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ITERATED TILTED ALGEBRAS

by

Ibrahim Ali Assem, B.Sc. (Eng.)

B.Sc. (Math.), M.Sc. (Math.)

A thesis submitted to the Faculty of
Graduate Studies and Research in partial
fulfillment of the requirements for the degree of

Doctor of Philosophy

Department of Mathematics and Statistics

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Ottawa, Canada

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The undersigned recommend to the Faculty of Graduate
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"Iterated tilted algebras"

submitted by Ibrahim Ali Assem, B.Sc. (Eng.), B.Sc. (Math.),
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ABSTRACT

Let k be a commutative field, and A a finite-dimensional k -algebra (associative, with 1). All our modules will be finite-dimensional. A module T_A is called a tilting module if:

(T1) $\text{pd } T_A \leq 1,$

(T2) $\text{Ext}_A^1(T, T) = 0,$

(T3) There is a short exact sequence $0 \rightarrow A_A \rightarrow T'_A \rightarrow T''_A \rightarrow 0,$

with T' and T'' direct sums of summands of T_A .

A tilting module T_A is called splitting if every indecomposable B -module N_B , where $B = \text{End } T_A$, is such that either $N \otimes_B T = 0$, or $\text{Tor}_1^B(N, T) = 0$. An iterated tilted algebra is a finite-dimensional k -algebra which can be reached from a hereditary algebra by a finite number of applications of functors associated to splitting tilting modules. Iterated tilted algebras of types A_n , B_n and C_n (that is, which can be reached from a tensor algebra $T(\Sigma)$, where the graph of the species Σ is A_n , B_n and C_n respectively) are studied here.

In order to be able to formulate the results in a proper setting, a notion of relation on a species is introduced. General

properties of iterated tilted algebras are studied, and iterated tilted algebras of types A_n , B_n and C_n are classified in terms of their bounden species and also of their Auslander-Reiten graphs.

Iterated tilted algebras of type A_n were already classified by Happel and the author in [1]. Throughout the text, the emphasis is put mainly on the use of the Auslander-Reiten graph to obtain the results.

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INTRODUCTION

Let k be a commutative field, and A a finite dimensional k -algebra (which we assume associative, and with an identity). All our modules will be finite-dimensional, and we shall denote by $\text{mod } A$ the category of finite-dimensional right A -modules.

The object of this thesis is to present an application of the theory of tilting modules. This notion has found, since its introduction by D. Happel and C.M. Ringel in [16], wide applications in the representation theory of algebras. Formally, a module T_A is called a tilting module if the following properties are satisfied:

$$(T1) \quad \text{pd } T_A \leq 1,$$

$$(T2) \quad \text{Ext}_A^1(T, T) = 0,$$

$$(T3) \quad \text{there is a short exact sequence } 0 \rightarrow A_A \rightarrow T'_A \rightarrow T''_A \rightarrow 0$$

with T' and T'' direct sums of summands of T_A .

Given a tilting module T_A , one may consider two full subcategories of $\text{mod } A$, namely the category $J(T_A)$ of all modules M_A generated by T_A , and the category $\mathcal{F}(T_A)$ of all modules M_A satisfying $\text{Hom}_A(T, M) = 0$. Then $(J(T_A), \mathcal{F}(T_A))$ is a torsion theory in $\text{mod } A$.

Let $B = \text{End } T_A$. There are two corresponding full subcategories of $\text{mod } B$, defined by:

$$\mathcal{X}(T_A) = \{N_B \mid N \otimes_B T = 0\},$$

$$\text{and } \mathcal{Y}(T_A) = \{N_B \mid \text{Tor}_1^{B'}(N, T) = 0\}.$$

Again, $(\mathcal{X}(T_A), \mathcal{Y}(T_A))$ is a torsion theory in $\text{mod } B$.

If T_A is a tilting module, and $B = \text{End } T_A$, then T_B is also a tilting module, and $A \simeq \text{End}_B T$, canonically. Also, we have equivalences of subcategories $\mathcal{X}(T_A) \simeq \mathcal{F}(T_A)$ and $\mathcal{Y}(T_A) \simeq \mathcal{T}(T_A)$. This result was essentially proved by Brenner and Butler in [8], with a rather different formulation and under a more restrictive definition of a tilting module. In the general case, it is due to Happel and Ringel [16].

In case the algebra A is hereditary, $B = \text{End } T_A$ is called a tilted algebra. Then the torsion theory $(\mathcal{X}(T_A), \mathcal{Y}(T_A))$ is splitting, that is to say, every indecomposable B -module is either in $\mathcal{X}(T_A)$ or in $\mathcal{Y}(T_A)$ (cf. [16]).

This, however, is no longer true if A is not hereditary. We are thus led to define a splitting tilting module to be a tilting module T_A such that the torsion theory $(\mathcal{X}(T_A), \mathcal{Y}(T_A))$ splits in $\text{mod } B$, where $B = \text{End } T_A$. It is natural to ask what type of algebras can be obtained as endomorphism rings of splitting tilting modules. We thus introduce the notion of an iterated tilted algebra (called "generalized tilted" in [1]). This is a finite-dimensional

k -algebra which can be reached from a hereditary algebra by a finite number of applications of the functors attached to a splitting tilting module.

Our aim will be to give a complete classification of those iterated tilted algebras which are of types A_n , B_n and C_n (that is, which can be reached from all tensor algebras $T(\Sigma)$, where the graph G_Σ of the species Σ is A_n , B_n and C_n). The case A_n was already considered by D. Happel and the author in [1].

In order to formulate the results in a proper setting, it was found necessary to define an appropriate notion of relation on species. Let k be a commutative field, and $\Sigma = (F_i, {}^*M_j)_{i,j \in I}$ be a k -species, of tensor algebra $T(\Sigma)$. An ideal R of $T(\Sigma)$ contained in $(\text{rad } T(\Sigma))^2$ will be called a relation ideal. We shall see that on a perfect field k , any finite-dimensional basic k -algebra can be written as $T(\Sigma)/R$, where Σ is a k -species, and R a relation ideal of $T(\Sigma)$. Such algebras will be called graph algebras. Writing ${}^cM_j = F_i T(\Sigma) F_j$, and ${}_iR_j = F_i R F_j$, it is clear that $R = \bigoplus_{i,j} {}_iR_j$ can be alternatively defined by the bimodule epimorphisms ${}_i\rho_j: {}^cM_j \rightarrow {}^cM_j / {}_iR_j$.

We thus prove the following:

Theorem (A): Let k be a perfect field. A connected basic finite-dimensional k -algebra is iterated tilted of type A_n if

and only if it is given by a k -species $\Sigma = (F, M)$ with
 relation ideal $R = \bigoplus_{i,j} R_{ij}$ such that:

- (α_1) the graph G of Σ is free,
- (α_2) for each vertex i of G , $R_{ii} = E$, where E is a (skew) field, finite dimensional over k , and, for each arrow $i \rightarrow j$
 $M_{ij} = E$ with the canonical operations,
- (α_3) all relations are zero-relations of length two,
- (α_4) each vertex has at most four neighbours,
- (α_5) if a vertex l has four neighbours, then G contains a full connected subgraph of the form:

with the zero-relations $i_2 M_{l l} \otimes M_{l l} i_1$ and $i_4 M_{l l} \otimes M_{l l} i_3$,

- (α_6) if a vertex l has three neighbours, then G contains a full connected subgraph of one of the forms:

with the zero relation $i_3 M_{l l} \otimes M_{l l} i_1$.

For B_n , we have a similar result:

Theorem (B): Let k be a perfect field. A connected basic finite-dimensional k -algebra is iterated tilted of type B_n if and only if it is given by a k -species $\Sigma = (F_i, M_{ij})_{i,j \in I}$ with relation ideal $R = \bigoplus_{i,j} R_{ij}$ such that:

(β_1) the graph G of Σ is a tree,

(β_2) there is a vertex i_0 such that $F_{i_0} = F$, and for all $i \neq i_0$, $F_i = E$, where E and F are two (skew) fields, finite dimensional over k and such that $\dim_F E = \dim E_F = 2$. Also, if $M_{ij} \neq 0$, then $M_{ij} = E$ if $i = i_0$, $M_{ij} = E_F$ if $j = i_0$, and $M_{ij} = E E$ otherwise, with the canonical operations,

(β_3) i_0 has at most two neighbours, and, if it is so, then $i \rightarrow i_0 \rightarrow j$, and there is a relation on the subspecies

$$\begin{array}{ccccc} & \begin{array}{c} E \\ \leftarrow F \end{array} & & \begin{array}{c} F \\ \leftarrow E \end{array} & \\ & E & \xrightarrow{\quad} & F & \xrightarrow{\quad} & E \\ & & \searrow \mu & & \nearrow & \end{array}$$

where μ is an epimorphism $\mu: E_F \otimes_F E \rightarrow E E$ such that the left and right actions on the image are induced by the ordinary multiplication of E ,

(β_4) all relations are of length two, and the only relations besides (μ) are the zero-relations,

(β_5) each vertex of G has at most four neighbours,

(β_6) if a vertex l has four neighbours, then G contains a full connected subgraph of the form:

with the zero-relations $i_2^M l \otimes l i_1^M$ and $i_4^M l \otimes l i_3^M$,

(β_7) if a vertex l has three neighbours, then G contains a full connected subgraph of one of the forms:

with the zero-relation $i_3^M l \otimes l i_1^M$.

The corresponding statement for C_n is similar, and follows from the preceding one by simply interchanging the fields E and F :

Theorem (C): Let k be a perfect field. A connected basic finite-dimensional k -algebra is iterated tilted of type C_n if and only if it is given by a k -species $\Sigma = (F_i, i_j^M, i, j \in I)$ with relation ideal $R = \bigoplus_{i,j \in I} R_{ij}$ such that:

(γ_1) the graph G of Σ is a tree,

(γ_2) there is a vertex i_0 such that $F_{i_0} = E$, and for all $i \neq i_0$, $F_i = F$, where E and F are two (skew) fields, finite-dimensional over k , and such that $\dim_F E = \dim E_F = 2$. Also, if $M_{ij} \neq 0$, then $M_{ij} = {}^E E_F$ if $j = i_0$, $M_{ij} = {}^E F$ if $i = i_0$, and $M_{ij} = {}^F F$ otherwise, with the canonical operations,

(γ_3) i_0 has at most two neighbours and, if it is so, then $i \rightarrow i_0 \rightarrow j$ and there is a relation on the subspecies:

where π is an epimorphism $\pi: {}^E E_F \otimes {}^E E_F \rightarrow {}^F F$ such that the left and right actions on the image are induced by the ordinary multiplication of F ,

(γ_4) all relations are of length zero, and the only relations besides (π) are the zero-relations,

(γ_5) each vertex of G has at most four neighbours,

(γ_6) if a vertex l has four neighbours, then G contains a full connected subgraph of the form:

with the zero-relations $M_{i_2 l} \otimes M_{l i_1}$ and $M_{i_4 l} \otimes M_{l i_3}$

(γ_7) if a vertex l has three neighbours, then G contains a full connected subgraph of one of the forms:

with the zero-relation $M_{i_3 l} \otimes M_{l i_1}$.

In all three cases, the main tool in proving the necessity of these conditions is the observation that most local properties of the Auslander-Reiten graph of an algebra remain unchanged when applying the tilting process. In the converse part, we actually construct a series starting from an algebra satisfying the stated conditions and ultimately reaching a hereditary algebra by generalizing the so-called APR-tilts (cf. [5]). In the course of the proof, another characterization of these algebras is given, this time in terms of their Auslander-Reiten graphs.

This dissertation is divided into five chapters. In Chapter (I), we introduce some notation and quote the theorems that will be used in the sequel. Chapter (II) consists in the description of our notion of relation on a species. In Chapter (III), we define the iterated tilted algebras, and study their general properties.

Chapters (IV) and (V) contain the above-mentioned classification theorems.

We wish to put our emphasis not only on the particular results obtained, but also on the use of the Auslander-Reiten graph to obtain these results.

CHAPTER I:
PRELIMINARIES

In this introductory chapter, we shall fix the notations and collect together the definitions and known results (without proofs) that will be needed in the sequel.

1. NOTATIONS:

Let k be a commutative field, and A a finite-dimensional k -algebra (associative, with an identity). All A -modules will be finite-dimensional (right) modules, and homomorphisms will be written on the opposite side of the scalars. The category of finite-dimensional A -modules will be denoted by $\text{mod } A$.

Given a module M_A , we shall let $\text{End } M$ be its endomorphism ring, $\text{rad } M$ its radical, $\text{soc } M$ its socle, $|M|$ its length, and $\text{pd } M$ its projective dimension. We shall also let $\text{add } M$ denote the (additive) full subcategory of $\text{mod } A$ of those modules which are direct sums of summands of M .

Most of the time, we shall choose a fixed ordering $S(1), \dots, S(n)$ of the simple A -modules, and denote by $P(i)$ and $I(i)$ respectively, the projective cover and the injective hull of $S(i)$ ($1 \leq i \leq n$).

The dimension-vector $\dim M$ of an A -module M is defined as follows: $(\dim M)_i$ is the number of composition factors of M isomorphic to $S(i)$ in any fixed composition series of M . The n -tuple $\dim M$ is usually considered as an element of $\mathbb{Q}^n$ or $\mathbb{R}^n$.
 Note that:

$$(\dim M)_i = |\operatorname{Hom}_A(P(i), M)| = |\operatorname{End} P(i) \operatorname{Hom}_A(M, I(i))|$$

We shall let D denote the standard duality with respect to the base field k , that is, for a module M ,

$$D(M) = \operatorname{Hom}_k(M, k).$$

2. SPECIES AND GRAPHS:

(2.1) Following [15], we shall define a k -species $\Sigma = (F_i, {}^M_{ij})_{i,j \in I}$ to consist of a finite set $(F_i)_{i \in I}$ of (skew) fields F_i , with k as a common central subfield such that $[F_i : k] < \infty$, and a finite set of $F_i - F_j$ bimodules ${}^M_{ij}$, finite-dimensional over k , on which k acts centrally.

To the species Σ , we can associate a k -algebra, called its tensor algebra as follows (cf. [10] or [11]): first, define the semisimple k -algebra $\Lambda = \prod_{i \in I} F_i$ and consider $M = \bigoplus_{i,j} {}^M_{ij}$ as a $\Lambda - \Lambda$ bimodule where the Λ -action is given by the projection

maps $\Lambda \rightarrow F_1$. Now, let $M^{(0)} = \Lambda$, $M^{(1)} = M$ and inductively $M^{(t)} = M^{(t-1)} \otimes M$. Define:

$$T(\Sigma) = \bigoplus_{t \geq 0} M^{(t)}$$

as a k -vector space. $T(\Sigma)$ becomes a k -algebra if we define the multiplication through the canonical isomorphisms $M^{(r)} \otimes M^{(s)} \rightarrow M^{(r+s)}$ and extend by distributivity. It is a well-known fact that $T(\Sigma)$ is hereditary.

Conversely, let A be a finite-dimensional k -algebra, which is basic in the sense that its semisimple part $\Lambda = A/\text{rad } A$ is the product of (skew) fields F_i ($i \in I$), then the Λ - Λ bimodule $M_{\Lambda \Lambda} = \text{rad } A / (\text{rad } A)^2$ can be uniquely decomposed as a direct sum of F_i - F_j bimodules on which k acts centrally, $M = \bigoplus_{i,j} M_{ij}$. Thus we have defined a k -species $\Sigma_A = (F_i, M_{ij})_{i,j \in I}$. Clearly, $\Sigma_{T(\Sigma)} = \Sigma$.

We can also associate to the species Σ an (oriented valued) graph G_Σ as follows: the graph consists of a finite set of points, or vertices, one corresponding to each element of I , and a set of ordered pairs (i,j) , called arrows or oriented edges, associated to each non-zero bimodule M_{ij} . We attach to each arrow the pair of natural numbers (d_{ij}, d_{ji}) , where $d_{ij} = \dim(M_{ij})_{F_j}$ and

$d_{ji} = \dim_{F_i} (M_{ij})$; such a pair is called a valuation, and the valued arrow is denoted by:

$$i \xrightarrow{(d_{ij}, d_{ji})} j.$$

Notice that the k -algebra $T(\Sigma)$ is finite-dimensional if and only if G_Σ contains no oriented cycles.

(2.2) Given a k -species $\Sigma = (F_i, {}^iM_j)_{i,j \in I}$, we define a (finite-dimensional) representation of Σ to be an object $\underline{X} = (X_i, {}_j\varphi_i)$ where to each $i \in I$ we associate a (finite-dimensional) F_i -vector space X_i , and to each arrow $i \rightarrow j$ of G_Σ , we associate an F_j -linear map ${}_j\varphi_i: X_i \otimes {}^iM_j \rightarrow X_j$.

A morphism $\underline{f}$ from the representation $\underline{X} = (X_i, {}_j\varphi_i)$ to the representation $\underline{Y} = (Y_i, {}_j\psi_i)$ consists in a family of F_i -linear maps $f_i: X_i \rightarrow Y_i$ such that, for each arrow $i \rightarrow j$, the square:

$$\begin{array}{ccc} X_i \otimes {}^iM_j & \xrightarrow{{}_j\varphi_i} & X_j \\ f_i \otimes 1 \downarrow & & \downarrow f_j \\ Y_i \otimes {}^iM_j & \xrightarrow{{}_j\psi_i} & Y_j \end{array}$$

is commutative.

It is well-known that the category $\mathcal{f}(\Sigma)$ of all finite-dimensional representations of Σ is equivalent to the category $\text{mod } T(\Sigma)$.

(2.3) A valued graph $(G, \underline{d})$ (cf. [10] or [12]) is a finite set G of points, or vertices, together with a mapping $\underline{d}: G \times G \rightarrow \mathbb{N}$ called a valuation such that:

$$(1) \ d_{ii} = 0 \quad \text{for all } i \in G,$$

(2) there exist (non-zero) natural numbers f_i such that:

$$d_{ij} f_j = d_{ji} f_i$$

for all $i, j \in G$.

A pair (i, j) such that $d_{ij} \neq 0$ (or, equivalently, $d_{ji} \neq 0$) is called an edge, and the points i, j are called neighbours. We shall denote an edge by:

$$i \xrightarrow{(d_{ij}, d_{ji})} j$$

If $d_{ij} = d_{ji} = 1$, we write simply $i \xrightarrow{\quad} j$.

An orientation of $(G, \underline{d})$ is given by prescribing, for each edge (i, j) of $(G, \underline{d})$, an order (indicated by an arrow $i \xrightarrow{\quad} j$).

An orientation is admissible provided there is no oriented cycle.

A k-modulation $\underline{m}$ of an oriented valued graph $(G, \underline{d})$ is a set of (skew) fields $(F_i)_{i \in G}$, finite-dimensional over the common central subfield k , together with a set $(M_{ij})_{i, j \in G}$ of $F_i - F_j$ bimodules on which k acts centrally such that:

$$\dim ({}_i M_j)_{F_j} = d_{ij},$$

$$\dim_{F_i} ({}_i M_j) = d_{ji},$$

and ${}_j M_i$ is the dual of ${}_i M_j$ in the sense that we have bimodule isomorphisms:

$${}_j M_i \cong \text{Hom}_{F_i} ({}_i M_j, F_i) \cong \text{Hom}_{F_j} ({}_i M_j, F_j).$$

A representation of the oriented valued graph $(G, \underline{d})$ with modulation $\underline{m}$ is defined to be a representation of the species $((F)_{i \in G}, ({}_i M_j)_{i \rightarrow j})$. The category of all such representations will be denoted by $\mathcal{F}(G, \underline{m})$.

If $\underline{X} = (X_i, {}_j \varphi_i)$ is a representation in $\mathcal{F}(G, \underline{m})$, the support $\text{Supp } \underline{X}$ of $\underline{X}$ is the full subgraph of G consisting of the set of vertices $\{i \in G \mid X_i \neq 0\}$, and all arrows between these vertices.

(2.4) A finite-dimensional k -algebra A is said to be of finite representation type provided $\text{mod } A$ has only finitely many non-isomorphic indecomposable objects.

Also, A is called of tame representation type if it is not of finite representation type and, for any dimension d , there is a finite number of embedding functors $F_i: \text{mod } k[T] \rightarrow \text{mod } A$ such that all but a finite number of indecomposable A -modules of dimension

d are of the form $F_i(M)$, for some i , and some indecomposable $k[T]$ -module M .

Let $(G, \underline{d})$ be an oriented valued graph with modulation m , then the category $\mathcal{F}(G, m)$ is said to be of finite or tame type according as the tensor algebra of the associated species is of finite or tame type. We have the:

Theorem ([10], [12]): Let $(G, \underline{d})$ be a valued graph with modulation m , then:

a) $\mathcal{F}(G, m)$ is of finite representation type if and only if

$(G, \underline{d})$ is a Dynkin graph:

b) $f(G, m)$ is of tame representation type if and only if

(G, d) is an Euclidean graph:

3. AUSLANDER-REITEN SEQUENCES AND GRAPHS:

(3.1) Let A be a finite-dimensional k -algebra. We shall call section (respectively, retraction) a right invertible (respectively, left invertible) morphism of $\text{mod } A$. We have the following. ([2]):

Theorem (Auslander-Reiten): Let N_A be an indecomposable non-projective A -module. There exists a non-split short exact sequence:

$$0 \rightarrow L_A \xrightarrow{f} M_A \xrightarrow{g} N_A \rightarrow 0$$

with the following properties:

- a) L_A is indecomposable,
- b) If $u: N' \rightarrow N$ is not a retraction, then there exists $\bar{u}: N' \rightarrow M$ such that $g\bar{u} = u$,

c) if $v: L \rightarrow L'$ is not a section, then there exists $\bar{v}: M \rightarrow L'$ such that $\bar{v}f = v$.

Also, such a sequence is unique up to isomorphism. ||

The sequence of the theorem is called an Auslander-Reiten sequence (or almost split sequence) ending with N . By duality, given a non-injective indecomposable A -module L_A , there also exists an Auslander-Reiten sequence starting with L :

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0.$$

Given a module X_A , let us define its transpose $\text{Tr}X$: take a minimal projective presentation of X :

$$P_1 \xrightarrow{p} P_0 \rightarrow X \rightarrow 0,$$

apply the functor $\text{Hom}_A(-, A)$, and define $\text{Tr}X$ to be the cokernel of $\text{Hom}_A(p, A)$:

$$\text{Hom}_A(P_0, A) \xrightarrow{\text{Hom}_A(p, A)} \text{Hom}_A(P_1, A) \rightarrow \text{Tr}X \rightarrow 0.$$

Then, given an Auslander-Reiten sequence:

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0,$$

we have $D\text{Tr} N \cong L$ (and $\text{Tr}D L \cong N$). L is called the translate of N . We shall denote this translation by $\tau = D\text{Tr}$, and also write $\tau^{-1} = \text{Tr}D$.

