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CROSSING NUMBERS OF CERTAIN FAMILIES OF GRAPHS

by

GELASIO SALAZAR M.Sc.

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfilment of
the requirements for the degree of
Doctor of Philosophy

Department of Mathematics and Statistics

Carleton University

June 1997

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submitted by
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in partial fulfilment of the requirements
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Abstract

Crossing numbers of graphs are in general very difficult to compute. The exact crossing numbers of very few families of graphs are known exactly. In this thesis we determine the crossing numbers of certain families of graphs, and we also obtain some asymptotic estimates for the crossing numbers of certain drawings.

One long standing conjecture is that the crossing number of the Cartesian product $C_m \times C_n$ of the cycles of sizes m and n is $(m-2)n$, for each $n \geq m \geq 3$. Prior to this thesis, this conjecture had been proved only for the cases $\min\{m, n\} \leq 5$. In this thesis we follow the approach by Richter and Thomassen, and find an asymptotic estimate for the number of crossings in large classes of drawings of $C_m \times C_n$. Methods used by earlier authors are extended to show that the crossing numbers of important classes of drawings of $C_m \times C_n$ are exactly $(m-2)n$. The crossing number of $C_6 \times C_n$ is shown to be $4n$, as conjectured.

We also show that the crossing number of the Generalized Petersen Graph $P(3k+h, 3)$ is $k+h$ if h is either 0 or 2, and that the crossing number of $P(3k+1, 3)$ is $k+3$, for each $k \geq 3$, with the single exception of $P(9, 3)$, whose crossing number is 2. Some of these results had been previously claimed by Fiorini, but a serious mistake in his argument invalidates the main results in his discussion.

For a related problem, we show that asymptotically the minimum number of intersections in a collection of n pairwise nondisjoint simple closed curves with at most M pairwise intersections is about $2\binom{n}{2}$.

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I dedicate this thesis to the memory of my uncle Alejandro

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List of Symbols

	Page
G	1
(u, v)	1
$cr(G)$	1
$\mathcal{D}$	1
$cr(\mathcal{D})$	1
C_n	7
$C_m \times C_n$	7
$P(m, \ell)$	9
$i^*(\mathcal{C}_1, \mathcal{C}_2)$	13
$i^*(m, n)$	13
$(\mathcal{C}_1, \mathcal{C}_2)$	13
$cr_M(C_m \times C_n)$	14
$i(n)$	14
$i_M^*(\mathcal{C}_1, \mathcal{C}_2)$	21
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I	24
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b_i	91
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$\mathcal{F}_i^{m\mathcal{D}}$	100
$\mathcal{R}_b$	101
$\mathcal{R}_r$	101
ρ	102
β	102
τ	102
ϵ	102
$G_{\mathcal{R}_r}$	102
$H_{\mathcal{R}_r}$	102
$V(H_{\mathcal{R}_r})$	102
$E(H_{\mathcal{R}_r})$	102
$\mathcal{R}$	108
Y_j	109
$\mathcal{P}_r$	109
R	120
$\overline{R}$	120

List of Terminology

	Page
Arc	55
Blue-disjoint drawing	46
Clean edge	91
Crossing (intersection)	1
Crossing number	1
Crossing of type I	75
Crossing of type II	75
Dominated edge	91
Drawing	1
Force	49
Good drawing	1
Interior (of an arc)	55
Intersecting Family	37
Intersecting ($< m$)-family	37
($< m$)-family	37
M -crossing number	19
(m, n) -mesh	13
$(m, n)_M$ -mesh	20
n -drawing	81
Principal cycle	81
Really-optimal drawing	55
Red-disjoint drawing	46
r -type I	109
r -type II	109
Tangential (intersection)	1

Chapter 1

Introduction

1.1 The Crossing Number of a Graph

A *drawing* of a graph G is a subset of the plane where the vertices of G are represented as distinct points and each edge (u, v) of G is represented as an open arc A containing no vertices, so that the ends of A are the points that represent u and v .

It is easy to see that every graph G can be drawn in such a way that every two edges in G intersect each other only a finite number of times. All drawings of graphs considered in this thesis will be assumed to satisfy this additional property.

The *crossing number* $cr(G)$ of a graph G is the minimum number of pairwise intersections of edges among all drawings of G . A drawing $\mathcal{D}$ of G is *optimal* if the number $cr(\mathcal{D})$ of pairwise intersections of edges in $\mathcal{D}$ satisfies the equality $cr(\mathcal{D}) = cr(G)$.

Figure 1.1 shows three different drawings of the 4-cycle C_4 . In the first drawing no edges are intersected, and in each of the other two drawings there are exactly two edges that intersect. The intersection in the second drawing is a *crossing*, and the intersection in the third drawing is a *tangential* intersection.

A drawing of a graph G is *good* if the following conditions are satisfied:

- (i) Each intersection is a crossing rather than tangential.
- (ii) No edge intersects itself.
- (iii) No two adjacent edges have an intersection.
- (iv) No two edges have more than one intersection.
- (v) No three edges have a common intersection.

As illustrated in Figure 1.2, it is easy to modify a drawing of a graph to obtain a good drawing, without increasing the number of crossings. Hence the crossing number of a graph G is attained in good drawings of G . Therefore, whenever we consider an optimal drawing $\mathcal{D}$ of a graph G , we assume that $\mathcal{D}$ is a good drawing of G .

Figure 1.3 shows drawings of $K_{3,3}$ and K_5 , each with one crossing. Using the Jordan–Schönflies Theorem, it is straightforward to prove that every drawing of $K_{3,3}$ or K_5 has at least one crossing. In his famous paper of 1930, Kuratowski [Ku] showed that these two graphs are basically the only obstacles for drawing a graph without any crossings.

Theorem 1.1 [Kuratowski, 1930] *The crossing number of a graph G is zero if and only if G contains no subdivision of $K_{3,3}$ or K_5 .*

Kuratowski’s Theorem settles the question of which graphs can be drawn in the plane without crossings. However, no such characterization is yet known for the graphs that can be drawn with a specified number of crossings.

Finding the exact crossing number of a given graph is, for most cases of interest, a daunting task. This is particularly true from the computational point of view since, as Garey and Johnson have proved [GJ], calculating the crossing

Three different drawings of C_4 .

Figure 1.1

(i) Eliminating a tangential crossing.

(ii) Eliminating a self-intersection of an edge.

Figure 1.2

(iii) Eliminating a crossing of two adjacent edges.

(iv) Eliminating multiple crossings between edges.

(v) Eliminating a common point of more than two edges.

How to modify a drawing to satisfy conditions (i) - (v), without increasing the number of crossings.

Figure 1.2 (continued)

Drawings of K_5 and $K_{3,3}$, each with one crossing.

Figure 1.3

number of a graph is an NP-complete problem. Therefore, every serious effort to find the crossing number of a graph with more than just a few vertices must involve arguments that exploit special features of the structure of the particular graph.

The usual way to calculate the crossing number of a graph G consists of exhibiting a drawing $\mathcal{D}$ of G , and then showing that every drawing of $\mathcal{D}$ has at least as many crossings as $\mathcal{D}$, so that $cr(\mathcal{D}) = cr(G)$. Since exhibiting a drawing of any given graph is a straightforward task, the difficult part of the calculation is to prove that the presented drawing is indeed optimal.

Therefore, the efforts to compute the crossing number of graphs consist in most cases of presenting *ad hoc* techniques to provide tight lower bounds for crossing numbers. However, for most cases of interest the best known lower bound for the number of crossings does not coincide with the number of crossings in the best known drawing. Indeed, for very few families of graphs are the crossing numbers known exactly. In the next section we give a survey of those families of graphs for which the exact crossing number is known, and we also present the best known lower bounds for the crossing numbers of some families of graphs of particular interest.

1.2 Crossing Numbers of Particular Families of Graphs

For very few families of graphs are the crossing numbers known exactly. In this section we give a survey of those families whose crossing number is known, and we also present the best lower bounds known for the crossing numbers of some families of graphs of particular interest.

1.2.1 Products of Graphs with Cycles

The *product* $G \times C_n$ of a graph G and a cycle C_n of length n is the graph obtained by taking n copies of G and joining corresponding vertices in a cyclic fashion. Figure 1.4 shows a drawing of the product $C_5 \times C_7$ of the 5-cycle with the 7-cycle. This drawing has 21 crossings. This drawing can be easily extended to produce a drawing of $C_m \times C_n$ with exactly $(m - 2)n$ crossings (with m pairwise disjoint blue concentric cycles and n pairwise disjoint red cycles). Harary et al. [HKS] have conjectured that this is the actual crossing number of $C_m \times C_n$, for each m, n satisfying $3 \leq m \leq n$.

The conjecture has been proved for $m = 3, 4$, and 5 for each $n \geq m$, [HKS], [RB], [BR], [DR], [RT1], [KRS], and for the special cases $C_6 \times C_6$, $C_6 \times C_7$, and $C_7 \times C_7$ [ARRo1, ARRo2]. In Chapter 5 of this thesis we prove that this conjecture also holds for $m = 6$ for arbitrary n . In Chapters 2 and 4 we give arguments that support the general conjecture. In Chapter 2 we show that, for each fixed m , the crossing numbers of large classes of drawings of $C_m \times C_n$ are not too different from the conjectured value of $cr(C_m \times C_n)$. On the other hand, in Chapter 4 we show if $n \geq m \geq 3$, then every drawing of $C_m \times C_n$ where either the n m -cycles are pairwise disjoint or the m n -cycles are pairwise disjoint has at least $(m - 2)n$ crossings. The general conjecture $cr(C_m \times C_n)$ remains open for all other values of m and n .

Based on the work by Richter and Thomassen [RT1], Shahrokhi et al. [SSSV] have shown that $cr(C_m \times C_n) \geq 3mn/5$ if $n \leq 5(m-1)/4$, and $cr(C_m \times C_n) \geq mn/9$ if $n > 5(m-1)/4$. These bounds are obtained using that $cr(C_5 \times C_5) = 15$, and

A drawing of $C_5 \times C_7$ with 21 crossings. The 7 5-cycles are coloured red, and the 5 7-cycles are coloured blue.

Figure 1.4

they can be slightly improved by using the equality $cr(C_7 \times C_7) = 35$ proved in [ARRo2].

Beineke and Ringeisen [BR] have also investigated the products of cycles with graphs of order four. In this direction, they established that the crossing number of $K_4 \times C_n$ is $3n$ for $n \geq 3$, and that the crossing number of $K_{1,3} \times C_n$ is 1 for $n = 3$, 2 for $n = 4$, 4 for $n = 5$, and n for $n \geq 6$.

1.2.2 Generalized Petersen Graphs

The Generalized Petersen Graph $P(m, \ell)$ is constructed in the following way. Let $C = (v_0, v_1, \dots, v_{m-1})$ be an m -cycle. Take m vertices $v'_0, v'_1, \dots, v'_{m-1}$ not in C , and join v_i to v'_i by an edge for each i , and also join v'_i to $v'_{i+\ell}$ for each i , where indices are read modulo m . The resulting graph is the *Generalized Petersen Graph* $P(m, \ell)$. Thus the classical Petersen Graph is $P(5, 2)$ (see Fig. 1.5).

Exoo et al. [EHK] proved that $cr(P(m, 2))$ is 0 for each even $m \geq 4$, and $cr(P(m, 2))$ is 3 for each odd $m \geq 7$. They also proved that the crossing numbers of $P(3, 2)$ and $P(5, 2)$ are 0 and 2, respectively.

The crossing number of $P(m, 3)$ was calculated in [F1], [MR], and [YY] for small values of m . Fiorini [F1] claimed to have determined the crossing number of $P(3k + h, 3)$ for $h = 0$ and $h = 2$, and he also claimed to have found a lower bound for the crossing number of $P(3k + 1, 3)$. However, as we show in Chapter 6, a serious error in his paper invalidates his claims. In this thesis we show that $cr(P(3k + h, 3)) = k + h$ if $h \in \{0, 2\}$, for each $k \geq 3$ (as Fiorini claimed), and, further, that $cr(P(3k + 1, 3)) = k + 3$.

As Fiorini noted, there are many isomorphism relations between the collection

A drawing of $P(5,2)$, the classical Petersen Graph. This drawing has 5 crossings, and is far from optimal, since $P(5,2)$ can be drawn with exactly two crossings.

Figure 1.5

of all Generalized Petersen Graphs, and this allows us to establish the crossing numbers of many Generalized Petersen Graphs from the known values of $P(m, 2)$ and $P(m, 3)$. These isomorphism relations are given in Chapter 6 below. The crossing number of $P(m, \ell)$ remains unknown for all other values of m and ℓ .

1.2.3 Complete Graphs

The crossing number of the complete graph K_n remains an open problem for each $n \geq 11$. It has been conjectured that the crossing number of K_n is

$$\frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor,$$

but this remains unproved for each $n \geq 11$.

The best result known in this direction is that, asymptotically, the crossing number of K_n is between $\frac{3}{10} \binom{n}{4}$ and $\frac{3}{8} \binom{n}{4}$ [K].

1.2.4 Complete Bipartite Graphs

Another long standing conjecture is that the crossing number of $K_{m,n}$ is

$$\left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor,$$

for each pair m, n of integers. The best general lower bound known is that $cr(K_{m,n}) \geq \frac{1}{5} m(m-1) \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$ for each $m \geq 5$ [K].

The conjecture was proved by Kleitman [K] for the case $\min\{m, n\} \leq 6$. and by Woodall [W] for a few special cases when $\min\{m, n\} \in \{7, 8\}$. The general conjecture remains open.

1.2.5 The n -cube

The n -cube Q^n is the graph whose vertices are the $\{0, 1\}$ -vectors of size n , where two vertices are joined by an edge if they have all but one entry in common. Madej [Ma] gives a general construction that shows $cr(Q^n) \leq 4^n/6 +$ smaller order terms. Erdős and Guy [EG] conjectured that $cr(Q^n) = (5/32)4^n +$ smaller order terms, but the best result known in this direction is that $cr(Q^n) \geq 4^n/20 +$ smaller order terms [SV].

1.3 Contributions in this Thesis to the Theory of Crossing Numbers

In this section we describe briefly the contributions to the theory of crossing numbers presented in this thesis. The results in this thesis are related to the crossing numbers of two families of graphs, namely the Cartesian product of cycles and the Generalized Petersen Graphs $P(n, 3)$.

1.3.1 On the Crossing Number of $C_m \times C_n$ and a related problem

Most of the results given in this thesis related to the crossing number of $C_m \times C_n$ are based on the approach of Richter and Thomassen [RT1]. Before outlining the new results given in this thesis, we provide a little background on the ideas introduced in [RT1].

As mentioned in 1.2.1, Harary et al. [HKS] conjectured that the crossing number of $C_m \times C_n$ is $(m - 2)n$ for all m, n satisfying $3 \leq m \leq n$. This has been verified for each $n \geq m$, if $3 \leq m \leq 5$. This has been achieved by calculating

the crossing number of $C_m \times C_m$ for each $m \in \{3, 4, 5\}$. and then performing an induction. Calculating the crossing number of $C_3 \times C_3$ is not very difficult [HKS], [RT3], but the calculation of the crossing number of $C_4 \times C_4$ [DR] makes use of remarkably more involved arguments. Beineke and Ringeisen wrote in 1980, "... it appears to be quite difficult to determine even the crossing number of $C_5 \times C_5$ ". However, in their paper of 1995, Richter and Thomassen [RT1] computed $cr(C_5 \times C_5)$ using a very elegant argument.

To compute the crossing number of $C_5 \times C_5$, Richter and Thomassen introduce (m, n) -meshes. An (m, n) -mesh is a pair $(\mathcal{C}_1, \mathcal{C}_2)$ of families of closed curves, such that each curve in the first family intersects each curve in the second family. Since each m -cycle has a vertex in common with each n -cycle in $C_m \times C_n$, it follows that in each drawing of $C_m \times C_n$ the closed curve representing an m -cycle intersects the closed curve representing an n -cycle. Thus, the underlying collection of cycles in every drawing of $C_m \times C_n$ is an (m, n) -mesh.

Thus it is a trivial observation that $cr(C_m \times C_n) \geq i^*(m, n) - mn$, where $i^*(m, n)$ is the minimum number of total intersections in a (m, n) -mesh. As Richter and Thomassen show, $i^*(5, 5) = 40$, and so $cr(C_5 \times C_5) \geq 40 - 25 = 15$. On the other hand, there are drawings of $C_5 \times C_5$ with exactly 15 crossings. It follows that the crossing number of $C_5 \times C_5$ is 15, in agreement with the general conjecture.

Using the same methods, Anderson et al. [ARRo1, ARRo2] have shown that $cr(C_6 \times C_6) = 24$ and $cr(C_7 \times C_7) = 35$. These values are also in agreement with the general conjecture.

In Chapter 2 we use the approach of Richter and Thomassen to the problem of the crossing number of $C_m \times C_n$ to show that the crossing numbers of large classes of drawings of $C_m \times C_n$ behave asymptotically as in the general conjecture.

It is clear that there are drawings of $C_m \times C_n$ where the m n -cycles cross each other an arbitrarily large number of times. However, we can focus our attention to those drawings where no two n -cycles intersect each other in more than M points, where M is any fixed positive integer. Let $cr_M(C_m \times C_n)$ be the minimum crossing number among all these drawings of $C_m \times C_n$. The main result of Chapter 2 is that

$$\lim_{n \rightarrow \infty} \frac{cr_M(C_m \times C_n)}{(m-2)n} = 1.$$

Actually, we can prove a slightly stronger result, established in Theorem 2.7, which basically states that either the crossing number of $C_m \times C_n$ is not very different from the conjectured value, or else the crossing numbers are attained in drawings where some pair of n -cycles intersect an arbitrarily large number of times. This last possibility is rather unnatural, and this reinforces our optimism in the correctness of the general conjecture.

On a related problem on the intersections of curve systems

As pointed out in [RT1], the following question arises naturally from the study of (m, n) -meshes: What is the minimum number $i(n)$ of intersections in a

collection of n pairwise nondisjoint simple closed curves? Trivially, $i(n) \geq \binom{n}{2}$, and Richter and Thomassen show that $\lim_{n \rightarrow \infty} i(n) \geq \frac{3}{2} \binom{n}{2}$, and that $\lim_{n \rightarrow \infty} i(n) \leq 2 \binom{n}{2}$.

In Chapter 3 we study this problem under an additional condition. Let $i_m(n)$ denote the minimum number of intersections in a collection of n pairwise nondisjoint simple closed curves, no two of which intersect in m points or more. The main result of Chapter 3 is that $\lim_{n \rightarrow \infty} i_m(n) = 2 \binom{n}{2}$.

Drawings of $C_m \times C_n$ with the n m -cycles pairwise disjoint or the m n -cycles pairwise disjoint

In Chapter 4 we prove that if $n \geq m \geq 3$, then every drawing of $C_m \times C_n$ in which either the n m -cycles are pairwise disjoint or the m n -cycles are pairwise disjoint has at least $(m-2)n$ crossings. Besides its intrinsic relevance as a support for the general conjecture, this result is very important in connection with the calculation of the exact crossing number of $C_m \times C_n$ for certain values of m and n .

As mentioned above, the crossing number of $C_m \times C_n$ is known exactly for every $n \geq m$, for each m satisfying $3 \leq m \leq 5$. The cases $m = 3, 4$, and 5 were solved in [RB], [BR], and [KRS] respectively, using an induction with base cases $cr(C_3 \times C_3) = 3$, $cr(C_4 \times C_4) = 8$, and $cr(C_5 \times C_5) = 15$ respectively. In each of [RB], [BR], and [KRS], an important part of the inductive step is to deal with the case where the n m -cycles are pairwise disjoint. A similar observation holds in Chapter 5 in this thesis, where we establish that $cr(C_6 \times C_n) = 4n$, as conjectured. Roughly speaking, half of the difficulty in these inductions lies in the case where

the n m -cycles are pairwise disjoint, and the other half lies in the case where some pair of m -cycles intersect. Thus, it is reasonable to expect that the main result in Chapter 4 will be of great help in the computation of the exact values of $cr(C_m \times C_n)$ for more values of m and n .

The crossing number of $C_6 \times C_n$

In Chapter 5 we show that the crossing number of $C_6 \times C_n$ is $4n$, in agreement with the general conjecture for the crossing number of $C_m \times C_n$. As in [RB], [BR], and [KRS], we proceed by induction. The base case is $cr(C_6 \times C_6) = 24$, proved in [ARRo1]. The technique we use is a generalization of the one used in [KRS], which consists of dividing the inductive steps into two cases, depending on whether or not two m -cycles cross each other in the given drawing of $C_6 \times C_n$.

1.3.2 The Crossing Number of the Generalized Petersen Graph $P(n, 3)$

In a paper published in 1986, Fiorini [F1] claimed to have determined that the crossing number of the Generalized Petersen Graphs $P(3k + h, 3)$ is $k + h$ for $h \in \{0, 2\}$, and he also claimed that the crossing number of $P(3k + 1, 3)$ is between $k + 1$ and $k + 3$. Unfortunately, a serious error in his argument invalidates these results. In Chapter 6 we calculate the crossing number of $P(3k + h, 3)$, for each $h \in \{0, 1, 2\}$. We make use of some of the interesting techniques developed by Fiorini, and we also devise certain constructions required for the computation of the exact crossing number of $P(3k + 1, 3)$.

1.4 More on Crossing Numbers

There is an extensive bibliography on crossing number problems [V]. In this section we overview some results that are not directly involved with the exact calculation of crossing numbers. For more information on crossing number problems, we refer the reader to the surveys [EG], [RT2], and [SSV].

The *rectilinear* crossing number of a graph G is the minimum number of crossings in a drawing of G where each edge is represented as a rectilinear segment. Not surprisingly, rectilinear crossing numbers are very difficult to calculate, and most of the work in this direction has to do with finding nontrivial lower bounds for rectilinear crossing numbers [J], [E], [T1], [BD1], [BD2].

Most of the time one is interested in drawings of graphs with the minimum possible number of crossings, but we can also raise the opposite question: what is the maximum number of crossings in a good drawing of a given graph? Harborth gives partial answers to this question [H1], [H2], for the cases where the graph under consideration is the n -cube or the n -cycle.

A graph G is k -critical if its crossing number is at least k and, for each edge e in G , the crossing number of $G - e$ is less than k . Much attention has been devoted to k -critical graphs. One of the strongest results in this direction is the statement, proved by Richter and Thomassen [RT3], that the crossing number of a k -critical graph is at most $2.5k + 16$. In a related result, Vitray [Vi] showed that every 2-critical graph, with the exception of $C_3 \times C_3$, has crossing number 2.

Finally, let us mention that some research has been done on drawings in other surfaces than the plane. For instance, Archdeacon proved an analogous version

of Kuratowski's Theorem for the projective plane [A]. Harary et al. [HKS] proved that the crossing number of $C_m \times C_n$ is arbitrarily high (for sufficiently large m and n). Since $C_m \times C_n$ can be drawn in the torus without any crossings, this shows that there is no trivial relation between the crossing numbers in the plane and in other surfaces. Shahrokhi et al. [SSSV] have found general lower bounds for the crossing number of $C_m \times C_n$ in the projective plane and in the Klein bottle. Guy and Jenkins [GJ] obtained lower bounds for the crossing number of $K_{m,n}$ in the torus, and Richter and Širáň [RS] calculated the crossing number of $K_{3,n}$ in an arbitrary surface.

Chapter 2

An Asymptotic Estimate for the Crossing Number of $C_m \times C_n$

2.1 Introduction

As mentioned in Section 1.3, Harary et al. [HKS] conjectured that the crossing number of $C_m \times C_n$ is $(m-2)n$, for all m, n satisfying $3 \leq m \leq n$. This has been proved only for the cases $m = 3, 4$, and 5 [RB], [DR], [KRS], and for the special cases $C_6 \times C_6$ [ARRo1] and $C_7 \times C_7$ [ARRo2], and in Chapter 5 below we prove it for every n if $m = 6$.

In this chapter we show that there is good reason to believe that the conjecture is correct, based on the asymptotic behaviour of the minimum crossing numbers of large classes of drawings of $C_m \times C_n$.

Let $cr_M(C_m \times C_n)$ denote the minimum number of crossings in a drawing of $C_m \times C_n$ such that the induced drawings of any two of the m n -cycles have no more than M intersections, where M is any fixed integer. We call $cr_M(C_m \times C_n)$ the M -crossing number of $C_m \times C_n$. The main result in this chapter is the following.

Theorem 2.1. *Let M be fixed. Then, for any $m \geq 3$,*

$$\lim_{n \rightarrow \infty} \frac{cr_M(C_m \times C_n)}{(m-2)n} = 1.$$

The strategy to prove Theorem 2.1 is as follows. In Section 2.2 we define $(m, n)_M$ -meshes and state Theorem 2.2. and establish that Theorem 2.1 follows from Theorem 2.2. In Section 2.3 we show that Lemma 2.3, a result on $(3, n)_M$ -meshes, implies Theorem 2.2, and in Section 2.4 we give a proof of Lemma 2.3.

In Section 2.5 we argue that there is reason to believe that $cr_M(C_m \times C_n)$ is not different from $cr(C_m \times C_n)$. In Section 2.6 we give some final remarks, including Theorem 2.7, another result that reinforces our optimism in the correctness of the general conjecture for $cr(C_m \times C_n)$.

2.2 $(m, n)_M$ -meshes

As Richter and Thomassen [RT1] showed in their work on intersections of systems of curves, the problem of determining the crossing number of $C_m \times C_n$ is closely related to the computation of the minimum number $i^*(m, n)$ of intersections in an (m, n) -mesh. An (m, n) -mesh (C_1, C_2) is a pair of collections C_1 and C_2 of closed curves such that each curve in C_1 intersects each curve in C_2 in at least one point, where $|C_1| = m$ and $|C_2| = n$. It is a trivial observation that $cr(C_m \times C_n) \geq i^*(m, n) - mn$. If the exact value of $i^*(m, n)$ is known for a pair (m, n) , this inequality can be used to obtain a lower bound on the crossing number of $C_m \times C_n$. Moreover, if there is a drawing $\mathcal{D}$ of $C_m \times C_n$ with exactly $i^*(m, n) - mn$ crossings, this shows that the crossing number of $C_m \times C_n$ is attained in $\mathcal{D}$.

Based on this approach, Richter and Thomassen [RT1] showed that the crossing number of $C_5 \times C_5$ is 15. Following the same ideas Anderson et al. [ARRo1, ARRo2] proved that the crossing numbers of $C_6 \times C_6$ and $C_7 \times C_7$ are 24 and 35 respectively. All these values are in agreement with the conjecture $cr(C_m \times C_n) =$

$(m - 2)n$, and it seems reasonable to expect that the same methods will help to compute $cr(C_m \times C_n)$ for more values of m and n , although the amount of work involved in the computation of $i^*(m, n)$ seems to increase considerably as m and n become larger.

We will focus our attention on (m, n) -meshes (C_1, C_2) such that each pair of curves in C_1 intersect each other in at most M points, where M is any fixed integer. A mesh satisfying this property will be called an $(m, n)_M$ -mesh. It is easy to check that if (C_1, C_2) is an $(m, n)_M$ -mesh whose total number of intersections $i_M^*(C_1, C_2)$ is minimum among all $(m, n)_M$ -meshes, then all the curves in $C_1 \cup C_2$ are simple. Therefore we will restrict our attention to meshes that satisfy this property.

Theorem 2.2. *Let M be fixed. Then, for any $m \geq 3$,*

$$\lim_{n \rightarrow \infty} \frac{i_M^*(m, n) - mn}{(m - 2)n} = 1.$$

We defer the proof of Theorem 2.2 for the moment, and show that Theorem 2.1 follows from Theorem 2.2.

Proof that Theorem 2.2 implies Theorem 2.1. Since $cr_M(C_m \times C_n) \geq i_M^*(m, n) - mn$, it follows from Theorem 2.2 that $cr_M(C_m \times C_n)/((m - 2)n)$ cannot converge to a number smaller than one. On the other hand, $cr_M(C_m \times C_n) \leq (m - 2)n$, since there are drawings of $C_m \times C_n$ with $(m - 2)n$ crossings in which the m n -cycles are pairwise disjoint. Hence $cr_M(C_m \times C_n)/((m - 2)n)$ converges to 1. $\square$

2.3 Theorem 2.2 follows from Lemma 2.3

The basic idea for the proof of Theorem 2.2 is the following. We establish in Lemma 2.4 that there is a number B' such that some curve in the second collection of each $(m, B' + 1)_M$ -mesh has at least $2m - 2$ intersections with the other curves in the mesh. This is enough to prove Theorem 2.2, since it follows that if (C_1, C_2) is an $(m, n)_M$ -mesh with $n > B'$, then, for each subfamily C'_2 of C_2 with $|C'_2| = B'$, each element in $C_2 \setminus C'_2$ has at least $2m - 2$ intersections with the curves in $C_1 \cup C'_2$. Hence, if n is very large compared to B' , the quotient $(i_M^*(C_1, C_2) - mn)/((m-2)n)$ is either bigger than 1 or arbitrarily close to 1.

