Plane Spanners of Degree Eight

by

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Abstract

Given a set of points P , we calculate the Delaunay triangulation of P , labeled $DT(P)$. We describe an algorithm that selects a subset of edges from $DT(P)$ to form the graph $D8(P)$. We then prove that $D8(P)$ has a constant spanning ratio of ≈ 4.414 with respect to the complete graph, and a maximum degree of 8.

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Table of Contents

Abstract	ii
Acknowledgments	iii
Table of Contents	iv
List of Tables	vi
List of Figures	vii
1 Introduction	1
1.1 Previous Work	3
1.2 Overview	5
2 General Definitions	7
2.1 Graphs	7
2.1.1 T-Spanners	8
2.1.2 Delaunay Triangulations	9
3 Building D8(P)	11
3.1 Preliminaries	11
3.2 The Algorithm	15
3.2.1 Constructing D8(P)	15
3.2.2 AddIncident(L)	16

3.2.3	AddCanonical(p,r)	17
4	D8(P) is Degree 8	20
4.1	Introduction	20
4.2	The Edges of E_A	21
4.3	The Edges of E_{CAN}	21
4.3.1	Cone Types	21
4.3.2	Cones in Neighbourhoods	23
4.3.3	Cones in Shared Triangles	27
4.3.4	Empty Cones	30
4.4	Charging Scheme for D8(P)	31
4.5	Proving the Degree of D8(P)	38
4.5.1	Preliminaries	38
4.5.2	Edges in Boundary Cones	38
4.5.3	Edges in Shared Triangles	42
4.5.4	The Degree of D8(P)	43
5	D8(P) is a Spanner	46
5.1	Introduction and Preliminaries	46
5.2	Spanning Ratio of Ideal Paths	48
5.3	Lengths of Paths in $D8(P)$	63
5.3.1	Introduction	63
5.3.2	Preliminaries	64
5.3.3	Paths in D8(P)	66
5.4	The Spanning Ratio of D8(P)	74
6	Conclusion and Future Work	80
	List of References	82

List of Tables

1	Known results for bounded degree planar spanners.	4

List of Figures

1	An example of how cones of a point are labeled.	12
2	The <i>bisector distance</i> from $[pq] = pq' $, where q' lies on the bisector of the cone.	13
3	The cone neighbourhood N_i^p	14
4	Canonical edges.	14
5	$AddIncident(L)$ selects edge (p, r) for E_A	16
6	The graph Can_0^p , based on $(p, r) \in E_A$, in red.	17
7	$AddCanonical(p, r)$	19
8	Regions and cones.	22
9	Properties of convex quadrilaterals in $DT(P)$	25
10	Consecutive canonical edges have an angle facing p of at least $2\pi/3$, and thus if $(p, r) \notin E_A$, there is at least one empty cone between them in $D8(P)$	26
11	Lemma 4.3.10	27
12	A triangle shared by N_0^p and $N_3^{p'}$	28
13	$q \in C_0^p \cap C_3^{p'}$ violates the empty circle property of Delaunay triangulations.	29
14	The case if $[wz] > [yz]$	34
15	Vertex s is an inner vertex of $Can_0^{(p,r)}$, $s \neq r$. Charge C_2^s	34
16	Vertex s is neither the first nor last vertex of $Can_0^{(p,r)}$, $s \neq r$	35
17	Vertex s is neither the first nor last vertex of $Can_0^{(p,r)}$, $s \neq r$	35
18	$(p, r) \in E_A$, canonical edges with endpoint r	36
19	(y, z) is the last canonical edge in $Can_0^{(p,r)}$, and is charged to C_4^z	36
20	(y, z) is the last canonical edge in $Can_0^{(p,r)}$, and is charged to C_4^z	37

21	(w, y) is charged to y in place of (y, z)	37
22	(w, y) is charged to the empty cone of w	38
23	There must be an edge of E_A between (y, z) and (y', z)	40
24	If (y, z) is not in E_A , there must be a neighbour of y or z in $\triangle(yxz)$	41
25	Only (p, s) can be added to E_{CAN} in Step 4c with apex q	43
26	Only (p', s) can be added to E_{CAN} in Step 4c with apex q	43
27	A degree 8 vertex in $D8(P)$. The red edges belong to E_{CAN} , while the black edges belong to E_A . The light edges are edges of $DT(P)$ that may or may not be in $D8(P)$	45
28	The canonical triangle T_{pq} in blue. Note, a possible canonical path (in red) does not need to be inside the canonical triangle.	47
29	The Inscribed Angle Theorem	49
30	Length of arc between r and q	50
31	If there is a point t , then (p, r) is not an edge in $DT(P)$	51
32	Circle $O_{r,a,q}$ contains u	51
33	A circle through a and p contains u or v	53
34	Lemma 5.2.6.	56
35	If $ pa \geq pq $, apply Lemma 5.2.6.	58
36	If the vertex a is in $\triangle(prq)$, we proceed by induction from r to a	58
37	Proceed by induction from a to q	59
38	$ r_a a \frac{\alpha_1}{\sin \alpha_1} < r_a a \frac{\alpha}{\sin \alpha}$ and $ a_q q \frac{\alpha_2}{\sin \alpha_2} < a_q q \frac{\alpha}{\sin \alpha}$	60
39	The distance $ r_a a < r_q a_q $	61
40	The distances across the arcs are $ r_q a_q \frac{\alpha}{\sin \alpha} + a_q q \frac{\alpha}{\sin \alpha} = r_q q \frac{\alpha}{\sin \alpha}$	61
41	Lemma 5.2.8.	62
42	An example arrangement of vertices.	64
43	Any edge with endpoint p in N_i^p clockwise from (p, q) must be longer than $[pq]$	65
44	y cannot be in C_i^z	66
45	If the final edge of Can_0^p is in C_0^y , we do <i>not</i> add it to E_{CAN}	67

46	Ideal paths from p to y and z to y	68
47	The path corresponding to Equation 5.3.1, Lemma 5.3.4.	69
48	The path corresponding to Equations 5.3.2 and 5.3.3, in Lemma 5.3.4.	69
49	The green path is the length of the path we can prove, the red path is how much path we have to pay for it.	71
50	Notice that the inductive path takes the longer path between a_2 and b_2	71
51	$ a_1s \geq sb_1 $	72
52	$ a_1s \geq sb_1 $	72
53	$AddCanonical(p, r)$	77

Chapter 1

Introduction

We want to go for a drive in our car. We are at home, which is our starting point, and we have a specific destination in mind.

Between these two locations are a system of roads. These roads intersect each other at discrete locations. Each road segment takes us from one location to another, whether that location is the start of our journey, the end of our journey, or an intersection along the way.

The structure of this network of roads shapes the way we interact with them in a specific way. When giving directions, we do not bother the details of how to drive. We don't say "stay between the lines, obey traffic laws, don't run in to other cars." Rather, we give a set of simple instructions. We say, for instance, take road 1 until it intersects road 2. Turn right on road 2 and drive until it intersects road 3. Take a left on road 3 until you come to the destination. We may give details such as how long to expect to drive down any given road, which is a rough estimate of the length of the road. Additionally, often without thinking, if we are giving directions or planning a route, we give the shortest path possible.

Thinking of roads in this way allows us to communicate directions simply, and understand directions when given to us. It allows us to package a lot of useful information in a few short details. In this way, without being conscious of it, we have modeled the system of roads as a graph.

Informally a graph consists of a set of points, and links between them that we refer to as edges. The above example deals with points that are physical locations, and has edges that

have a distance associated with them. This makes what we refer to as a *geometric graph*, which we define more formally in Chapter 2.

In the case of a road network, we are mainly concerned with traversing the graph, and we often wish to traverse it in a manner that will minimize the distance traveled. If the path from source to destination consists of a single, straight road, then the distance traveled is what we call the *Euclidean distance*, which is the shortest distance between two points. If there was a straight road between every possible pair of locations (a highly impractical solution), then the road network would consist of the *complete graph*, and we could always guarantee that the distance we travel between two locations is the shortest possible.

Of course such a road network is prohibitively expensive and impractical. Though we don't always have such a path, we try to approximate it as much as possible. A road network is good if the path between source and destination only deviates a reasonable amount from the shortest distance possible. If we can make guarantees that we never deviate from the shortest distance more than some number, say t , times the Euclidean distance, our road network would be a *spanning graph* of the complete graph, or a t -spanner.

There are other methods to evaluate a road network. Cost factors in highly; as such we would like an efficient road network, where we in a sense minimize the number of roads while still maintaining favourable spanning properties. Additionally, at each intersection or location, we would like to minimize the number of roads that start or end there. It limits congestion, and avoids the need for a large number of expensive traffic lights. In general we do not wish to have more a small number of road segments meeting at an intersection. In graph terminology, we would refer to the number of roads that start or end at a location as the *degree* of that location. Clearly to be a good road network, we want favourable spanning properties *and* a bounded degree.

Clearly there are other graph applications with similar needs. Computer networks need to allow packets to travel from one location to another in an efficient manner. Having too many paths going through the same location (high degree) means increased congestion and/or expensive infrastructure to handle the higher volume. Limiting the degree of a node

while maintaining favourable routing conditions gives us an efficient network that minimizes costs.

In this thesis we focus on the construction and analysis of bounded degree spanning graphs.

We take the paper of Bonichon *et al.* [1], who takes a graph known as the half- θ -6 graph, and creates a subgraph that is a bounded degree planar spanner. We take the techniques used and apply them on another graph, the Delaunay triangulation. In applying to the Delaunay triangulation, we borrow amply from results of Bose *et. al* [2], as well as Kanj *et al.* [3], both of whom construct bounded degree planar subgraphs of the Delaunay triangulation which are still t -spanners. Using a combination of these methods, we construct a subgraph of the Delaunay triangulation that has a maximum degree of 8 and has the best spanning ratio of any known bounded degree planar graph with degree less than 14.

1.1 Previous Work

The original spanning ratio result was on Delaunay triangulations in the L1 metric by Chew [4], who proved a spanning ratio of $\sqrt{10}$. Dobkin *at al.* [5] found a result of the Delaunay triangulation of the L2 metric, getting a ratio of $\frac{1+\sqrt{5}}{2}\pi$ using the Delaunay triangulations relation to the Voronoi diagram. This was then improved by Keil [6] to $\frac{2\pi}{3\cos(\frac{\pi}{6})}$, and finally taken to its current known spanning ratio of 1.998 by Xia [7].

Chew [4] conjectured the lower bound on the spanning ratio of the Delaunay triangulation was $\frac{\pi}{2}$, until Bose *et. al* [8] proved that almost all Delaunay triangulations had a spanning ratio of greater than $\frac{\pi}{2}$, by proving a lower bound of 1.5846. Xia [9] recently found a point set with a spanning ratio of 1.5932, and conjectured that the tight bound would be found at the limit of a point set in a similar configuration, when the number of points approached infinity.

There are also a number of results on bounded degree planar spanners. We give a brief overview of some of the results in Table 1.

Paper	Degree	Stretch Factor
Bose <i>et al.</i> [10]	27	$(\pi + 1)C_O \approx 8.27$
Li & Wang [11]	23	$(1 + \pi \sin(\frac{\pi}{4}))C_O \approx 6.44$
Bose <i>et al.</i> [12]	17	$(\frac{1+\sqrt{3}+3\pi}{2} + 2\pi \sin(\pi/12))C_O \approx 23.58$
Kanj <i>et al.</i> [3]	14	$(1 + \frac{2\pi}{14 \cos(\pi/14)})C_O \approx 2.92$
Bose <i>et al.</i> [2]	7	$(\frac{1}{1-2 \tan(\pi/8)})C_O \approx 11.65$
Bose <i>et al.</i> [2]	6	$(\frac{1}{(1-\tan(\pi/7)(1+1/\cos(\pi/14)))})C_O \approx 81.66$
Bonichon <i>et al.</i> [1]	9	6
Bonichon <i>et al.</i> [1]	6	6
Bonichon <i>et al.</i> [13]	4	$\sqrt{4 + 2\sqrt{2}}(19 + 29\sqrt{2}) \approx 156.82$
This paper	8	$(1 + \frac{2\pi}{6 \cos(\pi/6)})C_O \approx 4.41$
Kanj <i>et al.</i> [3]	14	$(1 + \frac{2\pi}{14 \cos(\pi/14)})C_O \approx 2.92$

$$C_O = 1.998 [7]$$

Table 1: Known results for bounded degree planar spanners.

Bounded degree planar spanners are often obtained by taking a subset of edges of an existing planar spanner that has bounded degree, while maintaining spanning properties. Using this technique, the properties of the underlying graph become very important. For instance, if we have a subgraph that is a t' -spanner, and the original graph was a t -spanner, then certainly $t' \geq t$. Thus, if we want a subgraph with a good spanning ratio, we want to take a subgraph of a graph with a good spanning ratio. We note how in Table 1 that most of the results are subgraphs of the Delaunay triangulation, which has the best known spanning ratio of any planar graph, currently 1.998 [7]. We note that Bonichon *et al.* [1] have good results using the half- θ 6 graph, which has a spanning ratio of 2.