If N is indecomposable, a map $g: M \rightarrow N$ is called right almost split if:

- 1) g is not a retraction,
- 2) if $u: N' \rightarrow N$ is not a retraction, then there exists $\bar{u}: N' \rightarrow M$ such that $g\bar{u} = u$.

If in addition, M is of smallest possible length, then g is called minimal right almost split. For every N , there always exists a minimal right almost split map, and it is unique up to isomorphism:

- 1) If N is not projective, the map $g: M \rightarrow N$ is minimal right almost split if and only if:

$$0 \rightarrow \text{Ker } g \rightarrow M \xrightarrow{g} N \rightarrow 0$$

is an Auslander-Reitén sequence,

- 2) If N is projective, the map $g: M \rightarrow N$ is minimal right almost split if and only if it is isomorphic to the inclusion map of $\text{rad } N$ into N .

Minimal left almost split maps are defined and constructed in the obvious dual manner.

(3.2) Following [3], a map $f: M \rightarrow N$ will be called irreducible if:

- 1) f is neither a section, nor a retraction,
- 2) for any factorization $f = gh$, either h is a section or g is a retraction.

An irreducible map is always either a monomorphism or an epimorphism. If $M \neq 0$, $f: M \rightarrow N$ is irreducible if and only if there exists $f': M' \rightarrow N$ such that $(f, f'): M \oplus M' \rightarrow N$ is minimal right almost split. Dually, if $N \neq 0$, $f: M \rightarrow N$ is irreducible if and only if there exists $f': M \rightarrow N'$ such that $\begin{pmatrix} f \\ f' \end{pmatrix}: M \rightarrow N \oplus N'$ is minimal left almost split.

We shall need the following property of irreducible maps: if $g: M \rightarrow N$ is an irreducible epimorphism, then its kernel is indecomposable. Dually, if $f: L \rightarrow M$ is an irreducible monomorphism, its cokernel is indecomposable.

(3.3) Let M, N be indecomposable A -modules, and $\text{rad}(M, N)$ be the set of the non-invertible homomorphisms from M to N . If M, N are not indecomposable, say have decompositions $M = \bigoplus_i M_i$ and $N = \bigoplus_j N_j$ with indecomposable summands M_i and N_j , we define

$$\text{rad}(M, N) = \bigoplus_{i,j} \text{rad}(M_i, N_j)$$

(where we use the identification $\text{Hom}_A(M, N) = \bigoplus_{i,j} \text{Hom}_A(M_i, N_j)$). Let $(\text{rad}(M, N))^2$ be the set of all homomorphisms $f: M \rightarrow N$ which can

be factored as $f = gh$, where $h \in \text{rad}(M, L)$ and $g \in \text{rad}(L, N)$ for some A -module L , and consider $\text{Irr}(M, N) = \text{rad}(M, N) / (\text{rad}(M, N))^2$. For indecomposable modules M, N this is an $\text{End } N - \text{End } M$ bimodule which actually is annihilated by the radicals of $\text{End } N$ and $\text{End } M$. It is called the bimodule of irreducible maps (cf. [19]).

Let us put:

$$a_{MN} = |\text{Irr}(M, N)_{\text{End } M}|$$

and

$$a'_{MN} = |_{\text{End } N} \text{Irr}(M, N)|.$$

Observe that in case M is not injective, a'_{MN} equals the multiplicity of N in the middle term of an Auslander-Reiten sequence starting with M , while, if N is not projective, a_{MN} equals the multiplicity of M in the middle term of an Auslander-Reiten sequence ending with N .

(3.4) The Auslander-Reiten graph Γ_A of the algebra A is defined as follows: the vertices are the isomorphism classes $[M]$ of indecomposable A -modules, and there is an arrow $[M] \rightarrow [N]$ provided there exists an irreducible map $M \rightarrow N$. Finally, this arrow is endowed with the valuation (a_{MN}, a'_{MN}) (where a_{MN} and a'_{MN} are as defined in (3.3)).

Since an endomorphism is never irreducible, an Auslander-Reiten graph has no loops. The Auslander-Reiten graph Γ_A is endowed with

the (partially defined) translation τ and, as seen in (3.2), if, for some vertex z , τz is defined, then the set of targets of the arrows $\tau z \rightarrow y$ coincides with the set of sources of the arrows $y \rightarrow z$, and this set is finite. This means in particular that each arrow $\alpha: y \rightarrow z$ such that τz is defined is paired with an arrow $\alpha\alpha: \tau z \rightarrow y$.

Of course, Γ_A is finite precisely when A is of finite representation type, and then its connected components correspond just to the primitive central idempotents in A . Accordingly, A is called connected if its only central idempotents are 0 and 1.

One may also regard the Auslander-Reiten graph Γ_A as part of a two-dimensional simplicial complex (cf. [18]) with edges both the underlying edges of the arrows, as well as additional edges $[L] \rightarrow [N]$ for each Auslander-Reiten sequence $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ and with triangles of the form;

in case M_1 is an indecomposable direct summand of M in the just mentioned Auslander-Reiten sequence. Thus we have a topological structure on Γ_A .

Finally, we define a path $x_0 \rightarrow x_1 \rightarrow \dots \rightarrow x_m$ in Γ_A to be a sectional path provided $x_{i-1} \neq \tau x_{i+1}$ for all $1 \leq i \leq m-1$.

4. TILTING MODULES AND TILTED ALGEBRAS:

(4.1) Before defining tilting modules, let us recall the definition of a torsion theory in $\text{mod } A$ (cf. [21]). A torsion theory in $\text{mod } A$ is a pair $(\mathcal{T}, \mathcal{F})$ of full subcategories such that:

- (i) $\text{Hom}_A(T, F) = 0$ for all $T \in \mathcal{T}$ and $F \in \mathcal{F}$,
- (ii) $\text{Hom}_A(X, F) = 0$ for all $F \in \mathcal{F}$, implies $X \in \mathcal{T}$,
- (iii) $\text{Hom}_A(T, X) = 0$ for all $T \in \mathcal{T}$, implies $X \in \mathcal{F}$.

$\mathcal{T}$ is called a torsion class, and its objects are torsion modules, while $\mathcal{F}$ is a torsion-free class consisting of torsion-free modules. Clearly, $\mathcal{T}$ is closed under quotients and extensions, and $\mathcal{F}$ under submodules and extensions. Given a torsion theory $(\mathcal{T}, \mathcal{F})$, any module M has a unique maximal torsion submodule $t(M)$, and $M/t(M)$ is torsion-free. The torsion theory $(\mathcal{T}, \mathcal{F})$ is splitting when, for any module M , $t(M)$ is a direct summand of M .

(4.2) Let A be a finite-dimensional k -algebra, a module T_A is called a tilting module (cf. [16]) provided the following three properties are satisfied:

- (T1) There exists a short exact sequence $0 \rightarrow P_A \rightarrow Q_A \rightarrow T_A \rightarrow 0$ with P, Q projective (thus, $\text{pd } T_A \leq 1$).

(T2) $\text{Ext}_A^1(T, T) = 0$.

(T3) There exists a short exact sequence $0 \rightarrow A_A \rightarrow T'_A \rightarrow T''_A \rightarrow 0$ with $T', T'' \in \text{add } T_A$.

The third defining condition is difficult to verify in practice.

However (cf. [7]), a module T_A is a tilting module if and only if it satisfies (T1), (T2), and:

(T3') The number of isomorphism classes of indecomposable summands of T equals the number of isomorphism classes of simple A -modules.

Moreover, we have:

Proposition (cf. [7]): Let the A -module T be such that $\text{pd } T_A \leq 1$ and $\text{Ext}_A^1(T, T) = 0$. Then there is some $X \in \text{mod } A$ such that $T \oplus X$ is a tilting module. $\parallel$

Let now T_A be a tilting module and $B = \text{End } T_A$. We consider the following two full subcategories of $\text{mod } A$: $\mathcal{J}(T_A)$ which is the full subcategory of all modules generated by T_A (or, equivalently, of all modules M_A such that $\text{Ext}_A^1(T, M) = 0$), and $\mathcal{F}(T_A)$, which is the full subcategory of all modules M_A such that $\text{Hom}_A(T, M) = 0$. Always the pair $(\mathcal{J}(T_A), \mathcal{F}(T_A))$ forms a torsion theory for $\text{mod } A$.

There are two corresponding full subcategories of $\text{mod } B$ defined by:

$$\mathcal{X}(T_A) = \{N_B \mid N_B \otimes_B T_A = 0\},$$

and

$$\mathcal{U}(T_A) = \{N_B \mid \text{Tor}_1^B(N_B, T_A) = 0\}.$$

Again $(\mathcal{X}(T_A), \mathcal{U}(T_A))$ forms a torsion theory for $\text{mod } B$. In fact, we have the following:

Theorem of Brenner-Butler (cf. [16], [7]): Let T_A be a tilting module, and $B = \text{End } T_A$. Then also ${}_B T$ is a tilting module and $A = \text{End } {}_B T$, canonically. Moreover the subcategories $\mathcal{F}(T_A)$ and $\mathcal{U}(T_A)$ are equivalent under the restrictions of the functors $\text{Hom}_A(T, -)$ and $- \otimes_B T$, which are mutually inverse to each other, and similarly, the subcategories $\mathcal{F}(T_A)$ and $\mathcal{X}(T_A)$ are equivalent under the restrictions of the functors $\text{Ext}_A^1(T, -)$ and $\text{Tor}_1^B(-, T)$ which are again mutually inverse to each other. $\parallel$

We also have the:

Connecting lemma (cf. [16], [7]): Let T_A be a tilting module, with $B = \text{End } T_A$. Let I_A be an indecomposable injective module, and P_A be a projective with $\text{soc } I_A \cong P_A / \text{rad } P_A$. Then

$$\tau^{-1} \text{Hom}_A(T, I) = \text{Ext}_A^1(T, P). \quad \parallel$$

The following lemma is useful in practical computations:

Lemma (cf. [16]): If $\text{pd } X_A \leq 1$, then:

$$\text{DExt}_A^1(X, M) \cong \text{Hom}_A(M, \tau X). \quad ||$$

In the general case, however, when the projective dimension of X may be arbitrary, we have the Auslander-Reiten formula $\text{DExt}_A^1(X, M) = D \overline{\text{Hom}}_A(M, \tau X) = D \underline{\text{Hom}}_A(\tau^{-1} M, X)$, where $\underline{\text{Hom}}$ (respectively, $\overline{\text{Hom}}$) denotes the factor group modulo the subgroup of those homomorphisms which factor through a projective (respectively, injective) module (cf. [2]).

We also have (cf. [7]):

Theorem: Let A be a finite-dimensional k -algebra, T_A a tilting module, and $B = \text{End } T_A$, then:

$$\text{gl.dim. } B \leq \text{gl.dim. } A + 1. \quad ||$$

Most of the time, we shall consider tilting modules which are multiplicity-free (that is, such that their indecomposable summands are non-isomorphic).

(4.3) If $B = \text{End } T_A$ for a tilting module T_A over a hereditary algebra A , then B is called a tilted algebra.

Theorem (1): Let A be a hereditary algebra, T_A be a tilting module, and $B = \text{End } T_A$. Then $(\mathcal{X}(T_A), \mathcal{Y}(T_A))$ is a splitting torsion theory in $\text{mod } B$. $||$

Another characterization of tilted algebras is by means of their complete slices. Namely, let B be a finite-dimensional algebra. A complete slice S is a set of indecomposable modules

with the following properties:

(S1) There is no chain of irreducible maps

$$S_0 \rightarrow S_1 \rightarrow \dots \rightarrow S_m \rightarrow S_0$$

with all the $S_i \in S$.

(S2) If $X_0 \rightarrow X_1 \rightarrow \dots \rightarrow X_m$ is a chain of irreducible maps between indecomposable modules, and $X_0, X_m \in S$, then $X_i \in S$ for all $0 \leq i \leq m$.

(S3) Given any indecomposable module X , S contains precisely one module from the orbit $\{\tau^z X \mid z \in \mathbb{Z}\}$.

Then:

Theorem (2): Let B be a finite-dimensional algebra. Assume some component of the Auslander-Reiten graph of B contains all indecomposable projective modules, and also some complete slice S . Then $T = \bigoplus_{U \in S} U$ is a tilting module, and $\text{End } T_B$ is hereditary (thus B is a tilted algebra). $\parallel$

In fact, if A is a connected hereditary algebra, a complete slice in Γ_B is given by $\{\text{Hom}_A(T, I(i)) \mid 1 \leq i \leq n\}$, where $\{I(i) \mid 1 \leq i \leq n\}$ is a set of representatives of the isomorphism classes of the indecomposable injective A -modules.

A representation with full support is called sincere. We have the:

Theorem (3): Let B be an algebra of finite representation type, such that Γ_B has no oriented cycles. If B has an indecomposable sincere representation N_B , then B is a tilted algebra, and N_B is actually a faithful module. $\parallel$

Let us also mention the following application of the theory of tilted algebras:

Theorem (4): Let B be a finite-dimensional algebra of finite representation type, such that Γ_B has no oriented cycles. Let N_B and N'_B be indecomposable B -modules. Then $\dim N_B = \dim N'_B$ implies $N_B \cong N'_B$. $\parallel$

(For the last four results, see [16]).

(4.4) Consider the algebra A of all $n \times n$ lower triangular matrices:

$$A = \begin{bmatrix} k & 0 & 0 & \dots & 0 & 0 \\ k & k & 0 & \dots & 0 & 0 \\ k & k & k & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ k & k & k & \dots & k & 0 \\ k & k & k & \dots & k & k \end{bmatrix}$$

A is the tensor algebra of the graph A_n with the linear orientation:

and the obvious modulation.

Then (cf. [17]), the tilted algebras of A are precisely the full connected subgraphs of the following "genealogical tree", with

all possible zero-relations of the form $N \otimes M$ (cf. Chapter (II)):

This result will be generalized in Chapter (IV).

CHAPTER II:
BOUNDED SPECIES

In this chapter, we shall define a natural notion of relation on a species, thus generalizing the notion of a bounden quiver. This is necessary in order to formulate the main results in an appropriate setting. As an illustration, we shall list all the tilted algebras of F_4 , and describe the relations in each case.

1. BOUNDED SPECIES:

(1.1). Let k be a commutative field, and $\Sigma = (F_i, {}_iM_j)_{i,j \in I}$ a k -species. We recall that there is an arrow $i \rightarrow j$ in the graph G_Σ of Σ whenever there exists a bimodule ${}_iM_j$, and a path of length t : $i = i_0, i_1, \dots, i_t = j$ whenever we have arrows $i_{s-1} \rightarrow i_s$ for $1 \leq s \leq t$. Let thus $i, j \in I$ be arbitrary, and

$$\rho = (i \rightarrow i_1 \rightarrow i_2 \rightarrow \dots \rightarrow i_t = j)$$

be a path. We shall call bimodule of the path ρ the $F_i - F_j$ bimodule:

$${}_iM_j^\rho = {}_iM_{i_1} \otimes {}_{i_1}M_{i_2} \otimes \dots \otimes {}_{i_{t-1}}M_j.$$

Let $P(i, j)$ be the set of all paths from i to j , we define the

(composed) bimodule from i to j to be the $F_i - F_j$ bimodule:

$${}_i M_j^c = \bigoplus_{p \in \mathbb{P}(i,j)} {}_i M_j^p.$$

Also, we shall agree to put ${}_i M_j^c = 0$ whenever there is no path from i to j .

Let us consider the k -vector space $(\prod_{i \in I} F_i) \otimes (\bigoplus_{i,j \in I} {}_i M_j^c)$. The tensor product defines on this set an algebra structure, which is none other than that of $T(\Sigma)$, the tensor algebra of the species Σ (in fact, we have ${}_i M_j^c = F_i T(\Sigma) F_j$).

(1.2). Definition: Let $\Sigma = (F_i, {}_i M_j)_{i,j \in I}$ be a k -species.

An ideal R of $T(\Sigma)$ is called a relation ideal whenever it is contained in $(\text{rad } T(\Sigma))^2$. The factor algebra $A = T(\Sigma)/R$ is then said to be given by the k -species Σ with relations R , or the bounden k -species (Σ, R) .

Thus, given a pair $(i,j) \in I \times I$, the relation ideal R defines an $F_i - F_j$ subbimodule ${}_i R_j$ of ${}_i M_j^c$ by:

$${}_i R_j = F_i R F_j,$$

each ${}_i R_j$ is called a relation on Σ . Clearly, $R = \bigoplus_{i,j \in I} {}_i R_j$.

A relation ${}_i R_j$ is of length ℓ if ${}_i R_j \subseteq (\text{rad } T(\Sigma))^\ell$ but ${}_i R_j \not\subseteq (\text{rad } T(\Sigma))^{\ell+1}$. It is minimal if it is of length ℓ , and the

shortest path from i to j is of length ℓ .

For a given pair of points $(i, j) \in I \times I$, ${}_i R_j$ is a zero-relation if there exists a subset $P \subseteq \mathcal{P}(i, j)$ of the set of paths from i to j such that:

$${}_i R_j = \bigoplus_{P \in \mathcal{P}(i, j)} {}_i M_j^P.$$

When $P = \mathcal{P}(i, j)$, ${}_i R_j = {}_i M_j^C$ is called a strict zero-relation.

If $\mathcal{P}(i, j) = \{P_1, P_2\}$ and there exists a common $F_i - F_j$ subbimodule ${}_i N_j$ of ${}_i M_j^{P_1}$ and ${}_i M_j^{P_2}$, then ${}_i R_j$ is a commutativity relation if ${}_i R_j$ is the diagonal subbimodule of ${}_i N_j \oplus {}_i N_j$. It is a strict commutativity relation when ${}_i M_j^{P_1} \cong {}_i M_j^{P_2}$ and ${}_i R_j$ is the diagonal subbimodule of ${}_i M_j^{P_1} \oplus {}_i M_j^{P_2}$.

Observe also that the subbimodule ${}_i R_j$ of ${}_i M_j^C$ determines uniquely a bimodule epimorphism ${}_i \rho_j: {}_i M_j^C \rightarrow {}_i M_j^C / {}_i R_j$ which can be used to define the relation ${}_i R_j$.

The relations will be represented in the graph G_Σ of the species Σ as follows: if ${}_i R_j$ is a (minimal) relation from the vertex i to the vertex j , it is denoted by a dotted line from i to j , attached to which is the map ${}_i \rho_j$.

(1.3) We shall now characterize those k -algebras which are given by bounden k -species:

Proposition: Every finite-dimensional basic algebra A on a perfect field k is given by a bounden k -species (Σ, R) .

Proof: Let k be a perfect field, and A a finite-dimensional basic k -algebra. Then $\Lambda = A/\text{rad } A$ is separable (cf. [14], p. 472) and, by Wedderburn's theorem, the canonical surjection $A \rightarrow \Lambda$ splits (cf. [9], p. 491, or [14], p. 471), that is:

$$A = \Lambda + \text{rad } A, \quad \Lambda \cap \text{rad } A = 0.$$

Now, if we write Λ as $\Lambda = \prod_{i \in I} F_i$, and decompose the Λ - Λ bimodule $M = \text{rad } A/(\text{rad } A)^2$ in F_i - F_j bimodules $M = \bigoplus_{i,j} M_{i,j}$, we obtain a species $\Sigma = (F_i, M_{i,j})_{i,j \in I}$. We want to define a surjective algebra homomorphism $\varphi: T(\Sigma) \rightarrow A$. It suffices of course to define φ on $\Lambda \oplus M$ since Λ and M generate $T(\Sigma)$ as a k -algebra. Let $\xi_1, \dots, \xi_m$ be a basis of the k -vector space M consisting of a union of bases of the $M_{i,j}$, and let us choose a basis $x_1, \dots, x_m$ of some supplementary k -subspace of $(\text{rad } A)^2$ in $\text{rad } A$. We then define, for each primitive idempotent e_i of Λ (that is, e_i is the identity of the field F_i):

$$\varphi(e_i) = e_i$$

where, on the right, Λ is considered as a k -subspace of A . This defines φ on Λ . We also put:

$$\varphi(\xi_j) = x_j$$

for $1 \leq j \leq n$. This defines a unique algebra homomorphism

$\varphi: T(\Sigma) \rightarrow A$ which is clearly surjective. It is also clear, by construction, that $\text{Ker } \varphi = R \subseteq (\text{rad } T(\Sigma))^2$. $\parallel$

In other words, on a perfect field k , every finite-dimensional k -algebra is Morita equivalent to an algebra given by a bounden species (Σ, R) . Such algebras will be called graph algebras.