We do not prove Lemma 2.4 directly, but show that it is a consequence of Lemma 2.3, which deals with the special case where $m = 3$ and each curve in the second mesh is tangent to each curve in the first mesh. If C and D are curves in $\mathbf{R}^2$ which intersect at a point p , we say that they are *tangent* at p if the rotation scheme around p consists of two subarcs of C followed by two subarcs of D ; otherwise we say that C and D *cross* at p .

Lemma 2.3. *Let $m \geq 3$ be fixed. Then there exists an integer $B = B(m)$ such that, for any $(3, B + 1)_M$ -mesh (C_1, C_2) with each curve in C_2 tangent to each curve in C_1 , some element C in C_2 satisfies $\sum_{D \in (C_2 \setminus \{C\})} |C \cap D| \geq m - 2$.*

We prove Lemma 2.3 in the next section, and devote the rest of the present section to show that Theorem 2.2 follows from Lemma 2.3. Our first observation is that Lemma 2.3 implies the following.

Lemma 2.4. *Let $m \geq 3$ be fixed. Then there exists an integer B' such that, for*

any $(m, B' + 1)_M$ -mesh (C_1, C_2) , some curve C in C_2 satisfies

$$\sum_{D \in C_1 \cup (C_2 \setminus \{C\})} |C \cap D| \geq 2m - 2.$$

Proof. First we note that if a curve C in C_2 is tangent to no more than two curves in C_1 , then C has at least $m + (m - 2) = 2m - 2$ intersections with the curves in C_1 . Since in this case the conclusion of Lemma 2.4 follows, we can assume that each curve in C_2 is tangent to at least three curves in C_1 .

Let B be as in the statement of Lemma 2.3. Let (C_1, C_2) be an $(m, \binom{m}{3}B + 1)_M$ -mesh. Then there is a collection of at least $B + 1$ curves in C_2 tangent to the same three curves in C_1 , and so by Lemma 2.3 there exists a curve C in C_2 such that $\sum_{D \in C_2 \setminus \{C\}} |C \cap D| \geq m - 2$. Since each curve in C_2 intersects every curve in C_1 , C has at least m intersections with the curves in C_1 , and so the total number of intersections of C with the curves in $C_1 \cup (C_2 \setminus \{C\})$ is at least $2m - 2$. $\square$

Proof that Lemma 2.3 implies Theorem 2.2. From the work in this section it follows that Lemma 2.3 implies Lemma 2.4. Now we show that Theorem 2.2 follows from Lemma 2.4. If (C_1, C_2) is an $(m, n)_M$ -mesh with $n > B'$, then an easy induction, using Lemma 2.4, shows that there are at least $mB' + (2m - 2)(n - B')$ intersections in (C_1, C_2) . Therefore for $n > B'$, $i_M^*(m, n) \geq mB' + (2m - 2)(n - B') = mn + (m - 2)(n - B')$, and so $(i_M^*(m, n) - mn)/((m - 2)n) \geq (1 - B'/n)$. On the other hand, the drawings constructed to show that $C_m \times C_n \leq (m - 2)n$ also show that $i_M^*(m, n) \leq mn + (m - 2)n$, and so $(i_M^*(m, n) - mn)/((m - 2)n) \leq 1$. Therefore $(1 - B'/n) \leq i_M^*(C_1, C_2)/((m - 2)n) \leq 1$ for every $n > B'$. Since B' is fixed, $1 - B'/n$ approaches 1 as n approaches infinity, and so Theorem 2.2 follows. $\square$

2.4 Proof of Lemma 2.3

To prove Lemma 2.3 we need the following technical lemma, which will be proved later in this section.

Lemma 2.5. *Let I_1, I_2, I_3 be pairwise disjoint arcs in the plane, and let $\mathcal{C}$ be a collection of simple closed curves. Suppose that each curve in $\mathcal{C}$ is tangent to each I_j on the same side of I_j for each $j \in \{1, 2, 3\}$. Let m be any integer. Then there exists a B'' such that, if $|\mathcal{C}| \geq B''$, there is a C in $\mathcal{C}$ with at least $m - 2$ intersections with curves in $\mathcal{C} \setminus \{C\}$.*

There are eight ways in which the curves in $\mathcal{C}$ may be tangent to each I_j ; e.g., each curve may be tangent to I_1 and I_3 on the left, and I_2 on the right.

Now we show that Lemma 2.3 follows from Lemma 2.5.

Proof of Lemma 2.3. Let $\mathcal{C}_1 = \{C_1, C_2, C_3\}$ and $\mathcal{C}_2$ be collections of curves such that $(\mathcal{C}_1, \mathcal{C}_2)$ is a $(|\mathcal{C}_1|, |\mathcal{C}_2|)_M$ -mesh, and such that each curve in $\mathcal{C}_2$ is tangent to each curve in $\mathcal{C}_1$.

It is easy to see that each curve C_i in $\mathcal{C}_1$ can be partitioned into a collection $\mathcal{I}_i = \{I_{i,j}\}_{j=1}^{T(i)}$ of $T(i)$ homeomorphs of $I = [0, 1]$ such that if $I_{i,j} \cap I_{k,l} \neq \emptyset$, then $I_{i,j}$ and $I_{k,l}$ meet in exactly one common end and no curve in $\mathcal{C}_2$ touches both $I_{i,j}$ and $I_{k,l}$. By assigning a direction to each interval $I_{i,j}$ in $\mathcal{I}_1 \cup \mathcal{I}_2 \cup \mathcal{I}_3$ it is possible to say whether a given curve in $\mathcal{C}_2$ which is tangent to $I_{i,j}$ touches $I_{i,j}$ on the left or on the right side of $I_{i,j}$.

We claim that $T(i)$ can be chosen to be at most $12M$. First we note that each curve in $\mathcal{C}_1$ is crossed by the other two curves in $\mathcal{C}_1$ in at most $2M$ points. Thus

each curve in $\mathcal{C}_1$ is divided into at most $2M$ *primary* intervals by these intersection points. It is possible, however, that a primary interval of a curve in $\mathcal{C}_1$ has both end points in common with a primary interval of a (possibly the same) curve in $\mathcal{C}_1$. Thus, to avoid this conflict we subdivide each primary interval into two *secondary* intervals. Therefore it follows that each curve in $\mathcal{C}_1$ can be divided into at most $4M$ intervals to satisfy the condition that if $I_{i,j} \cap I_{k,l} \neq \emptyset$ then $I_{i,j}$ and $I_{k,l}$ meet in exactly one common end (if some curve in $\mathcal{C}_1$ is not intersected by any other curve in $\mathcal{C}_1$, then we subdivide it into at least three secondary intervals, to satisfy this last condition).

Now in order to satisfy the condition that if $I_{i,j} \cap I_{k,l} \neq \emptyset$, then no curve in $\mathcal{C}_2$ touches both $I_{i,j}$ and $I_{k,l}$, it clearly suffices to subdivide each of the at most $4M$ secondary intervals into at most three subintervals (namely a middle subinterval that contains all the intersections with the curves in $\mathcal{C}_2$ plus two subintervals, each of which has a common end with the middle one). Therefore each curve C_i in $\mathcal{C}_1$ can be subdivided in $T(i) \leq 12M$ intervals to meet the required conditions.

Let B'' be as in Lemma 2.5, and let $B' = (8(12M)^3)B'' + 1$. Since there are $(2T(1)) (2T(2)) (2T(3)) \leq 8(12M)^3$ different ways of choosing one interval from each of $\mathcal{I}_1, \mathcal{I}_2$, and $\mathcal{I}_3$ together with a side for each of the intervals, it follows that there are intervals $I_{1,j_1}, I_{2,j_2}, I_{3,j_3}$, such that at least $B'' + 1$ curves in $\mathcal{C}_2$ touch these intervals on the same side of each of I_{1,j_1}, I_{2,j_2} , and I_{3,j_3} . Thus from Lemma 2.5 there exists a curve $C \in \mathcal{C}_2$ that has at least $m - 2$ intersections with the elements in $\mathcal{C}_2 \setminus \{C\}$, and so Lemma 2.3 follows. $\square$

We finish this section with a proof of Lemma 2.5.

Proof of Lemma 2.5. It is easy to construct homeomorphs J_1, J_2, J_3 of $I = [0, 1]$

such that $D_i = I_i \cup J_i$ is a simple closed curve for each i and the following conditions are satisfied:

- (1) Each element in $\mathcal{C}$ is tangent to each D_i .
- (2) No curve in $\mathcal{C}$ and no D_j is contained in the closed disc bounded by D_i if $i \neq j$.
- (3) The curves D_i and D_j are disjoint if $i \neq j$.

We prove that if $|\mathcal{C}| \geq m + 3(\lceil \frac{m}{2} \rceil - 1)$, then there exists a $C \in \mathcal{C}$ with at least $m - 2$ intersections with the curves in $\mathcal{C} \setminus \{C\}$.

Let $|\mathcal{C}| \geq m + 3(\lceil \frac{m}{2} \rceil - 1)$, and let us suppose that there is a subcollection $\mathcal{C}'$ of $\mathcal{C}$ with at least $\lceil \frac{m}{2} \rceil$ elements in $\mathcal{C}$ whose bounded discs contain the same curve D_i . It follows that any two elements in $\mathcal{C}'$ have at least two points in common, and so each element C in $\mathcal{C}'$ has at least $2(\lceil \frac{m}{2} \rceil - 1) \geq m - 2$ intersections with the elements in $\mathcal{C}' \setminus C$. Since in such a case we are done, we assume that there exist at most $3(\lceil \frac{m}{2} \rceil - 1)$ elements in $\mathcal{C}$ whose bounded discs contain a D_i , and so that there exist at least m elements in $\mathcal{C}$ whose bounded discs do not contain any D_i . In this case the desired result follows from our next lemma, and so this finishes the proof of Lemma 2.5. $\square$

Lemma 2.6. *Let $\mathcal{C}$ be a collection of simple closed curves, let D_1, D_2, D_3 be simple closed curves such that Properties (1), (2), and (3) above are satisfied, and suppose that no D_i is contained in the closed disc bounded by C for any C in $\mathcal{C}$. Let $m \geq 3$ be an integer, and suppose that $|\mathcal{C}| \geq m$. Then there exists a curve C in $\mathcal{C}$ with at least $m - 2$ intersections with elements in $\mathcal{C} \setminus \{C\}$.*

Proof. If $|C \cap C'| \neq \emptyset$ for all C, C' in $\mathcal{C}$, then the lemma follows, since then each

curve in $\mathcal{C}$ has at least $m - 1$ intersections with the elements in $\mathcal{C} \setminus \{C\}$. Therefore we assume that there exist C, C' in $\mathcal{C}$ such that $C \cap C' = \emptyset$.

There are exactly eight components in $\mathbf{R}^2 \setminus (D_1 \cup D_2 \cup D_3 \cup C \cup C')$. Five of these components are the discs bounded by the five curves, and the boundary of each of the other three components has empty intersection with D_i for exactly one i . Let H_i be the component whose boundary does not intersect D_i , and let C'' belong to $\mathcal{C} \setminus \{C, C'\}$. Since C'' is tangent to D_1, D_2 , and D_3 , it must intersect H_i for two different values of i , and this can only happen if C'' intersects either C or C' in at least two points. Therefore either C or C' has at least $2\lceil \frac{|\mathcal{C}|-2}{2} \rceil \geq m - 2$ intersections with the elements in $\mathcal{C} \setminus \{C, C'\}$. $\square$

2.5 $cr(C_m \times C_n)$ versus $cr_M(C_m \times C_n)$

As Richter and Thomassen point out in their paper [RT1], we cannot expect to compute $cr(C_m \times C_n)$ from $i^*(m, n)$ for all values of m and n . Indeed. Richter and Thomassen show that $i^*(3, n) = 3n + \lceil \frac{n+3}{4} \rceil + \lceil \frac{n+4}{4} \rceil$ for each $n > 3$. Thus, the lower bound for $cr(C_3 \times C_n)$ that we can obtain from $i^*(3, n)$ is that $cr(C_3 \times C_n) \geq i^*(3, n) - 3n = \lceil \frac{n+3}{4} \rceil + \lceil \frac{n+4}{4} \rceil$. This lower bound is about half the actual value for $cr(C_3 \times C_n)$, namely n .

Since $i^*(3, n) - 3n$ is smaller than $cr(C_3 \times C_n)$, it follows that the optimal $(3, n)$ -meshes cannot be extended to drawings of $C_3 \times C_n$ by assigning vertices to some of the intersection points. The reason is that in an optimal $(3, n)$ -mesh the cyclic condition for the order of the intersections of the curves in the first collection with the curves in the second collection is seriously violated. Thus, not only can we not make an optimal $(3, n)$ -mesh to be the underlying system of curves of a

drawing of $C_3 \times C_n$, but an optimal $(3, n)$ -mesh is far from satisfying the required cyclic intersection conditions for such a drawing.

The most awkward property of optimal $(3, n)$ -meshes is that the number of intersections between the 3 curves is arbitrarily large, for sufficiently large n . This possibility is automatically eliminated if we consider only $(m, n)_M$ -meshes: this is the reason for which we obtain that, for fixed m , $(i_M^*(m, n) - mn)/((m - 2)n)$ approaches 1 as n approaches infinity, even though $(i^*(m, n) - mn)/((m - 2)n)$ seems to converge to values smaller than 1 (it converges to $1/2$ for $m = 3$, and Richter and Thomassen argue that it also converges to $1/2$ for $m = 4$).

We expect this non-extendability property of optimal (m, n) -meshes to generalize to bigger (in fact, to all) values of m , since it is difficult to believe that the strong cyclic requirements of $C_m \times C_n$ would be satisfied by drawings where the m n -curves intersect each other in an arbitrarily large number of points. In other words, we do expect the values of $i^*(m, n) - mn$, for each fixed m , to be substantially smaller than $cr(C_m \times C_n)$, but we do not expect that the optimal meshes will be extendable to drawings of $C_m \times C_n$, for large values of n . Thus, we conjecture that the bounding condition for the number of intersections between curves in the first collection is not restrictive at all, that is, we conjecture that $cr_M(C_m \times C_n) = cr(C_m \times C_n)$.

2.6 Conclusions

As we mentioned in 2.1, a proof of the general conjecture $cr(C_m \times C_n) = (m - 2)n$ seems hopeless at this moment. Theorem 2.1 reinforces that conjecture, as does the following result, which is a straightforward consequence of Lemma 2.4.

Theorem 2.7. *Let $m \geq 3$ be a given integer, and suppose that there exists an infinite subsequence $\{n_1, n_2, \dots\}$ of $\{1, 2, \dots\}$ such that $cr(C_m \times C_{n_j}) < (m - 2)n_j$ for each n_j . Then either*

- (i) *there exists an integer $r(m)$ such that $cr(C_m \times C_{n_j}) \geq (m - 2)n_j - r(m)$ for each n_j , or*
- (ii) *given any integer s there exists an n_j such that there is an optimal drawing $\mathcal{D}$ of $C_m \times C_{n_j}$ where two of the m n_j -cycles intersect each other in more than s points.*

Basically, Theorem 2.7 says that if the crossing number of $C_m \times C_n$ is less than as conjectured, then either the actual and the conjectured crossing numbers are not very different or else some crossing numbers are attained in drawings where two n -cycles intersect each other an arbitrarily large number of times. This last condition is rather unappealing, and this increases our optimism in the correctness of the general conjecture.

The techniques introduced by Richter and Thomassen to compute $cr(C_m \times C_n)$ from $i^*(m, n)$ are essential for the calculation of $cr(C_m \times C_n)$ for all the values of $m \geq 5$ for which this crossing number is known. However, as noted in their paper [RT1], these methods cannot be used to obtain the crossing number of $C_m \times C_n$ for arbitrary values of m and n . This is illustrated by the discrepancy between $i^*(3, n) - 3n$ and $cr(C_3 \times C_n)$: for large values of n , $i^*(3, n) - 3n$ is approximately $n/2$, and the crossing number of $C_3 \times C_n$ is equal to n for every $n \geq 3$ [RB]. We remark that this is not in contradiction with the asymptotic behaviour of $cr_M(C_m \times C_n)/((m - 2)n)$ given by Theorem 2.1, since not every (m, n) -mesh is an $(m, n)_M$ -mesh.

Chapter 3

On a Question Raised by Richter and Thomassen

3.1 Introduction

In their work on intersections of curve systems [RT1], Richter and Thomassen raised the problem of evaluating $i(n)$, the minimum number of intersections in a collection of n simple closed curves in the plane, any two of which intersect, and where no point in the plane belongs to more than two curves in the collection. As pointed out in their paper, it is not difficult to show that $\frac{3}{2} \leq \lim_{n \rightarrow \infty} i(n)/\binom{n}{2} \leq 2$. Our main goal in this chapter is to prove that this limit is equal to two, if we add the condition that any two curves intersect each other in less than m points, where m is fixed but can be arbitrarily large. Thus if we let $i_m(n)$ to be the minimum number of intersections in a family of n simple closed curves which intersect each other at least once and in less than m points, we prove the following.

Theorem 3.1. *For each integer $m > 2$, $\lim_{n \rightarrow \infty} i_m(n)/\binom{n}{2} = 2$.*

The following argument shows that this limit exists and is not bigger than 2. A standard counting argument shows that that $i_m(n) \geq (\frac{n}{n-2})i_m(n-1)$, and

so $i_m(n)/\binom{n}{2} \geq i_m(n-1)/\binom{n-1}{2}$. Therefore $\{i_m(n)/\binom{n}{2}\}_{n=1}^{\infty}$ is a nondecreasing sequence. On the other hand, Figure 3.1 shows that, for any given n , it is possible to draw n simple closed curves that pairwise intersect in exactly two points, and so $i_m(n)/\binom{n}{2} \leq 2$ for any $m > 2$.

This chapter is organized as follows. In Section 3.2 we show how to obtain the lower bound for $\lim_{n \rightarrow \infty} i(n)/\binom{n}{2}$ given in [RT1]. In Section 3.3 we show that Theorem 3.1 follows from Lemmas 3.4 and 3.5. In Section 3.4 we give a proof of Lemma 3.4, and Section 3.5 contains a proof of Lemma 3.5. In Section 3.6 we make some final remarks.

3.2 The bound obtained by Richter and Thomassen

In this section we work out the details involved in the derivation of the inequality $\lim_{n \rightarrow \infty} i(n)/\binom{n}{2} \geq 3/2$, established in [RT1].

The last section in [RT1] concludes with the observation that the inequalities

$$\frac{3}{2} \leq \lim_{n \rightarrow \infty} \frac{i(n)}{\binom{n}{2}} \leq 2$$

hold. As explained in Section 3.1, the limit $\lim_{n \rightarrow \infty} \frac{i_m(n)}{\binom{n}{2}}$ exists, and is at most 2, for each $m > 2$. The same argument shows that $\lim_{n \rightarrow \infty} \frac{i(n)}{\binom{n}{2}}$ exists, and is at most 2. To prove that $\lim_{n \rightarrow \infty} i(n)/\binom{n}{2}$ is at least 3/2, Richter and Thomassen invoke the inequalities $i(2m) \geq 3m^2 - 3m + 2$ and $i(2m+1) \geq 3m^2 + 1$, for each $m \geq 3$. These inequalities are easily checked, using an induction with the base case $i(6) = 20$, and the inequality $i(n) \geq \frac{n}{n-2}i(n-1)$. The equality $i(6) = 20$ is established in [RT1]. The proofs of these facts are omitted in [RT1] but, as

How to draw a collection of n simple closed curves, such that each two curves in the collection intersect, and such that the total number

of intersections in the collection is $2 \binom{n}{2}$.

Figure 3.1

mentioned there, it is not difficult to prove them. We give later in this section a detailed proof of these equalities, but first we show how it follows from the inequality $i(2m) \geq 3m^2 - 3m + 2$ that $\lim_{n \rightarrow \infty} i(n)/\binom{n}{2} \geq 3/2$.

Since $i(n)/\binom{n}{2}$ is increasing, in order to prove that $\lim_{n \rightarrow \infty} i(n)/\binom{n}{2} \geq 3/2$ it suffices to show that the limit of the sequence $\{i(2m)/\binom{2m}{2}\}_{m=3}^{\infty}$ is at least $3/2$. Since $i(2m) \geq 3m^2 - 3m + 2$, we have

$$\lim_{m \rightarrow \infty} \frac{i(2m)}{\binom{2m}{2}} \geq \lim_{m \rightarrow \infty} \frac{3m^2 - 3m + 2}{\frac{(2m)(2m-1)}{2}} = \lim_{m \rightarrow \infty} \frac{6m^2 - 6m + 4}{4m^2 - 2m} = \frac{3}{2}.$$

Now we prove that $i(5) = 12$ and $i(6) = 20$.

Lemma 3.2. $i(5) = 12$.

Proof. Let $\mathcal{C} = \{C_1, C_2, C_3, C_4, C_5\}$ be a collection of pairwise nondisjoint simple closed curves. First we show that $i(\mathcal{C}) > 10$. If $i(\mathcal{C} \setminus \{C_i\}) > 6$ for each $C_i \in \mathcal{C}$, then some pair of curves in $\mathcal{C}$ intersects more than once, and so $i(\mathcal{C}) > \binom{5}{2} = 10$. Now suppose that there is a curve C_i in $\mathcal{C}$ such that $i(\mathcal{C} \setminus \{C_i\}) = 6$. We can assume without loss of generality that $i(\mathcal{C} \setminus \{C_5\}) = 6$. It is easy to check that, since $i(\mathcal{C} \setminus \{C_5\}) = 6$, the curves in $\mathcal{C} \setminus \{C_5\}$ must be drawn in such a way that the boundary of each component of $\mathbf{R}^2 \setminus (\mathcal{C} \setminus \{C_5\})$ intersects at most 3 curves in $\mathcal{C}$. Therefore C_5 must intersect two components of $\mathbf{R}^2 \setminus (\mathcal{C} \setminus \{C_5\})$, and so C_5 crosses some curve in $\mathcal{C}$. Thus, it follows that $i(\mathcal{C}) > 10$.

Now suppose that $i(\mathcal{C}) = 11$. Then exactly one pair (say (C_1, C_2)) of curves in $\mathcal{C}$ intersects in two points, and all the other pairs have a tangential intersection. This implies that each of C_3, C_4 , and C_5 is in the same one of four components of $\mathbf{R}^2 \setminus (C_1 \cup C_2)$. But in such a case we could easily remove one intersection between

C_1 and C_2 (as illustrated in Fig. 3.2), and the obtained drawing would have only 10 intersections, a contradiction. $\square$

Lemma 3.3. $i(6) = 20$.

Proof. Let $\mathcal{C} = \{C_1, C_2, C_3, C_4, C_5, C_6\}$ be a collection of curves that pairwise intersect at least once. We show that the total number of intersections $i(\mathcal{C})$ among the curves in $\mathcal{C}$ is at least 20.

First we show that we can assume that $i(\mathcal{C} \setminus \{C_i\}) = 12$ for some curve $C_i \in \mathcal{C}$. By Lemma 3.2, $i(\mathcal{C} \setminus \{C_i\}) \geq 12$ for each $C_i \in \mathcal{C}$. Suppose that $i(\mathcal{C} \setminus \{C_i\}) \geq 13$ for every $C_i \in \mathcal{C}$. Since

$$i(\mathcal{C}) = \frac{1}{4} \sum_{C_i \in \mathcal{C}} i(\mathcal{C} \setminus \{C_i\}),$$

it follows that $i(\mathcal{C}) \geq \frac{6 \cdot 13}{4} = 19.5 > 19$, and so $i(\mathcal{C}) = 20$. Therefore we assume that $i(\mathcal{C} \setminus \{C_i\}) = 12$ for some $C_i \in \mathcal{C}$. Without loss of generality, we assume that $i(\mathcal{C} \setminus \{C_6\}) = 12$.

Now we show that we can assume that the curves in $\mathcal{C} \setminus \{C_6\}$ are drawn as in Fig. 3.3. By the work in the proof of Lemma 3.2, it follows that there are two pairs (say (C_i, C_j) and (C_k, C_l)) of curves in $\mathcal{C} \setminus \{C_6\}$ that intersect in two points each, and all other pairs intersect tangentially. If $\{C_i, C_j\} \cap \{C_k, C_l\} = \emptyset$, then the three curves in $(\mathcal{C} \setminus \{C_6\}) \setminus \{C_i, C_j\}$ are in the same component of $\mathbb{R}^2 \setminus (C_i \cup C_j)$. As in the proof of Lemma 3.2, this implies that we can eliminate one of the two crossings between C_i and C_j , contradicting the fact that $i(5) = 12$. Therefore $\{C_i, C_j\} \cap \{C_k, C_l\}$ must consist of exactly one curve, and so we can assume without any loss of generality that $i = 1, j = k = 2$, and $l = 3$. Therefore C_2 intersects each of C_1 and C_3 in two points, and all the other intersections are

If C_3 , C_4 , and C_5 were all in the same component of $\mathbb{R}^2 \setminus (C_1 \cup C_2)$, then we could easily remove one intersection between C_1 and C_2

Figure 3.2

If the total number of intersections among the five curves is 12, then we can assume without any loss of generality that they are drawn as shown above.

Figure 3.3

tangential. It is not difficult to check that it follows that we can assume without any loss of generality that the curves in $\mathcal{C} \setminus \{C_6\}$ are drawn as in Figure 3.3.

It is easy to verify that, since C_6 intersects every curve in $\mathcal{C} \setminus \{C_6\}$, then C_6 intersects at least 4 components of $\mathbb{R}^2 \setminus (C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5)$. It follows that C_6 has at least 8 intersections with the curves in $\mathcal{C} \setminus \{C_6\}$. Since $i(\mathcal{C} \setminus \{C_6\}) = 12$, it follows that $i(\mathcal{C}) \geq 20$. $\square$

3.3 Decomposition of Theorem 3.1

An *intersecting family* $\mathcal{C}$ is a set of simple closed curves in the plane such that any two curves in $\mathcal{C}$ have a point in common. A $(< m)$ -*family* is a collection of simple closed curves in the plane which pairwise intersect in less than m points. An *intersecting* $(< m)$ -*family* is both an intersecting family and a $(< m)$ -family. We remark that in all these definitions we assume that no point of the plane is in more than two simple closed curves in the collection.

Now we state Lemmas 3.4 and 3.5.

Lemma 3.4. *Suppose there exist $\epsilon > 0$ and $m > 2$ such that $\lim_{n \rightarrow \infty} i_m(n)/\binom{n}{2} < 2 - \epsilon$. Then there exist constants $q \geq 2m$ and $s > 0$ such that, for any $N > q$, there is an $n \geq N$ and an intersecting $(< m)$ -family $\{C_1, C_2, \dots, C_q, C_{q+1}, \dots, C_n\}$ so that every element in a $2m$ -subset of $\{C_1, C_2, \dots, C_q\}$ is tangent to every curve in the same subset of $\{C_{q+1}, \dots, C_n\}$ of size at least $s(n - q)$.*

Lemma 3.5. *For any integer $m > 2$ there exists a number $B(m)$ such that if $\mathcal{C}_1$ is an intersecting $(< m)$ -family of size $2m$ and $\mathcal{C}_2$ is any $(< m)$ -family such that every curve in $\mathcal{C}_2$ is tangent to every curve in $\mathcal{C}_1$, then $|\mathcal{C}_2| \leq B(m)$.*

Lemmas 3.4 and 3.5 will be proved in Sections 3.4 and 3.5 respectively. They are all we need in order to prove Theorem 3.1.

Proof of Theorem 3.1. If $\lim_{n \rightarrow \infty} i_m(n)/\binom{n}{2} \neq 2$, then $\lim_{n \rightarrow \infty} i_m(n)/\binom{n}{2} < 2 - \epsilon$ for some $\epsilon > 0$. Let q and s be as given in Lemma 3.4 and let N be such that $B(m) < s(N - q)$. By Lemma 3.4, for some $n \geq N$ there exists an intersecting $(< m)$ -family $\mathcal{C}_1$ of size $2m$ and a $(< m)$ -family $\mathcal{C}_2$ of size at least $s(n - q)$ such that every curve in $\mathcal{C}_2$ is tangent to every curve in $\mathcal{C}_1$. Since $s(n - q) \geq s(N - q) > B(m)$, this contradicts Lemma 3.5. $\square$

3.4 Proof of Lemma 3.4

Fix ϵ so that $0 < \epsilon < 1$. Let $\mathcal{C} = \{C_1, C_2, \dots, C_n\}$ be an intersecting $(< m)$ -family such that the total number $i_m(\mathcal{C})$ of intersections among the curves in $\mathcal{C}$ is less than $(2 - \epsilon)\binom{n}{2}$. We assume the elements of $\mathcal{C}$ are ordered $C_1, C_2, \dots$ so that, for each i , the number of elements of $\mathcal{C}$ tangent to C_i is at least the number of elements of $\mathcal{C}$ tangent to C_{i+1} .