The research into bounded degree spanners begins with Bose *et al.*, [10] and a subgraph of the Delaunay triangulation that has a maximum degree of 27. Li & Wang [11] achieve a degree 23 graph with an improved spanning ratio, in which we see the degree bound reduced from 27 to 23 while maintaining respectable spanning properties. Bose *et al.*, [12] responded with a degree 17 graph, at the cost of an increase in the spanning ratio.

Then, notably, Kanj *et al.* [3] improve the degree to 14 while having the lowest spanning ratio yet, ≈ 2.92 . While their claim to the lowest spanning ratio of bounded degree planar graph still stands, their degree bound of 14 has been subsequently improved upon. Bose *et al.* [2] improved the degree bound on Delaunay triangulation subgraphs to 7 and 6, but at the cost of an inflated spanning ratio of ≈ 11.65 and ≈ 54.22 respectively. Bonichon *et al.* [1] managed similar degree bounds, most notably matching the degree bound of 6, while also managing to improve the spanning ratio to 6. It is notable that in their paper, Bonichon *et al.* eschew the seemingly traditional practice of sub-graphing the Delaunay triangulation, and instead opting to use the half- θ -6 graph. The natural partitioning of edges of the half- θ -6 graph make it something of a natural for selecting a bounded degree subgraph.

Bonichon *et al.* [13] additionally obtain the lowest degree planar spanner to date, with a degree bound of 4. However they pay a moderate price in their spanning ratio of ≈ 156.82 .

Our results take a philosophy similar to Bonichon *et al.* [1], in that there is value in attempting to optimize both degree and spanning ratio. As such obtain a respectable degree 8, with a highly competitive spanning ratio. It is the lowest spanning ratio of any graph of degree less than 14. Kanj *et al.* [3] have the best known spanning ratio of bounded degree planar graphs for degree 14 and under. They obtain a degree of 14 by dividing the area around a vertex into 14 equal angled "cones". We take a similar approach, dividing the area around a vertex into 6 cones. Our spanning ratio matches that of Kanj and Perkovic [3] when corrected for cone angle, although our degree is 2 higher than our number of cones. The similarities are not simply superficial; our initial analysis of path length in a bounded degree Delaunay subgraph is almost completely modeled from their analysis.

1.2 Overview

In Chapter 2, we provide general definitions of terms that will be used throughout this thesis. In Chapter 3 we describe how to select a subset of the edges of the graph $DT(P)$ to form the graph $D8(P)$. In Chapter 4 we prove that $D8(P)$ has a maximum degree of 8. In Chapter

5 we bound the spanning ratio of $D8(P)$ with respect to $DT(P)$. Since $DT(P)$ is a spanner of the complete Euclidean graph, this makes $D8(P)$ a spanner of the complete Euclidean graph as well. Finally in Chapter 6 we draw conclusions based on our work, and examine some potential directions that related research might extend.

Chapter 2

General Definitions

Now that we have looked into the background of the problem, we will give precise definitions of the terms we will be using.

2.1 Graphs

Definition 2.1.1. *A graph $G = (V, E)$ consists of a finite set of vertices V , and a set E of edges, where each edge is an unordered pair of vertices. We also use the alternative notation of $G(V)$, which means a graph G induced on the vertices V .*

Definition 2.1.2. *A graph is called planar if it can be drawn in the 2-dimensional plane as a set of points (vertices) and line segments (edges) such that no two edges cross except at a common endpoint. Such a drawing is called an embedding.*

Definition 2.1.3. *A plane graph is a graph that has been embedded in the plane such that no two edges cross.*

Plane graphs may have curved edges, but for the purposes of this thesis we will assume all edges are straight line segments.

Definition 2.1.4. *A geometric graph is a graph in which the edges and vertices are*

associated with geometric objects or measurements.

Definition 2.1.5. *A Euclidean graph is a geometric graph in which the vertices represent points in the plane, and the edges are assigned lengths equal to the Euclidean distance between those points.*

All graphs discussed in this thesis are Euclidean graphs.

Definition 2.1.6. *Consider a graph $G = (V, E)$. The degree of a vertex $v \in V$ is the number of edges $e \in E$ incident to v . Alternatively it is the number of edges $e \in E$ that v is the endpoint of.*

As discussed earlier, we are interested in graphs where the degree of any vertex is not too high. Formally:

Definition 2.1.7. *A bounded degree graph is a graph $G = (V, E)$ where, for any $v \in V$, the degree of v is no greater than some value d .*

2.1.1 T-Spanners

A t -spanner is a Euclidean graph that has the property that the shortest path between any two vertices is no more than a constant t times longer than the Euclidean distance between those vertices. Formally we say:

Definition 2.1.8. *Let $t \geq 1$ be any real number. Then a Euclidean graph G is a t -spanner if, for any two vertices p and q in the graph, the shortest distance from p to q through the edges of G , denoted $\delta_G(p, q)$, is no more than t times the Euclidean distance from p to q . That is:*

$$\frac{\delta_G(p, q)}{|pq|} \leq t$$

2.1.2 Delaunay Triangulations

One graph that is of particular interest to us is the Delaunay triangulation. The Delaunay triangulation forms the basis of the bounded-degree spanning graphs we construct here.

The following definitions assume general position of a point set P . Thus no three points are on a line, and no four points are on a circle.

Definition 2.1.9. *A triangulation of a point set P is a plane graph of the vertex set P such that we cannot add another edge between vertices of P without inducing an edge crossing. If we assume all edges are straight line segments, then every face of the graph is a triangle, with the possible exception of the outer face.*

Of special consideration to us is the *Delaunay triangulation*. The Delaunay triangulation of a set of points P is denoted $DT(P)$. It is a triangulated Euclidean graph that has many favourable properties. There are several equivalent constructions of the Delaunay triangulation, but there are two in particular that are the most conducive to our analysis.

- For every triple of points of the set P , if the unique circle through these three points is empty of points of P , then we connect the three points such that the edges form a triangle.
- For every pair of points x, y of the set P , if there exists a circle through x and y that is empty of any points of P , then we connect x and y with an edge.

This construction enables us to define a property of Delaunay triangulations that is of particular importance to this thesis.

Definition 2.1.10. *An edge e of a graph $G(P)$ has the empty circle property if there exists a circle through the endpoints of e that does not contain any other point of P .*

As we can see, every edge of a graph $DT(P)$ has the empty circle property.

It is worth reviewing that, for all of its favourable properties, the degree of the Delaunay triangulation is unbounded.

Chapter 3

Building $D8(P)$

Given as input a set P of n points in the plane, we present an algorithm for building a bounded degree plane graph with maximum degree 8 and constant spanning ratio denoted $D8(P)$.

The graph $D8(P)$ is a subgraph of the Delaunay triangulation of P , denoted $DT(P)$. While $DT(P)$ has the best known spanning ratio of any planar graph, with a spanning ratio of 1.998 [7], it does not have a bound on the degree of a vertex. Thus the focus of our algorithm is to compute a bounded degree spanning subgraph of $DT(P)$ that is still a spanner.

We start with some preliminaries, then present the algorithm.

3.1 Preliminaries

We assume general position of P ; i.e, no three points are on a line, no four points are on a circle, and no two points form a line with slope 0, $\sqrt{3}$ or $-\sqrt{3}$.

In our algorithm, we use the concept of *cones*. Cones are a partitioning of the space around every point $p \in P$ with rays, where the rays form the boundary of consecutive cones of p . See Figure 1.

We describe the precise partitioning by cones used by our algorithm. The first ray from p to infinity is parallel to the x -axis in the positive (right) direction. We then define 5

Figure 1: An example of how cones of a point are labeled.

additional rays, each starting at p and going to infinity, at angular intervals of $\pi/3$. Thus for every point $p \in P$, we partition the surrounding space into 6 cones at angular intervals of $\pi/3$ radians each.

We number the cones starting with the topmost cone (the cone that is defined by rays at $\pi/3$ and $2\pi/3$ to the x -axis) as C_0 . If we are referring to a specific cone on a specific point p we label it C_i^p , where i is the cone index. We then number the cones by increments of one in the clockwise direction. Thus C_0 is bounded on one side by C_1 and bounded on the other side by C_5 . See Figure 1.

When referring to a cone in relation to another cone, we use an offset that is modulo 6. Thus for cone i , where $i = 5$, when we refer to cone C_{i+2}^p , we are referring to cone C_1^p .

By our general position assumption we note that no point of P lies on the boundary of a cone.

In addition to Euclidean distance, there is another distance metric we use, which we call the bisector distance.

Definition 3.1.1. Given an arbitrary edge (p, q) , that lies in C_0^p , which implies that it

Figure 2: The bisector distance from $[pq] = |pq'|$, where q' lies on the bisector of the cone.

is also in C_3^q , the bisector length of (p, q) is the distance from p to the orthogonal projection of q onto the bisector of C_0^p . Denote this length as $[pq]$.

Note that the bisector of C_3^q is parallel to the bisector of C_0^p , and thus $[pq]$ is the same whether measured from p or q , and thus each edge of $DT(P)$ has a unique bisector length (dependent only on the orientation of the cones). See Figure 2 for an illustration.

Henceforth, any reference made to distance is to the bisector distance. When referring to Euclidean distance we state it explicitly, and use the notation $|pq|$.

Definition 3.1.2. Let $\{q_0, q_1, \dots, q_{d-1}\}$ be the sequence of all neighbours of p in $DT(P)$ in consecutive clockwise order. The neighbourhood N_p , with apex p , is the graph with the vertex set $\{p, q_0, q_1, \dots, q_{d-1}\}$ and the edge set $\{(p, q_j)\} \cup \{(q_j, q_{j+1})\}, 0 \leq j \leq d-1$, with all values mod d . The edges $\{(q_j, q_{j+1})\}$ are called canonical edges. N_i^p is the subgraph of N_p induced by all the vertices of N_p in C_i^p , including p . This is called the **cone neighbourhood** of p .

See Figures 3, 4.

Figure 3: The cone neighbourhood N_i^p .

Figure 4: Canonical edges.

3.2 The Algorithm

The algorithm *ConstructD8(P)* takes as input a point set P and returns the bounded degree graph $D8(P)$, with vertex set P and edge set E .

The algorithm calls two subroutines, *AddIncident()* and *AddCanonical()*. *AddIncident()* selects a set of edges E_A . For each edge (p, r) of E_A , we call *AddCanonical(p, r)* which adds edges to the set E_{CAN} . Note that both E_A and E_{CAN} are a subset of the edges in $DT(P)$.

The final graph $D8(P)$ consists of the vertex set P and the union of edge sets E_A and E_{CAN} .

We now give the details of the algorithm.

3.2.1 Constructing D8(P)

We have as input a set of points in the plane P , and it provides as output a graph $D8(P)$.

Algorithm: **ConstructD8(P)**

INPUT: Set P of n points in the plane in general position.

OUTPUT: $D8(P)$: spanning subgraph of $DT(P)$ with maximum vertex degree 8 and a constant

Step 1: We compute the Delaunay triangulation $DT(P)$ of the point set P .

Step 2: Sort all the edges of $DT(P)$ by their bisector length, into a set L , in non-decreasing order.

Step 3: Initialize the set $E_A = \emptyset$.

Step 4: Call the function *AddIncident(L)* with L as the argument. *AddIncident()* selects and returns the subset E_A of the edges of L .

Step 5: Iterate over the edges in E_A in sorted order. For each edge (p, r) in E_A call *AddCanonical(p, r)* and *AddCanonical(r, p)*. These two calls add edges to the set

Figure 5: $AddIncident(L)$ selects edge (p, r) for E_A .

E_{CAN} .

Step 6: Return $D8(P) = (P, E_A \cup E_{CAN})$.

3.2.2 AddIncident(L)

Algorithm: AddIncident(L)

INPUT: L : set of edges of $DT(P)$ sorted by bisector distance.

OUTPUT: E_A : a subset of edges of $DT(P)$.

Step 1: Initialize the set $E_A = \emptyset$.

Step 2: For each $(p, q) \in L$, in non-decreasing order, do:

- (a) Let i be the cone of p containing q . If E_A has no edges with endpoint p in N_i^p , and if E_A has no edges with endpoint q in N_{i+3}^q , then we add (p, q) to E_A .

Step 3: return E_A .

Note that this algorithm adds, for any vertex $p \in P$, at most one edge per cone of p .

Thus the graph $G = (P, E_A)$ has maximum degree 6.

Figure 6: The graph Can_0^p , based on $(p, r) \in E_A$, in red.

However, it is not clear that $G = (P, E_A)$ is a spanner. $AddCanonical()$ will add edges to the graph giving us a constant spanning ratio while slightly increasing the degree to 8.