(1.4) Example: We shall see that, for instance, the matrix algebra

$$A = \begin{bmatrix} C & 0 & 0 \\ C & R & 0 \\ C & C & C \end{bmatrix} = \left\{ \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ x & d & e \end{bmatrix} \mid c \in R, a, b, d, e, x \in C \right\}$$

(with the ordinary matrix operations) is given by a relation on the tame R -species Σ given by the Euclidean graph $\tilde{C}_2$:

$$C \xrightarrow[\underset{C}{C}]{\overset{C}{C}R} R \xrightarrow[\underset{R}{R}]{\overset{C}{C}C} C.$$

The composed bimodule from the first vertex to the last is

${}^C C R \otimes {}^C R C$. Let us consider the multiplication map:

$$\mu: {}^C C R \otimes {}^C R C \rightarrow {}^C C C$$

given by $\mu(x \otimes y) = xy$ (for $x, y \in \mathbb{C}$). This is clearly a bimodule epimorphism. In fact, $\text{Ker } \mu = {}_{\mathbb{C}}\mathbb{C}_{\mathbb{C}}$ (where the right $\mathbb{C}$ -action is given by the conjugation $-$). Indeed, it is seen that the elements:

$$u = \frac{1}{2} (1 \otimes 1 - i \otimes i),$$

$$v = \frac{1}{2} (1 \otimes 1 + i \otimes i),$$

form a basis of ${}_{\mathbb{C}}\mathbb{C}_R \otimes {}_R\mathbb{C}_{\mathbb{C}}$, and that

$$ui = \frac{1}{2} (1 \otimes i + i \otimes 1) = iu,$$

$$vi = \frac{1}{2} (1 \otimes i - i \otimes 1) = -iv.$$

Hence ${}_{\mathbb{C}}\mathbb{C}_R \otimes {}_R\mathbb{C}_{\mathbb{C}} \cong {}_{\mathbb{C}}\mathbb{C}_{\mathbb{C}} \oplus {}_{\mathbb{C}}\mathbb{C}_{\mathbb{C}}$, where the first summand is generated by u , and the second by v (cf. [13]). Moreover, the canonical projection ${}_{\mathbb{C}}\mathbb{C}_R \otimes {}_R\mathbb{C}_{\mathbb{C}} \cong {}_{\mathbb{C}}\mathbb{C}_{\mathbb{C}} \oplus {}_{\mathbb{C}}\mathbb{C}_{\mathbb{C}} \rightarrow {}_{\mathbb{C}}\mathbb{C}_{\mathbb{C}}$ is precisely μ . μ defines thus a relation on Σ . We shall call A the resulting graph algebra, whose bounden graph G_A is:

We shall now interpret A in terms of matrices. The tensor algebra of Σ is:

$$T(\Sigma) = \begin{bmatrix} \mathbb{C} & 0 & 0 \\ \mathbb{R}\mathbb{C} & \mathbb{R} & 0 \\ \mathbb{C}\mathbb{C} \oplus \mathbb{C}\mathbb{C} & \mathbb{C}\mathbb{R} & \mathbb{C} \end{bmatrix} = \left\{ \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ (x,y) & d & e \end{bmatrix} \mid \begin{array}{l} c \in \mathbb{R}, a, b, d, \\ e, x, y \in \mathbb{C} \end{array} \right\}$$

(here, we have used the identification $\mathbb{C}\mathbb{R} \otimes \mathbb{R}\mathbb{C} \cong \mathbb{C}\mathbb{C} \oplus \mathbb{C}\mathbb{C}$,

and written (x,y) for $xu + yv$), with the multiplication given by:

$$\begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ (x,y) & d & e \end{bmatrix} \begin{bmatrix} a' & 0 & 0 \\ b' & c' & 0 \\ (x',y') & d' & e' \end{bmatrix} = \begin{bmatrix} aa' & 0 & 0 \\ ba' + cb' & cc' & 0 \\ (X,Y) & dc' + ed' & ee' \end{bmatrix}$$

where $X = xa' + ex' + db'$,

and $Y = ya' + ey' + d\bar{b}'$.

Hence, A is the matrix algebra:

$$A = \begin{bmatrix} \mathbb{C} & 0 & 0 \\ \mathbb{R}\mathbb{C} & \mathbb{R} & 0 \\ \mathbb{C}\mathbb{C} & \mathbb{C}\mathbb{R} & \mathbb{C} \end{bmatrix} = \left\{ \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ x & d & e \end{bmatrix} \mid c \in \mathbb{R}, a, b, x, d, e \in \mathbb{C} \right\}$$

with the ordinary matrix multiplication. Moreover, $A = T(\Sigma)/R$,

where the relation ideal R of $T(\Sigma)$ is the set of all matrices

of the form:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ (0,y) & 0 & 0 \end{bmatrix}, \text{ with } y \in \mathbb{C}.$$

Let us observe that the multiplication map μ is the only bimodule epimorphism from ${}^{\mathbb{C}}R \otimes {}^R\mathbb{C}$ to a one-dimensional $\mathbb{C}$ - $\mathbb{C}$ bimodule such that both the left and right $\mathbb{C}$ -actions on the image are given by the ordinary multiplication of complex numbers. Indeed, we have seen that

$${}^{\mathbb{C}}R \otimes {}^R\mathbb{C} \cong \mathbb{C}u \oplus \mathbb{C}v$$

where $u.z = zu$, and $v.z = \bar{z}v$, for $z \in \mathbb{C}$. Thus, a one-dimensional epimorphic image of ${}^{\mathbb{C}}R \otimes {}^R\mathbb{C}$ is of the form $\mathbb{C}(\alpha u + \beta v)$ (for $\alpha, \beta \in \mathbb{C}$), and the right $\mathbb{C}$ -action is given by:

$$(\alpha u + \beta v).z = \alpha zu + \beta \bar{z}v \quad (\text{for } z \in \mathbb{C}).$$

In order that the right $\mathbb{C}$ -action be given by the ordinary multiplication of complex numbers, it is thus necessary that $\beta = 0$.

Finally, let us remark that A is of finite representation type and, in fact, is a tilted algebra of B_3 .

(1.5) Representations of bounden k-species: Let (Σ, R) be a bounden k-species. If $\underline{X} = (X_i, {}_i\varphi_j)_{i,j \in I}$ is an ordinary representation of Σ , let us take two points $i, j \in I$ and a path $p = (i \rightarrow i_1 \rightarrow \dots \rightarrow i_t = j)$ from i to j . We obtain a map

$${}_{j\varphi_i}^p : X_i \otimes {}_iM_j^p \rightarrow X_j$$

defined to be the composition:

$$j\varphi_i^P = j\varphi_{i_{t-1}} (i_{t-1}\varphi_{i_{t-2}} \otimes 1) \dots (i_2\varphi_{i_1} \otimes 1 \otimes \dots \otimes 1) (i_1\varphi_i \otimes 1 \otimes \dots \otimes 1)$$

and, consequently, a summation map:

$$j\varphi_i^c = \bigoplus_{P \in \mathcal{P}(i,j)} j\varphi_i^P : X_i \otimes iM_j^c \rightarrow X_j.$$

The representation $\underline{X}$ is said to be bound by R , or to be a bound representation, if the associated $T(\Sigma)$ -module is annihilated by the ideal $R = \bigoplus_{i,j} iR_j$, in other words, if

$$j\varphi_i^c (X_i \otimes iR_j) = 0$$

for all i, j (cf. [10]). The equivalence functor $\text{mod } T(\Sigma) \rightarrow \mathcal{F}(\Sigma)$ induces then, by passing to the cokernels, an equivalence between $\text{mod } (T(\Sigma)/R)$ and the category $\mathcal{F}(\Sigma, R)$ of all representations of Σ bound by R .

We shall now construct a set of representatives of the indecomposable projective bound representations of (Σ, R) . Let $i \in I$, and define $\underline{P}(i)$ by:

$$\underline{P}(i)_i = F_i,$$

$$\text{and } \underline{P}(i)_j = iM_j^c / iR_j \quad \text{for } j \neq i.$$

Note that, if there is no path from i to j , ${}_i M_j^C = 0$ and hence $\underline{P}(i)_j = 0$. The mapping ${}_j \varphi_\ell: \underline{P}(i)_\ell \otimes {}_\ell M_j \rightarrow \underline{P}(i)_j$ is the canonical mapping $({}_i M_\ell^C / R_\ell) \otimes {}_\ell M_j \rightarrow {}_i M_j^C / R_j$. Now, for every bound representation $\underline{X}$, we clearly have $\text{Hom}(\underline{P}(i), \underline{X}) \cong X_i$. In particular, the functor $\text{Hom}(\underline{P}(i), -)$ is exact, and $\underline{P}(i)$ is projective.

Its top is identified with the simple representation $\underline{S}(i)$ defined by $\underline{S}(i)_i = F_i$, and $\underline{S}(i)_j = 0$ for all $j \neq i$. Since the $\underline{S}(i)$ obviously yield all possible isomorphism classes of simple (bound) representations, the $\underline{P}(i)$ form a complete list of representatives of the indecomposable projective bound representations of (Σ, R) .

The indecomposable injective bound representations $\underline{I}(i)$ can be constructed in the same manner.

Notice that our construction allows to recover the ${}_i R_j$ from the maps between the indecomposable projectives. Indeed, for $i, j \in I$, ${}_i R_j$ is the kernel of the canonical epimorphism from ${}_i M_j^C$ to $\underline{P}(i)_j = \text{Hom}(\underline{P}(j), \underline{P}(i))$.

(1.6) Two particular relations: Let k be a perfect field, E and F be (skew) fields, finite-dimensional over k , and such that $\dim_F E = \dim E_F = 2$. We shall be particularly interested in studying the one-dimensional epimorphic images of the bimodules ${}_E F \otimes {}_F E$ and ${}_F E \otimes {}_E F$. Such images will define respectively

relations on the two species:

$$\begin{array}{l} E \xrightarrow[E F]{E} F \xrightarrow[F E]{E} E \\ \text{and} \\ F \xrightarrow[F E]{E} E \xrightarrow[E F]{E} F. \end{array}$$

It is clear that such relations do always exist. For instance, one may define an epimorphism $\mu_0: E_F^E \otimes F_E^E \longrightarrow E_E^E$ by using the multiplication map $\mu_0: x \otimes y \mapsto xy$ (where $x, y \in E$), while in the second case, an epimorphism $\pi_0: F_E^E \otimes E_F^E \longrightarrow F_F^F$ is given by:

$$\pi_0((x_1 + y_1 e) \otimes (x_2 + y_2 e)) = x_1 x_2 + y_1 y_2$$

where $\{1, e\}$ is a left and right F -basis of E , and $x_1, x_2, y_1, y_2 \in F$.

In general, however, there may be a rather large amount of different epimorphic images of $E_F^E \otimes F_E^E$ or $F_E^E \otimes E_F^E$ of the form E_E^E or F_F^F , respectively. For example, $H_C^H \otimes C_H^H \simeq H^H H^H \oplus H^H H^H$ and $R_C^C \otimes C_R^C \simeq R^R R^R \oplus R^R R^R$. Nevertheless, in some cases, these images are such that the resulting graph algebras are isomorphic.

Lemma: Let (Σ, R) be a bounden k -species satisfying the conditions (β_1) through (β_7) of Theorem (B) in the Introduction (respectively, (γ_1) through (γ_7) of Theorem (C)) and such that the field F is central in E . Then, for each value of the epimorphism

μ in (β_3) (respectively, π in (γ_3)), the resulting graph algebras are isomorphic.

Proof: Let (Σ, R) satisfy the conditions (β_1) through (β_7) and be such that F is central in E . Then, if $A(\mu)$ is the graph algebra of (Σ, R) corresponding to some value of μ , we claim that $A(\mu) \cong A(\mu_0)$, where μ_0 is the multiplication map.

Notice that the bimodule structure on the image E_E^E of μ can be assumed to be given on the left by the usual multiplication of E , and on the right, by a ring homomorphism $\lambda: E \rightarrow \text{End } E \cong E$. Since E is a field, λ is in fact an automorphism and, F being central in E , $\lambda|_F = 1_F$. Finally, when $\mu = \mu_0$, λ is the identity map. We shall write $\lambda(x) = \bar{x}$ for $x \in E$.

Giving the graph of Σ an admissible orientation, the algebra $A(\mu)$ can be identified to a lower triangular $n \times n$ matrix algebra whose diagonal entries are the F_i , and the (i, j) -entry is the bimodule ${}^M_{i,j}^C / {}_i R_j$. Thus, the multiplication of $A(\mu)$ is given by $[a_{ij}][b_{ij}] = [c_{ij}]$, where c_{ij} is zero if there is no non-zero path from i to j , it is the ordinary matrix product $\sum_{h=1}^n a_{ih} b_{hj}$ if there is a non-zero path from i to j on which i_0 does not lie, and $\sum_{h=1}^{i_0} a_{ih} \bar{b}_{hj} + \sum_{h=i_0+1}^n a_{ih} b_{hj}$ if i_0 lies on a non-zero path from i to j .

Let us define $\varphi: A(\mu) \rightarrow A(\mu_0)$ by $[a_{ij}] \mapsto [a'_{ij}]$, where a'_{ij} is $\bar{a}_{ij}$ if $i_0 \geq i \geq j$, and a_{ij} otherwise. Then φ is an algebra homomorphism: indeed, it suffices to check that it preserves the multiplication of matrices, and this is immediate from the definition of the multiplication of $A(\mu)$. Similarly, using the inverse field automorphism $\lambda^{-1}: E \rightarrow E$, we can define another algebra homomorphism $\psi: A(\mu_0) \rightarrow A(\mu)$ which is clearly the inverse of φ . This completes the proof of our claim. The second part of the lemma is proved in the same manner. $\parallel$

2. THE TILTED ALGEBRAS OF F_4 :

We shall now, as an illustration, represent all the (basic) tilted algebras of the Dynkin graph F_4 by bounden k -species, where k is a perfect field. In any such algebra, only two fields appear, which we shall denote by E and F , both of them are finite dimensional over the common central subfield k , and E is a (left and right) two dimensional F -space. Four types of relations occur in these tilted algebras, namely:

- a. the zero-relations,
- b. the commutativity relations,
- c. the relation (μ) : is defined on the subspecies $E \rightarrow F \rightarrow E$ by a bimodule epimorphism $\mu: {}^E_F \otimes {}^F_E \rightarrow {}^E_E$ such that the left and right E -actions on the image are given by the multiplication

of the field E .

d. the relation (π) : is defined on the subspecies

$F \rightarrow E \rightarrow F$, by a bimodule epimorphism $\pi: {}_F^E E \otimes {}_E^F F \rightarrow {}_F^F F$

such that the left and right F -actions on the image are given by the multiplication of the field F .

Observe that, if F is central in E , then, in each of the bounden species listed below, the resulting graph algebras are isomorphic for each value of μ and π (by Lemma (1.6)).

The algebras are now listed (non-oriented edges can be oriented arbitrarily):

1. $F \xrightarrow{\quad} F \xleftarrow{\quad} E \xleftarrow{\quad} E$,

2. $F \xrightarrow{\quad} F \xrightarrow{\quad} E \xrightarrow{\quad} E$,

3. $E \xrightarrow{\quad} E \xleftarrow{\quad} F \xleftarrow{\quad} F$,

4. $E \xrightarrow{\quad} E \xrightarrow{\quad} F \xrightarrow{\quad} F$,

5. $F \xleftarrow{\quad} F \xleftarrow{\quad} E \xleftarrow{\quad} E$,

CHAPTER (III):

ITERATED TILTED ALGEBRAS

We shall now define the iterated tilted algebras (which are called "generalized tilted" in [1]) and study some of their properties.

1. DEFINITION AND EXAMPLES:

Let A be a finite dimensional k -algebra, and T_A a tilting module with $\text{End } T_A = B$. We recall that if A is hereditary, then the torsion theory $(\mathcal{X}(T_A), \mathcal{Y}(T_A))$ in $\text{mod } B$ is splitting. This is not necessarily the case if A is not hereditary. Let us indeed consider the following example:

(1.1) Example: Let A be the graph algebra given by the bounden k -species

$$k \overset{\cdot}{\leftarrow} k \leftarrow k \rightarrow k \rightarrow k$$

(with the natural bimodules ${}_i M_j = {}_k k$, and where the dotted line indicates a zero-relation).

Then the tilting module T_A whose (non-isomorphic) indecomposable summands have the dimension-vectors (00001) , (01100) , (01111) , (11000) and (01000) is such that neither of the torsion theories $(\mathcal{T}(T_A), \mathcal{F}(T_A))$ and $(\mathcal{X}(T_A), \mathcal{Y}(T_A))$ splits. On the other hand, replacing the summand (01000) by its translate (10000) , we obtain a new tilting module

T'_A such that $(\mathcal{X}(T'_A), \mathcal{U}(T'_A))$ is a splitting torsion theory.

(1.2) The preceding example motivates the following definitions:

Definition (1.2.1): Let A be a finite dimensional k -algebra, a tilting module T_A is called splitting if the torsion theory $(\mathcal{X}(T_A), \mathcal{U}(T_A))$ is splitting in the category $\text{mod } B$, where $B = \text{End } T_A$.

In other words, T_A is splitting if every indecomposable B -module N_B is such that either $N \otimes T = 0$, or $\text{Tor}_1^B(N, T) = 0$.

Definition (1.2.2): Given a hereditary finite dimensional k -algebra A , a going-down tilting series from A , $(A_i, T_{A_i})_{i \in \mathbb{N}}$ is defined to consist of a family of k -algebras A_i , and a family of tilting modules T_{A_i} such that:

- (1) $A_0 = A$,
- (2) $A_{i+1} = \text{End } T_{A_i}$,
- (3) the tilting modules T_{A_i} are splitting.

Definition (1.2.3): A finite dimensional k -algebra B will be called iterated tilted if there exists a hereditary algebra A , a going-down tilting series $(A_i, T_{A_i})_{i \in \mathbb{N}}$ from A , and an $m \in \mathbb{N}$ such that $B \cong A_m$.

B will be called iterated tilted of type Δ , for a (non-oriented) valued graph Δ , if A is the tensor algebra of an oriented valued graph, with non-oriented underlying graph Δ .

(1.3) Example: Let A_0 be the algebra of all $n \times n$ lower triangular matrices:

$$A_0 = \begin{bmatrix} k & 0 & 0 & \dots & 0 & 0 \\ k & k & 0 & \dots & 0 & 0 \\ k & k & k & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ k & k & k & \dots & k & 0 \\ k & k & k & \dots & k & k \end{bmatrix}$$

Its graph is A_n , with the linear orientation:

and the obvious modulation. Then the module:

$$T_{A_0} = P(1) \oplus P(n) \oplus \left(\bigoplus_{i=2}^{n-1} P(n)/P(1) \right)$$

is a tilting module, which is automatically splitting, since A_0 is hereditary.

$A_1 = \text{End } T_{A_0}$ has the bounden graph:

with the inherited modulation, and where the dotted line indicates a zero-relation. Thus, the underlying graph is still A_n , but we have added one zero-relation. In fact, A_1 has global dimension two. We now take the module:

$$T_{A_1} = P(1) \oplus P(2) \oplus P(n) \oplus \left(\bigoplus_{i=3}^{n-1} P(n)/P(i) \right)$$

which is easily seen to be a splitting tilting module. Its endomorphism ring $A_2 = \text{End } T_{A_1}$ has the bounden graph:

with the inherited modulation, and two zero-relations shown by dotted lines. Here, $\text{gl. dim. } A_2 = 3$ (thus, in particular, A_2 is not a tilted algebra). We next take the splitting tilting module:

$$T_{A_2} = P(1) \oplus P(2) \oplus P(3) \oplus P(n) \oplus \left(\bigoplus_{i=4}^{n-1} P(n)/P(i) \right)$$

then the endomorphism ring $A_3 = \text{End } T_{A_2}$ has the bounden graph:

Repeating the same process in an obvious inductive manner, we obtain the algebra A_{n-2} with the bounden graph:

that is to say, $A_{n-2} = A_0 / (\text{rad } A_0)^2$ has zero square-radical. Its Auslander-Reiten graph $\Gamma_{A_{n-2}}$ is:

We thus have $\text{gl. dim. } A_{n-2} = n-1$.

We claim that it is not possible to go further. We shall need the following lemma:

Lemma: Let A be a finite dimensional k -algebra and T_A be a tilting module. Then, every indecomposable projective-injective module P_A is a direct summand of T_A .

Proof: Indeed, P_A being injective, $\text{Ext}_A^1(T, P) = 0$, hence $P \in \mathcal{J}(T_A)$, thus is generated by T_A . But every epimorphism with projective codomain is a retraction, and, since moreover P is indecomposable; it is necessarily a direct summand of T . ||

Thus, all the projective-injectives $P(i)$ ($i > 1$) must be direct summands of any (multiplicity-free) tilting module $T_{A_{n-2}}$. (T1) implies that no $S(i)$ (for $i \geq 3$) is a direct summand of $T_{A_{n-2}}$, and (T3), that either $P(1)$ or $S(2)$ is a summand of $T_{A_{n-2}}$. If $S(2)$ is a summand of $T_{A_{n-2}}$, then $T_{A_{n-2}} = S(2) \oplus (\bigoplus_{i \geq 1} P(i))$ and it is clear that $\text{End } T_{A_{n-2}}$ has more indecomposables than A_{n-2} , thus, $T_{A_{n-2}}$ cannot be a splitting tilting module. The only remaining possibility is to take $T_{A_{n-2}} = \bigoplus_{i=1}^n P(i)$: this is the trivial tilting module, and the going-down tilting series stops at this point.

This example shows in particular that there exist iterated tilted algebras with an arbitrary global dimension.

(1.4) It is not hard to see that, in contrast to the algebra of example (1.3), the (tilted) algebra of example (1.1) is such that the endomorphism rings of all its tilting modules are again tilted algebras. It is thus interesting to have a criterion saying when it is possible to obtain from a tilted algebra an iterated tilted algebra which is not tilted.

Definition: A tilted algebra B is minimal if, for every tilting module T_B , $\text{End } T_B$ is tilted.

Let B be a (connected) tilted algebra of finite representation type, and S a complete slice in its Auslander-Reiten graph. There always exists at least one source $[M]$ on S . If M is not injective, replacing $[M]$ by $[\tau^{-1}M]$ and every arrow $\alpha: [M] \rightarrow [N]$ with $[N] \in S$ by the corresponding arrow $\sigma^{-1}\alpha: [N] \rightarrow [\tau^{-1}M]$, we obtain a new complete slice. Applying this process repeatedly, we ultimately obtain a complete slice S_r such that, whenever we substitute any point $[X] \in S_r$ by $[\tau^{-1}X]$, we no longer have a complete slice. In particular, all sources on S_r are injective. We shall call this slice right-extremal. A point $[X]$ in Γ_B is on the right (respectively, on the left) of S_r if there exists an oriented path of positive length from a point $[M] \in S_r$ to $[X]$ (respectively, from $[X]$ to a point $[M] \in S_r$). We then have the following proposition, due to D. Happel:

Proposition: Let B be a (connected) tilted algebra of finite representation type. If all the points on the right of the right extremal slice S_r have projective dimension equal to 2, then B is minimal.

