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Loss of Rotation of a Rotating Two-Body Tethered Spacecraft – Analytical and Experimental Techniques

by

Frederick William Schultz

A thesis submitted to
The Faculty of Graduate Studies and Research
In partial fulfilment of the requirements for the degree of
Master of Engineering

Department of Mechanical and Aerospace Engineering
Ottawa-Carleton Institute
For Mechanical and Aerospace Engineering

Carleton University
Ottawa, Ontario
Canada

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submitted by

Frederick William Schultz

in partial fulfilment of the requirements for the degree of
Master of Engineering

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ABSTRACT

An ionospheric space science mission called BOLAS (Bi-static Observations with Low-Altitude Satellites) is currently under development. The BOLAS spacecraft comprises two tethered sub-satellites that rotate end-over-end in the orbit plane to maintain a nominal 100 metre separation of the sub-satellites. Mission success depends on maintaining a sufficient in-orbit-plane rotation rate. Gravity gradient forces excite longitudinal cyclical stretching of the tether. The rotation rate is potentially reduced by energy dissipation from material damping of the longitudinal stretching motion and by aerodynamic forces acting on the rotating configuration.

This thesis first develops non-linear equations of orbital motion. Simulation results of the equations show the dependence of rotation rate on tether material stiffness and damping. Also, the thermal environment is modelled to establish the range of temperatures of the tether in orbit. Large-scale experimental tests designed to emulate the in-orbit dynamic state are done in which the tether (SPECTRA-1000[®]) is suspended horizontally and attached to a pendulum supported sub-satellite mass. The stiffness and structural damping properties are deduced from a model of the test configuration that accounts for tether tension variation due to gravitational cable sag effects. Small-scale laboratory tests of tether samples at cryogenic temperatures are also conducted to establish the temperature dependence of tether material properties. Tether stiffness and damping properties at orbital temperatures are then estimated from the experimental data from both tests and applied to the equations of orbital motion. The effect of aerodynamic forces on BOLAS rotation rate is modelled. The loss of rotation from both gravity gradient and aerodynamic sources is combined to estimate a total BOLAS loss of rotation.

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LIST OF SYMBOLS

Due to duplication of symbols from various fields of engineering, the definition of symbols is performed separately in each section.

Chapter 1

1. Introduction

This thesis was done in support of the proposed BOLAS (Bi-static Observations with Low-Altitude Satellites) science mission. This mission involves a satellite comprised of two sub-satellites connected by a long tether. In this chapter previous tether missions are described to illustrate the operational and research status of tether technology. The BOLAS mission is described to provide background for the technical work of this thesis. Lastly the specific objectives and the scope of the thesis are outlined.

1.1. Space Missions Using Tethers

Tethers offer many unique features for use in space, including: (a) connecting two or more space instruments to enable simultaneous multi-point science measurement of the space ionosphere or upper atmosphere, (b) electro-dynamic power generation, and (c) a means of deploying sub-satellites to higher or lower orbits (including de-orbit) from a central space station. Space missions for which tethers are suited are described in various literature sources. Table 1-1 summarises recently proposed applications [Carroll and Oldson (1995), Vigneron, Jablonski and Tyc (1996)].

Table 1-1. Potential Uses of Space Tethers.**Tethers for Space Science**

- Atmospheric Tether Missions
 - AIRSEDS* (Atmospheric Ionospheric Research Using a Small Expendable Deployed Satellite)
 - BOLAS† (Bi-static Observations with Low Altitude Satellites)
- Multiple Tethered Satellites for Ionospheric Studies

Momentum Exchange Tethers for Space Transportation

- Space Shuttle Towing of the International Space Station
- Satellite Orbit Boosting or De-boosting
- LEO to GEO Tether Transportation System
 - YES‡ (Young Engineer's Satellite)
- Tether Systems for Lunar Missions

Tethered Reentry Capsules

- STEPS§ (Station Tethered Express Payload System)

Electrodynamic Tether Power Generation

- Electro-dynamic Tether Power Generation for the International Space Station

Electrodynamic Tether Propulsion

- Reboost of the International Space Station using Electro-dynamic Tethers
 - ProSEDS* (Propulsive Small Expendable Deployer System)
- Electro-dynamic Tether Upper Stages
 - STEP-AIRSEDS* (Space Transfer using Electro-dynamic Propulsion)
- Electro-dynamic Tethers for Jovian Exploration

Tether Technology Development

- ASTOR* (Advanced Safety Tether Operation and Reliability)

Tethers for Military Applications

- TiPS** (Tether Physics and Survivability); ATEx** (Advanced Tether Experiment)

Artificial Gravity

*NASA/Michigan Technic; † CSA/NASA/CRC/Bristol; ‡ ESA/Delta-Utec; § NASA/Tether Applications;

** Naval Research Laboratory

A list of tethered spacecraft missions flown to date is shown in Table 1-2 [Cosmo and Lorenzini(1997)]. The table indicates that there have been 17 flights involving tethers;

consequently tethers are viable for operational commitment in certain applications. For some applications, such as rotating configurations, further analysis and flight demonstrations is required.

Table 1-2: History of Space Tether Missions.

Name	Date	Orbit	Tether Length	Comments
Gemini 11	1967	LEO	30 m	rotating configurationz (about tether normal axis), 0.15 cpm
Gemini 12	1967	LEO	30 m	local vertical, stable swing
H-9M-69	1980	sub-orbital	500 m	partial deployment
S-520-2	1981	sub-orbital	500 m	partial deployment
Charge-1	1983	sub-orbital	500 m	full deployment
Charge-2	1984	sub-orbital	500 m	full deployment
ECHO-7	1988	sub-orbital	?	magnetic field aligned
Oedipus-A*	1989	sub-orbital	958 m	spinning configuration (about tether longitudinal axis), 42 cpm
Charge-2B	1992	sub-orbital	500 m	full deployment
TSS-1 [†]	1992	LEO	<1 km	electro-dynamic power generation, partial deployment, retrieved
SEDS-1 [‡]	1993	LEO	20 km	downward deployment, swing & cut
PMG [§]	1993	LEO	500 m	electro-dynamic orbital boost, upward deployment
SEDS-2 [‡]	1994	LEO	20 km	local vertical stable, downward deployment, deployment control
Oedipus-C*	1995	sub-orbital	1 km	spinning configuration (about tether longitudinal axis), 6 cpm
TSS-1R [†]	1996	LEO	19.6 km	electro-dynamic power generation, severed
TiPS**	1996	LEO	4 km	long life tether (since 06/96)
ATEx ^{††}	1999	LEO	6.05 km	partial deployment, jettisoned

* Observations of Electric-field Distribution in the Ionospheric Plasma - a Unique Strategy;
[†] Tethered Satellite System; [‡] Small Expendable Deployer System; [§] Plasma Motor Generator;
 ** Tether Physics and Survivability; ^{††} Advanced Tether Experiment;

The first tests of tethered spacecraft were performed in the 1960's on the Gemini 11 and Gemini 12 missions when a 30 metre DRACO tether was used to demonstrate the rotation and gravity gradient stabilisation of tethers [Lang and Nolting(1967)].

Analysis and simulation of the dynamic motion of space tethers has been ongoing since the first Gemini flights. Due to the short duration of most of these missions (less than one week) many of the models in existence have focused on modelling tether dynamics in the short term [Tai and Loh (1965), Barnum and Evans (1976), Misra and Modi (1986, 1996{with Kalantasis and Pradhan}), Arnold (1987), Steiner, Stendl and Troger (1994), Netzer and Kane (1995), Vigneron, Jablonski, and Tyc (1996 {with Han, Modi and Misra}), 1997{with Chandrashaker}), Vigneron, Jablonski, Tyc and Schultz (2000), Cochran, Cho, Cheng and Cicci (1998)]. Chapters 2 and 3 review some developments of these models. The exception to these short duration missions is the TiPS spacecraft, which has been in continuous flight since deployment on 20 June 1996. Much of the TiPS flight data is in agreement with current models but TiPS has experienced two areas of motion that are not well predicted. Specifically the level of atmospheric drag is higher than expected and there has been unexpected damping of TiPS spacecraft's librational oscillations. It is difficult to predict aspects of long duration tethered spacecraft missions because current models are not refined enough.

Regardless of the level of model sophistication, predicting the performance of tethers in space depends on an accurate prediction of the tether's material properties. The properties of various space tethers have been analysed [Hallman, Justen and Plescher (1995)]. Complex composite structures like the TSS conductive tether or the SEDS single-braided tether were found to have non-linear material properties and a memory for loading and thermal histories [Angrilli, Bianchini, Fanti and Debei (1995)]. Considering the large tether lengths involved in recent missions, experimental determinations of material properties for these missions were performed on greatly scaled down tether lengths and different end-masses than the space applications being investigated [He and Powell (1990)]. The validity of using the results of these scaled down tests has been much discussed within the space tether community [Angrilli, Bianchini, Fanti and Debei (1995), Bergamaschi, Carlette and Lion Stoppato (1995)].

Ground tests of tether properties are further complicated by laboratory environmental considerations. These are discussed in Chapter 4. The effect of gravity on tension variation in ground testing long tethers is important; a tether supported at discrete points on earth will have a gravity induced longitudinal tension variation. While a long tether in space will also experience some tension variation, the variation in a vertical ground tested tether is far greater than the tension variation occurring in orbit. A tether vertically suspended in a gravity field has a tension variation from top to bottom that is equal to the weight of the tether. The tension variation in a horizontally hung tether depends on the amount of sag allowed in that tether; in general, this variation can be much less than that found in a vertically hung tether and can approach the variation found in orbit. This is shown in Chapter 4. In designing ground testing procedures for space tethers careful consideration must be made to select an appropriate test configuration that minimises the tension variation in a long tether; doing so will provide a more accurate resemblance to flight conditions.

1.2. The BOLAS Mission

The proposed BOLAS (Bistatic Observations with Low Altitude Satellites) ionospheric science mission would investigate the peak of the ionospheric F region to better understand the irregularities, instabilities and dynamics of this layer with the eventual end goal of improving communications transmission and reception [James, et al (1997)]. The BOLAS (Figure 1-1) uses two sub-satellites connected by a non-conductive tether made from a combination of Spectra 1000[®] and acrylic. A nominal 100 metre separation of the sub-satellites is maintained by this tether. The planned six month mission would fly in a high orbital inclination (103°), 350 x 780 km, orbit and would be active during passes over Canada's auroral region. The configuration would rotate (cartwheel) in its orbit plane at 10-20 times the orbit rate of 0.011 cpm to spatially survey the ionosphere. The BOLAS would use spin stabilisation as its primary means of attitude stabilisation; there would be no on-board thrusters but some adjustment of the rotation rate and/or orientation is possible by adjusting the tether length or by using a magnetic coil.

Months after Launch	Argument of Perigee, ω	Perigee Location (approximate)	Mass of BOLAS-1 (aft body)	70 kg
0	335°		Mass of BOLAS-2 (forward body)	72 kg
0.8	270°	south pole	Tether Length (final configuration)	100 m
1.9	180°	equator	Rotation Rate	0.1 - 0.2 cpm
3	90°	north pole	Orbital Rate	0.011 cpm
4	0°	equator	Orbital Inclination	103°
5.1	270°	south pole	Perigee Altitude	350 km
6.2	180°	equator	Apogee Altitude	780 km
			Motion of Ascending Node ($\dot{\Omega} - \dot{\Omega}_n$)	20 deg/month

Figure 1-1. The BOLAS concept.

The tether tested for the BOLAS mission is equivalent to the tether used for the TiPS mission. The outer layer is braided SPECTRA-1000® giving strength, the core is acrylic

yarn which will puff the braided SPECTRA-1000[®] out to a 2 millimetre diameter giving it a larger cross section to improve its resistance to debris and small micrometeoroids.

Figure 1-2. SPECTRA-1000[®]/acrylic tether used in TiPS mission.

Figure 1-3. TiPS tether modulus of elasticity versus applied tension showing non-linear nature of the tethers modulus of elasticity. Tether material is braided SPECTRA-1000[®]/acrylic.

The tether has a linear density of 0.00136 kg/m. SPECTRA-1000[®] is a registered trademark of Allied-Signal, Inc., for the high-strength synthetic fibres the company produces from extended-chain polyethylene (ECPE). Key properties include low weight

and high strength, as well as resistance to impact, moisture, abrasion, chemicals, and puncture. The tether's axial stiffness has been shown to be non-linear (Figure 1-3). This non-linear behaviour is attributed to an increasing friction force between fibres as the braid is pulled tighter.

Figure 1-4. BOLAS spacecraft configuration showing instrument booms and solar panels.

The tether is initially wound on a deployment reel in the BOLAS-2 sub-satellite and threaded through two STACER[®] booms, one on each sub-satellite, to the tether retrieval reel located on the BOLAS-1 sub-satellite (Figure 1-4).

Launched as a secondary payload on a Delta II launcher the BOLAS-1 sub-satellite would be jettisoned from the Delta II 2nd stage and deploy forward and downward, towards the earth, 240 m of a thin tether and then 100 m of a thick tether. The combined initial separation is 340 m (Figure 1-5). Gravitational torques impart a rotation of the sub-satellites relative to the combined BOLAS/ Delta II 2nd stage centre of mass.

After the tether deployment is completed, the BOLAS-2 sub-satellite would be jettisoned from the Delta II 2nd stage and the thin tether retrieved by a mechanism in the BOLAS-1

sub-satellite. In the final configuration only 100 m of thick tether would separate the two sub-satellites. As the tether is retrieved, conservation of angular momentum would increase the in-plane rotation rate to within the target 0.1-0.2 cpm range. Once the 100 m separation is achieved the sub-satellite booms would be deployed to achieve the final configuration.

Figure 1-5. The BOLAS tether deployment showing spin-up of in-plane rotation.

1.2.1. BOLAS Loss of Rotation

Throughout the mission the in-plane rotation of the BOLAS must be maintained to allow the science instruments aboard the BOLAS to view a range of configurations of space plasma and magnetic fields. The design baseline in the BOLAS mission is to establish the rotation at the outset, and to accept a small loss of rotation (5%) over the first six months.

The pre-flight quantitative prediction of loss of rotation for BOLAS is then critical for the success of the mission. At this point there is no direct long-term flight experience or flight data with the rotating BOLAS configuration (i.e. two sub-satellites separated by a long tether). Other rotating spacecraft for which there is long-term rotational flight data

have configurations quite different from BOLAS (e.g., a rotating sphere, or a rotating central body with long booms) and past modelling of them cannot be directly used for pre-flight calculations for BOLAS.

Thus the objective of this thesis is eventually to establish with confidence through analysis, ground test and possibly modification of BOLAS parameters, that the rotation rate of BOLAS will remain within 5% of its initial value during the first six months of the mission.

The environmental mechanisms that have resulted in loss of rotation of other spacecraft include [Vigneron (1973), Hughes (1986)]: (a) structural energy dissipation by gravity-gradient induced deformations, (b) atmospheric drag, (c) solar pressure together with thermal-induced deformation and (d) eddy current damping. Mechanisms (c) and (d) are insignificant [Schultz and Vigneron (1998)]. Mechanisms (a) and (b) are recognised to be the significant mechanisms for BOLAS loss of rotation.

1.2.2. Structural Energy Dissipation by Gravity Gradient Induced Deformations

As the BOLAS configuration cartwheels in its orbit plane the orientation of the longitudinal tether axis changes with respect to the Earth. Newton's inverse square law of gravitation causes equal masses at different distances from the earth to experience unequal forces and accelerations. This is known as a gravity gradient. When the longitudinal tether axis is stably aligned perpendicular to the nadir-zenith line, as shown in Figure 1-6a, the sub-satellites are an equal distance from the earth and the gravitational forces exerted on both sub-satellites are equal. There is no gravitationally induced tether elongation from the end-bodies since there are no longitudinal forces acting on the tether ends.

When the longitudinal tether axis is stably aligned along any direction other than perpendicular to the nadir-zenith line, the gravitational forces exerted on the sub-satellites are unequal. Components of these unequal forces resolved along the tether axis cause elongation of the tether.

For a stable, non-rotating configuration the maximum force that the gravity gradient can produce in the tether (and hence maximum tether elongation) occurs when the tether axis is aligned with the vertical, as shown in Figure 1-6b. When the configuration rotates the tether would undergo oscillating elongation; internal material damping mechanisms would cause energy dissipation resulting in a reduction of the rate of in-plane rotation. Due to material damping the maximum elongation of the tether is slightly shifted from the vertical, as shown in Figure 1-6c. This shift is examined in Chapter 2.

Figure 1-6. Gravity gradient forces depend on tether orientation: (a) No gravity gradient forces occur when tether orientation is stable perpendicular to the nadir-zenith line; (b) gravity gradient forces causing tether elongation occur when tether oriented is along nadir-zenith line; (c) when rotating the maximum elongation from gravity gradient occurs at a shift from nadir-zenith line.

1.2.3. Atmospheric Drag

As demonstrated in Chapter 3, the rarefied atmosphere at the BOLAS' orbital altitudes presents unique challenges for predicting drag. Atmospheric drag on tethers affects tether shape, orbital life and rotation rate. Out-of-plane drag components can induce out-of-plane tether libration [Cosmo and Lorenzini (1997), King-Hele (1987)]. For long tethered configurations the drag on the tether can be more significant than the drag on the sub-satellites. This can lead to shaping of the tether profile. The tether profile of the TiPS mission may be due to such drag forces (Figure 1-7).

DAY: 173 14:57:46:17

Figure 1-7. Bow shape taken by TiPS 4 km tether (sub-satellites not visible) (Image filtered from original photograph taken from earth in 1997).

Figure 1-8. Rotation of the BOLAS causes different velocities of sub-satellites relative to the atmosphere (velocity vectors not to scale).

The area of the BOLAS tether is small compared to the area of the sub-satellites. Hence the drag of the sub-satellites must be considered in studying loss of rotation. Specifically, as the BOLAS rotates in its orbit plane the rotation causes the sub-satellites to experience different relative wind velocities and hence different drag forces (Figure 1-8). The difference in drag forces produces a moment that causes loss of rotation.

1.3. Scope of Thesis

It is necessary to model the BOLAS long term rotation dynamics to understand whether the gravity gradient damping forces and/or atmospheric drag forces would cause a loss of rotation large enough to jeopardise mission success. Since the in-flight tether physical properties must be confidently known to obtain valid results from the models, experimental testing of the tether is also necessary. Within resource constraints the experimental design for measuring the tether physical properties must closely duplicate the conditions that the BOLAS spacecraft would experience during flight. Accordingly, the objectives of this thesis are as follows:

- (a) to describe a theoretical model of and parametric study of loss of rotation due to gravity gradient and atmospheric drag forces of a tethered spacecraft with two sub-satellites (sub-satellites having similar masses);
- (b) to describe the experimental determination of key tether properties, performed at the Canadian Space Agency, St. Hubert, Quebec and at the National Research Centre, Montreal Road Campus, Ottawa, Ontario;
- (c) to give a quantitative estimate of the loss of rotation of the BOLAS spacecraft, based on extrapolation to orbital conditions of the tether properties obtained from the ground test data.

Chapter 2 describes the theoretical model of loss of rotation of a tethered two-body spacecraft due to gravity gradient forces (based on structural and viscous damping models) which has been developed by Schultz and Vigneron (1998). Results of computer simulations based on the BOLAS spacecraft mission profile are presented. An analysis of

orbital temperatures that would be experienced by the BOLAS tether is also presented in Chapter 2. Chapter 3 describes the theoretical loss of rotation of a tethered two-body spacecraft due to atmospheric drag.

Chapter 4 describes the experimental hardware and procedures performed on long tethers at the Canadian Space Agency, St. Hubert to determine the tether stiffness and damping with a detailed description of the unique methods used in testing a long horizontally hung tether. As well, Chapter 4 presents two models used in analysis of the long tether test data: (a) rod model and (b) cable model. Chapter 5 reports the analysis of the experimental results from St. Hubert and compares the rod and cable models, showing the cable model to provide a better resolution of the data.

Chapter 6 describes the experimental analysis and reports the results of cryogenically testing the material properties of short tether samples at the National Research Council, Ottawa. Chapter 7 predicts the overall loss of rotation that the BOLAS would experience based on extrapolation of the St Hubert long tether testing to orbital conditions. Chapter 7 will also discuss the impact of these results on the BOLAS mission. Finally, Chapter 8 provides conclusions and recommendations for future work.

2. Theoretical Loss of Rotation due to Gravity Gradient Forces

2.1. Introduction

The fundamental motions of gravity captured tethered two-body systems have been analysed extensively in the literature. The literature is reviewed in Appendix A, however the bulk of it deals only with short-term motion of tethered systems. Long term rotational motion of a tethered two-body system has not been extensively studied. A review of the fundamental motions of gravity captured tethered two-body systems are presented in Appendix B to this thesis. The five motions presented in Appendix B are (a) longitudinal, (b) libration, (c) transverse, (d) skiprope and (e) pendulous (end-body rocking).

The BOLAS is not gravity captured but is continuously rotating, end-over-end, in the orbit plane. This rotation is viewed as the dominant motion and is similar to libration of a gravity captured configuration; the same nomenclature used to characterise libration can be used to characterise rotation. The four fundamental motions, other than libration, of a

gravity captured configuration are, generally speaking, also present in a rotating configuration; they would be superimposed upon the dominating rotating motion and have different magnitudes than for the case of simple libration.

This chapter examines the equations of motion of a tethered two-body configuration rotating in its orbit plane. Gravity gradient excitation of longitudinal motion of the tether is considered and the transverse, skiprope and pendulous motions are ignored. This chapter summarises the results of computational parametric analysis of the equations of motion with specific application to the BOLAS mission.

2.1.1. Dynamic Modelling of Tethered Two-body Satellites

A chronological presentation of the development of dynamic theory of tethered spacecraft is presented at Appendix A. A summary is presented here with emphasis on work that impacts the development of the dynamic model in this chapter.

Generally, tethered satellites systems have been modelled as early as 1958, with increasing complexity. The earliest models focused on planar, point mass models in circular orbits. In these models the overall satellite centre of gravity described a circular orbit around a central body, the sub-satellites were modelled as point masses connected by a massless tether and the angular motion of rotation was constrained to one plane. This plane of rotation may or may not lie coplanar with the plane of the orbit. These models eventually led to three dimensional models incorporating rigid end-bodies with massive visco-elastic tethers in eccentric orbits.

From the found literature, up until the late 1990's typically only short-term simulations of tether dynamics were performed, on the order of 5 minutes to 1 day. This was due to the cost of computer processing. As well, the phenomena being studied, such as deployment and retrieval, are physically performed in short time frames.

2.1.1.1. Planar vs. Three Dimensional Modelling

Developing a cost-effective model to study long term loss of rotation of a tethered satellite requires a trade-off between high-cost, precise results and low-cost, approximate

results. Simpler modelling, such as planar modelling, adopts simplifying assumptions that constrain rotational motion to lie in a plane relative to the orbit plane. More complex modelling, such as three dimensional modelling, allows variation of the orientation of the plane of rotation relative to the orbit plane.

Misra, Lanoix, Modi and Jablonski (1998) have developed a model for long term orbit and attitude propagation that predicts trajectory, attitude, temperature and longitudinal vibrations of the system. The model accounts for atmospheric lift and drag, the asphericity of the Earth, radiation, lunisolar attraction, tether temperature and tether damping. The model was able to simulate deployment and station keeping of the TiPS spacecraft. Simulations performed with the MATLAB[®] software package were time-consuming. Further work is needed to reduce computation time and improve damping models.

Various studies have been published that indicate certain simplifications in modelling loss of rotation can be made. In the mid 1970's Bainum and Evans (1974, 1975, 1976) independently derived the three dimensional equations of motion for a fully deployed tethered system with point masses and constant rotation rate in a torque free environment. This confirmed the earlier finding of Kane and Robe (1967) and Robe (1968) that, to the first order, the in-plane motion decouples from the out of plane motion. As well, Bainum and Evans report that for small perturbations of the tether's orientation out of it's plane of rotation, the system will tend to rotate in a plane inclined to the original plane of rotation without affecting the rotation rate or equilibrium tether length. Thus, while modelling out-of-plane motion is required for precise attitude determination, it will not add significantly to the study of loss of rotation rate.

Lastly, Misra, Lanoix, Modi and Jablonski (1998) showed that the out-of-plane perturbations are small (± 1 degree) for the BOLAS mission. Thus use of a planar model is reasonable for estimating the loss of rotation in the study of the BOLAS mission.

2.1.1.2. Sources of Damping Affecting Loss of Rotation

Sources of material and mechanical damping are illustrated in the schematic shown in Figure 2-1.

Figure 2-1. Sources of material and mechanical damping in the BOLAS.

Two sources of material damping for the BOLAS are (a) material damping associated with the longitudinal stretching and relaxing of the BOLAS tether (identified as source 1 in Figure 2-1) and (b) damping due to the reels (identified as source 2 in Figure 2-1) which can come from two different interactions:

- (a) material damping within the built-up layers of the tether that are wound onto the reel and are subject to compression/relaxation as the tether tension varies; and
- (b) mechanical damping due to the mechanical tolerances within the drive/braking mechanism of the reel.

2.1.1.3. Selection of Damping Model

The foregoing sections lead us to adopt a planar, point mass model with only the tether longitudinal deformation being modelled. It remains to specify the damping mechanism

to be used with the longitudinal deformation of the tether. It is well known that the accurate simulation of internal damping in structures is difficult due to the difficulty in selecting the correct model(s) and of obtaining reasonable values for the material constants.

The nature of the damping mechanism to use for the BOLAS tether is complicated by the complex structure of the tether. Interwoven strands of braided SPECTRA-1000® and acrylic yarn are in contact; as tension is applied to the tether, the contact surfaces of the strands slide one upon the other and would exhibit a surface friction energy dissipation. Also as the applied tension increases the braid is pulled tighter and the inter-fibre contact forces would increase, causing even higher surface friction effects. Damping mechanisms internal to the individual strands or fibres are also present. This combination of internal material damping and external frictional damping can lead to a complex model for representing tether damping properties.

Figure 2-2. Proposed BOLAS thick tether showing braided structure of SPECTRA-1000® and acrylic yarn.

A review of literature on damping mechanisms reveals three classic mathematical treatments of damping processes are available. These are (a) internal viscous damping, (b) external viscous damping and (c) internal structural damping (also known as hysteretic damping) [Bishop (1956)]. An internal viscous damping force is proportional to a rate of elongation, say du/dt , where u is the overall elongation. The damping force,

F_d , would be expressed $F_d = C du/dt$ where C is a viscous proportionality constant. An internal structural damping force takes into account the damped frequency of oscillation, ω_d , such that $F_d = b/\omega_d du/dt$, where b is a structural damping proportionality constant. The choice of which model to use is on the outset not obvious. Since the damped natural frequency, ω_d , is expected to be relatively constant for a given set of material properties, the structural damping term b/ω_d can also be expected to be relatively constant. It is therefore possible to model loss of rotation based on the application of an equivalent viscous damping coefficient, C , where, if the true nature of the damping is internal viscous then C remains the internal viscous damping coefficient; if the true nature of the damping is internal structural then $C = b/\omega_d$ would be applied. Due to the varying nature of damping with temperature the equivalent viscous damping coefficient is expected to be a function of temperature.

The objective of the loss of rotation study is to provide confidence that the BOLAS will maintain a sufficiently constant rotation rate over its six month mission, with loss of rotation rate no more than 5% of the initial rotation rate. Due to the ambiguity of the tether's damping mechanism the damping model that produces the highest loss of rotation should be eventually adopted. This allows for a conservative design and ensures that the "worst case" damping situation is explored.

Other researchers have modelled damping in tethers. Two classes of tethers distinctly arise in literature based on differences in their fundamental construction. These two classes are (a) conducting and (b) non-conducting. The BOLAS tether is an example of a non-conducting tether; it is composed entirely of non-conducting materials. The TSS-1 tether is an example of a conducting tether. Non-conducting tethers have been modelled using internal viscous models. Specifically Misra, Lanoix, Modi and Jablonski (1998) and Alfriend, Barnds, Coffey, and Stuhrenburg (1995), have used an internal viscous model of the TiPS tether and the BOLAS tether. The TSS research team has used complex models to describe the damping processes within the conducting, multi-layered tether used on that mission. Figure 2-3 shows a tether properties model used in the analysis of the TSS missions. Their model incorporated variable longitudinal elastic

modulus with Maxwell and Kelvin-Voigt visco-elastic models to account for damping and creep effects as well as thermal elongation models Angrilli, Bianchini, Da Forno and Fanti (1995). A simplified version using equivalent internal viscous and friction elements is used for global modelling .

The overall strategy for modelling damping affecting loss of rotation is to determine an equivalent viscous damping coefficient for longitudinal motion of the tether and use this coefficient in the equations of motion, modelling only the longitudinal tether motion. This equivalent viscous damping coefficient can be based on a true viscous damping mechanism or a structural (hysteretic) damping mechanism which takes into consideration the frequency of oscillation of the system. This equivalent viscous damping coefficient can also include, in the one parameter, the damping associated with source 2 in Figure 2-1, damping due to the deployer and retriever reels.

Figure 2-3. TSS-1 simplified sample length model [Angrilli, Bianchini, Da Forno and Fanti (1995)].

The nature of damping processes for tethers has been much discussed in literature. In 1963, using planar, point mass models with a massless, visco-elastic tether, Paul (1963) established that internal viscous tether damping has considerably less effect on rotation rate than a non-linear, time independent structural damping mechanism. However, He and Powell (1990) report that structural damping yields less damping in space; using ground test data for the TSS-1 tether, experimental results tend to favour the combination of internal and external viscous damping mechanisms. Bergamaschi, Carlette, and Lion Stoppato (1995) report that flight analysis of TSS-1 pitch and roll damping results suggest that tether damping may be structural.

There appears to be no clear evidence how to select any one particular damping model and the approach to use an equivalent viscous damping coefficient and then adopt the model that gives the “worst case” estimate for the loss of rotation is warranted.

2.2. Model of Loss of Rotation due to Gravity Gradient Forces

This section summarises the derivation of equations performed by Schultz and Vigneron (1998) that model the loss of rotation rate due to the earth's gravity field acting on the deformable configuration. Based on the rationale developed in section 2.1.1, the BOLAS spacecraft will be modelled as two mass points connected by a massless, longitudinally extensible tether. The tether is considered elastic with either internal viscous or structural damping. Since BOLAS would rotate in the orbit plane (cartwheel), this orientation of spacecraft relative to orbit will be modelled. The loss-of rotation mechanism also has its maximum effect in the cartwheel orientation. To simplify the analysis, the orbit will be taken to be circular.

The following derivation is a Newtonian formulation of the equations of motion. The general three dimensional vector equations are derived first and then specialised for the planar case. Tyc (1996) has derived similar equations using variational methods and also includes deployment terms. Tyc's variational formulation agrees with the equations of motion formulated in this section.

2.2.1. General Vector Equations

The end masses are depicted in Figure 2-4 and Figure 2-5 relative to a geocentric inertial reference basis $[O_e \mathbf{e}_i]$, where $\mathbf{e}_1$ is in the direction of Aries and $\mathbf{e}_3$ is along the earth's North Pole. The end masses are idealised as mass points, with the aft and forward payloads having masses m_1 and m_2 respectively. The tether is shown as partially deployed, and the mass of the deployed portion is assumed negligible relative to the end masses. The force balance equations for the end masses are:

$$m_1 \ddot{\mathbf{R}}_1 = \mathbf{F}_{g1} + \mathbf{N}, \quad (2-1)$$

$$m_2 \ddot{\mathbf{R}}_2 = \mathbf{F}_{g2} - \mathbf{N}. \quad (2-2)$$

Figure 2-4. Forces and earth referenced variables for a tethered two-body spacecraft in earth orbit.

In the above, $\mathbf{R}_1$ and $\mathbf{R}_2$ are the positions of the aft and forward mass points, $\mathbf{F}_{1g}$ and $\mathbf{F}_{2g}$ are gravitational forces, and N is tether tension. The superscript circles represent vector differentiation with respect to the inertial frame.

From Figure 2-5:

$$\mathbf{R}_1 = \mathbf{R} + \mathbf{r}_1, \quad (2-3)$$

$$\mathbf{R}_2 = \mathbf{R} + \mathbf{r}_2. \quad (2-4)$$

For two orbiting mass points connected by a long tether, gravity gradient forces shift the centre of gravity so that it is lower than the centre of mass; for short tethers this shift is negligible and is not applied here [Cosmo and Lorenzini, (1997)]. The vector $\mathbf{R}$ is the position of the centre of mass of the composite configuration. The vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ are referenced from the centre of mass, and thus:

$$m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2 = \mathbf{0}. \quad (2-5)$$

Figure 2-5. Mass centre and relative co-ordinates of a tethered two-body satellite in earth orbit.

The gravitational force can be represented in the form of a series expansion in terms of products $(\mathbf{r}_i \cdot \mathbf{R}) / (\mathbf{R} \cdot \mathbf{R})$ to third order [Hughes (1) (1986)], and is:

$$\mathbf{F}_{gi} = -\frac{\mu m_i}{R^3} \mathbf{R} + \frac{\mu m_i}{R^3} \{3(\mathbf{e}_r \cdot \mathbf{r}_i) \mathbf{e}_r - \mathbf{r}_i\} \quad (2-6)$$

where $i = 1$ or 2 represents aft or forward mass point, respectively. In the above, $R = \sqrt{\mathbf{R} \cdot \mathbf{R}}$, and $\mathbf{e}_r$ is a vector along the trajectory radius, that is, $\mathbf{e}_r = \mathbf{R} / R$. Substitution of eqns (2-3), (2-4), and (2-6) into eqn (2-1) leads to:

$$m_1 \ddot{\mathbf{R}} + m_1 \ddot{\mathbf{r}}_1 = -\frac{\mu m_1}{R^3} \mathbf{R} + \frac{\mu m_1}{R^3} \{3(\mathbf{e}_r \cdot \mathbf{r}_1) \cdot \mathbf{e}_r - \mathbf{r}_1\} + \mathbf{N}. \quad (2-7)$$

Similarly, eqn (2-2) becomes:

$$m_2 \ddot{\mathbf{R}} + m_2 \ddot{\mathbf{r}}_2 = -\frac{\mu m_2}{R^3} \mathbf{R} + \frac{\mu m_2}{R^3} \{3(\mathbf{e}_r \cdot \mathbf{r}_2) \cdot \mathbf{e}_r - \mathbf{r}_2\} - \mathbf{N}. \quad (2-8)$$

Addition of the above two equations and use of eqn (2-5) results in:

$$\ddot{\mathbf{R}} = -\frac{\mu}{R^3} \mathbf{R}. \quad (2-9)$$

Equation (2-9) enables calculation of the trajectory of the mass centre. Equation (2-9) is uncoupled from $\mathbf{r}_1$ and $\mathbf{r}_2$ and thus $\mathbf{R}$ may be calculated separately from rotational dynamics. Combining eqn (2-9) with eqn (2-8) yields:

$$m_2 \ddot{\mathbf{r}}_2 = \frac{\mu m_2}{R^3} \{3(\mathbf{e}_r \cdot \mathbf{r}_2) \mathbf{e}_r - \mathbf{r}_2\} - \mathbf{N}. \quad (2-10)$$

The separation vector, $\mathbf{r}$, between the two mass points is shown in Figure 2-5 and is:

$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1.$$

Combining this with eqn (2-5):

$$\mathbf{r}_2 = \frac{m_1}{m_1 + m_2} \mathbf{r} . \quad (2-11)$$

Substitution of eqn (2-11) into eqn (2-10) then leads to the following differential equation for the separation vector:

$$m_e \ddot{\mathbf{r}} = \frac{\mu m_e}{R^3} \{3(\mathbf{e}_r \cdot \mathbf{r})\mathbf{e}_r - \mathbf{r}\} - \mathbf{N} \quad (2-12)$$

where m_e is the effective mass:

$$m_e = \frac{m_1 m_2}{m_1 + m_2} .$$

Equation (2-12) provides for the calculation of $\mathbf{r}$, and regards R and $\mathbf{e}_r$ as input functions that are obtained separately by solution of eqns (2-9) and (2-10). Equation (2-12) is exact and applies in three dimensions. As $\mathbf{r}$ need not be constant, it accommodates longitudinal stretch and/or deployment including direction changes.