3.2.3 AddCanonical(p,r)

We require the following definition:

Definition 3.2.1. Let $Can_0^{(p,r)}$ be the subgraph consisting of the ordered subsequence of canonical edges (s, t) of N_0^p in clockwise order around apex p such that $[ps] \geq [pr]$ and $[pt] \geq [pr]$. We call $Can_0^{(p,r)}$ a canonical subgraph. A vertex that is the first or last vertex of $Can_0^{(p,r)}$ is called an end vertex of $Can_0^{(p,r)}$. A vertex that is not the first or last vertex in $Can_0^{(p,r)}$ is called an inner vertex of $Can_0^{(p,r)}$. Vertex r is called the anchor of $Can_0^{(p,r)}$. See Figure 6.

Algorithm: AddCanonical(p,r)

INPUT: (p, r) , an edge of E_A .

OUTPUT: A set of edges that are a subset of the edges of $DT(P)$. All edges generated by calls to $AddCanonical()$ form the set E_{CAN} .

Step 1: Without loss of generality, let (p, r) be an edge of E_A , where $r \in C_0^p$.

Step 2: If there are at least three edges in $Can_0^{(p,r)}$, then for every canonical edge (s, t) in $Can_0^{(p,r)}$ that is not the first or last edge in the ordered subsequence of canonical edges $Can_0^{(p,r)}$, we add (s, t) to E_{CAN} .

Step 3: If the anchor r is the first or last vertex in $Can_0^{(p,r)}$, and there is more than one edge in $Can_0^{(p,r)}$, then add the edge of $Can_0^{(p,r)}$ with endpoint r to E_{CAN} . See Figure 7b.

Step 4: Consider the first and last canonical edge in $Can_0^{(p,r)}$. Since the conditions for the first and last canonical edge are symmetric, we only describe how to process the last canonical edge (y, z) . There are three possibilities.

- (a) If $(y, z) \in N_5^z$ we add (y, z) to E_{CAN} . See Figure 7c.
- (b) If $(y, z) \in N_4^z$ and N_4^z does not have an edge with endpoint z in E_A , then we add (y, z) to E_{CAN} . See Figure 7d
- (c) If $(y, z) \in N_4^z$ and there is an edge with endpoint z in $E_A \cap N_4^z \setminus (y, z)$, then there is exactly one canonical edge of z with endpoint y in N_4^z . We label this edge (w, y) and add it to E_{CAN} . See Figure 7e.

That concludes the steps in the construction of $D8(P)$. In Chapter 4 we will prove that the maximum degree of this graph is at most 8, and in Chapter 5 we will prove that its spanning ratio is bounded by a constant.

Figure 7: $AddCanonical(p, r)$

Chapter 4

D8(P) is Degree 8

4.1 Introduction

In this chapter we prove that the graph $D8(P)$ has a maximum degree of 8. We use a simple charging argument. Each edge (p, q) of $D8(P)$ is charged once to p and once to q . Thus the total charge on a vertex is equal to the degree of that vertex. We show that no vertex is charged more than 8.

In Section 4.2 we show that after the addition of the edges of E_A , the total charge on each vertex is at most 6.

In Section 4.3 we show that by adding the edges of E_{CAN} , the total charge of each vertex does not exceed 8. Showing this is considerably more involved. To track the total number of charges on a vertex, we associate each charge with a cone. By limiting the charges to each cone, we can limit the total charge on a vertex, and thus the degree.

In Section 4.3.1 we characterize the cones into distinct types. In Section 4.3.2 we look at cones in cone neighbourhoods, and find cones that are unique to a neighbourhood. In Section 4.3.3 we look at where the possibility of cones that are shared between two neighbourhoods exists, and how we resolve this issue. In Section 4.3.4, we establish what cones are available to be charged by edges of E_{CAN} .

In Section 4.4 we provide the charging scheme, in terms of what cones of the endpoints are charged. In Section 4.5 we look at and resolve some ambiguous areas of the charging

scheme, and prove the degree of $D8(P)$.

4.2 The Edges of E_A

Lemma 4.2.1. *An arbitrary vertex p of the graph $D8(P)$ is charged at most 6 by edges of E_A .*

Proof. It follows from Step 2a in algorithm $AddIncident()$, that for each cone C_i^p , at most one edge (p, q) is added to E_A for each cone with apex p . $\square$

4.3 The Edges of E_{CAN}

4.3.1 Cone Types

Our charging argument hinges on which cones are charged and how many edges are charged in each one. We must therefore track the cones that have been charged. With this in mind we provide the following definitions:

Definition 4.3.1. *For an arbitrary cone neighbourhood N_i^p we define the region of N_i^p as the polygonal region bounded by the canonical edges of N_i^p and the first and last edge of N_i^p with endpoint p . See Figure 8a.*

Definition 4.3.2. *Let (a, b) be the first edge and let (y, z) be the last edge in a canonical subgraph $Can_i^{(p,r)}$ (Definition 3.2.1). We define the region of $Can_i^{(p,r)}$ as the polygonal region bounded by the canonical edges of N_i^p between (a, b) and (y, z) inclusive, and the edges (p, a) and (p, z) . See Figure 8b.*

We provide the following definitions regarding the placement of cones in regions. Both of the following definitions also extend to regions of a cone neighbourhood.

Figure 8: Regions and cones.

Definition 4.3.3. Let $B(s, \epsilon)$ be a ball with center s and radius $\epsilon > 0$. Consider a cone C_j^s of a point s in $Can_i^{(p,r)}$. If there exists an $\epsilon > 0$ such that $B(s, \epsilon) \cap C_j^s$ is inside the region of $Can_i^{(p,r)}$, then we call this an internal cone of $Can_i^{(p,r)}$. Alternatively we say C_j^s is in $Can_i^{(p,r)}$.

Definition 4.3.4. Consider a cone C_j^s of a point s in $Can_i^{(p,r)}$. If for all $\epsilon > 0$, $B(s, \epsilon) \cap C_j^s$ is partially but not entirely in the region of $Can_i^{(p,r)}$, then we call this a boundary cone of $Can_i^{(p,r)}$. Alternatively we say C_j^s is on the boundary of $Can_i^{(p,r)}$.

Definition 4.3.5. A cone with vertex s as endpoint is empty if no edge of E_A or E_{CAN} incident to s is in the cone.

In order to properly count the number of charges to a particular vertex, in this section we associate each charge of an edge to a particular cone of a vertex. By summing the charges to each cone surrounding a vertex, we are able to prove a maximum degree of 8 for any vertex of $D8(P)$.

When referring to an angle formed by three points, we refer to the smaller of the two angles (that is, the angle that is $< \pi$) unless otherwise stated. When referring to a circle through three points p_1, p_2 , and p_3 , we use the notation O_{p_1, p_2, p_3} .

4.3.2 Cones in Neighbourhoods

We consider the edge (p, r) of E_A , where without loss of generality, r is in C_0^p . In this section we show the location of cones in the region of $Can_0^{(p,r)}$, so we may charge edges of E_{CAN} to them. To facilitate this we introduce another variation on the concept of the neighbourhood of a vertex:

Definition 4.3.6. Consider the cone neighbourhood N_i^p with the vertex set $\{p, q_0, q_1, \dots, q_{m-1}\}$, where $\{q_0, q_1, \dots, q_{m-1}\}$ are listed in clockwise order around p . A restricted

neighbourhood $N_p^{(q_j, q_k)}$ is the subgraph of N_i^p induced on the vertex set $\{p, q_j, q_{j+1}, \dots, q_k\}$, $0 \leq j \leq k \leq m-1$.

Now we illustrate some of the geometric properties of restricted neighbourhoods in $DT(P)$.

Lemma 4.3.7. *Consider the arbitrary restricted neighbourhood $N_p^{(r, q)}$. Each vertex $x \in N_p^{(r, q)} \setminus \{p, r, q\}$ is in the circle $O_{p, r, q}$ through p , r , and q .*

Proof. Since (p, x) is an edge in $DT(P)$, we can draw a disk through p and x that is empty of points of P . In particular, neither r nor q is in this disk. Hence the sum of the angles $\angle(prx)$ and $\angle(pqx)$ which lie on opposite sides of the same chord is smaller than π , and the sum of the other two angles $\angle(rqx)$ and $\angle(rpq)$ in the quadrilateral $(prxq)$ is greater than π . That implies x is inside $O_{p, r, q}$. $\square$

Lemma 4.3.8. *Consider a cone neighbourhood $N_p^{(r, q)}$ in cone C_i^p . Let (p, x) be an edge in $N_p^{(r, q)}$ where $x \neq r$ and $x \neq q$. Then angle $\angle(qxr) \geq \pi - \angle(qpr)$. Since the cone angle is $\pi/3$, we have that $\angle(qxr) > 2\pi/3$.*

Proof. We know by Lemma 4.3.7 that x lies inside the circle through p , r and q , which we label $O_{p, r, q}$. The angle $\angle(qxr)$ is minimized when x is on $O_{p, r, q}$. When x is on $O_{p, r, q}$, $\angle rxq = \pi - \angle(qpr)$, since the two angles lie on the same chord (r, q) . Therefore $\angle(rqx) \geq \pi - \angle(qpr)$. Since both q and r are in the same cone C_i^p , and the cone angle is $\pi/3$, the $\angle(qxr) > 2\pi/3$. $\square$

Which leads to the corollary:

Corollary 4.3.9. *Let s be an inner vertex of $Can_i^{(p, r)}$ that is not the anchor. Then there is at least one empty cone of s in $Can_i^{(p, r)}$.*

(a) When corners of a quadrilateral lie on a circle, opposite angles will sum to π .

(b) When one corner of a quadrilateral lies inside a circle, that corner and its opposite angles will sum to $> \pi$.

(c) When one corner of a quadrilateral lies outside a circle, that corner and its opposite angle will sum to $< \pi$.

Figure 9: Properties of convex quadrilaterals in $DT(P)$.

Figure 10: Consecutive canonical edges have an angle facing p of at least $2\pi/3$, and thus if $(p, r) \notin E_A$, there is at least one empty cone between them in $D8(P)$.

Proof. Since s is not the anchor, any internal cone of $Can_i^{(p,r)}$ on vertex s is empty, and by Lemma 4.3.8, there is at least one internal cone of $Can_i^{(p,r)}$ on vertex s . Therefore there is at least one empty internal cone on s in the region of $Can_i^{(p,r)}$. See Figure 10. $\square$

Lemma 4.3.10. Consider the edge (p, r) in E_A , and without loss of generality let r be in C_0^p . If r is an inner anchor of $Can_i^{(p,r)}$, then cones C_2^r and C_4^r are empty and in the region of $Can_i^{(p,r)}$.

Proof. Since (p, r) is in C_0^p , it must also be in C_3^r , and thus it is in neither C_2^r or C_4^r .

Assume that q , r and s are in consecutive order in $Can_i^{(p,r)}$. Thus $Can_i^{(p,r)}$ contains canonical edges (q, r) and (r, s) .

Recall that for every vertex x in $Can_i^{(p,r)}$, $[px] \geq [pr]$. Thus $[ps] \geq [pr]$ and $[pq] \geq [pr]$, which means that s and q are above the horizontal line through r in C_0^p . Since C_2^r and C_4^r lie below the horizontal line through r , they cannot contain the edges (q, r) and (r, s) . Since

Figure 11: Lemma 4.3.10

$\triangle(pqr)$ and $\triangle(prs)$ are triangles in $DT(P)$, C_2^r and C_4^r are empty and inside $Can_i^{(p,r)}$. See Figure 11 □

4.3.3 Cones in Shared Triangles

We will show the location of uncharged cones in the special case of overlapping regions. A set of regions overlap when at least one triangle of $DT(P)$ is contained in the intersection of all the regions in the set.

Lemma 4.3.11. *Consider the triangle $\triangle(pp's)$ in $DT(P)$. Let p' and s be in C_0^p , and let p and s be in $C_3^{p'}$. Then $\angle(p'sp) > 2\pi/3$.*

Proof. Without loss of generality, assume that s is left of directed line segment (p, p') . Consider the parallelogram formed by $C_0^p \cap C_3^{p'}$. Let a be the left intersection and b be the right intersection of C_0^p and $C_3^{p'}$. Thus s is in $\triangle(app')$. Note that $\angle(pp'a) + \angle(app') = \pi/3$. Thus $\angle(pp's) + \angle(spp') < \pi/3$, which implies that $\angle(p'sp) > 2\pi/3$. See Figure 12. □

Lemma 4.3.12. *Let $\triangle(pp's)$ be a triangle in $DT(P)$. Then $\triangle(pp's)$ can belong to cone neighbourhoods of at most two of p , p' and s .*

Figure 12: A triangle shared by N_0^p and $N_3^{p'}$.

