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UMI

ANALYSIS AND CONTROL SYSTEM DESIGN OF AN INNOVATIVE TUNED-ROTOR INSTRUMENT

By

JOHN CAMERON OWER, B.A.Sc, M.A.Sc

A thesis submitted to
The Faculty of Graduate Studies and Research
in partial fulfillment of
the requirements for the degree of
Doctor of Philosophy

Ottawa-Carleton Institute
for Mechanical and Aerospace Engineering

Department of
Mechanical and Aerospace Engineering

Carleton University

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**ANALYSIS AND CONTROL SYSTEM DESIGN OF AN INNOVATIVE TUNED-
ROTOR INSTRUMENT**

submitted by

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In partial fulfillment of the requirements for the degree of Doctor of Philosophy

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Abstract

For many years, the tuned-rotor gyroscope has been an integral attitude sensor in spacecraft attitude control systems. This thesis considers the control of an innovative form of such a mechanism in which the rotor size is increased a thousand-fold. By steering the tilt and varying the spin rate of the rotor in such a device, it can be used as a tilting rotor momentum wheel, providing an actuator for high accuracy spacecraft attitude control. To investigate the frequent operation at de-tuned speeds and varying tilts, the non-linear equations of motion of the system dynamics are developed. Using this model, the characteristic modes are derived and a new tuning condition is proposed. The torque needed to maintain the rotor at a fixed tilt is shown to be a function of the spin rate, tilt angles and inertial properties of the gimbal and rotor combination. It is demonstrated how the additional torque required beyond this predicted value is proportional to the satellite body angular rates. This finding facilitates the use of the proposed tuned rotor instrument in the dual role as actuator and inertial sensor. Finally, by representing the two rotor tilt angles and two tilt control torques as complex quantities, the tilt dynamics of a spinning rotor is considered in the form of a single-complex-input, single-complex-output (SCISCO) transfer function. Classical single-input, single-output (SISO) control design and analysis techniques are then generalized for use with SCISCO systems. The effectiveness of this approach is demonstrated for control design of this class of gyroscopic systems. In fact this method is applicable to all two-input, two output systems whose inputs and outputs are related in quadrature.

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Nomenclature

The following is a list of symbols used throughout this document:

$\mathcal{F}_i$	i^{th} reference frame for the rotation sequence used to describe the rotor attitude in terms of gimbal referenced coordinates
$\mathcal{F}_i'$	i^{th} reference frame for the rotation sequence used to described the rotor attitude in terms of case referenced coordinates
θ_x	Angle of rotation of the gimbal about the gimbal x-axis
θ_y	Angle of rotation of the rotor about the gimbal y-axis
θ_z	Angle of rotation of motor shaft with respect to the case
$\dot{\theta}_{ref_i}$	motor spin rate command
ϕ_x	Tilt angle of spin axis about x-axis of case frame
ϕ_y	Tilt angle of spin axis about y-axis of case frame
ϕ_z	Angle of rotation of the rotor about the tilted spin axis in case coordinates
ϕ_{ref_x}	case-referenced tilt command about x-axis
ϕ_{ref_y}	case-referenced tilt command about y-axis
$\dot{\phi}_{ref_z}$	rotor spin rate command
$\bar{\omega}_b$	Angular velocity of the base (spacecraft body), expressed in the base body frame
$\bar{\omega}_r$	Angular velocity of the tuned-rotor instrument rotor, expressed in rotor frame

$\vec{\omega}_g$	Angular velocity of the tuned-rotor instrument gimbal, expressed in the gimbal frame
$\vec{\omega}_m$	Angular velocity of the tuned-rotor instrument motor inertia, expressed in the motor frame
ω_r	Spin rate of the rotor in the linearized dynamics model
ω_{b_x}	Angular velocity of the external body with respect to the inertial frame, expressed about the x-axis of case frame
ω_{b_y}	Angular velocity of the external body with respect to the inertial frame, expressed about the y-axis of case frame
ω_{b_z}	Angular velocity of the external body with respect to the inertial frame, expressed about the z-axis of the case frame
C_{ϕ_x}	Cosine of ϕ_x
S_{ϕ_y}	Sine of ϕ_y
$(\phi_x)_x$	Rotation matrix about the x-axis for ϕ_x
$(\phi_y)_y$	Rotation matrix about the y-axis for ϕ_y
T_{g_x}	Generalized control torque about gimbal x-axis
T_{g_y}	Generalized control torque about gimbal y-axis
T_{g_z}	Generalized control torque about the gimbal z-axis
T_{c_x}	Control torque about the case-referenced x-axis
T_{c_y}	Control torque about the case-referenced y-axis
T_{c_z}	Control torque about the rotor spin axis
I_{r_z}	Rotor inertia about the spin axis
I_{r_x}	Rotor inertia about the transverse x-axis
I_{r_y}	Rotor inertia about the transverse y-axis
I_{r_t}	Rotor transverse axis inertia when $I_{r_x} = I_{r_y}$

I'_r	Modified rotor transverse axis inertia, i.e. $I'_r = \frac{2I_r + I_{s1}}{2}$
I_{s1}	Gimbal inertia about the spin axis
I_{s1}	Gimbal inertia about the transverse x-axis
I_{s2}	Gimbal inertia about the transverse y-axis
I_{s1}	Gimbal transverse axis inertia when $I_{s1} = I_{s2}$
K_x	x-axis flexure spring constant
K_y	y-axis flexure spring constant
K	Flexure spring constant when both axes are equal
C_x	Gimbal cross-flexure x-axis damping coefficient
C_y	Gimbal cross-flexure y-axis damping coefficient
c_x	Gimbal cross-flexure damping coefficient when $C_x = C_y$
T	System kinetic energy
V	System potential energy
L	System Lagrangian
$\mathcal{L}$	Laplace transform operator
J	Jacobian matrix used to relate the gimbal rates to the rotor angular velocity
J'	Jacobian matrix used to relate the case-referenced coordinates rates to the rotor angular velocity
$\mathbf{x}$	State vector of gimbal-referenced angles, used to describe the general non-linear dynamic model for a tuned-rotor instrument
$\mathbf{M}(\mathbf{x})$	Inertia matrix of the gimbal-referenced dynamic model for the non-linear tuned-rotor instrument dynamics
$\mathbf{C}$	Damping matrix of the gimbal-referenced dynamic model for the non-linear tuned-rotor instrument dynamics
$\mathbf{K}$	Stiffness matrix of the gimbal-referenced dynamic model for the non-linear tuned-rotor instrument dynamics

$\mathbf{T}_g$	Vector of gimbal torques used to describe the gimbal-referenced dynamic model of the non-linear tuned-rotor instrument dynamics
$\mathbf{F}_{nl}$	Vector of non-linear torques used to describe the gimbal-referenced dynamic model of the non-linear tuned-rotor instrument dynamics
α	Traditional tuning condition for a tuned-rotor gyroscope
ω_n	Nutation frequency of a tuned-rotor instrument
ω_{n_c}	Nutation frequency of a tuned-rotor instrument obtained from the Complete Model
ω_p	Precession frequency of a tuned-rotor instrument
β	$(K_x - K_y - 2J\omega_i^2)$ normalized with respect to $2I_{\tau} - I_{s_i}$
γ	I_{s_i} normalized with respect to $2I_{\tau} - I_{s_i}$
γ_z	I_{s_z} normalized with respect to $2I_{\tau} - I_{s_i}$
θ	Complex gimbal tilt angle ($\theta = \theta_x - j\theta_y$)
$\bar{\theta}$	Conjugate of complex gimbal tilt angle ($\bar{\theta} = \theta_x + j\theta_y$)
ϕ	Complex case-referenced tilt angle ($\phi = \phi_x - j\phi_y$)
$\bar{\phi}$	Conjugate of complex case-referenced tilt angle ($\bar{\phi} = \phi_x + j\phi_y$)
T_g	Complex gimbal-referenced torque ($T_g = T_{g_i} + jT_{g_y}$)
$\bar{T}_g$	Conjugate of complex gimbal-referenced torque ($\bar{T}_g = T_{g_i} - jT_{g_y}$)
T_c	Complex case-referenced control torque ($T_c = T_{c_i} - jT_{c_y}$)
$\bar{T}_c$	Conjugate of complex case-referenced control torque ($\bar{T}_c = T_{c_i} + jT_{c_y}$)
ω_b	Complex external body angular rate ($\omega_b = \omega_{b_i} - j\omega_{b_y}$)
$\bar{\omega}_b$	Conjugate of complex external body angular rate ($\bar{\omega}_b = \omega_{b_i} + j\omega_{b_y}$)

$\hat{G}_{\mathcal{A}}(s)$	Traditional model, case-referenced transfer function relating $\phi(s)$ to $T_z(s)$
$\hat{G}_{\mathcal{A}z}(s)$	Traditional model, case-referenced transfer function relating $\phi(s)$ to $\bar{T}_c(s - 2j\omega_s)$
$\hat{G}_{\mathcal{A}}(s)$	Traditional model, case-referenced transfer function relating $\phi(s)$ to $\omega_b(s)$
$\hat{G}_{\mathcal{A}z}(s)$	Traditional model, case-referenced transfer function relating $\phi(s)$ to $\bar{\omega}_z(s - 2j\omega_s)$
$\mathbf{G}_{\mathcal{A}1}(s)$	Complete model, gimbal-referenced transfer function relating $\theta(s)$ to $T_s(s)$
$\mathbf{G}_{\mathcal{A}2}(s)$	Complete model, gimbal-referenced transfer function relating $\theta(s)$ to $\bar{T}_s(s - 2j\omega_s)$
$\mathbf{G}_{\bullet 1}(s)$	Complete model, gimbal-referenced transfer function relating $\theta(s)$ to $\omega_b(s)$
$\mathbf{G}_{\bullet 2}(s)$	Complete model, gimbal-referenced transfer function relating $\theta(s)$ to $\bar{\omega}_b(s - 2j\omega_s)$
$\mathbf{d}(s)$	Complete model gimbal-referenced characteristic polynomial
$G_{\mathcal{A}}(s)$	Complete model, case-referenced transfer function relating $\theta(s)$ to $T_g(s)$
$\tilde{G}_{\mathcal{A}}(s)$	Reduced-order, Complete model case-referenced transfer function relating $\phi(s)$ to $T_z(s)$
$G_{\mathcal{A}z}(s)$	Complete model, case-referenced transfer function relating $\phi(s)$ to $\bar{T}_z(s - 2j\omega_s)$

$G_c(s)$	Complete model case-referenced transfer function relating $\phi(s)$ to $\omega_s(s)$
$G_{c_z}(s)$	Complete model case-referenced transfer function relating $\phi(s)$ to $\bar{\omega}_z(s - 2j\omega_s)$
$d(s)$	Complete model case-referenced characteristic polynomial
$\mathbf{x}_s$	State vector for gimbal-referenced transverse-axes dynamics model, i.e. $\mathbf{x}_s = [\theta_x \quad \theta_y \quad \dot{\theta}_x \quad \dot{\theta}_y]^T$
$\mathbf{u}_s$	Input vector for gimbal-referenced transverse-axes dynamics model, i.e. $\mathbf{u}_s = [T_{z_x} \quad T_{z_y}]^T$
$\mathbf{A}_s$	Gimbal referenced transverse-axes dynamics model state matrix
$\mathbf{B}_s$	Gimbal referenced transverse-axes dynamics model control matrix
$K_{\hat{\alpha}n}$	De-tuned stiffness of the Complete model of a tuned-rotor instrument
$\hat{K}_{\hat{\alpha}n}$	De-tuned stiffness of the Traditional model of a tuned-rotor instrument
$P_{\hat{\alpha}n}(s)$	Single-input, single-output system dynamics transfer function
$P_c(s)$	Single-input, single-output forward path controller transfer function
$Q(s)$	Single-input, single-output feedback path controller or sensor transfer function
$\mathbf{P}_{\hat{\alpha}n}(s)$	Two-input, two-output transfer function matrix describing system dynamics
$\mathbf{P}_c(s)$	Two-input, two-output transfer function matrix describing forward-path controller
$\mathbf{Q}(s)$	Two-input, two-output transfer function matrix describing feedback-path controller or sensor function

$\mathbf{P}_{cl}(s)$	Closed-loop transfer function matrix for a two-input, two-output control system, consisting of forward path elements $\mathbf{P}_{fn}(s)$ and $\mathbf{P}_c(s)$ and feedback path elements $\mathbf{Q}(s)$.
$\mathbf{y}(s)$	Vector of two system outputs for general two-input, two-output control system
$\mathbf{r}(s)$	Vector of two reference inputs for general two-input, two-output control system
$\mathbf{e}(s)$	Vector of two error terms for general two-input, two-output control system
$\mathbf{u}(s)$	Vector of two control inputs for general two-input, two-output control system
$\mathbf{G}_{fn}(s)$	Two-input, two-output transfer function matrix describing the decoupled system dynamics
G_{fn1}, G_{fn2}	Two SCISCO transfer functions contained in $\mathbf{G}_{fn}$, describing system dynamics
$\bar{G}_{fn}$	Complex conjugate of the SCISCO transfer function describing the system dynamics
$\mathbf{G}_c(s)$	Two-input, two-output transfer function matrix describing the decoupled forward path controller
G_{c1}, G_{c2}	Two SCISCO transfer functions contained in $\mathbf{G}_c$, describing the forward path controller dynamics
$\bar{G}_c$	Complex conjugate of the SCISCO transfer function describing the forward path controller
$\mathbf{H}(s)$	Two-input, two-output transfer function matrix describing the decoupled feedback controller or sensor function

H_1, H_2	Two SCISCO transfer functions contained in $\mathbf{H}$ describing feedback path controller or sensor dynamics
$\bar{H}_i$	Complex conjugate of the SCISCO transfer function describing the feedback path controller or sensor function
$\mathbf{W}$	2×2 transformation matrix used to diagonalize $\mathbf{P}_{\text{dyn}}$
$\mathbf{G}_{cl}(s)$	Closed-loop transfer function matrix for a diagonalized two-input, two-output feedback control system consisting of forward path elements $\mathbf{G}_{\text{dyn}}(s)$ and $\mathbf{G}_c(s)$ and feedback transfer function $\mathbf{H}(s)$.
$\mathbf{y}'(s)$	Vector of two system outputs for a de-coupled two-input, two-output control system
$\mathbf{r}'(s)$	Vector of two reference inputs for a de-coupled two-input, two-output control system
$\mathbf{e}'(s)$	Vector of two error terms for a de-coupled two-input, two-output control system
$\mathbf{u}'(s)$	Vector of two control inputs for a de-coupled two-input, two-output control system
A_i	Magnitude of the vector from the i^{th} zero to root locus corresponding to a desired value of s
B_j	Magnitude of the vector from the j^{th} pole to root locus corresponding to a desired value of s
α_i	Phase angle describing the orientation of the vector from the i^{th} zero to the root locus corresponding to a desired value of s
β_j	Phase angle describing the orientation of the vector from the j^{th} pole to the root locus corresponding to a desired value of s
$\tilde{K}$	Complex gain with one of four values, i.e. $\tilde{K} = (j)^q K$, $q = 0, 1, 2 \text{ or } 3$
μ_i	Orientation of the i^{th} asymptote in a root locus diagram

σ_c	Location of the centre of asymptotes for a root locus diagram
i_{coil_x}	Coil current on x-axis torquer
i_{coil_y}	Coil current on y-axis torquer
i_{coil}	Complex coil current, $(i_{coil} = i_{coil_x} + j i_{coil_y})$

Chapter 1

Introduction

The earliest satellites launched into low earth orbit weighed only 50 kg. and in today's terminology would be considered small. However over the period from the early 1960's to the 1980s. the prototypical satellite became progressively heavier with masses of a few thousand kilograms and payloads which were correspondingly more complex. It was not until the late 1980s that the industrial and scientific research communities recognized the need for smaller satellites.

The recent interest in small satellites is due to a number of factors. By the 1980's, with the increased complexity of satellites, the design cycle from conception to launch had increased from 3-5 years to 5-10 years [1]. There was a tendency to solve multiple problems by integrating several payloads on a single satellite. With this increased complexity and high value of payloads, there was a greater emphasis placed on using high reliability components. Exacting manufacturing and integration procedures further increased costs. Despite these measures, failures do occur with further loss of money and time in replacing the desired payloads on orbit.

By the late 1980s. spacecraft technology had evolved to the point where sub-system components weighed and cost less, compared with similar elements of the previous two decades. Digital memory and data processing capabilities which were implemented on a 50 Kg micro-satellite in 1989 were comparable to the most complex satellites of ten years

previous, with similar trends continuing to today [2]. Costs for small satellites were reduced with "piggy-back" launches and the introduction of launch vehicles such as Pegasus [3].

Small satellite is a term used to describe a spacecraft with mass of 10 to 500 Kg. They are further categorized as either micro-satellites (10-50 Kg) or mini-satellites (50-500 Kg). As noted by Ward [1], Fleeter *et al* [2] and Ovchinnikov [4], factors discriminating them from other satellites are cost, development time and accepted risk. Typically, the cost for design, manufacture, integration, test and launch is \$5-\$10 million with the duration of the projects being 1.5 to 2.5 years. In an effort to save money, one or more of the following measures are taken: use of non-redundant designs, use of lower grade components and adoption of a less stringent quality assurance philosophy. Small satellites are used for a wide variety of applications such as scientific experiments, technology demonstrations and ecological and environmental monitoring. The lower cost has allowed smaller companies and universities to become involved in the design, manufacture and launch of micro-satellites.

Early small satellites, particularly micro-satellites, carried payloads, which required non-exacting pointing capabilities. This was due not so much to the desired scope of the mission as it was to the available forms of attitude determination and control systems (ADCS) hardware. The very restricted mass, volume and power budgets of a small spacecraft led to the use of passive or semi-passive attitude control systems (ACS), particularly for micro-satellites. Recently, the demand has increased for small satellites to carry relatively complex payloads, which require the same accurate pointing, and multiple mission modes as large spacecraft systems. This places a demand on attitude control actuators, sensors and electronics which will produce a high accuracy ADCS while meeting a stringent mass budget.

Ideally, actuator and sensor technologies are required whose size and mass can be scaled to the particular spacecraft without the performance being penalized. Unfortunately for most sensor and actuator systems, the associated mechanisms and electronics cannot be scaled in such a manner. The GyroWheel system [5] being developed by Carleton University and Bristol Aerospace Corporation is unique as it offers the potential to meet the performance, mass and cost requirements of both large and small spacecraft. Its actuator capability is like that of a double-gimbaled momentum wheel. It consists of a drive motor, which is coupled to a rotor through a two-degree-of-freedom flexure suspension, similar to that used in a tuned-rotor gyroscope (TRG). Although primarily an actuator, the use of a mechanical design similar to a TRG provides the capability to measure external angular rates. The result is a momentum actuator for accurate 3-axis attitude control of spacecraft, which is lighter and simpler than currently available systems and provides an external rate sensing capability. The use of one lightweight mechanism to provide 3-axis actuation and 2-axis rate sensing is unique in the field of spacecraft attitude control.

Examination of the GyroWheel concept has prompted research into what is termed the "General Tuned-Rotor Instrument Control Problem". This involves control of a tuned-rotor instrument (TRI) at non-zero tilts and de-tuned speeds to vary the rotor momentum while still performing its traditional role of extracting external rates. Study of this problem has resulted in the following findings:

- a) A new "Complete" SCISCO transfer function model of a tuned-rotor instrument has been developed. Use of this model has revealed the presence of fixed modes in tuned-rotor instrument dynamics and an improvement on the traditional method of tuning gyroscopes.
- b) For the class of dynamic systems for which two degrees of freedom can be combined in quadrature (e.g. gyroscopes, AC-machines), their dynamics can be described in terms of single-complex-input, single-complex-output (SCISCO)

transfer functions. A generalized approach for applying Classical control tools to SCISCO systems has been proposed in this study which facilitates effective design and analysis of gyroscopic control systems.

- c) A method for extracting external rates from a general tuned-rotor instrument has been developed and demonstrated for simultaneous momentum management/attitude sensing operation.
- d) Through investigation of the non-linear dynamics of a tuned-rotor instrument, it has been shown that direct spin control of the device's rotor is far superior to the traditional motor shaft control method.

1.1 Scope of Thesis Document

This document is divided into three parts: the first describes relevant past research and presents the tuned-rotor instrument control problem; the second deals with research into the dynamic modeling, control and derivation of external body rates for a tuned-rotor instrument; and the third describes the application of these ideas to the design and simulation testing of an initial prototype of the GyroWheel system.

Part I: Review of Research of Gimbaled-Rotor Devices and their Control

- i) **Chapter 2: Overview of Momentum Management and the GyroWheel system.**

The concept of spacecraft attitude control using momentum management is presented. Details of the GyroWheel tuned-rotor instrument mechanical layout and principle of operation are presented.

- ii) **Chapter 3: Detailed review of past research of tuned-rotor gyroscopes and other gimbaled rotor devices.**

Past research into modeling methods, dynamic analyses and control system designs for tuned-rotor gyroscopes and gimbaled momentum wheels are examined.

- iii) **Chapter 4: The tuned-rotor instrument control problem**

Based on past research, justification for conducting research of this problem is provided.

Part II: Research into Control of Tuned-Rotor Gyroscopic Devices

- i) **Chapter 5: Derivation of Tuned-Rotor Instrument Dynamics Models**

A non-linear dynamic model of a tuned-rotor instrument is derived. This 3-body model consists of the motor shaft, gimbal and rotor dynamics being acted upon by external rates and generalized control torques. A linearized form of this model is derived and then converted into a new "Complete Linear Model" of the tuned-rotor instrument.

- ii) **Chapter 6: Analysis of Dynamic Models**

Analytical expressions for the poles and zeros of the new Complete Linear Model are derived. They show that there are four characteristic modes of a tuned-rotor instrument, two of which are fixed if closed loop control is employed. Analysis of the Complete Linear Model also reveals a more effective tuning condition for tuned-rotor gyroscopes.

iii) **Chapter 7: Classical Control of SCISCO Systems**

The concept of the SCISCO transfer function is introduced and its associated properties developed. Classical Single-Input, Single-Output (SISO) control design and analysis techniques are generalized for SCISCO systems, with the modification of old rules and introduction of some new ones. Systematic approaches to applying classical SISO design tools and concepts such as root locus, Nyquist stability and frequency domain design techniques to SCISCO transfer functions are presented. Application of these ideas to the nutation damper design of a tuned-rotor gyroscope is presented.

iv) **Chapter 8: Derivation of External Rate Sensing Concept for Tuned-Rotor Instruments**

Using the derived linear dynamics model, a new algorithm for extracting external rates is developed for the unique situation where a tuned-rotor instrument is simultaneously operating as an attitude sensor and momentum management device. It is shown how the control torque required to point the rotor along a prescribed path is a function of the spin rate, the tilt angles and rates and the gimbal inertia and stiffness characteristics. The additional torque beyond this predicted level is proportional to the satellite angular rates. Based on an error analysis of the algorithm, the sub-system requirements necessary to achieve a desirable rate sensing accuracy are presented.

Part III: Application of Research Findings to GyroWheel Design**i) Chapter 9: The Effect of Spin Axis Control on the Non-Linear Dynamics of a Tuned-Rotor Instrument**

The open-loop response of the non-linear dynamics of the GyroWheel tuned-rotor instrument is obtained through simulation. The subsequent effect of introducing spin-axis control on the dynamic behaviour is presented. It is demonstrated how direct rotor spin control is superior to the traditional method of motor spin control for the general tuned-rotor instrument control problem.

ii) Chapter 10: GyroWheel Transverse Axis Control Design

Using the proposed SCISCO control system design techniques, three different transverse-axis control systems are developed for the GyroWheel. The GyroWheel performance with each of these controllers is simulated using a non-linear simulation model including the spin and transverse-axis control system. It is demonstrated that the high frequency gimbal flutter, which occurs when the rotor is tilted, can affect external rate sensing. The gimbal flutter interacts with the control system to alter the torque required to hold the spin axis at a specified non-zero tilt. The Complete Model is used to analyse this problem, leading to a filter design that remedies this problem.

iii) Chapter 11: GyroWheel External Rate Sensing Design

The issues involved in implementing the external rate-sensing algorithm are discussed. The performance of the GyroWheel external rate sensing is demonstrated through two simulation studies.

Chapter 2

Overview of Momentum Management and the GyroWheel System

Doubled-gimbaled and steerable momentum wheels have been used in geosynchronous spacecraft for a number of years. They have proven to be very effective actuators as part of a momentum-management control strategy. However, they are quite complex and heavy. The introduction of the GyroWheel tuned-rotor instrument concept by Staley *et al.* [5] addresses the issues of mass and complexity with a novel gimbal mechanism. Using a modified version of a tuned-rotor gyroscope mechanical configuration, the device functions as both a steerable momentum wheel and a 2-axis external rate sensor. This concept is unique, as it is the first attitude control device to measure external rates of the spacecraft body as well as provide momentum control.

The following section provides background in two areas:

- a) Overview of momentum management
- b) Description of the GyroWheel system.

2.1 Momentum Management

This section is an overview of momentum management based on a discussion of the subject given in [5]. Attitude control of spacecraft involves pointing a satellite at the earth, other celestial bodies or other spacecraft. The most common method of spacecraft attitude control is through the direct control of angular momentum. This is achieved by maintaining the spacecraft's momentum in a non-zero state through one of two means: spinning the spacecraft body or including spinning bodies within the spacecraft [5].

a) Spin-Stabilized Spacecraft

The simplest spin-stabilized method is the passive control technique where the entire spacecraft is spun so that its angular momentum vector remains fixed in inertial space [6]. A variation of this is dual-spin stabilization where the spacecraft has two sections spinning at different rates about the same axis. The first, termed the rotor is rotated at a high rate to provide angular momentum and gyroscopic stiffness. The second section termed the platform or stator is de-spun to keep one axis pointed toward the earth or another celestial body. The main advantages of spin-stabilized spacecraft are: simplicity of control system, the resulting thermally benign environment for the spacecraft and the use of spinning motion for sensor scanning. One disadvantage is the need to accurately control the spacecraft mass properties to ensure the desired spin direction. Although the spinning motion provides gyroscopic stiffness which helps resist disturbance torques, it also creates the need for high thruster torques (and greater fuel demand) when the momentum vector of the spacecraft is re-oriented [6].

b) Three-Axis or Body Stabilized Spacecraft

For this method, the main body of the spacecraft does not spin. Instead, internal rotors provide the angular momentum. Generally, the spacecraft can be manoeuvred more easily than spin stabilized systems and can achieve higher pointing accuracies. However such control systems are generally more complex, more expensive and have more failure modes [6].

The general tuned-rotor instrument examined in this thesis is intended for 3-axis stabilized spacecraft in either a geosynchronous earth orbit (GEO) or low-earth orbit (LEO) mission.

Momentum-based attitude control methods provide the following benefits [7]:

- i) Flight proven concept
- ii) Operation is independent of the orbit or environmental conditions
- iii) High pointing accuracies are achievable
- iv) Reliance on expendable propellant is reduced or eliminated.

2.1.1 Application of Momentum Management in LEO and GEO Spacecraft

The roles of momentum management differ only slightly between geosynchronous and low-earth orbits. Applications of momentum management for these two types of missions are:

a) Geosynchronous Satellites

These satellites are used primarily for telecommunications. The spacecraft is maintained in a near-circular orbit above the equator at an altitude, which results in an orbit period of twenty-four hours. This ensures that the satellite remains fixed in position with respect to a point on the Earth's equator. There are a number of uses for momentum management, not only once the spacecraft is in its final orbit but also during orbit acquisition. These are :

i) Transfer into Geosynchronous Orbit

The spacecraft is transferred to its final geosynchronous orbit via a highly elliptical transfer orbit. This transfer orbit is initiated through the thrust of the launch vehicle third stage. During this transfer, the spacecraft is spin stabilized. As thrust is applied, nutation can occur. Based on attitude information obtained from sensors such as gyroscopes, the momentum state of the internal rotors can be controlled so that the nutation is removed.

ii) Acquisition of Final Geosynchronous Orbit

To insert the spacecraft into its final orbit, the spacecraft rotation rate must be removed using thrusters before normal pointing can begin. This is based on sensory information from sun sensors, earth sensors and gyroscopes. Under some conditions, it is possible for the spacecraft to switch directly to its non-spinning on-orbit state through the transfer of spacecraft body angular momentum to the internal rotor momentum.

iii) On-orbit Pointing

Once on-orbit, the attitude control system must maintain pointing in the presence of environmental disturbance torques, the most significant being that due to solar radiation pressure. As this disturbance is small, it is possible to provide full three-axis control while actually measuring

attitude-pointing errors on only two axes. The yaw control axis occurs along the line joining the spacecraft with the earth's centre. Rotations about this axis are difficult to measure on a continuous basis. However, if the rotor angular momentum along an orthogonal axis is sufficiently large, the yaw rotation error can be constrained to remain within an acceptable envelope, through a form of gyrocompassing.

Direct control of the other two axes, the roll and pitch axes, is accomplished by managing the internal angular momentum state. Thus, for example, by torquing between the spacecraft body and a pitch rotor, the rotor can be made to change speed in one direction and the spacecraft will change speed in the opposite direction. Similar control actuation can be obtained by simply tilting a spinning rotor in the direction of the desired torque. The increase in total angular momentum along the desired axis through an augmentation of the rotor angular momentum along this same axis is equivalent to applying a control torque [5].

b) **Low-Earth Orbit Satellites**

Low-earth orbit satellites are used for a variety of applications from remote sensing, aeronomy (examination of the earth's atmosphere), navigation and communication. Usually, the LEO satellite is injected directly into its final orbit by the launch vehicle, removing the need for a transfer orbit.

At a much lower altitude than GEO spacecraft, the Earth's magnetic field is sufficiently intense that it can be used as both an attitude-sensing source (like a magnetic compass) or as a source of actuation. In addition, the orbital angular velocity is very much larger than in a geosynchronous orbit. Therefore, under normal modes of operation, the yaw error can be constrained with a smaller rotor angular momentum. However, during orbit adjust manoeuvres, it is necessary to

measure all three attitude angles just as in the case of the geosynchronous satellite.

2.1.1.1 Momentum Management Actuators

Presently, there are two main methods employed to obtain high accuracy three-axis attitude control [5,8]. Each of these methods involves the control of stored angular momentum.

a) Multiple-Reaction Wheel Configuration

Three reaction wheels are mounted within the spacecraft in an orthogonal triad, with the total angular momentum magnitude and direction controlled through the variation of the spin rate in each wheel. To provide redundancy, an additional wheel must be supplied for each axis, adding considerable cost and weight. A simpler redundant configuration involves using four wheels mounted in a skewed configuration. In the event of a single wheel failure, the remaining three wheels can be used without degraded performance. The multi-reaction wheel configuration is generally heavy, complex, and particularly undesirable for small spacecraft given the required prime and redundant sets of wheels, each driven by an individual electronics box.

b) Double-Gimballed Momentum Wheel

A rotor mounted on a two-axis gimbal system can replace a complement of three orthogonal mounted reaction wheels. The angular momentum vector can be steered through the control of the gimbal angles and the spin rate of the rotor. A fully redundant system is achieved with two of these gimballed wheels. Figure

2-1 shows a typical double-gimballed wheel [9]. The rotor is mounted on a table whose two axes of tilt are controlled through low-level control of three linear jackscrews. Stepper motors drive the jackscrews, with their extension sized to produce a ± 6 -degree rotation about the two axes orthogonal to the rotor spin axis.

This approach appears superior to the multiple-wheel concept as comparable three-axis pointing control can be achieved with a single wheel. However, the jackscrew actuators operate in discrete steps, which limits the overall pointing accuracy and puts a premium on good nutation control. The double-gimballed wheel is also complex, employing a high number of mechanisms that results in increased mass and cost and lower reliability. Consequently, this approach is particularly ill suited for small spacecraft.

Figure 2-1: Double-Gimbaled Momentum Wheel [9]

2.2 The GyroWheel System

The GyroWheel system provides the same function as the double-gimbaled momentum wheel discussed above, but without its attendant disadvantages of high mass, cost and complexity. The following section describes the GyroWheel mechanical configuration, principle of operation, and method used for the determination of external rates. Finally, the performance requirements for the steerable momentum wheel and external rate sensing capabilities are provided.

2.2.1 System Configuration

An isometric view of the GyroWheel system is shown in Figure 2-2, with a cut-away isometric view shown in Figure 2-3. The system layout drawing is shown in Figure 2-4. The system consists of five main elements

- i) Spin Motor
- ii) Gimbal Assembly
- iii) Rotor
- iv) Tilt Sensors
- v) Torque Coils

Examining Figure 2-4, the GyroWheel rotor is coupled to the drive shaft of the spin motor through the gimbal assembly. Adjacent to the rotor is a stationary circular wall, termed the pick-off stand. Positioned at 90 degree intervals around the pick-off stand are the angular pick-offs. Within the rotor is an annular cavity with permanent magnets attached to the inner and outer wall. Mounted to the stationary housing and arrayed at 90 degree intervals are torque coils, which protrude into the rotor cavity. More detailed descriptions of the six components, and their unique characteristics, are provided below.

a) Spin Motor

A brushless DC motor spins the gimbal and the rotor. The motor is designed as part of the GyroWheel base, with the stator integrated into the stationary housing. The permanent magnets are mounted on the drive shaft, with the shaft supported by two duplex pair bearings.

b) Gimbal Assembly

The gimbal assembly shown in Figure 2-5 consists of a hollow cylindrical section with two pairs of crossed-flexure pivots. The axis of rotation of the two pairs of pivots is orthogonal to each other and the drive shaft. The drive shaft is coupled to the inner gimbal ring through a web structure and the outer gimbal is coupled to the rotor via the rotor-mounting flange. As shown in Figure 2-6, the cross-flexure pivot consists of two thin beam-like strips, mounted in a cross-like configuration. As the body attached to the top face rotates relative the bottom face, each of the two flexures will bend such that the intersection point of the two flexures is maintained at the center of rotation.

The gimbal assembly has been designed to accept deflections of 10 degrees about each crossed-flexure pivot axis. This is far larger than the typical maximum deflection of 0.1 degrees found in tuned-rotor gyros.

c) Rotor

The dimensions of the rotor are sized to maximize rotor spin-axis mass-moment-of-inertia while minimizing mass. Near the outer diameter of the rotor is an annular cavity where two permanent magnet rings are mounted on the inner and outer walls. To accommodate the 7-degree spin axis tilt required for momentum

vector steering, the outer surface of the rotor has the shape of a spherical section. This ensures that the gap between the rotor surface and the angle pick-offs is constant as the spin axis is tilted. The requirement for the constant gap is driven by the tilt sensor design.

d) Tilt Sensor

The tilt sensor functions as both a momentum axis tilt sensor and rotor spin rate sensor. A unique tilt sensor design is required for the following reasons:

- i) Unlike a tuned-rotor gyro, the tilt cannot be determined by measuring the gap between the bottom of the rotor and the stationary case because of the large tilt angles.
- ii) As shown in the discussion of dynamic control of tuned-rotor instruments in Chapter 9, the rotor spin rate rather than the motor spin rate must be controlled in order to obtain a good external rate-sensing estimate. Therefore, a resolver or other tachometer-like device on the input motor shaft is not a suitable spin rate sensor. The tilt sensor design allows the rotor spin rate to be measured even when the spin axis is tilted with respect to null.

Machined around the circumference of the rotor outer surface is a saw tooth pattern as shown in Figure 2-2 and Figure 2-3. Tilt sensor heads are positioned at 90, 180, 270 and 360 degrees around the tilt-sensor stand. The gap between the sensor head and rotor surface increases as the leading edge of each triangular pattern traverses the sensor and then decreases as the trailing edge is detected. The sensor detects the gap change using a current carrying coil. The size of the gap between the surface of the rotor and this coil will influence its current level.

This can be seen in Figure 2-7. The current waveform is not an ideal square wave due to lagging effects of the combined inductance and resistance in the coil.

The relative difference in readings between opposing sensor heads is a measure of tilt about an axis orthogonal to the line joining the heads and the spin axis. The waveform outputs of a sensor pair for a zero, positive and negative tilt are shown in Figure 2-8.

Examining Figure 2-8, one can see the principle of operation of the tilt sensor.

i) Case 1: Zero Tilt

When the tilt is zero, the rotor passes the sensor head such that the saw tooth pattern is traversed at its midpoint. For the zero tilt case, the width of the saw tooth is equal to the space between consecutive peaks for both the positive and negative sensors. The resulting sensors output a positive and negative voltage square-wave which when added together produces a zero voltage.

ii) Case 2: Positive Tilt

Here the rotor is tilted up at the “+ve” sensor head and down at the “-ve” sensor head. Therefore, the period during which the “-ve” sensor has a positive voltage is greater than in the zero-tilt case. Correspondingly, the period during which “-ve” sensor has a negative voltage is less than in the zero-tilt case. Adding the output waveforms of each sensor results in a square wave pattern with an average voltage that is positive and proportional to the tilt.

iii) Case 3: Negative Tilt

In this case, the output waveform produced from the sum of the positive and negative waveforms is a square wave with a negative average voltage.

Spin rate can also be determined using the tilt sensors. Based on the tilt of the rotor, the time taken between consecutive leading edges of the sensor pattern can be used to obtain spin rate.

It should be noted that both the tilt and spin rate measurements have a discrete output unlike traditional tilt and spin rate measurements. The update frequency of the tilt and spin measurement is dependent on the spin rate and the number of saw tooth patterns machined on the rotor outer surface.

e) Torque Coils

Figure 2-3 and Figure 2-4 shows the torquer coil, which is attached to the stationary housing and is positioned between inner and outer permanent magnet rings. By applying a voltage across the coils, the resulting current creates a magnetic dipole moment. The permanent magnets are radially magnetized with the resulting magnetic field interacting with the magnetic dipole moment to create an axial (parallel to spin axis) force on the rotor near its outer edge. By applying the opposite signed voltage on a coil positioned on the opposite side of the rotor, a torque can be applied on the rotor about either one of the two axes perpendicular to its spin axis. As the forces applied on opposite sides of the rotor are equal and have opposite sign, there is no net axial force acting on the rotor.

The torque scale factor relates coil current to the torque generated by the torquer. The large tilts of the GyroWheel rotor increase the likelihood that the torque scale

factor will not be constant over the ± 7 degree tilt operating range. Accurate characterization of the torque scale factor is critical to the accuracy of the external rate sensing capability.

Figure 2-2: The GyroWheel System

Figure 2-3: Internal View of GyroWheel System

Figure 2-4: GyroWheel System Layout

Figure 2-5: GyroWheel Gimbal Assembly

Figure 2-6: View of Crossed-Flexure Pivot in GyroWheel Gimbal

Figure 2-7: Tilt Sensor Waveforms

Figure 2-8: Combined Waveforms from a Pair of Tilt Sensors

2.2.2 Principle of Operation

2.2.2.1 Steerable Momentum Wheel

The nominal spin rate of the GyroWheel is chosen to coincide with the tuned speed. The concept of tuned speed is described in Section 3.1, with the mathematical basis for the tuning condition shown in Section 6.4. To vary the magnitude of the rotor momentum vector, the rotor spin rate is controlled over a range of ± 15 percent of the tuned speed. To vary the orientation of the momentum vector with respect to the GyroWheel case, the tilt of the spin axis is varied over a range of ± 7 degrees about the two orthogonal tilt axes. The system will frequently operate in a de-tuned condition and at a non-zero tilt. To avoid spin rate saturation or tilt saturation, the ACS of the spacecraft will periodically, or on a continuous basis dump momentum. This is achieved by generating external torques on the spacecraft using thrusters or magnetic torque rods. At the same time, the GyroWheel can be commanded back to its nominal momentum state.

Finally, from the point of view of failure tolerance, the GyroWheel system is superior to other gimballed wheels. In case of a failure of the torquer coils, the rotor spin axis will cone inwards until the tilt is at null. At this point, the GyroWheel will function like a fixed momentum wheel with zero tilt, which still provides gyroscopic stabilization of the spacecraft about two axes. In this configuration, the influence of the flexible suspension in the presence of disturbance torques will have to be examined.

2.2.2.2 External Rate Sensing

In a tuned-rotor gyroscope, the tilt angle is regulated at zero tilt and the torque required to maintain the rotor at this position is proportional to the angular velocity of the external body. In the GyroWheel, the torque coils acting within the equivalent of a TRG re-

balance control loop will be used to steer the rotor spin axis according to the demand of the spacecraft ACS.

As described in Chapters 8 and 11, the external rates of the general tuned-rotor instrument are determined based on an accurate characterization of the torques required to steer the rotor momentum vector along a time-varying tilt and spin-rate profile. The difference between the measured control torque and the predicted torque is a measure of the external rates of the spacecraft base body.

2.2.3 Performance Requirements

Listed below are the performance goals for both the momentum management and external rate sensing capabilities for an engineering model GyroWheel:

a) Momentum Management

i)	Nominal momentum:	1.0 N·m·sec
ii)	Bias momentum range:	0.85 – 1.15 N·m·sec
iii)	Nominal spin rate (tuned speed):	1500 rpm
iv)	spin rate range:	1300 – 1700 rpm
v)	spin axis tilt range:	± 7 deg about each transverse axis
vi)	control torque:	0.04 Nm about each transverse axis
vii)	spin control bandwidth:	10 Hz
viii)	Transverse-axes control bandwidth:	10 Hz

b) External Rate Sensing

i)	Rate measurement accuracy:	0.1 degree/hour about two axes
ii)	Rate measurement range:	To be determined (degree/hour)
ii)	Sensor output bandwidth:	0.1 Hz

There are a couple of points to note regarding these requirements. When operating at its nominal spin rate and zero tilt, the GyroWheel is operating in the same manner as a traditional tuned-rotor gyroscope. The GyroWheel spin and transverse-axes control bandwidths have been specified to be two orders of magnitude higher than the rate sensing bandwidth. This separation between rate sensing and spin/tilt control bandwidths has been made to ensure that only the steady state behaviour of the spin and transverse-axes control systems influence the rate-sensing estimate.

Chapter 3

Review of Gimbaled-Rotor Devices

The overview of the GyroWheel tuned-rotor instrument in Chapter 2 provides a useful reference in critically examining past research into gimbaled-rotor devices and their use in spacecraft attitude control systems. The following chapter examines the related technologies of tuned-rotor gyroscopes and steerable momentum wheels from three points-of-view: dynamic modeling, control design approaches, and methods for obtaining external rate estimates. This provides insight into those areas that must be further investigated in order to address the General Tuned-Rotor Instrument Control Problem.

3.1 Gimbaled-Rotor Dynamics

Given the similarity of the GyroWheel mechanical system with that of a TRG, an appreciation of tuned-rotor gyro dynamics is important in deriving an appropriate model for control system design. A tuned-rotor gyro, shown schematically in Figure 3-1 consists of a rotor, spun by a drive motor through a gimbal assembly made up of a gimbal ring and two flexure joints. The axes of the two pairs of flexure pivots are orthogonal and intersect at a common point on the spin axis. The gimbal assembly functions like a Hooke's joint, facilitating the transmission of rotary motion from the spin motor to the rotor when the input and output spin axes are not parallel. The orientation of the rotor is measured with electromagnetic pickoffs. The gap between the rotor and case surface is

proportional to the tilt of the rotor and can be adjusted through torques generated by a set of torquer coils. In a strap-down TRG, the torques required to maintain the rotor at the null orientation are a measure of the external rates of the body on which the sensor is mounted.

To illustrate the idea of the “tuning condition”, the motion of the gimbal during the operation of a tuned-rotor gyroscope is shown in Figure 3-2. The rotor spin axis is held at a non-zero tilt with respect to the input spin axis. At the start of an input rotation, the inner flexure pivot is deflected and the outer flexure pivot is undeflected. After a quarter rotation of the input, the inner flexure is now undeflected and the outer flexure is twisted. Over the course of one input rotation, both the inner and outer flexure will exhibit two cycles of twisting and relaxing. Their displacement profile with respect to time will be sinusoidal with the frequency of oscillation being twice the spin frequency. The inner flexure deflection response will be ninety degrees out of phase with the outer flexure displacement response. The two displacement profiles can be considered to be related in quadrature.

The oscillations about the inner and outer gimbal produce a fluttering motion of the gimbal ring. It is this fluttering motion, which can be exploited to dynamically tune the gyro. Due to the positive spring rate of the flexure pivot torsional springs, cyclic torques result. The fluttering motion in combination with the angular momentum of the gimbal about its spin axis also produces a gyroscopic precession torque. This precession torque is proportional to the deflection of the rotor spin axis and as such can be viewed as a negative spring rate. The system becomes dynamically tuned when the positive spring rate and negative spring are designed to be of equal value. When in this tuned condition, the rotor behaves as a freely spinning body, remaining inertially fixed regardless of the motion of its case.

Figure 3-1: Schematic Diagram of a Tuned-Rotor Gyroscope

Figure 3-2: Gimbal Motion in a Tuned-Rotor Gyroscope [10]

Dynamic models of tuned-rotor gyroscopes [11,12,13,14] describe the mechanical system in the schematic form shown in Figure 3-3. The TRG consists of three bodies: a rotor, a gimbal and a motor shaft/rotor. The flexure pivots and input drive bearings provide constraints such that there are three degrees of freedom: motion about the two flexure pivots and about the input spin axis. A further simplification is made by assuming that the input rate is constant. This is based on the observation that dynamic coupling between the motion of the rotor about the spin axis and the tilt of this axis is negligible when the spin axis tilt is small. This reduces the dynamic model to one describing the motion of the rotor and gimbal with two degrees of freedom. For the purposes of later investigations in this dissertation, such a model will be termed a “two-body” dynamic model.

Figure 3-3: Constituent Bodies of Tuned-Rotor Gyroscope [13]

Until recently, all two-body dynamic models presented in the literature were linear. In fact these models were derived using linear kinematics, rather than linearizing the non-linear dynamics model. Lipman [11] developed one of the first linear dynamic models of a tuned-rotor gyroscope. Assuming small angular motions of the spin axis about null, he employed a Lagrangian dynamics approach to derive two second-order linear differential equations, describing the motion of the gyro rotor with respect to its case. However, he did not include the inertial effects of the gimbal in this treatment. Howe and Savet [15] included the effect of the gimbal dynamics in their model and presented the first version of the tuning condition for tuned-rotor gyroscopes.

Craig [12] extended this modeling approach, determining the equations of motion of a multi-gimbal TRG, including damping coefficients for the torsional flexures and coulomb friction drag torques. A transfer function model of a TRG was developed which can be used for both rebalance loop servo design and analysis of errors caused by angular inputs acting at twice the spin frequency. This model has served as a benchmark for tuned-rotor gyro analysis and control system design. In addition, Craig extended the tuning condition to include the effect of multiple gimbals.

The models used to design control systems for tuned rotor gyroscopes generally assume that the system is tuned. The gimbal dynamics are ignored with the model representing a freely spinning rotor. Describing the motion of the spinning body in a body-fixed frame, the rotor dynamics can be described using Euler's equation [16]:

$$\begin{aligned} I_x \dot{\omega}_x - \omega_y \omega_z (I_y - I_z) &= T_x \\ I_y \dot{\omega}_y - \omega_x \omega_z (I_z - I_x) &= T_y \\ I_z \dot{\omega}_z - \omega_x \omega_y (I_x - I_y) &= T_z \end{aligned} \tag{3-1}$$

where

$\omega_{r_x}, \omega_{r_y}, \omega_{r_z}$	angular velocities of rotor expressed in x, y and z axis of its body frame
$I_{r_x}, I_{r_y}, I_{r_z}$	mass moment of inertia, expressed in the rotor body frame (principal axis of inertia assumed coincident with body frame)
$T_{r_x}, T_{r_y}, T_{r_z}$	control torques acting on rotor, expressed in body frame

Assuming that the inertias about the two non-spin axes (I_{r_x}, I_{r_y}) are equal, the equations in (3-1) simplify to the form:

$$\begin{aligned}
 I_{r_z} \dot{\omega}_{r_x} - (I_{r_z} - I_{r_x}) \omega_{r_y} \omega_{r_z} &= T_{r_x} \\
 I_{r_z} \dot{\omega}_{r_y} - (I_{r_z} - I_{r_x}) \omega_{r_x} \omega_{r_z} &= T_{r_y} \\
 I_{r_z} \dot{\omega}_{r_z} &= T_{r_z}
 \end{aligned}
 \quad \begin{array}{l} \text{transverse-axes dynamics} \\ \text{spin dynamics} \end{array}
 \quad (3-2)$$

where

$$I_{r_z} = I_{r_x} = I_{r_y} \quad (3-3)$$

If the rotor spin rate ω_{r_z} is held constant at the tuned speed ω_{tuned} , the first two equations in (3-2), describing the motion of the rotor spin axis tilt become decoupled from the motion of the rotor about the spin axis. The problem of control can then be partitioned into two parts: transverse-axes control and spin control.

Lipman [11] makes use of a device proposed by Thomson [17] for the analysis of the motion of freely spinning rigid bodies. For the transverse-axes dynamics described in [17], Thomson proposes the "complex method of analysis on small angular motions of a

spinning rigid body." Defining the complex rotor angular velocity ω_r and the complex control torque T_r as

$$\begin{aligned}\omega_r &= \omega_{r_1} - j\omega_{r_2} \\ T_r &= T_{r_1} - jT_{r_2}\end{aligned}\tag{3-4}$$

the first momentum axis equation in (3-2) can be added to $j = \sqrt{-1}$ times the second momentum axis equation in (3-2). The result is the following complex differential equation

$$I_{\dot{r}} \dot{\omega}_r - j(I_{\dot{r}_2} - I_{\dot{r}_1}) \omega_{r_{conj}} \omega_r = T_r\tag{3-5}$$

There are a number of benefits to expressing the transverse-axes dynamics in such a form. The number of dynamic states has been reduced in half, by retaining only one element of each complex conjugate pair of dynamic roots. Analysis and manipulation of a single differential equation with complex coefficients is also far easier than working with a system of two differential equations with real coefficients. Craig [12] extended Lipman's work, showing that a single complex differential equation could be derived, which included the effect of gimbal and flexure dynamics.

If the effects of flexure stiffness and gimbal dynamics are included in the transverse-axes dynamic equations of (3-2), the linearized equations of motion as detailed in [12] take the following general form:

$$m_1 \ddot{\theta}_x - c \dot{\theta}_y + k_1 \theta_x = T_{g_1}(t)\tag{3-6}$$

$$m_2 \ddot{\theta}_y - c \dot{\theta}_x - k_2 \theta_y = T_{g_2}(t)\tag{3-7}$$

where (θ_x, θ_y) are the gimbal angles, (T_{g_1}, T_{g_2}) are the generalized control torques acting about the two gimbal angles, (m_1, m_2) are inertia coefficients, c is the common damping term and (k_1, k_2) are stiffness terms. The stiffness, damping and inertia terms are a

function of the rotor and gimbal mass properties, the flexure stiffness and the spin rate. For such a coupled, two degree-of-freedom system, there will be two complex-conjugate pairs of eigenvalues. These are evident when observing the motion of the system in the spinning body frame. A more useful form for the dynamics is one that describes the motion in terms of case-referenced tilt angles of the momentum axis. By transforming these equations into a form expressed in non-spinning coordinates, most analyses [12] show that there are two characteristic roots; one at the nutation frequency and the other at zero frequency.