2.2.2. Planar Equation for Force Balance

The foregoing equations will now be specialised by using the following considerations:

- (a) The tether is considered to be fully deployed and the unstretched length is ℓ ;
- (b) The tether is considered elastic with viscous damping, so that $|\mathbf{r}| \neq \ell$ in general;
- (c) The orbit is assumed circular and the BOLAS is rotating (cartwheeling) in the orbit plane. For ease of algebra the orbit is taken to be in the $\mathbf{e}_1 - \mathbf{e}_2$ plane (although in the planned mission BOLAS will have an inclination of 103°); and
- (d) The mass points and tether are also in the orbit plane (the $\mathbf{e}_1 - \mathbf{e}_2$ plane).

The orbit of the mass centre is depicted in Figure 2-6. The radius, R , is constant, and the true anomaly, γ , is measured from $O\mathbf{e}_3$. γ , increases uniformly with time:

$$\gamma = n_0 t \quad (2-13)$$

where n_0 is the orbit rate:

$$n_0 = \sqrt{\mu/R^3}.$$

Figure 2-6. Co-ordinates for a circular orbit.

Figure 2-7. BOLAS in the Orbit Plane.

A basis $[\mathbf{a}]$ is assigned to describe the orientation of the configuration, with $\mathbf{a}_1$ parallel to the separation vector, and $\mathbf{a}_2$ in the orbit plane, as is depicted in Figure 2-7. The bases $[\mathbf{a}]$ and $[\mathbf{e}]$ are related through the expression:

$$[\mathbf{a}] = \begin{bmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} [\mathbf{e}] \quad (2-14)$$

where $[\mathbf{a}] = [\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3]^T$, $[\mathbf{e}] = [\mathbf{e}_1 \ \mathbf{e}_2 \ \mathbf{e}_3]^T$, and α are depicted in Figure 2-7.

Either α or γ can be used to describe the rotational position of the tether in the orbit plane, as is clear in Figure 2-7. It proves most convenient to regard γ as the main variable. From the figure:

$$\alpha = \gamma + n_0 t. \quad (2-15)$$

The angular velocity components in the $[\mathbf{a}]$ basis are:

$$\omega = [\mathbf{a}]^T \omega \quad (2-16)$$

where ω is a 3x1 column matrix $\{\omega_1 \ \omega_2 \ \omega_3\}^T$. Then:

$$\omega = \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \dot{\alpha} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \dot{\gamma} + n_0 \end{Bmatrix}. \quad (2-17)$$

As $\mathbf{a}_1$ lies along the separation vector, $\mathbf{r}$:

$$\mathbf{r} = [\mathbf{a}]^T \mathbf{r} = [\mathbf{a}]^T \begin{Bmatrix} x \\ 0 \\ 0 \end{Bmatrix} \quad (2-18)$$

where $\mathbf{r}$ is a 3x1 column matrix, with the first element equal to $x = |\mathbf{r}|$. The vector $\mathbf{N}$ has the following component form:

$$\mathbf{N} = [\mathbf{a}]^T \mathbf{N} = [\mathbf{a}]^T \begin{Bmatrix} N_1 \\ 0 \\ 0 \end{Bmatrix} \quad (2-19)$$

where $\mathbf{N}$ is a column matrix and N_1 is the scalar tension.

From Figure 2-6 and Figure 2-7, it is evident that $\mathbf{e}_r$ has the following components in the $[\mathbf{a}]$ basis:

$$\mathbf{e}_r = [\mathbf{a}]^T \xi_a = [\mathbf{a}]^T \begin{Bmatrix} \cos \gamma \\ -\sin \gamma \\ 0 \end{Bmatrix}. \quad (2-20)$$

Next, the vector equation (2-12) will be resolved into the $[\mathbf{a}]$ basis. It will be first resolved to a matrix form, and then to the final scalar form. The vector differentiation is resolved to matrix form using:

$$\dot{\mathbf{r}} = [\mathbf{a}]^T (\dot{r} + \tilde{\omega} r) \quad (2-21)$$

where $\tilde{\omega}$ derives from the vector cross product, and is:

$$\tilde{\omega} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}.$$

Combining eqn (2-12) with the matrix variables of eqns (2-16), (2-18), and (2-19), and use of eqn (2-21), leads to:

$$m_e(\ddot{r} + \tilde{\omega} r + 2\tilde{\omega} \dot{r} + \tilde{\omega} \tilde{\omega} r) = \frac{m_e \mu}{R^3} \{3(r^T \xi_a) \xi_a - r\} - N. \quad (2-22)$$

The reduction to scalar variables of the column matrices yields two scalar equations:

$$m_e \ddot{x} = \left\{ \frac{m_e \mu}{R^3} (3 \cos^2 \gamma - 1) + \omega_3^2 \right\} x - N_1, \quad (2-23)$$

$$x \dot{\omega}_3 + 2 \dot{x} \omega_3 = -3 \frac{\mu}{R^3} x \sin \gamma \cos \gamma. \quad (2-24)$$

Equations (2-23) and (2-24) are the scalar differential equations of force balance in the orbit plane. They are valid for all forms of x (deploying, stretching, or fully deployed states).

2.2.3. Force-Displacement Relationship of Longitudinal Deformation

The relation between tether tension and longitudinal deformation of the tether and boundary supports is assumed elastic and with damping:

$$N_1 = k(x - \ell) + C \dot{x} \quad (2-25)$$

where k is the stiffness, C is the equivalent viscous damping parameter discussed in section 2.1.1.3 and ℓ is unstrained tether length which is constant. The value of k could encompass flexibility of the tether material and any flexibility in the deployer and retriever mechanism. Stiffness of the tether material is:

$$k = \frac{EA}{\ell} \quad (2-26)$$

where E is Young's Modulus and A is the tether cross-sectional area. The deformation, u , will be used instead of x :

$$u = x - \ell. \quad (2-27)$$

The form of the parameter C is dependent on the type of material damping model used. Two common models are structural damping and viscous damping. For viscous damping the parameter C is a constant coefficient. For structural damping the parameter C is expressed:

$$C = b/\omega_d \quad (2-28)$$

and has an inverse relationship with the driving frequency (or damped natural frequency) of longitudinal stretch, ω_d , and where b is a constant structural damping coefficient.

The function in eqn (2-23) that involves $\cos^2 \gamma$ may be rewritten as:

$$3\cos^2 \gamma - 1 = \frac{3}{2}(1 + \cos 2\gamma) - 1 = \frac{1}{2} + \frac{3}{2}\cos 2\gamma \quad (2-29)$$

Combining eqns (2-25) and (2-29) with eqn (2-23) and rearranging yields:

$$m_e \ddot{u} + C \dot{u} + \left\{ \frac{EA}{\ell} - m_e \left(\omega_3^2 + \frac{n_0^2}{2} \right) - \frac{3}{2} m_e n_0^2 \cos 2\gamma \right\} u = m_e \ell \left\{ \left(\omega_3^2 + \frac{n_0^2}{2} \right) + \frac{3}{2} n_0^2 \cos 2\gamma \right\} \quad (2-30)$$

Dividing eqn (2-30) through by m_e yields:

$$\ddot{u} + 2\zeta_n \omega_n \dot{u} + \left\{ \omega_n^2 - \left(\omega_3^2 + \frac{n_0^2}{2} \right) - \frac{3}{2} n_0^2 \cos 2\gamma \right\} u = \left(\omega_3^2 + \frac{n_0^2}{2} \right) \ell + \frac{3}{2} n_0^2 \ell \cos 2\gamma \quad (2-31)$$

where ω_n is the natural frequency of the longitudinal stretch mode:

$$\omega_n = \sqrt{EA/m_e \ell} \quad (2-32)$$

and ζ_n is the damping ratio of the longitudinal stretch mode:

$$\zeta_n = C/2m_e \omega_n \quad (2-33)$$

Combination of eqn (2-27) with eqn (2-24) yields:

$$(\ell + u)\dot{\omega}_3 + 2\dot{u}\omega_3 = -\frac{3}{2}n_0^2(\ell + u)\sin 2\gamma \quad (2-34)$$

A kinematic relationship between ω_3 and $\dot{\gamma}$ is contained in eqn (2-17), and is:

$$\dot{\gamma} = \omega_3 - n_0 \quad (2-35)$$

Equations (2-31), (2-34), and (2-35) are a complete set of intercoupled differential equations which model the variables $u(t)$, $\gamma(t)$, and $\omega_3(t)$. The following points are noted:

- (a) Equation (2-30) or (2-31) models the longitudinal stretch of the tether in response to the forcing functions on the right hand side. The mass associated with the longitudinal vibration is thus the effective mass, m_e . The forcing function is essentially the sum of a constant (which causes constant stretch) plus a periodic term which stretches the tether at 2 x per cycle of γ . The effective stiffness of the system is $\left\{ \frac{EA}{\ell} - m_e n_0^2 \left(\frac{1}{2} + \frac{3}{2} \cos 2\gamma \right) - m_e \omega_3^2 \right\}$ and is less than the material stiffness of the string, EA/ℓ . It varies with $\cos 2\gamma$ which could give rise to parametric instability as described by Bolotin (1964).
- (b) Equation (2-34) and (2-35) model the rotation in the orbit plane, with intercoupling terms that account for the tether stretch. Note that if $u \equiv 0$ (no stretch) then eqns (2-34) and (2-35) result in:

$$\ddot{\gamma} + \frac{3}{2} n_0^2 \sin 2\gamma = 0. \quad (2-36)$$

- (c) The equations are non-linear and 'exact' since no terms have been discarded [after the third order expansion of gravitational force is performed in eqn (2-6)].
- (d) It is possible to increase the rotation rate of a tethered two-body configuration by simple retrieval of a short length of tether and allowing conservation of angular momentum to increase the rotation rate. An alternative method is to performed controlled pumping (retrieve-deploy) of the tether to take advantages of the gravity gradient and increase

rotation rate. A control strategy for performing this pumping action is not yet designed but Equation (2-31) can serve as the basis for study of this topic if $u(t)$ is interpreted as a function to be designed.

2.3. Simulation of Equations of Motion

The fourth order system of differential equations defined by eqns (2-31), (2-34) and (2-35) are non-linear and an exact analytical solution is not possible. Numerical solutions of the BOLAS in-plane rotation equations are possible. While there are many numerical solution methods available for solving this system of equations, a constant step-size fourth order Runge-Kutta method was chosen for implementation.

2.3.1. Conversion to First Order Equations

In order to perform a numerical integration of this system the three equations must be converted to a system of four first order differential equations. The system state is defined by using the state variables γ , ω_3 , u and $\dot{u}$. The following variable substitutions are made to facilitate computer implementation:

$$\begin{aligned}x_1 &= \gamma \\x_2 &= \omega_3 \\x_3 &= u \\x_4 &= \dot{u}\end{aligned}\tag{2-37}$$

Differentiating eqn (2-37) with respect to time yields:

$$\begin{aligned}\dot{x}_1 &= \dot{\gamma} \\ \dot{x}_2 &= \dot{\omega}_3 \\ \dot{x}_3 &= \dot{u} = x_4 \\ \dot{x}_4 &= \ddot{u}\end{aligned}\tag{2-38}$$

Examination of eqns (2-31), (2-34) and (2-35) show the functional relations, f_i between the state variables and their differentials have the following form:

$$\begin{aligned}
 \dot{x}_1 &= f_1(x_2) \\
 \dot{x}_2 &= f_2(x_1, x_2, x_3, x_4) \\
 \dot{x}_3 &= f_3(x_4) \\
 \dot{x}_4 &= f_4(x_1, x_2, x_3, x_4)
 \end{aligned} \tag{2-39}$$

Substituting for $\dot{\gamma}$ and ω_3 in eqn (2-35) for the substituted terms $\dot{x}_1$ and x_2 yields:

$$\boxed{\dot{x}_1 = x_2 - n_0} \tag{2-40}$$

Substituting for γ , ω_3 , $\dot{\omega}_3$, u and $\dot{u}$ in eqn (2-34) for the substituted terms, respectively, x_1 , x_2 , $\dot{x}_2$, x_3 and x_4 , yields:

$$(\ell + x_3)\dot{x}_2 + 2x_2x_4 = -\frac{3}{2}n_0^2(\ell + x_3)\sin 2x_1$$

which can be rearranged to give:

$$\boxed{\dot{x}_2 = -2\frac{x_2x_4}{\ell + x_3} - \frac{3}{2}n_0^2\sin 2x_1} \tag{2-41}$$

By definition of our state variables:

$$\boxed{\dot{x}_3 = x_4} \tag{2-42}$$

Substituting for γ , ω_3 , u and $\dot{u}$ and $\ddot{u}$ in eqn (2-31) for the substituted terms, respectively, x_1 , x_2 , x_3 and x_4 , and $\dot{x}_4$ yields:

$$\dot{x}_4 + 2\zeta_n\omega_n x_4 + \left\{ \omega_n^2 - \left(x_2^2 + \frac{n_0^2}{2} \right) - \frac{3}{2}n_0^2 \cos 2x_1 \right\} x_3 = \left(x_2^2 + \frac{n_0^2}{2} \right) \ell + \frac{3}{2}n_0^2 \ell \cos 2x_1$$

which can be rearranged to give:

$$\dot{x}_4 = -2\zeta_n \omega_n x_4 - \left\{ \omega_n^2 - \left(x_2^2 + \frac{n_0^2}{2} \right) - \frac{3}{2} n_0^2 \cos 2x_1 \right\} x_3 + \left(x_2^2 + \frac{n_0^2}{2} \right) \ell + \frac{3}{2} n_0^2 \ell \cos 2x_1. \quad (2-43)$$

Equations (2-40) through (2-43) define a system of four first order differential equations that represent equations (2-31), (2-34) and (2-35). The system of first order equations can now be numerically integrated using a standard fourth order Runge-Kutta numerical integration procedures.

2.3.2. Implementation

A compiled ANSI standard C language program was written and implemented on three different computer platforms: a Pentium 120 MHz based system, a Pentium III 450 MHz based system and a Silicon Graphics R10000 system. Runtime of the numerical routine is highly dependant on the degree of loss of rotation. Generally:

- (a) Runtime is longer where the loss of rotation is comparatively low, for example: when the tether viscous damping (ζ_n) is low or when the BOLAS rotation rate is significantly lower than the tether resonant frequency. Under these conditions the simulation time-step-size is required to be 10^{-2} seconds or smaller for simulating long periods of mission time. Typically one month of mission time could be simulated in 2-3 hours on either the 120 MHz Pentium or Silicon Graphics platforms and ½ hour on the 450 MHz Pentium platform. For BOLAS tether properties and mission parameters shown in Table 2-1, the loss of rotation simulations fall into this category; and
- (b) Runtime is shorter where the loss of rotation is comparatively high, for example: when the tether damping ratio (ζ_n) is high or when the BOLAS rotation rate is near the tether resonant frequency. Under these conditions a simulation time step size of 1 second gives results similar to smaller time step sizes - one month of mission time could be simulated in 2-3 minutes. Even though BOLAS tether properties and rotation rate from Table 2-1 do not fall into this category, these simulations were performed to complete the parametric study shown below.

2.3.3. Numerical Solution for Loss of Rotation

The BOLAS mission is six months long. For purposes of discussing the numerical simulation results a reference mission is defined from Table 2-1. Values used are

$$\ell = 100[m]; \quad m_e = 35.5[kg]; \quad EA = 200[N] \rightarrow \omega_n = 0.2374[rad / s]; \quad \zeta_n = 0.1;$$

$$n_0 = 0.011[cpm] \equiv 0.0011519[rad / s]; \quad (\omega_3)_0 = 10n_0 = 0.11[cpm] \equiv 0.011519[rad / s].$$

Table 2-1. Parameters for BOLAS.

Symbol	Parameter	Value	Units
m_1	Mass of BOLAS-1 (aft body)	70	kg
m_2	Mass of BOLAS-2 (forward body)	72	kg
m_e	Effective mass	35.5	kg
ℓ	Tether Length	100	m
ω_3	Rotation Rate (range)	0.1 to 0.2 0.01 to 0.02	cpm rad/s
n_0	Orbital Rate	0.011 0.00115	cpm rad/s
<u>Thin Tether</u>			
EA	Longitudinal Stiffness	200-15,000	N
ρ	Mass/Length	0.00033	kg/m
ω_n	Natural Frequency (approx)	2.3 to 20 0.23 to 2.0	cpm rad/s
ζ_n	Damping Ratio (range)	5-10	%
<u>Thick Tether</u>			
EA	Longitudinal Stiffness	150-10,000	N
ρ	Mass/Length	0.00136	kg/m
ω_n	Natural Frequency(approx)	1.4 to 15 0.14 to 1.5	cpm rad/s
ζ_n	Damping Ratio (range)	10	%
	Tether breaking strength	90	N

Figure 2-8. Rotation rate ω_3 (top) and tether longitudinal elongation $\bar{u}$ (bottom) at start of mission and at 30 days for $\zeta_n = 0.1$; $n_0 = 0.0011519$ [rad / s]; $(\omega_3)_0 = 0.023038$ [rad / s]; $\omega_n = 0.2374$ [rad / s].

Figure 2-8 shows a simulation result for the reference mission that demonstrates rotation rate (ω_3 vs t) and tether longitudinal deformation (u vs t) at the beginning of the mission compared to 30 days into the mission. The explanation of Figure 2-8 (physical and

analytical) is developed more fully in section 2.4 below. The cases in the figures depict steady responses; i.e. initial conditions have been chosen to avoid initial transients.

Figure 2-9 shows rotation loss for the reference mission, and a high loss case where the tether properties are: $\omega_n = 0.03[\text{rad} / \text{s}]$; $\zeta_n = 1$.

Figure 2-9. Average rotation rate ($\bar{S}$) vs. mission time for two natural frequencies: $\omega_n = 0.03[\text{rad} / \text{s}]$ at $\zeta_n = 1$ and $\omega_n = 0.2374[\text{rad} / \text{s}]$ at $\zeta_n = 0.1$; $n_0 = 0.0011519[\text{rad} / \text{s}]$; $(\omega_3)_0 = 0.023038[\text{rad} / \text{s}]$.

2.3.4. Parametric Study of Tether Properties EA and ζ_n

The elastic and damping properties of the tether have a significant impact on the loss of rotation of the BOLAS spacecraft. To properly design BOLAS it is important to understand how significant are these properties to the loss of rotation. Using the numerical solution for eqns (2-31), (2-34) and (2-35) a parametric study of the effect of tether properties on loss of rotation is shown in Figure 2-10. The orbit rate, $n_0 = 0.0011519[\text{rad} / \text{s}]$, and the initial rotation rate of BOLAS, $(\omega_3)_0 = 20n_0$, are maintained constant for this figure. The varied parameters are the tether properties: ζ_n and EA (in the form $\omega_n = \sqrt{EA/m_e \ell}$ where $\ell = 100[m]$; $m_e = 35.5[kg]$).

Figure 2-10. Loss of Rotation in 30 days as a function of ω_n and ζ_n for $\bar{S}_0 = 0.023038[\text{rad / s}]$; $n_0 = 0.0011519[\text{rad / s}]$.

Figure 2-10 shows that for visco-elastic properties of the BOLAS reference mission (i.e. thick tether with $EA = 200[N]$, $\omega_n = 0.2374[\text{rad / s}]$ and $\zeta_n = 0.1$) the loss of rotation is about 0.12% over 30 days and is thus low enough to maintain a sufficient rotation rate throughout the 180 day mission. However, there is a degree of uncertainty in the parameters of the BOLAS tether and associated flexibility in the deployer and retriever. If ω_n were an order of magnitude less, and/or ζ_n were higher, a loss rate as high as 10-20% per month would be experienced. To have a higher confidence that the BOLAS mission will maintain a sufficient rotation rate the material properties influencing both ω_n and ζ_n (i.e. axial stiffness, EA , and the equivalent viscous damping, C) must be more confidently evaluated. The temperature dependence of these two material properties cannot be ignored and an evaluation of the orbital temperature extremes for the BOLAS reference mission is performed at the end of this chapter.

2.4. Analytical Analysis of Equations of Motion

2.4.1. Domains of Motion

For very small u/ℓ it is useful to study the "zero-order approximation" where $u \equiv 0$ and $\dot{u} \equiv 0$ in eqn (2-34). eqn (2-34) and (2-35) then reduce to:

$$\ddot{\gamma} + \frac{3}{2}n_0^2 \sin 2\gamma = 0. \quad (2-44)$$

Multiply eqn (2-44) by $\dot{\gamma}$ yields:

$$\ddot{\gamma}\dot{\gamma} + \frac{3}{2}n_0^2\dot{\gamma} \sin 2\gamma = 0.$$

This converts to $\frac{d}{dt} \left[\frac{\dot{\gamma}^2}{2} - \frac{3}{4}n_0^2 \cos 2\gamma \right] = 0$ which, on integrating yields:

$$\frac{\dot{\gamma}^2}{2} - \frac{3}{4}n_0^2 \cos 2\gamma = E \quad (2-45)$$

where E is a constant for $u \equiv 0$. In eqn (2-45) $\frac{\dot{\gamma}^2}{2}$ is kinetic energy, $-\frac{3}{4}n_0^2 \cos 2\gamma$ is the potential energy and E is the total energy. The solution to eqn (2-45) can be visualised in terms of the $\gamma - \dot{\gamma}$ phase plane shown in Figure 2-11.

The below phase plane is a well-known result from rigid-body-spacecraft analysis. If the equations are solved with $u \neq 0$ then E will change very slowly due to energy dissipation of the $2\zeta_n \omega_n u$ damping term of eqn (2-31). In summary, there are three domains of motion.

Figure 2-11. Phase-Plane $\gamma - \dot{\gamma}$.

Gravity-captured State, $E < 0$

Gravity captured libration motion

- $\gamma = 0^\circ$ - may be shown to be stable;
- $\gamma = 90^\circ$ - may be shown to be unstable;
- $\gamma = 180^\circ$ - may be shown to be stable;
- $\gamma = 270^\circ$ - may be shown to be unstable.

Slow rotation State, $E > 0$ and small

Slow rotation, near gravity-captured state. Gravity capture is averted if:
 $S_0 \geq \sqrt{3/2}n_0 \geq 1.224n_0$ where S_0 is an initially-imparted rotation rate.

Fast Rotation State, $E \gg 0$

Suppose that an initial condition is specified on rate at a state where $\cos 2\gamma = 0$ (i.e. at $\gamma = 45^\circ$; $\dot{\gamma}(45^\circ) = S_0$ where S_0 is the initial rotation rate. From eqn (2-45):

$$E = \frac{S_0^2}{2}.$$

eqn (2-45) then becomes:

$$\dot{\gamma} = S_0 \sqrt{1 + \frac{3}{2} \frac{n_0^2}{S_0^2} \cos 2\gamma}$$

which for small n_0/S_0 this approximates to:

$$\dot{\gamma} = S_0 \left[1 + \frac{3}{4} \frac{n_0^2}{S_0^2} \cos 2\gamma \right] + \sigma \left(\frac{n_0^4}{S_0^4} \right), \quad (2-46)$$

Where $\sigma \left(\frac{n_0^4}{S_0^4} \right)$ is the remainder of a Taylor series expansion. For BOLAS (Fig. 1-1):

$S_0 = 0.1$ to 0.2 [cpm] and $n_0 = 0.011$ [cpm] then $S_0 = 10n_0$ to $20n_0$ [cpm] range and BOLAS is clearly in the $E \gg 0$ domain. For $S_0 = 10n_0$:

$$\dot{\gamma} = S_0 \left[1 + \frac{3}{4} \left(\frac{1}{10} \right)^2 \cos 2\gamma \right] = S_0 [1 + 0.0075 \cos 2\gamma].$$

For $S_0 = 20n_0$:

$$\dot{\gamma} = S_0 [1 + 0.001875 \cos 2\gamma].$$

The above form of $\dot{\gamma}$ is in evidence in Figure 2-8. Since this thesis deals only with rotating two-body systems well above the gravity capture threshold, the $E \gg 0$ domain will be explored further.

2.4.2. Gravity Gradient Torque Effects on Loss of Rotation

This section presents an approximate expression for $u(t)$ to first order of approximation developed by Schultz and Vigneron (1998). The derivation is not performed here in its entirety but may be found in the reference. The approximation is performed by

introducing a small parameter, ε , applicable solely in the $E \gg 0$ domain. The parameter ε , is defined to be:

$$\varepsilon = \left(\frac{n_0}{\bar{S}} \right)^2 \quad (2-47)$$

where $\bar{S}$ is an average rotation rate of the two-body configuration and n_0 is the orbit rate. When $(n_0/\bar{S})$ is sufficiently small (the average rotation rate, $\bar{S}$, is somewhat above gravity capture, i.e. $(n_0/\bar{S}) < \sqrt{2/3} = 0.8$) the elongation can be approximately expressed as:

$$u(t) = \bar{u} + \varepsilon \frac{u_{10}}{D} \cos(2\bar{S}t - \phi). \quad (2-48)$$

The variable $\bar{u}$ is an average elongation and

$$\bar{u} = \frac{\ell \left\{ (\bar{S} + n_0)^2 + \frac{n_0^2}{2} \right\}}{\left\{ \omega_n^2 - (\bar{S} + n_0)^2 - \frac{n_0^2}{2} \right\}}, \quad (2-49)$$

$$\sin \phi = \frac{2\zeta_n}{D} \left(\frac{2\bar{S}}{\omega_n} \right). \quad (2-50)$$

The numerator u_{10} is:

$$u_{10} = 3(\ell + \bar{u}) \left(1 + \frac{n_0}{2\bar{S}} \right) \frac{\bar{S}^2}{\omega_n^2}, \quad (2-51)$$

and the denominator D is:

$$D = \sqrt{\left\{ \left[1 - \left(\frac{\bar{S} + n_0}{\omega_n} \right)^2 - \frac{1}{2} \left(\frac{n_0}{\omega_n} \right)^2 \right] - \left(\frac{2\bar{S}}{\omega_n} \right)^2 \right\}^2 + 4\zeta_n^2 \left(\frac{2\bar{S}}{\omega_n} \right)^2}. \quad (2-52)$$

Figure 2-12 depicts the solution of elongation u , from eqn (2-48), added to the total unstretched tether length. The gravity gradient force on a tethered two-body configuration varies with angle γ and causes the tether to stretch/contract as the configuration rotates. The pattern of deformation over one cycle appears as a 'bulge' that is displaced by an angle $\phi/2$ from the nadir-zenith line. The angle $\phi/2$ depends on the damping ratio (ζ_n) and is zero if $\zeta_n = 0$.

Figure 2-12. Tether length vs $\bar{\gamma}$ showing the shift in point of maximum elongation of the tether from the local vertical.

A heuristic, physical explanation for loss of rotation is evident in Figure 2-12. Basically the gravity gradient forces exert a torque on the BOLAS configuration at each and every value of γ except $\gamma = 0, \frac{1}{2}\pi, \pi$ and $\frac{3}{2}\pi$. If the deformation, u , were zero then the average torques over π radians of γ would cancel to zero because of symmetry of the gravity gradient about the nadir-zenith line. Also, if ϕ were zero for non-zero u , symmetry would result in an average torque of zero over a γ rotation of π radians. However, in general, $u \neq 0$, $\phi \neq 0$ and the bulge shape of $z = \ell + u$ is not symmetric relative to the nadir-zenith line. Thus, for a γ rotation of π radians the resulting average torque is non-zero and in a direction which retards the motion.

2.5. Impact of Tether Temperature on Loss of Rotation

The tether longitudinal stiffness and longitudinal damping are functions of temperature. Estimating the loss of rotation due to gravity gradient forces requires knowledge of how the axial stiffness, EA , and the equivalent viscous damping coefficient, C , change with temperature. Knowledge of the tether's temperature in orbit is required to apply the material relationships as a function of temperature. The following section presents an analysis of the orbital temperature extremes that the BOLAS tether will experience.

2.5.1. Tether Orbital Temperature

Orbital temperatures of surfaces depend on surface absorptivity, α , surface emissivity, ε , orientation of the orbit and position in the orbit. Non-rotating tethered configurations with KEVLAR ($\alpha = 0.42; \varepsilon = 0.68$) and DYNEEMA ($\alpha = 0.071; \varepsilon = 0.54$) tethers were theoretically shown to have extremes of temperature ranging from -123°C to -13°C (KEVLAR) and -163°C to -98°C (DYNEEMA) for a high inclination orbit ($i = 62.8^{\circ}$) and 260 km altitude [Hallman, Justen and Plescher (1995)]. The BOLAS mission has been theoretically shown to have temperature swings of -151°C to -88°C using a SPECTRA-1000/acrylic tether ($\alpha = 0.1; \varepsilon = 0.625$) [Misra, Lanoix, Modi, and Jablonski (1998)].

2.5.1.1. Tether Models for Temperature Determination

Due to the complexity of internal heat transfer processes of the braided tether an accurate model of these internal processes is difficult. Figure 2-13a shows the outside fibres of the tether receiving incident radiation, causing the fibre to warm. The outer fibres emit radiation directly to space but also to the tether fibres immediately inside the outer layer. This second layer of fibres will then re-emit radiation back to the first layer as well as to the third layer of fibres deeper in the tether. Subsequent layers of fibres are thus irradiated and the interior of the tether eventually approaches the temperature of the outside surface. A possible method to simulate this is shown in Figure 2-13b.

The Concentric Shell model allows an unsteady state solution in which each shell has an outer surface temperature and an inner surface temperature. Conduction accounts for heat transfer between inner and outer surfaces of each shell while radiative heat transfer would occur between shells. Experimental measurement of outer layer surface temperatures during unsteady state heating would result in the calculation of an equivalent number of layers, n , which provides the appropriate response for a mathematical model.

Figure 2-13. Radiative Heat Transfer within Tether (a) braided fibres in layers (b) possible modelling method using concentric shells.

Figure 2-14. Tether cylinder model.

Alternatively the tether can be modelled as a solid cylinder with uniform temperature as shown in Figure 2-14. Here the density, ρ , tether radius, r , heat capacity, c_p and thermal conductivity, k , can be used to determine the unsteady state temperature profile within the tether cylinder. For a high rate of heat conduction the temperature of the cylinder can be modelled as being at a uniform temperature, T .

For a non-rotating tether of length L with adiabatic ends the unsteady state heat equation is:

$$\underbrace{c_p \rho \pi r^2 L \frac{dT}{dt}}_{\text{Rate of change}} = \underbrace{2rL\alpha J_s F_s}_{\text{Solar}} + \underbrace{2rL\alpha\alpha J_s K F_e}_{\text{Earth's albedo}} + \underbrace{2rL\epsilon F_e J_p}_{\text{Earth's thermal}} - \underbrace{2\pi r L \epsilon \sigma T^4}_{\text{Radiative Cooling}} \quad (2-53)$$

The BOLAS mission properties applicable to eqn (2-53) are shown in Table 2-2.

In eqn (2-53) the contribution of thermal radiation from the Earth is applied to the tether with the absorptivity being equal to the emissivity. This assumes the spectrum of the thermal radiation is ideal blackbody radiation (less the absorption of certain wavelengths due to the Earth's atmosphere).

Equation (2-53) simplifies to:

$$\frac{dT}{dt} = \frac{2}{c_p \rho \pi r^2} [\alpha J_s F_s + \alpha\alpha J_s K F_e + \epsilon F_e J_p - \pi \epsilon \sigma T^4] \quad (2-54)$$

For a gravity captured configuration the earth view factor F_e would be more-or-less constant (for a circular orbit). F_e for a rotating configuration undergoes variation, which for the BOLAS would fall in the range 0.13-1.28. The solar view factor F_s also varies, in the range 0-1, depending on tether orientation.

Convective heat transfer from atmospheric molecules is negligible for orbits above, say, 300 km altitude. Table 2-3 shows the aerodynamic heat flux for various orbit altitudes. At 350 km altitude aerodynamic heating is less than 1/100 of the solar and Earth's radiation fluxes combined and 1/12 of the Earth's thermal radiation flux. To determine tether temperature extremes the special case of a noon/midnight orbit is examined.

Table 2-2. BOLAS parameters applicable to thermal modelling.

Variable	Property	Value
r	Radius [m]	0.001 [*]
	Linear Density [kg/m]	0.00136 [*]
ρ	Volume Density [kg/m ³]	433
c_p	Heat Capacity [J/kg/K]	1393 [†]
α	Spectral Absorptance	0.05-0.15 [†]
ϵ	Spectral Emittance	0.5-0.75 [†]
J_s	Solar radiation flux at Earth [W/m ²]	1371 $\pm$ 5 [‡]
J_p	Earth thermal radiation flux, [W/m ²]	237 $\pm$ 21 [‡]
a	Earth Albedo	0.34 [‡]
F_s	Solar View Factor	0 - 1 [‡]
F_e	Earth View Factor	[350 km alt] 0.15-1.28 ^{‡§} [780 km alt] 0.13-0.94 ^{‡§}
K	Shading Factor	0-1 [‡]

* NRL/Carroll; † Carroll/Lanoix; ‡ Fortescue and Stark (1995);

§ low value is for tether axis parallel with nadir-zenith line, high value perpendicular to nadir-zenith line.

Table 2-3. Aerodynamic heating of Earth satellites [Fortescue and Stark(1995)]

Altitude [km]	Aerodynamic Heat Flux [W/m ²]	
	Circular Orbit	Eccentric Orbit (e=0.13)
100	170 000	17 000
150	1300	130
200	160	16
250	39	3.9
300	15	1.5
350	7	0.7
400	3.7	0.4
450	2.2	0.2
500	1.2	0.1

2.5.2. Temperature Variation in a Noon/Midnight Elliptic Orbit

Figure 2-15 shows an elliptical Noon/Midnight orbit. Perigee is at 350 km altitude and, for purposes of establishing the maximum tether temperature, lies on the Earth-Sun line. Apogee is in the Earth's shadow at 780 km altitude. For the satellite remaining in one stationary spot in the satellite orbit the maximum tether temperature will occur at perigee when the tether is aligned perpendicular to the nadir-zenith line. At perigee, the solar view can be described as a function of the angle β made by the tether longitudinal axis and the perpendicular to the nadir-zenith line. For such situation the solar view is described by:

$$F_s = |\cos \beta|. \quad (2-55)$$

Figure 2-15. BOLAS noon/midnight orbit showing perigee and apogee occurring in the plane of the ecliptic (a) view looking down on the plane of the ecliptic from the northern hemisphere, (b) view from a point lying in the plane of the ecliptic.

Figure 2-16. Earth view factor F_e as a function of the parameter β , the angle between the tether longitudinal axis and the perpendicular to the nadir-zenith line [Derived after Fortescue and Stark (1995)].

Figure 2-17. Earth view factor F_e for 350 and 780 km altitudes.

The earth view is a complex function of the angle, β , and is shown in Figure 2-16. Figure 2-17 shows specific relationships for earth view at 350 km and 780 km for tether angle β . Functional relationships for the curves in Figure 2-17 can be developed and used along with eqn (2-55) to numerically integrate eqn (2-54).

Typical temperature profiles of a rotating tethered configuration are shown in Figure 2-18. Figure 2-19 shows average tether temperature (in perigee and apogee positions) as a function of the parameters α and ε . Figure 2-18 shows a maximum average temperature at noontime perigee is -58°C while the minimum average temperature at midnight apogee is -103°C ($\alpha = 0.1; \varepsilon = 0.625$). For the same absorptivity and emissivity Misra, Lanoix, Modi, and Jablonski (1998) report a minimum of -151°C and a maximum of -88°C using a thermal model incorporated into the model of dynamic motion. It is expected that the simple model of eqn (2-54) will overestimate the maximum tether temperature and underestimate the minimum tether temperature. However, both estimates are higher than those reported by Misra, Lanoix, Modi, and Jablonski (1998).

Figure 2-18. Oscillating temperature profiles for rotating tether showing extremes of tether temperature experienced for tether properties $\alpha=0.1$, $\varepsilon=0.625$, $\bar{S} = 10n_0$; absorptivity of Earth thermal radiation is the same as emissivity; transitions from temperature extremes are shown.

Figure 2-19. Tether temperature extremes parameterised by absorptivity (α) and emissivity (ϵ); absorptivity of Earth thermal radiation is the same as emissivity.

