finite element procedure and a computer program to obtain the earthquake response of the dam-reservoir problem, using the frequency domain analysis.

1.4 Format of the Thesis

The thesis has been divided into six chapters as follows.

Chapter 1 introduces the problem, reviews the past work, and describes the objectives and scope of the present work and the format of the thesis.

Chapter 2 deals with the hydrodynamic rigid dam-reservoir problem. Classical solutions for the hydrodynamic pressures in reservoirs with idealized boundaries are derived first. This is followed by an investigation of the mathematical representation of an infinite reservoir by a reservoir of finite length. A radiation damping boundary condition, developed by the author, is described next and it is shown that the suggested boundary condition improves the accuracy of finite length representation of the infinite reservoir. Finally, a finite element procedure for the analysis of the reservoir problem is presented. The method is then used to obtain the complex frequency response values for the reservoir.

Chapter 3 deals with the finite element solution of the dam-reservoir problem using a time domain analysis. The finite element formulation of a gravity dam, using triangular finite elements, is presented first, followed by a formulation for the coupled system. Finally, a technique for solving the problem in the time domain is described.

Chapter 4 is devoted to a presentation of the frequency domain procedure developed by the author for the complete finite element dam-reservoir system.

Results for a few dam-reservoir problems using both time and frequency domain analyses are presented in Chapter 5.

Chapter 6 contains a summary, conclusions of the present study, and recommendations for future work.

CHAPTER 2

HYDRODYNAMIC PRESSURES ON A RIGID DAM2.1 Introduction

The small amplitude vibrations of a reservoir impounded by a rigid dam and subjected to ground motion can be represented by the classical two-dimensional wave equation. The derivation of this equation is presented briefly in this chapter.

Classical solutions exist for the wave equation when the reservoir boundaries are simple. These solutions are briefly reviewed for both the infinite and finite reservoirs.

When the reservoir boundaries are not simple, for example, if the dam face is inclined or the reservoir bed is sloped, classical solutions can no longer be derived and some form of numerical approximation methods must be used. This provides the motivation for a finite element solution for the dam reservoir problem. However, before attempting a finite element discretization, it is necessary to model the energy loss in the outgoing waves in an infinite reservoir. The Sommerfeld radiation damping condition has been used in the past for a similar problem⁽²⁾. It is shown that the proposed radiation condition fails to model the energy loss correctly and the results obtained from the application of such a condition may be widely different from those obtained by the classical method. A modified radiation condition is then developed and examples are presented to show that the method leads to solutions which are in excellent agree-

ment with the classical solution for an infinite reservoir for the whole range of frequencies from zero to the second natural frequency of the reservoir. Later in this study (Chapter 4), this modified radiation condition developed by the author is used in combination with a frequency domain analysis to obtain the solution to flexible dam reservoir problems.

A finite element formulation in which the nodal pressures are assumed to be the unknowns is then developed and the results obtained by the application of the method to the solution of rigid dam-reservoir problem are compared with known classical solutions.

2.2 Equation of Motion

The general fluid motion is governed by the well-known Navier-Stokes equations^(2,16). A simplified form of these equations is applicable to the case of the vibration of a reservoir impounded by a rigid dam. The simplification results from the following assumptions which are reasonable for the problem being considered:

- 1) The vibration problem is two-dimensional; the motion in any vertical plane perpendicular to the axis of the dam being similar.
- 2) The viscosity of water is negligible.
- 3) The displacements and velocities of the particles of the water are small.
- 4) The effect of free surface waves is negligible.

Now, if

p = hydrodynamic pressure in excess of the static pressure due to gravity
 u, v = x and y components respectively of the velocity of a particle of water
 a_x, a_y = x and y components respectively of the acceleration
 and ρ = mass density of water.

then, referring to Fig. 2.1, the equilibrium of forces in the x-direction acting on a small element $dx dy$ gives

$$p dy - (p + \frac{\partial p}{\partial x} dx) dy - \rho dx dy a_x = 0$$

or

$$\frac{\partial p}{\partial x} = -\rho a_x \quad (2.1a)$$

Similarly, for equilibrium in the y-direction,

$$\frac{\partial p}{\partial y} = -\rho a_y \quad (2.1b)$$

The x-component of acceleration is given by

$$a_x = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt}$$

which because of the assumption of small vibrations reduces to

$$a_x = \frac{\partial u}{\partial t} \quad (2.2a)$$

In a similar manner

$$a_y = \frac{\partial v}{\partial t} \quad (2.2b)$$

Substitution of Eqns. (2.2a) and (2.2b) into Eqns. (2.1a) and (2.1b) gives

Fig. 2.1. Equilibrium of hydrodynamic forces.

$$\frac{\partial p}{\partial x} = - \rho \frac{\partial u}{\partial t} \quad (2.3a)$$

$$\frac{\partial p}{\partial y} = - \rho \frac{\partial v}{\partial t} \quad (2.3b)$$

The net amount of water entering the element in x-direction per unit of time is

$$- (\rho u + \frac{\partial \rho u}{\partial x} dx) dy + \rho u dy$$

which becomes equal to

$$- \rho \frac{\partial u}{\partial x} dx dy \quad (2.4a)$$

when, within an element, ρ is considered a variable of time only.

In a similar manner, the net amount of water entering the element in the y-direction per unit of time is

$$- \rho \frac{\partial v}{\partial y} dx dy \quad (2.4b)$$

- The continuity condition requires that the net amount of water entering the element per unit of time must equal the rate of change of mass.

This gives

$$- \rho \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dx dy = \frac{\partial \rho}{\partial t} dx dy$$

or

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = - \frac{1}{\rho} \frac{\partial \rho}{\partial t} \quad (2.5)$$

By definition, the bulk modulus of a fluid, k , is given by

$$k = \frac{\partial p}{\partial \rho / \rho}$$

Substitution of the above relation in Eqn. (2.5) gives

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = - \frac{1}{k} \frac{\partial p}{\partial t} \quad (2.6)$$

A combination of Eqns. (2.3a), (2.3b) and (2.6) gives the following differential equation

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} \quad (2.7)$$

in which, $c = \sqrt{k/\rho}$ is the velocity of sound in water. Eqn. (2.7) is the governing equation of motion.

2.3 Classical Solutions

2.3.1 Infinite reservoir - vertical rigid dam system subject to horizontal harmonic excitation

Consider a rigid dam of height H with a vertical upstream face and a reservoir which extends to infinity in the upstream direction. If the effect of surface waves is neglected, and the rigid dam is vibrating with horizontal acceleration equal to $e^{i\omega t}$, the following boundary conditions apply (Fig. 2.2):

- 1) The hydrodynamic pressure at the surface of the water is zero, i.e.

$$p(x, H, t) = 0$$

- 2) The vertical acceleration of the particles of water at the base of the reservoir is zero, i.e.

$$\frac{\partial p}{\partial y}(x, 0, t) = 0$$

Fig. 2.2. Boundary conditions for rigid dam-infinite reservoir system.

- 3) The hydrodynamic pressure at infinity is zero, i.e.

$$p(\infty, y, t) = 0$$

- 4) The horizontal acceleration of the particles of water in contact with the dam face is equal to the acceleration of the dam, i.e.

$$\frac{\partial p}{\partial x}(0, y, t) = -\rho e^{i\omega t}$$

It is a property of linear time invariant systems that when the excitation is steady-state simple harmonic motion (without beginning or end), then the response is also steady-state simple harmonic motion with the same frequency⁽⁷⁾. Therefore, the solution of Eqn. (2.7) subject to the above boundary conditions is

$$p(x, y, t) = \bar{p}(x, y, \omega) e^{i\omega t} \quad (2.8)$$

Now, if

$$\bar{p}(x, y, \omega) = X(x) \cdot Y(y) \quad (2.9)$$

Eqn. (2.7) gives

$$Y \frac{d^2 X}{dx^2} + X \frac{d^2 Y}{dy^2} = -\frac{\omega^2}{c^2} XY$$

or

$$\frac{1}{X} \frac{d^2 X}{dx^2} + \frac{\omega^2}{c^2} = -\frac{1}{Y} \frac{d^2 Y}{dy^2} = m^2 \quad (2.10)$$

where m^2 is a constant.

One of the two equations represented by Eqn. (2.10) is

$$\frac{d^2 Y}{dy^2} + m^2 Y = 0 \quad (2.11)$$

which has a general solution of the type

$$Y = B_1 \cos my + B_2 \sin my \quad (2.12)$$

Boundary condition (2) requires that B_2 be zero. Application of boundary condition (1) then gives the following value for m

$$m = \lambda_n = \frac{(2n-1)\pi}{2H} \quad ; n=1,2,3,\dots,\infty \quad (2.13)$$

The solution for Y becomes

$$Y = B_1 \cos \lambda_n y \quad (2.14)$$

Eqns. (2.10) and (2.13) give

$$\frac{1}{X} \frac{d^2 X}{dx^2} + \frac{\omega^2}{c^2} = \lambda_n^2$$

or

$$\frac{d^2 X}{dx^2} - q_n^2 X = 0 \quad (2.15)$$

in which

$$q_n = \sqrt{\lambda_n^2 - \frac{\omega^2}{c^2}} \quad (2.16)$$

The differential Eqn. (2.15) has a general solution of the form

$$X = D_1 e^{-q_n x} + D_2 e^{q_n x} \quad (2.17)$$

Now, if it is assumed that q_n is real, boundary condition (3) requires that D_2 be zero.

Substitution of Eqns. (2.14) and (2.17) into Eqn. (2.8) gives the following expression

$$p = \sum_{n=1}^{\infty} A_n e^{-q_n x} \cos \lambda_n y e^{i\omega t} \quad (2.18)$$

Application of boundary condition (4) to Eqn. (2.18) gives

$$-\rho e^{i\omega t} = \sum_{n=1}^{\infty} -A_n q_n \cos \lambda_n y e^{i\omega t}$$

or

$$\rho = \sum_{n=1}^{\infty} A_n q_n \cos \lambda_n y$$

On multiplying the two sides of the above equation by $(\cos \lambda_\ell y)$ and integrating from $y=0$ to $y=H$, the following relationship is obtained

$$\rho \int_0^H \cos \lambda_\ell y \, dy = \sum_{n=1}^{\infty} A_n q_n \int_0^H \cos \lambda_\ell y \cos \lambda_n y \, dy \quad (2.19)$$

It can be proved that

$$\int_0^H \cos \lambda_\ell y \, dy = \frac{1}{\lambda_\ell} (-1)^{\ell+1} \quad (2.20)$$

and

$$\int_0^H \cos \lambda_\ell y \cos \lambda_n y \, dy = \begin{cases} 0 & \text{if } n \neq \ell \\ \frac{H}{2} & \text{if } n = \ell \end{cases} \quad (2.21)$$

From Eqns. (2.19), (2.20) and (2.21)

$$A_n = \frac{4\rho}{\pi} \frac{(-1)^{n+1}}{(2n-1)} \frac{1}{q_n} \quad (2.22)$$

Substitution of Eqn. (2.22) into Eqn. (2.18) yields the following expression for the hydrodynamic pressure

$$p(x,y,t) = \frac{4\rho}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)} \frac{1}{q_n} e^{-q_n x} \cos \lambda_n y e^{i\omega t} \quad (2.23)$$

It can be noticed from Eqn. (2.16) that the value of q_n may be real or imaginary, depending on the reservoir properties (H and c), the excitation frequency (ω), and the subscript (n). The solution $e^{q_n x}$ is bounded when q_n is imaginary and cannot be rejected on the grounds of unboundedness. However, the expression

$$e^{q_n x} e^{i\omega t} = e^{i(\bar{q}_n x + \omega t)}$$

is of the form $F(x+ct)$ which represents a wave moving inwards from infinity. On physical grounds, this is not admissible and hence the solution $e^{q_n x}$ must be rejected.

If it is assumed that n_1 is the smallest value of n which makes q_n real, Eqn. (2.23) can be re-written as

$$p = \frac{4\rho}{\pi} \sum_{n=1}^{n_1-1} \frac{(-1)^{n+1}}{(2n-1)} \frac{1}{i\bar{q}_n} e^{-i\bar{q}_n x} \cos \lambda_n y e^{i\omega t} + \frac{4\rho}{\pi} \sum_{n=n_1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)} \frac{1}{q_n} e^{-q_n x} \cos \lambda_n y e^{i\omega t} \quad (2.24)$$

in which

$$\bar{q}_n = \sqrt{\frac{\omega^2}{c^2} - \lambda_n^2} = q_n / i \quad (2.25)$$

Now, if the excitation is the real part of $e^{i\omega t}$, the pressure is given by the real part of Eqn. (2.24) which is

$$\begin{aligned}
p = \frac{4\rho}{\pi} & \left[\sum_{n=1}^{n_1-1} \frac{(-1)^n}{(2n-1)} \frac{1}{\bar{q}_n} \sin \bar{q}_n x \cos \lambda_n y \right. \\
& + \sum_{n=n_1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)} \frac{1}{\bar{q}_n} e^{-q_n x} \cos \lambda_n y \left. \right] \cos \omega t \\
& + \frac{4\rho}{\pi} \left[\sum_{n=1}^{n_1-1} \frac{(-1)^{n+1}}{(2n-1)} \frac{1}{\bar{q}_n} \cos \bar{q}_n x \cos \lambda_n y \right] \sin \omega t \quad (2.26)
\end{aligned}$$

The pressure at the dam face is

$$\begin{aligned}
p(0, y, t) = \frac{4\rho}{\pi} & \left[\sum_{n=n_1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)} \frac{1}{\bar{q}_n} \cos \lambda_n y \right] \cos \omega t \\
& + \frac{4\rho}{\pi} \left[\sum_{n=1}^{n_1-1} \frac{(-1)^{n+1}}{(2n-1)} \frac{1}{\bar{q}_n} \cos \lambda_n y \right] \sin \omega t \quad (2.27)
\end{aligned}$$

or

$$p(0, y, t) = \bar{p}_I \cos \omega t + \bar{p}_O \sin \omega t \quad (2.28)$$

where

$$\bar{p}_I = \text{in-phase pressure} = \frac{4\rho}{\pi} \sum_{n=n_1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)} \frac{1}{\bar{q}_n} \cos \lambda_n y \quad (2.29)$$

$$\bar{p}_O = \text{out-of-phase pressure} = \frac{4\rho}{\pi} \sum_{n=1}^{n_1-1} \frac{(-1)^{n+1}}{(2n-1)} \frac{1}{\bar{q}_n} \cos \lambda_n y \quad (2.30)$$

It is apparent from Eqn. (2.27) that the hydrodynamic pressures become infinite when

$$\bar{q}_n = 0$$

or

$$\frac{(2n-1)\pi}{2H} = \frac{\omega}{c}$$

Frequencies which satisfy the above relation are called the natural frequencies of the reservoir and are given by

$$\omega_n = \frac{(2n-1)\pi c}{2H}, \quad ; n=1,2,3,\dots,\infty \quad (2.31)$$

From Eqns. (2.13), (2.16) and (2.31)

$$q_n = \frac{1}{c} \sqrt{\omega_n^2 - \omega^2} \quad (2.32)$$

Thus, q_n is imaginary if $\omega > \omega_n$.

If the excitation frequency (ω) is less than the fundamental frequency of the reservoir (ω_1), i.e. $n_1=1$, Eqn. (2.27) becomes

$$p(0,y,t) = \frac{4\rho}{\pi} \cos \omega t \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)} \frac{1}{q_n} \cos \lambda_n y \quad (2.33)$$

which is the well-known Westergaard solution⁽⁶⁾. It can be concluded that the Westergaard solution is valid only if the excitation frequency (ω) is less than the fundamental frequency of the reservoir (ω_1).

2.3.2 Finite reservoir - vertical rigid dam system subject to horizontal harmonic excitation

(a) Far end stationary: Consider a rigid dam of height H with vertical upstream face and a reservoir of length L in the upstream direction. Neglecting the effect of surface waves, and considering the far end of the reservoir fixed, the boundary conditions can be written as (Fig. 2.3)

$$(1) \quad p(x,H,t) = 0$$

$$(2) \quad \frac{\partial p}{\partial y}(x,0,t) = 0$$

Fig. 2.3. Boundary conditions for rigid dam-finite reservoir system.
(fixed far end)

$$(3) \quad \frac{\partial p}{\partial x} (L, y, t) = 0$$

$$(4) \quad \frac{\partial p}{\partial x} (0, y, t) = -\rho e^{i\omega t}$$

Equations (2.8) through (2.17) of the previous section apply to this case too. Application of boundary condition (3) to Eqn. (2.17) gives

$$D_2 = D_1 e^{-2q_n L} \quad (2.34)$$

Substitution of D_2 from Eqn. (2.34) into Eqn. (2.17) leads to

$$X = D_1 [e^{-q_n x} + e^{-q_n (2L-x)}] \quad (2.35)$$

On substituting Eqns. (2.14) and (2.35) into Eqn. (2.8), the hydrodynamic pressure can be expressed as

$$p = \sum_{n=1}^{\infty} A_n [e^{-q_n x} + e^{-q_n (2L-x)}] \cos \lambda_n y e^{i\omega t} \quad (2.36)$$

Application of boundary condition (4) gives

$$-\rho = \sum_{n=1}^{\infty} -A_n q_n [1 - e^{-2q_n L}] \cos \lambda_n y \quad (2.37)$$

On multiplying the two sides of Eqn. (2.37) by $\cos \lambda_\ell y$ and integrating from $y=0$ to $y=H$, the following expression is obtained for A_n

$$A_n = \frac{4\rho}{\pi} \frac{(-1)^{n+1}}{(2n-1)} \frac{1}{q_n (1 - e^{-2q_n L})} \quad (2.38)$$

Substituting A_n from Eqn. (2.38) into Eqn. (2.36)

$$p = \frac{4\rho}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)} \frac{1}{q_n} \frac{e^{q_n(L-x)} + e^{-q_n(L-x)}}{e^{q_n L} - e^{-q_n L}} \cos \lambda_n y e^{i\omega t} \quad (2.39)$$

If n_1 is the smallest value of n which makes q_n real, Eqn. (2.39) can be rewritten as

$$p = \frac{4\rho}{\pi} \sum_{n=1}^{n_1-1} \frac{(-1)^{n+1}}{(2n-1)} \frac{1}{i\bar{q}_n} \frac{e^{i\bar{q}_n(L-x)} + e^{-i\bar{q}_n(L-x)}}{e^{i\bar{q}_n L} - e^{-i\bar{q}_n L}} \cos \lambda_n y e^{i\omega t} \\ + \frac{4\rho}{\pi} \sum_{n=n_1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)} \frac{1}{q_n} \frac{e^{q_n(L-x)} + e^{-q_n(L-x)}}{e^{q_n L} - e^{-q_n L}} \cos \lambda_n y e^{i\omega t}$$

or

$$p = \frac{4\rho}{\pi} \left[\sum_{n=1}^{n_1-1} \frac{(-1)^n}{(2n-1)} \frac{1}{\bar{q}_n} \frac{\cos \bar{q}_n(L-x)}{\sin \bar{q}_n L} \cos \lambda_n y \right. \\ \left. + \sum_{n=n_1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)} \frac{1}{q_n} \frac{\cosh q_n(L-x)}{\sinh q_n L} \cos \lambda_n y \right] e^{i\omega t} \quad (2.40)$$

in which

$$\bar{q}_n = q_n/i$$

It is apparent from Eqn. (2.40) that the hydrodynamic pressure is always in phase with the excitation, whatever the value of the exciting frequency. Also, if $n_1=1$, i.e. ω is less than ω_1 , the first term of Eqn. (2.40) vanishes. In addition, if L is large enough and the excitation is the real part of $e^{i\omega t}$, the hydrodynamic pressure at the dam face, i.e. $x=0$, is given by

$$p = \frac{4\rho}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)} \frac{1}{q_n} \cos \lambda_n y \cos \omega t \quad (2.41)$$

which is the Westergaard solution for infinite reservoirs⁽⁶⁾.