However Staley *et al.* [13] note that by retaining all terms in the dynamics equations through the transformation process, there are in fact four unique characteristic frequencies. The first of these modes occurs at nearly twice the spin frequency. The second is at the nutation frequency as identified in previous analyses. A third appears as a beat frequency caused by the close proximity of the nutation frequency to the twice-spin frequency terms. The final mode occurs at a very low frequency, but not zero frequency as the simplified linear model in [12] indicates.

Little work has been reported in the literature on the derivation of detailed non-linear simulation models of TRGs. There are some challenges in achieving accurate simulation models of such systems. It is difficult to reproduce some drift characteristics found in hardware tests. The difficulties in achieving perfect mass balancing (Mansour and Lacchini [18]) due to manufacturing and assembly tolerances cannot always be reflected in the appropriate choice of mass parameters in a simulation model. However, Cain [14] has developed a detailed non-linear “two-body” simulation model of the CSG-2 TRG, which produced responses extremely close to that obtained from hardware testing of this device. An important result from simulation studies of this model is that the precession frequency varies with spin-axis tilt. Simulating the dynamics of a CSG-2 tuned rotor gyroscope, the precession frequency was found to increase from 0.013 Hz at 1-degree tilt to 0.57 Hz at 10 degrees. More recently, Cain [19] developed a “three-body” simulation

model of a tuned-rotor gyroscope with no rebalance or spin control system active. The motion of the motor shaft and its coupling with the gimbal and rotor dynamics is accounted for. He observed that the precession frequency decreases slightly as the tilt of the rotor spin axis increases, rather than increasing as was found with the two-body model.

As shown in the work by Kalley [20] and Shamma and Michaelis [21], simplified linear models have been used in the design of double-gimbaled momentum wheels (DGMW). Despite the fact that the mechanical concept of the GyroWheel can be thought of as a double gimbaled wheel, the gimbal joints in the DGMW suspension are active with complex drive and transmission dynamics. Consequently, these analyses provide little insight into effective ways of modeling the GyroWheel dynamics.

3.2 Control of Gimbaled-Rotor Devices

Despite the fact that inertial sensors such as TRGs have been an integral component of spacecraft attitude control systems for many years, there is little available literature discussing their control, particularly in the last decade. This may be due in part to the reticence of the few companies who manufacture TRGs to publish their rebalance control loop designs as they are considered competition-sensitive information. It can also be attributed to the fact that TRGs are not as widely used as they were twenty years ago. The development of ring-laser and fibre-optic gyros has made TRGs less popular.

In addition to proposing the "complex method of analysis of small angular motions of spinning rigid bodies", Lipman [11] is one of the earliest authors to discuss compensation methods for a rebalance loop. Representing the linearized momentum-axis dynamics by a single linear complex differential equation, he noted that the root locus plot of the compensated system could be used to aid in the understanding of the closed loop system

dynamics. The idea of plotting the root locus of the momentum-axis dynamics complex transfer function was explored further by Kao *et al.* [22]. Through a trial-and-error controller selection process, they examine the resulting closed loop poles in the root locus diagram of a loop gain transfer function with complex coefficients.

The use of frequency domain control design of the transverse-axes dynamics in the form of a complex transfer function has not been reported in the literature. However another physical system that lends itself to modeling in the form of a transfer function with complex coefficients is the current control of multi-phase AC-machines. Briz *et al.* [23] use both root locus and complex vector frequency response functions to examine the performance of controllers. The emphasis in this study is more on estimating performance than on using these tools to choose suitable controllers and examine the system stability

More recently, both Staley [24] and Bird [25] discussed potential control systems for the rebalance loop of the CSG-2 tuned-rotor gyroscope. Expressing the simplified TRG dynamics in a form similar to (3-5), Staley [24] developed controllers for the equivalent single-input, single-output rebalance control system. In addition, he introduced the concept of precessional and nutation controllers. Noting the wide separation between the nutation and precessional frequencies of the TRG system, he proposed that the compensators for these two system poles could be designed separately. Although neither classical nor modern control methods were used to design the proposed controllers, the basis for phase compensation for the nutation controller is explained in terms of an analogy with the Nyquist stability criterion. Bird described two control designs, one of a similar nature to that proposed in [24] and another, which acts to eliminate the dynamic coupling between the two momentum axes. This was achieved through the selection of cross-axis compensators that provide near perfect decoupling at low frequencies. No detailed explanations are provided of the basis for the controller selection. Frequency responses were used to illustrate the direct and cross axis responses to commanded inputs

on either momentum axis. The discussion of relative stability was limited to describing the nature of the closed loop poles of the system.

Research efforts to improve rebalance controller designs have focused on removing the effect of nutation dynamics using inverse dynamic controllers. Wu [26] proposed the use of adaptive inverse dynamics control to eliminate the nutation resonance in TRGs. The objective was to increase the closed-loop bandwidth above the nutation frequency in order to improve the jitter control. Kalman filtering was employed to perform on-line parameter identification. Like Bird [25], Shingu and Otsuki [27] proposed de-coupling the interaction between the two momentum axes. They noted that the torque required to produce the de-coupling action is proportional to the angular accelerations of the case about the two momentum axes. They introduced a second set of torquer coils in the electromechanical design that served to generate the decoupling torques. By adding second coils termed "sub-torquers", both angular rate and acceleration can be sensed. The main torquer current is used to measure the external angular rates with the "sub-torquer" current used to measure external angular accelerations.

In TRGs, the issue of how to design an effective and stable control system over a wide gimbal angular travel is not a significant issue, but it could become one for the generalized tuned-rotor instrument. Larger angular tilts make the small angle approximations used in the tuned-rotor gyroscope models less valid. An alternative is to apply the non-linear control system design technique of feedback linearization. This approach has been used successfully for the control of manipulator arms (Spong and Vidyasagar [28]), control moment gyros (Dzielski *et al.* [29]) and steerable momentum wheels (Bender and Brehove [30]).

3.3 External Rate Sensing Using Tuned Rotor Gyroscopes

Used in a strap-down mode, TRGs can measure spacecraft angular rates. With a known initial condition for spacecraft attitude, these rates can be integrated over time to produce a continuous estimate of the spacecraft attitude about two axes. This measurement can then be used in a 3-axis stabilized attitude control system (Wertz [31]). Due to the lack of an initial reference and gyro drift and bias errors, the position reference must be updated periodically using an external reference sensor such as a star tracker, sun sensor or horizon sensor.

The principle by which external rates are extracted from the rate gyro can be seen by examining the following equations of motion of the transverse axes, including the effect of the case angular velocity [12]:

$$\begin{aligned} I_r \ddot{\phi}_x - I_r \omega_s \dot{\phi}_y - I_r \dot{\omega}_{b_x} + I_r \omega_s \omega_{b_y} &= T_{\tau_x} \\ I_r \ddot{\phi}_y - I_r \omega_s \dot{\phi}_x - I_r \dot{\omega}_{b_y} - I_r \omega_s \omega_{b_x} &= T_{\tau_y} \end{aligned} \quad (3-8)$$

where

- I_r rotor inertia about the spin axis
- I_r rotor inertia about the transverse x or y axes
- ϕ_x case-referenced tilt angle of momentum axis about x-axis of case frame
- ϕ_y case-referenced tilt angle of momentum axis about y-axis of case frame
- ω_{b_x} angular velocity of the spacecraft with respect to inertial frame, expressed about the x-axis of case frame
- ω_{b_y} angular velocity of the spacecraft with respect to inertial frame, expressed about the y-axis of case frame

- ω_s instantaneous spin rate of the motor shaft
- T_{z_x} case-referenced control torque acting about the x-axis of case frame
- T_{z_y} case-referenced control torque acting about the y-axis of case frame

A rebalance control system is used to generate the control torques (T_{z_x}, T_{z_y}) such that the case referenced tilt angles (ϕ_x, ϕ_y) remain at null. The bandwidth of rebalance control systems is high, generally greater than 10 Hz.

The purpose of a tuned-rotor gyro is to estimate the angular velocity of the base body about the two non-spin axes of the gyro case frame. The external rates can be determined based on the assumption that the torques induced on the gyro rotor by the motion of the base body will be cyclic and of a frequency lower than the gyro control system bandwidth. Given the frequency separation between the base motion and the control system bandwidth, the first three terms in each of the two equations can be ignored. As a result, the equations of motion simplify to:

$$\begin{aligned} I_r \omega_s \omega_{b_x} &\approx T_{z_y} \\ -I_r \omega_s \omega_{b_y} &\approx T_{z_x} \end{aligned} \quad (3-9)$$

Solving for the external rates yields:

$$\begin{bmatrix} \omega_{b_x} \\ \omega_{b_y} \end{bmatrix} \approx \frac{1}{I_r \omega_s} \begin{bmatrix} T_{z_y} \\ -T_{z_x} \end{bmatrix} \quad (3-10)$$

The control torques are proportional to the measured current in each torquer coil and can be determined using the following relation

$$\begin{bmatrix} T_{z_x} \\ T_{z_y} \end{bmatrix} = K_{\tau q} \begin{bmatrix} i_{z_x} \\ i_{z_y} \end{bmatrix} \quad (3-11)$$

where $K_{\tau q}$ describes the factor by which the torquers convert coil current to torque. It is based on the interaction of the magnetic dipole moment created by the current in the coil and its interaction with the radial magnetic field created by the inner and outer permanent magnets on the gyro rotor. State-of-the art tuned-rotor gyroscopes achieve accuracies as high as 0.005 deg/hr [32].

A good example of the use of TRGs within a 3-axis control system is the ODIN small spacecraft. In its astronomy mission mode, continuous use of the star tracker within the attitude control system is impractical given the long integration times and the potential for the sensor to be occulted by the Earth. Instead, a complement of three TRGs is used as the primary attitude sensor with periodic calibrations from a star tracker (David [33]). Based on long-term tests, Cain *et al.* [34] demonstrated the low rate drift that occurs in the CSG-2 strap-down rate-gyro. Based on a star tracker update period of 15 minutes, the gyroscopic drift reached a maximum of 2.4 arc seconds on either axis, with this increasing to just over 10 arc seconds for an update period of one hour. This performance illustrates how TRGs are a suitable sensor for high accuracy pointing tasks of systems like ODIN.

Chapter 4

The Tuned-Rotor Instrument Control Problem

The general tuned-rotor instrument as described in [5] offers unique capabilities to the design of spacecraft attitude determination and control systems. It consists of a tuned-rotor gyroscope-like mechanical configuration but has a larger rotor, increased tilt and spin-rate dynamic range and modified torquer and tilt sensors. The changes in the mechanical configuration allow the device to function as both a steerable momentum wheel and an external rate sensor. The challenge of simultaneously extracting external rates while controlling the rotor spin-axis tilt and rate is termed the “Tuned-Rotor Instrument Control Problem”.

The following section provides a detailed statement of the nature of the proposed Tuned-Rotor Instrument Control Problem. Based on previous research and industrial applications of tuned-rotor gyroscopes, justification for pursuing this problem is provided. Finally, the merits and potential applications of this research are outlined.

4.1 Problem Statement

Examination of the past research and industrial applications of tuned-rotor gyroscopes has revealed four areas where new research is required in order to address the “Tuned Rotor Instrument Control Problem.”

a) **Derivation and Analysis of a Complete Transfer Function Model of a Tuned-Rotor Instrument**

A “Complete” SCISCO transfer function model of the dynamics is required to examine the influence of the tuned-rotor-instrument dynamic characteristic modes on control design. Through a unique formulation, all four dynamic modes of the system are included, with the effect of the gimbal flutter action in a tuned-rotor instrument also included. This model is used to determine the influence of the four characteristic modes on tuned-rotor instrument behaviour. It is also used to derive a new, more accurate tuning condition for tuned-rotor instruments.

b) **Theorems for Applying Classical Control Techniques to Single-Complex-Input, Single-Complex-Output (SCISCO) Systems**

A systematic approach for developing controllers for the transverse-axes of tuned-rotor instruments and examining robust stability and performance is required. The transverse-axes dynamics is in a class of two-input, two-output systems, for which the inputs and outputs can be combined in quadrature. This thesis describes these systems in a single-complex input, single-complex-output (SCISCO) form. It is proposed that single-input, single-output (SISO) classical control design and analysis techniques can be generalized for application to SCISCO systems. SISO techniques considered in this study include root locus, the Nyquist stability criterion and frequency-response design.

c) **The Influence of Tuned-Rotor Instrument Three-Body Dynamics on Spin and Transverse-Axes Control**

A detailed “three-body” non-linear model of the tuned-rotor instrument dynamics operating on a rotating base is derived in this study. It is of a similar nature to that derived by Cain [19]. It is implemented in a non-real time simulator, along with a spin axis control system, transverse-axes control system and external rate-sensing algorithm. This simulation model facilitates an investigation into the significance of coupling between the spin and transverse-axes dynamics on the design of tuned-rotor instrument control systems. The three-body model allows a comparison to be made between traditional motor spin control and direct rotor spin control. The impact of these choices on transverse-axes control is then examined.

d) **Derivation and Demonstration of External Rate Sensing Concept for a General Tuned-Rotor Instrument**

The original concept developed to estimate the rates of an external body using a tuned-rotor gyroscope is based on the restriction that the rotor is held at or near null. This study proposes a more general method for extracting external rates from the motion of a tuned-rotor instrument, simultaneously operating as a momentum management actuator. This method accounts for the operation in a de-tuned condition and with the spin axis tilted at a significant non-zero angle, yet requires no additional sensor information beyond what is presently used in tuned-rotor gyroscopes. This concept is demonstrated for the GyroWheel system using the simulation model of the non-linear dynamics of the tuned-rotor instrument with spin and transverse axes control active.

Finally, the merits of the new analysis techniques and dynamic models are demonstrated through their application to the GyroWheel design.

4.2 Discussion of Thesis Problem

The relevance of considering the Tuned-Rotor Instrument Control Problem is presented in the following section. Advances in the state-of-the-art are discussed with respect to the four features described in Section 4.1.

4.2.1 Derivation and Analysis of a Complete Transfer Function Model For a Tuned-Rotor Instrument

Although linear models of tuned-rotor gyroscopes have been developed, notably those by Craig [12] and Staley and Cain [13], they lack some features that appear significant for the study of the general tuned-rotor instrument control problem. In addition, conclusions drawn from past analyses of these models appear almost contradictory in some cases.

The transfer function model derived by Craig [12] has served as a benchmark for tuned-rotor gyroscope analysis for more than twenty-five years. In this model, only two dynamic modes are identified: a nutation and precession frequency. However Staley [13] identifies two similar modes and two new modes. Of the two new modes, one is at a frequency of twice the spin rate. The second appears to be a beat frequency due to the difference between twice the spin rate and the nutation mode. The results of Craig and Staley appear contradictory until Craig's analysis is examined more closely. Craig [12] defines the transfer function relating the complex spin-axis tilt angle $\Theta(s)$ to the complex case-referenced torque $M_{\text{xy}}(s)$ as the following:

$$\Theta_x(s) = \frac{\bar{\tau}_1(s - 2jN)}{\tau_1(s)\bar{\tau}_1(s - 2jN) - \tau_2(s)\bar{\tau}_2(s - 2jN)} M_x(s) \quad (4.1)$$

$\tau_1(s)$ and $\tau_2(s)$ are second order transfer functions dependent on the gimbal and rotor inertias as well as the gimbal stiffness and damping characteristics. $\bar{\tau}_1(s)$ and $\bar{\tau}_2(s)$ are the complex conjugates of the transfer functions $\tau_1(s)$ and $\tau_2(s)$ respectively. Finally, N is the spin rate of the gyro rotor. The denominator of the transfer function is a fourth order polynomial, like that derived by Staley [25]. However Craig notes that in a practical design [35], gimbal inertias are much smaller than rotor inertias, allowing one to entirely neglect the presence of the term $\tau_2(s)\bar{\tau}_2(s - 2jN)$. This reduces the characteristic polynomial to second order with the two remaining roots being the nutation frequency and precession frequency. Evidently, a more complete transfer function model must be developed without the approximations made by Craig.

A transfer function model relating the tilt angles of the spin axis to the case-referenced control torques is required, which reflects the four characteristic roots identified by Staley [13]. This transfer function will facilitate the examination of the relative effect of these four roots. For example, Staley through analysis of a linear tuned-rotor gyro model and Cain [14] through the use of a non-linear simulation of an open-loop tuned-rotor gyro, report the presence of a “beat” frequency mode. Such a transfer function could help to explain why this “beat” frequency mode appears to have had no de-stabilizing effect on previous tuned-rotor gyro rebalance control system designs. This mode is at a relatively low frequency, one that could easily be inside typical rebalance control system bandwidths.

Although the proposed GyroWheel system will operate at non-zero tilts and de-tuned conditions, there may be instances where it operates solely like a tuned-rotor gyro. In

these cases, the tilt will be commanded to zero and the spin rate set to the tuned speed. Examination of the literature suggests that the tuning condition for tuned-rotor gyroscopes proposed by Savet [15] and later generalized by Craig [12] for multi-gimbal systems should be re-examined. The principle of the tuning condition for tuned-rotor gyroscopes was derived by determining the condition by which the precession frequency is set to zero. However, Staley [13] and Cain [14,19] have both shown that the spin axis of a tuned-rotor gyro will precess very slowly around null when set at an initial non-zero tilt. This suggests that the combination of spin rate, flexure stiffness and gimbal inertia could be chosen in a slightly different proportion such that a more effective tuned condition is achieved.

4.2.2 Control System Analysis and Design

Of the limited discussion of tuned-rotor gyroscope control in the literature, the methods for developing control systems fall into one of two categories: determination of the controllers from simple dynamic analysis or design using inverse dynamics methods. Based on simple dynamic analysis, Kao et al. [36], Bird [25] and Shingu *et al.* [27] all propose different types of transverse-axes controllers. However, the rationale for choosing such forms is given as trial and error, or not at all. An exception is Staley's work [25] where three forms of transverse-axes controllers are derived through manipulation of the characteristic polynomial of a TRG.

A more modern approach is that of Wu [26] who shows that closed-loop bandwidths above the nutation frequency are possible when the nutation dynamics are stabilized through inverse dynamic control. Despite the fact that the system dynamics are relatively constant with no variation in spin rate and very small tilt angles, such an approach still relies on a Kalman Filter to estimate the system dynamic model. This is not as appropriate for systems such as tuned-rotor instruments where there are significant

variations in both the tilt angle and spin rate. A more robust control approach would be desirable.

The two aspects of past work on TRG control which have not been addressed in any detail are:

a) Systematic design approach

There is very little discussion of a systematic approach to control system design of the transverse-axes dynamics. Such a method should provide ways to

- i) Conceptualize various suitable controllers through methods other than trial and error.
- ii) Incorporate performance objectives into the selection process of these controllers.
- iii) Consider relative stability during the design process.

b) Sensitivity of Control Design to Modeling Uncertainty and Operating Condition Variation

The lack of discussion concerning robustness can be attributed to the fact that tuned-rotor gyro dynamics change very little over their limited operating range. Therefore, the degree of model parameter uncertainty is small. This differs from the general tuned-rotor instrument control problem where the spin rate and tilt angles vary considerably.

Given the two-input, two-output characteristic of the transverse-axes dynamics, one approach is to apply one of the recognized multivariable feedback design methods. There

are a number of methods available for performing multi-variable feedback design. The Nyquist-like techniques [37,38] extend the single-input, single-output classical control system design tools of root-locus and Nyquist stability to multi-input, multi-output systems. Alternatively the relatively new method of H_∞ optimal control [39,40] can also be applied.

However, the structure of the transverse-axes dynamics of the tuned-rotor instrument offers an alternative approach that is more powerful, yet simple to apply. Previous analyses by Lipman [11], Kao et al. [36] and Staley [24] have demonstrated that the transverse-axes dynamics can be examined in the form of a single-complex-input, single-complex output (SCISCO) system. Yet, there has been little discussion of applying classical single-input, single-output (SISO) control approaches to this form of transfer function. Both Lipman [11] and Kao et al. [36] plotted the root locus of a SCISCO transfer function to show the effect of a re-balance controller on the resulting closed loop poles. In each case, the root locus was used as a device to plot the closed loop poles rather than a systematic technique to design a controller and examine relative stability.

In other engineering fields where dynamic systems can be described in terms of a SCISCO system, there is also little discussion of developing control systems with this form. The study by Briz et al. [23] is one of the few to apply frequency response methods to the analysis of a SCISCO transfer function. They examine the performance of the current regulator for a multi-phase AC load. Here again, the frequency responses are used to predict performance rather than as a tool to develop controllers and examine stability.

In each of these studies of systems described as a SCISCO transfer function, there is no discussion of the appropriate rules or limitations for applying classical control tools to SCISCO transfer functions. One of the greatest strengths of classical root locus and frequency response methods is the capability to conceptualize appropriate compensators

and examine relative stability. None of these studies have demonstrated this feature. To a degree, the following thesis attempts to address these shortcomings.

4.2.3 The Influence of Tuned-Rotor Instrument Non-Linear Dynamics on Spin and Transverse Axes Control

Cain's derivation of a non-linear dynamic model of a tuned-rotor gyroscope [14] is the first such model reported in the literature. He makes the important observation that a significant change occurs in the precession frequency of the system when the tilt of the momentum axis is large (1-5 degrees). At 5 degrees tilt, the precession frequency is several orders of magnitude larger than that predicted in Staley's linear analysis [13]. However, this model only considers the motion of two bodies: the gimbal and rotor. The motor shaft rate is assumed to be constant. Examining the motion of a Hooke's joint, it is apparent that as the tilt increases, the flutter motion of the gimbal also increases. Consequently, it is unlikely that the input shaft can be maintained at a perfectly constant rate as the tilt increases, in the presence of increasing flutter torque.

To account for the effect of the coupling between the gimbal and input shaft, Cain [19] developed a three-body model. With no spin axis control, the precession frequency decreases very gradually as the spin axis tilt is increased over a range of zero to eight degrees. However, as the tilt increases, there is an increasing component of oscillation in the motor spin rate due to the gimbal flutter, which occurs at twice the spin rate. The open loop behaviour of the non-linear dynamics of the two-body and three-body dynamic models appear to represent the two extremes of motor spin control of a tuned-rotor gyroscope. The open loop simulation response of the two-body model represents perfect control of the input spin rate, holding it at a constant value. The open-loop simulation

response of the three-body model is the case where there is no control action on the input spin rate.

As described in Section 3.1, typical tuned-rotor gyroscope control designs partition the problem into spin axis and transverse-axes control. For small spin-axis tilts, the spin axis dynamics is considered decoupled from the transverse-axes dynamics. Based on this assumption, spin control systems are simply designed to have a high bandwidth, ensuring that the spin rate is held at the constant tuned speed. Cain's studies of the two-body [14] and three-body [19] non-linear dynamics of an open-loop tuned-rotor gyro prompt questions about the effect of spin control when spin axis tilts are large (up to 5 degrees). If the motor spin control loop bandwidth is set at or above the flutter frequency, then the spin controller should track the spin rate, rejecting the flutter disturbance torques. In turn, the transverse-axes dynamics should behave in a similar manner to that observed with Cain's two-body model [14].

However, designing the spin control system to make the transverse-axes dynamics behave like a two-body model may be undesirable. The high precession frequencies observed when the spin axis tilt is large mean that the transverse-axes controller will require a large torque in order to counter precession and hold the spin axis at a desired tilt. For this reason, various methods of spin control are examined in this study using a "three-body model" of a tuned-rotor instrument. Such a study using a non-linear dynamics model of the motion of the three bodies within a tuned-rotor instrument has yet to be reported in the literature.

4.2.4 Tuned-Rotor Instrument External Rate Sensing

As described in Section 3.3, typical tuned-rotor gyroscopes can measure the external rates of the spacecraft body about two axes, based only on the current in the re-balance loop

torquer coils and the constant rotor momentum [33,34]. Although the mechanical design of the general tuned-rotor instrument considered in this study is similar to a TRG, there are aspects of its operating regime that do not permit external rates to be estimated in the traditional way. These are:

- a) The GyroWheel frequently operates in a de-tuned condition and therefore, the rotor spin axis will not remain fixed in inertial space as the case moves.
- b) Unlike the TRG, the transverse-axes angles are commanded to follow a time varying, non-zero tilt, with this tilt being significant at certain moments in its operation.

Therefore, in a tuned-rotor instrument the control torque is generated to counter the following secular and periodic loads:

- i) The effective de-tuned flexure stiffness
- ii) The gyroscopic torques resulting from the rate of change of the spin axis tilt as the GyroWheel momentum vector is steered along the command trajectory.
- iii) Torques resulting from acceleration/deceleration of the spin axis tilt as the GyroWheel is commanded to track time-varying momentum profiles.
- iv) Disturbance terms like the "flutter torque" which become larger as the spin-axis tilt of the device is increased.
- v) Torques induced on the tuned-rotor instrument due to spacecraft motion.

The effect of these torques must be considered in the development of any general tuned-rotor instrument external rate sensing capability. This problem has yet to be considered in the literature.

4.3 Merits of Investigating the Tuned-Rotor Instrument Control Problem

The research performed in this thesis offers some significant benefits not only in the development of the GyroWheel system but also in terms of understanding general tuned-rotor instrument dynamics and formulating control systems for them. Listed below are proposed benefits of the research presented in this dissertation:

- a) The application of classical control methods to SCISCO transfer functions provides insight into choosing compensator designs that would not otherwise be apparent with traditional MIMO methods. Greater visibility into the interaction of the controller with specific characteristics of the dynamics (e.g. particular poles) is possible than with other modern control methods. The proposed approach is applicable not only to tuned-rotor instruments, but to any two-input, two-output system whose inputs and outputs can be combined in quadrature. This method could potentially be applied to the attitude control of spin-stabilized spacecraft and current control in AC-machines.
- b) By demonstrating that frequency response methods can be applied to SCISCO systems, it opens the possibility of applying modern H_∞ SISO control design techniques such as that described by Doyle *et al.* [40]. This will require a re-examination of the requirement in this approach that the transfer functions have real-valued coefficients.
- c) The proposed complete SCISCO transfer function model can serve as a new general model for tuned-rotor instrument control system design and dynamic analysis.

- d) Application of the proposed alternate tuning condition for tuned-rotor instruments may result in a reduction of drift in commercial devices.
- e) The use of a flexure suspension as a double gimbal for a steerable momentum wheel is unique. An equally novel aspect of the GyroWheel is the proposal that spacecraft rates can be derived based on the commanded and sensed GyroWheel motion. The prospect that a device can serve as both an actuator and a sensor is an attractive one, particularly for small satellite attitude control where mass of the ADCS components is at a premium. The objective in this research is to obtain a sensing capability with a maximum drift rate of 0.1 degree/hour. This is a stringent requirement considering that few TRGs have accuracy's of better than 0.005 deg/hour [32].
- f) The simulator consisting of the non-linear dynamics, spin and transverse-axes control systems and the external rate sensing capability can serve as a general tool to predict general tuned-rotor instrument behaviour. It can also be used to plan the calibration of actual tuned-rotor instrument hardware.

Chapter 5

Derivation of Tuned-Rotor Instrument Dynamic Models

To examine the control system and external rate sensing concepts for the tuned-rotor instrument, both a general non-linear and linear model are required. The non-linear model derived here describes the motion of the three bodies making up the tuned-rotor instrument: the motor, gimbal and rotor. Like Cain's model of this type of device [19], it does not assume that the motor shaft rate is constant. This model will support the examination of two important features of the general tuned-rotor instrument:

- i) The influence of spin control on the behaviour of the transverse-axes dynamics.
- ii) The effect of spin axis tilt of up to 5 degrees on the control system and external rate sensing algorithms.

In this study, the linearized model of the transverse-axes dynamics is developed in the form of a single-complex-input, single-complex-output (SCISCO) transfer function, relating the complex tilt angle to the complex control torque. This unique transfer function model is termed the "Complete Model" as it more accurately reflects the modes and low-frequency gain of a tuned-rotor instrument. These improvements come about by including gimbal flutter effects previously ignored in traditional tuned-rotor gyroscope

transfer function models. The complete model facilitates improved control system and external rate sensing algorithm development. For comparison, the “Traditional” form of a tuned-rotor gyroscope model is also derived.

The starting point for the development of the non-linear model dynamics model is the kinematics of the tuned-rotor instrument.

5.1 Kinematics of Tuned-Rotor Instruments

Two sets of coordinates will be used to describe the motion of the tuned-rotor instrument: spinning coordinates and case-referenced coordinates. The following sections detail their derivation and the relationship between quantities in these two coordinate systems.

5.1.1 Kinematics Based on Spinning Coordinates

a) Position

The spinning coordinates describe the motion of the tuned rotor instrument in terms of the input motor shaft angle and the two gimbal angles. They represent the motion about the three degrees of freedom of the system and are termed spinning coordinates because the two gimbal angles are referenced to the spinning motor shaft. They are the most suitable set of generalized coordinates for deriving the equations of motion because each coordinate represents a specific mechanical degree-of-freedom of the system.

The tuned rotor instrument, as shown in Figure 3-3, consists of three bodies: a motor shaft/inner gimbal ring combination, a gimbal and a rotor. Four body-fixed reference frames ($\mathcal{F}_o, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$) are attached to the case, motor shaft, gimbal and rotor bodies

respectively. The reference frames $(\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ are denoted by the orthogonal triads (x, y, z) , (x', y', z') , (x'', y'', z'') and (x''', y''', z''') respectively as shown in Figure 5-1. . Figure 5-2 shows the three spinning coordinates and their drive axes in relation to the physical components of the tuned-rotor instrument motor. Attached to the spacecraft body is the case frame $\mathcal{F}_0$ as shown in Figure 5-1 and Figure 5-2. The gimbal is connected to the motor shaft by a pair of torsional springs aligned along the x -axis of the gimbal frame $\mathcal{F}_2$. The rotor is in turn connected to the gimbal ring by a second pair of torsional springs aligned along the y -axis of the gimbal frame $\mathcal{F}_3$.

Figure 5-1: Gimbal-Referenced Coordinates and Reference Frames

Figure 5-2: Gimbal-Referenced Coordinates and Drive Axes

Motion of the three bodies is constrained except about the following three gimbal-type angles:

- θ_z Motion of the motor shaft with respect to the case, described as a rotation about the z -axis of the frame $\mathcal{F}_1$ with respect to frame $\mathcal{F}_0$.
- θ_x Motion of the gimbal with respect to the motor shaft, described as a rotation about the x -axis of the frame $\mathcal{F}_2$ with respect to frame $\mathcal{F}_1$.
- θ_y Motion of the rotor with respect to the gimbal ring, described as a rotation about the y -axis of the frame $\mathcal{F}_3$ with respect to frame $\mathcal{F}_2$.

Based on the definitions of the three gimbal angles, the matrices transforming a vector in the $\mathcal{F}_i$ frame to its corresponding form in the $\mathcal{F}_{i-1}$ frame are given below:

$$(\theta_x)_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_{\theta_x} & S_{\theta_x} \\ 0 & -S_{\theta_x} & C_{\theta_x} \end{bmatrix} \quad (5-1)$$

$$(\theta_y)_y = \begin{bmatrix} C_{\theta_y} & 0 & -S_{\theta_y} \\ 0 & 1 & 0 \\ S_{\theta_y} & 0 & C_{\theta_y} \end{bmatrix} \quad (5-2)$$

$$(\theta_z)_z = \begin{bmatrix} C_{\theta_z} & S_{\theta_z} & 0 \\ -S_{\theta_z} & C_{\theta_z} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (5-3)$$

Based on Figure 5-1, the attitude of the rotor can be described by the following gimbal-referenced rotation sequence: a rotation θ_z about the z-axis, followed by a rotation θ_x about the x-axis and then a rotation θ_y about the y-axis. The transformation matrix $\mathbf{A}$ describes the rotation sequence and is defined below:

$$\mathbf{A} = (\theta_y)_y (\theta_x)_x (\theta_z)_z \quad (5-4)$$

Substituting the definitions of $(\theta_x)_x$, $(\theta_y)_y$ and $(\theta_z)_z$, the rotation matrix $\mathbf{A}$ takes the following form:

$$\mathbf{A} = \begin{bmatrix} C_{\theta_x} C_{\theta_z} - S_{\theta_x} S_{\theta_z} S_{\theta_y} & C_{\theta_x} S_{\theta_z} + S_{\theta_x} S_{\theta_z} C_{\theta_y} & -C_{\theta_x} S_{\theta_y} \\ -C_{\theta_x} S_{\theta_z} & C_{\theta_x} C_{\theta_z} & S_{\theta_x} \\ S_{\theta_x} C_{\theta_z} + S_{\theta_x} C_{\theta_z} S_{\theta_y} & S_{\theta_x} S_{\theta_z} - S_{\theta_x} C_{\theta_z} C_{\theta_y} & C_{\theta_x} C_{\theta_y} \end{bmatrix} \quad (5-5)$$

b) Angular Velocity

The angular velocities of each body can now be expressed in terms of the three gimbal angles and their first derivative with respect to time. The transformation matrix $\mathbf{A}$

describes the attitude of the rotor with respect to a fixed base (the gyro case) as expressed in the rotor body frame. To determine the angular velocity of the rotor, the effect of the base motion on the angular velocities of the motor shaft, gimbal and rotor will be included. This additional effect is necessary to derive a dynamical model, which can be used to examine external rate sensing. It is assumed that the case is fixed with respect to the spacecraft body on which it sits. The spacecraft angular velocity is respect to the inertial frame and is expressed in the spacecraft body frame $\mathcal{F}_o$:

$$\bar{\omega}_b = \begin{bmatrix} \omega_{b_1} \\ \omega_{b_2} \end{bmatrix} \quad (5-6)$$

The angular velocity of the motor shaft expressed in its body frame $\mathcal{F}_i$ is:

$$\bar{\omega}_m = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_z \end{bmatrix} + (\theta_z)_z \bar{\omega}_b \quad (5-7)$$

Substituting (5-3) and (5-6) into (5-7), the motor shaft angular velocity takes the following form:

$$\bar{\omega}_m = \begin{bmatrix} C_{\theta_z} \omega_{b_1} - S_{\theta_z} \omega_{b_2} \\ -S_{\theta_z} \omega_{b_1} - C_{\theta_z} \omega_{b_2} \\ \omega_{b_z} + \dot{\theta}_z \end{bmatrix} \quad (5-8)$$

The gimbal angular velocity expressed in its body frame $\mathcal{F}_2$ is a function of the base and motor shaft angular velocities where:

$$\bar{\omega}_g = \begin{bmatrix} \dot{\theta}_x \\ 0 \\ 0 \end{bmatrix} + (\theta_z)_x \bar{\omega}_m \quad (5-9)$$

Substituting (5-1) and (5-8) into (5-9) yields the following expression for the following gimbal angular velocity:

$$\bar{\omega}_g = \begin{bmatrix} \dot{\theta}_x - C_{\theta_z} \omega_{b_x} - S_{\theta_z} \omega_{b_z} \\ C_{\theta_x} (-S_{\theta_z} \omega_{b_x} - C_{\theta_z} \omega_{b_z}) - S_{\theta_x} (\omega_{b_z} + \dot{\theta}_z) \\ -S_{\theta_x} (-S_{\theta_z} \omega_{b_x} + C_{\theta_z} \omega_{b_z}) - C_{\theta_x} (\omega_{b_z} + \dot{\theta}_z) \end{bmatrix} \quad (5-10)$$

Finally, the rotor angular velocity expressed in its body frame $\mathcal{F}_3$ can be expressed in terms of the gimbal angular velocity and third gimbal angular rate $\dot{\theta}_y$ where:

$$\bar{\omega}_r = \begin{bmatrix} 0 \\ \dot{\theta}_y \\ 0 \end{bmatrix} + (\theta_y)_y \bar{\omega}_g \quad (5-11)$$

Substituting (5-2) and (5-10) into (5-11) yields the following expression for the rotor angular velocity:

$$\bar{\omega}_r = \begin{bmatrix} (C_{\theta_x} C_{\theta_z} - S_{\theta_x} S_{\theta_z} S_{\theta_y}) \omega_{b_x} + (C_{\theta_x} S_{\theta_z} - S_{\theta_x} S_{\theta_z} C_{\theta_y}) \omega_{b_z} - C_{\theta_x} \dot{\theta}_x - C_{\theta_x} S_{\theta_y} (\omega_{b_z} + \dot{\theta}_z) \\ C_{\theta_x} (-S_{\theta_z} \omega_{b_x} + C_{\theta_z} \omega_{b_z}) - S_{\theta_x} (\omega_{b_z} + \dot{\theta}_z) - \dot{\theta}_y \\ (S_{\theta_x} C_{\theta_z} - S_{\theta_x} C_{\theta_z} S_{\theta_y}) \omega_{b_x} + (S_{\theta_x} S_{\theta_z} - S_{\theta_x} C_{\theta_z} C_{\theta_y}) \omega_{b_z} + S_{\theta_x} \dot{\theta}_x + C_{\theta_x} C_{\theta_y} (\omega_{b_z} + \dot{\theta}_z) \end{bmatrix} \quad (5-12)$$

Ignoring the effect of base angular velocity, the expression describing the rotor angular velocity simplifies to:

$$\bar{\omega}_r = \begin{bmatrix} C_{\theta_x} \dot{\theta}_x - C_{\theta_x} S_{\theta_y} \dot{\theta}_z \\ \dot{\theta}_y - S_{\theta_x} \dot{\theta}_z \\ S_{\theta_x} \dot{\theta}_x + C_{\theta_x} C_{\theta_y} \dot{\theta}_z \end{bmatrix} \quad (5-13)$$

Alternatively, the rotor angular velocity can be expressed in terms of a Jacobian matrix $\mathbf{J}$ and the vector of gimbal angular rates where:

$$\bar{\omega}_r = \mathbf{J} \begin{bmatrix} \dot{\theta}_x \\ \dot{\theta}_y \\ \dot{\theta}_z \end{bmatrix} \quad (5-14)$$

The Jacobian matrix $\mathbf{J}$ is described below:

$$\mathbf{J} = \begin{bmatrix} C_{\theta_x} & 0 & -C_{\theta_x} S_{\theta_y} \\ 0 & 1 & S_{\theta_x} \\ S_{\theta_x} & 0 & C_{\theta_x} C_{\theta_y} \end{bmatrix} \quad (5-15)$$

5.1.2 Kinematics Based on Case-Referenced Coordinates

Case-referenced coordinates are now used to describe the motion of the rotor with respect to the fixed case. These coordinates are suitable for examining the control of a tuned-rotor instrument because the control torques are generated and the rotor tilt angles are measured with respect to the case. Therefore, these coordinates become the natural ones for control law formulation.

a) Angular Position

Figure 5-3 shows the reference frames used to develop the relations describing the rotor position and angular velocity in terms of case-referenced coordinates. The reference frames $(\mathcal{F}_0', \mathcal{F}_1', \mathcal{F}_2', \mathcal{F}_3')$ are denoted by the orthogonal triads (x, y, z) , (x', y', z') , (x'', y'', z'') and (x''', y''', z''') respectively as shown in Figure 5-3. Note that the frames $\mathcal{F}_0'$ and $\mathcal{F}_3'$ are coincident with the reference frames $\mathcal{F}_0$ and $\mathcal{F}_3$ used in the development of the spinning coordinates model. Using case referenced coordinates, the attitude of the rotor with respect to the case is described in terms of a rotation ϕ_x about the x-axis of $\mathcal{F}_1'$, then a rotation ϕ_y about the y-axis of $\mathcal{F}_2'$, and then finally a rotation ϕ_z about the z-axis of $\mathcal{F}_3'$.

The transformation matrix $\mathbf{A}'$ describes the rotation sequence and is defined below:

$$\mathbf{A}' = (\phi_z)_z (\phi_y)_y (\phi_x)_x \quad (5-16)$$

The following relations give the three constituent rotation matrices:

$$(\phi_x)_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_{\phi_x} & S_{\phi_x} \\ 0 & -S_{\phi_x} & C_{\phi_x} \end{bmatrix} \quad (5-17)$$

$$(\phi_y)_y = \begin{bmatrix} C_{\phi_y} & 0 & -S_{\phi_y} \\ 0 & 1 & 0 \\ S_{\phi_y} & 0 & C_{\phi_y} \end{bmatrix} \quad (5-18)$$

$$(\phi_z)_z = \begin{bmatrix} C_{\phi_z} & S_{\phi_z} & 0 \\ -S_{\phi_z} & C_{\phi_z} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (5-19)$$

Substituting the expressions for $(\phi_x)_x$, $(\phi_y)_y$ and $(\phi_z)_z$ into (5-16) yields the following rotation matrix:

$$\mathbf{A}' = \begin{bmatrix} C_{\phi_x} C_{\phi_z} & S_{\phi_x} S_{\phi_z} C_{\phi_y} + C_{\phi_x} S_{\phi_y} & -C_{\phi_x} S_{\phi_z} C_{\phi_y} - S_{\phi_x} S_{\phi_y} \\ -C_{\phi_x} S_{\phi_z} & -S_{\phi_x} S_{\phi_z} C_{\phi_y} - C_{\phi_x} S_{\phi_y} & S_{\phi_x} S_{\phi_z} C_{\phi_y} + C_{\phi_x} S_{\phi_y} \\ S_{\phi_x} & -S_{\phi_x} C_{\phi_y} & C_{\phi_x} C_{\phi_y} \end{bmatrix} \quad (5-20)$$

Figure 5-3: Case-Referenced Coordinates and Reference Frames

b) **Angular Rates**

The angular velocity of the rotor can be obtained by transforming the effect of the rate of change of each case-referenced angle to the rotor body-referenced frame. For the derivation of the case-referenced angular velocities, the effect of base angular rates will be ignored. In this study, the base angular rates are only required for the derivation of the overall dynamic model where gimbal referenced coordinates are used. The expression for the rotor angular velocity expressed in terms of the case-referenced angular rates is:

$$\omega'_r = \begin{bmatrix} 0 \\ 0 \\ \dot{\phi}_z \end{bmatrix} - (\phi_z)_z \begin{bmatrix} 0 \\ \dot{\phi}_y \\ 0 \end{bmatrix} - (\phi_z)_z (\phi_y)_y \begin{bmatrix} \dot{\phi}_x \\ 0 \\ 0 \end{bmatrix} \quad (5-21)$$

Evaluating this expression yields the angular velocity of the rotor in terms of the three case-referenced angular rates:

$$\omega'_r = \begin{bmatrix} C_{\phi_y} C_{\phi_z} \dot{\phi}_x + S_{\phi_z} \dot{\phi}_y \\ -C_{\phi_y} S_{\phi_z} \dot{\phi}_x + C_{\phi_z} \dot{\phi}_y \\ \dot{\phi}_z + S_{\phi_y} \dot{\phi}_x \end{bmatrix} \quad (5-22)$$

Alternatively, the rotor angular velocity can be expressed in terms of a Jacobian matrix $\mathbf{J}'$ and the vector of case-referenced angular rates where:

$$\omega'_r = \mathbf{J}' \begin{bmatrix} \dot{\phi}_x \\ \dot{\phi}_y \\ \dot{\phi}_z \end{bmatrix} \quad (5-23)$$

The case-referenced Jacobian matrix is defined as:

$$\mathbf{J}' = \begin{bmatrix} C_{\phi_y} C_{\phi_z} & S_{\phi_z} & 0 \\ -C_{\phi_y} S_{\phi_z} & C_{\phi_z} & 0 \\ S_{\phi_y} & 0 & 1 \end{bmatrix} \quad (5-24)$$

5.1.3 Relationship Between Gimbal-Referenced and Case-Referenced Angles, Rates and Torques

As discussed in the previous two sections, the gimbal referenced coordinates are required for the derivation of the dynamic model, while the case referenced coordinates are required for control system formulation. As shown in the following sections, once the dynamic model is developed and linearized, it will be transformed into case referenced coordinates. As a result, relationships between the gimbal and case-referenced coordinates are needed. The relationships for angles and rates first developed by Cain [14] are presented here for completeness. Using the gimbal referenced and case-referenced Jacobian matrices defined in Sections 5.1.1 and 5.1.2 respectively, a relationship between the gimbal and case-referenced torques is also presented. The relationships are summarized below with the descriptions of the derivations described in Appendix B.1.

a) Position

The case-referenced angles are related to the gimbal angles by the following relationships

$$\begin{aligned} \tan \phi_x &= \frac{S_{\phi_y} C_{\phi_z} C_{\phi_z} - S_{\phi_z} S_{\phi_z}}{C_{\phi_y} C_{\phi_z}} \\ \sin \phi_y &= S_{\phi_z} C_{\phi_z} + S_{\phi_y} C_{\phi_z} S_{\phi_z} \\ \tan \phi_z &= -\frac{C_{\phi_y} S_{\phi_z}}{S_{\phi_y} S_{\phi_z} S_{\phi_z} - C_{\phi_y} C_{\phi_z}} \end{aligned} \quad (5-25)$$

When the tilt of the spin axis is small and the motor spin rate is at a constant value of ω_s , equation (5-25) simplifies to:

$$\begin{bmatrix} \dot{\phi}_x \\ \dot{\phi}_y \end{bmatrix} = \begin{bmatrix} C_{\omega_s} & -S_{\omega_s} \\ S_{\omega_s} & C_{\omega_s} \end{bmatrix} \begin{bmatrix} \dot{\theta}_x \\ \dot{\theta}_y \end{bmatrix} \quad (5-26)$$

b) Rates

The relationship between the gimbal angular rates and the rate of change of the case-referenced angles is

$$\begin{bmatrix} \dot{\phi}_x \\ \dot{\phi}_y \\ \dot{\phi}_z \end{bmatrix} = \frac{1}{C_{\phi_z}} \begin{bmatrix} C_{\phi_z} C_{\phi_y} & -S_{\phi_z} & -S_{\phi_z} S_{\phi_y} - C_{\phi_z} C_{\phi_x} S_{\phi_y} \\ C_{\phi_z} S_{\phi_y} C_{\phi_x} & C_{\phi_z} C_{\phi_x} & C_{\phi_z} C_{\phi_x} S_{\phi_y} - C_{\phi_z} S_{\phi_x} C_{\phi_y} S_{\phi_y} \\ -C_{\phi_z} S_{\phi_x} - S_{\phi_z} C_{\phi_x} C_{\phi_y} & S_{\phi_z} S_{\phi_x} & S_{\phi_z} C_{\phi_x} C_{\phi_y} S_{\phi_y} - S_{\phi_z} S_{\phi_x} S_{\phi_y} - C_{\phi_z} C_{\phi_x} C_{\phi_y} \end{bmatrix} \begin{bmatrix} \dot{\theta}_x \\ \dot{\theta}_y \\ \dot{\theta}_z \end{bmatrix} \quad (5-27)$$

Assuming small tilt angles and a constant motor spin rate ($\dot{\theta}_z = \dot{\phi}_z = \omega_s$) the above expression simplifies to the following form:

$$\begin{bmatrix} \dot{\phi}_x \\ \dot{\phi}_y \end{bmatrix} = \begin{bmatrix} C_{\omega_s} & -S_{\omega_s} \\ S_{\omega_s} & C_{\omega_s} \end{bmatrix} \begin{bmatrix} \dot{\theta}_x \\ \dot{\theta}_y \end{bmatrix} - \omega_s \begin{bmatrix} S_{\omega_s} & C_{\omega_s} \\ -C_{\omega_s} & S_{\omega_s} \end{bmatrix} \begin{bmatrix} \theta_x \\ \theta_y \end{bmatrix} \quad (5-28)$$

a) Torques

The objective here is to determine the relationship between case-referenced and gimbal referenced torques. The case-referenced torques are defined as:

T_{c_i} Control torque about the x-axis of $\mathcal{F}_o'$

- T_{c_1} Control torque about the y-axis of $\mathcal{F}_1'$
- T_{c_2} Control torque about the tilted rotor shaft; described as a torque about the z-axis of the frame $\mathcal{F}_2'$.

The torques T_{c_1} and T_{c_2} are the transverse axes control torques acting from the case about the two orthogonal tilt axes. The torque T_{c_3} is the rotor spin control torque acting about tilted rotor spin axis.

The gimbal- referenced torques consist of:

- T_{g_1} Generalized torque about the x-axis of $\mathcal{F}_1$
- T_{g_2} Generalized torque about the y-axis of $\mathcal{F}_2$
- T_{g_3} Generalized torque about the motor shaft; described as a torque about the z-axis of the frame $\mathcal{F}_3$.