Instead of modelling absorptivity equal to emissivity for thermal radiation from the Earth eqn (2-53) can be rewritten with the absorptivity, α , applied to Earth's thermal radiation:

$$\underbrace{c_p \rho \pi r^2 L \frac{dT}{dt}}_{\text{Rate of change}} = \underbrace{2rL\alpha J_s F_s}_{\text{Solar}} + \underbrace{2rL\alpha a J_s K F_e}_{\text{Earth's albedo}} + \underbrace{2rL\alpha F_e J_p}_{\substack{\text{Earth's thermal;} \\ \alpha \text{ used instead} \\ \text{of } \epsilon}} - \underbrace{2\pi r L \epsilon \sigma T^4}_{\text{Radiative Cooling}}. \quad (2-56)$$

This can be written:

$$\frac{dT}{dt} = \frac{2\epsilon}{c_p \rho \pi r} \left[\frac{\alpha}{\epsilon} (J_s F_s + a J_s K F_e + F_e J_p) - \pi \sigma T^4 \right]. \quad (2-57)$$

Figure 2-20 shows typical temperature profiles of a rotating tethered configuration based on eqn (2-57). Figure 2-20 shows a maximum average temperature is at noontime perigee and is -84°C while the minimum average temperature is at midnight apogee and is -166°C ($\alpha = 0.1; \epsilon = 0.625$).

Figure 2-20. Oscillating temperature profiles for rotating tether showing extremes of tether temperature experienced for tether properties $\alpha = 0.1$, $\epsilon = 0.625$; absorptivity of Earth thermal radiation is the same as absorptivity of other radiation; transitions from temperature extremes are shown.

Figure 2-21. Tether temperature extremes parameterised by absorptivity to emissivity ratio (α/ϵ) and emissivity (ϵ); absorptivity of Earth thermal radiation is the same as absorptivity of other radiation.

Figure 2-22. Estimate of orbital temperature profile of the noon-midnight orbit. Perigee and apogee profiles are patched. Maximum temperature is -81°C . Minimum temperature is -149°C .

Figure 2-21 shows average tether temperature (in perigee and apogee positions) as a function of the parameters α/ε and ε . Figure 2-22 shows a composite profile patched together from the perigee and apogee profiles from Figure 2-20. From Figure 2-22 the temperature range that the tether will experience in orbit is -149°C to -82°C .

3. Theoretical Loss of Rotation due to Atmospheric Forces

This chapter examines the effects of atmospheric forces on the loss of rotation of a tethered two-body satellite. General atmospheric effects on satellites are examined in Appendix C with specific emphasis on describing the variable nature of the atmosphere and factors that influence the interaction between satellite surfaces and the atmospheric molecules. Equations from Appendix C are used in Section 3.1 to develop a planar analysis of the atmospheric forces affecting loss of rotation. The BOLAS is examined in section 3.2 with estimates made on its expected loss of rotation due to atmospheric drag.

3.1. Modelling Loss of Rotation due to Atmospheric Forces

The end-over-end rotation of BOLAS is subject to loss of rotation due to atmospheric drag that acts on the two end-bodies and on the tether. It is important to understand the magnitude of this phenomenon to judge whether it is a factor in BOLAS loss of rotation.

3.1.1. Modelling Assumptions

The following assumptions are made and further discussed in section 3.2.5:

- (a) Axis of rotation is always perpendicular to the direction of orbit motion;
- (b) Atmospheric velocities out of the orbit plane (due to atmospheric rotation with the earth) are negligible;
- (c) Aerodynamic cross-sections of the end-bodies are much greater than the aerodynamic cross-section of the tether;
- (d) Aerodynamic cross-sections of the end-bodies are equal to each other;
- (e) Masses of the two end-bodies are equal;
- (f) Flow regime of the BOLAS orbit is free molecular flow;
- (g) Atmospheric density at both ends of BOLAS is equal due to an aerodynamically short tether;
- (h) No wake effects occur when one end-body is in the aerodynamic shadow of the other; and
- (j) The tether is inextensible and inflexible.

3.1.2. General Approach

The BOLAS orbit is planned to be an elliptical orbit with a perigee of 350 km and an apogee of 780 km. This means that the atmospheric density as well as the orbital velocity will vary throughout the flight. Developing equations of motion that will be able to predict the loss of rotation, due to drag, at any point in the orbit is more complex than developing a method of estimating the average loss of rotation, say, in one orbit. As well, precise modelling is subject to the precision of the atmospheric density model used. In order to reduce the complexity of modelling the loss of rotation due to atmospheric drag for a circular orbit will be developed. The velocity and density variations that result from an elliptical orbit can be introduced into the model of a circular orbit as known functions of time.

The local co-ordinate axes are established as shown in Figure 3-1. The e_1 axis (denoted with the subscript, x , in equations) is in the direction of the free-stream free molecular flow and hence lies in the orbit plane. The e_2 axis is perpendicular to the e_1 axis and also lies in the plane of the orbit. For a circular orbit the e_2 axis is on the nadir-zenith line. The e_3 axis is perpendicular to the orbit plane, completing a right hand co-ordinate system.

Figure 3-1. BOLAS rotation in the orbit plane.

3.1.3. Two-body Analytical Expression

Applying a drag force from eqns (C-12) to each of the rotating end-bodies of BOLAS gives:

$$D_1 = \frac{1}{2} \rho_1 V_1^2 F A_1 C_D; D_2 = \frac{1}{2} \rho_2 V_2^2 F A_2 C_D \quad (3-1)$$

where F is an atmospheric rotation correction factor, as outlined in Appendix C.

For BOLAS rotation in the orbit plane the component and total velocities of the first end-body is:

$$(V_1)_x = V - \frac{\omega_3 \ell}{2} \sin \gamma; (V_1)_y = \frac{\omega_3 \ell}{2} \cos \gamma \quad (3-2)$$

$$V_1^2 = (V_1)_x^2 + (V_1)_y^2 = \left(V - \frac{\omega_3 \ell}{2} \sin \gamma \right)^2 + \left(\frac{\omega_3 \ell}{2} \cos \gamma \right)^2 = V^2 - \omega_3 \ell \sin \gamma + \frac{\omega_3^2 \ell^2}{4}$$

Similarly, for the second end-body:

$$(V_2)_x = V + \frac{\omega_3 \ell}{2} \sin \gamma; (V_2)_y = -\frac{\omega_3 \ell}{2} \cos \gamma \quad (3-3)$$

$$V_2^2 = (V_2)_x^2 + (V_2)_y^2 = \left(V + \frac{\omega_3 \ell}{2} \sin \gamma \right)^2 + \left(-\frac{\omega_3 \ell}{2} \cos \gamma \right)^2 = V^2 + \omega_3 \ell \sin \gamma + \frac{\omega_3^2 \ell^2}{4}$$

Figure 3-2. Velocity and drag components acting on a body in orbit.

The direction of the velocity vectors (see Figure 3-2) of each end-body are given by θ_1 and θ_2 where:

$$\begin{aligned}
 \tan \theta_1 &= \frac{\omega_3 \ell \cos \gamma}{2V - \omega_3 \ell \sin \gamma} & \tan \theta_2 &= \frac{-\omega_3 \ell \cos \gamma}{2V + \omega_3 \ell \sin \gamma} \\
 \sin \theta_1 &= \frac{\omega_3 \ell \cos \gamma}{2\sqrt{V^2 - \omega_3 \ell \sin \gamma + \frac{\omega_3^2 \ell^2}{4}}} & \sin \theta_2 &= \frac{-\omega_3 \ell \cos \gamma}{2\sqrt{V^2 + \omega_3 \ell \sin \gamma + \frac{\omega_3^2 \ell^2}{4}}} \\
 \cos \theta_1 &= \frac{2V - \omega_3 \ell \sin \gamma}{2\sqrt{V^2 - \omega_3 \ell \sin \gamma + \frac{\omega_3^2 \ell^2}{4}}} & \cos \theta_1 &= \frac{2V + \omega_3 \ell \sin \gamma}{2\sqrt{V^2 + \omega_3 \ell \sin \gamma + \frac{\omega_3^2 \ell^2}{4}}}
 \end{aligned} \tag{3-4}$$

The drag force on each end-body acts directly opposite to the velocity vector of each end-body, and expressing the drag vectors as a force magnitude applied at angles $\angle(\theta_1 + \pi)$ or $\angle(\theta_2 + \pi)$:

$$\begin{aligned}
 D_1 &= \frac{1}{2} C_D \rho_1 V_1^2 F A_1 \angle(\theta_1 + \pi) = -\frac{1}{2} C_D \rho_1 V_1^2 F A_1 \angle \theta_1 \\
 D_2 &= \frac{1}{2} C_D \rho_2 V_2^2 F A_2 \angle(\theta_2 + \pi) = -\frac{1}{2} C_D \rho_2 V_2^2 F A_2 \angle \theta_2
 \end{aligned} \tag{3-5}$$

which resolves to the component drag forces:

$$\begin{aligned}
 (D_1)_x &= -\frac{1}{2} C_D \rho_1 V_1^2 F A_1 \frac{(V_1)_x}{V_1} = -\frac{1}{2} C_D \rho_1 F A_1 V_1 (V_1)_x \\
 (D_1)_y &= -\frac{1}{2} C_D \rho_1 V_1^2 F A_1 \frac{(V_1)_y}{V_1} = -\frac{1}{2} C_D \rho_1 F A_1 V_1 (V_1)_y \\
 (D_2)_x &= -\frac{1}{2} C_D \rho_2 V_2^2 F A_2 \frac{(V_2)_x}{V_2} = -\frac{1}{2} C_D \rho_2 F A_2 V_2 (V_2)_x \\
 (D_2)_y &= -\frac{1}{2} C_D \rho_2 V_2^2 F A_2 \frac{(V_2)_y}{V_2} = -\frac{1}{2} C_D \rho_2 F A_2 V_2 (V_2)_y
 \end{aligned} \tag{3-6}$$

The sum of the component forces is:

$$\sum_{k=1}^2 (D_k)_x = -\frac{1}{2} C_D F (\rho_1 A_1 V_1 (V_1)_x + \rho_2 A_2 V_2 (V_2)_x) \quad (3-7)$$

$$\sum_{k=1}^2 (D_k)_y = -\frac{1}{2} C_D F (\rho_1 A_1 V_1 (V_1)_y + \rho_2 A_2 V_2 (V_2)_y)$$

If the tether length is sufficiently short then $\rho_1 = \rho_2 = \rho$ where ρ is a function solely of altitude. Also, with similar aerodynamic cross-sectional area for the two end-bodies then $A_1 = A_2 = A$ and eqn (3-7) becomes:

$$\sum_{k=1}^2 (D_k)_x = -\frac{1}{2} C_D F \rho A (V_1 (V_1)_x + V_2 (V_2)_x) \quad (3-8)$$

$$\sum_{k=1}^2 (D_k)_y = -\frac{1}{2} C_D F \rho A (V_1 (V_1)_y + V_2 (V_2)_y)$$

Neglecting the tether mass and with similar end-body masses $m_1 = m_2 = m$ the total mass to which the component drag forces are applied are: $m_T = m_1 + m_2 = 2m$.

Applying $m_T = 2m$ the overall acceleration acting on BOLAS centre of mass, due to atmospheric drag is:

$$a_x = -\frac{1}{4m} C_D F \rho A (V_1 (V_1)_x + V_2 (V_2)_x) \quad (3-9)$$

$$a_y = -\frac{1}{4m} C_D F \rho A (V_1 (V_1)_y + V_2 (V_2)_y)$$

(Note that there is a non-zero y component. Examination of the term $(V_1 (V_1)_y + V_2 (V_2)_y)$ indicates the following two cases can be made depending on whether $\omega_3 > 0$ or $\omega_3 < 0$:

$$\left\{ \omega_3 > 0 \Rightarrow (V_1(V_1)_y + V_2(V_2)_y) < 0 \text{ and } a_y > 0 \right\}, \text{ or}$$

$$\left\{ \omega_3 < 0 \Rightarrow (V_1(V_1)_y + V_2(V_2)_y) > 0 \text{ and } a_y < 0 \right\}.$$

The relative accelerations of the end-bodies, with respect to BOLAS centre of mass, are:

$$\begin{aligned} (a_1)_{x,cm} &= -\frac{1}{2m}C_D F \rho A V_1(V_1)_x + \frac{1}{4m}C_D F \rho A (V_1(V_1)_x + V_2(V_2)_x) \\ &= -\frac{1}{4m}C_D F \rho A (V_1(V_1)_x - V_2(V_2)_x) \end{aligned} \quad (3-10a)$$

$$\begin{aligned} (a_1)_{y,cm} &= -\frac{1}{2m}C_D F \rho A V_1(V_1)_y + \frac{1}{4m}C_D F \rho A (V_1(V_1)_y + V_2(V_2)_y) \\ &= -\frac{1}{4m}C_D F \rho A (V_1(V_1)_y - V_2(V_2)_y) \end{aligned} \quad (3-10b)$$

$$\begin{aligned} (a_2)_{x,cm} &= -\frac{1}{2m}C_D F \rho A V_2(V_2)_x + \frac{1}{4m}C_D F \rho A (V_1(V_1)_x + V_2(V_2)_x) \\ &= \frac{1}{4m}C_D F \rho A (V_1(V_1)_x - V_2(V_2)_x) \end{aligned} \quad (3-10c)$$

$$\begin{aligned} (a_2)_{y,cm} &= -\frac{1}{2m}C_D F \rho A V_2(V_2)_y + \frac{1}{4m}C_D F \rho A (V_1(V_1)_y + V_2(V_2)_y) \\ &= \frac{1}{4m}C_D F \rho A (V_1(V_1)_y - V_2(V_2)_y) \end{aligned} \quad (3-10d)$$

The relative accelerations of eqns (3-10) can be expressed as a force, P , on each end-body, with respect to BOLAS centre of mass:

$$(P_1)_{x,cm} = -\frac{1}{4}C_D F \rho A (V_1(V_1)_x - V_2(V_2)_x); (P_1)_{y,cm} = -\frac{1}{4}C_D F \rho A (V_1(V_1)_y - V_2(V_2)_y) \quad (3-11)$$

$$(P_2)_{x,cm} = \frac{1}{4}C_D F \rho A (V_1(V_1)_x - V_2(V_2)_x); (P_2)_{y,cm} = \frac{1}{4}C_D F \rho A (V_1(V_1)_y - V_2(V_2)_y)$$

eqns (3-11) shows that there exists a force couple that causes a moment about the BOLAS centre of mass. Evaluating the terms $V_1(V_1)_x - V_2(V_2)_x$ and $V_1(V_1)_y - V_2(V_2)_y$ and substituting from eqns (3-2) and (3-3) for V_1 , $(V_1)_x$, $(V_1)_y$, V_2 , $(V_2)_x$ and $(V_2)_y$ yields:

$$\begin{aligned} V_1(V_1)_x - V_2(V_2)_x &= \sqrt{V^2 - \omega_3 \ell \sin \gamma + \frac{\omega_3^2 \ell^2}{4}} \left(V - \frac{\omega_3 \ell}{2} \sin \gamma \right) \\ &\quad - \sqrt{V^2 + \omega_3 \ell \sin \gamma + \frac{\omega_3^2 \ell^2}{4}} \left(V + \frac{\omega_3 \ell}{2} \sin \gamma \right) \\ V_1(V_1)_y - V_2(V_2)_y &= \sqrt{V^2 - \omega_3 \ell \sin \gamma + \frac{\omega_3^2 \ell^2}{4}} \left(\frac{\omega_3 \ell}{2} \cos \gamma \right) \\ &\quad + \sqrt{V^2 + \omega_3 \ell \sin \gamma + \frac{\omega_3^2 \ell^2}{4}} \left(\frac{\omega_3 \ell}{2} \cos \gamma \right) \end{aligned} \quad (3-12)$$

The relative magnitude of the orbital velocity, V , to the velocity due to rotation, $\omega \ell / 2$, is

such that $V / \sqrt{V^2 - \omega_3 \ell \sin \gamma + \frac{\omega_3^2 \ell^2}{4}} \cong 1$ and $V / \sqrt{V^2 + \omega_3 \ell \sin \gamma + \frac{\omega_3^2 \ell^2}{4}} \cong 1$ (0.002%

error) and the terms $\sqrt{V^2 - \omega_3 \ell \sin \gamma + \frac{\omega_3^2 \ell^2}{4}}$ and $\sqrt{V^2 + \omega_3 \ell \sin \gamma + \frac{\omega_3^2 \ell^2}{4}}$ are replaced

with V . Equation (3-12) simplifies to yield:

$$V_1(V_1)_x - V_2(V_2)_x = V \left(V - \frac{\omega_3 \ell}{2} \sin \gamma \right) - V \left(V + \frac{\omega_3 \ell}{2} \sin \gamma \right) = -\omega_3 \ell V \sin \gamma \quad (3-13)$$

$$V_1(V_1)_y - V_2(V_2)_y = V \left(\frac{\omega_3 \ell}{2} \cos \gamma \right) + V \left(\frac{\omega_3 \ell}{2} \cos \gamma \right) = \omega_3 \ell V \cos \gamma$$

Substituting from eqn (3-13) for $V_1(V_1)_x - V_2(V_2)_x$ and $V_1(V_1)_y - V_2(V_2)_y$ in eqn (3-11) gives:

$$(P_1)_{x,\beta} = \frac{1}{4} C_D F \rho A \omega_3 \ell V \sin \gamma; \quad (P_1)_{y,\beta} = -\frac{1}{4} C_D F \rho A \omega_3 \ell V \cos \gamma \quad (3-14)$$

$$(P_2)_{x,\beta} = -\frac{1}{4} C_D F \rho A \omega_3 \ell V \sin \gamma; \quad (P_2)_{y,\beta} = \frac{1}{4} C_D F \rho A \omega_3 \ell V \cos \gamma$$

Figure 3-3. Resultant forces on end-bodies.

The angle of these forces in the BOLAS centre of mass reference are denoted β_1 and β_2 where:

$$\tan \beta_1 = \frac{(P_1)_{y,\beta_1}}{(P_1)_{x,\beta_1}} = \frac{-\cos \gamma}{\sin \gamma} = -\cot \gamma; \quad \tan \beta_2 = \frac{(P_2)_{y,\beta_2}}{(P_2)_{x,\beta_2}} = \frac{\cos \gamma}{-\sin \gamma} = -\cot \gamma, \text{ and}$$

$$\beta_1 = \gamma - \frac{\pi}{2}; \quad \beta_2 = \gamma + \frac{\pi}{2} \quad (3-15)$$

as shown in Figure 3-3.

The force vectors P_1 and P_2 relative to the BOLAS centre of mass reference can be written:

$$P_1 = \frac{1}{4} C_D F \rho A \omega_3 \ell V \angle \left(\gamma - \frac{\pi}{2} \right); \quad P_2 = \frac{1}{4} C_D F \rho A \omega_3 \ell V \angle \left(\gamma + \frac{\pi}{2} \right) \quad (3-16)$$

The components of these forces resolved parallel, $(P_i)_\parallel$, and perpendicular, $(P_i)_\perp$, to the tether are:

$$\begin{aligned} (P_1)_\parallel &= \frac{1}{4} C_D F \rho A \omega_3 \ell V \cos \left(-\frac{\pi}{2} \right) = 0; \\ (P_1)_\perp &= \frac{1}{4} C_D F \rho A \omega_3 \ell V \sin \left(-\frac{\pi}{2} \right) = -\frac{1}{4} C_D F \rho A \omega_3 \ell V; \\ (P_2)_\parallel &= \frac{1}{4} C_D F \rho A \omega_3 \ell V \cos \left(\frac{\pi}{2} \right) = 0; \\ (P_2)_\perp &= \frac{1}{4} C_D F \rho A \omega_3 \ell V \sin \left(\frac{\pi}{2} \right) = \frac{1}{4} C_D F \rho A \omega_3 \ell V \end{aligned} \quad (3-17)$$

The loss of rotation of BOLAS due to atmospheric drag is modelled as:

$$\sum_{k=1}^2 M_k = I \dot{\omega}_3 \quad (3-18)$$

where M_k are the moments acting on BOLAS and I is the moment of inertia about the centre of mass. Modelling the end-bodies as point masses I is:

$$I = m_1 \left(\frac{\ell}{2} \right)^2 + m_2 \left(\frac{\ell}{2} \right)^2 = \frac{m \ell^2}{2} \quad (3-19)$$

The total moment acting on BOLAS is:

$$\sum_{k=1}^2 M_k = (P_1)_\perp \frac{\ell}{2} - (P_2)_\perp \frac{\ell}{2} = -\frac{1}{4} C_D F \rho A \omega_3 \ell^2 V \quad (3-20)$$

From which eqn (3-18) becomes:

$$\dot{\omega}_3 = -\frac{1}{2m} C_D F \rho A V \omega_3 \quad (3-21)$$

In eqn (3-21) ω_3 is the BOLAS rotation relative to the orbital co-ordinates, V is the BOLAS orbital velocity, ρ is the gas density at the BOLAS orbit altitude and varies with this altitude, A is the aerodynamic cross-sectional area of each (assumed equal) of the BOLAS end-bodies, m is the mass of each (assumed equal) of the BOLAS end-bodies, F is a correction term for atmospheric rotation as given in eqn (C-13). The effect of the terms V and ρ will be examined in detail below.

Figure 3-4. Atmospheric Density at High Solar Activity.

3.1.4. Atmospheric Density Variations

The density of earth's atmosphere varies considerably with altitude, solar activity cycle and day/night conditions of the orbit. BOLAS is expected to fly during a high point of the 11 year solar activity cycle. Typical atmospheric density for high solar activity is given in Figure 3-4 above. BOLAS orbit apogee is 780[km] and perigee is 350[km] for which the density variation is 0.071 - 17.3[ng/m³] (for daylight solar activity). Figure 3-5 shows the density and orbital velocity variation through one BOLAS orbit.

Figure 3-5. Density and velocity profile through BOLAS orbit.

Equation (3-21) can be separated as follows:

$$\frac{d\omega}{\omega} = -\frac{1}{2m} C_D F \rho A V dt \quad (3-22)$$

Integrating for constant C_D , F , m and A gives:

$$\ln\left(\frac{\omega(t)}{\omega_0}\right) = -\frac{1}{2m} C_D F A \int_0^t \rho V dt \quad (3-23)$$

From the relationships shown in Figure 3-5 the integral $\int_0^t \rho V dt$ used in eqn (3-23) may

be evaluated numerically for one BOLAS orbit as: $\int_{apogee}^{apogee} \rho V dt = 1.772 \times 10^{-4} [kg / m^2]$ for

daylight solar activity. For night the integral expression is $\int_{apogee}^{apogee} \rho V dt = 7.455 \times 10^{-5} [kg / m^2]$.

Equation (3-23) can then be written in terms of the number of BOLAS orbits, N , as:

$$\ln\left(\frac{\omega(N)}{\omega_0}\right) = -\frac{1.772 \times 10^{-4}}{2m} C_D F A N, \text{ or} \quad (3-24a)$$

$$\frac{\omega(N)}{\omega_0} = e^{-\frac{1.772 \times 10^{-4}}{2m} C_D F A N} \text{ (day)} \quad (3-24b)$$

and for night:

$$\ln\left(\frac{\omega(N)}{\omega_0}\right) = -\frac{7.455 \times 10^{-5}}{2m} C_D F A N \quad (3-24c)$$

$$\frac{\omega(N)}{\omega_0} = e^{-\frac{7.455 \times 10^{-5}}{2m} C_D F A N} \text{ (night)} \quad (3-24d)$$

3.2. Estimate of Loss of Rotation due to Drag

Equations (3-24) can be used to determine the loss in rotation rate due to atmospheric drag. In this section the constant coefficients: C_D , F , m and A are evaluated for the BOLAS reference mission.

3.2.1. Estimation of Drag Coefficient, C_D

The exact geometry of the BOLAS end-bodies is not yet known. From Figure C-2 the BOLAS is seen to operate in a free molecular flow regime (end-bodies have an assumed characteristic length of 1 [m]). Assuming that the BOLAS end-bodies are roughly cylindrical (see Table C-3) with surfaces that provide diffuse re-emission of gas molecules the drag coefficient for a tumbling cylinder can be evaluated as outlined in Appendix C. From the discussion presented in Appendix C the accommodation coefficient is selected to be $\alpha = 0.95$ and T_s/T_i is assumed small giving $r \cong 0.24$. For an assumed cylindrical geometry of the BOLAS end-bodies with $\ell/d = 2$ the drag coefficient can be evaluated for a cylinder tumbling end-over-end:

$$C_D = 2(1 + 0.44r) = 2.21.$$

3.2.2. Estimation of Aerodynamic Cross-sectional Area, A

Table C-4 outlines equations for estimating the cross-sectional area, however, the exact geometry of BOLAS end-bodies is not yet specified and Table C-4 cannot be applied. A conservative estimate of the aerodynamic cross-sectional area of each end-body will be used and the area is selected as $A = 1.0[m^2]$.

3.2.3. Mass of end-bodies, m

The current estimates of BOLAS end-body masses are 70 and 72 [kg]. For these calculations of loss of rotation due to atmospheric drag the average mass $m = 71[kg]$ will be used as the mass for each end-body.

3.2.4. Atmospheric Rotation Parameter, F

Evaluating eqn (C-13) with the known BOLAS parameters $r_p = 350[km]$, $V_p = 7.821[km/s]$, $\omega_a = 7.29 \times 10^{-5}[rad/s]$, $i = 103^\circ$ gives $F = 1.014$.

For a reference mission of 180 days and neglecting any changes in the orbital parameters i , a and e , eqn (3-24b) gives a high (day) estimate of 0.8% loss of rotation. The low (night) estimate based on eqn (3-24d) is 0.3% loss of rotation. The simple average of these two is 0.55% loss of rotation.

3.2.5. Impact of Assumptions

The assumptions stated are now reviewed to check their applicability and impact on the model herein presented.

3.2.5.1. BOLAS out-of plane motion neglected

Misra, Lanoix, Modi and Jablonski (1998) have modelled a tethered two-body system that includes out-of-plane forces. In their report the out-of-plane motion for the BOLAS was evaluated as a ± 1 degree libration from which the error introduced is at most 2%.

3.2.5.2. Out-of-plane atmospheric velocity components neglected

The velocity vector of the free-stream, free molecular flow is not parallel to the orbit path. This is due to the rotation of the earth's atmosphere. Out-of-plane forces arise from this out-of-plane atmospheric motion (cross-drag). From eqn (C-7) the magnitude (V_3) of the atmospheric rotation velocity in the out-of-plane direction (perpendicular to orbital inclination i) is:

$$V_3 = r\omega \cos \phi \sin i. \quad (3-26)$$

From eqns (C-10) and (3-26) it is possible to calculate the in-plane and out-of-plane velocities of a satellite with respect to the free-stream atmospheric flow. The well-known equation of dynamic pressure, $q = \frac{1}{2} \rho V^2$, can be applied to show the relative forces from in-plane and out-of-plane atmospheric velocities. Figure 3-6 shows a comparison of in-plane and out-of-plane dynamic pressures for equatorial and polar region perigees for the

BOLAS satellite. The in-plane dynamic pressure is 10^2 - 10^3 time greater than the out-of-plane dynamic pressure.

Figure 3-6. In-plane out-of-plane dynamic pressures, $\frac{1}{2}\rho V^2$, expected for the BOLAS tethered satellite. Equatorial perigee and perigee at further polar extent (polar region) are shown [$i = 103[\text{deg}]$; altitude of perigee=350 km; altitude of apogee=800 km; atmosphere uniformly rotating at rotation rate of the Earth, density based on daytime density from Fig. 3-4].

3.2.5.3. End-body Aerodynamic Cross-sections predominant

The tether cross-sectional diameter determines the amount of increase in de-rotating moment attributable to the tether. For a TiPS type tether of length 100[m] and 1[mm] thickness, the cross-sectional area of the tether is 0.1[m²], which is 5% of the total area used in the above equations. For a heavier tether this percentage will increase. A BOLAS end-body cross-sectional area lower than the 1[m²] each used above, will also cause a greater tether impact but overall lower loss of rotation.

3.2.5.4. Equal Aerodynamic cross-section of end-bodies

Currently the two end-bodies are specified to have much of the same equipment. If there will exist a difference between the two end-body aerodynamic cross-sections, the de-rotating moment will increase due to the larger differential of forces acting on the end-bodies (similar to a weathervane effect where the large drag of the tail orients the vane into the prevailing wind). A full analysis of this should be performed and is possibly the subject of future work.

3.2.5.5. Mass of the two end-bodies are equal

The BOLAS end-bodies are currently specified as 70 and 72 [kg]. Any difference between the masses will cause the BOLAS centre of mass to shift from the centre of the tether. This will generally (but not always), cause an increase in the de-rotation moment. A full analysis of this should be performed and is possible the subject of future work.

3.2.5.6. Flow regime of the BOLAS orbit is free molecular flow

Figure C-2, in Appendix C, shows the flow regimes experienced by satellites based on a characteristic length and the altitude at which they are flying. Even though the BOLAS tether will be 100 [m] long in the deployed configuration, the characteristic length of the end-bodies is assumed to be approximately 1 [m]. Figure C-2 indicates that free molecular flow may be assumed for this dimension for the entire range of BOLAS orbital altitudes (350 - 780 [km]).

3.2.5.7. Equal atmospheric density at both ends of BOLAS

At an altitude of 350[km] the gas density changes approximately 0.2% for a 100[m] change in altitude. At 780[km] the gas density changes approximately 0.1% for a 100[m] change in altitude. If the length of this tether were, say, 4[km], the atmospheric density variation over 4[km] at 350[km] altitude would be 6%.

3.2.5.8. No wake effects

The rotation of BOLAS has a swept perimeter of approximately 314[m]. Taking a characteristic length of the end-bodies to be approximately 1 [m], in free molecular flow the affected portion of the swept perimeter is only 0.3%.

3.2.5.9. Inextensible tether

The proceeding derivation uses the assumption that the tether is inextensible. In reality the tether is extensible and flexible; the tether length will increase due to centrifugal and gravity gradient forces. This increase in length will increase the moment arm of the drag forces acting on the sub-satellites but also cause an increase in moment of inertia of the two-body configuration. Comparison eqns (3-19) and (3-20) show that the moments and moment of inertia vary with ℓ^2 . The loss of rotation estimate is insensitive to elongation of the tether due to tension.

3.3. Significance to BOLAS Mission

Equations (3-24b) and (3-24d) summarise the effect of atmospheric drag on BOLAS. The forecasted loss of rotation, due to this mechanism, for the six month mission is 0.55% based on equal time in day and night. This estimate is dependent on the solar activity occurring during flight and will vary depending on the exact nature of the solar activity. This estimate is conservative in that it uses an atmospheric density profile consistent with high solar activity. An increase in this estimate can arise from a final BOLAS configuration that has dissimilar masses or dissimilar aerodynamic cross-section of the two sub-satellites.

Misra, Lanoix, Modi and Jablonski (1998) have incorporated atmospheric drag forces in their detailed simulation model of the BOLAS dynamics. Lanoix (1998) reports a forecast loss of rotation due to drag forces of 3.4% over six months. Their simulation uses an exospheric temperature of 1100K for establishing the atmospheric properties that produce atmospheric drag. This temperature is essentially an average of the day and

night exospheric temperatures for periods of high solar activity. Lanoix models the sub-satellites as square prisms (as opposed to spherical sub-satellites used above) with dissimilar masses (76.3 [kg] and 74.8 [kg]); his analysis includes lift and drag on both the sub-satellites and the tether. Lanoix also uses the Jacchia 77 atmospheric model that takes into account oblateness of the earth's atmosphere. Due to the complexity of the Lanoix code it is not possible to understand fully the cause of the higher estimate obtained by it. A full reconciliation of the two estimates is left the subject of future work.

In summary, the loss of rotation of the BOLAS due to atmospheric drag would be about 0.6% for the six month mission.

4. Experimental Determination of Tether Mechanical Properties

As established in chapter 2 the gravity gradient induced excitation of longitudinal motion results in a dissipation of energy due to the longitudinal damping properties of the tether. A high amount of longitudinal damping, ζ_n , will dissipate energy more quickly and may result in unacceptable loss of rotation. The tether's longitudinal stiffness and the effective mass of the system determine the natural longitudinal frequency of the tether, $\omega_n = \sqrt{EA/m_e \ell}$, where E is Young's Modulus, A is the tether cross-sectional area, m_e is the effective mass of the system, m_e and ℓ is the unstrained tether length. The relationship between these key parameters and loss of rotation is summarised in Figure 2-9. Of the key parameters m_e and ℓ are not difficult to measure. However EA and the equivalent viscous damping, ζ_n , are not easily characterised due to the multi-stranded construction of the tether and its non-linear constituent polymer material.

There are two broad strategies of possible experimental determination of EA and ζ_n :

- (a) Small Scale Testing. This is testing of short tether samples using standard laboratory equipment available at facilities such as the National Research Council or Carleton University. This has the advantage that it is easily implemented using existing equipment and recognised methodology. As well, current analytical equipment allows the determination of tether material properties at temperatures similar to the orbital temperatures that the BOLAS would experience. However, testing of short tether samples alone does not take into account the BOLAS tether/hardware interactions that would introduce other sources of damping into the BOLAS loss of rotation dynamics; and
- (b) Large Scale Testing. This is testing in which the BOLAS hardware, such as the tether deployer and/or tether retriever, are set up in a laboratory and tested together with the tether with a test method that emulates the full-scale in orbit conditions as closely as is feasible. This has the advantage of categorising the equivalent material properties as they would apply to the BOLAS configuration and take into account mechanical sources of damping. However, such a unique large scale rig set-up would have to be fabricated and a suitable laboratory of sufficient size found to accommodate the test rig. As well, such a large scale test could not feasibly be brought to the colder temperature levels that the BOLAS would experience in orbit.

The results from the above two test strategies compliment each other. To predict on-orbit mechanical properties of the tether two separate analyses of tether properties were performed. Large scale testing was performed at the Canadian Space Agency (CSA)-St Hubert in ambient conditions to measure longitudinal elastic and damping properties of the tether at lengths ranging from 8 to 38 metres with tension ranges of 1 to 5 newtons. Tension variation in the tether was minimised by suspending the tether horizontally and supporting it at intermediate points. In this manner the orbital configuration of the BOLAS was matched as closely as possible, within the space limitations allowed at CSA.

Small scale testing was performed at the National Research Council (NRC) - Montreal Road Campus, Ottawa. Here the material properties of the tether at lengths of approximately 5mm was determined at cryogenic temperatures with a Thermo Dynamic Mechanical Analyser (TDMA). Elastic and damping properties of the tether was determined for temperatures from -103°C to -3°C. Room temperature testing was not conducted due to limitations of the test method.

It is convenient to describe the implementation and results from these two test strategies separately in this thesis. In this chapter general considerations are described. In Chapter 5 the large scale testing will be described. In Chapter 6 the small scale testing at cryogenic temperatures will be described. In Chapter 7 the results of large and small scale testing will be integrated with the analytical model results of chapter 2 to arrive at an estimate of loss of rotation of the BOLAS in orbit.

4.1. Testing Considerations and Material Modelling of Tethers

Of great importance in establishing test procedures to determine the effects of the laboratory environment compared to the orbital environment in which the tether will function. Differences between a ground test environment and orbital conditions include (a) gravity, (b) scale of testing, (c) temperature variations, (d) humidity and (e) presence of air. In a ground test laboratory (f) load history and (g) load variation rate also influence the test results. Lastly, time based factors such as tether degradation due to molecular impingement or radiation can change the properties of the tether.

4.1.1. Effects of gravitational force

In orbit the tension distribution due to gravity of a rotating tethered two-body configuration (Figure 4-1a) varies approximately according to:

$$N_{o,r} \approx m_t L \omega^2 + \mu g_h L \left(\frac{L}{R+h} \right) \left(\frac{x}{L} \right) \left(1 - \frac{x}{L} \right) \cos \gamma \quad (4-1)$$

where m_e is the effective mass, ω is the rotation rate, R is the radius of the earth, h is the altitude above the earth's surface, g_h is the gravitational acceleration at altitude h , μ is the linear density of the tether, γ is the angle between the tether longitudinal axis the local vertical, L is the length of the tether and x is the distance in the tether direction.

A tether hung vertically in ground-based testing (Figure 4-1b) has a linear tension distribution:

$$N_{g,y} = \mu g x \quad (4-2)$$

where g is the acceleration due to gravity at the earth's surface.

Figure 4-1. Tension distribution profiles due to gravity in a tether for various configurations (a) orbiting rotating tether with end-bodies (b) vertically hung tether on earth's surface with bottom end load (c) horizontally suspended tether on earth's surface.

