REDUCED COMPLEXITY FREQUENCY DOMAIN TURBO EQUALIZATION

by

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Abstract

This thesis investigates the performance of turbo equalizers based on interference canceller (IC) that has been implemented in the frequency domain (FD-IC), as opposed to the more typical time domain implementation (TD-IC). A computational complexity analysis shows that the FD-IC has much lower complexity when equalizing channels with long delay spreads, while still maintaining the excellent performance of the TD-IC. A novel, sub-optimal, reduced-complexity hybrid frequency/time domain interference canceller (hybrid IC), which is based on implementing part of the IC in the frequency domain (the P -filter), and part in the time domain (the q -filter), is proposed. The filter coefficients of the hybrid IC are derived according to the minimum mean squared error criterion. It is shown by means of computer simulation that turbo equalization using the hybrid IC can achieve respectable performance in terms of the bit and frame error rates, without the need for performing FFT's at each iteration, thereby providing reduced computational complexity. Although the hybrid IC is sub-optimal as compared to the FD-IC, in some cases it is shown that it can even provide slightly better performance than the FD-IC.

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List of Acronyms and Abbreviations

APP	A Posteriori Probability
AWGN	Additive White Gaussian Noise
BER	Bit Error Rate
dB	Decibel
DFE	Decision Feedback Equalizer
DFT	Discrete Fourier Transform
FER	Frame Error Rate
FFT	Fast Fourier Transform
FD-IC	Frequency Domain Interference Canceller
Hz	Hertz
IC	Interference Canceller
IFFT	Inverse Fast Fourier Transform
ISI	Intersymbol Interference
LE	Linear Equalizer
MAP	Maximum A <i>Posteriori</i> Probability
MCU	Metric Calculation Unit
MF	Matched Filter
MLSE	Maximum Likelihood Sequence Estimation
MMSE	Minimum Mean Square Error
MMSE-LE	Minimum Mean Square Error Linear Equalizer

QAM	Quadrature Amplitude Modulation
QPSK	Quadrature Phase Shift Keying
SNR	Signal to Noise Ratio
TCM	Trellis Coded Modulation
TD-IC	Time Domain Interference Canceller

Chapter 1

INTRODUCTION

1.1 Background

In wireless communication systems, the transmitted signals simultaneously travel over multipaths to the receiver, with the signal arriving along each path suffering from different attenuations and propagation delays. This multipaths interference causes time dispersion of the transmitted signals, results in intersymbol interference (ISI), or if the transmission environment is time-variant, frequency selective fading. To mitigate the detrimental effects of ISI, equalization is widely employed.

Traditionally, equalization techniques include the linear equalizer, decision feedback equalizer (DFE), maximum likelihood sequence estimation (MLSE), etc. During the past decade, significant attention has been placed on turbo equalization. Turbo equalization was motivated by the iterative decoding algorithm for turbo codes [1]-[6] and was first proposed by Douillard et al. in 1995 [7]. Traditionally, equalization and channel decoding are considered separately [8]-[10]. However, with turbo equalization, the equalization and channel decoding are combined through an iterative process and jointly optimized so that the whole receiver performance is enhanced. The performance of turbo equalization has been shown to be near optimum over many transmission channels. So far, turbo equalization has been proposed for many applications, such as in wireless cellular systems [11], diversity channels [12] and trellis coded modulation (TCM) [13, 14]. For the equalizer, different equalization schemes are employed, such as maximum *a posteriori* probability (MAP) equalization

[7, 15], linear equalizer (LE) [16] and interference canceller (IC) [17]. Since turbo equalization based on the IC has low computational complexity, much research has been placed on it [17]-[20].

1.2 Thesis motivation

IC-based turbo equalization was originally proposed in by Glavieux et al. [17] in 1997. The IC consists of two filters, the p -filter and the q -filter. The purpose of the p -filter is to enhance the received signal-to-noise ratio (SNR), and the purpose of the q -filter is to eliminate the ISI. The input to the P -filter is the received samples from the output of a filter matched to the transmitted pulse shape. Ideally, the input to the q -filter would be the actual transmitted symbols, but in practice these symbols are not known at the receiver. With IC-based turbo equalization, the received data is first processed with a linear equalizer, which partially suppresses the ISI. The equalizer output is passed to the decoder which decodes the convolutional code. Using the output of the decoder it is possible to locally generate estimates of the transmitted data. These estimates are used as the input to the q -filter of the IC to more effectively eliminate the ISI. The IC output is then passed back to the decoder, which again decodes the convolutional code, this time more reliably since there is less residual ISI in the input. This process is repeated in an iterative (turbo) fashion. With the originally proposed system the IC was implemented in the time domain, but more recently Tuchler and Hagenauer proposed a frequency domain implementation [21]. With this system, the two filters are implemented in the frequency domain, using the fast Fourier transform (FFT) to convert to and from the frequency domain. For channels with long delay spreads, the frequency domain system has lower implementation complexity than the

time-domain version. However, since the frequency-domain IC requires an FFT and an inverse FFT (IFFT) for each iteration of the turbo equalization loop, the complexity still remains quite high. It is desirable to further reduce the complexity by eliminating the need for FFT's within the loop. In this thesis we propose a novel hybrid frequency/time domain interference canceller (hybrid IC), which is based on implementing the P -filter in the frequency domain, and the q -filter in the time domain. By carefully calculating the optimal coefficients of these two filters, respectable performance is possible without the need for performing FFT's at each iteration.

1.3 Thesis objectives

The first objective of this thesis is to investigate whether the IC-based turbo equalization in frequency domain can improve the system performance over channels with long delay spreads, sometimes which was not investigated in the literature [21, 22].

The second objective is to find an approach to further reduce the computational complexity of IC-based turbo equalizers. If q -filter in the IC operating in time domain only uses the several largest pairs of q -filter coefficients instead of all coefficients when the length of channel response is long and the majority of its values are very small, what will happen to the whole performance? A hybrid reduced complexity IC, which makes use of both advantages of time domain and frequency domain and simplifies the receiver's implementation, is proposed in this thesis.

The third objective is to investigate the performances of this hybrid IC over both static ISI channels and frequency selective fading channels.

1.4 Thesis contributions

In this thesis, there are several contributions as follows:

1. Simulations verify that IC-based turbo equalization in the frequency domain can improve bit error rate (BER) and frame error rate (FER) performance for channels with long delay spreads.
2. The filter coefficients in the proposed hybrid IC are derived according to the minimum mean squared error criterion.
3. The proposed hybrid-IC is simulated over static ISI and frequency selective fading channels. Simulations of the proposed scheme show that the performance is similar to the frequency-domain IC (FD-IC) but with lower complexity.
4. A detailed computational complexity analysis verifies that the operation of the q -filter in time domain is easier than in the frequency domain.

1.5 Thesis Organization

Chapter Two will describe the communication system model, including the transmitter, the channel and the receiver. It will also review frequency domain linear equalization.

Chapter Three describes the IC-based turbo equalization in the frequency domain. The coefficients of P -filter in the frequency domain and q -filter in the time domain for the proposed hybrid IC will be derived. At the end of the chapter, the computational complexity of LE and IC in the time and frequency domain, and the computational complexity of the time domain IC (TD-IC), FD-IC and hybrid IC will be compared.

Chapter Four provides simulation results of the FD-IC and the hybrid IC over static ISI and frequency selective fading channels. This is accompanied by a discussion of the results and a comparison of the performance of the two schemes.

The conclusion will be summarized in Chapter Five along with a discussion on topics for possible future studies.

Chapter 2

SYSTEM MODEL AND REVIEW OF FREQUENCY DOMAIN LINEAR EQUALIZERS

2.1 Introduction

In this chapter we present a simple channel model for transmission of coded data over an ISI channel, using a time domain linear equalizer to encounter the ISI. A brief overview of frequency domain linear equalization is also introduced. A block diagram of the communication system is shown in Figure 2-1. To improve the reliability of data transmission, a channel coding scheme is used, which adds controlled redundancy to the data. The coded data is fed into a transmit filter, which produces an analog baseband signal. The signal is then passed to a modulator to convert it to a desired frequency band using some modulation technique.

Figure 2-1. Block diagram of the communication system.

The passband signal from the transmitter is then transmitted over the channel. For a wireless channel, important concerns are multipath propagation and frequency and time selective fading with time varying characteristics. Generally, the channel frequency response is not flat and will therefore cause ISI.

The received signal first goes to a coherent demodulator. The demodulated signal passes through the receive filter, which is matched to the transmit filter. The filtered signal is then sampled at the symbol rate to obtain the decision statistics for the symbol. These symbols are passed through an equalizer to remove the unwanted ISI while maximizing the signal-to-noise ratio. Finally, the equalized samples are fed into the channel decoder to estimate the transmitted message.

Mathematically, it is convenient to model the transmit filter, modulator, ISI channel, receive filter, demodulator and sampler as a discrete-time ISI channel. The transmission system then can be simplified as the composition of the transmitter, discrete-time ISI channel and receiver. These components are described in Sections 2.2, 2.3 and 2.4, respectively. The receiver discussed in Section 2.4 is based on using a time-domain linear equalizer. The use of a frequency domain linear equalizer is discussed in Section 2.5.

2.2 Transmitter

The transmitter is made up of a convolutional encoder, interleaver, and signal mapper, as shown in Figure 2-2.

Figure 2-2. Structure of the transmitter.

2.2.1 Convolutional Encoder

In the transmitter, a convolutional encoder is employed since the implementation of convolutional codes is well known and its soft-output decoding is useful for turbo equalization schemes. N_a message symbols with n_a message bits per symbol enter the encoder that produces N_c code symbols with n_c code bits per symbol. If the total memory of the encoder is m bits then the number of states is 2^m . The initial state is set to zero and to make the final state of encoder to be zero, a total of $(K_c - 1)$ extra zero symbols are fed into the encoder after the message, where $K_c = m + 1$ is the constraint length of the convolutional code. Therefore, the total codeword length is $N_c = N_a + K_c - 1$ code symbols.

The message word can be expressed as $\mathbf{a} = [a_0 \ a_1 \ \dots \ a_{N_a-1}]$, with each message symbol $a_n \in \{0, 1, \dots, 2^{n_a} - 1\}$ for $0 \leq n \leq N_a - 1$. It is assumed that the message symbols are all independent, and that each of the 2^{n_a} possible values for each symbol is equally likely to occur. That is, $\Pr \{a_n = a\} = \frac{1}{2^{n_a}}$, for all $a \in \{0, 1, \dots, 2^{n_a} - 1\}$. The codeword can be expressed as $\mathbf{c} = [c_0 \ c_1 \ \dots \ c_{N_c-1}]$, with each code symbol $c_n \in \{0, 1, \dots, 2^{n_c} - 1\}$.

2.2.2 Interleaver

The code symbols are fed into a random symbol interleaver, which rearranges the order of the code symbols, but keeps each symbol intact. The output of the interleaver has the same number of symbols and the same number of bits in each symbol compared with the input. The output of the interleaver is $\mathbf{d} = [d_0 \ d_1 \ \dots \ d_{N_c-1}]$ with each symbol

given by $d_n = c_{IL[n]}$, where $IL[\bullet]$ represents the interleaver mapping, which is a randomly selected permutation of the integers $\{0, 1, \dots, N_c-1\}$. Although randomly selected, both the transmitter and the receiver know the mapping.

2.2.3 Symbol Mapper

The interleaved symbols then go into the symbol mapper to be transformed to modulation symbols, which are points in a M -ary signal constellation, where $M = 2^{n_c}$. The output of the symbol mapper is $\mathbf{v} = [v_0 \ v_1 \ \dots \ v_{N_c-1}]$ with each modulation symbol given by $v_n = SM[d_n]$ where $SM[d_n]$ is the complex-valued point in the signal constellation corresponding to data symbol d_n .

The modulation symbols are used to modulate the carrier symbol for transmission over the ISI channel.

2.3 Discrete-time ISI Channel Model

The discrete-time ISI channel model incorporates the transmitter's pulse shaping filter and modulator, the analog ISI channel, and the receiver's demodulator, receive filter and symbol-rate signal sampler, as shown in Figure 2-3.

Figure 2-3. Discrete-time ISI channel model.

2.3.1 Pulse shaping filter

To permit transmission over an analog channel, the complex equivalent lowpass signal

$$v(t) = \sum_{n=0}^{N_c-1} v_n h_T(t - nT_s) \quad (2-1)$$

is generated, where $h_T(t)$ is the desired pulse shape, and T_s is the symbol period. To limit the spectral utilization of the transmitted signal, a band-limited pulse shape is often used, such as that produced by a root-raised-cosine filter, which has a frequency response of

$$H_T(f) = \begin{cases} 1, & \text{if } |f| < \frac{(1-\alpha)}{2T_s} \\ \cos\left(\frac{T_s}{2\alpha} \left[|f| - \frac{(1-\alpha)}{2T_s} \right]\right), & \text{if } \frac{(1-\alpha)}{2T_s} \leq |f| \leq \frac{(1+\alpha)}{2T_s}, \\ 0, & \text{if } |f| > \frac{(1+\alpha)}{2T_s} \end{cases} \quad (2-2)$$

where α is called the roll-off factor. It indicates how much bandwidth is being used over the ideal minimum bandwidth of $1/T_s$.

2.3.2 Modulator

The baseband signal is transmitted over the physical channel with a carrier frequency of f_c , so the transmitted passband signal is

$$v_c(t) = \text{Re}\{v(t)\sqrt{2}e^{j2\pi f_c t}\}. \quad (2-3)$$

2.3.3 Physical ISI channel

The transmitted signal travels over multiple propagation paths to the receiver with different attenuation factors and propagation delays along each path. The received signal is modeled as

$$r_c(t) = \text{Re} \left\{ \sum_{i=1}^{N_p} \beta_i v(t - \tau_i) \sqrt{2} e^{j2\pi f_c t} \right\} + w_c(t), \quad (2-4)$$

where β_i represents the attenuation and phase rotation along path i , and τ_i is the corresponding propagation delay. N_p is the number of propagation paths, and $w_c(t)$ represents the additive noise. For each path, β_i is modeled as a complex Gaussian random variable, with zero mean and a variance of $\sigma_i^2 = E[|\beta_i|^2]$. In this thesis we assume that perfect knowledge of β_i and τ_i is available to the receiver as side information. Both σ_i^2 , τ_i and N_p are parameters of the specific channel model, some examples of which are given in Chapter 4. The noise, $w_c(t)$, is modeled as a zero mean stationary white Gaussian random process, with a single-sided power spectral density of N_0 .