Although no applied torques act about these axes, gimbal torques can be considered as “generalized torques.” They represent the equivalent effect of the case-referenced control torques as seen about the three gimbal axes. Gimbal torques are required in order to formulate the equations of motion of the tuned rotor instrument. Based on the discussion in Appendix A.1, the gimbal torques are shown to be a function of the gimbal-referenced angles, case-referenced angles and the case-referenced torques where:

$$\begin{aligned}
T_{s_x} &= \frac{1}{C_{\phi_y}} \left\{ C_{\phi_z} C_{\phi_z} T_{c_x} + C_{\phi_z} C_{\phi_z} S_{\phi_z} T_{c_y} - (C_{\phi_z} S_{\phi_z} C_{\phi_z} - S_{\phi_z} C_{\phi_z}) T_{c_z} \right\} \\
T_{s_y} &= \frac{1}{C_{\phi_y}} \left\{ -S_{\phi_z} T_{c_x} + C_{\phi_z} C_{\phi_z} T_{c_y} + S_{\phi_z} S_{\phi_z} T_{c_z} \right\} \\
T_{s_z} &= \frac{1}{C_{\phi_y}} \left\{ - (C_{\phi_z} S_{\phi_z} C_{\phi_z} + S_{\phi_z} S_{\phi_z}) T_{c_x} - (C_{\phi_z} S_{\phi_z} C_{\phi_z} S_{\phi_z} - S_{\phi_z} S_{\phi_z} C_{\phi_z}) T_{c_y} \right. \\
&\quad \left. - (C_{\phi_z} S_{\phi_z} S_{\phi_z} C_{\phi_z} - S_{\phi_z} S_{\phi_z} S_{\phi_z} - C_{\phi_z} C_{\phi_z} C_{\phi_z}) T_{c_z} \right\}
\end{aligned} \tag{5-29}$$

For the situation where the tilt angles are small and the spin rate is a constant value ω_s , the gimbal torques simplify to the form:

$$\begin{aligned}
T_{s_x} &= C_{\omega_s} T_{c_x} + S_{\omega_s} T_{c_y} \\
T_{s_y} &= -S_{\omega_s} T_{c_x} + C_{\omega_s} T_{c_y} \\
T_{s_z} &= T_{c_z} = 0
\end{aligned} \tag{5-30}$$

5.2 Derivation of a Three-Body, Non-Linear Dynamic Model in Spinning Coordinates

Cain [14] demonstrated the effectiveness of using either Lagrange's equations or a Newton-Euler formulation to derive the equations of motion of a tuned-rotor gyroscope. In this study, Lagrange's equations were used to derive the equations of motion of the tuned-rotor instrument. This choice was made based on the involved trigonometry required to describe the tuned-rotor instrument system kinematics and the fact that internal constraint loads (e.g. forces acting on the flexures) were not required in this problem. In formulating the Lagrangian of the tuned-rotor gyro system depicted in Figure 5-2, it was assumed that reference frames $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ are coincident with the principal axes of inertia of the motor shaft, gimbal and rotor bodies respectively. The

kinetic energy T is a function of the angular velocities and mass properties of the three constituent bodies:

$$T = \frac{1}{2} [\bar{\omega}_m^T I_m \bar{\omega}_m + \bar{\omega}_g^T I_g \bar{\omega}_g + \bar{\omega}_r^T I_r \bar{\omega}_r] \quad (5-31)$$

The motor, gimbal and rotor inertia matrices are defined below:

$$I_m = \begin{bmatrix} I_{m_x} & 0 & 0 \\ 0 & I_{m_y} & 0 \\ 0 & 0 & I_{m_z} \end{bmatrix} \quad (5-32)$$

$$I_g = \begin{bmatrix} I_{g_x} & 0 & 0 \\ 0 & I_{g_y} & 0 \\ 0 & 0 & I_{g_z} \end{bmatrix} \quad (5-33)$$

$$I_r = \begin{bmatrix} I_{r_x} & 0 & 0 \\ 0 & I_{r_y} & 0 \\ 0 & 0 & I_{r_z} \end{bmatrix} \quad (5-34)$$

Substituting (5-32), (5-33) and (5-34) into (5-31), yields the following expression for T :

$$T = \frac{1}{2} (I_{m_x} \omega_{m_x}^2 + I_{m_y} \omega_{m_y}^2 + I_{m_z} \omega_{m_z}^2 + I_{g_x} \omega_{g_x}^2 + I_{g_y} \omega_{g_y}^2 + I_{g_z} \omega_{g_z}^2 + I_{r_x} \omega_{r_x}^2 + I_{r_y} \omega_{r_y}^2 + I_{r_z} \omega_{r_z}^2) \quad (5-35)$$

The potential energy function of the system is dependent on the two pairs of flexure pivot stiffness (K_x, K_y) and the two gimbal angles (θ_x, θ_y) as shown below:

$$V = \frac{1}{2} (K_x \theta_x^2 + K_y \theta_y^2) \quad (5-36)$$

The Lagrangian of the system is defined as:

$$L = T - V \quad (5-37)$$

The three equations of motion can now be determined by applying Lagrange's equation where:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_x} \right) - \frac{\partial L}{\partial \theta_x} = Q_{s_x} \quad (5-38)$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_y} \right) - \frac{\partial L}{\partial \theta_y} = Q_{s_y} \quad (5-39)$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_z} \right) - \frac{\partial L}{\partial \theta_z} = Q_{s_z} \quad (5-40)$$

Q_{s_x} , Q_{s_y} , and Q_{s_z} are the generalized torques acting about the three gimbal axes. As shown below, each of these generalized torques consist of a generalized control torque and in the case of the flexure pivot degrees of freedom, a structural damping torque.

$$\begin{aligned} Q_{s_x} &= T_{g_x} - C_x \dot{\theta}_x \\ Q_{s_y} &= T_{g_y} - C_y \dot{\theta}_y \\ Q_{s_z} &= T_{g_z} \end{aligned} \quad (5-41)$$

C_x and C_y are the damping coefficients of the two crossed-flexure pivot pairs.

The partial derivatives of the Lagrangian with respect to the generalized coordinates and their first derivatives with respect to time can be evaluated by noting the following relationship:

$$\begin{aligned} \frac{\partial L}{\partial \dot{\theta}_x} = & \frac{\partial L}{\partial \omega_{m_z}} \frac{\partial \omega_{m_z}}{\partial \dot{\theta}_x} + \frac{\partial L}{\partial \omega_{g_x}} \frac{\partial \omega_{g_x}}{\partial \dot{\theta}_x} + \frac{\partial L}{\partial \omega_{g_y}} \frac{\partial \omega_{g_y}}{\partial \dot{\theta}_x} + \frac{\partial L}{\partial \omega_{g_z}} \frac{\partial \omega_{g_z}}{\partial \dot{\theta}_x} \\ & + \frac{\partial L}{\partial \omega_{r_x}} \frac{\partial \omega_{r_x}}{\partial \dot{\theta}_x} + \frac{\partial L}{\partial \omega_{r_y}} \frac{\partial \omega_{r_y}}{\partial \dot{\theta}_x} + \frac{\partial L}{\partial \omega_{r_z}} \frac{\partial \omega_{r_z}}{\partial \dot{\theta}_x} \end{aligned} \quad (5-42)$$

The other partial derivative terms $\frac{\partial L}{\partial \theta_x}$, $\frac{\partial L}{\partial \dot{\theta}_y}$, $\frac{\partial L}{\partial \theta_y}$, $\frac{\partial L}{\partial \dot{\theta}_z}$ and $\frac{\partial L}{\partial \theta_z}$ can be determined in a similar manner. Evaluation of the above partial derivative terms is aided by the use of the following partial derivatives $\frac{\delta \bar{\omega}_b}{\delta \dot{\theta}_x}$, $\frac{\delta \bar{\omega}_b}{\delta \dot{\theta}_y}$, $\frac{\delta \bar{\omega}_m}{\delta \dot{\theta}_x}$, $\frac{\delta \bar{\omega}_m}{\delta \dot{\theta}_y}$, $\frac{\delta \bar{\omega}_g}{\delta \dot{\theta}_x}$, $\frac{\delta \bar{\omega}_g}{\delta \dot{\theta}_y}$, $\frac{\delta \bar{\omega}_r}{\delta \dot{\theta}_x}$ and $\frac{\delta \bar{\omega}_r}{\delta \dot{\theta}_y}$ given in Appendix A.2.

Evaluating equations (5-38), (5-39) and (5-40), the three equations of motion can be expressed in the following form:

$$\mathbf{M}(\mathbf{x}) \ddot{\mathbf{x}} + \mathbf{C} \dot{\mathbf{x}} + \mathbf{K} \mathbf{x} = \mathbf{T}_g + \mathbf{F}_{nl}(\mathbf{x}, \dot{\mathbf{x}}, \bar{\omega}_b, \dot{\bar{\omega}}_b) \quad (5-43)$$

The gimbal angles $\mathbf{x}$, gimbal torques $\mathbf{T}_g$, external rates $\bar{\omega}_b$, inertia matrix $\mathbf{M}$, stiffness matrix $\mathbf{K}$, damping matrix $\mathbf{C}$ and non-linear torques $\mathbf{F}_{nl}$ are given below:

$$\mathbf{x} = [\theta_x \quad \theta_y \quad \theta_z]^T \quad (5-44)$$

$$\mathbf{T}_g = [T_{g_x} \quad T_{g_y} \quad T_{g_z}]^T \quad (5-45)$$

$$\bar{\omega}_b = [\omega_{b_x} \quad \omega_{b_y} \quad \omega_{b_z}]^T$$

$$\mathbf{F}_{nl}(\mathbf{x}, \dot{\mathbf{x}}, \bar{\omega}_b, \dot{\bar{\omega}}_b) = \begin{bmatrix} f_{nl_1}(\mathbf{x}, \dot{\mathbf{x}}, \bar{\omega}_b, \dot{\bar{\omega}}_b) \\ f_{nl_2}(\mathbf{x}, \dot{\mathbf{x}}, \bar{\omega}_b, \dot{\bar{\omega}}_b) \\ f_{nl_3}(\mathbf{x}, \dot{\mathbf{x}}, \bar{\omega}_b, \dot{\bar{\omega}}_b) \end{bmatrix} \quad (5-46)$$

$$\mathbf{M}(\mathbf{x}) = \begin{bmatrix} I_z & 0 & \frac{1}{2}(I_z - I_x)C_{\bar{\theta}_1}S_{2\bar{\theta}_1} \\ 0 & I_x & I_x S_{\bar{\theta}_1} \\ \frac{1}{2}(I_z - I_x)C_{\bar{\theta}_1}S_{2\bar{\theta}_1} & I_x S_{\bar{\theta}_1} & I_z \end{bmatrix} \quad (5-47)$$

$$\mathbf{C} = \begin{bmatrix} C_x & 0 & 0 \\ 0 & C_y & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (5-48)$$

$$\mathbf{K} = \begin{bmatrix} K_x & 0 & 0 \\ 0 & K_y & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (5-49)$$

$$I_z = I_x C_{\bar{\theta}_1}^2 + I_y S_{\bar{\theta}_1}^2 + I_{g_z} \quad (5-50)$$

$$I_x = I_x S_{\bar{\theta}_1}^2 - I_y + C_{\bar{\theta}_1}^2 I_z - I_{g_x} - I_{g_z} \quad (5-51)$$

$$I_y = I_x C_{\bar{\theta}_1}^2 S_{\bar{\theta}_1}^2 - I_x S_{\bar{\theta}_1}^2 - I_z C_{\bar{\theta}_1}^2 C_{\bar{\theta}_1}^2 - I_{g_y} S_{\bar{\theta}_1}^2 - I_{g_z} C_{\bar{\theta}_1}^2 + I_{m_z} \quad (5-52)$$

The detailed expressions for the non-linear torque functions $f_{n_i}(\mathbf{x}, \dot{\mathbf{x}}, \bar{\omega}_b, \dot{\bar{\omega}}_b)$ are provided in Appendix A.3. It should be noted that the linear terms involving the angular acceleration $\dot{\bar{\omega}}_b$ have been included in the functions f_{n_i} .

For the purposes of simulation, the equations of motion can be described in terms of the following expression giving the acceleration of the three generalized coordinates:

$$\ddot{\mathbf{x}} = \mathbf{M}(\mathbf{x})^{-1} [-\mathbf{C}\dot{\mathbf{x}} - \mathbf{K}\mathbf{x}] + \mathbf{M}(\mathbf{x})^{-1} \mathbf{F}_{n_i}(\mathbf{x}, \dot{\mathbf{x}}, \bar{\omega}_b, \dot{\bar{\omega}}_b) + \mathbf{M}(\mathbf{x})^{-1} \mathbf{T}_g \quad (5-53)$$

Equation (5-53) consists of three terms: the linear spring/damper torques, the non-linear inertial torques and the generalized applied torques. Note also that the tuned-rotor instrument inertia matrix $\mathbf{M}$ is dependent on the gimbal angles.

5.3 Linearized Dynamic Model in Gimbal-Referenced Coordinates

A linear form of the tuned-rotor instrument equations of motion derived in Section 5.2 is required for the purposes of both dynamic and control system analysis. This can be obtained through the linearization process often used in conjunction with Lypanuov's linearization method of stability analysis [41]. Consider a dynamic system described in the following form:

$$\dot{\mathbf{x}} = \mathbf{F}_{nl}(\mathbf{x}, \mathbf{u}) \quad (5-55)$$

$\mathbf{x}$ is an n -vector of system states and $\mathbf{u}$ is a p -vector of system inputs. Assuming that $\mathbf{F}_{nl}(\mathbf{x}, \mathbf{u})$ is continuously differentiable, such a non-autonomous system can be written in terms of a Taylor series expansion of the form:

$$\dot{\mathbf{x}} = \left(\frac{\partial \mathbf{F}_{nl}}{\partial \mathbf{x}} \right)_{\mathbf{x}=\mathbf{0}} \mathbf{x} + \left(\frac{\partial \mathbf{F}_{nl}}{\partial \mathbf{u}} \right)_{\mathbf{x}=\mathbf{0}, \mathbf{u}=\mathbf{0}} \mathbf{u} + \mathbf{F}_{h.o.}(\mathbf{x}, \mathbf{u}) \quad (5-56)$$

$\left(\frac{\partial \mathbf{F}_{nl}}{\partial \mathbf{x}} \right)_{\mathbf{x}=\mathbf{0}}$ is the Jacobian of $\mathbf{F}_{nl}$ with respect to $\mathbf{x}$, $\left(\frac{\partial \mathbf{F}_{nl}}{\partial \mathbf{u}} \right)_{\mathbf{x}=\mathbf{0}, \mathbf{u}=\mathbf{0}}$ is the Jacobian of $\mathbf{F}_{nl}$

with respect to $\mathbf{u}$, while $\mathbf{F}_{h.o.}(\mathbf{x}, \mathbf{u})$ represents higher order terms in $\mathbf{x}$ and $\mathbf{u}$. It is assumed that the natural equilibrium point about which the linearization occurs is $\mathbf{x} = \mathbf{0}$. Ignoring the higher order terms, equation (5-56) can be expressed in the standard form of a linear state-space model:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad (5-57)$$

The state matrix $\mathbf{A}$ and control matrix $\mathbf{B}$ are given by the following partial derivatives:

$$\begin{aligned} \mathbf{A} &= \left(\frac{\partial \mathbf{F}_{nl}}{\partial \mathbf{x}} \right)_{\mathbf{x}=0} \\ \mathbf{B} &= \left(\frac{\partial \mathbf{F}_{nl}}{\partial \mathbf{u}} \right)_{\mathbf{x}=0, \mathbf{u}=0} \end{aligned} \quad (5-58)$$

This approach is now applied to the linearization of the tuned-rotor instrument equations of motion given in equation (5-43). The linearization occurs about the nominal operating condition of zero spin axis tilt with a constant spin rate ω_s i.e.

$$\begin{aligned} \mathbf{x}_0 &= \begin{bmatrix} \theta_x \\ \theta_y \\ \theta_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \omega_s t \end{bmatrix} \\ \dot{\mathbf{x}}_0 &= \begin{bmatrix} \dot{\theta}_x \\ \dot{\theta}_y \\ \dot{\theta}_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \omega_s \end{bmatrix} \end{aligned} \quad (5-59)$$

Equation (5-43) can be expressed as a combination of linear terms and a Taylor series expansion where

$$\begin{aligned} \left(\frac{\partial (\mathbf{M}(\mathbf{x})\ddot{\mathbf{x}})}{\partial \ddot{\mathbf{x}}} \right)_{\mathbf{x}=\mathbf{x}_0} \ddot{\mathbf{x}} + \mathbf{C} \dot{\mathbf{x}} + \mathbf{K} \mathbf{x} = \mathbf{T}_g + \left(\frac{\partial \mathbf{F}_{nl}}{\partial \mathbf{x}} \right)_{\substack{\mathbf{x}=\mathbf{x}_0, \mathbf{x}=\mathbf{x}_0, \\ \dot{\omega}_b=0, \ddot{\omega}_b=0}} \mathbf{x} - \left(\frac{\partial \mathbf{F}_{nl}}{\partial \dot{\mathbf{x}}} \right)_{\substack{\mathbf{x}=\mathbf{x}_0, \mathbf{x}=\mathbf{x}_0, \\ \dot{\omega}_b=0, \ddot{\omega}_b=0}} \dot{\mathbf{x}} \\ - \left(\frac{\partial \mathbf{F}_{nl}}{\partial \dot{\omega}_b} \right)_{\substack{\mathbf{x}=\mathbf{x}_0, \mathbf{x}=\mathbf{x}_0, \\ \dot{\omega}_b=0, \ddot{\omega}_b=0}} \dot{\omega}_b - \left(\frac{\partial \mathbf{F}_{nl}}{\partial \ddot{\omega}_b} \right)_{\substack{\mathbf{x}=\mathbf{x}_0, \mathbf{x}=\mathbf{x}_0, \\ \dot{\omega}_b=0, \ddot{\omega}_b=0}} \ddot{\omega}_b \quad (5-60) \\ + \mathbf{F}_{h.o.t.}(\mathbf{x}, \dot{\mathbf{x}}, \ddot{\omega}_b, \ddot{\omega}_b) \end{aligned}$$

Evaluating each of the partial derivative terms in (5-60) yields

$$\left(\frac{\partial(\mathbf{M}(\mathbf{x})\ddot{\mathbf{x}})}{\partial \mathbf{x}} \right)_{(\mathbf{x}=\mathbf{x}_0)} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} \ddot{\mathbf{x}} \quad (5-61)$$

$$\left(\frac{\partial \mathbf{F}_{ri}}{\partial \mathbf{x}} \right)_{\substack{\mathbf{x}=0, \dot{\mathbf{x}}=0, \\ \ddot{\mathbf{x}}=0}} = \begin{bmatrix} (I_{r_z} - I_{r_y} - I_{g_y} - 2J)\omega_z^2 & 0 & 0 \\ 0 & (I_{r_z} - I_{r_x})\omega_z^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (5-62)$$

$$\left(\frac{\partial \mathbf{F}_{ri}}{\partial \mathbf{x}} \right)_{\substack{\mathbf{x}=0, \dot{\mathbf{x}}=0, \\ \ddot{\mathbf{x}}=0}} = \begin{bmatrix} C_x & (I_{r_z} - I_{r_x} - I_{r_y})\omega_z & 0 \\ (I_{r_z} - I_{r_x} - I_{r_y})\omega_z & C_y & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (5-63)$$

$$\left(\frac{\partial \mathbf{F}_{ri}}{\partial \omega_b} \right)_{(\mathbf{x}=0, \dot{\mathbf{x}}=0, \ddot{\mathbf{x}}=0)} = \begin{bmatrix} (I_{r_x} + I_{g_x})C_{\omega_z} & (I_{r_x} + I_{g_x})S_{\omega_z} & 0 \\ -I_{r_y}S_{\omega_z} & -I_{r_y}C_{\omega_z} & 0 \\ 0 & 0 & (I_{m_z} + I_{g_z} - I_{r_z}) \end{bmatrix} \quad (5-64)$$

$$\left(\frac{\partial \mathbf{F}_{ri}}{\partial \omega_b} \right)_{\mathbf{x}=0, \dot{\mathbf{x}}=0, \ddot{\mathbf{x}}=0} = \omega_z \cdot \begin{bmatrix} -(I_{r_z} - I_{r_x} - I_{r_y} + I_{g_x} - I_{g_y} - I_{g_z})S_{\omega_z} & -(I_{r_z} - I_{r_x} - I_{r_y} - I_{g_x} - I_{g_y} - I_{g_z})C_{\omega_z} & 0 \\ -(I_{r_z} - I_{r_x} + I_{r_y})C_{\omega_z} & -(I_{r_z} - I_{r_x} - I_{r_y})S_{\omega_z} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (5-65)$$

Considering only the first-order terms of each partial derivative given above, equation (5-60) simplifies to the following three linearized equations of motion:

$$\begin{aligned}
& (I_{r_z} + I_{g_z})\ddot{\theta}_x - C_x\dot{\theta}_x + (I_{r_z} - I_{r_x} - I_{r_y})\omega_s\dot{\theta}_y + \left[(I_{r_z} - I_{r_x} + I_{g_y} - 2J)\omega_s^2 + K_x \right] \theta_x \\
& \quad - (I_{r_z} - I_{g_z})C_{\omega_s}\dot{\omega}_{b_x} - (I_{r_z} - I_{g_z})S_{\omega_s}\dot{\omega}_{b_y} \\
& \quad - \omega_s (I_{r_z} - I_{r_x} - I_{r_y} - I_{g_z} - I_{g_y} + I_{g_z}) (S_{\omega_s}\dot{\omega}_{b_x} - C_{\omega_s}\dot{\omega}_{b_y}) = T_{g_x}
\end{aligned} \tag{5-66}$$

$$\begin{aligned}
& I_{r_y}\ddot{\theta}_y + C_y\dot{\theta}_y - (I_{r_z} - I_{r_x} - I_{r_y})\omega_s\dot{\theta}_x + \left[(I_{r_z} - I_{r_x})\omega_s^2 + K_y \right] \theta_y \\
& \quad - I_{r_y}S_{\omega_s}\dot{\omega}_{b_x} - I_{r_y}C_{\omega_s}\dot{\omega}_{b_y} - \omega_s (I_{r_z} - I_{r_x} - I_{r_y}) (C_{\omega_s}\dot{\omega}_{b_x} + S_{\omega_s}\dot{\omega}_{b_y}) = T_{g_y}
\end{aligned} \tag{5-67}$$

$$(I_{m_z} + I_{g_z} - I_{r_z})(\ddot{\theta}_z - \dot{\omega}_{b_z}) = T_{g_z} \tag{5-68}$$

It should be noted that in choosing the operating condition, it was assumed that the spin rate is held at constant, near-tuned speed. Consequently, $\ddot{\theta}_z$ should be zero, with equation (5-68) reducing to the following trivial form: $T_{g_z} = 0$. However, equation (5-68) will be kept in this form. For small spin axis tilts, the motion of the rotor about its spin axis is decoupled from the transverse-axes motion. The spin rate is purely a function of the spin axis inertias and the applied motor shaft torque. Equation (5-68) can be used to design a spin-axis control system.

Examining equations (5-66) and (5-67), the transverse-axes dynamics is strongly tied to the spin rate. The linearization process shows that the transverse-axes dynamics will not be affected as greatly by the small variations of the spin rate ω_s , as it will by the large average value of ω_s , particularly for terms involving ω_s^2 . By assuming a constant, non-zero spin rate, equations (5-66) and (5-67) can be used to study the transverse-axes motion

Further discussion of the nature of coupling between the spin and transverse-axes dynamics is given in Chapter 9 during the examination of the behaviour of the non-linear dynamics of the GyroWheel tuned-rotor instrument.

5.4 Linear Dynamic Models Expressed in Case-Referenced Coordinates

In order to analyse or design control systems for the transverse-axis dynamics of a tuned-rotor instrument, the model must be expressed in terms of the output being controlled (tilt angles with respect to the case) and control input (torques from torquer coils). Consequently, equations (5-66) and (5-67) must be transformed into expressions in terms of these case-referenced quantities. This is achieved by employing the relationships relating case-referenced variables to gimbal referenced variables outlined in Section 5.1.3.

Two linear dynamic models of the transverse-axes dynamics of a tuned-rotor instrument are derived in this section. To facilitate the efficient transformation of the equations of motion into a form expressed in terms of case-referenced quantities, the idea of a single complex tilt angle and a single complex control torque is introduced. The linear equations of motion are then expressed in the form of a single differential equation with complex coefficients. In this simpler form, analysis and transformation of the equations into non-spinning coordinates can be performed more easily. Finally, the case-referenced complex equation of motion is expressed in two different SCISCO transfer function representations. The models are summarized below:

- a) The first model is termed the “**Traditional Model**” and is of the form normally used for tuned-rotor gyroscope analysis. It contains both constant and periodic inertia, damping and stiffness terms varying at a frequency of twice the spin rate.

For linear control system analysis and design, the periodic inertia, damping and stiffness coefficients must be omitted. In Chapter 6, it will be shown how the omission of these periodic terms removes some important low-frequency characteristics of the dynamics.

- b) The second, “**Complete Model**” is of a form with no periodic inertia, damping and stiffness terms. Consequently, the complete linear model can be used for control system design without loss of any important low frequency characteristics of the system.

5.4.1 Traditional Model Of Transverse-Axis Dynamics

Before beginning the transformation process, the following assumptions were made to simplify the model without loss of accuracy.

- a) The rotor transverse inertias were assumed to be equal and set to the value I_{r_t} . Similarly, the gimbal transverse inertias were assumed to be equal and set to the value I_{g_t} .

$$\begin{aligned} I_{r_x} &= I_{r_y} = I_{r_t} \\ I_{g_x} &= I_{g_y} = I_{g_t} \end{aligned}$$

- b) The spin inertias for the rotor and gimbal are re-assigned the variable names I_{r_s} and I_{g_s} to more clearly distinguish between spin (sub-subscript “s”) and transverse (sub-subscript “t”).
- c) The gimbal damping factors C_x and C_y were set equal and assigned the common gimbal damping term c_g .

Based on these assumptions, the transverse-axes equations of motion (5-66) and (5-67) simplify to following form:

$$\begin{aligned} (I_r - I_{g_i})\ddot{\theta}_x - c_g \dot{\theta}_x - (I_r - 2I_r)\omega_s \dot{\theta}_y + [(I_r - I_r - I_{g_i} - 2J)\omega_s^2 + K_x]\theta_x \\ - (I_r + I_{g_i})C_{\omega_r}\dot{\omega}_{b_i} + (I_r + I_{g_i})S_{\omega_r}\dot{\omega}_{b_y} - \omega_s(I_r + I_{g_i})(S_{\omega_r}\omega_{b_i} - C_{\omega_r}\omega_{b_y}) = T_{g_x} \end{aligned} \quad (5-69)$$

$$\begin{aligned} I_r \ddot{\theta}_y - c_g \dot{\theta}_y - (I_r - 2I_r)\omega_s \dot{\theta}_x - [(I_r - I_r)\omega_s^2 + K_y]\theta_y \\ - I_r S_{\omega_r}\dot{\omega}_{b_i} - I_r C_{\omega_r}\dot{\omega}_{b_y} - \omega_s I_r (C_{\omega_r}\omega_{b_i} + S_{\omega_r}\omega_{b_y}) = T_{g_y} \end{aligned} \quad (5-70)$$

To express equations (5-69) and (5-70) in the combined form of a single complex differential equation, the following complex variables are defined:

$$\theta = \theta_x + j\theta_y \quad (5-71)$$

$$T_g = T_{g_x} + jT_{g_y} \quad (5-72)$$

$$\omega_b = \omega_{b_i} - j\omega_{b_y} \quad (5-73)$$

The gimbal angles, gimbal-referenced torques and base-body rates are then expressed in terms of these quantities where:

$$\theta_x = \frac{1}{2}(\theta + \bar{\theta}) \quad (5-74)$$

$$\theta_y = \frac{j}{2}(\bar{\theta} - \theta) \quad (5-75)$$

$$T_{g_x} = \frac{1}{2}(T_g + \bar{T}_g) \quad (5-76)$$

$$T_{g_y} = \frac{j}{2}(\bar{T}_g - T_g) \quad (5-77)$$

$$\omega_{b_i} = \frac{1}{2}(\omega_b + \bar{\omega}_b) \quad (5-78)$$

$$\omega_{b_s} = \frac{1}{2}(\bar{\omega}_b - \omega_b) \quad (5-79)$$

Equation (5-69) is now added to $j = \sqrt{-1}$ times equation (5-70). By substituting equations (5-74) to (5-79) into this result, the following single differential equation with complex coefficients is formed:

$$\begin{aligned} \frac{1}{2}(2I_r - I_{g_i})\ddot{\theta} - \frac{1}{2}[2c_s - j(I_r - 2I_r)\omega_s]\dot{\theta} - \frac{1}{2}[(2I_r - 2I_r - I_{g_i} - 2J)\omega_s^2 - K_x + K_y]\theta \\ + \frac{1}{2}(C_{\omega_{r_i}} - jS_{\omega_{r_i}})[(2I_r + I_{g_i})\dot{\omega}_b - j(2I_r - I_{g_i})\omega_s\omega_b] - \\ + \frac{1}{2}I_{g_i}\ddot{\bar{\theta}} - \frac{1}{2}[(I_{g_i} - 2J)\omega_s^2 - K_x - K_y]\bar{\theta} \\ - \frac{1}{2}(C_{\omega_{r_i}} - jS_{\omega_{r_i}})[I_{g_i}\dot{\bar{\omega}}_b - jI_{g_i}\omega_s\bar{\omega}_b] = T_s \end{aligned} \quad (5-80)$$

To facilitate later analysis and to highlight some previously recognized characteristics of tuned-rotor instruments, the following quantities are now introduced:

$$\alpha = \frac{K_x + K_y - 2J\omega_s^2}{2I_r - I_{g_i}} \quad (5-81)$$

$$\beta = \frac{K_x - K_y - 2J\omega_s^2}{2I_r - I_{g_i}} \quad (5-82)$$

$$\gamma = \frac{I_{g_i}}{2I_r - I_{g_i}} \quad (5-83)$$

$$\omega_n = \frac{2I_r}{2I_r + I_{g_i}}\omega_s \quad (5-84)$$

$$\gamma_z = \frac{I_{g_i}}{2I_r - I_{g_i}} \quad (5-85)$$

$$c_s' = \frac{c_s}{2I_{r_i} - I_{s_i}} \quad (5-86)$$

$$I_{r_i}' = I_{r_i} - \frac{I_{s_i}}{2} \quad (5-87)$$

ω_n represents the nutation frequency of a tuned-rotor instrument and when $\alpha = 0$, the transverse-axes dynamics are considered "tuned".

Substituting the expressions for α , β , γ , γ_2 , ω_n and c_s' into equation (5-80), the transverse-axis dynamic equation takes the form

$$\begin{aligned} \ddot{\theta} + [2c_s' - j(\omega_n - 2(1-\gamma))\omega_s] \dot{\theta} + [\omega_s\omega_n - (2\gamma-1)\omega_s^2 - \alpha] \theta \\ + e^{j\omega_s t} [\dot{\bar{\omega}}_s - j(\omega_n - \gamma_2\omega_s)\bar{\omega}_s] + \\ + \gamma\ddot{\bar{\theta}} + (\gamma\omega_s^2 - \beta)\bar{\theta} - e^{j\omega_s t} [\gamma\ddot{\bar{\omega}}_s - j\gamma_2\omega_s\bar{\omega}_s] = \frac{2}{(2I_{r_i} - I_{s_i})} T_s \end{aligned} \quad (5-88)$$

This equation represents the response of both the complex angle θ and its complex conjugate $\bar{\theta}$ to the complex control torque T_s , the base rate ω_s and its complex conjugate $\bar{\omega}_s$.

A complex equation of motion of the transverse-axis dynamics expressed in terms of the case-referenced angles (ϕ_x, ϕ_y) and torques (T_{τ_x}, T_{τ_y}) can now be found. From equation (5-26), the relationship between the two gimbal angles $[\theta_x, \theta_y]^T$ and case-referenced tilt angles $[\phi_x, \phi_y]^T$ is repeated below:

$$\begin{bmatrix} \theta_x \\ \theta_y \end{bmatrix} = \begin{bmatrix} C_{\omega_r} & S_{\omega_r} \\ -S_{\omega_r} & C_{\omega_r} \end{bmatrix} \begin{bmatrix} \phi_x \\ \phi_y \end{bmatrix}$$

The complex, case-referenced spin-axis tilt $\phi(s)$ is introduced as:

$$\phi = \phi_x + j\phi_y \quad (5-89)$$

Based on (5-26) and (5-89), the corresponding relation between the complex case-reference tilt angle $\phi(t)$ and the complex gimbal angle $\theta(t)$ is

$$\begin{aligned} \phi(t) &= (C_{\omega_r} - jS_{\omega_r})\theta(t) \\ &= e^{j\omega_r t} \theta(t) \end{aligned} \quad (5-90)$$

Similarly the conjugate of the complex case-referenced tilt angle $\bar{\phi}$ can be expressed in terms of the conjugate of the complex gimbal referenced angle $\bar{\theta}$. Taking the complex conjugate of the expression in (5-90) results in the following expression:

$$\begin{aligned} \bar{\phi}(t) &= (C_{\omega_r} + jS_{\omega_r})\bar{\theta}(t) \\ &= e^{-j\omega_r t} \bar{\theta}(t) \end{aligned} \quad (5-91)$$

Finally, the complex case-referenced torque T_c is defined as

$$T_c = T_{c_x} + jT_{c_y}$$

The relationship between the case-referenced torques and the gimbal-referenced torques described in (5-30) can be expressed in the following complex form:

$$T_g = e^{-j\omega_s t} T_c \quad (5-92)$$

Taking the first and second derivatives of θ and $\bar{\theta}$ with respect to time yields the expressions:

$$\begin{aligned} \dot{\theta} &= e^{-j\omega_s t} \dot{\phi} - j\omega_s e^{-j\omega_s t} \phi \\ \dot{\bar{\theta}} &= e^{j\omega_s t} \dot{\bar{\phi}} + j\omega_s e^{j\omega_s t} \bar{\phi} \\ \ddot{\theta} &= e^{-j\omega_s t} \ddot{\phi} - 2j\omega_s e^{-j\omega_s t} \dot{\phi} - \omega_s^2 e^{-j\omega_s t} \phi \\ \ddot{\bar{\theta}} &= e^{j\omega_s t} \ddot{\bar{\phi}} + 2j\omega_s e^{j\omega_s t} \dot{\bar{\phi}} - \omega_s^2 e^{j\omega_s t} \bar{\phi} \end{aligned} \quad (5-93)$$

Substituting (5-89), (5-91), (5-92) and (5-93) into (5-88), yields the following equation of motion of the transverse-axes dynamics expressed in terms of the case-referenced tilt angle $\phi(t)$ and its complex conjugate $\bar{\phi}(t)$:

$$\begin{aligned} e^{-j\omega_s t} \left\{ \ddot{\phi} - \left[2c_g' - j(\omega_n + 2\gamma\omega_s) \right] \dot{\phi} - \left(\alpha - 2j\omega_s c_g' \right) \phi \right. \\ \left. - \dot{\omega}_b - j(\omega_n + \gamma_2\omega_s)\omega_b \right. \\ \left. - e^{2j\omega_s t} \left[\left(\gamma\ddot{\bar{\phi}} - 2j\gamma\omega_s\dot{\bar{\phi}} - \beta\bar{\phi} \right) + \left(\gamma\dot{\bar{\omega}}_b + j\gamma_2\omega_s\bar{\omega}_b \right) \right] \right\} = e^{-j\omega_s t} \frac{T_c}{I_r} \end{aligned} \quad (5-94)$$

Eliminating the factor $e^{-j\omega_s t}$, the transverse-axes equation of motion simplifies to the following form:

$$\begin{aligned} \ddot{\phi} - \left[2c_g' - j(\omega_n - 2\gamma\omega_s) \right] \dot{\phi} + \left(\alpha - 2j\omega_s c_g' \right) \phi + \dot{\omega}_b - j(\omega_n - \gamma_2\omega_s)\omega_b \\ - e^{2j\omega_s t} \left[\left(\gamma\ddot{\bar{\phi}} - 2j\gamma\omega_s\dot{\bar{\phi}} - \beta\bar{\phi} \right) + \left(\gamma\dot{\bar{\omega}}_b + j\gamma_2\omega_s\bar{\omega}_b \right) \right] = \frac{T_c}{I_r} \end{aligned} \quad (5-95)$$

To obtain the case-referenced system model expressed in terms of the real-valued tilt angles and control torques, the left and right sides of equation (5-95) can be expanded out into real and complex quantities. The real terms on the left and right sides of (5-95) can be equated to obtain the first of the case-referenced transverse-axes equations of motion. Similarly, the complex terms on the left and right sides of (5-95) can be equated to obtain the second case-referenced transverse-axes equation of motion. The result is the following system of equations:

$$\mathbf{M}_z \begin{bmatrix} \ddot{\phi}_x \\ \ddot{\phi}_y \end{bmatrix} - \mathbf{D}_z \begin{bmatrix} \dot{\phi}_x \\ \dot{\phi}_y \end{bmatrix} + \mathbf{K}_z \begin{bmatrix} \phi_x \\ \phi_y \end{bmatrix} - \mathbf{M}_b \begin{bmatrix} \dot{\omega}_{b_x} \\ \dot{\omega}_{b_y} \end{bmatrix} - \mathbf{D}_b \begin{bmatrix} \omega_{b_x} \\ \omega_{b_y} \end{bmatrix} = \frac{2}{2I_{\tau} + I_{st}} \begin{bmatrix} T_{\tau_x} \\ T_{\tau_y} \end{bmatrix} \quad (5-96)$$

The case-referenced inertia matrices $\mathbf{M}_z$ and $\mathbf{M}_b$, damping matrices $\mathbf{D}_z$ and $\mathbf{D}_b$ and stiffness matrix $\mathbf{K}_z$ are given below:

$$\begin{aligned} \mathbf{M}_z &= \begin{bmatrix} 1 - \gamma C_{2\omega_s} & \gamma S_{2\omega_s} \\ \gamma S_{2\omega_s} & 1 - \gamma C_{2\omega_s} \end{bmatrix} \\ \mathbf{D}_z &= \begin{bmatrix} 2c_g' - 2\gamma\omega_s S_{2\omega_s} & (\omega_n - 2\gamma\omega_s) - 2\gamma\omega_s C_{2\omega_s} \\ -(\omega_n - 2\gamma\omega_s) + 2\gamma\omega_s C_{2\omega_s} & 2c_g' + 2\gamma\omega_s S_{2\omega_s} \end{bmatrix} \\ \mathbf{K}_z &= \begin{bmatrix} \alpha + \beta C_{2\omega_s} & 2\omega_s c_g' + \beta S_{2\omega_s} \\ -2\omega_s c_g' + \beta S_{2\omega_s} & \alpha - \beta C_{2\omega_s} \end{bmatrix} \\ \mathbf{M}_b &= \begin{bmatrix} 1 + \gamma C_{2\omega_s} & \gamma S_{2\omega_s} \\ \gamma S_{2\omega_s} & 1 - \gamma C_{2\omega_s} \end{bmatrix} \\ \mathbf{D}_b &= \begin{bmatrix} -\gamma_2\omega_s S_{2\omega_s} & (\omega_n + \gamma_2\omega_s) + \gamma_2\omega_s C_{2\omega_s} \\ -(\omega_n + \gamma_2\omega_s) - \gamma_2\omega_s C_{2\omega_s} & \gamma_2\omega_s S_{2\omega_s} \end{bmatrix} \end{aligned} \quad (5-97)$$

Examining equations (5-96) and (5-97), the most important thing to note is the periodic component in each of the mass, damping and stiffness matrices. In this form, the full model cannot be used for linear control system analysis or design. Typically for tuned-rotor gyroscope rebalance control system design, the periodic components are ignored with the model simplifying to the following form:

$$\begin{aligned}
 \ddot{\phi}_x - 2c_g' \dot{\phi}_x - \alpha \phi_x - (\omega_n - 2\gamma\omega_s) \dot{\phi}_y + 2c_g' \omega_s \phi_y \\
 - \dot{\omega}_{b_x} + (\omega_n - \gamma_2 \omega_s) \omega_{b_x} &= \frac{T_{c_x}}{I_{r_x}} \\
 \ddot{\phi}_y - 2c_g' \dot{\phi}_y - \alpha \phi_y - (\omega_n - 2\gamma\omega_s) \dot{\phi}_x - 2c_g' \omega_s \phi_x \\
 - \dot{\omega}_{b_y} - (\omega_n - \gamma_2 \omega_s) \omega_{b_y} &= \frac{T_{c_y}}{I_{r_y}}
 \end{aligned} \tag{5-98}$$

The disadvantage of this approach is that some information is lost. In fact analysis in Chapter 7 will show that surprisingly, the low-frequency behaviour of the model changes slightly when the high frequency periodic terms at $2\omega_s$ are discarded. For the purposes of this study, the model derived above will be termed the “Traditional Model”. The “Complete Model” derived in the following section will address some of the shortcomings of the Traditional Model.

Finally, the equations of motion can be expressed in transfer function form. The SCISCO transfer function is developed first by taking the Laplace transform of (5-95). The SCISCO transfer function relating the tilt angle $\phi(s)$ to the complex torque $T_c(s)$, the base angular rate $\omega_b(s)$, the complex conjugate of the tilt $\bar{\phi}(s - 2j\omega_s)$, and complex conjugate of the base angular rate $\bar{\omega}_b(s - 2j\omega_s)$ is of the form:

$$\phi(s) = \hat{G}_{\phi T_c}(s) T_c(s) + \hat{G}_{\phi \omega_b}(s) \omega_b(s) + \hat{G}_{\phi \bar{\phi}}(s) \bar{\phi}(s - 2j\omega_s) + \hat{G}_{\phi \bar{\omega}_b}(s) \bar{\omega}_b(s - 2j\omega_s) \tag{5-99}$$

The four transfer functions characterizing the system are given below:

$$\hat{G}_{x1}(s) = \frac{1}{I'_r \left\{ s^2 + \left(2c'_g - j(\omega_n + 2\gamma\omega_s) \right) s + \left(\alpha - 2jc'_g\omega_s \right) \right\}} \quad (5-100)$$

$$\hat{G}_{x1}(s) = \frac{-(s - j(\omega_n + \gamma_2\omega_s))}{\left\{ s^2 + \left(2c'_g - j(\omega_n - 2\gamma\omega_s) \right) s - \left(\alpha - 2jc'_g\omega_s \right) \right\}} \quad (5-101)$$

$$\hat{G}_{x2} = \frac{\gamma s^2 - 2j\gamma\omega_s s + \beta}{I'_r \left\{ s^2 - \left(2c'_g - j(\omega_n - 2\gamma\omega_s) \right) s - \left(\alpha - 2jc'_g\omega_s \right) \right\}} \quad (5-102)$$

$$\hat{G}_{x2} = \frac{\gamma s - j\omega_s(\gamma_2 - 2\gamma)}{\left\{ s^2 - \left(2c'_g - j(\omega_n - 2\gamma\omega_s) \right) s - \left(\alpha - 2jc'_g\omega_s \right) \right\}} \quad (5-103)$$

The standard form of the two-real input, two-real output transfer function can be determined by taking the Laplace transform of (5-98). Ignoring the terms involving $\bar{\phi}(s - 2j\omega_s)$ and $\bar{\omega}_2(s - 2j\omega_s)$, this results in the following two equations:

$$\begin{aligned} \left(s^2 + 2c'_g s - \alpha \right) \phi_x(s) + \left((\omega_n + 2\gamma\omega_s) s - 2c'_g\omega_s \right) \phi_y(s) \\ - s\omega_{b_x}(s) + (\omega_n + \gamma_2\omega_s)\omega_{b_y}(s) &= \frac{T_{c_1}}{I'_r}, \\ \left(-(\omega_n + 2\gamma\omega_s) s - 2c'_g\omega_s \right) \phi_x(s) + \left(s^2 + 2c'_g s + \alpha \right) \phi_y(s) \\ - s\omega_{b_y}(s) - (\omega_n - \gamma_2\omega_s)\omega_{b_x}(s) &= \frac{T_{c_2}}{I'_r}, \end{aligned} \quad (5-104)$$

Expressed in matrix notation, equation (5-104) takes the form:

$$\begin{bmatrix} s^2 + 2c_g' s + \alpha & (\omega_n + 2\gamma\omega_s)s + 2c_g' \omega_s \\ -(\omega_n - 2\gamma\omega_s)s - 2c_g' \omega_s & s^2 + 2c_g' s + \alpha \end{bmatrix} \begin{bmatrix} \phi_x(s) \\ \phi_y(s) \end{bmatrix} = \frac{1}{I_\tau'} \begin{bmatrix} T_{\tau_x}(s) \\ T_{\tau_y}(s) \end{bmatrix} \quad (5-105)$$

$$\begin{bmatrix} s & (\omega_n + \gamma\omega_s) \\ -(\omega_n - \gamma\omega_s) & s \end{bmatrix} \begin{bmatrix} \omega_{b_x}(s) \\ \omega_{b_y}(s) \end{bmatrix}$$

Solving for $[\phi_x(s) \ \phi_y(s)]^T$ in terms of $[T_{\tau_x}(s) \ T_{\tau_y}(s)]^T$ and $[\omega_{b_x}(s) \ \omega_{b_y}(s)]^T$ yields the following transfer function expression:

$$\begin{bmatrix} \phi_x(s) \\ \phi_y(s) \end{bmatrix} = \begin{bmatrix} \frac{s^2 + 2c_g' s + \alpha}{d(s)} & \frac{-(\omega_n + 2\gamma\omega_s)s - 2c_g' \omega_s}{d(s)} \\ \frac{(\omega_n - 2\gamma\omega_s)s + 2c_g' \omega_s}{d(s)} & \frac{s^2 + 2c_g' s + \alpha}{d(s)} \end{bmatrix} \begin{bmatrix} T_{\tau_x}(s) \\ T_{\tau_y}(s) \end{bmatrix} \quad (5-106)$$

$$\begin{bmatrix} \frac{s}{d(s)} & \frac{(\omega_n + 2\gamma\omega_s)}{d(s)} \\ \frac{-(\omega_n - 2\gamma\omega_s)}{d(s)} & \frac{s}{d(s)} \end{bmatrix} \begin{bmatrix} \omega_{b_x} \\ \omega_{b_y} \end{bmatrix}$$

The characteristic polynomial is given by the following expression:

$$d(s) = I_\tau' \left[\left(s^2 + 2c_g' s + \alpha \right)^2 + \left((\omega_n + 2\gamma\omega_s)s + 2c_g' \omega_s \right)^2 \right] \quad (5-107)$$

5.4.2 A New Complete Model of Transverse-Axes Dynamics

The model described in equation (5-106) is typical of the models used for tuned-rotor gyro control system design. As discussed in the previous section, it ignores the effect of the periodic inertia, damping and stiffness terms. Examining equations (5-96) and (5-97), it is not clear how important the periodic coefficients are. Staley [21] has demonstrated

that the transverse-axes dynamics of a tuned-rotor gyroscope exhibits four distinct modes when motion is viewed from the fixed case. Examining the denominator of the Traditional transfer function model in equation (5-107), there is a 4th order denominator polynomial. At most, there are two pairs of complex conjugate roots, indicating two distinct oscillatory modes. Evidently, the omission of these periodic terms has an effect on the modes of the system.

To solve these equations in a different manner, Floquet's theory of systems with periodic coefficients was considered [42]. Floquet examined differential equations of the form:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

The state matrix $\mathbf{A}$ above is considered to be periodic, i.e. $\mathbf{A}(t + T) = \mathbf{A}(t)$. He proposed the theorem that such systems can only be solved by first introducing a transformation of variables, yielding a linear homogenous system with constant coefficients. But the desired transformed system is already known for this problem; it is the equations of motion expressed in terms of gimbal-referenced coordinates as in equations (5-69) and (5-70).

Examining equation (5-95) provides a second indication of an alternative approach to deriving a transfer function model in case-referenced coordinates. It contains terms involving the complex conjugates of the case-referenced angle ϕ and base angular rate ω_b . Retaining the complex conjugate terms $\bar{\phi}(s - 2j\omega_s)$ and $\bar{\omega}_b(s - 2j\omega_s)$, as was done in deriving equation (5-96), the equivalent two differential equations expressed in terms of the real angles ϕ_x and ϕ_y have periodic terms in each of the inertia, damping and stiffness matrices. However, if the $\bar{\phi}(s - 2j\omega_s)$ and $\bar{\omega}_b(s - 2j\omega_s)$ terms are ignored in (5-95), the resulting state-space model of (5-96) has no periodic terms in the inertia, damping and stiffness matrices.