Let q be a fixed integer. It is reasonable to expect that if q is small compared to n , then C_q will be tangent to a relatively large number of curves. The following lemma makes this statement precise.

Lemma 3.6. *If $n > \frac{4q}{\epsilon}$, then C_q is tangent to at least $\frac{\epsilon}{2}(n - q)$ curves in $\mathcal{C}$.*

Proof. Suppose that there are less than $\frac{\epsilon}{2}(n - q)$ curves tangent to C_q . Since no curve C_i with $i > q$ is tangent to more elements in $\mathcal{C}$ than C_q , it follows that C_i is not tangent to at least $(1 - \frac{\epsilon}{2})(n - q)$ curves in $\mathcal{C}$, for each $i > q$. Thus, each

curve in $\{C_{q+1}, \dots, C_n\}$ is not tangent to at least $(1 - \frac{\epsilon}{2})(n - q)$ curves in C , and since there are $n - q$ curves in $\{C_{q+1}, \dots, C_n\}$, it follows that there are at least $\frac{1}{2}(n - q)^2(1 - \frac{\epsilon}{2})$ pairs of nontangent curves in C . Therefore the total number $i(C)$ of intersections among the elements of C is at least $\binom{n}{2} + \frac{1}{2}(n - q)^2(1 - \frac{\epsilon}{2})$. Since $i(C) \leq (2 - \epsilon)\binom{n}{2}$, this yields $(1 - \epsilon)\binom{n}{2} \geq \frac{1}{2}(n - q)^2(1 - \frac{\epsilon}{2})$. It is straightforward to check that this inequality does not hold if $n > \frac{4q}{\epsilon}$. $\square$

Let us suppose that $n > \frac{4q}{\epsilon}$. It follows from Lemma 3.6 and the ordering condition for the elements in C that the number of curves tangent to C_i is at least $\frac{\epsilon}{2}(n - q)$ for each $i \leq q$. It is natural to expect that if q is very small compared to n , then a large portion of the curves tangent to C_i will belong to $\{C_{q+1}, \dots, C_n\}$, for each $i \leq q$. It is indeed not difficult to show, for instance, that if n is large enough, then there are at least $\frac{\epsilon}{4}(n - q)$ curves in $\{C_{q+1}, \dots, C_n\}$ tangent to C_i for $1 \leq i \leq q$, and so C will be assumed to be large enough to guarantee this property. Then we can let H_i be a collection of $\lceil \frac{\epsilon}{4}(n - q) \rceil$ curves in $\{C_{q+1}, \dots, C_n\}$ tangent to C_i , for each i satisfying $1 \leq i \leq q$. Each set H_i can be assumed to be nonempty by making n appropriately large.

In order to finish the proof of Lemma 3.4, it is enough to show that if q and n are adequately large, then there is a collection $\{H_i\}_{i=1}^{2m}$ of size $2m$ whose elements have a set of size at least $s(n - q)$ as their common intersection, where s does not depend on either n or q and $0 < s < 1$. This is accomplished by the following lemma, since, using its notation, all we need is to make $q \geq q_0(\frac{\epsilon}{4}, 2m)$ and $n \geq N_0(\frac{\epsilon}{4}, 2m) + q$. This finishes the proof of Lemma 3.4. $\square$

Lemma 3.7. *Let R be an integer, and let t be such that $0 < t < 1$. Then there exist integers $N_0(t, R), q_0(t, R)$ and a positive number $s_0(t, R)$ such that, for any*

$N \geq N_0$ and $q \geq q_0$, any q -collection of $\lceil tN \rceil$ -subsets of an N -set has R elements whose intersection contains at least $s_0 N$ elements.

Proof. It is enough to show that the statement is true if $R = 2^k$, for all positive integers k . We do this by induction on k . First let us suppose that $R = 2$. Let $L = \lceil t^{-1} \rceil$. Since $0 < t < 1$, $L \geq 2$. We will show that we can take $N_0 = L^4$, $q_0 = L + 1$, and $s_0 = L^{-4}$. Let $M = \lceil tN \rceil$, and let $H_1, H_2, \dots, H_{L+1}$ be M -subsets of an N -set S , for some $N \geq N_0$. Suppose that $|H_i \cap H_j| < L^{-4}N$ for all i, j satisfying $1 \leq i < j \leq L$. Then, for $i = 1, 2, \dots, L$, $|H_i \setminus \cup_{j \leq L, j \neq i} H_j| > M - LL^{-4}N = M - L^{-3}N$. Hence $|\cup_{i=1}^L H_i| \geq LM - L^{-2}N \geq N - L^{-2}N$. Now let us consider H_{L+1} . This set has at least $M - L^{-2}N$ elements in common with $\cup_{i=1}^L H_i$. It follows that there exists some H_i , for $1 \leq i \leq L$, such that $|H_i \cap H_{L+1}| \geq ML^{-1} - L^{-3}N \geq L^{-2}N - L^{-3}N \geq L^{-4}N$, as desired.

Now let us suppose that the statement is true for $R = 2^k$. We show that we can take

$$N_0(t, 2^{k+1}) = \overline{N}_0 = \max\{N_0(s_0(t, 2^k), 2), N_0(t, 2^k)\},$$

$$q_0(t, 2^{k+1}) = \overline{q}_0 = q_0(s_0(t, 2^k), 2) \cdot q_0(t, 2^k),$$

and
$$s_0(t, 2^{k+1}) = s_0(s_0(t, 2^k), 2).$$

Let $\mathcal{H} = \{H_1, H_2, \dots, H_{\overline{q}_0}\}$ be a collection of M -subsets of an N -set S , for $N \geq \overline{N}_0$, where $M = \lceil tN \rceil$. Let $\mathcal{H}_1, \dots, \mathcal{H}_{q_0(s_0(t, 2^k), 2)}$ be a partition of $\mathcal{H}$ into $q_0(t, 2^k)$ -sets. Since $N \geq \overline{N}_0 \geq N_0(t, 2^k)$, for each i satisfying $1 \leq i \leq q_0(s_0(t, 2^k), 2)$, there exists a set $\overline{H}_i \subset S$ of size at least $s_0(t, 2^k)N$ contained in 2^k elements of $\mathcal{H}_i$. Since $N \geq \overline{N}_0 \geq N_0(s_0(t, 2^k), 2)$, there exist different numbers

i_0, i_1 such that $\overline{H}_{i_0} \cap \overline{H}_{i_1} \geq s_0(s_0(t, 2^k), 2)N$. Hence the 2^{k+1} sets which are elements of $\mathcal{H}_{i_0} \cup \mathcal{H}_{i_1}$ have at least $s_0(s_0(t, 2^k), 2)N$ elements in common. $\square$

3.5 Proof of Lemma 3.5

Let $\mathcal{C}_1 = \{C_1, C_2, \dots, C_{2m}\}$ be an intersecting ($< m$)-family, and let $\mathcal{C}_2 = \{C'_1, C'_2, \dots, C'_r\}$ be a ($< m$)-family such that every curve in $\mathcal{C}_2$ is tangent to every curve in $\mathcal{C}_1$. The number of faces in the embedding of $\mathcal{C}_1$ is bounded as a function of m , and, since each $C'_i \in \mathcal{C}_2$ is contained in the closure of one of these faces, it is enough to show that the number of elements of $\mathcal{C}_2$ in a given face F cannot be arbitrarily large. The boundary $\partial(F)$ of F is a simple closed curve, and the intersection of $\partial(F)$ with each $C_i \in \mathcal{C}_1$ is a disjoint collection of simple closed arcs. It is not difficult to see that it is enough to prove that the number of elements in $\mathcal{C}_2$ which touch each element in a given collection $\{A_1, A_2, \dots, A_{2m}\}$ of arcs cannot be arbitrarily large, where A_i is a subarc of C_i contained in $\partial(F)$. Let the collection $\{A_1, A_2, \dots, A_{2m}\}$ be fixed. We can assume without any loss of generality that as we go along $\partial(F)$ we find these arcs in the given cyclic order. We will show the number of elements in $\mathcal{C}_2$ which touch each element in $\{A_1, \dots, A_{2m}\}$ cannot be bigger than one, and this will finish the proof.

Let $\mathcal{C}'_2$ be the subcollection of $\mathcal{C}_2$ which consists of all the curves which touch each arc in $\{A_1, \dots, A_{2m}\}$ exactly once. Since every curve in $\mathcal{C}_2$ is simple, each curve in $\mathcal{C}'_2$ touches the arcs in the given cyclic order. Let us suppose that C'_i, C'_j are different curves in $\mathcal{C}'_2$, and let $I_{i,k}, I_{j,k}$ be the arcs of C'_i and C'_j respectively which go from A_k to A_{k+1} , and let $a_{l,k}$ be the point where C'_l meets A_k for $l \in \{i, j\}$. If we go along $\partial(F)$ in one of its two possible orientations, we find these

end points in one of two possible cyclic orders, namely $(a_{i,k}, a_{j,k}, a_{i,k+1}, a_{j,k+1})$ and $(a_{i,k}, a_{j,k}, a_{j,k+1}, a_{i,k+1})$. In the first case $I_{i,k} \cap I_{j,k} \neq \emptyset$, and in the second case either both $I_{i,k-1} \cap I_{j,k}$ and $I_{i,k+1} \cap I_{j,k}$ are nonempty or both $I_{i,k} \cap I_{j,k-1}$ and $I_{i,k} \cap I_{j,k+1}$ are nonempty. Therefore for each k either $I_{i,k} \cap C'_j \neq \emptyset$ or $I_{j,k} \cap C'_i \neq \emptyset$. This implies that $|C'_i \cap C'_j| \geq m$, a contradiction. Thus there is at most one element in C'_2 . $\square$

3.6 Conclusions

It follows from Theorem 3.1 that if we denote by $i_C(n)$ the minimum number of intersections in a collection of n pairwise nondisjoint circles, then $\lim_{n \rightarrow \infty} i_C(n) / \binom{n}{2} = 2$. This can be extended to $\lim_{n \rightarrow \infty} i_{\mathcal{F}}(n) / \binom{n}{2} = 2$, where $i_{\mathcal{F}}(n)$ denotes the minimum number of intersections in a collection of n pairwise nondisjoint curves, each of which is similar to some element in a fixed finite family $\mathcal{F}$ of simple closed curves, where each curve in $\mathcal{F}$ is the boundary of a closed convex set.

The convexity condition for the elements in the family $\mathcal{F}$ above is enough to guarantee that there is a fixed number $m(\mathcal{F})$ such that any two curves similar to elements in $\mathcal{F}$ have at most $m(\mathcal{F})$ intersections, and so Theorem 3.1 can be applied. It is easy to construct a simple closed curve C similar to two curves that intersect in an infinite number of points; therefore some condition, such as the convexity condition above, must be imposed on the curves in $\mathcal{F}$ in order to guarantee that the number of intersections between two curves similar to curves in $\mathcal{F}$ cannot be arbitrarily large. This raises an interesting question, from the case where $\mathcal{F}$ consists of a single curve: Is it possible to characterize the simple closed curves C such that the maximum number of intersections between two curves

similar to C is bounded? Sufficient conditions, such as the convexity condition in the above paragraph, are easy to obtain, but it seems to be very difficult to give a complete characterization of the curves that satisfy this property.

Theorem 3.1 also gives a condition that has to be satisfied if $\lim_{n \rightarrow \infty} i(n)/\binom{n}{2} < 2$. Suppose that the limit is $L < 2$. For any integer m_0 , there is a collection $\mathcal{C}_{m_0}$ such that $i(\mathcal{C}_{m_0}) \leq L \binom{|\mathcal{C}_{m_0}|}{2}$, and such that two curves in $\mathcal{C}_{m_0}$ intersect each other in at least m_0 points. This seemingly unnatural condition reinforces our conjecture that $\lim_{n \rightarrow \infty} i(n)/\binom{n}{2} = 2$.

Lemma 3.7 seems to be an interesting result by itself, and we conjecture that it can be extended in the following natural way.

Conjecture. *Let R be an integer, and let s, t be such that $0 < s < t < 1$. Then there exist integers $N_0(s, t, R)$ and $q_0(s, t, R)$ such that, for any $N \geq N_0$ and $q \geq q_0$, any q -collection of $\lceil tN \rceil$ -subsets of an N -set has R elements whose intersection contains at least sN elements.*

As in Lemma 3.7, the critical step seems to be the case $R = 2$.

Chapter 4

Drawings of $C_m \times C_n$ with one Disjoint Family

4.1 Introduction

In Chapter 2 we presented an important result regarding the asymptotic behaviour of the crossing numbers of certain drawings of $C_m \times C_n$, namely those drawings in which the number of intersections between any of the two m n -cycles is bounded by an arbitrary integer M . Our goal in this chapter is to establish the minimum number of crossings in a drawing of $C_m \times C_n$ in which one of the two main families of cycles is a pairwise disjoint collection.

Theorem 4.1. *Let m, n be such that $n \geq m \geq 3$. Then every drawing of $C_m \times C_n$ such that either the n m -cycles are pairwise disjoint or the m n -cycles are pairwise disjoint has at least $(m - 2)n$ crossings.*

As mentioned in Chapter 2, for each m, n satisfying $n \geq m \geq 3$ it is easy to give drawings of $C_m \times C_n$ with $(m - 2)n$ crossings, and Harary et al. [HKS] conjectured that this is the actual crossing number. For each m, n satisfying $n > m$, all the drawings known of $C_m \times C_n$ with exactly $(m - 2)n$ crossings have the n m -cycles pairwise disjoint. Similarly, if $m \geq 5$, then in all such drawings

$m - 4$ of the n -cycles are pairwise disjoint. Thus, Theorem 4.1 represents an interesting complement to our knowledge of $cr(C_m \times C_n)$, since it establishes that no drawing in which either the n m -cycles are pairwise disjoint or the m n -cycles are pairwise disjoint has less than $(m - 2)n$ crossings.

Besides its value as an independent result, Theorem 4.1 is very important in connection with the calculation of the exact crossing number of $C_m \times C_n$ for certain values of m and n . As mentioned in Chapter 1, the crossing number of $C_m \times C_n$ is known exactly for every $n \geq m$, for each m satisfying $3 \leq m \leq 5$. The cases $m = 3, 4$, and 5 were solved in [RB], [BR], and [KRS] respectively, using an induction with base cases $cr(C_3 \times C_3) = 3$, $cr(C_4 \times C_4) = 8$, and $cr(C_5 \times C_5) = 15$ respectively. In each of [RB], [BR], and [KRS], an important part of the inductive step is to deal with the case where the n m -cycles are pairwise disjoint. The same is true in Chapter 5 in this thesis, where we establish that $cr(C_6 \times C_n) = 4n$, as conjectured. Roughly speaking, half of the difficulty in these inductions lies in the case where the n m -cycles are pairwise disjoint, and the other half lies in the case where some pair of m -cycles intersect.

This chapter is organized as follows. In Section 4.2 we state Lemmas 4.2 and 4.3 and Proposition 4.4. Lemmas 4.2 and 4.3 imply Theorem 4.1, and Proposition 4.4 is the main ingredient of the proofs of Lemmas 4.2 and 4.3. In Section 4.3 we prove Lemmas 4.2 and 4.3, and in Section 4.4 we prove Proposition 4.4. Section 4.5 contains some final remarks.

4.2 Decomposition of Theorem 4.1

To help comprehension, we colour the edges of $C_m \times C_n$ as follows: colour the m n -cycles blue and the n m -cycles red. Thus, if $m \neq n$, the graphs $C_m \times C_n$ and $C_n \times C_m$ are coloured in a different way. That is, even though these graphs are isomorphic, the relative order of m and n does matter for our present discussion. We remark that although most of the time we deal with graphs $C_p \times C_q$ where $q \geq p$, in general we do not assume that this inequality holds.

A drawing $\mathcal{D}$ of $C_m \times C_n$ is a *red-disjoint drawing* if the red cycles are pairwise disjoint in $\mathcal{D}$. Similarly, $\mathcal{D}$ is a *blue-disjoint drawing* if the blue cycles are pairwise disjoint. For brevity, we refer to red-disjoint drawings as *rd-drawings* and to blue-disjoint drawings as *bd-drawings*. The minimum number of crossings in an *rd-drawing* of $C_m \times C_n$ is denoted by $cr_{rd}(C_m \times C_n)$, and the minimum number of crossings in a *bd-drawing* of $C_m \times C_n$ is denoted by $cr_{bd}(C_m \times C_n)$. Using this notation, Theorem 4.1 can be written in the following way.

Theorem 4.1. *Let m, n be such that $n \geq m \geq 3$. Then $cr_{rd}(C_m \times C_n) \geq (m-2)n$ and $cr_{bd}(C_m \times C_n) \geq (m-2)n$.*

We prove the two inequalities of Theorem 4.1 separately, as Lemmas 4.2 and 4.3.

Lemma 4.2. *Let m, n be such that $n \geq m \geq 3$. Then $cr_{bd}(C_m \times C_n) \geq (m-2)n$.*

Lemma 4.3. *Let m, n be such that $n \geq m \geq 3$. Then $cr_{rd}(C_m \times C_n) \geq (m-2)n$.*

The following result is the heart of the proofs of Lemmas 4.2 and 4.3. In a

given drawing of $C_m \times C_n$, we say that a cycle C *separates* the cycles C' and C'' if C' and C'' are contained in different components of $\mathbb{R}^2 \setminus C$.

Proposition 4.4. *Suppose that $\min\{p, q\} \geq 3$. Let $\mathcal{D}$ be a drawing of $C_p \times C_q$, such that the q p -cycles are pairwise disjoint and no p -cycle separates two p -cycles. Then $cr(\mathcal{D}) \geq (p-2)q$.*

We remark that in Proposition 4.4 we do not assume that $p \geq q$, as we often do when dealing with the graph $C_p \times C_q$.

The proof of Proposition 4.4 is presented in Section 4.4, and in the next section we use this result to prove Lemmas 4.2 and 4.3.

4.3 Proofs of Lemmas 4.2 and 4.3

We now prove Lemmas 4.2 and 4.3.

Proof of Lemma 4.2. We prove this lemma by induction on $m+n$, the base case being $m = n = 3$. To see that $cr_{bd}(C_3 \times C_3) \geq 3$, we note that $cr(C_3 \times C_3) = 3$ (see [HKS] or [RT1]), and that $cr_{bd}(C_p \times C_q) \geq cr(C_p \times C_q)$ for all integers p, q .

Let m, n be such that $n \geq m \geq 3$, where m and n are not both equal to three, and suppose that $cr_{bd}(C_{m'} \times C_{n'}) \geq (m'-2)n'$ for all m', n' satisfying $n' \geq m' \geq 3$ and $m' + n' < m + n$. Let $\mathcal{D}$ be a bd -drawing of $C_m \times C_n$. To complete the proof, we show that $cr(\mathcal{D}) \geq (m-2)n$.

First let us suppose that no blue cycle separates two blue cycles in $\mathcal{D}$. In this case we can apply Proposition 4.4 with $p = n$ and $q = m$ to obtain that $cr(\mathcal{D}) \geq (n-2)m$. Now since $n \geq m$, $(n-2)m \geq (m-2)n$, and so $cr(\mathcal{D}) \geq (m-2)n$.

Now we consider the case in which a blue cycle B separates two blue cycles in $\mathcal{D}$. In this case B crosses every red cycle, and so B has at least n crossings. Thus, if $m = 3$, then $cr(\mathcal{D}) \geq n$, as required. Now suppose that $m > 3$. Let $\mathcal{D}'$ be the drawing of $C_{m-1} \times C_n$ obtained by removing B from $\mathcal{D}$. Since B has at least n crossings, $cr(\mathcal{D}) \geq cr(\mathcal{D}') + n$. By the induction hypothesis, $cr(\mathcal{D}') \geq (m-3)n$, and so $cr(\mathcal{D}) \geq (m-3)n + n = (m-2)n$. $\square$

Proof of Lemma 4.3. We note that if $n = m$, then the blue cycles and the red cycles are exchangeable, and so it follows from Lemma 4.2 that $cr_{rd}(C_m \times C_m) \geq (m-2)m$ for every $m \geq 3$. For each fixed m , we prove that $cr_{rd}(C_m \times C_n) \geq (m-2)n$ by induction on n , the base case being $cr_{rd}(C_m \times C_m) \geq (m-2)m$. Thus, we let $n > m$, and assume that $cr_{rd}(C_m \times C_{n-1}) \geq (m-2)(n-1)$. Let $\mathcal{D}$ be an rd -drawing of $C_m \times C_n$. We finish the proof by showing that $cr(\mathcal{D}) \geq (m-2)n$.

First we note if some red cycle R separates two red cycles, then every blue cycle crosses R , and so Lemma 4.3 follows easily from the induction hypothesis. Therefore we may assume that no red cycle separates two red cycles. Then Proposition 4.4 applies with $p = m$ and $q = n$, and so we obtain $cr(\mathcal{D}) \geq (m-2)n$. $\square$

4.4 Proof of Proposition 4.4

We now prove Proposition 4.4. The proof of this result is largely based in the proof of Lemma 3 in [KRS], which establishes Proposition 4.4 for the special case $q \geq p$, $p = 5$.

Proof of Proposition 4.4. The red p -cycles are naturally cyclically ordered as $R_1, R_2, \dots, R_q$, so that each of the blue q -cycles has an edge that joins a vertex

of R_{i-1} with a vertex of R_i , for $i = 1, 2, \dots, q$, where the indices are read modulo q .

Let H_i denote the subgraph of $C_p \times C_q$ induced by the vertices in $R_{i-1} \cup R_i$, for each $i = 1, 2, \dots, q$. Thus, H_i has $2p$ red edges and p blue edges, every red edge is in two of the H_i , and every blue edge is in exactly one of the H_i .

As in [KRS], we define the *force* $f(H_i)$ of H_i as the total number of crossings of the following types:

- (1) a crossing of a blue edge in H_i with an edge in $R_i \cup R_{i+1}$;
- (2) a crossing of a blue edge in H_{i-1} with an edge in H_i ;
- (3) a crossing of a blue edge in H_i with a blue edge in H_i ; and
- (4) a self-intersection of R_i .

It is not difficult to check that no crossing counted in $f(H_i)$ is counted in $f(H_j)$ if $i \neq j$. Thus it follows that the number $cr(\mathcal{D})$ of crossings in $\mathcal{D}$ is at least $\sum_{1 \leq i \leq q} f(H_i)$. Therefore, in order to complete the proof of Proposition 4.4, it suffices to show that, for each $i = 1, 2, \dots, q$, $f(H_i) \geq p - 2$. It is a trivial observation that by the symmetry it suffices to show that $f(H_1) \geq p - 2$.

Now let $I = I_{H_1, R_2}$ be the collection of blue edges of H_1 that intersect R_2 . Let $f_I(H_1)$ be the total number of crossings that contribute to $f(H_1)$ that involve a blue edge of H_1 and a red edge of R_2 , and let $f_{\bar{I}}(H_1)$ be the total number of crossings that contribute to $f(H_1)$ that do not contribute to $f_I(H_1)$.

Thus, $f(H_1) = f_I(H_1) + f_{\bar{I}}(H_1)$, and so it suffices to show that $f_I(H_1) + f_{\bar{I}}(H_1) \geq p - 2$. We claim that the statements in Claims 4.5 and 4.6 below combine to prove this last inequality. For if $|I_{H_1, R_2}| \geq p - 2$, then, by Claim 4.5, $f_I(H_1) \geq 2(p - 2) > p - 2$, since $p \geq 3$. On the other hand, if $|I_{H_1, R_2}| < p - 2$,

then by Claims 4.5 and 4.6 we obtain $f_I(H_1) + f_{\bar{I}}(H_1) \geq p - 2 + |I_{H_1, R_2}| \geq p - 2$.

Thus Claims 4.5 and 4.6 complete the proof of Proposition 4.4. $\square$

Claim 4.5. $f_I(H_1) \geq 2|I_{H_1, R_2}|$.

Proof. Since no two of R_q, R_1 , and R_2 intersect and no one separates the other two, each edge in I_{H_1, R_2} crosses R_2 at least twice. $\square$

Claim 4.6. Suppose that $|I_{H_1, R_2}| < p - 2$. Then $f_{\bar{I}}(H_1) \geq (p - 2) - |I_{H_1, R_2}|$.

Proof. Let H'_1 denote the subgraph $H_1 - I_{H_1, R_2}$. Note that the cycles R_n and R_1 are still contained in H'_1 , but these cycles may contain some vertices of degree two (subdivisions) in H'_1 . Since any such subdivisions are irrelevant to the discussion, we eliminate (i.e., unsubdivide) them from H'_1 , and call H''_1 the obtained graph. The cycles R_q and R_1 induce naturally two cycles R''_q and R''_1 in H''_1 . We note that each of R''_q and R''_1 has exactly $p - |I_{H_1, R_2}| \geq 3$ vertices. As in H_1 , we colour the edges in R''_q and R''_1 red, and we colour all the other edges in H''_1 blue.

Now let $e = uv$ be any blue edge of H''_1 , with $u \in V(R''_q)$ and $v \in V(R''_1)$. As we go along e from v to u , let a be that first point in e that is in R''_q (u is such a point), and let b be the last point in e before a that is in R''_1 (v is such a point). Let τe denote the segment of e from a to b . It is not difficult to see that if $b \neq v$, then e crosses R''_1 at b . Let $\tau H''_1$ denote the subset of the plane consisting of R''_n, R''_1 , and all the τe , for e a blue edge of H''_1 .

We note that there are exactly $p - |I_{H_1, R_2}|$ blue edges going from R''_1 to R_2 . Since R_2 is disjoint from $\tau H''_1$ and is not separated by either R''_q or R''_1 from the other, there is a region F of $\tau H''_1$ that contains R_2 and whose boundary contains some of the blue part of $\tau H''_1$.

Let us suppose that there are exactly k vertices on R_1'' incident with F . Then there must be at least $(p - |I_{H_1, R_2}|) - k$ crossings of blue edges of H_2 with the boundary of this region. Clearly, these crossings contribute to $f_{\overline{T}}(H_1)$. Thus, $f_{\overline{T}}(H_1) \geq (p - |I_{H_1, R_2}|) - k$. Consequently, if $k \leq 2$, then $f_{\overline{T}}(H_1) \geq (p - 2) - |I_{H_1, R_2}|$, as required.

We now consider the possibility that $k \geq 3$. Since we have already found $(p - |I_{H_1, R_2}|) - k$ crossings that contribute to $f_{\overline{T}}(H_1)$, it follows that we must find an additional $((p - 2) - |I_{H_1, R_2}|) - ((p - |I_{H_1, R_2}|) - k) = k - 2$ crossings that contribute to $f_{\overline{T}}(H_1)$. We note that the boundary of F is a simple closed curve whose intersection with R_1'' is a connected segment.

Let s_1 be the point where we start as we go along the portion P of the boundary of F in R_1'' . As we go along P , we traverse edges and possibly self-intersections of R_1'' , and we go through vertices $w_1, w_2, \dots, w_l$ of R_1'' , in this order, and finally finish at a point s_2 . We remark that s_1 and s_2 need not be vertices of R_1'' , but in any case $l \geq k - 2$. The vertices $w_1, \dots, w_l$ are in the interior of the segment of R_1'' that is in the boundary of F .

For each $i = 1, \dots, l$, let e_i be the blue edge from R_q'' to w_i . If $b_i = w_i$, then the segment τe_i goes from w_i to R_q'' , and so τe_i must go through the region F . But this is impossible, since in such a case we would have incorrectly identified F . Therefore $b_i \neq w_i$, and so e_i crosses R_1'' at b_i .

Therefore, there are an additional $l \geq k - 2$ crossings that contribute to $f_{\overline{T}}(H_1)$, and so $f_{\overline{T}}(H_1) \geq ((p - |I_{H_1, R_2}|) - k) + (k - 2) = (p - 2) - |I_{H_1, R_2}|$, as required. $\square$

4.5 Conclusions

As mentioned in Section 4.1, dealing with the case where the n m -cycles are pairwise disjoint is a substantial part of the argument to show inductively that $C_m \times C_n = (m - 2)n$, for $m = 3, 4$, and 5 . This is also true for the proof that $C_6 \times C_n = 4n$, established in Chapter 5. Anderson, Richter, and Rodney [ARRo2] have shown that $cr(C_7 \times C_7) = 35$ (as conjectured), and so it seems plausible that if a proof of the conjecture $cr(C_7 \times C_n) = 5n$ ever comes up, it will be a major relief that the case where the n 7 -cycles are pairwise disjoint has been dealt with in Theorem 4.1. Of course, a similar observation holds for a potential proof of the conjecture $cr(C_8 \times C_n) = 6n$, and of the general conjecture $cr(C_m \times C_n) = (m - 2)n$.