2.3.4 Demodulator

The demodulated signal is

$$\begin{aligned} r_o(t) &= r_c(t) \sqrt{2} e^{-j2\pi f_c t} \\ &= \frac{1}{2} \left(\sum_{i=1}^{N_p} \left[\beta_i v(t - \tau_i) \sqrt{2} e^{j2\pi f_c t} + \beta_i^* v^*(t - \tau_i) \sqrt{2} e^{-j2\pi f_c t} \right] \right) \sqrt{2} e^{-j2\pi f_c t} \\ &= \sum_{i=1}^{N_p} \beta_i v(t - \tau_i) + \sum_{i=1}^{N_p} \beta_i^* v^*(t - \tau_i) e^{-j4\pi f_c t} + w_o(t) \\ &= \sum_{i=1}^{N_p} \beta_i v(t - \tau_i) + \sum_{i=1}^{N_p} \beta_i^* v^*(t - \tau_i) e^{-j4\pi f_c t} + w_o(t) \end{aligned} \quad (2-5)$$

where $w_o(t)$ is the demodulated noise process. The first term in the right-hand side of the Equation (2-5) is the desired data-bearing signal, and the second is a high-frequency

component that will be suppressed by the receive filter, therefore the second term in Equation (2-5) will not be considered.

2.3.5 Receive filter

In this communication system, the receive filter, with impulse response $h_R(t)$, is matched to the transmitted pulse shape, so that $h_R(t) = h_T^*(-t)$. The filtered received signal is

$$\begin{aligned}
 r(t) &= \int_{-\infty}^{+\infty} r_o(t-\tau)h_R(\tau)d\tau \\
 &= \int_{-\infty}^{+\infty} \sum_{i=1}^{N_p} \beta_i v(t-\tau-\tau_i)h_R(\tau)d\tau + \int_{-\infty}^{+\infty} w_o(t-\tau)h_R(\tau)d\tau \\
 &= \sum_{i=1}^{N_p} \beta_i \int_{-\infty}^{+\infty} v(t-\tau-\tau_i)h_R(\tau)d\tau + w(t) \\
 &= \sum_{i=1}^{N_p} \beta_i \int_{-\infty}^{\infty} \sum_{n=0}^{N_c-1} v_n h_T(t-\tau-\tau_i-nT_s)h_R(\tau)d\tau + w(t) \\
 &= \sum_{n=0}^{N_c-1} v_n \sum_{i=1}^{N_p} \beta_i h_{TR}(t-nT_s-\tau_i) + w(t), \tag{2-6}
 \end{aligned}$$

where $h_{TR}(t) = \int_{-\infty}^{+\infty} h_T(t-\tau)h_R(\tau)d\tau$ is the combined impulse response of the transmit and

receive filters and $w(t) = \int_{-\infty}^{+\infty} w_o(t-\tau)h_R(\tau)d\tau$ is the filtered noise.

2.3.6 Symbol-rate symbol sampler

The received signal is sampled at the symbol rate, with a timing offset of $\tau_0 = \min_i \tau_i$. The received samples are given by

$$\begin{aligned}
r_n &= r(nT_s + \tau_0) \\
&= \sum_{m=0}^{N_c-1} v_m \sum_{l=1}^{N_p} \beta_l h_{lTR}(nT_s - mT_s - \tau_l + \tau_0) + w(nT_s + \tau_0) \\
&= \sum_{l=-\infty}^{+\infty} h_l v_{n-l} + w_n,
\end{aligned} \tag{2-7}$$

where

$$h_l = \sum_{i=1}^{N_p} \beta_i h_{iTR}(lT_s - (\tau_i - \tau_0)) \tag{2-8}$$

are the equivalent discrete-time ISI channel coefficients, and $\{w_n\}$ are the noise samples, which are independent complex Gaussian random variables with zero mean and $E[|w_n|^2] = N_0$ if $h_R(t)$ is such that $\int_{-\infty}^{+\infty} |h_R(t)|^2 dt = 1$.

In practice, the channel delay spread is finite, and $h_{lTR}(t)$ is typically truncated to some finite duration. As such, we can assume that h_l is non-zero only for $l \in \{0, 1, \dots, L\}$, for some L . Therefore Equation (2-7) can be rewritten as

$$r_n = \sum_{l=0}^L h_l v_{n-l} + w_n. \tag{2-9}$$

It is clear from Equation (2-9) that each transmitted symbol affect $L+1$ different received samples, and that each received sample depends on $L+1$ different transmitted symbols. Intersymbol interference is therefore present. The received samples are passed to rest of the receiver for further processing.

2.4 Receiver

The receiver is made up of an equalizer, a deinterleaver and a decoder as shown in Figure 2-4.

2.4.1 Equalizer

Figure 2-4. Structure of the receiver.

To remove the ISI, several different methods of equalization have been proposed, such as LE, DFE, MLSE and MAP equalization. Both MLSE and MAP equalization are optimal techniques but have high complexity, so LE and DFE are generally used in practice. In this thesis we study turbo equalization, and since it has been shown that LE generally gives better performance than DFE when used with turbo equalization [23], with lower complexity, only LE is considered in this thesis.

The linear equalizer employs a transversal filter. Although in theory it is necessary to use an infinite number of coefficients to remove all the ISI, in practice the number of coefficients are generally chosen to be finite, and depends on the length of channel impulse response.

For a transversal filter with $2K+1$ taps, the equalizer output is

$$z_n = \sum_{k=-K}^{+K} p_k r_{n-k} \quad (2-10)$$

where $\{p_k\}$ are the filter coefficients (tap weights). The filter coefficients are typically chosen according to the minimum mean square error (MMSE) criterion. That is, $\{p_k\}$

are chosen to minimize the mean of the square of the error between the equalizer output and the corresponding transmitted symbol, $E[|z_n - v_n|^2]$. Under this criterion the filter coefficients are given as the solutions to the $2K+1$ equations [8].

$$\sum_{k=-K}^K p_k \left[\sigma_v^2 \sum_{a=0}^L h_a^* h_{l-k+a} + N_0 \delta_{l-k} \right] = \sigma_v^2 h_{-l}^*, \quad (2-11)$$

for $l \in \{-K, -K+1, \dots, K-1, K\}$, where $\delta_{\bullet}$ is the Kronecker delta, and $\sigma_v^2 = E[|v_n|^2]$ is the average transmitted energy per symbol. In Equation (2-11) we define $h_l = 0$ for $l \notin \{0, 1, \dots, L\}$.

2.4.2 Deinterleaver

The equalized samples $\mathbf{z} = [z_0 \ z_1 \ \dots \ z_{N_c-1}]$ are fed into a symbol deinterleaver, which rearranges the order of the samples so that they are in the same order as the original code symbols. The output of the deinterleaver is $\mathbf{y} = [y_0 \ y_1 \ \dots \ y_{N_c-1}]$, with each sample given by $y_n = z_{IL^{-1}[n]}$ where $IL^{-1}[\bullet]$ represents the deinterleaver mapping, which is the inverse permutation of the selected permutation used in the transmitter (that is $IL^{-1}[IL(n)] = n$).

2.4.3 MAP Decoder

The convolutional code is decoded using a maximum *a posteriori* probability (MAP) decoder, which is based on minimizing the probability of a message bit error. The decoder calculates the *a posteriori* probabilities (APP's) $\Pr\{a_n = a | \mathbf{y}\}$ for all $n \in \{0, 1, \dots, N_a - 1\}$ and $a \in \{0, 1, \dots, 2^{n_a} - 1\}$. The decoder output for each message symbol is the possible value of a that maximizes the APP's. That is

$$\hat{a}_n = \arg \max_a \Pr\{a_n = a \mid \mathbf{y}\}. \quad (2-12)$$

Calculation of the APP's is performed efficiently using the APP algorithm by Chang & Hancock [24], which is described in the context of decoding convolutional codes by Bahl et al. [25].

Implementation of this algorithm involves the use of branch metrics, which are the likelihood functions for each deinterleaved equalizer output symbol for each of the possible transmitted code symbols, $\gamma_n(c) = f(y_n \mid c_n = c)$, where y_n is the output of the deinterleaver after equalizer. Taking into account the symbol interleaving at the transmitter, and the deinterleaver at the receiver, we have

$$\gamma_n(c) = f(z_{IL^{-1}[n]} \mid d_{IL^{-1}[n]} = c) = f(z_{IL^{-1}[n]} \mid v_{IL^{-1}[n]} = SM[c]) \quad (2-13)$$

To calculate the branch metrics we therefore need an expression for $f(z_n \mid v_n = v)$. Under the typical assumption that the residual ISI after equalization has a Gaussian distribution, we have

$$f(z_n \mid v_n = v) = \frac{1}{\pi\sigma_z^2} \exp\left\{-\frac{1}{\sigma_z^2} |z_n - \mu_z(v)|^2\right\} \quad (2-14)$$

where the mean is

$$\mu_z(v) = E[z_n \mid v_n = v] \quad (2-15)$$

and the variance is

$$\sigma_z^2 = E[|z_n - \mu_z(v)|^2 \mid v_n = v]. \quad (2-16)$$

For the linear equalizer described in Section 2.4.1, we have

$$\mu_z(v) = v \sum_{k=-K}^K p_k h_{-k} \quad (2-17)$$

and

$$\sigma_z^2 = \sigma_v^2 \left(\sum_{k=-K}^K p_k h_{-k} \right)^* \left(1 - \sum_{k=-K}^K p_k h_{-k} \right). \quad (2-18)$$

where $\sigma_v^2 = E[|v_n|^2]$ is the average transmitted energy per symbol.

2.5 Frequency Domain Linear Equalization

For channels with long delay spread, the complexity of the time-domain linear equalizer can be quite large since a large number of filter taps are required (the number of taps must be several times greater than the length of the channel impulse response). In such situations it was shown by Falconer et al. that complexity can be reduced by implementing the equalizer in the frequency domain [29].

A block diagram of a frequency domain linear equalizer (FD-LE) is shown in Figure 2.5. The received samples, r_n are converted to the frequency domain with a fast Fourier transform (FFT). The frequency domain samples R_k are multiplied by the filter coefficients P_k , giving Z_k . These are transformed back to the time domain using an inverse FFT, giving the equalizer output samples z_n . The optimal filter coefficients are found according the MMSE criterion, and are given by [29]

Figure 2-5. Frequency domain linear equalizer.

$$P_k = \frac{\sigma_v^2 H_k^*}{\sigma_v^2 |H_k|^2 + N_0} \quad \forall k \in \{0, 1, \dots, N-1\}, \quad (2-19)$$

where

$$H_k = \sum_{n=0}^{N-1} h_n e^{-j2\pi kn/N} \quad (2-20)$$

are the Discrete Fourier Transform (DFT) of the discrete-time ISI channel impulse response, and $N = N_c$ is the DFT block length. In this thesis, it is assumed that a radix-2 FFT is used to compute the DFT. These coefficients are only optimal under the assumption that the ISI results from a circular convolution of the transmitted data symbols with the channel impulse response, instead of a linear convolution as is actually the case. To support this assumption it is necessary for the transmitter to place a “cyclic-prefix” at the beginning of each block, whereby the last L symbols of the block, $[v_{N-L} \ v_{N-L+1} \ \dots \ v_{N-1}]$, are duplicated and transmitted immediately prior to the transmission of the block. By doing so, the linear convolution of the channel is equivalent to a circular convolution.

After the inverse FFT the equalized samples are deinterleaved and passed to the MAP decoder. This processing is done in the same manner as for the time-domain linear equalization, except that when calculating the branch metrics according to Equation (2-14), the mean and variance are given by (as shown in Appendix A),

$$\mu_z(v) = vq_0 \quad (2-21)$$

and

$$\sigma_z^2 = \sigma_v^2 q_0^* (1 - q_0) \quad (2-22)$$

$$\text{with } q_0 = \frac{1}{N} \sum_{k=0}^{N-1} P_k H_k. \quad (2-23)$$

Chapter 3

FREQUENCY DOMAIN TURBO EQUALIZATION

3.1 Introduction to Turbo Equalization

The receiver structures presented in Sections 2.4 and 2.5 are inherently suboptimal because they treat the equalization and decoding as two separate tasks. An optimal receiver requires that the ISI and channel coding be treated as a single code, jointly decoded. When interleaving is used, such an optimal receiver has an enormous complexity.

Motivated by the discovery of turbo codes in 1993 [1], turbo equalization was proposed in 1995 by Douillard et al. [7]. Based on the use of a MAP equalizer and a MAP decoder through an iterative decoding process, its performance has proved to be near optimum in eliminating the effect of ISI.

However, since the complexity of the MAP equalizer is still very high, particularly for channels with long delay spreads, the application of this structure is limited. As a consequence, it is necessary to find more practical alternative methods to reduce the prohibitive complexity. An approach based on an interference canceller instead of the MAP equalizer was proposed in 1997 by Glavieux et al. [17], and shown to have significantly reduced complexity. In this thesis, IC-based turbo equalization is studied further.

3.1.1 Interference Canceller

A block diagram of an IC is shown in Figure 3-1, and the output is given by

$$z_n = \sum_{k=-L}^0 p_k r_{n-k} - \sum_{\substack{k=-L \\ k \neq 0}}^L q_k v_{n-k} , \quad (3-1)$$

where the filter coefficients, $\{p_k\}$ and $\{q_k\}$, are determined according to the MMSE criterion to be [17]

$$p_k = \frac{\sigma_v^2 h_{-k}^*}{\sigma_v^2 E_h + N_0} \quad (3-2)$$

and

$$q_k = \sum_{l=-L}^0 p_l h_{k-l} , \quad (3-3)$$

where $E_h = \sum_{l=0}^L |h_l|^2$ is the overall channel attenuation.

By substituting Equation (2-9) for r_{n-k} in Equation (3-1), it is clear that

$$z_n = \frac{\sigma_v^2 E_h}{\sigma_v^2 E_h + N_0} v_n + \sum_{k=-L}^0 p_k w_{n-k} , \quad (3-4)$$

Figure 3-1. Block diagram of an interference canceller.

so the ISI is completely removed. Of course, use of Equation (3-1) is only possible if $v_{n-L}, \dots, v_{n-1}, v_{n+1}, \dots, v_{n+L}$ are known, which is not the case in practice. Fortunately, this problem can be addressed by using turbo equalization.

3.1.2 IC-based turbo equalization

A block diagram of an IC-based turbo equalizer is shown in Figure 3-2. The received samples are first processed using a linear equalizer, as described in Section 2.4.1. The equalizer output samples are deinterleaved and used as input to the MAP decoder which decodes the convolutional code. This processing is the same as described in Section 2.4. However, the MAP decoder is modified slightly to produce not only the *a posteriori* probabilities (APP's) of the message symbols, but also the APP's of the code symbols,

$$\varphi_n(c) = \Pr\{c_n = c \mid \mathbf{y}\}, \quad (3-5)$$

for $n \in \{0, 1, \dots, N_c - 1\}$ and $c \in \{0, 1, \dots, 2^{n_c} - 1\}$.

Figure 3-2. Block diagram of IC-based turbo equalizer.

The code symbol APP's are interleaved using the same permutation selected at the transmitter, giving

$$\phi_n(d) = \varphi_{ll[n]}(d), \quad (3-6)$$

for $d \in \{0, 1, \dots, 2^{n_c} - 1\}$. The interleaved APP's are then used to generate soft symbols according to

$$\bar{v}_n = \sum_{d=0}^{M-1} SM[d] \phi_n(d). \quad (3-7)$$

These soft symbols are estimates of the transmitted symbols, generated at the receiver based on knowledge of the transmitted data as determined by the MAP decoder.