From Irvine (1981) a tether suspended horizontally in ground-based testing (Figure 4-1c) has a tension distribution:

$$N_{g,x} = H \cosh \frac{\mu g}{H} \left(x - \frac{\ell}{2} \right) \quad (4-3)$$

where H is a constant horizontal component of tension and ℓ is the horizontal span between support points. H is determined from the solution of the following:

$$\frac{\mu g L}{2H} - \sinh\left(\frac{\mu g \ell}{2H}\right) = 0. \quad (4-4)$$

Figure 4-2. Maximum tension in tether for orbital, vertical ground-based and horizontal ground-based configurations. Horizontal ground-based configurations are parameterised by the ratio of span length to tether length, ℓ/L ; $\mu = 0.00136[kg/m]$; $m_e = 35.5[kg]$; $\omega = 0.023028[rad/s]$. Vertical ground-based configuration has no end-mass load.

Figure 4-2 shows the maximum tether tension for the rotating BOLAS configuration is similar to the tension for a horizontal ground-based configuration. The horizontal ground test has maximum tension depending on how much sag is allowed in the tether. This is parameterised through the ratio of the horizontal span to the tether length ℓ/L ; the closer the value of ℓ/L to unity, the higher the maximum tension and the less sag there is in the hanging tether.

Figure 4-3 shows the variation in tension (difference between maximum tension and minimum tension) for an orbital tether configuration is also considerably lower than for the vertical ground configuration. However, the horizontal ground configuration can have tension variation similar to that of an orbiting tether again depending on how much

sag is allowed in the tether. For a 100 m tether the tension variation for the horizontal configurations shown in Figure 4-3 vary from only 1.5 to 20 times the tension variation in the orbital configuration.

Figure 4-3. Tension variation in tethers for orbital, vertical ground-based and horizontal ground-based configurations. Horizontal ground-based configurations are parameterised by the ratio of span length to tether length, ℓ/L ; $\mu = 0.00136[kg/m]$; no end-masses.

4.1.2. Effect of Scale of Testing

Tethers as long as 20 km have been deployed in orbit. It is not possible to accurately measure thermomechanical properties of such lengths of tethers and tests at reduced lengths must be carried out and extrapolated to the mission length.

4.1.3. Effect of Thermal Variations

In orbit the tether will experience temperatures nominally ranging from -149°C to -82°C ; this temperature is dependent on tether surface properties (see section 2.5). Such

temperature variations will certainly cause non-negligible variations in the mechanical properties of the tether.

4.1.4. Effect of Humidity and Presence of Air

Angrilli, Bianchini, Fanti and Debei (1995) found that humidity and presence of air in the laboratory environment had a negligible impact for a vertically hung tether test configuration. It was thought that surface property changes may be produced by changes in ambient humidity. Air may have a possible lubricating effect between fibres or between layers in a composite tether. Also, air produces drag on a moving specimen that could contribute to the measured damping properties.

4.1.5. Effects of Load Variation Rate

The complex construction of tethers causes an observed phenomena called stick/slip between layers of a composite construction. At quick load variation rates Angrilli, Bianchini, Fanti and Debei (1995) found that tension impulses arose due to stick/slip in tethers with length/diameter ratios greater than 1000.

4.1.6. Effect of Load History

The thermomechanical properties of braided and composite tethers show a great dependency on the thermomechanical load history of the tether. Virgin tethers have shown a reduction of as much as 50% in the axial stiffness after the internal composite structure has been allowed to settle. Damping ratio has been, at times, observed to be more than 30% higher in virgin tethers.

4.2. Material Models of Tether Mechanical Properties-Literature Sources

Comprehensive modelling of the mechanical properties of the tether is complicated by the complex structure of the tether. Many mathematical models, ranging in complexity, have been developed to represent tethers in space. He and Powell (1990) have used a

simple linear elastic model with material damping (structural and viscous) and no bending resistance. The model used a linear stress-strain relationship.

Figure 4-4 shows a more complex model used in the analysis of tether properties for the TSS missions. The model incorporated variable longitudinal elastic modulus with Maxwell and Kelvin-Voigt visco-elastic models to account for creep effects as well as thermal elongation models [Fanti (1995)].

Figure 4-4. TSS-1 sample length modelling [Fanti(1995)].

The relative contributions of microscopic material properties and macroscopic braided construction to the overall mechanical properties are not separated in the model of loss of rotation presented in chapter 2 and a prediction of combined elastic and damping properties is sought.

4.3. Prediction of Orbital Tether Mechanical Properties For BOLAS

The prediction of orbital tether thermomechanical properties to use with the loss of rotation model developed in chapter 2 requires that ground test results be corrected for

(a) tether length (b) tension variation (c) temperature and (d) endbody attachment. Ideally tether lengths equal to the configured on-orbit tether length would be tested. This is not possible and shorter lengths must be tested and scale of testing accounted for. As discussed above on-orbit tension variation will be low and the tension variation in a ground tested tether must be accounted for. Orbital temperatures for BOLAS have been shown to range, conservatively, from -149°C to -82°C . Performing cryogenic mechanical testing of long tethers is not possible within the scope of this thesis. Results from ground test of long tethers at ambient temperature must be corrected to reflect the orbital temperatures.

5. Large Scale Testing For BOLAS

This chapter presents the theory and results of the large scale testing of the BOLAS tether/hardware at the Canadian Space Agency – St Hubert facility. Section 5.1 introduces the original concept and evolution of strategy for performing the large scale testing. Section 5.2 presents the mechanical design of the large scale experimental test rig. In section 5.3 equations of motion for the test rig are developed in which the tether is modelled both as a rod and as a cable. As well, section 5.3 presents the method of analysis used to reduce the raw data obtained during experimentation. Lastly, section 5.4 presents the results of the experimental data and gives estimates of axial stiffness, EA , and equivalent viscous damping coefficient, C .

5.1. Introduction and Strategy

The initial intent for the large scale testing was to simulate the in-orbit vibration of the BOLAS configuration on the ground with the in-orbit centrifugal force being created by a pendulum mass. The in-orbit equation of motion for constant from Eq. (2-31) is:

$$\ddot{u} + 2\zeta_n \omega_n \dot{u} + \left\{ \omega_n^2 - \left(\omega_3^2 + \frac{n_0^2}{2} \right) - \frac{3}{2} n_0^2 \cos 2\gamma \right\} u = \left(\omega_3^2 + \frac{n_0^2}{2} \right) \ell + \frac{3}{2} n_0^2 \ell \cos 2\gamma.$$

Variation in ω_3 and γ in the above equation cause the coefficients of this second order equation to vary only slightly. If these coefficients were exactly constant, the equation has exactly the same form as a simple derivation of the equation of motion of the horizontal large scale experimental test rig, from Eq. (5-19a) below:

$$\ddot{\phi} + \frac{C}{m_p} \dot{\phi} + \frac{EA_0}{L_0} + \frac{g}{\ell_p} \left(\frac{\mu L_0}{2m_p} \frac{(X-r)}{X} + 1 \right) \phi = \frac{EA_0 (X-r-L_0)}{m_p \ell_p L_0}.$$

To reproduce as closely as possible the BOLAS tethered configuration for ground testing it was originally envisioned to make a full scale rig using 100 m of tether, equal to that used in the final BOLAS configuration. It was not possible to find an appropriate facility that would house 100 m of a horizontally suspended tether. Ultimately a reduced length test rig using 38 m of tether was fabricated. Analysis of the test data for a horizontally suspended tether proved to be more complex than originally envisioned due to tension variation caused by the sag of the tether.

Analysis of the experimental data allowed the researchers to characterise the tether's longitudinal material properties at the ambient conditions of the test facility. The influences of tether length, suspension interval length and end attachment were investigated at tensions bounding the expected orbital tensions in the BOLAS tether.

5.2. Mechanical Design of Experimental Equipment

5.2.1. Canadian Space Agency High Bay Configuration

The High Bay at the Canadian Space Agency's (St. Hubert) was selected for performing long tether testing as it had the largest open area of the various sites reviewed. A large open area was required to maximize the length of tether that could be tested.

Figure 5-1 shows the experimental configuration in the CSA High Bay. The area at the pendulum mass and instruments is denoted the “near end” and the area at the end attachment is denoted the “far end”. This nomenclature naturally arises with respect to the operator of the experiment who works mostly with the instruments and the pendulum mass, at the near end. However, the origin of the coordinate system used for description and analysis of the setup is at the end attachment point of the tether, at the far end. The x axis is defined as being horizontal and positive in the direction from the end attachment to the pendulum mass. The y axis is defined as being in the vertical direction with positive being up. The z axis completes a right hand system of coordinate axes.

The experimental setup was comprised of a 35 kg dumbbell shaped pendulum mass supported by two wires giving an overall 3.325 metre radius of pendulum motion. At the near end the tether was attached to a load cell mounted on the pendulum mass. The tether was run horizontally and was suspended at intervals to reduce the tension variation in the tether. The far end of the tether was attached to either a fixed or reeled end attachment mechanism that anchored the far end of the tether.

An experiment would be performed by setting up a particular experimental configuration, starting the pendulum mass in motion and recording the tension data transmitted by the load cell with the LMS data acquisition equipment. Analysis of data was performed separately from the experiments.

5.2.2. Apparatus Specifications

Components for the long tether testing consisted of (a) Pendulum mass, (b) Load Cell, (c) Pendulum support and support wires, (d) Tether supports, (e) End Attachment (two types: fixed and wound reel), (f) Tether, (g) Load cell power supply with voltmeter, and (h) LMS Modal Analysis Equipment.

5.2.2.1. Pendulum Mass

Design of the pendulum mass/wire support system was driven by the need to minimise sources of energy dissipation, allowing test results to more accurately reflect the energy dissipation (i.e. damping properties) of only the tether. The pendulum mass was designed

Figure 5-2. Assembly drawing of pendulum mass.

Figure 5-3. Side view of pendulum mass showing ruler to measure static equilibrium position of centre of mass (White tether highlighted in black).

to a nominal 35 kg mass, that being the estimated effective mass of the BOLAS spacecraft. It comprised of two 35 lb weights connected by a shaft enclosed by a bearing housing from which the mass was supported by two high tensile steel wires. Mounting locations for a load cell and accelerometers were provided for on the bearing housing although in testing only the load cell was used. After fabrication of all components the connecting shaft was machined to achieve an overall mass of 35 kg. Low viscosity sewing machine oil was used to lubricate the bearings to reduce energy dissipation.

5.2.2.2. Load Cell

The load cell selected for measuring tether tension was a TTI with an upper measurement limit of 7.5 Newtons (at 50% allowable overrange). Figure 5-4 shows the calibration curve for the load cell using 10 VDC excitation. Figure 5-5 shows a view of the pendulum mass with the load cell attached to the bearing housing and the tether attached to the load cell. The tether was attached to the load cell by using a threaded cylindrical adaptor.

Figure 5-4. Load cell calibration curve for large scale testing at Canadian Space Agency - St Hubert (10 VDC excitation voltage).

Figure 5-5. Pendulum mass showing load cell positioning with tether attached.

5.2.2.3. Pendulum Support and Support Wires

The design of the pendulum support and support wires was driven, again, by the need to minimise energy dissipation. Figure 5-6 shows the suspension of the pendulum mass from the overhead beam by the support wires. High tensile piano wire was selected to approximate the inextensible case for the supporting wires thereby minimising interference with the pendulum/tether dynamics. Crucial in maintaining low energy dissipation was the manner in which the wires were attached to the pendulum and to the support beam.

The wires were soldered into end-stops upon which the support forces were distributed. A specially designed clamp from which the wires were supported was attached to the overhead beam. Special threaded plugs were used to attach the wires to the pendulum bearing housing. While the pendulum was in motion a small amount of flexure of the

wire occurs at their entry into the soldered end-stops; this flexure is a very minor source of energy dissipation. Low pendulum damping is established in section 5.7.1.

Figure 5-6. Pendulum mass showing support wires and beam clamp.

5.2.2.4. Tether Support Lines

The tether was supported at intermediate points between the pendulum mass and the end attachment. Simple clamps, with extensions to provide proper z-direction spacing, were attached to an overhead beam. From the clamp extensions lines of light-weight waxed dental floss were used to support the tether [Figure 5-7]. Waxed dental floss was chosen for this application due to its high strength/weight ratio and commercial availability. As well, the wax coating proved to be invaluable when attaching the support lines to the tether. The wax coating provided sufficient self-adhesion of the dental floss so that it was possible to loop the dental floss around the tether and use a single overhand knot to secure the dental floss to the tether. This simplicity in attaching the support line to the tether greatly reduced the time to align all supported points to the same horizontal level. Tether support lengths throughout the experimentation were kept constant. However for

tests of the 19.560 metre long tether, with fixed end attachment, the number of tether supports used was varied to determine the effect of support length distance.

Figure 5-7. View of the tether supported by vertical line (waxed dental floss).

5.2.2.5. End Attachment

To anchor the far end of the tether two attachments designated (a) "Fixed End Attachment" and (b) "Reeled End Attachment", were used. These are shown in Figure 5-8 and Figure 5-9, respectively. The end attachments were clamped to a column and provided a firm anchor point for the far end of the tether. Adjustment of the tether tension was performed using a screw-type tensioner that moved the far end of the tether in the x direction. The tether was drawn axially through a hole in the cylinder and out the side where the tether was knotted to keep it in place. Winding of the tether material on the reeled end attachment is shown in Figure 5-10.

Figure 5-8. Fixed End Attachment.

Figure 5-9. Reeled End Attachment.

Figure 5-10. Reeled End Attachment showing tether wound on reel.

5.2.2.6. Tether

The tether was supplied by Tether Technologies which is located in San Diego, California. The total tether length tested was varied from 1.630 to 37.549 metres using the fixed end attachment and 1.394 to 31.33 metres using the reeled end attachment. The increments in tether length were determined by the 6 metre spacing of the columns onto which the end attachment could be mounted. As well, minor variation in initial tether length occurred from experiment to experiment for the reeled end attachment. For the reeled end attachment compression of the wound tether at higher tensions caused a 0.025 metre variation of initial tether length from experiment to experiment before techniques were developed to avoid this.

5.2.2.7. Load Cell Power Supply and Voltmeter

The load cell power supply and voltmeter was provided by CSA.

5.2.2.8. LMS Modal Analysis Equipment

The Modal Analysis Equipment was provided by CSA. It comprises the Leuven Measurement System (LMS) software on a Hewlett Packard workstation with supporting front end hardware.

Figure 5-11. Equipment layout at near end of experimental setup.

5.3. Model of Tether and Pendulum

This section presents increasingly complex dynamic models of the experimental system described in preceding sections.

The first model presented neglects the influence of gravity on tether creating tether sag. The Lagrangian method is used to develop linear equations of motion based on an elastic rod in longitudinal deformation. Subsequent models of dynamic motion include the influence of gravity on the tether. The formulation includes gravity induced sag and treats the tether as a cable without bending stiffness. First examined is the case of static equilibrium, second is the case of a single span tether in dynamic longitudinal

deformation about static equilibrium and lastly the case of a multi-span tether in dynamic longitudinal deformation.

5.3.1. Model With Tether Sag Neglected (Rod Model)

In this model the tether is modelled as a longitudinally deforming straight rod (spring with mass) and neglects any sag due to gravity induced in the tether. The mass of the tether is taken into account for determination of kinetic and potential energies used in the Lagrangian derivation.

Figure 5-12. Definition diagram of a single span tether/pendulum experimental configuration modelled as a straight rod in longitudinal deformation.

The rod is fixed at one end (point A) and attached to the pendulum at the other end (point B). The tether is always a straight line between points A and B. The centre of gravity of the tether is at the midpoint of the uniform tether of unstrained length L_0 and is subject to motion in the plane due to motion of the pendulum. The tether undergoes a longitudinal deformation of u over its entire length. The length from the fixed point to

the centre of gravity is $(L_0 + u)/2$. Due to motion the tether has a general inclination to the horizontal defined as the angle θ . The kinetic energy of the tether is

$$K_t = \frac{1}{2} \mu L_0 \left[\left(\frac{L_0 + u}{2} \right)^2 \dot{\theta}^2 + \left(\frac{\dot{u}}{2} \right)^2 \right] \text{ and the kinetic energy of the pendulum mass } K_p \text{ is}$$

given by $K_p = \frac{1}{2} m_p \ell_p^2 \dot{\phi}^2$. The total kinetic energy K_t of the system is:

$$K_t = \frac{1}{2} m_p \ell_p^2 \dot{\phi}^2 + \frac{1}{8} \mu L_0 \left[(L_0 + u)^2 \dot{\theta}^2 + \dot{u}^2 \right] \quad (5-1)$$

From the geometry of the setup $\theta = \tan^{-1} \left\{ \frac{\ell_p (1 - \cos \phi)}{X - \ell_p \sin \phi} \right\}$ and

$$\dot{\theta} = \frac{\ell_p (X \sin \phi - \ell_p + \ell_p \cos \phi)}{X^2 - 2X\ell_p \sin \phi + 2\ell_p^2 (1 - \cos \phi)} \dot{\phi}. \text{ For simplification in notation we write}$$

$\dot{\theta} = g(\phi) \dot{\phi}$ where:

$$g(\phi) = \frac{\ell_p (X \sin \phi - \ell_p + \ell_p \cos \phi)}{X^2 - 2X\ell_p \sin \phi + 2\ell_p^2 (1 - \cos \phi)} \quad (5-2)$$

Now the elongation u is

$$u = \sqrt{(X - \ell_p \sin \phi - r \cos \theta)^2 + (\ell_p (1 - \cos \phi) - r \sin \theta)^2} - L_0 \quad (5-3)$$

but

$$\sin \theta = S(\phi) = \frac{\ell_p (1 - \cos \phi) (X^2 - 2X\ell_p \sin \phi + \ell_p^2 (1 - \cos^2 \phi))}{(X - \ell_p \sin \phi) (X^2 - 2X\ell_p \sin \phi + 2\ell_p^2 (1 - \cos \phi))}, \quad (5-4)$$

$$\cos \theta = C(\phi) = \frac{\sqrt{(X^2 - 2X\ell_p \sin \phi + \ell_p^2 (1 - \cos^2 \phi))}}{\sqrt{(X^2 - 2X\ell_p \sin \phi + 2\ell_p^2 (1 - \cos \phi))}}$$

Substituting Eqs. (5-4) into Eq. (5-3):

$$u = f(\phi) - L_0; \dot{u} = f_\phi(\phi)\dot{\phi} \quad (5-5)$$

where

$$f(\phi) = \sqrt{\left(X - \ell_p \sin \phi - r \sqrt{\frac{(X^2 - 2X\ell_p \sin \phi + \ell_p^2(1 - \cos^2 \phi))}{(X^2 - 2X\ell_p \sin \phi + 2\ell_p^2(1 - \cos \phi))}} \right)^2 + \left(\ell_p(1 - \cos \phi) - r \sqrt{\frac{\ell(1 - \cos \phi)(X^2 - 2X\ell_p \sin \phi + \ell_p^2(1 - \cos^2 \phi))}{(X - \ell_p \sin \phi)(X^2 - 2X\ell_p \sin \phi + 2\ell_p^2(1 - \cos \phi))}} \right)^2} \quad (5-6)$$

The total kinetic energy K_s of the system is then

$$K_s = \frac{1}{2} \left\{ m_p \ell_p^2 + \frac{1}{4} \mu L_0 \left[(f(\phi)g(\phi))^2 + f_\phi^2(\phi) \right] \right\} \dot{\phi}^2. \quad (5-7)$$

The potential energy U_t of the tether is the strain potential energy plus the gravitational potential energy and

$$U_t = \frac{1}{2} \frac{EA_0}{L_0} u^2 + \frac{1}{2} \mu g L_0 (L_0 + u) \sin \theta. \quad (5-8)$$

The potential energy of the pendulum U_p is determined from the horizontal reference and

$$U_p = m_p g \ell_p (1 - \cos \phi). \quad (5-9)$$

The total potential energy U_s of the system is

$$U_s = \frac{1}{2} \frac{EA_0}{L_0} u^2 + \frac{1}{2} \mu g L_0 (L_0 + u) \sin \theta + m_p g \ell_p (1 - \cos \phi). \quad (5-10)$$

or

$$U_s = \frac{1}{2} \frac{EA_0}{L_0} [f(\phi) - L_0]^2 + \frac{1}{2} \mu g L_0 f(\phi) S(\phi) + m_p g \ell_p (1 - \cos \phi). \quad (5-11)$$

The non-conserved work due to damping using an equivalent viscous damping, C is:

$$\delta W_{nc} = -C \frac{\partial u}{\partial t} \delta u. \quad (5-12)$$

From the previous definition of u it is possible to find $\partial u / \partial \phi$ and δu but these are too cumbersome to relate immediately.

Since $u = f(\phi) - L_0$ and L_0 is constant $\frac{\partial u}{\partial \phi} = \frac{\partial f(\phi)}{\partial \phi} = f_\phi(\phi)$ from which is written $\delta u = f_\phi(\phi) \delta \phi$. Equation (5-12) becomes:

$$\delta W_{nc} = -C f_\phi^2(\phi) \frac{\partial \phi}{\partial t} \delta \phi. \quad (5-13)$$

The virtual work is a function of the generalised force F_ϕ :

$$\delta W_{nc} = F_\phi \delta \phi = -C f_\phi^2(\phi) \frac{\partial \phi}{\partial t} \delta \phi, \quad (5-14)$$

and the generalised force for Lagrange's method becomes

$$F_\phi = -C f_\phi^2(\phi) \frac{\partial \phi}{\partial t}. \quad (5-15)$$

Applying Lagrange's method of formulating the equation of motion:

$$\frac{d}{dt} \left(\frac{\partial K_s}{\partial \dot{\phi}} \right) - \frac{\partial K_s}{\partial \phi} = - \frac{\partial U_s}{\partial \phi} + F_\phi. \quad (5-16)$$

Evaluating the various terms for Eq. (5-15):

$$\begin{aligned}
 \frac{d}{dt} \left(\frac{\partial K_s}{\partial \dot{\phi}} \right) &= \frac{d}{dt} \left(\frac{\partial}{\partial \dot{\phi}} \left[\frac{1}{2} \left\{ m_p \ell_p^2 + \frac{1}{4} \mu L_0 \left[(f(\phi)g(\phi))^2 + f_\phi^2(\phi) \right] \right\} \dot{\phi}^2 \right] \right) \\
 &= \frac{d}{dt} \left(\left\{ m_p \ell_p^2 + \frac{1}{4} \mu L_0 \left[(f(\phi)g(\phi))^2 + f_\phi^2(\phi) \right] \right\} \dot{\phi} \right) \\
 &= \left\{ m_p \ell_p^2 + \frac{1}{4} \mu L_0 \left[(f(\phi)g(\phi))^2 + f_\phi^2(\phi) \right] \right\} \ddot{\phi} \\
 &\quad + \frac{1}{2} \mu L_0 \left\{ f(\phi)g(\phi) \left[f(\phi)g_\phi(\phi) + g(\phi)f_\phi(\phi) \right] + f_\phi(\phi)f_{\phi\phi}(\phi) \right\} \dot{\phi}^2
 \end{aligned}$$

Dropping the non-linear term in $\dot{\phi}^2$ gives

$$\frac{d}{dt} \left(\frac{\partial K_s}{\partial \dot{\phi}} \right) = \left\{ m_p \ell_p^2 + \frac{1}{4} \mu L_0 \left[(f(\phi)g(\phi))^2 + f_\phi^2(\phi) \right] \right\} \ddot{\phi}$$

$$\begin{aligned}
 \frac{\partial U_s}{\partial \phi} &= \frac{\partial}{\partial \phi} \left\{ \frac{1}{2} \frac{EA_0}{L_0} [f(\phi) - L_0]^2 + \frac{1}{2} \mu g L_0 f(\phi) S(\phi) + m_p g \ell_p (1 - \cos \phi) \right\} \\
 &= \left\{ \frac{1}{2} \frac{EA_0}{L_0} \frac{\partial}{\partial \phi} [f(\phi) - L_0]^2 + \frac{1}{2} \mu g L_0 \frac{\partial}{\partial \phi} \{f(\phi) S(\phi)\} + m_p g \ell_p \frac{\partial}{\partial \phi} (1 - \cos \phi) \right\} \\
 &= \frac{EA_0}{L_0} (f(\phi) - L_0) \frac{\partial f(\phi)}{\partial \phi} + \frac{1}{2} \mu g L_0 \{f_\phi(\phi) S(\phi) + f(\phi) S_\phi(\phi)\} + m_p g \ell_p \sin \phi
 \end{aligned}$$

Substituting these terms and Eq. (5-15) into Eq. (5-16) and rearranging yields:

$$\begin{aligned}
 &\left\{ m_p \ell_p^2 + \frac{1}{4} \mu L_0 \left[(f(\phi)g(\phi))^2 + f_\phi^2(\phi) \right] \right\} \ddot{\phi} + C f_\phi^2(\phi) \dot{\phi} \\
 &= -\frac{EA_0}{L_0} (f(\phi) - L_0) f_\phi(\phi) - \frac{1}{2} \mu L_0 \{f_\phi(\phi) S(\phi) + f(\phi) S_\phi(\phi)\} - m_p g \ell_p \sin \phi
 \end{aligned} \tag{5-17}$$

We now evaluate the various function combinations of $f(\phi)$, $g(\phi)$, $S(\phi)$ and their derivatives. To facilitate a linearization of Eq. (5-17) various Taylor series expansions of component functions are shown in Table 5-1.

Table 5-1. Taylor series expansion terms of Eq. (5-17) component functions.

Function	0 th order term	1 st order term	2 nd order term
$(f(\phi)g(\phi))^2$	0	0	$\frac{(X-r)^2}{X^2} \ell_p^2 \phi^2$
$f(\phi)g(\phi) \left(\begin{matrix} f_p(\phi)g(\phi) \\ + f(\phi)g_p(\phi) \end{matrix} \right)$	0	$\frac{(X-r)^2}{X^2} \ell_p^2 \phi$	$\frac{3(X-r)(X-3r)}{2X^3} \ell_p^3 \phi^2$
$f_p^2(\phi)$	ℓ_p^2	$-\frac{r^2 \ell_p^2}{X(X-r)} \phi$	$\frac{1}{4} \frac{r^4 \ell_p^2}{X^2(X-r)^2} \phi^2$
$(f(\phi) - L_0)f_p(\phi)$	$-\ell_p(X-r-L_0)$	$\ell_p^2 \phi$	$\frac{\ell_p \phi^2}{2} (X-r-L_0)$
$f_p(\phi)S(\phi) + f(\phi)S_p(\phi)$	0	$\frac{(X-r)}{X} \ell_p \phi$	$-\frac{3}{2} \frac{r \ell_p^2 \phi^2}{X^2}$
$\sin \phi$	0	ϕ	0
$\cos \phi$	1	0	$-\frac{\phi^2}{2}$

Performing a first order Taylor series expansion of Eq. (5-17) yields:

$$\left(m_p + \frac{1}{4} \mu L_0 \right) \ell_p^2 \ddot{\phi} + C \ell_p^2 \dot{\phi} = -\frac{EA_0}{L_0} (-\ell_p(X-r-L_0) + \ell_p^2 \phi) - \frac{1}{2} \mu g L_0 \frac{(X-r)}{X} \ell_p \phi - m_p g \ell_p \phi \quad (5-18)$$

Grouping like terms Eq. (5-18) becomes:

$$\ddot{\phi} + \frac{C}{\left(m_p + \frac{1}{4} \mu L_0 \right)} \dot{\phi} + \frac{\frac{EA_0}{L_0} + \frac{g}{\ell_p} \left(\frac{\mu L_0}{2} \frac{(X-r)}{X} + m_p \right)}{\left(m_p + \frac{1}{4} \mu L_0 \right)} \phi = \frac{EA_0(X-r-L_0)}{\left(m_p + \frac{1}{4} \mu L_0 \right) \ell_p L_0} \quad (5-19)$$

Equation (5-19) is a second order constant coefficient differential equation. The static

displacement is seen to be $\phi_0 = \frac{EA_0}{\ell_p L_0} (X-r-L_0) \left/ \left[\frac{EA_0}{L_0} + \frac{g}{\ell_p} \left(m_p + \frac{\mu L_0}{2} \frac{(X-r)}{X} \right) \right] \right.$.

The natural frequency is $\omega_n = \sqrt{\frac{EA_0}{(m_p + \frac{1}{4}\mu L_0)L_0} + \frac{g}{(m_p + \frac{1}{4}\mu L_0)\ell_p} \left(\frac{\mu L_0}{2} \frac{(X-r)}{X} + m_p \right)}$.

The term $m_p + \frac{1}{4}\mu L_0$ is an effective mass of the pendulum combined with one quarter of the mass of the tether. The pendulum mass dominates since $m_p / \frac{1}{4}\mu L_0 > 2700$. With the pendulum mass as dominant the tether component can be ignored and the equation of motion is:

$$\ddot{\phi} + \frac{C}{m_p} \dot{\phi} + \frac{EA_0}{L_0} + \frac{g}{\ell_p} \left(\frac{\mu L_0}{2m_p} \frac{(X-r)}{X} + 1 \right) \phi = \frac{EA_0(X-r-L_0)}{m_p \ell_p L_0} \quad (5-19a)$$

Equation (5-19a) is of the form $\ddot{\phi} + 2B\dot{\phi} + \omega_n^2\phi = 0$ where: $B = \frac{C}{2m_p}$ and

$$\omega_n = \sqrt{\frac{EA_0}{m_p L_0} + \frac{g}{\ell_p} \left(\frac{\mu L_0}{2m_p} \frac{(X-r)}{X} + 1 \right)}.$$
 For a damped oscillatory system the well know

solution is $\phi = De^{-Bt} \sin(\omega_d t + \delta)$ where $\omega_d = \sqrt{\omega_n^2 - B^2}$, $D = \sqrt{\phi_0 + \frac{\dot{\phi}_0}{\omega_n}}$ and

$$\delta = \tan^{-1} \frac{\phi_0 \omega_n}{\dot{\phi}_0}.$$

5.3.2. Model With Tether Sag Included (Cable Model)

5.3.2.1. Static Equilibrium of a Single Span

An extensible tether has a modulus of elasticity E and an unstrained cross-sectional area A_0 . The tether is supported between fixed point A and point B on the pendulum mass as shown in Figure 5-13. The Cartesian co-ordinates of these two points are $(0,0)$ and $(x_0|_B, y_0|_B)$ respectively. Point C is the point of suspension of the pendulum and has Cartesian co-ordinates (X, ℓ_p) .

Figure 5-13. Definition diagram of a single span tether/pendulum experimental configuration showing equilibrium of an element and boundary conditions for static equilibrium of an elastic tether.

For the static case the pendulum mass is not in motion. Thus the span of the tether is $x_0|_B = \ell$ and the vertical separation of the two tether endpoints is $y_0|_B$. The unstrained tether has a length L_0 . A point P on the unstrained (unloaded) tether has a Lagrangian co-ordinate s that is the length along the tether from point A to the point P . Gravity acts on the mass of the tether giving a total weight of $\mu g L_0$ where μ is the constant linear density of the unstrained tether. The weight of the tether from point A to P is $\mu g s$. Under this weight the point P moves to the new position P' .

The point P' has Cartesian co-ordinates (x, y) and Lagrangian co-ordinate p which is the length along the strained (loaded) tether from point A to point P' . The distance s along the unstrained tether has deformed to a new distance p along the strained tether. The unstrained tether length L_0 has elongated under strain to the strained tether length L .

Variable strain along the tether length will cause the density to vary. The total weight of the tether is $\mu(p)gL$ where $\mu(p)$ the tether density as a function of co-ordinate p . The weight of the tether from point A to point P' is $\int_0^P \mu(p)gdp$. Since the weight of the tether from point A to point P in the unstrained co-ordinates is the same as the weight of the tether from point A to point P' in the strained co-ordinates $\int_0^P \mu(p)gdp = \mu g s$.

The subscript annotation 0 adopted for this section indicates static equilibrium values of those terms so subscripted. The free-body analysis of the pendulum mass is:

$$\sum F_x = m_p \ddot{x} = T \sin \phi_0 - N_0|_B \cos \theta_0 \quad (5-20)$$

$$\sum F_y = m_p \ddot{y} = m_p g + N_0|_B \sin \theta_0 - T \cos \phi_0 \quad (5-21)$$

where T is the tension in the pendulum support and $N_0|_B$ is the tether tension exerted onto the pendulum. The angle of this tether tension is θ_0 to the horizontal as shown in Figure 5-13. Based on the shown geometry the distance $y_0|_B$ is dependent on the two angles θ_0 and ϕ_0 :

$$y_0|_B = \ell_p (1 - \cos \phi_0) - r \sin \theta_0 \quad (5-22)$$

For static equilibrium all forces are in equilibrium and

$$0 = T \sin \phi_0 - N_0|_B \cos \theta_0 \quad (5-23)$$

$$0 = m_p g + N_0|_B \sin \theta_0 - T \cos \phi_0 \quad (5-24)$$

The static equilibrium of an element of the tether shown in Figure 5-13 has varying tension $N_0(s)$ along the length of the tether due to gravity. This static equilibrium can be analysed to determine the equilibrium shape of the suspended tether [Irvine (1981)]. The vertical equilibrium requires:

$$-N_0 \frac{dy_0}{dp} - \mu(p)g\Delta p + \left[N_0 \frac{dy_0}{dp} + \frac{d}{dp} \left(N_0 \frac{dy_0}{dp} \right) \Delta p \right] = 0$$

$$-\mu(p)g\Delta p + \frac{d}{dp} \left(N_0 \frac{dy_0}{dp} \right) \Delta p = 0 \text{ and :}$$

$$\frac{d}{dp} \left(N_0 \frac{dy_0}{dp} \right) = \mu(p)g. \quad (5-25)$$

Integrating Eq. (5-25) for static equilibrium: $\int_0^p \frac{d}{dp} \left(N_0 \frac{dy_0}{dp} \right) dp = \int_0^p \mu(p)g dp$.

Since $\int_0^p \mu(p)g dp = \int_0^s \mu g ds = \mu g s$ and a reaction force at A of magnitude V_0 is expressed

$N_0 \frac{dy_0}{dp} \Big|_0 = -V_0$ the integrals become:

$$N_0 \frac{dy_0}{dp} = \mu g s - V_0. \quad (5-26)$$

The horizontal equilibrium requires:

$$N_0 \frac{dx_0}{dp} - \left[N_0 \frac{dx_0}{dp} + \frac{d}{dp} \left(N_0 \frac{dx_0}{dp} \right) \Delta p \right] = 0 \text{ and}$$

$$\frac{d}{dp} \left(N_0 \frac{dx_0}{dp} \right) = 0. \quad (5-27)$$

Equation (5-27) may be integrated directly to get

$$N_0 \frac{dx_0}{dp} = H_0. \quad (5-28)$$

A constitutive equation relating the tension to the strain experienced can be based on Hooke's Law:

$$N_0 = EA_0 \left(\frac{dp}{ds} - 1 \right). \quad (5-29)$$

The geometric boundary conditions at the tether supports A and B are $y_0|_A = 0$, $p_A = 0$, $s_A = 0$, $y_0|_B = \ell_p(1 - \cos \phi) - r \sin \theta$, $p_B = L$, $s_A = L_0$. The geometric constraint for the strained tether is $dx_0^2 + dy_0^2 = dp^2$ or

$$\left(\frac{dx_0}{dp} \right)^2 + \left(\frac{dy_0}{dp} \right)^2 = 1. \quad (5-30)$$

Squaring Eqs. (5-28) and (5-26) and adding the results together yields

$$\left(\frac{dx_0}{dp} \right)^2 + \left(\frac{dy_0}{dp} \right)^2 = \frac{H_0^2}{T_0^2} + \frac{(\mu g s - V_0)^2}{T_0^2}. \quad (5-31)$$

Comparing Eq. (5-30) to Eq. (5-31) we can immediately replace the L.H.S. of Eq. (5-31) with unity and multiply by N_0^2 to get $N_0^2 = H_0^2 + (\mu g s - V_0)^2$. Finally tension N_0 is expressed as a function of s :

$$N_0(s) = \sqrt{H_0^2 + (\mu g s - V_0)^2}.$$
(5-32)

To relate the s coordinate to the x_0 coordinate we write:

$$\frac{dx_0}{ds} = \frac{dx_0}{dp} \frac{dp}{ds}. \quad (5-33)$$

Substituting Eq. (5-32) to substitute for T_0 in Eq. (5-28) we write:

$$\frac{dx_0}{dp} = \frac{H_0}{\sqrt{H_0^2 + (\mu g s - V_0)^2}}. \quad (5-34)$$

Solving Eq. (5-29) for dp/ds :

$$\frac{N_0}{EA_0} + 1 = \frac{dp}{ds} = \frac{\sqrt{H_0^2 + (\mu g s - V_0)^2}}{EA_0} + 1. \quad (5-35)$$

Substituting Eqs. (5-34) and (5-35) into Eq. (5-33) yields:

$$\frac{dx_0}{ds} = \frac{H_0}{EA_0} + \frac{H_0}{\sqrt{H_0^2 + (\mu g s - V_0)^2}}. \quad (5-36)$$

Integrating Eq. (5-36) by separation of variables:

$$x_0(s) = \frac{H_0 s}{EA_0} + \frac{H_0}{\mu g} \left[\sinh^{-1} \left(\frac{V_0}{H_0} \right) + \sinh^{-1} \left(\frac{\mu g s - V_0}{H_0} \right) \right]. \quad (5-37)$$

In order to relate the s co-ordinate to the y_0 co-ordinate we write:

$$\frac{dy_0}{ds} = \frac{dy_0}{dp} \frac{dp}{ds}. \quad (5-38)$$

From Eqs. (5-26) and (5-32):

$$\frac{dy_0}{dp} = \frac{\mu g s - V_0}{\sqrt{H_0^2 + (\mu g s - V_0)^2}}. \quad (5-39)$$

Substituting Eqs. (5-35) and (5-39) into Eq. (5-38) yields:

$$\frac{dy_0}{ds} = \frac{\mu g s - V_0}{EA_0} + \frac{\mu g s - V_0}{\sqrt{H_0^2 + (\mu g s - V_0)^2}}. \quad (5-40)$$

Equation (5-40) is integrated through separation of variables :

$$y_0(s) = \frac{\mu g s^2}{2EA_0} - \frac{V_0 s}{EA_0} + \frac{H_0}{\mu g} \left[\sqrt{1 + \left(\frac{\mu g s - V_0}{H_0} \right)^2} - \sqrt{1 + \left(\frac{V_0}{H_0} \right)^2} \right]. \quad (5-41)$$

The derivation has been for the general case described by Figure 5-13. The equations for x_0 , y_0 and N_0 are expressed in terms of V_0 , H_0 and s .