Based on these observations, the case-referenced equations of motion were re-derived by first eliminating the complex conjugate term $\bar{\theta}$ in the gimbal-referenced complex differential equation in (5-88). To produce this form, a second complex differential equation describing the transverse-axes dynamics in terms of gimbal-referenced coordinates was created through the addition of equation (5-69) to $-j$ times equation (5-70). One approach is to repeat the algebraic manipulations performed to obtain the expression in equation (5-88). A simpler method is to note the results of Section 7.1. The desired second equation can be obtained by simply taking the complex conjugate of equation (5-88), i.e.

$$\begin{aligned} \ddot{\bar{\theta}} - [2c'_g - j(\omega_n - 2(1-\gamma))\omega_s] \dot{\bar{\theta}} + [\omega_s \omega_n + (2\gamma - 1) - \alpha] \bar{\theta} \\ + e^{j\omega_s t} [\dot{\bar{\omega}}_b - j(\omega_n - \gamma_2 \omega_s) \bar{\omega}_b] + \\ - \gamma \ddot{\theta} - (\gamma \omega_s^2 - \beta) \theta + e^{-j\omega_s t} [\gamma \dot{\omega}_b - j\gamma_2 \omega_s \omega_b] = \frac{\bar{T}_g}{I_{\bar{r}}} \end{aligned} \quad (5-108)$$

With equations (5-88) and (5-108) both describing the motion of θ and $\bar{\theta}$, an expression involving only θ can now be found. This is most easily facilitated by first taking the Laplace transform of equations (5-88) and (5-108). Noting the presence of the terms with coefficients $e^{j\omega_s t}$ and $e^{-j\omega_s t}$ in (5-108), the following Laplace transforms are required:

$$\begin{aligned} \mathcal{L}(e^{j\omega_s t} f(t)) &= F(s - j\omega_s) \\ \mathcal{L}\left(e^{j\omega_s t} \frac{df'(t)}{dt}\right) &= (s - j\omega_s)F(s - j\omega_s) \end{aligned} \quad (5-109)$$

Taking the Laplace transform of equations (5-88) and (5-108) results in the following two equations:

$$n_\theta(s)\theta(s) - n_{\bar{\theta}}(s)\bar{\theta}(s) + n_\omega(s)\omega_b(s + j\omega_s) + n_{\bar{\omega}}(s)\bar{\omega}_b(s - j\omega_s) = \frac{T_g}{I_r} \quad (5-110)$$

$$\bar{n}_{\xi}(s)\theta(s) + \bar{n}_{\xi}(s)\bar{\theta}(s) + \bar{n}_{\omega}(s)\omega_b(s + j\omega_r) - \bar{n}_{\omega}(s)\bar{\omega}_b(s - j\omega_r) = \frac{\bar{T}_g}{I_g'} \quad (5-111)$$

Expressions for each of the polynomials n_{θ} , n_{ξ} , n_{ω} and $n_{\bar{\omega}}$ are provided below:

$$\begin{aligned} n_{\xi} &= s^2 + \left[2c_g' - j[\omega_n - 2(1-\gamma)\omega_r] \right] s + [\omega_r\omega_n + (2\gamma - 1)\omega_r^2 + \alpha] \\ n_{\bar{\xi}} &= \gamma s^2 + (\gamma\omega_r^2 - \beta) \\ n_{\omega} &= s - j(\omega_n - \omega_r) - j\gamma_2\omega_r \\ n_{\bar{\omega}} &= \gamma s - j\omega_r(\gamma_2 - \gamma) \end{aligned} \quad (5-112)$$

The following expression for $\bar{\theta}$ can be determined from (5-111):

$$\bar{\theta}(s) = \frac{-1}{\bar{n}_{\xi}(s)} \left\{ \bar{n}_{\xi}(s)\theta(s) + \bar{n}_{\xi}(s)\omega_b(s + j\omega_r) - \bar{n}_{\omega}(s)\bar{\omega}_b(s - j\omega_r) - \frac{\bar{T}_g(s)}{I_g'} \right\} \quad (5-113)$$

Substituting (5-113) into (5-110) and solving for $\theta(s)$ yields the desired expression for $\theta(s)$.

$$\begin{aligned} \theta(s) &= \mathbf{G}_{\mathbf{11}}(s)T_g(s) - \mathbf{G}_{\mathbf{12}}(s)\bar{T}_g(s) \\ &\quad \mathbf{G}_{\mathbf{21}}(s)\omega_b(s + j\omega_r) + \mathbf{G}_{\mathbf{22}}(s)\bar{\omega}_b(s - j\omega_r) \end{aligned} \quad (5-114)$$

where

$$\begin{aligned}\mathbf{G}_{d1}(s) &= \frac{\bar{n}_{\bar{\theta}}(s)}{I_{\bar{\tau}}' \mathbf{d}(s)} \\ \mathbf{G}_{d2}(s) &= \frac{-n_{\bar{\theta}}(s)}{I_{\bar{\tau}}' \mathbf{d}(s)}\end{aligned}\tag{5-115}$$

$$\begin{aligned}\mathbf{G}_{\bullet 1}(s) &= \frac{\bar{n}_{\bar{\omega}}(s)n_{\bar{\theta}}(s) - n_{\bar{\omega}}(s)\bar{n}_{\bar{\theta}}(s)}{\mathbf{d}(s)} \\ \mathbf{G}_{\bullet 2}(s) &= \frac{\bar{n}_{\omega}(s)n_{\bar{\theta}}(s) - \bar{n}_{\bar{\theta}}(s)n_{\bar{\omega}}(s)}{\mathbf{d}(s)}\end{aligned}$$

$$\mathbf{d}(s) = n_{\bar{\theta}}(s)\bar{n}_{\bar{\theta}}(s) - n_{\bar{\theta}}(s)^2\tag{5-116}$$

Equation (5-114) represents the complex equation of motion of the transverse-axes dynamics expressed in terms of the complex gimbal angle $\theta(s)$, the complex gimbal-referenced torque $T_g(s)$ and the complex base angular rate $\omega_s(s)$. The remaining task is to express the complex gimbal angle $\theta(s)$ in terms of the complex case-referenced tilt angle $\phi(s)$ and the complex gimbal-referenced torque $T_g(s)$ in terms of the complex case-referenced torque $T_c(s)$. Taking the Laplace transform of equation (5-90) yields:

$$\phi(s) = \theta(s - j\omega_s)\tag{5-117}$$

Similarly, taking the Laplace transform of (5-92) results in the following two relations:

$$\begin{aligned}T_g(s) &= T_c(s + j\omega_s) \\ \bar{T}_g(s) &= \bar{T}_c(s - j\omega_s)\end{aligned}\tag{5-118}$$

First, the expressions for $T_g(s)$ and $\bar{T}_g(s)$ in equation (5-118) are substituted into (5-114), resulting in the following modified transfer function relationship:

$$\begin{aligned}\theta(s) = & \mathbf{G}_{d1}(s)T_c(s-j\omega_s) + \mathbf{G}_{d2}(s)\bar{T}_c(s-j\omega_s) \\ & + \mathbf{G}_{e1}(s)\omega_b(s+j\omega_s) + \mathbf{G}_{e2}(s)\bar{\omega}_b(s-j\omega_s)\end{aligned}\quad (5-119)$$

$\phi(s)$ can now be determined by noting (5-117) and evaluating $\theta(s-j\omega_s)$. The expression for $\phi(s)$ takes the form:

$$\begin{aligned}\phi(s) = \theta(s-j\omega_s) = & \mathbf{G}_{d1}(s-j\omega_s)T_c(s) + \mathbf{G}_{d2}(s-j\omega_s)\bar{T}_c(s-2j\omega_s) \\ & - \mathbf{G}_{e1}(s-j\omega_s)\omega_b(s) + \mathbf{G}_{e2}(s-j\omega_s)\bar{\omega}_b(s-2j\omega_s)\end{aligned}\quad (5-120)$$

The case-referenced tuned-rotor instrument dynamics transfer functions $G_{d1}(s)$, $G_{d2}(s)$ and the case-referenced external rate dynamics transfer functions $G_{e1}(s)$, $G_{e2}(s)$ are now defined as:

$$\begin{aligned}G_{d1}(s) &= \mathbf{G}_{d1}(s-j\omega_s) \\ G_{d2}(s) &= \mathbf{G}_{d2}(s-j\omega_s) \\ G_{e1}(s) &= \mathbf{G}_{e1}(s-j\omega_s) \\ G_{e2}(s) &= \mathbf{G}_{e2}(s-j\omega_s)\end{aligned}\quad (5-121)$$

Equation (5-120) then takes the following form:

$$\begin{aligned}\phi(s) = \theta(s-j\omega_s) = & G_{d1}(s)T_c(s) + G_{d2}(s)\bar{T}_c(s-2j\omega_s) \\ & - G_{e1}(s)\omega_b(s) + G_{e2}(s)\bar{\omega}_b(s-2j\omega_s)\end{aligned}\quad (5-122)$$

After evaluating $\bar{n}_3(s - j\omega_s)$, $n_6(s - j\omega_s)$, $n_{\bar{6}}(s - j\omega_s)$, $n_{\omega}(s - j\omega_s)$, $n_{\bar{\omega}}(s - j\omega_s)$, $\bar{n}_2(s - j\omega_s)$ and $\bar{n}_{\bar{2}}(s - j\omega_s)$, the case-referenced tuned-rotor instrument dynamic transfer functions $G_{d1}(s)$ and $G_{d2}(s)$ take the following form:

$$G_{d1}(s) = \frac{\bar{n}_3(s - j\omega_s)}{I_r' [n_3(s - j\omega_s)\bar{n}_6(s - j\omega_s) - n_{\bar{6}}(s - j\omega_s)^2]} \quad (5-123)$$

$$= \frac{s^2 - (2c_g' - j(\omega_n - 2(2 - \gamma)\omega_s))s - (\alpha - j2c_g'\omega_s + 2\omega_s\omega_n - 4\omega_s^2(1 - \gamma))}{I_r' d(s)}$$

$$G_{d2}(s) = \frac{-n_{\bar{3}}(s - j\omega_s)}{I_r' [n_6(s - j\omega_s)\bar{n}_3(s - j\omega_s) - n_{\bar{3}}(s - j\omega_s)^2]} \quad (5-124)$$

$$= \frac{-\gamma s^2 - (1 - \gamma)\omega_s^2 - \beta}{I_r' d(s)}$$

where

$$d(s) = [s^2 - (2c_g' - j(\omega_n - 2\gamma\omega_s))s + (\alpha - j2c_g'\omega_s)] \cdot$$

$$[s^2 + (2c_g' - j(\omega_n - 2(2 - \gamma)\omega_s))s + (\alpha - j2c_g'\omega_s - 2\omega_n\omega_s - 4\omega_s^2(1 - \gamma))] -$$

$$[\gamma s^2 + (1 - \gamma)\omega_s^2 + \beta]^2 \quad (5-125)$$

Similarly, the case-referenced external rate transfer functions $G_{e1}(s)$ and $G_{e2}(s)$ take the form

$$G_{e1}(s) = \frac{1}{d(s)} \cdot \left\{ \begin{aligned} &[\gamma s - j\omega_s(\gamma_2 - \gamma)] \cdot [\gamma s^2 + (1 - \gamma)\omega_s^2 + \beta] - \\ &[s - j(\omega_n - \omega_s) - j\gamma_2\omega_s] \cdot \\ &[s^2 + (2c_g' + j(\omega_n - 2(2 - \gamma)\omega_s))s + (\alpha - j2c_g'\omega_s + 2\omega_n\omega_s - 4\omega_s^2(1 - \gamma))] \end{aligned} \right\} \quad (5-126)$$

$$G_{e2}(s) = \frac{1}{d(s)} \cdot \left\{ \begin{aligned} &[s - j(\omega_n - \omega_s) - j\gamma_2\omega_s] \cdot [\gamma s^2 + (1 - \gamma)\omega_s^2 + \beta] - \\ &[\gamma s - j\omega_s(\gamma_2 - \gamma)] \cdot \\ &[s^2 + (2c_g' - j(\omega_n - 2(2 - \gamma)\omega_s))s + (\alpha - j2c_g'\omega_s - 2\omega_n\omega_s - 4\omega_s^2(1 - \gamma))] \end{aligned} \right\} \quad (5-127)$$

Due to the large number of terms, the characteristic polynomial $d(s)$ and the numerator polynomials of $G_{e1}(s)$ and $G_{e2}(s)$ have not been expanded out further than the form shown in (5-125), (5-126) and (5-127) respectively. When using these transfer functions for control system analysis, the polynomial coefficients were evaluated using the commercial analysis tool MATLAB [43].

In summary, the Complete Model of the transverse-axes dynamics is given by the relationship

$$\begin{aligned} \phi(s) = & G_{x1}(s)T_c(s) - G_{x2}(s)\bar{T}_c(s - 2j\omega_s) \\ & + G_{e1}(s)\omega_b(s) + G_{e2}(s)\bar{\omega}_b(s - 2j\omega_s) \end{aligned} \quad (5-128)$$

The response of the complex case-referenced transverse-axis tilt $\phi(s)$ is a function of the complex control torque $T_c(s)$, the periodic control torque $\bar{T}_c(s - 2j\omega_s)$ occurring at twice

the spin frequency ω_s , the complex base angular rate $\omega_s(s)$ and the periodic external rate $\bar{\omega}_s(s - 2j\omega_s)$. Comparing this Complete SCISCO model with the Traditional SCISCO model given in (5-99), the following observations can be made:

- i) In the traditional model, the complex tilt angle $\phi(s)$ is a function of $\bar{\phi}(s - 2j\omega_s)$. Consequently, $\phi(s)$ is dependent on a transfer function with periodic coefficients.

However, in the Complete model, the dependency of $\phi(s)$ on $\bar{\phi}(s)$ has been removed. $\phi(s)$ has been expressed as a function of both $T_c(s)$ and the **periodic input** $\bar{T}(s - 2j\omega_s)$.

- ii) The characteristic polynomial of the Traditional model is of order 2, indicating two system poles. The Complete Model has a characteristic polynomial of order 4, indicating four system poles.

Conversion of the complex transfer function into its equivalent two-input, two output, real-valued form reveals a system with an 8th order characteristic polynomial made up of the roots of the complex transfer function denominator and their complex conjugates. Given that the system should still be 4th order when converted to non-spinning coordinates, this model appears to be a non-minimal realization of the gimbal referenced dynamics when expressed in non-spinning coordinates. This is not the case and the reasoning for this is discussed further in Chapter 7.

5.5 Summary

A non-linear dynamic model of the 3-body motion of a tuned-rotor gyroscope has been developed. As shown in the later chapters, this model will serve as a valuable tool to understand the interaction between spin axis and transverse axes dynamics and the influence that the control system has on this relationship. After linearizing this model about a nominal zero tilt and tuned spin rate, two linear dynamic models of the transverse-axes dynamics expressed in case-referenced coordinates were derived. The first, termed the Traditional model is shown to have periodic mass, damping and stiffness coefficients. In this form it is unsuitable for linear analysis and control design, unless these terms are ignored. However, this simplification eliminates aspects of the tuned-rotor instrument dynamics. The second linear model, termed the Complete Model has no periodic coefficients, instead having constant and periodic inputs.

In Chapter 9, it is shown that accurate knowledge of the stiffness and damping characteristics of the system is critical for estimation of angular rates of the external body on which the a rate sensor sits. With the periodic components in the Traditional Model, these two characteristics cannot be judged directly from the matrices in (5-97). The Complete Model allows these characteristics to be examined directly.

Chapter 6

Analysis of Tuned-Rotor Instrument Dynamics

A new Complete Model of the transverse-axes dynamics of a tuned rotor instrument was developed in Chapter 5. Of interest is how this modified model may alter the approach to control of the transverse-axes dynamics of a tuned-rotor instrument. In this chapter, the poles and zeros and low frequency gain of the SCISCO transfer functions of the Traditional and Complete models are determined. To allow for parametric analysis, the poles and zeros are expressed in terms of the system mechanical parameters. By omitting the periodic coefficients, the Traditional Model is shown to exhibit a different low frequency behaviour and a different set of poles as compared with the Complete Model. Although the Complete model is shown to have four distinct modes, two of these modes become fixed when case-referenced torques are applied to the system as would occur in the rebalance loop of a tuned-rotor gyroscope.

To confirm the findings of the linear analysis, the closed-form solution to the linear transverse-axes dynamics is determined and the response of the system examined. For the purposes of external rate sensing, the notion of dynamic stiffness of a tuned-rotor gyroscope is introduced. Finally, a new tuning condition is proposed, based on the observation that a tuned-rotor gyroscope exhibits a very small precession mode when the traditional tuning condition is applied.

6.1 Poles and Zeros of Linearized Dynamics Model

The objective of this analysis was to determine analytical expressions for the poles and zeros of the Traditional and Complete SCISCO transfer functions models. The expressions for the SCISCO transfer functions; particularly those of the Complete Model are very complex and not conducive to determining poles and zeros analytically. To avoid this problem, the poles and zeros of the gimbal-referenced transfer function were determined first and then the roots were transformed to case-referenced coordinates.

6.1.1 Traditional Transfer Function Model

The SCISCO transfer function model of the transverse-axis dynamics is given in equation (5-99). For this analysis, the periodic terms $\bar{\phi}(s-2j\omega_s)$ and $\bar{\omega}_b(s-2j\omega_s)$ will be ignored as they represent a high frequency effect at twice the spin frequency. Consequently, equation (5-99) simplifies to the following form:

$$\begin{aligned} \phi(s) = & \frac{1}{I_{\eta}' \left\{ s^2 - \left(2c_g' - j(\omega_n + 2\gamma\omega_s) \right) s + \left(\alpha - 2jc_g'\omega_s \right) \right\}} T_z(s) \\ & - \frac{s - j(\omega_n + \gamma_z\omega_s)}{I_{\eta}' \left\{ s^2 - \left(2c_g' - j(\omega_n - 2\gamma\omega_s) \right) s + \left(\alpha - 2jc_g'\omega_s \right) \right\}} \omega_b(s) \end{aligned} \quad (6-1)$$

The poles of this system are:

$$s = -\left\{c_s' - j \frac{(\omega_n - 2\gamma\omega_s)}{2}\right\} \pm \frac{j\omega_n}{2} \sqrt{1 - \frac{4\gamma\omega_s}{\omega_n} - \frac{4\gamma^2\omega_s^2}{\omega_n^2} - \frac{4(\alpha - 2j\omega_s c_s')}{\omega_n^2} - \frac{4c_s'^2}{\omega_n^2} - \frac{4jc_s'}{\omega_n} - \frac{4jc_s'\gamma\omega_s}{\omega_n^2}} \quad (6-2)$$

With the exception of the first term, each other term within the square root sign is a small quantity compared with unity. Therefore, the expression within the square root can be simplified by noting that

$$\sqrt{1-x} = 1 + \frac{x}{2} - \frac{x^2}{8} - \dots, \quad -1 \leq x \leq 1 \quad (6-3)$$

Expanding out the term within the square root sign up to the second order terms in (6-3), equation (6-2) takes the form

$$s = -\left[c_s' - j \frac{(\omega_n - 2\gamma\omega_s)}{2}\right] \pm j \left[\frac{\omega_n}{2} - \gamma\omega_s - \frac{\alpha - 2j\omega_s c_s'}{\omega_n} - \frac{c_s'^2}{\omega_n} - jc_s' - j \frac{c_s'\gamma\omega_s}{\omega_n} \right]$$

Therefore, the two values of s are the following system poles p_1 and p_2 :

$$\begin{aligned} p_1 &= -c_s' \left[\frac{\omega_s(2-\gamma)}{\omega_n} \right] - j \left[\frac{\alpha}{\omega_n} - \frac{c_s'^2}{\omega_n} \right] \\ p_2 &= -c_s' \left[2 - \frac{\omega_s(2-\gamma)}{\omega_n} \right] - j \left[\omega_n - 2\gamma\omega_s + \frac{\alpha}{\omega_n} - \frac{c_s'^2}{\omega_n} \right] \end{aligned} \quad (6-4)$$

The damping term for each of the two poles is initially ignored. The first pole has a very small magnitude and can be termed the precession pole. When the system is not tuned ($\alpha \neq 0$), this mode represents a coning motion of the spin axis about the null tilt position. The second frequency describes the nutation motion of the rotor about its spin axis, and is shifted from the previously defined nutation frequency ω_n by $2\gamma\omega_s + \frac{\alpha}{\omega_n}$.

The transmission zero of the second transfer function in equation (6-1), representing the effect of the external rate $\omega_s(s)$ on the tilt angle $\phi(s)$ is given by

$$z_1 = -j(\omega_n - \gamma_z\omega_s)$$

Therefore, the zero of this transfer function is almost coincident with the pole near the nutation frequency ω_n .

6.1.2 New Complete Transfer Function Model

Like the examination of the Traditional Model, the transfer functions involving the periodic terms $\bar{T}_z(s - 2j\omega_s)$ and $\bar{\omega}_b(s - 2j\omega_s)$ will not be considered. The model in equation (6-118) simplifies to the form

$$\phi(s) = G_{zi}(s)T_z(s) + G_{ei}(s)\omega_s(s) \quad (6-5)$$

Omission of the periodic input terms is not an indication that they are trivial. As shown in Chapter 10, they can interact with a feedback controller to produce changes in the secular closed-loop behaviour of transverse-axes control system.

As noted in Chapter 5, the numerator and denominator expressions for $G_{d1}(s)$ and $G_{e1}(s)$ are complex polynomials with a great many terms. Determining expressions for the poles and zeros directly from these two transfer functions is a laborious exercise. Instead, the approach employed by Staley [13] was used here. This method involves finding the poles and zeros of the gimbal referenced transfer functions $H_{d1}(s)$ and $H_{e1}(s)$ and then transforming the roots to case-referenced coordinates.

To determine analytical expressions for the poles and zeros of the system, it will be assumed that flexure damping is negligible.

6.1.2.1 Transfer Function $G_{d1}(s)$

First, the transverse-axes dynamics transfer function $\mathbf{G}_{d1}(s)$, which relates gimbal-referenced angles to gimbal-referenced torques, is considered. Based on equations (6-105) and (6-108), $\mathbf{G}_{d1}(s)$ can be expressed in the following form:

$$\mathbf{G}_{d1}(s) = \frac{s^2 - \left(2c_g' - j(\omega_n - 2(1-\gamma)\omega_s)\right)s - (\omega_s\omega_n - (2\gamma-1)\omega_s^2 - \alpha)}{I_\tau' \mathbf{d}(s)} \quad (6-6)$$

The gimbal-referenced characteristic polynomial is given by the following expression:

$$\begin{aligned} \mathbf{d}(s) = & \begin{bmatrix} s^2 - \left(2c_g' - j(\omega_n - 2(1-\gamma)\omega_s)\right)s - (\omega_s\omega_n - (2\gamma-1)\omega_s^2 - \alpha) \\ s^2 + \left(2c_g' - j(\omega_n - 2(1-\gamma)\omega_s)\right)s - (\omega_s\omega_n - (2\gamma-1)\omega_s^2 - \alpha) \\ -\gamma s^2 - (\gamma\omega_s^2 - \beta) \end{bmatrix} \quad (6-7) \end{aligned}$$

The denominator polynomial is considered first. As in Staley's analysis, $\mathbf{d}(s)$ can be expressed in terms of the factor $s^2 + \omega_s^2$, where

$$\begin{aligned} \mathbf{d}(s) = & \left[1 - \gamma^2 \right] (s^2 - \omega_s^2)^2 \\ & - \left[\omega_n (\omega_n - 2\omega_s) + 2(\alpha - \gamma\beta) + 4\gamma\omega_s (\omega_n - \omega_s) + 4\gamma^2 \omega_s^2 \right] (s^2 - \omega_s^2) \\ & - \left[2\alpha\omega_s (\omega_n - 2\omega_s) - \alpha^2 - \beta^2 + 4\alpha\gamma\omega_s^2 \right] \end{aligned} \quad (6-8)$$

This simplifies the complexity of the polynomial $\mathbf{d}(s)$ considerably. The approach used to solve the roots of $\mathbf{d}(s)$ is outlined below:

- i) First, the roots $(s^2 - \omega_s^2)$ of $\mathbf{d}(s)$ were determined. The expressions for these two roots were simplified by employing the approximation $\sqrt{1-x} = 1 - x - \frac{x^2}{2} - \dots$ when x is small.
- ii) From the two roots $(s^2 - \omega_s^2)$ of $\mathbf{d}(s)$, the four roots s of $\mathbf{d}(s)$ were determined using the same type of approximation.
- iii) The four roots of $d(s)$ were found by noting that $d(s) = \mathbf{d}(s - j\omega_s)$. Therefore the four roots of $d(s)$ are simply the four roots of $\mathbf{d}(s)$, each less $j\omega_s$.

The resulting four poles of $d(s)$ are

$$\begin{aligned}
 p_1 &= j \left[2\omega_s - \frac{\alpha}{\omega_n} - \frac{(\alpha^2 - \beta^2)}{2\omega_s \omega_n (\omega_n - 2\omega_s)} - \frac{2\alpha\gamma\omega_s}{\omega_n (\omega_n - 2\omega_s)} \right] \\
 p_2 &= j \left[\omega_n - \frac{(\alpha - \gamma\beta)}{(\omega_n - \omega_s)} - 2\gamma\omega_s - \frac{\alpha\omega_s}{\omega_n (\omega_n - \omega_s)} - \frac{2\alpha\gamma\omega_s^2}{\omega_n (\omega_n - \omega_s)(\omega_n - 2\omega_s)} \right. \\
 &\quad \left. - \frac{(\alpha^2 - \beta^2)}{2\omega_n (\omega_n - \omega_s)(\omega_n - 2\omega_s)} + \frac{\gamma^2\omega_n (\omega_n - 2\omega_s)}{2(\omega_n - \omega_s)} \right] \\
 p_3 &= j \left[(2\omega_s - \omega_n) - \frac{(\alpha - \gamma\beta)}{(\omega_n - \omega_s)} - 2\gamma\omega_s - \frac{\alpha\omega_s}{\omega_n (\omega_n - \omega_s)} - \frac{2\alpha\gamma\omega_s^2}{\omega_n (\omega_n - \omega_s)(\omega_n - 2\omega_s)} \right. \\
 &\quad \left. - \frac{(\alpha^2 - \beta^2)}{2\omega_n (\omega_n - \omega_s)(\omega_n - 2\omega_s)} - \frac{\gamma^2\omega_n (\omega_n - 2\omega_s)}{2(\omega_n - \omega_s)} \right] \\
 p_4 &= j \left[-\frac{\alpha}{\omega_n} - \frac{(\alpha^2 - \beta^2)}{2\omega_s \omega_n (\omega_n - 2\omega_s)} - \frac{2\alpha\gamma\omega_s}{\omega_n (\omega_n - 2\omega_s)} \right]
 \end{aligned} \tag{6-9}$$

When the traditional tuning condition $\alpha = 0$ is applied, the poles of the system simplify to the following:

$$\begin{aligned}
 p_1 &= j \left[2\omega_s - \frac{\beta^2}{2\omega_s \omega_n (\omega_n - 2\omega_s)} \right] \\
 p_2 &= j \left[\omega_n - \frac{\gamma\beta}{(\omega_n - \omega_s)} - 2\gamma\omega_s - \frac{\beta^2}{2\omega_n (\omega_n - \omega_s)(\omega_n - 2\omega_s)} \right. \\
 &\quad \left. - \frac{\gamma^2\omega_n (\omega_n - 2\omega_s)}{2(\omega_n - \omega_s)^2} \right] \\
 p_3 &= j \left[(2\omega_s - \omega_n) + \frac{\gamma\beta}{(\omega_n - \omega_s)} - 2\gamma\omega_s - \frac{\beta^2}{2\omega_n (\omega_n - \omega_s)(\omega_n - 2\omega_s)} \right. \\
 &\quad \left. - \frac{\gamma^2\omega_n (\omega_n - 2\omega_s)}{2(\omega_n - \omega_s)} \right] \\
 p_4 &= j \left[-\frac{\beta^2}{2\omega_s \omega_n (\omega_n - 2\omega_s)} \right]
 \end{aligned} \tag{6-10}$$

A similar approach can be used to determine the zeros of the numerator of the case-referenced transfer function $G_{z1}(s)$. The numerator of $\mathbf{G}_{z1}(s)$ described in (6-6) can be expressed in terms of the factor $(s - j\omega_z)$, resulting in the following expression:

$$\bar{n}_z(s) = \left\{ (s - j\omega_z)^2 - j(\omega_n - 2\gamma\omega_z)(s - j\omega_z) - \alpha \right\} \quad (6-11)$$

Using the same methods employed to determine the system poles, the two case-referenced zeros are:

$$\begin{aligned} z_1 &= j \left(2\omega_z - \frac{\gamma^2 \omega_z^2}{\omega_n^2} - \frac{\alpha}{\omega_n} \right) \\ z_2 &= j \left((2\omega_z - \omega_n) - 2\gamma\omega_z - \frac{\gamma^2 \omega_z^2}{\omega_n} - \frac{\alpha}{\omega_n} \right) \end{aligned} \quad (6-12)$$

The detailed derivation of both the poles and zeros of $G_{z1}(s)$ is documented in Appendix B.1.

Examining the expressions for the two zeros and four poles, the following observations can be made:

a) Four Distinct Poles

The case-referenced characteristic polynomial has four distinct roots as first identified by Staley [21]. Due to a sign error in the development of the expressions for the four system poles in [21], the expressions in (6-9) are slightly different. The first pole p_1 occurs at very nearly twice the spin frequency. The second pole p_2 occurs near the nutation frequency ω_n . The third pole appears to be a mode like a beat frequency, created by the close proximity of the nutation

frequency ω_r and a mode at close to twice the spin frequency ($2\omega_r$). The final root is extremely small and represents a precession frequency with a magnitude several orders of magnitude smaller than the first three roots.

b) Non-zero Precession Pole when Operating at Traditional Tuned Frequency

Most traditional analyses of tuned-rotor gyro systems describe the transverse-axes dynamics as exhibiting only two frequencies: a nutation frequency and a precession frequency as given by the poles p_1 and p_2 in equation (6-4). When the system is tuned using the Traditional model, $p_1 = 0$. Of the four characteristic frequencies identified in this analysis of the Complete model, the roots p_2 and p_3 are similar to the nutation and precession frequencies identified in the Traditional model. However, examining (6-10) the precession pole p_2 is not zero after applying the traditional tuning condition $\alpha = 0$. The rotor, rather than behaving as a freely spinning body, will precess about the null axis when the spin axis is set at a non-zero tilt. This low-frequency motion will appear as a small drift. To completely eliminate this precession term, an alternative tuning condition is required. This will be addressed later in Section 6.4.

c) Beat and Twice-Spin Frequency Poles

As noted by Staley [21], the poles p_1 and p_3 of the Complete Model are not normally considered in the transverse-axes dynamic control problem of tuned-rotor gyroscopes. The pole p_3 is of particular concern as it occurs well below the spin frequency ω_r and as such could be inside the desired bandwidth of the transverse-axes controller. This could limit the maximum achievable bandwidth of the transverse-axes control system.

With the exception of the non-linear simulation study by Cain [23], examinations of the tested or simulated responses of tuned-rotor gyros have not identified the presence of the poles p_1 and p_3 in the measured responses. The fact that the pole p_1 was not identified as a dynamic mode is understandable. It is shifted only a small amount from $2\omega_s$. Usually oscillations observed at twice the spin frequency are attributed to the gimbal flutter, which occurs when the rotor spin axis is tilted with respect to null. A good reason for having not observed the effect of the pole p_3 in past test data is not as obvious. Examination of the overall transfer function $G_{x1}(s)$ and its zeros provides a potential explanation as explained in d) below.

d) Fixed-Modes in $G_{x1}(s)$

Comparing the poles and zeros of the transverse-axes dynamics transfer function $G_{x1}(s)$ for the Complete Model, the zero z_1 and pole p_1 are nearly coincident at a frequency of roughly twice the spin frequency ($2\omega_s$). In addition, the zero z_2 is nearly coincident with the pole p_3 at the beat frequency ($2\omega_s - \omega_n$). As a result, the poles p_1 and p_3 can be considered fixed modes, having little influence on the control of the transverse-axes dynamics. This may explain why these two modes have not been observed in most studies of tuned rotor gyroscopes with active rebalance control. However, when the tuned-rotor instrument is perturbed from null and has no control or external torques acting on it, its resulting response will exhibit all four frequencies. This will be demonstrated in Section 6.2.

In summary, the case-referenced transverse-axes transfer function $G_{x1}(s)$ can be expressed in the following form:

$$G_{d1}(s) = \frac{(s - z_1)(s - z_2)}{I_{\tau}'(s - p_1)(s - p_2)(s - p_3)(s - p_4)} \quad (6-13)$$

Noting the near cancellation of p_1 with z_1 and p_3 with z_2 , this transfer function can be approximated by the simpler transfer function $\tilde{G}_{d1}(s)$ given below

$$\tilde{G}_{d1}(s) \approx \frac{1}{I_{\tau}'(s - j\omega_{n_2})(s - j\omega_p)} \quad (6-14)$$

ω_{n_2} and ω_p represent the nutation pole and precession pole respectively and are defined by the relations below:

$$\begin{aligned} \omega_{n_2} &= -j \cdot p_2 \\ \omega_p &= -j \cdot p_4 \end{aligned} \quad (6-15)$$

Now consider the nature of the root locus of the transfer function $G_{d1}(s)$ in equation (6-13) when combined with a controller $G_c(s)$. It is likely that since p_1 is not exactly equal to z_1 and z_2 is not exactly equal to p_3 , there will be loci from p_1 to z_1 and z_2 to p_3 when closed loop control is applied. It remains to be shown that these two loci will be in the left-half plane. If this is always true, then the effect of poles p_1 and p_3 can be ignored.

Assuming negligible flexure damping, each of the zeros z_1 and poles p_1 sit on the imaginary axis. The case of simple proportional control of the transverse-axes dynamics is now considered. In Chapter 7, it will be shown that when using the Traditional Model, which contains only poles like p_2 and p_4 , it is not possible to stabilize both the nutation pole p_2 and the precession pole p_4 with simple proportional control. Dynamic control is

usually introduced to stabilize the nutation pole p_2 . Of interest in the following discussion is stability of the very short loci between p_1 and z_1 and between z_2 and p_3 .

As noted by Wang *et al.* [44], when using proportional control, the poles and zeros must alternate along the imaginary axis in order for the system to be stable. Figure 6-1 shows the effect of the positioning of zeros z_1 and z_2 with respect to poles p_1 and p_3 . When the zero frequency z_1 is above the pole frequency p_1 , the root loci passes into the right half of the s-plane before meeting back at z_1 . Similarly when the zero frequency z_2 is greater than the pole frequency p_3 , the root loci from p_3 passes into the right half of the s-plane before meeting back at z_2 . There is only one case where all the loci except that from the nutation pole will be stable. This occurs in Case 4 where the following conditions hold

$$\|p_1\| \geq \|z_1\| \text{ and } \|p_3\| \geq \|z_2\| \quad (6-16)$$

It remains to be demonstrated that the poles p_1 and p_3 and zeros z_1 and z_2 always satisfy condition (6-16). Examining the expressions for the pole p_1 and zero z_1 in equations (6-9) and (6-12) respectively, the zero will have a smaller magnitude than the pole. Therefore part 1 of condition (6-16) is satisfied. After examination of the expressions for the pole p_3 and the zero z_2 , it is not apparent which term is generally larger. For this case, a numerical check was performed using the mechanical characteristics of the GyroWheel tuned rotor instrument. The magnitude of the pole p_3 is marginally greater than that of the zero z_2 .

Figure 6-1: Root Locus Diagrams of $G_z(s)G_{d1}(s)$ where $G_z(s) = K$, Loci shown for four cases of positioning of z_1 relative to p_2 and z_2 relative to p_1

6.1.2.2 External Rate Dynamics Transfer Function $G_{e1}(s)$

The poles of $G_{e1}(s)$ are already known, as they are the same as those of $G_{d1}(s)$. The zeros of $G_{e1}(s)$ will be found by first determining the zeros of the gimbal referenced transfer function $\mathbf{G}_{\bullet 1}(s)$. Based on equations (6-105) and (6-108), $\mathbf{G}_{\bullet 1}(s)$ has the form:

$$\mathbf{G}_{\bullet,1}(s) = \frac{\bar{n}_{\omega}(s)n_{\xi}(s) - n_{\omega}(s)\bar{n}_{\xi}(s)}{\mathbf{d}(s)} \quad (6-17)$$

$$\begin{aligned} \bar{n}_{\omega}n_{\xi} - n_{\omega}\bar{n}_{\xi} = & [\gamma s - j\omega_s(\gamma_z - \gamma)] \cdot [\gamma s^2 + (\gamma\omega_s^2 + \beta)] - [s - j(\omega_n - \omega_s) - j\gamma_z\omega_s] \cdot \\ & \left[s^2 - \left[2c_g' + j[\omega_n - 2(1-\gamma)\omega_s] \right] s + [\omega_s\omega_n + (2\gamma - 1)\omega_s^2 - \alpha] \right] \end{aligned} \quad (6-18)$$

Examining the right hand side of (6-18), $\bar{n}_{\omega}(s)n_{\xi}(s)$ is noticeably smaller than $-n_{\omega}(s)\bar{n}_{\xi}(s)$ because of the presence of the small parameters γ and γ_z in both $\bar{n}_{\omega}$ and n_{ξ} . Based on this observation, $\mathbf{G}_{\bullet,1}(s)$ can be approximated by the following simplified transfer function:

$$\mathbf{G}_{\bullet,1}(s) \approx -\frac{n_{\omega}(s)\bar{n}_{\xi}(s)}{\mathbf{d}(s)} \quad (6-19)$$

Noting equation (5-121), the case-referenced transfer function $G_e(s)$ is found by evaluating $\mathbf{G}_{\bullet,1}(s - j\omega_s)$. This results in the following case-referenced transfer function:

$$G_e(s) \approx -\frac{n_{\omega}(s - j\omega_s)\bar{n}_{\xi}(s - j\omega_s)}{\mathbf{d}(s - j\omega_s)}$$

The first two zeros of $G_e(s)$ are the roots of $\bar{n}_{\xi}(s - j\omega_s)$ and are given in (6-12). They are displayed below for completeness:

$$\begin{aligned} z_1 &= j \left[2\omega_s + \frac{\gamma^2\omega_s^2}{\omega_n^2} + \frac{\alpha}{\omega_n} \right] \\ z_2 &= j \left[(2\omega_s - \omega_n) - 2\gamma\omega_s - \frac{\gamma^2\omega_s^2}{\omega_n^2} - \frac{\alpha}{\omega_n} \right] \end{aligned}$$

To determine the root of $n_\omega(s - j\omega_s)$, the root of $n_\omega(s)$ is first found to be the following form:

$$s = j(\omega_n - \omega_s) - \gamma_z \omega_s \quad (6-20)$$

Evaluating $n_\omega(s - j\omega_s)$, the third of the three zeros of $G_{el}(s)$ is given by the following expression:

$$z_3 = j(\omega_n - \gamma_z \omega_s) \quad (6-21)$$

Overall the external rate dynamics transfer function $G_{el}(s)$ can be described by the following expression:

$$G_{el}(s) = \frac{(s - z_1)(s - z_2)(s - z_3)}{(s - p_1)(s - p_2)(s - p_3)(s - p_4)} \quad (6-22)$$

Examining (6-21), the zero z_3 is almost coincident with the pole p_2 . Noting that z_1 approximately cancels p_1 and z_2 approximately cancels p_3 ; the transfer function $G_{el}(s)$ has the following approximate form:

$$G_{el}(s) = \frac{1}{(s - p_4)} \quad (6-24)$$

If the system was tuned in a manner that makes the precession pole p_4 zero, the transfer function would further simplify to

$$G_{el}(s) = \frac{1}{s} \quad (6-25)$$

To demonstrate that the short loci from p_2 to z_2 is entirely in the left hand of the s-plane when closed loop control is active, it must be shown that the pole p_2 is slightly larger than the zero z_2 . The pole is shifted from the nutation frequency by the dominant term $2\gamma\omega_s$ and some smaller terms, while the zero is shifted from ω_n by $2\gamma_2\omega_s$. Examining the definitions of γ and γ_2 , the ratio of these two terms is

$$\frac{2\gamma}{\gamma_2} = \frac{2I_{s_i}}{I_{s_r}} \quad (6-26)$$

As shown in [19], the transverse inertia of a symmetric spinning disc is bounded by the following condition:

$$I_{s_r} > I_{s_i} \geq \frac{1}{2}I_{s_r} \quad (6-27)$$

Consequently, the ratio in (6-26) has a lower bound of 1, i.e.

$$\frac{2\gamma}{\gamma_2} \geq 1 \quad (6-28)$$

Therefore, z_2 must have a smaller magnitude than the pole p_2 . The resulting root loci of the external body dynamics transfer function $G_{e1}(s)$ will be as shown in Figure 6-2.

Figure 6-2: Root Locus Diagrams of $G_z(s)G_e(s)$ where $G_z(s) = K$

6.2 Solution to the Transverse-axes Dynamics

Equations of Motion

A solution to the transverse-axis dynamics equations presented in (6-66) and (6-67) is now derived. The closed-form time response of the case-referenced angles ϕ_x and ϕ_y to constant torque inputs T_{x_0} and T_{y_0} and initial conditions ϕ_{x_0} and ϕ_{y_0} is presented. This solution serves the following purposes:

- a) The closed-form solution provides a tool to test the observations made in the examination of the system poles and zeros in Section 6.1. It is used to confirm that the pole p_2 at the nearly twice spin frequency and p_3 at the beat frequency

are fixed modes when case-referenced torques are applied to a tuned-rotor instrument.

- b) The closed-form solution allows for quick generation of linear time responses for the following cases: free response and response to constant applied torques. When the system is tuned, the precession period is just below 20,000 seconds. The time taken to generate the response of the system over one precession period is approximately one minute with the closed-form solution. This duration is many orders of magnitude less than the hours required to generate the time response of the complete non-linear dynamic model using numerical integration.
- c) The closed-form solution can be used to verify the behaviour of the non-linear simulation model for small tilt angles, as it is the exact solution.

To determine the solution, the linear gimbal-referenced equations of motion of the transverse-axes dynamics given in (5-69) and (5-70) were first converted into a state space model of the following form:

$$\dot{\mathbf{x}}_g = \mathbf{A}_g \mathbf{x}_g - \mathbf{B}_g \mathbf{u}_g \quad (6-29)$$

The base body rates ($\bar{\omega}_b$) and gimbal damping (c_g) have been ignored. The state vector $\mathbf{x}_g$, input vector $\mathbf{u}_g$, state matrix $\mathbf{A}_g$ and control matrix $\mathbf{B}_g$ are given below:

$$\mathbf{x}_g = [\theta_x \quad \theta_y \quad \dot{\theta}_x \quad \dot{\theta}_y]^T \quad (6-30)$$

$$\mathbf{u}_g = [T_{c_x} \quad T_{c_y}]^T \quad (6-31)$$

$$\mathbf{A}_g = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \bar{k}_x & 0 & 0 & \bar{c}_1 \\ 0 & \bar{k}_y & \bar{c}_2 & 0 \end{bmatrix} \quad (6-32)$$

$$\mathbf{B}_g = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{\cos \omega_z t}{I_{r_1} - I_{s_1}} & \frac{\sin \omega_z t}{I_{r_1} - I_{s_1}} \\ \frac{-\sin \omega_z t}{I_{r_2}} & \frac{\cos \omega_z t}{I_{r_2}} \end{bmatrix} \quad (6-33)$$

$$\bar{k}_1 = \frac{(I_{r_1} - I_{r_2} - I_{s_1} - 2J)\omega_z^2 - K_x}{I_{r_1} - I_{s_1}} \quad (6-34)$$

$$\bar{k}_2 = \frac{(I_{r_1} - I_{r_2})\omega_z^2 - K_x}{I_{r_1}} \quad (6-35)$$

$$\bar{c}_1 = \frac{(I_{r_1} - I_{r_2} - I_{s_1})\omega_z}{I_{r_1} - I_{s_1}} \quad (6-36)$$

$$\bar{c}_2 = \frac{-(I_{r_1} - I_{r_2} - I_{r_2})\omega_z}{I_{r_1}} \quad (6-37)$$

Note that in this form, the gimbal referenced torques T_{s_1} and T_{s_2} have already been replaced by the equivalent case-referenced torques T_{c_1} and T_{c_2} , noting the relationship in (5-30). A solution to the state-space equation in terms of the initial state vector $\mathbf{x}_g(0)$ and constant case-referenced applied torque vector $\mathbf{u}_g$ is desired. The general solution to a system described in equation (6-29) is given by the following equation:

$$\mathbf{x}_g(t) = e^{\mathbf{A}_g t} \mathbf{x}_g(t_0) + \int_0^t e^{\mathbf{A}_g(t-\tau)} \mathbf{B}_g \mathbf{u}_g(\tau) d\tau \quad (6-38)$$

$e^{\mathbf{A}_g t}$ is the system's state transition matrix. The solution $\mathbf{x}_g(t)$ provides a response of the two gimbal angles θ_x and θ_y . Once this is obtained, the response of the two case-referenced angles ϕ_x and ϕ_y can be determined by noting the following relation:

$$\begin{bmatrix} \phi_x \\ \phi_y \end{bmatrix} = \begin{bmatrix} C_{\omega_{x^*}} & -S_{\omega_{x^*}} \\ S_{\omega_{x^*}} & C_{\omega_{x^*}} \end{bmatrix} \begin{bmatrix} \theta_x \\ \theta_y \end{bmatrix}$$

As documented in Appendix B.2, the resulting response of the case-referenced angles $\phi_x(t)$ and $\phi_y(t)$ as a function of the initial tilt angles $\phi_x(0)$ and $\phi_y(0)$, initial tilt rates $\dot{\phi}_x(0)$ and $\dot{\phi}_y(0)$ and constant applied torques T_{z_x} and T_{z_y} can be found to be

$$\begin{aligned} \phi_x(t) = & \sum_{j=1}^4 S_{z_{xj}} \cos \omega_{xj} t \phi_x(0) - \sum_{j=1}^4 S_{z_{xj}} \sin \omega_{xj} t \dot{\phi}_x(0) \\ & - \sum_{j=1}^4 R_{z_{xj}} \sin \omega_{xj} t \phi_x(0) - \sum_{j=1}^4 R_{z_{xj}} \cos \omega_{xj} t \dot{\phi}_x(0) \\ & - D_{z_{x0}} - \sum_{j=1}^4 D_{z_{xj}} \cos \omega_{xj} t - D_{z_{x5}} \cos 2\omega_{xj} t - T_{z_{x1}} \\ & - \sum_{j=1}^4 D_{z_{xj}} \sin \omega_{xj} t - D_{z_{x5}} \sin 2\omega_{xj} t - T_{z_{x2}} \end{aligned} \quad (6-39)$$

$$\begin{aligned} \phi_y(t) = & \sum_{j=1}^4 S_{z_{yj}} \sin \omega_{yj} t \phi_y(0) - \sum_{j=1}^4 S_{z_{yj}} \cos \omega_{yj} t \dot{\phi}_y(0) \\ & - \sum_{j=1}^4 R_{z_{yj}} \cos \omega_{yj} t \phi_y(0) + \sum_{j=1}^4 R_{z_{yj}} \sin \omega_{yj} t \dot{\phi}_y(0) \\ & - \sum_{j=1}^4 D_{z_{yj}} \sin \omega_{yj} t - D_{z_{y5}} \sin 2\omega_{yj} t - T_{z_{y1}} \\ & - D_{z_{y0}} - \sum_{j=1}^4 D_{z_{yj}} \cos \omega_{yj} t - D_{z_{y5}} \cos 2\omega_{yj} t - T_{z_{y2}} \end{aligned} \quad (6-40)$$

ω_{xj} are the four characteristic modes of the transverse-axis dynamics as viewed from the case frame. These four frequencies are the same four values determined from the case-referenced SCISCO transfer function in Section 6.1. The approach to deriving this

response as well as the expressions for the four modes ω_i , position influence coefficients $S_{i,j}$, rate influence coefficients $R_{i,j}$ and compliance coefficients $D_{i,j}$ and $D_{z,j}$ are documented in Appendix B.2.

To illustrate the nature of the free and forced portions of the response of a tuned-rotor instrument, the characteristics of the GyroWheel system can be used. The inertia and stiffness characteristics are listed in Table 6-1. The parameters have been chosen to ensure that the tuning condition is satisfied. Using these parameters, the four characteristic frequencies and relevant transient response parameters are:

$$\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{bmatrix} = \begin{bmatrix} 46.279 \\ 3.7083 \\ 49.987 \\ -5.286 \times 10^{-5} \end{bmatrix} \text{ (Hz)}$$

$$D = \begin{bmatrix} 2.99 \times 10^3 & -3.32 \times 10^{-3} & -7.51 \times 10^{-5} & 1.09 \times 10^1 & -2.99 \times 10^3 & -1.09 \times 10^1 \\ 2.99 \times 10^3 & -3.51 \times 10^{-3} & 7.95 \times 10^{-5} & -1.09 \times 10^1 & -2.99 \times 10^3 & 1.09 \times 10^1 \end{bmatrix} \left(\frac{\text{rad}}{\text{N} \cdot \text{m}} \right)$$

$$S = \begin{bmatrix} 5.418 \times 10^{-3} & 1.226 \times 10^{-3} & -1.666 \times 10^{-3} & 4.587 \times 10^{-1} \\ -5.384 \times 10^{-3} & 1.219 \times 10^{-3} & -1.675 \times 10^{-3} & -4.612 \times 10^{-1} \end{bmatrix}$$

$$R = \begin{bmatrix} 3.453 \times 10^{-3} & -7.818 \times 10^{-6} & -1.247 \times 10^{-5} & -3.433 \times 10^{-3} \\ 3.425 \times 10^{-3} & 7.754 \times 10^{-6} & 1.252 \times 10^{-5} & -3.445 \times 10^{-3} \end{bmatrix}$$

The first frequency ω_1 represents the nutation frequency of the system. The second frequency ω_2 acts like a beat frequency, being the difference between the nutation frequency ω_1 and twice the spin frequency $2\omega_s$. The third frequency is very close the twice the spin frequency and the fourth mode ω_4 is the low frequency precession mode.

Although the tuning condition has been used to choose the mechanical parameters and tuned speed, there is no zero frequency. The tuning condition has not resulted in the ω_z frequency being nulled.

The following comments can be made concerning the free and forced responses of the linearized transverse-axes dynamics:

a) Free Response

Using (6-39) and (6-40), the free response of the system was generated for the case where the spin axis tilt is initially set at 1 degree about the case-referenced x-axis. The response was generated for 18,500 seconds, which is just less than the precession period of 18918 seconds. Figure 6-3 shows the response of the tilt about the case-referenced x-axis plotted against the tilt about the case-referenced y-axis. The response is dominated by the clockwise coning motion of the spin axis about null at the frequency of ω_z . Figure 6-4 shows the response of both case-referenced tilt angles with respect to time in greater detail over the first 0.5 seconds. The nutation mode ω_y is the dominant term, with its effect modulated by the beat frequency ω_z . The influence of the mode ω_x appears to be minor.

b) Forced Response

Examining equation (6-39) and (6-40), the forced portion of the time response consists of a steady state and periodic component. The periodic component consists of terms due to the four characteristic modes and a term at exactly twice the spin frequency. Examining the magnitudes of the coefficients $D_{i,j}$, the response of the system can be approximated by the expression

$$\phi_x \approx (D_{10} + D_{12} \cos \omega_s t) T_{c_x} \quad (6-41)$$

Noting that D_{10} and D_{12} are almost equal the response simplifies further to

$$\theta_x' = (D_{10} (1 - \cos \omega_s t)) T_{c_x}$$

The coefficient D_{10} can be considered as a compliance, which determines the amount of torque required to tilt the spin axis away from null.

The coefficients D_{13} and D_{15} are almost identical in magnitude and opposite in sign. These coefficients indicate the influence of the characteristic mode ω_3 at nearly $2\omega_3$ and the forced component at exactly $2\omega_3$. The sum of the D_{13} and D_{15} coefficients results in a gain approximately $1/10^{\text{th}}$ of the magnitude of the nutation frequency coefficient D_{11} . As a result, the nutation frequency ω_1 is noticeable in the response, with the beat frequency ω_2 and twice spin mode ω_3 having less of an influence as predicted by the analysis in Section 6.1.

The above observations are illustrated in Figure 6-5 and Figure 6-6 for the case where a torque was applied to cause the precession to occur from an initial 1-degree tilt about null. Based on the compliance D_{10} , a torque T_{c_x} of $5.84 \times 10^{-6} \text{ Nm}$ is required to tilt the spin axis 1 degree about the case x-axis.

Figure 6-5 shows how the application of the torque shifts the centre of the coning motion of spin axis from null to a tilt of 1 degree about the case x-axis. In this response, the coning motion is again plotted for 18500 seconds in order to illustrate the direction of the coning motion. The response of each case angle with respect to time is shown in greater detail in Figure 6-6. The oscillation at the

nutations frequency is most dominant with the beat frequency having a lesser effect.

Finally, Figure 6-7 and Figure 6-8 show the response of the transverse-axes dynamics with the spin axis initially tilted 1 degree about the case-referenced x-axis and a constant torque T_{x_1} of $5.84 \times 10^{-6} \text{ Nm}$ applied. This corresponds to the situation where a transverse-axes controller would be required to hold the spin axis at a constant tilt, stopping the tendency to precess at the frequency ω_z . Applying this torque about the case-referenced x-axis offsets the case-referenced x-axis spring stiffness $\frac{1}{D_{x_0}}$ when tilted 1 degree. Examining Figure 6-7, the spin axis tilt is held at 1 degree about the case-referenced x-axis, with no coning occurring.

Mechanical Parameter	Symbol	Baseline Design Value
Rotor spin-axis inertia	I_{r_z}	$6.402 \times 10^{-3} \text{ kg-m}^2$
Rotor transverse-axis inertia	I_{r_x}, I_{r_y}	$3.458 \times 10^{-3} \text{ kg-m}^2$
Gimbal spin-axis inertia	I_{g_z}	$1.8047 \times 10^{-3} \text{ kg-m}^2$
Gimbal transverse-axes inertia	I_{g_x}, I_{g_y}	$1.2758 \times 10^{-3} \text{ kg-m}^2$
Flexure pivot torsional stiffness	K_x, K_y	$9.201 \times 10^{-2} \text{ N-m/radian}$
Tuned speed	ω_s	157.0412 radian/sec

Table 6-1: Mechanical Parameters of GyroWheel System

Figure 6-3: Time Response of the Case-referenced x-axis tilt (ϕ_x) versus the Case referenced y-axis tilt (ϕ_y) for Open-Loop GyroWheel over 18500 second period, Initial Tilt of 1 degree about the case-referenced x-axis

Figure 6-4: Time Response of the Case-referenced x-axis tilt (ϕ_x) and the Case referenced y-axis tilt (ϕ_y) for Open-Loop GyroWheel over initial 0.5 second period, Initial Tilt of 1 degree about the case-referenced x-axis

Figure 6-5: Time Response of the Case-referenced x-axis tilt (ϕ_x) versus the Case referenced y-axis tilt (ϕ_y) for Open-Loop GyroWheel over initial 18500 second period, Zero Initial Tilt, Constant Applied Torque T_{cx} of $5.84 \times 10^{-6} \text{ Nm}$ about the Case Referenced x-axis

Figure 6-6: Time Response of the Case-referenced x-axis tilt (ϕ_x) and the Case referenced y-axis tilt (ϕ_y) for Open-Loop GyroWheel over initial 0.5 second period, Zero Initial Tilt, Constant Applied Torque T_{cx} of 5.84×10^{-6} Nm about the Case-referenced x-axis

Figure 6-7: Time Response of the Case-referenced x-axis tilt (ϕ_x) versus the Case-referenced y-axis tilt (ϕ_y) for Open-Loop GyroWheel over initial 18500 second period, Initial Tilt of 1 degree about the Case-referenced x-axis, Constant Applied Torque T_{α} of $5.84 \times 10^{-6} \text{ Nm}$ about the Case-referenced x-axis

Figure 6-8: Time Response of the Case-referenced x-axis tilt (ϕ_x) versus the Case referenced y-axis tilt (ϕ_y) for Open-Loop GyroWheel over initial 18500 second period, Initial Tilt of 1 degree about the case-referenced x-axis, Constant Applied Torque T_{cx} of $5.84 \times 10^{-6} \text{ Nm}$ about the Case-referenced x-axis

6.3 Low Frequency Dynamics of a Tuned-Rotor Instrument

In a tuned-rotor gyroscope, the tilt of the rotor with respect to its fixed case is regulated about null. The torque required to maintain this zero tilt is a direct measure of the external angular rates of the instrument's case about the two orthogonal axes lying in the rotor plane. This torque is due only to the motion of the external body because there are no spring or damping torques for the rebalance torquers to overcome. The absence of other torques is due to the tuned condition of the gyro, which balances the negative gyroscopic stiffness of the precessing gimbal against the spring rates of the two pairs crossed-flexures.