By using the soft symbol $\bar{v}_{n-k}$ in place of v_{n-k} in Equation (3-1), the interference canceller can be implemented. Although the transmitted symbols are not known, if the soft symbols are reasonably accurate estimates, the IC will be reasonably effective at removing the ISI.

The IC is applied to the received data, and the IC output, $\{z_n\}$, is deinterleaved and passed to the MAP decoder. When calculating the branch metrics based on the IC output, Equation (2-14) is used, with the sample mean and variance given by (as shown in Appendix A)

$$\mu_z(v) = vq_0 \quad (3-8)$$

and

$$\sigma_z^2 = vq_0^*(1 - q_0), \quad (3-9)$$

where $q_0 = \frac{\sigma_v^2 E_h}{\sigma_v^2 E_h + N_0}$.

The MAP decoder output, $\varphi_n(c)$, could again be interleaved and fed back to the IC, and the process could be repeated in an iterative fashion. With each iteration the soft symbols become more accurate estimates of v_n , so the IC become more effective at eliminating the ISI. Turbo equalization is therefore a very useful technique for processing signals corrupted by ISI [17].

3.2 Turbo Equalization Using Frequency Domain IC

From Equation (3-1) the output of the time-domain IC is

$$\begin{aligned} z_n &= \sum_{k=-L}^0 p_k r_{n-k} - \sum_{\substack{k=-L \\ k \neq 0}}^L q_k v_{n-k} \\ &= \sum_{k=-L}^0 p_k r_{n-k} - \sum_{k=-L}^L q_k v_{n-k} + q_0 v_n. \end{aligned} \quad (3-10)$$

If the two filters were implemented in the frequency domain, the output of the IC would be of the form

$$z_n = \frac{1}{N} \sum_{k=0}^{N-1} (P_k R_k - Q_k V_k) e^{j2\pi kn/N} + q_0 v_n, \quad (3-11)$$

where $\{R_k\}$ and $\{V_k\}$ are the DFT's of $\{r_n\}$ and $\{v_n\}$, respectively, and $N = N_c$ is the DFT block length. By applying the MMSE criterion, it is shown in Appendix A that the optimal filter coefficients are

$$P_k = \frac{\sigma_v^2 H_k^*}{\sigma_v^2 E_h + N_0}, \quad (3-12)$$

$$Q_k = P_k H_k = \frac{\sigma_v^2 |H_k|^2}{\sigma_v^2 E_h + N_0}, \quad (3-13)$$

and

$$q_0 = \frac{\sigma_v^2 E_h}{\sigma_v^2 E_h + N_0}, \quad (3-14)$$

where $H_k = \sum_{n=0}^{N-1} h_n e^{-j2\pi kn/N}$ is the DFT of the channel coefficients.

Equation (3-11) can be manipulated slightly to be

$$\begin{aligned} z_n &= \frac{1}{N} \sum_{k=0}^{N-1} (P_k R_k - Q_k V_k) e^{j2\pi kn/N} + \frac{1}{N} \sum_{k=0}^{N-1} q_0 V_k e^{j2\pi kn/N} \\ &= \frac{1}{N} \sum_{k=0}^{N-1} (P_k R_k - (Q_k - q_0) V_k) e^{j2\pi kn/N} \\ &= \frac{1}{N} \sum_{k=0}^{N-1} (P_k R_k - \tilde{Q}_k V_k) e^{j2\pi kn/N}, \end{aligned} \quad (3-15)$$

where

$$\tilde{Q}_k = Q_k - q_0 = \frac{\sigma_v^2 |H_k|^2}{\sigma_v^2 E_h + N_0} - \frac{\sigma_v^2 E_h}{\sigma_v^2 E_h + N_0} = \frac{\sigma_v^2 (|H_k|^2 - E_h)}{\sigma_v^2 E_h + N_0}. \quad (3-16)$$

Turbo equalization is performed as described in Section 3.1, except that the interference canceller shown in Figure 3-2 is implemented in the frequency domain as in Figure 3-3.

When calculating the branch metrics according to Equation (2.14), the MCU uses (as shown in Appendix A), a mean of

$$\mu_z(v) = q_0 v \quad (3-17)$$

and a variance of

$$\sigma_z^2 = \sigma_v^2 q_0^* (1 - q_0), \quad (3-18)$$

$$\text{where } q_0 = \frac{\sigma_v^2 E_h}{\sigma_v^2 E_h + N_0}.$$

Figure 3-3. Structure of the frequency domain IC.

3.3 Reduced Complexity Hybrid Frequency/Time Domain IC

3.3.1 Motivation

As will be discussed in Section 3.4, implementation of the IC in the frequency domain instead of the time domain leads to a substantial reduction in the system complexity. However, careful examination of Figures 3-2 and 3-3 reveals that one FFT and one IFFT operation are required for each iteration of the turbo equalizer. Since FFT's are not trivial operations, it is desirable to avoid their use within the iterative loop. Let us review the coefficients of the q -filter in the time domain. Observing Equation (3-3), if most of values of h_k are zero, then most of the q_k coefficients will be zero. Similarly, if most of values of h_k are small, then most of the q_k coefficients will be near to zero. In other words, the energy of some taps will be much greater than the total energy of the rest of the taps. In this situation, the system can be simplified so that

Figure 3-4. Structure of the hybrid IC.

only the few largest q_k are applied and the rest of the q_k are ignored. In such case the q -filter can be implemented in the time domain with reasonable complexity.

Therefore, the proposed hybrid frequency /time domain IC shown in Figure 3-4 is introduced. The P -filter still operates in the frequency domain but the q -filter operates in the time domain. In the proposed model, the P -filter and q -filter will not have the same coefficients as presented in Equation (3-12) and (3-3). These coefficients will depend on how many q -filter taps are chosen to use. If the numbers of taps are chosen equal to the channel delay spread, the P -filter and q -filter will have the same values as (3-12) and (3-3), respectively.

3.3.2 Coefficients of the hybrid IC

The hybrid IC is made up of two parts, the P -filter operating in frequency domain and the q -filter operating in the time domain. The output of the IC is

$$z_n = \frac{1}{N} \sum_{k=0}^{N-1} P_k R_k e^{j2\pi kn/N} - \sum_{i=1}^{N_t} (q_{M_i} v_{n-M_i} + q_{-M_i} v_{n+M_i}), \quad (3-19)$$

where N_l is the number of feedback tap pairs and $\{M_i \mid i \in \{1, 2, \dots, N_l\}\}$ are the indices of the desired feedback taps. The system performance will depend on N_l , in that larger N_l should provide better performance, but with higher computational complexity. Typically N_l is less than or equal to 3. The N_l largest values of q_k and their positions can be found using Equation (3-3). These chosen q_k are rearranged in ascending order according to the values and give them to q_{M_i} for $\{M_i \mid i \in \{1, 2, \dots, N_l\}\}$.

Using the mean square error criterion to get the optimum coefficients, it is shown in Appendix B that

$$P_l = \frac{H_l^* \sigma_v^2}{|H_l|^2 \sigma_v^2 + N_0} \left[1 + \sum_{i=1}^{N_l} (q_{M_i} e^{-j2\pi M_i l / N} + q_{-M_i} e^{j2\pi M_i l / N}) \right] \quad (3-20)$$

and

$$q_m = \frac{1}{N} \sum_{k=0}^{N-1} P_k H_k e^{j2\pi km / N} \quad (3-21)$$

for $l \in \{0, 1, \dots, N-1\}$ and $m \in \{M_1, M_2, \dots, M_{N_l}\}$. Substituting Equation (3-20) into (3-21) gives

$$\begin{aligned} q_m &= \frac{1}{N} \sum_{k=0}^{N-1} \frac{H_k^* \sigma_v^2}{|H_k|^2 \sigma_v^2 + N_0} \left[1 + \sum_{i=1}^{N_l} (q_{M_i} e^{-j2\pi M_i k / N} + q_{-M_i} e^{j2\pi M_i k / N}) \right] H_k e^{j2\pi km / N} \\ &= x_m + \sum_{i=1}^{N_l} (q_{M_i} x_{m-M_i} + q_{-M_i} x_{m+M_i}), \end{aligned} \quad (3-22)$$

where

$$x_m = \frac{1}{N} \sum_{k=0}^{N-1} \frac{|H_k|^2 \sigma_v^2}{|H_k|^2 \sigma_v^2 + N_0} e^{j2\pi km / N} \quad (3-23)$$

is the inverse DFT of

$$X_k = \frac{|H_k|^2 \sigma_v^2}{|H_k|^2 \sigma_v^2 + N_0}. \quad (3-24)$$

By defining

$$\mathbf{q} = [q_{M_1} \cdots q_{M_{N_t}} \quad q_{-M_1} \cdots q_{-M_{N_t}}]^T, \quad (3-25)$$

$$\mathbf{x} = [x_{M_1} \cdots x_{M_{N_t}} \quad x_{-M_1} \cdots x_{-M_{N_t}}]^T, \quad (3-26)$$

and

$$\mathbf{C} = \begin{bmatrix} x_{M_1-M_1} & x_{M_1-M_2} & \cdots & x_{M_1-M_{N_t-1}} & x_{M_1-M_{N_t}} & x_{M_1+M_1} & x_{M_1+M_2} & \cdots & x_{M_1+M_{N_t-1}} & x_{M_1+M_{N_t}} \\ \vdots & & & & & & & & & \\ x_{M_{N_t}-M_1} & x_{M_{N_t}-M_2} & \cdots & x_{M_{N_t}-M_{N_t-1}} & x_{M_{N_t}-M_{N_t}} & x_{M_{N_t}+M_1} & x_{M_{N_t}+M_2} & \cdots & x_{M_{N_t}+M_{N_t-1}} & x_{M_{N_t}+M_{N_t}} \\ x_{-M_1-M_1} & x_{-M_1-M_2} & \cdots & x_{-M_1-M_{N_t-1}} & x_{-M_1-M_{N_t}} & x_{-M_1+M_1} & x_{-M_1+M_2} & \cdots & x_{-M_1+M_{N_t-1}} & x_{-M_1+M_{N_t}} \\ \vdots & & & & & & & & & \\ x_{-M_{N_t}-M_1} & x_{-M_{N_t}-M_2} & \cdots & x_{-M_{N_t}-M_{N_t-1}} & x_{-M_{N_t}-M_{N_t}} & x_{-M_{N_t}+M_1} & x_{-M_{N_t}+M_2} & \cdots & x_{-M_{N_t}+M_{N_t-1}} & x_{-M_{N_t}+M_{N_t}} \end{bmatrix}, \quad (3-27)$$

Equation (3-22) can be written as

$$\mathbf{q} = \mathbf{x} + \mathbf{C}\mathbf{q}, \quad (3-28)$$

To simplify the calculation of $\mathbf{C}$, $\mathbf{C}$ can be expressed as

$$\mathbf{C} = \begin{bmatrix} \mathbf{F} & \mathbf{E} \\ \mathbf{E}^* & \mathbf{F}^* \end{bmatrix}, \quad (3-29)$$

where

$$\mathbf{E} = \begin{bmatrix} x_{M_1+M_1} & x_{M_1+M_2} & \cdots & x_{M_1+M_{N_t-1}} & x_{M_1+M_{N_t}} \\ \vdots & & & & \\ x_{M_{N_t}+M_1} & x_{M_{N_t}+M_2} & \cdots & x_{M_{N_t}+M_{N_t-1}} & x_{M_{N_t}+M_{N_t}} \end{bmatrix} \quad (3-30)$$

and

$$\mathbf{F} = \begin{bmatrix} x_{M_1-M_1} & x_{M_1-M_2} & \cdots & x_{M_1-M_{N_t-1}} & x_{M_1-M_{N_t}} \\ \vdots & & & & \\ x_{M_{N_t}-M_1} & x_{M_{N_t}-M_2} & \cdots & x_{M_{N_t}-M_{N_t-1}} & x_{M_{N_t}-M_{N_t}} \end{bmatrix}. \quad (3-31)$$

By defining $\mathbf{q}_1 = [q_{M_1} \cdots q_{M_{N_t}}]^T$, $\mathbf{q}_2 = [q_{-M_1} \cdots q_{-M_{N_t}}]^T$, $\mathbf{x}_1 = [x_{M_1} \cdots x_{M_{N_t}}]^T$ and $\mathbf{x}_2 = [x_{-M_1} \cdots x_{-M_{N_t}}]^T$, then Equation (3-28) can be rewritten as

$$\begin{bmatrix} \mathbf{I} - \mathbf{F} & -\mathbf{E} \\ -\mathbf{E}^* & \mathbf{I} - \mathbf{F}^* \end{bmatrix} \begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}. \quad (3-32)$$

From Equation (3-32), we obtain

$$\mathbf{q}_1 = \left[(\mathbf{I} - \mathbf{F}) - \mathbf{E}(\mathbf{I} - \mathbf{F}^*)^{-1} \mathbf{E}^* \right]^{-1} \left[\mathbf{x}_1 + \mathbf{E}(\mathbf{I} - \mathbf{F}^*)^{-1} \mathbf{x}_2 \right] \quad (3-33)$$

and

$$\mathbf{q}_2 = \left[(\mathbf{I} - \mathbf{F}^*) - \mathbf{E}^*(\mathbf{I} - \mathbf{F})^{-1} \mathbf{E} \right]^{-1} \left[\mathbf{E}^*(\mathbf{I} - \mathbf{F})^{-1} \mathbf{x}_1 + \mathbf{x}_2 \right]. \quad (3-34)$$

Since it can be proved that $q_{-m} = q_m^*$ according to Equation (3-21), actually, calculating either $\mathbf{q}_1$ or $\mathbf{q}_2$ is sufficient. The P-filter coefficients can be calculated according to Equation (3-20).

3.3.3 Mean and Variance of the IC output

When calculating the branch metrics according to Equation (2-14), the MCU uses (as shown in Appendix B), a mean of

$$\mu_z(v) = q_0 v \quad (3-35)$$

and a variance of

$$\sigma_z^2 = \sigma_v^2 q_0^* (1 - q_0) \quad (3-36)$$

where q_0 is given by Equation (3-21).

3.4 Complexity Comparison

In this section we analyze the complexity of various components of the equalizer, including the time-domain linear equalizer (TD-LE), frequency domain linear equalizer (FD-LE), time-domain IC (TD-IC), frequency-domain IC (FD-IC) and hybrid IC, in terms of the number of multiplications and additions. In this thesis, calculating complexities for the LE, the FD-IC and the hybrid reduced complexity IC is based on the assumption that a radix-2 FFT is used to compute the DFT. The application of FFT is a consideration of its speed advantage over DFT. The number of complex multiplications in a direct DFT implementation is of the order of N^2 , where N is the DFT block length. If N is an integer power of 2 and a radix-2 FFT is used, then only $\frac{N}{2} \log_2 N$ complex multiplications are needed. If only considering the multiplication factor, it is obvious that the speed advantage is approximately:

$$\frac{N^2}{\frac{N}{2} \log_2 N} = \frac{2N}{\log_2 N}$$

As the transform length increases, the advantage will be more obvious. For instance, when $N = 16$, the ratio is 8; when $N = 1024$, the ratio will be approximately 205.