Proof. Suppose, for the sake of contradiction, that triangle $\triangle(pp's)$ is in a cone neighbourhood of p , and a cone neighbourhood of p' , and a cone neighbourhood of s . Without loss of generality, let p' and s be in C_0^p . This means that p' and s must be in $C_3^{p'}$. By Lemma 4.3.11 angle $\angle(psp') > 2\pi/3$. Therefore p and p' cannot be in the same cone neighbourhood of s . $\square$

Corollary 4.3.13. *A triangle can be shared by at most 2 cone neighbourhoods.*

Proof. Follows from Lemma 4.3.12. $\square$

Which leads to the following definition:

Definition 4.3.14. *If $\triangle(pp's)$ occurs in exactly two cone neighbourhoods of p, p' and s , then we refer to it as a shared triangle. If $\triangle(pp's)$ is in cone neighbourhoods of p and p' , then (p, p') is referred to as the base of the shared triangle.*

Corollary 4.3.15. *In a shared triangle $\triangle(pp's)$ with base (p, p') , s has two empty cones internal to $\triangle(pp's)$.*

Proof. Without loss of generality, assume that $p' \in C_0^p$ and $p \in C_3^{p'}$. Then $p \in C_3^s$ and $p' \in C_0^s$. Then either C_2^s and C_1^s , or C_5^s and C_4^s are internal to $\triangle(pp's)$ and thus cannot

Figure 13: $q \in C_0^p \cap C_3^{p'}$ violates the empty circle property of Delaunay triangulations.

contain any edges with endpoint s . □

We show two cone neighbourhoods can share at most one triangle:

Lemma 4.3.16. *Each Delaunay edge is the base of at most 1 shared triangle.*

Proof. By contradiction.

Consider the shared triangle $\triangle(pp's)$ with base (p, p') . Without loss of generality, assume p' is in C_0^p , and p is in $C_3^{p'}$. Since (p, p') is an edge in exactly two triangles, let $\triangle(pp'q)$ be the other triangle with edge (p, p') , and assume that q is in both C_0^p and $C_3^{p'}$.

By Lemma 4.3.11 both angles $\angle(p'sp)$ and $\angle(pqp')$ are greater than $2\pi/3$. Thus their sum is greater than $4\pi/3$. But by the empty circle property of Delaunay triangulations, $\angle(p'sp) + \angle(pqp')$ must be less than π , which is a contradiction. □

Corollary 4.3.17. *Two cone neighbourhoods can share at most one triangle.*

4.3.4 Empty Cones

Lemma 4.3.18. *Consider the edge (p, r) in E_A . Without loss of generality let r be in C_0^p . Each inner vertex s of $Can_0^{(p,r)}$ that is not the anchor has at least one unique empty cone in the region of $Can_0^{(p,r)}$.*

Proof. If s is not part of a shared triangle, we know by Corollary 4.3.9 that s has an empty cone internal to $Can_0^{(p,r)}$.

Consider the shared triangle $\triangle(pp's)$, and without loss of generality, let p' be in C_0^p . Assume that there is an edge (p, r) of E_A in C_0^p , and an edge (p', r') of E_A in $C_3^{p'}$. Thus both sets $Can_0^{(p,r)}$ and $Can_3^{(p',r')}$ are well-defined.

By Corollary 4.3.15 there are two empty cones of s internal to $\triangle(pp's)$. The empty cone adjacent to (p', s) is the empty cone of s in the region of $Can_0^{(p,r)}$, and the empty cone adjacent to (p, s) is the empty cone of s in the region of $Can_3^{(p',r')}$.

Thus any inner vertex s of an arbitrary canonical subgraph $Can_0^{(p,r)}$ that is not the anchor, has a unique empty cone that is in the region of $Can_0^{(p,r)}$.

□

Lemma 4.3.19. *Consider the edge (p, r) in E_A . Without loss of generality let r be in C_0^p . Let $z \neq r$ be an end vertex in $Can_0^{(p,r)}$. By symmetry, let z be the last vertex. Let y be the neighbour of z in $Can_0^{(p,r)}$. If y is in C_5^z , then C_4^z is an empty cone internal to the region of $Can_0^{(p,r)}$.*

Proof. Triangle $\triangle(pyz)$ is a triangle in $DT(P)$. Since (p, z) is in C_3^z and (y, z) is in C_5^z , C_4^z will have no edges in $DT(P)$ with endpoint z . Since both E_A and E_{CAN} are subsets of the edges of $DT(P)$, C_4^z will not contain any edges of E_A and E_{CAN} with endpoint s , and thus is empty.

□

4.4 Charging Scheme for $D8(P)$

Recall that $D8(P)$ consists of the point set P and the edge set $E_A \cup E_{CAN}$.

We present the charging scheme that will allow us to prove that $D8(P)$ has a maximum degree 8. Each edge charges one cone for each endpoint. Therefore the total charge on the cones of a vertex will be exactly the degree of the vertex. We will show that no vertex will be charged more than 8.

We begin with the charging scheme for the edges of E_A . Consider an edge (p, r) of E_A , where without loss of generality r is in C_0^p and p is in C_3^r .

E_A Charge 1: An edge (p, r) of E_A charges C_0^p once and C_3^r once.

We now show to which cones the endpoints of the edges in E_{CAN} are charged. Consider an arbitrary cone neighbourhood N_i^p , and without loss of generality let $i = 0$. We consider a call to $AddCanonical(p, r)$, where (p, r) is an edge in E_A . Without loss of generality, let r be in C_0^p . By outlining the algorithm $AddCanonical(p, r)$ in its entirety, and for each step of $AddCanonical(p, r)$ that adds edges to E_{CAN} , we describe which cones of the endpoints an edge is charged to.

Without loss of generality, let (p, r) be an edge of E_A , where $r \in C_0^p$. Let $Can_0^{(p,r)}$ be the subgraph consisting of the ordered subsequence of canonical edges (s, t) of N_0^p in clockwise order around apex p such that $[ps] \geq [pr]$ and $[pt] \geq [pr]$. We call $Can_0^{(p,r)}$ a canonical subgraph.

Step 1: If the anchor r is the first or last vertex in $Can_0^{(p,r)}$, and there is more than one edge in $Can_0^{(p,r)}$, then add the edge of $Can_0^{(p,r)}$ with endpoint r to E_{CAN} . See Figure 7b.

Step 2: *If there are at least three edges in $Can_0^{(p,r)}$, then for every canonical edge (s, t) in $Can_0^{(p,r)}$ that is not the first or last edge in the ordered subsequence of canonical edges $Can_0^{(p,r)}$, we add (s, t) to E_{CAN} .*

The edge (s, t) is charged once to s and once to t . Since the charging scheme is the same for both s and t , without loss of generality we only describe how to charge s .

Charge vertex s :

- (a) If s is the anchor (thus $s = r$), then by Lemma 4.3.10, C_2^r and C_4^r are empty cones inside $Can_0^{(p,r)}$. If t is left of directed line segment pr , charge (r, t) to C_4^r . If t is right of pr , charge (r, t) to C_2^r . See Figure 18.
- (b) If $s \neq r$ then by Lemma 4.3.18, s has an empty cone C_j^s inside $Can_0^{(p,r)}$. Charge (s, t) once to C_j^s . See Figures 15, 16, 17.

Step 3: *Consider the first and last canonical edge in $Can_0^{(p,r)}$. Since the conditions for the first and last canonical edge are symmetric, we only describe how to process the last canonical edge (y, z) . There are three possibilities.*

- (a) *If $(y, z) \in C_5^z$, add (y, z) to E_{CAN} . See Figure 7c.*
- (b) *If $(y, z) \in N_4^z$ and N_4^z does not have an edge with endpoint z in E_A , then we add (y, z) to E_{CAN} . See Figure 7d*

Charge vertex y :

- i If y is the anchor, then C_2^y is empty and inside $Can_0^{(p,r)}$ by Lemma 4.3.10. Charge (y, z) to C_2^y . Figure 18.
- ii Otherwise y is not the first or last vertex in $Can_0^{(p,r)}$, and by Corollary 4.3.9 has an empty cone C_j^y inside $Can_0^{(p,r)}$. Charge (y, z) to C_j^y . Figures 15, 16, 17.

Charge vertex z :

- iii Step 4a: (y, z) is in C_5^z . By Lemma 4.3.19 C_4^z is empty and inside $Can_0^{(p,r)}$. Charge (y, z) to C_4^z . Figure 20.
- iv Step 4b: (y, z) is in C_4^z , and C_4^z does not contain an edge of E_A with endpoint z . Note C_4^z is a boundary cone of $Can_0^{(p,r)}$. Charge (y, z) to C_4^z . Figure 19.

- (c) If $(y, z) \in C_4^z$ and there is an edge $(u, z), u \neq y$ of E_A in C_4^z , then there is one canonical edge of z with endpoint y in C_4^z . Label the edge (w, y) and add it to E_{CAN} . See Figure 7e.

Charge vertex y :

- i If $y = r$, then C_2^y is empty and inside $Can_0^{(p,r)}$ by Lemma 4.3.10. Charge (w, y) to C_2^y . Figure 18.
- ii Otherwise y is not the first or last vertex in $Can_0^{(p,r)}$, and by Corollary 4.3.9 has an empty cone C_j^y internal to $Can_0^{(p,r)}$. Charge (w, y) to C_j^y . Figure 21.

Charge vertex w :

- iii If $w = u$ ((z, u) in E_A), then C_2^w is empty and inside $Can_4^{(z,u)}$ by Lemma 4.4.1. Charge (w, y) to C_2^w . Figure 18.
- iv If $w \neq u$, then w is not the first or last vertex in $Can_4^{(z,u)}$, and by Corollary 4.3.9 has an empty cone C_j^w inside $Can_4^{(z,u)}$. Charge (w, y) to C_j^w . Figure 22.

Step 3(c)iii makes use of the following lemma:

Lemma 4.4.1. *Assume that on a call to $AddCanonical(p, r)$, where (p, r) is in C_0^p , we add edge (w, y) to E_{CAN} in Step 4c. Let (y, z) be the last edge in $Can_0^{(p,r)}$, and assume that (w, z) is in E_A . Then C_2^w is empty and inside $Can_4^{(z,u)}$.*

Proof. To prove this we shall establish that $[yz] \geq [wz]$. This, together with Lemma 4.3.10 implies that C_2^w is empty and inside $Can_4^{(z,u)}$.

We prove by contradiction, thus assume that $[wz] > [yz]$. See Figure 14. This means that $AddIncident(L)$ examined (y, z) before (w, z) , and thus $C_4^z \cap E_A$ was empty of edges with endpoint z when (y, z) was examined by $AddIncident(L)$. Since (y, z) was not added to E_A , y must have had an edge of E_A in C_1^y with endpoint y that was shorter than (y, z) .

Figure 14: The case if $[wz] > [yz]$

Figure 15: Vertex s is an inner vertex of $Can_0^{(p,r)}$, $s \neq r$. Charge C_2^s .

Since $\triangle(pyz)$ is a triangle in $DT(P)$, and $p \in C_3^y$, there cannot be an edge with endpoint y in C_1^y clockwise from (y, z) . In the counter-clockwise direction from (y, z) , we have the $\triangle(ywz) \in DT(P)$. However, since $[wz] > [yz]$, (w, y) cannot be in C_1^y . Thus C_1^y contained no edge of E_A with endpoint y when (y, z) was examined by $AddIncident(L)$. Which means if $[wz] > [yz]$, then (y, z) would have been added to E_A by $AddIncident(L)$. But we know (z, w) is in E_A , therefore it must be that $[yz] \geq [wz]$, which implies that C_2^w is empty and by Lemma 4.3.10 is inside $Can_4^{(z,u)}$. See Figure 14. $\square$

Figure 16: Vertex s is neither the first nor last vertex of $\text{Can}_0^{(p,r)}$, $s \neq r$.

Figure 17: Vertex s is neither the first nor last vertex of $\text{Can}_0^{(p,r)}$, $s \neq r$.

Figure 18: $(p, r) \in E_A$, canonical edges with endpoint r .

Figure 19: (y, z) is the last canonical edge in $\text{Can}_0^{(p,r)}$, and is charged to C_4^z .

Figure 20: (y, z) is the last canonical edge in $\text{Can}_0^{(p,r)}$, and is charged to C_4^z .

Figure 21: (w, y) is charged to y in place of (y, z) .

Figure 22: (w, y) is charged to the empty cone of w .

4.5 Proving the Degree of D8(P)

4.5.1 Preliminaries

The charging argument of the previous section establishes where charges are made. In this section we show a limit to how many edges can be charged to the different cones.

4.5.2 Edges in Boundary Cones

All edges added to E_{CAN} are charged to internal cones, with the exception of the edge that is added in $AddCanonical(p, \cdot)$ Step 4b to C_4^z , which is to a boundary cone. Since C_4^z is on the boundary of $Can_0^{(p,r)}$, it may also be the boundary cone of a different cone neighbourhood.

Lemma 4.5.1. *The boundary cone C_4^z of $Can_0^{(p,r)}$ charged in $AddCanonical(p, r)$ Step 4b cannot be the internal cone of a different canonical subgraph .*

Proof. C_4^z is inside a canonical subgraph , then y must be the apex of said canonical subgraph. That implies that both shared neighbours of z and y must be in C_1^y . But shared neighbour

p is in C_3^y , thus C_4^z cannot be inside a canonical subgraph . $\square$

This implies that C_4^z may only be shared with a different canonical subgraph as a boundary cone. Thus the only other edge of E_{CAN} that can be charged to C_4^z must be added in some call to $AddCanonical(\cdot, \cdot)$ Step 4b. We prove here this is impossible.