Proof: Let $T_B = \bigoplus_i T(i)$ be an arbitrary tilting module, with the $T(i)$ indecomposables. Since all the points on the right of S_r have their projective dimension equal to 2, no indecomposable summand $T(i)$ of T lies on the right of S_r . Let $M(a)$ be the indecomposable B -modules lying on S_r , and $M_B = \bigoplus_a M(a)$ the slice module of S_r , then M_B is a tilting module and $A = \text{End } M_B$ is hereditary. Now it is clear (and it follows from Theorem (2.4) of [7]) that $U_A = D(M)$ is a tilting module, and moreover:

$$\begin{aligned} M_B &\cong \text{Hom}_{A^{\text{op}}} (A, M_B) \\ &\cong \text{Hom}_{A^{\text{op}}} (U_A, D(A)) \end{aligned}$$

implies that $M(a) = \text{Hom}_A(U, I(a))$, where the $I(a)$ are the indecomposable injective A -modules. But this implies that the indecomposable modules in $\mathcal{U}(U_A)$ are precisely those B -modules not lying on the right of S_r , and those in $\mathcal{I}(U_A)$ are those B -modules lying on the right of S_r . In particular, all the $T(i)_B$ lie in $\mathcal{U}(U_A)$, that is, can be written in the form $\text{Hom}_A(U_A, T(i)_A)$ for some

$\bar{T}(i)_A$. We claim that $\bar{T}_A = \bigoplus_i \bar{T}(i)_A$ is a tilting module. Since A is hereditary, it suffices to check that, for any i, j ,

$$\text{Ext}_A^1(\bar{T}(i), \bar{T}(j)) = 0$$

and this obviously follows from:

$$\text{Ext}_A^1(\bar{T}(i), \bar{T}(j)) = \text{Ext}_B^1(T(i), T(j)).$$

Thus, $\bar{T}_A$ is a tilting module. Since clearly $\text{End } \bar{T}_A \cong \text{End } T_B$, it follows that $\text{End } T_B$ is a tilted algebra. $\parallel$

(1.5) We shall apply the preceding criterion to calculate all going-down tilting series starting from F_4 , thus, all iterated tilted algebras of type F_4 :

Theorem: Let k be a perfect field. The finite dimensional connected basic k -algebra B is iterated tilted of type F_4 if and only if B is either:

- (i) One of the tilted algebras of type F_4 listed in (II-2), or:
- (ii) One of the following two graph algebras of global dimension three:

where E, F are two (skew) fields, finite-dimensional over the common central subfield k , and such that $\dim_{E_F} E = \dim_F E = 2$.

Proof: Applying the criterion of Proposition (1.4), we see that only two of the tilted algebras of (II-2) are not minimal. Namely, these are the algebras:

a) The algebra B with bounden graph:

has the Auslander-Reiten graph:

(where modules are represented by their dimension vectors).

The only module on the right of the right extremal slice whose projective dimension is not two is the simple injective (1000). Observe that, by the proof of Proposition (1.4), any tilting module whose endomorphism ring is not tilted has (1000) as a summand. Also, the projective-injective (0011) is a summand of any tilting module. There

remains four indecomposables whose projective dimension is one.

Notice that we cannot have (0110) as a summand of such a tilting module, since there is an extension between (0110) and (1000). Also, (0001) and (0010) cannot appear simultaneously. Thus there are only two (multiplicity-free) tilting modules:

$$T'_B = (0010) \oplus (0011) \oplus (1110) \oplus (1000),$$

$$\text{and } T''_B = (0001) \oplus (0011) \oplus (1110) \oplus (1000).$$

The first one has endomorphism ring:

which is again a tilted algebra, while $C = \text{End } T''_B$ is

and this is not a tilted algebra. It has the following Auslander-Reiten graph:

We shall now find all (multiplicity-free) tilting modules on C . Any tilting module must have as summands the two projective-injectives

(1100) and (0011). On the other hand, (0001), (0010) and (0120) have a projective dimension larger than one. There remains to consider the three modules (1000), (0100) and (0110). We have thus only two (multiplicity-free) tilting modules:

$$T'_C = (1100) \oplus (0011) \oplus (0110) \oplus (0100),$$

and $T''_C = (1100) \oplus (0011) \oplus (0110) \oplus (1000).$

The first one is not splitting: indeed $\text{End } T'_C$ is

$$\begin{array}{ccccccc} E & \xleftarrow{\quad} & E & \xleftarrow{\quad} & F & \xleftarrow{\quad} & F \\ & & & & \text{---} & & \\ & & & & 0 & & \end{array}$$

which has more indecomposables than C . The second one is the trivial tilting module. Thus the tilting series stops. Observe that $\text{gl. dim. } C = 3$.

b) The algebra B with bounden graph:

$$\begin{array}{ccccccc} E & \longrightarrow & E & \longrightarrow & F & \longrightarrow & F \\ & & & & \text{---} & & \\ & & & & 0 & & \end{array}$$

has the Auslander-Reiten graph:

Here, the simple injective (1000) with projective dimension one lies on the right of the right-~~extremal~~ slice. Let T be a tilting module with this simple injective as a summand. We know that (0011) is necessarily a summand of T . Also, (0120) cannot be a summand, since it has an extension with (1000). Finally, (0001) and (0010) cannot be both summands of T . Thus, there are only two (multiplicity-free) tilting modules on B , namely:

$$T'_B = (0010) \oplus (0011) \oplus (1120) \oplus (1000),$$

$$\text{and } T''_B = (0001) \oplus (0011) \oplus (1120) \oplus (1000).$$

The first one has endomorphism ring

$$\begin{array}{ccccccc} F & \longleftarrow & F & \longleftarrow & E & \longleftarrow & E \\ & & & & \text{---} & & \\ & & & & 0 & & \end{array}$$

which is again a tilted algebra, while $C = \text{End } T''_B$ is:

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \text{---} & & \\ F & \longleftarrow & F & \longleftarrow & E & \longleftarrow & E \\ & & & & \text{---} & & \\ & & & & 0 & & \end{array}$$

and has the following Auslander-Reiten graph:

As in a), there are only two possible (multiplicity-free) tilting modules, namely:

$$T'_C = (0100) \oplus (1100) \oplus (0210) \oplus (0011),$$

$$\text{and } T''_C = (1000) \oplus (1100) \oplus (0210) \oplus (0011).$$

The first one is not splitting: indeed, $\text{End } T'_C$ is

which has more indecomposables than C . The second one is the trivial tilting module. So the tilting series stops. Again, $\text{gl. dim } C = 3$. ||

2. GENERAL PROPERTIES:

(2.1) Our aim will be to characterize the iterated tilted algebras of types $\Delta = A_n$, B_n and C_n . In all these cases, Δ is a Dynkin graph. Clearly, every iterated tilted algebra of type Δ , where Δ is a Dynkin graph, is of finite representation type.

(2.2) B is an iterated tilted (respectively, tilted) algebra of type Δ , for any valued graph Δ , if and only if B^{op} is iterated tilted (respectively, tilted) of the same type.

(2.3) Let G_A be the bounden graph of the iterated tilted algebra A , and G'_A be a full connected subgraph of G_A such that:

(i) There exists an admissible orientation of G_A such that there is no oriented path from i to j , for every $i \in G'_A$, $j \notin G'_A$.

(ii) There is no relation between a point $i \in G'_A$ and a point $j \notin G'_A$, or within G'_A itself.

Then, for every change of orientation in G'_A only, (the arrows in $G_A \setminus G'_A$ being unchanged), the resulting bounden graph represents an iterated tilted algebra of the same type as A .

Indeed, such a change of orientation is given by a partial Coxeter functor, that is to say, by a tilting module T_A such that both torsion theories $(\mathcal{T}(T_A), \mathcal{F}(T_A))$ and $(\mathcal{X}(T_A), \mathcal{Y}(T_A))$ split (cf. [5], [6] or [10]).

(2.4) Let A be a finite dimensional k -algebra, T_A a splitting tilting module, and $B = \text{End } T_A$. Then the Auslander-Reiten sequences of $\text{mod } B$ can be characterized in terms of those of $\text{mod } A$. The following proof is due to Bongartz [7], who used it only for the case of tilted algebras. However, his arguments remain valid in the general situation.

Let $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ be an Auslander-Reiten sequence in $\text{mod } B$, we have three possible cases:

(i) $N \in \mathcal{Y}(T_A)$: Then L and M also belong to $\mathcal{Y}(T_A)$, and the given sequence is an Auslander-Reiten sequence in the subcategory $\mathcal{Y}(T_A)$,

(ii) $L \in \mathcal{X}(T_A)$: then M and N also belong to $\mathcal{X}(T_A)$, and the given sequence is an Auslander-Reiten sequence in the subcategory $\mathcal{X}(T_A)$,

(iii) $L \in \mathcal{U}(T_A)$ and $N \in \mathcal{X}(T_A)$: this case is more complicated: we first prove that L_B is the image of an injective A -module under $\text{Hom}_A(T, -)$. Let $L' \in \mathcal{J}(T_A)$ be such that $L_B = \text{Hom}_A(T, L')$, and I be the injective envelope of L' in $\text{mod } A$. We have:

$$\begin{aligned} \text{Ext}_A^1(I/L', L') &\simeq \text{Ext}_B^1(\text{Hom}_A(T, I/L'), \text{Hom}_A(T, L')) \\ &= \text{Ext}_B^1(\text{Hom}_A(T, I/L'), L) \\ &\simeq D \underline{\text{Hom}}_B(\tau^{-1}L, \text{Hom}_A(T, I/L')) \\ &= D \underline{\text{Hom}}_B(N, \text{Hom}_A(T, I/L')) \\ &= 0. \end{aligned}$$

For, $N \in \mathcal{X}(T_A)$ is torsion, while $\text{Hom}_A(T, I/L') \in \mathcal{U}(T_A)$ is torsion-free, and thus $\text{Hom}_B(N, \text{Hom}_A(T, I/L')) = 0$. Therefore the short exact sequence $0 \rightarrow L' \rightarrow I \rightarrow I/L' \rightarrow 0$ splits and, since I is the injective envelope of L' , we have $L' = I$, which proves our claim.

Lemma: Let S be a simple A -module, P its projective cover, and I its injective envelope.

a) A morphism $\text{Hom}_A(T, f): \text{Hom}_A(T, I) \rightarrow \text{Hom}_A(T, K)$ with $K \in \mathcal{J}(T_A)$ is irreducible in $\text{mod } B$ if and only if f is isomorphic to the composition map $I \xrightarrow{p} I/S \xrightarrow{r} K$, for a retraction r , and the canonical map $p: I \rightarrow I/S$.

b) A morphism $\text{Ext}_A^1(T, g): \text{Ext}_A^1(T, R) \rightarrow \text{Ext}_A^1(T, P)$ with $P, R \in \mathcal{J}(T_A)$ is irreducible in $\text{mod } B$ if and only if g is isomorphic to the compo-

sition map $R \xrightarrow{s} \text{rad } P \xrightarrow{i} P$, for a section s , and the canonical inclusion $i: \text{rad } P \rightarrow P$.

Proof: We shall only prove a). It is well-known (cf. [4]) that $\text{Hom}_A(T, f)$ is irreducible in $\text{mod } B$ if and only if $\text{Hom}_A(T, f)$ is irreducible in $\text{mod } B/\mathcal{I}(T_A)$, where $\text{mod } B/\mathcal{I}(T_A)$ is the category obtained from $\text{mod } B$ by replacing the spaces of morphisms by their quotients modulo the morphisms of $\text{mod } B$ which factor through a module in $\mathcal{I}(T_A)$. Since the torsion theory $(\mathcal{I}(T_A), \mathcal{U}(T_A))$ is splitting, we have equivalences $\mathcal{I}(T_A) \cong \mathcal{U}(T_A) \cong \text{mod } B/\mathcal{I}(T_A)$. Hence $\text{Hom}_A(T, f)$ is irreducible in $\text{mod } B$ if and only if f is irreducible in $\mathcal{I}(T_A)$. Clearly, the canonical projection $p: I \rightarrow I/S$ remains irreducible in $\mathcal{I}(T_A)$, and conversely, an irreducible map $f: I \rightarrow K$ in $\mathcal{I}(T_A)$ factors through the canonical projection. $\parallel$

Theorem: Let A be a finite dimensional k -algebra, T_A a splitting tilting module, and $B = \text{End } T_A$. Let

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$$

be an Auslander-Reiten sequence in $\text{mod } B$. We have three possible cases:

- (i) It is an Auslander-Reiten sequence in $\mathcal{U}(T_A)$.
- (ii) It is an Auslander-Reiten sequence in $\mathcal{I}(T_A)$.
- (iii) It is a connecting sequence, that is, a sequence of the form:

$$0 \rightarrow \text{Hom}_A(T, I) \rightarrow \text{Hom}_A(T, I/\text{soc } I) \oplus \text{Ext}_A^1(T, \text{rad } P) \rightarrow \text{Ext}_A^1(T, P) \rightarrow 0$$

where P is a projective not in $\text{add } T_A$, and I the injective envelope of $P/\text{rad } P$.

Proof: By the preceding discussion, if the sequence does not lie completely in $\mathfrak{X}(T_A)$ or $\mathfrak{U}(T_A)$, it must start with a term of the form $\text{Hom}_A(T, I)$ with I_A injective. But the connecting lemma says that:

$$\tau^{-1} \text{Hom}_A(T, I) = \text{Ext}_A^1(T, P)$$

where P is the projective cover of $\text{soc } I$ (and, of course, $P \notin \text{add } T_A$).

The given sequence is thus of the form:

$$0 \rightarrow \text{Hom}_A(T, I) \rightarrow X \oplus Y \rightarrow \text{Ext}_A^1(T, P) \rightarrow 0$$

with $X \in \mathfrak{X}(T_A)$, $Y \in \mathfrak{U}(T_A)$. The preceding lemma gives $Y = \text{Hom}_A(T, I/\text{soc } I)$ and $X = \text{Ext}_A^1(T, \text{rad } P)$. $\parallel$

(2.5) Lemma: Let B be an iterated tilted algebra of type Δ , where Δ is a Dynkin graph, then Γ_B has no oriented cycles.

Proof: We shall use induction on the length of a going-down' tilting series used to reach B . The result being obvious for any Dynkin graph, let us assume it for the iterated tilted algebra A and let T_A be a splitting tilting module such that $B = \text{End } T_A$.

Let $M_0 \rightarrow M_1 \rightarrow \dots \rightarrow M_m \rightarrow M_0$ be an oriented cycle in Γ_B . If

all the M_i ($0 \leq i \leq m$) belong to $\mathcal{X}(T_A)$ (respectively, $\mathcal{U}(T_A)$), this cycle is the image of an oriented cycle in $\mathcal{X}(T_A)$ (respectively, $\mathcal{U}(T_A)$), because A is of finite representation type, and this contradicts the hypothesis that Γ_A has no oriented cycles. Thus there exists an $0 \leq i \leq m$ such that $M_i \in \mathcal{X}(T_A)$ and $M_{i+1} \in \mathcal{U}(T_A)$ (where we agree to set $M_{m+1} = M_0$). But then $\text{Hom}_B(M_i, M_{i+1}) \neq 0$, and this is impossible, because M_i is torsion while M_{i+1} is torsion-free. $\parallel$

The absence of oriented cycles in Γ_B implies the following properties:

- (1) Indecomposable B -modules are uniquely determined by their dimension-vectors..
- (2) Any indecomposable B -module M can be written as $M = \tau^{-t}P$, where $t \geq 0$, and P_B is an indecomposable projective B -module.
- (3) We have, besides the existence of complete slices, another criterion to recognize the tilted algebras among the iterated tilted: if B has an indecomposable sincere representation (and hence B has an indecomposable faithful module), then B is a tilted algebra.

This criterion is far from being necessary: for instance, the graph algebra given by the bounden k -species:

(with the natural bimodules $M_{ij} = k_k$, and the indicated zero-relation) is a tilted algebra of A_4 , since its bounden graph can be embedded in the "genealogical tree" (cf. (I.4.4)) but has no indecomposable sincere representation.

(2.6) Theorem: Let B be an iterated tilted algebra of type Δ , where Δ is a Dynkin graph, then Γ_B is simply connected.

Proof: The proof is again done by induction on the length of a going-down tilting series used to reach B . The result being trivial for a Dynkin graph, let A be an iterated tilted algebra with Γ_A simply connected and T_A a splitting tilting module such that $B = \text{End } T_A$.

Let us assume that Γ_B is not simply connected, there must exist in Γ_B a closed (non-oriented) path w which is not null-homotopic. We can of course assume w to be of minimal length. The absence of oriented cycles implies the existence of sources and sinks on w . One can suppose that every sink corresponds to a projective module: indeed, if the sink $[M]$ is such that M is not projective, we can replace $[M]$ by $[\tau M]$, and each arrow $\alpha: [X] \rightarrow [M]$ on the path w by the corresponding arrow $\sigma\alpha: [\tau M] \rightarrow [X]$: this

process does not affect the length of w . On the other hand, applying repeatedly this process, we cannot reach another point of the original path w : this follows from the minimality of w , and the fact that it is not null-homotopic.

Let now $[X]$ be a point on w which is not a sink. Then there exists an oriented path from $[X]$ to some sink $[P]$. But P is projective, hence $P \in \mathcal{U}(T_A)$, thus $X \in \mathcal{U}(T_A)$, and consequently all the modules on w belong to $\mathcal{U}(T_A)$. Let $[N(1)], \dots, [N(r)]$ be the sources on w , and put $\bar{N}(j)_A = N(j) \otimes_B T_A$ for $1 \leq j \leq r$.

By Theorem (2.4), if $\bar{N}(j)_A$ is not injective, the Auslander-Reiten sequence starting with $N(j)_B = \text{Hom}_A(T, \bar{N}(j))$ lies in $\mathcal{U}(T_A)$. We may thus replace $[N(j)]$ by $[\tau^{-1} N(j)]$, and each arrow $\beta: [N(j)] \rightarrow [Y]$ on the path w by $\sigma^{-1} \beta: [Y] \rightarrow [\tau^{-1} N(j)]$, thus obtaining a new path, homotopic to w , of the same length, and still lying in $\mathcal{U}(T_A)$. Applying this process as many times as necessary, we obtain a path w' in $\mathcal{U}(T_A)$, homotopic to w , of the same length, and with all sources $[J(j)]_B$ such that $\bar{J}(j)_A = J(j) \otimes_B T_A$ is injective in $\text{mod } A$ (for all j). Observe that w' may have sinks which do not correspond to projectives, what is important for our purposes, however, is that w' still lies in $\mathcal{U}(T_A)$.

Applying the functor $-\otimes_B T_A$ to return to $\text{mod } A$, let $\bar{w}'$ be the closed path corresponding to w' (such a path exists, since A is

of finite representation type). Now $\bar{J}(j)_A$ is injective, and $\bar{w}'$ is null-homotopic (because, by the induction hypothesis, Γ_A is simply connected) hence $\bar{w}'$ cannot be minimal: in particular, there exists on $\bar{w}'$ a single arrow $[\bar{J}(j)_A] \rightarrow [X_A]$ with source $[\bar{J}(j)]$. Clearly, the minimality of w' implies that $X \notin \mathcal{T}(T_A)$. However $\bar{J}(j)_A \rightarrow X_A$ is an irreducible map with injective domain, hence is a surjection on a direct summand of $\bar{J}(j)/\text{soc } \bar{J}(j)$. Now $\bar{J}(j) \in \mathcal{T}(T_A)$ which is a torsion class, hence $X \in \mathcal{T}(T_A)$, a contradiction which completes the proof. $\parallel$

(2.7) Prefactors and postfactors: Let A be a finite dimensional k -algebra, and $\text{ind } A$ the full subcategory of all its (finite dimensional) indecomposable modules.

Definition: Let $M \in \text{ind } A$, we consider the two sets:

$$P_+(M) = \{N \in \text{ind } A \mid \text{Hom}_A(N, M) \neq 0, \text{Hom}_A(N, \tau M) = 0\}$$

called the set of prefactors of M , and

$$P_-(M) = \{L \in \text{ind } A \mid \text{Hom}_A(M, L) \neq 0, \text{Hom}_A(\tau^{-1}M, L) = 0\}$$

called the set of postfactors of M .

What is the interest of the preceding sets? Let T_A be a tilting module, and let i, j be two vertices on the bounden graph G_B of $B = \text{End } T_A$ such that there exists a non-zero path from i to j .

Then, if $T(i)$ and $T(j)$ are the corresponding indecomposable summands of T_A , there exists a non-trivial map $T(j) \rightarrow T(i)$.

Moreover:

$$\text{Hom}_A(T(j), T(i)) = D \text{Ext}_A^1(T(i), T(j)) = 0.$$

Thus $T(j) \in P_+(T(i))$. Similarly, $T(i) \in P_-(T(j))$.

(2.8) A final remark that can be made concerns the possibility of finding a bound for the components of the dimension-vectors.

Lemma: Let A be a finite dimensional k -algebra, and $m \in \mathbb{N}$ be such that, for any two indecomposable A -modules X, Y , we have:

$$|\text{Hom}_A(X, Y)_{\text{End } X}| \leq m,$$

$$\text{and } |\text{End } Y^{\text{Hom}_A(X, Y)}| \leq m.$$

Then, if T_A is a splitting tilting module with $B = \text{End } T_A$, and M_B is an indecomposable B -module, we have:

$$\dim M_i \leq m$$

for every vertex i of the bounden graph of B .

Proof: There are two possibilities for M :

(i) $M \in \mathcal{U}(T_A)$, then $M_B = \text{Hom}_A(T, \bar{M})$, with $\bar{M} \in \mathcal{J}(T_A)$:

$$M_i \simeq \text{Hom}_B(P(i), M)$$

$$= \text{Hom}_B(P(i), \text{Hom}_A(T, \bar{M}))$$

$$\simeq \text{Hom}_B(P(i) \otimes_B T, \bar{M})$$

$$\simeq \text{Hom}_A(T(i), \bar{M}).$$

Therefore:

$$\dim M_1 = |\operatorname{Hom}_A(T(1), \bar{M})_{\operatorname{End} T(1)}| \leq m.$$

(11) $M \in \mathcal{X}(T_A)$, then $M_B = \operatorname{Ext}_A^1(T, \bar{M})$, with $\bar{M} \in \mathcal{X}(T_A)$:

$$M_1 \cong \operatorname{Hom}_B(P(1), \operatorname{Ext}_A^1(T, \bar{M}))$$

$$\cong \operatorname{Ext}_A^1(P(1) \otimes_B T, \bar{M})$$

$$= \operatorname{Ext}_A^1(T(1), \bar{M})$$

$$\cong {}^D \operatorname{Hom}_A(\bar{M}, \tau T(1)).$$

Therefore:

$$\dim M_1 = |\operatorname{End} T(1) \operatorname{Hom}_A(\bar{M}, \tau T(1))| \leq m. \quad ||$$

CHAPTER (IV):
 ITERATED TILTED ALGEBRAS
 OF TYPE A_n .

In this chapter, we shall give a classification of the iterated tilted algebras of type A_n in terms of their bounden species, and also in terms of their Auslander-Reiten graphs. We shall assume, throughout this chapter, that the underlying field k is perfect, that all our algebras are connected and basic, and all our tilting modules are multiplicity-free.