Introducing the end conditions at point B into Eqs. (5-37) and (5-41) yields:

$$x_0|_B = \ell = X - \ell_p \sin \phi_0 - r \cos \theta_0 = \frac{H_0 L_0}{EA_0} + \frac{H_0}{\mu g} \left[\sinh^{-1} \left(\frac{V_0}{H_0} \right) + \sinh^{-1} \left(\frac{\mu g L_0 - V_0}{H_0} \right) \right] \quad (5-42)$$

$$y_0|_B = \ell_p (1 - \cos \phi_0) - r \sin \theta_0 = \left(\frac{\mu g L_0}{2} - V_0 \right) \frac{L_0}{EA_0} + \frac{H_0}{\mu g} \left[\sqrt{1 + \left(\frac{\mu g L_0 - V_0}{H_0} \right)^2} - \sqrt{1 + \left(\frac{V_0}{H_0} \right)^2} \right] \quad (5-43)$$

Substituting the trigonometric relations of Eqs. (D-22) and (D-23) from Appendix D into Eqs. (5-42) and (5-43) yields:

$$X - \ell_p \sin \phi_0 - \frac{r H_0}{\sqrt{H_0^2 + (\mu g L_0 - V_0)^2}} = \frac{H_0 L_0}{EA_0} + \frac{H_0}{\mu g} \left[\sinh^{-1} \left(\frac{V_0}{H_0} \right) + \sinh^{-1} \left(\frac{\mu g L_0 - V_0}{H_0} \right) \right] \quad (5-44)$$

$$\ell_p (1 - \cos \phi_0) - \frac{r (\mu g L_0 - V_0)}{\sqrt{H_0^2 + (\mu g L_0 - V_0)^2}} = \left(\frac{\mu g L_0}{2} - V_0 \right) \frac{L_0}{EA_0} + \frac{H_0}{\mu g} \left[\sqrt{1 + \left(\frac{\mu g L_0 - V_0}{H_0} \right)^2} - \sqrt{1 + \left(\frac{V_0}{H_0} \right)^2} \right] \quad (5-45)$$

Equations (5-44) and (5-45) will be used later in the derivation of tether tension for the dynamic case. Further use of the trigonometric relationships of (D-30) and (D-31) from Appendix D yields:

$$0 = X - \frac{\ell_p H_0}{\sqrt{H_0^2 + (m_p g + \mu g L_0 - V_0)^2}} - \frac{r H_0}{\sqrt{H_0^2 + (\mu g L_0 - V_0)^2}} - \frac{H_0 L_0}{EA_0} - \frac{H_0}{\mu g} \left[\sinh^{-1} \left(\frac{V_0}{H_0} \right) + \sinh^{-1} \left(\frac{\mu g L_0 - V_0}{H_0} \right) \right] \quad (5-46)$$

$$0 = \ell_p \left(\frac{m_p g + \mu g L_0 - V_0}{\sqrt{H^2 + (m_p g + \mu g L_0 - V_0)^2}} - 1 \right) + \frac{r(\mu g L_0 - V_0)}{\sqrt{H_0^2 + (\mu g L_0 - V_0)^2}} - \frac{V_0 L_0}{EA_0} + \frac{\mu g L_0^2}{2EA_0} + \frac{H_0}{\mu g} \left[\sqrt{1 + \left(\frac{\mu g L_0 - V_0}{H_0} \right)^2} - \sqrt{1 + \left(\frac{V_0}{H_0} \right)^2} \right] \quad (5-47)$$

Equations (5-46) and (5-47) represent a system of two equations with two unknowns: V_0 and H_0 . For any given set of system parameters X , ℓ_p , m_p , r , μ , L_0 and constant EA_0 these two equations can be solved for the static equilibrium values of V_0 and H_0 . With these values known Eqs. (5-32), (5-37) and (5-41) are a set of parametric equations giving the values of $N_0(s)$, $x_0(s)$ and $y_0(s)$, respectively, for any $s \leq L_0$.

Special Case $y_0|_B = 0$

For the special sub-case where the two supports points A and B are level and substituting $y_0|_B = 0$ for the L.H.S. of Eq. (5-42) yields $V_0 = \mu g L_0 / 2$. Substituting $V_0 = \mu g L_0 / 2$ into Eq. (5-37) and retaining the L.H.S. of Eq. (5-37) simply as ℓ gives:

$$\frac{\mu g L_0}{2H_0} - \sinh \left(\frac{\mu g \ell}{2H_0} - \frac{\mu g L_0}{2EA_0} \right) = 0, \quad (5-48)$$

where $y_0|_B = 0$; $V_0 = \frac{\mu g L_0}{2}$.

Equations (5-48) can be solved for the value of the constant horizontal component of tension, H_0 .

The component of the gravitational force of the pendulum mass is $-m_p g \ell_p \sin \phi$. The component of the tension force $N|_B$, due to tether stiffness is that component of $N|_B$ that is resolved along the vector perpendicular to the line BC . Also, the component of force $F|_B$, due to damping of the dynamic tether, is resolved along the same vector. The direction of CB is given by the vector $\{-\sin \phi \quad -\cos \phi\}^T$. The direction perpendicular to this is $\{-\cos \phi \quad \sin \phi\}^T$. The direction vector of the forces $N|_B$ and $F|_B$ at the pendulum is $\{-\cos \theta \quad -\sin \theta\}^T$. The components of $N|_B$ and $F|_B$ along $\{-\cos \phi \quad \sin \phi\}^T$ are given by the dot product of the direction vector of $N|_B$ and $F|_B$ with $\{-\cos \phi \quad \sin \phi\}^T$.

Thus the total force from the tether providing a moment on the pendulum is:

$$(N|_B + F|_B)(\cos \phi \cos \theta - \sin \phi \sin \theta) = (N|_B + F|_B) \cos(\phi + \theta). \quad (5-50)$$

The length at which to apply this force is ℓ_p since the force acts on the centre of mass of the pendulum. Thus:

$$m_p \ell_p^2 \ddot{\phi} + m_p g \ell_p \sin \phi - (N|_B + F|_B) \ell_p \cos(\phi + \theta) = 0. \quad (5-51)$$

Equation (5-51) represents the general equation of motion for the pendulum mass of the pendulum/tether system, neglecting the kinetic and gravitational potential energies of the tether. The variable $N|_B$ represents the force exerted on the pendulum by the tether due to longitudinal stiffness of the tether. The variable $F|_B$ represents the force exerted on the pendulum by the tether due to damping forces arising in the tether due to dynamic motion.

Damping of tether

The strain through the tether varies over its unstrained length ($0 \leq s \leq L_0$):

$$\varepsilon = \varepsilon(s) = \frac{du}{ds}. \quad (5-52)$$

where $u(s) = p(s) - s$ and is the elongation of the strained tether at a point s along the unstrained length of the tether. Integrating the strain over the total length of the tether yields the total elongation of the tether:

$$u|_{s=L_0} = \int_0^{L_0} \varepsilon(s) ds. \quad (5-53)$$

Substituting $\varepsilon(s) = \frac{N(s)}{EA(s)}$ into Eq. (5-53) yields:

$$u|_{s=L_0} = \int_0^{L_0} \frac{N(s)}{EA(s)} ds. \quad (5-54)$$

For the tether we then write for the equivalent viscous damping coefficient, C , the damping force $F|_B$:

$$F|_B = C \frac{d u|_{s=L_0}}{dt} = C \frac{d}{dt} \left[\int_0^{L_0} \frac{T(s)}{EA(s)} ds \right] \quad (5-55)$$

The angle ϕ denotes the angular position of the pendulum mass. Equation (5-55) may be expanded as a function of $\dot{\phi}$:

$$F|_B = C \frac{d u|_{s=L_0}}{dt} = C \frac{d}{d\phi} \left[\int_0^{L_0} \frac{N(s)}{EA(s)} ds \right] \dot{\phi}. \quad (5-56)$$

Incorporating this damping force term into Eq. (5-51) yields:

$$m_p \ell_p^2 \ddot{\phi} - C \frac{d}{d\phi} \left[\int_0^{L_0} \frac{N(s)}{EA(s)} ds \right] \ell_p \cos(\phi + \theta) \dot{\phi} + m_p g \ell_p \sin \phi - N|_B \ell_p \cos(\phi + \theta) = 0 \quad (5-57)$$

Linearization of Equation of Motion

Equation (5-57) is a non-linear differential equation in ϕ . The variables $N(s)$, $EA(s)$, θ and $N|_B$ have non-linear relationships with ϕ . Expansion and linear approximations of these relationships about the position of static equilibrium can be performed in order to produce a linear equation of motion.

The variables $N(s)$, $EA(s)$, θ and $N|_B$ are related to ϕ through simultaneous parametric equations that give values of the horizontal and vertical components of tension along the tether. For any pendulum position not at static equilibrium the zero subscript is dropped from the variables H_0 , V_0 , N_0 , ϕ_0 and θ_0 . Equations (21)-(23), (29)-31) and (5-45)-(5-46) are written for the non-equilibrium case:

$$\tan \theta = \frac{\mu g L_0 - V}{H}, \cos \theta = \frac{H}{\sqrt{H^2 + (\mu g L_0 - V)^2}}, \sin \theta = \frac{\mu g L_0 - V}{\sqrt{H^2 + (\mu g L_0 - V)^2}}, \quad (5-58)$$

$$\tan \phi = \frac{H}{m_p g + \mu g L_0 - V}, \cos \phi = \frac{m_p g + \mu g L_0 - V}{\sqrt{H^2 + (m_p g + \mu g L_0 - V)^2}}, \quad (5-59)$$

$$\sin \phi = \frac{H}{\sqrt{H^2 + (m_p g + \mu g L_0 - V)^2}},$$

$$0 = X - \ell_p \sin \phi - \frac{rH}{\sqrt{H^2 + (\mu g L_0 - V)^2}} - \frac{HL_0}{EA_0} - \frac{H}{\mu g} \left[\sinh^{-1} \left(\frac{V}{H} \right) + \sinh^{-1} \left(\frac{\mu g L_0 - V}{H} \right) \right] \quad (5-60)$$

and:

$$0 = \ell_p (1 - \cos \phi) - \frac{r(\mu g L_0 - V)}{\sqrt{H^2 + (\mu g L_0 - V)^2}} - \left[\frac{\mu g L_0}{2} - V \right] \frac{L_0}{EA_0} - \frac{1}{\mu g} \left[\frac{\sqrt{H^2 + (\mu g L_0 - V)^2}}{-\sqrt{H^2 + V^2}} \right] \quad (5-61)$$

for which the tension along the tether is given as a function of s :

$$N(s) = \sqrt{H^2 + (\mu g s - V)^2}. \quad (5-62)$$

These static equilibrium values are found through the solution of Eqs. (5-46) and (5-47):

$$0 = X - \frac{\ell_p H_0}{\sqrt{H_0^2 + (m_p g + \mu g L_0 - V_0)^2}} - \frac{r H_0}{\sqrt{H_0^2 + (V_0 - \mu g L_0)^2}} - \frac{H_0 L_0}{EA_0} - \frac{H_0}{\mu g} \left[\sinh^{-1} \left(\frac{V_0}{H_0} \right) + \sinh^{-1} \left(\frac{\mu g L_0 - V_0}{H_0} \right) \right] \quad (5-46)$$

$$0 = \ell_p \left(\frac{m_p g + \mu g L_0 - V_0}{\sqrt{H_0^2 + (m_p g + \mu g L_0 - V_0)^2}} - 1 \right) + \frac{r(\mu g L_0 - V_0)}{\sqrt{H_0^2 + (\mu g L_0 - V_0)^2}} - \frac{V_0 L_0}{EA_0} + \frac{\mu g L_0^2}{2EA_0} + \frac{H_0}{\mu g} \left[\sqrt{1 + \left(\frac{\mu g L_0 - V_0}{H_0} \right)^2} - \sqrt{1 + \left(\frac{V_0}{H_0} \right)^2} \right] \quad (5-47)$$

for which, from Eq. (5-32):

$$N_0(s) = \sqrt{H_0^2 + (\mu g s - V_0)^2} \quad (5-32)$$

The value of the tether tension at the endpoints $A(s = 0)$ and $B(s = L_0)$ are:

$$N_0|_A = \sqrt{H_0^2 + V_0^2} \quad (5-63)$$

and:

$$N_0|_B = \sqrt{H_0^2 + (V_0 - \mu g L_0)^2} \quad (5-64)$$

The static equilibrium values of the angles are given by

$$\tan \theta_0 = \frac{\mu g L_0 - V_0}{H_0},$$

and:

$$\tan \phi_0 = \frac{H_0}{m_p g + \mu g L_0 - V_0}.$$

For small motion of the pendulum about the position of static equilibrium the following substitutions are used

$$\phi = \phi_0 + \hat{\phi},$$

$$\theta = \theta_0 + \hat{\theta},$$

$$N = N_0 + \hat{N},$$

$$H = H_0 + \hat{H},$$

$$V = V_0 + \hat{V}.$$

(5-65)

Various derivatives required by the equation of motion become:

q	$\dot{q}$	$\ddot{q}$
$\phi = \phi_0 + \hat{\phi}$	$\dot{\phi} = \dot{\hat{\phi}}$	$\ddot{\phi} = \ddot{\hat{\phi}}$
$\theta = \theta_0 + \hat{\theta}$	$\dot{\theta} = \dot{\hat{\theta}}$	$\ddot{\theta} = \ddot{\hat{\theta}}$
$N = N_0 + \hat{N}$	$\dot{N} = \dot{\hat{N}}$	$\ddot{N} = \ddot{\hat{N}}$
$H = H_0 + \hat{H}$	$\dot{H} = \dot{\hat{H}}$	$\ddot{H} = \ddot{\hat{H}}$
$V = V_0 + \hat{V}$	$\dot{V} = \dot{\hat{V}}$	$\ddot{V} = \ddot{\hat{V}}$

(5-66)

Substituting Eqs. (5-65) and (5-66) into Eqs. (5-57), (5-46), (5-47) and (5-32) yields, respectively:

$$0 = m_p \ell_p^2 \ddot{\hat{\phi}} - C \frac{d}{d\phi} \left[\int_0^{L_0} \frac{N_0 + \hat{N}}{EA(s)} ds \right] \ell_p \cos(\phi_0 + \hat{\phi} + \theta_0 + \hat{\theta}) \dot{\hat{\phi}} + m_p g \ell_p \sin(\phi_0 + \hat{\phi}) - (N_0|_B + \hat{N}|_B) \ell_p \cos(\phi_0 + \hat{\phi} + \theta_0 + \hat{\theta}) \quad (5-67)$$

$$0 = X - \ell_p \sin(\phi_0 + \hat{\phi}) - \frac{r(H_0 + \hat{H})}{\sqrt{(H_0 + \hat{H})^2 + (V_0 + \hat{V} - \mu g L_0)^2}} - \frac{(H_0 + \hat{H})L_0}{EA_0} - \frac{H_0 + \hat{H}}{\mu g} \left[\sinh^{-1} \left(\frac{V_0 + \hat{V}}{H_0 + \hat{H}} \right) - \sinh^{-1} \left(\frac{V_0 + \hat{V} - \mu g L_0}{H_0 + \hat{H}} \right) \right], \quad (5-68)$$

$$0 = \ell_p (\cos(\phi_0 + \hat{\phi}) - 1) - \frac{r(V_0 + \hat{V} - \mu g L_0)}{\sqrt{(H_0 + \hat{H})^2 + (V_0 + \hat{V} - \mu g L_0)^2}} - \left(V_0 + \hat{V} - \frac{\mu g L_0}{2} \right) \frac{L_0}{EA_0} - \frac{1}{\mu g} \left[\sqrt{(H_0 + \hat{H})^2 + (V_0 + \hat{V})^2} - \sqrt{(H_0 + \hat{H})^2 + (V_0 + \hat{V} - \mu g L_0)^2} \right], \quad (5-69)$$

and:

$$N_0 + \hat{N} = \sqrt{(H_0 + \hat{H})^2 + (V_0 + \hat{V} - \mu g s)^2}. \quad (5-70)$$

The term $m_p g \ell_p \sin(\phi_0 + \hat{\phi})$ in Eq. (5-67) can be expanded to give

$$m_p g \ell_p \sin(\phi_0 + \hat{\phi}) = m_p g \ell_p (\sin \phi_0 + \hat{\phi} \cos \phi_0 + \sigma(\hat{\phi}^2)). \quad (5-71)$$

The term $(N_0|_B + \hat{N}|_B) \ell_p \cos(\phi_0 + \hat{\phi} + \theta_0 + \hat{\theta})$ in Eq. (5-67) can be expanded by rearranging Eqs. (5-32) and (D-21) after substituting the definitions of Eq. (5-65)

$$\hat{N}|_B = \sqrt{(H_0 + \hat{H})^2 + (\mu g L_0 - V_0 - \hat{V})^2} - N_0|_B, \quad (5-72)$$

$$\hat{\theta} = \tan^{-1} \left(\frac{\mu g L_0 - V_0 - \hat{V}}{H_0 + \hat{H}} \right) - \theta_0, \quad (5-73)$$

then substituting into the term $(N_0|_B + \hat{N}|_B) \ell_p \cos(\phi_0 + \hat{\phi} + \theta_0 + \hat{\theta})$ to give:

$$\sqrt{(H_0 + \hat{H})^2 + (\mu g L_0 - V_0 - \hat{V})^2} \ell_p \cos \left[\phi_0 + \hat{\phi} + \tan^{-1} \left(\frac{\mu g L_0 - V_0 - \hat{V}}{H_0 + \hat{H}} \right) \right]. \quad (5-74)$$

The Taylor series expansion of Eq. (5-74), w.r.t $\hat{\phi}$, $\hat{H}$ and $\hat{V}$ is:

$$\begin{aligned} (N_0|_B + \hat{N}|_B) \ell_p \cos(\phi_0 + \hat{\phi} + \theta_0 + \hat{\theta}) &= N_0|_B \ell_p \cos(\phi_0 + \theta_0) - N_0|_B \ell_p \sin(\phi_0 + \theta_0) \hat{\phi} \\ &+ [(\mu g L_0 - V_0) \sin(\phi_0 + \theta_0) + H_0 \cos(\phi_0 + \theta_0)] \frac{\ell_p}{T_0|_B} \hat{H} \\ &+ [H_0 \sin(\phi_0 + \theta_0) + (\mu g L_0 - V_0) \cos(\phi_0 + \theta_0)] \frac{\ell_p}{T_0|_B} \hat{V} + \sigma(\hat{\phi}^2, \hat{H}^2, \hat{V}^2) \end{aligned} \quad (5-75)$$

The Taylor series expansion of Eq. (5-68) w.r.t $\hat{\phi}$, $\hat{H}$ and $\hat{V}$ is gives:

$$\begin{aligned} 0 &= X - \ell_p \sin \phi_0 - \frac{r H_0}{\sqrt{H_0^2 + (\mu g L_0 - V_0)^2}} - \frac{H_0 L_0}{E A_0} \\ &- \frac{H_0}{\mu g} \left[\sinh^{-1} \left(\frac{V_0}{H_0} \right) - \sinh^{-1} \left(\frac{V_0 - \mu g L_0}{H_0} \right) \right] - \{ \ell_p \cos \phi_0 \} \hat{\phi} \\ &+ \left\{ \frac{r H_0^2}{[H_0^2 + (\mu g L_0 - V_0)^2]^{3/2}} + \frac{1}{\mu g} \left[\frac{\frac{V_0}{\sqrt{H_0^2 + V_0^2}} + \frac{\mu g (L_0 - r) - V_0}{\sqrt{H_0^2 + (\mu g L_0 - V_0)^2}}}{-\sinh^{-1} \left(\frac{V_0}{H_0} \right) + \sinh^{-1} \left(\frac{V_0 - \mu g L_0}{H_0} \right)} \right] - \frac{L_0}{E A_0} \right\} \hat{H} \\ &+ \left\{ \frac{r H_0 (V_0 - \mu g L_0)}{[H_0^2 + (\mu g L_0 - V_0)^2]^{3/2}} - \frac{H_0}{\mu g} \left[\frac{1}{\sqrt{H_0^2 + V_0^2}} - \frac{1}{\sqrt{H_0^2 + (\mu g L_0 - V_0)^2}} \right] \right\} \hat{V} \\ &+ \sigma(\hat{\phi}^2, \hat{H}^2, \hat{V}^2) \end{aligned} \quad (5-76)$$

From Eq. (5-46) the first five terms on the R.H.S. of Eq. (5-76) are zero:

$$0 = X - \ell_p \sin \phi_0 - \frac{r H_0}{\sqrt{H_0^2 + (\mu g L_0 - V_0)^2}} - \frac{H_0 L_0}{E A_0} - \frac{H_0}{\mu g} \left[\sinh^{-1} \left(\frac{V_0}{H_0} \right) - \sinh^{-1} \left(\frac{V_0 - \mu g L_0}{H_0} \right) \right]$$

and making the substitutions from Eqs. (5-63) and (5-64), $\sqrt{H_0^2 + V_0^2} = N_0|_A$ and $\sqrt{H_0^2 + (V_0 - \mu g L_0)^2} = N_0|_B$ Eq. (5-76) becomes:

$$0 = \left\{ \frac{r H_0 (V_0 - \mu g L_0)}{[N_0|_B]^3} - \frac{H_0}{\mu g} \left[\frac{1}{N_0|_A} - \frac{1}{N_0|_B} \right] \right\} \hat{V} - \{ \ell_p \cos \phi_0 \} \hat{\phi} + \left\{ \frac{r H_0^2}{[N_0|_B]^3} + \frac{1}{\mu g} \left[\frac{\frac{V_0}{N_0|_A} + \frac{\mu g (L_0 - r) - V_0}{N_0|_B}}{-\sinh^{-1} \left(\frac{V_0}{H_0} \right) + \sinh^{-1} \left(\frac{V_0 - \mu g L_0}{H_0} \right)} \right] - \frac{L_0}{E A_0} \right\} \hat{H} + \sigma(\hat{\phi}^2, \hat{H}^2, \hat{V}^2) \quad (5-77)$$

The Taylor series expansion of Eq. (5-69) w.r.t $\hat{\phi}$, $\hat{H}$ and $\hat{V}$ is gives

$$0 = \ell_p (\cos \phi_0 - 1) - \frac{r(V_0 - \mu g L_0)}{\sqrt{H_0^2 + (\mu g L_0 - V_0)^2}} - \left(V_0 - \frac{\mu g L_0}{2} \right) \frac{L_0}{E A_0} - \frac{1}{\mu g} \left[\sqrt{H_0^2 + V_0^2} - \sqrt{H_0^2 + (\mu g L_0 - V_0)^2} \right] - \{ \ell_p \sin \phi_0 \} \hat{\phi} + \left\{ \frac{r H_0 (V_0 - \mu g L_0)}{[H_0^2 + (\mu g L_0 - V_0)^2]^{3/2}} + \frac{1}{\mu g} \left[\frac{\frac{V_0^2}{H_0 \sqrt{H_0^2 + V_0^2}} - \frac{(\mu g L_0 - V_0)^2}{H_0 \sqrt{H_0^2 + (\mu g L_0 - V_0)^2}}}{+\frac{\sqrt{H_0^2 + V_0^2}}{H_0} - \frac{\sqrt{H_0^2 + (\mu g L_0 - V_0)^2}}{H_0}} \right] \right\} \hat{H} - \left\{ \frac{r}{\sqrt{H_0^2 + (\mu g L_0 - V_0)^2}} + \frac{1}{\mu g} \left[\frac{V_0}{\sqrt{H_0^2 + V_0^2}} + \frac{\mu g L_0 - V_0}{\sqrt{H_0^2 + (\mu g L_0 - V_0)^2}} \right] \right\} \hat{V} + \sigma(\hat{\phi}^2, \hat{H}^2, \hat{V}^2) \quad (5-78)$$

According to Eq. (5-47) the first line of Eq. (5-78) is equal to zero, and substituting Eqs. (5-63) and (5-64) for $\sqrt{H_0^2 + V_0^2} = N_0|_A$ and $\sqrt{H_0^2 + (V_0 - \mu g L_0)^2} = N_0|_B$ Eq. (5-78) becomes:

$$0 = \left\{ \frac{rH_0(V_0 - \mu g L_0)}{[N_0|_B]^3} - \frac{1}{\mu g} \left[-\frac{V_0^2}{H_0 N_0|_A} + \frac{(\mu g L_0 - V_0)^2}{H_0 N_0|_B} - \frac{N_0|_A}{H_0} + \frac{N_0|_B}{H_0} \right] \right\} \hat{H} - \left\{ \frac{r}{N_0|_B} + \frac{1}{\mu g} \left[\frac{V_0}{N_0|_A} + \frac{\mu g L_0 - V_0}{N_0|_B} \right] \right\} \hat{V} - \{\ell_p \sin \phi_0\} \hat{\phi} + \sigma(\hat{\phi}^2, \hat{H}^2, \hat{V}^2) \quad (5-79)$$

Ignoring the higher order terms, Eqs. (5-77) and (5-79) are linear in $\hat{\phi}$, $\hat{H}$ and $\hat{V}$ and have the form:

$$0 = f_1 \hat{\phi} + h_1 \hat{H} + v_1 \hat{V}$$

$$0 = f_2 \hat{\phi} + h_2 \hat{H} + v_2 \hat{V} \quad (5-80)$$

where:

$$f_1 = -\{\ell_p \cos \phi_0\}$$

$$h_1 = \frac{rH_0^2}{[N_0|_B]^3} + \frac{1}{\mu g} \left[\frac{V_0}{N_0|_A} + \frac{\mu g(L_0 - r) - V_0}{N_0|_B} - \sinh^{-1} \left(\frac{V_0}{H_0} \right) + \sinh^{-1} \left(\frac{V_0 - \mu g L_0}{H_0} \right) \right] - \frac{L_0}{EA_0} \quad (5-81)$$

$$v_1 = \frac{rH_0(V_0 - \mu g L_0)}{[N_0|_B]^3} - \frac{H_0}{\mu g} \left[\frac{1}{N_0|_A} - \frac{1}{N_0|_B} \right]$$

$$f_2 = -\{\ell_p \sin \phi_0\}$$

$$h_2 = \frac{rH_0(V_0 - \mu g L_0)}{[N_0|_B]^3} + \frac{1}{\mu g} \left[\frac{V_0^2}{H_0 N_0|_A} - \frac{(\mu g L_0 - V_0)^2}{H_0 N_0|_B} + \frac{N_0|_A}{H_0} - \frac{N_0|_B}{H_0} \right]$$

$$v_2 = -\frac{r}{N_0|_B} - \frac{1}{\mu g} \left[\frac{V_0}{N_0|_A} + \frac{\mu g L_0 - V_0}{N_0|_B} \right]$$

Solving Eqs. (5-80) for $\hat{H}$ and $\hat{V}$ as functions of $\hat{\phi}$ gives:

$$\hat{H} = C_1 \hat{\phi},$$

$$\hat{V} = C_2 \hat{\phi}, \quad (5-82)$$

where:

$$C_1 = \frac{v_2 f_1 - v_1 f_2}{h_2 v_1 - h_1 v_2}, \quad (5-83)$$

$$C_2 = \frac{h_1 f_2 - h_2 f_1}{h_2 v_1 - h_1 v_2}.$$

Substituting Eq. (5-83) into Eq. (5-75) yields:

$$\begin{aligned} (N_0|_B + \hat{N}|_B) \ell_p \cos(\phi_0 + \hat{\phi} + \theta_0 + \hat{\theta}) &= N_0|_B \ell_p \cos(\phi_0 + \theta_0) \\ + \hat{\phi} &\left[\begin{aligned} &-N_0|_B \ell_p \sin(\phi_0 + \theta_0) + [(\mu g L_0 - V_0) \sin(\phi_0 + \theta_0) + H_0 \cos(\phi_0 + \theta_0)] \frac{\ell_p}{N_0|_B} C_1 \\ &+ [H_0 \sin(\phi_0 + \theta_0) + (\mu g L_0 - V_0) \cos(\phi_0 + \theta_0)] \frac{\ell_p}{N_0|_B} C_2 \end{aligned} \right] \end{aligned} \quad (5-84)$$

The damping term is simply linearized by treating the term $\frac{d}{d\phi} \left[\int_0^{L_0} \frac{N_0 + \hat{N}}{EA(s)} ds \right]$ as a constant to be evaluated from experimental data. The Taylor series expansion of the damping term to achieve a linear ordinary differential equation is :

$$\begin{aligned} &C \frac{d}{d\phi} \left[\int_0^{L_0} \frac{N_0 + \hat{N}}{EA(s)} ds \right] \ell_p \cos(\phi_0 + \hat{\phi} + \theta_0 + \hat{\theta}) \\ &= C \frac{d}{d\phi} \left[\int_0^{L_0} \frac{N_0 + \hat{N}}{EA(s)} ds \right] \ell_p \cos(\phi_0 + \theta_0) \end{aligned} \quad (5-85)$$

Substituting Eqs. (5-71) (5-84) and (5-85) into Eq. (5-67) gives:

$$\begin{aligned}
 0 = & m_p \ell_p^2 \ddot{\hat{\phi}} - C \frac{d}{d\phi} \left[\int_0^{L_0} \frac{N_0 + \hat{N}}{EA(s)} ds \right] \ell_p \cos(\phi_0 + \theta_0) \dot{\hat{\phi}} \\
 & + m_p g \ell_p (\sin \phi_0 + \hat{\phi} \cos \phi_0) - N_0|_B \ell_p \cos(\phi_0 + \theta_0) \\
 & - \hat{\phi} \left[-N_0|_B \ell_p \sin(\phi_0 + \theta_0) + [(\mu g L_0 - V_0) \sin(\phi_0 + \theta_0) + H_0 \cos(\phi_0 + \theta_0)] \frac{\ell_p}{N_0|_B} C_1 \right. \\
 & \left. + [H_0 \sin(\phi_0 + \theta_0) + (\mu g L_0 - V_0) \cos(\phi_0 + \theta_0)] \frac{\ell_p}{N_0|_B} C_2 \right]
 \end{aligned} \quad (5-86)$$

Dividing Eq. (5-86) by $m_p \ell_p^2$ and rearranging yields:

$$\begin{aligned}
 & \ddot{\hat{\phi}} - \dot{\hat{\phi}} \frac{C}{m_p \ell_p} \cos(\phi_0 + \theta_0) \frac{d}{d\phi} \left[\int_0^{L_0} \frac{N_0 + \hat{N}}{EA(s)} ds \right] \\
 & - \hat{\phi} \left[-\frac{g}{\ell_p} \cos \phi_0 - \frac{N_0|_B \sin(\phi_0 + \theta_0)}{m_p \ell_p} \right. \\
 & \left. + \frac{C_1}{m_p \ell_p T_0|_B} [(\mu g L_0 - V_0) \sin(\phi_0 + \theta_0) + H_0 \cos(\phi_0 + \theta_0)] \right. \\
 & \left. + \frac{C_2}{m_p \ell_p T_0|_B} [H_0 \sin(\phi_0 + \theta_0) + (\mu g L_0 - V_0) \cos(\phi_0 + \theta_0)] \right] \\
 & = \frac{N_0|_B \cos(\phi_0 + \theta_0)}{m_p \ell_p} - \frac{g}{\ell_p} \sin \phi_0
 \end{aligned} \quad (5-87)$$

Rearranging Eq. (D-17) yields:

$$N_0|_B (\cos \phi_0 \cos \theta_0 - \sin \theta_0 \sin \phi_0) = m_p g \sin \phi_0. \quad (5-88)$$

But $(\cos \phi_0 \cos \theta_0 - \sin \theta_0 \sin \phi_0) = \cos(\phi_0 + \theta_0)$ and Eq. (5-88) becomes:

$$0 = N_0|_B \cos(\phi_0 + \theta_0) - m_p g \sin \phi_0. \quad (5-89)$$

Dividing Eq. (5-89) by $m_p \ell_p$ yields:

$$0 = -\frac{g}{\ell_p} \sin \phi_0 + \frac{N_0|_B}{m_p \ell_p} \cos(\phi_0 + \theta_0). \quad (5-90)$$

Equation (5-90) shows the R.H.S. of Eq. (5-87) is zero and the final equation of motion is:

$$0 = \ddot{\hat{\phi}} - \dot{\hat{\phi}} \frac{C}{m_p \ell_p} \cos(\phi_0 + \theta_0) \frac{d}{d\phi} \left[\int_0^{\ell_0} \frac{N_0 + \hat{N}}{EA(s)} ds \right] - \left[\begin{aligned} & -\frac{g}{\ell_p} \cos \phi_0 - \frac{N_0|_B \sin(\phi_0 + \theta_0)}{m_p \ell_p} \\ & -\hat{\phi} \left[\frac{C_1}{m_p \ell_p N_0|_B} [(\mu g L_0 - V_0) \sin(\phi_0 + \theta_0) + H_0 \cos(\phi_0 + \theta_0)] \right. \\ & \left. + \frac{C_2}{m_p \ell_p N_0|_B} [H_0 \sin(\phi_0 + \theta_0) + (\mu g L_0 - V_0) \cos(\phi_0 + \theta_0)] \right] \end{aligned} \right]. \quad (5-91)$$

Equation (5-91) is of the form:

$$\ddot{\hat{\phi}} + 2B\dot{\hat{\phi}} + \omega_n^2 \hat{\phi} = 0. \quad (5-92)$$

where:

$$\omega_n = \left[\begin{aligned} & \frac{N_0|_B \sin(\phi_0 + \theta_0)}{m_p \ell_p} - \frac{C_2}{m_p \ell_p N_0|_B} [H_0 \sin(\phi_0 + \theta_0) + (\mu g L_0 - V_0) \cos(\phi_0 + \theta_0)] \\ & - \frac{C_1}{m_p \ell_p N_0|_B} [(\mu g L_0 - V_0) \sin(\phi_0 + \theta_0) + H_0 \cos(\phi_0 + \theta_0)] + \frac{g}{\ell_p} \cos \phi_0 \end{aligned} \right]^{\frac{1}{2}} \quad (5-93)$$

$$B = -\frac{C}{2m_p} \frac{\cos(\phi_0 + \theta_0)}{\ell_p} \frac{d}{d\phi} \left[\int_0^{\ell_0} \frac{N_0 + \hat{N}}{EA(s)} ds \right] \quad (5-94)$$

which, for a damped oscillatory system has the well know solution:

$$\hat{\phi} = D e^{-Bt} \sin(\omega_d t + \delta). \quad (5-95)$$

where $\omega_d = \sqrt{\omega_n^2 - B^2}$; $D = \sqrt{\hat{\phi}_0 + \frac{\dot{\hat{\phi}}_0}{\omega_n}}$ and $\delta = \tan^{-1} \frac{\hat{\phi}_0 \omega_n}{\dot{\hat{\phi}}_0}$. For initial conditions

$\hat{\phi}(0) = \hat{\phi}_0$ and $\dot{\hat{\phi}}(0) = \dot{\hat{\phi}}_0$ motion is:

$$\hat{\phi} = e^{\frac{C}{2m_p} \frac{\cos(\phi_0 + \theta_0)}{\ell_p} \frac{d}{d\phi} \left[\int_0^{\ell_0} \frac{N_0 + \hat{N}}{EA(s)} ds \right] t} \sqrt{\hat{\phi}_0 + \frac{\dot{\hat{\phi}}_0}{\omega_n}} \sin \left(\sqrt{\omega_n^2 - B^2} t + \tan^{-1} \frac{\hat{\phi}_0 \omega_n}{\dot{\hat{\phi}}_0} \right), \quad (5-96)$$

$$\omega_d = \sqrt{\omega_n^2 - \frac{C^2}{4m_p^2} \frac{\cos^2(\phi_0 + \theta_0)}{\ell_p^2} \left\{ \frac{d}{d\phi} \left[\int_0^{\ell_0} \frac{N_0 + \hat{N}}{EA(s)} ds \right] \right\}^2}. \quad (5-97)$$

For the equivalent viscous damping coefficient, C , based on a structural damping model, or $C = b/\omega_d$, it is seen that the term ω_d appears on both sides of Eq. (5-97); solving for ω_d based on a structural damping model yields:

$$\omega_d = \frac{1}{2} \left(\sqrt{\omega_n^2 + \frac{b}{m_p} \frac{\cos(\phi_0 + \theta_0)}{\ell_p} \frac{d}{d\phi} \left[\int_0^{\ell_0} \frac{N_0 + \hat{N}}{EA(s)} ds \right]} + \sqrt{\omega_n^2 - \frac{b}{m_p} \frac{\cos(\phi_0 + \theta_0)}{\ell_p} \frac{d}{d\phi} \left[\int_0^{\ell_0} \frac{N_0 + \hat{N}}{EA(s)} ds \right]} \right). \quad (5-98)$$

5.3.2.3. Dynamic Motion of a Multi-Span

The variation of tension in the horizontally hanging tether is reduced by supporting the tether at intermediate locations between the points A and B .