One FFT needs $\frac{1}{2} \log_2 N$ complex multiplications and $\log_2 N$ additions per symbol. One complex multiplication requires four real multiplications and two additions. The numbers of real multiplications and real additions for one FFT is shown in Table 3-1, where CM , CA , RM and RA stand for complex multiplication, complex addition, real multiplication and real addition, respectively.

	RM	RA
$\frac{1}{2}\log_2 N \times CM$	$2\log_2 N$	$\log_2 N$
$\log_2 N \times CA$		$2\log_2 N$
total per FFT	$2\log_2 N$	$3\log_2 N$

Table 3-1. Complexity per symbol of one FFT.

	RM	RA
$(2K + 1) \times CM$	$4(2K + 1)$	$2(2K + 1)$
$(2K) \times CA$		$2(2K)$
total	$8K + 4 (= 32L + 4)$	$8K + 2 (= 32L + 2)$

Table 3-2. Complexity per symbol of the TD-LE.

	RM	RA
$2 \times \text{FFT}$	$4\log_2 N$	$6\log_2 N$
$1 \times CM$	4	2
total	$4\log_2 N + 4$	$6\log_2 N + 2$

Table 3-3. Complexity per symbol of the FD-LE.

3.4.1 Complexity of LE

In the time domain, we assume that the total number of taps of the LE is $2K + 1$, with $K = 4L$, typically, where L is channel impulse response length. The complexity is shown in Table 3-2. In the frequency domain one FFT and one IFFT are needed, as well as one complex multiplication per symbol to apply the filter coefficients. Its computational complexity per symbol is shown in Table 3-3.

3.4.2 Complexity of IC

For the IC in the time domain, the p -filter has $L+1$ taps and the q -filter has $2L$ taps. One extra addition per symbol is required to subtract the q -filter output from the p -filter output. The complexity is shown in Table 3-4 and 3-5. In the frequency domain, the P -filter and the Q -filter each require one FFT and one complex multiplication to apply the filter coefficients. A complex addition is required to calculate the difference between the two filters, and an IFFT is needed to convert back to the time domain. The complexity is shown in Table 3-6 and 3-7.

To get an idea of the total amount of work that must be done by the receiver for the purposes of equalization for a given number of iterations, note that a LE (either time or frequency domain) is always used for the first iteration, and an IC (either time or frequency domain) is always used on the second and subsequent iterations. Since the output of the p -filter (P -filter) does not change with each iteration, its output need only be calculated once, in the second iteration, and stored for reuse in the third and subsequent iterations. For $N_i \geq 2$, where N_i is the number of iteration, the total equalization complexity is as shown in Table 3-8 and 3-9 for the time and frequency domain, respectively.

		RM	RA
p -filter	$(L+1) \times CM$	$4(L+1)$	$2(L+1)$
	$L \times CA$		$2L$
total		$4L+4$	$4L+2$

Table 3-4. Complexity of the p -filter in the TD-IC.

		RM	RA
q -filter	$(2L) \times CM$	$4(2L)$	$2(2L)$
	$(2L-1) \times CA$		$2(2L-1)$
extra CA			2
total		$8L$	$8L$

Table 3-5. Complexity of the q -filter in the TD-IC.

		RM	RA
P -filter	$1 \times \text{FFT}$	$2\log_2 N$	$3\log_2 N$
	$1 \times CM$	4	2
total		$2\log_2 N + 4$	$3\log_2 N + 2$

Table 3-6. Complexity of the P -filter in the FD-IC.

		RM	RA
q -filter	$1 \times \text{FFT}$	$2 \log_2 N$	$3 \log_2 N$
	$1 \times CM$	4	2
extra CA	$1 \times CA$		2
IFFT	$1 \times \text{FFT}$	$2 \log_2 N$	$3 \log_2 N$
total		$4 \log_2 N + 4$	$6 \log_2 N + 4$

Table 3-7. Complexity of the Q -filter in the FD-IC.

	RM	RA
TD-LE	$32L + 4$	$32L + 2$
p -filter	$4L + 4$	$4L + 2$
q -filter	$8L(N_i - 1)$	$8L(N_i - 1)$
total	$8LN_i + 28L + 8$	$8LN_i + 28L + 4$

Table 3-8. Complexity of the TD-IC.

	RM	RA
FD-LE	$4 \log_2 N + 4$	$6 \log_2 N + 2$
P -filter	$2 \log_2 N + 4$	$3 \log_2 N + 2$
Q -filter	$(4 \log_2 N + 4)(N_i - 1)$	$(6 \log_2 N + 4)(N_i - 1)$
total	$(4 \log_2 N + 4)N_i + 2 \log_2 N + 4$	$(6 \log_2 N + 4)N_i + 3 \log_2 N$

Table 3-9. Complexity of the FD-IC.

3.4.3 Complexity of the hybrid IC

The complexity of the hybrid IC is dependent on the number of pairs of taps that are used in the q -filter, N_t . The complexity of the P -filter is shown in Table 3-10, and the complexity of the q -filter is shown in Table 3-11. The total equalization complexity for $N_t \geq 2$ iteration is shown in Table 3-12.

3.4.4 Comparison

The total equalization complexity for the three schemes (TD-IC, FD-IC and hybrid IC) is summarized in Table 3-13. Let us consider a numerical example. When $L = 9$, and $N_t = 5$, the complexity of TD-IC is 620 *RM* and 616 *RA*. From this example, we see that the complexity is pretty high for both multiplications and additions. If L becomes larger, the complexity will be even worse. Compare this with the complexity of FD-IC. When $N = 1024$, $N_t = 5$, the complexity of FD-IC is 244 *RM* and 350 *RA*. We can see the complexity of FD-IC is lower than TD-IC. Similarly, let us compute the complexity of the hybrid IC. When $N=1024$, $N_t = 5$ and $N_t = 2$, the complexity of the hybrid IC is 152 *RM* and 188 *RA*. We can see the complexity of the hybrid IC is lower than FD-IC, and much lower than TD-IC.

The computational complexity of the FD-IC and the hybrid IC are shown in Figure 3-5 as a function of the block size (N). It is assumed that the hybrid IC uses one pair of coefficients ($N_t = 1$) and five iterations are done ($N_t = 5$). Also shown is the complexity of TD-IC (with $L = 4$), which does not depend on the block size. Increasing L will increase the complexity of TD-IC, and increasing N_t will increase the complexity of the hybrid IC which eventually will be more complex than FD-IC.

		RM	RA
p -filter	$2 \times \text{FFT}$	$4 \log_2 N$	$6 \log_2 N$
	$1 \times CM$	4	2
total		$4 \log_2 N + 4$	$6 \log_2 N + 2$

Table 3-10. Complexity of the P -filter in the hybrid IC.

		RM	RA
q -filter	$2N_t \times CM$	$4 \times 2N_t$	$2 \times 2N_t$
	$(2N_t - 1) \times CA$		$2 \times (2N_t - 1)$
extra CA			2
total		$8N_t$	$8N_t$

Table 3-11. Complexity of the q -filter in the hybrid IC.

	RM	RA
FD-LE	$4 \log_2 N + 4$	$6 \log_2 N + 2$
P -filter	$4 \log_2 N + 4$	$6 \log_2 N + 2$
q -filter	$8N_t(N_t - 1)$	$8N_t(N_t - 1)$
total	$8 \log_2 N + 8 + 8N_t(N_t - 1)$	$12 \log_2 N + 4 + 8N_t(N_t - 1)$

Table 3-12. Complexity of the hybrid IC.

	RM	RA
TD-IC	$8LN_i + 28L + 8$	$8LN_i + 28L + 4$
FD-IC	$(4 \log_2 N + 4)N_i + 2 \log_2 N + 4$	$(6 \log_2 N + 4)N_i + 3 \log_2 N$
hybrid-IC	$8 \log_2 N + 8 + 8N_i(N_i - 1)$	$12 \log_2 N + 4 + 8N_i(N_i - 1)$

Table 3-13. Complexity comparison.

Increasing N_i will magnify the differences between the systems.

Note that in all the analysis of this section we have ignored the complexity involved in calculating the filter coefficients. In particular, calculating the coefficients for the hybrid IC can require significant effort, since an IFFT and two matrix inversions are required (although the inversion are for small $N_i \times N_i$ matrices). However, depending on the rate at which the channel coefficients are changing (the fading rate), it may only be necessary to calculate the coefficients once for every several transmitted blocks. Therefore, the additional complexity has been omitted.

Figure 3-5. Computational complexity of TD-IC, FD-IC and hybrid IC.

Chapter 4

SIMULATION RESULTS

In this chapter, simulation results for IC-based turbo equalization in the frequency domain, called FD-IC, are presented. One purpose of these simulations is to verify that operating in the frequency domain is as effective at eliminating ISI as in the time domain. The other purpose is to test the performance of the proposed hybrid reduced complexity system derived in Chapter Three.

For all simulations, message blocks of length $N_a = 1020$ binary symbols are encoded using a rate 1/2 convolutional code with generator (23, 35) (expressed in octal), giving $N_c = 1024$. The number of simulation block is more than 10000 at BER of 10^{-5} and may be decreased at BER more than 10^{-5} . The symbol interleaver uses a randomly selected permutation and the interleaved code symbols are transmitted over the discrete-time ISI channel. Root raised cosine pulse shaping is used by the transmit filter with a roll-off factor of $\alpha = 0.1$.

In this chapter, Section 4.1 investigates the performances for FD-IC over static ISI channels. Section 4.2 discusses the performance of FD-IC over Rayleigh frequency selective fading channels. Section 4.3 inspects the proposed hybrid reduced complexity IC over static ISI and Rayleigh frequency-selective fading channels.

4.1 Performance of FD-IC over Static ISI Channels

For the FD-IC turbo equalizer, the LE is employed at the first iteration. For the following iterations, the FD-IC is used. The FD-IC model is shown in Figure 3-3. The

performance curves of BER and FER versus E_b/N_0 are plotted for the system model.

To show the performance degradation due to ISI, the BER and FER of an ISI-free coded data transmission system is also plotted. Although the advantage of frequency domain equalization, in terms of complexity, is evident, primarily in channels with long delay spreads, it is useful to begin with a short delay spread channel.

4.1.1 Proakis B channel

The impulse and frequency responses of the Proakis B channel [8] are shown in Figure 4-1. As shown in Figure 4-2, the BER performance of the FD-IC improves with each iteration of the equalizer. The system suffers a 8.5 dB ISI penalty at a BER of 10^{-4} after the first iteration compared with the performance with no-ISI. After the second iteration, the ISI penalty is reduced to about 2.5 dB. Further iterative processing gives an overall improvement of 8 dB at a BER of 10^{-4} , and almost completely eliminates the ISI. In Figure 4-2 the BER performance of the TD-IC is also plotted [31], and we can see that there is little difference between the performance of TD-IC and FD-IC.

As shown in Figure 4-3, the FER performance also improves with each iteration of the equalizer. After five iterations, the ISI penalty is reduced by 8.5 dB at a FER of 10^{-2} compared to the first iteration.

4.1.2 Proakis C channel

The impulse and frequency responses of the Proakis C channel [8] are shown in Figure 4-4. As shown in Figure 4-5, the BER performance improves with each iteration of the equalizer. The system suffers a 17 dB ISI penalty at a BER of 10^{-4} after the first

(a) Impulse response.

(b) Frequency response.

Figure 4-1. Impulse response (a) and frequency response (b) of the Proakis B channel. T is the symbol duration.

Figure 4-2. BER performance of the FD-IC over the Proakis B channel.

Figure 4-3. FER performance of the FD-IC over the Proakis B channel.

iteration, which is reduced to 8.5 dB after five iterations. Unlike for the Proakis B channel, turbo equalization is not able to completely eliminate the ISI for the Proakis C channel since the Proakis C channel is much more severe than the Proakis B channel. In Figure 4-5, the BER performance of the TD-IC over Proakis C channel is also plotted and again we can see that there is little difference between the performance of TD-IC and FD-IC, except of high SNR where TD-IC appears to suffer from an error floor. As shown in Figure 4-6, performance on terms of the FER is similar to that of the BER.

4.1.3 Two-tap channel with a long delay spread

As shown in Section 3.4, time domain equalization requires many multiplication operations for channels with a long delay spread. For example, for the channel in Figure 4-7, at least several hundred multiplications are needed per symbol. The high computational complexity limits the application of time domain equalization, so it is wiser to use frequency domain equalization instead.

The impulse and frequency responses of a channel with a delay spread of 100 symbols are shown in the Figure 4-7. The results in Figure 4-8 show that the BER performance of the FD-IC improves with each iteration. The system suffers from a 3.5 dB ISI penalty at a BER of 10^{-4} after the first iteration, and after the third iteration, the ISI is completely eliminated for E_b/N_0 greater than 3 dB. Similar results hold with regard to the FER, as shown in Figure 4-9. Frequency domain turbo equalization is clearly able to handle channels with long delay spreads and with low complexity.

(a) Impulse response.

(b) Frequency response.

Figure 4-4. Impulse response (a) and frequency response (b) of the Proakis C channel.

Figure 4-5. BER performance of the FD-IC over the Proakis C channel.

Figure 4-6. FER performance of the FD-IC over the Proakis C channel.

(a) Impulse response.

(b) Frequency response.

Figure 4-7 Impulse (a) and frequency (b) response of a 2-tap channel with long delay spread.

Figure 4-8. BER performance of the FD-IC over 2-tap channel with long delay spread.

Figure 4-9. FER performance of the FD-IC over 2-tap channel with long delay spread.

4.2 Performance over Rayleigh Frequency Selective Fading Channels

In this section, we investigate the BER and FER performances of the FD-IC over Rayleigh frequency-selective fading channels, such as the SUI-5, SUI-6 and 10-tap channels.

4.2.1 SUI-5 and SUI-6 channels

The standardization of fixed broadband wireless access (BWA) systems is carried out by the IEEE 802.16 Working Group [30]. They consider six different radio channel models for BWA in three terrain categories. In this thesis, the SUI-5 channel and SUI-6 channel models with long delay spreads and with the use of omnidirectional antenna in terrain type A are considered. Their delay spread profiles are shown in Figure 4-10 and Figure 4-11. Both of these channels have three paths ($N_p=3$) with different attenuation (β_i) and delay spread (τ_i) for each path. β_i is modeled as a complex Gaussian random variable, with zero mean and a variance of $\sigma_i^2 = E[|\beta_i|^2]$. When a symbol rate of 1 Mbaud is assumed, σ_i^2 and τ_i of the SUI-5 and SUI-6 channels are as shown in Table 4-1.

The BER performance of the FD-IC with QPSK for the SUI-5 channel is shown in Figure 4-12. Although there is a 1 dB iteration gain from using two iterations instead of one, further gains are not possible and a 1 dB ISI penalty remains after five iterations compared to the matched filter bound (MFB). The MFB represents the performance of an optimal receiver for the same channel and same transmitted signal, except that the symbol rate has been drastically reduced so that ISI does not occur. The FER results in Figure 4-13 show that the FD-IC with two iterations can reach the MFB. Similar results

Figure 4-10. Delay spread profile for SUI-5 channel.

Figure 4-11. Delay spread profile for SUI-6 channel.