It is apparent from Eqn. (2.40) that resonance takes place if one of q_n , $\bar{q}_n$, $\sinh q_n L$, or $\sin \bar{q}_n L$ is zero. The first three conditions give the same natural frequencies as in the case of the infinite reservoir, i.e.

$$\omega_n = \frac{(2n-1)\pi c}{2H} \quad ; n=1,2,\dots,\infty \quad (2.31)$$

The condition $\sin \bar{q}_n L = 0$ gives

$$\bar{q}_n L = m\pi \quad ; m=0,1,2,\dots,\infty$$

or

$$\omega_{nm} = \frac{(2n-1)\pi c}{2H} \sqrt{1 + \left(\frac{m/(2n-1)}{L/2H}\right)^2} \quad (2.42)$$

where ω_{nm} is a natural frequency of the finite reservoir. It is apparent from Eqns. (2.31) and (2.42) that the fundamental frequencies of the finite and infinite reservoirs are the same, and are given by

$$\omega_1 = \frac{\pi c}{2H} \quad (2.43)$$

(b) Far end moving with the ground acceleration: Consider a rigid dam of height H with vertical upstream face and a reservoir of length L in the upstream direction. Neglecting the effect of surface waves, and considering that the far end of the reservoir moving with the ground acceleration, the boundary conditions can be written as (Fig. 2.4):

$$(1) \quad p(x,H,t) = 0$$

Fig. 2.4. Boundary conditions for rigid dam-finite reservoir system.
(far end moving with the ground acceleration)

$$(2) \quad \frac{\partial p}{\partial y} (x, 0, t) = 0$$

$$(3) \quad \frac{\partial p}{\partial x} (L, y, t) = -\rho e^{i\omega t}$$

$$(4) \quad \frac{\partial p}{\partial x} (0, y, t) = -\rho e^{i\omega t}$$

Equations (2.8) through (2.17) of the previous section apply to this case too.

Application of boundary conditions (3) and (4) to Eqn. (2.17) gives

$$D_2 = D_1 \frac{e^{-q_n L} - 1}{e^{q_n L} - 1} \quad (2.44)$$

Substitution of D_2 from Eqn. (2.44) into Eqn. (2.17) leads to

$$\chi = D_1 \left[e^{-q_n x} + \frac{e^{-q_n L} - 1}{e^{q_n L} - 1} e^{q_n x} \right] \quad (2.45)$$

On substituting Eqns. (2.14) and (2.45) into Eqn. (2.8), the hydrodynamic pressure can be expressed as

$$p = \sum_{n=1}^{\infty} A_n \left[e^{-q_n x} + \frac{e^{-q_n L} - 1}{e^{q_n L} - 1} e^{q_n x} \right] \cos \lambda_n y e^{i\omega t} \quad (2.46)$$

Application of boundary condition (4) gives

$$-\rho = \sum_{n=1}^{\infty} A_n q_n \left[\frac{e^{q_n L} - e^{-q_n L}}{e^{q_n L} - 1} \right] \cos \lambda_n y \quad (2.47)$$

On multiplying the two sides of Eqn. (2.47) by $\cos \lambda_\ell y$ and integrating from $y=0$ to $y=H$, the following expression is obtained for A_n

$$A_n = \frac{4\rho}{\pi} \frac{(-1)^{n+1}}{(2n-1)} \frac{1}{q_n} \frac{e^{q_n L} - 1}{e^{q_n L} - e^{-q_n L}} \quad (2.48)$$

Substituting A_n from Eqn. (2.48) into Eqn. (2.46) and assuming that n_1 is the smallest value of n which makes q_n real, the hydrodynamic pressure can be expressed as

$$p = \frac{4\rho}{\pi} \left[\sum_{n=1}^{n_1-1} \frac{(-1)^n}{(2n-1)} \frac{1}{\bar{q}_n} \frac{\cos \bar{q}_n (L-x) - \cos \bar{q}_n x}{\sin \bar{q}_n L} \cos \lambda_n y \right. \\ \left. + \sum_{n=n_1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)} \frac{1}{q_n} \frac{\cosh q_n (L-x) - \cosh q_n x}{\sinh q_n L} \cos \lambda_n y \right] e^{i\omega t} \quad (2.49)$$

It is apparent from Eqn. (2.49) that the hydrodynamic pressure is always in phase with the excitation, whatever the value of the exciting frequency. Also, if $n_1=1$, i.e. ω is less than ω_1 , the first term of Eqn. (2.49) vanishes. In addition, if L is large enough and the excitation is the real part of the $e^{i\omega t}$, the hydrodynamic pressure at the dam face, i.e. $x=0$, is given by

$$p = \frac{4\rho}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)} \frac{1}{q_n} \cos \lambda_n y \cos \omega t \quad (2.50)$$

which is the Westergaard solution for infinite reservoirs⁽⁶⁾

2.4 Representation of an Infinite Reservoir by a Finite Reservoir

An infinite reservoir must be represented by a reservoir of finite length before it can be modelled using finite elements. For an excitation frequency (ω) less than the fundamental frequency of

the reservoir (ω_1), and different values of reservoir length (L), Table 2.1 and Fig. 2.5 show a comparison between the classical solutions of the infinite and finite reservoirs. It can be concluded that a finite reservoir of length equal to twice the height can represent the infinite reservoir.

When the excitation frequency is greater than the fundamental frequency (ω_1), a finite length reservoir will in general fail to represent the energy loss in the outgoing waves that is characteristic of infinite reservoirs. It has been suggested that by a suitable choice of the boundary condition at the truncation boundary it may be possible to model such energy loss. Zienkiewicz and Newton⁽²⁾ have described one such boundary condition based on the assumption that at a sufficient distance from the dam face, the water waves can be considered as plane waves. A plane wave can be represented by an equation of the form

$$p = F(x-ct)$$

The above equation will give

$$\frac{\partial p}{\partial x} \equiv \frac{\partial p}{\partial n} = - \frac{1}{c} \frac{\partial p}{\partial t} \quad (2.51)$$

where n is the outward normal at the truncation boundary. Eqn. (2.51) represents the well-known Sommerfeld radiation condition. This condition introduces a damping in the system and models the loss of energy in the outgoing waves. This section examines the effectiveness of the Sommerfeld radiation condition in representing the energy loss in an infinite reservoir. The solutions obtained from a finite reservoir

Table 2.1. Classical solutions for finite and infinite reservoirs.

$\frac{y}{H}$	HYDRODYNAMIC PRESSURE ($\bar{p}/\rho H$)			
	INFINITE RESERVOIR	FINITE RESERVOIR*		
		L=H	L=2H	L=3H
1.000	0	0	0	0
0.875	.2950	.3236	.2969	.2951
0.750	.4773	.5337	.4810	.4776
0.625	.6142	.6960	.6196	.6146
0.500	.7187	.8228	.7256	.7192
0.375	.7966	.9189	.8047	.7972
0.250	.8507	.9867	.8597	.8514
0.125	.8827	1.0270	.8922	.8834
0.0	.8932	1.0400	.9030	.8939

$\omega=10$ rad/s

$\omega_1=18.535$ rad/s

* Far end is fixed.

Fig. 2.5. Classical solution for infinite and finite reservoir problem.

with a radiation damping condition applied to the far end are compared with the classical solution for an infinite reservoir and it is shown that the correlation is not good. A modified radiation damping condition is then derived and is seen to give much improved results.

Consider a rigid dam of height H with a vertical upstream face impounding a reservoir of length L . When the effect of surface waves is neglected and a radiation condition is applied to the far end of the reservoir, the following boundary conditions apply (Fig. 2.6):

$$(1) \quad p(x, H, t) = 0$$

$$(2) \quad \frac{\partial p}{\partial y}(x, 0, t) = 0$$

$$(3) \quad \frac{\partial p}{\partial x}(L, y, t) = -\frac{1}{c} \frac{\partial p}{\partial t}$$

$$(4) \quad \frac{\partial p}{\partial x}(0, y, t) = -\rho e^{i\omega t}$$

Eqns. (2.8) through (2.17) of Section 2.3.1 apply in this case too. Substitution of Eqns. (2.14) and (2.17) into Eqn. (2.8) gives the following expression for the pressure

$$p = \sum_{n=1}^{\infty} (D_{1n} e^{-q_n x} + D_{2n} e^{q_n x}) \cos \lambda_n y e^{i\omega t} \quad (2.52)$$

Application of boundary condition (3) gives

$$D_{2n} = D_{1n} e^{-2q_n L} Z_n \quad (2.53)$$

in which

$$Z_n = \frac{q_n - i \frac{\omega}{c}}{q_n + i \frac{\omega}{c}} \quad (2.54)$$

Fig. 2.6. Boundary conditions for rigid dam-finite reservoir system. (radiation damping at far end)

The expression for p then becomes

$$p = \sum_{n=1}^{\infty} A_n [e^{-q_n x} + Z_n e^{-q_n (2L-x)}] \cos \lambda_n y e^{i\omega t} \quad (2.55)$$

Application of boundary condition (4) gives

$$-\rho = \sum_{n=1}^{\infty} -A_n q_n (1 - Z_n e^{-2q_n L}) \cos \lambda_n y \quad (2.56)$$

On multiplying the two sides of Eqn. (2.56) by $\cos \lambda_\ell y$ and integrating from $y=0$ to $y=H$, the following value is obtained for A_n

$$A_n = \frac{4\rho}{\pi} \frac{(-1)^{n+1}}{(2n-1)} \frac{1}{q_n} \frac{1}{1 - Z_n e^{-2q_n L}} \quad (2.57)$$

On substituting A_n from Eqn. (2.57) into Eqn. (2.55), the hydrodynamic pressure can be expressed as

$$p = \frac{4\rho}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)} \frac{1}{q_n} \frac{e^{q_n(L-x)} + Z_n e^{-q_n(L-x)}}{e^{q_n L} - Z_n e^{-q_n L}} \cos \lambda_n y e^{i\omega t} \quad (2.58)$$

or

$$p = \bar{p} e^{i\omega t} \quad (2.59)$$

or

$$p = (\bar{p}_R + i\bar{p}_I)(\cos \omega t + i \sin \omega t) \quad (2.60)$$

in which $\bar{p}_R$ is the real part of $\bar{p}$ and $\bar{p}_I$ is the imaginary part of $\bar{p}$.

Now, if the excitation is the real part of $e^{i\omega t}$, i.e. $\cos \omega t$, the pressure is the real part of Eqn. (2.60) and is given by

$$p = \bar{p}_R \cos \omega t - \bar{p}_I \sin \omega t$$

or

$$p = \bar{p}_S \cos \omega t + \bar{p}_O \sin \omega t \quad (2.61)$$

in which

$$\bar{p}_S = \bar{p}_R = \text{in-phase pressure}$$

$$\bar{p}_O = -\bar{p}_I = \text{out-of-phase pressure}$$

A computer program was developed to calculate the in-phase and out-of-phase pressures given in Eqn. (2.61). Using this program, the in-phase and out-of-phase pressures on the dam face were obtained for finite reservoirs with different length to height ratios and compared with the corresponding pressures in an infinite reservoir. A comparison of results is presented in Table 2.2 and Fig. 2.7. From these results, it can be concluded that the radiation damping condition does not yield good approximation for the infinite reservoir for excitation frequencies between the first and second natural frequencies of the infinite reservoir.

It is apparent that if an infinite reservoir is to be represented by a reservoir of finite length, a more accurate radiation damping condition is required. To investigate the possibility of deriving an improved radiation damping condition, consider the classical solution for the rigid dam and infinite reservoir presented in Section 2.3.1. Eqn. (2.24) can be rewritten in the form

$$p = \left[\sum_{n=1}^{n_1-1} \bar{A}_n e^{-i\bar{q}_n x} \cos \lambda_n y + \sum_{n=n_1}^{\infty} A_n e^{-q_n x} \cos \lambda_n y \right] e^{i\omega t}$$

Table 2.2a. Effect of radiation conditions on classical solutions. ($\omega < \omega_1$)

$\frac{y}{H}$	HYDRODYNAMIC PRESSURE ($\bar{p}/\rho H$) ON DAM FACE					
	IN-PHASE			OUT OF PHASE		
	INFINITE RESERVOIR	FINITE RESERVOIR (1)	FINITE RESERVOIR (2)	INFINITE RESERVOIR	FINITE RESERVOIR (1)	FINITE RESERVOIR (2)
1.000	0	0	0	0	0	0
.875	.2950	.2958	.2969	0	0	0
.750	.4773	.4788	.4810	0	0	0
.625	.6142	.6165	.6196	0	0	0
.500	.7187	.7216	.7256	0	0	0
.375	.7966	.7999	.8047	0	0	0
.250	.8507	.8544	.8597	0	0	0
.125	.8827	.8866	.8922	0	0	0
0.000	.8932	.8973	.9030	0	0	0

(1) Using the Sommerfeld radiation condition.

(2) Using the modified radiation condition.

$$\omega_1 = 18.535 \text{ rad/sec}$$

$$\omega/\omega_1 = 0.5395$$

$$L = 2H$$

Table 2.2b. Effect of radiation conditions on classical solutions. ($\omega_1 < \omega < \omega_2$)

$\frac{y}{H}$	HYDRODYNAMIC PRESSURE ($\bar{p}/\rho H$) ON DAM FACE					
	IN-PHASE			OUT OF PHASE		
	INFINITE RESERVOIR	FINITE RESERVOIR (1)	FINITE RESERVOIR (2)	INFINITE RESERVOIR	FINITE RESERVOIR (1)	FINITE RESERVOIR (2)
1.000	0	0	0	0	0	0
.875	.1135	.0852	.1135	.1747	.1231	.1747
.750	.1185	.0629	.1185	.3427	.2414	.3427
.625	.0884	.0077	.0884	.4975	.3505	.4975
.500	.0437	-.0589	.0437	.6333	.4461	.6333
.375	-.0029	-.1236	-.0029	.7446	.5246	.7446
.250	-.0425	-.1765	-.0425	.8274	.5829	.8274
.125	-.0687	-.2110	-.0687	.8784	.6188	.8784
0.000	-.0779	-.2230	-.0779	.8956	.6309	.8956

(1) Using the Sommerfeld radiation condition.

(2) Using the modified radiation condition.

$$\omega_1 = 18.535 \text{ rad/sec}$$

$$\omega/\omega_1 = 1.3488$$

$$L = 2H$$

Table 2.2c. Effect of radiation conditions on classical solutions. ($\omega_1 < \omega < \omega_2$)

$\frac{y}{H}$	HYDRODYNAMIC PRESSURE ($\bar{p}/\rho H$) ON DAM FACE					
	IN-PHASE			OUT OF PHASE		
	INFINITE RESERVOIR	FINITE RESERVOIR (1)	FINITE RESERVOIR (2)	INFINITE RESERVOIR	FINITE RESERVOIR (1)	FINITE RESERVOIR (2)
1.000	0	0	0	0	0	0
.875	.1177	.1477	.1177	.1243	.1249	.1243
.750	.1247	.1836	.1247	.2437	.2451	.2437
.625	.0944	.1798	.0944	.3538	.3558	.3538
.500	.0476	.1564	.0476	.4504	.4528	.4504
.375	.0	.1257	.0	.5296	.5324	.5296
.250	-.0449	.0972	-.0449	.5884	.5916	.5884
.125	-.0735	.0774	-.0735	.6247	.6281	.6247
0.000	-.0835	.0704	-.0835	.6369	.6404	.6369

(1) Using the Sommerfeld radiation condition.

(2) Using the modified radiation condition.

$$\omega_1 = 18.535 \text{ rad/sec}$$

$$\omega/\omega_1 = 1.6186$$

$$L = 2H$$

Table 2.2d. Effect of radiation conditions on classical solutions. ($\omega > \omega_2$)

$\frac{y}{H}$	HYDRODYNAMIC PRESSURE ($\bar{p}/\rho H$) ON DAM FACE					
	IN-PHASE			OUT OF PHASE		
	INFINITE RESERVOIR	FINITE RESERVOIR (1)	FINITE RESERVOIR (2)	INFINITE RESERVOIR	FINITE RESERVOIR (1)	FINITE RESERVOIR (2)
1.000	0	0	0	0	0	0
.875	.0888	.0971	.0747	.0860	.0739	.0931
.750	.0540	.0682	.0306	.1543	.1340	.1661
.625	-.0087	.0071	-.0335	.1927	.1708	.2051
.500	-.0511	-.0381	-.0689	.1981	.1818	.2070
.375	-.0499	-.0432	-.0549	.1775	.1719	.1800
.250	-.0127	-.0132	-.0030	.1450	.1516	.1402
.125	.0312	.0250	.0522	.1169	.1329	.1064
0.000	.0503	.0420	.0756	.1060	.1255	.0933

(1) Using the Sommerfeld radiation condition.

(2) Using the modified radiation condition.

$$\omega_1 = 18.535 \text{ rad/sec}$$

$$\omega/\omega_1 = 4.3162$$

$$L = 2H$$

Fig. 2.7a. Effect of radiation damping conditions on classical solutions.

Fig. 2.7b. Effect of radiation damping conditions on classical solutions.

Fig. 2.7c. Effect of radiation damping conditions on classical solution.

Fig. 2.7d. Effect of radiation damping conditions on classical solution.