Lemma 4.5.2. *Consider the edge (p, r) of E_A in C_i^p , and without loss of generality, let $i = 0$. Let (y, z) be the last edge in $Can_0^{(p,r)}$, and let z be the last vertex in $Can_0^{(p,r)}$. Assume that (y, z) was added to E_{CAN} in a call to $AddCanonical(p, r)$ Step 4b, and thus by Charge iv is charged to the cone C_4^z . Then (y, z) is the only edge in $D8(P)$ charged to C_4^z .*

Proof. Edge (y, z) is added to E_{CAN} in a call to $AddCanonical(p, r)$ Step 4b, only if there is no edge of E_A in C_4^z . Thus C_4^z is not charged by an edge of E_A by E_A Charge 1.

Cone C_4^z is a boundary cone of $Can_0^{(p,r)}$. By Lemma 4.5.1, C_4^z cannot be an internal cone of another canonical subgraph. Thus C_4^z can only be a boundary cone of any canonical subgraph. Since $AddCanonical(\cdot, \cdot)$ Step 4b is the only call that adds an edge to E_{CAN} that is charged to a boundary cone, only another call to $AddCanonical(\cdot, \cdot)$ Step 4b can charge an additional edge to C_4^z .

Assume we have edges (y, z) and (y', z) in $Can_i^{(p,r)}$ and $Can_j^{(p',r')}$ respectively. Without loss of generality, assume that z is the first vertex in $Can_j^{(p',r')}$ and the last vertex in $Can_i^{(p,r)}$, and assume (y, z) and (y', z) occupy the same cone C_k^z . For both (y, z) and (y', z) to be added in $AddCanonical(\cdot, \cdot)$ Step 4b, it must be that $k = i - 2 = j + 2$. Without loss of generality let $i = 0, k = 4$ and $j = 2$, and z is in C_0^p and $C_2^{p'}$.

We know that both $\triangle(pyz)$ and $\triangle(p'y'z)$ are triangles in $DT(P)$. Since $(y, z) \in C_1^y$, and $(y, p) \in C_3^y$, there is no edge in C_1^y clockwise from (y, z) . Symmetrically, there is no edge in $C_1^{y'}$ counter-clockwise from (y', z) . Thus y must have a neighbour closer than z in C_1^y counter-clockwise from (y, z) , and y' must have a neighbour closer than z in $C_1^{y'}$ clockwise from (y', z) . See Figure 23.

We consider the shorter of (y, z) and (y', z) , ties broken arbitrarily. Without loss of

Figure 23: There must be an edge of E_A between (y, z) and (y', z) .

Cone Legend:

Figure 24: If (y, z) is not in E_A , there must be a neighbour of y or z in $\triangle(yxz)$.

generality we will assume $[yz] < [y'z]$. We therefore know that $y' \notin C_1^y$. So there must be a vertex t in C_1^y that is a neighbour of y and closer to y than z counter-clockwise from (y, z) . Since $z \in C_1^y$ and y' is not, the counter-clockwise cone boundary of C_1^y must intersect (y', z) at a point which we will call x . Therefore t must be in triangle $\triangle(xyz)$. See Figure 24.

Within $\triangle(xyz)$ we take the closest vertex to z and call it u . (u, z) must be an edge in $DT(P)$, and C_1^u is bounded on one side by (y, z) , and bounded on the other side by (y', z) , and thus z is the closest point to u in C_1^u . Which means that (u, z) would have been added to E_A in $AddIncident()$, which means that C_4^z has an edge $(u, z) \in E_A$. Since there is an edge of E_A in C_4^z , neither (y, z) nor (y', z) would have been added to E_{CAN} in calls to $AddCanonical(\cdot, \cdot)$ Step 4b, and neither would be charged to C_4^z .

□

This leads to the following corollary:

Corollary 4.5.3. *Assume an edge (y, z) is added to E_{CAN} in $AddCanonical(p, r)$ Step 4b, and charged to a boundary cone C_4^z . Then of all the edges in $D8(P)$, only (y, z) is charged*

to C_4^z .

4.5.3 Edges in Shared Triangles

The shared triangle is the only scenario where the internal cones of two separate cone neighbourhoods are adjacent on the same vertex.

Consider a shared triangle $\triangle(pp's)$ with base (p, p') , and assume that p' is in C_0^p , $(p, r) \in E_A$ is in C_0^p , and $(p', r') \in E_A$ is in $C_3^{p'}$. Vertex s has adjacent cones inside $Can_0^{(p,r)}$ and $Can_3^{(p',r')}$. We prove a limit on the number of canonical edges of p and p' that were added to E_{CAN} and charged to cones of s inside $Can_0^{(p,r)}$ and $Can_3^{(p',r')}$.

Lemma 4.5.4. *If (s, p') was added to E_{CAN} and charged to the empty cone of s inside $Can_0^{(p,r)}$, then (s, p) will not be charged to the empty cone of s inside $Can_3^{(p',r')}$.*

Proof. Assume that (p', s) was added to E_{CAN} by a call to $AddCanonical(p, r)$. That implies that (p', s) is not the first or last edge of $Can_i^{(p,r)}$. Thus we know by Lemma 4.3.16 that (p, s) must be the last edge in $Can_3^{(p',r')}$, which implies that it is not added by $AddCanonical(p', r')$.

Otherwise assume that (p, p') is a canonical edge of q , and (p, s) was added to E_{CAN} on a call to $AddCanonical(q, \cdot)$ in Step 4c. This implies that (p, p') is in C_5^q . See Figure 25. There are two possible ways to add (p', s) so that it is charged to the cone of s inside $Can_3^{(p',r')}$. We show that neither occurs:

1. $AddCanonical(q, \cdot)$ adds (p', s) to E_{CAN} in Step 4c. This implies that (p, p') is in C_4^q . See Figure 26. However, (p, s) was added in Step 4c, which means that (p, p') is in C_5^q , which is a contradiction. Thus both edges cannot be added by calls to $AddCanonical(q, \cdot)$, Step 4c.
2. $AddCanonical(p, r)$ adds (p', s) to E_{CAN} . The shared neighbour q of p and p' is not in C_0^p , and thus (p', s) is the last canonical edge in $Can_0^{(p,r)}$. Thus (p', s) is not added to E_{CAN} by a call to $AddCanonical(p, r)$ (by omission, Step 4).

Figure 25: Only (p, s) can be added to E_{CAN} in Step 4c with apex q .

Figure 26: Only (p', s) can be added to E_{CAN} in Step 4c with apex q .

□

Corollary 4.5.5. *If the empty cone of s inside $Can_0^{(p,r)}$ is charged twice by edges of $Can_0^{(p,r)}$, then the empty cone of s inside $Can_3^{(p',r')}$ is charged at most once by edges of $Can_3^{(p',r')}$.*

4.5.4 The Degree of D8(P)

Lemma 4.5.6. *The edges of E_A and E_{CAN} are never charged to the same cone.*

Proof. The edges of E_A are charged directly to the cone they occupy on each endpoint. We

know from the charging argument that the edges of E_{CAN} are charged to empty cones, or to a cone that does not contain an edge of E_A . Thus the edges of E_{CAN} and E_A are never charged to the same cone. $\square$

Lemma 4.5.7. *Consider a cone C_i^s of a vertex s in $D8(P)$ that is charged twice by edges of E_{CAN} . Then the two neighbouring cones C_{i-1}^s and C_{i+1}^s are charged at most once by edges of $D8(P)$.*

Proof. Consider an edge (p, r) of E_A in C_i^p . A vertex s in $Can_i^{(p,r)}$ can have a cone that is charged twice by edges of E_{CAN} only if it is not the first or last vertex in $Can_i^{(p,r)}$. Additionally, r is not charged twice in any internal cone of $Can_i^{(p,r)}$. Thus assume s is not the first or last vertex in $Can_i^{(p,r)}$, and $s \neq r$.

Each cone C_{i-1}^s and C_{i+1}^s is either an empty internal cone of $Can_i^{(p,r)}$, or a boundary cone containing a canonical edge of $Can_i^{(p,r)}$ with endpoint s . We will consider C_{i+1}^s since the cases are symmetric.

If C_{i+1}^s is an empty internal cone of $Can_i^{(p,r)}$, then it is only charged for an edge if s is on a shared triangle $\triangle(pp's)$ and s is not on the base. In this case C_{i+1}^s is charged for at most one edge of E_{CAN} by Lemma 4.5.5.

Otherwise C_{i+1}^s contains a canonical edge in $Can_i^{(p,r)}$. By our charging scheme we know only empty cones are double charged, and by Lemma 4.5.6 no cone is charged for both an edge of E_A and an edge of E_{CAN} . Thus C_{i+1}^s is either charged for an edge of E_A , an edge of E_{CAN} , or it is not charged. $\square$

Theorem 4.5.8. *The maximum degree of $D8(P)$ is at most 8.*

Proof. Each edge (p, r) of E_A is charged once to the cone of p containing r and once to the cone of r containing p . By Lemma 4.2.1, no cone is charged more than once by edges of E_A .

No edge of E_{CAN} is charged to a cone that is charged by an edge of E_A by Lemma 4.5.6.

By Lemma 4.5.7, if a cone of a vertex s of $D8(P)$ is charged twice, then its neighbouring

cones are charged at most once. This implies that there are at most 3 double charged cones on any vertex s in $D8(P)$. Assume that we have a vertex s with 3 cones that have been charged twice. A cone of s that is charged twice is an internal cone of some cone neighbourhood N_i^P by our charging argument. Thus s is endpoint to two canonical edges (q, s) and (s, t) in N_i^P . Note that $\angle(qst) > 2\pi/3$ by Lemma 4.3.8, and this angle contains the cone of s that is charged twice. Thus to have 3 cones charged twice, the total angle around s would need to be $> 2\pi$, which is impossible. Thus there are at most two double charged cones on s , which gives us a maximum degree of 8.

☐

See Figure 27 for an example of a point set P for which the maximum degree of $D8(P)$ is equal to 8.

Chapter 5

D8(P) is a Spanner

5.1 Introduction and Preliminaries

We will prove that $D8(P)$ is a spanner of $DT(P)$ with a spanning ratio of $(1 + \frac{\theta}{\sin \theta}) = (1 + \frac{2\pi}{3\sqrt{3}}) \approx 2.21$, thus making it a $(1 + \frac{2\pi}{3\sqrt{3}}) \cdot D_T$ -spanner of the complete geometric graph, where D_T is the spanning ratio of the Delaunay triangulation. As of this writing, the current best bound of the spanning ratio of the Delaunay triangulation is 1.998 [7], which makes $D8(P)$ approximately a 4.42-spanner of the complete graph.

We begin by defining the ideal path, and proving the spanning ratio of an ideal path with respect to the graph $DT(P)$.

Before we define the ideal path, we first prove the following:

Lemma 5.1.1. *Let (p, r) be an edge in E_A in the cone C_i^p . Then $Can_i^{(p,r)}$ forms a path.*

Proof. We prove by contradiction. Note that $Can_i^{(p,r)}$ is a collection of paths. Assume that there are at least two paths in this collection. Without loss of generality, let $i = 0$. Let (a, b) and (y, z) be the first and last edge respectively in $Can_0^{(p,r)}$. Thus of all the vertices in $N_0^p \setminus \{p\}$ between a and z there exists at least one consecutive subset T where for each $t_j \in T, 0 \leq j < |T|, [pt_j] < [pr]$. We consider the vertex $t_k \in T, [pt_k] \leq [pt_j]$, for all $t_j \in T, 0 \leq j < |T|$. Since $[pt_k] < [pr]$, $AddIncident(L)$ examined (p, t_k) before (p, r) . Thus

Figure 28: The canonical triangle T_{pq} in blue. Note, a possible canonical path (in red) does not need to be inside the canonical triangle.

when (p, t_k) was examined, C_i^p contained no edges of E_A with endpoint p . Since (p, t_k) was not added to E_A , there must have been an edge of E_A with endpoint t_k in $C_3^{t_k}$. However, we know $[pt_{k-1}] \leq [pt_k]$ and $[pt_{k+1}] \leq [pt_k]$ (whether or not t_{k-1} and t_{k+1} are in T). Thus neither (t_k, t_{k-1}) nor (t_k, t_{k+1}) can be in $C_3^{t_k}$. Since $\triangle(pt_k t_{k-1})$ and $\triangle(pt_k t_{k+1})$ are triangles in $DT(P)$, the only edge with endpoint t_k in $C_3^{t_k}$ is (p, t_k) . This means that (p, t_k) would have been added to E_A instead of (p, r) , which is a contradiction. $\square$

Definition 5.1.2. Consider an edge (p, r) in C_i^p in E_A , and the graph $Can_i^{(p,r)}$. An ideal path is a simple path from p to any vertex in $Can_i^{(p,r)}$ using the edges of $(p, r) \cup Can_i^{(p,r)}$.