SUI-5	σ_i^2	0.706	0.223	0.0706
	τ_i	0	4	10
SUI-6	σ_i^2	0.877	0.0877	0.0349
	τ_i	0	14	20

Table 4-1. Parameters for the SUI-5 and SUI-6 channels.

occur with the SUI-6 channel, as shown in Figure 4-14 and 4-15. Since the effect of ISI can be eliminated in terms of the FER but not the BER, this suggests that, for frames with errors, the average number of bits in error per frame is larger than for the MFB system, although the number of frames with errors is the same. The likely cause of this is that, although the FD-IC is usually able to completely eliminate the ISI, in some cases, such as when the overall channel attenuation is large, iterative processing introduces additional errors.

For the SUI-5 channel we also investigate the performance of the FD-IC when 16-QAM is used with Ungerboeck's 16-state rate-3/4 trellis-coded modulation (TCM) code [31]. The BER results shown in Figure 4-16 are similar to that of the QPSK system, but the iteration gain is slight larger (2 dB) as is the ISI penalty after five iterations (1.5 dB). The FER results shown in Figure 4-17, on the other hand, are notably different. The ISI penalty reduces slowly with each iteration, and even after five iterations there remains a penalty of 0.5 dB.

Overall, the BER performance of the FD-IC shows that turbo equalization does not give the dramatic improvement for the SUI-5 and SUI-6 channels that was evident in the static ISI channels. Nonetheless, the ISI penalty after turbo equalization is quite small, particularly in terms of the FER. Furthermore, it is evident that only two iterations are needed to realize all the benefits of turbo equalization. The only exception is for the FER of the 16-QAM system which does show improvement with each iteration.

Figure 4-12. BER of the FD-IC for QPSK over the SUI-5 channel.

Figure 4-13. FER of the FD-IC for QPSK over the SUI-5 channel.

Figure 4-14. BER performance of the FD-IC for QPSK over the SUI-6 channel.

Figure 4-15. FER performance of the FD-IC for QPSK over the SUI-6 channel.

Figure 4-16. BER performance of the FD-IC for 16-QAM over the SUI-5 channel.

Figure 4-17. FER performance of the original FD-IC for 16-QAM over the SUI-5 channel.

4.2.2 10-tap fading channel

The 10-tap fading channel model consists of ten channel taps, each tap with randomly selected amplitude with a variance of 0.1. Referring to Section 2.3.3, there are delay spread $\tau_i = (i-1)T_s$ and variance $\sigma_i^2 = 0.1$ for $i \in \{1, 2, \dots, 10\}$. We use both QPSK and 16-QAM to investigate the BER and FER performance of the FD-IC.

The BER performance for QPSK can be seen in Figure 4-18. The performance improves with the first three iterations, but then starts to get worse. However, as Figure 4-19 shown, the FER performance continues to improve with each iteration, and after the fourth iteration, the FER performance can almost approach the MFB. This interesting result can be seen more clearly when ten iterations are used, as shown in Figure 4-20 and Figure 4-21. From Figure 4-20, the BER performance becomes worse and worse after the third iteration. Nevertheless, the FER performance still gets better and better as the number of iterations increases. A similar phenomenon can be found for 16-QAM TCM over the same 10-tap channel from the Figure 4-22 and 4-23. Although a small number of message blocks may behave very badly due to the fading channel, causing the whole BER performance to become worse, the overall number of block errors decreases by the iterative process.

In addition, these results show that the FD-IC is much better at combating the ISI when QPSK is used than when 16-QAM is. Although the iterative processing gain is comparable for the two modulation schemes, the residual ISI penalty is much worse for 16-QAM.

Figure 4-18. BER performance of the FD-IC for QPSK over 10-tap fading channel.

Figure 4-19. FER performance of the FD-IC for QPSK over 10-tap fading channel.

Figure 4-20. BER performance of the FD-IC for QPSK over 10 taps fading channel (with 10th iteration).

Figure 4-21. FER performance of the FD-IC for QPSK over 10 taps fading channel (with 10th iteration).

Figure 4-22. BER performance of the FD-IC for 16-QAM over 10-tap fading channel.

Figure 4-23. FER performance of the FD-IC for 16-QAM over 10-tap fading channel.

4.3 Performance of the Hybrid IC

In terms of complexity, the advantage of using the hybrid IC instead of the FD-IC is mostly apparent when a large number of iterations are required. However, the BER results in the previous section suggest that only two, or perhaps three, iterations are required to realize all the iterative processing gain. Nonetheless, there are cases when more iterations are beneficial, such as when the FER is of interest, or for the static Proakis B and C channels.

In this section, we investigate the BER and FER performances of the hybrid reduced complexity IC over static ISI channel, such as the Proakis B and C channels, and Rayleigh frequency-selective fading channels, such as the SUI-5 and 10-tap channels.

4.3.1 Performance over static ISI channels

Figure 4-24 and 4-25 show that BER and FER performance of the hybrid IC system with one pair of taps ($N_t = 1$) for the static Proakis B channel. Also shown is the performance of the original FD-IC after five iterations. As can be seen the hybrid IC system gives almost identical performance as the FD-IC, but with much lower complexity (102 vs. 262 multiplications, and 135 vs. 375 additions). Since $L = 2$ for this channel, increasing N_t to 2 for the hybrid IC is equivalent to using all the taps, so the performance is then identical to the FD-IC.

The BER and FER results for the Proakis C channel are shown in Figure 4-26 and 4-27 for the hybrid IC with 1, 2, 3, and all four pairs of taps, which is the same as (FD-IC). These surprising results show that better results are achieved when only one or two

Figure 4-24. BER of the hybrid IC for QPSK over the Proakis B channel.

Figure 4-25. FER of the hybrid IC for QPSK over the Proakis B channel.

Figure 4-26. BER of the hybrid IC for QPSK over the Proakis C channel.

Figure 4-27. FER of the hybrid IC for QPSK over the Proakis C channel.

taps are used instead of all of them. The hybrid IC is therefore capable of not only providing lower complexity, but also better results. We can explain this case by analysing the structure of FD-IC and the hybrid scheme. The P -filter of FD-IC enhances the SNR and the Q -filter eliminates the ISI. However, the P -filter in the hybrid scheme not only takes on the task as in the FD-IC, but also eliminates the ISI that the q -filter cannot remove. On the other hand, if there are errors from previous iteration, the P -filter in the hybrid scheme can easily handle a small number of pairs than a large number of pairs. We may also explain this by analyzing Equation (B-2). In Equation (B-2), it is assumed that v_n is fed back to the IC, although it might never happen. When the feedback v_n contains error, the optimizing q -filter for $J = E[|z_n - v_n|^2]$ is more prone to error if the filter is longer.

4.3.2 Performance over the SUI-5 channel

Figure 4-28 and 4-29 show the BER and FER of the hybrid IC with $N_t = 1$ for the SUI-5 channel with QPSK. Also shown is the performance of the original FD-IC. In terms of the BER there is very little difference between the hybrid IC and the FD-IC, but in terms of the FER the hybrid IC is about 1 dB worse. If only the BER is of interest, then it is advantageous to use the hybrid scheme since it has lower complexity, even though only two iterations are required.

4.3.3 Performance over 10-tap fading channel

The BER and FER of the hybrid IC after five iterations with one, two, three and all 9 pairs of taps are shown in Figure 4-30 and 4-31 for the 10-tap fading channel with QPSK. Increasing the number of taps leads to better performance, and there is little

difference between using 3 and 9 pairs in terms of the BER. However, in terms of the FER there is about a 0.8 dB penalty for using 3 pairs instead of all 9. There is a clear disadvantage to using fewer pairs.

When 16-QAM is used instead of QPSK, the BER results shown in Figure 4-32 indicates that there is little difference between using 1, 2, 3 pairs. Furthermore all three of these choices for the number of pairs give better BER performance than the FD-IC. This is not the case when the FER is examined, as shown I Figure 4-33. However, there is little difference between using 2, 3, or 9 pairs.

Figure 4-28. BER of the hybrid IC for QPSK over the SUI-5 channel.

Figure 4-29. FER of the hybrid IC for QPSK over the SUI-5 channel.

Figure 4-30. BER of the hybrid IC for QPSK over the 10-tap fading channel.

Figure 4-31. FER of the hybrid IC for QPSK over the 10-tap fading channel.

Figure 4-32. BER of the hybrid IC for 16-QAM TCM over the 10-tap fading channel.

Figure 4-33. FER of the hybrid IC for 16-QAM TCM over the 10-tap fading channel.

Chapter 5

CONCLUSIONS AND FUTURE RESEARCH

5.1 Summary and Discussion

The work in this thesis has investigated the performance of IC-based turbo equalization in the frequency domain over static and fading ISI channels in terms of the BER and FER. A hybrid reduced complexity IC has been proposed and its performances and computational complexity compared with that of TD-IC and FD-IC.

The FD-IC can improve the BER and FER performance over static ISI channels. For the long delay spread channel, the TD-IC requires prohibitively high complexity but the FD-IC can perform well and with low complexity.

For frequency selective fading channel, the BER performance of the FD-IC shows that turbo equalization does not give the same dramatic improvement over the SUI-5 and SUI-6 channels. However, the FER performance of FD-IC presents pretty good behaviour over these two channels. For the 10-taps fading channel, after the third iteration, the BER performance will become worse and worse, but FER still performs well. It can be found that although for a few message blocks the performance may be very poor due to the fading channel, causing the overall BER to be bad, the number of frame errors gets less by the iterative process. The FD-IC consists of two filters, the P -filter, and the Q -filter. The P -filter enhances the SNR and the Q -filter eliminates the ISI. However, if there are many errors from the previous iteration, instead of

eliminating the ISI, the Q -filter will actually introduce more, causing error propagation. Therefore, the FD-IC turbo equalization works well for FER over fading channels.

It is known that ISI not only spreads out the signal energy but also adds the interference in the symbols. Fortunately, the proposed hybrid IC can effectively solve the problem when the powers in a few channel taps are dominant. The P -filter in the hybrid scheme not only takes on the same task as in the FD-IC but also eliminates the ISI that the q -filter cannot remove. When only a small number of taps are used in the q -filter, the P -filter does more of the work in suppressing the ISI so the hybrid is less susceptible to error propagation. On the other hand, when the number of taps is small, the hybrid IC is more like a linear equalizer, and therefore less effective at suppressing the interference. These two contradictory forces govern the performance of the hybrid IC. These issues can be seen clearly from the simulation results of the Proakis B and C channels. For the Proakis B channel, the hybrid scheme can achieve the same performance as the FD-IC. For the Proakis C channel, the hybrid scheme with two pairs can perform even better than the FD-IC. So, the hybrid scheme can work well over the static channels. For the fading channels, the BER performance of hybrid scheme performs differently depending on the channel and modulation scheme. For the SUI-5 channel, the BER and the FER performances do not improve after the third iteration. The BER degradation of the hybrid IC is very small compared with the FD-IC, but the FER performance of the hybrid IC cannot reach the MFB and so is a little bit worse than the FD-IC. For the 10-tap fading channel, the hybrid scheme for QPSK and 16-QAM TCM can work well for FER but not for BER. The BER will work reasonably well as the number of pairs increase. However, in the meantime, some frame of

message block may have large or very small bit errors so that the average bit error will not be determined as FER. Although in some situation, the BER cannot improve much, but the FER can perform well even in the SUI channel. When the FER is more concerned, the turbo equalization can improve the performance dramatically.

This hybrid scheme is aimed on further reducing the complexity. Since the characteristics of frequency domain make the computational complexity lower than in the time domain, especially for longer channel delay spreads, the comparison of the computational complexity of the TD-IC, FD-IC and the hybrid scheme shows that the proposed hybrid IC can have lower complexity than both TD-IC and FD-IC.

5.2 Further Research

The transmission systems investigated so far assume that the channel impulse response is perfectly known as is the noise variance. In practice, the channel impulse response and the noise variance are seldom known in advance and are often time varying. One solution is to employ adaptive schemes to estimate the channel impulse response and the noise variance, for instance, based on transmitting training sequences. In the first iteration, the training data can be used to estimate the initial coefficients of the LE. In the following iteration, the coefficients in the IC will be updated by adaptive techniques.

In the further research, turbo codes can be considered instead of convolutional code. Since in the no-ISI circumstances, turbo coding has shown to provide near Shannon-limit error correcting performance. Therefore, the use of turbo codes with turbo equalization may achieve better performance.

In this thesis, the delay of interleaver and the IC are ignored. In practice, these delays will affect the whole system performance, especially for long message blocks. This will be another factor that further research will take account of.

Finally, the proposed model with various system parameter combinations over more complicated time-variant channels will perform.

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Appendix A Coefficients, Mean and Variance for the Original IC Turbo Equalization

A.1 Coefficients of the original IC

The output of a time-domain IC is from Equation (3-1) [17]

$$\begin{aligned} z_n &= \sum_{k=-L}^0 p_k r_{n-k} - \sum_{\substack{k=-L \\ k \neq 0}}^L q_k v_{n-k} \\ &= \sum_{k=-L}^0 p_k r_{n-k} - \sum_{k=-L}^L q_k v_{n-k} + q_0 v_n. \end{aligned} \quad (\text{A-1})$$

A corresponding frequency domain IC would have output given by

$$z_n = \frac{1}{N} \sum_{k=0}^{N-1} (P_k R_k - Q_k V_k) e^{j2\pi kn/N} + q_0 v_n, \quad (\text{A-2})$$

where, from the definition of the DFT,

$$q_0 = \frac{1}{N} \sum_{k=0}^{N-1} Q_k, \quad (\text{A-3})$$

and N is the length of the DFT block. The optimum filter coefficients are obtained according to the minimum mean square error criterion. The mean square error is

$$J = E[|z_n - v_n|^2], \quad (\text{A-4})$$

and the coefficients are found by solving the set of linear equations arising when the partial derivating of J with respect to P_l and Q_l are set to zero.