At a large distance from the dam, the second term of the above equation vanishes, and the pressure is given by

$$p = \sum_{n=1}^{n_1-1} \bar{A}_n e^{-i\bar{q}_n x} \cos \lambda_n y e^{i\omega t} = \sum_{n=1}^{n_1-1} p_n \quad (2.62)$$

From Eqn. (2.62), it is easily proved that

$$p \frac{\partial p}{\partial x} = - \frac{1}{\omega} \sum_{n=1}^{n_1-1} \bar{q}_n p_n \frac{\partial p}{\partial t} \quad (2.63)$$

When x is set equal to L , Eqn. (2.63) represents the exact radiation damping condition. For $n_1=2$, i.e. when the excitation frequency has a value lying between the first and second natural frequencies,

Eqn. (2.62) gives

$$p = \sum_{n=1}^1 p_n = p_1$$

Hence, Eqn. (2.63) becomes

$$p \frac{\partial p}{\partial x} = - \frac{1}{\omega} \bar{q}_1 p \frac{\partial p}{\partial t}$$

or

$$\frac{\partial p}{\partial x} = - \frac{\bar{q}_1}{\omega} \frac{\partial p}{\partial t}$$

or

$$\frac{\partial p}{\partial x} = - \frac{1}{c} \sqrt{1 - \left(\frac{\omega_1}{\omega}\right)^2} \frac{\partial p}{\partial t} \quad (2.64)$$

which is the exact radiation damping condition for $\omega_1 < \omega < \omega_2$. It is apparent from Eqn. (2.64) that the modified radiation damping condition is frequency-dependent.

In conclusion, it can be said that the following condition should be applied at the truncation boundary

$$\frac{\partial p}{\partial n} = 0 \quad ; \quad \omega < \omega_1 \quad (2.65a)$$

$$\frac{\partial p}{\partial n} = - \frac{1}{c} \sqrt{1 - \left(\frac{\omega_1}{\omega}\right)^2} \frac{\partial p}{\partial t} \quad ; \quad \omega > \omega_1 \quad (2.65b)$$

Apparently, the above condition is not exact when the excitation frequency is greater than ω_2 . For high excitation frequencies, Eqn. (2.65b) takes the same form as in Eqn. (2.51) which is suitable for high frequency responses.

The hydrodynamic pressure obtained by applying the modified boundary condition represented by Eqn. (2.65) will be given by an expression similar to Eqn. (2.58) except that Z_n will now be as follows

$$Z_n = \frac{q_n - \frac{i\omega}{c} \sqrt{1 - \left(\frac{\omega_1}{\omega}\right)^2}}{q_n + \frac{i\omega}{c} \sqrt{1 - \left(\frac{\omega_1}{\omega}\right)^2}} \quad ; \quad \omega > \omega_1 \quad (2.66a)$$

$$Z_n = 1 \quad ; \quad \omega < \omega_1 \quad (2.66b)$$

A computer program was developed to calculate the hydrodynamic pressures in a finite reservoir with condition (2.65) applied to the truncation boundary. Using this program, the pressure values were calculated for reservoirs with different length to height ratios and different values of the excitation frequency. A comparison of these

results and those obtained from the classical solution of an infinite reservoir is presented in Table 2.2 and Fig. 2.7. It is apparent that the modified radiation condition gives much better results than the normal Sommerfeld condition for a range of excitation frequencies from 0 to ω_2 . For high excitation frequencies, the correlation with the classical solution for infinite reservoirs is not as good but may still be satisfactory for calculation of response to an earthquake motion.

2.5 Finite Element Analysis of the Reservoir Problem

2.5.1 Finite element discretization

The finite element formulation of a field problem can be obtained by two different methods. The first of these is a variational approach, and the second is based on a weighted residual or Galerkin process⁽¹⁾. The formulation based on the variational principle is used in this study and is described below.

For approximate solution of the reservoir problem, it will be assumed that the reservoir is subdivided into finite elements in each of which the pressure is given by:

$$p = [N_i \ N_j \ \dots] \begin{Bmatrix} p_i \\ p_j \\ \vdots \\ \vdots \end{Bmatrix} = [N]\{p^e\} \quad (2.67)$$

where $\{p^e\}$ is a vector representing the value of p at the element nodal points, and $[N]$ contains functions of the spatial coordinates

called the 'shape functions'. It should be noticed that the nodal values $\{p^e\}$ are functions of time. The pressure p is governed by the wave equation

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = 0 \quad (2.7)$$

subject to the following boundary conditions:

- i)* p is prescribed on the boundary S_1
- ii) $\frac{\partial p}{\partial n}$ is prescribed on the boundary S_2

where S_1 and S_2 together enclose the domain under consideration which is the reservoir in this case.

Equation (2.7) represents the Euler's condition for the minimization of a functional^(1,5). The variation of this functional can be written as

$$\delta I = \iint_A \delta p \left[\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} \right] dx dy = 0 \quad (2.68)$$

Using Green's theorem for a double integral⁽⁹⁾, the following relationships are obtained

$$\begin{aligned} \iint_A \delta p \frac{\partial^2 p}{\partial x^2} dx dy &= - \iint_A \frac{\partial p}{\partial x} \frac{\partial}{\partial x} (\delta p) dx dy + \int_S \frac{\partial p}{\partial x} \delta p dy \\ &= - \frac{1}{2} \delta \iint_A \left(\frac{\partial p}{\partial x} \right)^2 dx dy + \int_S \frac{\partial p}{\partial x} \delta p dy \end{aligned} \quad (2.69)$$

and

$$\iint_A \delta p \frac{\partial^2 p}{\partial y^2} dx dy = - \frac{1}{2} \delta \iint_A \left(\frac{\partial p}{\partial y} \right)^2 dx dy - \int_S \frac{\partial p}{\partial y} \delta p dx \quad (2.70)$$

in which A represents the domain and S its boundary, and the integrals around S are taken in an anti-clockwise direction. Referring to Fig. 2.8, if n and S represent respectively the outward normal and the tangent to the boundary, and α is the angle between the normal and the x-axis, the transformation of coordinates can be expressed as

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} \sin \alpha & -\cos \alpha \\ -\cos \alpha & -\sin \alpha \end{bmatrix} \begin{Bmatrix} S \\ n \end{Bmatrix}$$

Thus

$$\frac{\partial x}{\partial n} dS = -\cos \alpha dS = dy \quad (2.71a)$$

and

$$\frac{\partial y}{\partial n} dS = -\sin \alpha dS = -dx \quad (2.71b)$$

Also,

$$\frac{\partial p}{\partial n} = \frac{\partial p}{\partial x} \frac{\partial x}{\partial n} + \frac{\partial p}{\partial y} \frac{\partial y}{\partial n} \quad (2.72)$$

Integration of the above equation around the boundary S, and substitution of Eqns. (2.71a) and (2.71b) gives

$$\int_S \frac{\partial p}{\partial n} dS = \int_S \frac{\partial p}{\partial x} dy - \int_S \frac{\partial p}{\partial y} dx \quad (2.73)$$

On adding Eqns. (2.69) and (2.70) and using Eqn. (2.73), the following relationship is obtained

$$\begin{aligned} \iint_A \delta p \left[\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} \right] dx dy &= -\frac{1}{2} \delta \iint_A \left[\left(\frac{\partial p}{\partial x} \right)^2 + \left(\frac{\partial p}{\partial y} \right)^2 \right] dx dy \\ &+ \int_S \frac{\partial p}{\partial n} \delta p dS \end{aligned} \quad (2.74)$$

Fig. 2.8. Transformation of axes.

The third integral in Eqn. (2.68) can be written as

$$\iint_A \delta p \frac{\partial^2 p}{\partial t^2} dx dy = \iint_A \frac{\partial^2}{\partial t^2} (p \delta p) dx dy = \frac{1}{2} \delta \iint_A \frac{\partial^2}{\partial t^2} p^2 dx dy \quad (2.75)$$

Using the relations given in Eqns. (2.74) and (2.75), Eqn. (2.68) can be written as

$$\begin{aligned} \delta I = & -\frac{1}{2} \delta \iint_A \left[\left(\frac{\partial p}{\partial x} \right)^2 + \left(\frac{\partial p}{\partial y} \right)^2 \right] dx dy - \frac{1}{2c^2} \delta \frac{\partial^2}{\partial t^2} \iint_A p^2 dx dy \\ & + \delta \int_S \frac{\partial p}{\partial n} p dS = 0 \end{aligned} \quad (2.76)$$

If p is chosen such that the boundary condition on S_1 is identically satisfied, δp is zero on S_1 and the last integral in Eqn. (2.76) can be restricted to boundary S_2 , hence the functional I becomes

$$\begin{aligned} I = & \frac{1}{2} \iint_A \left[\left(\frac{\partial p}{\partial x} \right)^2 + \left(\frac{\partial p}{\partial y} \right)^2 \right] dx dy + \frac{1}{2c^2} \frac{\partial^2}{\partial t^2} \iint_A p^2 dx dy \\ & - \int_{S_2} p \frac{\partial p}{\partial n} dS \end{aligned} \quad (2.77)$$

The solution of the problem can thus be obtained by minimizing the functional given by Eqn. (2.77).

In Eqn. (2.67), if $[N]$ is chosen such that p is continuous between the elements, then all three integrals of Eqn. (2.77) can be evaluated separately for each element and finally summed. From Eqn. (2.67), the following expressions are obtained for the temporal and spatial derivatives of p :

$$\begin{aligned}
 \frac{\partial^2 p}{\partial t^2} &= [N]\{\ddot{p}^e\} \\
 \frac{\partial p}{\partial x} &= \frac{\partial}{\partial x} [N]\{p^e\} = [B_1]\{p^e\} \\
 \frac{\partial p}{\partial y} &= \frac{\partial}{\partial y} [N]\{p^e\} = [B_2]\{p^e\}
 \end{aligned}
 \quad \left. \vphantom{\begin{aligned} \frac{\partial^2 p}{\partial t^2} &= [N]\{\ddot{p}^e\} \\ \frac{\partial p}{\partial x} &= \frac{\partial}{\partial x} [N]\{p^e\} = [B_1]\{p^e\} \\ \frac{\partial p}{\partial y} &= \frac{\partial}{\partial y} [N]\{p^e\} = [B_2]\{p^e\} \end{aligned}} \right\} \quad (2.78)$$

Using Eqns. (2.67) and (2.78), Eqn. (2.77) can be written as

$$\begin{aligned}
 I^e &= \frac{1}{2} \iint_{A_e} \{p^e\}^T [B_1]^T [B_1] \{p^e\} dx dy \\
 &+ \frac{1}{2} \iint_{A_e} \{p^e\}^T [B_2]^T [B_2] \{p^e\} dx dy \\
 &+ \frac{1}{2c^2} \frac{\partial^2}{\partial t^2} \iint_{A_e} \{p^e\}^T [N]^T [N] \{p^e\} dx dy \\
 &- \int_{S_{2e}} \{p^e\}^T [N]^T \frac{\partial p}{\partial n} dS
 \end{aligned}
 \quad (2.79)$$

It can be proved that

$$\begin{aligned}
 \frac{\partial}{\partial \{p^e\}} \{p^e\}^T [N]^T &= [N]^T \\
 \frac{\partial}{\partial \{p^e\}} \frac{1}{2} \{p^e\}^T [B]^T [B] \{p^e\} &= [B]^T [B] \{p^e\}
 \end{aligned}
 \quad \left. \vphantom{\begin{aligned} \frac{\partial}{\partial \{p^e\}} \{p^e\}^T [N]^T &= [N]^T \\ \frac{\partial}{\partial \{p^e\}} \frac{1}{2} \{p^e\}^T [B]^T [B] \{p^e\} &= [B]^T [B] \{p^e\} \end{aligned}} \right\} \quad (2.80)$$

Minimizing the functional I^e with respect to the unknowns $\{p^e\}$ and using the relations given in Eqn. (2.80), one obtains

$$\frac{\partial I^e}{\partial \{p^e\}} = \iint_{A_e} ([B_1]^T [B_1] + [B_2]^T [B_2]) dx dy \{p^e\}$$

$$+ \frac{1}{c^2} \iint_{A_e} [N]^T [N] dx dy \{\ddot{p}^e\} - \int_{S_{2e}} [N]^T \frac{\partial p}{\partial n} dS = 0 \quad (2.81)$$

or

$$[G^e]\{p^e\} + [H^e]\{\ddot{p}^e\} = \{F^e\} \quad (2.82)$$

in which

$$[G^e] = \iint_{A_e} ([B_1]^T [B_1] + [B_2]^T [B_2]) dx dy \quad (2.83)$$

$$[H^e] = \frac{1}{c^2} \iint_{A_e} [N]^T [N] dx dy \quad (2.84)$$

and

$$\{F^e\} = \int_{S_{2e}} [N]^T \frac{\partial p}{\partial n} dS \quad (2.85)$$

Assembling Eqns. (2.82) for all elements of the region gives

$$[G]\{p\} + [H]\{\ddot{p}\} = \{F\} \quad (2.86)$$

The boundary conditions for the normal pressure gradient are incorporated through Eqn. (2.85). Mathematical representation of the various boundary conditions to be encountered in practical problems is discussed in the next section.

2.5.2 Boundary conditions

- (a) Free surface: If the surface waves are neglected, the hydrodynamic pressure at the surface must be zero, i.e.

$$p = 0$$

- (b) Specified acceleration: When the acceleration of the water particles is prescribed, the related condition can be expressed as

$$\frac{\partial p}{\partial n} = -\rho \ddot{\delta}_n \quad (2.87)$$

where $\ddot{\delta}_n$ is the outward normal acceleration of the particles on the boundary.

- (c) Truncation boundary: In the finite element modelling, the infinite fluid region must be truncated at some sufficiently large distance from the region of interest. Along the truncation line, suitable boundary conditions must be imposed. There are two practical cases for such boundaries depending upon the direction of advancing wave which may be perpendicular or parallel to the truncation boundary. For the case of wave parallel to the truncation line, the boundary condition is simply

$$\frac{\partial p}{\partial n} = 0$$

For the case of a wave perpendicular to the truncation line, the radiation damping condition is imposed. If the radiation condition proposed by Zienkiewicz and Newton⁽²⁾, given by Eqn. (2.51), is applied

$$\begin{aligned} \{F^e\} &= -\frac{1}{c} \int_{S_{2e}} [N]^T \frac{\partial p}{\partial t} dS \\ &= -\frac{1}{c} \int_{S_{2e}} [N]^T [N] \{\dot{p}^e\} dS \end{aligned}$$

or

$$\{F^e\} = -[T^e]\{\dot{p}^e\} \quad (2.88)$$

where

$$[T^e] = \frac{1}{c} \int_{S_{2e}} [N]^T [N] dS \quad (2.89)$$

for those boundary elements lying on the truncation line, and

$$[T^e] = [0]$$

for the other elements.

Assembling all elements, Eqn. (2.86) becomes

$$[G]\{p\} + [T]\{\dot{p}\} + [H]\{\ddot{p}\} = \{F\} \quad (2.90)$$

If the modified radiation damping condition, given by Eqn. (2.65) is applied, the element damping matrix $[T^e]$ is given by

$$[T^e] = [0] \quad : \text{for } \omega \leq \omega_1 \quad (2.91a)$$

$$[T^e] = \frac{\sqrt{1 - \left(\frac{\omega_1}{\omega}\right)^2}}{c} \int_{S_{2e}} [N]^T [N] dS : \text{for } \omega > \omega_1 \quad (2.91b)$$

2.5.3 Two-dimensional triangular elements

The three node triangular elements are used in this study to discretize the continuous fluid region. The characteristics of such elements are well described in the literature^(1,23) and are only briefly reviewed here.

The triangular element, Fig. 2.9, has three nodes i, J and m with coordinates (x_i, y_i) , (x_j, y_j) and (x_m, y_m) referred to the

Fig. 2.9. A linear triangular element.

centroidal axes x and y . Now, if

$$\begin{aligned} a_i &= x_j y_m - x_m y_j ; b_i = y_j - y_m ; c_i = x_m - x_j \\ a_j &= x_m y_i - x_i y_m ; b_j = y_m - y_i ; c_j = x_i - x_m \\ a_m &= x_i y_j - x_j y_i ; b_m = y_i - y_j ; c_m = x_j - x_i \end{aligned} \quad (2.92)$$

then the area of the triangular element is given by

$$\Delta = \frac{1}{2} \text{ABS} \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{vmatrix} = \frac{1}{2} \text{ABS}(a_i + a_j + a_m) \quad (2.93)$$

If the pressure is assumed to vary linearly within the element, it can be expressed as

$$p = [1 \quad x \quad y] \begin{Bmatrix} \alpha_i \\ \alpha_j \\ \alpha_m \end{Bmatrix} \quad (2.94)$$

Substitution of the nodal pressures and coordinates into Eqn. (2.94)

leads to the following matrix equation

$$\begin{Bmatrix} p_i \\ p_j \\ p_m \end{Bmatrix} = \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{bmatrix} \begin{Bmatrix} \alpha_i \\ \alpha_j \\ \alpha_m \end{Bmatrix} \quad (2.95)$$

Eqns. (2.95) can be solved for $(\alpha_i, \alpha_j, \alpha_m)$ to give

$$\begin{Bmatrix} \alpha_i \\ \alpha_j \\ \alpha_m \end{Bmatrix} = \frac{1}{2\Delta} \begin{bmatrix} a_i & a_j & a_m \\ b_i & b_j & b_m \\ c_i & c_j & c_m \end{bmatrix} \begin{Bmatrix} p_i \\ p_j \\ p_m \end{Bmatrix} \quad (2.96)$$

Substitution of Eqn. (2.96) into Eqn. (2.94) gives

$$p = \frac{1}{2\Delta} \begin{bmatrix} a_i + b_i x + c_i y & a_j + b_j x + c_j y & a_m + b_m x + c_m y \end{bmatrix} \begin{Bmatrix} p_i \\ p_j \\ p_m \end{Bmatrix}$$

or

$$p = [N_i \ N_j \ N_m] \begin{Bmatrix} p_i \\ p_j \\ p_m \end{Bmatrix} = [N] \{p^e\} \quad (2.97)$$

where N_i , N_j and N_m are the shape functions defined in Eqn. (2.67).

Matrices $[B_1]$ and $[B_2]$, defined in Eqns. (2.78), are now given by

$$[B_1] = \frac{\partial}{\partial x} [N] = \frac{1}{2\Delta} [b_i \ b_j \ b_m] \quad (2.98)$$

$$[B_2] = \frac{\partial}{\partial y} [N] = \frac{1}{2\Delta} [c_i \ c_j \ c_m] \quad (2.99)$$

and the element 'stiffness' matrix $[G^e]$, defined in Eqn. (2.83), takes the form

$$[G^e] = \frac{1}{4\Delta} \begin{bmatrix} b_i b_i + c_i c_i & b_i b_j + c_i c_j & b_i b_m + c_i c_m \\ & b_j b_j + c_j c_j & b_j b_m + c_j c_m \\ \text{sym} & & b_m b_m + c_m c_m \end{bmatrix} \quad (2.100)$$

It can be proved that

$$\iint_A N_i N_j \, dx \, dy = \begin{cases} \frac{\Delta}{6}, & \text{if } i=j \\ \frac{\Delta}{12}, & \text{if } i \neq j \end{cases}$$

The element 'mass' matrix $[H^e]$, defined in Eqn. (2.84), therefore takes the form

$$[H^e] = \frac{\Delta}{12c} \begin{bmatrix} 2 & 1 & 1 \\ & 2 & 1 \\ \text{sym} & & 2 \end{bmatrix} \quad (2.101)$$

The frequency independent element 'damping' matrix $[T^e]$, defined in Eqn. (2.89) is given by

$$[T^e] = \frac{L}{6c} \begin{bmatrix} 2 & 1 & 0 \\ & 2 & 0 \\ \text{sym} & & 0 \end{bmatrix} \quad (2.102)$$

and the frequency dependent matrix, defined in Eqn. (2.91), takes the form

$$[T^e] = [0] \quad ; \text{ for } \omega < \omega_1 \quad (2.103a)$$

$$[T^e] = \frac{L \sqrt{1 - \left(\frac{\omega_1}{\omega}\right)^2}}{6c} \begin{bmatrix} 2 & 1 & 0 \\ & 2 & 0 \\ \text{sym} & & 0 \end{bmatrix} ; \text{ for } \omega > \omega_1 \quad (2.103b)$$

where L is the length of side $i-j$ lying on the truncation boundary.