1. THE PREFACTOR AND POSTFACTOR SETS:

(1.1) Let A be a finite dimensional k -algebra, and Γ_A its Auslander-Reiten graph. We shall be particularly interested in the following properties of Γ_A :

(AR) There are at most two arrows in Γ_A with a prescribed source or target.

Let M, N be two indecomposable A -modules, then:

$$|\text{Irr}(M, N)_{\text{End } M}| \leq 1$$

$$\text{and } |\text{End } N \text{ Irr}(M, N)| \leq 1.$$

(B) If P_A is an indecomposable projective, and there is an irreducible map $R \rightarrow P$, then there is at most one irreducible map of codomain R . Dually, if I_A is an indecomposable injective, and there is an irreducible map $I \rightarrow J$, then there is at most one irreducible map of domain J .

Obviously, the initials (AR) stand for Auslander-Reiten: the condition states that for any point $[M] \in \Gamma_A$, there are at most two arrows entering or leaving $[M]$. The second part of the condition (AR) asserts that the valuation of all arrows of Γ_A is (1,1). Similarly, (B) stands for branching: the condition says indeed that we cannot have full connected subgraphs of Γ_A of the forms:

with P_A indecomposable projective, and I_A indecomposable injective.

Then we have:

(1.2) Proposition: Let A be a finite dimensional k -algebra of finite representation type satisfying (AR) and (B), then, for every indecomposable A -module M , each of $\mathcal{P}_+(M)$ and $\mathcal{P}_-(M)$ consists of the modules lying on a union of two sectional paths of Γ_A intersecting at $[M]$.

Proof: We start by constructing two sectional paths of Γ_A intersecting at $[M]$. By (AR), we know that there exist at most two arrows of target $[M]$: let $f_1: E_1 \rightarrow M$ and $g_1: F_1 \rightarrow M$ be the corresponding irreducible maps. If M is simple projective, $E_1 = F_1 = 0$ (and then, obviously, $\mathcal{P}_+(M) = \{M\}$). Otherwise, we can assume that $E_1 \neq 0$.

If M is projective, (B) implies that there is at most one arrow of target $[E_1]$; if M is not projective, there exists an irreducible map $\sigma f_1: \tau M \rightarrow E_1$, and we have thus at most one other arrow of target $[E_1]$. That is to say, there is always at most one arrow of target $[E_1]$ whose source is not $[\tau M]$. If such an arrow exists, we let $f_2: E_2 \rightarrow E_1$ be the corresponding irreducible map. Otherwise the construction stops.

Inductively, assume that E_i has been defined for all $i \leq t$. By the same reasoning as above, there exists at most one arrow of target $[E_t]$ whose source is not $[\tau E_{t-1}]$. If such an arrow exists, we call $f_{t+1}: E_{t+1} \rightarrow E_t$ the corresponding irreducible map. The process stops when there is only one irreducible map into E_{t-1} . The set $(E_t)_{t \geq 1}$, together with M , defines by construction a sectional path of Γ_A , which will be denoted by $f(f_1)$.

Similarly, g_1 defines a sectional path $f(g_1)$.

We claim that $P_+(M) = f(f_1) \cup f(g_1)$. Let thus N_A be an indecomposable A -module such that there is a non-zero map $h: N \rightarrow M$, but no non-zero map from N to τM . We claim that $[N] \in f(f_1) \cup f(g_1)$.

Either h is an isomorphism, or else it factors through the right almost split map $[g_1 \ f_1]: F_1 \oplus E_1 \rightarrow M$. Thus we may assume that there exists a non-zero map $h_1: N \rightarrow E_1$ such that $h = f_1 h_1$.

Next, if h_1 is not an isomorphism, it factors through the right almost split map $[0 \ f_1 \ f_2]: \tau M \oplus E_2 \rightarrow E_1$. But it cannot factor through τM , by hypothesis, hence there exists a non-zero map $h_2: N \rightarrow E_2$ such that $h_1 = f_2 h_2$.

Inductively, either $N \cong E_i$ for some $i \leq t$, or else, by the same argument, there exists an $h_{t+1}: N \rightarrow E_{t+1}$ such that $h_t = f_{t+1} h_{t+1}$. Since for some m we have $E_m = 0$, it follows that $[N] \in f(f_1)$.

Similarly, if there exists a non-zero map $h'_1: N \rightarrow F_1$ such that $h = g_1 h'_1$, we have $[N] \in f(g_1)$.

There remains to show that if $[E_t] \in f(f_1)$, then $[E_t] \in P_+(M)$, that is, that the map $f_1 f_2 \dots f_t: E_t \rightarrow M$ is non-zero. Indeed, let t be minimal such that $f_1 f_2 \dots f_t = 0$. Since f_1 is an irreducible map, it is either a monomorphism or an epimorphism. The minimality of t implies that f_1 is in fact an epimorphism. Hence

$\text{Ker } f_1$ is indecomposable (because it is the kernel of an irreducible epimorphism). On the other hand, $f_1(f_2 \dots f_t) = 0$ implies the existence of a map $g: E_t \rightarrow \text{Ker } f_1$ such that the following diagram is commutative:

$$\begin{array}{ccccccc}
 0 & \rightarrow & \text{Ker } f_1 & \xrightarrow{k} & E_1 & \xrightarrow{f_1} & M \rightarrow 0 \\
 & & \uparrow g & \nearrow & & & \\
 & & E_t & & & \xrightarrow{f_2 \dots f_t} &
 \end{array}$$

(where k denotes the canonical inclusion $\text{Ker } f_1 \rightarrow E_1$). But we know that there is only one map from E_t to E_1 , namely $f_2 \dots f_t$. Therefore $\text{Ker } f_1 \cong E_s$ for some $s \leq t$. Now we cannot have $s < t$, for otherwise $f_1 \dots f_s = 0$, which contradicts the minimality of t . On the other hand, if $s = t$, the sequence

$$0 \rightarrow E_t \xrightarrow{f_2 \dots f_t} E_1 \xrightarrow{f_1} M \rightarrow 0$$

is exact and thus $\text{Ext}_A^1(M, E_t) \neq 0$. But $\text{Hom}_A(E_t, M) = 0$ implies that:

$$\text{Ext}_A^1(M, E_t) \cong D \overline{\text{Hom}}_A(E_t, M) = 0$$

another contradiction which proves our claim. Similarly, $f(g_1) \in P_+(M)$.

The dual result for $P_-(M)$ is shown in the same manner. ||

(1.3) Remarks: with the notations of (1.2):

1. Let $[N_1], [N_2] \in \mathcal{L}(f_1)$, then the preceding construction shows that either $\text{Hom}_A(N_1, N_2) \neq 0$, or $\text{Hom}_A(N_2, N_1) \neq 0$.

2. Let A be an algebra satisfying the conditions of Proposition (1.2), T_A be a tilting module, and $B = \text{End } T_A$. Let $T(i)$ be the indecomposable summand of T corresponding by $\text{Hom}_A(T, -)$ to the B -projective indecomposable $P(i)$, there are at most four sectional paths intersecting in $[T(i)]$ which we shall denote by f_1, f_2, f_3, f_4 . For any other indecomposable summand $T(j)$, if $\text{Hom}_A(T(i), T(j)) \neq 0$, $[T(j)] \in P_-(T(i))$, and if $\text{Hom}_A(T(j), T(i)) = 0$, $[T(j)] \in P_+(T(i))$. These two cases are shown in the following figures:

Let j and h be two neighbours of i in the graph G_B of B . Either $[T(i)], [T(j)], [T(h)]$ belong to the same sectional path, or not. This gives two cases, represented as shown in the above figures. In the first case, the map $T(h) \rightarrow T(i) \rightarrow T(j)$ is non-zero: indeed, $[T(h)]$ and $[T(i)]$ both belong to the same

sectional path on $[T(j)]$. In the second case, $[T(j)] \rightarrow [T(i)] \rightarrow [T(h)]$ is zero, since $[T(j)]$ is not on the sectional path on $[T(h)]$ on which $[T(i)]$ lies. We shall use these remarks in the proof of the main theorem.

(1.4) Corollary: Let A be an algebra satisfying the conditions of Proposition (1.2), T_A be a splitting tilting module, and $B = \text{End } T_A$. Then the Auslander-Reiten graph Γ_B of B satisfies (B).

Proof: Assume that we have a full connected subgraph of Γ_B of the form:

$$\begin{array}{ccc} [X_1] & \searrow & \\ & [R] \rightarrow [P] & \\ [X_2] & \nearrow & \end{array}$$

with P_B indecomposable projective. Then $P \in \mathcal{U}(T_A)$, hence $R, X_1, X_2 \in \mathcal{U}(T_A)$. Let us apply the functor $-\otimes_B T_A$ to return to $\text{mod } A$. $P_B \otimes_B T_A = T'_A$ is an indecomposable summand of T . Also, this functor being exact in $\mathcal{U}(T_A)$, $R \otimes T \rightarrow T'$ is a monomorphism. Hence the maps $f_i: X_i \otimes T \rightarrow R \otimes T \rightarrow T'$ ($i = 1, 2$) are non-zero. Now, there is no non-zero map from $R \otimes T$ to $\tau T'$, for,

$$\text{Hom}_A(R \otimes T, \tau T') = D \text{Ext}_A^1(T', R \otimes T) = 0.$$

By Proposition (1.2), $[R \otimes T]$ belongs to one of the two sectional paths on $[T]$, which we shall denote by ρ . Thus,

$[X_1 \otimes T]$ and $[X_2 \otimes T]$, for the same reason, belong to $\mathcal{f}$, and hence there exists a non-zero map from $X_1 \otimes T$ to $X_2 \otimes T$ (or from $X_2 \otimes T$ to $X_1 \otimes T$). But then, returning to $\text{mod } B$ via the functor $\text{Hom}_A(T, -)$, we obtain a non-zero map from X_1 to X_2 (or from X_2 to X_1), an absurdity. This shows the first half of (B), the second half is dual. $\parallel$

2. THE MAIN THEOREM:

(2.1) We start by defining a set of properties:

Definition: Let k be a commutative field. A k -species

$\Sigma = (F_i, {}_iM_j)_{i,j \in I}$ with relation ideal $R = \bigoplus_{i,j} {}_iR_j$ is said to satisfy the properties (α) if it satisfies:

(α_1) the bounden graph G of (Σ, R) is a tree,

(α_2) for each vertex i , $F_i = E$, where E is a (skew) field, finite dimensional over the central subfield k , and, for each

arrow $i \rightarrow j$, ${}_iM_j = {}_E E$ with the canonical operations,

(α_3) all relations are zero-relations of length two,

(α_4) each vertex has at most four neighbours,

(α_5) if a vertex l has four neighbours, then G contains a full connected subgraph of the form:

with the indicated zero-relation,

(α_6) if a vertex l has three neighbours, then G contains a full connected subgraph of one of the forms:

with the indicated zero-relation.

The main theorem can now be stated as:

Theorem: Let k be a perfect field, and B a finite-dimensional k -algebra, then the following statements are equivalent:

- (a) B is iterated tilted of type A_n ,
- (b) the bounden species (Σ, R) of B satisfies the properties (α) ,
- (c) the Auslander-Reiten graph Γ_B of B is finite, simply connected, and satisfies (AR) and (B).

The proof of the theorem will be divided into three parts.

(2.2) Lemma: Let B be an iterated tilted algebra of type A_n , then the bounden species (Σ, R) of A satisfies the properties (α) and moreover the Auslander-Reiten graph Γ_B of B satisfies (AR) and (B).

Proof: The proof will be done by induction on the length of a going-down tilting series needed to reach B . Since all the statements are trivial for a hereditary algebra of graph A_n , we let A be an iterated tilted algebra satisfying the stated properties, and T_A be a splitting tilting module such that $B = \text{End } T_A$.

By Corollary (1.4), Γ_B satisfies (B). We now show that (Σ, R) satisfies (α) .

(α_1) The graph G of (Σ, R) is a tree:

For, if not, there is a full subgraph G' of G which is a (non-oriented) cycle. Since G has no oriented cycles, G' always contains at least one source and one sink. We may then consider two cases for G' :

Case (1): there are on G' two paths from i to j connected by a relation:

then $\text{Hom}_B(P(j), P(i)) \neq 0$, and it follows that in Γ_A , $T(j)$ lies on one of the two sectional paths ending at $T(i)$. Now, by our assumption on G' , $T(j)$ should in fact lie on both sectional paths ending at $T(i)$. This, however, is impossible. Hence, the only relations are linear relations.

We shall consider the second case after proving the properties (α_2) and (α_3) :

(α_2) For each vertex i , $F_i = E$, where E is a (skew) field, finite dimensional over the central subfield k , and, for each arrow $i \rightarrow j$, $M_j = {}_E E$ with the canonical operations.

This follows at once from the fact that the algebra A satisfies these properties, and from the construction of the sectional paths.

(α_3) All relations are zero-relations of length two.

For, let $i_1 \leftarrow i_2 \leftarrow \dots \leftarrow i_r$ be a minimal relation of length $r \geq 3$. Let $P(i_1), \dots, P(i_r)$ be the indecomposable projective B -modules, and $T(i_1), \dots, T(i_r)$ be the indecomposable summands of T_A corresponding to the above vertices: using the sectional paths determined by $T(i_2)$, we see that $r = 3$, and that $\text{Hom}_A(T(i_1), T(i_3)) = 0$.

(α_1) G is a tree, second case:

Case (2): There is a zero-relation on at least one of the non-oriented paths from the source i to the sink j : then there are two irreducible maps with domain $P(j)$ and codomains M_1 and M_2 , and $\text{rad } P(i)$ is decomposable, with indecomposable summands R_1 and R_2 . Since Γ_A is path-connected, there exist paths w_t from M_t to R_t ($t = 1, 2$). However, w_1 and w_2 are clearly non homotopic, since they factor over exactly one indecomposable summand of $\text{rad } P(i)$, a contradiction to Theorem (III.2.6).

(α_4) Each vertex has at most four neighbours:

Indeed, let i be a vertex of G , then, if j is a neighbour of i , $[T(j)]$ has to lie on one of the four sectional paths intersecting at $[T(i)]$.

(α_5) If a vertex ℓ has four neighbours i_1, i_2, i_3 and i_4 , we have a full connected subgraph of G of the form:

with the indicated zero-relations.

Indeed, considering the corresponding summands $T(i_t)$ ($1 \leq t \leq 4$), necessarily, each of $[T(i_1)]$, $[T(i_2)]$, $[T(i_3)]$ and $[T(i_4)]$ lies on a different sectional path determined by $[T(\ell)]$. We then just need to apply the remarks in (1.3).

(α_6) If a vertex ℓ has three neighbours i_1, i_2 and i_3 , we have a full connected subgraph of one of the forms:

with the indicated zero-relation.

For, necessarily, two of the $[T(l_t)]$ ($1 \leq t \leq 3$) are on the same sectional path as $[T(l)]$, while the third one is not.

There only remains to prove that B satisfies (AR). Let us first observe that for any indecomposable M_B and vertex i of the graph of B , we have $\dim M_i \leq 1$.

(i) Assume first that M is projective. By (α_1) , $\text{rad } M_B$ has at most two indecomposable summands, therefore there are at most two irreducible maps with codomain M .

(ii) Similarly, if M is injective, there are at most two irreducible maps with domain M .

(iii) Consider an Auslander-Reiten sequence:

$$0 \rightarrow L \rightarrow \bigoplus_{i=1}^r M^{(i)} \rightarrow N \rightarrow 0$$

with the $M^{(i)}$ indecomposables. Thus, $\dim \text{Hom}_B(L, N) = r-1$.

(a) If $\tau^{-1}L = N \in \mathcal{U}(T_A)$, we have $M^{(i)} \in \mathcal{U}(T_A)$ and hence $\dim \text{Hom}_A(\underbrace{L \otimes T}_B, \underbrace{N \otimes T}_B) = r-1$. The induction hypothesis implies that $r \leq 2$.

(b) Dually, $r \leq 2$ if $L \in \mathcal{X}(T_A)$.

(c) If $L \in \mathcal{U}(T_A)$ and $N \in \mathcal{X}(T_A)$, we can assume that $M^{(1)}$ is projective, and one of the $M^{(i)}$ is injective, otherwise we get a contradiction to a) or b) (namely, if the middle term has no projective summand, and $r \geq 3$, there are at least three irreducible maps with codomain $L \in \mathcal{U}(T_A)$, dually there must be an injective summand).

Assume that $M^{(1)}$ is not injective, say $M^{(1)} = P(i)$ and $M^{(2)} = I(j)$. The existence of the non-zero map $L \rightarrow I(j)$ implies that $L_j \neq 0$ and, since L is a direct summand of $\text{rad } P(i)$, we have $P(i)_j \neq 0$. But we also have $I(j)_j \neq 0$. Therefore $N_j \neq 0$, which contradicts the fact that N is a direct summand of $I(j)/\text{soc } I(j)$. Hence $M^{(1)}$ is projective-injective.

Now, let $G' = \text{Supp } M^{(1)}$. Since G is a tree, G' is of the form $j \rightsquigarrow \dots \rightsquigarrow i$ and there is no relation on G' . Furthermore, for any vertex l of G not in G' , there is no non-zero path from G' to l or from l to G' (in other words, G' is maximal for the property of not containing zero-relations). Therefore, $I(j)/\text{soc } I(j)$ and $\text{rad } P(i)$ are indecomposable and defined on G' . Thus, the given Auslander-Reiten sequence contains only modules defined on G' and is in fact given as follows:

$$\begin{array}{ccccc}
 & & M^{(1)} = P(i) = I(j) & & \\
 & \nearrow & & \searrow & \\
 L & & & & N \\
 & \searrow & & \nearrow & \\
 & & M^{(2)} = L/S(j) & &
 \end{array}$$

This completes the proof of (AR) and hence of the lemma. $\parallel$

(2.3) Lemma: Let B be a finite dimensional k -algebra whose bounden species (Σ, R) satisfies the properties (α) , then B is iterated tilted of type A_n , where n is the number of vertices in the graph G_B of (Σ, R) .

Proof: We can assume that there is a simple sink i_0 in G_B , that is, a sink with only one arrow going in, otherwise we form the opposite algebra. Now, let $\bar{w}$ be the minimal non-oriented path in G_B starting from i_0 and ending at the middle vertex i_1 of a zero-relation. By applying reflection functors and, if necessary, by passing to the opposite algebra, we can in fact assume that we have the following situation:

where G'_B, G''_B may be empty, and w is the linear oriented path from i_1 to i_0 . Let r be the number of vertices in w , and choose a numbering of the vertices of G_B such that $i_0 = 1, \dots, i_1 = r$. We shall construct a module T_B such that:

- a) T_B is a tilting module,
- b) the torsion theory $(\mathcal{T}(T_B), \mathcal{F}(T_B))$ is splitting,
- c) the bounden species (Σ', R') of $A = \text{End } T_B$ satisfies again the properties (α) ,
- d) there is exactly one relation less.

This means that we can again apply the same process to A and obtain ultimately a species satisfying (α) , but with no zero-relations, hence of type A_n , and the assertion will be proved. Define $T(i) = P(r)/P(r-i)$ for $1 \leq i \leq r-1$, and $T(i) = P(i)$ for $r \leq i \leq n$. Then, obviously, the $T(i)$ are indecomposable. Put $T_B = \bigoplus_{i=1}^n T(i)$. We shall show that T_B satisfies the required conditions. But first note that the Auslander-Reiten graph Γ_B of B has actually the form:

We note that, by construction and by the assumption on G_B , we have that $P(r)/P(r-1)$ is a direct summand of $\text{rad } P(s)$.

a) T_B is a tilting module.

(T1) If $r \leq i \leq n$, then $\text{pd } T(i) = 0$. If $1 \leq i \leq r-1$,

$T(i) = P(r)/P(r-i)$ has the projective resolution

$$0 \rightarrow P(r-1) \rightarrow P(r) \rightarrow P(r)/P(r-1) \rightarrow 0$$

and thus $\text{pd } T(i) = 1$.

(T2) It suffices to show that $\text{Ext}_B^1(T(i), T(j)) = 0$ for $1 \leq i \leq r-1$ and $r \leq j \leq n$. Now, since $\text{pd } T_B \leq 1$, $\text{Ext}_B^1(T(i), T(j)) = D \text{Hom}_B(T(j), \tau T(i))$. But $\text{Hom}_B(T(j), \tau T(i)) = 0$ by construction, since the support of $\tau T(i)$ is in fact $w \setminus \{r\}$.

(T3) We must show the existence of a short exact sequence

$0 \rightarrow P_B \rightarrow T'_B \rightarrow T''_B \rightarrow 0$, with $T', T'' \in \text{add } T_B$ for every indecomposable projective P_B . If $P_B = P(j)$, with $r \leq j \leq n$, there is nothing to show; while, if $P_B = P(i)$, with $1 \leq i \leq r-1$, the required sequence is, as in (T1),

$$0 \rightarrow P(i) \rightarrow P(r) \rightarrow P(r)/P(i) \rightarrow 0.$$

b) The torsion theory $(\mathcal{T}(T_B), \mathcal{F}(T_B))$ is splitting:

Let first M_B be an indecomposable such that $M_j = 0$ for all $r \leq j \leq n$, then $\text{Hom}_B(T, M) = 0$: indeed, it is clear that $\text{Hom}_B(P(j), M) = 0$ for all $r \leq j \leq n$, and on the other hand, if, for some $1 \leq i \leq r-1$, there exists a non-zero map $T(i) = P(r)/P(r-1) \rightarrow M$, then we also have a non-zero map $P(r) \rightarrow M$, a contradiction. Thus $M \in \mathcal{F}(T_B)$. Next, let M_B be an indecomposable such that $M_j \neq 0$ for some $r \leq j \leq n$, then:

$$\begin{aligned}
 \text{Ext}_B^1(T, M) &= \bigoplus_{i=1}^{r-1} \text{Ext}_B^1(T(i), M) \\
 &= \bigoplus_{i=1}^{r-1} D \text{Hom}_B(M, \tau T(i)) \\
 &= 0
 \end{aligned}$$

by construction. Hence $M_B \in \mathcal{J}(T_B)$. Thus:

$$\mathcal{J}(T_B) = \{M_B \mid M_j = 0 \text{ for all } r \leq j \leq n\}$$

$$\mathcal{J}(T_B) = \{M_B \mid M_j \neq 0 \text{ for some } r \leq j \leq n\}$$

c) The bounden species (Σ', R') of $A = \text{End } T_B$ satisfies (α) :

Indeed, G_A has the form:

where $G'_A = G'_B$, $G''_A = G''_B$ (for, the vertices lying inside G'_A and G''_A correspond to indecomposable projectives). Obviously, there are no maps from $P(r)/P(i)$ ($1 \leq i \leq r-1$) to projectives corresponding to vertices in G'_B , and similarly no maps from projectives corresponding to vertices in G''_B to $P(r)/P(i)$ ($1 \leq i \leq r-1$). On the other hand, all maps from projectives corresponding to vertices in G'_B to $P(r)/P(i)$ ($1 \leq i \leq r-1$) factor over $P(r)$, and all maps from $P(r)/P(i)$ ($1 \leq i \leq r-1$) to projectives corresponding to vertices in G''_B must factor over $P(s)$. This shows that G_A has the above form.