A.1.1 Coefficients of Q -filter

Setting $\frac{\partial J}{\partial Q_l} = 0$ is equivalent to setting

$$E\left[(z_n - v_n) \frac{\partial}{\partial Q_l} (z_n - v_n)^*\right] = 0. \quad (\text{A-5})$$

Since

$$\begin{aligned}\frac{\partial}{\partial Q_l}(z_n - v_n) &= \frac{1}{N} \sum_{k=0}^{N-1} \frac{\partial}{\partial Q_l} (P_k R_k - Q_k V_k) e^{j2\pi kn/N} + v_n \sum_{k=0}^{N-1} \frac{\partial q_0}{\partial Q_l} \\ &= -\frac{1}{N} V_l e^{j2\pi nl/N} + \frac{1}{N} v_n,\end{aligned}\quad (\text{A-6})$$

this is equivalent to

$$E\left[(z_n - v_n) \left(-\frac{1}{N} V_l^* e^{-j2\pi nl/N} + \frac{1}{N} v_n^*\right)\right] = 0 \quad (\text{A-7})$$

or

$$E[(z_n - v_n) V_l^*] e^{-j2\pi nl/N} = E[(z_n - v_n) v_n^*]. \quad (\text{A-8})$$

The left-hand side of Equation (A-8) is

$$\begin{aligned}E[(z_n - v_n) V_l^*] e^{-j2\pi nl/N} &= E\left[\left(\frac{1}{N} \sum_{k=0}^{N-1} (P_k R_k - Q_k V_k) e^{j2\pi kn/N} + q_0 v_n - v_n\right) V_l^*\right] e^{-j2\pi nl/N} \\ &= \frac{1}{N} \sum_{k=0}^{N-1} (P_k E[R_k V_l^*] - Q_k E[V_k V_l^*]) e^{j2\pi(k-l)n/N} + (q_0 - 1) E[v_n V_l^*] e^{-j2\pi nl/N}. \quad (\text{A-9})\end{aligned}$$

Since $R_k = H_k V_k + W_k$, $V_l = \sum_{m=0}^{N-1} v_m e^{-j2\pi ml/N}$ and $E[W_k] = 0$, this becomes

$$\begin{aligned}E[(z_n - v_n) V_l^*] e^{-j2\pi nl/N} &= \frac{1}{N} \sum_{k=0}^{N-1} (P_k H_k - Q_k) E[V_k V_l^*] e^{j2\pi(k-l)n/N} + (q_0 - 1) \sum_{m=0}^{N-1} E[v_n v_m^*] e^{j2\pi(m-n)/N}. \quad (\text{A-10})\end{aligned}$$

Since $E[v_n v_m^*] = \sigma_v^2 \delta_{n-m}$ and $E[V_k V_l^*] = \sigma_v^2 N \delta_{k-l}$, this becomes

$$E[(z_n - v_n) V_l^*] e^{-j2\pi nl/N}$$

$$\begin{aligned}
&= \frac{1}{N} \sum_{k=0}^{N-1} \sigma_v^2 N \delta_{k-l} (P_k H_k - Q_k) e^{j2\pi(k-l)n/N} + (q_0 - 1) \sum_{m=0}^{N-1} \sigma_v^2 \delta_{m-n} e^{j2\pi m(m-n)/N} \\
&= \sigma_v^2 (P_l H_l - Q_l) + \sigma_v^2 (q_0 - 1).
\end{aligned} \tag{A-11}$$

The right-hand side of Equation (A-8) is

$$\begin{aligned}
E[(z_n - v_n) v_n^*] &= E \left[\left[\sum_{k=0}^{N-1} \frac{1}{N} (P_k R_k - Q_k V_k) e^{j2\pi kn/N} + (q_0 - 1) v_n \right] v_n^* \right] \\
&= \frac{1}{N} \left[\sum_{k=0}^{N-1} (P_k E[R_k v_n^*] - Q_k E[V_k v_n^*]) e^{j2\pi kn/N} \right] + (q_0 - 1) E[|v_n|^2] \\
&= \frac{1}{N} \left[\sum_{k=0}^{N-1} (P_k H_k - Q_k) E[V_k v_n^*] e^{j2\pi kn/N} \right] + (q_0 - 1) \sigma_v^2.
\end{aligned} \tag{A-12}$$

Since $E[V_k v_n^*] = \sigma_v^2 e^{-j2\pi kn/N}$, this becomes

$$\begin{aligned}
E[(z_n - v_n) v_n^*] &= \frac{1}{N} \sum_{k=0}^{N-1} (P_k H_k - Q_k) \sigma_v^2 + (q_0 - 1) \sigma_v^2 \\
&= \sigma_v^2 \left[\frac{1}{N} \sum_{k=0}^{N-1} P_k H_k - \frac{1}{N} \sum_{k=0}^{N-1} Q_k + (q_0 - 1) \right] \\
&= \sigma_v^2 \left[\frac{1}{N} \sum_{k=0}^{N-1} P_k H_k - q_0 + (q_0 - 1) \right] \\
&= \sigma_v^2 \left[\frac{1}{N} \sum_{k=0}^{N-1} P_k H_k - 1 \right].
\end{aligned} \tag{A-13}$$

Equating Equation (A-11) to Equation (A-13) gives

$$\sigma_v^2 (P_l H_l - Q_l) + \sigma_v^2 (q_0 - 1) = \sigma_v^2 \left[\frac{1}{N} \sum_{k=0}^{N-1} P_k H_k - 1 \right]$$

or

$$Q_l - q_0 = P_l H_l - \frac{1}{N} \sum_{k=0}^{N-1} P_k H_k$$

or

$$Q_l - \frac{1}{N} \sum_{k=0}^{N-1} Q_k = P_l H_l - \frac{1}{N} \sum_{k=0}^{N-1} P_k H_k . \quad (\text{A-14})$$

So, a valid solution is

$$Q_l = P_l H_l . \quad (\text{A-15})$$

A.1.2 Coefficients of P -filter

Setting $\frac{\partial J}{\partial P_l} = 0$ is equivalent to

$$E \left[(z_n - v_n) \frac{\partial}{\partial P_l} (z_n - v_n)^* \right] = 0 . \quad (\text{A-16})$$

Since

$$\begin{aligned} \frac{\partial}{\partial P_l} (z_n - v_n) &= \frac{1}{N} \sum_{k=0}^{N-1} \frac{\partial}{\partial P_l} (P_k R_k - Q_k V_k) e^{j2\pi kn/N} \\ &= \frac{1}{N} \sum_{k=0}^{N-1} R_k \delta_{k-l} e^{j2\pi kn/N} \\ &= \frac{1}{N} R_l e^{j2\pi ml/N} , \end{aligned} \quad (\text{A-17})$$

this is equivalent to

$$E \left[(z_n - v_n) \frac{1}{N} R_l^* e^{-j2\pi ml/N} \right] = 0$$

or

$$E[z_n R_l^*] = E[v_n R_l^*] . \quad (\text{A-18})$$

The left-hand side of (A-18) is

$$\begin{aligned}
E[z_n R_l^*] &= E\left[\frac{1}{N} \sum_{k=0}^{N-1} (P_k R_k - Q_k V_k) e^{j2\pi kn/N} R_l^*\right] + q_0 E[v_n R_l^*] \\
&= \frac{1}{N} \sum_{k=0}^{N-1} (P_k E[R_k R_l^*] - Q_k E[V_k R_l^*]) e^{j2\pi kn/N} + q_0 E[v_n (H_l^* V_l^* + W_l^*)] \\
&= \frac{1}{N} \sum_{k=0}^{N-1} (P_k E[(H_k V_k + W_k)(H_l^* V_l^* + W_l^*)] - Q_k E[V_k (H_l^* V_l^* + W_l^*)]) e^{j2\pi kn/N} \\
&\quad + q_0 H_l^* E[v_n V_l^*] \\
&= \frac{1}{N} \sum_{k=0}^{N-1} [P_k (H_k H_l^* E[V_k V_l^*] + E[W_k W_l^*]) - Q_k H_l^* E[V_k V_l^*]] e^{j2\pi kn/N} \\
&\quad + q_0 H_l^* E[v_n V_l^*] \quad . \quad (\text{A-19})
\end{aligned}$$

Since $E[V_k V_l^*] = \sigma_v^2 N \delta_{k-l}$ and $E[W_k W_l^*] = N_0 N \delta_{k-l}$, this becomes

$$\begin{aligned}
E[z_n R_l^*] &= \frac{1}{N} \sum_{k=0}^{N-1} [P_l (H_l H_l^* \sigma_v^2 + N_0) N \delta_{k-l} - Q_k H_l^* \sigma_v^2 N \delta_{k-l}] e^{j2\pi kn/N} \\
&\quad + q_0 H_l^* \sigma_v^2 e^{j2\pi nl/N} \\
&= [P_l (|H_l|^2 \sigma_v^2 + N_0) - Q_l H_l^* \sigma_v^2 + q_0 H_l^* \sigma_v^2] e^{j2\pi nl/N} \quad . \quad (\text{A-20})
\end{aligned}$$

The right-hand side of Equation (A-18) is

$$\begin{aligned}
E[v_n R_l^*] &= H_l^* E[v_n V_l^*] + E[v_n W_l^*] \\
&= H_l^* \sigma_v^2 e^{j2\pi nl/N} \quad . \quad (\text{A-21})
\end{aligned}$$

Combining Equation (A-20) and (A-21) gives

$$[P_l (|H_l|^2 \sigma_v^2 + N_0) - Q_l H_l^* \sigma_v^2 + q_0 H_l^* \sigma_v^2] e^{j2\pi nl/N} = H_l^* \sigma_v^2 e^{j2\pi nl/N} \quad . \quad (\text{A-22})$$

Substituting $Q_l = P_l H_l$ gives

$$P_l N_0 + q_0 H_l^* \sigma_v^2 = H_l^* \sigma_v^2$$

or

$$P_l = \frac{\sigma_v^2 H_l^*}{N_0} (1 - q_0) . \quad (\text{A-23})$$

However

$$\begin{aligned} q_0 &= \frac{1}{N} \sum_{l=0}^{N-1} Q_l = \frac{1}{N} \sum_{l=0}^{N-1} P_l H_l \\ &= \frac{\sigma_v^2}{N_0} (1 - q_0) \frac{1}{N} \sum_{l=0}^{N-1} |H_l|^2 . \end{aligned} \quad (\text{A-24})$$

Define

$$E_h = \frac{1}{N} \sum_{l=0}^{N-1} |H_l|^2 \quad (\text{A-25})$$

as the overall channel attenuation, so

$$q_0 = \frac{\frac{\sigma_v^2}{N_0} E_h}{\frac{\sigma_v^2}{N_0} E_h + 1} = \frac{\sigma_v^2 E_h}{\sigma_v^2 E_h + N_0} . \quad (\text{A-26})$$

Then

$$P_l = \frac{\sigma_v^2 H_l^*}{\sigma_v^2 E_h + N_0} \quad (\text{A-27})$$

and

$$Q_l = \frac{\sigma_v^2 |H_l|^2}{\sigma_v^2 E_h + N_0} . \quad (\text{A-28})$$

A.2 Mean and Variance

To find the mean and variance of z_n , conditioned on $v_n = v$, note that since

$$Q_l = P_l H_l \text{ and } R_l = H_l V_l + W_l ,$$

$$P_l R_l - Q_l V_l = P_l (H_l V_l + W_l) - (P_l H_l) V_l = P_l W_l$$

so,

$$z_n = \frac{1}{N} \sum_{k=0}^{N-1} P_l W_l e^{j2\pi kn/N} + q_0 v_n. \quad (\text{A-29})$$

The mean is therefore

$$\mu_z(v) = E[z_n | v_n = v] = q_0 v. \quad (\text{A-30})$$

The variance is

$$\begin{aligned} \sigma_z^2 &= E[|z_n|^2 | v_n = v] - |\mu_z(v)|^2 \\ &= E\left[\left|\frac{1}{N} \sum_{k=0}^{N-1} P_k W_k e^{j2\pi kn/N}\right|^2\right] + |q_0|^2 |E[v_n | v_n = v]|^2 - |q_0|^2 |v|^2 \\ &= \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} P_k P_l^* E[W_k W_l^*] e^{j2\pi(k-l)n/N} \\ &= \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} P_k P_l^* N_0 N \delta_{k-l} e^{j2\pi(k-l)n/N} \\ &= N_0 \frac{1}{N} \sum_{k=0}^{N-1} |P_k|^2 \\ &= N_0 \left(\frac{\sigma_v^2}{\sigma_v^2 E_h + N_0} \right)^2 \frac{1}{N} \sum_{k=0}^{N-1} |H_k|^2 \\ &= N_0 \left(\frac{\sigma_v^2}{\sigma_v^2 E_h + N_0} \right) \left(\frac{\sigma_v^2 E_h}{\sigma_v^2 E_h + N_0} \right) \\ &= \sigma_v^2 (1 - q_0) q_0, \end{aligned} \quad (\text{A-31})$$

$$\text{where } q_0 = \frac{\sigma_v^2 E_h}{\sigma_v^2 E_h + N_0}.$$

Appendix B Coefficients, Mean and Variance for Hybrid Reduced Complexity Turbo Equalization

B.1 Coefficients of the Hybrid IC

The output of the hybrid IC is

$$z_n = \frac{1}{N} \sum_{k=0}^{N-1} P_k R_k e^{j2\pi kn/N} - \sum_{i=1}^{N_t} (q_{M_i} v_{n-M_i} + q_{-M_i} v_{n+M_i}), \quad (\text{B-1})$$

where, $\{M_i \mid i \in \{1, 2, \dots, N_t\}\}$, are the indices of the desired feedback taps. The MMSE is used to determine the optimum coefficients. The mean square error is

$$J = E[|z_n - v_n|^2], \quad (\text{B-2})$$

and the coefficients are determined by solving the set of linear equations when the partial derivating of J with respect to P_l and q_m are set to zero.

B.1.1 P -filter coefficients

Setting $\frac{\partial J}{\partial P_l} = 0$ is equivalent to

$$E\left[(z_n - v_n) \frac{\partial}{\partial P_l} ((z_n - v_n)^*)\right] = 0. \quad (\text{B-3})$$

Since from (A-17)

$$\frac{\partial}{\partial P_l} (z_n - v_n) = \frac{1}{N} R_l e^{j2\pi nl/N}, \quad (\text{B-4})$$

this is equivalent to

$$E\left[(z_n - v_n) \frac{1}{N} R_l^* e^{-j2\pi nl/N}\right] = 0 \quad (\text{B-5})$$

or

$$E[z_n R_l^*] = E[v_n R_l^*]. \quad (\text{B-6})$$

The left-hand side of Equation (B-6) is

$$E[z_n R_l^*] = E\left[\sum_{k=0}^{N-1} \frac{1}{N} P_k R_k R_l^* e^{j2\pi kn/N} - \sum_{i=1}^{N_l} q_{M_i} v_{n-M_i} R_l^* - \sum_{i=1}^{N_l} q_{-M_i} v_{n+M_i} R_l^*\right]. \quad (\text{B-7})$$

The first term of Equation (B-7) is

$$\begin{aligned} E\left[\sum_{k=0}^{N-1} \frac{1}{N} P_k R_k R_l^* e^{j2\pi kn/N}\right] &= E\left[\sum_{k=0}^{N-1} \frac{1}{N} P_k (H_k V_k + W_k)(H_l^* V_l^* + W_l^*) e^{j2\pi kn/N}\right] \\ &= \frac{1}{N} \sum_{k=0}^{N-1} P_k (H_k H_l^* E[V_k V_l^*] + E[W_k W_l^*]) e^{j2\pi kn/N} \\ &= \frac{1}{N} \sum_{k=0}^{N-1} P_k (H_k H_l^* \sigma_v^2 N \delta_{k-l} + N_0 N \delta_{k-l}) e^{j2\pi kn/N} \\ &= P_l (H_l H_l^* \sigma_v^2 + N_0) e^{j2\pi nl/N}. \end{aligned} \quad (\text{B-8})$$