The 'consistent load' vector $\{F^e\}$, given by Eqn. (2.85), can be derived as follows. The integration in Eqn. (2.85) is to be carried out only on those boundaries (S_2) where the pressure gradient $\partial p / \partial n$ is prescribed. When this pressure gradient is related to the normal acceleration by Eqn. (2.87), the 'load' vector takes the form

$$\{F^e\} = - \rho \int_{S_{2e}} [N]^T \ddot{\delta}_n \, dS \quad (2.104)$$

When the pressure is prescribed at a node, the gradient cannot be prescribed and one equation corresponding to that node must be deleted.

If the acceleration on the boundary is defined in terms of its nodal values as follows

$$\delta = [N^S] \{\delta^e\} \quad (2.105)$$

where

$$\{\delta^e\} = [\delta_{ix} \ \delta_{iy} \ \delta_{jx} \ \delta_{jy} \ \delta_{mx} \ \delta_{my}]^T \quad (2.106)$$

and if $[R]$ is the matrix which transforms the displacements along the coordinates (x,y) to the outward normal displacement, δn , the load vector $\{F^e\}$ becomes

$$\{F^e\} = - \rho \int_{S_{2e}} [N]^T [R] [N^S] ds \{\ddot{\delta}^e\} \quad (2.107)$$

If i and j are the two nodal points lying on the boundary, j being on an anti-clockwise direction with respect to i , and if the shape functions N^S are linear, the vector $\{F^e\}$ can be obtained as follows.

Referring to Fig. 2.8, $[R]$ can be defined by

$$[R] = [-\cos\alpha \ -\sin\alpha] = \frac{1}{L} [y_j - y_i \ x_i - x_j] \quad (2.108)$$

Along the side $i-j$, $[N]$ and $[N^S]$ are defined by

$$[N] = [1 - \frac{S}{L} \quad \frac{S}{L} \quad 0] \quad (2.109a)$$

$$[N^S] = \begin{bmatrix} 1 - \frac{S}{L} & 0 & \frac{S}{L} & 0 & 0 & 0 \\ 0 & 1 - \frac{S}{L} & 0 & \frac{S}{L} & 0 & 0 \end{bmatrix} \quad (2.109b)$$

where L is the length of side $i-j$.

Substitution of Eqns. (2.108) and (2.109) into Eqn. (2.107) gives

$$\{F^e\} = -\rho [Q^e] \{\ddot{\delta}^e\} \quad (2.110)$$

in which

$$[Q^e] = \begin{bmatrix} \frac{y_j - y_i}{3} & \frac{x_i - x_j}{3} & \frac{y_j - y_i}{6} & \frac{x_i - x_j}{6} & 0 & 0 \\ \frac{y_j - y_i}{6} & \frac{x_i - x_j}{6} & \frac{y_j - y_i}{3} & \frac{x_i - x_j}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (2.111)$$

Assembling $[Q^e]$ for all elements on boundary, Eqn. (2.90) becomes

$$[G]\{p\} + [T]\{\dot{p}\} + [H]\{\ddot{p}\} = -\rho [Q]\{\ddot{\delta}\} \quad (2.112)$$

2.5.4 Natural frequencies of the reservoir

It is apparent from Eqns. (2.91) and (2.103) that if a frequency dependent radiation condition is to be used, the fundamental frequency of the reservoir must be obtained before the response of an infinite reservoir to a harmonic excitation with frequency greater than the fundamental, can be evaluated. This means that an eigenvalue problem of the type

$$[[G] - \omega^2 [H]]\{p\} = \{0\}$$

must be solved. Matrices $[G]$ and $[H]$ are banded matrices with relatively small band widths. The technique proposed by Bathe and Wilson⁽¹⁵⁾ seems appropriate for the solution of the above eigenvalue problem. This technique depends on the evaluation of the characteristic polynomial using Gauss elimination followed by the calculation of eigenvalues by the secant iteration procedure. A brief description

of the method is presented below.

Let the eigenvalue problem be defined as,

$$[A - \mu B]\{v\} = \{0\} \quad (2.113)$$

in which A and B are banded and symmetrical. The characteristic polynomial of the problem is given by

$$P(\mu) = \det(A - \mu B) \quad (2.114)$$

The polynomial is a function of μ and graphs as shown in Fig. 2.10.

The polynomial can be evaluated for a particular value of μ using Gauss elimination. Thus, if the matrix $[A - \mu B]$ is factorized as

$$[A - \mu B] = [L][D][L]^T \quad (2.115)$$

where $[L]$ is a lower triangular matrix with all diagonal elements equal to unity and $[D]$ is a diagonal matrix, then the polynomial $P(\mu)$ is the product of the diagonal elements in $[D]$, i.e.

$$P(\mu) = \prod_{i=1}^n d_{ii} \quad (2.116)$$

where d_{ii} is the diagonal element i of matrix $[D]$, and the sign $\prod_{i=1}^n$ represents a repeated product over $i=1$ to $i=n$, where n is the number of degrees of freedom of the system.

An iterative procedure is used to find the zeros of the polynomial which correspond to the eigenvalues of the system. Let μ_{k-2} and μ_{k-1} be two approximations to the smallest eigenvalue λ_1 where $\mu_{k-2} < \mu_{k-1} < \lambda_1$ (Fig. 2.10). Then, using the accelerated secant

Fig. 2.10. A typical characteristic polynomial.

Fig. 2.11. Iteration for second eigenvalue.

iteration, the next approximation to λ is obtained as follows

$$\mu_k = \mu_{k-1} + \eta \frac{P(\mu_{k-1})}{P(\mu_{k-2}) - P(\mu_{k-1})} (\mu_{k-1} - \mu_{k-2}) \quad (2.117)$$

where η is a constant. When $\eta=1$, the method reduces to the regular secant iteration, in which $\mu_k \rightarrow \lambda_1$ as $k \rightarrow \infty$. Once a suitable convergence criterion has been defined, an acceptable value for λ_1 can be determined after a number of iterations.

The triangular factorization at μ_k yields the number of eigenvalues smaller than μ_k . This number is equal to the number of negative diagonal elements in matrix $[D]$.

To obtain any other eigenvalue, say λ_{j+1} , one can use instead of $P(\mu)$ in Eqn. (2.114), the deflated polynomial $P_j(\mu)$ given by

$$P_j(\mu) = \frac{P(\mu)}{\prod_{i=1}^j (\mu - \lambda_i)} \quad (2.118)$$

where $\prod_{i=1}^j$ represents a repeated product over $i=1$ to $i=j$. Fig. 2.11 shows the iteration for the second eigenvalue λ_2 using Eqn. (2.118).

It should be noticed that the matrix $[A - \mu B]$ contains certain degrees of freedom, along which the variable under consideration, such as a displacement or a pressure, has prescribed value. The corresponding equations should therefore be deleted before evaluating the polynomial $P(\mu)$. This is indirectly accomplished in this work by a method similar to that proposed by Payne and Irons⁽²⁰⁾ as follows.

If i is a degree of freedom along which the variable under consideration has a known value, the corresponding diagonal element a_{ii} in matrix $[A]$ is multiplied by a large number, say $C=10^{20}$. This process is similar to deleting the i^{th} equation when a Gauss elimination is used. The diagonal element d_{ii} in matrix $[D]$ will have the large fictitious value ($C \cdot a_{ii}$). Such an element is skipped during the multiplication of the diagonal elements of matrix $[D]$.

2.5.5 Steady state response of a reservoir subjected to harmonic ground motion

The equations of motion of the reservoir are given by Eqn.

(2.112)

$$[G]\{p\} + [T]\{\dot{p}\} + [H]\{\ddot{p}\} = -\rho [Q]\{\ddot{\delta}\} \quad (2.112)$$

If the vector of acceleration is represented by

$$\{\ddot{\delta}\} = \{\ddot{\delta}\} e^{i\omega t} \quad (2.119)$$

the steady state response can be expressed as

$$\left. \begin{aligned} \{p\} &= \{\bar{p}\} e^{i\omega t} \\ \{\dot{p}\} &= i\omega \{\bar{p}\} e^{i\omega t} \\ \{\ddot{p}\} &= -\omega^2 \{\bar{p}\} e^{i\omega t} \end{aligned} \right\} \quad (2.120)$$

and

Substitution of Eqns. (2.119) and (2.120) into Eqn. (2.112) gives

$$[[G] + i\omega[T] - \omega^2[H]] \{\bar{p}\} = -\rho [Q]\{\ddot{\delta}\}$$

or

$$[FK]\{\bar{p}\} = \{\bar{F}\} \quad (2.121)$$

By solving the system of equations given by Eqn. (2.121), the pressures $\{\bar{p}\}$ can be determined.

Given the nodal coordinates, element data, boundary data, water density, velocity of sound in water, and the excitation frequency, solution of the problem is obtained in the following steps.

1. Form element matrices

$$[G^e] \quad , \text{ Eqn. (2.100)}$$

$$[H^e] \quad , \text{ Eqn. (2.101)}$$

and save. At the same time, assemble the global matrices $[G]$ and $[H]$.

2. Solve the eigenvalue problem

$$[[G]-\omega^2[H]]\{p\} = \{0\}$$

by the method described in Section 2.5.4 to obtain the fundamental frequency ω_1 ; having determined ω_1 obtain the element damping matrices $[T^e]$ by using Eqn. (2.103).

3. For the assembly of the global matrix $[FK]$ given in Eqn. (2.121), consider the vector $\{\bar{p}\}$ to be of the type

$$\{\bar{p}\} = [\bar{p}_{1R} \bar{p}_{1I} \bar{p}_{2R} \bar{p}_{2I} \bar{p}_{iR} \bar{p}_{iI} \bar{p}_{NR} \bar{p}_{NI}]^T \quad (2.122)$$

in which subscript i represents the node number, and R and I indicate the real and imaginary parts of $\bar{p}$ respectively. Matrix $[FK]$ is the assembly of the element matrices $[FK^e]$ which are derived as follows.

At the element level

$$[FK^e]\{\bar{p}^e\} = \{\bar{F}^e\}$$

or

$$[[G^e] + i\omega[T^e] - \omega^2[H^e]]\{\bar{p}_R^e + i\bar{p}_I^e\} = \{\bar{F}^e\}$$

or

$$\begin{bmatrix} [G^e] - \omega^2[H^e] & -\omega[T^e] \\ -\omega[T^e] & -[G^e] + \omega^2[H^e] \end{bmatrix} \begin{Bmatrix} \bar{p}_{iR} \\ \bar{p}_{jR} \\ \bar{p}_{mR} \\ \bar{p}_{iI} \\ \bar{p}_{jI} \\ \bar{p}_{mI} \end{Bmatrix} = \begin{Bmatrix} \bar{F}_i \\ \bar{F}_j \\ \bar{F}_m \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (2.123)$$

Now, form $[FK]$ by assembling the element matrices $[FK^e]$. Since matrices $[FK^e]$ are symmetric, matrix $[FK]$ is a banded and symmetric.

4. Assemble vector $\{\bar{F}\}$ from the given boundary conditions.
5. Modify matrix $[FK]$ and vector $\{\bar{F}\}$ to take into account the known zero pressures by the method of Payne and Irons⁽²⁰⁾. Then, solve the system of Eqns. (2.121) by Gauss elimination to determine the hydrodynamic pressure vector $\{\bar{p}\}$.

2.6 Results of Analysis

The finite element approach described in the previous section was applied to the solution of a series of reservoir vibration problems. In this section, the results obtained are compared with classical solutions wherever such solutions are available. The

following data was used in the solutions presented here:

Height of dam = $H = 400$ feet

Water density = $\rho = 1.9379$ lb mass/ft³

Velocity of sound in water $\approx c = 4720$ ft/sec

2.6.1 Hydrodynamic pressures for different excitation frequencies

In order to check the convergence of the finite element solution, three different mesh divisions, as shown in Fig. 2.12, were used. The length of the reservoir was taken as twice its height. Solutions were obtained for different values of the excitation frequency. In each case, the modified radiation damping condition given by Eqn. (2.65) was used.

A comparison of the results with those obtained from the classical solutions of the infinite reservoir is presented in Tables 2.3 and in Figs. 2.13. In each case, the finite element solution converges to the exact solution as the mesh is made finer. Mesh 3, in general, gives results that are in very good agreement with the classical solutions. Even Mesh 2 gives fairly good results.

As expected for the case of the excitation frequency less than ω_1 , the pressures are entirely in phase with the excitation. For ω greater than ω_1 both in-phase and out-of-phase components exist. It should be noted that the phase angle varies along the height of the dam. This means that the maximum pressures along the height do not occur at the same instance of time.

(a) MESH (1)

(b) MESH (2)

(c) MESH (3)

Fig. 2.12. Finite element models of reservoir for vertical dam face.

Table 2.3a. Convergence of the finite element solution.
 $(\omega < \omega_1)$

$\frac{y}{H}$	HYDRODYNAMIC PRESSURE ($\bar{p}/\rho H$)							
	CLASSICAL (INFINITE RESERVOIR)		MESH (1)		MESH (2)		MESH (3)	
	IN- PHASE	OUT-PHASE	IN- PHASE	OUT- PHASE	IN- PHASE	OUT- PHASE	IN- PHASE	OUT- PHASE
1.000	0	0	0	0	0	0	0	0
.875	.2950	0			.2529	0	.2689	0
.750	.4773	0	.3874	0	.4435	0	.4609	0
.625	.6026	0			.5897	0	.6050	0
.500	.7187	0	.6404	0	.7013	0	.7142	0
.375	.7966	0			.7842	0	.7952	0
.250	.8508	0	.7856	0	.8418	0	.8514	0
.125	.8827	0			.8761	0	.8850	0
0	.8932	0	.8356	0	.8883	0	.8970	0

$$\omega_1 = 18.535 \text{ rad/sec}$$

$$\omega/\omega_1 = 0.5395$$

$$L = 2H$$

Table 2.3b. Convergence of the finite element solution.
 $(\omega > \omega_1)$

$\frac{y}{H}$	HYDRODYNAMIC PRESSURE ($\bar{p}/\rho H$)							
	CLASSICAL (INFINITE RESERVOIR)		MESH (1)		MESH (2)		MESH (3)	
	IN- PHASE	OUT-PHASE	IN- PHASE	OUT- PHASE	IN- PHASE	OUT- PHASE	IN- PHASE	OUT- PHASE
1.000	0	0	0	0	0	0	0	0
.875	.1322	.0827			.0920	.0815	.1064	.0813
.750	.1472	.1622	.0731	.1809	.1184	.1605	.1329	.1600
.625	.1165	.2354			.1006	.2334	.1112	.2327
.500	.0623	.2997	.0161	.3303	.0562	.2972	.0627	.2965
.375	.0007	.3524			.0010	.3450	.0040	.3491
.250	-.0541	.3917	-.0851	.4295	-.0515	.3904	-.0498	.3892
.125	-.0915	.4158			-.0906	.4180	-.0893	.4164
0	-.1049	.4239	-.1534	.4764	-.1094	.4332	-.1071	.4317

$$\omega_1 = 18.535 \text{ rad/sec}$$

$$\omega/\omega_1 = 2.1581$$

$$L = 2H$$

Table 2.3c. Convergence of the finite element solution.
($\omega > \omega_1$)

$\frac{y}{H}$	HYDRODYNAMIC PRESSURE ($\bar{p}/\rho H$)							
	CLASSICAL (INFINITE RESERVOIR)		MESH (1)		MESH (2)		MESH (3)	
	IN- PHASE	OUT-PHASE	IN- PHASE	OUT- PHASE	IN- PHASE	OUT- PHASE	IN- PHASE	OUT- PHASE
1.000	0	0	0	0	0	0	0	0
.875	.0888	.0860			.0404	.1084	.0546	.0345
.750	.0539	.1543	-.0334	.0617	.0141	.1938	.0289	.1516
.625	-.0088	.1926			-.0372	.2423	-.0226	.1882
.500	-.0511	.1982	-.1602	.0473	-.0753	.2538	-.0562	.1930
.375	-.0499	.1775			-.0802	.2389	-.0516	.1744
.250	-.0126	.1450	-.2358	-.0134	-.0564	.2134	-.0154	.1464
.125	.0312	.1170			-.0263	.1922	.0255	.1236
0	.0503	.1060	-.2519	-.0471	-.0164	.1863	.0413	.1175

$$\omega_1 = 18.535 \text{ rad/sec}$$

$$\omega/\omega_1 = 4.3162$$

$$L = 2H$$

Fig. 2.13a. Finite element solution convergence.

Fig. 2.13b. Finite element solution convergence.

Fig. 2.13c. Finite element solution convergence.

2.6.2 Natural frequencies of a finite reservoir

The natural frequencies of a finite reservoir can be calculated using the finite element method as described in Section 2.5.4. The method was used to calculate the first ten natural frequencies of a finite reservoir of a length equal to twice its height. Finite element models, shown in Figs. 2.12b and 2.12c, were used. The results are compared in Table 2.4 with the exact values obtained from Eqn. (2.42). The agreement between the two solutions is satisfactory for both mesh divisions.

It should be noted that the natural frequencies calculated by the finite element model are those of the finite reservoir and not those of the infinite one. For example, it can be noticed from Table 2.4 that the fourth and ninth natural frequencies of the finite reservoir are the second and the third natural frequencies of the infinite reservoir, respectively. The finite reservoir has, in this case, two additional natural frequencies between the first and second of the infinite reservoir, and four additional frequencies between the second and the third.

2.6.3 Response to vertical excitation

For a harmonic vertical motion of the ground, the hydrodynamic pressures are calculated by the finite element method for different values of excitation frequency. The mesh division shown in Fig. 2.12b was used in the calculations. The results are compared in Table 2.5 and Fig. 2.14 with the classical solution⁽⁷⁾ which can be expressed as

Table 2.4. Natural frequencies of a finite reservoir.