Chapter 5

The Crossing Number of $C_6 \times C_n$

5.1 Introduction

As mentioned in Section 1.4, Harary et al. [HKS] conjectured that the crossing number of $C_m \times C_n$ is $(m-2)n$, for all m, n satisfying $3 \leq m \leq n$. This conjecture has been verified for the cases $m = 3, 4$, and 5 , for each $n \geq m$ [RB], [BR], [DR], [RT1], [KRS]. Our goal in this chapter is to show that this conjecture also holds for the case $m = 6$.

Theorem 5.1. *The crossing number of $C_6 \times C_n$ is $4n$, for every $n \geq 6$.*

As we mentioned in Chapter 1, there are drawings of $C_m \times C_n$ with exactly $(m-2)n$ crossings, for every m, n satisfying $3 \leq m \leq n$. Thus, the difficult part of the Theorem 5.1 is the inequality $cr(C_6 \times C_n) \geq 4n$. We prove this by induction on n , as in [RB], [BR], and [KRS]. The strategy is as follows. The base case is $n = 6$, proved in [ARRol]. Let $\mathcal{D}$ be an optimal drawing of $C_6 \times C_n$, where $n \geq 7$, and suppose that the statement of Theorem 5.1 holds for $C_6 \times C_{n-1}$. We show that if two 6-cycles cross each other in $\mathcal{D}$, then there is an (not necessarily different from $\mathcal{D}$) optimal drawing $\mathcal{D}'$ of $C_6 \times C_n$ in which a 6-cycle is crossed at least four times; thus in this case the inequality $cr(C_6 \times C_n) \geq 4n$ follows easily from the

induction hypothesis. On the other hand, if the n 6-cycles are pairwise disjoint, then it follows from Theorem 4.1 that there are at least $4n$ crossings in $\mathcal{D}$. Thus, in order to deal with the inductive step we only need to analyze the case where two 6-cycles cross each other in $\mathcal{D}$.

This chapter is organized as follows. In Section 5.2 we show that Theorem 5.1 follows from Theorem 5.2, which corresponds to the unsolved case in the inductive step mentioned in the last paragraph. In Section 5.3 we outline the proof of Theorem 5.2. In Section 5.4 we establish a technical lemma used in the proof of Theorem 5.2, and in Section 5.5 we prove Theorem 5.2 for one particular case. Section 5.6 contains the proof of Theorem 5.2. Section 5.7 contains some final remarks.

5.2 Decomposition of Theorem 5.1

As in [KRS], it is convenient for our subsequent work to colour the edges in $C_6 \times C_n$ red and blue, so that the edges of the n 6-cycles are red and the edges of the 6 n -cycles are blue. We remark that it follows from the definition of $C_6 \times C_n$ that no two distinct cycles of the same colour have a vertex in common, and that every two cycles of distinct colour have exactly one vertex in common.

Our proof of Theorem 5.1 is based on observations about the number of crossings of the edges of certain cycles. We often make no distinction between a cycle and its corresponding closed curve in a drawing of $C_6 \times C_n$, if no confusion arises. However, if we say that a cycle C is *crossed* in a drawing $\mathcal{D}$ of $C_6 \times C_n$, it must be understood that an *edge* of C is crossed in $\mathcal{D}$. If $\mathcal{D}$ is the only drawing considered in a discussion, we omit reference to $\mathcal{D}$ and simply speak of the crossings

of a cycle or of an edge.

An *arc* A is a homeomorph of $[0, 1]$ contained in a drawing. As with cycles, we say that an arc A in a drawing $\mathcal{D}$ is *crossed* at the point r if an edge crosses A at r , and omit reference to $\mathcal{D}$ if no confusion arises. If p and q are the end points of A , $A \setminus \{p, q\}$ is the *interior* of A .

An optimal drawing of $C_6 \times C_n$ is *really optimal* if the number of crossings among red cycles is least among all optimal drawings of $C_6 \times C_n$.

We claim that Theorem 5.1 follows from Theorem 4.1 and the following result.

Theorem 5.2. *Let $\mathcal{D}$ be a really optimal drawing of $C_6 \times C_n$. Suppose that two red cycles cross each other in $\mathcal{D}$. Then there is an optimal drawing of $C_6 \times C_n$ in which some red cycle has at least four crossings.*

Proof of Theorem 5.1. As mentioned in Chapter 1, there are drawings of $C_6 \times C_n$ with exactly $4n$ crossings. Therefore $cr(C_6 \times C_n) \leq 4n$. We prove the reverse inequality by induction on n . The base case, $cr(C_6 \times C_6) \geq 24$, is proved in [ARR01]. Let $n \geq 7$ and suppose that $cr(C_6 \times C_{n-1}) \geq 4(n-1)$. Let $\mathcal{D}$ be a really optimal drawing of $C_6 \times C_n$. If no two red cycles cross each other, then the number of crossings $cr(\mathcal{D})$ in $\mathcal{D}$ is at least $4n$ by Theorem 4.1. On the other hand, by Theorem 5.2, if two red cycles cross each other, then a red cycle R has at least four crossings in an optimal drawing $\mathcal{D}'$. By the induction hypothesis, the drawing $\mathcal{D}''$ of $C_6 \times C_{n-1}$ obtained by deleting R from $\mathcal{D}'$ has at least $4(n-1)$ crossings. Hence $cr(C_6 \times C_n) = cr(\mathcal{D}') \geq cr(\mathcal{D}'') + 4 \geq 4(n-1) + 4 = 4n$. $\square$

5.3 Outline of the proof of Theorem 5.2

The basic idea for the proof of Theorem 5.2 is a generalization of an argument used in [KRS]. In that paper, Klešć et al. show that if two 5-cycles cross each other in an optimal drawing of $C_5 \times C_n$, then one of these 5-cycles has at least three crossings. The argument is as follows. Suppose that the 5-cycles R_1 and R_2 cross each other in an optimal drawing of $C_5 \times C_n$. Since no two 5-cycles have a vertex in common, R_1 and R_2 cross in at least two points. Hence, it suffices to show that one of R_1 and R_2 has an additional crossing. The existence of this additional crossing is easily demonstrated, since if each of these 5-cycles had only these two crossings then we could remove the crossings, contradicting the optimality of the drawing.

We remark that this argument can be used in the case $m = 6$ to show that if two 6-cycles R_1 and R_2 cross each other in $\mathcal{D}$, then one of R_1 and R_2 has at least *three* crossings in $\mathcal{D}$. Unfortunately, this is not enough, since to prove Theorem 5.2 we need to show that some 6-cycle has at least *four* crossings in some optimal drawing.

In our case, we show that if two 6-cycles cross each other in $\mathcal{D}$, then *some* 6-cycle has at least four crossings in an (not necessarily different from $\mathcal{D}$) optimal drawing $\mathcal{D}'$. The essence of the proof of Theorem 5.2 is the construction of the drawing $\mathcal{D}'$. This drawing is obtained in the following way.

First we obtain some detailed information of what the drawing $\mathcal{D}$ must look like, under the assumption that two 6-cycles R_1 and R_2 cross each other and neither of them has at least four crossings. Based on Lemma 5.3, Proposition 5.4.

How two red cycles R_1 and R_2 are drawn in a drawing of $C_6 \times C_n$, if neither of them has at least four crossings.

Figure 5.1

(a) The original drawing D .

(b) The drawing D' obtained by re-drawing e_{u_1, u_2}

Figure 5.2

and Claims 5.5 and 5.6, we show that we can assume without loss of generality that R_1 and R_2 are drawn as in Figure 5.1. In Figure 5.1, one of f_1 and g_1 is a crossing and the other one is a vertex, one of f_2 and g_2 is a crossing and the other one is a vertex, and no edges other than those shown intersect the open discs bounded by R_1 and R_2 . The cycles R_1 and R_2 cross each other in the points p and q , and p and q divide each R_i into two arcs A_i and B_i , as shown in Figure 5.1.

The next step consists of showing that there are adjacent vertices u_1 in B_1 and u_2 in B_2 , that do not appear in Figure 5.1, such that the edge e_{u_1, u_2} that joins u_1 and u_2 can be re-drawn in such a way that (i) e_{u_1, u_2} crosses each of R_1 and R_2 , and (ii) the drawing $\mathcal{D}'$ obtained by re-drawing e_{u_1, u_2} is also optimal. The re-drawing of e_{u_1, u_2} to obtain $\mathcal{D}'$ is illustrated in Figure 5.2.

Since each R_i had three crossings in $\mathcal{D}$, it follows that each R_i has four crossings in $\mathcal{D}'$. Hence some red cycle has at least four crossings in an optimal drawing $\mathcal{D}'$, as required.

5.4 Red cycles with fewer than four crossings

Our first step towards the proof of Theorem 5.2 is a characterization of the drawings where a given red cycle has fewer than four crossings.

Lemma 5.3. *Let $\mathcal{D}$ be a drawing of $C_6 \times C_n$. Let R be a red cycle with fewer than four crossings. Suppose that there are different components C_1 and C_2 of $\mathbb{R}^2 \setminus R$ such that each of C_1 and C_2 contains at least one vertex. Then the following statements hold.*

- (i) One of $\mathcal{C}_1$ and $\mathcal{C}_2$ contains exactly one vertex v .
- (ii) Both red edges incident with v cross R .
- (iii) One blue edge incident with v crosses R , and the other blue edge incident with v is incident with a vertex in R .
- (iv) R has three crossings, none of which is a self-crossing.
- (v) The component of $\mathbf{R}^2 \setminus R$ that contains v is intersected only by v and by the four edges incident with v .

Lemma 5.3 is illustrated in Figure 5.3, for the case where the unbounded component of $\mathbf{R}^2 \setminus R$ is the component that contains more than one vertex.

Proof. We divide this proof in two steps. In Step 1 we prove (i), and in Step 2 we prove (ii), (iii), (iv), and (v).

STEP 1. Seeking a contradiction, suppose that each $\mathcal{C}_i$ contains two different vertices u_i and v_i . Each vertex is in one red cycle and one blue cycle, and no two vertices have more than one monochromatic cycle in common. Therefore, for each i , at least three different monochromatic cycles $\{D_{i,1}, D_{i,2}, D_{i,3}\}$ intersect $\mathcal{C}_i$, and at least one of these cycles, say $D_{i,1}$, is red.

We claim that there is an i such that each of $D_{i,1}, D_{i,2}$, and $D_{i,3}$, crosses R . For suppose that there are j, k such that all the edges in $D_{1,j}$ are contained in $\mathcal{C}_1$ and all the edges in $D_{2,k}$ are contained in $\mathcal{C}_2$. Clearly $D_{1,j}$ and $D_{2,k}$ are of the same colour, since every two cycles of different colours have a common vertex. It follows that, possibly with the exception of R , each cycle with colour different from that of $D_{1,j}$ and $D_{2,k}$ crosses R , since every such cycle has a common vertex with each of $D_{1,j}$ and $D_{2,k}$. Since there are at least six cycles of each colour, this

The points p_1 , p_2 , and p_3 are crossings, and v and w are vertices. The only edges that intersect the open disc bounded by R are the edges e_1 , e_2 , e_3 , and e_w .

Figure 5.3

contradicts the hypothesis that R is crossed at most three times.

Thus we can assume without any loss of generality that each of $D_{1,1}$, $D_{1,2}$, and $D_{1,3}$, crosses R . Since $D_{1,1}$ is red, it has no vertices in common with R . Thus $D_{1,1}$ crosses R at least twice. Since both $D_{1,2}$ and $D_{1,3}$ cross R , it follows that R has at least four crossings in total. This contradicts the assumption that R has fewer than four crossings. Therefore one of C_1 and C_2 contains exactly one vertex.

STEP 2. We can assume without loss of generality that C_1 is the component in $\{C_1, C_2\}$ that contains exactly one vertex v . Since there are no vertices other than v in C_1 , both red edges incident with v must cross R . This proves (ii).

By (ii) R crosses two red edges, and since by hypothesis R has fewer than four crossings in total, it follows that R crosses at most one blue edge. On the other hand, one of the blue edges incident with v must cross R , since v cannot be adjacent to more than one vertex in R . Hence (iii) follows.

The statement in (iv) follows immediately from (ii) and (iii).

Finally, we prove (v). It is easy to see that if C_1 is intersected by an edge e not incident with v , then R is crossed by e . It follows that no such edge e can intersect C_1 , since R has three crossings with edges incident with v , and by assumption R has fewer than four crossings. Therefore (v) follows. $\square$

5.5 Self-crossing red cycles

Our aim in this section is to prove Theorem 5.2 for the case where one of R_1 and R_2 has a self-crossing.

Proposition 5.4. *Let $\mathcal{D}$ be an optimal drawing of $C_6 \times C_n$. Suppose that the red cycles R_1 and R_2 cross each other in $\mathcal{D}$, and suppose that one of R_1 and R_2 crosses itself in $\mathcal{D}$. Then one of R_1 and R_2 has at least four crossings in $\mathcal{D}$.*

Proof. By symmetry we can assume that R_1 has a self-crossing. Each pair of red cycles cross each other in an even number of points. Therefore if R_1 and R_2 cross each other in more than two points then we are done, and so we assume that R_1 and R_2 cross each other in exactly two points p and q . If one of R_1 and R_2 self-crosses more than once then it has at least four crossings in total, and so we also assume that neither R_1 nor R_2 crosses itself more than once.

The points p and q divide R_i into two curves A_i and B_i , for each i . One of A_1 and B_1 , say A_1 , is an arc, and the other one has a self-crossing. The curve B_1 contains at least two vertices, since otherwise the good-drawing condition for $\mathcal{D}$ would be violated (see Figure 5.4).

Since the interiors of A_2 and B_2 are contained in different components of $\mathbf{R}^2 \setminus R_1$, it follows from Statement (v) of Lemma 5.3 that either one of A_2 and B_2 does not contain any vertex, or R_1 has at least four crossings. Since in the latter case we are done, we assume that A_2 does not contain any vertex.

Suppose that A_1 contains more than one vertex. Since B_1 contains at least two vertices, and A_1 and B_1 are contained in different components of $\mathbf{R}^2 \setminus R_2$, it

If B_1 contains fewer than two vertices, then we can remove the self-crossing r of R_1 to obtain a drawing D' with fewer crossings than D .

Figure 5.4

follows from Statement (i) of Lemma 5.3 that R_2 has at least four crossings. Since in this case we are done, we assume that A_1 contains at most one vertex. If A_1 does not contain any vertex, then A_1 and A_2 are contained in edges that cross each other more than once. Since this violates the good-drawing property of $\mathcal{D}$, we conclude that A_1 contains exactly one vertex v_1 .

We modify $\mathcal{D}$ to obtain a drawing $\mathcal{D}''$ of $C_6 \times C_n$ in the following way. Let p'' and q'' be points in B_2 , contained in small neighbourhoods of p and q respectively. Substitute A_2 by an arc A_2'' very close to A_1 , so that the end points of A_2'' are p'' and q'' . This is illustrated in Figure 5.5. It is easy to see that we can draw A_2'' close enough to A_1 , so that an edge e crosses A_2'' only if either e crosses A_1 or e is a blue edge incident with v_1 . Let $\mathcal{D}''$ be the drawing thus obtained. Clearly, R_1 and R_2 do not cross each other in $\mathcal{D}''$.

The arc A_2'' must be crossed at least twice, since otherwise $\mathcal{D}''$ would have fewer crossings than $\mathcal{D}$, contradicting the optimality of $\mathcal{D}$. If A_2'' is crossed by an edge that also crosses A_1 , then R_1 has at least four crossings in total. Since in this case we are done, we assume that A_2'' is crossed only by the two blue edges incident with v_1 . Then $\mathcal{D}$ and $\mathcal{D}''$ have the same number of crossings. On the other hand, $\mathcal{D}''$ has fewer red-red crossings than $\mathcal{D}$, since the two crossings of A_2'' are blue-red crossings. This contradicts the real-optimality of $\mathcal{D}$, since $\mathcal{D}''$ is also optimal. $\square$

How to modify D to obtain the drawing D' .

Figure 5.5

5.6 Proof of Theorem 5.2

We prove Theorem 5.2 in two steps. The first step consists of obtaining a detailed picture of what the drawing $\mathcal{D}$ must look like if neither of R_1 and R_2 has at least four crossings. Indeed, we show that we can assume without loss of generality that the situation is as in Figure 5.1. In the second step of the proof we show, using this information about $\mathcal{D}$, that we can guarantee the existence of an optimal drawing $\mathcal{D}'$ where some red cycle has at least four crossings. The drawing $\mathcal{D}'$ is obtained, as explained in Section 5.3, by re-drawing an edge that joins a vertex in R_1 to a vertex in R_2 .

Proof of Theorem 5.2. Let $\mathcal{D}$ be a really optimal drawing of $C_6 \times C_n$, and let R_1, R_2 be red cycles that cross each other in $\mathcal{D}$. Let us suppose that both R_1 and R_2 are crossed fewer than four times in $\mathcal{D}$. As explained above, this proof is divided in two steps.

STEP 1. In this step we obtain a detailed picture of the properties of the drawing $\mathcal{D}$ that follow from the assumption that both R_1 and R_2 have fewer than four crossings.

By Proposition 5.4, neither of R_1 and R_2 has a self-crossing. Since R_1 and R_2 cross each other in an even number of points, it follows that they cross each other in exactly two points p and q . Let A_i and B_i be the arcs with end points p and q contained in R_i , for each $i \in \{1, 2\}$.

Claim 5.5. *Each of A_1, A_2, B_1 , and B_2 contains at least one vertex.*

Proof. At least one of A_i and B_i contains a vertex for each i , since each R_i

contains six vertices. If two arcs in $\{A_1, A_2, B_1, B_2\}$ contain no vertices, then the edges that contain these two arcs cross each other more than once. Since this would violate the good-drawing condition for $\mathcal{D}$, it follows that at most one of A_1, A_2, B_1 , and B_2 contains no vertices.

Suppose that one of these four arcs, say A_1 , contains no vertices. We show that this implies that there is an optimal drawing $\mathcal{D}''$ of $C_6 \times C_n$ with fewer red-red crossings than $\mathcal{D}$, contradicting the real-optimality of $\mathcal{D}$.

As argued above, A_1 is the only arc in $\{A_1, A_2, B_1, B_2\}$ that contains no vertices, and so each of A_2 and B_2 contains at least one vertex. Since the interiors of A_2 and B_2 are contained in distinct components of $\mathbf{R}^2 \setminus R_1$, it follows from Statement (i) in Lemma 5.3 that one of A_2 and B_2 , say A_2 , contains exactly one vertex v_2 .

Now we obtain from $\mathcal{D}$ a drawing $\mathcal{D}''$ in the following way. Let p'' and q'' be points in B_1 very close to p and q respectively. Substitute A_1 by an arc A_1'' very close to A_2 , so that p'' and q'' are the ends of A_1'' . This is illustrated in Figure 5.6. It is easy to see that we can draw A_1'' close enough to A_2 , so that an edge e crosses A_1'' only if e crosses A_2 or if it is incident with v_2 . Let $\mathcal{D}''$ be the drawing thus obtained. It follows from the optimality of $\mathcal{D}$ that A_1'' is crossed at least twice.

Let C_{v_2} be the component of $\mathbf{R}^2 \setminus R_1$ (in $\mathcal{D}$) that contains v_2 . It follows from Statement (ii) in Lemma 5.3 that the only edges that intersect C_{v_2} are the four edges incident with v_2 . Since every edge that crosses A_2 intersects C_{v_2} , it follows that A_2 is not crossed by any edge. Therefore, since A_1'' is crossed at least twice, it follows that A_1'' is crossed once by each blue edge incident with v_2 . Thus $\mathcal{D}''$ is also optimal. However, $\mathcal{D}''$ has fewer red-red crossings than $\mathcal{D}$, contradicting the real-optimality of $\mathcal{D}$. $\square$

How to obtain a drawing D'' with fewer crossings than D , if A_1 contains no vertices.

Figure 5.6

By Claim 5.5, each of A_1 and B_1 contains at least one vertex. Since the interiors of A_1 and B_1 are contained in different components of $\mathbf{R}^2 \setminus R_2$, it follows from Statement (i) in Lemma 5.3 that one of these arcs, say A_1 , contains exactly one vertex v_1 . By an analogous argument we can assume that A_2 contains exactly one vertex v_2 .

Let $\mathcal{D}_{R_1 \cup R_2}$ be the drawing of R_1 and R_2 induced by $\mathcal{D}$. We denote by F_A, F_B, F_{12} , and F_{21} the faces in $\mathcal{D}_{R_1 \cup R_2}$ bounded by the pairs of arcs $\{A_1, A_2\}$, $\{B_1, B_2\}$, $\{A_1, B_2\}$, and $\{A_2, B_1\}$, respectively. By using a standard Riemann stereographic projection argument we can assume that F_B is the unbounded face of $\mathcal{D}_{R_1 \cup R_2}$, as shown in Figure 5.7.

Let C_{v_1} be the component of $\mathbf{R}^2 \setminus R_2$ that contains v_1 . Since by assumption R_2 has fewer than four crossings, it follows from Statement (i) in Lemma 5.3 that v_1 is the only vertex contained in C_{v_1} . An analogous argument shows that v_2 is the only vertex contained in C_{v_2} , where C_{v_2} denotes the component of $\mathbf{R}^2 \setminus R_1$ that contains v_2 . We note that C_{v_1} consists of the union of the faces F_A and F_{12} with the interior of the arc A_1 . Similarly, C_{v_2} consists of the union of F_A and F_{21} with the interior of the arc A_2 .

We claim that F_A is not intersected by any edge.

Claim 5.6. *No edge intersects F_A .*

Proof. We show that if an edge intersects F_A , then we can modify $\mathcal{D}$ to obtain an optimal drawing $\mathcal{D}'$ with fewer red–red crossings than $\mathcal{D}$, contradicting the real–optimality of $\mathcal{D}$.

Suppose that F_A is intersected by some edge. Since F_A is contained in both C_{v_2} and C_{v_1} , it follows from Statement (i) in Lemma 5.3 that the only edges that

The four faces in $D_{R_1 \cup R_2}$

Figure 5.7

What the drawing D would look like, if F_A was intersected by some edge e .

Figure 5.8

can intersect F_A are the blue edges incident with *both* v_1 and v_2 . Thus, v_1 and v_2 must be joined by a blue edge e that intersects F_A , and no edge other than e intersects F_A .

Since e is incident with both v_1 and v_2 , and A_1 and A_2 form the boundary of F_A , it follows from the good-drawing property of $\mathcal{D}$ that e is contained in F_A . By Statement (iv) in Lemma 5.3, the other blue edge e_1 incident with v_1 crosses R_2 at a point q_2 , and the other blue edge e_2 incident with v_2 crosses R_1 at a point q_1 (see Figure 5.8).

Let p_{B_1} and q_{B_1} be points in B_1 contained in small neighbourhoods of p and q respectively. Similarly, let p_{B_2} and q_{B_2} be points in B_2 contained in small neighbourhoods of p and q respectively. Now we explain how to modify $\mathcal{D}$ to obtain the drawing $\mathcal{D}''$.

Delete the small subarcs of B_1 going from p_{B_1} to p and from q_{B_1} to q , and delete the small subarcs of B_2 going from q_{B_2} to q and from p_{B_2} to p . Also delete A_1, A_2, e , and the pieces of e_1 and e_2 contained in F_{12} and F_{21} respectively.

Join p_{B_1} and q_{B_1} by an arc A_1'' contained in F_{21} , and join p_{B_2} and q_{B_2} by an arc A_2'' contained in F_{12} . Let v_1'' and v_2'' be (new) vertices contained in A_1'' and A_2'' respectively. Join v_1'' and v_2'' by an edge e'' . Join v_1'' to q_2 by an arc a_1'' , so that the only edge that crosses a_1'' is an edge in A_2'' . Similarly, join v_2'' to q_1 by an arc a_2'' , so that the only edge that crosses a_2'' is an edge in A_1'' . Let $\mathcal{D}''$ be the drawing thus obtained (see Figure 5.9).

It is not difficult to see that the only crossings in $\mathcal{D}$ that are not present in $\mathcal{D}''$ are p and q . Similarly, it is not difficult to check that the only crossings in $\mathcal{D}''$ that are not present in $\mathcal{D}$ are the point r_1 where a_1'' crosses A_2'' and the point r_2 where a_2'' crosses A_1'' . Hence $\mathcal{D}''$ has the same number of crossings as $\mathcal{D}$. On the

How to obtain D'' from D . The dotted lines are pieces of edges present in D but not in D'' .

Figure 5.9

If u_1 is in b_1 , then u_1 and u_2 are in different components of D_{v_2} , and so e_{u_1, u_2} crosses D_{v_2} .

Figure 5.10

other hand, $\mathcal{D}''$ has two fewer red-red crossings than $\mathcal{D}$, since p and q are red-red crossings and r_1 and r_2 are red-blue crossings. This violates the real-optimality of $\mathcal{D}$. $\square$

By Statement (ii) in Lemma 5.3, the only edges that intersect $F_A \cup F_{21}$ are the blue edges incident with v_1 , and by Claim 5.6 neither of these edges intersects F_A . Therefore, the only edges that intersect F_{21} are a blue edge B_{v_1, w_2} joining v_1 to a vertex w_2 in R_2 , and a blue edge B_{v_1} incident with v_1 that crosses R_2 at a point t_2 . A similar argument shows that the only edges that intersect F_{21} are the blue edge B_{v_2, w_1} joining v_2 to a vertex w_1 in R_1 , and the blue edge B_{v_2} incident with v_2 that crosses R_1 at a point t_1 .

This situation is illustrated in Figure 5.1, for the case where F_B is the unbounded face in $\mathcal{D}_{R_1 \cup R_2}$ (see the Remark after the proof of Claim 5.5). Note that in Figure 5.1, one of f_1 and g_1 corresponds to the vertex w_1 , and the other one corresponds to the crossing point t_1 . Similarly, one of f_2 and g_2 corresponds to the vertex w_2 , and the other one corresponds to the crossing point t_2 .

By the definition of $C_6 \times C_n$, if two vertices in different red cycles R and R' are adjacent, then every vertex in R is adjacent to a vertex in R' . Since the vertex v_1 in R_1 is adjacent to the vertex w_2 in R_2 , it follows that every vertex in R_1 is adjacent to a vertex in R_2 . In particular, since v_1 and v_2 are the only vertices in A_1 and A_2 respectively, each vertex in B_1 different from w_1 is adjacent to a vertex in B_2 different from w_2 .

STEP 2. So far we have not found any contradiction from the assumption that each of R_1 and R_2 has fewer than four crossings. Indeed, all we have established is a detailed picture of what the drawing $\mathcal{D}$ must look like under these conditions.

Our next step is to establish the existence of the edge e_{u_1, u_2} mentioned in Section 5.3.

Claim 5.7. *Let D_{v_1} and D_{v_2} be the blue cycles that contain v_1 and v_2 respectively. Then there is an edge e_{u_1, u_2} joining vertices u_1 and u_2 in B_1 and B_2 , respectively, such that e_{u_1, u_2} crosses at least two edges not in $D_{v_1} \cup D_{v_2}$.*

We defer the proof of Claim 5.7 for the moment, and use this result to finish the proof of Theorem 5.2.

Let b_i the subarc of B_i going from w_i to t_i , for each $i \in \{1, 2\}$. It is straightforward to check that if u_1 is in b_1 , then u_1 and u_2 are in different components of D_{v_2} , and so e_{u_1, u_2} must cross an edge in D_{v_2} (see Figure 5.10). A similar argument shows that if u_2 is in b_2 , then e_{u_1, u_2} crosses an edge in D_{v_1} .

Now re-draw e_{u_1, u_2} in the following way to obtain a drawing $\mathcal{D}'$. Let e_{u_1, u_2} pass through the faces F_{12}, F_A , and F_{21} , so that e_{u_1, u_2} crosses each A_i exactly once. It is not difficult to check that if u_1 is not in b_1 , then e_{u_1, u_2} can be drawn without crossing D_{v_2} (see Figure 5.11). Similarly, if u_2 is not in b_2 , then e_{u_1, u_2} can be drawn without crossing D_{v_1} .

Let us say that a crossing of e_{u_1, u_2} (in either $\mathcal{D}$ or $\mathcal{D}'$) is *of type I* if it involves e_{u_1, u_2} and an edge in $D_{v_1} \cup D_{v_2}$. If a crossing of e_{u_1, u_2} is not of type I, then we say it is *of type II*.

Now we show that (a) the number of crossings of type I in $\mathcal{D}'$ is not bigger than the number of crossings of type I in $\mathcal{D}$, and (b) the number of crossings of type II in $\mathcal{D}'$ is not bigger than the number of crossings of type II in $\mathcal{D}$. This finishes the proof of Theorem 5.2, since it follows that $\mathcal{D}'$ is also optimal, and each of R_1 and R_2 has four crossings in $\mathcal{D}'$.

If u_1 is not in b_1 , then e_{u_1, u_2} can be re-drawn without crossing D_{v_2} . Therefore e_{u_1, u_2} crosses D_{v_2} in D' only if u_1 is in b_1 .