However, general tuned-rotor instruments frequently operate in a de-tuned state and at time varying non-zero tilts. The torque required to steer the tilt of the spin axis to a non-zero value is due to the external rates and the torque required to overcome the "de-tuned stiffness" and effective case-referenced damping. As the spring torque and damping torque cannot be measured directly, accurate knowledge of the case-referenced 'dynamic stiffness' and damping are essential to determining good estimates of the external rates.

6.3.1 Dynamic Stiffness

Equation (6-14) provides an approximation for the case-referenced SCISCO transfer function model of the transverse-axes dynamics. The approximate transfer function has the following form:

$$\phi(s) = \frac{1}{I_r' (s - j\omega_{\tau_c}) (s - j\omega_p)} T_c(s)$$

The following expression relates the complex tilt angle ϕ to the complex applied torque T_c at steady state:

$$\phi = \frac{1}{K_{\dot{\phi}n}} T_c \quad (6-42)$$

$K_{\dot{\phi}n}$ is termed the 'dynamic stiffness' of a tuned-rotor instrument transverse-axes dynamics and given by the expression below:

$$K_{\dot{\phi}n} = I_{\tau}' \omega_n \omega_p \quad (6-43)$$

It is dependent on the offsetting effects of the flexure stiffness and the negative gyroscopic spring stiffness that varies with spin rate.

Examining the expression for the modified nutation frequency ω_n in (6-15), it can be approximated by ω_n . Substituting the expression for ω_n given in (5-84), the dynamic stiffness is given by the following expression:

$$K_{\dot{\phi}n} = I_{\tau}' \omega_s \omega_p$$

Noting that $I_{\tau}' \omega_s$ is the magnitude of the angular momentum of the rotor (h), the above expression further simplifies to the following:

$$K_{\dot{\phi}n} = h \cdot \omega_p \quad (6-44)$$

The dynamic stiffness is a critical parameter in the external rate-sensing concept described in Chapter 8. This expression summarizes the observation that for a spinning

rotor of momentum h tilted ϕ away from null, the precession of its spin axis at the frequency ω_p can be countered by a torque $K_{\phi n}\phi$.

6.3.2 Effect of Crossed-Flexure Damping

A low degree of damping can exist in the crossed-flexures of the gimbal assembly. Generally, in tuned-rotor gyro designs, the damping is minimized. As demonstrated below, cross-flexure damping affects the case-referenced stiffness and tuning condition, making it undesirable.

a) Case-Referenced Damping and Stiffness

The SCISCO transfer function form of the Traditional Model provides a useful tool to examine the effect of crossed-flexure damping at low frequencies. To determine the effect of damping on the tuned-rotor instrument dynamics, the effect of the base dynamics $\omega_b(s)$ will be ignored. As the focus of this analysis is on low-frequency effects of damping, the effect of the periodic terms $\bar{\phi}(s-2j\omega_s)$ and $\bar{\omega}_s(s-2j\omega_s)$ at $2\omega_s$ will also be ignored. Based on these assumptions, the SCISCO transfer function model in (5-96) reduces to

$$\phi(s) = \frac{1}{I_{\tau}' \left\{ s^2 + \left(2c_g' - j(\omega_n + 2\gamma\omega_s) \right) s - \left(\alpha - 2jc_g'\omega_s \right) \right\}} T_z(s)$$

There are two damping-like coefficients of s in the characteristic polynomial. The term $-jI_{\tau}'(\omega_n + 2\gamma\omega_s)$ is simply the effect of gyroscopic coupling and does not represent a true damping effect. The gimbal referenced damping term $2I_{\tau}'c_g'$ which is equivalent to

$2c_g$ will provide a real component to both the precession pole p_1 and nutation pole p_2 as indicated by the expressions for these two poles in equation (6-4). The consequence of cross-flexure damping is that if a tuned rotor instrument is set at a non-zero tilt, the momentum axis will cone toward the origin in a spiraling manner even if it is tuned.

The stiffness term $I_r'(\alpha - 2jc_g'\omega_s)$ consists of two components. $I_r'\alpha$ represents the de-tuned stiffness of the system. The presence of the complex factor j in the second stiffness term $-2I_r'c_g'\omega_s$ indicates this term can be considered a cross-axis stiffness. To illustrate this notion, the torque required to hold the rotor at a non-zero tilt is expressed in the non-complex form given below:

$$\begin{bmatrix} T_x \\ T_y \end{bmatrix} = \left(\frac{2I_r - I_{st}}{2} \right) \begin{bmatrix} \alpha & 2c_g'\omega_s \\ -2c_g'\omega_s & \alpha \end{bmatrix} \begin{bmatrix} \phi_x \\ \phi_y \end{bmatrix}$$

The introduction of crossed-flexure damping has introduced a cross-axis stiffness of magnitude $2c_g'\omega_s$. The system can be tuned in the traditional sense ($\alpha = 0$), yet still require a torque to hold the spin axis at a non-zero tilt.

For the GyroWheel system at its most severe de-tuned speed, the stiffness $I_r'\alpha$ is $2.28 \times 10^{-2} \frac{\text{N} \cdot \text{m}}{\text{rad}}$. Typical crossed-flexure damping values for the GyroWheel system are 1.0×10^{-5} to $1.0 \times 10^{-4} \frac{\text{N} \cdot \text{m} \cdot \text{sec}}{\text{rad}}$ when measured at a pressure of 1 atmosphere. The damping is due primarily to windage. For a nominal operating speed of 157.04 radians/second, the resulting stiffness is 1.57×10^{-3} to $1.57 \times 10^{-2} \frac{\text{N} \cdot \text{m}}{\text{rad}}$. This is close to the maximum de-tuned stiffness of the GyroWheel. For this reason, it is desirable to

evacuate the chamber in which the gyro operates, thereby reducing the effective damping down to an insignificant level.

b) Tuning Condition

The classical tuning condition is applied by operating the tuned-rotor gyroscope at a spin rate such that $\alpha = 0$. Significant crossed-flexure damping compromises the tuning condition for two reasons. First, the damping causes the momentum axis to precess in toward the null axis for any initial non-zero tilt. Secondly, the damping creates cross-axis stiffness terms. There is no tuning condition that can achieve zero stiffness, simply through the selection of an appropriate spin rate ω_s . In summary, a tuned-rotor gyroscopic device cannot be tuned when significant crossed-flexure damping exists.

6.4 A New Tuning Condition for Tuned-Rotor Instruments

Although the GyroWheel concept involves operating the system at a non-tuned speed, it may operate with zero spin-axis tilt and at the tuned speed to obtain the highest possible rate sensing accuracy. Therefore, it is worthwhile to re-examine the traditional tuning condition to determine if more effective tuning can be achieved. Since the introduction of the tuned-rotor gyroscope concept, these instruments have been dynamically tuned using the same condition. The tuning condition is chosen so as to null the precession frequency (ω_p) of the device.

To review the basis of the classic tuning condition, the poles of the Traditional Model given in equation (6-4) are re-examined with the cross-flexure damping c_g' set to zero. The following relation defines the precession frequency of the Traditional Model:

$$\omega_{p_z} = -\frac{\alpha}{\omega_n} \quad (6-45)$$

To null this frequency, the parameter α defined in (5-81) must be set to zero. This places the following condition on the flexure stiffness, gimbal inertias and spin rate:

$$K_x - K_y - 2J\omega_s^2 = 0 \quad (6-46)$$

This condition will be referred to as the “Traditional Tuning Condition”. By satisfying this condition, the intent is to fix the spin axis in inertial space.

With the introduction of the Complete Model in Chapter 5, it has been shown that the precession frequency of a tuned-rotor instrument is slightly different from that determined with the Traditional Model. As shown in Section 6.1, the following expression provides an approximation for the precession frequency (ω_p) of the Complete Model:

$$\omega_p = -j \left[\frac{\alpha}{\omega_n} + \frac{\alpha^2 - \beta^2}{2\omega_s \omega_n (\omega_n - 2\omega_s)} - \frac{2\alpha\gamma\omega_s}{\omega_n (\omega_n - 2\omega_s)} \right]$$

Applying the Traditional Tuning Condition of $\alpha = 0$ still results in a non-zero precession frequency. Without changing the composite gimbal inertia J and flexure stiffness' K_x and K_y , a new spin rate ω_s is desired that zeros the precession frequency ω_p . From the above expression for the precession pole ω_p , the new tuning condition can be achieved by satisfying the following condition:

$$2\alpha\omega_s(\omega_n - 2\omega_s) - (\alpha^2 - \beta^2) - 4\alpha\gamma\omega_s^2 = 0 \quad (6-47)$$

It would be desirable to express the tuning condition in a form similar to equation (6-46). This can be done by first noting that the left-hand side of equation (6-47) can be rearranged as a quadratic expression in terms of α as shown below:

$$\alpha^2 + 2[\omega_r(\omega_n - 2\omega_r) + 2\gamma\omega_r^2]\alpha - \beta^2 = 0 \quad (6-48)$$

The two roots of this equation are given below:

$$\alpha = -[\omega_r(\omega_n - 2\omega_r) + 2\gamma\omega_r^2] \pm \sqrt{1 + \frac{4\gamma\omega_r}{\omega_n - 2\omega_r} - \frac{4\gamma^2\omega_r^2}{(\omega_n - 2\omega_r)^2} - \frac{\beta^2}{\omega_r^2(\omega_n - 2\omega_r)^2}} \quad (6-49)$$

Noting that the magnitude of the last three terms within the square root sign are small with respect to unity, the square root expression can be expressed in terms of a binomial expansion. Considering terms in the expansion up to the third order, but ignoring the small magnitude ones, equation (6-49) simplifies to the following form:

$$\begin{aligned} \alpha &= -[\omega_r(\omega_n - 2\omega_r) + 2\gamma\omega_r^2] \\ &= -[\omega_r(\omega_n - 2\omega_r) + 2\gamma\omega_r^2 + \frac{\beta^2}{2\omega_r(\omega_n - 2\omega_r)} - \frac{\gamma\beta^2}{(\omega_n - 2\omega_r)^2}] \end{aligned} \quad (6-50)$$

The two roots are then

$$\begin{aligned} \alpha &= \frac{\beta^2}{2\omega_r(\omega_n - 2\omega_r)} - \frac{\gamma\beta^2}{(\omega_n - 2\omega_r)^2} \\ \alpha &= -2\omega_r(\omega_n - 2\omega_r) - 4\gamma\omega_r^2 - \frac{\beta^2}{2\omega_r(\omega_n - 2\omega_r)} - \frac{\gamma\beta^2}{(\omega_n - 2\omega_r)^2} \end{aligned} \quad (6-51)$$

The desired solution for α is a small perturbation from the previous tuning condition $\alpha = 0$. Therefore the smaller of the two roots in (6-51) is desired. Rearranging the first equation in (6-51), the new tuning condition can be expressed in the following form:

$$\alpha - \frac{\beta^2}{2\omega_r(\omega_n - 2\omega_r)} - \frac{\gamma\beta^2}{(\omega_n - 2\omega_r)^2} = 0 \quad (6-52)$$

In this study, the flexure stiffnesses K_x and K_y are designed to be equal. For this reason and to obtain a compact form of the tuning condition, it is assumed that

$$K_x = K_y = K$$

Noting the definitions for α , β and ω_n , (6-52) can be expressed in the following form:

$$K - J\omega_r^2 \left(1 - \left(\frac{J}{I_{tune}} - \frac{2J \cdot I_{z_t}}{I_{tune}^2} \right) \right) = 0 \quad (6-53)$$

The tuning inertia I_{tune} is defined by the expression below:

$$I_{tune} = 2(I_{r_t} - I_{z_t} - I_{r_z}) \quad (6-54)$$

The new term $\left(\frac{J}{I_{tune}} - \frac{2J \cdot I_{z_t}}{I_{tune}^2} \right)$ in the tuning condition is a small negative quantity of magnitude less than unity. As a result, the tuned speed will be slightly larger than that obtained using the Traditional Tuning Condition. The new tuning condition in equation (6-53) is called the "Complete Tuning Condition" in this document.

To illustrate the effect of using the Complete Tuning Condition, the operation of the GyroWheel device with no transverse-axes control is considered. The tuned speed was determined using the Traditional Tuning Condition and the Complete Tuning condition. The tuned speeds were determined using the system mechanical characteristics documented Table 6-1. Table 6-2 shows the tuned speed and resulting characteristic modes of the transverse-axes dynamics for the Traditional and Complete tuning conditions. Applying the Complete Tuning Condition, the tuned speed increases by 0.2 percent of the Traditional tuned speed (0.3 radians/second). Although this is a small increase, it is well within the resolution of a typical spin controller. The three non-zero characteristic modes of the transverse-axes dynamics change by only a small amount with the new tuned speed, while the fourth mode approaches zero.

It should be noted that the derivation of the Complete Tuning Condition is based on an approximation of the precession frequency ω_p . To illustrate the accuracy of the Complete Tuning Condition, the tuned speed was obtained by determining the spin rate at which the precession pole of the Complete Model characteristic polynomial goes to zero. This was obtained using the numerical method of iteration as shown in Appendix B.4. The result was a tuned speed of 157.3272, which is negligibly higher than the result obtained using the Complete Tuning Condition in equation (6-53).

Figure 6-9 shows the response of the GyroWheel, with initial tilt of 0.1 degrees about the case-referenced x-axis and the spin rate set at the tuned speed. The plot shows the tilt about the case referenced x-axis plotted against the tilt about the case referenced y-axis. The simulation was performed for 20,000 seconds. Using the Traditional Tuning Condition, the spin axis cones slightly more than one rotation over this 20000-second period. Using the Complete Tuning Condition, there is a small motion of the spin axis about the 0.1-degree x-axis tilt over this same period. The small motion of the spin axis is due to gimbal flutter and nutation. It should be noted that the jagged points on the

response using the Complete Tuning Condition are due to an insufficiently small plot interval.

Tuned Speed and Characteristic Modes	Modes Using Traditional Tuning Condition (radians/second)	Modes Using Complete Tuning Condition (radians/second)
ω_{tuned}	157.0412	157.32705
ω_1	3.1408×10^{-2}	3.1465×10^{-2}
ω_2	2.9078×10^{-2}	2.9131×10^{-2}
ω_3	2.3300×10^1	2.3343×10^1
ω_4	-3.3212×10^{-4}	-1.908×10^{-4}

Table 6-2: Characteristic Modes of GyroWheel Using Traditional and Complete Tuning Conditions

Figure 6-9: Response of Case-referenced x-axis tilt ϕ_x versus Case-referenced y-axis tilt ϕ_y for Open Loop GyroWheel, Initial tilt of 0.1 degrees about the Case-referenced x-axis, Responses shown for Traditional Tuning Condition and the Complete Tuning Condition, Closed-Form Solution Used

6.5 Summary

The Complete Model of the transverse-axes dynamics of a tuned-rotor instrument derived in Chapter 5 has been shown to be a superior representation to that of the Traditional Model. The SCISCO form of the Complete Model was used to demonstrate that not only does the transverse-axes dynamics exhibit four dynamic modes as Staley [21] had shown before, but two of these modes are fixed when closed-loop control is employed. A closed-form solution to the transverse-axes equations of motion was derived which served to illustrate the relative influences of the four characteristic modes. The concepts of case-referenced dynamic stiffness and damping were introduced. These characteristics will be shown to be key factors for the extraction of external rates from the motion of a general tuned-rotor instrument. Finally, the Complete Model was used to derive a more accurate tuning condition, which should eliminate drift from both tuned-rotor gyroscopes and future general tuned-rotor instruments such as the GyroWheel.

Chapter 7

New Theorems for the Application of Classical Design Techniques to the Control of Single-Complex-Input, Single-Complex-Output (SCISCO) Systems

The primary focus of tuned-rotor dynamic analysis is on the behaviour of the two spin axis tilt angles in response to the two rebalance torques. Recognizing that the two inputs (torques) and two outputs (tilt angles) are in quadrature, the transverse-axes dynamics can be expressed compactly as a single differential equation in terms of a single complex tilt angle and single complex rebalance torque. This complex formulation has proved particularly useful for analysis purposes. Describing the dynamics as a single-complex-input, single-complex-output system is appropriate for not only tuned-rotor gyroscopes, but also other two-input, two-output systems where both the inputs and outputs are in quadrature.

For the purposes of this study, such a single-complex-input, single-complex-output system will be termed a SCISCO system. Although analysis of such systems is documented in the literature, there has been limited discussion on the application of systematic control approaches to SCISCO systems or on the concept of relative stability for this form. In the following chapter it is demonstrated that with some small

modifications, classical single-input, single-output control system design methods can be applied to SCISCO transfer functions to systematically develop controllers and examine stability. Classical techniques such as Root locus, Nyquist plots and frequency-domain design can be applied almost directly to this class of two-input, two output systems, with the ease associated with single-input, single-output procedures.

The introduction of the complex angle and torques to reduce the transverse-axes dynamics to a single complex differential equation may initially appear only as an ingenious device. The basis for this result comes from the decoupled form of this class of system. Before describing the application of the classical control techniques to SCISCO systems, this chapter begins by introducing the SCISCO control problem and showing its relation to its equivalent 2-input, 2-output form.

7.1 SCISCO Control System

The objective in this chapter is to show how controllers for 2-input, 2-output systems whose inputs and outputs are in quadrature can be designed using the equivalent SCISCO form. Consider the SCISCO transfer function representation of the Traditional model of the transverse-axes dynamics as given in equation (5-122). Ignoring the effect of the base motion (ω_b) and the periodic tilt effects ($\bar{\phi}(s - 2j\omega_s)$), the SCISCO transfer function relationship describing the tilt dynamics is given by the following expression

$$\phi(s) = G_{\phi\eta_i}(s) T_c(s) \quad (7-1)$$

$$G_{\phi\eta_i}(s) = \frac{1}{I_r' \left\{ s^2 + 2c_s' s - \alpha - j \left[(\omega_\pi + 2\gamma\omega_s) s - 2c_s' \omega_s \right] \right\}} \quad (7-2)$$

The objective in this chapter is to show how SCISCO forward path ($G_{\varepsilon_1}(s)$) and feedback path ($H_1(s)$) compensators can be chosen to provide both stability and desired performance for the SCISCO dynamics transfer function. The SCISCO control problem is shown in Figure 7-1 below.

Figure 7-1: SCISCO Control System Block Diagram

The following section describes the relationship between the SCISCO control system above and its equivalent 2-input, 2-output form. In addition, properties of the SCISCO control system form above are outlined. Sections 7.2 and 7.3 will deal with how $G_{\varepsilon_1}(s)$ and $H_1(s)$ can be chosen using either root locus or frequency domain design techniques.

7.1.1 Relationship Between SCISCO and 2-Input, 2-Output Dynamics Representations

In Chapter 5, traditional and complete models of the transverse-axes dynamics were derived by noting that the two differential equations describing the motion of the transverse-axes dynamics could be combined into the form of a single differential equation with complex coefficients. The equations were combined in quadrature, i.e.

adding the first equation to j times the second equation. The more formal relationship between the 2-input, 2-output form and the SCISCO form can be found by determining the transformation that diagonalizes the 2-input, 2-output transfer function matrix.

For the purposes of developing the idea of a SCISCO control system, the Traditional model will be used. Although the following ideas apply equally well to the Complete Model, the Traditional model is simpler form to deal with. Equation (5-106) is re-written below where the dynamics are described in the form of a 2-input, 2-output transfer function matrix $\mathbf{P}_{\dot{x}\pi}$, which relates the case-referenced tilt angles (ϕ_x, ϕ_y) to the case-referenced control torques (T_{z_x}, T_{z_y}) . This is summarized in the following form:

$$\mathbf{y}(s) = \mathbf{P}_{\dot{x}\pi}(s) \mathbf{u}(s) \quad (7-3)$$

$$\mathbf{y} = [\phi_x \quad \phi_y]^T \quad (7-4)$$

$$\mathbf{u} = [T_{z_x} \quad T_{z_y}]^T \quad (7-5)$$

$$\mathbf{P}_{\dot{x}\pi}(s) = \frac{1}{d(s)} \begin{bmatrix} s^2 + 2c_g' s + \alpha & -(\omega_n - 2\gamma\omega_s)s - 2c_g' \omega_s \\ (\omega_n - 2\gamma\omega_s)s - 2c_g' \omega_s & s^2 + 2c_g' s + \alpha \end{bmatrix} \quad (7-6)$$

$$d(s) = I_z' \left\{ \left(s^2 + 2c_g' s + \alpha \right)^2 - \left((\omega_n - 2\gamma\omega_s)s + 2c_g' \omega_s \right)^2 \right\} \quad (7-7)$$

A transformation matrix $\mathbf{W}$ is now determined which diagonalizes $\mathbf{P}_{\dot{x}\pi}$ into the following form:

$$\mathbf{G}_{\dot{x}\pi}(s) = \mathbf{W}^{-1} \mathbf{P}_{\dot{x}\pi}(s) \mathbf{W} = \begin{bmatrix} G_{\dot{x}\pi_1}(s) & 0 \\ 0 & G_{\dot{x}\pi_2}(s) \end{bmatrix} \quad (7-8)$$

After taking the determinant of $\mathbf{P}_{\alpha n}(s)$, the eigenvalues of the transfer function matrix are found to be:

$$\lambda_{1,2} = \left(s^2 + 2c_g' s + \alpha \right) = j \left[(\omega_n - 2\gamma\omega_s) s + 2c_g' \omega_s \right] \quad (7-9)$$

Solving the eigenvectors of the system, the transformation matrix $\mathbf{W}$ and its inverse $\mathbf{W}^{-1}$ are found to be of the form:

$$\mathbf{W} = \begin{bmatrix} 1 & 1 \\ -j & j \end{bmatrix} \quad \mathbf{W}^{-1} = \frac{1}{2} \begin{bmatrix} 1 & j \\ 1 & -j \end{bmatrix} \quad (7-10)$$

The transformed input and output vectors take the following form:

$$\begin{aligned} \mathbf{u}'(s) &= \mathbf{W}^{-1} \begin{bmatrix} T_{\alpha} \\ T_{\alpha} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} T_z \\ \bar{T}_z \end{bmatrix} \\ \mathbf{y}'(s) &= \mathbf{W}^{-1} \begin{bmatrix} \phi_z \\ \phi_y \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \phi \\ \bar{\phi} \end{bmatrix} \end{aligned} \quad (7-11)$$

where the complex control torque input T_z , complex tilt angle output ϕ and their complex conjugates are given by the following relations:

$$\begin{aligned} T_z &= T_{\alpha} - jT_{\beta} \\ \bar{T} &= T_{\alpha} + jT_{\beta} \\ \phi &= \phi_z + j\phi_y \\ \bar{\phi} &= \phi_z - j\phi_y \end{aligned} \quad (7.12)$$

The diagonalized transfer function matrix $\mathbf{G}_{\dot{\mathbf{x}}n}(s)$ is of the form:

$$\mathbf{G}_{\dot{\mathbf{x}}n}(s) = \begin{bmatrix} G_{\dot{\mathbf{x}}n_1}(s) & 0 \\ 0 & G_{\dot{\mathbf{x}}n_2}(s) \end{bmatrix} \quad (7-13)$$

where $G_{\dot{\mathbf{x}}n_1}(s)$ and the characteristic polynomial $d(s)$ are defined as

$$G_{\dot{\mathbf{x}}n_1}(s) = \frac{s^2 + 2c'_s s + \alpha - j[(\omega_n - 2\gamma\omega_s)s - 2c'_s \omega_s]}{d(s)} \quad (7-14)$$

$$G_{\dot{\mathbf{x}}n_2}(s) = \frac{s^2 + 2c'_s s + \alpha + j[(\omega_n - 2\gamma\omega_s)s - 2c'_s \omega_s]}{d(s)} \quad (7-15)$$

$$d(s) = I'_r \left[(s^2 + 2c'_s s + \alpha) - j[(\omega_n - 2\gamma\omega_s)s - 2c'_s \omega_s] \right] \quad (7-16)$$

The expression for $\mathbf{G}_{\dot{\mathbf{x}}n}(s)$ can be further simplified by expressing $d(s)$ in the form:

$$d(s) = I'_r \left[\left(s^2 + 2c'_s s + \alpha \right) - j[(\omega_n - 2\gamma\omega_s)s - 2c'_s \omega_s] \right] \cdot \left[\left(s^2 + 2c'_s s + \alpha \right) + j[(\omega_n - 2\gamma\omega_s)s - 2c'_s \omega_s] \right] \quad (7-17)$$

Substituting (7-17) into (7-13) yields the diagonalized dynamics transfer function:

$$\mathbf{G}_{\dot{\mathbf{x}}n}(s) = \begin{bmatrix} G_{\dot{\mathbf{x}}n_1}(s) & 0 \\ 0 & G_{\dot{\mathbf{x}}n_2}(s) \end{bmatrix} \quad (7-18)$$

where

$$\begin{aligned}
 G_{\dot{\phi}_x}(s) &= \frac{1}{I_s' \left\{ s^2 - 2c_s' s - \alpha - j \left[(\omega_n - 2\gamma\omega_s) s - 2c_s' \omega_s \right] \right\}} \\
 G_{\dot{\phi}_y}(s) &= \frac{1}{I_s' \left\{ s^2 - 2c_s' s - \alpha - j \left[(\omega_n - 2\gamma\omega_s) s - 2c_s' \omega_s \right] \right\}}
 \end{aligned} \tag{7-19}$$

The first transfer function $G_{\dot{\phi}_x}(s)$ relates the complex angle $\phi_x - j\phi_y$ to the applied complex control torque $T_x - jT_y$ and is of the same form as that shown in (7-2). Similarly, the second transfer function $G_{\dot{\phi}_y}(s)$ relates the conjugate of the complex angle $\phi(s)$ to the conjugate of the complex control torque $T_x(s)$. Overall, this leads to the following properties of SCISCO transfer functions:

- The SCISCO transfer functions $G_{\dot{\phi}_x}(s)$ and $G_{\dot{\phi}_y}(s)$ are complex conjugate transfer functions, i.e. the like-order coefficients of s in the two polynomials are complex conjugates.
- As the denominators of the two transfer functions $G_{\dot{\phi}_x}(s)$ and $G_{\dot{\phi}_y}(s)$ are complex conjugates, the poles of $G_{\dot{\phi}_x}(s)$ will be the complex conjugates of the poles of $G_{\dot{\phi}_y}(s)$.
- The transformation $\mathbf{W}^{-1}$ performs the equivalent of combining the inputs $\mathbf{u} = \begin{bmatrix} T_x & T_y \end{bmatrix}^T$ and outputs $\mathbf{y} = \begin{bmatrix} \phi_x & \phi_y \end{bmatrix}^T$ in quadrature. As such, it can be used for other two-input, two-output systems whose inputs and outputs can be combined in quadrature to produce the equivalent SCISCO form of dynamics. Although not shown in this document, the transformation $\mathbf{W}^{-1}\mathbf{G}_{\dot{\phi}}\mathbf{W}$ can be applied to the control of a current regulator for a multi-phase AC load [23] or the

roll/yaw control of the attitude dynamics of a bias-momentum spacecraft [45]. In each of these cases, the properties described in a) and b) above are again exhibited.

7.1.2 SCISCO Control Problem

The objective in this chapter is to use the SCISCO transfer functions to design forward path and feedback compensation to yield desired stability and performance. As the two SCISCO transfer functions $G_{\dot{\phi}n_1}(s)$ and $G_{\dot{\phi}n_2}(s)$ are decoupled, it might appear that control of the overall system can be accomplished by designing separate controllers for each of $G_{\dot{\phi}n_1}(s)$ and $G_{\dot{\phi}n_2}(s)$. In fact, the properties of the two SCISCO transfer functions representing the system dynamics allow for a simpler solution. The control block diagrams for the two decoupled SCISCO systems shown in Figure 7-2 illustrate this idea. The complex tilt angle reference ϕ_{ref} , complex tracking error ϕ_{err} and complex feedback angle ϕ_b are defined as the following:

$$\phi_{ref}(s) = \phi_{ref_1}(s) - j\phi_{ref_2}(s) \quad (7-20)$$

$$\phi_b(s) = H_1(s)\phi(s) \quad (7-21)$$

$$\phi_{err}(s) = \phi_{ref}(s) - \phi_b(s) \quad (7-22)$$

The forward path controller $G_{\dot{\phi}}(s)$ and feedback controller $H_1(s)$ are chosen to stabilize $G_{\dot{\phi}n_1}(s)$ and minimize tracking error of $(\phi_{ref} - \phi)$ with a desired bandwidth. Similarly $G_{\dot{\phi}}(s)$ and $H_2(s)$ are chosen to stabilize $G_{\dot{\phi}n_2}(s)$ and achieve the same level of tracking error $(\bar{\phi}_{ref} - \bar{\phi})$ with the same bandwidth as the first decoupled loop.

Figure 7-2: Decoupled SCISCO Feedback Control Loops

The control torques T_c and $\bar{T}_c$ are generated according to the following relations:

$$T_c(s) = G_{c_1}(s)\phi_{err}(s) \quad (7-23)$$

$$\bar{T}_c(s) = \bar{G}_{c_2}(s)\bar{\phi}_{err}(s) \quad (7-24)$$

Taking the complex conjugate of the expression in (7-24) yields

$$T_c(s) = \bar{G}_{c_2}(s)\phi_{err}(s) \quad (7-25)$$

Comparing (7-23) and (7-25),

$$G_c(s) = \bar{G}_c(s) \quad (7-26)$$

Using the same argument, it can be shown that

$$H_1(s) = \bar{H}_2(s) \quad (7-27)$$

The consequence of these observations is that only one of the SCISCO control loops in Figure 7-2 need be analysed to determine the control system for the entire system.

In this study, the complex forward path and feedback compensation terms $G_c(s)$ and $H_1(s)$ will be designed using a variation on classical single-input, single-output control theory. Once these terms are chosen, the only remaining problem is to determine the equivalent two-input, two-output form, expressed in terms of real inputs and outputs. The overall control system block diagram for the system in decoupled SCISCO form is shown in Figure 7-3 where the following signals and transfer functions are defined:

$$\begin{aligned} \mathbf{r}' &= \begin{bmatrix} \bar{\phi}_{ref} & \bar{\phi}_{ref}^T \end{bmatrix} \\ \mathbf{e}' &= \begin{bmatrix} \bar{\phi}_{err} & \bar{\phi}_{err}^T \end{bmatrix} \\ \mathbf{y}'_{fb} &= \begin{bmatrix} \bar{\phi}_{fb} & \bar{\phi}_{fb}^T \end{bmatrix} \end{aligned} \quad (7-28)$$

$$\mathbf{G}_c(s) = \begin{bmatrix} \bar{G}_c(s) & 0 \\ 0 & \bar{G}_c(s) \end{bmatrix} \quad (7-29)$$

$$\mathbf{H} = \begin{bmatrix} \bar{H}_1(s) & 0 \\ 0 & \bar{H}_1(s) \end{bmatrix} \quad (7-30)$$

Figure 7-3: Combined SCISCO Block Diagram of Control System

Noting equations (7-11), the following signals for the coupled 2 real-input, 2 real-output representation of the transverse-axes dynamics can be found by applying the transformation $\mathbf{W}$

$$\begin{aligned}
 \mathbf{r} &= \begin{bmatrix} \phi_{ref_x} \\ \phi_{ref_y} \end{bmatrix} = \mathbf{W} \mathbf{r}' \\
 \mathbf{e} &= \begin{bmatrix} \phi_{ref_x} - \phi_x \\ \phi_{ref_y} - \phi_y \end{bmatrix} = \mathbf{W} \mathbf{e}' \\
 \mathbf{u} &= \begin{bmatrix} T_{c_x} \\ T_{c_y} \end{bmatrix} = \mathbf{W} \mathbf{u}' \\
 \mathbf{y} &= \begin{bmatrix} \phi_x \\ \phi_y \end{bmatrix} = \mathbf{W} \mathbf{y}'
 \end{aligned} \tag{7-31}$$

The resulting 2 real-input, 2 real-output control system block diagram is shown in Figure 7-4 with the forward path and feedback path controller matrices defined below:

$$\mathbf{P}_c(s) = \frac{1}{2} \begin{bmatrix} G_{c_1} - \bar{G}_{c_1} & j(G_{c_1} - \bar{G}_{c_1}) \\ -j(G_{c_1} - \bar{G}_{c_1}) & G_{c_1} - \bar{G}_{c_1} \end{bmatrix} \tag{7-32}$$

$$Q(s) = \frac{1}{2} \begin{bmatrix} H_1 - \bar{H}_1 & j(H_1 - \bar{H}_1) \\ -j(H_1 - \bar{H}_1) & H_1 - \bar{H}_1 \end{bmatrix} \quad (7-33)$$

Figure 7-4: Two-Input, Two-Output Feedback Control System

7.1.3 Properties of SCISCO Systems

Before proceeding to apply control system design methods to SCISCO transfer functions, it is appropriate to examine some of their properties and illustrate the equivalent characteristics for real-valued, two-input, two-output transfer functions.

a) Transmission Zeros of Complex-Coefficient Transfer Functions

In single-input, single-output systems as shown in Figure 7-5, zeros in the loop gain function $P_c(s)P_{dyn}(s)Q(s)$ are transmission zeros. For the following loop gain function:

$$P_c(s)P_{dyn}(s)Q(s) = \frac{K(s - z_1)(s - z_2) \cdots (s - z_n)}{(s - p_1)(s - p_2) \cdots (s - p_n)} \quad (7-34)$$

the closed loop poles are the roots of characteristic polynomial given below:

$$d(s) = K(s - z_1)(s - z_2) \cdots (s - z_m) - (s - p_1)(s - p_2) \cdots (s - p_n) \quad (7-35)$$

For a very high gain K , m closed loop poles will tend to the locations of the m open loop zeros.

Figure 7-5: Single Input, Single Output Feedback Control System

In multiple-input, multiple-output systems such as that shown in Figure 7-4, the zeros of any of the four transfer functions relating the inputs to outputs within the loop gain function $P_c(s)P_{fn}(s)Q(s)$ are not the transmission zeros of such a system. However, modeling the transverse-axes dynamics in the form of a SCISCO transfer function (using either $G_{\hat{x},\hat{y}_1}(s)$ or $G_{\hat{x},\hat{y}_2}(s)$), for either pair of $G_c(s)$ and H_c , the zeros of the loop gain function $G_c G_{\hat{x},\hat{y}_1} H_c$ will be transmission zeros of the system. This follows using the same argument used to describe transmission zeros for SISO systems above.

- b) **The coefficients of the resulting controller transfer functions $P_c(s)$ and $Q(s)$ are always real-valued**

In some instances, the system dynamics of an n -input, n -output system is decoupled using the approach used in equation (7-8). However, when the compensation terms $G_c(s)$ and $H(s)$ are transformed back into the non-decoupled space, some of the resulting coefficients of the transfer functions within $P_c(s)$ and $Q(s)$ are often complex numbers and cannot be implemented directly. In systems where the coupling is weak, only the real-valued terms are considered in the controller structure [39]. However, for the transverse axes dynamics of the tuned-rotor instrument and other two-input, two-output systems, which can be diagonalized using the transformation defined in equation (7-8), the coefficients of the resulting controller transfer functions $P_c(s)$ and $Q(s)$ are always real-valued.

This can be seen by examining the structure of the transfer function matrices $P_c(s)$ and $Q(s)$ in equations (7-32) and (7-33) respectively. The sum of polynomials with complex coefficients yields a polynomial with only real coefficients. Therefore the diagonal elements of $P_c(s)$ and $Q(s)$ will be transfer functions with only real coefficients. The difference of two polynomials with complex coefficients will be a polynomial with no real-valued coefficients. Consequently, $(G_{\hat{e}} - \bar{G}_{\hat{e}})$ and $(H_{\hat{e}} - \bar{H}_{\hat{e}})$ will be transfer functions with only imaginary-valued coefficients while $j(G_{\hat{e}} - \bar{G}_{\hat{e}})$ and $j(H_{\hat{e}} - \bar{H}_{\hat{e}})$ will have only real-valued coefficients.

c) **Closed-Loop Transfer Function of SCISCO System**

It has been shown that only one of the two compensators $G_{\dot{z}_i}(s)$ and $G_{\dot{z}_i}(s)$ need be chosen in order to design the complete control system. Therefore, the control problem simplifies to designing $G_{\dot{z}_i}(s)$ for the SCISCO control system shown in the block diagram in Figure 7-2. The controller $G_{\dot{z}_i}(s)$ can be chosen such that the closed loop transfer function relating the complex output $y_i'(s)$ to the complex input $r_i'(s)$ takes the form:

$$G_{\dot{z}_i}(s) = \frac{G_{\dot{z}_i}(s)G_{\dot{z}_{in}}(s)}{1 - G_{\dot{z}_i}(s)G_{\dot{z}_{in}}(s)H_i(s)} \quad (7-36)$$

Similarly, the closed loop transfer function relating $\bar{y}_i'(s)$ to $\bar{r}_i'(s)$ is:

$$\bar{G}_{\dot{z}_i}(s) = \frac{\bar{G}_{\dot{z}_i}(s)\bar{G}_{\dot{z}_{in}}(s)}{1 - \bar{G}_{\dot{z}_i}(s)\bar{G}_{\dot{z}_{in}}(s)\bar{H}_i(s)} \quad (7-37)$$

Combining these two relations, the overall equation describing the closed-loop momentum dynamics expressed in the transformed space is given below:

$$\begin{bmatrix} y_i'(s) \\ \bar{y}_i'(s) \end{bmatrix} = \mathbf{G}_{ci}(s) \begin{bmatrix} r_i'(s) \\ \bar{r}_i'(s) \end{bmatrix} \quad (7-38)$$

where

$$\mathbf{G}_{ci} = \begin{bmatrix} G_{\dot{z}_i} & 0 \\ 0 & \bar{G}_{\dot{z}_i} \end{bmatrix} \quad (7-39)$$

The closed loop poles in the transformed system should consist of the union of the poles of $G_{ci}(s)$ and $\bar{G}_{ci}(s)$, since the poles must be the same in the transformed and untransformed spaces. This is confirmed by obtaining the closed-loop relationship between the outputs y and reference inputs r , which has the form:

$$\begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = \mathbf{P}_{ci}(s) \begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix} \quad (7-40)$$

The closed-loop transfer function matrix $\mathbf{P}_{ci}(s)$ is given by the following expression:

$$\mathbf{P}_{ci}(s) = \mathbf{W} \mathbf{G}_{ci}(s) \mathbf{W}^{-1} \quad (7-41)$$

Substituting equations (7-39) and (7-10) into (7-41), the overall closed loop system transfer function matrix is:

$$\mathbf{P}_{ci}(s) = \frac{1}{2} \begin{bmatrix} G_{ci} - \bar{G}_{ci} & j(G_{ci} - \bar{G}_{ci}) \\ -j(G_{ci} - \bar{G}_{ci}) & G_{ci} - \bar{G}_{ci} \end{bmatrix} \quad (7-42)$$

where

$$\begin{aligned} G_{ci} - \bar{G}_{ci} &= \frac{G_{ci} G_{\phi n_i} (1 - \bar{G}_{ci} \bar{G}_{\phi n_i} \bar{H}_i) - \bar{G}_{ci} \bar{G}_{\phi n_i} (1 - G_{ci} G_{\phi n_i} H_i)}{(1 + G_{ci} G_{\phi n_i} H_i)(1 - \bar{G}_{ci} \bar{G}_{\phi n_i} \bar{H}_i)} \\ G_{ci} - \bar{G}_{ci} &= \frac{G_{ci} G_{\phi n_i} (1 - \bar{G}_{ci} \bar{G}_{\phi n_i} \bar{H}_i) - \bar{G}_{ci} \bar{G}_{\phi n_i} (1 + G_{ci} G_{\phi n_i} H_i)}{(1 + G_{ci} G_{\phi n_i} H_i)(1 - \bar{G}_{ci} \bar{G}_{\phi n_i} \bar{H}_i)} \end{aligned} \quad (7-43)$$

Therefore the closed loop poles of the system are the union of the poles of G_{ci} and $\bar{G}_{ci}$.

This result is significant as it means that by choosing $G_{ci}(s)$ to yield the desired poles of

the closed-loop SCISCO transfer function G_{z_i} , the closed loop poles of the overall two-input, two-output system will be the poles of G_{z_i} and their complex conjugates.

d) Use of Complex Transfer Function for Control Design to Meet Performance Objectives

The SCISCO transfer function G_{z_i} relates the transformed complex output y'_i to the complex control input u'_i . y'_i and u'_i are linear combinations of the outputs (y_1, y_2) and inputs (u_1, u_2) respectively, and as such do not necessarily have any physical meaning. In many systems after diagonalizing the dynamics, it is sometimes difficult to design a controller based on the transformed variable rather than the desired untransformed one. Generally, the transverse-axes dynamics studied here is an exception to this rule.

Examining the closed loop transfer function matrix $\mathbf{P}_{z_i}(s)$, it can be seen that the diagonal entries relating y_i to u_i are simply the average of $G_{z_i}(s)$ and $\bar{G}_{z_i}(s)$. This can be expressed in the following alternate form:

$$\frac{G_{z_i}(s) + \bar{G}_{z_i}(s)}{2} = \text{Re}(G_{z_i}(s)) \quad (7-44)$$

$G_{z_i}(s)$ has no significant imaginary component up to the frequency at which it rolls off (closed loop bandwidth). Therefore $G_{z_i}(j\omega) \approx \text{Re}(G_{z_i}(j\omega))$ up to the closed-loop bandwidth. Therefore, a SCISCO controller $G_c(s)$ can be designed for the SCISCO transfer function $G_{z_i}(s)$ to meet performance objectives such as bandwidth or overshoot, and similar characteristics will be realized in the equivalent 2 real input, 2 real output system. For this guideline to hold true, there are some restrictions on the

maximum achievable bandwidth and type of controller used. These limitations are discussed in Section 7.3.2.2.

e) Cross Coupling in Closed Loop System

For a SCISCO control system characterized by plant dynamics $G_{\dot{\alpha}\eta_1}(s)$, forward path controller $G_c(s)$ and feedback controller $H_f(s)$, the equivalent 2-real input, 2-real output closed-loop transfer function matrix is given in equation (7-43). Examining equation (7-43), the cross-coupling transfer function matrix is defined as $\frac{j}{2}(G_{z\dot{\alpha}} - \bar{G}_{z\dot{\alpha}})$. In the control designs examined later in this chapter, the feedback transfer function H_f will be chosen to have no complex coefficients (i.e. $H_f = \bar{H}_f$). Consequently, in equation (7-43), $\frac{j}{2}(G_{z\dot{\alpha}} - \bar{G}_{z\dot{\alpha}})$ simplifies to

$$\frac{j}{2}(G_{z\dot{\alpha}} - \bar{G}_{z\dot{\alpha}}) = \frac{j}{2} \frac{(G_c G_{\dot{\alpha}\eta_1} - \bar{G}_c \bar{G}_{\dot{\alpha}\eta_1})}{(1 + G_c G_{\dot{\alpha}\eta_1} H_f)(1 - \bar{G}_c \bar{G}_{\dot{\alpha}\eta_1} \bar{H}_f)} \quad (7-45)$$

This can be expressed in the even more compact form:

$$\frac{j}{2}(G_{z\dot{\alpha}} - \bar{G}_{z\dot{\alpha}}) = \frac{-j \operatorname{Im}(G_c G_{\dot{\alpha}\eta_1})}{(1 + G_c G_{\dot{\alpha}\eta_1} H_f)(1 - \bar{G}_c \bar{G}_{\dot{\alpha}\eta_1} \bar{H}_f)} \quad (7-46)$$

Therefore, the degree of cross coupling can be limited by minimizing the imaginary component of $G_c G_{\dot{\alpha}\eta_1}$. The imaginary component of $G_{\dot{\alpha}\eta_1}$ is due to the poles occurring at the dynamic modes of the system, e.g. the nutation and precession frequency of the tuned-rotor instrument. Therefore, to limit the degree of cross-coupling, the compensator $G_c(s)$ must be chosen to cancel or nearly cancel these modes in $G_{\dot{\alpha}\eta_1}(s)$.

7.2 Root Locus Method for a SCISCO Transfer Function

The objective of this particular part of the study was not to simply determine the closed loop poles of a SCISCO transfer function. There are a number of commercial software tools that can be used to perform this task (e.g. MATLAB [43]). Instead, the objective is to show how root locus diagrams can be used to systematically choose SCISCO controllers ($G_c(s)$) that will stabilize the SCISCO plant transfer function ($G_{\text{sys}}(s)$). To achieve this, the traditional rules for generating root loci must be extended somewhat. Examples are given showing how a number of controllers can be chosen to stabilize the nutation mode in the transverse-axes dynamics of a tuned-rotor instrument.

7.2.1 Guidelines for Determining Root Loci of SCISCO Transfer Functions

The general rules commonly used for root locus diagrams can be applied to the analysis and design of SCISCO transfer functions. However, two unique features of SCISCO systems must be considered when using them.

a) Use of Complex Gain $\tilde{K}$

As SCISCO transfer functions have at least one complex coefficient in the denominator, it is necessary to consider the possibility of a complex control gain $\tilde{K}$:

$$\tilde{K} = (j)^q K, \quad q = 0, 1, 2 \text{ or } 3, \quad K \in \mathbb{R} \quad (7-47)$$

where the gain can have one of four possible values.

The need to consider a complex gain can be seen by considering the precession control of a gyroscopic system. To make a spinning body precess, the torque must be applied about an axis orthogonal to both the precession and spin axes. Therefore, when designing gyro rebalance control systems expressed in the form of Figure 7-4 and equation (7-31), the control law is often in the cross-coupled form described below:

$$\begin{bmatrix} T_{z_1} \\ T_{z_2} \end{bmatrix} = G_z(s) \begin{bmatrix} \phi_{ref} - \phi_z \\ -(\phi_{ref} - \phi_z) \end{bmatrix} \quad (7-48)$$

Adding T_{z_1} to j times T_{z_2} results in the following expression:

$$T_z = -jG_z(s)(\phi_{ref}(s) - \phi(s)) \quad (7-49)$$

It relates the complex control torque $T_z(s)$ to the complex tracking error $\phi_{ref} - \phi$.

The presence of this complex gain means that some of the rules governing the generation and interpretation of root locus diagrams described in Appendix C.2 must be re-considered. Use of a complex gain in the loop gain function in equation (C-6) introduces not only an amplification/attenuation, but also a phase shift equal to

$$\phi = q \left(\frac{\pi}{2} \right) \quad (7-50)$$

- b) **Pole/Zeros of SCISCO transfer functions are not necessarily symmetric about the real axis**

Unlike transfer functions with real-valued coefficients, complex transfer functions can contain a complex pole or zero without the presence of its complex conjugate. Therefore, root loci of SCISCO systems are not symmetric about the real axis.

These two key differences lead to slight modifications of the angle condition rule and rules governing orientation and origin of asymptotes. These modifications are described in the context of the SCISCO system shown in Figure 7-1.

- a) **Modified Angle Condition for SCISCO Systems**

Noting the effect on phase of a complex gain, the angle condition given in equation (C-11) must be modified to

$$\text{phase}(G_c G_{\text{dyn}} H_c) = q \left(\frac{\pi}{2} \right) - \sum_{i=1}^m \alpha_i - \sum_{j=1}^n \beta_j = \pm (2k-1)\pi \quad (7-51)$$

where q is the order of the complex gain $\tilde{K} = (j)^q K$.

- b) **Direction of Asymptotes for SCISCO Systems**

Consider a SISO loop gain function of the form of equation (C-6) with numerator order m and denominator order n . In a root locus diagram, there are as many branches as there are open loop poles. A branch indicates the loci of the real and imaginary component of a closed loop pole as the loop gain is increased. In a system where $m = n$, as the loop gain approaches infinity, each branch will tend to one of the open loop zeros. For a loop gain function where $n > m$, m branches will tend to the m zeros as the gain approaches

infinity, while $(n-m)$ branches will tend to an infinite magnitude along one of the $(n-m)$ asymptotes.

For a SISO system, the direction of the asymptotes must satisfy the following angle condition

$$\mu_i = \frac{\pm(2i+1)\pi}{n-m} \quad i = 1, 2, \dots, (n-m) \quad (7-52)$$

where μ_i is the angle describing the orientation of the i^{th} asymptote. The angles are uniformly distributed over 360 degrees. The above condition is found using the angle condition and assumes that the gain of the transfer function in equation (C-6) is a positive real number.

For a SCISCO loop gain function, the origin and orientation of these asymptotes are affected by complex gains. To consider the effect of the four possible complex gains, the rationale behind (7-52) is examined. As one of the loci branches approaches infinity along an asymptote, the location of the closed-loop pole on this locus must satisfy the angle condition. The real and imaginary component of this closed loop pole will be much larger than that of any of the open poles and zeros. As seen in Figure 7-6, the angle describing the vector from each pole or zero to this closed loop pole will approach the same value of μ . For this case, the angle condition can be expressed as

$$m \cdot \mu - n \cdot \mu = \pm(2i+1)\pi \quad (7-53)$$

Now consider the case where the control gain is not a positive real number, i.e.

$$\tilde{K} = jK$$

The angle condition governing the position of the closed loop pole must be modified to the following form:

$$\frac{\pi}{2} - m \cdot \mu - n \cdot \mu = \pm(2i+1)\pi$$

This can be rearranged in the following form:

$$\mu_i = \frac{\pm(2i+1)\pi}{n-m} - \frac{\pi}{2(n-m)} \quad \text{if } \tilde{K} = jK \quad (7-54)$$

Overall, for the general complex gain $\tilde{K} = j^q K$, the asymptote angles are given by the following relationship

$$\mu_i = \frac{\pm(2i+1)\pi}{n-m} - \frac{q\pi}{2(n-m)} \quad (7-55)$$

Figure 7-6: Illustration of Asymptote Angle when s is very large

c) **Origin of Asymptotes**

For all transfer functions with real-valued coefficients, the origin of the asymptotes will be on the real axis and located at

$$\sigma_c = \frac{\sum_{j=1}^n p_j - \sum_{i=1}^m z_i}{n - m} \quad (7-56)$$

For any SISO transfer function with real-valued coefficients, if any z_i or p_j are complex, its complex conjugate must also be present as a zero or pole of the numerator or denominator polynomials respectively. When summing the poles or zeros in (7-56), the complex components will cancel, resulting in σ_c being a real number, and the origin of the asymptotes always lying on the real axis of the s-plane.