Freq. No.	NATURAL FREQUENCY (rad/sec)				
	Classical*			Finite Element	
	n	m	Value	Mesh (2)	Mesh (3)
1	1	0	18.535	18.564	18.565
2	1	1	26.213	26.724	26.474
3	1	2	41.446	43.852	42.569
4	2	0	55.606	56.380	56.395
5	2	1	58.614	60.757	60.077
6	2	2	66.830	65.339	61.691
7	1	4	76.423	73.291	70.351
8	2	3	78.639	89.596	82.724
9	3	0	92.677	93.095	85.974
10	1	5	94.512	96.464	96.351

* Eqn. (2.42)

Table 2.5. Hydrodynamic pressure due to vertical harmonic excitations.

$\frac{y}{H}$	HYDRODYNAMIC PRESSURE ($\bar{p}/\rho H$)							
	$\beta=0$		$\beta=0.5395$		$\beta=1.6186$		$\beta=4.3162$	
	Classical	F.E.	Classical	F.E.	Classical	F.E.	Classical	F.E.
1.00	0	0	0	0	0	0	0	0
.875	.125	.125	.1885	.1881	-.1489	-.1422	.1365	.109
.750	.250	.250	.3749	.3743	-.2828	-.2717	.1664	.158
.625	.375	.375	.5571	.5567	-.3884	-.3750	.0946	.110
.500	.500	.500	.7330	.7328	-.4551	-.4413	-.0412	-.006
.375	.625	.625	.9007	.9010	-.4762	-.4636	-.1491	-.120
.250	.750	.750	1.0580	1.0590	-.4497	-.4394	-.1562	-.157
.125	.875	.875	1.2040	1.2060	-.3780	-.3711	-.0577	-.093
0	1.000	1.000	1.3364	1.3406	-.2686	-.2655	.0799	.032

$$\omega_1 = 18.535 \text{ rad/sec}$$

$$\beta = \omega / \omega_1$$

Fig. 2.14. Hydrodynamic pressures due to vertical excitation.

$$\bar{p}(x, y, \omega) = \frac{\rho c}{\omega} \frac{\sin \frac{\omega}{c} (H-y)}{\cos \frac{\omega H}{c}} \quad (2.124)$$

The agreement between the two solutions is satisfactory.

2.6.4 Effect of the inclination of the upstream face of the dam

The finite element method is used to compute the hydrodynamic pressures on a rigid dam of height 400 ft. with its upstream face inclined at 15° to the vertical. The finite element model used in the computations is shown in Fig. 2.15. The results obtained for different values of the horizontal excitation are shown in Table 2.6 and Fig. 2.16. For $\omega=0$, which represents incompressible water, the exact solution given by Chwang⁽¹³⁾ is also presented. The correlation of the results is very good. No classical solution exists for an inclined dam face and compressible water.

The effect of varying inclination is studied by repeating the response calculations for different values of θ using a frequency ratio $(\omega/\pi c/2H)$ equal to 0.75. The results are presented in Fig. 2.17 and Table 2.7. It is seen that as the angle of inclination increases the hydrodynamic pressures decrease significantly.

The response of a reservoir with an inclined dam face to a vertical ground motion is also studied. The results are presented in Table 2.8. It is seen that as θ increases the hydrodynamic pressure decreases slightly. Comparing the hydrodynamic pressures in Tables 2.7 and 2.8, it can be noticed that the pressures due to vertical excitation are

Fig. 2.15. Finite element model of a reservoir for inclined dam face.

Table 2.6. Hydrodynamic pressures for an inclined upstream dam face⁽¹⁾ due to horizontal harmonic ground motion.

$\frac{y}{H}$	EXCITATION FREQUENCY $(\omega/\omega_1)^{(2)}$				
	0	.3166	.4750	.6333	.7500
	HYDRODYNAMIC PRESSURE $(\bar{p}/\rho H)$				
1.00	0	0	0	0	0
.875	.1875	.1933	.2020	.2181	.2367
.750	.3274	.3395	.3576	.3914	.4305
.625	.4319	.4501	.4773	.5287	.5884
.500	.5083	.5320	.5676	.6351	.7140
.375	.5609	.5891	.6318	.7130	.8084
.250	.5919	.6236	.6716	.7634	.8716
.125	.6018	.6356	.6870	.7856	.9023
0	.5893	.6239	.6765	.7777	.8970

(1) $\theta=15^\circ$

(2) $\omega_1=18.949$ rad/sec (F.E.)

Fig. 2.16. Hydrodynamic pressure for an inclined dam face.

Fig. 2.17. Effect of inclination of dam face on hydrodynamic pressures. (horizontal harmonic excitation)

Table 2.7. Effect of inclination of dam face on hydrodynamic pressures. (horizontal harmonic motion)*

$\frac{y}{H}$	DAM FACE INCLINATION θ					
	0	15	30	45	60	75
	HYDRODYNAMIC PRESSURE ($\bar{p}/\rho H$)					
1.00	0	0	0	0	0	0
.875	.3895	.2367	.1784	.1253	.0755	.0336
.750	.5546	.4305	.3319	.2422	.1543	.0703
.625	.7507	.5884	.4613	.3462	.2322	.1119
.500	.9059	.7140	.5660	.4345	.3034	.1572
.375	1.025	.8084	.6442	.5014	.3614	.2019
.250	1.109	.8716	.6937	.5423	.3983	.2375
.125	1.160	.9023	.7106	.5507	.4039	.2497
0	1.180	.8979	.6885	.5151	.3586	.2000

* $\frac{\omega}{\pi c/2H} = 0.75$

Table 2.8. Effect of inclination of dam face on hydrodynamic pressures. (vertical harmonic ground motion)*

$\frac{y}{H}$	DAM FACE INCLINATION θ					
	0	15	30	45	60	75
	HYDRODYNAMIC PRESSURE ($\bar{p}/\rho H$)					
1.00	0	0	0	0	0	0
.875	.3236	.2498	.1960	.1553	.1296	.1243
.750	.6411	.5160	.4233	.3485	.2903	.2590
.625	.9453	.7875	.6698	.5719	.4861	.4178
.500	1.2300	1.0540	.9243	.8163	.7156	.6148
.375	1.4880	1.3050	1.1760	1.0710	.9720	.8601
.250	1.7160	1.5340	1.4130	1.3220	1.2430	1.1550
.125	1.9080	1.7520	1.6240	1.5530	1.5090	1.4820
0	2.0600	1.8890	1.7930	1.7450	1.7360	1.7820

$$\frac{\omega}{\pi c/2H} = 0.75$$

2

much greater than those due to horizontal excitation, for the same frequency and inclination. As pointed out in Ref. (40), these pressures calculated on the assumption of a rigid reservoir bed may not be realistic, and the deformability of the bed should therefore be taken into account. The question is not pursued further here.

The fundamental frequencies for different inclinations of dam face, as calculated by the finite element method, are presented in Table 2.9. It is seen that as the angle of inclination increases the fundamental frequency increases slightly.

Table 2.9. Effect of inclination of upstream face on fundamental frequency* of the reservoir.

θ	$\frac{\omega_1}{(\omega_1)_{\theta=0}} = \omega_1 \frac{2H}{\pi c}$
0	1.0015
15	1.0223
30	1.0346
45	1.0415
60	1.0445
75	1.0441

* Calculated by the finite element method.

CHAPTER 3

DAM-RESERVOIR PROBLEM, TIME DOMAIN ANALYSIS

3.1 Introduction

The main objective of this chapter is to present a procedure for obtaining the response of a flexible dam-reservoir system to an arbitrary horizontal ground excitation using a time domain analysis. The method used generally follows that described by Sharan^(3,4). The analysis is presented so that the results can be compared with those obtained from the frequency domain analysis developed by the author and presented in Chapter 4. A brief review of the finite element formulation of a flexible gravity dam using linear displacement, triangular finite elements, and the solution of the coupled system of equations of motion of the dam-reservoir system is presented, followed by the technique of solution in the time domain mentioned above.

As mentioned earlier, the time domain analysis has certain drawbacks. These can be summarized as follows:

- 1) The technique uses the Sommerfeld radiation damping condition to model the infinite reservoirs. It has been shown in Chapter 2 that this condition fails to model the energy loss correctly.
- 2) The results of the present study show that the procedure is computationally inefficient for practical problems.

3.2 Finite Element Formulation of Gravity Dams

A gravity dam can be idealized as a plane stress or plane strain problem depending upon whether the dam consists of an assemblage of monoliths or is constructed as a single unit.

The formulation of the plane strain or plane stress problem using the finite element method is well developed and has been used quite often. Details of the formulation may be found in many references (1, 3, 23). The method depends on dividing the continuum into small elements. The displacements $\{u\}$ within an element are defined in terms of the nodal displacements of the element $\{\delta^e\}$ and the shape functions $[N^S]$ as follows

$$\{u\} = [N^S] \{\delta^e\} \quad (3.1)$$

In order for the finite element solution to converge to the exact solution, the shape functions must satisfy certain criteria⁽¹⁾. Once appropriate shape functions have been chosen, the various components of strains $\{\epsilon\}$ in the element are obtained by taking proper derivatives of the shape functions. The resulting relationship may be expressed as:

$$\{\epsilon\} = [B] \{\delta^e\} \quad (3.2)$$

If a general elastic behaviour is assumed, the relationship between stresses $\{\sigma\}$ and strains $\{\epsilon\}$ is given by

$$\{\sigma\} = [D] \{\epsilon\} \quad (3.3)$$

where $[D]$ is an elasticity matrix which depends upon the properties of the material.

Minimization of the potential energy with respect to the nodal displacements $\{\delta\}$, then leads to the following set of simultaneous equations

$$[K]\{\delta\} = \{F\} \quad (3.4)$$

where the components of matrix $[K]$ and vector $\{F\}$ are obtained as

$$[K] = \sum_e [K^e] \quad (3.5)$$

and

$$\{F\} = \sum_e \{F^e\} \quad (3.6)$$

where $[K^e]$ and $\{F^e\}$ are element matrices, and the summation extends over all elements. The element matrices are given by the following expression:

$$[K^e] = \int_{V_e} [B]^T [D] [B] dv_e \quad (3.7)$$

and

$$\{F^e\} = \int_{V_e} [N^s]^T \{q_b\} dv_e + \int_{S_e} [N^s]^T \{q_s\} dS_e + \{R_c^e\} \quad (3.8)$$

where

$\{q_b\}$ is the vector of body force components

$\{q_s\}$ is the vector of applied surface loads, and

$\{R_c^e\}$ is the list of nodal concentrated forces.

Matrix $[K]$ in Eqn. (3.4) is the familiar stiffness matrix and vector $\{F\}$ is the consistent load vector.

When the displacements of an elastic solid body do not vary with time (static case), the force-displacement relationship is given by Eqn. (3.4). However, if the displacements vary with time, there are two types of additional forces which act on a solid body. One is the inertia force and the other is the force due to frictional resistance called damping force. Thus, the total distributed body force is given by

$$\{q_b\} = \{q_g\} + \{q_m\} + \{q_d\} \quad (3.9)$$

where subscripts g, m and d stand for gravity, inertia and damping forces, respectively. Since the response to only the dynamic forces is considered, the gravity load vector $\{q_g\}$ in Eqn. (3.9) and vector $\{R_c^e\}$ in Eqn. (3.8) are not included in the analysis.

The inertia force may be replaced by the following expression using d'Alembert principle:

$$\{q_m\} = - \rho_s \frac{\partial^2}{\partial t^2} \{u\} \quad (3.10)$$

in which ρ_s is the mass density of the material.

If the damping force is considered to be linear and viscous, then

$$\{q_d\} = - \mu_s \frac{\partial}{\partial t} \{u\} \quad (3.11)$$

where μ_s is the damping constant of the system. Thus, the element nodal forces arising from body forces are given by

$$\begin{aligned}
\{F_b^e\} &= \int_{V_e} [N^S]^T \{q_b\} dV_e \\
&= \int_{V_e} [N^S]^T \{q_m\} dV_e + \int_{V_e} [N^S]^T \{q_d\} dV_e
\end{aligned} \tag{3.12}$$

Using Eqns. (3.1), (3.10) and (3.11), Eqn. (3.12) can be written as

$$\{F_b^e\} = - \int_{V_e} [N^S]^T \rho_s [N^S] dV_e \frac{\partial^2}{\partial t^2} \{\delta^e\} - \int_{V_e} [N^S]^T \mu_s [N^S] dV_e \frac{\partial}{\partial t} \{\delta^e\} \tag{3.13}$$

Substitution of Eqn. (3.13) into Eqn. (3.4) gives the following general equation of motion:

$$[K]\{\delta\} + [C]\{\dot{\delta}\} + [M]\{\ddot{\delta}\} = \{F\} \tag{3.14}$$

where

$$\{\dot{\delta}\} = \frac{\partial}{\partial t} \{\delta\} \tag{3.15a}$$

$$\{\ddot{\delta}\} = \frac{\partial^2}{\partial t^2} \{\delta\} \tag{3.15b}$$

Matrices $[M]$ and $[C]$ are the consistent mass and consistent damping matrices, respectively. The elements of these matrices may be obtained by summing the contributions for each element. The contribution of a particular element is given by

$$[M^e] = \int_{V_e} [N^S]^T \rho_s [N^S] dV_e \tag{3.16}$$

$$[C^e] = \int_{V_e} [N^S]^T \mu_s [N^S] dV_e \tag{3.17}$$

3.3 Triangular Finite Element Characteristics

3.3.1 Shape functions

The triangular elements, Fig. 2.9, selected for this study, have two translation components of displacements at each node. If the variation of displacement is assumed to be linear, the two components of displacements u and v at any point may be expressed in terms of the six nodal displacements through the shape functions $[N^S]$. The shape functions, as defined in Eqn. (3.1), are

$$[N^S] = [IN_i \quad IN_j \quad IN_m] \quad (3.18)$$

where I is a 2×2 identity matrix and N_i , N_j and N_m are given in Eqn. (2.97).

3.3.2 Strains and stresses

For a problem in two dimensional space, the strain components at any point may be expressed as⁽¹⁷⁾:

$$\{\epsilon\} = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix} \quad (3.19)$$

Hence, matrix $[B]$ in Eqn. (3.2) is given by:

$$[B] = \frac{1}{2\Delta} \begin{bmatrix} b_i & 0 & b_j & 0 & b_m & 0 \\ 0 & c_i & 0 & c_j & 0 & c_m \\ c_i & b_i & c_j & b_j & c_m & b_m \end{bmatrix} \quad (3.20)$$

The relationship between stresses and strains given by Eqn. (3.3) takes the form

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [D] \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad (3.21)$$

where, for plane stress case,

$$[D] = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \quad (3.22)$$

and for the plane strain case,

$$[D] = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & 0 \\ \frac{\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix} \quad (3.23)$$

In the above equations,

E = modulus of elasticity

and ν = Poisson's ratio.

3.3.3 Stiffness matrix

Once matrices $[B]$ and $[D]$ have been established, the stiffness matrix $[K]$ may be obtained by using Eqns. (3.5) and (3.7). Since matrices $[B]$ and $[D]$ are independent of x and y , the element stiffness matrix, Eqn. (3.7), is given by

$$[K^e] = [B]^T [D] [B] t_o \Delta \quad (3.24)$$

in which t_o and Δ are the thickness and area of the element, respectively.

3.3.4 Mass matrix

The element mass matrix $[M^e]$, given by Eqn. (3.16), reduces to the following form:

$$[M^e] = \frac{\rho_s t_o \Delta}{12} \begin{bmatrix} 2I & I & I \\ & 2I & I \\ \text{sym} & & 2I \end{bmatrix} \quad (3.25)$$

in which I is a 2x2 identity matrix.

The mass matrix may be approximated by 'lumping' the total mass of an element at its nodes. In that case, the mass matrix becomes diagonal resulting in a significant reduction in the computational efforts. The lumped mass matrix is given by:

$$[M^e] = \frac{\rho_s t_o \Delta}{3} \begin{bmatrix} I & 0 & 0 \\ & I & 0 \\ \text{sym} & & I \end{bmatrix} \quad (3.26)$$

3.3.5 Damping matrix

The element damping matrix $[C^e]$ can be evaluated by using Eqn. (3.17). However, in practice, the parameter (μ_s) is difficult to evaluate properly. Hence, a method very commonly used for considering damping in structural dynamics is to use a Rayleigh damping matrix of the form

$$[C] = a[M] + b[K] \quad (3.27)$$

In Eqn. (3.27), a and b are factors which can be determined from values of damping ratios (λ_1 and λ_2) prescribed for any two modes of vibration, say the first and second modes. The set of two simultaneous equations used to obtain a and b is

$$\lambda_n = \frac{a}{2\omega_n} + \frac{b\omega_n}{2} \quad ; n=1,2 \quad (3.28)$$

The damping matrix given by Eqn. (3.27) is used in the present analysis. When a Rayleigh damping is used, the higher modes have considerably larger damping when compared to the lower modes. Hence, a very fine finite element mesh, which is generally necessary to include the effects of higher modes of vibration, is not needed. In the present analysis, the values of Rayleigh coefficients are calculated by using Eqn. (3.28). The first two natural frequencies of the dam required in Eqn. (3.28) are calculated by the algorithm presented in Section 2.5.4.

3.3.6 Consistent load vector

In the present analysis, the only external forces acting on the dam are the hydrodynamic forces. Therefore, the surface load vector in Eqn. (3.8) can be written as

$$\{F^e\} = \int_{S_e} [N^S]^T \{R'\} p \, dS_e \quad (3.29)$$

where $\{R'\}$ is a transformation matrix which transforms a vector normal to the dam surface directed inwards, to vectors parallel to the Cartesian coordinates x and y . Referring to Fig. 2.8, it can be

noticed that $\{R'\}$ is the transpose of matrix $[R]$ of Eqn. (2.108).

Substitution of Eqn. (2.97) into Eqn. (3.29) gives

$$\{F^e\} = \int_{S_e} [N^s]^T [R]^T [N] \{p^e\} dS_e$$

or

$$\{F^e\} = [S^e] \{p^e\} \quad (3.30)$$

in which

$$[S^e] = \int_{S_e} [N^s]^T [R]^T [N] dS_e \quad (3.31)$$

The resulting equations of motion of the dam can be expressed

as

$$[K]\{\delta\} + [C]\{\dot{\delta}\} + [M]\{\ddot{\delta}\} = [S]\{p\} \quad (3.32)$$

in which

$$[S] = \sum_e [S^e] \quad (3.33)$$

On comparing Eqns. (2.107) and (3.31), it is apparent that

$$[Q^e] = [S^e]^T \quad (3.34)$$

3.4 Equations of Motion for the Coupled Dam-Reservoir System

Equations of motion for the dam are given by Eqn. (3.32).