We also introduce the concept of the *canonical triangle*, as seen in the paper by Bose *et al.* [14]:

Definition 5.1.3. Consider a vertex q in the cone C_i^p of vertex p . Take the line through q perpendicular to the bisector of C_i^p . Let a and b be the intersection points of this line with the cone C_i^p . Then the canonical triangle T_{pq} is the triangle $\triangle(pab)$. See Figure 28

Consider an edge (p, r) in C_i^p in E_A , and the graph $Can_i^{(p,r)}$. We will prove that the length of the ideal path from p to q is not greater than $|pa| + \frac{\theta}{\sin \theta}|aq|$, where a is the corner of the canonical triangle to the side of (p, q) that has r , and $\theta = \pi/3$ is the cone angle.

We then use ideal paths to prove there exists a path with bounded spanning ratio between

any two vertices p and q in $D8(P)$, where (p, q) is an edge in $DT(P)$. We prove a bound on the length of the path from p to q of $|pa| + \frac{\theta}{\sin \theta}|aq|$.

We note that the distance $|pa| + \frac{\theta}{\sin \theta}|aq|$ is with respect to the canonical triangle T_{pq} rather than the Euclidean distance $|pq|$. To finish the proof we show that:

$$|pa| + \frac{\theta}{\sin \theta}|aq| \leq (1 + \frac{\theta}{\sin \theta})|pq|.$$

5.2 Spanning Ratio of Ideal Paths

We begin with a couple of well-known geometric lemmas. The first is an observation regarding the relative lengths of convex paths, when one resides inside the other.

Lemma 5.2.1. *If a convex body C is contained within another convex body C' , then the perimeter of C' is longer than C . [15], page 42.*

The next lemma is a well known result traditionally called “The Inscribed Angle Theorem”.

Lemma 5.2.2. *Consider 3 points p, q, s on the boundary of a circle O with center o , such that $\angle(pqs) = \alpha$. Let A be the arc of O from p to s that does not go through q , and let $\overline{A}$ be the arc of O from p to s through q . Then the angle $\angle(pos)$ facing A is equal to 2α . Further, the angles $\angle(pqs)$ facing A is the same for any point p that is on $\overline{A}$.*

That allows us to establish this result:

Lemma 5.2.3. *Let O be a circle through points p and q and r in clockwise order, and let α denote the angle $\angle(qpr)$. Then the length of the arc from q to r on the boundary of $D_{p,q,r}$ is*

$$\frac{\alpha}{\sin \alpha}|qr|$$

Figure 29: The Inscribed Angle Theorem

Proof. From the center point of O , the angle between q and s is 2α by Lemma 5.2.2. Thus the arc length between q and r is $2\alpha R$, where R is the radius of O . Also, $|qr| = 2\sin\alpha R$, which means $R = \frac{|qr|}{2\sin\alpha}$. Thus the arc length between q and r is equal to:

$$\begin{aligned} 2\alpha R &= \frac{2\alpha}{2\sin\alpha} |qr| \\ &= \frac{\alpha}{\sin\alpha} |qr| \end{aligned}$$

which completes the proof. See Figure 30. □

Lemma 5.2.4. Consider the restricted neighbourhood $N_p^{(r,q)}$ in $DT(P)$ in the cone C_i^p . Let $O_{p,r,q}$ be the circle through the points p , q , and r . Then there are no points of P in $O_{p,r,q}$ to the side of (p,r) that does not contain q . Likewise there are no points of P in $O_{p,r,q}$ to the side of (p,q) that does not contain r .

Figure 30: Length of arc between r and q .

Proof. Since the cases are symmetric, we prove that there are no points of P in the region R of $O_{p,r,q}$ to the side of (p,r) that does not contain q . We prove by contradiction. Thus assume there is a point t in R . Then the circle $O_{p,t,r}$ contains q and the circle $O_{p,r,q}$ contains t , thus there is no circle through p and r that is empty of points of P . Thus (p,r) cannot be a Delaunay edge, which is a contradiction to our definition of restricted neighbourhood. See Figure 31. □

Lemma 5.2.5. *Consider the restricted neighbourhood $N_p^{(r,q)}$ in cone C_i^p . Let rq be the directed line from r to q , and assume there are no neighbours of p in $N_p^{(r,q)}$ right of rq . If (r,q) is not an edge in $N_p^{(r,q)}$, then there is a vertex $a \in N_p^{(r,q)}$ such that the circle $O_{r,a,q}$ is empty of vertices of P left of rq .*

Proof. We prove by contradiction, thus assume that we have found a vertex a left of rq such that $O_{r,a,q}$ is empty of vertices of P left of rq , and a is not in $N_p^{(r,q)}$. Note vertex a must exist, otherwise (r,q) is on the convex hull and thus in $N_p^{(r,q)}$. Since the region of $N_p^{(r,q)}$ is empty of vertices of P , a must be outside of $N_p^{(r,q)}$.

We look at two cases:

Figure 31: If there is a point t , then (p, r) is not an edge in $DT(P)$.

Figure 32: Circle $O_{r,a,q}$ contains u .

1. a is outside of $O_{p,r,q}$: Since (r, q) is not in $DT(P)$, there is at least one vertex u in $N_p^{(r,q)} \setminus \{p, r, q\}$. By Lemma 4.3.7 and our initial assumption that $N_p^{(r,q)}$ contains no neighbours of p to the right of rq , u must be in $O_{p,r,q}$ to the left of rq . Since a is outside of $O_{p,r,q}$, the arc of $O_{r,a,q}$ to the left of rq contains the arc of $O_{p,r,q}$ to the left of rq . Thus u is in $O_{r,a,q}$ to the left of rq , which is a contradiction to our selection of vertex a . See Figure 32.
2. a is inside $O_{p,r,q}$: Since $\angle(rpq) < \pi/3$ (since it is in a cone), and a is inside $O_{p,r,q}$, a must be positioned radially between two consecutive edges with endpoint p in $N_p^{(r,q)}$. Call these edges (p, u) , and (p, v) . Note that $\triangle(puv)$ is a triangle in $DT(P)$, and thus the circle $O_{p,u,v}$ does not contain a by the empty circle property of the Delaunay triangulation. This implies that, since p, u, a , and v form a convex quadrilateral with p and a across the diagonal, any circle through p and a must contain at least one of u or v .

Since a is inside $O_{p,r,q}$, $O_{r,a,q}$ contains p . Thus we can draw the circle O_1 through a and p tangent to $O_{r,a,q}$. The portion of O_1 to the left of rq is contained in $O_{r,a,q}$, and thus does not contain any points of P . But any circle through a and p must contain at least one of u and v , and u and v are to the left of rq , which is a contradiction. See Figure 33

Thus, if (r, q) is not an edge in $N_p^{(r,q)}$, there is a neighbour a of p in $N_p^{(r,q)}$ such that $O_{r,a,q}$ is empty of vertices of P left of rq . See Figure 33.

□

We now turn to a lemma from the paper of Bose and Keil [16] that tells us the length of a path between two points in the Delaunay triangulation of a set of vertices. We provide a slightly modified and truncated version that suits our needs. The lemma of Bose and Keil does not provide an explicit construction. We apply the lemma to a restricted neighbourhood, and are able to provide a construction of the path along with an upper bound on its length.

Figure 33: A circle through a and p contains u or v .

Lemma 5.2.6. Consider the restricted neighbourhood $N_p^{(r,q)}$ in $DT(P)$ in the cone C_i^p . Let $\alpha = \angle(rpq) < \pi/3$. If no point of P lies in the triangle $\triangle(prq)$ then there is a path from r to q in $DT(P)$, using canonical edges of p , whose length satisfies:

$$\delta(r, q) \leq |rq| \frac{\alpha}{\sin \alpha}$$

Proof. Let o be the center of $O_{p,r,q}$, and let $\beta = \angle(roq) = 2\alpha$.

Lemma 5.2.4 and the assumption that no vertices of P lie in the triangle $\triangle(prq)$ imply that there are no vertices of P in $O_{p,r,q}$ to the right of directed line segment rq .

We proceed by induction on the number of vertices in $N_p^{(r,q)}$. If there are only 3 vertices in $N_p^{(r,q)}$, then (r, q) is an edge in $DT(P)$, and the path from r to q has length $|rq| < |rq| \frac{\alpha}{\sin \alpha}$ and we are done.

Now assume that the inductive hypothesis holds for all restricted neighbourhoods with fewer vertices than $N_p^{(r,q)}$. Assume $N_p^{(r,q)}$ has more than 3 vertices, otherwise we are done by the same argument as above.

Lemma 5.2.4 tells us that there is a vertex a in $N_p^{(r,q)}$ where $O_{r,a,q}$ is empty of vertices of P left of rq .

Let O_1 be the circle through r and a with center o_1 on the line segment (o, r) . Let O_2 be

the circle through a and q whose center o_2 lies on the line segment (o, q) . Let $\alpha_1 = \angle(ro_1a)$ and let $\alpha_2 = \angle(ao_2q)$. $N_p^{(r,a)}$ and $N_p^{(a,q)}$ have fewer vertices than $N_p^{(r,q)}$, and O_1 is empty of vertices of P to the right of directed segment ra , and O_2 is empty of vertices of P to the right of directed line segment aq . Thus by the inductive hypothesis:

$$\begin{aligned}\delta(r, q) &= \delta(r, a) + \delta(a, q) \\ &= |ra| \frac{\alpha_1}{\sin \alpha_1} + |aq| \frac{\alpha_2}{\sin \alpha_2}\end{aligned}$$

Let $r' \neq r$ be the intersection of O_1 and rq , and let $q' \neq q$ be the intersection of O_2 and rq . Since $\beta < \pi$, O_1 and O_2 overlap. Let O_3 be the circle through q' and r' with center o_3 on the intersection of the line segment between o_1 and r' and the line segment between o_2 and q' . See Figure 34.

Triangles $\triangle(roq)$, $\triangle(ro_1r')$, $\triangle(q'o_2q)$, and $\triangle(q'o_3r')$ are all similar isosceles triangles. Thus by Lemmas 5.2.2 and 5.2.3 the length of the arc of O_1 left of rq is $|rr'| \frac{\alpha}{\sin \alpha}$, the length of the arc of O_2 left of rq is $|q'q| \frac{\alpha}{\sin \alpha}$, and the length of the arc of O_3 left of rq is $|q'r'| \frac{\alpha}{\sin \alpha}$.

Note that O_3 is completely contained in the intersections of O_1 and O_2 . Let A_1 be the arc of O_1 left of rq from a to r' , and let A_2 be the arc of O_2 left of rq from a to q' . Note that $A_1 \cap A_2$ is a convex shape from q' to r' that contains the arc of O_3 left of rq . Thus $|A_1 \cap A_2| \geq |q'r'| \frac{\alpha}{\sin \alpha}$ by convexity (Lemma 5.2.1).

We observe that:

$$\begin{aligned}
\delta(r, q) &= \delta(r, a) + \delta(a, q) \\
&= |ra| \frac{\alpha_1}{\sin \alpha_1} + |aq| \frac{\alpha_2}{\sin \alpha_2} \\
&= |rr'| \frac{\alpha}{\sin \alpha} + |q'q| \frac{\alpha}{\sin \alpha} - |A_1 \cap A_2| \\
&\leq |rr'| \frac{\alpha}{\sin \alpha} + |q'q| \frac{\alpha}{\sin \alpha} - |q'r'| \frac{\alpha}{\sin \alpha} \\
&= |rq| \frac{\alpha}{\sin \alpha}
\end{aligned}$$

as required. □

Now we prove the following:

Lemma 5.2.7. *Consider the restricted neighbourhood $N_p^{(r,q)}$ and without loss of generality let $N_p^{(r,q)}$ be in C_0^p . Let $\alpha = \angle(rpq)$. Let $r_q \neq p$ be the point where the line through p and r intersects the canonical triangle T_{pq} . Let $q_r \neq p$ be the point where the edge (p, q) intersects T_{pr} . If $[pr]$ is the shortest edge of all edges in $N_p^{(r,q)}$ with endpoint p , then the distance from r to q using the canonical edges of p in $N_p^{(r,q)}$ is at most:*

$$\max\{|rr_q|, |q_rq|\} + |r_qq| \frac{\alpha}{\sin \alpha}$$

Proof. Let $\delta(r, q)$ be the length of the path between r and q in $N_p^{(r,q)}$. We will prove by induction on the number of vertices in $N_p^{(r,q)}$.

If there are only three vertices in $N_p^{(r,q)}$, then (r, q) is an edge in $N_p^{(r,q)}$. Since $|rq| \leq \max\{|rr_q|, |q_rq|\} + |r_qq| \frac{\alpha}{\sin \alpha}$, we are done.