However, when either the numerator or denominator polynomials have complex coefficients, the numerator or denominator polynomial can possess a single complex root without its complex conjugate also being contained in the transfer function. **Therefore, σ_c can be a complex number.**

d) **Departure Angle of Loci Originating From Complex Open Loop Poles**

For SCISCO systems, the modified angle condition must now be applied in determining the angle of departure from an open-loop pole on the imaginary axis.

7.2.2 Application of SCISCO Root Locus Method to Nutation Damping Control of a Tuned-Rotor Instrument

The following section outlines how to apply the root locus method to the design of a controller for a SCISCO transfer function. The example of designing a nutation damper for the transverse-axes dynamics of a tuned-rotor instrument is chosen. Proportional and then dynamic methods are discussed. In addition, the idea of partitioning the control into precession and nutation controllers is introduced.

7.2.2.1 Proportional Control

Consider the first of the two SCISCO transfer functions given in equation (7-19) describing the transverse-axes dynamics:

$$G_{\phi n}(s) = \frac{1}{I_r' \left\{ s^2 + 2c_s' s - \alpha - j \left[(\omega_n + 2\gamma\omega_r) s - 2c_s' \omega_r \right] \right\}}$$

For the purposes of illustrating the concepts of SCISCO root locus, the following assumptions are made:

- The damping coefficient c_d is set to zero based on the reasons provided in Section 6.3.2.
- The system is tuned according to the classic tuning condition shown in equation (6-46). Tuning is achieved by setting the parameter α to zero.
- Based on the system characteristics of the GyroWheel system given in Table 6-1, the normalized inertia parameter γ is much smaller than 1 and can be ignored.

Based on the above assumptions, the simplified transverse-axes SCISCO transfer function takes the form:

$$G_{\dot{\omega}_n}(s) = \frac{1}{I_r' s(s - j\omega_n)}$$

To simplify the presentation of the following concepts, the nutation frequency ω_n is assumed to be 1 radian/second and the rotor transverse inertia I_r' is set to 1 $\text{kg}\cdot\text{m}^2$. The resulting transfer function simplifies to the following:

$$G_{\dot{\omega}_n}(s) = \frac{1}{s(s - j)} \quad (7-57)$$

The two poles of the system are:

$$p_1 = 0$$

$$p_2 = j$$

In the following discussion, the subscripts 1 and 2 will refer to the precession pole and nutation pole at 0 and j respectively. Now consider the application of general complex proportional control of the form:

$$G_{\dot{\omega}_n}(s) = j^q K, \quad q = 0, 1, 2 \text{ or } 3$$

The case where $q = 1$, and the gain is complex is examined first. The following is a brief description of how the response is constructed:

- a) As there are two open loop poles and no transmission zeros, there are two asymptotes. Applying the asymptote angle condition given in (7-55) for the case

where the gain is jK , the angles of the two asymptotes are determined from the relation

$$\mu_i = \frac{\pm(2i+1)\pi}{2} - \frac{\pi}{4}$$

Therefore the two asymptote angles are at $\frac{3\pi}{4}$ and $-\frac{\pi}{4}$ radians (145 degrees and -45 degrees)

- b) Applying (7-56), the origin of the asymptotes is at

$$\sigma_c = \frac{j}{2}$$

This shows how the origin of the asymptotes does not necessarily lie on the real axis for SCISCO transfer functions.

- c) To determine the departure angles of the loci from the precession and nutation poles respectively, the angle condition is applied for the case where the gain is jK . With no transmission zeros, this takes the form

$$\frac{\pi}{2} - (\beta_1 - \beta_2) = \pi$$

Therefore the condition on the departure angle of the nutation and precession poles is:

$$(\beta_1 - \beta_2) = -\frac{\pi}{2}$$

First, consider the loci originating from the precession pole p_1 at $s = 0$.

$\beta_1 = -\frac{\pi}{2}$ and consequently, the departure angle β_1 is 0 radians. For the loci

near the open loop nutation pole p_2 at $s = j$, the angle β_2 is $\frac{\pi}{2}$ radians. Using

the above angle condition, the departure angle of the nutation root loci is $-\pi$ radians.

For Case 2 ($\tilde{K} = jK$), the resulting root locus is shown in Figure 7-7 below. It can be seen that the locus of the closed-loop nutation pole is stabilized through the application of the gain jK . However, the closed-loop precession pole becomes unstable through the application of this gain.

Figure 7-7 also shows the root loci for the other three gain cases. It can be seen that as the order q of the complex gain $j^q K$ is incremented in steps of one, the asymptotes rotate in increments of 90 degrees. Similarly, the departure angles of the loci from the precession and nutation poles change by increments of 90 degrees. Application of the gain $j^q K$ does not stabilize either of the poles. In cases 2, 3 and 4, only one of the two poles is stabilized. It is evident that a form of dynamic compensation is required for each of the four cases, which will provide unique shifts of the departure angle of each locus.

Figure 7-7: Root Loci of the Transverse-Axes Control Loop for Four Cases of Proportional Control

7.2.2.2 Dynamic Control

The potential dynamic controllers that stabilize the transverse-axes dynamics can be considered in terms of the four gain cases.

a) Design Case 1: Proportional Gain $\tilde{K} = j^0 K$

Examining Figure 7-7, a pole/zero combination is required which will produce a $\frac{\pi}{2}$ radian shift in the departure angle from the nutation pole and a $-\frac{\pi}{2}$ degree shift in the departure angle from the precession pole. Positioning a zero on the imaginary axis between the precession and nutation pole modifies both departure angles appropriately. The controller has the form:

$$G_{\tilde{K}}(s) = K(s - z_1)$$

The zero z_1 is chosen to be near the nutation pole, where $z_1 = 0.8j$. Applying the angle condition for a positive real gain yields the following relation:

$$\alpha_1 - (\beta_1 - \beta_2) = \pi$$

For the locus originating at the precession pole shown in Figure 7-7, the following angles are known.

$$\alpha_1 = -\frac{\pi}{2}$$

$$\beta_2 = -\frac{\pi}{2}$$

Solving the departure angle of the precession pole locus, yields $\beta_1 = \pi$ which is desired.

Examining the locus originating from the nutation pole in Figure 7-7, the following angles are known:

$$\alpha_1 = \frac{\pi}{2}$$

$$\beta_1 = \frac{\pi}{2}$$

Solving the departure angle of the nutation pole locus yields $\beta_2 = \pi$ which is desired.

Therefore, the introduction of the zero appears suitable. However, the above compensator is not physically realizable. A more suitable form of the compensator is:

$$G_c(s) = K \frac{(s - z_1)}{(s - p_2)}$$

Introduction of the pole p_2 solves the problem of realizability. It is chosen to lie on the real-axis and to be 10 times larger in magnitude than the compensator zero so that there is no appreciable change in the departure angles of the precession and nutation poles, i.e.

$$p_2 = j10 \cdot z_1$$

The compensator zero is chosen to be closer to the nutation pole than the precession pole to reduce the effect of coupling. However, it is chosen sufficiently far away to account for the variation in the nutation frequency that occurs as the spin rate is varied in a device such as the GyroWheel.

The resulting root locus diagram is shown in Figure 7-8 below. Consider the design objective outlined in Section 3.3. A bandwidth of 10 Hz is desired, which is 0.217 radians/second after normalization with respect to the nutation frequency (46.31 Hz). Therefore, the dominant closed loop pole with real magnitude of slightly greater than 0.2 is desired. This corresponds to the case where:

$$G_{\ddot{z}}(s) = 0.26 \frac{(s - 0.8j)}{(0.125s + 1)}$$

The resulting closed loop poles are:

$$p_1 = -0.215 - j0.012$$

$$p_2 = -0.053 - j0.99$$

$$p_3 = -7.73 - j0.007$$

This design has introduced a damping factor of 0.053 to the nutation pole.

Figure 7-8: Root Locus Diagram of Nutation Damper Design for Transverse-axes

Dynamics when gain of $j^0 K$ is used, $G_{c_1}(s) = 0.22 \cdot j \frac{(1.25s - j)}{(0.125s + 1)}$

b) Design Case 2: Proportional Gain $\tilde{K} = j^0 K$

Figure 7-7 shows the root locus plot for the proportional control case where $\tilde{K} = j^0 K$. A compensator is required which will adjust the departure angle of the locus originating at p_1 by $\frac{\pi}{2}$ to π radians while not appreciably changing the departure angle of the locus from p_2 . To produce a change of greater than $\frac{\pi}{2}$ radians, an additional pole or zero

must be introduced in the right half of the s -plane. This means introducing either an unstable open loop pole or making the system non-minimum phase. The latter option is more desirable.

To shift the departure angle of the precession pole locus, the zero must be placed on the positive real axis. Unfortunately, introducing the zero in this location shifts the departure angle of the nutation pole such that the locus tends more towards the right half of the s -plane. This effect can be offset somewhat by introducing a pole on the negative real axis. The overall compensator is of the form:

$$G_z(s) = jK \frac{(s - z_1)}{(s - p_3)}$$

This pole has no effect on the departure angle of the precession pole locus but has the effect of drawing the nutation pole locus back into the left half of the s -plane. To confirm that the precession pole has been stabilized, the angle condition for gain case jK can be used where:

$$\alpha_1 - (\beta_1 + \beta_2 - \beta_3) = \frac{\pi}{2}$$

Examining the locus near the precession pole in Figure 7-7, the following angles are known:

$$\alpha_1 = \pi$$

$$\beta_2 = -\frac{\pi}{2}$$

$$\beta_3 = 0$$

Solving for β_1 from the angle condition above yields π radians, which is the desired orientation.

The remaining task is to choose the magnitudes of the zero and pole on the real axis. This can be done by considering the effect of the zero and pole on the locus originating from the nutation pole. For this loci, the angle $\beta_1 = \frac{\pi}{2}$ and the angle condition simplifies to a form giving the departure angle of the nutation pole locus as:

$$\beta_2 = \pi - (\alpha_1 - \beta_3)$$

To ensure that this locus departs into the left side of the s-plane, we desire the following:

$$\frac{\pi}{2} < \beta_2 < \pi$$

Therefore, the angles α_1 and β_3 should be chosen such that

$$|\alpha_1 - \beta_3| < \frac{\pi}{2}$$

Now assume that the magnitude of z_1 and p_3 are the same. Under the assumption that

$$\beta_3 = \pi - \alpha_1$$

the angle condition simplifies to:

$$\alpha_1 \leq \frac{\pi}{4}$$

Therefore, the magnitude of z_1 and p_3 must each be less than the magnitude of the nutation pole p_2 to satisfy this angle condition. To set the departure angle of the nutation pole p_2 to the desired value of π , z_1 and p_3 must be positioned very close to the origin. However, the close proximity of these terms to the precession pole would affect the low-frequency response of the system. A more suitable compromise is to set the magnitudes

of z_1 and p_3 to $\frac{p_1}{2}$ which results in an adequate departure angle of the locus of $\frac{3\pi}{4}$.

This lies in between the desired bounds of $\frac{\pi}{2}$ and π described above.

The resulting root locus diagram is shown in Figure 7-9. The control gain was again chosen to achieve a dominant closed-loop pole with real magnitude of approximately 0.217. Examining the root locus, it can be seen that the fastest the precession pole can be set to is approximately 0.15 radians/second. Above this gain, the imaginary component of the dominant pole is less than half the real component, i.e. a damping ratio of less than 0.707 results. The resulting controller is:

$$G_c(s) = 0.1j \frac{(2s-1)}{(2s-1)}$$

The resulting closed loop poles are:

$$p_1 = -0.15 - j0.076$$

$$p_2 = -0.07 - j0.906$$

$$p_3 = -0.28 - j0.17$$

Comparing the results of Design Case 2 with Design Case 1, the non-minimum phase controller has produced a slightly better damping factor of 0.07 on the closed loop nutation pole.

Figure 7-9: Root Locus Diagram of Nutation Damper Design for Transverse-axes

Dynamics when gain of $j^1 K$ is used, $G_c(s) = 0.1j \frac{(2s-1)}{(2s+1)}$

c) **Design Case 3: Proportional Gain $\tilde{K} = j^2 K$**

Using a proportional controller $j^2 K$, the departure angles of the precession and nutation pole are $\frac{\pi}{2}$ and $-\frac{\pi}{2}$ radians as shown in Figure 7-7. To shift the departure angles of the nutation and precession poles approximately $\frac{\pi}{2}$ and $-\frac{\pi}{2}$ radians respectively, a lag compensator of the form

$$G_c(s) = \frac{j^2 K}{s - p_3}$$

can be applied, where the compensator pole p_3 is positioned in between these two plant poles, near the imaginary axis. This can be confirmed by applying the angle condition for the gain case of $j^2 K$. Denoting the phase angle contribution from p_3 by β_3 , the angle condition of (7-51) simplifies to

$$2\left(\frac{\pi}{2}\right) - 0 - (\beta_1 - \beta_2 - \beta_3) = \pi$$

or

$$\beta_1 - \beta_2 + \beta_3 = 0$$

For a compensator pole p_3 on the imaginary axis, the departure angles of p_1 and p_2 both become π radians, which is desirable. However, if the compensator pole p_3 is placed directly on the imaginary axis, its loci will be drawn into the right half of the s-plane for any positive gain K . This follows by applying the above angle condition, with the resulting departure angle for p_3 being 0 radians. Therefore, p_3 must have a negative real-component. It must be chosen sufficiently large such that for the gain required to

adequately “pull” the nutation and precession poles into the left half of the s-plane, p_3 is not drawn too close to the imaginary axis. The compromise choice of compensator is

$$G_a = \frac{j^2 K}{s + 0.5 - 0.5j}$$

The resulting root locus diagram is shown in Figure 7-10 below. The objective is to draw the precession pole into the left half of the s-plane such that its real magnitude is approximately 0.2 radians/second. Examining Figure 7-10, when the real component of the precession pole reaches -0.2 , the compensator pole p_3 is drawn almost to the imaginary axis. A compromise choice of gain ($K = 0.07$) was made which results in the following closed loop poles:

$$p_1 = -0.11 - j0.078$$

$$p_2 = -0.11 - j0.92$$

$$p_3 = -0.29 - j0.5$$

Of note in this design case is the improvement in nutation damping introduced, with the damping factor being almost double that obtained with either Design Case 1 or 2.

Figure 7-10: Root Locus Diagram of Nutation Damper Design for Transverse-axes

Dynamics when gain of $j^2 K$ is used, $G_c(s) = \frac{0.07 j^2}{(s + 0.5 - 0.5j)}$

d) Design Case 4: Proportional Gain $\tilde{K} = j^3 K$

Based on the plot of $G_{\dot{\alpha}}(s)G_{\dot{\alpha}n}(s)$ in Figure 7-7, $G_{\dot{\alpha}n}(s)$ should be chosen to shift the departure angle of the nutation pole by between $\frac{\pi}{2}$ and π radians. This will direct the nutation pole locus into the left side of the s-plane, while not affecting the precession pole. By placing compensator poles or zeros on the negative real axis, the departure angle of p_1 is unaffected, yet the departure angle of the locus from p_2 can be shifted.

For the control gain $j^3 K$, applying the angle condition to the locus originating from p_2 results in the relation:

$$-\frac{\pi}{2} - \sum_{i=1}^n \alpha_i - \left(\beta_1 + \beta_2 + \sum_{j=3}^{n-2} \beta_j \right) = \pi$$

The terms $\sum_{i=1}^n \alpha_i$ and $\sum_{j=3}^{n-2} \beta_j$ represent the effect of n additional zeros and n additional poles on the departure angle of the locus from p_2 . Noting that $\beta_1 = \frac{\pi}{2}$, the nutation pole departure angle is given by:

$$\beta_2 = \sum_{i=1}^n \alpha_i - \sum_{j=3}^n \beta_j$$

Examining Figure 7-7, a compensator zero on the negative real axis will contribute 0 to $\frac{\pi}{2}$ radians phase angle shift at the origin of the p_2 locus, depending on how close it is to the imaginary axis. Similarly, a compensator pole on the negative real axis will contribute 0 to $-\frac{\pi}{2}$ radians phase angle shift depending on how close it is to the imaginary axis. Therefore at least two identical compensator poles or two identical compensator zeros will provide the desired π radian phase angle shift.

The option of introducing two or more compensator poles on the real axis is more desirable. Introduction of the zeros will make the system more susceptible to noise due to an increased bandwidth. For two compensator poles to introduce a π radian phase angle shift at the origin of the p_z locus, they must be almost at the origin of the s-plane. This is not desirable, as it will affect the locus of the precession pole. It is preferable to position them far enough to the left of p_z to ensure its locus is unaffected. Therefore, three or more compensator poles are required. To ensure no interaction of this compensator with the precession pole loci, four poles are chosen, resulting in a compensator of the form:

$$G_z(s) = \frac{j^3 K}{(s - p_z)^4}$$

Each of the four poles will contribute a $\frac{\pi}{4}$ phase angle shift at the origin of the p_z locus.

The resulting departure angle of the p_z locus is:

$$\beta_z = -\sum_{j=3}^6 \beta_j = -4\left(\frac{\pi}{4}\right) = \pi$$

Four new poles have been added to the system. It is relevant to examine their loci. As there are no transmission zeros, there will be 6 asymptotes. Their centre is located at:

$$\sigma_z = \frac{\sum_{j=1}^6 p_j}{6} = -\frac{4}{6} - \frac{j}{6}$$

From equation (7-55), the asymptote angles can be found from the relation:

$$\mu_i = \frac{(2i+1)}{n} \pi - \frac{\pi}{2n} = \frac{(2i+1)}{6} \pi - \frac{\pi}{12}$$

Therefore, the asymptotes have the following orientations:

$$\mu = \left[\frac{\pi}{12} \quad \frac{5\pi}{12} \quad \frac{9\pi}{12} \quad \frac{13\pi}{12} \quad \frac{17\pi}{12} \quad \frac{21\pi}{12} \right]$$

The resulting root locus is shown in Figure 7-11. The 6 loci tend toward the 6 predicted asymptotes. Like the minimum phase compensator design, there is a limit of approximately -0.15 on the maximum real component of the closed-loop precession pole. The control gain is chosen to ensure the damping factor on the closed-loop precession pole is greater than 0.707 while maximizing the real component. The resulting closed loop poles are:

$$\begin{aligned} p_1 &= -0.145 - j0.0843 \\ p_2 &= -0.026 + j1.002 \end{aligned}$$

The nutation-pole damping factor of 0.026 is the poorest of the four design options. As shown in Figure 7-12, the nutation pole is not drawn significantly away from the imaginary axis.

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Figure 7-12: Root Locus Diagram of Nutation Damper Design for Transverse-axes

Dynamics when gain of $j^3 K$ is used, $G_c(s) = \frac{0.15 \cdot j^3}{(s+1)^4}$, Enlarged view

near precession pole p_1 and nutation pole p_2 .

d) Comparison of Time Responses for the Four Design Cases

In order to examine the time response of the four design options, the complex transfer functions must be transformed into their non-complex form. Applying equation (7-32), the four controllers in non-complex form are:

$$\text{i) Design Case 1: } \mathbf{P}_c(s) = 0.156 \begin{bmatrix} \frac{s}{0.1s-1} & \frac{-0.8}{0.1s+1} \\ \frac{0.8}{0.1s-1} & \frac{s}{0.1s-1} \end{bmatrix}$$

$$\text{ii) Design Case 2: } \mathbf{P}_c(s) = 0.1 \begin{bmatrix} 0 & \frac{(2s-1)}{(2s-1)} \\ -\frac{(2s-1)}{(2s-1)} & 0 \end{bmatrix}$$

$$\text{iii) Design Case 3: } \mathbf{P}_c(s) = -0.035 \begin{bmatrix} \frac{(2s-1)}{s^2+s+0.5} & \frac{1}{s^2-s+0.5} \\ \frac{-1}{s^2-s+0.5} & \frac{(2s-1)}{s^2-s+0.5} \end{bmatrix}$$

$$\text{iii) Design Case 4: } \mathbf{P}_c(s) = 0.1 \begin{bmatrix} 0 & -\frac{1}{(s-1)^2} \\ \frac{1}{(s-1)^2} & 0 \end{bmatrix}$$

To compare the performance of these designs, linear time responses were generated for the case where a step input $\phi_{z_{ref}}$ is applied. Figure 7-13 shows the resulting responses of the two tilt angles ϕ_x and ϕ_y . Figure 7-14 shows the response after the main transient has settled. As predicted, the response using design case 1 yields the fastest response, with a

time constant of approximately 8 seconds. It should be noted that the dynamics have been scaled by the nutation frequency of 291 radians/second. In addition, this design is the most effective in decoupling the transverse-axes dynamics, with the peak transient of ϕ_y being the smallest.

Design case 3, employing the lag design with positive feedback produces the best nutation damping as illustrated in Figure 7-14. The negative feature of this design is the very significant coupling during the initial transient, with the uncommanded tilt angle ϕ_x , reaching a peak amplitude of 60 percent of the commanded tilt angle ϕ_x . This can be attributed to the introduction of another complex pole as part of the nutation damper. Examining equation (7-46), greater cross-coupling occurs as the term $\text{Im}(G_z G_{\dot{\phi}_x})$ increases. Introduction of the pole at $-0.5 - j0.5$ will increase the magnitude of $\text{Im}(G_z G_{\dot{\phi}_x})$. This differs from design case 1 where the introduction of the zero at $j0.8$ almost cancels the effect of the nutation pole at j , thereby reducing the magnitude of $\text{Im}(G_z G_{\dot{\phi}_x})$.

The non-minimum phase design (case 2) produces the second best damping of the nutation pole with the lag design (design case 4) producing the worst damping. The effect of the non-minimum phase design is seen in the response of design case 2 where the ϕ_x initially moves in the opposite direction to the command before rising to the desired set point.

Figure 7-13: Linear Time Response of Transverse-axes angles (ϕ_x, ϕ_y) to a step input of 1 degree about the case-referenced x-axis; responses shown for three nutation damper designs

Figure 7-14: Linear Time Response of Transverse-axes angles (ϕ_x, ϕ_y) to a step input of 1 degree about the case-referenced x-axis; enlarged view of last 50 seconds of the response, shown for three nutation damper designs

7.2.2.3 Nutation and Precession Controllers

In all of the above dynamic control problems, the main task was to determine a method of stabilizing the nutation dynamics. Poles/zeros for the nutation dampers were chosen so as not to affect the precession pole loci. The compensator poles/zeros had magnitudes of the order of the nutation frequency rather than the desired closed loop bandwidth ($0.1\omega_n - 0.2\omega_n$). Finally, the control gain was chosen to achieve the desired dominant closed loop pole (bandwidth).

Therefore, the control problem was effectively divided into choosing a nutation damper and a proportional precessional controller. This approach of partitioning the two problems will be used in later discussions of the transverse-axes control system in Chapter 10.

7.3 Frequency Response Methods for SCISCO Systems

The study of the root locus method has shown the effectiveness of using SCISCO transfer functions for control system design. In applying the root locus method to design stabilizing compensation, the focus was on phase angle compensation. One aspect of the design process that was not dealt with was relative stability. Both of these issues are topics that are particularly well suited to classical frequency response methods.

The following section focuses on the application of classical frequency response methods to a SCISCO transfer function. It is shown how Nyquist plots and the Nyquist criterion can be applied to such systems. Finally the use of Bode plots for design and judging relative stability is demonstrated.

7.3.1 Nyquist Stability for SCISCO Systems

The objective of this part of the analysis was to determine whether the Nyquist Stability Criterion could be applied directly to SCISCO loop gain functions such as that describing the transverse-axes dynamics control loop for a tuned-rotor instrument. The overall conclusions are that the Nyquist Stability Criterion can be applied directly to SCISCO systems. However, the concept of the Simplified Nyquist Criterion and phase margin must be modified. As a point of reference, Appendix C.3 provides the Nyquist Stability Criterion, the Simplified Nyquist Criterion and definitions for gain and phase margins for SISO systems.

7.3.1.1 Application of Nyquist Criterion to SCISCO Transfer Functions

The following section provides some examples of Nyquist plots of SCISCO transfer functions and the application of the Nyquist Stability Criterion. These examples involve using a simple proportional controller but are important as they provide insight on how ultimately to use frequency response tools effectively when dealing with SCISCO transfer functions.

Figure 7-15 shows the Nyquist diagrams for the SCISCO loop gain function $G_{\hat{z}}(s)G_{\hat{z}v\hat{n}_1}(s)H_{\hat{z}}(s)$ of the transverse-axes control loop. $G_{\hat{z}}(s)$ is set to be one of the four complex control gains, the feedback transfer function $H_{\hat{z}}(s)$ is set to unity and $G_{\hat{z}v\hat{n}_1}(s)$ is given by the transverse-axes dynamics transfer function given in equation (7-57). The following are some comments concerning the construction of the plots:

- a) The solid line portion of the plot of $G_c(j\omega)G_{dyn}(j\omega)H_1(j\omega)$ is for $j\omega$ over the interval from 0 to ∞ . The dashed line portion of the plot of $G_c(j\omega)G_{dyn}(j\omega)H_1(j\omega)$ is for $j\omega$ over the interval from 0 to $-\infty$.
- b) The portions of the plot of $G_c(j\omega)G_{dyn}(j\omega)H_1(j\omega)$ over the intervals $0^+ \rightarrow 0^+$ and $\omega_n^- \rightarrow \omega_n^-$ consist of semi-circular paths of infinite radius. They correspond to interval where the $j\omega$ passes over the indentations in the Nyquist contour at the precession pole $j0$ and nutation pole $j\omega_n$.
- c) The non-semi-circular portions of the path of $G_c(j\omega)G_{dyn}(j\omega)H_1(j\omega)$ on the diagram actually lie exactly on the real axis for gain cases j^2K and j^2K or on the imaginary axis for the gain cases j^1K and j^3K . The points at frequencies of 0^+ , 0^+ , ω_n^- and ω_n^- have been drawn away from the axis to more clearly show the nature of the path of $G_c(j\omega)G_{dyn}(j\omega)H_1(j\omega)$.

Examining the Nyquist plot for the gain case j^2K , $G_cG_{dyn}H_1$ does not encircle the -1 point on the real axis, but passes directly over it. Therefore, the dynamics are on the verge of instability as shown in root locus plot of this system. For the gain case j^2K , the dynamics are also on the verge of instability, with $G_cG_{dyn}H_1$ passing directly over the -1 point. If the gain K is increased sufficiently, the local minimum point between the frequencies of 0 and ω_n^- will increase beyond the magnitude of -1 . For this gain or higher, the plot of $G_cG_{dyn}H_1$ will encircle the -1 point, with the system now being unstable. Examining the Nyquist plots for the gain cases j^1K and j^3K , the -1 point is encircled once showing the system to be unstable.

Case 1: $G_z(j\omega) = j^2 K$

$$G_z(j\omega)G_{\hat{s}\hat{n}}(j\omega)H_z(j\omega) = \frac{-K}{\omega(\omega - \omega_n)}$$

Case 2: $G_z(j\omega) = j^2 K$

$$G_z(j\omega)G_{\hat{s}\hat{n}}(j\omega)H_z(j\omega) = \frac{-jK}{\omega(\omega - \omega_n)}$$

Figure 7-15: Nyquist Plots of Transverse-axes Control Loop Gain Function for 4 Cases of Proportional Control

Case 3: $G_z(j\omega) = j^2 K$

$$G_z(j\omega)G_{\hat{\omega}_n}(j\omega)H_1(j\omega) = \frac{K}{\omega(\omega - \omega_n)}$$

d) **Case 4:** $G_z(j\omega) = j^3 K$

$$G_z(j\omega)G_{\hat{\omega}_n}(j\omega)H_1(j\omega) = \frac{jK}{\omega(\omega - \omega_n)}$$

Figure 7-15: Nyquist Plots of Transverse-axes Control Loop Gain Function for 4 Cases of Proportional Control (cont'd)

7.3.1.2 Simplified Nyquist Criterion for SCISCO Loop Gain Functions

Although the transverse-axis dynamics has no unstable open loop poles, the Simplified Nyquist Criterion cannot be applied directly to SCISCO transfer functions. This is due to the fact that frequency domain information is lost when only one of the SCISCO transfer functions is used. This can be illustrated by re-examining Case 2 shown in Section 7.3.1.1 where the proportional gain is increased to a much larger value. The modified Nyquist diagram is shown in Figure 7-16. It can be seen from this figure that the -1 point is still encircled once, indicating that the system is unstable. However, if the Simplified Nyquist Criterion is applied to this system, it will be judged stable. There is only one traversal of the Nyquist plot into the unit circle as frequency is increased from 0 to $-\infty$. $G_c G_{\hat{x}\hat{n}} H_c$ passes to the right of the -1 point, indicating the system is stable. However in this instance, it is the response over the frequency spectrum 0^+ to $-\infty$ which indicates the system is unstable.

To demonstrate this, consider the plot of $G_c G_{\hat{x}\hat{n}} H_c$ as the frequency is increased in magnitude from 0^+ to $-\infty$. The Nyquist contour is now being traversed in a counterclockwise direction. Correspondingly, if the plot of $G_c G_{\hat{x}\hat{n}} H_c$ enters the unit circle, it must do so to the left of the -1 point for the system to be stable. Re-examining the Nyquist plot in Figure 7-16, as the frequency is increased in magnitude from 0^+ to $-\infty$, $G_c G_{\hat{x}\hat{n}} H_c$ traverses the unit circle between the frequencies of 0^+ and $-\infty$. This traversal occurs to the right of the -1 point. This indicates that the precession pole of the system is unstable.

An alternative way to view this instability is to examine the Nyquist plot of $\bar{G}_c(j\omega)\bar{G}_{\hat{x}\hat{n}}(j\omega)\bar{H}_c(j\omega)$ shown in Figure 7-17. For the frequency spectrum over the entire Nyquist contour, the -1 point is encircled once indicating one unstable closed-loop

pole. Examining the response over the positive frequency spectrum ($0 \rightarrow +\infty$), the Nyquist plot crosses into the unit circle once at a frequency between 0 and $-\infty$. Applying the Simplified Nyquist Criterion for SISO systems, this system is unstable as the crossover occurs to the right of the -1 point.

Based on these observations, the Simplified Nyquist Criterion for SCISCO systems can be summarized as follows:

Simplified Nyquist Criterion for SCISCO Systems

If $G_c G_{\phi n} H_c$ has no open-loop unstable poles, the closed loop system is stable if and only if

- a) The -1 point lies to the left of the polar plot of **both** $G_c(j\omega)G_{\phi n}(j\omega)H_c(j\omega)$ and $\bar{G}_c(j\omega)\bar{G}_{\phi n}(j\omega)\bar{H}_c(j\omega)$ when moving along the Nyquist contour in the direction of increasing ω from 0 to ∞ **or**
- b) The -1 point lies to the left of the polar plot of $G_c(j\omega)G_{\phi n}(j\omega)H_c(j\omega)$ when moving along the contour in the direction of increasing ω from 0 to ∞ and the -1 point lies to the right of the polar plot of $G_c(j\omega)G_{\phi n}(j\omega)H_c(j\omega)$ when moving along this plot in the direction of decreasing ω from 0^+ to $-\infty$.

In summary, when using frequency response methods for stability analysis of SCISCO systems, one must examine both $G_c G_{\phi n} H_c$ and $\bar{G}_c \bar{G}_{\phi n} \bar{H}_c$ over the positive frequency spectrum or examine $G_c G_{\phi n} H_c$ over both the positive and negative frequency spectrums.

Figure 7-16: Nyquist Plot of Transverse-axes Control Loop Gain Function of

$$G_c G_{\dot{\omega}_n} H_1 = \frac{jK}{s(s - j\omega_n)}, \text{ Control Gain set at a high value for Case 2.}$$

Figure 7-17: Nyquist Plot of Transverse-axes Control Loop Gain Function of

$$\bar{G}_c \bar{G}_{\dot{\omega}_n} \bar{H}_1 = \frac{-jK}{s(s + j\omega_n)}, \text{ Control Gain set at a high value for Case 2.}$$

7.3.1.3 Relative Stability for SCISCO Transfer Functions

In classical control design of a SISO system, the controller is chosen to ensure that the polar plot of $P_c P_{\phi n} Q$ passes sufficiently far to the right of the -1 point on the Nyquist diagram. As described in Appendix C.3, gain and phase margin are used to determine the degree to which $P_c P_{\phi n} Q$ passes to the right of the -1 point on the Nyquist diagram. Based on the Simplified Nyquist Criterion for SCISCO systems outlined in Section 7.3.1.2, the following is a modified phase margin criteria for such systems:

Modified Phase Margin Criteria for SCISCO Systems ($G_c G_{\phi n} H_1$)

There are two ways to examine the phase margins of a SCISCO system

Option 1: Examine $G_c(j\omega)G_{\phi n}(j\omega)H_1(j\omega)$ and $\bar{G}_c(j\omega)\bar{G}_{\phi n}(j\omega)\bar{H}_1(j\omega)$

At each crossover frequency ω_{c_1} of $G_c(j\omega)G_{\phi n}(j\omega)H_1(j\omega)$ into the unit circle, the phase margin ϕ_{m_1} is the negative phase shift or clockwise rotation that causes the plot of $G_c(j\omega)G_{\phi n}(j\omega)H_1(j\omega)$ to pass through the -1 point on the real axis. At each crossover frequency $\bar{\omega}_{c_1}$ of $\bar{G}_c(j\omega)\bar{G}_{\phi n}(j\omega)\bar{H}_1(j\omega)$ into the unit circle, the phase margin $\bar{\phi}_{m_1}$ is the negative phase shift or clockwise rotation that causes the plot of $\bar{G}_c(j\omega)\bar{G}_{\phi n}(j\omega)\bar{H}_1(j\omega)$ to pass through the -1 point on the real axis, i.e.

$$\phi_{m_1} = \pi - \text{phase}\left(G_c(j\omega_{c_1})G_{\phi n}(j\omega_{c_1})H_1(j\omega_{c_1})\right)$$

$$\bar{\phi}_{m_1} = \pi - \text{phase}\left(\bar{G}_c(j\bar{\omega}_{c_1})\bar{G}_{\phi n}(j\bar{\omega}_{c_1})\bar{H}_1(j\bar{\omega}_{c_1})\right)$$

Option 2: Check $G_c(j\omega)G_{\hat{p}n}(j\omega)H_1(j\omega)$ and $G_c(-j\omega)G_{\hat{p}n}(-j\omega)H_1(-j\omega)$

At each crossover frequency ω_{c_1} of $G_c(j\omega)G_{\hat{p}n}(j\omega)H_1(j\omega)$ into the unit circle, the phase margin ϕ_{m_1} is the negative phase shift or clockwise rotation that causes the plot of $G_c(j\omega)G_{\hat{p}n}(j\omega)H_1(j\omega)$ to pass through the -1 point on the real axis. At each crossover frequency $\bar{\omega}_{c_1}$ of $G_c(-j\omega)G_{\hat{p}n}(-j\omega)H_1(-j\omega)$, the phase margin $\bar{\phi}_{m_1}$ is the positive phase shift or counter-clockwise rotation, which causes the plot of $G_c(-j\omega)G_{\hat{p}n}(-j\omega)H_1(-j\omega)$ to pass through the -1 point on the real axis.

$$\phi_{m_1} = \pi + \text{phase}\left(G_c(j\omega_{c_1})G_{\hat{p}n}(j\omega_{c_1})H_1(j\omega_{c_1})\right)$$

$$\bar{\phi}_{m_1} = \pi - \text{phase}\left(G_c(-j\bar{\omega}_{c_1})G_{\hat{p}n}(-j\bar{\omega}_{c_1})H_1(-j\bar{\omega}_{c_1})\right)$$

Figure 7-18b) shows how the phase margin $\bar{\phi}_{m_1}$ is interpreted for $G_c(-j\omega)G_{\hat{p}n}(-j\omega)H_1(-j\omega)$. The different phase margin equation for $\bar{\phi}_{m_1}$ in the two options above reflects the difference between the frequency responses of $\bar{G}_c(j\omega)\bar{G}_{\hat{p}n}(j\omega)\bar{H}_1(j\omega)$ and $G_c(-j\omega)G_{\hat{p}n}(-j\omega)H_1(-j\omega)$. As shown in Appendix C 1, the magnitude of the frequency responses of $\bar{G}_c(j\omega)\bar{G}_{\hat{p}n}(j\omega)\bar{H}_1(j\omega)$ and $G_c(-j\omega)G_{\hat{p}n}(-j\omega)H_1(-j\omega)$ are the same, while their phase angle responses are opposite in sign. This accounts for the opposite signs in the calculation of $\bar{\phi}_{m_1}$ in the two options

The use of Option 2 is now discussed for the response of Case 2 as shown in Figure 7-16. $G_z(j\omega)G_{dyn}(j\omega)H_z(j\omega)$ has one crossover into the unit circle at a frequency between ω_n^- and ∞ . Applying the first part of the phase margin definition yields a phase margin of ϕ_m of $\frac{\pi}{2}$ radians indicating a good stability margin near the frequency of ω_n . $G_z(-j\omega)G_{dyn}(-j\omega)H_z(-j\omega)$ has one crossover into the unit circle at a frequency between 0^- and $-\infty$. Applying the second part of the phase margin definition yields a phase margin of $\bar{\phi}_m$ of $-\frac{\pi}{2}$ radians indicating an unstable pole with a large negative stability margin near the precession frequency of 0 . The gain margin at each of these crossover frequencies is infinite as the polar plot meets the real axis only at the origin.

Figure 7-18: Polar Plots of $P_z(j\omega)P_{dyn}(j\omega)Q(j\omega)$ and $P_z(-j\omega)P_{dyn}(-j\omega)Q(-j\omega)$

7.3.2 Bode Plots

After first inspection of the Simplified Nyquist Criterion for SCISCO transfer functions, it may not appear to offer any significant advantages over the standard Nyquist Criterion. The frequency response of $G_{\dot{z}}(j\omega)G_{\dot{x}\dot{y}}(j\omega)H_{\dot{z}}(j\omega)$ must be evaluated over both the $j\omega$ and $-j\omega$ spectrum. Alternatively, the frequency response of both $G_{\dot{z}}(j\omega)G_{\dot{x}\dot{y}}(j\omega)H_{\dot{z}}(j\omega)$ and $\bar{G}_{\dot{z}}(j\omega)\bar{G}_{\dot{x}\dot{y}}(j\omega)\bar{H}_{\dot{z}}(j\omega)$ must be examined. Despite having to look at two distinct frequency responses, the following section shows how SCISCO Bode plots are particularly effective for determination of relative stability and control system design of 2-input, 2-output systems such as the tuned-rotor instrument. They can be used for shaping the frequency response of a system loop gain function in order to achieve both stability and performance goals. The shaping of the response can be performed using either classical or modern robust control techniques.

The Simplified Nyquist Criterion for SCISCO systems and the corresponding methods of measuring relative stability on Nyquist diagrams can be applied directly to Bode plots illustrating the magnitude and phase of $G_{\dot{z}}(j\omega)G_{\dot{x}\dot{y}}(j\omega)H_{\dot{z}}(j\omega)$. Frequency response methods for SCISCO systems also provide an added advantage over SCISCO root locus diagrams by providing a measure of the degree of coupling in the design.

7.3.2.1 Characteristics of the Bode Plot of a SCISCO Transfer Function

In the discussion of the root locus method, it was noted that for SCISCO loop gain functions, the loci of the complex conjugate of this function is found by taking the mirror image about the real-axis of the s -plane. Such a relation between the frequency response of a SCISCO transfer function and its complex conjugate also exists.

To illustrate this point, the frequency response of the transverse-axes dynamics is determined based on the transfer function given in equation (C-6). The spin rate is below the tuned speed at 1300 rpm, resulting in nutation and precession frequencies of -0.0262 radians/second and 291 radians/second respectively. The Bode plots of $G_{\dot{\phi}\dot{\eta}}(j\omega)$ and $G_{\dot{\phi}\dot{\eta}}(-j\omega)$ are shown in Figure 7-19. Examining $G_{\dot{\phi}\dot{\eta}}(j\omega)$, the nutation frequency mode at ω_n can be seen at the only local peak in the magnitude response. A corresponding 180-degree phase shift occurs at this frequency. The plot of magnitude plot $G_{\dot{\phi}\dot{\eta}}(-j\omega)$ is identical to $G_{\dot{\phi}\dot{\eta}}(j\omega)$ except at the nutation frequency and precession frequencies. The $G_{\dot{\phi}\dot{\eta}}(-j\omega)$ plot has no peak at the nutation frequency yet has a pronounced peak at the precession frequency. The reason for this can be seen by examining the dynamics transfer function. Over the positive frequency spectrum, $(j\omega - j\omega_p)$ increases monotonically. Therefore, there is no discontinuity indicating the precession mode in the plot of $G_{\dot{\phi}\dot{\eta}}(j\omega)$. Similarly, $(j\omega - j\omega_n)$ decreases monotonically over the negative frequency spectrum. This explains the absence of a local peak in the magnitude response of $G_{\dot{\phi}\dot{\eta}}(-j\omega)$ at the nutation frequency.

One of the shortcomings of using the root locus method for SCISCO systems is that it provides limited visibility into the resulting cross coupling, which occurs in the equivalent two-input, two-output system. This was illustrated in the discussion of Design Case 3 in Section 7.2.2.2. The effect of cross coupling was judged after examining the time response of the closed loop, two-input, two-output system. It is preferable to judge coupling using the SCISCO form before examining the resulting time response. Frequency response methods afford better visibility into the effect of the compensator choice $G_c(j\omega)$ on the resulting coupling. This can be done by examining the response of $\text{Im}(G_c G_{\dot{\phi}\dot{\eta}})$. Design cases 3 and 4 in the following section illustrate how frequency

response methods of SCISCO systems can be used to help achieve limited coupling in the overall two-input, two output system.

Figure 7-19: Bode Plots of $G_{\dot{\varphi}_{n1}}(j\omega)$ and $G_{\dot{\varphi}_{n1}}(-j\omega)$ for the Transverse-axes Dynamics of the GyroWheel Tuned-Rotor Instrument

7.3.2.2 Application of Frequency Response Design Method to Nutation Damping Control of a Tuned-Rotor Instrument

The problem of designing a nutation damper for the transverse-axes dynamics of a tuned-rotor instrument is now reconsidered using frequency response methods. As with the root locus design examples, the feedback transfer function $H_1(j\omega) = 1$. The tuned-condition case is examined where the SCISCO transfer function $G_{dyn_1}(s)$ is

$$G_{dyn_1}(s) = \frac{1}{s(s-j)}$$

Figure 7-20 shows the open loop frequency response of $G_c(j\omega)G_{dyn_1}(j\omega)H_1(j\omega)$ for the four potential gain cases

$$G_c(j\omega) = j^q K, \quad q = 0, 1, 2 \text{ or } 3$$

Selection of four compensator designs is discussed; with the starting point being the four possible proportional gain cases above.

a) Design Case 1: Complex Lead Compensation

The plot of $G_c(j\omega)G_{dyn_1}(j\omega)H_1(j\omega)$ with $G_c(j\omega) = j^0 K$ is shown in Figure 7-20. It has crossover frequencies of 0.2 and 1 radians/second, with the phase margins being 180 degrees and 0 degrees at these two respective frequencies. These margins indicate that both the precession and nutation poles are on the verge of instability. The plot of $\bar{G}_c(j\omega)\bar{G}_{dyn_1}(j\omega)\bar{H}_1(j\omega)$ shown in Figure 7-21 indicates a single cross-over frequency at 0.2 radians/second with a phase margin

of 0 degrees. The nutation frequency peak does not show on the plot of $\bar{G}_c(j\omega)\bar{G}_{dyn}(j\omega)\bar{H}_1(j\omega)$.

A dynamic compensator is required which will produce a phase angle shift of -90 degrees at the first crossover frequency of $G_c(j\omega)G_{dyn}(j\omega)H_1(j\omega)$ and a 90 -degree phase shift near the second crossover just above the nutation frequency. The following complex lead compensator

$$G_c(s) = K \frac{(\tau_z s - j)}{(\tau_p s - 1)}$$

produces the desired phase shifts below and above the nutation frequency as shown in Figure 7-22. In fact, the zero produces both of the desired phase shifts with the lag term chosen to limit the effect of noise, with the corner frequency (τ_p^{-1}) chosen to be a decade above the lead corner frequency (τ_z^{-1}). The gain K is chosen to be 0.22 to produce a first crossover frequency at the desired bandwidth of 0.22 radians/second. Applying this compensator results in a stable transverse-axis dynamics as shown in Figure 7-23. The phase margin at the first crossover frequency of 0.22 radians/second is now 90 degrees with the phase margin at the second crossover also being 90 degrees. Examining the open-loop frequency response of $\bar{G}_c(j\omega)\bar{G}_{dyn}(j\omega)\bar{H}_1(j\omega)$ in Figure 7-24, there is a phase margin of 90 degrees at the single crossover frequency of 0.22 radians/second.

b) Design Case 2: Non-minimum Phase Lead Compensation

Figure 7-20 shows the frequency response of $G_c(j\omega)G_{dyn}(j\omega)H_1(j\omega)$ when $G_c(j\omega) = j^2 K$. The phase margins at the first and second crossover frequencies

are -90 and 90 degrees. From this plot, it may appear that if the gain K is increased sufficiently, there will be only one crossover at a frequency above the nutation frequency with the phase margin at this point being 90 degrees. For such a gain, the system would appear stable. However, examining the response of $\bar{G}_c(j\omega)\bar{G}_{\dot{\phi}_n}(j\omega)\bar{H}_1(j\omega)$ in Figure 7-21. It can be seen that for any non-zero gain K , the system will be unstable, with a phase margin of -90 degrees.

Compensation is required which will provide a phase shift of 90 degrees in the 0.2 radians/second frequency region of $G_c(j\omega)G_{\dot{\phi}_n}(j\omega)H_1(j\omega)$, yet not change the desired phase angle of -90 degrees just above the nutation frequency of 1 radian/second. In addition, this compensator must be chosen such that the gain characteristic around the desired bandwidth remains unchanged. The requirements call for a lead compensator, but with unity gain over the frequency range. This can be accomplished using a non-minimum phase lead compensator where

$$G_c(s) = jK \frac{(\tau_z s - 1)}{(\tau_p s - 1)}$$

Setting $\tau_z = \tau_p$, $G_c(j\omega)$ has a constant magnitude of 1 . However, by choosing a non-minimum phase zero, the phase angle will be ± 180 degrees at very low frequencies. As the frequency increases, the lead and lag terms will both contribute an equally decreasing phase angle until at high frequencies just above the nutation frequency, the phase angle contribution tends to zero degrees. It is not possible to achieve a phase angle shift of 180 degrees at a frequency of 0.2 radians/second and zero degrees just above the nutation frequency. A compromise is to shift the phase angle 135 degrees at the precession frequency of

0.2 radians/second and 75 degrees just above the nutation frequency. This is shown in Figure 7-22.

The resulting open loop frequency response is shown in Figure 7-23. The phase margin at the first normalized crossover frequency of 0.11 is 75 degrees. At the second normalized crossover frequency, the phase margin is an adequate 40 degrees. To complete the examination of the stability margins, the plot of $\bar{G}_c(j\omega)\bar{G}_{\dot{\alpha}}(j\omega)\bar{H}_l(j\omega)$ is examined in Figure 7-24. The phase margin at the first crossover frequency of 0.1 radians/second is 60 degrees. Using this form of compensation, the desired normalized bandwidth of 0.2 radians/second cannot be met.

c) **Design Case 3: Complex Lag Compensation**

Examining Figure 7-20, with $G_c(j\omega) = j^2K$, $G_c(j\omega)G_{\dot{\alpha}}(j\omega)H_l(j\omega)$ has phase margins of 0 degrees and 180 degrees at the crossover frequencies of 0.2 and 1 radians/second respectively. Like Design Case 1, these margins indicate that both the precession and nutation poles are on the verge of instability. The plot of $\bar{G}_c(j\omega)\bar{G}_{\dot{\alpha}}(j\omega)\bar{H}_l(j\omega)$ shows a phase margin of 180 degrees at the single crossover frequency at 0.2 radians/second.

A dynamic compensator is required which will produce a phase angle shift of 90 degrees at the first crossover frequency of $G_c(j\omega)G_{\dot{\alpha}}(j\omega)H_l(j\omega)$ and a -90-degree phase shift near the second crossover just above the nutation frequency. The following complex lag compensator

$$G_c = \frac{j^2K}{s+a-jb}$$

produces a shift of up to 90 degrees at low frequencies due to the pole at $-a - jb$, and a -90 degree shift at a frequencies above b radians/second. The frequency b must be chosen between the desired first crossover frequency and the nutation frequency. If the damping term a is set to zero, this will introduce a second undamped mode at b and increase the magnitude of $G_c(j\omega)G_{\dot{\alpha}}(j\omega)H_1(j\omega)$ in this frequency range, which is undesirable. Therefore the compensator pole must have significant damping such that the first crossover frequency occurs prior to the nutation frequency. The compromise choice of compensator is

$$G_c = \frac{j^2 0.07}{s - 0.5 - j0.5}$$

The open loop frequency response $G_c(j\omega)G_{\dot{\alpha}}(j\omega)H_1(j\omega)$ is shown in Figure 7-22, with the phase margins at the two crossover frequencies of 0.1 radians/second and 1 radian/second being approximately 40 degrees in each case. The damping at the nutation frequency is superior to the first two compensator design cases. The first crossover frequency cannot be increased above 0.1 radians/second because the pole in $G_c(j\omega)$ increases the magnitude of $G_c(j\omega)G_{\dot{\alpha}}(j\omega)H_1(j\omega)$ over the frequency spectrum just below the nutation frequency. One of the negative consequences of introducing the complex pole at $-a - jb$ is that increases the magnitude of $\text{Im}(G_c(j\omega)G_{\dot{\alpha}}(j\omega))$, thereby increasing the coupling of the command tilt axis with the uncommanded one. This is illustrated in e) below.

d) **Design Case 4: Lag Compensation**

With $G_{\ddot{z}}(j\omega) = j^3 K$, Figure 7-20 shows the frequency response of $G_{\ddot{z}}(j\omega)G_{\ddot{z}n_1}(j\omega)H_1(j\omega)$. The phase margins at the first and second crossover frequencies are 90 degrees and -90 degrees. Dynamic compensation is required that shifts the phase angle at the second crossover, without affecting the phase or magnitude at the first crossover frequency. This requirement can be met using lag compensation of the following form:

$$G_{\ddot{z}}(j\omega) = j^3 K \frac{1}{(\tau_p s - 1)^n}$$

Unlike the application of most lag filters used to roll off the magnitude response at high frequencies, the filter in this case is required mainly for phase compensation.

The order n and corner frequency $\frac{1}{\tau_p}$ of the lag filter must be chosen to produce

a phase shift of 180 degrees near the normalized frequency of 1. In addition, it must have little effect on the magnitude and phase characteristic near the desired normalized bandwidth of 0.2. A second order filter could produce this phase shift, however its corner frequency would have to be at or lower than the desired bandwidth, which is undesirable. A 4th order filter was selected.