Figure 5.11

The edge e_{u_1, u_2} crosses D_{v_2} in $\mathcal{D}'$ only if u_1 is in b_1 . On the other hand, if u_1 is in b_1 then e_{u_1, u_2} crosses D_{v_2} in $\mathcal{D}$. Therefore e_{u_1, u_2} crosses D_{v_2} in $\mathcal{D}'$ only if e_{u_1, u_2} crosses D_{v_2} in $\mathcal{D}$. An analogous argument shows that e_{u_1, u_2} crosses D_{v_1} in $\mathcal{D}'$ only if e_{u_1, u_2} crosses D_{v_1} in $\mathcal{D}$. Hence (a) follows.

To prove (b), we note that there are exactly two crossings of type II in $\mathcal{D}'$, namely the points where e_{u_1, u_2} crosses A_1 and A_2 . On the other hand, by Claim 5.7 there are at least two crossings of type II in $\mathcal{D}$. Therefore the number of crossings of type II in $\mathcal{D}'$ is not bigger than the number of crossings of type II in $\mathcal{D}$. $\square$

Proof of Claim 5.7. Since R_1 contains six vertices and A_1 has only one vertex, B_1 contains exactly five vertices. Let y_1, u_1 and z_1 be vertices in B_1 distinct from w_1 , ordered in such a way that as we go from p to q along B_1 we find y_1, u_1 , and z_1 in this order. Let y_2, u_2 , and z_2 be the vertices in B_2 adjacent to y_1, u_1 , and z_1 respectively. By the remark at the end of Step 1, none of y_2, u_2 , and z_2 is equal to w_2 .

Let e_{u_1, u_2} be the edge that joins u_1 and u_2 . To finish the proof of Claim 5.7, we show that e_{u_1, u_2} crosses at least two edges in neither D_{v_1} nor D_{v_2} .

Let D_{y_1}, D_{z_1} be the blue cycles containing y_1 and z_1 respectively. If D_{y_1} crosses e_{u_1, u_2} , then it does so at least twice, since D_{y_1} crosses neither R_1 nor R_2 . Similarly, if D_{z_1} crosses e_{u_1, u_2} then it does so in at least two points. Since in either case Claim 5.7 follows, we assume that e_{u_1, u_2} crosses neither D_{y_1} nor D_{z_1} .

Every red cycle has a common vertex with each of D_{y_1} and D_{z_1} . In particular, each of R_3, R_4, R_5 , and R_6 has a common vertex with each of D_{y_1} and D_{z_1} . It is easy to check that it follows that each of R_3, R_4, R_5 , and R_6 crosses e_{u_1, u_2} , since

neither R_1 nor R_2 is crossed by a red cycle in $\{R_3, R_4, R_5, R_6\}$, and e_{u_1, u_2} crosses neither D_{y_1} nor D_{z_1} . Thus in this case e_{u_1, u_2} crosses at least four red cycles, and so Claim 5.7 follows. $\square$

5.7 Comments

Computing the exact crossing number of $C_m \times C_n$ has proved to be a very difficult task. However, as argued in Chapters 2 and 4, it is possible to support the general conjecture $cr(C_m \times C_n) = (m - 2)n$, based on the asymptotic behaviour of $cr(C_m \times C_n)$ and on the number of crossings of some very natural families of drawings of $C_m \times C_n$, namely those drawings where the n m -cycles are pairwise disjoint.

Anderson et al. [ARRo2] have proved that the crossing number of $C_7 \times C_7$ is 35. This is also in agreement with the general conjecture for $cr(C_m \times C_n)$. It seems reasonable to expect that this result, together with Theorem 4.1 and the techniques developed above, could be used to calculate $cr(C_7 \times C_n)$. However, there are some new difficulties to be overcome, mainly because we do not have a smooth extension of Lemma 5.3 for red cycles with fewer than five crossings.

For instance, if a red cycle R_1 crosses itself and crosses some red cycle R_2 in an (assumed) optimal drawing of $C_7 \times C_n$, then we have to show that either R_1 has at least 5 crossings in total or that some red cycle has at least 5 crossings in total. It is not difficult to see that such an argument would be remarkably more involved than the one used to prove Proposition 5.4, the equivalent of this case in the analysis of drawings of $C_6 \times C_n$. Another possibility that would need to be analyzed in a lot more detail than in the present chapter is the case where two red

cycles R_1 and R_2 cross each other exactly twice. In our discussion in Section 5.6 we were able to obtain a very detailed picture of what such a drawing must look like if the involved graph is $C_6 \times C_n$, but if we are considering drawings of $C_7 \times C_n$ it seems that we would need to analyze a number of possibilities separately.

In conclusion, we do expect that the techniques developed above can be generalized to show that $cr(C_7 \times C_n) = 35$ (with the help of Theorem 4.1), but we anticipate that such a proof would involve a lot more case analysis than the one we have presented to prove that $cr(C_6 \times C_n) = 4n$.

Chapter 6

The Crossing Number of $P(n, 3)$

6.1 Introduction

A few years ago Fiorini [F1] claimed to have determined the crossing numbers of certain families of Generalized Petersen Graphs. Unfortunately, his paper contains one serious mistake that invalidates the principal results. Our aim in this chapter is to present correct proofs of the main statements in Fiorini's paper, and to extend these results by determining the crossing numbers of a family of graphs for which Fiorini had only claimed upper and lower bounds.

The main result of this chapter is the following.

Theorem 6.1. *The crossing number of the Generalized Petersen Graph $P(3k + h, 3)$ is $k + h$ if $h \in \{0, 2\}$ and $k + 3$ if $h = 1$, for each $k \geq 3$, with the single exception of $P(9, 3)$, whose crossing number is 2.*

As mentioned in Section 1.3, the Generalized Petersen Graph $P(m, \ell)$ is obtained in the following way. Let $C = (v_0, \dots, v_{m-1})$ be an m -cycle, and let $v'_0, \dots, v'_{m-1}$ be vertices not in C . Let ℓ be a positive integer. For each i , join v_i to v'_i by an edge e_i^s , and join v'_i to $v'_{i+\ell}$ by an edge (indices are read modulo ℓ).

The graph thus obtained is the *Generalized Petersen Graph* $P(m, \ell)$. The cycle C is the *principal cycle* of $P(m, \ell)$, and the edges e_i^s are the *spokes* of the graph.

To prove Theorem 6.1 we need to analyze certain drawings of graphs related to $P(m, \ell)$. The *derived Generalized Petersen Graph* $P'(m, \ell)$ is the graph obtained by contracting the spokes of $P(m, \ell)$. The m -cycle in $P'(m, \ell)$ whose edges correspond to the edges in the principal cycle of $P(m, \ell)$ will also be called the *principal cycle* of $P'(m, \ell)$. Figure 6.1 shows drawings of the Generalized Petersen Graph $P(8, 3)$ and its corresponding Derived Generalized Petersen Graph $P'(8, 3)$.

A drawing of $P'(m, 3)$ is an n -drawing if the edge rotation around each vertex consists of two edges in the principal cycle followed by two edges not in the principal cycle. Thus the drawing of $P'(8, 3)$ in Figure 6.1(b) is not an n -drawing, but Figure 6.2 shows an n -drawing of the same graph. We denote by $cr_n(P'(m, 3))$ the minimum number of crossings in an n -drawing of $P'(m, 3)$. We remark that an n -drawing is not, in general, a good drawing.

We prove Theorem 6.1 by induction on k for each h . As shown below, it is straightforward to deal with the inductive step if we consider drawings of $P(3k + h, 3)$ where at least one spoke is crossed. On the other hand, if we take a drawing of $P(3k + h, 3)$ where no spoke is crossed, then there is no simple way to make use of the induction hypothesis. When faced with a drawing of this kind we proceed to contract the spokes, thus obtaining an n -drawing of $P'(3k + h, 3)$ with the same number of crossings as the drawing of $P(3k + h, 3)$. Finally we show, by a separate inductive process, that $cr_n(P'(3k + h, 3))$ is at least $k + h$ if $h \in \{0, 2\}$, and at least $k + 3$ if $h = 1$.

This chapter is organized as follows. In Section 6.2 we reduce the proof of Theorem 6.1 to the proof of Theorem 6.2, which gives a lower bound for

(a) A drawing of $P(8,3)$

(b) A drawing of $P'(8,3)$

Figure 6.1

An n -drawing of $P'(8,3)$

Figure 6.2

$cr_n(P'(3k + h, 3))$. Most of the rest of the chapter is devoted to the proof of Theorem 6.2. In Section 6.3 we describe the strategy to prove Theorem 6.2. Section 6.4 contains the first part of the proof of Theorem 6.2, and in Section 6.5 we carry out a construction that is central for the rest of the proof. Sections 6.6, 6.7, and 6.8 deal with the final part of the proof for the specific cases $h = 0, 2$, and 1 , respectively. In Section 6.9 we indicate the error that invalidates the main results in [F1]. Section 6.10 contains some final remarks.

6.2 Proof of Theorem 6.1

Since there is a lack of symmetry in the role of h in Theorem 6.1, we define a function $f : \{0, 1, 2\} \rightarrow \{0, 2, 3\}$ by the rules $f(0) = 0$, $f(1) = 3$, and $f(2) = 2$. Using this notation, Theorem 6.1 can be written in the following way.

Theorem 6.1. *The crossing number of the Generalized Petersen Graph $P(3k + h, 3)$ is $k + f(h)$, for each $k \geq 3$, with the single exception of $P(9, 3)$, whose crossing number is 2.*

Proof of Theorem 6.1. As proved in [F1], the crossing number of $P(9, 3)$ is 2. The drawing in Figure 6.3 shows that there are drawings of $P(3k, 3)$ with exactly k crossings for each $k \geq 4$. Similarly, the drawings in Figures 6.4 and 6.5 show that there are drawings of $P(3k + 1, 3)$ and $P(3k + 2, 3)$ with exactly $k + 3$ and $k + 2$ crossings respectively, for each $k \geq 3$. Therefore $cr(P(3k + h, 3)) \leq k + f(h)$, for each $k \geq 3$. It follows that to prove Theorem 6.1 it suffices to show that $cr(P(3k + h, 3)) \geq k + f(h)$, for each $k \geq 3$, if $h \in \{1, 2\}$, and that $cr(P(3k, 3)) \geq k$ for each $k \geq 4$.

A drawing of $P(3k,3)$ with k crossings

Figure 6.3

A drawing of $P(3k+1,3)$ with $k+3$ crossings

Figure 6.4

A drawing of $P(3k+2,3)$ with $k+2$ crossings

Figure 6.5

We prove these two last inequalities by induction on k . The base cases are $cr(P(10,3)) = 6$ for $h = 1$, $cr(P(11,3)) = 5$ for $h = 2$, and $cr(P(12,3)) = 4$ for $h = 0$; these have been verified by computer by Yuanshen Yan [YY].

Let $k \geq 4$ if $h \in \{1, 2\}$ and let $k \geq 5$ if $h = 0$, and suppose that the inequality $cr(P(3(k-1) + h, 3)) \geq (k-1) + f(h)$ holds for every h . We work on two cases separately.

CASE I. *At least one spoke is crossed in $\mathcal{D}$.* First we note that the graph obtained by deleting three consecutive spokes in $P(3k + h, 3)$ is a subdivision of $P(3(k-1) + h, 3)$ (see Figure 6.6).

Suppose that the spoke e_i^s is crossed in $\mathcal{D}$. By deleting the spokes e_{i-1}^s, e_i^s , and e_{i+1}^s , we get a drawing $\mathcal{D}'$ of a subdivision of $P(3(k-1) + h, 3)$ with fewer crossings than $\mathcal{D}$. Since by the induction hypothesis $\mathcal{D}'$ has at least $(k-1) + f(h)$ crossings, it follows that $\mathcal{D}$ has at least $(k-1) + f(h) + 1 = k + f(h)$ crossings.

CASE II. *No spokes are crossed in $\mathcal{D}$.* If we contract topologically the spokes in $\mathcal{D}$ (see Figure 6.7), then we obtain an n -drawing $\mathcal{D}'$ of $P'(3k + h, 3)$ with the same number of crossings as $\mathcal{D}$. Hence we are done, since by Theorem 6.2 below $\mathcal{D}'$ has at least $k + f(h)$ crossings. $\square$

Theorem 6.2. $cr_n(P'(3k + h, 3)) \geq k + f(h)$, for each $k \geq 3$.

We remark that in the statement of Theorem 6.2 we do not require k to be at least 4 if $h = 0$. Thus, even though the crossing number of $P(9, 3)$ is 2, the minimum number of crossings in an n -drawing of $P'(9, 3)$ is at least 3.

In the next section we outline the proof of Theorem 6.2.

Deleting three consecutive spokes of $P(8,3)$ yields a subdivision of $P(5,3)$. In general, deleting three consecutive spokes of $P(3k+h,3)$ yields a subdivision of $p(3(k-1) + h,3)$.

Figure 6.6

(a) A drawing of $P(8,3)$ where no spoke is crossed.

(b) By contracting topologically the spokes in the drawing of $P(8,3)$ in (a), we obtain an n -drawing of $P'(8,3)$ with the same number of crossings as the drawing of $P(8,3)$.

Figure 6.7

6.3 Strategy of the Proof of Theorem 6.2

The strategy to prove Theorem 6.2 is as follows. We proceed by induction on k for each h . We assume that the statement in Theorem 6.2 is true for $k' < k$, and take an n -drawing $\mathcal{D}$ of $P'(3k + h, 3)$. We show that if $\mathcal{D}$ does not have at least $k + f(h)$ crossings, then we can construct a family $\mathcal{F}$ of edges not in the principal cycle, with the property that there is a one-to-three map from $\mathcal{F}$ to the collection of edges in the principal cycle, such that no edge is in the image of two different edges in $\mathcal{F}$. This induces a partition of a collection of $3|\mathcal{F}|$ edges in the principal cycle into 3-sets, and since there are exactly $3k + h < 3(k + 1)$ edges in the principal cycle, it follows that $|\mathcal{F}| < k + 1$. Finally we show that this inequality, together with inequalities given in Lemmas 6.8 and 6.10, implies that the crossing number of $\mathcal{D}$ is at least one more than $cr_n(P(3(k - 1) + h, 3))$.

The proof of Theorem 6.2 is divided in three parts. The first two parts, given in Sections 6.4 and 6.5, apply to all three possible values of h . In Section 6.4 we present the induction hypothesis, and the objective of Section 6.5 is to develop the necessary tools to obtain and deal with the collection $\mathcal{F}$ described above. Finally, in Sections 6.6, 6.7, and 6.8 we complete the proof for the cases $h = 0, 2$, and 1 , respectively. The arguments in Section 6.8 are remarkably more elaborate than those used in Sections 6.6 and 6.7. Perhaps this is related to the fact that the crossing number of $P(3k + h, 3)$ is equal to $k + h$ if $h \in \{0, 2\}$, and equal to $k + h + 2$ if $h = 1$.

We remark that using Sections 6.4 and 6.5 it is straightforward to prove that $cr(P(3k + 1, 3)) \geq k + 1$ (this inequality constitutes Lemma 6.12). This is.

perhaps not coincidentally, the lower bound that Fiorini claimed to have proved for $cr(P(3k + 1, 3))$.

6.4 First Part of the Proof of Theorem 6.2

We prove Theorem 6.2 in three parts for each h . All the results in this section and in Section 6.5 are valid for every $h \in \{0, 1, 2\}$.

The proof of Theorem 6.2 is by induction on k , for each h . The base cases are obtained in the following way. Since any given n -drawing of $P'(3k + h, 3)$ can be extended to a drawing of $P(3k + h, 3)$ by splitting the vertices to create spokes, it follows that $cr_n(P'(3k + h, 3)) \geq cr(P(3k + h, 3))$. Therefore the base cases $cr(P(10, 3)) = 6$ and $cr(P(11, 3)) = 5$ used in the proof of Theorem 6.1 can be used to obtain $cr_n(P'(10, 3)) \geq 6$ and $cr_n(P'(11, 3)) \geq 4$. On the other hand, we show in Appendix A that $cr_n(P'(9, 3)) = 3$. These three inequalities constitute the base cases for our present induction, for the values $h = 1, 2$, and 0 , respectively. Thus in order to finish the proof of Theorem 6.2 it suffices to prove the following.

Inductive Claim. *Let $k \geq 4$ be an integer. Suppose that $cr_n(P'(3(k - 1) + h, 3)) \geq (k - 1) + f(h)$, for each $h \in \{0, 1, 2\}$. Then $cr_n(P'(3k + h, 3)) \geq k + f(h)$.*

The hypotheses in this claim will be assumed to hold for the rest of this paper. We prove the Inductive Claim by taking a fixed n -drawing $\mathcal{D}$ of $P'(3k + h, 3)$, and showing that its number of crossings $cr(\mathcal{D})$ is at least $k + f(h)$.

6.5 The Main Construction

It will be very convenient for our subsequent work to colour the edges in the drawing $\mathcal{D}$ of $P'(3k+h, 3)$. This graph has two kinds of edges, namely the edges in the principal cycle, which will be coloured blue, and those which correspond to edges joining vertices v'_i in $P(3k+h, 3)$, which will be coloured red. We denote by B the $(3k+h)$ -cycle that consists of all blue edges.

Each red edge r joins two vertices that are joined by a path $P(r)$ of length 3 consisting of blue edges. We say that the blue edges in $P(r)$ are *dominated* by the red edge r . We denote by $C(r)$ the 4-cycle obtained by adding r to its dominated path $P(r)$.

An edge in $\mathcal{D}$ is *clean* if it is not crossed by any other edge in the drawing.

We denote the vertices in $P'(3k+h, 3)$ by v_i , for $0 \leq i \leq 3k+h-1$. If we use the notation (v_i, v_j) to indicate the edge that joins v_i to v_j , then each blue edge is (v_i, v_{i+1}) for some i , and each red edge is (v_i, v_{i+3}) for some i . For brevity, we denote by b_i the blue edge (v_i, v_{i+1}) , and we denote by r_i the red edge (v_i, v_{i+3}) . Thus, the blue 3-path dominated by the red edge r_i is $P(r_i) = (v_i, v_{i+1}, v_{i+2}, v_{i+3})$ (see Figure 6.8). We remark that indices are always read modulo $3k+h$.

The main objectives of this section are the construction and analysis of the collection $\mathcal{F}$ mentioned in Section 6.3. We emphasize that this family is constructed under the assumption that the Inductive Claim does not hold, as otherwise Theorem 6.2 follows. First we establish Propositions 6.3, 6.4, 6.5, 6.6 and 6.7 to obtain information about clean red edges and their dominated edges. Once we prove these results, we describe an inductive process from which $\mathcal{F}$, a collection of clean

The blue 3-path $P(r_i) = (v_i, v_{i+1}, v_{i+2}, v_{i+3})$ is the path dominated by r_i .

Figure 6.8

The drawing D' of $P'(3(k-1) + h, 3)$ is not an n -drawing, but we can easily modify D' in a small neighbourhood of v to obtain an n -drawing of $P'(3(k-1) + h, 3)$ with one more crossing than D' .

Figure 6.9

red edges, is obtained. Finally, we give in Lemmas 6.8 and 6.10 two important inequalities involving the size of $\mathcal{F}$.

Proposition 6.3. *Suppose that the Inductive Claim does not hold. Let r be a clean red edge in $\mathcal{D}$. Then $P(r)$ is crossed at most once, and if it is crossed, then it is crossed by a red edge adjacent to r .*

Proof. Let r_i be a clean red edge in $\mathcal{D}$. It is not difficult to check that if we delete the edges in $P(r_i)$ and then contract r_i , then we obtain a drawing $\mathcal{D}'$ of $P'(3(k-1) + h, 3)$. Depending on the rotations around v_i and v_{i+3} , $\mathcal{D}'$ might or might not be an n -drawing. If $\mathcal{D}'$ is an n -drawing, then $P(r_i)$ can have no crossings, as otherwise the Inductive Claim would hold. Therefore we can assume that $\mathcal{D}'$ is not an n -drawing.

Since $\mathcal{D}'$ is not an n -drawing, the rotation around the vertex of contraction v is red-blue-red-blue. In this case, adding a single crossing in a small neighbourhood of v produces an n -drawing of $P'(3(k-1) + h, 3)$ (see Figure 6.9). It follows that $P(r_i)$ has at most one crossing, since otherwise the Inductive Claim would hold. Hence $P(r_i)$ has at most one crossing.

Now we show that if $P(r_i)$ has one crossing, then it is crossed by an edge adjacent to r_i .

Since $P'(3k + h, 3)$ is 4-regular, for each j there exists an open neighbourhood $\mathcal{N}(v_j)$ of v_j such that the drawing $\mathcal{D}$ restricted to $\mathcal{N}(v_j)$ consists of four rays going out from v_j .

The drawing $\mathcal{D}$ induces a drawing $\mathcal{D}_{C(r_i)}$ of $C(r_i)$. Since r_i is clean, it is contained in the boundary of two open faces F_1 and F_2 of $\mathcal{D}_{C(r_i)}$ (see Figure 6.10).

The two faces of the drawing $D_{C(r_i)}$.

Figure 6.10

The small subarc of r_{i-3} contained in $N(v_i)$ is also contained in F_1 .

Figure 6.11

6.10). The edge r_{i-3} is incident with v_i , and so it intersects either $\mathcal{N}(v_i) \cap F_1$ or $\mathcal{N}(v_i) \cap F_2$. We can assume without loss of generality that r_{i-3} intersects $\mathcal{N}(v_i) \cap F_1$. This is illustrated in Figure 6.11.

Let us suppose that $P(r_i)$ has one crossing. As argued above, in this case the rotation around the contracted vertex in $\mathcal{D}'$ is red–blue–red–blue. It is not difficult to check that this rotation scheme in $\mathcal{D}'$ implies that r_{i+3} intersects $\mathcal{N}(v_{i+3}) \cap F_2$ (see Figure 6.12).

Now we show that the edge that crosses $P(r_i)$ is red. It is easy to see that there is a red path P from v_i to v_{i+3} that does not contain r_i . This path contains both r_{i-3} and r_{i+3} , and since r_{i-3} intersects F_1 and r_{i+3} intersects F_2 , P must cross $P(r_i)$. Since all the edges in P are red, it follows that the edge that crosses $P(r_i)$ is red.

Finally, we show that $P(r_i)$ is crossed by one of r_{i-3} and r_{i+3} . Suppose that $P(r_i)$ is crossed by a red edge different from r_{i-3} and r_{i+3} . Then r_{i-3} and r_{i+3} are contained in F_1 and F_2 respectively, and so each of F_1 and F_2 contains at least one vertex. We claim that this implies that a blue edge crosses $P(r_i)$.

Suppose that there are vertices u_1 and u_2 contained in F_1 and F_2 respectively. Each of u_1 and u_2 is incident with two blue edges, and since the blue edges form a cycle, there are edge-disjoint blue paths P_B and P'_B joining u_1 to u_2 . This is illustrated in Figure 6.13. Since only two of the blue edges incident with vertices in $C(r_i)$ are not in $C(r_i)$, it follows that one of P_B and P'_B crosses an edge in $P(r_i)$. $\square$

The proof of the following result is contained in the proof of Proposition 6.5. This statement is a useful characterization of the drawing of $C(\tau)$ induced by $\mathcal{D}$,

Since the rotation scheme around the contracted vertex v in D' is red-blue-red-blue, it follows that the small subarc of r_{i+3} contained in $N(v_{i+3})$ is also contained in F_2 .

Figure 6.12

If the vertices u_1 and u_2 are contained in F_1 and F_2 respectively, then an edge in either P_B or $P_{B'}$ must cross $P(r_i)$.

Figure 6.13

for each clean red edge r .

Proposition 6.4. *Suppose that the Inductive Claim does not hold. Let r be a clean red edge. Then the cycle $C(r)$ has no self-crossings, and all the vertices not in $C(r)$ are contained in the same component of $\mathbf{R}^2 \setminus C(r)$.*

Let r_i be a clean red edge. As we traverse $C(r_i)$ from v_i to v_{i+3} via $P(r_i)$, we find one of the faces of $C(r_i)$ to our right, and the other one to our left. We call these the *right* and *left* faces of $C(r_i)$, and we denote them by $F_{i,R}$ and $F_{i,L}$ respectively.

If $P = (v_i, v_{i+1}, v_{i+2}, v_{i+3})$ is a 3-path, then (v_{i+1}, v_{i+2}) is the *central* edge of P .

Proposition 6.5. *Suppose that the Inductive Claim does not hold. Let r_i, r_j be different clean red edges in $\mathcal{D}$. Then $P(r_i)$ and $P(r_j)$ have at most one common edge. If $P(r_i)$ and $P(r_j)$ have a common edge b , then b is not the central edge of either $P(r_i)$ nor $P(r_j)$. Thus, j is either $i - 2$ or $i + 2$.*

Proof. It suffices to show that the central edge of neither of $P(r_i)$ and $P(r_j)$ is contained in the other path.

Suppose that the central edge of $P(r_i) = (v_i, v_{i+1}, v_{i+2}, v_{i+3})$ is in $P(r_j)$. Since $r_i \neq r_j$, the paths $P(r_i)$ and $P(r_j)$ are not equal, and so j is either $i - 1$ or $i + 1$. For the sake of definiteness, we assume $j = i + 1$. Therefore $P(r_j) = (v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4})$.

By Proposition 6.4 neither $C(r_i)$ nor $C(r_j)$ has a self-crossing, and it follows from Proposition 6.3 that they do not cross each other. Hence $F_{j,R}$ is contained in either $F_{i,R}$ or $F_{i,L}$.

Let us suppose that $F_{j,R}$ is contained in $F_{i,R}$ (see Figure 6.14). By Proposition 6.3, r_{i+1} must be contained in $F_{i,R}$, and so it follows from Propositions 6.3 and 6.4 that also r_{i+2} is contained in $F_{i,R}$. An analogous argument shows that r_{i-2} is contained in $F_{j,L}$. But this is impossible, since as we go along $P(r_i)$ we cannot find r_{i+2} both to the right and to the left of v_{i+2} . A similar contradiction is found by assuming that $F_{j,R}$ is contained in $F_{i,L}$. $\square$

For any 2-path (v_i, v_{i+1}, v_{i+2}) the vertex v_{i+1} is the *interior* vertex of the path.

Proposition 6.6. *Suppose that $P(r_i)$ and $P(r_j)$ have exactly one common edge b . Then the red edge going from the interior vertex of $P(r_i) - b$ to the interior vertex of $P(r_j) - b$ crosses a blue edge.*

Proof. Let r_i, r_j be clean red edges, and suppose that the Inductive Claim does not hold. By Proposition 6.5, j is either $i - 2$ or $i + 2$. For the sake of definiteness we assume that $j = i + 2$.

It follows from Proposition 6.3 that $C(r_i)$ and $C(r_{i+2})$ do not cross each other, and by Proposition 6.4 neither of these cycles has a self-crossing. Hence $F_{j,R}$ is contained in either $F_{i,L}$ or $F_{i,R}$.

Let us suppose that $F_{j,R}$ is contained in $F_{i,R}$. By Proposition 6.3, r_{i+2} must be contained in $F_{i,R}$, and so it follows from Propositions 6.3 and 6.4 that also r_{i+1} is contained in $F_{i,R}$. An analogous argument shows that r_{i+1} is also contained in $F_{j,L}$. Therefore, as we go from v_i to v_{i+5} along $P(r_i) \cup P(r_j)$, we find r_{i+1} to the right of v_{i+1} and then to the left of v_{i+4} . Since the blue edges form a cycle, it follows that r_{i+1} crosses a blue edge. A similar conclusion is obtained by

The drawing of r_i , r_j , $P(r_i)$, and $P(r_j)$, when $C_{j,R}$ is contained in $C_{i,R}$.

Figure 6.14

assuming that $F_{j,R}$ is contained in $F_{i,L}$. Thus we are done, since r_{i+1} joins the interior vertices of $P(r_i) - b$ and $P(r_j) - b$. $\square$

As we explained in Section 6.3, our goal is, based on the assumption that the Inductive Claim does not hold, to find a family $\mathcal{F}$ of clean red edges such that there is a one-to-three map from $\mathcal{F}$ to the edges in the blue cycle, with the property that no blue edge is in the image set of two different edges in $\mathcal{F}$. The family $\mathcal{F}$ is obtained at the end of an inductive process that we now describe.