Otherwise assume there are more than 3 vertices in $N_p^{(r,q)}$. Consider the edge $(p, a) \in N_p^{(r,q)}$, such that $[pa] \leq [pt]$, for all $(p, t) \in N_p^{(r,q)} \setminus \{r, q\}$. Let $\alpha_1 = \angle(rpa)$. We consider two

Figure 34: Lemma 5.2.6.

cases:

1. If $[pa] > [pq]$, then $[pr]$ and $[pq]$ are the shortest edges in $N_p^{(r,q)}$, which implies that there are no points in $\triangle(prq)$. Thus from Lemma 5.2.6, the length of the path from r to q is at most $|rq|\frac{\alpha}{\sin \alpha}$. We have

$$\begin{aligned}\delta(r, q) &\leq |rq|\frac{\alpha}{\sin \alpha} \\ &\leq |rr_q| + |r_qq|\frac{\alpha}{\sin \alpha}\end{aligned}$$

by convexity (Lemma 5.2.1) and thus

$$\begin{aligned}\delta(r, q) &\leq |rr_q| + |r_qq|\frac{\alpha}{\sin \alpha} \\ &\leq \max\{|rr_q|, |q_rq|\} + |r_qq|\frac{\alpha}{\sin \alpha}\end{aligned}$$

which satisfies the inductive hypothesis. See Figure 35

2. If $[pa] < [pq]$, let $\alpha_1 = \angle(rpa)$ and $\alpha_2 = \angle(apq)$. Since $[pr] \leq [pa]$ we can apply the inductive hypothesis on $N_p^{(r,a)}$. Let r_a be the point where the line through p and r intersects the horizontal line through a , and let a_r be the point where the line through p and a intersects the horizontal line through r . See Figure 36. Then by the inductive hypothesis:

$$\delta(r, a) = \max\{|rr_a|, |a_ra|\} + |r_aa|\frac{\alpha_1}{\sin \alpha_1}$$

Since $[pa] \leq [pq]$ we can apply the inductive hypothesis on $N_p^{(a,q)}$. Let $a_q \neq p$ be the

Figure 35: If $|pa| \geq |pq|$, apply Lemma 5.2.6.

Figure 36: If the vertex a is in $\triangle(prq)$, we proceed by induction from r to a .

Figure 37: Proceed by induction from a to q .

point where the line through a and p exits T_{pq} , and let $q_a \neq p$ be the point where (p, q) intersects T_{pa} , and let $\alpha_2 = \angle(apq)$. See Figure 37. Then by the inductive hypothesis:

$$\delta(a, q) = \max\{|aa_q|, |q_aq|\} + |a_qq| \frac{\alpha_2}{\sin \alpha_2}$$

We have $\alpha_1 \leq \alpha$ and $\alpha_2 \leq \alpha$ and $\frac{\alpha}{\sin \alpha}$ is an increasing function for the interval $0 < \alpha < \pi/3$. Also note that $|pa_q| \leq \max\{|pr_q|, |pq|\}$. Thus:

$$\begin{aligned}
\delta(r, q) &= \max\{|rr_a|, |a_r a|\} + \max\{|aa_q|, |q_a q|\} + |r_a a| \frac{\alpha_1}{\sin \alpha_1} + |a_q q| \frac{\alpha_2}{\sin \alpha_2} \\
&\leq \max\{|rr_q|, |q_r q|\} + |r_a a| \frac{\alpha_1}{\sin \alpha_1} + |a_q q| \frac{\alpha_2}{\sin \alpha_2} \\
&\leq \max\{|rr_q|, |q_r q|\} + |r_q a_q| \frac{\alpha_1}{\sin \alpha_1} + |a_q q| \frac{\alpha_2}{\sin \alpha_2} \\
&\leq \max\{|rr_q|, |q_r q|\} + |r_q a_q| \frac{\alpha}{\sin \alpha} + |a_q q| \frac{\alpha}{\sin \alpha} \\
&\leq \max\{|rr_q|, |q_r q|\} + |r_q q| \frac{\alpha}{\sin \alpha}
\end{aligned}$$

Figures 38, 39, 40.

Lemma 5.2.8. *Consider the edge (p, r) in E_A , located in Can_i^p , and the associated*

Figure 39: The distance $|r_a a| < |r_q a_q|$

Figure 40: The distances across the arcs are $|r_q a_q| \frac{\alpha}{\sin \alpha} + |a_q q| \frac{\alpha}{\sin \alpha} = |r_q q| \frac{\alpha}{\sin \alpha}$

Figure 41: Lemma 5.2.8.

canonical subgraph $\text{Can}_i^{(p,r)}$. Without loss of generality, assume that $i = 0$. The length of the ideal path from p to any vertex q in $\text{Can}_0^{(p,r)}$ is:

$$\delta(r, q) \leq |pa| + \frac{\theta}{\sin \theta} |aq|$$

where a is the corner of T_{pq} such that $r \in \triangle(pqa)$, and $\theta = \pi/3$ is the angle of the cones.

Proof. By Lemma 5.2.7 the path from r to q is no greater than $\max\{|rr_q|, |q_rq|\} + |r_qq| \frac{\alpha}{\sin \alpha}$, where $\alpha = \angle(rpq)$. We have

$$|pr| + \max\{|rr_q|, |q_rq|\} \leq |pa|$$

Since $\frac{\alpha}{\sin \alpha}$ is an increasing function in the interval $0 \leq \alpha \leq \pi/3$, $\theta > \alpha$ and $|aq| > |r_qq|$,

we have

$$\begin{aligned}\delta(p, q) &\leq |pr| + \max\{|rr_q|, |q_rq|\} + |r_qq| \frac{\alpha}{\sin \alpha} \\ &\leq |pa| + |aq| \frac{\theta}{\sin \theta}\end{aligned}$$

as required. □

See Figure 41.

5.3 Lengths of Paths in $D8(P)$

5.3.1 Introduction

In this section we prove the following theorem:

Theorem 5.4.2: *For any edge $(p, q) \in DT(P)$, there is a path in $D8(P)$ from p to q with length at most $(1 + \frac{\theta}{\sin \theta}) |pq|$, where $\theta = \pi/3$ is the cone width. Thus $D8(P)$ is a $(1 + \frac{\theta}{\sin \theta})D_T$ -spanner of the complete graph, where D_T is the spanning ratio of the Delaunay triangulation (currently 1.998 [7])*

We first prove some supporting lemmas. Then we list the types of paths we can have between p and q in $D8(P)$, where (p, q) is an edge in $DT(P)$. Then we prove a bound of

$$|pa| + \frac{\theta}{\sin \theta} |aq|$$

on each type of path, where (p, a) and (a, q) are line segments on the border of the canonical triangle T_{pq} . The final type of path is the inductive path, and it encapsulates the other cases.

At this point our spanning ratio is with respect to T_{pq} . We then prove that $D8(P)$ is a

Figure 42: An example arrangement of vertices.

spanner with respect to the Euclidean distance $|pq|$.

5.3.2 Preliminaries

We consider an edge $(p, q) \in DT(P)$. If $(p, q) \in D8(P)$ then the length of the path from p to q in $D8(P)$ is $|pq| \leq (1 + \frac{\theta}{\sin \theta}) |pq|$, as required.

Thus we assume $(p, q) \notin D8(P)$. Without loss of generality we assume q is in C_p^p . Since $(p, q) \notin D8(P)$, there is an edge (p, r) of E_A in Can_0^p or (q, u) in Can_3^q (or both), where $[pr] \leq [pq]$ and $[qu] \leq [pq]$. Otherwise p and q would have agreed to (p, q) , and it would have been added to E_A in $AddIncident(L)$. Without loss of generality we shall assume there is the edge $(p, r) \in E_A$, $[pr] \leq [pq]$, and that (p, q) is clockwise from (p, r) around p .

Let s be the vertex such that s is a neighbour of q in $N_p^{(p,r)}$ and $s \neq p$ (but possibly $s = r$). Let a be the upper left corner of T_{pq} , and b be the upper right corner. Let $\alpha = \angle(rpq)$ and $\theta = \pi/3$ be the width of the cones. See Figure 42 for an example construction.

Lemma 5.3.1. *Recall that $(p, r) \in E_A$, where $r \in C_i^p$. Then there is an ideal path from p to any vertex q in $Can_i^{(p,r)}$, where q is not an end vertex of Can_i^p .*

Proof. In the algorithm $AddCanonical(p, r)$, we add every canonical edge of p in $Can_i^{(p,r)}$ that is not the first or last edge. By Lemma 5.3.1, the edges of $Can_i^{(p,r)}$ form a path. Thus

there is the ideal path from p to any vertex q in $Can_i^{(p,r)}$ that is not the first or last vertex. $\square$

Lemma 5.3.2. *Let r and q be two consecutive neighbours of p , in an arbitrary cone C_i^p . Without loss of generality, let (p, q) be clockwise from (p, r) in the cone C_i^p . If q is in C_i^r , then all edges with endpoint p in C_i^p that appear after (p, q) in clockwise order are longer than $[pq]$.*

Proof. By Lemma 4.3.8, any edge (p, t) clockwise from (p, q) in C_i^p is such that the angle $\angle(rqt) > 2\pi/3$. Since (r, q) is in C_i^r , it is at an angle of at least $\pi/3$ from the positive x -axis. Since $\angle(rqt) > 2\pi/3$, the edge (r, t) must be at an angle > 0 with respect to the positive x -axis. Thus $[pt] > [pq]$, for all (p, t) clockwise from (p, q) in C_i^p . See Figure 43. $\square$

By Lemma 5.3.1, there is the ideal path from p to any vertex in $Can_0^{(p,r)}$ that is not the first or last vertex. We now turn our attention to the first or last vertex in $Can_0^{(p,r)}$. Because the cases are symmetric, we focus on the last vertex, which we designate z . If $z = r$, the

path from p to z is trivial, thus we assume $z \neq r$. Let y be the neighbour of z in $Can_0^{(p,r)}$. By Lemma 5.3.3, (y, z) cannot be in C_0^z . Thus (y, z) can be in C_5^z , C_4^z , or C_3^z .

Case 2: Edge (y, z) is in C_4^z . There are three possibilities.

- Case 3: Edge (y, z) is in C_3^z . Then (y, z) was not added to E_{CAN} . See Figure 45.

In Case 1, Case 2a, and Case 2b there is an ideal path from p to q . Thus Lemma 5.2.8 tells us there is a path from p to q not longer than $|pa| + \frac{\theta}{\sin \theta}|aq|$.

Figure 46: Ideal paths from p to y and z to y .

In Case 2c, we have two ideal paths that meet at y . As in the case of a single ideal path, the sum of the lengths of these two paths is not more than $|pa| + \frac{\theta}{\sin \theta}|aq|$. The following lemma proves this claim:

Lemma 5.3.4. *Consider the edge (p, r) in E_A in the graph $D8(P)$, r in C_0^p . Let (y, z) be the last edge in $Can_0^{(p,r)}$, and let (y, z) be in C_4^z . Let (z, u) be an edge in E_A in C_4^z . Assume there is an ideal path from p to y in C_0^p , and an ideal path from z to y in C_4^z . Let a be the top left corner of T_{pz} . We prove an upper bound on the length $\delta(p, z)$:*

$$\delta(p, z) \leq |pa| + \frac{\theta}{\sin \theta}|az|$$

Proof. Let a_1 be the top left corner of T_{py} , and let b_2 be the top right corner of T_{zy} (as seen from apex z). Note that T_{zy} lies in C_z^4 . Since (y, z) is the *last* edge in $Can_0^{(p,r)}$, we note that the ideal path from p to y is to the side of (p, y) that contains r and does not contain z . Similarly, the ideal path from z to y is to the side of (y, z) that contains u and does not contain p . See Figure 46. By Lemma 5.3.1, the length of the path from p to z in $D8(P)$ is:

Figure 47: The path corresponding to Equation 5.3.1, Lemma 5.3.4.

Figure 48: The path corresponding to Equations 5.3.2 and 5.3.3, in Lemma 5.3.4.

$$\begin{aligned}
\delta_{D8(P)}(p, z) &\leq \delta_{D8(P)}(p, y) + \delta_{D8(P)}(z, y) \\
&\leq |pa_1| + \frac{\theta}{\sin \theta} |a_1 s| + |zb_2| + \frac{\theta}{\sin \theta} |b_2 y| \\
&\leq |pa_1| + \frac{\theta}{\sin \theta} |a_1 s| + |b_2 y| + \frac{\theta}{\sin \theta} |zb_2| \tag{5.3.1}
\end{aligned}$$

$$\leq (|pa_1| + |b_2 y|) + \frac{\theta}{\sin \theta} (|a_1 y| + |zb_2|) \tag{5.3.2}$$

$$= |pa| + \frac{\theta}{\sin \theta} |az| \tag{5.3.3}$$

Inequality 5.3.1 holds because $\frac{\theta}{\sin \theta} > 1$, and $|b_2 y| \leq |zb_2|$, since $|zb_2|$ is the longest possible line segment in T_{zy} . See Figures 46, 47, and 48. $\square$

In Case 3 there is no edge from y to z . We prove the length of the path from p to z in Case 3 by induction, as part of the main lemma of this section:

Lemma 5.3.5. *Consider the edge (p, r) in E_A in the graph $D8(P)$. Without loss of generality, let r be in C_0^p . Let a and b be the top left corner and top right corner respectively of T_{pq} . For any edge $(p, q) \in DT(P)$, there exists a path from p to q in $D8(P)$ that is not longer than:*

$$\max\{|pa| + \frac{\theta}{\sin \theta} |aq|, |pb| + \frac{\theta}{\sin \theta} |bq|\}$$

Proof. Let $\delta(p, q)$ be the shortest path from p to q in $D8(P)$. We do a proof by induction on the size of the canonical triangle T_{pq} .