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Next, the relations which took place inside G'_B and G''_B remain the same: suppose there was a zero-relation ending at r and starting in G'_B at some vertex i' , then we get obviously a zero-relation $i' \xrightarrow{\quad} r$ in G_A . If the zero-relation ending at r started in G''_B at i'' , we get a zero-relation $i'' \xrightarrow{\quad} r$. We now claim that there are no new relations. Such new relations can only start in G''_B and end at some vertex $1 \leq i \leq r-1$, or else start at some vertex $1 \leq i \leq r-1$ and end in G'_B . Suppose $i' \in G'_B$ is such that there is a zero-relation from the vertex $1 \leq i \leq r-1$ to i' . Since every mapping $P(i') \rightarrow P(r)/P(i)$ factors over $P(r)$, such a relation must already exist between i' and r , hence inside G'_B . Similarly, if $i'' \in G''_B$ is such that there exists a relation from i'' to $1 \leq i \leq r-1$, $\text{Hom}_B(P(r)/P(i), P(i'')) = 0$ and $\text{Hom}_B(P(s), P(i'')) \neq 0$ give $\text{Hom}_B(P(r), P(i'')) = 0$, and this gives one of the zero-relations already considered before.

d) If m is the number of relations in G_B , then the number of relations in G_A is $m-1$:

Indeed, c) shows that the number of relations in G_A is not greater than m , but now, by construction, there is no relation between $r-1$ and s , that is, we have removed exactly one relation.

This completes the proof of the lemma. ||

(2.4) Proof of the theorem: Lemma (2.2) shows that (a) implies (b) and (c) (taking into account the Theorem (III.2.6)). On the other hand, Lemma (2.3) is just the proof that (b) implies (a). There only remains to show that (c) implies (b).

Let thus B be an algebra of finite representation type, such that Γ_B is simply connected, and satisfies (AR) and (B). We claim that the bounden species (Σ, R) of B satisfies the properties (α) . Consider indeed the trivial tilting module $T_B = B_B$. Applying the proof of Lemma (2.2) to T_B , we see that the bounden species of $\text{End } T_B = B$ satisfies the properties (α) . $\parallel$

3. SOME PROPERTIES OF THE ITERATED TILTED ALGEBRAS OF TYPE A_n

(3.1) Let us first observe that the algebras given by the bounden graphs of (α_5) and (α_6) do indeed appear, in fact they are even tilted algebras of A_5 and A_4 , respectively.

(3.2) Corollary: Let B be an iterated tilted algebra of type A_n , then any algebra B' whose bounden graph is a full connected subgraph of the bounden graph of B is also iterated tilted of type A_n .

Proof: Indeed, any full connected subgraph of a bounden graph satisfying the properties (α) also satisfies the properties (α) . $\parallel$

(3.3) We shall now give a lower bound for the global dimension of an iterated tilted algebra of type A_n . Let thus $\Sigma = (F_i, M_{ij})_{i,j \in I}$ be a k -species satisfying the properties (α) , and G its graph. For a vertex $i \in G$, we define the set of successors of i , Σ_i , to be the set of those vertices $j \in G$ such that there exists a non-zero path (of non-negative length) from i to j . Dually, the set Π_i of predecessors of i will be the set of those vertices $h \in G$ such that there exists a non-zero path (of non-negative length) from h to i .

Thus, if E is a (skew) field, finite dimensional over the central subfield k , and $F_i = E$ for all $i \in G$, $P(i)$ is the representation defined by $P(i)_j = E$ if and only if $j \in \Sigma_i$ (and the obvious maps between the coordinate vector spaces). Similarly, $I(i)$ is given by $I(i)_h = E$ if and only if $h \in \Pi_i$.

Let $j \longleftarrow i$ be a given arrow. Then G has the form:

We define $P_j(i)$ and $P^j(i)$ by:

$$P_j(i)_l = E \quad \text{if and only if} \quad l \in \Sigma_i \cap \Sigma_j,$$

$$\text{and } P^j(i)_l = E \quad \text{if and only if} \quad l \in \Sigma_i \cap G_i.$$

This, together with the obvious maps between the coordinate vector spaces, defines two indecomposable representations such that the sequence:

$$0 \rightarrow P_j(1) \rightarrow P(1) \rightarrow P^j(1) \rightarrow 0$$

is exact. It is clear that $P^j(1)$ and $P_j(1)$ have respective tops $S(1)$ and $S(j)$, and that $P_j(1) \subseteq \text{rad } P(1)$.

Similarly, one defines two indecomposable representations

$I^1(j)$ and $I_1(j)$ by:

$$I^1(j)_\ell = E \quad \text{if and only if } \ell \in \pi_1 \cap \pi_j,$$

$$\text{and } I_1(j)_\ell = E \quad \text{if and only if } \ell \in \pi_j \cap G_j;$$

We again have a short exact sequence:

$$0 \rightarrow I_1(j) \rightarrow I(j) \rightarrow I^1(j) \rightarrow 0$$

and $\text{soc } I_1(j) = S(j)$, while $\text{soc } I^1(j) = S(1)$. Note that $I^1(j)$ is an epimorphic image of $I(j)/\text{soc } I(j)$.

Proposition: Let B be an iterated tilted algebra of type A_n , then, if its bounden graph G_B contains a full connected subgraph of the form:

we have $\text{gl. dim. } B \geq t-1$.

Proof: We shall exhibit a B-module of projective dimension at least $t-1$. Since the species of B satisfies (α) , G_B has the form:

We notice that $P_{i-1}(1) = P^{i-2}(i-1)$ for all $3 \leq i \leq t$: this follows from the fact that $\Sigma_1 \cap \Sigma_{i-1} = \Sigma_1 \cap G_{i-1} = \Sigma_{i-1} \cap G_{i-1}$. Thus, the short exact sequences:

$$0 \rightarrow P_{i-1}(1) \rightarrow P(1) \rightarrow P^{i-1}(1) \rightarrow 0$$

where $2 \leq i \leq t$, combine to give the projective resolution:

$$0 \rightarrow P_1(2) \rightarrow P(2) \rightarrow P(3) \rightarrow \dots \rightarrow P(t-1) \rightarrow P(t) \rightarrow P^{t-1}(t) \rightarrow 0$$

Now, $P_1(2) \neq 0$ (since it has a simple top), Therefore $\text{pd } P^{t-1}(t) \geq t-1$. ||

(3.4) It is natural to ask for a criterion allowing to recognize the tilted algebras among the iterated tilted. The following result gives a necessary condition:

Proposition: Let B be an iterated tilted algebra of type A_n , then, if B is tilted, its bounden graph G_B contains no full connected subgraph of the form G_t :

with the indicated zero-relations, where $t \geq 4$, all edges between 3 and $t-2$ can be oriented arbitrarily, and there are no other zero-relations between 2 and $t-1$.

Proof: We shall suppose that G_B contains such a G_t as a full connected subgraph and show that this implies that Γ_B contains no complete slice.

(i) $I(1)$ lies on the left of the sectional path Γ from $P(t-1)$ to $P(t)$.

We start by constructing a sequence of non-zero mappings defining an oriented path from $I(1)$ to $P(t)$.

Recall that, by the argument in (3.3), we have an epimorphism $I(1) \rightarrow I^2(1)$ and a monomorphism $P_{t-1}(t) \rightarrow P(t)$. We shall now construct a representation M such that:

- a) M is indecomposable,
- b) $I^2(1)$ is a subrepresentation of M ,
- c) $P_{t-1}(t)$ is an epimorphic image of M .

Let indeed B be given by the species $\Sigma = (F_1, \dots, M_j)_{1, j \in I}$ with relation ideal R , and let the (skew) field E be as in (α_2) . Then put $M_j = E$ if and only if j satisfies one of the following three conditions:

- 1) $2 \leq j \leq t-1$
- 2) $j \in \pi_1 \cap \pi_2$
- 3) $j \in \Sigma_t \cap \Sigma_{t-1}$

This, together with the obvious maps between the coordinate vector spaces, defines obviously the wanted representation M . We thus have an oriented path in Γ_B :

$$I(1) \rightarrow I^2(1) \rightarrow M \rightarrow P_{t-1}(t) \rightarrow P(t).$$

Let us remark that $M_{t-1} \neq 0$, and $I(1)_{t-1} = 0$ imply that $M \not\cong I(1)$. Finally, $\text{Hom}_B(P(t-1), P(t)) \neq 0$ gives $P(t-1) \in \mathcal{P}_+(P(t))$. Consequently, the fact that the mapping $P(t-1) \rightarrow P(t)$ factors through M implies that M lies on the sectional path $\mathcal{f}$ from $P(t-1)$ to $P(t)$, and $I(1)$ on its left.

(ii) Γ_B contains no complete slice:

If S is a complete slice in Γ_B , it cannot contain modules lying on the left of $\mathcal{f}$. For, let $N \in S$ lie on the left of $\mathcal{f}$, we thus have an oriented path in Γ_B : $N \rightarrow \dots \rightarrow L$, with $L \in \mathcal{f}$. We may obviously assume that L is chosen such that, for every L' between N and L on this path, we have $L' \notin \mathcal{f}$.

Since S is a complete slice, it must contain a module of the form $\tau^{-m}P(t)$ (with $m \geq 0$). We then have a chain of irreducible maps:

$$N \rightarrow \dots \rightarrow L_{-1} \rightarrow L \rightarrow L_1 \rightarrow \dots \rightarrow P(t) \rightarrow \dots \rightarrow \tau^{-m}P(t).$$

Since N and $\tau^{-m}P(t)$ belong to S , all the intermediate modules are on S . In particular, we necessarily have $m = 0$. Now $L \in \mathcal{L}$, and $P(t) \in \mathcal{L}$, therefore $L_1 \in \mathcal{L}$. However, we also have $L_{-1} \notin \mathcal{L}$, thus, by (AR), $L_{-1} = \tau L_1$, and this contradicts the fact that S is a complete slice.

This argument shows that no complete slice S of Γ_B contains modules lying on the left of $\mathcal{L}$. In particular, it cannot contain $I(1)$, hence any module of the form $\tau^l I(1)$ (with $l \geq 0$). Therefore, Γ_B contains no complete slice. $\parallel$

Observe that there are iterated tilted algebras of type A_n which are not tilted and whose global dimension is two. An example is the algebra of the bounden graph:

(3.5) Finally, we shall illustrate the construction of Lemma (2.3) on one particular example, namely, we shall construct a tilting series opposite to the one of Example (III.1.3). Let A_0 be the algebra of $n \times n$ lower triangular matrices over k , and $B_0 = A_0 / (\text{rad } A_0)^2$. The bounden graph of B_0 is:

Its Auslander-Reiten graph Γ_B is:

We have seen in (III.1.3) that there are exactly two multiplicity-free tilting modules on B_0 , namely the trivial one and $T_{B_0} = S(2) \oplus (\bigoplus_{i \geq 2} P(i))$, the latter being precisely the tilting module constructed in Lemma (2.3). $B_1 = \text{End } T_{B_0}$ has the bounden graph:

that is, to say, we have removed exactly one zero-relation. Γ_{B_1} is:

Following again the construction of Lemma (2.3), we take the tilting module $T_{B_1} = P(3)/P(1) \oplus P(3)/P(2) \oplus (\bigoplus_{i \geq 3} P(i))$, whose

endomorphism ring $B_2 = \text{End } T_{B_1}$ has the bounden graph:

Repeating the same construction, we ultimately obtain

$$B_{n-2} = A_0.$$

CHAPTER (V):

ITERATED TILTED ALGEBRAS OF TYPES B_n and C_n

We shall now classify the iterated tilted algebras of types B_n and C_n over a perfect field in terms of their bounden species, and also in terms of their Auslander-Reiten graphs. Again in this chapter, we shall assume the underlying field k to be perfect, all our algebras to be connected and basic and our tilting modules to be multiplicity-free.

1. THE PREFACTOR AND POSTFACTOR SETS IN THE CASE B_n :

(1.1) Definition: Let A be a finite dimensional k -algebra and Γ_A its Auslander-Reiten graph. A point x of Γ_A is a border point if:

- (i) there exists at most one arrow α of source x ,
- (ii) there exists at most one arrow β of target x ,
- (iii) if α and β both exist, then $\beta = \sigma\alpha$.

The following properties of Γ_A will play the same role as their analogues in Chapter (IV):

(AR) There are at most two arrows in Γ_A with a prescribed source or target.

Let M, N be two indecomposable A -modules, then:

$$|\text{Irr}(M, N)_{\text{End } M}| \leq 2,$$

$$\text{and } |\text{End } N \text{ Irr}(M, N)| \leq 2.$$

Moreover, there exists a unique τ -orbit θ of Γ_A entirely consisting of border points (and called the reflecting border) such that, if $\text{Irr}(M, N) \neq 0$, then:

$$|\text{Irr}(M, N)_{\text{End } M}| = 2 \quad \text{if and only if } [M] \in \theta,$$

$$\text{and } |\text{End } N \text{ Irr}(M, N)| = 2 \quad \text{if and only if } [N] \in \theta$$

(B) If P_A is an indecomposable projective module, and $[R] \rightarrow [P]$ is an arrow of Γ_A , there is at most one arrow of target $[R]$. Dually, if I_A is an indecomposable injective module, and $[I] \rightarrow [J]$ is an arrow of Γ_A , then there is at most one arrow of source $[J]$.

Here, the second part of the condition (AR) is a description of the valuation of Γ_A : it asserts the existence of a unique τ -orbit θ of Γ_A entirely consisting of border points and such that:

a) If N is not projective, then the Auslander-Reiten sequence ending with N :

$$0 \rightarrow \tau N \rightarrow \bigoplus_{i=1}^r L_i \rightarrow N \rightarrow 0$$

(with the L_i indecomposables) has at most three middle terms, at most two of them being non-isomorphic. A given indecomposable M such that there is an arrow $[M] \rightarrow [N]$ in Γ_A has multiplicity two as a middle term (that is, $M \cong L_1 \cong L_2$) if and only if $[M] \in \theta$.

b) If M is not injective, then the Auslander-Reiten sequence starting with M :

$$0 \rightarrow M \rightarrow \bigoplus_{j=1}^s K_j \rightarrow \tau^{-1} M \rightarrow 0$$

(with the K_j indecomposables) has at most three middle terms, at most two of them being non-isomorphic. A given indecomposable N such that there is an arrow $[M] \rightarrow [N]$ in Γ_A has multiplicity two as a middle term (that is, $N \cong K_1 \cong K_2$) if and only if $[N] \in \theta$.

Observe that (AR) implies immediately that for any two non-isomorphic indecomposable A -modules X and Y , we have:

$$|\text{Hom}_A(X, Y)_{\text{End } X}| \leq 2,$$

and $|\text{Hom}_A(X, Y)_{\text{End } Y}| \leq 2.$

By Lemma (III.2.8), this implies at once that if T_A is a splitting tilting module with $B = \text{End } T_A$, and M_B is an indecomposable B -module, we have:

$$\dim M_i \leq 2$$

for any vertex i of the bounden graph of B .

(1.2) Proposition: Let A be a finite dimensional k -algebra of finite representation type satisfying (AR) and (B), then, for every indecomposable M_A , each of $\rho_+(M)$ and $\rho_-(M)$ consists of the modules lying on a union of three sectional paths of Γ_A , two of them intersecting in $[M]$.

Proof: Let M_A be an indecomposable module, we know that there are at most two arrows of Γ_A of target $[M]$: let $f_1: E \rightarrow M$ and $g_1: F_1 \rightarrow M$ be the corresponding irreducible maps. If M is simple projective, $E_1 = F_1 = 0$ (and then $\rho_+(M) = \{[M]\}$). Otherwise we can assume that $E_1 \neq 0$. We have then two cases:

(1) E_1 is indecomposable: then there exist at most two arrows of target $[E_1]$. If M is projective, (B) implies that there is at most one. If M is not projective, there exists an arrow $[\tau M] \rightarrow [E_1]$ and we have at most one other arrow. In both cases, we have at most one arrow of target $[E_1]$ whose source is not $[\tau M]$.

If such an arrow exists, we shall call $f_2: E_2 \rightarrow E_1$ the corresponding irreducible map. Otherwise the construction stops.

(2) $E_1 = E'_1 \oplus E''_1$ with $[E'_1] \in \mathcal{B}$, the construction also stops.

Inductively, assume that E_i has been constructed for all $i \leq t$, we again have two cases:

(1) E_t is indecomposable: as before, we have at most one arrow of target $[E_t]$ whose source is not $[\tau E_{t-1}]$. If such an arrow exists, we shall call $f_{t+1}^{(0)}: E_{t+1} \rightarrow E_t$ the corresponding irreducible map.

(2) $E_t = E'_t \oplus E''_t$ with $[E'_t] \in \mathcal{B}$: thus, if E_{t-1} is projective, there is no arrow of target $[E'_t]$ (by definition of a border point): thus the construction stops. If E_{t-1} is not projective, let us put $E_{t+1}^{(0)} = \tau E_{t-1}$, and $f_{t+1}^{(0)} = \sigma f_t^{(0)}: E_{t+1}^{(0)} \rightarrow E'_t$. Continuing the construction as before, we obtain a family $(E_{t+1}^{(s)}, f_{t+1}^{(s)})_{s \geq 0}$. If there exists an $s \geq 0$ such that $\text{Hom}_A(E_{t+1}^{(s)}, \tau M) = 0$, we let s_0 be the smallest such s , and put $E_{t+1} = E_{t+1}^{(s_0)}$, and $f_{t+1} = f_{t+1}^{(0)} f_{t+1}^{(1)} \dots f_{t+1}^{(s_0)}$.

The set $f(f_1) = \{[M]\} \cup \{[E_t] \mid t \geq 1\}$ (where, of course, we have to replace E_t by E'_t , if $E_t = E'_t \oplus E'_t$, with $[E'_t] \in \mathcal{B}$) is thus, by construction, a union of two sectional paths. Similarly, g_1 defines a sectional path $f(g_1)$. The uniqueness of the border $\mathcal{B}$ implies indeed that at most one of $f(f_1)$ and $f(g_1)$ consists of two disjoint sectional paths. The construction may be represented in the following figure:

Let now $N \in \mathcal{P}_+(M)$ and $h: N \rightarrow M$ be a non-zero map. Then, either h is an isomorphism, or else it factors by the right almost split map $[g_1 \ f_1]: F_1 \oplus E_1 \rightarrow M$. We might thus assume that $h = f_1 h_1$ for some non-zero map $h_1: N \rightarrow E_1$.

We then have two cases to consider: if $E_1 = E'_1 \oplus E'_1$, with $[E'_1] \in \mathcal{B}$, we have at most one arrow of target $[E'_1]$, whose source is necessarily $[\tau M]$. Let us assume that the two irreducible maps: $p_i h_1: N \rightarrow E_1 \rightarrow E'_1$ ($i = 1, 2$) (where p_1, p_2 are the canonical projections) factor through τM , then $h_1 = (p_1 + p_2)h_1$ factors through τM , hence h itself factors through τM , and this contradicts the hypothesis. Thus, one of the $p_i h_1$ is an isomorphism,

and $N \cong E_1'$. If E_1 is indecomposable, and h_1 is not an isomorphism, it factors through the right almost split map $\tau M \oplus E_2 \rightarrow E_1$.

But it cannot factor through τM , by hypothesis, then, if $E_2 \neq 0$,

there exists a non-zero map $h_2: N \rightarrow E_2$ such that $h_1 = f_2 h_2$.

By induction, either $N \cong E_i$ for some $i \leq t$, or there

exists a non-zero map $h_{t+1}: N \rightarrow E_{t+1}$ such that $h_t = f_{t+1} h_{t+1}$:

if $E_t = E_t' \oplus E_t'$ with $[E_t'] \in \mathcal{A}$ and if neither of the maps

$p_i h_t: N \rightarrow E_t'$ ($i = 1, 2$) is an isomorphism, they factor through

$E_{t+1}^{(0)}$. Hence the same is true for $h_t = (p_1 + p_2) h_t$. If

$\text{Hom}_A(E_{t+1}^{(s)}, \tau M) \neq 0$, we cannot have $N \cong E_{t+1}^{(s)}$; since by hypothesis

$N \in \mathcal{P}_+(M)$; we must then go up to E_{t+1} : that is to say, there

exists a non-zero map $h_{t+1}: N \rightarrow E_{t+1}$ such that $h_t = f_{t+1} h_{t+1}$.

This is still more evident if E_t is indecomposable.

Since for some m we have $E_m = 0$, it follows that $[N] \in \mathcal{I}(f_1)$.

Similarly, if there exists a non-zero map $h_1': N \rightarrow F_1$ with $h = g_1 h_1'$,

we have $[N] \in \mathcal{I}(g_1)$.

Finally, to show that $\mathcal{P}_+(M) \supseteq \mathcal{I}(f_1) \cup \mathcal{I}(g_1)$, we must prove

that, if $[E_t] \in \mathcal{I}(f_1)$, then the map $f_1 f_2 \dots f_t: E_t \rightarrow M$ is

non-zero. Indeed, let us assume that t is minimal such that

$f_1 f_2 \dots f_t = 0$. Since f_1 is an irreducible map, it is either a

monomorphism or an epimorphism. But the minimality of t implies

that $f_2 \dots f_t \neq 0$, hence f_1 is an epimorphism. Therefore $K = \text{Ker } f_1$ is indecomposable, since it is the kernel of an irreducible epimorphism. On the other hand, $f_1(f_2 \dots f_t) = 0$ implies the existence of a map $g: E_t \rightarrow K$ such that $f_2 \dots f_t$ is the composition of g and of the canonical inclusion $k: K \rightarrow E_1$.

$$\begin{array}{ccccccc} 0 & \rightarrow & K & \xrightarrow{k} & E_1 & \xrightarrow{f_1} & M \rightarrow 0 \\ & & \uparrow g & & \uparrow f_2 \dots f_t & & \\ & & E_t & & & & \end{array}$$

Now, we have two possibilities: either $f_1 f_2 \dots f_t$ does not factor over θ , in which case the proof is identical to the case A_n , or else it does factor over θ , which is the case we are going to consider now.