Let us suppose that the Inductive Claim does not hold, and let $\mathcal{R}_c$ denote the set of all clean red edges. We set $\mathcal{F}_1^0 = \mathcal{F}_2^0 = \emptyset$ and $\mathcal{F}_3^0 = \mathcal{R}_c$, and we assume that after m steps we have partitioned the set of clean red edges into collections $\mathcal{F}_1^m, \mathcal{F}_2^m$, and $\mathcal{F}_3^m$, with $\mathcal{F}_3^m$ not empty. The $(m+1)$ -th step consists of taking one edge r in $\mathcal{F}_3^m$ and letting $\mathcal{F}_1^{m+1} = \mathcal{F}_1^m \cup \{r\}$. Then we define $\mathcal{F}_2^{m+1}$ as the union of $\mathcal{F}_2^m$ with the set of edges in $\mathcal{F}_3^m \setminus \{r\}$ which have a dominated edge that is also dominated by r , and we let $\mathcal{F}_3^{m+1} = \mathcal{R}_c \setminus (\mathcal{F}_1^{m+1} \cup \mathcal{F}_2^{m+1})$. Since at every step we take at least one element from the third collection, there exists an integer $m_{\mathcal{D}}$ such that $\mathcal{F}_3^{m_{\mathcal{D}}} = \emptyset$, and so we stop the process after $m_{\mathcal{D}}$ steps, and set $\mathcal{F} = \mathcal{F}_1^{m_{\mathcal{D}}}$.

Remark. Whenever we make reference to $\mathcal{F}$, we suppose that the Inductive Claim does not hold, since $\mathcal{F}$ is constructed under this assumption.

Remark. If the Inductive Claim does not hold, then $\mathcal{F}$ has at least one element, since there is at least one clean red edge. To see that there is at least one clean red edge, we note that if every red edge is crossed, then there are at least $\lceil (3k + h)/2 \rceil$ crossings, and this number is at least $k + f(h)$ for every $k \geq 4$.

The main property of $\mathcal{F}$ is established in our next result.

Proposition 6.7. *There exists a one-to-three map from $\mathcal{F}$ to the set of blue edges, with the property that no blue edge belongs to the image sets of different elements in $\mathcal{F}$.*

Proof. We associate to each edge r in $\mathcal{F}$ the three edges in $P(r)$. It follows from the construction of $\mathcal{F}$ that no two elements in $\mathcal{F}$ dominate a common edge. $\square$

Since there are exactly $3k + h < 3(k + 1)$ blue edges, Proposition 6.7 implies the following.

Lemma 6.8. *There are at most k elements in $\mathcal{F}$.*

The last two results in this section involve inequalities which relate the size of $\mathcal{F}$ to the sizes of the collections in a partition $\{\mathcal{R}_b, \mathcal{R}_r, \mathcal{R}_c\}$ of the set $\mathcal{R}$ of red edges. We let $\mathcal{R}_b$ be the subset of $\mathcal{R}$ consisting of all red edges which cross a blue edge, we define $\mathcal{R}_r$ as the set of all red edges which do not cross any blue edge but that cross at least one red edge, and we let $\mathcal{R}_c$ be the set of all clean red edges. We use the following inequality in the proof of Lemma 6.10.

Lemma 6.9. $|\mathcal{F}| \geq |\mathcal{R}_c| - |\mathcal{R}_b|$.

Proof. First we note that $|\mathcal{R}_c| = |\mathcal{F}_1^{m_D}| + |\mathcal{F}_2^{m_D}|$, and since $\mathcal{F}_1^{m_D} = \mathcal{F}$, it is enough to prove that $|\mathcal{F}_2^{m_D}| \leq |\mathcal{R}_b|$. We prove this last inequality by exhibiting an injection $\mathcal{I}$ from $\mathcal{F}_2^{m_D}$ to $\mathcal{R}_b$.

Let r_i be an edge in $\mathcal{F}_2^{m_D}$, and let m_0 be the smallest integer such that $r_i \in \mathcal{F}_2^{m_0}$. By construction, exactly one of r_{i-2} and r_{i+2} is in $\mathcal{F}_1^{m_0}$. If r_{i-2} is in

$\mathcal{F}_1^{m_0}$. then we let $\mathcal{I}(r_i) = r_{i-1}$; if r_{i+2} is in $\mathcal{F}_1^{m_0}$, then we let $\mathcal{I}(r_i) = r_{i+1}$. We claim that if $\mathcal{I}(r_i) = r_{i-1}$, then (a) r_{i-1} is in $\mathcal{R}_b$, and (b) r_{i-1} cannot be $\mathcal{I}(r_j)$ for an edge $r_j \neq r_i$. Similarly, we claim that if $\mathcal{I}(r_i) = r_{i+1}$, then (a) r_{i+1} is in $\mathcal{R}_b$, and (b) r_{i+1} cannot be $\mathcal{I}(r_j)$ for an edge $r_j \neq r_i$.

Suppose that r_{i-2} is in $\mathcal{F}_1^{m_0}$, so that $\mathcal{I}(r_i) = r_{i-1}$. Since both r_i and r_{i-2} are clean, it follows from Proposition 6.6 that r_{i-1} is in $\mathcal{R}_b$. Now if r_{i-1} is $\mathcal{I}(r_j)$ for some r_j , then by definition r_j is in $\mathcal{F}_2^{m_D}$ and j is either i or $i-2$. It is clear that j cannot be $i-2$, since r_{i-2} is in $\mathcal{F}_1^{m_0}$. Therefore r_i is the only edge r_j such that $\mathcal{I}(r_j) = r_{i-1}$.

A similar argument shows that if r_{i+2} is in $\mathcal{F}_1^{m_0}$, then r_{i+1} is in $\mathcal{R}_b$ and r_i is the only edge r_j such that that $\mathcal{I}(r_j) = r_{i+1}$. $\square$

Our last result in this section gives a lower bound for $\mathcal{F}$ in terms of $k, cr(\mathcal{D})$, and certain quantities that depend on the intersections between the edges in $\mathcal{R}_b, \mathcal{R}_r$, and $\mathcal{R}_c$. We define ρ as the number of intersections between elements in $\mathcal{R}_b$ and elements in $\mathcal{R}_r$, we let β be the number of blue–blue crossings, and we denote by τ the number of intersections between elements in $\mathcal{R}_b$. We define the *excess* of an element in $\mathcal{R}_b$ as the total number of blue crossings it has minus one, and we denote by ϵ the sum of the excesses of all the elements in $\mathcal{R}_b$.

Let $G_{\mathcal{R}_r}$ be the graph whose vertex set $V(G_{\mathcal{R}_r})$ is the set of edges in $\mathcal{R}_r$, and in which two vertices are joined by an edge if and only if they cross. Finally, we let $H_{\mathcal{R}_r}$ be the subgraph of $G_{\mathcal{R}_r}$ induced by the set of non-isolated vertices in $G_{\mathcal{R}_r}$.

Lemma 6.10. $|\mathcal{F}| \geq 3k + h - 2cr(\mathcal{D}) + \rho + 2\beta + 2\tau + 2\epsilon + (2|E(H_{\mathcal{R}_r})| - |V(H_{\mathcal{R}_r})|).$

Proof. First we show that $|\mathcal{R}_b| = cr(\mathcal{D}) - |E(H_{\mathcal{R}_r})| - \tau - \beta - \tau - \epsilon$. The

number of red–red crossings is equal to $|E(H_{\mathcal{R}_r})| + \rho + \tau$, and there are exactly β blue–blue crossings. Therefore the number of blue–red crossings is equal to $cr(\mathcal{D}) - |E(H_{\mathcal{R}_r})| - \rho - \beta - \tau$. Since ϵ of these crossings occur on red edges which cross more than one blue edge, it follows that $|\mathcal{R}_b| = cr(\mathcal{D}) - |E(H_{\mathcal{R}_r})| - \rho - \beta - \tau - \epsilon$.

Since $\{\mathcal{R}_b, \mathcal{R}_r, \mathcal{R}_c\}$ is a partition of the set of red edges, $|\mathcal{R}_c| = 3k + h - (|\mathcal{R}_b| + |\mathcal{R}_r|)$, and so by Lemma 6.8 $|\mathcal{F}| \geq |\mathcal{R}_c| - |\mathcal{R}_b| = 3k + 1 - 2|\mathcal{R}_b| - |\mathcal{R}_r|$. It follows from the definition of $H_{\mathcal{R}_r}$ that there are at most $|V(H_{\mathcal{R}_r})| + \rho$ elements in $|\mathcal{R}_r|$, and so $|\mathcal{F}| \geq 3k + h - 2(cr(\mathcal{D}) - |E(H_{\mathcal{R}_b})| - \rho - \beta - \tau - \epsilon) - (|V(H_{\mathcal{R}_b})| + \rho) = 3k + h - 2cr(\mathcal{D}) + \rho + 2\beta + 2\tau + 2\epsilon + (2|E(H_{\mathcal{R}_b})| - |V(H_{\mathcal{R}_b})|)$. $\square$

Remark. If a red edge is crossed more than once, then one of $(2|E(H_{\mathcal{R}_b})| - |V(H_{\mathcal{R}_b})|)$, ρ , τ , and ϵ is positive, and so $|\mathcal{F}| > 3k + h - 2cr(\mathcal{D})$, since each one of $(2|E(H_{\mathcal{R}_b})| - |V(H_{\mathcal{R}_b})|)$, ρ , β , τ , and ϵ is nonnegative.

Lemma 6.10 is almost all we need in order to prove Theorem 6.2 for the cases $h = 0$ and $h = 2$; the final details are carried out in Sections 6.6 and 6.7. In the case $h = 1$ the arguments required to finish the proof of Theorem 6.2 are a lot more sophisticated than those used in Sections 6.6 and 6.7.

6.6 Conclusion of the Proof of Theorem 6.2 for the Case $h = 0$.

In order to finish the proof of Theorem 6.2 all we need to do is show that the Inductive Claim holds. If it does not hold, then $cr(\mathcal{D}) \leq k - 1$, and the family $\mathcal{F}$ can be constructed. Applying Lemma 6.10 we get $|\mathcal{F}| \geq k + 3$, contradicting Lemma 6.8. $\square$

6.7 Conclusion of the Proof of Theorem 6.2 for the Case $h = 2$.

As in Section 6.6, we need to prove that the Inductive Claim is true. So we assume it is false, and obtain a contradiction.

Since the Inductive Claim does not hold, $cr(\mathcal{D}) \leq k + 1$. By Lemma 6.10, $|\mathcal{F}| \geq k + (\rho + 2\beta + 2\tau + 2\epsilon) + (2|E(H_{\mathcal{R}_+})| - |V(H_{\mathcal{R}_+})|) \geq k$. Since by Lemma 6.8 $|\mathcal{F}| \leq k$, equality must hold, and so each of $\rho, \beta, \tau, \epsilon$, and $(2|E(H_{\mathcal{R}_+})| - |V(H_{\mathcal{R}_+})|)$ is equal to zero. By the remark after Lemma 6.10 it follows that no red edge is crossed more than once. But this is impossible, by Claim 6.11 below. $\square$

Claim 6.11. *Suppose that the Inductive Claim does not hold for the drawing $\mathcal{D}$ of $P'(3k + 2, 3)$, where $k \geq 4$. Suppose that $|\mathcal{F}| = k$, and suppose that $\beta = 0$. Then at least one red edge is crossed more than once.*

Proof. Each element in $\mathcal{F}$ dominates three blue edges, and no two blue edges are dominated by the same edge in $\mathcal{F}$. Since $|\mathcal{F}| = k$, there are exactly two blue edges not dominated by any edge in $\mathcal{F}$, and since $k > 3$, we conclude that there is at least one pair $\{r_i, r_j\}$ of adjacent clean red edges in $\mathcal{F}$.

Without any loss of generality we can assume that $j = i + 3$, and so $P(r_i) = (v_i, v_{i+1}, v_{i+2}, v_{i+3})$ and $P(r_{i+3}) = (v_{i+3}, v_{i+4}, v_{i+5}, v_{i+6})$ are the blue 3-paths dominated by r_i and $r_j = r_{i+3}$ respectively. We show that if the red edge r_{i+1} does not cross the blue cycle in at least two points, then it crosses both r_{i-1} and r_{i+2} .

Since $\beta = 0$ and $\mathcal{D}$ is an n -drawing, the edges r_i and r_{i+3} are in the same

component of $\mathbf{R}^2 \setminus B$ (see Figure 6.15). It follows from Propositions 6.3, 6.4, and 6.5 that none of r_{i-1} , r_{i+1} , and r_{i+2} is contained in the same component of $\mathbf{R}^2 \setminus B$ as r_i and r_{i+3} , and it also follows that either r_{i+1} crosses the blue cycle at least twice or it is contained in the component of $\mathbf{R}^2 \setminus B$ that does not contain r_i and r_{i+3} . Since in the first case we are done, we assume that r_{i+1} is contained in the component of $\mathbf{R}^2 \setminus B$ that does not contain r_i and r_{i+3} , and claim that r_{i+1} crosses both r_{i-1} and r_{i+2} (see Figure 6.16).

By Proposition 6.3, r_{i+1} does not cross any edge in $P(r_i) \cup P(r_{i+3})$. Since r_i is clean and $C(r_i)$ has no self-crossings, it follows that the cycle $C = (v_i, v_{i+1}, v_{i+4}, v_{i+3}, v_i)$ has no self-crossings. Since r_{i+1} does not cross the blue cycle, neither of v_{i-1} and v_{i+5} is contained in the same component of $\mathbf{R}^2 \setminus C$ as v_{i+2} . Therefore each of r_{i-1} and r_{i+2} crosses an edge in C . Since by Proposition 6.3 neither of these edges crosses an edge in $P(r_i) \cup P(r_{i+3})$, it follows that both r_{i-1} and r_{i+2} cross r_{i+1} . $\square$

6.8 Conclusion of the Proof of Theorem 6.2 for the Case $h = 1$.

The conclusion of the proof of Theorem 6.2 for the case $h = 1$ is remarkably more complicated than for the cases $h = 0$ and $h = 2$. This is somewhat natural, since our claim is that $\mathcal{D}$ has at least $k + h + 2$ crossings, as opposed to only $k - h$ crossings in the previous cases. Indeed, the framework developed so far is enough to prove the following.

The edges of r_i and r_{i+3} are in the same component of $\mathbb{R}^2 \setminus C_B$

Figure 6.15

The red edge r_{i+1} is contained in the component of $\mathbb{R}^2 \setminus C_B$ that does not contain $r_i \cup r_{i+3}$. As argued in the text, both r_{i-1} and r_{i+2} cross r_{i+1} .

Figure 6.16

Lemma 6.12. $cr_n(P(3k + 1, 1)) \geq k + 1$.

Proof. If we weaken the induction hypothesis in the first part of the proof of Theorem 6.2 to the assumption that $cr_n(P'(3(k - 1) + 1, 3)) \geq (k - 1) + 1 = k$, then the base case is still satisfied (since $cr_n(P'(10, 3)) \geq cr(P(10, 3)) = 6$). Thus it is enough to prove that $cr_n(P'(3k + 1, 3)) \geq k + 1$ for each $k \geq 4$.

Let us suppose that this last inequality does not hold. Then there is an n -drawing $\mathcal{D}'$ of $P'(3k + 1, 3)$ such that $cr(\mathcal{D}') = k - 1 + l$ for some $l \leq 1$. It is not difficult to check that the construction of $\mathcal{F}$ and the properties obtained for this family are also valid under the weaker induction hypothesis, and so Lemma 6.10 applies. Thus we obtain the inequality $|\mathcal{F}| \geq k + 3 - 2l + \rho + 2\beta + 2\tau + 2\epsilon + (2|E(H_{\mathcal{R}_\tau})| - |V(H_{\mathcal{R}_\tau})|)$. Since $l \leq 1$, and each of $\rho, \beta, \tau, \epsilon$, and $2|E(H_{\mathcal{R}_\tau})| - |V(H_{\mathcal{R}_\tau})|$ is nonnegative, this gives $|\mathcal{F}| \geq k + 1$, contradicting Lemma 6.8. $\square$

We remark that since the bound established is not the one given by Theorem 6.2, we still have to show that the Inductive Claim holds, and so we maintain our assumption that $\mathcal{D}$ is an n -drawing of $P'(3k + 1, 3)$.

As in Sections 6.6 and 6.7, we prove the Inductive Claim by assuming it does not hold, and then using the tools developed in Section 6.5 to get a contradiction about the size of $\mathcal{F}$. The strategy is as follows.

We show that if the Inductive Claim does not hold (so that $\mathcal{F}$ can be constructed), then, by Lemmas 6.8 and 6.10, β must be either 0 or 1. We analyze these possibilities separately, and in each case we argue that the lower bound for $|\mathcal{F}|$ given by Lemma 6.10 is in contradiction with Lemma 6.8. Thus the proof for $h = 1$ is not qualitatively different from the proofs for $h = 0$ and $h = 2$; the point

is that in the case $h = 1$ the arguments needed to show that, in contradiction with Lemma 6.8, $|\mathcal{F}|$ must be at least $k + 1$, require a much more detailed analysis.

First we show that $0 \leq \beta \leq 1$.

Proposition 6.13. *Suppose that the Inductive Claim does not hold. Then $\beta \leq 1$.*

Proof. By assumption the Inductive Claim does not hold, and so $cr(\mathcal{D}) \leq k + 2$. Applying Lemma 6.10 we get $|\mathcal{F}| \geq k - 3 + \rho + 2\beta + 2\tau + 2\epsilon + (2|E(H_{\mathcal{R}_b})| - |V(H_{\mathcal{R}_b})|)$. Since each of ρ, τ, ϵ , and $(2|E(H_{\mathcal{R}_b})| - |V(H_{\mathcal{R}_b})|)$ is nonnegative, we conclude that $|\mathcal{F}| \geq k - 3 + 2\beta$, and since $|\mathcal{F}| < k + 1$ by Lemma 6.8, we obtain $-3 + 2\beta < 1$, and so $\beta < 2$. $\square$

We analyze the cases $\beta = 0$ and $\beta = 1$ separately.

6.8.1 Case $\beta = 0$

We prove in this subsection that the Inductive Claim holds for each $k \geq 6$, if $\beta = 0$. The examination of the cases $k = 4$ and $k = 5$ involves a long case analysis, and since there are no important new arguments involved in these particular instances, we present the discussion of these two cases in Appendix B.

Some of the results established in this subsection are independent of the values of β and k . Whenever no restriction on β is made as a hypothesis for a statement, it must be understood that the statement holds for each nonnegative β . Similarly, if no restriction on k is mentioned in a statement, we understand that it holds for each $k \geq 4$.

The strategy to prove that the Inductive Claim holds in the case $\beta = 0$ is as follows. Let $\mathcal{R}$ be the collection of all red edges. For each edge $r \in \mathcal{R}$, we shall

partition $\mathcal{R} \setminus \{r\}$ into k 3-sets Y_j , with the property that the subgraph $B \cup Y_j$ induced by the blue cycle B and each Y_j contains at least one crossing. This gives k crossings for $\mathcal{D}$. Therefore, to prove the Inductive Claim for the case $\beta = 0$, it suffices to show that at least three crossings in $\mathcal{D}$ occur between edges that do not belong to the same $B \cup Y_j$.

Let us suppose that $\beta = 0$. For each red edge $r \in \mathcal{R}$, $\mathcal{R} \setminus \{r\}$ partitions naturally into sets of the form $\{r_i, r_{i+1}, r_{i+2}\}$. Let $\mathcal{P}_r$ denote this partition. A crossing in $\mathcal{D}$ is of *r-type I* if it occurs between edges in the same collection in $\mathcal{P}_r$ or if it occurs between an edge in $\mathcal{R} \setminus \{r\}$ and a blue edge. If a crossing in $\mathcal{D}$ is not of *r-type I*, then it is of *r-type II*.

Now we give conditions under which the Inductive Claim holds if $\beta = 0$.

Lemma 6.14. *Let us suppose that $\beta = 0$. Suppose that one of the following statements holds for some red edge r .*

(6.14.1) *r is crossed at least three times.*

(6.14.2) *r is crossed at least once, and at least two crossings occur between edges in different sets in $\mathcal{P}_r$.*

(6.14.3) *At least three crossings occur between edges in different sets in $\mathcal{P}_r$.*

Then the Inductive Claim holds.

Proof. Suppose that $\beta = 0$. Let r be a red edge. For any collection Y in $\mathcal{P}_r$, the subgraph induced by the edges in Y plus the blue edges is a subdivision of $K_{3,3}$ (see Figure 6.17). Since there are k collections in $\mathcal{P}_r$, it follows that there are at least k crossings of *r-type I*.

Now suppose that one of (6.14.1), (6.14.2), and (6.14.3) holds for r . It follows

The subgraph induced by the edges in Y plus the blue cycle is a subdivision of $K_{3,3}$.

Figure 6.17

If r_j and r_{j+3} are both clean, then r_{j+1} and r_{j+2} cross each other.

Figure 6.18

that there are at least three crossings of r -type II in $\mathcal{D}$. Hence there are at least $3k + 3$ crossings in $\mathcal{D}$. and so the Inductive Claim follows. $\square$

We prove in this subsection that the Inductive Claim holds if $\beta = 0$ for each $k \geq 6$, by showing that in this case one of (6.14.1), (6.14.2), and (6.14.3) holds for some red edge r . To prove this, we use Lemmas 6.15 and 6.16 below. The proof of each of these lemmas depends on a series of technical results, and so we defer their proofs for the moment and show how they imply the Inductive Claim for the case $k \geq 6$, if $\beta = 0$.

Lemma 6.15. *Suppose that there is a pair of adjacent clean red edges. Then at least one of the following statements holds for some i .*

(6.15.1) *The edge r_i is crossed at least three times.*

(6.15.2) *The edges r_{i-7} and r_{i-5} cross each other, r_{i-4} and r_{i-2} cross each other, and r_i has at least one crossing.*

(6.15.3) *The edges r_{i-1} and r_{i+1} cross each other, r_{i+2} and r_{i+4} cross each other, and r_{i+5} and r_{i+7} cross each other.*

Lemma 6.16. *Suppose that $\beta = 0$ and $k \geq 6$. Then either the Inductive Claim holds or there is a pair of adjacent clean red edges.*

Proof of the Inductive Claim for the case $\beta = 0$, for $k \geq 6$. By Lemma 6.16, it suffices to show that if there is a pair of adjacent clean red edges, then the Inductive Claim holds.

Suppose that there is a pair of adjacent clean red edges. Lemma 6.15 applies, and so one of (6.15.1), (6.15.2), and (6.15.3) holds. We show that one of (6.14.1),

(6.14.2), and (6.14.3) holds. It is obvious that (6.15.1) implies (6.14.1). To see that (6.14.2) follows from (6.15.2), we note that r_{i-7} and r_{i-5} are in different collections in $\mathcal{P}_{r_i}$, and r_{i-4} and r_{i-2} are in different collections in $\mathcal{P}_{r_i}$. Similarly, to see that (6.14.3) follows from (6.15.3), we note that r_{i-1} and r_{i+1} are in different collections in $\mathcal{P}_{r_i}$, r_{i+2} and r_{i+4} are in different collections in $\mathcal{P}_{r_i}$, and r_{i-5} and r_{i+7} are in different collections in $\mathcal{P}_{r_i}$. Hence one of (6.14.1), (6.14.2), and (6.14.3) holds. Since $\beta = 0$, by Lemma 6.14 the Inductive Claim follows. $\square$

We now turn to the proofs of the two technical facts, namely Lemmas 6.15 and 6.16.

Proof of Lemma 6.15. Suppose that there is a pair of adjacent clean red edges r_j and r_{j+3} . Proposition 6.17 below applies, and so one of (6.17.1), (6.17.2), and (6.17.3) holds.

If (b) holds in either (6.17.2) or (6.17.3), then there exists a red edge with at least three crossings, and so (6.15.1) holds. Thus we suppose that (a) holds in both (6.17.2) and (6.17.3).

If r_{j+6} has some crossing, then (6.15.2) follows from (6.17.2)(a) and (6.17.3)(a), by setting $i = j + 6$. Finally, suppose that r_{j+6} does not have any crossing. Applying Proposition 6.17 to the two pairs of adjacent red clean edges $\{r_j, r_{j+3}\}$ and $\{r_{j+3}, r_{j+6}\}$, we obtain from (6.17.2)(a) and (6.17.3)(a) that r_{j-1} crosses r_{j+1} , r_{j+2} crosses r_{j+4} , and r_{j+5} crosses r_{j+7} . Hence (6.15.3) follows by setting $i = j$. $\square$

The proof above makes use of the following result.

Proposition 6.17. *Suppose that r_j and r_{j+3} are adjacent clean red edges. Then the following statements hold.*

(6.17.1) *The edges r_{j+1} and r_{j+2} cross each other.*

(6.17.2) *Either (a) r_{j-1} crosses r_{j+1} , or (b) r_{j+1} is crossed at least three times.*

(6.17.3) *Either (a) r_{j+2} crosses r_{j+4} , or (b) r_{j+2} is crossed at least three times.*

Statements (6.17.1) and (6.17.2) are illustrated in Figures 6.18 and 6.19, respectively.

Proof. It is enough to show that (6.17.1) and (6.17.2) hold, since (6.17.3) follows from (6.17.2) by relabeling all the vertices accordingly to the rule $v_l \mapsto v_{6-l}$.

By Proposition 6.3, the path $P_{j,j+6} = P(r_j) \cup P(r_{j+3})$ has no self-crossings. Since $\mathcal{D}$ is an n -drawing, as we traverse $P_{j,j+6}$ from v_j to v_{j+6} we find r_j and r_{j+3} on the same side of v_{j+3} . Relabeling if necessary, we can assume that we find both r_j and r_{j+3} on the left side of v_{j+3} .

It follows from Propositions 6.3 and 6.4 that as we traverse $P_{j,j+6}$ from v_j to v_{j+6} , we find all the red edges incident with each of $v_{j+1}, v_{j+2}, v_{j+4}$, and v_{j+5} to the right of each of these vertices. In particular, we find r_{j+1} to the right of both v_{j+1} and v_{j+4} , and we find r_{j+2} to the right of both v_{j+2} and v_{j+5} .

By Proposition 6.3, $C(r_{j+1})$ does not cross itself, and no edge in $P_{j,j+6}$ crosses $C(r_{j+1})$.

Let p be a point in r_{j+2} such that the part of r_{j+2} from v_{j+2} to p does not cross any edge. It is not difficult to see that p and v_{j+5} are contained in different components of $\mathbb{R}^2 \setminus C(r_{j+1})$, since no edge in $P_{j,j+6}$ crosses $C(r_{j+1})$. It follows that r_{j+2} crosses $C(r_{j+1})$. Therefore r_{j+1} crosses r_{j+2} , since by Proposition 6.3 no edge in $P_{j,j+6}$ crosses r_{j+2} . Hence (6.17.1) holds.

Suppose that r_{j+1} crosses the blue cycle. It follows that r_{j+1} crosses at least two blue edges, since as we traverse $P_{j,j+6}$ from v_j to v_{j+6} we find r_{j+1} to the

- (a) If r_{j+1} does not cross the blue cycle, then r_{j-1} crosses r_{j+1} .

- (b) If r_{j+1} crosses the blue cycle, then r_{j+1} has at least three crossings.

Figure 6.19

right of both v_{j+1} and v_{j+4} . Since r_{j+1} also crosses r_{j+2} , it follows that r_{j+1} has at least three crossings in total, and so (b) in (6.17.2) holds.

Now suppose that r_{j+1} does not cross the blue cycle. Let p be a point in r_{j-1} such that the part of r_{j-1} that goes from v_{j+2} to p does not cross any edge. It is not difficult to check that v_{j-1} and p belong to different components of $\mathbf{R}^2 \setminus C(r_{j+1})$, since r_{j+1} does not cross the blue edge (v_{j-1}, v_j) . Hence r_{j-1} crosses $C(r_{j+1})$. On the other hand, by Proposition 6.3 no edge in $P_{j,j+6}$ crosses r_{j-1} , and so r_{j-1} must cross r_{j+1} . Therefore, if r_{j+1} does not cross the blue cycle, then (a) in (6.17.2) holds. $\square$

The proof of Lemma 6.16 depends on Propositions 6.18 and 6.19 below. The proof of Proposition 6.18 is straightforward, but the proof of Proposition 6.19 is long and it depends on two technical results, namely Propositions 6.20 and 6.21. After stating Proposition 6.19 we outline its proof, then we establish Propositions 6.20 and 6.21, and finally we give the proof of Proposition 6.19.

Proof of Lemma 6.16. Suppose that $\beta = 0$ and $k \geq 6$. If $\mathcal{F}$ can be constructed, then by Propositions 6.18 and 6.19 there is a pair of adjacent clean red edges. Thus we are done, since if $\mathcal{F}$ cannot be constructed then the Inductive Claim holds. $\square$

Proposition 6.18. *Suppose that $k \geq 6$ and $|\mathcal{F}| \geq k - 1$. Then there are two adjacent clean red edges.*

Proof. Suppose that $|\mathcal{F}| \geq k - 1$. We show that if no two edges in $\mathcal{F}$ are adjacent, then $k < 6$. This proves Proposition 6.18, since all the edges in $\mathcal{F}$ are clean.

Re-colour all blue edges, so that edges dominated by a red edge are colored green, and edges not dominated by any red edge are colored yellow. Green edges

come in 3-paths, and so it suffices to show that if no two such 3-paths have a common end, then $k < 6$.