The base case is when T_{pq} is the smallest canonical triangle. One instance of this occurs when there is an ideal path from p to q , as in Case 1, Case 2a, and Case 2b. Thus by Lemma 5.2.8:

$$\delta(p, q) \leq |pa| + \frac{\theta}{\sin \theta} |aq|.$$

Figure 49: The green path is the length of the path we can prove, the red path is how much path we have to pay for it.

Figure 50: Notice that the inductive path takes the longer path between a_2 and b_2 .

The other instance is Case 2c, where two ideal paths meet at a vertex. By Lemma 5.3.4 we have:

$$\delta(p, q) \leq |pa| + \frac{\theta}{\sin \theta} |aq|.$$

Since $|aq| \leq \max\{|aq|, |bq|\}$, the proof holds in all base cases.

In Case 3, q is the first or last vertex in $Can_0^{(p,r)}$. Since the cases are symmetric, consider when q is the last vertex, and assume it has a neighbour s in $Can_0^{(p,r)}$, such that the canonical edge (s, q) in N_0^p is in C_0^s . Thus (s, q) was not added to E_{CAN} on a call to $AddCanonical(p, r)$.

We break down T_{pq} into canonical triangles T_{ps} and T_{sq} . Call the upper left corner of T_{pq} a , and the upper right corner b . Also the upper left corner of T_{ps} is a_1 , the upper right corner of T_{sq} is a_2 , the upper right corner of T_{ps} is b_1 , and the upper right corner of T_{sq} is b_2 . Since (s, q) is in C_0^s , both T_{ps} and T_{sq} must be smaller than T_{pq} .

We note the following facts:

Fact 1: $|pa| = |pa_1| + |sa_2|$ and likewise $|pb| = |pb_1| + |sb_2|$

Fact 2: $|ab| = |a_1b_1| + |a_2b_2|$

Fact 3: $|aa_2| = |a_1s|$ and $|b_2b| = |sb_1|$

Fact 4: q is on the line (a_2, b_2)

Without loss of generality, assume the path from p to s is to the side of the line through p and s with a_1 (note that we are not assuming that $|a_1s| > |b_1s|$).

We extend the line (p, s) until it intersects (a_2, b_2) at a point we label s' . Since q is the last vertex in $Can_0^{(p,r)}$, q must be to the side of s' closer to b_2 .

Since $|pa| = |pb|$ and $|sa_2| = |sb_2|$, it is sufficient to prove:

$$|pa_1| + \frac{\theta}{\sin \theta} |a_1s| + |sa_2| + \frac{\theta}{\sin \theta} \max\{|a_2q|, |qb_2|\} \leq |pa| + \frac{\theta}{\sin \theta} \max\{|aq|, |bq|\}$$

By Fact 1 this is equivalent to:

$$\begin{aligned} \frac{\theta}{\sin \theta} |a_1s| + \frac{\theta}{\sin \theta} \max\{|a_2q|, |qb_2|\} &\leq \frac{\theta}{\sin \theta} \max\{|aq|, |bq|\} \\ |a_1s| + \max\{|a_2q|, |qb_2|\} &\leq \max\{|aq|, |qb|\} \end{aligned}$$

We consider two scenarios:

1. $|a_1s| \leq |sb_1|$: There are two sub-cases:

(a) $|qb| \geq |aq|$: If $|a_2q| \geq |qb_2|$, then:

$$\begin{aligned} |a_1s| + |a_2q| &\leq |aq| \\ &\leq |qb| \end{aligned}$$

as required. Otherwise, $|qb_2| > |a_2q|$, thus:

$$\begin{aligned} |a_1s| + |qb_2| &\leq |sb_1| + |qb_2| \\ &= |qb| \end{aligned}$$

as required.

(b) $|qb| < |aq|$: Together with $|a_1s| \leq |sb_1|$ implies that $|a_2q| > |qb_2|$. See Figure 52.

Then $|a_1s| + |a_2q| = |aq|$, as required.

2. $|a_1s| > |sb_1|$: Since q is radially to the right of (p, s) , $|aq| > |qb|$. It is also true that $|a_2q| > |qb_2|$. Thus, using Fact 3:

$$\begin{aligned} |a_1s| + |a_2q| &= |aa_2| + |a_2q| \\ &= |aq| \end{aligned}$$

as required. See Figure 51.

□

5.4 The Spanning Ratio of $D8(P)$

For an edge (p, q) in $DT(P)$, we have a bound on the length of the path in $D8(P)$. However, this bound is terms of the size of the canonical triangle T_{pq} , which is not the same as the Euclidean distance $|pq|$.

We prove that $(1 + \frac{\theta}{\sin \theta}) |pq|$ is actually an upper bound on $|pa| + \frac{\theta}{\sin \theta} |aq|$ here:

Claim 5.4.1.

$$\max\{|pa| + \frac{\theta}{\sin \theta} |aq|, |pb| + \frac{\theta}{\sin \theta} |bq|\} \leq \left(1 + \frac{\theta}{\sin \theta}\right) |pq|$$

Proof. Without loss of generality, we will assume that

$$\max\{|pa| + \frac{\theta}{\sin \theta}|aq|, |pb| + \frac{\theta}{\sin \theta}|bq|\} = |pa| + \frac{\theta}{\sin \theta}|aq|$$

Let

$$\lambda = \left(\frac{\theta}{\sin \theta} - 1 \right) (|pq| - |aq|)$$

We will show that:

$$|pa| + \frac{\theta}{\sin \theta}|aq| \leq |pa| + \frac{\theta}{\sin \theta}|aq| + \lambda \leq \left(1 + \frac{\theta}{\sin \theta} \right) |pq|$$

Since $|pq| \geq |pa|$ (by the sine law), and $\frac{\theta}{\sin \theta} > 1$, we get $\lambda \geq 0$. Thus

$$\begin{aligned} & |pa| + \frac{\theta}{\sin \theta}|aq| \\ & \leq |pa| + \frac{\theta}{\sin \theta}|aq| + \lambda \end{aligned}$$

It remains to be shown that:

$$\begin{aligned}
|pa| + \frac{\theta}{\sin \theta} |aq| + \lambda &\leq \left(1 + \frac{\theta}{\sin \theta}\right) |pq| \\
|pa| + \frac{\theta}{\sin \theta} |aq| + \left(\frac{\theta}{\sin \theta} - 1\right) (|pq| - |aq|) &\leq \left(1 + \frac{\theta}{\sin \theta}\right) |pq| \\
|pa| - |pq| + |aq| + \frac{\theta}{\sin \theta} (|aq| + |pq| - |aq|) &\leq \left(1 + \frac{\theta}{\sin \theta}\right) |pq| \\
|pa| - |pq| + |aq| + \frac{\theta}{\sin \theta} |pq| &\leq |pq| + \frac{\theta}{\sin \theta} |pq| \\
|pa| - |pq| + |aq| &\leq |pq| \\
|pa| + |aq| &\leq 2|pq|
\end{aligned}$$

Thus we must show that $|pa| + |aq| \leq 2|pq|$ holds true for all values of $\alpha = \angle(apq)$.

Let a' be the point to the side of (p, q) that contains a , such that $\triangle(a'pq)$ is an equilateral triangle. Thus

$$|pa'| + |a'q| = 2|pq|.$$

See Figure 53b. We will prove that

$$|pa| + |aq| \leq |pa'| + |a'q| = 2|pq|.$$

Note that $\angle(paq) = \angle(pa'q) = \theta$. That implies that the circle $O_{pa'q}$ through p , a' and q also goes through a . See Figure 53c.

To better analyze the problem, we rotate, translate, and scale p , q , a and a' such that $q = (-\sin \theta, 0)$, $p = (\sin \theta, 0)$, and $a' = (0, -1.5)$. Let a be any point on $O_{pqa'}$ below the line through p and q . Let $E(p, q, d)$, where $d = 2|pq|$, represent an ellipse with focal points p and q such that for each point b on the boundary of $E(p, q, d)$, $|pb| + |bq| = d = 2|pq|$. Note the center of $O_{pqa'}$ is $(0, -0.5)$, and $O_{pqa'}$ has a radius of 1. See Figure 53d. The equation for $O_{pqa'}$ is:

(c) Circle $O_{pqa'}$ also goes through a .

(d) The intersection of $O_{pqa'}$ and $E(p, q, d)$.

Figure 53: $AddCanonical(p, r)$

$$x^2 + \left(y + \frac{1}{2}\right)^2 = 1$$

The equation for $E(p, q, d)$ is:

$$\begin{aligned} \frac{x^2}{a^2} + \frac{y^2}{b^2} &= 1 \\ \frac{x^2}{(2 \sin \theta)^2} + \frac{y^2}{\frac{3}{2}} &= 1 \\ \frac{x^2}{3} + \frac{4y^2}{9} &= 1 \end{aligned}$$

Thus we find the intersection of $O_{pqa'}$ and $E(p, q, d)$ by solving the following system of equations:

$$\begin{aligned} x^2 + \left(y + \frac{1}{2}\right)^2 &= 1 \\ \frac{x^2}{3} + \frac{4y^2}{9} &= 1 \end{aligned}$$

This gives us a single solution at $(0, -1.5)$.

Note that, when $\angle(aqp) = \pi/2$, $|pa| = 2$ and $|aq| = 2 \cos \theta = 1$. Thus $|pa| + |aq| = 3$.

We have $2|pq| = 2 * (2 \sin \theta) \approx 3.46$. Thus when $\angle(aqp) = \pi/2$, $|pa| + |aq| < 2|pq| = |pa'| + |a'q|$, which means that a is inside $E(p, q, d)$, which means all of $O_{pqa'}$ is inside $E(p, q, d)$, with the exception of $(0, -1.5)$. Thus for all points a on $O_{pqa'}$,

$$|pa| + |aq| \leq |pa'| + |a'q| = 2|pq|.$$

Which implies that:

$$\delta(p, q) \leq |pa| + \frac{\theta}{\sin \theta} |aq| \leq (1 + \frac{\theta}{\sin \theta}) |pq|$$

as required. □

We can now prove the main theorem:

Theorem 5.4.2. *For any edge $(p, q) \in DT(P)$, there is a path in $D8(P)$ from p to q with length at most $(1 + \frac{\theta}{\sin \theta}) |pq|$, where $\theta = \pi/3$ is the cone width. Thus $D8(P)$ is a $(1 + \frac{\theta}{\sin \theta})D_T$ -spanner of the complete graph, where D_T is the spanning ratio of the Delaunay triangulation (currently 1.998 [7]).*

Proof. Follows from Lemma 5.3.5 and Claim 5.4.1. □

Chapter 6

Conclusion and Future Work

We have given a construction for a graph $D8(P)$ that has a degree of 8, which is low enough to be practical in many applications, while still maintaining a competitive spanning ratio of ≈ 4.41 , which is the best known spanning ratio of a planar graph with a degree bound less than 14.

To build and analyze this graph, we borrow heavily from the techniques used by Kanj *et. al* [3] in their analysis of a bounded degree spanning subgraph of the Delaunay triangulation. Additionally we reuse techniques and analysis used by Bonichon *et. al* [1] in their degree 6 spanning subgraph of half- θ -6 graphs. The combination of these techniques gives a spanning ratio that, when adjusted for cone angle, matches the best known. We get an improvement on the spanning ratio of the graphs of Bonichon *et. al* [1], which shows the difference a strong underlying graph can make.

Although our construction and analysis involves a few more edge cases than either Bonichon *et. al* [1] or Kanj *et. al* [3], it does demonstrate the flexibility of the techniques involved. Perhaps they could be applied to other variations of graphs, or variations of the Delaunay triangulation or the half- θ -6 graph.

While incredibly useful, the analysis of Kanj *et. al* [3], and by extension our analysis, is quite technical. The paper by Bose *et. al* [2] provides the model for the algorithm that constructs $D8(P)$, and they use an algorithm with a subroutine similar to our *AddIncident()*. However, rather than sorting the edges by bisector distance, they use Euclidean distance.

This allows them a huge simplification in the analysis of the subsequent graph over what we achieve. Perhaps some similar simplification of the analysis of our graph is possible, either by modifying the construction, or by providing a more clever analysis technique.

In the paper by Bonichon et al, [1], they first develop a degree 9 graph, and then use that graph to transition into a degree 6 graph. Originally we had meant to follow in their footsteps and try something similar. That is, we wished to use $D8(P)$ as a transition into a degree 6 graph with a similar, if not identical, spanning ratio. As such a degree 6 graph is a possible next step.

Always of interest in the graph setting, a generalized or constrained version of this graph could prove interesting. Bose *et al.* [14] have applied constraints to the graph developed by Bonichon et al, [1]. Having borrowed many of their techniques and applied them successfully on a Delaunay triangulation, it seems apropos to attempt to apply their constrained techniques to our setting as well.

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