Figure 5-15, below, shows the tether with n supports (including the support point at point A but not including the support point at point B). In static equilibrium the distances between support points are X_{i0} through X_{n0} . Thus:

$$X = \sum_{i=1}^n X_{i0}, \quad (5-99)$$

The unstrained length of each tether segment between supported points is L_1 through L_n . The total unstrained length of the tether is the sum of the unstrained lengths of the individual tether segments:

$$L_0 = \sum_{i=1}^n L_i, \quad (5-100)$$

Figure 5-15. Definition diagram of a multi-span (N spans) tether/pendulum experimental configuration undergoing simple motion of the pendulum due to gravity and tension/damping in an elastic tether. Static equilibrium is shown.

The supports are adjusted so that the supporting lines hang exactly vertically when the system is at static equilibrium. All supported points (other than point B) are at the same level as point A . When the pendulum is in motion the supported points will move due to variation in the amount of sag in the tether. Vertical motion of the supported points is assumed negligible.

The shown vertical support reactions V_{i0} are the vertical reaction at the indicated points that are attributable solely to the i^{th} tether segment. The horizontal tension when the system is at static equilibrium is constant at all points of the tether and is denoted H_0 . Since the supported points are all level, from the special level case represented by Eq. (5-48) the constant static horizontal tension and vertical support reactions for segments 2..n are

$$0 = \sinh\left(\frac{\mu g X_i}{2H_0} - \frac{\mu g L_i}{2EA_0}\right) - \frac{\mu g L_i}{2H_0}, i = 2..n, \quad (5-101)$$

and:

$$V_{i0} = \frac{\mu g L_i}{2}, i = 2..n. \quad (5-102)$$

From Eqs. (5-46) and (5-47) the equations which describe the relationship between L_1 , H_0 and V_{10} we write:

$$0 = \frac{\ell_p H_0}{\sqrt{H_0^2 + (m_p g + \mu g L_1 - V_{10})^2}} - X_1 + \frac{r H_0}{\sqrt{H_0^2 + (V_{10} - \mu g L_1)^2}} + \frac{H_0 L_1}{EA_0} + \frac{H_0}{\mu g} \left[\sinh^{-1}\left(\frac{V_{10}}{H_0}\right) + \sinh^{-1}\left(\frac{\mu g L_1 - V_{10}}{H_0}\right) \right] \quad (5-103)$$

and:

$$0 = \ell_p + \frac{\ell_p (V_{10} - m_p g - \mu g L_1)}{\sqrt{H_0^2 + (m_p g + \mu g L_1 - V_{10})^2}} + \frac{r (V_{10} - \mu g L_1)}{\sqrt{H_0^2 + (\mu g L_1 - V_{10})^2}} + \frac{H_0}{\mu g} \left[\sqrt{1 + \left(\frac{V_{10}}{H_0}\right)^2} - \sqrt{1 + \left(\frac{\mu g L_1 - V_{10}}{H_0}\right)^2} \right] + \left(V_{10} - \frac{\mu g L_1}{2} \right) \frac{L_1}{EA_0} \quad (5-104)$$

The tension force exerted by the tether on the pendulum, in static equilibrium, is:

$$N_{10}|_B = \sqrt{H_0^2 + (\mu g L_1 - V_{10})^2} \quad (5-105)$$

from which the vertical equilibrium reaction at the first support is:

$$V_{10} = \mu g L_1 - \sqrt{N_{10}|_B^2 - H_0^2} \quad (5-106)$$

Given $r, \mu, g, m_p, \ell_p, L_0, X_{i0}; i = 1..n, N_{10}|_B$ the $n+2$ equations represented by Eqs. (5-100), (5-101), (5-103) and (5-104) can be solved for the values of the $n+2$ variables $H_0, EA_0, L_i; i = 1..n$. Equation (5-106) can then be used to calculate the value of V_{10} and Eqs. (5-102) used to calculate the values of $V_{i0}; i = 2..n$. The equilibrium values of ϕ_0 and θ_0 can be evaluated from the equations:

$$\phi_0 = \tan^{-1} \left(\frac{H_0}{m_p g + \mu g L_1 - V_{10}} \right), \quad (5-107)$$

$$\theta_0 = \tan^{-1} \left(\frac{\mu g L_1 - V_{10}}{H_0} \right). \quad (5-108)$$

The system is perturbed from the equilibrium state to a new angle ϕ as shown in Fig 5. Since the values of $L_i; i = 1..n$ do not change, changes in ϕ must move the horizontal position of the support points, changing the values of $X_i; i = 1..n$. For the given values of r, μ, g, m_p, ℓ_p , for known X and specified $\hat{\phi}$ the values of the $n+2$ variables $X_i; i = 1..n, H$ and V_i of the perturbed state can be solved for with $n+2$ Eqs. (5-109) to (5-112):

$$X = \sum_{i=1}^n X_i \quad (5-109)$$

$$0 = X_1 - \ell_p \sin \phi - \frac{rH}{\sqrt{H^2 + (\mu g L_1 - V_1)^2}} - \frac{HL_1}{EA_0} - \frac{H}{\mu g} \left[\sinh^{-1} \left(\frac{V_1}{H} \right) + \sinh^{-1} \left(\frac{\mu g L_1 - V_1}{H} \right) \right] \quad (5-110)$$

$$0 = \ell_p(1 - \cos \phi) - \frac{r(\mu g L_1 - V_1)}{\sqrt{H^2 + (\mu g L_1 - V_1)^2}} - \left[\frac{\mu g L_1}{2} - V_1 \right] \frac{L_1}{EA_0} - \frac{1}{\mu g} \left[\frac{\sqrt{H^2 + (\mu g L_1 - V_1)^2}}{-\sqrt{H^2 + V_1^2}} \right] \quad (5-111)$$

and

$$0 = \sinh \left(\frac{\mu g X_i}{2H} - \frac{\mu g L_i}{2EA_0} \right) - \frac{\mu g L_i}{2H}, \quad i = 2..n \quad (5-112)$$

Figure 5-16. Definition diagram of a multi-span (N spans) tether/pendulum experimental configuration perturbed from static equilibrium.

Ignoring the kinetic energy and gravitational potential energy of the tether the development of the equation of motion for a multi-span system is similar to the development of the equation of motion for a single span system. A moment balance of the pendulum mass yields the equation of motion

$$\begin{aligned}
0 = & \ddot{\phi} - \dot{\phi} \frac{C}{m_p \ell_p} \cos(\phi_0 + \theta_0) \frac{d}{d\phi} \left[\int_0^{L_0} \frac{N_0 + \hat{N}}{EA(s)} ds \right] \\
& - \dot{\phi} \left[-\frac{g}{\ell_p} \cos \phi_0 - \frac{N_{10}|_B \sin(\phi_0 + \theta_0)}{m_p \ell_p} \right. \\
& \left. + \frac{C_1}{m_p \ell_p N_{10}|_B} [(\mu g L_1 - V_{10}) \sin(\phi_0 + \theta_0) + H_0 \cos(\phi_0 + \theta_0)] \right. \\
& \left. + \frac{C_2}{m_p \ell_p N_{10}|_B} [H_0 \sin(\phi_0 + \theta_0) + (\mu g L_1 - V_{10}) \cos(\phi_0 + \theta_0)] \right].
\end{aligned} \tag{5-113}$$

where the damping term is taken for the whole tether, as indicated by the integral term. For constant EA_0 The integral of the damping term can be evaluated

$$\int_0^{L_0} \frac{N_0 + \hat{N}}{EA(s)} ds = \frac{1}{EA_0} \sum_{i=1}^n \int_0^{L_i} [N_{i0}(s_i) + \hat{N}_i(s_i)] ds_i, \tag{5-114}$$

where s_i is the local un-strained co-ordinate variable for the i^{th} tether segment.

Since $V_i; i = 1..n$ are known from the known $L_i; i = 1..n$ from the equations

$$V_i = \frac{\mu g L_i}{2}, \quad i = 2..n \tag{5-115}$$

and

$$V_1 = \mu g L_1 - \sqrt{N|_B^2 - H^2}, \tag{5-116}$$

the variation in tether tension of each segment can be expressed analytically as

$$N_i + \hat{N}_i = \sqrt{H^2 + (V_i - \mu g s_i)^2}, \tag{5-117}$$

where $0 \leq s_i \leq L_i$. Integrating Eq. (5-117) with respect to s_i yields

$$\int_0^{L_i} [N_{i0}(s) + \hat{N}_i(s)] ds = \frac{1}{2} \left(\begin{aligned} &L_i \sqrt{H^2 + (\mu g L_i - V_i)^2} - \frac{V_i}{\mu g} \left(\sqrt{H^2 + V_i^2} - \sqrt{H^2 + (\mu g L_i - V_i)^2} \right) \\ &+ \frac{H^2}{\mu g} \ln \left[\frac{\sqrt{H^2 + (\mu g L_i - V_i)^2} + \mu g L_i - V_i}{\sqrt{H^2 + V_i^2} - V_i} \right] \end{aligned} \right) \quad (5-118)$$

Equation (5-114) becomes:

$$\begin{aligned} &\int_0^{L_0} \frac{N_0 + \hat{N}}{EA(s)} ds \\ &= \frac{1}{2EA_0} \sum_{i=1}^n \int_0^{L_i} \left[\begin{aligned} &L_i \sqrt{H^2 + (\mu g L_i - V_i)^2} - \frac{V_i}{\mu g} \left(\sqrt{H^2 + V_i^2} - \sqrt{H^2 + (\mu g L_i - V_i)^2} \right) \\ &+ \frac{H^2}{\mu g} \ln \left[\frac{\sqrt{H^2 + (\mu g L_i - V_i)^2} + \mu g L_i - V_i}{\sqrt{H^2 + V_i^2} - V_i} \right] \end{aligned} \right] ds_i \end{aligned} \quad (5-119)$$

The equation of motion, Eq. (5-113), after substituting Eq. (5-119) has the form:

$$\ddot{\phi} + 2B\dot{\phi} + \omega_n^2 \hat{\phi} = 0, \quad (5-120)$$

where:

$$\omega_n = \left[\begin{aligned} &\frac{N_{10}|_B \sin(\phi_0 + \theta_0)}{m_p \ell_p} - \frac{C_2}{m_p \ell_p N_{10}|_B} [H_0 \sin(\phi_0 + \theta_0) + (\mu g L_1 - V_{10}) \cos(\phi_0 + \theta_0)] \\ &-\frac{C_1}{m_p \ell_p N_{10}|_B} [(\mu g L_1 - V_{10}) \sin(\phi_0 + \theta_0) + H_0 \cos(\phi_0 + \theta_0)] - \frac{g}{\ell_p} \cos \phi_0 \end{aligned} \right]^{\frac{1}{2}} \quad (5-121)$$

and

$$B = -\frac{C}{2m_p} \frac{\cos(\phi_0 + \theta_0)}{\ell_p} \frac{d}{d\phi} \left\{ \frac{1}{EA_0} \sum_{i=1}^n \int_0^{L_i} [N_{i0}(s_i) + \hat{N}_i(s_i)] ds_i \right\}. \quad (5-122)$$

The constants C_1 and C_2 for the multispan case are evaluated similarly to Eqs. (5-81) and (5-83):

$$C_1 = \frac{v_2 f_1 - v_1 f_2}{h_2 v_1 - h_1 v_2}, \quad (5-123)$$

$$C_2 = \frac{h_1 f_2 - h_2 f_1}{h_2 v_1 - h_1 v_2}.$$

where

$$f_1 = -\{\ell_p \cos \phi_0\}$$

$$h_1 = \frac{rH_0^2}{[N_{10|B}]^3} + \frac{1}{\mu g} \left[\frac{V_{10}}{N_{10|1}} + \frac{\mu g(L_1 - r) - V_{10}}{N_{10|B}} - \sinh^{-1} \left[\frac{V_{10}}{H_0} \right] + \sinh^{-1} \left[\frac{V_{10} - \mu g L_1}{H_0} \right] \right] - \frac{L_1}{EA_0}$$

$$v_1 = \frac{rH_0(V_{10} - \mu g L_1)}{[N_{10|B}]^3} - \frac{H_0}{\mu g} \left[\frac{1}{N_{10|1}} - \frac{1}{N_{10|B}} \right]$$

$$f_2 = -\{\ell_p \sin \phi_0\}$$

(5-124)

$$h_2 = \frac{rH_0(V_{10} - \mu g L_1)}{[N_{10|1}]^3} + \frac{1}{\mu g} \left[\frac{V_{10}^2}{H_0 N_{10|1}} - \frac{(\mu g L_1 - V_{10})^2}{H_0 N_{10|B}} + \frac{N_{10|1}}{H_0} - \frac{N_{10|B}}{H_0} \right]$$

$$v_2 = -\frac{r}{N_{10|B}} - \frac{1}{\mu g} \left[\frac{V_{10}}{N_{10|1}} + \frac{\mu g L_1 - V_{10}}{N_{10|B}} \right]$$

and

$$N_{10|1} = \sqrt{H_0^2 + V_{10}^2},$$

$$N_{10|B} = \sqrt{H_0^2 + (V_{10} - \mu g L_1)^2}. \quad (5-125)$$

From Eq. (5-122) the equivalent visous damping coefficient, C , can be calculated from experimental data using

$$C = -\frac{2Bm_p \ell_p}{\cos(\phi_0 + \theta_0)} \frac{1}{\frac{d}{d\phi} \left[\int_0^{L_0} \frac{N_0 + \hat{N}}{EA(s)} ds \right]}. \quad (5-126)$$

where ϕ_0 and θ_0 come from Eqs. (5-107) and (5-108), B is experimentally determined as outlined in section 5.5 below. The term $\frac{d}{d\phi} \left[\int_0^{L_0} \frac{N_0 + \hat{N}}{EA(s)} ds \right]$ is found numerically. This is through

- (a) Using Eqs. (5-100)-(5-105) to determine static equilibrium values of H_0 , $V_{i0}; i = 1..n$ and L_i .
- (b) Using Eqs. (5-109)-(5-112) to determine perturbed values of H and $V_i; i = 1..n$ for various ϕ .
- (c) From the values of H and $V_i; i = 1..n$ Eq. (5-119) is evaluated to provide numerical values of $\int_0^{L_0} \frac{N_0 + \hat{N}}{EA(s)} ds$ versus ϕ from which the derivative $\frac{d}{d\phi} \left[\int_0^{L_0} \frac{N_0 + \hat{N}}{EA(s)} ds \right]$ is determined by finding the least squares regression of a straight line of the curve in the region around $\hat{\phi} = 0$.

The natural frequency, ω_n , can be determined from the measured damped frequency, ω_d :

$$\omega_n = \sqrt{\omega_d^2 + B^2}. \quad (5-127)$$

With Eq. (5-127) providing a known value for ω_n , Eqs. (5-121), (5-123), (5-124) and (5-125) can also be used to evaluate another estimate for EA_0 for comparison with the value determined from Eqs. (5-100) to (5-105).

5.4. Experimental Data

Experimental data was acquired using the LMS modal analysis data acquisition through the following general procedure:

- (a) The end attachment (either fixed or reeled) was mounted on one of columns C1 to C7 as shown in Figure 5-1;
- (b) The tether length was adjusted to provide a nominal 2 Newton tether tension as measured at the load cell;
- (c) The tension fine adjustment on the end attachment was adjusted to bring the tether tension at the load cell to the desired static equilibrium tension;
- (d) The tether supports were checked to ensure that they were hanging vertically and at a uniform spacing along the length of the tether;
- (e) The static equilibrium span from the end attachment to the centreline of the pendulum mass was measured prior to setting the pendulum mass into motion;
- (f) The pendulum mass was set into motion and the LMS data acquisition system enabled to acquire tether tension data from the load cell on the pendulum mass (sample tether tension history is shown in Figure 5-17, below);
- (g) Once pendulum motion had stopped the static equilibrium span from the end attachment to the centreline of the pendulum mass was again measured to verify that the pendulum had returned to the pre-motion static equilibrium position; and
- (h) Typically the experiment was performed at least three times to verify the repeatability of the experiment.

Figure 5-17. Example of tether force history (at load cell) acquired using LMS data acquisition system; 6 tether supports with fixed end attachment used, as depicted; unstrained tether length $L_0 = 37.549[m]$; measured damped natural frequency $\omega_d = 2.375[\text{rad/s}]$.

5.4.1. Experimental Testing Configurations

During the period June-July, 1998 manufacture of the experimental equipment was performed at Carleton University and at the Canadian Space Agency, St Hubert facility. During August – October, 1998 testing of various test configurations was performed as outlined in

Table 5-3, below. Testing was performed by mounting the end attachment (either fixed or reeled) on one of seven building columns (C1 through C7 as shown in Figure 5-1). Data from tests taken with the end attachment on column C1 were found to be too widely scattered for proper analysis. Data taken from columns C2 through C7 are categorized into 10 different groupings based on the nominal tether length and end attachment used. These groupings are shown in Table 5-2.

Table 5-2. Data set naming convention for experimental configurations tested (long tether testing – Canadian Space Agency, St Hubert).

Group Name	Experimental Configuration
8 metre fixed	Unstretched tether length was 7.73 metre; fixed end attachment was mounted at column C2; tether supports at column C1.
8 metre reeled	Unstretched tether length varied from 7.5 to 7.525 metre; reeled end attachment was mounted at column C2; tether supports at column C1.
14 metre fixed	Unstretched tether length was 13.53 metre; fixed end attachment was mounted at column C3; tether supports at columns C1 and C2.
14 metre reeled	Unstretched tether length was 13.456 metre; reeled end attachment was mounted at column C3; tether supports at columns C1 and C2.
20 metre fixed	Unstretched tether length was 19.56 metre; fixed end attachment was mounted on column C4; three tether support configurations were tested: (a) tether supports at columns C1, C2 and C3, (b) tether supports at columns C1 and C3, (c) tether supports at columns C2 and C3.
26 metre fixed	Unstretched tether length was 25.66 metre; fixed end attachment was mounted at column C5; tether supports at columns C1, C2, C3 and C4.
26 metre reeled	Unstretched tether length was 25.327 metre; reeled end attachment was mounted at column C5; tether supports at columns C1, C2, C3 and C4.
32 metre fixed	Unstretched tether length was 31.527 metre; fixed end attachment was mounted at column C6; tether supports at columns C1, C2, C3, C4 and C5.
32 metre reeled	Unstretched tether length was 31.33 metre; reeled end attachment was mounted at column C6; tether supports at columns C1, C2, C3, C4 and C5.
38 metre fixed	Unstretched tether length was 37.549 metre; fixed end attachment was mounted at column C7; tether supports at columns C1, C2, C3, C4, C5 and C6.

Table 5-3. Summary of Experimental Configurations Tested at Canadian Space Agency – St Hubert.

Variable	Fixed End Attachment	Reeled End Attachment
Experiments Analysed	348	101
Tether Length	Six different lengths ranging from 7.730 to 37.549 metres	Four different lengths ranging from 7.500 to 31.330 metres
Tether Tension	0.35 to 5.36 Newtons with 54% of the data within the range 1.0 to 3.0 Newtons	1.16 to 5.68 Newtons with 63% of the data within the range 1.0 to 3.0 Newtons
Tether Support Interval Length	For most experiments the initial tether segment was 2.116 metres with 6 metre intermediate spacing and variable final segment (due to tensioning). For 19.560 metre long tether intermediate spacing was change to include 8.116, 6.000 and 12.000 metre variations	Initial tether segment was 2.116 metres with 6 metre intermediate spacing and variable final segment (due to tensioning and reel compression)

5.5. Analysis Techniques for Experimental Data

The objective of analysis of the experimental data is to extract damping and elastic coefficients. Various methods exist for the analysis of the experimental data from the long tether test setup. Equation of motion (5-19), for the rod model, and eqn (5-120) for the cable model are second order spring mass damper systems if the equation coefficients are considered constant. These are of the form:

$$\ddot{\hat{\phi}} + 2B\dot{\hat{\phi}} + \omega_n^2\hat{\phi} = 0,$$

which has the general solution from eqn (5-95):

$$\hat{\phi} = De^{-Bt} \sin(\omega_d t + \delta), \quad (5-95)$$

where $\omega_d = \sqrt{\omega_n^2 - B^2}$; $D = \sqrt{\hat{\phi}_0 + \frac{\dot{\hat{\phi}}_0}{\omega_n}}$ and $\delta = \tan^{-1} \frac{\dot{\hat{\phi}}_0 \omega_n}{\hat{\phi}_0}$.

Force data acquired from the load cell attached to the tether and to the pendulum mass will have the same damped natural frequency, ω_d , and exponential decay constant, B , as does the angular motion shown in eqn (5-95). The force data will oscillate about the static equilibrium force, $N_0|_B$, at the load cell and damping will occur. The force data must have the functional form of an underdamped: $N|_{B, \text{lower}} = N_0|_B - ae^{-Bt}$

$$N|_B = N_0|_B + ae^{-Bt} \sin(\omega_d t + \delta),$$

where a is constant. This functional form is shown in Figure 5-18.

Figure 5-18. Simply damped vibration system showing data points taken at local extents of oscillation to define points on upper and lower bounding curves.

In Figure 5-18 the damping is due to the coefficient B . In order to determine the B coefficient, experimental data need not be analyzed to provide coefficients associated with the (possibly complex) oscillation of the system. In order to find a good estimate for the coefficient B a least squares analysis can be performed that minimizes the error of selecting a value for B which describes the experimental data.

This analysis presents a least squares method for fitting the experimental data conforming to the two bounding equations for a damped oscillation:

$$N|_{B,u} = N_0|_B + ae^{-Bt} \text{ (upper bounding equation)} \quad (5-128)$$

$$N|_{B,l} = N_0|_B - ae^{-Bt} \text{ (lower bounding equation)} \quad (5-129)$$

For known upper curve and lower curve sample data: $(t_i, N|_{B,i})$ and $(t_j, N|_{B,j})$, respectively, the least squares formulation can be written:

$$S = \sum_{i=1}^n (N|_{B,i} - N|_{B,u})^2 + \sum_{j=1}^m (N|_{B,j} - N|_{B,l})^2 \quad (5-130)$$

There are n data points in the upper curve and m data points in the lower curve. Substituting eqns (5-128) and (5-129) into eqn (5-130) yields:

$$S = \sum_{i=1}^n (N|_{B,i} - N_0|_B - ae^{-Bt_i})^2 + \sum_{j=1}^m (N|_{B,j} - N_0|_B - ae^{-Bt_j})^2 \quad (5-131)$$

The least squares formulation is found by solving the system of equations described by:

$$\frac{dS}{dN_0|_B} = 0; \frac{dS}{da} = 0; \frac{dS}{dB} = 0$$

for the three unknowns coefficients $N_0|_B, a, B$. Differentiating eqn (5-131) with respect to the unknown coefficients yields:

$$\frac{dS}{dN_0|_B} = 2 \sum_{i=1}^n (N_0|_B + ae^{-Bt_i} - N|_{B,i}) + 2 \sum_{j=1}^m (N_0|_B - ae^{-Bt_j} - N|_{B,j}) \quad (5-132)$$

$$\frac{dS}{da} = 2 \sum_{i=1}^n (N_0|_B + ae^{-Bt_i} - N|_{B,i}) e^{-Bt_i} - 2 \sum_{j=1}^m (N_0|_B - ae^{-Bt_j} - N|_{B,j}) e^{-Bt_j} \quad (5-133)$$

$$\frac{dS}{dB} = 2a \sum_{i=1}^n t_i e^{-Bt_i} (N|_{B,i} - N_0|_B - ae^{-Bt_i}) - 2a \sum_{j=1}^m t_j e^{-Bt_j} (N|_{B,j} - N_0|_B + ae^{-Bt_j}) \quad (5-134)$$

Setting eqns (5-132)-(5-134) equal to zero and simplifying yields, respectively:

$$N_0|_B (n+m) + a \left[\sum_{i=1}^n e^{-Bt_i} - \sum_{j=1}^m e^{-Bt_j} \right] - \sum_{i=1}^n N|_{B,i} - \sum_{j=1}^m N|_{B,j} = 0 \quad (5-135)$$

$$N_0|_B \left[\sum_{i=1}^n e^{-Bt_i} - \sum_{j=1}^m e^{-Bt_j} \right] + a \left[\sum_{i=1}^n e^{-2Bt_i} + \sum_{j=1}^m e^{-2Bt_j} \right] - \sum_{i=1}^n N|_{B,i} e^{-Bt_i} + \sum_{j=1}^m N|_{B,j} e^{-Bt_j} = 0 \quad (5-136)$$

$$\sum_{i=1}^n N|_{B,i} t_i e^{-Bt_i} - \sum_{j=1}^m N|_{B,j} t_j e^{-Bt_j} - N_0|_B \left[\sum_{i=1}^n t_i e^{-Bt_i} - \sum_{j=1}^m t_j e^{-Bt_j} \right] - a \left[\sum_{i=1}^n t_i e^{-2Bt_i} + \sum_{j=1}^m t_j e^{-2Bt_j} \right] = 0 \quad (5-137)$$

Equation (5-135) can be solved for A as a function of the other two coefficients a, B :

$$N_0|_B = \frac{1}{n+m} \left\{ \sum_{i=1}^n N|_{B,i} + \sum_{j=1}^m N|_{B,j} - a \left[\sum_{i=1}^n e^{-Bt_i} - \sum_{j=1}^m e^{-Bt_j} \right] \right\} \quad (5-138)$$

Substituting eqn (5-138) into eqn (5-136) and solving for a as a function of the coefficient B yields:

$$a = \frac{\frac{1}{n+m} \left(\sum_{i=1}^n N|_{B,i} + \sum_{j=1}^m N|_{B,j} \right) \left(\sum_{i=1}^n e^{-Bt_i} - \sum_{j=1}^m e^{-Bt_j} \right) - \sum_{i=1}^n N|_{B,i} e^{-Bt_i} + \sum_{j=1}^m N|_{B,j} e^{-Bt_j}}{\frac{1}{n+m} \left(\sum_{i=1}^n e^{-Bt_i} - \sum_{j=1}^m e^{-Bt_j} \right)^2 - \sum_{i=1}^n e^{-2Bt_i} - \sum_{j=1}^m e^{-2Bt_j}} \quad (5-139)$$

Equations (5-138) and (5-139) can be used to substitute for $N_0|_B, a$ in eqn (5-137) to determine an equation solely in the coefficient B . When that substitution is made the resulting equation is a series of summations including exponential terms: $\sum_{i=1}^n e^{-Bt_i}$, $\sum_{j=1}^m e^{-Bt_j}$, $\sum_{i=1}^n e^{-2Bt_i}$, $\sum_{i=1}^n N|_{B,i} e^{-Bt_i}$, $\sum_{j=1}^m N|_{B,j} e^{-bt_j}$, $\sum_{i=1}^n N|_{B,i} t_i e^{-Bt_i}$, $\sum_{j=1}^m N|_{B,j} t_j e^{-bt_j}$, requiring a numerical evaluation to solve for the coefficient B .

Figure 5-19. Example of tether force profile showing upper and lower bounding damped exponential curves determined through least squares regression; circles show peaks used for regression (points used for regression are within bounds of ± 0.05 [N] of the static equilibrium tension); 6 tether supports used, as depicted; unstrained tether length $L_0 = 37.549$ [m]; measured damped natural frequency $\omega_d = 2.375$ [rad/s]; axial stiffness $EA_c = 605.3$ [N]; structural damping coefficient $b_{t,c} = 3.677$ [N/m].

Since a numerical method is required for the solution to these equations the following method for solution was used:

- (a) Select an initial value of coefficient B ;
- (b) Evaluate coefficient a using eqn (5-139);

- (c) Evaluate coefficient $N_0|_B$ using eqn (5-138);
- (d) Evaluate the L.H.S. of eqn (5-137) and compare to zero; and
- (e) Adjust the value of B iteratively to bring the left hand side of eqn (5-137) to zero.

MATLAB[®] routines were written to implement this procedure from which the least squares values of $N_0|_B$, a and B are evaluated. Expressions relating $N_0|_B$, a and B to equivalent viscous damping coefficient C are available from eqn (5-19), for the rod model, and eqn (5-126), for the cable model. Figure 5-19 shows an example of this analysis technique.

5.6. Tether Axial Stiffness Deduced From Data

For the experimental configurations listed in Table 5-2 above, values were determined for the axial stiffness, EA , of the tether using the rod and cable models.

Figure 5-20 shows the values for axial stiffness, EA_{avg} , which result in a fit of experimental data to the cable and rod models. Although there is some scatter of the data, possibly due to aging effects of the tether, the cable model values agree well with the data obtained by the Naval Research Laboratories for the TiPS mission tether [data taken from Figure 1-3 is shown as a black line in Figure 5-20]. The rod model, however, provides approximately 3-4 times higher estimates of EA_{avg} than both the cable model and the confirmatory data from the TiPS mission. This difference confirms the validity of using the cable model in preference to the rod model, even for high tether tensions (4-6 [N]).

The data in Figure 5-20a shows scatter that may be attributable to age effects of the tether during the period of testing shown in Table 5-4. Figure 5-21 shows a break-out of the data in Figure 5-20a based on chronology of testing.

Figure 5-20. Average tether axial stiffness, EA_{avg} , versus mean tether tension at pendulum, $N_0|_B$, for tethers of various lengths and end attachment. Average of 3-5 tests for two models (a) cable (b) rod.

Figure 5-21. Tether axial stiffness (cable model), $EA_{avg,cable}$, versus mean tether tension at pendulum, $N_{0|_B}$, for four different periods of testing, showing aging effects on the tether; (a) Day 1, (b) Days 9-10, (c) Days 16-17, (d) Days 22-23. Linear least squares fits are displayed as solid lines.

Table 5-4. Chronological order of configurations tested.

Day	Configuration Tested
1	14 m fixed
9-10	20 m fixed case 1
9-10	20 m fixed case 2
10	20 m fixed case 3
16	8 m fixed
17	8 m reeled
17	14 m reeled
22	26 m fixed
22	26 m reeled
22	32 m fixed
22	32 m reeled
23	38 m fixed

Figure 5-22. Linear least squares trend lines for $EA_{avg,cable}$, versus mean tether tension at pendulum, $N_{0|_B}$, for four different time periods of experimental data with extrapolated values to zero tether tension (at pendulum) shown.

Figure 5-21c and Figure 5-21d show less scatter among the data taken within time periods of 1.5 days. The linear least squares fits from Figure 5-21 are shown in Figure 5-22. The trend-lines for the periods (a) Day 1, (b) Days 9-10 and (c) Days 22-23 follow a pattern indicative of changes to the tether EA due to aging, load history (creep) or environmental phenomena. The tether loses axial stiffness with the passage of time. The trendline for data from Days 16-17 is contrary to the pattern displayed by the other three trendlines.

Figure 5-23. Tether axial stiffness (cable model), $EA_{avg,cable}$, versus mean tether tension at pendulum, $N_0|_B$, corrected to Day 0 of the experimental test period. Data for Days 16-17 is omitted.

This trendline data can be deemed to be suspect as it's extrapolated zero tension value is considerably negative. Discarding the Day 16-17 data set and forcing the other three data sets through the origin results in the following ageing correlation:

$$EA = 559.1N_0|_B e^{-0.0203t} \quad (5-140)$$

where t is given in days.

Figure 5-23 shows data from Figure 5-20a corrected with the exponential aging represented in eqn (5-140).

5.6.1. Axial Stiffness and Tether Length

An alternative explanation for the variation in data shown in Figure 5-20a is that there is some dependence on the tether length.

Figure 5-24. Product of axial stiffness, EA_{avg} , and unstretched tether length, L_0 , versus tether equilibrium force at the pendulum, $N_0|_B$ for various tether lengths.

Figure 5-24 shows the product $EA_{avg}L_0$ plotted versus the force at the pendulum load cell, $N_0|_B$. The result for the tether lengths plotted is very linear. From Figure 5-24 the following equation correlates axial stiffness with applied force and tether length:

$$EA = \frac{(9140 N_0|_B - 12170)}{L_0}. \quad (5-141)$$

In summary, the cable model must be used to obtain appropriate values of EA from the experimental data. There appears to be either an ageing influence on the value of EA or a dependence of EA on tether length, or a combination of both.

5.7. Tether Damping Deduced From Data

5.7.1. Damping Implicit in Pendulum Mass

A typical longitudinal acceleration profile of the pendulum mass without tether attached is shown in Figure 5-25.

Figure 5-25. Longitudinal acceleration profile of pendulum mass without tether attached showing upper and lower bounding damped exponential curves determined through least squares regression; $\ell_p = 3.325[m]$; $\omega_n = 1.72[rad/s]$; $C_p = 0.1667[N \cdot s/m]$.

The damping attributable to all sources including air drag and mechanical resistance is estimated to produce an external equivalent viscous damping coefficient for the pendulum mass of $C_p = 0.1667 [N \cdot s/m]$. Figure 5-25 shows typical longitudinal (x-direction) acceleration of the pendulum mass with no tether attached.

5.7.2. Tether Damping

The experimental results, once reduced to determine a structural damping coefficient as outlined in section 5.3.2, are shown in Figure 5-26.

Figure 5-26. Average tether structural damping coefficient, $b_{t,avg,cable}$, versus damped natural frequency, ω_d , for tethers of various lengths and end attachment. Reported $b_{t,avg}$ is average of 3-5 tests at a given setup (analysis based on cable model using calculation of EA from dynamic data; corrected for pendulum mass damping).

Here the structural damping coefficient is plotted versus the damped natural frequency of oscillation. This figure is a highlight of the large scale testing in that it shows that the tether's structural damping coefficient is readily predictable (at laboratory environmental conditions), based on knowing the damped natural frequency of loading. The results of Figure 5-26 play an important role in predicting the structural damping coefficient (and hence the equivalent viscous damping coefficient) for the orbiting BOLAS configuration.