Since $|\mathcal{F}| \geq k - 1$, by Proposition 6.7 there are at least $3(k - 1)$ green edges. Since there are $3k + 1$ blue edges in total, it follows that there are at most four yellow edges. Therefore, if there are no two green 3-paths with a common end, then there are at most four green 3-paths in total, and so $|\mathcal{F}| \leq 4$. Since $|\mathcal{F}| \geq k - 1$, this implies $k < 6$. $\square$

Proposition 6.19. *Suppose that $\beta = 0$ and $|\mathcal{F}| < k - 1$. Then there are two adjacent clean red edges.*

The proof of Proposition 6.19 makes use of Propositions 6.20 and 6.21 below, and we present it after establishing these results. The main ideas involved in the proof of Proposition 6.19 are easy to explain, and since they motivate the introduction of Propositions 6.20 and 6.21, we give an outline of the proof of Proposition 6.19 before stating these auxiliary claims.

Let R be the $(3k + 1)$ -cycle that consists of all red edges. It follows from Proposition 6.21 that, except perhaps for one or two exceptions, every edge in $\mathcal{R}_r$ is adjacent to an edge in $\mathcal{R}_b$. Therefore if we contract all (except perhaps for one or two) the edges in $\mathcal{R}_r$ to get a cycle $\bar{R}$, then two clean red edges are adjacent in $\bar{R}$ if and only if they are adjacent in R . Thus in order to show that there are two adjacent clean red edges in R it is enough to show that more than half the edges in $\bar{R}$ belong to $\mathcal{R}_c$. This will turn out to be a straightforward task once we establish from Proposition 6.20 certain inequalities involving the sizes of $\mathcal{F}$, $\mathcal{R}_b$, $\mathcal{R}_r$, and $\mathcal{R}_c$.

Proposition 6.20. *If $|\mathcal{F}| < k - 1$, then $\rho + (2|E(H_{\mathcal{R}_r})| - |V(H_{\mathcal{R}_r})|) \leq 1$.*

Proof. Since $\mathcal{F}$ can be constructed, the Inductive Claim does not hold. and so $cr(\mathcal{D}) \leq k + 2$. By Lemma 6.10, $|\mathcal{F}| \geq 3k + 1 - 2(k + 2) + \rho + (2|E(H_{\mathcal{R}_r})| - |V(H_{\mathcal{R}_r})|) = k - 3 + \rho + (2|E(H_{\mathcal{R}_r})| - |V(H_{\mathcal{R}_r})|)$. Since $|\mathcal{F}| < k - 1$, it follows that $\rho + (2|E(H_{\mathcal{R}_r})| - |V(H_{\mathcal{R}_r})|) \leq 1$. $\square$

Proposition 6.21. *Suppose that $\beta = 0$ and $|\mathcal{F}| < k - 1$. Then either:*

- (i) $(2|E(H_{\mathcal{R}_r})| - |V(H_{\mathcal{R}_r})|) = 0$ and $\rho = 0$, and each edge in $\mathcal{R}_r$ is adjacent to an edge in $\mathcal{R}_b$, or
- (ii) $(2|E(H_{\mathcal{R}_r})| - |V(H_{\mathcal{R}_r})|) = 0$ and $\rho = 1$, and each edge in $\mathcal{R}_r$, except perhaps for one edge, is adjacent to an edge in $\mathcal{R}_b$, or
- (iii) $(2|E(H_{\mathcal{R}_r})| - |V(H_{\mathcal{R}_r})|) = 1$ and $\rho = 0$, and each edge in $\mathcal{R}_r$, except perhaps for two adjacent edges in $\mathcal{R}_r$, is adjacent to an edge in $\mathcal{R}_b$.

Proof. By Proposition 6.20, $\rho + (2|E(H_{\mathcal{R}_r})| - |V(H_{\mathcal{R}_r})|) \leq 1$. Since $\rho \geq 0$, $(2|E(H_{\mathcal{R}_r})| - |V(H_{\mathcal{R}_r})|)$ must be either 0 or 1. We analyze these possibilities separately.

CASE $(2|E(H_{\mathcal{R}_r})| - |V(H_{\mathcal{R}_r})|) = 0$. It follows from Proposition 6.20 that $\rho \leq 1$. and so at most edge in $\mathcal{R}_r$ crosses an edge in $\mathcal{R}_b$. Let r be the only edge in $\mathcal{R}_r$ that crosses an edge in $\mathcal{R}_b$, if such an edge exists.

If $\mathcal{R}_r = \emptyset$ or $\mathcal{R}_r = \{r\}$, then (i) and (ii) trivially hold, and so we assume that there exists an edge in $\mathcal{R}_r$ different from r .

Since $(2|E(H_{\mathcal{R}_r})| - |V(H_{\mathcal{R}_r})|) = 0$, each edge in $\mathcal{R}_r$ crosses a unique edge in $\mathcal{R}_r$. To prove (i) and (ii) we show that, possibly with the exception of the edge r'

in $\mathcal{R}_\tau$ that crosses r (if r exists), each edge r_i in $\mathcal{R}_\tau$ is adjacent to an edge $r_{i'}$ in $\mathcal{R}_b$.

Let r_i be an edge in $\mathcal{R}_\tau \setminus \{r'\}$, and let r_j be the unique edge in $\mathcal{R}_\tau$ crossed by r_i . Since $r_i \neq r'$, r_j is not r , and so r_j is crossed only by r_i .

Since $\beta = 0$, the path $P_{j,j+3} = (v_j, v_{j+1}, v_{j+2}, v_{j+3})$ does not cross itself. On the other hand, r_j does not cross any blue edge, and so $C(r_j)$ has no self-crossings.

Since the blue cycle has no self-crossings and r_j crosses no blue edges, one of the components of $\mathbf{R}^2 \setminus C(r_j)$ does not contain any vertex. Since r_j intersects r_i , and r_i crosses no blue edges, it follows that r_i is incident with one of v_{j+1} and v_{j+2} . Let $r_{i'}$ be the other red edge incident with the vertex in $\{v_{j+1}, v_{j+2}\}$ incident with r_i . Since $\mathcal{D}$ is an n -drawing, $r_{i'}$ also intersects the component of $\mathbf{R}^2 \setminus C(r_j)$ that contains no vertices. Hence $r_{i'}$ crosses an edge in $C(r_j)$, since only one vertex incident with $r_{i'}$ is in $C(r_j)$. Since r_j is only crossed by r_i , it follows that the edge in $C(r_j)$ crossed by $r_{i'}$ is blue. Thus we are done, since $r_{i'}$ is in $\mathcal{R}_b$, and it is adjacent to r_i .

CASE $(2|E(H_{\mathcal{R}_\tau})| - |V(H_{\mathcal{R}_\tau})|) = 1$. In this case each edge in $\mathcal{R}_\tau$, except for one edge r , is crossed by exactly one element in $\mathcal{R}_\tau$ and by no other edges. The edge r is crossed by two edges r' and r'' , both in $\mathcal{R}_\tau$, and r is not crossed by any other edge.

To prove (iii), we show that (a) if r' and r'' are adjacent, then each edge $r_i \in \mathcal{R}_\tau \setminus \{r', r''\}$ is adjacent to an edge $r_{i'}$ in $\mathcal{R}_b$, and (b) if r' and r'' are not adjacent, then each edge r_i in $\mathcal{R}_\tau$ is adjacent to an edge $r_{i'}$ in $\mathcal{R}_b$.

Let r_i be an edge in $\mathcal{R}_\tau$, and let r_j be an edge that crosses r_i . As in Case $(2|E(H_{\mathcal{R}_\tau})| - |V(H_{\mathcal{R}_\tau})|) = 0$, the cycle $C(r_j)$ obtained by adding the path $P_{j,j+3} =$

$(v_j, v_{j+1}, v_{j+2}, v_{j+3})$ to r_j has no self crossings, and one component of $\mathbf{R}^2 \setminus C(r_j)$ contains no vertices. Hence r_i is incident with one of v_{j+1} and v_{j+2} . Let $r_{i'}$ be the other red edge incident with the vertex in $\{v_{j+1}, v_{j+2}\}$ incident with r_i .

The edge $r_{i'}$ intersects the component of $\mathbf{R}^2 \setminus C(r_j)$ that contains no vertices, since $\mathcal{D}$ is an n -drawing. Since only one vertex incident with $r_{i'}$ is in $C(r_j)$, it follows that $r_{i'}$ crosses an edge in $C(r_j)$.

Suppose that r' and r'' are adjacent. We show that if r_i is neither r' nor r'' , then $r_{i'}$ is in $\mathcal{R}_b$. If r_i is neither r' nor r'' , then r_j is not r , and so r_j is crossed only by r_i . Therefore the edge in $C(r_i)$ that crosses $r_{i'}$ must be blue, and so $r_{i'}$ is in $\mathcal{R}_b$.

Finally we show that if r' and r'' are not adjacent, then $r_{i'}$ is in $\mathcal{R}_b$. Suppose that r' and r'' are not adjacent. It is not difficult to check that if r_j is not r , then the edge in $C(r_j)$ crossed by $r_{i'}$ is blue, and so $r_{i'}$ is in $\mathcal{R}_b$. Thus we assume that r_j is r . In this case r_i is either r' or r'' , and so $r_{i'}$ is neither r' nor r'' , since r' and r'' are not adjacent. Therefore the edge in $C(r_j)$ crossed by $r_{i'}$ must be blue, since r is crossed only by r' and r'' . Hence $r_{i'}$ is in $\mathcal{R}_b$. $\square$

We are now ready to prove Proposition 6.19.

Proof of Proposition 6.19. First we establish two results that we use throughout the proof. Eq. (6.8.1) follows since $\{\mathcal{R}_r, \mathcal{R}_b, \mathcal{R}_c\}$ is a partition of the collection of red edges, and (6.8.2) follows from the definitions of $|V(H_{\mathcal{R}_r})|$ and ρ .

$$|\mathcal{R}_c| = 3k + 1 - (|\mathcal{R}_b| + |\mathcal{R}_r|) \quad (6.8.1)$$

$$|\mathcal{R}_r| \leq |V(H_{\mathcal{R}_r})| + \rho \quad (6.8.2)$$

Let R be the $(3k + 1)$ -cycle that consists of all red edges. By Proposition 6.20, $\rho \leq 1$. We analyze the cases $\rho = 0$ and $\rho = 1$ separately.

CASE $\rho = 1$. Since $|\mathcal{F}| < k - 1$, it follows from Lemmas 6.8 and 6.10 that both τ and ϵ are equal to zero. As stated in the proof of Lemma 6.10, $|\mathcal{R}_b| = cr(\mathcal{D}) - |E(H_{\mathcal{R}_\tau})| - \rho - \beta - \tau - \epsilon$, and so $|\mathcal{R}_b| = cr(\mathcal{D}) - |E(H_{\mathcal{R}_\tau})| - 1$. By Proposition 6.20, $(2|E(H_{\mathcal{R}_\tau})| - |V(H_{\mathcal{R}_\tau})|) = 0$, and so $|\mathcal{R}_b| = cr(\mathcal{D}) - (|V(H_{\mathcal{R}_\tau})|/2) - 1$. Since $\mathcal{F}$ has been constructed, the Inductive Claim does not hold, and so $cr(\mathcal{D}) \leq k + 2$. Therefore $|\mathcal{R}_b| \leq k + 1 - (|V(H_{\mathcal{R}_\tau})|/2)$. Using this last inequality and (6.8.1) and (6.8.2), we obtain

$$|\mathcal{R}_c| \geq 2k - 1 - (|V(H_{\mathcal{R}_\tau})|/2). \quad (6.8.3)$$

By Proposition 6.21, at most one edge in $\mathcal{R}_\tau$ is not adjacent to an edge in $\mathcal{R}_b$. Let $\overline{R}$ be the cycle obtained from R by contracting all the edges in $\mathcal{R}_\tau$, except for the edge that is not adjacent to an edge in $\mathcal{R}_b$, if such an edge exists.

Since every contracted edge was adjacent to an edge in $\mathcal{R}_b$, it follows that no edges in $\mathcal{R}_c$ not adjacent in R become adjacent in $\overline{R}$ because of the contraction of an edge. Thus two elements in $\mathcal{R}_c$ are adjacent in R if and only if they are adjacent in $\overline{R}$.

Let $|\overline{R}|$ be the number of edges in $\overline{R}$. If the number of edges in $\mathcal{R}_c$ is more than one half of $|\overline{R}|$, then there are two adjacent elements of $\mathcal{R}_c$ in $\overline{R}$, and so in R . Therefore to complete the proof of Proposition 6.19 for the case $\rho = 1$ it suffices to show that $|\mathcal{R}_c| > |\overline{R}|/2$.

Since there are at least $|V(H_{\mathcal{R}_\tau})|$ elements in $\mathcal{R}_\tau$, it follows that there are at most $3k + 1 - |V(H_{\mathcal{R}_\tau})|$ edges in $\overline{R}$. Thus $|\overline{R}| \leq 3k + 1 - |V(H_{\mathcal{R}_\tau})|$. Since

$2k - 1 - (|V(H_{\mathcal{R}_\tau})|/2) > (3k + 1 - |V(H_{\mathcal{R}_\tau})|)/2$ for each $k > 3$, it follows from (6.8.3) that $|\mathcal{R}_c| > |\overline{R}|/2$.

CASE $\rho = 0$. Since both β and ρ are zero, and τ and ϵ are nonnegative, the equality $|\mathcal{R}_b| = cr(\mathcal{D}) - |E(H_{\mathcal{R}_\tau})| - \rho - \beta - \tau - \epsilon$ given in the proof of Lemma 6.10 yields $|\mathcal{R}_b| \leq cr(\mathcal{D}) - |E(H_{\mathcal{R}_\tau})|$. Since $\mathcal{F}$ has been constructed, the Inductive Claim does not hold, and so $cr(\mathcal{D}) \leq k + 2$. Therefore $|\mathcal{R}_b| \leq k + 2 - |E(H_{\mathcal{R}_\tau})|$. Using (6.8.1) and (6.8.2) and this last inequality, we obtain

$$|\mathcal{R}_c| \geq 2k - 1 - |V(H_{\mathcal{R}_\tau})| + |E(H_{\mathcal{R}_\tau})|. \quad (6.8.4)$$

Let $\overline{R}$ be the cycle obtained from R by contracting all the elements in $\mathcal{R}_\tau$ except *one* edge that is not adjacent to an edge in $\mathcal{R}_b$, if such an edge exists.

By Proposition 6.20, either each edge in $\mathcal{R}_\tau$ is adjacent to an edge in $\mathcal{R}_b$, or there are exactly two edges r, r' in $\mathcal{R}_\tau$ not adjacent to an edge in $\mathcal{R}_b$, where r and r' are adjacent. It is not difficult to check that in either case two edges in $\mathcal{R}_c$ are adjacent in R if and only if they are adjacent in $\overline{R}$.

Now we show that at most $(2|E(H_{\mathcal{R}_\tau})| - |V(H_{\mathcal{R}_\tau})|)$ elements in $\mathcal{R}_\tau$ are not contracted in the process to obtain $\overline{R}$.

By Proposition 6.20, $(2|E(H_{\mathcal{R}_\tau})| - |V(H_{\mathcal{R}_\tau})|)$ is either 0 or 1. If $(2|E(H_{\mathcal{R}_\tau})| - |V(H_{\mathcal{R}_\tau})|) = 0$, then by Proposition 6.21 each edge in $\mathcal{R}_\tau$ is adjacent to an edge in $\mathcal{R}_b$, and so every edge in $\mathcal{R}_\tau$ is contracted to get $\overline{R}$. Now suppose that $(2|E(H_{\mathcal{R}_\tau})| - |V(H_{\mathcal{R}_\tau})|) = 1$. In this case, by Proposition 6.20 at most two edges in $\mathcal{R}_\tau$ are not adjacent to an edge in $\mathcal{R}_b$. If these edges exist, then they are adjacent, and exactly one of them is contracted to get $\overline{R}$. Thus, if $(2|E(H_{\mathcal{R}_\tau})| - |V(H_{\mathcal{R}_\tau})|) = 1$, then at most one element in $\mathcal{R}_\tau$ is not contracted to obtain $\overline{R}$.

Therefore at least $|\mathcal{R}_r| - (2|E(H_{\mathcal{R}_r})| - |V(H_{\mathcal{R}_r})|)$ elements in R are not in $\overline{R}$. and since $|R| = 3k+1$, it follows that $|\overline{R}| \leq 3k+1 - (|\mathcal{R}_r| - (2|E(H_{\mathcal{R}_r})| - |V(H_{\mathcal{R}_r})|))$. Since $\rho = 0$, there are no isolated vertices in $G_{\mathcal{R}_r}$, and so $|\mathcal{R}_r| = |V(H_{\mathcal{R}_r})|$. Therefore $|\overline{R}| \leq 3k+1 - (|V(H_{\mathcal{R}_r})| - (2|E(H_{\mathcal{R}_r})| - |V(H_{\mathcal{R}_r})|)) = 3k+1 + 2(|V(H_{\mathcal{R}_r})| - |E(H_{\mathcal{R}_r})|)$.

If more than half the elements in $\overline{R}$ are in $\mathcal{R}_c$, then there are two adjacent elements of $\mathcal{R}_c$ in $\overline{R}$, and so in R . Hence to finish the proof for the case $\rho = 1$ it is enough to prove that $|\mathcal{R}_c| > |\overline{R}|/2$.

First we note that $2k-1 - |V(H_{\mathcal{R}_r})| + |E(H_{\mathcal{R}_r})| > (3k+1 + 2(|V(H_{\mathcal{R}_r})| - |E(H_{\mathcal{R}_r})|))/2$, since $k > 3$. Using (6.8.4), this implies $|\mathcal{R}_c| > (3k+1 + 2(|V(H_{\mathcal{R}_r})| - |E(H_{\mathcal{R}_r})|))/2$. Thus we are done, since the right hand side of this inequality is not less than $(|\overline{R}|)/2$. $\square$

6.8.1 Case $\beta = 1$

As in the case $\beta = 0$, the special cases $k = 4$ and $k = 5$ require a separate discussion. We deal with these cases in Appendix B, and work in this subsection in the general case $k \geq 6$.

We assume that the Inductive Claim does not hold, and derive a contradiction.

Since $\beta = 1$, the blue cycle has exactly one self-crossing. Since the Inductive Claim does not hold, $|\mathcal{F}|$ can be constructed, and by Lemma 6.10 $|\mathcal{F}| \geq 3k+1 - 2cr(\mathcal{D}) + \rho + 2\beta + 2\tau + 2\epsilon + (2|E(H_{\mathcal{R}_r})| - |V(H_{\mathcal{R}_r})|)$. Using again the assumption that the Inductive Claim does not hold, $cr(\mathcal{D}) \leq k+2$, and so $|\mathcal{F}| \geq k-1$, since $\beta = 1$ and each of ρ, τ, ϵ , and $(2|E(H_{\mathcal{R}_r})| - |V(H_{\mathcal{R}_r})|)$ is nonnegative. Since $k \geq 6$, by Proposition 6.18 there are two adjacent clean red edges, and so Proposition

6.17 applies. Therefore at least one red edge is crossed more than once.

It follows from the definitions of ρ, τ, ϵ , and $H_{\mathcal{R}_\tau}$ that if each of ρ, τ, ϵ , and $(2|E(H_{\mathcal{R}_\tau})| - |V(H_{\mathcal{R}_\tau})|)$ is zero, then no red edge is crossed more than once. We conclude that in the present case $\rho + 2\tau + 2\epsilon + (2|E(H_{\mathcal{R}_\tau})| - |V(H_{\mathcal{R}_\tau})|) \geq 1$, and since $\beta = 1$, $\rho + 2\beta + 2\tau + 2\epsilon + (2|E(H_{\mathcal{R}_\tau})| - |V(H_{\mathcal{R}_\tau})|) \geq 3$.

By assumption, the Inductive Claim does not hold, and so $cr(\mathcal{D}) \leq k + 2$. Thus, applying Lemma 6.10, we obtain $|\mathcal{F}| \geq k$. By Proposition 6.7, there are at least $3k$ blue edges dominated by clean red edges, and by Proposition 6.3 none of these $3k$ blue edges is crossed by a blue edge. But this is impossible, since the assumption $\beta = 1$ implies that there are exactly $3k - 1$ blue edges not crossed by any blue edge. $\square$

6.9 The Error in Fiorini's Argument

Fiorini [F1] attempts an induction as follows. If some edge e of the blue cycle B is in a blue-blue crossing, then deleting e , two successive edges from B , and (as suggested in [F2]) contracting the red edge joining the resulting degree-3 vertices, yields a drawing of $P'(3(k-1)+h, 3)$. This is correct; what is not is the assumption that this drawing has fewer crossings than the drawing of $P'(3k+h, 3)$. What is missing is that the contracted edge should have no crossings.

6.10 Comments

An important motivation for investigating the crossing numbers of the Generalized Petersen Graphs is the following. It follows from the work in [B] and [L] that for the random cubic graph G , $cr(G) \geq k|V|^2$. However, for all families of cubic graphs for which the crossing number is known, the crossing number is proportional to $|V|$. Thus it would be interesting to find a family of cubic graphs whose crossing numbers are proportional to $|V|^2$.

Fiorini also claimed [F1] that the crossing number of the Derived Generalized Petersen Graph $P'(3k + h, 3)$ is $k + h$ for every $h \in \{0, 1, 2\}$. Unfortunately, the error pointed out in Section 6.9 also invalidates the proof of this statement. However, we expect that the techniques developed above can be used to show that the crossing number of $P'(3k + h, 3)$ is indeed $k + h$ for every h .

As noted in [F1], the graphs $P(3k+1, k)$, $P(3k+1, 2k+1)$, and $P(3k+1, 3k-2)$ have the same crossing number as $P(3k+1, 3)$, and each of $P(3k+2, k+3)$, $P(3k+2, 2k+1)$, and $P(3k+2, 3k-1)$ has the same crossing number as $P(3k+2, 3)$. Therefore it follows from Theorem 6.1 that $cr(P(3k+1, k)) = cr(P(3k+1, 2k+1)) = cr(P(3k+1, 3k-2)) = k+3$, and that $cr(P(3k+2, k+3)) = cr(P(3k+2, 2k+1)) = cr(P(3k+2, 3k-1)) = k+2$.

Exoo et al. [EHK] proved that $cr(P(m, 2))$ is 0 for each even $m \geq 4$, and $cr(P(m, 2))$ is 3 for each odd $m \geq 7$. They also proved that the crossing numbers of $P(3, 2)$ and $P(5, 2)$ are 0 and 2, respectively. Except for $P(m, 2)$, $P(m, 3)$, for the graphs in the above paragraph, and for a few isolated cases [F1, YY], the crossing number of the Generalized Petersen Graph $P(m, \ell)$ remains unknown.

APPENDIX A

Our aim in this appendix is to prove the following.

Proposition A.1. *Every n -drawing of $P'(9, 3)$ has at least three crossings.*

We remark that $P'(9, 3)$ consists of a blue 9-cycle B obtained by the concatenation of the blue edges $b_1 = (v_1, v_2), b_2 = (v_2, v_3), \dots, b_9 = (v_9, v_1)$, plus three red 3-cycles $R_{1,4,7}, R_{2,5,8}, R_{3,6,9}$. The edges of $R_{1,4,7}$ are the red edges $r_1 = (v_1, v_4), r_4 = (v_4, v_7)$, and $r_7 = (v_7, v_1)$. Similarly, the edges of $R_{2,5,8}$ are r_2, r_5 , and r_8 , and the edges of $R_{3,6,9}$ are r_3, r_6 , and r_9 .

Proof of Proposition A.1. Let $\mathcal{D}$ be an n -drawing of $P'(9, 3)$. If B has at least three self-crossings in $\mathcal{D}$ then we are done, and so we assume that B has at most two self-crossings. We analyze the cases in which B has exactly 0, 1, and 2 self-crossings separately.

CASE 1. *B has no self-crossings.*

Let H_1, H_2, H_3 denote the subgraphs of $P'(9, 3)$ induced by the edges in B plus the sets of edges $\{r_1, r_2, r_3\}, \{r_4, r_5, r_6\}$, and $\{r_7, r_8, r_9\}$ respectively. Each H_i is a subdivision of $K_{3,3}$ and so, for each H_i , two edges in H_i cross each other. For each crossing involving two edges in H_i , at least one of these two edges is red, since there are no blue-blue crossings. Since each red edge is in exactly one H_i , it follows immediately that $\mathcal{D}$ has at least three crossings in total.

CASE 2. *B has exactly one self-crossing.*

First we show that if $\mathcal{D}$ has either a red-blue crossing or a red-red crossing, then $\mathcal{D}$ has at least three crossings (as required). Finally, we prove that $\mathcal{D}$ must have either a red-blue crossing or a red-red crossing, since otherwise we obtain a contradiction.

If some red edge r crosses an edge in B , then the red cycle that contains r crosses B in at least two points, since $\mathcal{D}$ is an n -drawing. Thus, if there is a red-blue crossing, then there are at least two red-blue crossings, and so $\mathcal{D}$ has at least three crossings in total, since the self-crossing of B is blue-blue.

Now let us suppose that there is a red-red crossing. It suffices to show that in such a case there are at least two red-red crossings, since the self-crossing of $\mathcal{D}$ is blue-blue. First we note that the red-red crossing must involve edges in different red cycles, as otherwise the good-drawing condition of $\mathcal{D}$ is violated. But if two red cycles cross, then they do so in at least two points, and so there are at least two red-red crossings in $\mathcal{D}$.

Finally, we show that there must be either a red-blue crossing or a red-red crossing in $\mathcal{D}$. For suppose that $\mathcal{D}$ has neither a red-red crossing nor a red-blue crossing. Thus, $\mathcal{D}$ has exactly one crossing, namely the blue-blue self-crossing of B . If we split the vertices of $\mathcal{D}$ to create spokes, then we obtain a drawing of $P(9, 3)$ with exactly one crossing. But this is impossible, since $cr(P(9, 3)) = 2$ [F1].

CASE 3. *B has exactly two self-crossings.*

If some red edge is crossed then we are done, since the self-crossings of B are

blue–blue. Thus, we assume that no red edge is crossed in $\mathcal{D}$.

It is easy to check that if for some red cycle $R_{i,i+3,i+6}$ the other two red cycles are contained in different components of $\mathbf{R}^2 \setminus \{R_{i,i+3,i+6}\}$, then some blue edge must cross $R_{i,i+3,i+6}$ (the cycles in $\{R_{1,4,7}, R_{2,5,8}, R_{3,6,9}\} \setminus \{R_{i,i+3,i+6}\}$ are joined by three blue edges, each of whom must cross $R_{i,i+3,i+6}$). Thus, for each red cycle $R_{i,i+3,i+6}$, the other two red cycles are contained in the same component of $\mathbf{R}^2 \setminus \{R_{i,i+3,i+6}\}$. Moreover, by using a Riemann stereographic projection argument we can assume that, for each red cycle $R_{i,i+3,i+6}$, the other two red cycles are contained in the unbounded component of $\mathbf{R}^2 \setminus \{R_{i,i+3,i+6}\}$.

Now let $P_{1,3}$, $P_{4,6}$, and $P_{7,9}$ denote the blue 2-paths from v_1 to v_3 , from v_4 to v_7 , and from v_7 to v_9 respectively. Since there are exactly two blue–blue crossings, it follows that there are two paths in $\{P_{1,3}, P_{4,6}, P_{7,9}\}$ that do not cross each other in $\mathcal{D}$. By relabeling if necessary, we can assume that no edge in $P_{1,3}$ crosses an edge in $P_{4,6}$.

Let H be the subgraph of $P'(9, 3)$ that consists of the red edges plus the blue edges in $P_{1,3}$ and $P_{4,6}$. It is easy to see that there are six components in $\mathbf{R}^2 \setminus H$. Three of these components, say F_1, F_2 , and F_3 , are the open discs bounded by $R_{1,4,7}, R_{2,5,8}$, and $R_{3,6,9}$ respectively. Another component (say F_4) is bounded by $(v_1, v_2), (v_4, v_5)$, and two red arcs $A_{1,4}$ and $A_{2,5}$ from v_1 to v_4 and from v_2 to v_5 respectively. The fifth component F_5 is bounded by $(v_2, v_3), (v_5, v_6)$, and two red arcs $A'_{2,5}$ (different from $A_{2,5}$) from v_2 to v_5 , and $A'_{3,6}$ from v_3 to v_6 . The last component F_6 is bounded by $(v_1, v_2), (v_2, v_3), (v_4, v_5), (v_5, v_6)$, and red arcs $A'_{1,4}$ (different from $A_{1,4}$) from v_1 to v_4 , and $A'_{3,6}$ (different from $A_{3,6}$) from v_3 to v_6 . Without any loss of generality, we can assume that this last component is the unbounded component of $\mathbf{R}^2 \setminus H$. This situation is illustrated in Fig. 6.20.

Figure 6.20

It is not difficult to see that the edge b_3 must be contained in F_6 (as in Fig. 6.20), since otherwise the good-drawing condition of $\mathcal{D}$ is violated.