We first claim that the canonical inclusion $k: K \rightarrow E_1$ cannot factor over $f_2: E_2 \rightarrow E_1$. For, if this were the case, the non-zero map $k': K \rightarrow E_2$ would induce an amalgamated sum ("pushout"):

$$\begin{array}{ccccccc} 0 & \rightarrow & K & \xrightarrow{k} & E_1 & \xrightarrow{f_1} & M \rightarrow 0 \\ & & \downarrow k' & & \downarrow k'' & & \parallel \\ 0 & \rightarrow & E_2 & \rightarrow & L & \rightarrow & M \rightarrow 0 \end{array}$$

and the commutative square:

$$\begin{array}{ccc}
 K & \xrightarrow{k} & E_1 \\
 \downarrow k' & & \downarrow 1_{E_1} \\
 E_2 & \xrightarrow{f_2} & E_1
 \end{array}$$

would induce a unique map $\ell: L \rightarrow E_1$ such that $\ell k'' = 1_{E_1}$. In particular, k'' is a section, and E_1 is a direct summand of L . By the Krull-Schmidt theorem, this implies that the lower sequence does not split, and hence that $\text{Ext}_A^1(M, E_2) \neq 0$. This, however, is contradicted by:

$$\text{Ext}_A^1(M, E_2) = D \overline{\text{Hom}}_A(E_2, \tau M) = 0.$$

It follows that $k: K \rightarrow E_1$ must factor over the irreducible map $\sigma f_1: \tau M \rightarrow E_1$ corresponding to $f_1: E_1 \rightarrow M$. That is to say, there exists a map $i: K \rightarrow \tau M$ such that $k = (\sigma f_1)i$. In particular, since k is a monomorphism, so is i . But now $g: E_t \rightarrow K$ is a non-zero map, since $f_2 \dots f_t \neq 0$, hence $ig: E_t \rightarrow \tau M$ is also non-zero, and this contradicts the fact that $\text{Hom}_A(E_t, \tau M) = 0$.

This contradiction completes the proof of the proposition. $\parallel$

(1.3) Remarks: As in Chapter (IV), we have:

1. If $[N_1], [N_2] \in \mathcal{L}(f_1)$, either $\text{Hom}_A(N_1, N_2) \neq 0$, or else $\text{Hom}_A(N_2, N_1) \neq 0$.

2. Let A satisfy the conditions of Proposition (1.2), T_A be a tilting module, and $B = \text{End } T_A$. We assume that B is given by a bounden graph, and take i to be a vertex of this graph. If j, h are two neighbours of i , and $T(i), T(j), T(h)$ are the corresponding indecomposable summands of T_A , then either they belong to the same prefactor (postfactor) set, or not. These two cases are illustrated by the figures:

The map $T(h) \rightarrow T(i) \rightarrow T(j)$ is non-zero in the first case, and zero in the second.

(1.4) Corollary: Let A be an algebra satisfying the conditions of Proposition (1.2), T_A a splitting tilting module, and $B = \text{End } T_A$. Then the Auslander-Reiten graph Γ_B of B satisfies (B).

Proof: The proof is exactly the same as for the corresponding result in the case A_n . ||

(1.5) Lemma: Let B be an algebra given by a bounden tree G_B . If B' is a factor algebra of B whose graph is a full connected

subtree G'_B of G_B , then there exists a full, faithful and exact embedding $\varphi: \text{mod } B' \rightarrow \text{mod } B$ which is the only functor such that $\rho\varphi = 1$, where $\rho: \text{mod } B \rightarrow \text{mod } B'$ is the restriction functor. Moreover, if G'_B is maximal for the property of not containing zero-relations, φ preserves projectives and injectives.

Proof: The only such functor is defined as follows: given a B' -module $M_{B'}$, its image $\varphi(M_{B'})$ is defined by:

$$\varphi(M_{B'})_i = \begin{cases} (M_{B'})_i & \text{if } i \in G'_B \\ 0 & \text{if } i \notin G'_B \end{cases}$$

the rest is evident (cf. [19] p. 148). ||

2. THE MAIN THEOREM:

(2.1) Definition: Let k be a commutative field. A k -species $\Sigma = (F_i, {}_iM_j)_{i,j \in I}$ with relation ideal $R = \bigoplus_{i,j \in I} R_{ij}$ is said to satisfy the properties (β) if it satisfies:

(β_1) the bounden graph G of $(\Sigma, \bar{R})$ is a tree,

(β_2) there is a vertex i_0 such that $F_{i_0} = F$, and for all $i \neq i_0$, $F_i = E$, where E and F are two (skew) fields, finite dimensional over the common central subfield k , and such that $\dim_F E = \dim_E F = 2$. Also, if ${}_iM_j \neq 0$, then ${}_iM_j = {}_F E$ if $i = i_0$, ${}_iM_j = {}_E F$ if $j = i_0$ and ${}_iM_j = {}_E E$ otherwise, with the canonical operations,

(β_3) L_0 has at most two neighbours, and, if it is so, then $i \rightarrow i_0 \rightarrow j$, and there is a relation on the subspecies

$$E \xrightarrow[E \cdot F]{E} F \xrightarrow[F \cdot E]{E} E$$

μ

where μ is an epimorphism $\mu: \begin{smallmatrix} E \\ E \cdot F \end{smallmatrix} \otimes \begin{smallmatrix} F \\ F \cdot E \end{smallmatrix} \rightarrow \begin{smallmatrix} E \\ E \cdot E \end{smallmatrix}$ such that the left, and right actions on the image are induced by the ordinary multiplication of E ,

(β_4) all relations are of length two, and the only relations besides (μ) are the zero-relations,

(β_5) each vertex of G has at most four neighbours,

(β_6) if a vertex l has four neighbours, then G contains a full connected subgraph of the form:

with the indicated zero-relations,

(β_7) if a vertex l has three neighbours, there is a full connected subgraph of one of the forms

with the indicated zero-relation.

With this terminology, the main theorem can be stated as:

Theorem: Let k be a perfect field, and B be a finite dimensional k -algebra, then the following statements are equivalent:

- (a) B is iterated tilted of type B_n .
- (b) the bounden species (Σ, R) of B satisfies the properties (B),
- (c) the Auslander-Reiten graph Γ_B of B is finite, simply connected, and satisfies (AR) and (B).

The proof of the theorem will be, as in the case A_n , divided into three parts.

(2.2) Lemma: The bounden k -species of an iterated tilted algebra B of type B_n satisfies (B), and moreover, its Auslander-Reiten graph satisfies (AR) and (B).

Proof: We shall use induction on the length of a going-down tilting series needed to reach B . Since all the statements are trivially true for B_n , we let A be an iterated tilted algebra satisfying the stated properties, and T_A be a splitting tilting module such that $B = \text{End } T_A$. We already know, by Corollary (1.4), that Γ_B satisfies the branching condition (B). We shall now

show that the bounden species $(\Sigma, R)_0$ of B satisfies (β) :

(β_1) The graph G_B of (Σ, R) is a tree.

If G_B is not a tree, it has a full subgraph G'_B which is a (non-oriented) cycle. Since G_B does not contain oriented cycles, G'_B always contains at least one source and one sink. We shall thus consider three cases:

(i) Assume that there are on G' two paths from i to j connected by a relation: if this is the case

then $\text{Hom}_B(P(j), P(i)) \neq 0$, and hence in Γ_A the module $T(j)$ lies on one of the sectional paths over $T(i)$. Now, by our assumption on G' , there are two different oriented paths from $P(j)$ to $P(i)$, and this is impossible. Hence the only relations that may occur in G' are linear relations.

We shall consider the two remaining cases after proving the properties (β_2) , (β_3) and (β_4) .

(β_2) There is a vertex i_0 such that $F_{i_0} = F$, and, for all $i \neq i_0$, $F_i = E$ with $\dim_F E = \dim E_F = 2$ (where E and F are two (skew) fields finite dimensional over the common central sub-

field k). Also, if $M_{ij} \neq 0$, then $M_{ij} = \begin{smallmatrix} E \\ F E \end{smallmatrix}$ if $i = i_0$, $M_{ij} = \begin{smallmatrix} E \\ E F \end{smallmatrix}$ if $j = i_0$, and $M_{ij} = \begin{smallmatrix} E \\ E E \end{smallmatrix}$ otherwise.

Indeed, F must appear at least once in (Σ, R) , for, if $\text{End } T(i) = E$ for every indecomposable summand $T(i)$ of T , then the semisimple part of $B = \text{End } T_A$ contains only copies of E . In particular, $\text{End } T_B \not\cong A$. For the same reason, we cannot have more than one copy of F . The second assertion follows from the fact that if $M_{ij} \neq 0$, $M_{ij} \cong \text{End } P(i) \text{Hom}_B(P(j), P(i)) \text{End } P(j) \cong \text{End } T(i) \text{Hom}_A(T(j), T(i)) \text{End } T(j)$, and the construction of the sectional paths.

$(p_3) i_0$ has at most two neighbours, and, if it is so, then $i \rightarrow i_0 \rightarrow j$, and there is a relation on the subspecies

$$E \xrightarrow{\begin{smallmatrix} E \\ E F \end{smallmatrix}} P \xrightarrow{\begin{smallmatrix} E \\ F E \end{smallmatrix}} E$$

where μ is an epimorphism $\mu: \begin{smallmatrix} E \\ E F \end{smallmatrix} \otimes \begin{smallmatrix} E \\ F E \end{smallmatrix} \rightarrow \begin{smallmatrix} E \\ E E \end{smallmatrix}$ such that the left and right actions on the image are induced by the ordinary multiplication of E .

The first part is evident: indeed, if $T(i)$ is such that $\text{End } T(i) = F$, then, by (AR), $[T(i)]$ is a border point. Hence there exists exactly one sectional path ending at $[T(i)]$ and exactly one starting from $[T(i)]$, from which it follows that i has at

most two neighbours. On the other hand, these neighbours cannot be from the orbit β , hence their endomorphism rings are all equal to E . Now the subspecies $E \rightarrow F \rightarrow E$, if not bound, is of infinite representation type, we must then have a relation of length two. Let $T(h)$ and $T(j)$ be the indecomposable summands of T_A which correspond to the neighbours of F : it is clear by the construction of the sectional paths that

$$\text{End } T(h) \xrightarrow{\text{Hom}_A(T(j), T(h))} \text{End } T(j) \cong \begin{smallmatrix} E & \\ & E \end{smallmatrix},$$

and the left and right E -actions are induced by the ordinary multiplication of the field E . Thus the relation is defined by an epimorphism

$$\mu: \begin{smallmatrix} E & & E \\ E & F & F & E \\ & E & E & \end{smallmatrix} \rightarrow \begin{smallmatrix} E \\ E & E \end{smallmatrix} \quad \text{such that the left and right actions on the image are induced by the ordinary multiplication of } E.$$

(β_4) All relations are of length two, and the only relations besides (μ) are the zero-relations.

Let us take an arbitrary (linear) relation $i_1 \leftarrow i_2 \leftarrow \dots \leftarrow i_r$ of length r . Let $P(i_1), \dots, P(i_r)$ be the indecomposable projective B -modules, and $T(i_1), \dots, T(i_r)$ be the indecomposable summands of T_A corresponding to the above vertices: we then have mappings $T(i_1) \rightarrow T(i_2) \rightarrow \dots \rightarrow T(i_r)$ bound by a minimal relation. Using the prefactor and postfactor sets of $T(i_2)$, we see that $r = 3$. Moreover, we have only three possibilities:

1. $\text{Hom}_A(T(i_1), T(i_3)) = 0$: we have a zero-relation,

2. $\text{Hom}_A(T(i_1), T(i_3)) \cong_{E E}^{E E}$ and $\text{End } T(i_2) = F$: we then have the relation (μ) (cf. (β_3)),

3. $\text{Hom}_A(T(i_1), T(i_3)) \cong_{E E}^{E E}$ and $\text{End } T(i_2) = E$: we have no relation in this case.

(β_1) G_B is a tree, following:

(ii) As a second step, we show that there necessarily exists a zero-relation on the non-oriented cycle G'_B . Let us suppose indeed that we have only a linear relation (μ) on the cycle G'_B with m vertices (say):

then we claim that this cycle is of infinite representation type.

Let us indeed consider the Euclidean species Σ' :

(where $E_i = E$ for all $i = 1, 2, \dots, m$). It is of type $\tilde{A}_n$, so of infinite representation type (cf. Theorem (I.2.7)). Now, let

$\underline{X} = (X_i, \varphi_i)$ be an indecomposable representation of Σ' , we assert that such a representation induces an indecomposable representation

of G'_B . Indeed, let $Y_F = X_2 \otimes_{E_F} E_F$, and φ be the composition of the maps:

$$X_2 \otimes_{E_F} E_F \otimes_{F^E} E_F \xrightarrow{1 \otimes \varphi_2 \otimes 1} X_1 \otimes_{E_F} E_F \otimes_{F^E} E_F \xrightarrow{1 \otimes \mu} X_1 \otimes_{E^E} E_F \xrightarrow{\sim} X_1.$$

We have thus defined a representation

$$\underline{X}' = \left(\begin{array}{ccccc} & & X_1 & \xleftarrow{\varphi} & Y_F & \xleftarrow{1} & X_2 & & \\ & & & & & & & & \\ \underline{X}' & = & (X_m & \xrightarrow{\quad} & & & & & X_3) \end{array} \right)$$

of the subspecies G'_B . Clearly $\underline{X}'$ is indecomposable, because $\underline{X}$ is. Thus, to each $\underline{X}$ belonging to an infinite family of indecomposable representations of Σ' , we can associate an $\underline{X}'$, thus showing that G'_B is of infinite representation type.

(iii) We have shown that there exists at least one zero-relation on one of the two (non-oriented) paths from the source 1 to the sink j . Hence $\text{rad } P(i)$ is decomposable, with indecomposable summands R_1 and R_2 , and on the other hand there are two irreducible maps of domain $P(j)$ and respective codomains M_1 and M_2 . Since Γ_B is path connected, there exists a path w_1 from M_1 to R_1 ($i = 1, 2$). Now w_1 and w_2 are clearly non-homotopic, since they factor over exactly one indecomposable summand of $\text{rad } P(i)$ and this contradicts the fact that Γ_B is simply connected.

We have thus completed the proof of (β_1) .

(β_5) Every point of G_B has at most four neighbours.

Indeed, if i is a vertex of G_B and j a neighbour of i , $T(j)$ must be either a postfactor or a prefactor of $T(i)$.

(β_6) If a vertex l has four neighbours, then G contains a full connected subgraph of the form:

with the zero-relations $i_2 M_l \otimes M_l i_1$ and $i_4 M_l \otimes M_l i_3$.

Indeed, if l has four neighbours i_1, i_2, i_3 and i_4 , by passing to the corresponding indecomposable summands of T , necessarily each of the $T(i_t)$ ($1 \leq t \leq 4$) lies on exactly one of the sectional paths in the prefactor and postfactor sets determined by $T(l)$.

(β_7) If a vertex l has three neighbours i_1, i_2 , and i_3 , we have a full connected subgraph of G_B of one of the forms:

with the zero-relation $i_3 M_l \otimes M_l i_1$.

Indeed, two of the indecomposable summands $T(i_t)$ ($1 \leq t \leq 3$) are necessarily in the same prefactor (or postfactor) set as $T(1)$, while the third one is not.

There only remains to show:

(AR) There are at most two arrows in Γ_B with a prescribed source or target.

Let M, N be two indecomposable B -modules, then

$$|\text{Irr}(M, N)_{\text{End } M}| \leq 2,$$

$$\text{and } |\text{End } N^{\text{Irr}(M, N)}| \leq 2.$$

Moreover, the τ -orbit β of the unique indecomposable projective whose endomorphism ring is F consists entirely of border points, and is such that, if $\text{Irr}(M, N) \neq 0$, then

$$|\text{Irr}(M, N)_{\text{End } M}| = 2 \quad \text{if and only if } [M] \in \beta,$$

$$\text{and } |\text{End } N^{\text{Irr}(M, N)}| = 2 \quad \text{if and only if } [N] \in \beta.$$

(i) Let us first assume that M_B is projective, say $M = P(j)$, then, by (β_1) , $\text{rad } M_B$ has at most two non-isomorphic summands N' and N'' . Since $M \in \mathcal{U}(T_A)$, N' and N'' also belong to $\mathcal{U}(T_A)$. Passing to $\text{mod } A$ via the functor $- \otimes_{B/A} T$, we have that $N' \otimes T$, $N'' \otimes T$ and $P(j) \otimes T = T(j)$ all belong to $\tau(T_A)$, and moreover $T(j)$ is an indecomposable summand of T_A . Since we have at most two

linearly independent maps, from $N' \otimes T$ to $T(j)$ (respectively, $N'' \otimes T$ to $T(j)$), we have at most two linearly independent maps from N' to $P(j)$ (respectively, N'' to $P(j)$).

On the other hand, by construction of the prefactor sets, there exist two linearly independent maps from N' to $P(j)$ if and only if $N' \otimes T \in \mathcal{P}_+(T(j))$ is a border point of endomorphism ring F (and then, by (β_2) , $N'' \otimes T$ is necessarily of endomorphism ring E ; and hence there exists only one map from N'' to $P(j)$). But the equivalence $J(T_A) \cong U(T_A)$ implies that N' is a border point of endomorphism ring F , while N'' is not.

(ii) The same reasoning holds if M is injective.

(iii) Finally, consider the Auslander-Reiten sequence

$$0 \rightarrow L \rightarrow \bigoplus_{i=1}^r M^{(i)} \rightarrow N \rightarrow 0$$

with the $M^{(i)}$ indecomposables. It is clear that we have $r-1$ linearly independent homomorphisms from L to N .

a) If $N \in U(T_A)$, then $L, M^{(i)} \in U(T_A)$ and then there are $r-1$ linearly independent maps from $L \otimes_{B A} T_A$ to $N \otimes_{B A} T_A$. The induction hypothesis gives that $r \leq 3$ and $r = 3$ if and only if two linearly independent maps from $L \otimes_{B A} T_A$ to $N \otimes_{B A} T_A$ factor through the reflecting border of Γ_A . Hence two linearly independent maps from L to N factor through the border of Γ_B , and $M^{(1)} \cong M^{(2)}$ has endomorphism ring F .

b) The case $L \in \mathcal{I}(T_A)$ is treated in the obvious dual manner.

c) There remains the case $L \in \mathcal{U}(T_A)$ and $N \in \mathcal{I}(T_A)$. We can assume that $M^{(1)}$ is projective, and another indecomposable summand of $\bigoplus_{i=1}^r M^{(i)}$ is injective, otherwise we would obtain a contradiction to a) or b). Namely, if the middle term has no projective direct summand, and $r \geq 4$ (respectively, $r = 3$ and the three middle terms are non-isomorphic) we have four irreducible maps of codomain $L \in \mathcal{U}(T_A)$ (respectively, three irreducible maps with non-isomorphic sources). Similarly, we must have an injective summand.

Let us assume that this injective direct summand is different from $M^{(1)}$. That is, that $M^{(1)} = P(i)$ and $M^{(2)} = I(j)$, for $i, j \in G_B$. Since there exists a non-zero map from L to $I(j)$, we must have $L_j \neq 0$. On the other hand, L being a direct summand of $\text{rad } P(i)$, $P(i)_j \neq 0$. But we also have $I(j)_j \neq 0$. We claim that this implies $N_j \neq 0$. Since the dimensions of all the spaces concerned do not exceed two, this is indeed always the case except if $\dim L_j = \dim I(j)_j + \dim P(i)_j$, that is, if $\dim L_j = 2$, $\dim I(j)_j = 1 = \dim P(i)_j$. We would then have two linearly independent maps: $u_1, u_2: P(j) \rightarrow L$ and only one $P(j) \rightarrow P(i)$. Thus, combining $v: L \rightarrow M^{(1)} = P(i)$ with u_1, u_2 , we obtain $vu_1 = vu_2$ (up to a scalar multiple). But v is the inclusion of L as a direct sum-

mand of $\text{rad } P(i)$, in particular it is a monomorphism. Hence $u_1 = u_2$, an absurdity. Therefore $N_j \neq 0$. But this contradicts the fact that N is a direct summand of $I(j)/\text{soc } I(j)$. Consequently, $M^{(1)}$ is projective-injective, that is, $M^{(1)} = P(i) = I(j)$.

Let thus $G'_B = \text{Supp } M^{(1)}$. Since G_B is a tree, G'_B is of the form $j \leftarrow \dots \leftarrow i$ and is maximal for the property of not containing zero-relations. Hence $\text{rad } P(i)$ has only one indecomposable direct summand, and is defined on G'_B . Similarly, $I(j)/\text{soc } I(j)$ is indecomposable and defined on G'_B . Hence the given Auslander-Reiten sequence contains only modules defined on G'_B , and is in fact given as follows:

$$\begin{array}{ccccc}
 & & M^{(1)} = P(i) = I(j) & & \\
 & \nearrow & & \searrow & \\
 L & & & & N \\
 & \searrow & & \nearrow & \\
 & & M^{(2)} = L/S(j) & &
 \end{array}$$

Indeed, we have two cases: let i_0 be the vertex of G_B such that $F_{i_0} = F$.