Use of Eqns. (2.112) and (3.34) leads to the following equations for the motion of the reservoir:

$$[G]\{p\} + [T]\{\dot{p}\} + [H]\{\ddot{p}\} = -\rho [S]^T \{\ddot{\delta}\} \quad (3.35)$$

Eqns. (3.32) and (3.35) are the coupled equations of motion for the dam-reservoir system. Combining these two equations, the coupled equations of motion of the complete system can be expressed as follows:

$$\begin{bmatrix} [K] & -[S] \\ [0] & [G] \end{bmatrix} \begin{Bmatrix} \delta \\ p \end{Bmatrix} + \begin{bmatrix} [C] & [0] \\ [0] & [T] \end{bmatrix} \begin{Bmatrix} \dot{\delta} \\ \dot{p} \end{Bmatrix} + \begin{bmatrix} [M] & [0] \\ \rho[S]^T & [H] \end{bmatrix} \begin{Bmatrix} \ddot{\delta} \\ \ddot{p} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (3.36)$$

In Eqn. (3.36), although matrices $[K]$, $[C]$, $[M]$, $[G]$, $[T]$ and $[H]$ are symmetric and banded, the corresponding 'stiffness', 'damping' and 'mass' matrices of the coupled system are unsymmetric, unbanded and of order $(m+n)$ where m and n are the number of degrees of freedom for the fluid and dam system, respectively. As a result, the coupled system becomes computationally difficult to handle and the solution of Eqn. (3.36) requires a special procedure. A procedure for the direct integration of the equations in the time domain using step-by-step numerical integration method is presented below.

3.5 Step-by-Step Integration Method

The technique uses the constant average acceleration method which is a special case of the well known Newmark method^(3,4,18).

It is assumed that $\{\ddot{p}, \ddot{\delta}\}$ remain constant within a short time step from t to $t+\Delta t$ and are equal to the average of $\{\ddot{p}, \ddot{\delta}\}_t$ and $\{\ddot{p}, \ddot{\delta}\}_{t+\Delta t}$. Pressures and displacements and their time derivatives at the instant $t+\Delta t$ are then expressed as follows:

$$\{\dot{p}, \dot{\delta}\}_{t+\Delta t} = \{\dot{p}, \dot{\delta}\}_t + \frac{\Delta t}{2} \{\ddot{p}, \ddot{\delta}\}_t + \frac{\Delta t}{2} \{\ddot{p}, \ddot{\delta}\}_{t+\Delta t} \quad (3.37)$$

$$\{p, \delta\}_{t+\Delta t} = \{p, \delta\}_t + \Delta t \{\dot{p}, \dot{\delta}\}_t + \frac{\Delta t^2}{4} \{\ddot{p}, \ddot{\delta}\}_t + \frac{\Delta t^2}{4} \{\ddot{p}, \ddot{\delta}\}_{t+\Delta t} \quad (3.38)$$

At the particular instant $t + \Delta t$, Eqns. (3.32) and (3.35) may be written as

$$[K]\{\delta\}_{t+\Delta t} + [C]\{\dot{\delta}\}_{t+\Delta t} + [M]\{\ddot{\delta}\}_{t+\Delta t} - [S]\{p\}_{t+\Delta t} = \{0\} \quad (3.39)$$

$$[G]\{p\}_{t+\Delta t} + [T]\{\dot{p}\}_{t+\Delta t} + [H]\{\ddot{p}\}_{t+\Delta t} + [S]^T\{\delta\}_{t+\Delta t} = \{0\} \quad (3.40)$$

Substitution of Eqns. (3.37) and (3.38) into Eqns. (3.39) and (3.40) leads to the following equations:

$$[A]\{\ddot{\delta}\}_{t+\Delta t} - \frac{\Delta t^2}{4} [S]\{\ddot{p}\}_{t+\Delta t} = \{f_1\}_t \quad (3.41)$$

$$\rho[S]^T\{\ddot{\delta}\}_{t+\Delta t} + [B]\{\ddot{p}\}_{t+\Delta t} = \{f_2\}_t \quad (3.42)$$

where

$$[A] = \frac{\Delta t^2}{4} [K] + \frac{\Delta t}{2} [C] + [M] \quad (3.43)$$

$$[B] = \frac{\Delta t^2}{4} [G] + \frac{\Delta t}{2} [T] + [H] \quad (3.44)$$

$$\{f_1\}_t = -[K]\{a\}_t - [C]\{b\}_t + [S]\{c\}_t \quad (3.45)$$

$$\{f_2\}_t = -[G]\{c\}_t - [T]\{d\}_t \quad (3.46)$$

$$\{a\}_t = \{\delta\}_t + \Delta t\{\dot{\delta}\}_t + \frac{\Delta t^2}{4} \{\ddot{\delta}\}_t \quad (3.47)$$

$$\{b\}_t = \{\dot{\delta}\}_t + \frac{\Delta t}{2} \{\ddot{\delta}\}_t \quad (3.48)$$

$$\{c\}_t = \{p\}_t + \Delta t \{\dot{p}\}_t + \frac{\Delta t^2}{4} \{\ddot{p}\}_t \quad (3.49)$$

$$\{d\}_t = \{\dot{p}\}_t + \frac{\Delta t}{2} \{\ddot{p}\}_t \quad (3.50)$$

If Eqn. (3.42) is multiplied by $[B]^{-1}$ and the expression for $\{\ddot{p}\}_{t+\Delta t}$ thus obtained is substituted in Eqn. (3.41), the following equations result:

$$[X]\{\ddot{\delta}\}_{t+\Delta t} = \{f_3\}_t \quad (3.51)$$

$$[B]\{\ddot{p}\}_{t+\Delta t} = \{f_4\}_t \quad (3.52)$$

where,

$$[X] = [A] + \frac{\Delta t^2}{4} \rho [S][B]^{-1}[S]^T \quad (3.53)$$

$$\{f_3\}_t = \{f_1\}_t + \frac{\Delta t^2}{4} [S][B]^{-1}\{f_2\}_t \quad (3.54)$$

$$\{f_4\}_t = \{f_2\}_t - \rho [S]^T \{\ddot{\delta}\}_{t+\Delta t} \quad (3.55)$$

If the initial conditions are known, successive values of $\{p, \delta\}_{t+\Delta t}$ may be obtained by using Eqns. (3.41) through (3.55). If nodes are numbered in an appropriate manner, matrix $[X]$ can be made banded. Advantage is also taken of the fact that matrix $[S]$ has very few non-zero terms in reducing the computational effort.

3.6 Computer Program for the Earthquake Response of the Dam-Reservoir System

A computer program was developed to obtain the earthquake response of a dam-reservoir system on the basis of the analytical treatment presented in this chapter. The program considers the case of horizontal ground motion and horizontal bed slope of the reservoir. However, it can be modified to take into account the sloping bed as well as a vertical excitation. The main steps used in obtaining the earthquake response are as follows:

- 1) Input reservoir data.
- 2) Form and save the reservoir matrices $[G]$, $[H]$ and $[T]$.

These are the global matrices formed by the assembly of the following element matrices:

- i) $[G^e]$ Eqn. (2.100)
- ii) $[H^e]$ Eqn. (2.101)
- iii) $[T^e]$ Eqn. (2.102)

- 3) Using matrices $[G]$ and $[H]$, compute the fundamental frequency of the reservoir as described in Section 2.5.4.
- 4) Input dam data.
- 5) Form and save the dam matrices $[K]$ and $[M]$. The corresponding element matrices are:

- i) $[K^e]$ Eqn. (3.24)
- ii) $[M^e]$ In the present analysis, the lumped mass matrix, given in Eqn. (3.26), is used. However, the consistent mass matrix given in Eqn. (3.25) can also be used.

- 6) Compute the first two frequencies of the dam (ω_1 and ω_2), using matrices $[K]$ and $[M]$, as described in Section 2.5.4.
- 7) Form and save the Rayleigh damping matrix $[C]$, Eqn. (3.27).
- 8) It can be noticed from the coupled equations of motion given by Eqns. (3.32) and (3.35) that the coupled matrix $[S]$ has dimensions of m by n , where m and n are the numbers of degrees of freedom of the dam and the reservoir, respectively. Also, it is noted that only $2k$ by k elements of this matrix are non-zeros, where k is the number of common nodes between the dam and the reservoir. If the common nodes are assigned numbers that are consecutive, starting from 1 and going up to k , the matrix $[S]$ takes the form:

$$[S] = \begin{bmatrix} [S] & [0] \\ [0] & [0] \end{bmatrix} \quad (3.56)$$

where $[S]$ is a $2k \times k$ matrix. Thus, given the number of common nodes k , form and save matrix $[S]$.

- 9) Input earthquake data, that is, the time step, number of steps, and the ground acceleration at each time step.
- 10) Form the following matrices:
 - i) $[A]$ Eqn. (3.43)
 - ii) $[B]$ Eqn. (3.44)
 - iii) $[X]$ Eqn. (3.53)
- 11) Modify matrix $[B]$ by the method of Payne and Irons⁽²⁰⁾ to take into account the known zero pressures. Triangularize

- the modified matrix $[B]$ by Gauss elimination and save.
- 12) Modify matrix $[X]$ by the method of Payne and Irons⁽²⁰⁾ to take into account the known accelerations of the ground nodal points. Triangularize the modified matrix $[X]$ by Gauss elimination and save.
 - 13) Assume initial conditions such that $\{p, \dot{p}, \ddot{p}, \delta, \dot{\delta}, \ddot{\delta}\}$ at time $(-\Delta t)$ are zeros.
 - 14) For each time interval,
 - i) Compute $\{f_3\}_t$ Eqn. (3.54)
 - ii) Evaluate $\{\ddot{\delta}\}_{t+\Delta t}$ by back substitution Eqn. (3.51)
 - iii) Compute $\{f_4\}_t$ Eqn. (3.55)
 - iv) Evaluate $\{\ddot{p}\}_{t+\Delta t}$ by back substitution Eqn. (3.52)
 - v) Compute $\{p, \delta\}_{t+\Delta t}$, Eqn. (3.38)
 - vi) Evaluate stresses in dam, Eqns. (3.2) and (3.3)
 - vii) Scan stresses, displacements and pressures to establish their maximum values during the entire period of the earthquake.

Several dam-reservoir problems were solved by the time domain analysis method using the computer program described in this chapter and the results are presented in Chapter 5.

CHAPTER 4

DAM-RESERVOIR PROBLEM
FREQUENCY DOMAIN ANALYSIS4.1 Introduction

When a finite element analysis is used to solve a reservoir problem, the infinite reservoir must be represented by a reservoir of finite length. A realistic representation of the infinite reservoir, however, requires that the energy loss in the outgoing waves is modelled correctly. This can be done by introducing an appropriate boundary condition at the truncation boundary. It was shown in Chapter 2 that for obtaining better accuracy in the finite element analysis, the radiation condition at the truncation boundary should be frequency dependent. When such a boundary condition is used, a time domain analysis is not feasible and resort must be had to an analysis in the frequency domain. Further, the analytical results presented later in this study show that the time domain method of analysis is less efficient than the frequency domain analysis for practical problems. These reasons provide the motivation for choosing an analysis in the frequency domain rather than one in the time domain.

Previous studies^(9,10) which applied frequency domain analysis have modelled the dam as finite elements and considered the reservoir as a continuum. The limitation of this method is that it can not consider reservoir boundaries of arbitrary shape. For the analysis of

a general problem, it is therefore necessary to model the reservoir by finite elements.

This chapter describes a method of solving the dam-reservoir problem by using a finite element representation of the reservoir problem as discussed in Chapter 2 and by applying the technique of frequency domain analysis. The effectiveness of this method of analysis is in a large measure due to the highly efficient Fast Fourier algorithms developed over the recent years^(19,38). These algorithms cut down the computation time by several orders of magnitude and, in many cases, make the frequency domain analysis equal to or more efficient than the time domain analysis.

4.2 Equations of Motion

The formulation of the finite element equations of motion of the dam-reservoir system has been presented in Chapters 2 and 3. When the displacements of the dam nodal points relative to the ground displacements are considered, Eqn. (3.32) becomes⁽¹⁰⁾

$$[K]\{r\} + [C]\{\dot{r}\} + [M]\{\ddot{r}\} = -\{m^x\}\ddot{\delta}_g^x - \{m^y\}\ddot{\delta}_g^y + [S]\{p\} \quad (4.1)$$

where

$$\{r\} = [r_1^x r_1^y \ r_2^x r_2^y \ \dots \ r_i^x r_i^y \ \dots \ r_n^x r_n^y]^T \quad (4.2)$$

$$r_i^x = \delta_i^x - \delta_g^x \quad (4.3a)$$

$$r_i^y = \delta_i^y - \delta_g^y \quad (4.3b)$$

$$\delta_g^x = \text{the x-component of the ground displacement}$$

and

δ_g^y = the y-component of the ground displacement

Vectors $\{m^x\}$ and $\{m^y\}$ given in Eqn. (4.1) are defined as

$$\{m^x\} = [M_1 \ 0 \quad M_2 \ 0 \quad \dots \quad M_i \ 0 \quad \dots \quad M_n \ 0]^T \quad (4.4a)$$

$$\{m^y\} = [0 \ M_1 \quad 0 \ M_2 \quad \dots \quad 0 \ M_i \quad \dots \quad 0 \ M_n]^T \quad (4.4b)$$

where n is the number of nodal points of the dam, and M_i is the mass lumped at nodal point i .

The natural frequencies and mode shapes of the dam with an empty reservoir may be obtained by solving the eigenvalue problem

$$[[K] - \omega_i^2 [M]] \{\phi_i\} = \{0\} \quad (4.5)$$

where ω_i is the natural frequency i and ϕ_i is the corresponding mode shape of vibration. From Eqn. (4.5), any number of frequencies of the dam can be obtained by using the method described in Section 2.5.4. The vibration mode shape corresponding to a particular frequency is then obtained by substituting that frequency value in Eqn. (4.5) and solving for $\{\phi\}$. It should be noted that the elements of vector $\{\phi\}$ are not all independent; one of them must be assigned an arbitrary value and the remaining are then obtained in terms of this particular element.

The relative displacements of the dam including reservoir effects can be approximately represented by a linear combination of

the first M mode shapes of the dam as follows⁽¹⁸⁾

$$\{r(t)\} = \sum_{m=1}^M \{\phi_m\} Y_m(t) \quad (4.6)$$

in which $\{\phi_m\}$ is the shape of the m^{th} mode of vibration of the dam with the reservoir empty and Y_m is the generalized displacement in that mode. It should be noted that the expansion given by Eqn. (4.6) is exact if M is equal to the total number of degrees of freedom of the dam finite element system.

Substitution of Eqn. (4.6) into Eqn. (4.1) gives

$$\begin{aligned} [K] \sum_{m=1}^M \{\phi_m\} Y_m + [C] \sum_{m=1}^M \{\phi_m\} \dot{Y}_m + [M] \sum_{m=1}^M \{\phi_m\} \ddot{Y}_m \\ = - \{m^x\} \ddot{\delta}_g^x - \{m^y\} \ddot{\delta}_g^y + [S]\{p\} \end{aligned} \quad (4.7)$$

On multiplying Eqn. (4.7) by $\{\phi_i\}^T$ where $i=1$ to M , and noting that the vectors $\{\phi_i\}$ are orthogonal with respect to the mass and stiffness matrices, and also with respect to the damping matrix $[C]$ which is of Rayleigh type, the following M equations are obtained

$$K_m^* \ddot{Y}_m(t) + C_m^* \dot{Y}_m(t) + M_m^* \ddot{Y}_m(t) = P_m^*(t); \quad m=1,2,\dots,M \quad (4.8)$$

where

$$M_m^* = \{\phi_m\}^T [M] \{\phi_m\} \text{ is the generalized mass} \quad (4.9)$$

$$C_m^* = \{\phi_m\}^T [C] \{\phi_m\} = 2\lambda_m \omega_m M_m^* \text{ is the generalized damping} \quad (4.10)$$

$$K_m^* = \{\phi_m\}^T [K] \{\phi_m\} = \omega_m^2 M_m^* \text{ is the generalized stiffness} \quad (4.11)$$

$$P_m^*(t) = - \{\phi_m\}^T \{m^x\} \ddot{\delta}_g^x - \{\phi_m\}^T \{m^y\} \ddot{\delta}_g^y + \{\phi_m\}^T [S] \{p\} \quad (4.12)$$

is the generalized load and

λ_m is the damping ratio in the m^{th} mode.

Eqns. (4.8) and (3.35) are the coupled equations of motion of the complete dam-reservoir system.

4.3 Complex Frequency Response of the Dam-Reservoir System for Horizontal Excitation

Specializing of the equation of motion of the dam, given by Eqn. (4.8) for the case when the ground motion acts only in horizontal direction, gives

$$K_m^* \ddot{Y}_m(t) + C_m^* \dot{Y}_m(t) + M_m^* \ddot{Y}_m(t) = - \{\phi_m\}^T \{m^x\} \ddot{\delta}_g^x(t) + \{\phi_m\}^T [S] \{p(t)\} \quad (4.13)$$

When the ground acceleration is

$$\ddot{\delta}_g^x(t) = e^{i\omega t} \quad (4.14)$$

The steady state response for the generalized displacements becomes

$$Y_m(t) = \bar{Y}_m(\omega) e^{i\omega t} \quad (4.15)$$

and the accelerations of the dam nodal points become

$$\{\ddot{\delta}(t)\} = \{\bar{\ddot{\delta}}(\omega)\} e^{i\omega t} = \{\{\bar{A}_g\} - \omega^2 \sum_{k=1}^M \{\phi_k\} \bar{Y}_k(\omega)\} e^{i\omega t} \quad (4.16)$$

where

$$\{\bar{A}_g\} = [1 \ 0 \ 1 \ 0 \ 1 \ 0 \ \dots \ 1 \ 0] \quad (4.17)$$

The steady state response of the hydrodynamic pressures may be expressed as

$$\{p(t)\} = \{\bar{p}(\omega)\}e^{i\omega t} \quad (4.18)$$

where $\{\bar{p}(\omega)\}$ is the solution of Eqn. (2.121) for dam accelerations $\{\ddot{\delta}(\omega)\}$. Applying the principle of superposition, the complex frequency response of the pressures can be expressed as

$$\{\bar{p}(\omega)\} = \{\bar{p}_0(\omega)\} - \omega^2 \sum_{k=1}^M \bar{Y}_k \{\bar{p}_k(\omega)\} \quad (4.19)$$

where

$\{\bar{p}_0(\omega)\}$ is the solution of Eqn. (2.121) for dam acceleration $\{\bar{A}_g\}$ and $\{\bar{p}_k(\omega)\}$ is the solution of Eqn. (2.121) for dam acceleration $\{\phi_k\}$.

It is noticed in Eqn. (4.19) that the nodal pressures on the dam have been expressed in terms of the generalized coordinates $\bar{Y}_m(\omega)$.