Figure 5-27. Average tether structural damping coefficient, $b_{t,avg,cable}$, versus tether tension force at pendulum for tethers of various lengths and end attachment. Reported $b_{t,avg}$ is average of 3-5 tests at a given setup (analysis based on cable model using calculation of EA from dynamic data; corrected for pendulum mass damping).

In Figure 5-26 the structural damping coefficient is seen to vary consistently with the measured damped natural frequency, regardless of tether length, even though damped frequency is taken into account in the definition of structural damping. The physical

mechanism resulting in the relationship shown in Figure 5-26 is not speculated on in this thesis. Figure 5-27 shows the structural damping coefficient versus the static equilibrium tether tension force at the pendulum, for the various configurations used.

5.7.3. Effect of Tension Variation Due to Cable Sag

The analysis of the experimental data was performed with both the rod and cable models of the tether/pendulum system. Figure 5-28 shows the cable model provides a higher estimate for damping coefficient at low tether tensions.

Figure 5-28. Ratio of tether structural damping coefficients derived from cable model and rod model, $b_{t,avg,cable}/b_{t,avg,rod}$, versus tether tension force at pendulum for tethers of various lengths and end attachment (analysis uses calculation of EA from dynamic data; data corrected for pendulum mass damping).

At higher tether tensions the estimate provided by the rod model and the cable model are nearly equivalent, with the exception of the 8 metre data sets. Here the cable model estimate remains higher by a factor of approximately 1.3 instead of becoming equivalent. This is attributable to either (a) aging effects on the tether or (b) interaction with the short tether segment leading from column C1 to the pendulum mass. This latter interaction would be most pronounced with the 8 metre data sets.

5.7.4. Effect of Reel End Attachment versus Fixed End Attachment

Figure 5-29 compares damping coefficients evaluated from the experiments with the reeled end attachment and the fixed end attachment at four different tether lengths. Three of the four cases show negligible difference when using the reeled end attachment. Figure 5-29(b) shows some difference for the nominally 14 metre long tether. This is attributed to tether aging effects between reeled end and fixed end series of experiments. The reeled end experiments for Figure 5-29(b) were performed 16 days after the fixed end experiments whereas the experiments for Figure 5-29 (a), (c), (d) were, at most, 1 day apart. Thus the impact on damping coefficient of a reel at the end of a deployed tether is negligible when compared to the normal variation in the experimental system.

5.7.5. Effect of Tether Length

As with the axial stiffness there appears to be some dependence on tether length. Figure 5-30 shows the product $b_i L_0$ plotted versus the force at the pendulum load cell, $N_0|_B$. The result for the tether lengths plotted is very linear, for $N_0|_B > 1.5$. From Figure 5-30 the following equation correlates axial stiffness with applied force and tether length:

$$b_i = \frac{(311.8 N_0|_B - 464.8)}{L_0}. \quad (5-142)$$

Figure 5-29. Comparison of structural damping coefficient versus tether tension at pendulum mass for tether fixed end attachment and tether reeled end attachment at nominal tether lengths (a) 8 m, (b) 16 m, (c) 26 m, (d) 32 m.

Figure 5-30. Product of structural damping, b , and unstretched tether length, L_0 , versus tether equilibrium force at the pendulum, $N_{0|B}$ for various tether lengths.

5.7.6. Effect of Tether Support Distance

Three tether support variations were performed on the 19.56 [m] long tether (Table 5-5).

Table 5-5. Descriptions of Tether Support Variation Experiments.

Case	Description	Pictorial Representation (key for figures)
1	Fixed end attachment mounted on column C4. Three intermediate tether supports at columns C1, C2 and C3.	<p>Cable model analysis</p> <p>Rod model analysis</p>
2	Fixed end attachment mounted on column C4. Two intermediate tether supports at columns C2 and C3.	<p>Rod model analysis</p>
3	Fixed end attachment mounted on column C4. Three intermediate tether supports at columns C1 and C3.	<p>Cable model analysis</p> <p>Rod model analysis</p>

Figure 5-31. Comparison of tether structural damping coefficient b_t versus tether average tension at the pendulum $N_0|_B$ for various support configurations of a nominal 20 metre long tether analysed with two models (a) rod model (b) cable model.

Figure 5-31 shows the experimental relationship of the damping coefficient with the tether tension varies with the support configuration used. With more supports (giving less variation in tension and providing a more horizontal profile) the evaluated damping coefficients are higher for both the rod and cable models. The cable model marginally improves the fit of the data as determined by comparing the ratio of the damping values for case 1 (three supports) to those of case 3 (two supports, columns 1 and 3) for the rod and cable models.

Figure 5-32. Ratio of damping factor for case 1 (three supports) to damping factor for case 3 (two supports, C1 and C3), $b_{t,case 1}/b_{t,case 3}$, versus mean tether tension at pendulum, $N_0|_B$, for rod and cable models.

Figure 5-32 was generated by using a linear least squares regression of the data in Figure 5-31 for the tension range 1.8 to 3.4 [N]. Figure 5-32 shows that data analysis using the cable model provides a slightly better resolution of the experimental data than the rod model in the sense that the ratio $b_{t,case 1}/b_{t,case 3}$ is overall closer to unity, for the cable model, over the whole range of tensions. Again, the improvement shown by the cable model is more pronounced at lower tensions.

Figure 5-33. Comparison of tether structural damping coefficient b_t versus damped natural frequency ω_d for various support configurations of a nominal 20 metre long tether analysed with two models (a) rod model (b) cable model.

Figure 5-33 shows the experimental relationship of the damping coefficient with the natural frequency also varies with the support configuration used.

Figure 5-34. Ratio of damping factor for case 1 (three supports) to damping factor for case 3 (two supports, C1 and C3), $b_{t,case 1}/b_{t,case 3}$, versus damped frequency of oscillation ω_d , for rod and cable models.

Figure 5-34 was generated by using a quadratic least squares regression of the data in Figure 5-33 for the tension range 2.6 to 4.5 [N]. Figure 5-34 again shows that data analysis using the cable model provides a slightly better resolution of the experimental data than the rod model in the sense that the ratio $b_{t,case 1}/b_{t,case 3}$ is overall closer to unity, for the cable model, over the whole range of tensions. In the case of the correlation with damped frequency the improvement is better than given by the correlation with force. At the lower end of the range the rod model has an eight fold difference between case 1 and case 3 while the cable model provides a 3.5 times difference between these same two cases.

In summary a structural damping model for tether damping results in a better fit to the experimental data than does a viscous damping model. As well, the experimental fit obtained from using the cable model must be used to obtain appropriate values of the structural damping coefficient, b_t , in conjunction with the determination of axial stiffness, EA .

6. Testing of Short Tether Samples At Low Temperatures

The temperature extremes of the BOLAS mission vary from -149°C to -82°C . Large scale testing was performed at the ambient pressure and temperature conditions of the CSA High Bay as described in Chapter 5. The main intent of testing short tether samples was to establish the temperature sensitivity of the tether's material properties. The testing was done at the National Research Council's Montreal Road Campus, Ottawa.

6.1. Mechanical Models of Polymers

6.1.1. Models From Literature

Experimentally determined viscoelastic properties of the tether can be related to the in-orbit material constants used in the BOLAS dynamic model. Two simple mechanical analogies for the performance of polymers are found in Ferry (1970) and Read and Dean (1978), the Maxwell element and the Voigt (or Kelvin) element, as shown in Figure 6-1. Creep effects can be accounted for using the Maxwell element. Here only solids in simple extension will be examined and are typically modeled using the Voigt element:

$$N = C\dot{x} + kx, \quad (6-1)$$

where N is the total force, x is the displacement from the equilibrium position, b is an equivalent viscous damping constant and k is the spring constant where:

$$C = \eta A/L,$$

$$k = EA/L. \quad (6-2)$$

Figure 6-1. Simple mechanical analogies for visco-elastic properties of polymers (a) Maxwell Element, (b) Voigt Element.

In eqns (6-2) η is the polymer viscosity, E is Young's Modulus, A is the cross-sectional area and L is the length of the tether. Typically η and E will be functions of temperature and frequency of loading. Reducing eqn (6-1) to a relationship between longitudinal stress and strain yields:

$$\sigma = \eta\dot{\epsilon} + E\epsilon, \quad (6-3)$$

where σ is the longitudinal stress and ϵ is the longitudinal elongation.

Experimental methods for determining the viscoelastic properties of polymers typically report either:

- a. 'Loss Modulus' and 'Storage Modulus' (explained below); or
- b. 'Stress Amplitude' and 'Phase shift (δ)' (explained below).

The model parameters k and C can be determined from these experimentally reported values. Ferry (1970) shows that for linear viscoelastic behaviour a sinusoidally varying longitudinal strain of frequency ω :

$$\varepsilon = \varepsilon_0 \sin \omega t$$

$$\dot{\varepsilon} = \varepsilon_0 \omega \cos \omega t \quad (6-4)$$

produces a sinusoidally varying longitudinal stress:

$$\sigma = \varepsilon_0 (E' \sin \omega t + E'' \cos \omega t) \quad (6-5)$$

where E' is the Storage Modulus and E'' is the Loss Modulus. Substituting ε_0 of eqns (6-4) for ε of eqns (6-3) produces:

$$\sigma = \varepsilon_0 (E \sin \omega t + \omega \eta \cos \omega t) \quad (6-6)$$

Comparison of eqns (6-5) and (6-6) show that the Storage Modulus, E' , is also the Young's Modulus, E and that the Loss Modulus, E'' , is equal to $\omega \eta$:

$$E' = E; E'' = \omega \eta \quad (6-7)$$

Equation (6-5) can be written:

$$\sigma = \varepsilon_0 \sqrt{E'^2 + E''^2} \left(\frac{E'}{\sqrt{E'^2 + E''^2}} \sin \omega t + \frac{E''}{\sqrt{E'^2 + E''^2}} \cos \omega t \right). \quad (6-8)$$

The phase angle, δ , can be defined by the two equations:

$$\sin \delta = \frac{E''}{\sqrt{E'^2 + E''^2}}; \cos \delta = \frac{E'}{\sqrt{E'^2 + E''^2}}. \quad (6-9)$$

Equation (6-8) becomes:

$$\begin{aligned}\sigma &= R(\sin \delta \sin \omega t + \cos \delta \cos \omega t) \\ &= R(\cos(\omega t - \delta))\end{aligned}\quad (6-10)$$

where R is the amplitude of stress and δ is the phase shift of the stress from the applied strain:

$$R = \frac{\varepsilon_0 L}{A} \sqrt{k^2 + \omega^2 C^2}, \quad (6-11)$$

$$\tan \delta = \frac{k}{\omega C}.$$

Figure 6-2. Stress response to oscillating strain input.

For experimentally determined values of R and $\tan \delta$ or of E' and E'' (at a specified frequency, ω , and specified strain amplitude, ε_0) the material properties of the tether can be determined as:

$$EA = \frac{RA}{\varepsilon_0} \sin \delta = E'A, \quad (6-12)$$

$$\eta A = \frac{RA}{\omega \varepsilon_0} \cos \delta = \frac{E''A}{\omega}$$

In operation of a Dynamic Mechanical Analyser, the value of A , the cross-sectional area and L , is input into analyser by the user. The values of EA and ηA are specific for the

geometry of the tether used. Equations (6-12) can be used with eqns (6-2) to determine k and C for the length of tether used in the analysis, or for any tether length, L .

Since viscosity, η , and Young's modulus, E , are functions of temperature and frequency, ω , then k and C will also be functions of temperature and frequency of loading. In analysis using a Dynamic Thermal Mechanical Analyser the temperature and frequency of loading are varied. Liquid nitrogen is used to obtain cryogenic temperatures of the test sample.

6.1.2. Viscous Damping From DTMA Results

Examination of eqn (6-1) shows that the coefficient C is equivalent to the viscous damping coefficient in the equations of motion for the long tether testing experimental modeling.

6.1.3. Structural Damping

As discussed in Chapters 4 and 5 structural or hysteretic damping is modeled through inclusion of the frequency of oscillation such that:

$$C = \frac{b}{\omega} \quad (6-13)$$

Where b is the structural damping coefficient and ω is the frequency of loading. The structural damping coefficient can be calculated from the experimentally determined coefficient C and the known frequency of loading.

6.2. Dynamic Thermal Mechanical Analysis (DTMA)

A Rheometrics Scientific Solids Analyser (RSA II) at the National Research Council – Montreal Road facility, in Ottawa, Canada was used to perform variable frequency and variable temperature testing on nominally 22 mm long samples of the tether. The analyser is shown in Figure 6-3, below.

Figure 6-3. Rheometrics Scientific Solids Analyser (RSA II) similar to RSA II instrument at National Research Council - Montreal Road facility, building M-20.

6.3. DTMA Test Results

Temperature was varied from -130°C to 50°C and frequency of loading was varied from $0.126 - 12.6$ [rad/s]. A preset tension was applied to the tether and then a further oscillating tension applied at the frequency of loading. Three sets of experiments at three sample preset tensions were performed: (a) 0.04 ± 0.004 [N], (b) 2.23 ± 1.25 [N], and (c) 2.55 ± 1.58 [N]. The test results are shown in Figure 6-4, Figure 6-5 and Figure 6-6, respectively. Due to mechanical limitations of the DTMA it was not possible to obtain test data at both BOLAS equivalent mission tension, say 2.0 [N] and the BOLAS equivalent tension amplitude, say ± 0.01 [N]. The minimum frequency of loading for the DTMA was 0.126 [rad/s] which is 5-10 times faster than the predicted BOLAS axial loading frequency of $0.012 - 0.023$ [rad/s].

Figure 6-4. Temperature variation of (a) Axial stiffness, EA , (b) Structural damping coefficient, b , and (c) Viscous damping coefficient, C . Sample length 22.1 mm; tension = 0.04 ± 0.004 [N]. Least squares regression shown.

(a) Axial stiffness, EA , at 3 frequencies of loading.

(b) Structural damping coefficient, b , at 3 frequencies of loading.

(c) Viscous damping coefficient, C , at 3 frequencies of loading.

Figure 6-5. Temperature variation of (a) Axial stiffness, EA , (b) Structural damping coefficient, b , and (c) Viscous damping coefficient, C . Sample length 22.4 mm; tension = 2.23 ± 1.25 [N]. Least squares regression shown.

(a) Axial Stiffness, EA , at 3 frequencies of loading.

(b) Structural damping coefficient, b , at 3 frequencies of loading.

(c) Viscous damping coefficient, C , at 3 frequencies of loading.

Figure 6-6. Temperature variation of (a) Axial stiffness, EA , (b) Structural damping coefficient, b , and (c) Viscous damping coefficient, C . Sample length 22.1 mm; tension = 2.55 ± 1.58 [N]. Least squares regression shown.

6.3.1. Axial Stiffness, EA

For all three test cases, and at the frequency most similar to BOLAS loading frequencies, the axial stiffness, EA , slightly decreases with increasing temperature, which is typical of many polymeric materials. The linear slope of the three test cases at a loading frequency of 0.126 [rad/s] are shown in Table 6-1.

Table 6-1. Thermal coefficients of (a) axial stiffness (b) structural damping and (c) viscous damping for three test cases. Based on linear least squares regression of test cases with 0.126 [rad/s] frequency of loading.

Tension Range of Test Case [N]	0.04±0.004	2.23±1.25	2.55±1.58
Thermal Coefficient of Axial Stiffness [N/°C]	-1.65	-0.090	-0.006
Thermal Coefficient of Structural Damping [N/m/°C]	--	0.00038	0.00088
Thermal Coefficient of Viscous Damping [N-s/m/°C]	--	0.0030	0.0070

6.3.2. Structural Damping Coefficient, b

For the 2.23 [N] and 2.55 [N] tension test cases the structural damping coefficient, b , are virtually constant for the temperature range tested showing only slight increase as temperature increases. The 0.04 [N] test case showed a decrease and then increase with temperature. The 2.23 [N] and 2.55 [N] test cases also show a tendency for the coefficient to reach a limiting value at the 0.126 [rad/s] frequency. The linear increase of structural damping for the 2.23 [N] and 2.55 [N] test cases are shown in Table 6-1.

6.3.3. Viscous Damping Coefficient, C

The viscous damping coefficient also shows very little change with temperature at the 2.23 [N] and 2.55 [N] tests. Similar to the structural damping coefficient the 0.04 [N] tension test case shows a decrease and subsequent increase with temperature. The linear increase of viscous damping for the 2.23 [N] and 2.55 [N] test cases are shown in Table 6-1.

6.3.4. Structural Damping versus Viscous Damping

The structural damping model appears to provide a better resolution of the test data than does the viscous damping model, particularly when focusing on the 2.23 [N] and 2.55 [N] test cases. Structural damping results for these two cases are taken from Figure 6-5b and Figure 6-6b and combined into Figure 6-7. The structural damping coefficients found from the test data fall within one order of magnitude but the viscous damping coefficients determined from the same test data are separated by two orders of magnitude. The structural damping seems to reconcile the data better than the viscous damping model.

Figure 6-7. Comparison of results for (a) Structural damping coefficient, b , at 3 frequencies of loading and 2 tensions and (b) Viscous damping coefficient, C , at 3 frequencies of loading and 2 tensions.

7. BOLAS Loss of Rotation

An analysis of orbital tether temperatures for the BOLAS mission was performed at the end of chapter 2. The tether can be expected to have temperatures ranging from, nominally -149°C to -82°C with possible excursions due to variations in solar activity or changes in emissivity or absorptivity of the tether.

Chapter 5 has shown experimental results for axial stiffness, EA , and equivalent viscous damping coefficient, C (and hence ζ_n), at ambient conditions of 70% relative humidity and 25°C . Chapter 6 has shown experimental results for cryogenic testing of short tether lengths at orbital temperatures similar to those expected for the BOLAS mission. This chapter combines the results of Chapters 5 and 6 to estimate the tether mechanical properties at the orbital temperatures and then develops a parametric analysis of gravity gradient induced losses based on the orbital modelling of Chapter 2. Finally the atmospheric drag results of Chapter 3 are recalled and combined to estimate the total potential loss of rotation for BOLAS.

7.1. Loss Due to Gravity Gradient Forces

7.1.1. On-orbit Mechanical Properties

7.1.1.1. On-orbit Tether Damping Properties

Estimates of the tether structural damping properties over the nominal temperature range -153°C to 77°C are shown in Figure 7-1 and Figure 7-2. Estimates of a tether viscous damping coefficient are given in Figure 7-3.

Figure 7-1. Structural damping coefficient, $b_{t,cable}$, versus damped frequency of loading, ω_d , of the proposed BOLAS tether adjusted for three temperatures. Frequency range shown is based on experimentally tested frequency values. Temperature correction based on DTMA experiments at 2.23 [N] and 2.55 [N] tension.

Figure 7-2. Structural damping coefficient, $b_{t,cable}$, versus tether tension, $N_{0|B}$, of the proposed BOLAS tether adjusted for three temperatures. Frequency range shown is based on experimentally tested frequency values. Temperature correction based on DTMA experiments at 2.23 [N] and 2.55 [N] tension.

Figure 7-3. Viscous damping coefficient, $C_{t,cable}$, versus damped frequency of loading, ω_d , of the proposed BOLAS tether adjusted for three temperatures. Frequency range shown is based on experimentally tested frequency values. Temperature correction based on DTMA experiments at 2.23 [N] and 2.55 [N] tension.

The BOLAS is proposed to have rotation rates of 0.012-0.023 [rad/s]. This gives a loading frequency (of the longitudinal deformation of the tether) of 0.024-0.046 [rad/s]. These values of frequency are not within the experimental range, as seen in Figure 7-1 and Figure 7-3. The lowest frequency from the experiment is nominally 2.0 [rad/s]. The damping coefficients from these frequencies are shown in Table 7-1, below. Alternatively the nominal BOLAS tether tension value of 2 Newtons can be used to estimate the damping coefficients using Figure 7-2. These are also shown in Table 7-1, below. The higher value will be used to predict a “worst case” loss of rotation.

Table 7-1. Damping Coefficients Selected for BOLAS Loss of Rotation Estimate.

Temperature [°C]	Structural Damping Coefficient		Viscous Damping Coefficient
	$b_{t,cable}$ [N/m]		$C_{t,cable}$ [N-s/m]
	Based on Fig 7-1	Based on Fig 7-2	Based on Fig 7-3
-153	0.81	1.15	0.38
-83	0.89	1.27	0.42
77	1.08	1.54	0.51

7.1.1.2. On-orbit Tether Stiffness Properties

Chapter 5 data is used to produce the data in Figure 7-4. Figure 7-5 shows an expansion of the data from Figure 7-4 to illustrate the nominal 2 Newton in-flight tension range expected for the BOLAS. Both the tether ageing model and the tether length dependence model are shown in these two figures. Using the ageing model from Chapter 5 (Figure 5-23) the day 180 value for EA , at a nominal 2 Newton tension is 29 Newtons. Using the tether length dependence of EA from Chapter 5 (Figure 5-24) the value for EA , at a nominal 2 Newton tension is 53 Newtons.

Figure 7-4. Estimated axial stiffness, EA_{cable} , versus tension for BOLAS mission. Ageing of tether used to determine estimate for day 0 and day 180 of mission. Figure 5-24 correlation of $(EAL_0)_{cable}$ with tension used with a BOLAS tether length $L_0 = 100[m]$.

Figure 7-5. Estimated axial stiffness, EA_{cable} , versus tension for BOLAS mission for tension range 1.5 to 2.5 Newtons. Data expanded from Figure 7-4.

7.2. Estimated Loss of Rotation

7.2.1. Loss of Rotation Due to Gravity Gradient Forces

Prediction of loss of rotation can be performed through the use of the 4th order Runge-Kutta routine shown in Chapter 2. Table 7-2 examines viscous and structural damping models together with an axial stiffness derived from Figure 5-24 which takes into account the dependence of axial stiffness on tether length.

Table 7-2. Loss of rotation estimates of the BOLAS mission for various cases.

Case	Nominal Rotation Rate, ω_3^*	Damping		EA [N]	ζ_n	ω_n [rad/s]	% Loss of Rotation in 30 days
		Struc. $b_{t,cable}$ [N/m]	Visc. $C_{t,cable}$ [N-s/m]				
(a)	(b)	(c ₁)	(c ₂)	(d)	(e)	(f)	(g)
-153°C; structural damping	$20 n_0$	0.89	--	53	2.88	0.122	4.90%
-83°C; structural damping	$20 n_0$	0.98	--	53	3.18	0.122	4.61%
77°C; structural damping	$20 n_0$	1.18	--	53	3.86	0.122	4.02%
-153°C; viscous damping	$20 n_0$	--	0.38	53	0.044	0.122	0.36%
-83°C; viscous damping	$20 n_0$	--	0.42	53	0.049	0.122	0.40%
77°C; viscous damping	$20 n_0$	1.18	0.51	53	0.059	0.122	0.49%
* n_0 = earth rate = 0.011 [cpm] (0.0012 [rad/s])							

The results of Table 7-2 show that the loss of rotation of BOLAS can be significant. The structural damping model produces higher estimates of loss of rotation than the viscous damping model. Lower values of EA will produce a higher loss of rotation. A “worst case” estimate of the loss of rotation is 4.90% loss of rotation in 30 days. This has an equivalent 180 day loss of rotation of 29.67%.

It is important to understand that this loss of rotation estimate is a direct result of the stiffness and damping correlations with tether length shown in Figures 5-24 and 5-30.

The limit of applicability of these two correlations has not been determined. Common sense indicates that the express relationships shown in Figures 5-24 and 5-30 must not hold true for very long tether lengths. In the limit, from these correlations an infinitely long tether would have a zero axial stiffness and zero structural damping coefficient. Deviation from these correlations must occur; it is unknown whether this deviation starts at tether lengths less than 100 metres or at tether lengths greater than 100 metres. Thus the extrapolation of this correlation to the in-flight, 100 metre BOLAS tether length needs to be confirmed by ground testing a full length, 100 metre long tether. Ideally this testing would be performed at actual in-flight BOLAS tether temperatures.

7.2.2. Loss of Rotation Due to Aerodynamic Drag

From Chapter 3 the loss of rotation due to aerodynamic drag is approximately 0.55% for the 180 day mission period.

7.3. Total Loss of Rotation

Combining the estimates from sections 7.2.1 and 7.2.2, the total loss of rotation for the 180 days of the BOLAS mission will be $(29.67\% + 0.55\% =) 30.22\%$ using the current reference parameters. This estimate is dependent on the length and mechanical properties of the BOLAS tether as well as the level of solar activity that occurs during the mission period. If the reference parameters of the BOLAS are changed in the future, for example an increase in tether length, the estimate could increase and should be reviewed.

8. Conclusions and Future Work

8.1. Conclusions

There are potentially two significant mechanisms of rotation loss for the BOLAS. Firstly, gravity gradient induced structural energy dissipation related to the longitudinal stretch of the tether could induce a loss of up to 29.67% in 180 days. Secondly, aerodynamic drag would result in a loss of 0.55% in 180 days. Although not as low as the 5% loss of rotation initially envisioned, the combined loss of 30.22% is sufficiently low for the BOLAS to avoid gravity capture.

A substantial contribution has been made in the technology of gravity gradient induced damping in this thesis with the following additional conclusions made in this area:

- (a) Models of the BOLAS in-orbit dynamics have been obtained which are relatively uncomplicated yet serve to reveal the essential relationships between physical parameters (tether stiffness, damping) and rotation rate;
- (b) A large-scale horizontal tether experimental configuration was successful and provided for tether tension variation within 0.004 Newtons in a 37.5

metre long tether. Modelling that takes into account the tether sag is required in this type of experimental test. A model based on extensible cable literature was successfully developed and applied;

- (c) The experiment data on the SPECTRA-1000®/acrylic tether appears to be best predicted by a structural damping model, as opposed to a viscous damping model. The structural damping model provided reasonably good consistency over several test configurations;
- (d) There appears to be a dependence of the tether elastic and damping properties on the length of tether used; prior to designing the final BOLAS flight configuration full scale tests using the actual length of the BOLAS tether must be performed to confirm the tether length dependence of the tether elastic and damping properties;
- (e) The influence of temperature on the stiffness and damping properties of the tether material was measured and the associated secondary effects were incorporated; and
- (f) The tether appeared to experience a possible ageing effect in which its structural stiffness and damping properties changed a small amount with time. This may be a relaxation effect associated with the winding/storage/unwinding of the tether relative to the reel.

8.2. Future Work

Various items were identified as the focus for further work. There are three categories:

(a) expanded analytical modelling of aerodynamic forces effecting loss of rotation of BOLAS, (b) verification of the material model of the tether and (c) modifications to the experimental testing philosophy for identifying tether material properties.

8.2.1. Aerodynamic Modelling

Models of loss of rotation of the BOLAS spacecraft can be improved by improving the atmospheric drag models to determine (a) the effect of dissimilar masses of the sub-satellites and (b) the effect of dissimilar aerodynamic cross-section of the sub-satellites.

8.2.2. Verification of the Material Model of the Tether

The object of ground testing of space tethers is to determine their orbital properties for inclusion in orbital models of a tethered spacecraft. The ground testing approach can be modified to provide heightened confidence of the material model of the tether by:

- (a) Performing detailed tests of the tether material to verify its material characteristics including determination of creep and relaxation aging effects on the damping and elastic properties at orbital temperatures;
- (b) In parallel with ground testing applications fly a tether experiment to correlate the ground test results with actual data from an instrumented tethered spacecraft in an orbiting microgravity environment; and
- (c) Improve the long tether ground test configuration by: (i) providing additional tether support points to further reduce the tension variation in the tether, (ii) develop a spatially large, cryogenic facility that can accommodate long lengths of a tether and (iii) develop mechanical exciters in place of the pendulum tether stretching system to allow a wider frequency spectrum of oscillating tension loads than possible using the fixed length pendulum technique of this thesis.

8.2.3. Experimental Analytical Models

Refinements in the modelling of the experimental configuration can improve the timeliness and value of experimental results by:

- (a) Developing a completely analytical form for the cable model that does not rely on the numerical determination of the expression [from eqn (5-126)]:

$$\frac{d}{d\phi} \left[\int_0^{L_0} \frac{N_0 + \hat{N}}{EA(s)} ds \right]; \text{ and}$$

- (b) Simplifying the cable and/or rod models to find additional modes of vibration and incorporate modal analysis techniques in the analysis of the experimental data to identify the parameters for the higher modes of vibration.

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Chronological Development of Dynamic Models of Tethered Two-body Satellites

Introduction

Theoretical dynamics for tethered satellite systems can be traced in the available literature from 1958 through to the present with increasing complexity. The early planar, point mass models in circular orbits eventually led to three dimensional models incorporating rigid end-bodies with massive visco-elastic tethers. At any given time during the history of tethered satellite theory development there have been specific technological focuses that have spurred research in the direction that it has taken. In the early 1960's the main driver was generation of artificial gravity through the rotation of a space station tethered to its counterweight. Since then, a large range of tether applications has developed including atmospheric science missions, electricity generation, electro-magnetic propulsion and vibration isolation.

Throughout this development, the names of certain researchers will appear repeatedly, alone or as groups. Much of the refinement of tethered satellite dynamic models is due to the continued involvement of a core group of researchers who have maintained interest in this field over the decades.

The following is essentially a chronological listing of literature references with some description of their contribution to the field of tether dynamics. The reader is cautioned that this is not an exhaustive list. The reader is also cautioned that the author has not read all of the papers listed; the relevance of many of these papers has been determined from citations in other papers.

Late 1950's

Tethered satellites systems have been modelled as early as 1958. Using point masses for the end-bodies, massless tethers and rotation in the orbit plane Ordway [1], Robinson [2] and Moran [3] modelled the libration, longitudinal and transverse motions of tethered space stations rotating to generate artificial gravity. Their models were limited to small oscillations and neglected perturbations from orbital elements. As well it was established that the rotational and vibrational motion cannot be treated independently from orbital motion without violating the laws of conservation of energy and conservation of angular momentum [4].

1960's

This decade began by improving the planar models developed in the late 50's . By the end of this decade three-dimensional modelling with rigid end-bodies had been performed.

Garber [5] examined a planar model of an orbiting, long flexible wire, with no end-masses, for communications purposes. Chobotov [6] examined a planar model with constant orbital radius, constant orbital angular velocity and constant rotation rate. This study determined the effect of a viscous damped tether on the rotation rate and

established the effect of orbital angular velocity on the stability of the free motion of a rotating system.

Continuing to use planar, point mass models with a massless, visco-elastic tether, Paul [7] established that an internal viscous tether damping has considerably less effect on rotation rate than a non-linear, time independent hysteretic damping mechanism.

To investigate the roll stability of a rotating, tethered space station Tai, et al, [8, 9] extended the planar, point mass models to three dimensions. Targoff [10] also used a three dimensional model to determine the modal shapes and frequencies of an orbiting tether's transverse vibrations. The works of Kane, et al [11], Robe [12] and Anderson [13] introduce distributed masses for the end-bodies within the framework of a three-dimensional model. Kane and Robe [11, 12] show that first order torque effects may be neglected and in-plane motion is decoupled from out-of-plane motion; this identified the need to develop separate stability criteria for out-of-plane motion. Anderson [13] showed that the attitude response of tethered rotating space station is that of an undamped second order system coupled with rigid body characteristics and tether transverse mode effects.

1970's

This decade saw review of the planar models and refinement of the three dimensional models to better understand tether deployment dynamics and find means of controlling tether vibrations. By the end of the decade models were including aerodynamic drag effects in a rotating atmosphere and eccentric orbits.

Ebner [14] first reported the planar deployment dynamics of a tethered two—body configuration in which the pitch oscillations were approximated from conservation of angular momentum which then lead to an analysis of transverse vibrations.

At Howard University, Stabekis and Bainum [15] reverted back to a planar model to estimate the damping requirements of a tethered space station system. Bainum, et al [16]

continued to develop their model to include distributed endmasses and reported the required tether system damping and rotational damping of endmasses to achieve adequate damping in each mode. Bainum and Stuvier [17] further developed an algorithm to find deployment control forces for any given planar deployment scenario.

Next, the team of Bainum and Evans [18, 19, 20] independently derived the three dimensional equations of motion for a fully deployed tethered system with point masses and constant rotation rate in a torque free environment. This confirmed the earlier findings of Kane and Robe that, to the first order, the in-plane motion decouples from the out of plane motion. As well, Bainum and Evans report that for small perturbations of the tether's orientation out of it's plane of rotation, the system will tend to rotate in a plane inclined to the original plane of rotation without affecting the rotation rate or equilibrium cable length.

Concurrent with investigations at Howard University other researchers contributed additions to modelling space tethers. Beletskii and Novikova [21] produced for the first time analysis of both slack and taut tether motion. Nixon [22] developed a three-dimensional model with multiple tethers and determined the dynamic equilibrium states of such systems. Childs and Hardison [23] found that control of a moveable mass can provide wobble damping in all phases of operation including fixed stages of deployment and at extremes in rotation rate but did not consider varying the tether length while the system rotates.

Rupp [24] examined tension control laws for deployment and retrieval of tethered systems (neglecting transverse vibrations) from the Space Shuttle and identified a growth of pitch oscillations during retrieval. Baker [25] added aerodynamic drag effects in a rotating atmosphere to the deployment model. Kulla [26] presented models to determine the tether shape during deployment that included transverse vibrations but excluded longitudinal oscillations and out-of-plane motion.

Kane and Levinson [27] examined passive deployment procedures that require no actuator to control satellite motion; this work neglected any vibrations.

Modi and Misra [28, 29, 30] produced models of increasing complexity to examine a variety of phenomena including:

- (a) orbital perturbations of tethered satellites; orbital perturbations were found to depend more on longitudinal strain than transverse vibrations of similar dimensionless magnitude;
- (b) deployment dynamics in three-dimensions including longitudinal, transverse and libration motions; and
- (c) deployment solutions for the Space Shuttle based tethered system in three-dimensions including aerodynamics of a rotating atmosphere, eccentric orbits and tether attachment point offsets from the Space Shuttle centre of mass; aerodynamic drag was found to increase tether tension.

1980's

With the exception of Swan [31], who performed research on tethered systems in elliptical orbits, there was a hiatus in published research of tethered satellite theory in the early 1980's even though the H-9M-69, S-520-2, Charge-1 and Charge-2 sub-orbital missions flew in the period 1980-1984.

In 1986 Misra and Modi [32, 33] reviewed the dynamics and control issues of tethered satellites. The instigation of Tethered Satellite System (TSS) project spurred continued development of various models to understand the dynamics of satellites connected by a conductive tether for electromagnetic propulsion and/or power generation. Bergamaschi, et al [34, 35], researched tether damping using a viscous damping model for the TSS conductive tether. Graziani, et al [36], analyzed the end-mass effect on the transverse vibration.

Glaese [37] showed that deployment, spinup, despin and retrieval of tethered satellite should require no active control to damp lateral oscillations. Lemke, et al [38] showed that conventional means of attitude control of tethered satellites are not effective and derived the theory showing that attachment point offset variation can be used to obtain control torques for attitude control. Arnold [39] reviewed the behaviour of tethers in space environments.

Hardware development for the Small Expendable Deployer System (SEDS) also spurred research. Carroll and Alexander [40] reported on the development of the Small Expendable Deployer System for deploying tethered satellites.

The ECHO-7 the OEDIPUS-A sub-orbital tethered missions flew in 1988 and 1989 respectively.

1990's

This decade has seen the actual orbital flights of the OEDIPUS, TSS, SEDS, PMG, TiPS and ATeX missions from which many papers were published. These papers not only cover general dynamics of tethered systems but report hardware development, deployment and retrieval schemes, novel control methods and analysis of actual flight data. These will be summarized by category.

Dynamics

Early in the decade Powell and He [41] examined tether damping mechanisms and vibrational modes of longitudinal, transverse and skiprope motion using a three dimensional point mass model and visco-elastic continuum tether with no bending resistance in a circular orbit. Steiner, et al [42, 43, 44, 45, 46, 47] use a discretized massive tether that obeys a linear elastic Kelvin-Voight Law.

The OEDIPUS missions and proposed BOLAS mission have generated development of models for spinning tethered two-body configurations by Vigneron, Tyc, Jablonski, et al

[48, 49, 50, 51]. Tyc [52] produced a comprehensive study of the dynamics and stability of spinning flexible space tether systems.

Misra, et al [53, 54], included geometric tether strain and structural damping into a generalized three dimensional formulation for N-body tethered satellites using discretized representations of the tether. Notably, the analysis [55] of the TiPS and proposed BOLAS missions required simulator propagation to mission times of 6 months; previous dynamic simulations reviewed have propagated for mission times of up to approximately 1 day.