It is also readily seen that each of b_6 and b_9 must cross an edge in $\{b_1, b_2, b_3, b_4, b_5\}$. Thus it suffices to show that either b_7 or b_8 crosses an edge in $\mathcal{B} = \{b_1, b_2, b_3, b_4, b_5\}$.

If b_7 does not cross an edge in $\mathcal{B}$, then b_7 is contained in F_4 . Since v_9 is not in the boundary of F_4 , it follows that b_8 must cross an edge in $\mathcal{B}$. $\square$

APPENDIX B

In this appendix we prove that the Inductive Claim holds for $h = 1$ in the cases $k = 4$ and 5 . As proved in Section 6.8, we can assume that in each case b is either 0 or 1 . First we assume that $\beta = 0$ and show that the Inductive Claim holds for both values of k , and then we prove the same statement for the case $\beta = 1$. Before proceeding to analyze these cases separately, we make a few observations that hold independently of the value of β .

Proposition B.1. *Let $S = \{s_{i_1}, \dots, s_{i_m}\}$ and $T = \{t_{j_1}, \dots, t_{j_n}\}$ be disjoint sets of red edges, where $S \subseteq \mathcal{R}_b$ and each edge in T has at least one crossing. Then there are at least $|S| + \lceil |T|/2 \rceil$ crossings in $\mathcal{D}$ that are not blue–blue crossings.*

Proof. It follows easily from the definition of $\mathcal{R}_b$. $\square$

Proposition B.2. *If there is a pair of adjacent clean red edges, then the Inductive Claim holds.*

Proof. For the case $\beta = 0$, see the second paragraph in the Proof of the Inductive Claim for the case $\beta = 0$, for $k \geq 6$ (this part is independent of the value of k).

Now suppose that $\beta = 1$, and suppose that there is a pair of adjacent clean red edges. It is not difficult to see, by the same arguments used to prove Proposition 6.17, that some red edge has at least two crossings. Hence one of ρ, τ, ϵ , and $(2|E(H_{\mathcal{R}_\tau})| - |V(H_{\mathcal{R}_\tau})|)$ is at least one. Since $\beta = 1$, by Lemma 6.10 we obtain $|\mathcal{F}| \geq 3k + 4 - 2cr(\mathcal{D})$.

Since $\beta = 1$, two blue edges cross, and so by Proposition 6.3 at most $3k - 1$ blue edges can be dominated by a clean red edge. Thus, by Proposition 6.7,

$|\mathcal{F}| \leq k - 1$. Since $|\mathcal{F}| \geq 3k + 4 - 2cr(\mathcal{D})$, this implies $cr(\mathcal{D}) \geq k + 3$, and so the Inductive Claim holds. $\square$

Since if there are two adjacent clean red edges then the Inductive Claim holds, we assume the following.

Non-adjacency Condition. *No two clean red edges are adjacent.*

We claim that we can assume that $|\mathcal{F}| = k - 1$. First suppose that $\beta = 0$. By Proposition 6.19 and the Non-adjacency Condition, $|\mathcal{F}|$ is at least $k - 1$. Now if $\beta = 1$, then by Lemma 6.10 $|\mathcal{F}| \geq k - 1$. Hence in either case $|\mathcal{F}| \geq k - 1$. On the other hand, it is easy to prove, as in Proposition 6.18, that if $|\mathcal{F}| \geq k$, then there are two adjacent clean red edges (for every $k \geq 4$). Thus by the Non-adjacency Condition $|\mathcal{F}| \leq k - 1$. Therefore we can assume that $|\mathcal{F}| = k - 1$.

The strategy to prove that the Inductive Claim is true is the same for both $\beta = 0$ and 1, and for both possible values of k . In each case we assume that the Inductive Claim does not hold, so that the family $\mathcal{F}$ can be constructed. For both values of k we conclude, after some case analysis, that $cr(\mathcal{D}) \geq k + 3$, contradicting the assumption that the Inductive Claim is not true.

We investigate separately the cases $\beta = 0$ and $\beta = 1$.

Case $\beta = 0$

Since $\beta = 0$, the blue cycle B has no self-crossings. Let F_1, F_2 be the components of $\mathbb{R}^2 \setminus B$. We analyze individually the cases $k = 4$ and $k = 5$.

Case $k = 4$.

Since $|\mathcal{F}| = k - 1 = 3$, there are at least three pairwise nonadjacent red edges r_i, r_j , and r_l in $\mathcal{F}$. These three edges are clean, since every edge in $\mathcal{F}$ is clean. Since the dominated 3-paths of these red edges have no common edges, it follows that the shortest blue path that contains all of $P(r_i), P(r_j)$, and $P(r_l)$ has eleven edges. Relabeling if necessary, we can assume that $j = i + 4$ and $l = i + 8$.

Each edge in $\{r_i, r_j, r_l\}$ is contained in one of F_1 and F_2 , and so there are eight different ways to accommodate the edges in $\{r_i, r_j, r_l\}$ in the faces F_1 and F_2 . It is easy to check that it suffices to consider the following four cases, since the other four possibilities are obtained from these cases by relabeling the vertices according to the rule $v_m \rightarrow v_{11-m}$, or by exchanging F_1 and F_2 , or by a combination of these two transformations.

CASE 1. *Each of r_i, r_j , and r_l is in F_1 .* This case is illustrated in Figure 6.21. The n -drawing condition of $\mathcal{D}$ and Propositions 6.3 and 6.4 imply that each of $r_{i+1}, r_{i+3}, r_{i+5}, r_{i+7}, r_{i+10}$, and r_{i+11} is in $\mathcal{R}_b$. By Proposition 6.5, r_{i+9} has at least one crossing. Applying Proposition B.1 with $S = \{r_{i+1}, r_{i+3}, r_{i+5}, r_{i+7}, r_{i+10}, r_{i+11}\}$ and $T = \{r_{i+9}\}$, we obtain $cr(\mathcal{D}) \geq 7 = k + 3$.

CASE 2. *r_i is in F_1 , and r_j and r_l are in F_2 .* The n -drawing condition of $\mathcal{D}$ and Propositions 6.3 and 6.4 imply that each one of r_{i+2}, r_{i+5} , and r_{i+7} is in $\mathcal{R}_b$. It is also readily checked that at least one of r_{i+9} and r_{i+12} is in $\mathcal{R}_b$. Let r_m be an edge in $\{r_{i+9}, r_{i+12}\}$ such that $\{r_{i+9}, r_{i+12}\} \setminus \{r_m\}$ is in $\mathcal{R}_b$. By Proposition 6.5, each one of r_{i+1} and r_{i+3} has at least one crossing, and since r_{i+10} is adjacent to r_i and r_{i+8} is adjacent to r_{i+11} , the Non-adjacency Condition implies that each of r_{i+10} and r_{i+11} has at least one crossing. Applying Proposition B.1 with $S = \{r_{i+2}, r_{i+5}, r_{i+7}\} \cup (\{r_{i+9}, r_{i+12}\} \setminus \{r_m\})$ and $T = \{r_{i+1}, r_{i+3}, r_{i+10}, r_{i+11}\}$.

If each of r_i , r_j , and r_l is in F_1 , then the n -drawing condition implies that r_{i+3} is in R_b , that is, that r_{i+3} crosses the blue cycle. Similarly, r_{i+1} , r_{i+5} , r_{i+7} , r_{i+10} , and r_{i+11} are in R_b .

Figure 6.21

If r_i and r_j are in F_1 and r_l is in F_2 , then r_{i+6} is in R_b . Similarly, r_{i+1} and r_{i+3} are in R_b , and at least one of r_{i+9} and r_{i+12} is in R_b .

Figure 6.22

$r_m\}$. we obtain $cr(\mathcal{D}) \geq 7$.

CASE 3. r_i and r_j are in F_1 , and r_l is in F_2 . This case is illustrated in Figure 6.22. By Propositions 6.3 and 6.4 and the n -drawing property of $\mathcal{D}$, each one of r_{i+1}, r_{i+3} , and r_{i+6} is in $\mathcal{R}_b$, and at least one of r_{i+9} and r_{i+12} is in $\mathcal{R}_b$. Let r_m be an edge in an edge in $\{r_{i+9}, r_{i+12}\}$ such that $\{r_{i+9}, r_{i+12}\} \setminus \{r_m\}$ is in $\mathcal{R}_b$. By Proposition 6.5, both r_{i+7} and r_m are crossed at least once, and by the Non-adjacency Condition each of r_{i+5}, r_{i+10} , and r_{i+11} has at least one crossing. Applying Proposition B.1 with $S = \{r_{i+1}, r_{i+3}, r_{i+6}\} \cup (\{r_{i+9}, r_{i+12}\} \setminus \{r_m\})$ and $T = \{r_{i+5}, r_{i+7}, r_{i+10}, r_{i+11}, r_m\}$, we obtain $cr(\mathcal{D}) \geq 7$.

CASE 4. r_i and r_l are in F_1 , and r_j is in F_2 . In this case using Propositions 6.3 and 6.4 and the n -drawing property of $\mathcal{D}$, we obtain that $r_{i+2}, r_{i+6}, r_{i+10}$, and r_{i+11} are in $\mathcal{R}_b$. On the other hand, by Proposition 6.5, each of $r_{i+3}, r_{i+5}, r_{i+7}, r_{i+9}$, and r_{i+12} has at least one crossing, and so applying Proposition B.1 with $S = \{r_{i+2}, r_{i+6}, r_{i+10}, r_{i+11}\}$ and $T = \{r_{i+3}, r_{i+5}, r_{i+7}, r_{i+9}, r_{i+12}\}$, we obtain $cr(\mathcal{D}) \geq 7$.

Case $k = 5$.

Since $|\mathcal{F}| = k - 1$, there are four pairwise nonadjacent red edges r_i, r_j, r_l , and r_m in $\mathcal{F}$. These four edges are clean, since every edge in $\mathcal{F}$ is clean. The Non-adjacency Condition is satisfied only if the end points of each of the paths $P(r_i), P(r_j), P(r_l)$, and $P(r_m)$ are incident with edges not dominated by any of r_i, r_j, r_l , and r_m . Since the blue cycle has sixteen edges, we can assume that $j = i + 4, l = i + 8$, and $m = i + 12$. Each edge in $\{r_i, r_j, r_l, r_m\}$ is contained in one of F_1 and F_2 , and so there are sixteen possible ways in which the edges

r_i, r_j, r_l, r_m can be distributed in F_1 and F_2 . As in the case $k = 4$, we do not need to check all sixteen configurations. We examine four cases below, and it can be readily verified that each of the remaining twelve possibilities follows from one of these four cases by relabeling the vertices according to the rule $v_p \rightarrow v_{p+3q+1}$ for some integer q , or by exchanging F_1 and F_2 , or by a combination of these two modifications.

CASE 1. *Every edge in $\{r_i, r_j, r_l, r_m\}$ is in F_1 .* This case is illustrated in Figure 6.23. By Propositions 6.3 and 6.4 and the n -drawing property of $\mathcal{D}$, all the edges in $S = \{r_{i+1}, r_{i+3}, r_{i+5}, r_{i+7}, r_{i+9}, r_{i+11}, r_{i+13}, r_{i+15}\}$ are in $\mathcal{R}_b$, and so by Proposition B.1 (using $T = \emptyset$) the inequality $cr(\mathcal{D}) \geq 8$ follows.

CASE 2. *r_i is in F_1 , and r_j, r_l and r_m are in F_2 .* Using Propositions 6.3 and 6.4 and the n -drawing property of $\mathcal{D}$, we obtain that each of $r_{i+2}, r_{i+5}, r_{i+7}, r_{i+9}, r_{i+11}$, and r_{i+14} is in $\mathcal{R}_b$. By the Non-adjacency Condition, each of r_{i+1}, r_{i+13} , and r_{i+15} has at least one crossing. Using Proposition B.1 with $S = \{r_{i+2}, r_{i+5}, r_{i+7}, r_{i+9}, r_{i+11}, r_{i+14}\}$ and $T = \{r_{i+1}, r_{i+13}, r_{i+15}\}$, we obtain $cr(\mathcal{D}) \geq 8$.

CASE 3. *r_i and r_j are in F_1 , and r_l and r_m are in F_2 .* Using the n -drawing property of $\mathcal{D}$ and Propositions 6.3 and 6.4, each edge in $S = \{r_{i+1}, r_{i+3}, r_{i+6}, r_{i+9}, r_{i+11}, r_{i+14}\}$ is in $\mathcal{R}_b$. On the other hand, Proposition 6.5 implies that each of r_{i+5}, r_{i+7} , and r_{i+13} has at least one crossing, and so using Proposition B.1 with S as above and $T = \{r_{i+5}, r_{i+7}, r_{i+13}\}$, we obtain $cr(\mathcal{D}) \geq 8$.

CASE 4. *r_i and r_l are in F_1 , and r_j and r_m are in F_2 .* In this case by Propositions 6.3 and 6.4 and the n -drawing condition of $\mathcal{D}$, each one of $r_{i+2}, r_{i+6}, r_{i+10}$, and r_{i+14} is in $\mathcal{R}_b$, and by Proposition 6.5 each edge in $T = \{r_{i+1}, r_{i+3}, r_{i+5}, r_{i+7}, r_{i+9}, r_{i+11}, r_{i+13}\}$ has at least one crossing. Using Proposition B.1 with $S = \{r_{i+2}, r_{i+6},$

If each of r_i , r_j , r_l , and r_m is in F_1 , then r_{i+5} is in R_b .
 Similarly, each of r_{i+1} , r_{i+3} , r_{i+7} , r_{i+9} , r_{i+11} , r_{i+13} ,
 and r_{i+15} is in R_b .

Figure 6.23

If r_i and r_j are in F_1 and r_l is in F_3 , then r_{i+7} is in R_b .
 Similarly, each of r_{i+1} , r_{i+3} , r_{i+6} , r_{i+10} , and r_{i+12} is in R_b .

Figure 6.24

$r_{i+10}, r_{i+14}\}$, and T as above, we obtain $cr(\mathcal{D}) \geq 8$.

Case $\beta = 1$

First we prove the following.

Proposition B.3. *If the Inductive Claim does not hold, then no red edge has more than one crossing.*

Proof. We show that if some red edge has more than one crossing, then $|\mathcal{F}| \geq k$, contradicting the assumption that $|\mathcal{F}| \leq k - 1$.

Suppose that some red edge has more than one crossing. It follows from the definitions of ρ, τ, ϵ , and $H(\mathcal{R}_r)$ that one of ρ, τ, ϵ , and $2(|E(H_{\mathcal{R}_r})| - |V(H_{\mathcal{R}_r})|)$ is at least one. Since by assumption the Inductive Claim does not hold, $cr(\mathcal{D}) \leq k + 2$, and so applying Lemma 6.10 we obtain $|\mathcal{F}| \geq k$. $\square$

We show that the Inductive Claim must hold by making use of Proposition B.3, and in a few cases by obtaining directly the inequality $cr(\mathcal{D}) \geq k + 3$.

Since $\beta = 1$, the blue cycle has exactly one self-crossing. The two blue edges that cross cannot be adjacent, as otherwise we could easily remove the crossing without violating the n -drawing property. Therefore, if we delete these two blue edges, we obtain two blue paths P_B and P'_B , such that the number of edges in each of P_B and P'_B is at least 1.

Let $|P_B|$ and $|P'_B|$ denote the number of edges in P_B and P'_B respectively. Since there are $3k + 1$ blue edges in total, it follows that $|P_B| + |P'_B| = 3k - 1$.

Since no blue edge dominated by a clean red edge has a blue-blue crossing, it follows that if r is a red clean edge, then $P(r)$ is contained in either P_B or P'_B .

We analyze separately the possible values of k .

Case $k = 4$.

As in the case $\beta = 0$, the equality $|\mathcal{F}| = k - 1$ implies that there are at least three pairwise nonadjacent red edges r_i, r_j , and r_l in $\mathcal{F}$.

The blue cycle has exactly thirteen edges, and so $|P_B| + |P'_B| = 11$. We can assume that $P_B > P'_B$, and so $|P_B| \geq 6$. If $|P_B| = 6$, then P_B contains two adjacent dominated 3-paths. Since this violates the Non-adjacency Condition, $|P_B|$ must be at least 7. On the other hand, if $|P'_B| \leq 2$, then all three dominated 3-paths must be contained in P_B , and since $|P'_B| \geq 1$, we must have $|P_B| \leq 10$. It is easy to check that in such a case at least two of r_i, r_j , and r_l are adjacent. Since this contradicts the Non-adjacency condition, we conclude that $|P'_B| \geq 3$. Since $|P'_B| \leq 5$, P'_B contains exactly one 3-path dominated by one of r_i, r_j , and r_l , and the other dominated 3-paths are contained in P_B .

Since $|P_B| \geq 7$ and $|P'_B| \geq 3$, it follows that either $|P_B| = 7$ and $|P'_B| = 4$ or $|P_B| = 8$ and $|P'_B| = 3$. We analyze these cases separately.

Case $|P_B| = 7, |P'_B| = 4$.

We can assume without any loss of generality that $P(r_i)$ and $P(r_j)$ are contained in P_B , and that $P(r_l)$ is contained in P'_B . Since $|P_B| = 7$, the middle edge of P_B is in neither $P(r_i)$ nor $P(r_j)$. Since $|P'_B| = 4$, exactly one of the edges in P'_B is not contained in $P(r_l)$. Relabeling if necessary, we can assume that $j = i + 4$ and $l = i + 9$. Thus P_B joins v_i to v_{i+7} , and P'_B joins v_{i+8} to v_{i+12} , and the blue edges that cross each other are b_{i+7} and b_{i+12} .

Let F_1, F_2 , and F_3 be the components of $\mathbb{R}^2 \setminus B$. Each of P_B and P'_B is contained in the boundary of two of F_1, F_2 , and F_3 , and the boundary of each of these components contains either P_B or P'_B . We can assume without any loss of generality that P_B is contained in the boundary of F_1 and F_2 , and that P'_B is contained in the boundary of F_2 and F_3 . Thus each of r_i and r_j is contained in either F_1 or F_2 , and r_l is contained in either F_2 or F_3 . Hence we have to analyze eight different possibilities, according to which face contains each of r_i, r_j , and r_l .

CASE 1. r_i and r_j are contained in F_1 and r_l is contained in F_3 . This case is illustrated in Figure 6.24. In this case the n -drawing condition implies that each of $r_{i+1}, r_{i+3}, r_{i+6}, r_{i+7}, r_{i+10}$, and r_{i+12} is in $\mathcal{R}_b$. Using Proposition B.1 with $S = \{r_{i+1}, r_{i+3}, r_{i+6}, r_{i+7}, r_{i+10}, r_{i+12}\}$ and $T = \emptyset$, we obtain $cr(\mathcal{D}) \geq 7$, since there is one blue-blue crossing.

The remaining seven cases are handled in a common way. In each case we claim that some red edge has at least two crossings; in every case this follows from the n -drawing condition, and the fact that if an edge intersects both F_1 and F_3 then it must intersect the blue cycle at least twice. Since by Proposition B.3 this is all we need in order to show that the Inductive Claim holds, we just indicate in each of these seven cases which red edge has at least two crossings.

CASE 2. r_i and r_l are in F_2 , and r_j is in F_1 . The edge r_{i+7} has at least two crossings.

CASE 3. r_i is in F_2 , r_j is in F_1 , and r_l is in F_3 . The edge r_{i+12} has at least two crossings.

CASE 4. r_i, r_j , and r_l are in F_2 . The edge r_{i+11} has at least two crossings.

CASE 5. r_i and r_j are in F_2 , and r_l is in F_3 . The edge r_{i+6} has at least two crossings.

CASE 6. r_i and r_j are in F_1 , and r_l is in F_2 . The edge r_{i+7} has at least two crossings.

CASE 7. r_i is in F_1 , and r_j and r_l are in F_2 . The edge r_{i+10} has at least two crossings.

CASE 8. r_i is in F_1 , r_j is in F_2 , and r_l is in F_3 . The edge r_{i+6} has at least two crossings.

Case $|P_B| = 8, |P'_B| = 3$.

As in the case $|P_B| = 7, |P'_B| = 4$, we can assume without any loss of generality that $P(r_i)$ and $P(r_j)$ are contained in P_B , and that $P(r_l)$ is contained in P'_B . Let F_1, F_2 , and F_3 be the three components in $\mathbb{R}^2 \setminus B$. We can also assume that P_B is in the boundary of F_1 and F_2 , and that P_3 is in the boundary of F_2 and F_3 .

There is an m such that P_B joins v_m to v_{m+8} , and P'_B joins v_{m+9} to v_{m+12} . Exactly two edges in P_B are not dominated by any of $P(r_i)$ and $P(r_j)$. Relabeling if necessary, we get that these two edges are either b_m and b_{m+4} or b_{m+3} and b_{m+4} . Again by relabeling if necessary, in the first case we can assume that $i = m + 1$, and in the second case we can assume that $i = m$. We note that in the first case $j = i + 4$ and $l = i + 8$, and in the second case $j = i + 5$ and $l = i + 9$.

Before analyzing these two possibilities separately, we remark that in every case each of r_i and r_j is contained in either F_1 or F_2 , and r_l is contained in either F_2 or F_3 . Thus, for each possible distribution of r_i in P_B (depending on whether

$i = m$ or $i = m + 1$), we need to analyze eight possible different configurations, according to in which component of $\mathbf{R}^2 \setminus B$ is contained each of r_i, r_j , and r_l .

Subcase A. $i = m$.

First we analyze the following case.

CASE 1. r_i and r_j are in F_1 , and r_l is in F_3 . This case is illustrated in Figure 6.25. In this case the n -drawing property of $\mathcal{D}$ implies that each of $r_{i+2}, r_{i+3}, r_{i+6}, r_{i+8}, r_{i+10}$, and r_{i+12} is in $\mathcal{R}_b$. Therefore there are at least six blue-red crossings, and since there is a blue-blue crossing, it follows that $cr(\mathcal{D}) \geq 7$.

For each of the remaining seven cases we identify a red edge which intersects both F_1 and F_3 , due to the n -drawing property of $\mathcal{D}$. By Proposition B.3, this shows that the Inductive Claim is true.

CASE 2. r_i and r_l are in F_2 , and r_j is in F_1 . The edge r_{i+11} has at least two crossings.

CASE 3. r_i is in F_2 , r_j is in F_1 , and r_l is in F_3 . The edge r_{i+12} has at least two crossings.

CASE 4. r_i, r_j and r_l are in F_2 . The edge r_{i+7} has at least two crossings.

CASE 5. r_i and r_j are in F_2 , and r_l is in F_3 . The edge r_{i+12} has at least two crossings.

CASE 6. r_i and r_j are in F_1 , and r_l is in F_2 . The edge r_{i+10} has at least two crossings.

CASE 7. r_i is in F_1 , and r_j and r_l are in F_2 . The edge r_{i+7} has at least two crossings.

If r_i and r_j are in F_1 and r_{i+9} is in F_3 , then r_{i+3} is in R_b . Similarly, each of r_{i+2} , r_{i+6} , r_{i+8} , r_{i+10} , and r_{i+12} is in R_b .

Figure 6.25

If r_i and r_j are in F_1 , and r_l is in F_3 then each of r_{i+1} , r_{i+3} , r_{i+5} , r_{i+7} , r_{i+9} , and r_{i+11} is in R_b .

Figure 6.26

CASE 8. r_i is in F_1 , r_j is in F_2 , and r_l is in F_3 . The edge r_{i+6} has at least two crossings.

Subcase B. $i = m + 1$.

The following case requires a special analysis.

CASE 1. r_i and r_j are in F_1 , and r_l is in F_3 . This case is illustrated in Figure 6.26. In this case the n -drawing property of $\mathcal{D}$ implies that each of $r_{i+1}, r_{i+3}, r_{i+5}, r_{i+7}, r_{i+9}$, and r_{i+11} belongs to $\mathcal{R}_b$. Thus $cr(\mathcal{D}) \geq 7$, since there are at least six blue-red crossings and one blue-blue crossing.

We show that the Inductive Claim holds in the other seven cases by exhibiting a red edge that intersects F_1 and F_3 . This edge has at least two blue-red crossings, as follows from the n -drawing property of $\mathcal{D}$, and so we are done by Proposition B.3.

CASE 2. r_i and r_l are in F_2 , and r_j is in F_1 . The edge r_{i+7} has at least two crossings.

CASE 3. r_i is in F_2 , r_j is in F_1 , and r_l is in F_3 . The edge r_{i+11} has at least two crossings.

CASE 4. r_i, r_j and r_l are in F_2 . The edge r_{i+6} has at least two crossings.

CASE 5. r_i and r_j are in F_2 , and r_l is in F_3 . The edge r_{i+11} has at least two crossings.

CASE 6. r_i and r_j are in F_1 , and r_l is in F_2 . The edge r_{i+7} has at least two crossings.

CASE 7. r_i is in F_1 , and r_j and r_l are in F_2 . The edge r_{i+6} has at least two crossings.

CASE 8. r_i is in F_1 , r_j is in F_2 , and r_l is in F_3 . The edge r_{i+5} has at least two crossings.

Case $k = 5$.

Since $|\mathcal{F}| = k - 1$, there are four pairwise nonadjacent red edges r_i, r_j, r_l , and r_m in $\mathcal{R}$. Since the blue cycle has sixteen edges, two of which are contained in neither P_B nor P'_B , it follows that $|P_B| + |P'_B| = 14$. It is not difficult to verify that if either P_B or P'_B contains less than seven edges, then the Non-adjacency Condition is violated, and so we conclude that $|P_B| = |P'_B| = 7$.

Since there are exactly six edges in each of P_B and P'_B dominated by clean red edges, it follows from the Non-adjacency Condition that only the middle edge of each of P_B and P'_B is not dominated by any of r_i, r_j, r_l , and r_m . Thus we can assume, by relabeling the vertices if necessary, that $j = i + 4$, $l = i + 8$, and $m = i + 12$.

Let F_1, F_2 , and F_3 be the components of $R^2 \setminus B$. The boundaries of two of these components, say F_1 and F_2 , contain P_B , and the boundary of F_3 and the boundary of one of F_1 and F_2 contains P'_B . We can assume without any loss of generality that the boundaries of F_2 and F_3 contain P'_B .

Each of r_i and r_j is contained in either F_1 or F_2 , and each of r_l and r_m is contained in either F_2 or F_3 . Let us suppose that both r_j and r_l are contained in F_2 . It is easy to check that the n -drawing property of $\mathcal{D}$ implies that r_{i+6} crosses the blue cycle at least twice. Therefore in this case the Inductive Claim holds, by

Proposition B.3. The same kind of argument can be used to show that if both r_i and r_m are contained in F_2 then the Inductive Claim holds.

On the other hand, if r_j is contained in F_2 and r_l is contained in F_3 , then r_{i+5} crosses two blue edges, and so in this case the Inductive Claim also follows. It is readily verified that a similar argument shows that if (i) r_j is contained in F_1 and r_l is contained in F_2 , or (ii) r_i is contained in F_1 and r_m is contained in F_2 , or (iii) r_i is contained in F_2 and r_m is contained in F_3 , then by Proposition B.3 the Inductive Claim must hold. Therefore in each of the following cases the Inductive Claim is true:

- (a) Both r_j and r_l are contained in F_2 .
- (b) Both r_i and r_m are contained in F_2 .
- (c) r_j is contained in F_2 , and r_l is contained in F_3 .
- (d) r_j is contained in F_1 and r_l is contained in F_2 .
- (e) r_i is contained in F_1 and r_m is contained in F_2 .
- (f) r_i is contained in F_2 and r_m is contained in F_3 .

Each of r_i and r_j is contained in either F_1 or F_2 , and each of r_l and r_m is contained in either F_2 or F_3 , and it is readily checked that if neither of (a), (b), (c), (d), (e), and (f), holds, then both r_i and r_j are contained in F_1 and r_l and r_m are contained in F_3 (see Figure 6.27). We show that in this case $cr(\mathcal{D}) \geq 8 = k + 3$, thus finishing the proof of the Inductive Claim for the case $k = 5$.

Suppose that r_i and r_j are contained in F_1 , and that r_l and r_m are contained in F_3 . The n -drawing property of $\mathcal{D}$, together with Propositions 6.3 and 6.4, implies that each of $r_{i+1}, r_{i+3}, r_{i+5}, r_{i+7}, r_{i+9}, r_{i+11}, r_{i+13}$, and r_{i+15} is in $\mathcal{R}_b$. Hence there are at least eight blue-red crossings, and so $cr(\mathcal{D}) \geq 8$.

If r_i and r_j are in F_1 , and r_l and r_m are in F_3 , then r_{i+7} is in R_b . Similarly, each of r_{i+1} , r_{i+3} , r_{i+5} , r_{i+9} , r_{i+11} , r_{i+13} , and r_{i+15} is in R_b .

Figure 6.27

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IMAGE EVALUATION TEST TARGET (QA-3)

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