Substitution of Eqns. (4.14) through (4.19) into Eqn. (4.13) gives the following M equations

$$\begin{aligned} [K_m^* + i\omega C_m^* - \omega^2 M_m^*] \bar{Y}_m(\omega) = & - \{\phi_m\}^T \{m^x\} + \{\phi_m\}^T [S] \{\bar{p}_0(\omega)\} \\ & - \omega^2 \sum_{k=1}^M \bar{Y}_k(\omega) \{\bar{p}_k(\omega)\} \end{aligned} \quad (4.20)$$

Eqn. (4.20) may be arranged and expressed in matrix form as

$$\begin{bmatrix} E_{11}(\omega) & E_{12}(\omega) & \dots & E_{1M}(\omega) \\ E_{21}(\omega) & E_{22}(\omega) & \dots & E_{2M}(\omega) \\ \vdots & \vdots & \ddots & \vdots \\ E_{M1}(\omega) & E_{M2}(\omega) & \dots & E_{MM}(\omega) \end{bmatrix} \begin{Bmatrix} \bar{Y}_1(\omega) \\ \bar{Y}_2(\omega) \\ \vdots \\ \bar{Y}_M(\omega) \end{Bmatrix} = \begin{Bmatrix} L_1(\omega) \\ L_2(\omega) \\ \vdots \\ L_M(\omega) \end{Bmatrix} \quad (4.21)$$

where

$$E_{mk}(\omega) = \omega^2 \{\phi_m\}^T [S] \{\bar{p}_k(\omega)\} \quad , \quad m \neq k \quad (4.22a)$$

$$E_{mm}(\omega) = [K_m^* + i\omega C_m^* - \omega^2 M_m^*] + \omega^2 \{\phi_m\}^T [S] \{\bar{p}_m(\omega)\} \quad (4.22b)$$

$$L_m(\omega) = - \{\phi_m\}^T \{m^x\} + \{\phi_m\}^T [S] \{\bar{p}_0(\omega)\} \quad (4.22c)$$

and

$$m=1,2, \dots, M$$

$$k=1,2,\dots, M$$

The coefficient matrix $E(\omega)$ in Eqn. (4.21) is a frequency dependent matrix which relates the generalized displacement vector $\bar{Y}(\omega)$ with the corresponding generalized load $L(\omega)$. Unlike, in classical modal analysis, the matrix $E(\omega)$ is not diagonal because the displacement vectors $\{\phi_m\}$ are not the normal modes of the dam-reservoir system; they are simply the modes of the dam with the reservoir empty.

A solution of Eqn. (4.21) for all values of excitation frequency ω would give the complete frequency response functions for all the generalized displacements $\bar{Y}_m(\omega)$, $m=1,2,\dots,M$.

4.4 Response to Arbitrary Horizontal Ground Motion

For obtaining the response to an arbitrary excitation through the frequency domain analysis, the basic procedure is to express the excitation in terms of harmonic components, evaluate the response of the system to each component, and then superimpose the harmonic responses to obtain the total response⁽¹⁸⁾. When the excitation history

is $a(t)$ and $\bar{Y}(\omega)$ is the response of the system to excitation $e^{i\omega t}$, the harmonic synthesis is expressed mathematically as

$$Y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A(\omega) \cdot \bar{Y}(\omega) e^{i\omega t} d\omega \quad (4.23)$$

where

$$A(\omega) = \int_{-\infty}^{\infty} a(t) e^{-i\omega t} dt \quad (4.24)$$

Eqn. (4.24) is the Fourier transform of the excitation function $a(t)$, while Eqn. (4.23) is the inverse Fourier transform of the product of the frequency functions $A(\omega)$ and $\bar{Y}(\omega)$.

In order to make practical use of the frequency domain analysis, the integrals in Eqns. (4.23) and (4.24) must be evaluated by numerical techniques. This is done by converting the continuous Fourier transforms into Discrete Fourier Transforms (DFTs). The first step in using the DFTs is to assume that the excitation is periodic of period T_0 . This constitutes an approximation in the treatment of an arbitrary general excitation but is necessary in order to replace the infinite integrals of Eqns. (4.23) and (4.24) by a finite sum. The errors that this assumption involves and the method of eliminating or minimizing these errors will be discussed later. The selection of the excitation period, T_0 , also serves to define the lowest frequency that may be considered in the analysis. This lower frequency is given by

$$\omega_1 = \Delta\omega = \frac{2\pi}{T_0} \quad (4.25)$$

The excitation period is then divided into N equal time increments, each equal to Δt , and the excitation function is defined for the discrete times $t_m = m\Delta t$. When use is made of the above relationships, the DFTs of the continuous forms given in Eqns. (4.23) and (4.24) become

$$\{Y(m, \Delta t)\} = \frac{\Delta \omega}{2\pi} \sum_{n=0}^{N-1} \{A(n, \Delta \omega) \cdot \bar{Y}(n, \Delta \omega)\} e^{i 2\pi n m / N} \quad (4.26)$$

and

$$\{A(n, \Delta \omega)\} = \Delta t \sum_{m=0}^{N-1} \{a(m, \Delta t)\} e^{-i 2\pi n m / N} \quad (4.27)$$

The evaluation of the sums involved in the above two DFT equations is greatly simplified by the Fast Fourier Transform (FFT) technique⁽¹⁹⁾, which has made the frequency domain computationally competitive with the time domain analysis.

When using the DFT, it is important to note that it is based on the assumption that both the excitation and the response are periodic of period T_0 . This involves errors in obtaining the transient response of the system particularly when the damping in the system is small. The reason for these errors is simply that the system is not starting from rest due to the action of the excitations prior to the initial time. Two different techniques can be used to eliminate these errors. Both techniques are designed to ensure that the system starts from rest at the beginning of each period, that is, at time 0, $\pm T_0, \pm 2T_0$.

The first technique is known as a limiting process^(11,38). In this technique, the excitation period is elongated by adding a grace band of zero excitations to the end, to ensure that the system will

have a period of free vibration. Thus, when the system is damped and the period of the free vibration is long enough, the system comes to rest at the end of each period, which means that the calculated response is approximately the true transient response. It is clear from this discussion that this technique has two drawbacks. The addition of the zero excitations is computationally inefficient because it requires more core memory and computation time to obtain the response. Also, the method is not applicable to undamped systems.

The second technique, which is used in this work, was proposed by Meek and Veltsos⁽³⁸⁾. The first step in that method is to evaluate the exact response of the system to a periodic extension of the prescribed excitation. Next, a corrective periodic response is superimposed so as to bring the system to rest at the end of each excitation period. This ensures that a single period of the sum of the two periodic responses is identical to the response induced by one-time application of the sum of the prescribed excitation and the corrective excitation. By confining the corrective excitation to the last few time steps of the original excitation, the true transient response at all other time steps is obtained.

To apply this method to the problem being considered in this study which in effect consists of evaluating M single degrees of freedom systems, consider the k^{th} degree of freedom.

The periodic response given by Eqns. (4.26) and (4.27) can be rewritten as

$$Y_k^P(m, \Delta t) = \frac{\Delta \omega}{2\pi} \text{IDFT}\{\Delta t \cdot \text{DFT}\{a(m\Delta t)\} \cdot \bar{Y}_k(n, \Delta \omega)\} \quad (4.28)$$

in which DFT means Discrete Fourier Transform and IDFT is an Inverse Discrete Fourier Transform. The corrective method is then applied as described below.

In order to ensure that the system starts from rest at the beginning of each period, the displacement and velocity at the last step should equal zero. The values of the excitation function at the last two steps are therefore set equal to zero if they are not already zero (Fig. 4.1). Next, the periodic response $Y_k^P(m\Delta t)$ is obtained as given by Eqn. (4.28).

Now, let it be assumed that a corrective excitation consisting of two impulses R_1 and R_2 placed at steps $N-4$ and $N-3$, as shown in Fig. 4.1, is applied to the system. Let this corrective excitation be called $x^*(t)$. The response of the system to this corrective excitation can be obtained by using Eqn. (4.28) and replacing $a(t)$ by $x^*(t)$. Recognizing that the DFT of a unit impulse is a unit function⁽¹⁹⁾, the periodic response of the system for a unit impulse at time zero becomes

$$Y_0^{*P}(m, \Delta t) = \frac{\Delta \omega}{2\pi} \text{IDFT}\{\bar{Y}_k(n, \Delta \omega)\} \quad (4.29)$$

Using the time shifting property of DFT⁽¹⁹⁾, the responses at the last two steps due to unit impulse at time $(N-4)\Delta t$ are

$$Y_1^{*P}((N-2)\Delta t) = Y_0^{*P}(2\Delta t) \quad (4.30)$$

$$Y_1^{*P}((N-1)\Delta t) = Y_0^{*P}(3\Delta t) \quad (4.31)$$

Fig. 4.1. Steps of obtaining the response of the k^{th} generalized displacement.

Similarly, for a unit impulse at time $(N-3)\Delta t$, the last two responses are

$$Y_2^{*P}((N-2)\Delta t) = Y_0^{*P}(\Delta t) \quad (4.32)$$

$$Y_2^{*P}((N-1)\Delta t) = Y_0^{*P}(2\Delta t) \quad (4.33)$$

The values of the corrective impulses R_1 and R_2 are now obtained from the following equations

$$Y_k^P((N-2)\Delta t) + R_1 Y_0^{*P}(2\Delta t) + R_2 Y_0^{*P}(\Delta t) = 0 \quad (4.34)$$

$$Y_k^P((N-1)\Delta t) + R_1 Y_0^{*P}(3\Delta t) + R_2 Y_0^{*P}(2\Delta t) = 0 \quad (4.35)$$

The above equations are simply the conditions that the displacements at time $(N-2)\Delta t$ and $(N-1)\Delta t$ are set equal to zero, which ensures that the system starts from rest at the beginning of each period. The solution of Eqns. (4.34) and (4.35) yields the values of R_1 and R_2 . Using these values and the function Y_0^{*P} , the response Y_k^{*P} of the system to the corrective excitation $x^*(t)$ is obtained from the following equation

$$Y_k^{*P}(i\Delta t) = R_1 Y_0^{*P}(i_1\Delta t) + R_2 Y_0^{*P}(i_2\Delta t) \quad (4.36)$$

where

$$i_1 = i + 4 \quad (4.37a)$$

$$i_2 = i + 3$$

When the values of i_1 and i_2 are greater than $(N-1)$, they should be modified as follows:

$$i_1 = i + 4 - N$$

(4.37b)

$$i_2 = i + 3 - N$$

On superimposing the two periodic responses $Y_k^P(t)$ and $Y_k^{*P}(t)$, the true transient response $Y_k(t)$ of the k^{th} system becomes (Fig. 4.1)

$$Y_k(m, \Delta t) = Y_k^P(m, \Delta t) + Y_k^{*P}(m, \Delta t) \quad (4.38)$$

Having obtained the responses of the generalized displacements $Y_k(t)$; $k=1,2,\dots,M$, the relative displacements of the dam nodal points can be evaluated at each time step using Eqn. (4.6). In a similar manner, the stresses at each time step can be obtained by using Eqns. (3.2) and (3.3).

4.5 Computer Program for the Earthquake Response of the Dam-Reservoir System

A computer program was developed to obtain the earthquake response of the dam-reservoir system on the basis of the analytical treatment presented in this chapter. The program considers the case of horizontal ground motion and horizontal bed slope of the reservoir. However, it can be modified by simple changes to take into account the sloping bed as well as a vertical ground motion. The program was limited to include three mode shapes in the analysis. It is possible to include a larger number of modes by simple changes in the program. A previous study⁽²¹⁾ has shown; however, that consideration of such few mode shapes gives satisfactory results.

The main steps used in obtaining the earthquake response are:

- 1) Input the reservoir data.
- 2) Form and save the following element matrices:
 - i) $[G^e]$ Eqn. (2.100)
 - ii) $[H^e]$ Eqn. (2.101)

At the same time, assemble the corresponding global matrices $[G]$ and $[H]$.
- 3) Using the matrices $[G]$ and $[H]$, compute the fundamental frequency of the reservoir as described in Section 2.5.4.
- 4) Input dam data.
- 5) Form the dam matrices $[K]$ and $[M]$. The corresponding element matrices are:
 - i) $[K^e]$ Eqn. (3.24). A situation of either plane stress or plane strain can be considered.
 - ii) $[M^e]$ Eqn. (3.26)
- 6) Compute the first three natural frequencies of the dam, ω_{1d} , ω_{2d} and ω_{3d} , using matrices $[K]$ and $[M]$, as described in Section 2.5.4.
- 7) Using the first three natural frequencies of the dam and matrices $[K]$ and $[M]$, obtain the first three mode shapes of vibration ($\{\phi_1\}$, $\{\phi_2\}$, and $\{\phi_3\}$) by Eqn. (4.5). Normalize the mode shapes such that

$$M_m^* = \{\phi_m\}^T [M] \{\phi_m\} = 1.0 \quad (4.39)$$

- 8) Obtain the generalized masses, the generalized damping coefficients and the generalized stiffness coefficients. Because of the way the mode shapes have been normalized, the three generalized masses become

$$M_m^* = 1 \quad m=1,2,3 \quad \text{Eqn. (4.9)}$$

and the three generalized damping coefficients become

$$C_m^* = 2\lambda_m \omega_{md} \quad m=1,2,3 \quad \text{Eqn. (4.10)}$$

where λ_1 and λ_2 are prescribed values and λ_3 is calculated by using Eqn. (3.28) after obtaining the Rayleigh damping coefficients a and b as discussed in Section 3.3.5.

The three generalized stiffness coefficients are given by

$$K_m^* = \omega_{md}^2, \quad m=1,2,3 \quad \text{Eqn. (4.11)}$$

- 9) Form the non-zero elements of the coupled matrix $[S]$ by assembling the boundary element matrices $[S^e]$ given by Eqns. (3.34) and (2.111), and save.
- 10) Input the earthquake data which consists of the time step Δt , the total number of steps N , and the earthquake acceleration history.

It should be mentioned here that a proper choice of the time step, Δt , and the total number of steps, N , is very important for obtaining good results. The following considerations govern the selection of the time step, Δt , and the total number of steps, N ⁽¹¹⁾:

- 1) For good resolution of the time-variation of the response, it is recommended that Δt be less than

$$\frac{1}{4} \frac{\pi}{\omega_{3d}}$$

where ω_{3d} is the third natural frequency of the dam in rad/sec.

- ii) The frequency increment should be small enough to permit an accurate description of the frequency response functions especially near the resonant frequencies of the coupled system. It is recommended that $\Delta\omega$ be less than 1/50 of the first natural frequency of the dam which means that the number of steps, N , be greater than

$$\frac{100\pi}{\omega_{1d}} \frac{1}{\Delta t}$$

where ω_{1d} is the fundamental frequency of the dam.

- iii) The frequency range should include all significant frequencies contained in typical earthquake records. Most strong-motion accelerographs accurately record frequencies up to about 18 cps. Based on this consideration, it is recommended that Δt be less than 1/36 sec.
- 11) Obtain the complex frequency responses $\bar{Y}_k(m\Delta\omega)$; $k=1,2,3$ as described in Section 4.3, for the range of frequency steps from $m=0$ to $m=N/2$. The fact that the IDFT of the complex frequency response is a real function requires that the real part of $\bar{Y}_k(\omega)$ be an even function and the imaginary part be an

odd function⁽¹⁹⁾. This is done by folding the first $(N/2)$ real values of $\bar{Y}_k(\omega)$ about the $N/2$ step, and folding and flipping the first $(N/2)$ imaginary values of $\bar{Y}_k(\omega)$ about the $N/2$ step. In fact, these second $N/2$ values of $\bar{Y}_k(\omega)$ are the frequency responses for the range of negative frequencies since the frequency range extends periodically from $(-N/2 \Delta\omega)$ to $(N/2 \Delta\omega)$.

- 12) For each mode k , ($k=1,2,3$) repeat the following steps:
 - a) Obtain the periodic response Y_k^P as given by Eqn. (4.28) using the FFT algorithm.
 - b) Obtain the periodic response Y_O^P as given by Eqn. (4.29) using the FFT algorithm.
 - c) Use Eqns. (4.34) and (4.35) to obtain the corrective impulses R_1 and R_2 .
 - d) Compute the corrective periodic response Y_k^{*P} using Eqn. (4.36).
 - e) Superimpose the two periodic responses Y_k^P and Y_k^{*P} as given by Eqn. (4.38) to obtain the true transient response Y_k .
- 13) For each time step:
 - a) Calculate the relative displacements of the dam nodal points (Eqn. (4.6)).
 - b) Compute the element stresses as given by Eqns. (3.2) and (3.3).
 - c) Scan displacements and stresses in order to establish the maximum values during the entire period of excitation.

Several dam-reservoir problems were solved by the frequency domain analysis using the computer program described in this chapter and the results are presented in Chapter 5.

CHAPTER 5

ANALYTICAL RESULTS FOR THE DAM-RESERVOIR PROBLEM

5.1 Introduction

This chapter presents the results of analytical studies carried out using the time and frequency domain analyses described in Chapters 3 and 4. At the outset, two problems are solved to compare the relative efficiencies of the frequency and time domain methods. It is shown that although the results obtained by the two methods compare well, the time domain analysis is computationally inefficient for practical problems. A few other problems are then solved in the frequency domain and the results are presented. A study of the effect of the time step, Δt , and the total time, T_0 , is also carried out.

5.2 A Comparison of the Time Domain and Frequency Domain Analyses

Before attempting the solution of a practical problem, a problem of smaller size was solved by the time and frequency domain analysis methods to test the effectiveness of these methods. A rectangular dam of height 300 feet and width 90 feet, with a full reservoir extending to infinity, was considered. The finite element models of the dam and the reservoir are shown in Fig. 5.1. A plane strain situation was assumed and the following data were used:

Density of the material of the dam = $4.814 \text{ lb mass/ft}^3$

Modulus of elasticity = $0.72 \times 10^9 \text{ lb/ft}^2$

Poisson's ratio = 0.17

Fig. 5.1. Finite element model of dam-reservoir system

Damping ratios	= 5% of critical in the first two modes
Density of water	= 1.9379 lb mass/ft ³
Velocity of sound in water	= 4720 ft/sec

The system was subjected to a short sine wave ground acceleration of amplitude 10ft/sec^2 and frequency equal to 0.5 cps, as shown in Fig. 5.2. The excitation frequency selected was less than the fundamental frequency of the reservoir so that the error arising in the time domain analysis because of the use of the radiation condition could be avoided. The time step, Δt , was taken as 0.02 seconds. The total time of excitation was considered to be 20 seconds and the response was obtained for the first ten seconds. The horizontal and vertical relative crest displacements are shown in Figs. 5.3 and 5.4, respectively. It can be seen from these figures that good agreement exists between the results obtained by the time domain and frequency domain analyses. It is noticed also that the initial portion of the response is affected by transient disturbance but the response ultimately settles down to a steady sine curve as would be expected.