1) $i_0 \notin G'_B$, then $G'_B = (E \leftarrow E \leftarrow \dots \leftarrow E)$ is of type A_n , and the result is trivial,

2) $i_0 \in G'_B$, then $i_0 \neq i, j$, otherwise G'_B would be of type B_n and would not have an indecomposable projective-injective. But this means that G'_B is in fact of the form:

$$E \leftarrow E \leftarrow \dots \leftarrow E \leftarrow F \leftarrow E \leftarrow \dots \leftarrow E$$

Now, the representation $P(i) = I(j)$ is of the form:

$$P(i) = I(j) = (E_E \leftarrow E_E \leftarrow \dots \leftarrow E_E \xleftarrow{1} E_F \xleftarrow{1} E_E \leftarrow \dots \leftarrow E_E \leftarrow E_E)$$

hence

$$L = \text{rad } P(i) = (E_E \leftarrow E_E \leftarrow \dots \leftarrow E_E \xleftarrow{1} E_F \xleftarrow{1} E_E \leftarrow \dots \leftarrow E_E \leftarrow 0)$$

and

$$L/S(j) = (0 \leftarrow E_E \leftarrow \dots \leftarrow E_E \xleftarrow{1} E_F \xleftarrow{1} E_E \leftarrow \dots \leftarrow E_E \leftarrow 0)$$

which is obviously indecomposable except if i_0 is the only vertex between i and j , and finally

$$N = (0 \leftarrow E_E \leftarrow \dots \leftarrow E_E \xleftarrow{1} E_F \xleftarrow{1} E_E \leftarrow \dots \leftarrow E_E \leftarrow E_E)$$

In other words, we have obtained an Auslander-Reiten sequence with only two middle terms. There only remains the case where i_0 is the only vertex between i and j , that is, G'_B is:

$$E \leftarrow F \leftarrow E$$

It has the Auslander-Reiten graph:

(where the modules are represented by their dimension-vectors). The

Auslander-Reiten sequence of the central square has three middle terms, two of them being isomorphic border points of endomorphism ring F . By Lemma (1.5), however, a border point for G'_B is also a border point for G_B . This completes the proof of (AR), and hence of the lemma. $\parallel$

(2.3) Lemma: Let B be a finite-dimensional k -algebra whose bounden species (Σ, R) satisfies the properties (β) , then B is iterated tilted of type B_n , where n is the number of vertices in the graph G_B of the species Σ .

Proof: As in the case A_n , we can assume that we have the situation:

for a suitable numbering of the points, where G'_B, G''_B can be empty, and w is a linear oriented path from r to 1 , maximal with the property of not containing a relation. We shall construct a module T_B such that:

- T_B is a tilting module,
- $(\tau(T_B), \mathfrak{F}(T_B))$ is a splitting torsion theory,
- the bounden species of $A = \text{End } T_B$ satisfies (β) ,
- there is at most one relation less.

We can thus apply the same process to A and ultimately obtain a species satisfying (β) but without relations, hence of type B_n .

The subgraph w is linear, and without relations, hence of type

A_r or B_r . We thus have three possible cases to consider:

(1) G_B is of the form:

where $E_i = E$ for all i ,

(2) G_B is of the form:

where $E_i = E$ for $i \neq r$, and $F_r = F$. Here, G'_B is empty, by (β_3) .

(3) G_B is of the form:

where $F_1 = F$, and $E_i = E$ for $i \neq 1$.

In the first and third case, we define:

$$T(i) = P(r)/P(r-1) \quad \text{for } 1 \leq i \leq r-1$$

$$\text{and } T(j) = P(j) \quad \text{for } r \leq j \leq n.$$

In the second case, we define:

$$T(i) = P(r) \oplus P(r)/P(r-1) \quad \text{for } 1 \leq i \leq r-1$$

$$\text{and } T(j) = P(j) \quad \text{for } r \leq j \leq n.$$

Clearly, these modules are indecomposables. Let $T_B = \bigoplus_{i=1}^n T(i)$.

We note that Γ_B has the form:

Observe that $T(1)$ is a summand of $\text{rad } P(s)$.

a) T_B is a tilting module:

(T1) If $r \leq i \leq n$, then $\text{pd } T(i) = 0$. If $1 \leq i \leq r-1$, $T(i)$

has the following projective resolution in the first and third

cases:

$$0 \rightarrow P(r-1) \rightarrow P(r) \rightarrow T(i) \rightarrow 0$$

and the following in the second case:

$$0 \rightarrow P(r-1) \rightarrow P(r) \oplus P(r) \rightarrow T(i) \rightarrow 0$$

Thus, in all cases, $\text{pd } T(i) \leq 1$.

(T2) It suffices to show that $\text{Ext}_B^1(T(i), T(j)) = 0$ for $1 \leq i \leq r-1$ and $r \leq j \leq n$. Now, since $\text{pd } T_B \leq 1$, $\text{Ext}_B^1(T(i), T(j)) = D \text{Hom}_B(T(j), \tau T(i))$. But $\text{Hom}_B(T(j), \tau T(i)) = 0$ by construction, since the support of $\tau T(i)$ is in fact $w \setminus \{r\}$.

(T3) We shall exhibit a short exact sequence $0 \rightarrow P_B \rightarrow T'_B \rightarrow T''_B \rightarrow 0$, with $T', T'' \in \text{add } T_B$ for every indecomposable projective P_B . If $P_B = P(j)$ with $r \leq j \leq n$, there is nothing to show, while if $P_B = P(i)$ with $1 \leq i \leq r-1$, the required sequences are as in (T1).

b) The torsion theory $(\mathcal{T}(T_B), \mathcal{F}(T_B))$ is splitting:

Let first M_B be an indecomposable such that $M_j = 0$ for all $r \leq j \leq n$, then $\text{Hom}_B(T, M) = 0$: indeed, it is clear that $\text{Hom}_B(P(j), M) = 0$ for all $r \leq j \leq n$, and, on the other hand, if, for some $1 \leq i \leq r-1$, there exists a non-zero map from $T(i)$ to M , there also exists a non-zero map from $P(r)$ to M , a contradiction. Thus $M \in \mathcal{F}(T_B)$. Next, let M_B be an indecomposable module such that $M_j \neq 0$ for some $r \leq j \leq n$, then:

$$\begin{aligned} \text{Ext}_B^1(T, M) &= \bigoplus_{i=1}^{r-1} \text{Ext}_B^1(T(i), M) \\ &= \bigoplus_{i=1}^{r-1} D \text{Hom}_B(M, \tau T(i)) \\ &= 0 \end{aligned}$$

by construction. Hence $M_B \in \mathcal{T}(T_B)$. Thus:

$$\mathcal{F}(T_B) = \{M_B \mid M_j = 0 \text{ for all } r \leq j \leq n\},$$

$$\mathcal{I}(T_B) = \{M_B \mid M_j \neq 0 \text{ for some } r \leq j \leq n\}.$$

c) The bounden species of $A = \text{End } T_B$ satisfies (B):

Indeed, let G_A be the bounden graph of A , then, in the three different cases, G_A takes respectively the forms:

$$(1) \quad G'_A \quad \text{---} E_r \leftarrow E_{r-1} \leftarrow \dots \leftarrow E_1 \leftarrow E_s \quad G''_A$$

where $E_i = E$ for all i ,

$$(2) \quad F_r \leftarrow E_{r-1} \leftarrow \dots \leftarrow E_1 \leftarrow E_s \quad G''_A$$

where $E_i = E$ for $i \neq r$, and $F_r = F$,

$$(3) \quad G'_A \quad \text{---} E_r \leftarrow E_{r-1} \leftarrow \dots \leftarrow E_2 \leftarrow F_1 \leftarrow E_s \quad G''_A$$

where $E_i = E$ for $i \neq 1$, and $F_1 = F$.

In all three cases, $G'_A = G'_B$ and $G''_A = G''_B$ (for, the vertices lying inside G'_B and G''_B correspond to indecomposable projectives).

Obviously, there are no maps from the $T(i)$ ($1 \leq i \leq r-1$) to projectives corresponding to vertices in G'_B , and similarly, no maps

from projectives corresponding to vertices of G''_B to the $T(i)$

($1 \leq i \leq r-1$). On the other hand, all the mappings from the pro-

jectives corresponding to the vertices of G'_B to the $T(i)$ ($1 \leq i \leq r-1$)

must factor over $P(r)$, and all the mappings from the $T(i)$ to the projectives corresponding to the vertices of G_B'' must factor over $P(s)$. This shows that G_A has the above form.

Next, the relations which took place inside G_B^* and G_B'' remain the same: suppose there was a relation ending at r and starting in G_B' at i' , then we get obviously the same relation $i' \xrightarrow{\quad} r$ in G_A . If the relation ending at r started in G_B'' at i'' , we get the same relation $i'' \xrightarrow{\quad} r$ in G_A . We claim that there are no new relations: such relations can only start at a vertex $1 \leq i \leq r-1$ and end in G_A' , or else start at a vertex of G_A'' and end at a vertex $1 \leq i \leq r-1$. Let us suppose that we have a relation from $1 \leq i \leq r-1$ to $i' \in G_A'$. Since every mapping $P(i') \rightarrow T(i)$ factors over $P(r)$, such a relation must already exist between i' and r , hence inside G_B' . Similarly, if $i'' \in G_A''$ is such that there exists a relation from the vertex i'' to $1 \leq i \leq r-1$, every non-zero map $T(i) \rightarrow P(i'')$ gives a non-zero map $P(r) \rightarrow P(i'')$, hence such a relation already exists between r and i'' . Note that in the first and second case, there is no relation anymore between $r-1$ and s . In the third case, we have a subspecies of the form $E \leftarrow F \leftarrow E$. Now, $\text{Hom}_B(T(2), T(s)) \cong \begin{smallmatrix} E \\ E \end{smallmatrix} E$, since $T(2)$ and $T(s)$ have both their supports contained in $G_B = G_B \setminus \{1\}$, which is the

graph of an iterated tilted algebra of type A_n , so we have the relation (μ) .

a) There is at most one relation less:

In the first and second cases, there is in fact exactly one relation less. In the third, we have restored one map, but substituted a relation by another. Nevertheless, it is clear that in G_A , F has two neighbours, hence, applying the process once more to A , we would have exactly one relation less.

This completes the proof of the lemma. $\parallel$

(2.4) Proof of the theorem: Lemma (2.2) shows that (a) implies (b) and (c). On the other hand, Lemma (2.3) shows that (b) implies (a). In order to show that (c) implies (b), we let B be an algebra of finite representation type, with Γ_B simply connected, and satisfying (AR) and (B). Applying the proof of Lemma (2.2) to the trivial tilting module $T_B = B_B$, we see that the bounden species (Σ, R) of $\text{End } T_B = B$ satisfies (β) . $\parallel$

3. SOME PROPERTIES OF THE ITERATED TILTED ALGEBRAS OF TYPE B_n :

(3.1) Let us first remark that the algebras represented by the bounden graphs of (β_6) and (β_7) do indeed appear, in fact, they are even tilted algebras of A_4, B_4, A_5 or B_5 .

(3.2) Let (Σ, R) be a bounden k -species satisfying the properties (β) and such that the field F is central in E . Then, by Lemma (II.1.6), for each value of the epimorphism μ in (β_3) , the resulting graph algebras are isomorphic.

(3.3) Corollary: Let B be an iterated tilted algebra of type B_n , then any algebra B' whose bounden graph is a full connected subgraph of the bounden graph of B is also iterated tilted of type A_n or B_n .

Proof: For, any full connected subgraph of a bounden graph satisfying the properties (β) must satisfy the properties (α) or (β) . $\parallel$

(3.4) Let (Σ, R) be a bounden species whose graph $G_{(\Sigma, R)}$ satisfies the properties (β) . We recall that the indecomposable projective module $P(i)$ is defined by:

$$P(i)_1 = F_1$$

$$P(i)_j = M_{j1}^C / R_j \quad \text{if } j \in \Sigma_i, j \neq i.$$

(where Σ_i denotes, as in (IV-3), the set of successors of i)

and $P(i)_j = 0$ otherwise.

Let $j \longleftarrow i$ be an arrow, $G_{(\Sigma, R)}$ has the form:

we define $P_j(1)$ by:

$$P_j(1)_\ell = M_{ij}^C / R_j \quad \text{if } \ell \in \Sigma_i \cap \Sigma_j,$$

$$P_j(1)_\ell = 0 \quad \text{otherwise,}$$

(together with the obvious maps between the coordinate vector spaces), and also $P^j(1)$ by:

$$P^j(1)_i = F_i$$

$$P^j(1)_\ell = M_{ij}^C / R_j \quad \text{if } \ell \in \Sigma_i \cap G_i,$$

$$P^j(1)_\ell = 0 \quad \text{otherwise.}$$

This gives a short exact sequence:

$$0 \rightarrow P_j(1) \rightarrow P(1) \rightarrow P^j(1) \rightarrow 0$$

where $P^j(1)$ is indecomposable, with top $S(1)$, while $P_j(1)$ is contained in $\text{rad } P(1)$.

Similarly, one defines two representations $I^i(j)$ and $I_i(j)$ such that there is an exact sequence:

$$0 \rightarrow I_i(j) \rightarrow I(j) \rightarrow I^i(j) \rightarrow 0$$

Moreover, $\text{soc } I_i(j) = S(j)$, and $I^i(j)$ is thus an epimorphic image of $I(j)/\text{soc } I(j)$.

Proposition: Let B be an iterated tilted algebra of type B_n , then, if its bounden graph G_B contains a full connected subgraph of the form:

we have $\text{gl. dim. } B \geq t-1$ (Here, the relation from $i+1$ to $i-1$ ($2 \leq i \leq t-1$) is a zero relation if $F_i \neq F$, and (μ) if $F_i = F$).

Proof: We have already seen in the case A_n that if G_B contains a full connected subgraph of the above form, with $F_i = E$ for all i , then $\text{gl. dim. } B \geq t-1$. Thus, assume that $F_i = F$ for some $2 \leq i \leq t-1$. Then, G_B has the form:

We have of course $F_j = E$ for $j \neq i$. Notice that $P_i(i+1)$ is the representation

where the coordinate space $P_i(i+1)_\ell = E_E$ if $\ell \in G_{i-1}$ is a

successor of i , and 0 otherwise. The maps between the coordinate spaces are zero maps, or identity maps. Thus, clearly,

$$P_i(i+1) = P(i) \oplus P(i)/P_{i-1}(i)$$

On the other hand, we clearly have $P_{j-1}(j) = P^{j-2}(j-1)$ for all $j \neq 1$.

Thus we have short exact sequences:

$$0 \rightarrow P_{j-1}(j) \rightarrow P(j) \rightarrow P^{j-1}(j) \rightarrow 0$$

for $j \neq 1$, and a short exact sequence

$$0 \rightarrow P_{i-1}(i) \rightarrow P(i) \oplus P(i) \rightarrow P_i(i+1) \rightarrow 0$$

which combine to give the projective resolution:

$$0 \rightarrow P_1(2) \rightarrow P(2) \rightarrow \dots \rightarrow P(i-1) \rightarrow P(i) \oplus P(i) \rightarrow P(i+1) \rightarrow \dots \rightarrow P(t) \rightarrow P^{t-1}(t) \rightarrow 0.$$

Now, it is clear that $P_1(2) \neq 0$ (since it has a simple top).

Therefore $\text{pd } P^{t-1}(t) \geq t-1$.

The cases where $F_1 = F$ and $F_t = F$ are treated in the obvious similar manner. ||

(3.5) As in the case A_n , we have the following necessary condition for an iterated tilted algebra of type B_n to be tilted:

Proposition: Let B be an iterated tilted algebra of type B_n , then, if B is tilted, its bounden graph G_B contains no full connected subgraph of the form:

with two relations as indicated (where it is assumed that $t \geq 4$, that there are no other relations, and that arrows between 3 and $t-2$ can be oriented arbitrarily).

Proof: We already know from the case A_n that G_B cannot contain a full connected subgraph of the above form if $F_1 \neq F$ for all i . We shall thus assume that G_B contains a full connected subgraph of the above form with $F_{t-1} = F$, and the relation from t to $t-2$ is μ . We claim that $I(1)$ lies on the left of the sectional path ρ from $P(t-1)$ to $P(t)$.

We first construct a sequence of non-zero mappings defining an oriented path from $I(1)$ to $P(t)$. We already know that there exists an epimorphism $I(1) \rightarrow I^2(1)$, and a monomorphism $P_{t-1}(t) \rightarrow P(t)$. We shall construct a representation M such that:

- (a) M is indecomposable or equal to $P(t-1) \oplus P(t-1)$,
- (b) $I^2(1)$ is a subrepresentation of M ,
- (c) $P_{t-1}(t)$ is an epimorphic image of M .

Such a representation M is constructed as follows:

1. $M_{t-1} = E_F$
2. $M_i = E_E \oplus E_E$ if $2 \leq i \leq t-1$ is a successor of $t-1$,

3. $M_i = E_E$ if $2 \leq i \leq t-1$ is not a successor of $t-1$,
4. $M_i = E_E$ if i is a predecessor of 2 such that the path $1 \rightarrow 2 \rightarrow i$ is non-zero,
5. $M_i = 0$ otherwise.

The maps between the coordinate spaces are the zero maps, the identity maps, and the diagonal inclusion $\delta: E \rightarrow E \oplus E$ given by $\delta(x) = (x, x)$ for $x \in E$. Clearly, M is indecomposable except if all $2 \leq i \leq t-1$ are successors of $t-1$, and the only arrow of target 2 has source 3 , in which case $M \cong P(t-1) \oplus P(t-1)$. The two other required properties are obviously satisfied. We thus have an oriented path in Γ_B :

$$I(1) \rightarrow I^2(1) \rightarrow M \rightarrow P_{t-1}(t) \rightarrow P(t)$$

(or $I(1) \rightarrow I^2(1) \rightarrow P(t-1) \rightarrow P_{t-1}(t) \rightarrow P(t)$).

Again, $M_{t-1} \neq 0$ and $I(1)_{t-1} = 0$ imply that $M \not\sim I(1)$.

Moreover, $\text{Hom}_B(P(t-1), P(t)) \neq 0$ gives $P(t-1) \in \mathcal{P}_+(P(t))$, and

the fact that $\text{End } P(t-1) = F$ while $\text{End } P(t) = E$ shows that

$P(t-1)$ is a reflecting border point, while $P(t)$ is not. Consequently,

M lies on the sectional path $\mathcal{S}$ from $P(t-1)$ to $P(t)$ and $I(1)$

on its left. We have thus proved our claim.

As in the case A_n , this implies that Γ_B has no complete slice, and consequently the algebra B is not tilted. The other

cases are treated in the obvious similar manner. ||

Again, there are iterated tilted algebras of type B_n which are not tilted and whose global dimension is equal to 2. An example is the algebra of the bounden graph

4. ITERATED TILTED ALGEBRAS OF TYPE C_n :

The same method can be used to classify the iterated tilted algebras of type C_n . In this case, the properties (AR) and (B) of the Auslander-Reiten graph Γ_B of the finite dimensional k -algebra B will be as follows:

(AR) There are at most two arrows in Γ_B with a prescribed source or target.

Let M, N be two indecomposable B -modules, then:

$$|\text{Irr}(M, N)_{\text{End } M}| \leq 2,$$

$$\text{and } |\text{End } N \text{ Irr}(M, N)| \leq 2.$$

Moreover, there exists a unique τ -orbit θ' of Γ_B entirely consisting of border points and such that, if $\text{Irr}(M, N) \neq 0$, then:

$$|\text{End } N \text{ Irr}(M, N)| = 2 \quad \text{if and only if } [M] \in \theta',$$

$$\text{and } |\text{Irr}(M, N)_{\text{End } M}| = 2 \quad \text{if and only if } [N] \in \theta'.$$

(B) If P_B is an indecomposable projective, and $[R] \rightarrow [P]$ is an arrow of Γ_B , then there is at most one arrow of target $[R]$. Dually, if I_B is indecomposable injective, and $[I] \rightarrow [J]$ is an arrow of Γ_B , then there is at most one arrow of source $[J]$.

A k -species $\Sigma = (F_i, M_{ij})_{i,j \in I}$ with relation ideal $R = \bigoplus_{i,j} R_{ij}$ is said to satisfy the properties (γ) if it satisfies:

(γ_1) the bounden graph G of (Σ, R) is a tree,

(γ_2) there is a vertex i_0 such that $F_{i_0} = E$, and for all $i \neq i_0$, $F_i = F$, where E and F are two (skew) fields, finite dimensional over k , and such that $\dim_F E = \dim E_F = 2$. Also, if $M_{ij} \neq 0$, then $M_{ij} = {}^E E_F$ if $j = i_0$, $M_{ij} = {}^E F$ if $i = i_0$, and $M_{ij} = {}^F F$ otherwise, with the canonical operations,

(γ_3) i_0 has at most two neighbours, and, if it is so, then $i \rightarrow i_0 \rightarrow j$, and there is a relation on the subspecies

where π is an epimorphism $\pi: {}^E E_F \otimes {}^E F \rightarrow {}^F F$ such that the left and right actions on the image of π are induced by the ordinary multiplication of F ,

(γ_4) all relations are of length two, and the only relations besides (π) are the zero-relations,

(γ_5) each vertex of G has at most four neighbours,

(γ_6) if a vertex l has four neighbours, then G contains a full connected subgraph of the form:

with the indicated zero-relations,

(γ_7) if a vertex l has three neighbours, then G contains a full connected subgraph of one of the forms

with the indicated zero-relation.

If (Σ, R) is a bounden k -species satisfying the above properties (γ) and such that the field F is central in E , then, by Lemma (II.1.6), for each value of the epimorphism π in (γ_3), the resulting graph algebras are isomorphic.

Theorem: Let k be a perfect field, and B be a finite dimensional k -algebra, then the following statements are equivalent:

- (a) B is iterated tilted of type C_n ,
- (b) the bounden species (Σ, R) of B satisfies the properties (γ) ,
- (c) the Auslander-Reiten graph Γ_B of B is finite, simply connected, and satisfies (AR) and (B). $\parallel$

An obvious corollary is the following:

Corollary (1): There exists a one-to-one correspondence between iterated tilted algebras of types B_n and C_n , given by simply interchanging the fields E and F . $\parallel$

Also, as in the case B_n , we have:

Corollary (2): Let B be an iterated tilted algebra of type C_n , then any algebra B' whose bounden graph is a full connected subgraph of the bounden graph of B , is also iterated tilted of type C_n or A_n . $\parallel$

Proposition (1): Let B be an iterated tilted algebra of type C_n , then, if its bounden graph G_B contains a full connected subgraph of the form:

we have $gl. \dim. B \geq t-1$ (Here, the relation from $i+1$ to $i-1$

$(2 \leq i \leq t-1)$ is a zero-relation if $F_i \neq E$, and (π) if $F_i = E$). $\parallel$

Proposition (2): Let B be an iterated tilted algebra of type C_n , then, if B is tilted, its bounden graph G_B contains no full connected subgraph of the form:

with two relations as indicated (where it is assumed that $t \geq 4$, that there are no other relations, and that arrows between 3 and $t-2$ can be oriented arbitrarily.) ||

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