The second term of Equation (B-7) is

$$\begin{aligned} E\left[\sum_{i=1}^{N_l} q_{M_i} v_{n-M_i} R_l^*\right] &= E\left[\left(\sum_{i=1}^{N_l} q_{M_i} \left(\frac{1}{N} \sum_{k=0}^{N-1} V_k e^{j2\pi(n-M_i)k/N}\right) (H_l^* V_l^* + W_l^*)\right)\right] \\ &= \frac{1}{N} \sum_{i=1}^{N_l} q_{M_i} \sum_{k=0}^{N-1} (H_l^* E[V_k V_l^*] + E[W_k W_l^*]) e^{j2\pi(n-M_i)k/N} \\ &= \frac{1}{N} \sum_{i=1}^{N_l} q_{M_i} \sum_{k=0}^{N-1} H_l^* N \sigma_v^2 \delta_{k-l} e^{j2\pi(n-M_i)k/N} \\ &= H_l^* \sigma_v^2 e^{j2\pi nl/N} \sum_{i=1}^{N_l} q_{M_i} e^{-j2\pi M_i l/N}. \end{aligned} \quad (\text{B-9})$$

The third term of Equation (B-7) is

$$E\left[\sum_{i=1}^{N_l} q_{-M_i} v_{n+M_i} R_l^*\right] = E\left[\left(\sum_{i=1}^{N_l} q_{-M_i} \left(\frac{1}{N} \sum_{k=0}^{N-1} V_k e^{j2\pi(n+M_i)k/N}\right) (H_l^* V_l^* + W_l^*)\right)\right]$$

$$\begin{aligned}
&= \frac{1}{N} \sum_{i=1}^{N_l} q_{-M_i} \sum_{k=0}^{N-1} \left(H_l^* E[V_k V_l^*] + E[V_k W_l^*] \right) e^{j2\pi(n+M_i)k/N} \\
&= \frac{1}{N} \sum_{i=1}^{N_l} q_{-M_i} \sum_{k=0}^{N-1} H_l^* N \sigma_v^2 \delta_{k-l} e^{j2\pi(n+M_i)k/N} \\
&= H_l^* \sigma_v^2 e^{j2\pi nl/N} \sum_{i=1}^{N_l} q_{-M_i} e^{j2\pi M_i l/N} .
\end{aligned} \tag{B-10}$$

Therefore, Equation (B-7) is rewritten as

$$\begin{aligned}
E[z_n R_l^*] &= P_l (H_l H_l^* \sigma_v^2 + N_0) e^{j2\pi nl/N} - \\
&\quad H_l^* \sigma_v^2 e^{j2\pi nl/N} \sum_{i=1}^{N_l} \left(q_{M_i} e^{-j2\pi M_i l/N} + q_{-M_i} e^{j2\pi M_i l/N} \right).
\end{aligned} \tag{B-11}$$

And, the right-hand side of Equation (B-6) is

$$\begin{aligned}
E[v_n R_l^*] &= \frac{1}{N} E \left[\sum_{k=0}^{N-1} V_k e^{j2\pi kn/N} (H_l^* V_l^* + W_l^*) \right] \\
&= \frac{1}{N} \left[\sum_{k=0}^{N-1} H_l^* E[V_k V_l^*] e^{j2\pi kn/N} \right] \\
&= \frac{1}{N} \left[\sum_{k=0}^{N-1} H_l^* \sigma_v^2 N \delta_{k-l} e^{j2\pi kn/N} \right] \\
&= H_l^* \sigma_v^2 e^{j2\pi nl/N} .
\end{aligned} \tag{B-12}$$

Equating Equation (B-11) to Equation (B-12) gives

$$P_l (|H_l|^2 \sigma_v^2 + N_0) - H_l^* \sigma_v^2 \sum_{i=1}^{N_l} \left(q_{M_i} e^{-j2\pi M_i l/N} + q_{-M_i} e^{j2\pi M_i l/N} \right) = H_l^* \sigma_v^2$$

or

$$P_l = \frac{H_l^* \sigma_v^2}{|H_l|^2 \sigma_v^2 + N_0} \left[1 + \sum_{i=1}^{N_l} \left(q_{M_i} e^{-j2\pi M_i l/N} + q_{-M_i} e^{j2\pi M_i l/N} \right) \right]. \tag{B-13}$$

B.1.2 q -filter coefficients

Setting $\frac{\partial J}{\partial q_m} = 0$ is equivalent to

$$E\left[(z_n - v_n) \frac{\partial}{\partial q_m} ((z_n - v_n)^*)\right] = 0 \quad (\text{B-14})$$

Since

$$\frac{\partial}{\partial q_m} (z_n - v_n) = -v_{n-m}, \text{ for } m \in \{M_1, \dots, M_i, \dots, M_{N_i}\} \quad (\text{B-15})$$

this is equivalent to

$$E[(z_n - v_n)(-v_{n-m}^*)] = 0. \quad (\text{B-16})$$

And for $m \in \{M_1, \dots, M_i, \dots, M_{N_i}\}$

$$\begin{aligned} E[(z_n - v_n)(-v_{n-m}^*)] &= E\left[\left(\frac{1}{N} \sum_{k=0}^{N-1} P_k R_k e^{j2\pi kn/N} - \sum_{i=1}^{N_i} (q_{M_i} v_{n-M_i} + q_{-M_i} v_{n+M_i}) - v_n\right)(-v_{n-m}^*)\right] \\ &= -E\left[\frac{1}{N} \sum_{k=0}^{N-1} P_k R_k (v_{n-m}^*) e^{j2\pi kn/N}\right] + E\left[\sum_{i=1}^{N_i} (q_{M_i} v_{n-M_i} + q_{-M_i} v_{n+M_i}) v_{n-m}^*\right] \\ &= E\left[-\frac{1}{N} \sum_{k=0}^{N-1} P_k R_k v_{n-m}^* e^{j2\pi kn/N}\right] + q_m \sigma_v^2, \end{aligned} \quad (\text{B-17})$$

for $v_{n-m} = \frac{1}{N} \sum_{p=0}^{N-1} V_p e^{-j2\pi(n-m)p/N}$, then

$$\begin{aligned} E[(z_n - v_n)(-v_{n-m}^*)] &= E\left[-\frac{1}{N^2} \sum_{k=0}^{N-1} P_k (H_k V_k + W_k) \sum_{p=0}^{N-1} V_p^* e^{-j2\pi(n-m)p/N} e^{j2\pi kn/N}\right] + q_m \sigma_v^2 \\ &= -\frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{p=0}^{N-1} P_k H_k E[V_k V_p^*] e^{j2\pi mp/N} e^{j2\pi(k-p)n/N} + q_m \sigma_v^2 \\ &= -\frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{p=0}^{N-1} P_k H_k N \sigma_v^2 \delta_{k-p} e^{j2\pi mp/N} e^{j2\pi(k-p)n/N} + q_m \sigma_v^2 \end{aligned}$$

$$= -\sigma_v^2 \frac{1}{N} \sum_{k=0}^{N-1} P_k H_k e^{j2\pi mk/N} + q_m \sigma_v^2 = 0, \quad (\text{B-18})$$

so, for $m \in \{M_1, \dots, M_i, \dots, M_{N_i}\}$

$$q_m = \frac{1}{N} \sum_{k=0}^{N-1} P_k H_k e^{j2\pi km/N}. \quad (\text{B-19})$$

Similarly,

$$q_{-m} = \frac{1}{N} \sum_{k=0}^{N-1} P_k H_k e^{-j2\pi km/N}. \quad (\text{B-20})$$

Substituting Equation (B-13) into Equation (B-19) gives, for $m \in \{M_1, \dots, M_i, \dots, M_{N_i}\}$,

$$\begin{aligned} q_m &= \frac{1}{N} \sum_{k=0}^{N-1} \frac{H_k^* \sigma_v^2}{|H_k|^2 \sigma_v^2 + N_0} \left[1 + \sum_{i=1}^{N_i} (q_{M_i} e^{-j2\pi M_i k/N} + q_{-M_i} e^{j2\pi M_i k/N}) \right] H_k e^{j2\pi mk/N} \\ &= \frac{1}{N} \sum_{k=0}^{N-1} \frac{|H_k|^2 \sigma_v^2}{|H_k|^2 \sigma_v^2 + N_0} e^{j2\pi mk/N} \\ &\quad + \frac{1}{N} \sum_{i=1}^{N_i} \left[q_{M_i} \sum_{k=0}^{N-1} \frac{|H_k|^2 \sigma_v^2}{|H_k|^2 \sigma_v^2 + N_0} e^{j2\pi(m-M_i)k/N} + q_{-M_i} \sum_{k=0}^{N-1} \frac{|H_k|^2 \sigma_v^2}{|H_k|^2 \sigma_v^2 + N_0} e^{j2\pi(m+M_i)k/N} \right] \\ &= x_m + \sum_{i=1}^{N_i} (q_{M_i} x_{m-M_i} + q_{-M_i} x_{m+M_i}), \end{aligned} \quad (\text{B-21})$$

where

$$x_m = \frac{1}{N} \sum_{k=0}^{N-1} \frac{|H_k|^2 \sigma_v^2}{|H_k|^2 \sigma_v^2 + N_0} e^{j2\pi mk/N}. \quad (\text{B-22})$$

Based on Equation (B-21), for $m \in \{M_1, \dots, M_i, \dots, M_{N_i}\}$, there are the following expressions

$$\begin{cases} q_{M_1} = x_{M_1} + \sum_{i=1}^{N_t} (q_{M_i} x_{M_1-M_i} + q_{-M_i} x_{M_1+M_i}) \\ q_{M_2} = x_{M_2} + \sum_{i=1}^{N_t} (q_{M_i} x_{M_2-M_i} + q_{-M_i} x_{M_2+M_i}) \\ \vdots \\ q_{M_{N_t}} = x_{M_{N_t}} + \sum_{i=1}^{N_t} (q_{M_i} x_{M_{N_t}-M_i} + q_{-M_i} x_{M_{N_t}+M_i}) \\ q_{-M_1} = x_{-M_1} + \sum_{i=1}^{N_t} (q_{M_i} x_{-M_1-M_i} + q_{-M_i} x_{-M_1+M_i}) \\ \vdots \\ q_{-M_{N_t}} = x_{-M_{N_t}} + \sum_{i=1}^{N_t} (q_{M_i} x_{-M_{N_t}-M_i} + q_{-M_i} x_{-M_{N_t}+M_i}) \end{cases} \quad (\text{B-23})$$

Define the following vectors and matrix:

$$\mathbf{q} = [q_{M_1} \cdots q_{M_{N_t}} \quad q_{-M_1} \cdots q_{-M_{N_t}}]^T \quad (\text{B-24})$$

$$\mathbf{x} = [x_{M_1} \cdots x_{M_{N_t}} \quad x_{-M_1} \cdots x_{-M_{N_t}}]^T \quad (\text{B-25})$$

$$\mathbf{C} = \begin{bmatrix} x_{M_1-M_1} & x_{M_1-M_2} & \cdots & x_{M_1-M_{N_t-1}} & x_{M_1-M_{N_t}} & x_{M_1+M_1} & x_{M_1+M_2} & \cdots & x_{M_1+M_{N_t-1}} & x_{M_1+M_{N_t}} \\ \vdots & & & & & & & & & \\ x_{M_{N_t}-M_1} & x_{M_{N_t}-M_2} & \cdots & x_{M_{N_t}-M_{N_t-1}} & x_{M_{N_t}-M_{N_t}} & x_{M_{N_t}+M_1} & x_{M_{N_t}+M_2} & \cdots & x_{M_{N_t}+M_{N_t-1}} & x_{M_{N_t}+M_{N_t}} \\ x_{-M_1-M_1} & x_{-M_1-M_2} & \cdots & x_{-M_1-M_{N_t-1}} & x_{-M_1-M_{N_t}} & x_{-M_1+M_1} & x_{-M_1+M_2} & \cdots & x_{-M_1+M_{N_t-1}} & x_{-M_1+M_{N_t}} \\ \vdots & & & & & & & & & \\ x_{-M_{N_t}-M_1} & x_{-M_{N_t}-M_2} & \cdots & x_{-M_{N_t}-M_{N_t-1}} & x_{-M_{N_t}-M_{N_t}} & x_{-M_{N_t}+M_1} & x_{-M_{N_t}+M_2} & \cdots & x_{-M_{N_t}+M_{N_t-1}} & x_{-M_{N_t}+M_{N_t}} \end{bmatrix} \quad (\text{B-26})$$

then Equation (B-23) can be expressed as

$$\mathbf{q} = \mathbf{x} + \mathbf{C}\mathbf{q} \quad (\text{B-27})$$

or

$$\mathbf{q} = (\mathbf{I} - \mathbf{C})^{-1} \mathbf{x}. \quad (\text{B-28})$$

where $\mathbf{I}$ is a $2N_t \times 2N_t$ identity matrix.

To simplify the calculation of $\mathbf{C}$, it can be expressed as

$$\mathbf{C} = \begin{bmatrix} \mathbf{F} & \mathbf{E} \\ \mathbf{E}^* & \mathbf{F}^* \end{bmatrix}, \quad (\text{B-29})$$

where

$$\mathbf{E} = \begin{bmatrix} \mathbf{x}_{M_1+M_1} & \mathbf{x}_{M_1+M_2} & \cdots & \mathbf{x}_{M_1+M_{N_t-1}} & \mathbf{x}_{M_1+M_{N_t}} \\ \vdots & & & & \\ \mathbf{x}_{M_{N_t}+M_1} & \mathbf{x}_{M_{N_t}+M_2} & \cdots & \mathbf{x}_{M_{N_t}+M_{N_t-1}} & \mathbf{x}_{M_{N_t}+M_{N_t}} \end{bmatrix} \quad (\text{B-30})$$

and

$$\mathbf{F} = \begin{bmatrix} \mathbf{x}_{M_1-M_1} & \mathbf{x}_{M_1-M_2} & \cdots & \mathbf{x}_{M_1-M_{N_t-1}} & \mathbf{x}_{M_1-M_{N_t}} \\ \vdots & & & & \\ \mathbf{x}_{M_{N_t}-M_1} & \mathbf{x}_{M_{N_t}-M_2} & \cdots & \mathbf{x}_{M_{N_t}-M_{N_t-1}} & \mathbf{x}_{M_{N_t}-M_{N_t}} \end{bmatrix}. \quad (\text{B-31})$$

By defining

$$\mathbf{q}_1 = [q_{M_1} \cdots q_{M_{N_t}}]^T, \quad (\text{B-32})$$

$$\mathbf{q}_2 = [q_{-M_1} \cdots q_{-M_{N_t}}]^T, \quad (\text{B-33})$$

$$\mathbf{x}_1 = [x_{M_1} \cdots x_{M_{N_t}}]^T, \quad (\text{B-34})$$

and

$$\mathbf{x}_2 = [x_{-M_1} \cdots x_{-M_{N_t}}]^T, \quad (\text{B-35})$$

then Equation (B-27) can be rewritten as

$$\begin{bmatrix} \mathbf{I} - \mathbf{F} & -\mathbf{E} \\ -\mathbf{E}^* & \mathbf{I} - \mathbf{F}^* \end{bmatrix} \begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \quad (\text{B-36})$$

From Equation (B-36), we obtain

$$\mathbf{q}_1 = \left[(\mathbf{I} - \mathbf{F}) - \mathbf{E}(\mathbf{I} - \mathbf{F}^* - \mathbf{F}^*)^{-1} \mathbf{E}^* \right]^{-1} \left[\mathbf{x}_1 + \mathbf{E}(\mathbf{I} - \mathbf{F}^*)^{-1} \mathbf{x}_2 \right] \quad (\text{B-37})$$

$$\mathbf{q}_2 = \left[(\mathbf{I} - \mathbf{F}^*) - \mathbf{E}^*(\mathbf{I} - \mathbf{F})^{-1} \mathbf{E} \right]^{-1} \left[\mathbf{E}^*(\mathbf{I} - \mathbf{F})^{-1} \mathbf{x}_1 + \mathbf{x}_2 \right] \quad (\text{B-38})$$

Since it can be proved that $q_{-m} = q_m^*$ according to (B-19) and (B-20). So, actually, calculating either $\mathbf{q}_1$ or $\mathbf{q}_2$ is sufficient.