A trade-off was made between a Butterworth filter and a combination of simple lags as was used in the root locus analysis. Figure 7-25 shows the open loop frequency response of the transverse-axes dynamics with these the two filter options. In each case, the corner frequency was chosen to be at the nutation frequency. They both produce a -180 degree phase shift at the nutation frequency. Upon further comparison, the following observations can be made:

- i) The Butterworth filter has the advantage of having less attenuation of the magnitude and negative phase angle shift at the desired normalized crossover frequency of 0.2.
- ii) However with the Butterworth filter, the -180 degree phase shift is more sensitive to variations in the nutation frequency (due to a spin rate variation). In the case of the GyroWheel system, there is a ± 15 percent variation in the spin rate and nutation frequency. The resulting change in the phase shift at the nutation frequency will be ± 10 degrees with the 4th order simple lag and ± 15 degrees with the 4th order Butterworth filter.
- iii) The Butterworth Filter provides more damping of the nutation pole with the nutation peak being much broader with this option.
- iv) The Butterworth filter does not roll off the magnitude of the loop gain as well between the first crossover frequency and the nutation frequency. The effect will be more a more noticeable nutation oscillation in early portion of the transient because the peak at the nutation frequency is higher.
- v) By accentuating the peak in $G_{\dot{\alpha}}(j\omega)G_{\dot{\alpha}\eta}(j\omega)H_{\dot{\alpha}}(j\omega)$ at the nutation frequency with the Butterworth filter, the cross-coupling term will be greater. This can be explained by considering the expression for the closed-loop cross-coupling transfer function $\frac{j}{2}(G_{\dot{\alpha}} - \bar{G}_{\dot{\alpha}})$. As noted in Section 7.1.3, $G_{\dot{\alpha}} - \bar{G}_{\dot{\alpha}}$ can be made smaller by minimizing $\text{Im}(G_{\dot{\alpha}}(j\omega)G_{\dot{\alpha}\eta}(j\omega))$. Unlike the 4th order lag compensator, the Butterworth filter has a significant imaginary component to the poles. Consequently, the Butterworth Filter design will have more significant coupling than the 4th order lag compensator design.

e) **Description of Closed-Loop Frequency Response and Time Responses**

The closed-loop frequency response of the transfer function $G_{z_t}(j\omega)$ is shown for the four design cases in Figure 7-26. The bandwidths for the four designs appear approximately the same. Design case 3 differs from the other cases, exhibiting superior damping of the magnitude plot where it peaks at the nutation frequency. However, this design increases the magnitude to a value greater than 1 around 0.1 radians/second. This indicates that nutation damping will be good, but there will be more initial overshoot than the other designs. Design Case 2 shows a higher peak at the nutation frequency indicating a higher initial overshoot for the non-minimum phase lead design but the damping factor on the nutation pole is almost as good as Design Case 3. Design cases 1 and 4 have similar closed loop frequency responses but the lead design has a poorer damping factor on the nutation pole.

The frequency response of the closed-loop cross-coupling transfer function is shown in Figure 7-27. Design Case 1 shows better suppression of the low frequency cross coupling, although this lead compensator has poor damping of the nutation mode. The lag and non-minimum phase lead designs have poorer low-frequency cross coupling, with the non-minimum phase lead design showing a higher level of coupling at the nutation frequency. The poorest cross-coupling will occur with Design Case 3. This is due to the introduction of another complex pole at $-0.5 + j0.5$.

The observations drawn from the examination of the closed loop frequency responses are confirmed in the time responses of the transverse-axis control system to a step input $\phi_{ref,t}$ of 0.1 degrees from an initial null tilt. These responses are shown in Figure 7-13 and Figure 7-14. In addition, for the same simulation case, Figure 7-28 shows the difference between using a 4th order simple lag and a

4th order Butterworth filter. The simple lag option is superior, with a similar bandwidth but a better suppression of cross coupling as predicted.

Finally, in Section 7.1.3, it was noted that a SCISCO controller $G_{c_i}(s)$ can be designed for the SCISCO transfer function $G_{dyn_i}(s)$ to meet performance objectives such as bandwidth or overshoot, and the same characteristics will be realized in the equivalent 2 real input, 2 real output system. More specifically, it was predicted that $G_{z_i}(j\omega) \approx \text{Re}(G_{z_i}(j\omega))$ up to the closed-loop bandwidth. Figure 7-29 shows the plots of $G_{z_i}(j\omega)$ and $\text{Re}(G_{z_i}(j\omega))$ for the four design cases. For all but design case 3, $G_{z_i}(j\omega) \approx \text{Re}(G_{z_i}(j\omega))$ up to the closed-loop bandwidth of 0.1-0.2 radians/second. Design Case 3 differs from the other cases because it introduced a pole $(-0.5 \pm j0.5)$ with a significant imaginary component near the desired dominant pole position. As shown in Figure 7-27, this introduces more coupling between the commanded and uncommanded axis. The coupling could be reduced by decreasing the damping of the complex pole or increasing its imaginary magnitude. In either case, this option would come at the expense of lower control gain and lower bandwidth.

Figure 7-20: Nutation Damper Design for Transverse-axes Dynamics - Open Loop Frequency Response of $G_c(j\omega)G_{dyn_1}(j\omega)H_1(j\omega)$ for the four possible proportional control gains.

Figure 7-21: Nutation Damper Design for Transverse-axes Dynamics - Open Loop Frequency Response of $\bar{G}_c(j\omega)\bar{G}_{dyn}(j\omega)\bar{H}_1(j\omega)$ for the four possible proportional control gains.

Figure 7-22: Nutation Damper Design for Transverse-axes Dynamics - Open Loop Frequency Response of $G_a(j\omega)$ for four candidate dynamic compensator designs.

Figure 7-23: Nutation Damper Design for Transverse-axes Dynamics - Open Loop Frequency Response of $G_{\hat{z}}(j\omega)G_{dyn_1}(j\omega)H_1(j\omega)$ for the four dynamic compensator designs.

Figure 7-24: Nutation Damper Design for Transverse-axes Dynamics - Open Loop
Frequency Response of $\bar{G}_z(j\omega)\bar{G}_{dyn}(j\omega)\bar{H}_1(j\omega)$ for the four
dynamic compensator designs.

Figure 7-25: Nutation Damper Design for Transverse-axes Dynamics - Open Loop Frequency Response of $G_c(j\omega)G_{dyn}(j\omega)H_z(j\omega)$ for 4th order simple lag and 4th order Butterworth compensator designs

Figure 7-26: Nutation Damper Design for Transverse-axes Dynamics – Closed Loop Frequency Response of $G_z(j\omega)G_{dyn}(j\omega)H_z(j\omega)$ for four dynamic compensator designs.

Figure 7-27: Nutation Damper Design for Transverse-axes Dynamics – Closed Loop Frequency Response of Cross Coupling Transfer function $G_{c1} - \bar{G}_{c1}$ for four dynamic compensator designs.

Figure 7-28: Linear Time Response of Transverse-axes angles (ϕ_x, ϕ_y) to a step input of 1 degree about the case-referenced x-axis; responses shown for 4th order simple lag and 4th order Butterworth compensator design

Figure 7-29: Magnitude Frequency Response of $G_{c1}(j\omega)$ and $\text{Re}(G_{c1}(j\omega))$ for the four design cases

7.4 Summary

The basis for using single-complex-input, single-complex-output transfer (SCISCO) functions has been demonstrated with an associated set of properties relating them to general two-input, two-output systems. A modified set of rules for using root locus plots with SCISCO systems has been proposed to account for complex gains and asymmetries about the real axis. It has been shown that the Nyquist Stability Criterion can be applied directly to SCISCO transfer functions. A new form of the Simplified Nyquist Stability Criterion for SCISCO systems has been proposed which allows Bode plots to be used both for stability analysis and control system design.

The advantages of reducing a unique class of two-input, two output systems to a SCISCO form are:

- a) Simplification of the design/analysis problem, with the model order reduced by a factor of two.
- b) Convenient form for the application of SISO-like classical methods to develop controllers which provide the designer with a clearer view of how particular aspects of the dynamics are affected by control choices. Using traditional methods, it is unlikely that the three nutation damper designs could have been chosen as easily; particularly the lag and non-minimum phase lead designs. The use of the modified classical methods with SCISCO transfer functions provides insight into design that is not available with traditional MIMO methods.
- c) More direct way of judging relative stability than some MIMO methods.

Finally, the effective use of frequency response methods indicates that with some investigation, modern robust control methods can be applied to SCISCO transfer function.

Chapter 8

External Rate Sensing for a Tuned-Rotor Instrument

The general tuned-rotor instrument operates at spin rates other than the tuned speed and at spin-axis tilts away from null. This allows the momentum vector of the device to be controlled like a steerable momentum wheel. To support this capability, the angular travel of the transverse axes has been increased and the tilt sensor and torquer coils have been modified to accommodate this increased dynamic range. However, the mechanical design is still very similar to a traditional tuned-rotor gyroscope. If the general tuned-rotor instrument was operated at the tuned speed and regulated at zero tilt, external rates could be determined in the traditional manner. The more general problem of measuring external rates, while steering the rotor momentum vector is now addressed.

The method for determining external rates from a general tuned-rotor instrument is an extension of the traditional method, which relies on the measured torque required to hold the gyro rotor at null. When operating at de-tuned spin rates and non-zero tilts, the control system must generate additional torques to overcome dynamic stiffness and gyroscopic coupling torques. By relying on characterizations of these low-frequency characteristics, the external rates can be determined based on

- i) Measured control torque
- ii) Measured spin rate

- iii) Commanded tilt angles and rates.

The practical limitations of the concept are presented through an error analysis of the equation used to extract the external rates. This highlights the required accuracies of the above quantities in order to meet a commercial-grade accuracy of 0.1 degrees/hour.

8.1 Derivation of External Rate Sensing Relationship

The Traditional SCISCO transverse-axis dynamics transfer function model in equation (5-99) was used to derive the external rate sensing equation for a general tuned-rotor instrument. This model was used instead of the Complete model because it is somewhat simpler to manipulate and contains the crossed-flexure damping terms. The unique aspects of the Complete Model that influence the external rate estimate are then discussed.

The Traditional SCISCO transverse-axis dynamics transfer function model in equation (5-99) can be expressed in the following form:

$$\phi(s) = \hat{G}_{x_1}(s)T_z(s) - \hat{G}_{e1}\omega_z(s) - \hat{G}_{x2}(s)\bar{T}_z(s - 2j\omega_s) - \hat{G}_{e2}(s)\bar{\omega}(s - 2j\omega_s) \quad (8-1)$$

The Traditional model transfer functions $\hat{G}_{x1}$, $\hat{G}_{x2}$, $\hat{G}_{e1}$ and $\hat{G}_{e2}$ are listed below:

$$\hat{G}_{x1}(s) = \frac{1}{I'_x \left\{ s^2 - \left(2c'_g - j(\omega_n - 2\gamma\omega_s) \right) s - \left(\alpha - 2jc'_g\omega_s \right) \right\}} \quad (8-2)$$

$$\hat{G}_{e1}(s) = \frac{-(s - j(\omega_n - \gamma_z\omega_s))}{\left\{ s^2 - \left(2c'_g - j(\omega_n - 2\gamma\omega_s) \right) s - \left(\alpha - 2jc'_g\omega_s \right) \right\}} \quad (8-3)$$

$$\hat{G}_{x,z}(s) = \frac{\gamma s^2 - 2j\gamma\omega_s s + \beta}{I'_r \left\{ s^2 - \left(2c_g' - j(\omega_n + 2\gamma\omega_s) \right) s + \left(\alpha - 2jc_g'\omega_s \right) \right\}} \quad (8-4)$$

$$\hat{G}_{e,z}(s) = \frac{\gamma s - j\omega_s(\gamma_z - 2\gamma)}{\left\{ s^2 - \left(2c_g' - j(\omega_n + 2\gamma\omega_s) \right) s + \left(\alpha - 2jc_g'\omega_s \right) \right\}} \quad (8-5)$$

Initially, the effect of the periodic terms $\bar{T}_z(s - 2j\omega_s)$ and $\bar{\omega}_z(s - 2j\omega_s)$ are ignored in the development of the external rate sensing concept. They represent the influence of control torques and external rate induced torques occurring at twice the spin frequency due to gimbal flutter. They do have a small effect on the external rate-sensing estimate and are discussed in the treatment of GyroWheel design in Chapter 11. This reduced Traditional model can be expressed in the form of a block diagram as shown in Figure 8-1. The transfer function $\hat{G}_z(s)$ relates the external rates $\omega_s(s)$ to the torque $T_b(s)$ induced on the rotor by the moving base and is defined as the following expression:

$$\hat{G}_z(s) = \frac{T_b(s)}{\omega_s(s)} = \frac{\hat{G}_{e,z}(s)}{\hat{G}_{x,z}(s)} = -(s - j(\omega_n - \gamma_z\omega_s)) \quad (8-6)$$

The controller $G_c(s)$ is of the form discussed in Chapter 7. As shown in the following discussion, the specific type of compensator does not affect the external rate estimate.

In a tuned-rotor gyroscope, the rotor is a freely spinning body. The only torque required to hold the rotor spin axis at null is that induced by the motion of the body on which the tuned-rotor gyroscope sits. In the general tuned-rotor instrument, the control torque $T_c(s)$ is a function of two effects: the torque induced by the external rates $\omega_s(s)$ and the torque required to track the transverse-axes set-point $\phi_{ref}(s)$. Their effects can be determined independently and then combined to obtain the net control torque T_{net} .

Figure 8-1: Block Diagram of Transverse-axes Control System With Tuned-rotor Instrument Operating on a Moving Base

With the external rates $\omega_b(s)$ set to zero, the control torque $T_c(s)$ is termed $T_{c:}(s)$ and is related to the complex spin-axis tilt angle reference $\phi_{ref}(s)$ by the following transfer function relationship:

$$T_{c:}(s) = \frac{G_c(s)}{1 + G_c(s)\hat{G}_{J:}(s)} \phi_{ref}(s) \quad (8-7)$$

With the complex spin-axis tilt angle reference $\phi_{ref}(s)$ set to zero, the control torque $T_c(s)$ is termed $T_{c:}(s)$ and is a function of the external rates $\omega_b(s)$. $T_{c:}(s)$ is given by the following equation:

$$T_{c:}(s) = T_{net}(s) - \hat{G}_b(s)\omega_b(s) \quad (8-8)$$

For this case, the net torque applied to the rotor is given by the following expression:

$$T_{net}(s) = \frac{\hat{G}_b(s)}{1 - G_c(s)\hat{G}_{di}(s)}\omega_b(s) \quad (8-9)$$

Solving for the control torque $T_{\dot{\alpha}_i}(s)$ yields the following expression:

$$T_{\dot{\alpha}_i}(s) = -\frac{\hat{G}_b(s)G_c(s)\hat{G}_{di}(s)}{1 - G_c(s)\hat{G}_{di}(s)}\omega_b(s) \quad (8-10)$$

Applying the principle of superposition, the overall control torque $T_c(s)$ is the combination of $T_{\dot{\alpha}_i}$ and $T_{\dot{\alpha}_b}$:

$$\begin{aligned} T_c(s) &= T_{\dot{\alpha}_i}(s) - T_{\dot{\alpha}_b}(s) \\ &= \frac{G_c(s)\hat{G}_{di}(s)}{1 - G_c(s)\hat{G}_{di}(s)} \left[\frac{\phi_{ref}(s)}{\hat{G}_{di}(s)} - \hat{G}_b(s)\omega_b(s) \right] \end{aligned} \quad (8-11)$$

The closed-loop transfer function $G_{ci}(s)$, relating the momentum axis angle $\phi(s)$ to the commanded momentum axis angle $\phi_{ref}(s)$ is given below:

$$G_{ci}(s) = \frac{G_c(s)\hat{G}_{di}(s)}{1 - G_c(s)\hat{G}_{di}(s)}$$

Substituting the expressions for $\hat{G}_b(s)$ and $\hat{G}_{di}(s)$ into (8-11), the expression for $T_c(s)$ takes the following form:

$$T_z(s) = G_{zi}(s) \left(I_z - \frac{I_{sz}}{2} \right) \left\{ \left[s^2 - \left(2c_g' - j(\omega_n + 2\gamma\omega_s) \right) s - \left(\alpha - 2j\omega_s c_g' \right) \right] \phi_{ref}(s) - \left[(s - j(\omega_n + \gamma\omega_s)) \right] \omega_b(s) \right\} \quad (8-12)$$

For frequencies up to the closed-loop bandwidth of the tuned-rotor instrument transverse-axes controller, $G_{zi}(s)$ has a magnitude of 1, i.e. $\phi(s) = \phi_{ref}(s)$. Assuming that the desired bandwidth of the external rate measurement is less than the closed-loop tuned-rotor instrument bandwidth, the expression for the control torque simplifies to the following form:

$$T_z(s) \approx \left(I_z - \frac{I_{sz}}{2} \right) \left\{ \left[s^2 - \left(2c_g' - j(\omega_n + 2\gamma\omega_s) \right) s - \left(\alpha - 2j\omega_s c_g' \right) \right] \phi_{ref}(s) - \left[(s - j(\omega_n + \gamma\omega_s)) \right] \omega_b(s) \right\} \quad (8-13)$$

Taking the inverse Laplace transform of this expression yields

$$T_z \approx \left(I_z - \frac{I_{sz}}{2} \right) \left\{ \ddot{\phi}_{ref} - \left(2c_g' - j(\omega_n + 2\gamma\omega_s) \right) \dot{\phi}_{ref} + \left(\alpha - 2j\omega_s c_g' \right) \phi_{ref} - \dot{\omega}_b - j(\omega_n + \gamma\omega_s) \omega_b \right\} \quad (8-14)$$

Given the relatively low bandwidth of the spacecraft attitude control system commanding a device such as the GyroWheel, $\ddot{\phi}_{ref}$ can be ignored. Similarly, given the low-frequency nature of the spacecraft external rate profile, the acceleration term $\dot{\omega}_b$ can be ignored. Equation (8-14) can then be manipulated to provide the following expression for the

external rate ω_z as a function of the control torque T_c , commanded tilt ϕ_{ref} and commanded rate of change of tilt $\dot{\phi}_{ref}$.

$$\omega_z \approx \frac{1}{j(\omega_n + \gamma_z \omega_s)} \left\{ -\frac{2}{(2I_r - I_{z_t})} T_c + (\alpha - 2j\omega_s c_s') \phi_{c_{ref}} + (2c_s' - j(\omega_n + 2\gamma \omega_s)) \dot{\phi}_{c_{ref}} \right\}$$

Noting the definitions for ω_n and γ , the external rate expression simplifies to the following form:

$$\omega_z \approx \frac{j}{h_z} \left\{ T_z - (\bar{K}_{dyn} - 2j\omega_s c_s') \phi_{ref} - (c_s' - j(I_r - I_{z_t}) \omega_s) \dot{\phi}_{ref} \right\} \quad (8-15)$$

The Traditional dynamic stiffness $\hat{K}_{dyn}$ and the total tuned-rotor instrument momentum h_z are defined below:

$$h_z = \left(I_r - \frac{I_{z_t}}{2} \right) \omega_s \quad (8-16)$$

$$\hat{K}_{dyn} = \left(I_r - \frac{I_{z_t}}{2} \right) \alpha \quad (8-17)$$

The control torques T_{c_x} and T_{c_y} generated by the two torquers are obtained through the measurement of the current in their respective coils. The complex coil current i_{coil} is obtained by combining the coil currents i_{coil_x} and i_{coil_y} in quadrature.

$$i_{coil} = i_{coil_x} - j \cdot i_{coil_y} \quad (8-18)$$

The complex control torque T_c is proportional to the complex coil current by the torque scale factor K_{trq} as shown in the following relation:

$$T_c = K_{trq} i_{coil} \quad (8-19)$$

Substituting (8-19) into (8-15) yields the relation that can be used to obtain the external rates

$$\omega_b \approx \frac{j}{h_z} \left\{ K_{trq} i_{coil} - \left(\hat{K}_{dyn} - 2jc'_s \omega_s \right) \phi_{ref} - \left(c'_s - j(I_r + I_{s_t}) \omega_s \right) \dot{\phi}_{ref} \right\} \quad (8-20)$$

Applying the definitions for the complex quantities T_c , ϕ_{ref} and ω_b , equation (8-20) can be decomposed into the following two expressions for the estimated external rates ω_{b_x} and ω_{b_y} .

$$\begin{aligned} \omega_{b_x} &= \frac{1}{h_z} \left\{ -K_{trq} i_{coil_x} - 2c'_s \omega_s \phi_{ref_x} - \hat{K}_{dyn} \phi_{ref_x} - (I_r - I_{s_t}) \omega_s \dot{\phi}_{ref_x} - c'_s \dot{\phi}_{ref_x} \right\} \\ \omega_{b_y} &= \frac{1}{h_z} \left\{ K_{trq} i_{coil_y} - 2c'_s \omega_s \phi_{ref_y} - \hat{K}_{dyn} \phi_{ref_y} - (I_r + I_{s_t}) \omega_s \dot{\phi}_{ref_y} - c'_s \dot{\phi}_{ref_y} \right\} \end{aligned} \quad (8-21)$$

Therefore the external rates can be estimated from the measured control currents (i_{coil_x}, i_{coil_y}) , measured spin rate (ω_s) , tilt commands $(\phi_{ref_x}, \phi_{ref_y})$, tilt rate commands $(\dot{\phi}_{ref_x}, \dot{\phi}_{ref_y})$ and the estimated inertia, damping and dynamic stiffness characteristics.

When the commanded spin-axis tilt angles and rates are set to zero, the equations in (8-21) simplify to the following form:

$$\begin{aligned}\omega_{b_x} &= \frac{-K_{trq} i_{canl_y}}{h_z} \\ \omega_{b_y} &= \frac{K_{trq} i_{canl_x}}{h_z}\end{aligned}\tag{8-22}$$

These equations represent the traditional method for determining external rates in a tuned-rotor gyroscope.

8.2 System Requirements for External Rate Sensing

Based on the equations derived to estimate the external rates, the feasibility of producing an estimate of 0.1 degree/hour was determined using a static error analysis. First, the factors that contribute the most significant errors to the external rate estimate were identified. Based on the desired rate sensing accuracy, the requirements on the sensed variables and characterized system parameters were determined. The sub-system requirements for the GyroWheel were then chosen using the results of this error analysis.

8.2.1 Error Analysis

The following assumptions were made in the error analysis:

- a) The effect of gimbal damping is ignored.
- b) The effect of a $\dot{\phi}_{ref}$ error is ignored, as this term is not explicitly controlled in the transverse-axes control system.

This error is a consequence of the tracking error of ϕ_{ref} . Offset or bias errors in the $\dot{\phi}_{ref}$ tracking will have no effect on the $\dot{\phi}_{ref}$ error. Given the digital nature of the tilt position measurement, there should not be a significant amount of noise in the $\dot{\phi}_{ref}$ tracking error. In addition the overall rate estimate will be filtered.

- d) It is assumed that the machining tolerances of the cross-flexures, gimbal and rotor are sufficiently low that estimation errors for the characteristics $K, I_{r_z}, I_{r_x}, I_{s_z}, I_{s_x}$ are negligible in the context of external rate sensing.
- e) The following terms are considered to make a significant contribution to the overall error:
 - i) Measured torquer coil current i_{coil} .
 - ii) Estimated/calibrated torquer scale factor $K_{\sigma q}$.
 - iii) Tracking error of the momentum axis tilt angle ϕ_{ref} .
 - iv) Measured rotor spin rate ω_s .

The external rates ω_b can then be expressed in the form of a function

$$\omega_b = f(i_{coil}, K_{\sigma q}, \phi, \omega_s) \quad (8-23)$$

- f) In the derivation of the external rate sensing algorithm, it was assumed that within the bandwidth of the transverse-axes control system, the following is true:

$$\frac{G_c(j\omega)\hat{G}_{d1}(j\omega)}{1+G_c(j\omega)\hat{G}_{d1}(j\omega)} \approx 1$$

In the actual system there will be tracking errors due to effects such as tilt-sensor error. Consequently, for the purposes of the error analysis, the expression for the external rates described in (8-20) shall be expressed in terms of ϕ rather than ϕ_{ref} .

Ignoring the effect of cross-flexure gimbal damping, equation (8-20) simplifies to the following form:

$$\omega_b \approx \frac{j}{h_z} \left\{ K_{\tau q} i_{coul} - \hat{K}_{spin} \phi - j(I_{r_z} - I_{s_z}) \omega_s \dot{\phi} \right\} \quad (8-24)$$

Taking the variation of ω_b results in the expression for $\delta\omega_b$ given below:

$$\delta\omega_b = \frac{\partial\omega_b}{\partial K_{\tau q}} \delta K_{\tau q} + \frac{\partial\omega_b}{\partial i_{coul}} \delta i_{coul} - \frac{\partial\omega_b}{\partial \omega_s} \delta\omega_s + \frac{\partial\omega_b}{\partial \phi} \delta\phi \quad (8-25)$$

The partial derivatives $\frac{\partial\omega_b}{\partial K_{\tau q}}$, $\frac{\partial\omega_b}{\partial i_{coul}}$, $\frac{\partial\omega_b}{\partial \omega_s}$ and $\frac{\partial\omega_b}{\partial \phi}$ are given below:

$$\frac{\partial\omega_b}{\partial K_{\tau q}} = j \frac{i_{coul}}{h_z} \quad (8-26)$$

$$\frac{\partial\omega_b}{\partial i_{coul}} = j \frac{K_{\tau q}}{h_z} \quad (8-27)$$

$$\frac{\partial\omega_b}{\partial \omega_s} = -\frac{j}{I_{spin}} \left\{ \frac{K_{\tau q} i_{coul}}{\omega_s^2} - \frac{(K - J\omega_s^2)\phi}{\omega_s^2} \right\} \quad (8-28)$$

$$\frac{\partial\omega_b}{\partial \phi} = -\frac{j}{h_z} \left\{ (K - J\omega_s^2) \right\} \quad (8-29)$$

It is assumed that the errors $\delta K_{\tau q}$, δi_{coul} , $\delta\omega_s$ and $\delta\phi$ are independent. As a result, the overall rate sensing error can be expressed in the following form:

$$\delta\omega_b = \sqrt{\left(\frac{\partial\omega_b}{\partial K_{\tau q}}\delta K_{\tau q}\right)^2 + \left(\frac{\partial\omega_b}{\partial i_{\text{coil}}}\delta i_{\text{coil}}\right)^2 + \left(\frac{\partial\omega_b}{\partial \omega_s}\delta\omega_s\right)^2 + \left(\frac{\partial\omega_b}{\partial \phi}\delta\phi\right)^2} \quad (8-30)$$

It is assumed that each of the four error terms on the right hand side of (8-20) contribute equally to the overall error $\delta\omega_b$. Therefore, the requirements on the four errors $\delta K_{\tau q}$, δi_{coil} , $\delta\omega_s$, and $\delta\phi$ are a function of the desired rate sensing error $\delta\omega_b$ where

$$\begin{aligned} \delta K_{\tau q} &= \frac{\delta\omega_b}{2 \frac{\partial\omega_b}{\partial K_{\tau q}}} \\ \delta i_{\text{coil}} &= \frac{\delta\omega_b}{2 \frac{\partial\omega_b}{\partial i_{\text{coil}}}} \\ \delta\omega_s &= \frac{\delta\omega_b}{2 \frac{\partial\omega_b}{\partial \omega_s}} \\ \delta\phi &= \frac{\delta\omega_b}{2 \frac{\partial\omega_b}{\partial \phi}} \end{aligned} \quad (8-31)$$

8.2.2 GyroWheel Sub-System Requirements for External Rate Sensing

In Chapter 3, the required accuracy of the external rate sensing capability of the GyroWheel was identified as 0.1 degree/hour ($4.85 \times 10^{-7} \text{ rad/sec}$). This is approximately twenty times less accurate than the capability of state-of-the-art tuned-rotor gyroscopes. To determine the required accuracies of the torque scale factor ($K_{\tau q}$), measured current in the torquer coils (i_{coil}), tracking error of the transverse-axes axis

angles (ϕ) and tracking error of the spin rate (ω_s), the worst case operating condition was chosen. Examining the expressions for the maximum tolerable errors $\delta K_{\tau q}$, δi_{coil} , $\delta \omega_s$ and $\delta \phi$, they will be smallest when $\frac{\partial \omega_s}{\partial K_{\tau q}}$, $\frac{\partial \omega_s}{\partial i_{\text{coil}}}$, $\frac{\partial \omega_s}{\partial \phi}$ and $\frac{\partial \omega_s}{\partial \phi}$ are at their maximum. These four terms will be largest when the spin rate ω_s is at its minimum and the spin axis tilt angle and torquer coil current are largest. Therefore, the following operating condition is suitable to determine the desired sub-system requirements:

$$\begin{aligned}\phi &= 5 \text{ deg} \\ \omega_s &= 1300 \text{ rpm} \\ i_{\text{coil}} &= 1.989 \text{ mA}\end{aligned}$$

The coil current i_{coil} was obtained from the results of the non-linear simulations presented in Chapter 10 for the tilt and spin rate presented above. Based on these operating conditions and the mechanical and electrical characteristics of the GyroWheel, the maximum tolerable errors $\delta K_{\tau q}$, δi_{coil} , $\delta \omega_s$ and $\delta \phi$ are:

$$\begin{aligned}\delta K_{\tau q} &= 1.06 \times 10^{-4} \frac{\text{N} \cdot \text{m}}{\text{A}} \\ \delta i_{\text{coil}} &= 2.12 \times 10^{-7} \text{ A} \\ \delta \omega_s &= 2.38 \times 10^{-3} \frac{\text{rad}}{\text{sec}} (0.023 \text{ rpm}) \\ \delta \phi &= 9.24 \times 10^{-6} \text{ rad} (1.91 \text{ arc - sec})\end{aligned}$$

Based on the maximum values of $K_{\tau q}$, i_{coil} , ω_s and ϕ , the required accuracies expressed in terms of parts-per-million are

$$\begin{aligned} K_{trq} &: 106 \text{ ppm} \\ i_{cont} &: 11 \text{ ppm} \\ \omega_s &: 14 \text{ ppm} \\ \phi &: 106 \text{ ppm} \end{aligned}$$

Despite having an overall rate sensing requirement which is 20 times less stringent than the state-of-the-art tuned-rotor gyros, the requirements on the tracking errors for the spin rate, transverse-axis angles and sensed coil current are still very demanding. This is due to the fact that tilt angle range is far greater than that found in tuned rotor gyroscopes and the deviation of the spin rate from the tuned condition is large.

8.2.3 External Rate Sensing Bandwidth

In Section 8.1, the external rates were shown to be a function of the control torque and transverse-axes angle commands through a transfer function relationship. This simplified to an algebraic relation by making the following assumption:

$$\left\| 1 - \frac{G_c(j\omega)G_{d1}(j\omega)}{1 + G_c(j\omega)\bar{G}_{d1}(j\omega)} \right\| < \varepsilon, \quad \omega < \omega_{tac} \quad (8.32)$$

ω_{tac} is the bandwidth of the transverse-axis control system and ε is a small tolerance value. Therefore the upper bound on the tuned-rotor instrument rate sensing bandwidth is the tilt-axis angle control bandwidth.

In most spacecraft designs, the maximum bandwidth of the attitude control system (ACS) is roughly 0.01 Hz. The external rate sensing bandwidth must be greater than the ACS bandwidth, preferably by at least an order of magnitude. Denoting the ACS bandwidth

by ω_{acc} , the bandwidth of the external rate sensing capability ω_{ers} is bounded by the relation

$$10 \cdot \omega_{acc} < \omega_{ers} < \omega_{mac} \quad (8-33)$$

8.3 Summary

In a general tuned-rotor instrument, the control torque required to point the rotor along a prescribed path is a function of the spin rate, the tilt angles and rates as well as the gimbal inertia and stiffness characteristics. Assuming a high bandwidth rotor tilt control and small tilt angles, the additional torque generated in the torquers beyond this predicted level is proportional to the spacecraft body angular rates. Therefore external rates can be determined based on the three factors:

- i) Measured torquer coil current and spin rate
- ii) Commanded tilt angles and rates
- iii) Accurate characterizations of the torquer scale factor, gimbal and rotor inertia and cross-flexure stiffness.

It should be noted that the algorithm does not rely on direct measurement of the tilt angles and rates. Instead, it uses only the commanded values, with the implicit assumption that good asymptotic tracking of the spin axis tilt angles and rates is achievable. Requirements on these tracking errors are discussed in Chapter 10. If the measured tilt angles and rates were used instead of the commanded values, these signals would have to be filtered as they contain gimbal flutter oscillations at twice the spin frequency.

A preliminary error analysis of the concept reveals that for a commercial tuned-rotor instrument to achieve an accuracy of 0.1 degree/hour, the current and spin rate must be

measured with accuracy's of the order of 10 PPM. Given currently available sensing technologies, this is achievable.

Chapter 9

The Effect of Spin Control on Non-Linear Tuned-Rotor Instrument Dynamics

Thus far, the focus of modeling, analysis and control of tuned-rotor instruments in this study has been on the transverse-axes dynamics, particularly its linear form. By assuming that tilt angles are small and the motor spin rate can be held at a constant value, the spin and transverse-axes dynamics are decoupled and can be analysed separately. However tuned-rotor instruments such as the GyroWheel operate at tilts of up to 5 degrees. Given that greater coupling will occur between the spin and transverse-axes as the tilt of the rotor spin axis is increased, the following two issues are addressed in this chapter:

i) **Variation of tuned-rotor instrument characteristic modes with spin axis tilt.**

Before examining the effect of spin control, the open loop system is examined to determine the degree to which dynamic coupling between the spin and transverse-axes dynamics increases with increasing spin-axis tilt. The effect of coupling is examined by determining if any of the four characteristic modes of the transverse axes vary with tilt angle. The explicit solution to the response of the linear transverse-axes dynamics given in equations (6-39) and (6-40) is compared with the numerically simulated response of the non-linear dynamics.

ii) **Effect of spin control on tuned-rotor instrument characteristic modes.**

The oscillations, which occur in the transverse-axes dynamics at the four characteristic frequencies, must also be present in the spin axis dynamics. The spin axis controller must be designed to tightly track a rate demand. The degree to which the spin axis controller can suppress these characteristic modes will have a bearing on the transverse-axes non-linear dynamic behaviour. Traditionally, rotor spin is regulated through the direct control of the motor shaft spin rate. As rotor spin rate is the desired control variable in the tuned-rotor instrument control problem, the effect of direct rotor spin control is also considered.

The non-linear dynamics model presented in equation (6-52) was used to study these two effects. It was implemented in the form of a non-real-time simulation model described in Appendix D. A simple spin control system was introduced to examine how the dynamic characteristics vary with spin control bandwidth.

9.1 Free Behaviour of 3-Body Model

The open-loop behaviour of the non-linear dynamics was examined by performing simulations of the GyroWheel tuned-rotor instrument with an initial tilt angle on the case-referenced x-axis. Three initial tilts were considered: 0.1, 1.0 and 5 degrees. The smallest angle represents the largest tilt seen in a tuned-rotor gyroscope. The one-degree tilt represents a tilt at which the GyroWheel will frequently operate. The five-degree tilt represents the extreme angle at which the GyroWheel will operate. The mechanical system parameters of the GyroWheel were used in this study and are shown in Table 7-1.

Two GyroWheel spin rates were considered: the first was the classical tuned speed (1499.6 rpm) and second was the minimum spin rate of 1300 rpm. The first simulations

were performed for 20,000 seconds, which is marginally larger than the 18,925-second period of the precession mode predicted by the linear model. The second group of simulations for 1300-rpm operation was performed for 500 seconds, which is slightly larger than the 237-second precession period determined from the linear model.

The nature of the response as a function of tilt is now discussed in three main areas: spin rate response, low frequency response of the transverse-axes and high frequency response of the transverse-axes. The division between low and high frequency response is made because the linear analysis revealed the characteristic modes fall into two groups: a very low precession frequency and three other relatively high frequency modes.

9.1.1 Spin-Axis Response

With the inclusion of the motor shaft as a third body in the model, its spin rate is no longer constrained to a specified value. Figure 9-1, Figure 9-2 and Figure 9-3 show the response of both the motor shaft spin rate and the rotor spin rate for each of the three tilt cases, with the GyroWheel operating at the Traditional tuned speed.

At a tilt of 0.1 degrees, there is no perceptible variation in the rotor spin rate and an insignificant variation in the motor spin rate. Therefore, the underlying assumption of the linear model that the motor rate is constant and equal to the rotor rate is valid here. Increasing the tilt to 1 degree, the oscillations of the motor rate about the tuned speed of 1499.63 rpm are more noticeable, with a variation of 0.2 rpm from the tuned speed. The oscillations occur at roughly 50 Hz, which is twice the spin frequency. As with the 0.1-degree tilt case, the rotor rate has no perceptible variation about the tuned speed. At the maximum operating tilt of 5 degrees, the 50 Hz oscillation of the motor rate about the tuned speed increases to almost ± 6 rpm but the rotor spin rate remains relatively constant.

The 50 Hz mode corresponds to the frequency of the gimbal flutter. In this model, unlike the two-body model in [14], the motor shaft rate is not fixed at the tuned speed but is subject to inertial and spring torques passing through the first gimbal.

One unforeseen difficulty encountered in initializing these simulations was setting the initial spin rate. The non-linear dynamic model uses the gimbal referenced dynamic variables in its computations. However, the case-referenced coordinates are the most natural coordinate set in which to initialize the motion of the tune-rotor instrument. As seen in the following responses, the motor shaft rate oscillates about the assigned spin rate while the rotor spin rate is relatively constant. As shown in Figure 9-3, the motor spin rate must be initialized at the peak of its oscillation about the desired tuned speed in order to ensure that its mean rate is at the tuned speed.

Figure 9-1: Motor and Rotor Spin Rate Responses for Open Loop GyroWheel Operating at the Traditional Tuned Speed, Initial 0.1 degree tilt about the case-referenced x-axis

Figure 9-2: Motor and Rotor Spin Rate Responses for Open Loop GyroWheel Operating at the Traditional Tuned Speed, Initial 1.0 degree tilt about the case-referenced x-axis

Figure 9-3: Motor and Rotor Spin Rate Responses for Open Loop GyroWheel Operating at the Traditional Tuned Speed, Initial 5.0 degree tilt about the case-referenced x-axis

9.1.2 Low-Frequency Response of Transverse-axes Dynamics

Figure 9-4, Figure 9-5 and Figure 9-6 show the non-linear time response of the case-referenced tilt angles ϕ_x and ϕ_y over the 20,000 second run period for initial tilts of 0.1, 1.0 and 5.0 degrees. For comparison, the response of the explicit solution to the linear model (equations 7-38 and 7-39) is also plotted.

At 0.1 degrees tilt shown in Figure 9-4, the linear and non-linear dynamic responses are identical, with both having a precession period of 18,925 seconds. Increasing the tilt beyond this level reveals small changes in the precession period of the non-linear dynamics model. Examining Figure 9-5, for an initial tilt of 1 degree about the x-axis, the precession period has increased slightly to 20,100 seconds. With the initial tilt at the maximum value of 5 degrees, the precession period increases to more than double that predicted by the linear analysis (43,080 seconds) as shown in Figure 9-6. The spin axis now precesses in a counter-clockwise direction. The differences between the three tilt cases are more clearly seen in Figure 9-7 where the tilt angle about the case-referenced x-axis (ϕ_x) is plotted against the tilt angle about the case-referenced y-axis (ϕ_y). This illustrates the coning motion of the spin axis and the opposite direction of precession, which occurs when the initial tilt is increased to 5 degrees.

Although there is a decrease in the precession frequency as the spin-axis tilt is increased, this change is not nearly as severe as that found by Cain [17] using a non-linear, two-body simulation. The much smaller drop-off of precession frequency with tilt angle and the change in direction of precession for high tilt angles are the same observations made by Cain [19] in his study of a three-body model of a tuned-rotor instrument. It appears that by removing the constraint of a constant motor shaft rate, the gimbal inertial and spring torques are free to react on the motor shaft. The result is that the increase in precession frequency with increasing tilt is not nearly as pronounced as with the two-body model.

Figure 9-8 shows the response of the free system when the spin rate is dropped to 1300 rpm. The precession period of 237.3 seconds for the 0.1-degree tilt case appears identical to that calculated using equation (7-10) of the linear model. This period increases marginally to 237.4 seconds when the initial tilt is increased to 1 degree. The precession period increases slightly more to 240.45 seconds when the tilt is increased to the maximum of 5 degrees.

Figure 9-4: Spin-axis Tilt Responses (ϕ_x, ϕ_y) of the Open Loop GyroWheel Operating at the Traditional Tuned Speed, Initial 0.1 degree tilt about the case-referenced x-axis

Figure 9-5: Spin-axis tilt Responses (ϕ_x, ϕ_y) of the Open Loop GyroWheel Operating at the Traditional Tuned Speed, Initial 1.0 degree tilt about the case-referenced x-axis

Figure 9-6: Spin-axis tilt Responses (ϕ_x, ϕ_y) of the Open Loop GyroWheel Operating at the Traditional Tuned Speed, Initial 5.0 degree tilt about the case-referenced x-axis

Figure 9-7: Plot of Precession Motion of Spin axis tilt (ϕ_y versus ϕ_x) for the Open-Loop GyroWheel Operating at the Traditional Tuned Speed, Initial tilts of 0.1, 1.0 or 5.0 degrees about the case-referenced x-axis

Figure 9-8: Plot of Precession Motion of Spin Axis Tilt (ϕ_y versus ϕ_x) for the Open Loop GyroWheel Operating at Spin Rate of 1300 rpm, Initial tilts of 0.1, 1.0 or 5.0 degrees about the case-referenced x-axis

9.1.3 High-Frequency Response of Transverse-Axes Dynamics

The response of the open-loop GyroWheel is now re-examined for the case where it operates at the Traditional tuned speed and is initialized to tilts of 0.1, 1.0 or 5.0 degrees about the case-referenced x-axis. Figure 9-9, Figure 9-10 and Figure 9-11 show the simulation response of the case-referenced tilt angles (ϕ_x, ϕ_y) over the first one second of the 20,000-second simulation. The explicit solution to the linear equations of motion (equations (7-38) and (7-39)) is also plotted for the purposes of comparison.

At 0.1-degree tilt, the non-linear and linear responses appear identical. Two significant modes are evident in the response: an oscillation near 50 Hz modulated by a beat like frequency at 3-4 Hz. To confirm these frequencies, a Power Spectral Density (PSD) estimate of the simulated x-axis tilt was generated and compared with the PSD of the closed-form linear response of this same signal. Examining Figure 9-12, the responses are identical with three modes evident in the response: a mode at 3.55 Hz corresponding to the beat frequency, the nutation frequency mode at 46.4 Hz and a twice-spin frequency mode at 50.4 Hz. The precession frequency mode does not appear as the duration of the simulation response over which the PSD was taken is less than twice the precession period.

Increasing the tilt to 1.0 degree (Figure 9-10) results in a non-linear tilt response that still appears identical to that obtained using the linear model. Figure 9-13 shows the PSD of the nonlinear and linear responses of the x-axis tilt. Again the PSDs exhibit peaks at 3.55, 46.4 and 50.4 Hz, with the plots being almost identical up to 50 Hz. Even at a tilt of 5 degrees, the high frequency behaviour of the non-linear model still appears to be almost identical to the linear model. Examining Figure 9-11, after one second of simulation, the linear response on the y-axis is in phase with the linear model but shifted slightly below the non-linear response. This is likely due to the significantly lower precession frequency exhibited by the non-linear model when the transverse axis tilt is at 5 degrees. This is

confirmed by examining Figure 9-14 which shows that the PSD estimate of the linear and non-linear simulation response for initial 5 degree tilt are nearly identical.

9.1.4 Summary of Differences Between Linear and Non-Linear Dynamic Models

At high frequencies, the linear model appears to be a good approximation of the non-linear dynamics over the tilt operating range. The twice-spin rate, nutation and beat-like modes in the linear and non-linear models occur at the same frequency. The amplitude of the nutation oscillations as they are modulated by the beat frequency also appears the same over the tilt operating range.

The one difference between the models is the precession frequency. When operating at the Traditional Tuning Condition, the magnitude of the precession frequency observed in the non-linear simulation is very close to that determined with the linear model up to angles of 1 degree. However, when the tilt is further increased, the non-linear model precession frequency decreases more noticeably, being more than 50 percent lower than the predicted value at 5 degrees tilt. The percentage decrease in precession frequency is not nearly as noticeable when the system is operated at the de-tuned speed of 1300 rpm. This is due to the fact that the precession frequency at 1300 rpm is approximately two orders of magnitude higher than at the Traditional Tuning Condition.

Figure 9-9: Spin-axis tilt Response (ϕ_x, ϕ_y) of the Open Loop GyroWheel Operating at the Traditional Tuned Speed with initial 0.1 degree tilt about the case-referenced x-axis; plot shows first 1 second of simulation response

Figure 9-10: Spin-axis tilt Response (ϕ_x, ϕ_y) of the Open Loop GyroWheel Operating at the Traditional Tuned Speed with an initial 1.0 degree tilt about the case-referenced x-axis; plot shows first 1 second of simulation response

Figure 9-11: Spin-axis tilt Response (ϕ_x, ϕ_y) of the Open Loop GyroWheel Operating at the Traditional Tuned Speed with an initial 5.0 degree tilt about the case-referenced x-axis; plot shows first 1 second of simulation response

Figure 9-12: Power Spectral Density Estimate of the Spin-axis Tilt Response $(\phi_x(t))$, Open Loop GyroWheel Operating at Traditional Tuned Speed, Initial 0.1 degree tilt about the case-referenced x-axis

Figure 9-13: Power Spectral Density Estimate of the Spin-axis Tilt Response $(\phi_x(t))$, Open Loop GyroWheel Operating at Traditional Tuned Speed, Initial 1.0 degree Tilt about the case-referenced x-axis

Figure 9-14: Power Spectral Density Estimate of the Spin-axis Tilt Response $(\phi_x(t))$, Open Loop GyroWheel Operating at Traditional Tuned Speed, Initial 5.0 degree tilt about the case-referenced x-axis

9.2 Behaviour of 3-Body Model with Spin Control

In the linear dynamics model of the spin axis, the motor shaft rate was assumed to be held at a constant spin rate; the implicit assumption is that a very high bandwidth spin controller is active. However, hardware implementation constraints will limit the spin control bandwidth to a value in the 10 to 100 Hz range. By closing a control loop around the spin dynamics, the interaction between the spin and transverse-axes dynamics will be altered, potentially modifying the behaviour observed in Section 6.2.

The following section examines the degree to which spin control bandwidth affects the open-loop behaviour of the transverse-axes dynamics. An alternative spin control approach is then proposed where the rotor spin is controlled directly.

9.2.1 Motor Spin Control

To examine the effect of spin control bandwidth, a simple closed loop control system was used. This is shown in Figure 9-15. Table 9-1 lists the definition, symbol and numerical value of the parameters shown in the model. The motor spin rate command $\dot{\theta}_{z_{ref}}$ is compared against the sensed motor spin rate $\dot{\theta}_z$. The resulting error is passed through a proportional-plus integral controller. The resulting voltage demand V_{mot} is applied to a voltage-driven DC-motor. The motor torque $K_t i_{mot}$ is applied as the gimbal-referenced spin-axis torque T_{sz} in the non-linear dynamic simulation described in Appendix D.

The proportional-plus-integral controller was tuned to yield varying spin control bandwidths by scaling the proportional and integral gains. In order to study the effect of bandwidths as high as 1 kHz, the electrical time constant was set to $1/10^{\text{th}}$ the normal

value of 0.5 milliseconds. In the real motor spin control system, the electrical time constant would limit the achievable bandwidth to less than 400 Hz.

Figure 9-15: Block Diagram of Motor Spin Control System and Simplified Drive Dynamics

Spin Control System Parameter	Symbol	Numerical Value
Motor torque sensitivity	K_t	$0.0244 \frac{Nm}{A}$
Motor back-emf constant	K_{emf}	$0.0244 \frac{V \cdot sec}{rad}$
Motor winding resistance	R_{mot}	0.79 ohms
Motor winding electrical time constant	τ_{elec}	0.5 msec
Proportional control gain (for 30 Hz bandwidth)	K_p	$38.7 \frac{V \cdot sec}{rad}$
Integral control gain (for 30 Hz bandwidth)	K_i	$4.77 \frac{V \cdot sec^2}{rad}$

Table 9-1: Spin Axis Control System Parameters

Two different effects were examined:

- a) Effect of spin axis tilt angle on the precession of the rotor.
- b) Effect of motor spin control bandwidth on the precession of the rotor.

For a) above where spin-axis tilt was varied, a spin control bandwidth of 30 Hz was chosen to meet the GyroWheel requirement specified in Section 3.3

9.2.1.1 Effect of Spin Axis Tilt Variation

To examine the effect of spin axis tilt on tuned-rotor instrument dynamic behaviour, the spin axis was first set at an initial tilt of 0.1 degrees about the case-referenced x-axis. The spin rate command was set at the Traditional tuned speed. The free motion of the spin axis tilt was then observed with the motor spin control loop closed. The response of the x-axis tilt versus the y-axis tilt is shown in over the precession period predicted by the linear analysis (18925 seconds). The response is similar to that predicted by linear analysis. Over this period, the spin axis precesses slightly less than one rotation. In addition, the axis precesses gradually inwards towards null, indicating that the system is no longer tuned. There is a residual spring rate drawing the spin axis slowly into the null position.

Increasing the initial tilt to 1-degree results in more dramatic deviations of the response from that predicted by linear analysis. Figure 9-16 shows the spin axis now precessing in a counter-clockwise direction. At this increased initial tilt angle, there is a more pronounced component of motion towards null. When the tilt magnitude has coned inward to approximately 0.2 degrees, the precession direction switches from a counter-clockwise to clockwise direction. Beyond the point where the magnitude of tilt has

decreased to 0.1 degree, the subsequent response looks very similar to that of the first case where the tilt was initialized at 0.1 degrees.

Examining Figure 9-16, a similar trend is observed when the initial tilt is 5 degrees. While the rotor spin axis precesses, the inward component of motion along the radial axis is even greater than with the 1 degree tilt case. At approximately 0.2 degrees tilt, the precession motion switches from a counter-clockwise to clockwise direction. Again, once the tilt magnitude is near 0.1 degree, the response is very similar in form to that observed in the 0.1-degree initial tilt case.

In summary, with the exception of the case of the very small initial tilt (0.1 degrees), motor spin control drastically changes the precession behaviour of the system as compared with the open-loop case.

9.2.1.2 Effect of Motor Spin Control Bandwidth Variation

The simulation performed in the previous section was repeated with an initial tilt of 1 degree and the motor spin control bandwidth varied from 3 to 3000 Hz. As an extreme case, the response for the same scenario was also generated using a two-body model of the transverse-axes dynamics. For this model, the motor spin rate is assumed to be constant. Setting the spin rate to a constant value, the non-linear model described in Chapter 5 simplifies to a form equivalent to the two-body model developed by Cain [14]. This can be considered as a case where the spin control bandwidth is infinite.