The total CPU time required for obtaining the generalized displacements and the crest displacements by the frequency domain analysis was 3.37 minutes. In the time domain, the same problem took 8.7 minutes of CPU time. Although the above example does show the relative efficiency of the frequency domain analysis, a practical problem of a larger size was solved by the two methods to confirm this conclusion. The response of Pine Flat Dam to a sine wave excitation was obtained in this problem. The finite element models of the

Fig. 5.2. Ground acceleration history.

Fig. 5.3. Horizontal displacement of crest.

Fig. 5.4. Vertical displacement of crest.

dam and the reservoir are shown in Fig. 5.5. The data given in the previous example were used except that a plane stress situation was assumed and the Poisson's ratio was taken as 0.2.

A comparison between the crest displacements, obtained from the time and frequency domain analyses, is presented in Figs. 5.6 and 5.7. The two sets of results compare very well. The time domain algorithm took 76.2 minutes of CPU time for obtaining the crest displacements for 500 steps, i.e. for ten seconds of the earthquake, while the frequency domain algorithm took only 18.5 minutes of CPU time for obtaining the three generalized displacements as well as the crest displacements for 1000 time steps or a total period of 20 seconds. It is apparent that in terms of computational efficiency the frequency domain procedure is far superior to the time domain procedure described here.

5.3 Calculation of the Response of a Dam by the Frequency Domain Analysis

In order to apply the frequency domain analysis developed in Chapter 4 to a practical problem, a dam of height 300 feet and a vertical upstream face was considered. The finite element models of the dam and the reservoir are shown in Fig. 5.8. The height of the reservoir is 270 feet and its length is assumed to be infinite. The same dam-reservoir system was considered by Finn⁽³³⁾ to obtain the frequency response of the hydrodynamic pressures. He used a finite element model for the dam and considered the reservoir as a continuum.

Fig. 5.5. Finite element models of pine flat dam and its full reservoir

Fig. 5.6. Horizontal displacement of crest of pine flat dam.

Fig. 5.7. Vertical displacement of crest of pine flat dam.

Fig. 5.8. Finite element model for the dam-reservoir system.

A comparison will be made between Finn's results and the present results wherever possible. A plane stress situation was assumed and the following data were used:

Density of the material of the dam	= 4.814 lb mass/ft ³
Modulus of elasticity	= 0.72×10^9 lb/ft ²
Poisson's ratio	= 0.17
Damping ratios	= 5% of critical in the first two modes
Density of water	= 1.9379 lb mass/ft ³
Velocity of sound in water	= 4720 ft/sec

The horizontal component of El-Centro earthquake acceleration was selected as the excitation. To satisfy the recommendations given in Section 4.5, the time interval, Δt , and the total time of excitation, $T_0 = \Delta t \cdot N$, were chosen to be 0.01 sec. and 16 sec., respectively. Accordingly, the frequency step, $\Delta \omega$, becomes 0.3927 rad/sec. The choice of these parameters will be investigated in the next section. Fig. 5.9 shows the earthquake acceleration record for the period considered in this analysis, while Figs. 5.10a and 5.10b are the real and imaginary parts of the DFT of that acceleration record. It was stated earlier in Chapter 4 that most strong earthquake records contain frequencies up to 18 CPS. It can be seen from Figs. 5.10a and 5.10b that the above fact is true in the present case. It may be noted in reference to these figures that the frequency of 18 CPS corresponds to 288 frequency steps.

Fig. 5.9. Acceleration record - El Centro, California, Earthquake, May 1940.

Fig. 5.10a. Discrete Fourier transform of El-Centro earthquake acceleration history.

Fig. 5.10b. Discrete Fourier transform of El-Centro earthquake accelerations.

Fundamental frequency of the reservoir obtained by the present analysis is 27.493 rad/sec while the exact value from the classical solution, Eqn. (2.43), is 27.460 rad/sec. The first three natural frequencies of the dam with empty reservoir are given in Table 5.1 and compared with those obtained by Finn⁽³³⁾. It can be seen from this table that the present results compare well with those of Finn who used a much larger number of elements. The first three modes of vibration are shown in Figs. 5.11a through 5.11c.

The complex frequency response of the three generalized displacements were calculated for every second frequency step, i.e. every 0.7854 rad/sec. Figs. 5.12 through 5.14 show these complex frequency responses.

In order to study the effect of flexibility of the dam on the hydrodynamic pressures, the program used in this computation was designed to output the pressures for the cases of rigid dam and flexible dam at each frequency used to evaluate the complex frequency responses of the generalized displacements. Figs. 5.15a and 5.15b present a comparison between the hydrodynamic pressures in the rigid and flexible dam cases, for different values of the excitation frequency. It can be seen from these figures that the effect of the flexibility of the dam is significant. In order to compare the results with those obtained by Finn⁽³³⁾, the total hydrodynamic forces were calculated for the above two cases and a comparison is presented in Table 5.2. The agreement is satisfactory. Figs. 5.16a through 5.16c show the time series responses of the three generalized displacements.

Table 5.1. Natural frequencies of the dam with empty reservoir.

MODE	FREQUENCY (rad/sec)	
	Present	Finn*
1	33.557	32.98
2	73.565	73.96
3	88.339	89.04

* Reference (33)

Table 5.2. Total hydrodynamic force for rigid and flexible dams.

$\omega/\frac{\pi C}{2H}$	PRESENT ANALYSIS			FINN*		
	Hydrodynamic Force $ P_d /P_s$		$ P_d _{flex}$	Hydrodynamic Force $ P_d /P_s$		$ P_d _{flex}$
	Rigid	Flexible	$ P_d _{rigid}$	Rigid	Flexible	$ P_d _{rigid}$
0.744	.04831	.07198	1.500	0.050	0.076	1.520
1.516	.02844	.01318	0.463	0.028	0.013	0.464

* Reference (33)

$\omega_1 = 33.557 \text{ rad/sec}$

Fig. 5.11a. First mode of vibration of the dam.

Fig. 5.11b. Second mode of vibration of the dam.

Fig. 5.11c. Third mode of vibration of the dam.

Fig. 5.12a. Complex frequency response of FIRST generalized displacement.

Fig. 5.12b. Complex frequency response of FIRST generalized displacement.

Fig. 5.13a. Complex frequency response of SECOND generalized displacement.

Fig. 5.13b. Complex frequency response of SECOND generalized displacement.

Fig. 5.14a. Complex frequency response of THIRD generalized displacement.

Fig. 5.14b. Complex frequency response of THIRD generalized displacement.

Fig. 5.15a. Rigid vs flexible dam pressures.

Fig. 5.15b. Rigid vs flexible dam pressures.

Fig. 5.16a. Time response of FIRST generalized displacement.

Fig. 5.16b, Time response of SECOND generalized displacement.

Fig. 5.16c. Time response of THIRD generalized displacement.

Figs. 5.17a and 5.17b are the horizontal and vertical displacement histories of the crest, while Figs. 5.18a and 5.18b are the horizontal and vertical displacement histories of dam nodal point 36 which is located at the mid-height of the upstream face. The first 1596 steps in these figures represent the true response; the last four steps are not correct because of the manner in which the frequency analysis was formulated, and should be disregarded.

5.4 Effect of the Time Step and Total Time on the Response Analysis

It was mentioned earlier in Chapter 4 that a proper choice of the time step, Δt , and the total time of excitation, T_0 , is very important for obtaining good results. In order to study the effect of the time step, Δt , on the present analysis, the same problem described in the previous section (5.3) was solved again with a time step of 0.02 second instead of 0.01 second while the total time, T_0 , was kept as 16 seconds. The time responses of the three generalized displacements are presented in Figs. 5.19, 5.20 and 5.21, and compared with the corresponding responses for $\Delta t = 0.01$. It can be noticed from these figures that the responses have not been affected significantly with the change of Δt from 0.01 second to 0.02 seconds. In fact, the difference is hardly discernible from the figures.

In order to study the effect of the total time, T_0 , the same problem was considered again. The total time, T_0 , was chosen to be 8 seconds and the time step, Δt , was kept as 0.02 second. The results are presented in Figs. 5.22 through 5.25 and compared with those

Fig. 5.17a. Horizontal displacement of crest.

Fig. 5.17b. Vertical displacement of crest.

Fig. 5.18a. Horizontal displacement of nodal point (36).

Fig. 5.18b. Vertical displacement of nodal point (36).

Fig. 5.19. Effect of time step on the response of FIRST generalized displacement.

Fig. 5.20. Effect of time step on the response of the SECOND generalized displacement.

Fig. 5.21. Effect of the time step on the response of the THIRD generalized displacement.

Fig. 5.22. Effect of total time on the response of FIRST generalized displacement.

Fig. 5.23. Effect of total time on the response of SECOND generalized displacement.

Fig. 5.24. Effect of the total time on the response of THIRD generalized displacement.

Fig. 5.25a. Effect of total time on the horizontal displacement of crest.

Fig. 5.25b. Effect of total time on vertical displacement of crest.

results obtained for the case of T_0 equal to 16 seconds. It can be seen that the choice of T_0 does affect the response analysis. However, T_0 equal to 8 seconds can still be considered reasonable, as is seen from the results in Figs. 5.22 through 5.25.

5.5 Response of an Inclined Dam

To demonstrate the effectiveness of the method of the analysis in solution of a problem in which the face of the dam is inclined, an idealized earth dam of height 300 feet and base 900 feet was considered. The depth of water in reservoir was taken as 270 feet and the reservoir was considered to be infinite. The finite element models of the dam and the reservoir are shown in Fig. 5.26. A plane strain situation was assumed and the following data were considered:

Density of material of the dam	= 4.0373 lb mass/ft ³
Modulus of elasticity	= 1.1707×10^7 lb/ft ²
Poisson's ratio	= 0.45
Damping ratios	= 20% of critical in first two modes
Density of water	= 1.9379 lb mass/ft ³
Velocity of sound in water	= 4720 ft/sec

A similar problem has been solved by Clough and Chopra⁽³⁹⁾, who obtained the earthquake response of the dam, neglecting the effect of the reservoir interaction. A comparison will be made between Clough's results and the present results wherever possible.

Fig. 5.26. A finite element model for an earth dam-reservoir system.

The dam-reservoir system was considered to be subjected to the first eight seconds of the horizontal component of El-Centro earthquake accelerations shown in Fig. 5.9. The time step was taken as 0.02 seconds. Hence, the frequency step becomes

$$\Delta\omega = \frac{2\pi}{8} = 0.7854 \text{ rad/sec}$$

The fundamental frequency of the reservoir was found to be 28.675 rad/sec. As a comparison, the classical solution for vertical upstream yields a value of 27.46 rad/sec.

It has been shown in Section 2.6.4 that as the inclination of the dam face increases the fundamental frequency of the reservoir increases slightly. The first three natural frequencies of the dam with empty reservoir are compared in Table 5.3 with those obtained by Clough⁽³⁹⁾. It is seen from that table that the two results are identical. The first three modes of vibration of the dam are shown in Fig. 5.27. The complex frequency responses of the three generalized displacements are presented in Figs. 5.28 through 5.30. Figs. 5.31 through 5.32 present the time response of the three generalized displacements while Figs. 5.32a and 5.32b present the horizontal and vertical components of crest displacements.

Table 5.3. Natural frequencies of the dam.

MODE	FREQUENCY (rad/sec)	
	Present	Clough [*]
1	7.7144	7.71
2	12.5230	12.52
3	14.5980	14.60

* Reference (39)

a) FIRST Mode

b) SECOND Mode

c) THIRD Mode

Fig. 5.27. Free modes of vibration of the earth dam.

Fig. 5.28a. Complex frequency response of FIRST generalized displacement.

Fig. 5.28b. Complex frequency response of FIRST generalized displacement.

Fig. 5.29a. Complex frequency response of SECOND generalized displacement.

Fig. 5.29b. Complex frequency response of SECOND generalized displacement.

Fig. 5.30. Complex frequency response of THIRD generalized displacement.

Fig. 5.30b. Complex frequency response of THIRD generalized displacement.

Fig. 5.31. Time response of FIRST generalized displacement.

Fig. 5.32. Time response of SECOND generalized displacement.

Fig. 5.33. Time response of THIRD generalized displacement.

Fig. 5.34a. Horizontal displacement of crest.

Fig. 5.34b. Vertical displacement of crest.

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CHAPTER 6

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS
FOR FUTURE WORK6.1 Summary

The dynamic analysis of dams has received the attention of many investigators during the past few years due to the damages caused to some dams by recent earthquakes. Most of the previous investigations have shown that any dynamic analysis procedure, to be able to give reasonably accurate results, must consider the effect of the reservoir on the dynamic properties and the response of the dam. Due to the many complexities involved in the problem of the earthquake response of the dam-reservoir system, in general, a numerical procedure must be used to solve the problem. The main objective of this study has been to develop an accurate and efficient numerical procedure to solve the dam-reservoir problem using the finite element method. The following assumptions are made in the formulation of the finite element procedure.

- 1) The vibration problem is two-dimensional, the motion in any vertical plane perpendicular to the axis of the dam being similar.
- 2) Dam material is homogeneous and elastic.
- 3) The water is compressible and inviscous.
- 4) The effect of free surface waves of water is negligible.

The procedure developed in this study uses a complete finite element model for both the dam and the reservoir in which the nodal

displacements of the dam and the nodal pressure in the reservoir are considered to be the basic unknowns. The response calculations are carried out through a frequency domain analysis. To the knowledge of the author, this is the first attempt to use a frequency domain analysis in conjunction with a finite element representation of both the dam and the reservoir to solve the problem of coupled vibrations of the dam-reservoir system. The development of such a procedure has been approached as described below.

A complete study of the hydrodynamic reservoir problem is carried out first. Equations that govern the motion of water in the reservoir are reviewed first. This is followed by presentation of some classical solutions for the vibration problems of infinite and finite reservoirs - rigid dam systems with simple boundary conditions such as vertical dam face and horizontal reservoir bed.

In order to model an infinite reservoir by one of finite length, it is necessary to model the energy loss in the outgoing waves which are a characteristic of infinite reservoirs. The Sommerfeld radiation damping condition has been used in the past for a similar problem^(2,3). It is shown that this radiation condition fails to model the energy loss correctly and the results obtained from the application of such a condition may be widely different from those obtained by the classical method. A modified radiation condition is then developed and examples are presented to show that the method leads to solutions which are in excellent agreement with the classical solution for an infinite reservoir over a wide range of exciting frequencies.

Since the use of the radiation condition mentioned above allows the representation of an infinite reservoir by a finite one, the finite element formulation of the finite reservoir problem is then developed. Three node triangular finite elements are used. The pressure is assumed to vary linearly within each element. Several reservoir vibration problems are solved by the finite element procedure and, wherever possible, the results are compared with the classical solutions to demonstrate the effectiveness and accuracy of the method.

The next step in the formulation of the procedure is a review of the finite element formulation of the dam vibration problem. The coupled equations of motion of the dam-reservoir system are then obtained. In order to provide a comparison with the results obtained by the use of the proposed frequency domain analysis, a time domain analysis procedure based on a direct integration of the equations of motion is considered as described in Ref. (3). It is shown that the time domain procedure has two serious drawbacks. First, it depends on the application of the Sommerfeld radiation damping condition which, as shown in this study, fails to model the infinite reservoir correctly. Second, the solution of practical problems shows that the time domain analysis is computationally inefficient when compared to the frequency domain analysis.

A procedure for solving the coupled equations of the dam-reservoir system through a frequency domain analysis is then developed. The method can be summarized as follows:

- 1) The nodal displacements of the dam inclusive of the reservoir effects are expressed as a linear combination of the first few modes of vibration, say M modes, of the dam without the reservoir.

- 2) The above formulation is used to obtain the complex frequency response functions for the M generalized displacements of the dam by solving M coupled equations, for each given frequency. Unlike the classical modal analysis, the M equations are coupled. This is simply because the modes of vibrations used in the analysis are those of the dam only, and not of the dam-reservoir system. It should be noted that this representation is correct if all modes of dam vibration are included. However, reasonable accuracy can be obtained by considering only the first few mode shapes.
- 3) Having obtained the complex frequency response functions of the generalized displacements, the modal responses to an arbitrary ground motion are obtained by a Fourier synthesis. The effectiveness of the method of analysis is in a large measure due to the highly efficient Fast Fourier Transform algorithms developed over recent years. These algorithms cut down the computation time by several orders of magnitude and, in many cases, make the frequency domain analysis more efficient than the time domain analysis.
- 4) Having obtained the time history of generalized displacements, the displacements and stresses for the dam are obtained by standard transformations.

A computer program was developed to obtain the earthquake response of the dam-reservoir system using the proposed finite element frequency domain analysis.

Several dam-reservoir problems are solved and presented to demonstrate the effectiveness and accuracy of the proposed method.

6.2 Conclusions

As a result of the present work, the following major conclusions are drawn:

- 1) The finite element frequency domain analysis is a reasonably accurate and efficient method for obtaining the transient response of a general dam-reservoir system subjected to a horizontal component of earthquakes. The results obtained by the present method are found to be in excellent agreement with the limited number of previous results available. The present method removes several limitations of the existing methods. By using the present method, arbitrary geometries of reservoir and dam-reservoir interface can be considered. Also, it is possible to obtain the response for the whole period of earthquake with less computational effort than by any other available method.
- 2) The Sommerfeld radiation damping condition does not give an accurate representation of an infinite reservoir by a reservoir of finite length. The modified radiation damping condition developed here is successful in modelling the energy loss in outgoing waves over a wide range of exciting frequencies.

- 3) The present analysis confirms the conclusions of the past analyses that the compressibility of water and flexibility of the dam can not be ignored in the design of dams.
- 4) The hydrodynamic pressures on dams are significantly affected by the inclination of the dam face which may therefore have an important effect on the total system.
- 5) The fundamental frequency of a reservoir is increased slightly by increasing the inclination of the dam face.

6.3 Recommendations for Future Work

The following recommendations are made for future work.

- 1) The present computer program can be modified by simple changes to include the following items:
 - a) Vertical ground motion.
 - b) Flexibility and inclination of reservoir bed
 - c) Non-homogeneous dam material properties.
- 2) The extension of this work to a three-dimensional problem may be considered. Three-dimensional finite elements for the dam and the reservoir are available in the references^(1,3), and can be used in a finite element formulation of the problem.
- 3) The frequency domain analysis method proposed here can be used to carry out an in-depth parametric study to obtain the effect of all variables affecting the earthquake response of a dam-reservoir system.

- 4) For obtaining the hydrodynamic pressures on dams, the effect of viscosity of water has always been neglected in the past because the problem becomes intractable even when the interaction effect is ignored. It would be of interest to know the extent of error resulting from neglecting the viscosity.
- 5) The proposed method of analysis can have applications in several other areas of Civil Engineering such as the dynamic analysis of liquid retaining structures, submerged structures and floating structures.

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