B.2 Mean and Variance

The mean $\mu_z(v)$ and variance σ_z^2 are calculated as follows:

$$\begin{aligned} \mu_z(v) &= E[z_n | v_n = v] \\ &= E \left[\frac{1}{N} \sum_{k=0}^{N-1} P_k R_k e^{j2\pi k n / N} - \sum_{i=1}^{N_i} (q_{M_i} v_{n-M_i} + q_{-M_i} v_{n+M_i}) | v_n = v \right]. \end{aligned} \quad (\text{B-39})$$

For $M_i \neq 0$, $E[v_{n-M_i} | v_n = v] = 0$, $V_k = \sum_{m=0}^{N-1} v_m e^{-j2\pi k m / N}$ and $E[v_{n+M_i} | v_n = v] = 0$, so

$$\begin{aligned} \mu_z(v) &= E \left[\frac{1}{N} \sum_{k=0}^{N-1} P_k (H_k V_k + W_k) e^{j2\pi k n / N} | v_n = v \right] \\ &= \frac{1}{N} \sum_{k=0}^{N-1} P_k H_k \sum_{m=0}^{N-1} E[v_m | v_n = v] e^{j2\pi k (n-m) / N} \\ &= \frac{1}{N} \sum_{k=0}^{N-1} P_k H_k \sum_{m=0}^{N-1} v \delta_{n-m} e^{j2\pi k (n-m) / N} \\ &= v \frac{1}{N} \sum_{k=0}^{N-1} P_k H_k \\ &= v q_0. \end{aligned} \quad (\text{B-40})$$

The variance is

$$\sigma_z^2 = E[|z_n|^2 | v_n = v] - |\mu_z(v)|^2$$

$$= E \left[\left| \frac{1}{N} \sum_{k=0}^{N-1} P_k R_k e^{j2\pi kn/N} - \sum_{i=1}^{N_t} (q_{M_i} v_{n-M_i} + q_{-M_i} v_{n+M_i}) \right|^2 \mid v_n = v \right] - |\mu_z(v)|^2. \quad (\text{B-41})$$

And

$$\begin{aligned} & E \left[\left| \frac{1}{N} \sum_{k=0}^{N-1} P_k R_k e^{j2\pi kn/N} - \sum_{i=1}^{N_t} (q_{M_i} v_{n-M_i} + q_{-M_i} v_{n+M_i}) \right|^2 \mid v_n = v \right] \\ &= E \left[\frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} P_k R_k e^{j2\pi kn/N} P_l^* R_l^* e^{-j2\pi ml/N} \mid v_n = v \right] \\ &\quad - E \left[\frac{1}{N} \sum_{k=0}^{N-1} P_k R_k e^{j2\pi kn/N} \sum_{j=1}^{N_t} (q_{M_j}^* v_{n-M_j}^* + q_{-M_j}^* v_{n+M_j}^*) \mid v_n = v \right] \\ &\quad - E \left[\frac{1}{N} \sum_{l=0}^{N-1} P_l^* R_l^* e^{-j2\pi ml/N} \sum_{i=1}^{N_t} (q_{M_i} v_{n-M_i} + q_{-M_i} v_{n+M_i}) \mid v_n = v \right] \\ &\quad + E \left[\sum_{i=1}^{N_t} (q_{M_i} v_{n-M_i} + q_{-M_i} v_{n+M_i}) \sum_{j=1}^{N_t} (q_{M_j}^* v_{n-M_j}^* + q_{-M_j}^* v_{n+M_j}^*) \mid v_n = v \right]. \quad (\text{B-42}) \end{aligned}$$

The first term of Equation (B-42) is

$$\begin{aligned} & E \left[\frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} P_k R_k e^{j2\pi kn/N} P_l^* R_l^* e^{-j2\pi ml/N} \mid v_n = v \right] \\ &= E \left[\frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} P_k P_l^* (H_k V_k + W_k) (H_l^* V_l^* + W_l^*) e^{j2\pi(k-l)n/N} \mid v_n = v \right] \\ &= \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} P_k P_l^* [H_k H_l^* E[V_k V_l^* \mid v_n = v] + E[W_k W_l^*]] e^{j2\pi(k-l)n/N}. \quad (\text{B-43}) \end{aligned}$$

Since

$$E[V_k V_l^* \mid v_n = v] = \sum_{a=0}^{N-1} \sum_{b=0}^{N-1} E[v_a v_b^* \mid v_n = v] e^{j2\pi(lb-ak)/N}, \quad (\text{B-44})$$

and

$$\begin{aligned}
E[v_a v_b^* | v_n = v] &= \begin{cases} E[v_a | v_n = v] E[v_b^* | v_n = v] & \text{if } a \neq b \\ E[|v_n|^2 | v_n = v] & \text{if } a = b \end{cases} \\
&= E[|v_a|^2 | v_n = v] \delta_{a-b} \\
&= [|v|^2 \delta_{a-n} + \sigma_v^2 (1 - \delta_{a-n})] \delta_{a-b} \\
&= [\sigma_v^2 + (|v|^2 - \sigma_v^2) \delta_{a-n}] \delta_{a-b}, \tag{B-45}
\end{aligned}$$

Equation (B-43) becomes

$$\begin{aligned}
&E\left[\frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} P_k R_k e^{j2\pi kn/N} P_l^* R_l^* e^{-j2\pi ml/N} | v_n = v\right] \\
&= \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} P_k P_l^* \left[H_k H_l^* \sum_{a=0}^{N-1} \sum_{b=0}^{N-1} [\sigma_v^2 + (|v|^2 - \sigma_v^2) \delta_{a-n}] \delta_{a-b} e^{j2\pi(lb-ka)/N} \right. \\
&\quad \left. + E[W_k W_l^*] \right] e^{j2\pi(k-l)n/N} \\
&= \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} P_k P_l^* \left[H_k H_l^* \sum_{a=0}^{N-1} [\sigma_v^2 + (|v|^2 - \sigma_v^2) \delta_{a-n}] e^{j2\pi(l-k)a/N} + N_0 N \delta_{k-l} \right] e^{j2\pi(k-l)n/N} \\
&= \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} P_k P_l^* [H_k H_l^* \sigma_v^2 N \delta_{k-l} + H_k H_l^* (|v|^2 - \sigma_v^2) e^{j2\pi(l-k)n/N} + N_0 N \delta_{k-l}] e^{j2\pi(k-l)n/N} \\
&= \frac{1}{N} \sum_{k=0}^{N-1} P_k P_k^* (|H_k|^2 \sigma_v^2 + N_0) + (|v|^2 - \sigma_v^2) \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} P_k H_k P_l^* H_l^* \\
&= \frac{1}{N} \sum_{k=0}^{N-1} |P_k|^2 (|H_k|^2 \sigma_v^2 + N_0) + (|v|^2 - \sigma_v^2) |q_0|^2. \tag{B-46}
\end{aligned}$$

The second term of Equation (B-42) is

$$\begin{aligned}
&E\left[\frac{1}{N} \sum_{k=0}^{N-1} P_k R_k e^{j2\pi kn/N} \sum_{j=1}^{N_t} (q_{M_j}^* v_{n-M_j}^* + q_{-M_j}^* v_{n+M_j}^*) | v_n = v\right] \\
&= \frac{1}{N} \sum_{k=0}^{N-1} P_k \sum_{j=1}^{N_t} (q_{M_j}^* E[v_{n-M_j}^* R_k | v_n = v] + q_{-M_j}^* E[v_{n+M_j}^* R_k | v_n = v]) e^{j2\pi kn/N}. \tag{B-47}
\end{aligned}$$

Since

$$\begin{aligned}
E[R_k v_{n-M_j}^* | v_n = v] &= H_k E[v_k v_{n-M_j}^* | v_n = v] \\
&= H_k \sum_{m=0}^{N-1} E[v_m v_{n-M_j}^* | v_n = v] e^{-j2\pi km / N} \\
&= H_k \sum_{m=0}^{N-1} \sigma_v^2 \delta_{n-M_j-m} e^{-j2\pi km / N} \\
&= H_k \sigma_v^2 e^{j2\pi(M_j-n)k / N},
\end{aligned} \tag{B-48}$$

then Equation (B-47) is

$$\begin{aligned}
&E\left[\frac{1}{N} \sum_{k=0}^{N-1} P_k R_k e^{j2\pi kn / N} \sum_{j=1}^{N_t} (q_{M_j}^* v_{n-M_j}^* + q_{-M_j}^* v_{n+M_j}^*) | v_n = v\right] \\
&= \sigma_v^2 \frac{1}{N} \sum_{k=0}^{N-1} P_k H_k \sum_{j=1}^{N_t} (q_{M_j}^* e^{j2\pi M_j k / N} + q_{-M_j}^* e^{-j2\pi M_j k / N}) \\
&= \sigma_v^2 \frac{1}{N} \sum_{k=0}^{N-1} P_k H_k \sum_{j=1}^{N_t} (q_{-M_j} e^{j2\pi M_j k / N} + q_{M_j} e^{-j2\pi M_j k / N}).
\end{aligned} \tag{B-49}$$

Similarly, the third term of Equation (B-42) is

$$\begin{aligned}
&E\left[\frac{1}{N} \sum_{l=0}^{N-1} P_l^* R_l^* e^{-j2\pi ml / N} \sum_{i=1}^{N_t} (q_{M_i} v_{n-M_i} + q_{-M_i} v_{n+M_i}) | v_n = v\right] \\
&= \sigma_v^2 \frac{1}{N} \sum_{k=0}^{N-1} P_k^* H_k^* \sum_{j=1}^{N_t} (q_{-M_j} e^{j2\pi M_j k / N} + q_{M_j} e^{-j2\pi M_j k / N}).
\end{aligned} \tag{B-50}$$

The fourth term of Equation (B-42) is

$$\begin{aligned}
&E\left[\sum_{i=1}^{N_t} (q_{M_i} v_{n-M_i} + q_{-M_i} v_{n+M_i}) \sum_{j=1}^{N_t} (q_{M_j}^* v_{n-M_j}^* + q_{-M_j}^* v_{n+M_j}^*) | v_n = v\right] \\
&= \sum_{i=1}^{N_t} \sum_{j=1}^{N_t} E[q_{M_i} q_{M_j}^* v_{n-M_i} v_{n-M_j}^* + q_{-M_i} q_{-M_j}^* v_{n+M_i} v_{n+M_j}^* | v_n = v] \\
&= \sum_{i=1}^{N_t} (|q_{M_i}|^2 E[v_{n-M_i} | v_n = v] + |q_{-M_i}|^2 E[v_{n+M_i} | v_n = v])
\end{aligned}$$

$$= \sigma_v^2 \sum_{i=1}^{N_t} (|q_{M_i}|^2 + |q_{-M_i}|^2). \quad (\text{B-51})$$

Finally, based on Equation (B-46), (B-49), (B-50) and (B-51), the variance is written as

$$\begin{aligned} \sigma_z^2 &= \frac{1}{N} \sum_{k=0}^{N-1} |P_k|^2 (|H_k|^2 \sigma_v^2 + N_0) + (|v|^2 - \sigma_v^2) |q_0|^2 \\ &\quad - \sigma_v^2 \frac{1}{N} \sum_{k=0}^{N-1} P_k H_k \sum_{j=1}^{N_t} (q_{-M_j} e^{j2\pi M_j k / N} + q_{M_j} e^{-j2\pi M_j k / N}) \\ &\quad - \sigma_v^2 \frac{1}{N} \sum_{k=0}^{N-1} P_k^* H_k^* \sum_{j=1}^{N_t} (q_{-M_j} e^{j2\pi M_j k / N} + q_{M_j} e^{-j2\pi M_j k / N}) \\ &\quad + \sigma_v^2 \sum_{i=1}^{N_t} (|q_{M_i}|^2 + |q_{-M_i}|^2) - |v|^2 |q_0|^2 \\ &= \frac{1}{N} \sum_{k=0}^{N-1} P_k^* \left[P_k (|H_k|^2 \sigma_v^2 + N_0) - H_k^* \sigma_v^2 \sum_{i=1}^{N_t} (q_{M_i} e^{-j2\pi k M_i / N} + q_{-M_i} e^{j2\pi k M_i / N}) \right] \\ &\quad - \sigma_v^2 \sum_{j=1}^{N_t} \left(q_{-M_j} \frac{1}{N} \sum_{k=0}^{N-1} P_k H_k e^{j2\pi M_j k / N} + q_{M_j} \frac{1}{N} \sum_{k=0}^{N-1} P_k H_k e^{-j2\pi M_j k / N} \right) \\ &\quad + \sigma_v^2 \sum_{i=1}^{N_t} (|q_{M_i}|^2 + |q_{-M_i}|^2) - \sigma_v^2 |q_0|^2 \end{aligned} \quad (\text{B-52})$$

Since $P_k (|H_k|^2 \sigma_v^2 + N_0) - H_k^* \sigma_v^2 \sum_{i=1}^{N_t} (q_{M_i} e^{-j2\pi k M_i / N} + q_{-M_i} e^{j2\pi k M_i / N}) = H_k^* \sigma_v^2$ according to

Equation (B-13), $q_{M_j} = \frac{1}{N} \sum_{k=0}^{N-1} P_k H_k e^{j2\pi M_j k / N}$, $q_{-M_j} = \frac{1}{N} \sum_{k=0}^{N-1} P_k H_k e^{-j2\pi M_j k / N}$ and $q_{-m} = q_m^*$

according to Equation (B-19) and (B-20), giving

$$\begin{aligned} \sigma_z^2 &= \frac{1}{N} \sum_{k=0}^{N-1} P_k^* H_k^* \sigma_v^2 - \sigma_v^2 \sum_{i=1}^{N_t} (q_{M_i} q_{M_i}^* + q_{-M_i} q_{-M_i}^*) \\ &\quad + \sigma_v^2 \sum_{i=1}^{N_t} (|q_{M_i}|^2 + |q_{-M_i}|^2) - \sigma_v^2 |q_0|^2 \end{aligned}$$

$$\begin{aligned}
&= \sigma_v^2 q_0^* - \sigma_v^2 |q_0|^2 \\
&= \sigma_v^2 q_0^* (1 - q_0).
\end{aligned}
\tag{B-53}$$