Additional papers on tether dynamics published in this decade include: Beletskii, et al [56], Bergamaschi & Bonon [57], Cochran, et al [58], Kessler & Cicci [59], Krupa, et al [60], Luongo & Vestroni [61], Ohkami, et al [62], Pasca & Lorenzini [63], Quadrelli & Lorenzini [64], Vigneron, et al [65], Wang, et al [66], Warnock & Cochran [67] and Yu [68].

Control Schemes

The various papers published this decade examining different control schemes for tethered satellites and/or satellite control using tethers include: Anderson & Hagedorn [69], Ashenberg & Lorenzini [70], Banerjee & Kline-Schoder [71], Kane & Netzer [72, 73, 74], Kumar, et al [75, 76, 77, 78], Modi, et al [79, 80], Pasca [81], Pradhan, et al [82, 83], Steiner, et al [84], Thornburg & Powell [85, 86] and Vestroni, et al [87].

Deployment and Retrieval

The various papers published this decade examining deployment and retrieval of tethered satellites include: Banerjee [88], De Luca [89], Kane and Stoen [90], Kim & Vadali [91], Pradeep, et al [92, 93], and Seeman & Hagedorn [94].

Mission Analysis

Various papers published this decade performing analysis of the Tethered Satellite System missions include: Angrilli, et al [95], Beragamaschi, et al [96, 97, 98] and Lorenzini, et al [99]. Various papers published this decade performing analysis of the OEDIPUS missions include: Jablonski, et al [100], Tyc & Han [101].

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Appendix B

Fundamental Motions of Gravity Captured Tethered Systems

The fundamental motion of a tether with sub-satellites is best described by the motion of the simple dumbbell configuration of two sub-satellites connected by a tether. Forces acting on an orbiting dumbbell configuration are shown in Figure B-1. Since the acceleration due to gravity decreases non-linearly as altitude increases the centre of mass of an orbiting structure is higher than its centre of gravity. In earth orbit this difference becomes significant only when the orbiting structure has large lengths, as reported by Cosmo and Lorenzini (1997).

The lower satellite experiences greater gravitational force than centrifugal force while the upper satellite experiences greater centrifugal force than gravitational force. The resulting moment causes the configuration to have a stable orientation aligned with the local vertical. Motion away from the vertical orientation produces restoring forces that return the structure to the vertical orientation.

Figure B-1. Forces acting on a tethered satellite.

There are five categories of dynamic motion of tethered two-body satellites as shown in Figure B-2.

Longitudinal motion

Oscillating longitudinal stretching of the tether is caused by gravity gradient forces, deployment and retrieval forces and any spurious forces due to action of attitude control systems onboard the sub-satellites. The natural frequency of longitudinal oscillations has been shown to be dependent on tether length, tether stiffness and effective mass of the sub-satellites. Longitudinal tether motion affects loss of rotation through energy dissipation from tether damping.

Figure B-2. Oscillatory motions experienced by orbiting tethered configurations (a) longitudinal, (b) libration, (c) transverse, (d) skiprope, (e) pendulous (end-body rocking).

Libration

Libration of a dumbbell configuration can occur in both in-orbit-plane and out-of-orbit-plane directions (Figure B-3) when the dumbbell is perturbed from a stable vertical alignment. Gravity gradient restoring forces cause the configuration to return to vertical alignment.

Out-of-plane libration has a higher frequency than in-plane libration. The natural frequency for in-plane libration is $\sqrt{3}\omega$, where ω is the orbital angular velocity; the natural frequency for out-of-plane libration is 2ω . This is due to the centrifugal force vectors being radial for the in-plane case, and parallel for the out-of-plane case. This difference in the vector alignment causes the net force in the out-of-plane case to have a smaller axial component and a larger restoring component. These restoring forces grow with tether length, so libration frequencies are independent of the tether length. However, the axial force can become negative, for $\phi > 60^\circ$ or near the ends of retrograde in-plane librations $> 65.9^\circ$, causing slack tether to occur [Cosmo and Lorenzini (1997)].

Figure B-3. Libration of a tethered system in two directions (a) in orbit plane, (b) out of orbit plane.

Typically, in-plane libration is not close to resonance with any significant driving force. However, it is possible for out-of-plane libration to be in resonance with several forces including: (a) Out-of-plane components of aerodynamic forces (for non-equatorial elliptical orbits that see different air density in northward and southward passes); (b) Solar heating; or (c) Electrodynamic forces.

When in elliptical orbits, the variations in the orbital rate, ω , can cause tether librations that exert periodic moments on an initially uniformly rotating object. In highly elliptical orbits this can induce tumbling [Swan (1984)]. The back-and-forth, swinging motion of classical libration does not occur for the rotating two-body configuration. However, gravity gradient forces that lead to libration in gravity captured tethered spacecraft will cause oscillations in rotation rate of the rotating configuration [Schultz and Vigneron (1998)]. The relative magnitude of these oscillations will depend on how close the configuration is to gravity capture.

Transverse Motion

Transverse, or wavelike, motion is shown in Figure B-2c. This occurs through control motions of one or both of the sub-satellites. The effects of tether transverse motion are judged to have negligible contributions to the study of loss of rotation.

Skip-rope Motion

Shown in Figure B-2d the skip-rope motion is a rigid-body revolution of the tether between its two ends. It is actually a special case of superimposed in-plane and out-of-plane transverse motions where the vibration frequency and amplitude are equal for both planes but there is a 90° phase difference. Pure skip-rope dynamics has no damping since the tether elongation is constant. Skip-rope dynamics are prevented from occurring by any linear damping of in-plane transverse motion or if a frequency difference exists between the in-plane and out-of-plane transverse motions [Carroll and Alexander (1987)].

Pendulous (end body rocking)

Shown in Figure B-2e, pendulous motion occurs when the sub-satellites rock about their attachment points to the tether. The magnitude of this motion depends on the nature of the attachment of the sub-satellite to the tether; offsets between the tether attachment point and the end body centre of mass cause torques on the end body that induce this motion. Contributions to loss of rotation of a rotating configuration can occur if damping exists in the means attachment between tether and sub-satellite and a means of energy transfer to this motion also exists.

Atmosphere Effects on Satellites

Most satellites (95%) are subject to slow attrition by atmospheric drag which results in orbit decay and end-of-life. As well, for altitudes between 150 km-500 km there can be in addition to the drag, aerodynamic lift and moments. Aerodynamic moments will cause changes to the attitude of a satellite. Within the range of 120-2000 km altitude the atmosphere exerts a noticeable effect on satellites. Above 2000 km the effect of air drag is negligible. The difficulty in modelling aerodynamic effects is that the accuracy of predicting the upper atmospheric density is limited. It cannot be accurately predicted even one-year ahead to better than $\pm 10\%$.

Within the effected altitudes, aerodynamic forces affect all orbital elements as shown in Table C-1. A detailed discussion of these effects can be found in literature [for example King-Hele (1987)]. This Appendix presents theory for estimating aerodynamic effects including the kinetic theory of gases and atmospheric models. Equations shown in the Appendix are used in Chapter 3 of this thesis.

Table C-1. Orbital Perturbations of Earth Satellites (200-1500 km altitude).

	Secular (building up progressively)		Periodic (periods 1 month to 1 year)	
	Large (up to 100 km/day)	Small (< 2 km/day)	Moderate (~ 10 km/day)	Small (< 2 km/day)
Gravitational	Ω, ω	---	e	i, Ω, ω
Atmospheric	a, e	i	---	Ω, ω
Luni-Solar	---	---	---	a, e, i, Ω, ω
Only for low perigee satellites is the drag large; for slight drag a, e is shifted to the small column. Ω = right ascension of the ascending node; i = inclination; ω = argument of perigee; a = orbit semi-major axis; e = orbit eccentricity.				

Atmospheric Models and the Kinetic Theory of Gases

Gas consists of small individual molecules or atoms that are in a constant, random motion. If there is no macroscopic or bulk motion of the gas then the microscopic molecular velocities average to zero. For moving gas the molecular velocities average to the bulk velocity of the gas. Kinetic Theory can be used to calculate various properties of the gas including pressure, temperature, speed of sound and transport properties.

Molecular Mean Free Path and Satellite Flow Regimes

The molecular model used with the kinetic theory allows that a molecule has mass, an external force field and an internal structure. A billiard ball model of the molecule, with spherical molecular diameter d is the simplest model suitable for this purpose. Molecular mass m consists of the mass of all nuclei and electrons that constitute the molecule. This mass is used to determine the momentum mc and kinetic energy mc^2 of a molecule traveling at velocity c .

The electron field surrounding the nuclei of the molecule exerts a force into the space around the molecule. The true shape of the molecule is not spherical but the macroscopic thermodynamic properties of the gas can be reliably determined if the diameter is

adjusted by matching any one calculated macroscopic property to its experimental counterpart.

The internal structure of the molecule is important in determining how internal energy is stored in the molecule (Figure C-1). Here the molecule stores energy through the two mechanisms of rotation and vibration. At high temperatures significant amounts of energy are stored in these rotation and vibration modes and are not available for translational motion; not all the energy is present as kinetic energy.

The molecular mean free path, λ , is the average distance that a molecule travels between collisions. Knowledge of this parameter becomes important in determining the type of flow regime a satellite is flying in, which in turn depends on the size of the satellite.

Figure C-1. Intermolecular potential energy, $V(r)$, for spherical molecules.

As a molecule travels it will collide with other molecules. With spherical molecules of diameter d and considering a molecule density of n molecules per unit volume collisions are defined to occur when the centres of two molecules are a distance d apart. The sphere of influence of any single molecule is then a distance $2d$. The volume swept out by the sphere of influence of a molecule travelling at a velocity c is $\pi d^2 |c|$. The number

of collisions per unit time is $\pi d^2 |\mathbf{c}_{rel}| n$ where $\mathbf{c}_{rel} = \sqrt{2}\mathbf{c}$ is the average relative velocity of the other molecules which are also moving. The distance travelled per unit time is $|\mathbf{c}|$. The average distance travelled between collisions is the molecular mean free path and is:

$$\lambda = \frac{|\mathbf{c}|}{\pi d^2 |\mathbf{c}_{rel}| n} = \frac{1}{\sqrt{2} \pi d^2 n}.$$

For most orbits the molecular mean free path is large enough so that molecules leaving the surface of the satellite do not effect the molecules approaching the surface of a satellite. This situation is known as free molecular flow. Free molecular flow occurs when the molecular mean free path is approximately 10 times the length, ℓ , of the satellite (Knudsen number, $K = \lambda/\ell > 10$).

Figure C-2. Flow regimes experienced by satellites [King-Hele (1987)].

Impact and Reflection of Molecules from Surfaces

For determination of aerodynamic drag and moments the transfer of momentum between atmospheric molecules and the surface of a satellite must be determined. Momentum transfer occurs when molecules arrive at the surface of the satellite and when they leave. The assumption of free molecular flow allows these two cases to be calculated additively and to treat a geometrically complicated surface in terms of simpler constituent surfaces (there are two notable exceptions to this latter point: shadowing and multiple collisions).

The two limiting cases for modelling the impact and reflection of molecules from the surface of a satellite are called *specular* reflection and *diffuse* reflection.

Specular Reflection

Specular reflection has the molecule bounce off the surface of the satellite in a purely deterministic manner and is characterised by the following:

- (a) The molecule has no change in energy;
- (b) The angle of incidence is the same as the angle of reflection;
- (c) The incoming velocity vector, the outgoing velocity vector and the surface normal are coplanar; and
- (d) The incoming velocity is the same as the outgoing velocity.

The momentum transfer in specular reflection is twice the normal component of the incoming momentum and in a direction normal to the surface. Typically, few molecules are reflected via specular reflection.

Diffuse Reflection

The majority of molecules are partially accommodated by the surface onto which they impact; the molecules hit and stick for awhile before being re-emitted into space. By the time the molecule is re-emitted it has partially “forgotten” its original direction of motion

Figure C-3. Specular and diffuse molecular reflection [Hughes (2) (1986)].

and its original energy. The re-emission can be assumed to be *diffuse*, according to the Knudsen cosine law (the number of molecules emitted in directions making angles between θ and $\delta\theta$ with the normal to the surface is proportional to $\cos\theta\delta\theta$). The re-emitted molecules have a Maxwellian velocity distribution based on the temperature of re-emission, T_r . This temperature of re-emission is an uncertain quantity. The molecule may retain all of its incident temperature, T_i , or may take on entirely the temperature, T_s ,

of the surface to which it adheres for an instant. The parameter called the accommodation coefficient α takes this into account and is defined as

$$\alpha = \frac{T_i - T_r}{T_i - T_s}.$$

For $\alpha = 1$ there is complete accommodation and the molecules take on entirely the surface temperature and are re-emitted based on this temperature. For $\alpha = 0$ the molecules would be re-emitted without loss of energy. Determining the accommodation coefficient requires a model of the interaction of the surface molecules with the impacting gas molecules. Head-on collision of incident and surface atoms gives:

$$\alpha = \frac{4\mu}{(1 + \mu)^2}, \quad (\text{C-1})$$

where, in this case, μ is the ratio of the mass of gas atom to the mass of the surface atom (assumed to be less than unity). Equation (C-1) is an upper limit. Other models are available which have angular deflections different from diffuse reflection and have lower accommodation coefficients.

The surface molecule of the satellite that is to be used in calculation of the accommodation coefficient may not be obvious. Similar satellites plated with different metals, for example aluminium versus gold, have almost identical drag coefficients in orbit. Empirical results from orbital studies indicate that the accommodation coefficient has a value of 0.9 to 1.0; this is apparently due to a mono-layer of atomic oxygen which is acquired by the surface of a satellite as it moves through atomic oxygen rich orbits.

Density Variation with Altitude

Important in assessing the effect of aerodynamic forces on satellite motion is an understanding of how the density of the upper atmosphere changes with altitude, time, day/night cycle and solar activity.

Ideal Density Variation Based on Density Scale Height

A simple theory of density variation with altitude can be performed using the Ideal gas law that relates temperature T , pressure P , density ρ , average molecular weight M and gas constant $R = 8.314[J / K / gmol]$:

$$\frac{p}{\rho} = \frac{RT}{M}.$$

Assuming an spherical earth of radius r_E and an inverse-square gravitation field, for density scale height H and where ρ_0 is the density at a chosen reference altitude, $y = y_0$, the density variation with altitude can be written:

$$\rho = \rho_0 \exp\left(-\frac{y - y_0}{H}\right). \quad (C-2)$$

H , the density scale height, is related to constant scale height h :

$$\frac{1}{H} = \frac{1}{h} - \frac{2}{r_0}, \quad r_0 = r_E + y_0,$$

where $h = \frac{RT}{Mg}$ and is constant if the average molecular weight M of the gas is constant.

Since $h \ll r_0$ the difference between h and H is small. Equation (C-2) gives an ideal analytical expression. The actually density of the upper atmosphere varies due to the effects of: (a) an oblate atmosphere, (b) variation in atmosphere composition with altitude, (c) day-to-night density variations, (d) complex atmosphere rotation, and (e) solar activity.

Figure C-4 shows the concentrations of various gases at various altitudes. The change in constituent gases for the atmosphere causes a change in the average molecular weight and hence and change in the density scale height of eqn (C-2). Figure C-5 shows air density variation due to the 11 year solar cycle.

Figure C-4. Concentrations of nitrogen, oxygen, helium and hydrogen at orbital altitudes up to 2000 km (from CIRA 1972).

Figure C-5. Variations of air density with altitude from 150 to 1000 km for high and low solar activity (based on CIRA 1972).

Figure C-6. Aerodynamic lift, drag and turning Moment acting on a satellite.

Evaluation of Drag Coefficient, C_D

Aerodynamic forces that affect satellites include drag but also includes lift and moment. A satellite moving with velocity V relative to the atmosphere is subjected to aerodynamic forces. Various terms will be used that are typically applied to traditional aeronautical engineering of aircraft. The aerodynamic forces are separable into three distinct quantities: drag, lift and turning moment.

Drag

Acting in the direction of the relative wind, the typical expression for calculating aerodynamic drag forces is:

$$D = \frac{1}{2} C_D \rho V^2 A. \quad (\text{C-3})$$

where ρ is the density of the ambient air, A is a reference area frequently taken to be the cross-sectional area of the object perpendicular to the direction of motion, V is in the direction of the free-stream gas velocity and C_D is the drag coefficient. The equation is also sometimes written $D = C_D A q$ where q is the dynamic pressure and is $q = \frac{1}{2} \rho V^2$.

Lift

Lift acts perpendicular to the direction of the relative wind. The typical expression for calculating aerodynamic lifting forces is:

$$L = \frac{1}{2} C_L \rho V^2 A, \quad (\text{C-4a})$$

where C_L is the lift coefficient. This term is taken from an analogy with aircraft (as the turning moment is normally destabilizing, cylindrical satellites tumble end-over-end and the effect of Lift cancels out.).

Turning Moment

The turning moment acts about the centre of mass. The typical expression for calculating aerodynamic moment is:

$$M = \frac{1}{2} C_M \rho V^2 A, \quad (\text{C-4b})$$

where C_M is the moment coefficient. It arises due to a difference between the centre of pressure of the aerodynamic forces and the centre of gravity of the satellite (other turning moments arise from the gravity gradient, Earth's magnetic field, electrodynamic forces and other sources).

Drag Coefficient of a Sphere

Satellites above 200 km altitude experience free molecular flow. When the satellite speed greatly exceeds the mean molecular speed, the drag coefficient of any satellite, based on the cross-sectional area perpendicular to the direction of the relative wind is

$$C_D = 2 + W.$$

The first term (or "2") is directly due to the incident molecules. The W term is a parameter that depends on the satellite's shape, the accommodation coefficient of its

surfaces and the mode of reflection. For a diffusely reflecting sphere with a uniform accommodation coefficient values of C_D are given in Table C-2.

Table C-2. Values of C_D for a sphere in hyper-thermal flow, with diffuse re-emission and constant α [King-Hele(1987)].

α	0.6	0.8	0.9	0.95	0.98	1.0
C_D	2.56	2.40	2.28	2.21	2.14	2.07

Satellites at orbital altitudes of 200-300 km (high atomic oxygen content) have a drag coefficient usually taken to be 2.2, with a maximum error less than 10% if $0.8 \leq \alpha \leq 1.0$. At higher altitudes the molecular weight of the gas decreases (μ decreases) and the assumption of hyperthermal flow breaks down.

At higher altitudes the molecular weight decreases. At 800 km altitude the atmosphere is mostly helium (at solar minimum) with a molecular weight of about 5 giving $\mu = 0.3$ for surface atoms of atomic oxygen (molecular weight 16) and $\alpha = 0.7$, giving $C_D \approx 2.5$ by interpolation from Table C-2.

As the velocity of the gas molecules increase the drag coefficient will increase. For example, at an altitude of 800 km the orbital velocity, V , is 7.4 km/s ($e=0$) and the molecular velocity, V_m , is about 1.2 km/s (for scale height $h = 100 \sqrt{2gh} \approx 1.2$). The molecular speed ratio, V/V_m , is approximately 6 rather than ∞ ; the drag coefficient will be 1% greater than the hyperthermal value. At 800 km this is negligible but at, say 2000-6000 km altitude $V/V_m \approx 2$ and the drag coefficient will increase approximately 40%.

Drag Coefficients for Other Shapes

Drag coefficients for various other shapes are available from literature. Cook (1965) reports for convex bodies of various shapes based on a cross-sectional area perpendicular to the wind direction.

Table C-3. Drag Coefficients for various satellite geometries* [Cook (1965)].

Sphere	$C_D = 2(1 + \frac{4}{9}r)$
Cylinder with axis perpendicular to flow	$C_D = 2(1 + \frac{\pi}{6}r)$
Flat plate at incidence θ	$C_D = 2(1 + \frac{2}{3}r \sin \theta)$
Cylinder of length ℓ and diameter d tumbling end-over-end	$C_D = 2 \left\{ 1 + \frac{\pi^2(\ell + d)}{6(4\ell + \pi d)} r \right\}$
Tumbling cylinder $\ell/d = 2$	$C_D = 2(1 + 0.44r)$ (same as sphere)
Tumbling cylinder $\ell/d = 5$	$C_D = 2(1 + 0.43r)$

* For $r = \sqrt{1 - \alpha(1 - T_i/T_i)}$ and for $T_i/T_i = 0.006$.

From Table C-3 it is seen that a tumbling cylinder has virtually the same drag coefficient as a sphere. A good standard value for C_D is 2.2 since it is appropriate for a sphere or rotating convex body or cylinder.

For concave bodies detailed geometric analysis is required. This must take into consideration the possible shielding of one surface by another.

Drag Coefficient at Low Altitudes

The onset of slip-flow and hypersonic continuum flow occurs at altitudes below 120 km. The drag coefficient decreases when this happens (although the overall drag increases due to an increase in air density).

Estimate of Cross-sectional Area

The applicable area to use as cross-sectional area perpendicular to the wind direction may be complicated by either complicated geometry, rotation of the satellite or both. Examining a cylinder the two limiting cases of motion relative to the free-stream gas are given in Table C-4.

Table C-4. Aerodynamic reference area calculation on a cylinder based on the type of end-body motion [King-Hele (1987)].

Case		Mean cross-section
(a)	Traveling like an airplane propeller	ℓd
(b)	Tumbling end-over-end	$\frac{2}{\pi}(\ell d + \frac{1}{4}\pi d^2)$
(c)	Other direction between (a) and (b)	$\ell d(0.818 + 0.25d/\ell)$

Free-stream Velocity

Equation (C-3) requires the use of the free-stream gas velocity, V . This must be related to the orbital velocity v in order to apply the drag perturbing forces to prediction of orbital motion. The vector velocity v of the satellite relative to the Earth's centre is the vector sum of its velocity V relative to the free-stream and the velocity V_A of the atmosphere relative to the Earth's centre:

$$V = v - V_A. \quad (C-5)$$

The following analysis is for a simple west-to-east wind due to the Earth's rotation and no north-to-south (meridional) winds. The west-to-east rotation normally greatly exceeds the meridional winds.

For γ denoting the angle between V_A and v eqn (C-5) reduces to

$$V^2 = v^2 + V_A^2 - 2vV_A \cos \gamma. \quad (C-6)$$

The value of γ depends on the climb angle of the satellite. For most purposes it is suitable to approximate γ by taking it equal to the angle γ' between V_A and the horizontal component of v .

The atmosphere rotates with angular velocity w about the Earth's axis

$$V_A = rw \cos \phi, \quad (C-7)$$

where r is the distance from the Earth's centre and ϕ is the geocentric latitude.

Equation (C-6) becomes:

$$V^2 = v^2 + r^2 w^2 \cos^2 \phi - 2vrw \cos \phi \cos \gamma'. \quad (C-8)$$

Figure C-7. Spherical triangle connecting i , ϕ , γ' and argument of latitude u .

Figure C-7 shows the spherical triangle connecting ϕ and γ' with the inclination i : the argument of latitude $u(= \omega + \theta)$ is also shown. From spherical trigonometry:

$$\cos \phi \cos \gamma' = \cos i. \quad (C-9)$$

and eqn (C-7) becomes:

$$V^2 = v^2 \left(1 - \frac{rw}{v} \cos i \right)^2 + r^2 w^2 (\cos^2 \phi - \cos^2 i). \quad (\text{C-10})$$

Figure C-8. The factor F which takes account of atmospheric rotation, for perigee altitudes of 200 km (solid) or 400 km (dashed). For $i > 90^\circ$ $F(i)$ may be taken as $\{2 - F(180^\circ - i)\}$. For atmospheric rotation equal to Earth's.

The term $r^2 w^2 < 0.005 V^2$ and can be neglected. The term rw/v may be written $r_p w/v_p$ (subscript denotes perigee values) since the main effects of drag occur at altitudes not more than $2H$ above perigee, where r/v does not differ from r_p/v_p by more than 4% if $H/r_p < 0.01$. Equation (C-10) may be re-written:

$$V = v \left(1 - \frac{r_p w}{v_p} \cos i \right). \quad (\text{C-11})$$

Substituting eqn (C-10) into eqn (C-3) the drag force is written:

$$D = \frac{1}{2} F C_D \rho v^2 S. \quad (\text{C-12})$$

where

$$F = \left(1 - \frac{r_p w}{v_p} \cos i \right)^2. \quad (\text{C-13})$$

The variation in F for orbital altitudes of 200-400 km is shown in Figure C-8, using atmospheric rotation rate, w , equal to the Earth's angular velocity. In reality w can take the value of 0.8 to 1.3 rev/day, depending on altitude and local time at perigee.

Appendix D

Static Equilibrium of an Inextensible Tether

The following derivation of the static equilibrium case of an inextensible tether is used to develop certain trigonometric relationships referenced in the derivations of Chapter 5. The subscript annotation 0 adopted for this appendix indicates static equilibrium values of those terms so subscripted. The free-body analysis of the pendulum mass is:

$$\sum F_x = m_p \ddot{x} = T_0 \sin \phi_0 - N_0|_B \cos \theta_0 \quad (\text{D-1})$$

$$\sum F_y = m_p \ddot{y} = m_p g + N_0|_B \sin \theta_0 - T_0 \cos \phi_0 \quad (\text{D-2})$$

where T_0 is the tension in the pendulum support and $N_0|_B$ is the tether tension exerted onto the pendulum. The angle of this tether tension is θ_0 to the horizontal as shown in Figure D-1. Based on the shown geometry the distance $y_0|_B$ is dependent on the two angles θ_0 and ϕ_0 :

$$y_0|_B = \ell_p (1 - \cos \phi_0) - r \sin \theta_0 \quad (\text{D-3})$$

Figure D-1. Definition diagram of a single span tether/pendulum experimental configuration showing equilibrium of an element and boundary conditions for static equilibrium of an inextensible tether.

For static equilibrium all forces are in equilibrium and

$$0 = T_0 \sin \phi_0 - N_0|_B \cos \theta_0 \quad (\text{D-4})$$

$$0 = m_p g + N_0|_B \sin \theta_0 - T_0 \cos \phi_0 \quad (\text{D-5})$$

The static equilibrium of an element of the tether shown in Figure D-1 which has varying tension $N_0(s)$ due to gravity can be analyzed to determine the shape of the tether when in static equilibrium [Irvine(1981)]. The vertical equilibrium requires

$$-N_0 \frac{dy_0}{ds} - \mu g \Delta s + \left[N_0 \frac{dy_0}{ds} + \frac{d}{ds} \left(N_0 \frac{dy_0}{ds} \right) \Delta s \right] = 0$$

$$-\mu g \Delta s + \frac{d}{ds} \left(N_0 \frac{dy_0}{ds} \right) \Delta s = 0 \text{ and}$$

$$\frac{d}{ds} \left(N_0 \frac{dy_0}{ds} \right) = \mu g. \quad (\text{D-6})$$

In static equilibrium, for a reaction force at A of magnitude V_0 eqn (D-6) can be integrated to obtain

$$N_0 \frac{dy_0}{ds} = \mu g s - V_0. \quad (\text{D-7})$$

The horizontal equilibrium requires

$$N_0 \frac{dx_0}{ds} - \left[N_0 \frac{dx_0}{ds} + \frac{d}{ds} \left(N_0 \frac{dx_0}{ds} \right) \Delta s \right] = 0 \text{ and} \quad (\text{D-8})$$

$$\frac{d}{ds} \left(N_0 \frac{dx_0}{ds} \right) = 0.$$

Equation (D-8) may be integrated directly to get

$$N_0 \frac{dx_0}{ds} = H_0, \quad (\text{D-9})$$

where H_0 is the horizontal component of tether tension for static equilibrium, which is constant everywhere for this case (there are no horizontal loads).

The geometric boundary conditions at the tether supports A and B are $y_0|_A = 0$, $s|_A = 0$, $y_0|_B = \ell_p(1 - \cos \phi) - r \sin \theta$, $s|_B = L_0$. For the inextensible differential element the geometric constraint that must be satisfied is $dx^2 + dy^2 = ds^2$ or

$$\left(\frac{dx_0}{ds} \right)^2 + \left(\frac{dy_0}{ds} \right)^2 = 1. \quad (\text{D-10})$$

Squaring eqns (D-7) and (D-9) and adding the results together yields:

$$N_0^2 \left[\left(\frac{dx_0}{ds} \right)^2 + \left(\frac{dy_0}{ds} \right)^2 \right] = H_0^2 + (\mu g s - V_0)^2. \quad (\text{D-11})$$

Comparing eqns (D-10) and (D-11) leads us to write $N_0^2 = H_0^2 + (\mu g s - V_0)^2$. The tension $N_0(s)$ at any length s along the tether from point A is expressed as a function of s :

$$N_0(s) = \sqrt{H_0^2 + (\mu g s - V_0)^2}. \quad (\text{D-12})$$

Substituting eqn (D-12) for $T_0(s)$ in eqn (D-9) yields:

$$\frac{dx_0}{ds} = \frac{H_0}{\sqrt{H_0^2 + (\mu g s - V_0)^2}}. \quad (\text{D-13})$$

Integrating eqn (D-13) by separation of variables

$$dx_0 = \frac{H_0}{\sqrt{H_0^2 + (\mu g s - V_0)^2}} ds \text{ and then } \int_0^x dx_0 = \int_0^s \frac{H_0}{\sqrt{H_0^2 + (\mu g s - V_0)^2}} ds \text{ and}$$

$$x_0 = \frac{H_0}{\mu g} \left[\sinh^{-1} \left(\frac{V_0}{H_0} \right) - \sinh^{-1} \left(\frac{V_0 - \mu g s}{H_0} \right) \right]. \quad (\text{D-14})$$

From eqns (D-7) and (D-12) we write

$$\frac{dy_0}{ds} = \frac{\mu g s - V_0}{\sqrt{H_0^2 + (\mu g s - V_0)^2}}. \quad (\text{D-15})$$

Equation (D-15) is integrated through separation of variables

$$dy_0 = \frac{\mu g s - V_0}{\sqrt{H_0^2 + (\mu g s - V_0)^2}} ds \text{ and then } \int_0^{y_0} dy_0 = \int_0^s \frac{\mu g s - V_0}{\sqrt{H_0^2 + (\mu g s - V_0)^2}} ds \text{ and}$$

$$y_0 = \frac{H_0}{\mu g} \left[\sqrt{1 + \left(\frac{\mu g s - V_0}{H_0} \right)^2} - \sqrt{1 + \left(\frac{V_0}{H_0} \right)^2} \right]. \quad (D-16)$$

Eliminating T from eqns (D-4) and (D-5) yields

$$N_0|_B = \frac{m_p g \tan \phi_0}{\cos \theta_0 - \sin \theta_0 \tan \phi_0}. \quad (D-17)$$

Evaluating eqn (D-12) at point B

$$N_0|_B = \sqrt{H_0^2 + (V_0 - \mu g L_0)^2}. \quad (D-18)$$

Combining eqns (D-17) and (D-18)

$$\sqrt{H_0^2 + (V_0 - \mu g L_0)^2} = \frac{m_p g \tan \phi_0}{\cos \theta_0 - \sin \theta_0 \tan \phi_0}. \quad (D-19)$$

From eqns (D-13) and (D-15) the value of dy_0/dx_0 can be determined

$$\frac{dy_0}{dx_0} = \frac{dy_0}{ds} \frac{ds}{dx_0} = \frac{\mu g s - V_0}{H_0}. \quad (D-20)$$

As the value of $\tan \theta_0$ is simply $\left. \frac{dy_0}{dx_0} \right|_B$ then

$$\tan \theta_0 = \frac{\mu g L_0 - V_0}{H_0}. \quad (D-21)$$

The term $\mu g L_0$ is simply the total weight of the tether and is the absolute maximum value of the vertical reaction force V_0 at point A (if there is no vertical reaction force at point B). Therefore $\mu g L_0 - V_0 > 0$ and the value of θ_0 is positive which fits with the system of coordinate definitions from Figure D-1. Using various trigonometric identities we also write

$$\cos \theta_0 = \frac{H_0}{\sqrt{H_0^2 + (\mu g L_0 - V_0)^2}}. \quad (\text{D-22})$$

$$\sin \theta_0 = \frac{\mu g L_0 - V_0}{\sqrt{H_0^2 + (\mu g L_0 - V_0)^2}}. \quad (\text{D-23})$$

The above derivation has been for the general case described by Figure D-1. The equations for x_0 , y_0 , and T_0 are expressed in terms of V_0 , H_0 , and s . Introducing the end condition at the fixture of the tether to the pendulum (point B) in eqns (D-14) and (D-16) yields

$$X - \ell_p \sin \phi_0 - r \cos \theta_0 = \frac{H_0}{\mu g} \left[\sinh^{-1} \left(\frac{V_0}{H_0} \right) - \sinh^{-1} \left(\frac{V_0 - \mu g L_0}{H_0} \right) \right]. \quad (\text{D-24})$$

$$\ell_p (1 - \cos \phi_0) - r \sin \theta_0 = \frac{H_0}{\mu g} \left[\sqrt{1 + \left(\frac{\mu g L_0 - V_0}{H_0} \right)^2} - \sqrt{1 + \left(\frac{V_0}{H_0} \right)^2} \right]. \quad (\text{D-25})$$

Substituting the trigonometric relations of eqns (D-22) and (D-23) into eqns (D-19), (D-24) and (D-25) yields

$$H_0 + (V_0 - \mu g L_0) \tan \phi_0 = m_p g \tan \phi_0, \quad (\text{D-26})$$

$$X - \ell_p \sin \phi_0 - \frac{r H_0}{\sqrt{H_0^2 + (\mu g L_0 - V_0)^2}} = \frac{H_0}{\mu g} \left[\sinh^{-1} \left(\frac{V_0}{H_0} \right) - \sinh^{-1} \left(\frac{V_0 - \mu g L_0}{H_0} \right) \right], \quad (\text{D-27})$$

$$\ell_p (1 - \cos \phi_0) - \frac{r (\mu g L_0 - V_0)}{\sqrt{H_0^2 + (\mu g L_0 - V_0)^2}} = \frac{H_0}{\mu g} \left[\sqrt{1 + \left(\frac{V_0 - \mu g L_0}{H_0} \right)^2} - \sqrt{1 + \left(\frac{V_0}{H_0} \right)^2} \right]. \quad (\text{D-28})$$

Equation (D-26) can be rearranged to solve for

$$\tan \phi_0 = \frac{H_0}{m_p g + \mu g L_0 - V_0} \quad (\text{D-29})$$

$$\sin \phi_0 = \frac{H_0}{\sqrt{H_0^2 + (m_p g + \mu g L_0 - V_0)^2}} \quad (\text{D-30})$$

$$\cos \phi_0 = \frac{m_p g + \mu g L_0 - V_0}{\sqrt{H_0^2 + (m_p g + \mu g L_0 - V_0)^2}}. \quad (\text{D-31})$$

Equations (D-30) and (D-31) may be substituted into eqns (D-27) and (D-28) respectively to yield

$$0 = X - \frac{\ell_p H_0}{\sqrt{H_0^2 + (m_p g + \mu g L_0 - V_0)^2}} - \frac{r H_0}{\sqrt{H_0^2 + (\mu g L_0 - V_0)^2}}, \quad (\text{D-32})$$

$$- \frac{H_0}{\mu g} \left[\sinh^{-1} \left(\frac{V_0}{H_0} \right) - \sinh^{-1} \left(\frac{V_0 - \mu g L_0}{H_0} \right) \right]$$

$$0 = \ell_p \left(\frac{m_p g + \mu g L_0 - V_0}{\sqrt{H_0^2 + (m_p g + \mu g L_0 - V_0)^2}} - 1 \right) + \frac{r(\mu g L_0 - V_0)}{\sqrt{H_0^2 + (\mu g L_0 - V_0)^2}} \quad (\text{D-33})$$

$$+ \frac{H}{\mu g} \left[\sqrt{1 + \left(\frac{\mu g L_0 - V_0}{H_0} \right)^2} - \sqrt{1 + \left(\frac{V_0}{H_0} \right)^2} \right]$$

Equations (D-32) and (D-33) represent a system of two equations with two unknowns: V_0 and H_0 . For any given set of system parameters X , ℓ_p , m_p , r , μ and L_0 these two equations can be solved for the static equilibrium values of V_0 and H_0 . With these values known eqns (D-12), (D-14) and (D-16) are a set of parametric equations giving the values of N_0 , x_0 and y_0 for any $s \leq L_0$.