Figure 9-17 shows the response of the spin axis tilt angle ϕ_x plotted with respect to the spin axis tilt angle ϕ_y . The plots are shown for a duration of 545 seconds. This was the period over which the spin axis precessed one full rotation for the two-body model. Therefore for the two-body model, there is a 40-fold decrease in the precession period

from that predicted with the linear model. As the motor spin control bandwidth is decreased in the three-body model, the precession frequency decreases. In addition the component of precession motion along the radial direction toward null becomes more dominant.

The observations described above can be attributed to the coupling which exists between the motor shaft and the rotor because of the fluttering gimbal. The resulting cyclic torque acting on the motor shaft through the first gimbal can be considered as a disturbance torque on the spin axis. The spin controller attempts to suppress this disturbance as it tracks the spin rate set point. As the motor spin control bandwidth is increased, the control loop is better able to suppress the effect of this high frequency torque. Figure 9-18 shows the response of the motor rate from 1 to 1.1 seconds during the same simulation. For all four bandwidth cases, the average spin rate is at the desired classic tuned speed of 1499.6 rpm. However as the bandwidth of the spin controller is decreased, the amplitude of the rate oscillations increases. The control loop becomes less capable of rejecting the gimbal flutter torques.

As the rotor is coupled to the motor shaft through this fluttering gimbal, an increase of motor control bandwidth has an effect on the rotor motion. Figure 9-19 shows the response of the rotor spin rate (case-referenced spin rate $\dot{\phi}_r$) as a function of the motor spin control bandwidth. With increasing control bandwidth, the motor rate becomes more closely clamped to its set point, which forces the rotor spin rate oscillations at the flutter frequency to become larger.

In the absence of a spin control system, the gimbal flutter will induce a much larger amplitude oscillation of the motor spin rate than the rotor spin rate. This occurs because the GyroWheel rotor spin inertia is 350 times larger than the motor spin inertia. At low spin control bandwidths where control authority is minimal, the rotor spin rate exhibits lower amplitude oscillations at the flutter frequency than the motor spin rate. As the

motor control bandwidth increases, rotor spin rate oscillations increase, while motor spin rate oscillations decrease.

Figure 9-16: Plot of Precession Motion of Spin Axis (ϕ_y versus ϕ_x) for GyroWheel Operating with Closed-Loop Motor Spin Control (30 Hz bandwidth); Response generated for initial tilts of 0.1, 1.0 and 5.0 degrees about the Case-referenced x-axis, with Spin Rate at Traditional Tuned Speed.

Figure 9-17: Plot of Precession Motion of Spin Axis (ϕ_y versus ϕ_x) for GyroWheel Operating with Closed-Loop Motor Spin Control and Initial tilt of 1.0 degrees about the case-referenced x-axis; Response generated for Bandwidths of 3 – 3000 Hz with Spin rate at Traditional Tuned Speed.

Figure 9-18: Motor Spin Rate $(\dot{\theta}_{z_c})$ for GyroWheel Operating with Closed-Loop Motor Spin Control and initial tilt of 1.0 degrees about the case-referenced x-axis; Response generated for bandwidths of 3 – 3000 Hz, Spin rate at Traditional Tuned Speed.

Figure 9-19: Rotor Spin Rate $(\dot{\theta}_{z_r})$ for GyroWheel Operating with Closed-Loop Motor Spin Control and initial tilt of 1.0 degrees about the case-referenced x-axis; Response generated for bandwidths of 3 – 3000 Hz, Spin rate at Traditional Tuned Speed.

9.2.2 Rotor Spin Control

Some interesting conclusions can be drawn from the study of tilt angle and motor spin control bandwidth effects on the behaviour of the transverse-axes dynamics:

- a) At initial tilts as small as 0.1 degrees, the behaviour of the transverse-axes dynamics begins to deviate from that predicted by the linear model.
- b) When the spin axis is initially tilted away from null, a very high control bandwidth will reduce the tendency of it to spiral back in towards null.
- c) Higher motor spin control bandwidths result in a more oscillatory rotor spin rate profile, as the spin controller is better able to suppress the effect of gimbal flutter.
- d) Higher motor spin control bandwidths also result in a marked decrease in the precession period. This indicates that the torque required to hold the rotor at a non-zero tilt will increase as the spin control bandwidth is increased. This increases the power budget of the device. This increased torque will be discussed further in Chapter 10.

Based on these results, a low spin control bandwidth is desirable; the bandwidth should be lower than the lowest bandwidth presented here (3 Hz). However such a low bandwidth is undesirable from the point of view of using the device in a spacecraft attitude control system.

An alternative approach is to use rotor spin control. Motor spin control forces the relatively large rotor spin rate to oscillate while the small motor shaft inertia tracks the rate set point. Without any spin control, the rotor spin rate is relatively insensitive to gimbal flutter. It is desirable that with spin control active, the high frequency behaviour of the rotor spin rate is the same as it is in the open loop case. The spin rate controller should only alter the low-frequency behaviour. For these reasons, the rotor spin rate rather than the motor spin rate should be controlled.

To test this approach, the spin control system was modified and is shown in Figure 9-20. The case-referenced spin rate $\dot{\phi}_z$ was fed back and compared with the commanded spin rate $\dot{\phi}_z$. The torque generated by the motor is applied as a case-referenced torque $T_{\dot{z}}$ about the rotor spin axis.

Figure 9-20: Block Diagram of Rotor Spin Control System

The simulations presented in Section 9.2.1 were repeated with the rotor spin control bandwidth set at 30 Hz. The resulting responses are shown in Figure 9-21 and can be compared with the results obtained using motor spin control, shown in **Error! Reference source not found.** Note that the scales are different in **Error! Reference source not found.** and Figure 9-21 for the 1-degree and 5 degree initial tilt cases. This is due to the more severe inward coning motion observed with motor spin control. Overall, the following observations can be made:

- a) For an initial 0.1 degree tilt, the response is identical to that expected from the linear model, with the precession period being 18925 seconds. There is no apparent coning motion of the spin axis towards null.
- b) Increasing the initial tilt to 1 degree, the precession period decreases slightly, but there is no radial motion of the spin-axis towards null as was found with the motor spin control examples. The response is identical to the free response of the open-loop GyroWheel non-linear dynamics shown in Figure 9-7.
- c) With a tilt of 5 degrees, the transverse-axes dynamics no longer exhibits a steady precession. Like the motor control examples, there is a radial component to the precession motion, although its magnitude is 1-2 orders of magnitude smaller than that exhibited by the motor spin control response for this initial tilt of 5 degrees. In addition, the precession direction switches from a counter-clockwise to clockwise direction when the tilt magnitude decreases to approximately 4 degrees.

The use of rotor spin control appears more suitable than motor spin control. The variation of precession frequency with tilt is not nearly as severe as that observed when using motor spin control. Therefore the torques required to hold the rotor spin axis at a non-zero tilt will be much less than that required using a motor spin controller. Up to a 1 degree tilt, the responses are identical to that found with the open-loop non-linear dynamics model and also very similar to the linear dynamics model.

9.3 Summary

Simulations of the non-linear dynamics of the GyroWheel tuned-rotor instrument show that its high frequency dynamics are almost identical to the linearized version of the model. The main difference between the non-linear and linearized dynamics is the

precession frequency, which is exhibited when the systems are initialized at non-zero tilts. With the non-linear model, the precession frequency decreases as the tilt of the spin axis is increased.

Direct rotor spin control was found to be superior to motor spin control. It results in the transverse-axes dynamics exhibiting similar precession frequencies to that observed with no spin control applied. When the spin axis tilt is non-zero, rotor spin control results in lower precession frequencies than that exhibited when motor spin control is employed. This means that with the transverse-axes control system active, the torques required to hold the spin axis at a non-zero tilt will be much less than with motor spin control active.

Figure 9-21: Plot of Precession Motion of Spin Axis (ϕ_y versus ϕ_x) for GyroWheel Operating with Closed-Loop Rotor Spin Control and Initial tilt of 1.0 degrees about the Case-referenced x-axis; Response generated for 30 Hz bandwidth, Spin rate at Traditional Tuned Speed.

Chapter 10

GyroWheel Transverse-axes Control System Design

The general problem of controlling the transverse-axes motion of a gyroscopic device was addressed in Chapter 7. These principles can be applied to the GyroWheel system to achieve the desired bandwidth and stability margins. In addition, the selection of transverse axis controllers is also examined from the following two perspectives:

a) **Insensitivity of Rate Sensing Calibration to Transverse-axes Control System Design**

In the examination of spin control, the method employed was shown to have a significant impact on the precession frequency exhibited by the transverse-axes dynamics. As shown in equation (6-44), the precession frequency is directly related to the “dynamic stiffness” of the system. As shown in Chapter 8, this stiffness characteristic is a critical parameter in the estimation of external rates. The objective in this study is to make this tuned-rotor instrument dynamic stiffness easily understood, easily characterized and insensitive to the control design employed.

In this section, it is demonstrated how this characteristic can be affected when the high frequency dynamics of the GyroWheel interacts with the transverse-axes

control system. It is then shown how the introduction of notch filters in the transverse-axes controller can preserve the desired dynamic stiffness characteristic observed in the open-loop system.

b) **Effect of Model Variation and Uncertainty**

Unlike a tuned-rotor gyroscope, the GyroWheel operating conditions (spin rate and tilt) vary significantly as do some system parameters. The robustness of the performance and stability is not addressed formally in this study. However approaches for dealing with the variation of the spin rate and the subsequent variation of the nutation frequency are discussed.

10.1 Performance and Stability Considerations

The classical control design tools developed for SCISCO systems in Chapter 7 can now be applied to the transverse-axes control of the GyroWheel system. Three methods of stabilizing the nutation dynamics are applied directly to the GyroWheel. The resulting responses are then examined.

10.1.1 Transverse-Axis Controller Selection

In Chapter 7, the concepts of nutation damping and precession control were introduced. It was shown how a dynamic controller is required to stabilize either the nutation or precession pole. The approach taken in Chapter 7 was to first select a stabilizing damper for the nutation pole. The frequency response at the precession pole can then be shaped to achieve the desired bandwidth and low-frequency characteristics. By stabilizing the nutation dynamics first, the precession controller can be chosen solely from the point of view of system performance.

To examine the transverse-axes control system design, the SCISCCO transfer function of the complete model, (equation (6-14)) was used. This includes an accurate estimate of the precession frequency and is described in the following form:

$$\frac{\phi(s)}{T_z(s)} = G_{d1}(s) = \frac{1}{I_r'(s - j\omega_p)(s - j\omega_n)} \quad (10-1)$$

As shown in Chapter 6, the nutation and precession frequencies are a function of the spin rate ω_s . In the following section, it is assumed that the system is tuned using the new tuning condition developed in Chapter 6. Consequently, the system transfer function simplifies to the form:

$$\frac{\phi(s)}{T_z(s)} = G_{d1}(s) \approx \frac{1}{I_r' s(s - j\omega_n)} \quad (10-2)$$

Overall, the control system block diagram can be seen in Figure 10-1. The nutation damper $G_{nd}(s)$ is in series with the precession controller $G_{pc}(s)$. The nutation damper provides phase compensation to stabilize the nutation dynamics. The precession controller acts on the combination of the nutation damper and plant dynamics to provide the desired bandwidth.

Figure 10-1: Block Diagram of Transverse-axes Control System Shown in Complex Form

10.1.1.1 Nutation Damper

The nutation damper must be designed to stabilize the nutation pole ω_{n_2} contained in equation (10-3). Four types of nutation dampers were developed in Chapter 7 using root locus or frequency domain loop shaping. Of the four designs presented, three of the designs will be examined in detail in this section. Design Case 3 is not examined here. Although it produced very good nutation damping, it was found to produce a very high degree of coupling between the commanded and uncommanded axis. Re-expressing each of the remaining three compensator designs of Section 7.2.2.2 in terms of the GyroWheel nutation frequency, the following SCISCO nutation dampers can be applied to the GyroWheel dynamics:

a) Option 1: Lead Compensator

$$G_{nd}(s) = \frac{(1.25s - j\omega_n)}{(0.1s - \omega_n)} \quad (10-4)$$

b) Option 2: Non-minimum Phase Lead Compensator

$$G_{nd}(s) = \frac{j(2s - \omega_n)}{(2s - \omega_n)} \quad (10-5)$$

c) **Option 3: Lag Compensator**

$$G_{rd}(s) = \frac{j^3 \omega_n^4}{(s - \omega_n)^4} \quad (10-6)$$

10.1.1.2 Precession Controller

As demonstrated in Chapter 7, once a nutation damper is chosen which stabilizes the nutation dynamics, a simple proportional controller can be used to achieve the desired precession control performance criteria such as bandwidth and tracking error. Generally, precessional controllers in tuned rotor gyros are chosen as proportional or proportional-plus-integral compensators. However, there are two unique aspects of the GyroWheel, which must be considered in choosing this compensator:

- a) Unlike tuned-rotor gyroscopes, the transverse-axes angles are not regulated to zero. The GyroWheel must track a momentum demand where the rate of change of the momentum is equivalent to a torque imparted on the spacecraft. The rate of change of momentum of the GyroWheel expressed in the spacecraft body frame is denoted by $\dot{\mathbf{h}}_{gw}$. The following approximation relates the rate of change of GyroWheel momentum $\dot{\mathbf{h}}_{gw}$ to the magnitude of the rotor momentum h and the transverse angles ϕ_x and ϕ_y :

$$\dot{\mathbf{h}}_{gw} \approx h \begin{bmatrix} \dot{\phi}_y \\ -\dot{\phi}_x \\ 1 \end{bmatrix} \quad (10-7)$$

Therefore, the GyroWheel is required to be capable of tracking a ramp-type angular position command.

- b) Unlike tuned rotor gyroscopes, the GyroWheel system frequently operates in a de-tuned condition which changes the plant dynamics from a "type 1" system where ω_p is zero to a "type 0" system where ω_p is non-zero as shown in equation (10-1).

To track a ramp input command, the combined controller and plant dynamics loop gain function must be of type 2. When the GyroWheel is in the tuned condition, a proportional-plus-integral controller will achieve this requirement. When the system is de-tuned, the controller must contain two integrators. However, use of such a compensator would make it more difficult to stabilize the system.

Alternatively, the magnitude of the steady-state tracking error when using a proportional-plus-integral controller can be examined. Based on the control system block diagram shown in Figure 10-1, the steady state error is given by the following equation:

$$e_{ss} = \lim_{s \rightarrow 0} s \frac{1}{1 + G_{pc}(s)G_{nd}(s)G_{d1}(s)} \phi_{ref}(s) \quad (10-8)$$

Examining equations (10-4), (10-5) and (10-6), each nutation damper is type 0 and has unity gain. A proportional-plus-integral controller of the following form is used:

$$G_{pc}(s) = \frac{K_p s + K_i}{s} \quad (10-9)$$

Based on (10-1), (10-4) and (10-9), the expression for steady state error tracking error simplifies to the following:

$$e_{ss} = \frac{K_{\phi n}}{K_i} \dot{\phi}_{ref} \quad (10-10)$$

The dynamic stiffness is defined as

$$K_{dyn} = \frac{(2I_{r_i} + I_{g_i})\omega_p \omega_{n_z}}{2} \quad (10-11)$$

$\dot{\phi}_{ref}$ is the rate of change of the complex transverse-axes angular reference angle. The tracking error will decrease as the integral gain is increased. In this section and the following one, the integral gain K_i will be designed and the consequence of this choice on steady-state tracking error examined.

When designing the precessional controller, it is relevant to consider only the low frequency dynamics. With the high frequency nutation dynamics stabilized, the loop gain function $G_{nd}(s)G_{d1}(s)$ simplifies to

$$G_{nd}(s)G_{d1}(s) \approx \frac{1}{hs} \quad (10-12)$$

As described in Section 2.2.3, the desired bandwidth of the GyroWheel control system is approximately 5 Hz. To achieve this performance, the proportional and integral controller will be chosen such that the first unity gain crossover frequency ω_c is 5 Hz (31.4 radians/second). To ensure that there is adequate phase margin at this frequency, the numerator of the precessional controller $G_{pc}(s)$ will be chosen such that the positive phase shift is 60 degrees ($\frac{\pi}{3}$ radians). The zero of the proportional-plus-integral

compensator $\left(\frac{K_i}{K_p}\right)$ can then be chosen according to the relation

$$\frac{K_i}{K_p} = \frac{\omega_c}{\sqrt{3}} \quad (10-13)$$

The proportional plus integral compensator is rearranged in following the form:

$$G_{PC}(s) = \frac{K_i \left(\frac{K_p}{K_i} s + 1 \right)}{s}$$

The integral gain K_i can then be chosen such that the unity gain crossover occurs at ω_c .

For this to occur, K_i must satisfy the following relation

$$K_i = \frac{h \cdot \omega_c^2}{2} \quad (10-14)$$

For the following nominal GyroWheel parameters,

$$h = 1 \text{ Nm} \cdot \text{sec}$$

$$\omega_n = 291 \text{ rad / sec}$$

the resulting precessional controller is

$$G_{pc}(s) = \frac{27.2s - 493}{s}$$

10.1.2 Frequency Response Analysis

The open loop frequency response of the transverse-axes control loop can now be examined in Figure 10-2. All three designs have a crossover frequency of between 31 and 34 radians/second. There is a second crossover frequency around 300 radians/second due to the nutation pole at 291 radians/second. The phase and gain margins are listed below in Table 10-1.

The lead compensator design appears to have good phase margin at both crossover frequencies and a healthy gain margin. However both the lag and non-minimum phase lead designs have phase margins at the first crossover frequency which are lower than a desired level of 45 degrees. The earlier assumption that the high frequency nutation dynamics has no phase shifting effect at the desired crossover frequency of 5 Hz is not true for these designs. Both the lag and non-minimum phase lead nutation dampers contribute a negative phase shift of approximately 25 degrees at ω_c .

Taking note of this, the positive shift contributed by the proportional-plus-integral compensator numerator is increased from 60 degrees to approximately 70 degrees. This can be accomplished by setting the proportional and integral gains according to the following two relations:

$$\frac{K_i}{K_p} = \frac{\omega_c}{3} \quad (10-15)$$

$$K_i = \frac{h \cdot \omega_c^2}{\sqrt{10}} \quad (10-16)$$

The resulting precessional controller for the lag and non-minimum phase lead designs is given by the following transfer function:

$$G_{\pi}(s) = \frac{30.0s + 314}{s}$$

The modified gain and phase margins with the updated precession controller for the non-minimum phase lead compensator and lag compensator are given in Table 10-2 below with the open loop frequency responses shown in Figure 10-3.

With the precession controller chosen, the transverse-axes control tracking error for a ramp input can be examined. Examining equation (10-10), the tracking error will be at a maximum when the dynamic stiffness K_{dyn} is greatest and the integral gain K_i is lowest. The greatest value of K_{dyn} will occur when the GyroWheel is in an extreme de-tuned condition. At 1300 rpm, the dynamic stiffness increases to 2.29×10^{-2} Nm/radian. Of the three precession controller designs, the lag and the non-minimum phase lead designs have the lowest integral gain K_i at 314 Nm/radian-sec. The resulting steady state error is a very small 0.007 percent of the magnitude of the ramp input. With a maximum commanded transverse-axes angular rate of 0.01 radians/second, the maximum tracking error is 7.0×10^{-7} radians/second or 0.144 deg/hr. This is comparable to the desired external rate sensing error of 0.1 deg/hr. Therefore, for this first examination of the precession controller, a simple proportional-plus-integral controller is adequate for tracking.

Compensator Design	Phase Margin	Gain Margin
Lead Compensator	58.6 deg @ 30.6 rad/sec 80.0 deg @ 300 rad/sec	50 dB @ 233 rad/sec
Non-minimum Phase Lead Compensator	35.6 deg @ 34.8 rad/sec 42.7 deg @ 319 rad/sec	8.3 dB @ 126.2 rad/sec
Lag Compensator	35.3 deg @ 34 rad/sec 84.2 deg @ 300 rad/sec	10 dB @ 106.5 rad/sec 65 dB @ 700 rad/sec

Table 10-1: Gain and Phase Margins for First Iteration of Compensator Designs

Compensator Design	Phase Margin	Gain Margin
Lead Compensator	58.6 deg @ 30.6 rad/sec 80.0 deg @ 300 rad/sec	50 dB @ 233 rad/sec
Non-minimum Phase Lead Compensator	46.2 deg @ 35.3 rad/sec 42.7 deg @ 319 rad/sec	8.2 dB @ 137.4 rad/sec
Lag Compensator	46.2 deg @ 34.3 rad/sec 84.2 deg @ 300 rad/sec	9.8 dB @ 137.4 rad/sec 65 dB @ 700 rad/sec

Table 10-2: Gain and Phase Margins for Final Compensator Designs

Figure 10-2: Open Loop Frequency Response of Transverse-axes Control System for 3 Nutation Damper Designs; First Selection of Precession Controller with GyroWheel Operating at New Tuned Speed

Figure 10-3: Open Loop Frequency Response of Transverse-axes Control System for 3 Nutation Damper Designs; Final Selection of Precession Controller with GyroWheel operating at New Tuned Speed

10.1.3 Transverse-Axis Simulation Response

The linearized GyroWheel dynamics model was used to choose the transverse-axes control system. The effectiveness of the transverse control loop design can now be examined using the detailed non-linear simulation model of the GyroWheel described in Appendix D. Of interest is the response of the tilt angles and the control torques applied to maintain a constant tilt.

For the three different transverse controller designs, the response of the GyroWheel is examined for the following four simulation cases:

- a) Case 1: Initial tilt at null $(\phi_{x_{ref}}, \phi_{y_{ref}}) = (0, 0)$ degrees
Step command of 0.1 deg applied about x-axis for 10 seconds
- b) Case 2: Initial tilt at null $(\phi_{x_{ref}}, \phi_{y_{ref}}) = (0, 0)$ degrees
Step command of 0.1 degree applied about y-axis for 10 seconds
- c) Case 3: Initial tilt at null $(\phi_{x_{ref}}, \phi_{y_{ref}}) = (5, 0)$ degrees
Step command of 0.1 degree applied about x-axis for 10 seconds
- d) Case 4: Initial tilt at null $(\phi_{x_{ref}}, \phi_{y_{ref}}) = (0, 5)$ degrees
Step command of 0.1 degree applied about y-axis for 10 seconds

The spin rate of the system is commanded at the classic tuned condition. Cases 1 and 2 were chosen as the small spin-axis tilt angle is within the linear region of the dynamics as shown in Chapter 9. Cases 3 and 4 were chosen to examine the effect of the non-linear dynamics on the chosen control system when operating at large tilt angles.

a) Cases 1 and 2: Response of the GyroWheel When Operating Near Null

Figure 10-4 shows the response of the tilt angles to a step command of 0.1 deg on the x-axis. The transient responses are similar with the lead design rising up to the command slightly faster and having only a 10 percent overshoot. The rise time is approximately 0.05 seconds with the systems settling in 0.3 to 0.35 seconds. The lag and non-minimum phase lead designs are almost identical in response except for the initial motion. The non-minimum phase response ϕ_x initially moves in the negative direction, this being characteristic of non-minimum phase systems. The lag and non-minimum phase systems have a slightly larger overshoot of 20 to 25 percent. It should be noted that these responses are very similar to those obtained in the linear system examples shown in Chapter 7 with the exception that the addition of the proportional plus integral compensator yields a slight overshoot.

The response of the uncommanded tilt angle ϕ_y is best for the lead design. As predicted, the complex zero in this compensator almost cancels the nutation dynamics pole, significantly decoupling the system. The peak transient angle is 0.003 degrees (3 percent of the command) in this case. The non-minimum phase lead design is the poorest with the peak angle reaching 0.025 degrees (25 percent of the command).

Figure 10-5 shows the steady state response of the tilt angles ϕ_x and ϕ_y at the end of 10 seconds for this same simulation. For each of the three designs, the angles oscillate at a frequency of $2\omega_s$. The amplitude is smallest for the non-minimum phase lead design, with the flutter amplitude being slightly less than 0.0002 deg (± 0.2 percent variation from the command). The lag and lead designs have flutter oscillations of amplitude of just less than 0.004 deg (± 0.4 percent variation with respect to the command).

The control torques applied to the GyroWheel are presented in Figure 10-6. Of note here is the large initial transient torque T_{c_i} , which reaches $0.7 \text{ N} \cdot \text{m}$ for the lead design. The large torque is produced because the lead nutation damper produces a large initial transient torque to cancel the effect of the nutation dynamics and decouple the system. Examining the transverse-axes control torques over the first 0.1 seconds in Figure 10-7, it can be seen that this torque is almost two orders of magnitude larger than the T_{c_i} torque observed for the lag and non-minimum phase lead designs. A large transient torque is of note not only for sizing the torquer coils in the GyroWheel, but also in the context of external rate sensing. The concept relies on measuring the low-frequency torque required to make the GyroWheel track its tilt demand. It filters out the high frequency and transient torques. A torque of such a large magnitude cannot be completely attenuated and will have the undesirable effect of causing more spikes in the external rate estimate. One way to reduce these peak torques is to introduce a small amount of damping into complex lead by adding a small negative real component to the complex zero in the numerator of the lead type nutation damper.

Figure 10-8 and Figure 10-9 show the tilt angle and torque responses of the system to a step command of 0.1 deg on the y-axis. The responses are nearly identical to that observed for the x-axis command, but with the axes switched.

b) Cases 3 and 4: Response of the GyroWheel When Operating at Maximum Tilt

Figure 10-10 shows the spin-axis tilt response of the GyroWheel to a 0.1 deg step command on the case-referenced x-axis, with the system initially tilted 5 degrees about the x-axis. The general transient responses observed here are the same as that generated when operating about null. However the noticeable difference is the greater effect of the gimbal flutter on the response, with far more pronounced $2\omega_{\phi}$ oscillations occurring. The amplitude of the tilt oscillations has now increased to ± 0.01 degrees (0.2 percent of

the command) for the non-minimum phase design and ± 0.02 degrees (0.4 percent of the command) for the lag and lead designs. Although not discussed in this study, the impact of this flutter on the spacecraft dynamics must be considered during attitude control system design.

The general trend of the transient part of the torque responses shown in Figure 10-11 is the same as that observed when the GyroWheel is at a small tilt. For the case where the GyroWheel is initially set at a 5-degree tilt about the y-axis, Figure 10-12 and Figure 10-13 show the spin-axis tilt and control torque responses of the system to a 0.1 deg step change of tilt about the y-axis. The responses are identical to that observed for the x-command, but with the axes switched.

Figure 10-4: Case 1 – Transient Time Response (first 0.5 seconds) of Spin-axis tilt Angles (ϕ_x, ϕ_y) to a Step Command of 0.1 degrees about the x-axis for initial tilt of $(\phi_x, \phi_y) = (0, 0)$ deg, Spin rate at the Classic Tuned Condition.

Figure 10-5: Case 1 - Steady State Time Response (last 0.1 seconds) of Spin-Axis Tilt Angles (ϕ_x, ϕ_y) to a Step command of 0.1 degrees about the x-axis, Initial Tilt of $(\phi_x, \phi_y) = (0, 0)$ deg, Spin rate at the Classic Tuned Condition.

Figure 10-6: Case 1 - Transient Time Response (first 0.5 seconds) of Transverse-Axes Control Torques (T_{c_x}, T_{c_y}) to a Step command of 0.1 degrees about the x-axis, Initial tilt of $(\phi_x, \phi_y) = (0, 0)$ deg, Spin rate at the Classic Tuned Condition.

Figure 10-7: Case 1 – Transient Time Response (first 0.1 seconds) of Transverse-Axes Control Torques (T_{c_x}, T_{c_y}) to step command of 0.1 degrees about x-axis for initial tilt of $(\phi_x, \phi_y) = (0, 0)$ deg, Spin rate at the Classic Tuned Condition.

Figure 10-8: Case 2 – Transient Time Response (first 0.5 seconds) of Spin-Axis Tilt Angles (ϕ_x, ϕ_y) to a Step Command of 0.1 degrees about the y-axis, Initial tilt of $(\phi_x, \phi_y) = (0, 0)$ deg, Spin Rate at Classic Tuned Condition.

Figure 10-9: Case 2 – Transient Time Response (first 0.5 seconds) of Transverse-axes Control Torques (T_{cx}, T_{cy}) to a Step Command of 0.1 degrees about the y-axis, Initial tilt of $(\phi_x, \phi_y) = (0, 0)$ deg, Spin rate at the Classic Tuned Condition.

Figure 10-10: Case 3 - Transient Time Response (first 0.5 seconds) of Spin-Axis tilt Angles (ϕ_x, ϕ_y) to a Step Command of 0.1 degrees about the x-axis at 2 seconds, Initial tilt of $(\phi_x, \phi_y) = (5, 0)$ deg, Spin rate at the Classic Tuned Condition.

Figure 10-11: Case 3 - Transient Time Response (first 0.5 seconds) of Spin-Axis Tilt Control Torques (T_{c_i}, T_{c_y}) to a Step Command of 0.1 degrees about the x-axis at 2 seconds, Initial tilt of $(\phi_x, \phi_y) = (5, 0)$ deg, Spin rate at the Classic Tuned Condition.

Figure 10-12: Case 4 - Transient Time Response (first 0.5 seconds) of Spin-Axis Tilt Angles (ϕ_x, ϕ_y) to a Step Command of 0.1 degrees about y-axis at 2 seconds, Initial tilt of $(\phi_x, \phi_y) = (0, 5)$ deg, Spin rate at the Classic Tuned Condition.

Figure 10-13: Case 4 - Transient Time Response (first 0.5 seconds) of Spin-axis tilt Control Torques (T_{cx}, T_{cy}) to a Step Command of 0.1 degrees about the y-axis at 2 seconds, Initial tilt of $(\phi_x, \phi_y) = (5, 0)$ deg, Spin Rate at the Classic tuned condition.

10.2 Rate Sensing Calibration Issues

As demonstrated in the examination of spin control in Chapter 9, the control method had a significant impact on the relationship describing the transverse-axes torques required to hold the spin axis at a particular angle for a given spin rate. This stiffness characteristic is a critical parameter in the external rate-sensing algorithm. For the open-loop system, this relationship is dependent on system mechanical characteristics such as the flexure stiffness and gimbal inertia. Spin control has been chosen to maintain the observed open-loop characteristic. In this section it is demonstrated how the transverse-axes controller must also be modified in order to preserve this desired dynamic stiffness characteristic.

10.2.1 Impact of Nutation Damper Design on Closed-Loop GyroWheel Dynamic Stiffness

In the Section 10.1.3, the simulation responses of the GyroWheel were generated with the Classic Tuning Condition applied. According to the analysis of the linear dynamics model in Chapter 6, it is expected that a small torque will be required to hold the GyroWheel at a non-zero tilt despite the Classic Tuning Condition being met. The case where the GyroWheel was commanded to a tilt of 0.1 degrees from null is ideal for examination of the effect of the transverse controller on the dynamic stiffness characteristic. For tilts of 0.1 degrees or less, the transverse-axes dynamics behave as a linear system.

Figure 10-14 shows the steady state control torques required to hold the GyroWheel at a 0.1-degree tilt about the case-referenced x-axis. Using the non-minimum phase lead design, the amplitude of the control torque oscillations at the $2\omega_s$ frequency is $\pm 1.0 \times 10^{-4} \text{ N}\cdot\text{m}$. To illustrate more clearly the average torques, the control torques are

passed through a fourth-order Butterworth lowpass filter with a cut-off frequency of 10 Hz to attenuate the effect of the flutter at $2\omega_f$ (50 Hz). The resulting filtered control torque response is shown in Figure 10-15. Even with this filter, there is still some evidence of the flutter. The average torques on the x and y-axes are given in Table 10-3 below

Nutation Damper	$T_{\bar{c}_x}$ (N·m)	$T_{\bar{c}_y}$ (N·m)
Lead Comp.	5.04×10^{-7}	-2.43×10^{-7}
Non-min. Phase Lead Comp.	2.55×10^{-7}	-1.10×10^{-7}
Lag Comp.	5.74×10^{-7}	-1.5×10^{-7}

Table 10-3: Steady State Control Torques After Application of 0.1 degree step input on case referenced x or y axes

Based on the linear model and equation (6-44), the expected torques are 5.84×10^{-7} N·m about the x-axis and 0 N·m about the y-axis. The lag design provides the closest match with theory (1.7 percent error on $T_{\bar{c}_x}$), but has almost as large a component of torque on the y-axis. The non-minimum phase lead response differs most with theory having an error of 56 percent on the $T_{\bar{c}_x}$ torque. It is evident that the choice of nutation damper in the transverse-axes control system has a major effect on the average control torque required to hold the GyroWheel at a constant tilt.

There are two potential explanations why the average control torques do not match the values predicted by the linear model:

a) Control Torque Contains a Periodic term at $2\omega_s$

When the spin axis tilt angles reach their commanded values, the transverse axes control torques applied to the GyroWheel consist of both a constant and periodic term at $2\omega_s$. The periodic term occurs because the control torques are based on the measured spin axis tilt angles, which contain the flutter component. The steady state control torques T_{c_x} and T_{c_y} are shown in detail in Figure 10-16. They can be approximated by the following expressions:

$$\begin{aligned} T_{c_x}(t) &= T_{c_{x1}} - T_{c_{x2}} \sin(2\omega_s t - \sigma) \\ T_{c_y}(t) &= T_{c_{y1}} + T_{c_{y2}} \cos(2\omega_s t - \sigma) \end{aligned} \quad (10-17)$$

Noting that the amplitude of the torque flutter is equal on both axes, the above two torque terms can be expressed in the following complex form:

$$T_c(t) = T_{c_{n1}} - jT_{c_{n2}} e^{j(2\omega_s t - \sigma)} \quad (10-18)$$

$T_{c_{n1}}$ and $T_{c_{n2}}$ are defined below:

$$\begin{aligned} T_{c_{n1}} &= T_{c_{x1}} + jT_{c_{y1}} \\ T_{c_{n2}} &= T_{c_{x2}} = T_{c_{y2}} \end{aligned} \quad (10-19)$$

b) Effect of $2\omega_s$ Periodic Term in Complete Transfer Function Model

Ignoring the effect of external rate terms $\omega_b(s)$ and $\bar{\omega}_b(s - 2j\omega_s)$, the complete linear transfer function model of the GyroWheel given in equation (5-120) simplifies to the form:

$$\phi(s) = G_{d1}(s)T_c(s) + G_{d2}(s)\bar{T}_c(s - 2j\omega_s) \quad (10-20)$$

The design of the transverse axes control system has focused only on the first term in the above expression. If a signal such as that described in (10-18) is injected into the first transfer function $G_{d1}(s)$, the periodic component $jT_{c\eta 2}e^{j(2\omega_s t - \sigma)}$ on the right side of equation (10-18) will not shift the steady state component of ϕ . Only the secular term $T_{c\eta 1}$ can cause a change in the steady state value of ϕ . It was for this reason that the second term in the linear model involving the input $T_c(s - 2j\omega_s)$ was initially ignored. However, when the input contains a periodic component at $2\omega_s$, the second term on the right side of equation (10-20) must be considered.

When $\bar{T}_c(s - 2j\omega_s)$ is transformed into the time domain by taking the inverse Laplace transform, it takes the form $\bar{T}_c(t)$:

$$\bar{T}_c(t) = \mathcal{L}^{-1}\{\bar{T}_c(s - 2j\omega_s)\} = e^{2j\omega_s t}\bar{T}_c(t) \quad (10-21)$$

$\bar{T}_c(t)$ can be found by taking the complex conjugate of the right hand side of (10-18) where

$$\bar{T}_c(t) = \bar{T}_{c\eta 1} - jT_{c\eta 2}e^{-j(2\omega_s t - \sigma)} \quad (10-22)$$

Overall the complex input torque $\bar{T}_c$ acting on the transfer function $G_{d2}(s)$, expressed in the time domain is given by the following equation:

$$\bar{T}_c = e^{2j\omega_s t}\bar{T}_c(t) = e^{2j\omega_s t}\bar{T}_{c\eta 1} - jT_{c\eta 2}e^{-j\sigma} \quad (10-23)$$

The first term in (10-23) is a periodic torque at $2\omega_s$, while the second term is an offset torque whose real and complex magnitudes are dependent on the phase shift σ . Expanding out this complex torque into its constituent elements yields the following two real torques:

$$\begin{aligned}\bar{T}_{z_1} &= T_{z_{11}} \cos(2\omega_s t) + T_{z_{12}} \sin(2\omega_s t) - T_{z_{13}} \sin \sigma \\ \bar{T}_{z_2} &= T_{z_{21}} \sin(2\omega_s t) - T_{z_{22}} \cos(2\omega_s t) - T_{z_{23}} \cos \sigma\end{aligned}\quad (10-24)$$

In summary, when $T_z(t)$ contains a periodic component at twice the spin frequency, the term $\bar{T}_c(s - 2j\omega_s)$ will consist of both a periodic term at $2\omega_s$ and an offset torque. In this situation, the dynamic compliance of the tuned-rotor instrument is a function of both $G_{d1}(s)$ and $G_{d2}(s)$.

At low frequencies, the steady state complex tilt angle ϕ is given by the expression:

$$\phi(s) = G_{d1}(0)T_c(0) - G_{d2}(0)\bar{T}_c(-2j\omega_s) \quad (10.25)$$

To examine the relative effects of the terms $T_c(0)$ and $\bar{T}_c(-2j\omega_s)$, consider the steady state gains of the two transfer functions $G_{d1}(s)$ and $G_{d2}(s)$. Evaluating equations (5-123) and (5-124), $G_{d1}(s)$ and $G_{d2}(s)$ take the following form:

$$\begin{aligned}G_{d1}(s) &= \frac{\{s^2 - j[2\omega_s(1-\gamma) - \omega_n]s + (\omega_s\omega_n + (2\gamma-1)\omega_s^2 + \alpha)\}}{I_{\eta}'(s-j\omega_1)(s-j\omega_2)(s-j\omega_3)(s-j\omega_4)} \\ G_{d2}(s) &= \frac{-(\gamma s^2 - 2j\omega_s - \beta)}{I_{\eta}'(s-j\omega_1)(s-j\omega_2)(s-j\omega_3)(s-j\omega_4)}\end{aligned}\quad (10.26)$$

Evaluating $G_{x1}(s)$ and $G_{x2}(s)$ at steady state, the resulting gains $G_{x1}(0)$ and $G_{x2}(0)$ represent compliances and are of the following form:

$$\begin{aligned} G_{x1}(0) &= \frac{\omega_s \omega_n + (2\gamma - 1)\omega_s^2 - \alpha}{I_{\tau}' \omega_1 \omega_2 \omega_3 \omega_4} \\ G_{x2}(0) &= \frac{-\beta}{I_{\tau}' \omega_1 \omega_2 \omega_3 \omega_4} \end{aligned} \quad (10-27)$$

Ignoring the periodic components of the torque inputs $T_z(s)$ and $\bar{T}_z(s - 2j\omega_s)$, $T_z(0)$ and $\bar{T}_z(-2j\omega_s)$ can be found from equations (10-18) and (10-23).

$$\begin{aligned} T_z(0) &= T_{z_{ni}} \\ \bar{T}_z(-2j\omega_s) &= -jT_{z_{ni}} e^{-j\sigma} \end{aligned} \quad (10-28)$$

Therefore the steady-state complex tilt angle can be expressed as

$$\phi(0) = \frac{\omega_s \omega_n + (2\gamma - 1)\omega_s^2 - \alpha}{I_{\tau}' \omega_1 \omega_2 \omega_3 \omega_4} T_{z_{ni}} + \frac{\beta}{I_{\tau}' \omega_1 \omega_2 \omega_3 \omega_4} T_{z_{ni}} e^{-j\sigma} \quad (10-29)$$

This expression can now be used to explain how the different nutation damper designs resulted in different steady state control torques, none matching the expected result from linear theory. The design case with the lag nutation damper will be examined. Figure 10-16 shows a portion of the steady state control torque response shown in Figure 10-15, but over an even narrower window from 9.9 to 9.94 seconds. For this window of time and with the frequency $2\omega_s = 49.988\text{Hz}$, a complete cycle of $\sin 2\omega_s t$ or $\cos 2\omega_s t$ occurs at 9.922 seconds. It can be seen that for the lag design, $T_{z_{ni}}$ is almost in phase with $-\sin 2\omega_s t$ and $\bar{T}_{z_{ni}}$ is almost in phase with $\cos 2\omega_s t$. The lead and non-minimum phase

lead designs have slightly larger phase shifts. This figure also shows that the amplitude of the control torque oscillation is smallest for the lag design. This occurs because of the steeper roll-off of the nutation damper at high frequencies as shown in Figure 10-3. Beyond the nutation frequency, the fourth order lag attenuates the magnitude more severely than the lead or non-minimum phase lead designs.

Based on Figure 10-15 and Figure 10-16, the relevant parameters describing the steady state control torque for the lag design are

$$\begin{aligned}T_{z_{ii}} &= 5.74 \times 10^{-7} \text{ N} \cdot \text{m} \\T_{z_{vi}} &= 1.5 \times 10^{-7} \text{ N} \cdot \text{m} \\T_{z_{iz}} = T_{z_{vz}} &= 4.1 \times 10^{-5} \text{ N} \cdot \text{m} \\\sigma &= 5.4 \text{ deg}\end{aligned}$$

Evaluating equation (10-29) based on the GyroWheel characteristics, the resulting complex angle $\phi(0)$ is

$$\phi(0) = 1.716 \times 10^{-3} - j4.60 \times 10^{-6} \text{ rad}$$

Expressed in terms of the two real components, the spin axis tilt expressed in degrees is

$$\begin{aligned}\phi_x &= 1.007 \times 10^{-1} \text{ deg} \\\phi_y &= 2.000 \times 10^{-4} \text{ deg}\end{aligned}$$

These tilt angles are very near the tilt of $\phi_x = 0.1 \text{ deg}, \phi_y = 0.0 \text{ deg}$ observed in the simulation.

If the second term in equation (10-29) is ignored, the resulting complex angle $\phi(0)$ is

$$\phi(0) = 1.75 \times 10^{-3} - j4.48 \times 10^{-4} \text{ rad}$$

Expressed in terms of the two real components, the spin axis tilt expressed in degrees is

$$\phi_x = 0.098 \text{ deg}$$

$$\phi_y = -0.026 \text{ deg}$$

There is a 2 percent error on the x-axis and an error of 26 percent of the command on the y-axis, introduced by the control system.

This example illustrates how the Complete model of a tuned-rotor instrument can be used to more accurately explain the behaviour of the system. The simulations have shown how various control systems produce different magnitudes of jitter torque $T_{x,2}$ and phase shift σ . The example has shown how these terms have an impact on the dynamic stiffness of the system. The following section will address how the transverse axis control system can be modified to reduce the effect of the gimbal flutter torque on the dynamic stiffness of the system.

Figure 10-14: Case 1 – Steady State Time Response (last 0.1 seconds) of Transverse-axes Control Torques (T_{c_x}, T_{c_y}) to a Step Command of 0.1 degrees about the x-axis, Initial tilt of $(\phi_x, \phi_y) = (0, 0)$ deg, Spin Rate at the Classic Tuned Condition.

Figure 10-15: Case 1 – Filtered Steady State Time Response (last 0.1 seconds) of Transverse-axes Control Torques (T_{cx}, T_{cy}) to a Step Command of 0.1 Degrees About the x-axis, Initial Tilt of $(\phi_x, \phi_y) = (0, 0)$ deg, Spin rate at the Classic Tuned Condition.

Figure 10-16: Case 1 – Filtered Steady State Time Response (last 0.1 seconds) of Spin-axis tilt Angles (ϕ_x, ϕ_y) to a Step command of 0.1 degrees About the x-axis, Initial tilt of $(\phi_x, \phi_y) = (0,0)$ deg, Spin rate at classic tuned condition

10.2.2 Improving Rate Sensing Calibration Through Notch Filtering

The preceding discussion illustrated how the twice-spin-frequency dynamics in the GyroWheel are a significant term when calibrating the system dynamic stiffness. The degree to which the gimbal flutter affected dynamic stiffness was found to be dependent on the nature of the transverse-axis controller chosen. In addition, each transverse axis controller design introduced some cross-coupling in the stiffness characteristic, i.e. tilting the spin axis about one transverse axis requires torques from both torquers.

It is desirable to make the dynamic stiffness of the system insensitive to the transverse axis controller design. To do this, the effect of gimbal flutter at the frequency of $2\omega_s$, must be limited within the transverse axis control system. The approach adopted here is to apply a notch filter at $2\omega_s$ on the sensed tilt angles. The only disadvantage of such an approach is that the spin frequency ω_s varies during the GyroWheel operation. Consequently, the notch frequency must be varied based on the measured spin rate.

A simple notch filter of the following form was introduced in series with the precession and nutation damper terms as shown in Figure 10-17.

$$G_{notch}(s) = \frac{s^2 + 4\omega_s^2}{s^2 + \frac{2\omega_s}{Q_{notch}} - 4\omega_s^2}$$

The only parameter that must be set is the attenuation factor Q_{notch} . As Q_{notch} is lowered, the frequency band over which the notch attenuates the signal becomes greater. If the intention is for the notch to function over a range of spin rates, then a lower Q_{notch} is desirable, however the phase shift will occur over a broader band. Conversely, a very high Q_{notch} will put a premium on accurate spin rate sensing but not affect the nutation

damper design as much. Figure 10-18 shows the effect of Q_{notch} on the magnitude and phase response of $G_{notch}(s)$ for a normalized notch frequency of 1.

Figure 10-17: Transverse-axes Control System Using Notch Filter Shown in Complex Form

Of interest is the effect the notch filter has on the nutation damper designs. Given that the nutation frequency is roughly $\frac{I_{r_1}}{I_t} \omega_s$, the ratio of $\frac{2\omega_s}{\omega_n}$ is $\frac{2I_{r_1}}{I_t}$. Applying the

GyroWheel parameters, $\frac{2I_{r_1}}{I_t}$ is 1.078..

With the nutation frequency at a normalized frequency of 0.9275, a negative phase shift of 19 degrees occurs at ω_n even with Q_{notch} set at 20. However, with this value of Q_{notch} , an attenuation of at least 20 dB occurs over a very narrow frequency band of ± 0.25 percent of the notch frequency. This appears to make the approach overly sensitive to system parameter variation. Decreasing the Q_{notch} down to 5 increases the negative phase shift to -55 degrees, but widens the band over which the notch provides 20 dB attenuation to ± 1.0 percent of the notch frequency. In terms of ensuring sufficient

attenuation of the $2\omega_n$ terms, this is a better option. However, the desired phase shift at the nutation frequency will be noticeably affected, and the nutation damper design will have to be modified.

Figure 10-18: Frequency Response of Simple Notch Filter for Varying Q_{notch}

10.2.3 Modified Nutation Damper Design

The nutation damper acts to stabilize the nutation dynamics by adjusting the phase at the nutation frequency. Introduction of a notch filter with a moderate $Q_{notch} = 5$ modifies the phase at ω_n by -55 degrees. A phase shift of +55 degrees must be made in each of the nutation damper designs to account for this. The feasibility of making this adjustment for each of the three designs is discussed below.

a) Option 1: Lead Compensator

Examining the open loop frequency response of the transverse-axes control designs in Figure 10-3, it can be seen that the phase angle of the response of the lead design is at an optimum -90 degrees at the nutation frequency. Given the structure of the compensator, the -180 degrees phase shift that occurs just prior to the nutation frequency cannot be adjusted to a higher value.

$$G_c(s) = \frac{(1.25s - j\omega_n)}{(0.1s + \omega_n)} \quad (10-30)$$

The phase at the nutation frequency will have to remain at -145 degrees. The alternative is to employ a notch filter with a higher Q_{notch} . For the purposes of examining the rate sensing calibration of the dynamic stiffness and testing the effect of the notch filter, the same notch filter and lead compensator will be used as above.

b) Option 2: Non-minimum Phase Lead Compensator

As described in Chapter 7, the non-minimum phase lead cannot provide the optimum 180-degree phase shift just prior to the nutation frequency. The phase

shift is only -135 degrees. The addition of the notch filter means that the combined phase shift of the nutation damper and notch filter at the nutation frequency is now -190 degrees. This results in a phase angle of -100 degrees at the nutation frequency, which is more than adequate. Therefore, the addition of the notch filter should actually improve the response of the system when the nutation damper is a non-minimum phase lead.

c) **Option 3: Lag Compensator**

The lag compensator achieved a -180 degree phase shift at the nutation frequency through the combined effect of four simple lags. The phase shift introduced by the damper nutation must be adjusted by -55 degrees. This is accomplished by shifting the lag cut-off frequency from ω_n to a somewhat higher frequency.

After adjusting introducing the notch filter and adjusting the lag design cut-off frequency, the modified open loop frequency response of the transverse-axis control loop is shown in Figure 10-19. Both the lag and non-minimum phase designs have phase margins of approximately 90 degrees at the second gain crossover near the nutation frequency. The lead design has a poor phase margin of only 35 degrees.

Figure 10-19: Open Loop Frequency Response of Transverse-axes Control System for 3 Nutation Damper Designs, Final Selection of Precession Controller, Modified Nutation Damper Designs with Notch Filter, GyroWheel operating at New Tuned Speed

10.2.4 Modified Transverse-Axes Response

Figure 10-20 and Figure 10-22 show the spin-axis tilt response of the GyroWheel to step commands of 0.1 degrees on the x or y axis, with the system initially at zero tilt and the spin rate at the classic tuned speed. The responses are similar to those shown in Figure 10-4 and Figure 10-8 where the notch filter was not used. The main difference between the two responses is the poorer nutation damping when the notch filter is active, particularly for the lead-type nutation damper. It is evident that the lead compensator design would perform better using a notch filter with increased Q_{notch} .

Of interest is the steady state torque observed about the case-referenced x and y-axes for simulation cases 1 and 2 with the notch filter active. Figure 10-21 and Figure 10-23 show the steady state torques T_{c_x} and T_{c_y} for simulation cases 1 and 2. The expected torques (T_{c_x}, T_{c_y}) are $(5.839 \times 10^{-7}, 0) \text{ N}\cdot\text{m}$ for a 0.1 deg step input on the case-referenced x-axis and $(0, 5.839 \times 10^{-7}) \text{ N}\cdot\text{m}$ for a 0.1 degree step input on the case referenced y-axis. The average torques observed in the simulations are summarized in Table 10-4 below:

Nutation Damper Design	0.1 deg. step input on x T_{c_x}, T_{c_y} (Nm)	0.1 deg. step input on y T_{c_x}, T_{c_y} (Nm)
Lead Compensator	$5.832 \times 10^{-7}, -5.173 \times 10^{-10}$	$5.176 \times 10^{-10}, 5.832 \times 10^{-7}$
Non-minimum Phase Lead Compensator	$5.829 \times 10^{-7}, 3.417 \times 10^{-10}$	$-3.416 \times 10^{-10}, 5.829 \times 10^{-7}$
Lead Compensator	$5.832 \times 10^{-7}, -1.051 \times 10^{-10}$	$1.052 \times 10^{-10}, 5.832 \times 10^{-7}$

Table 10-4: Steady State Control Torques After Application of 0.1 degree step input on case referenced x or y axes, Notch Filter Included in Design

The errors with respect to the expected torques are now of the order of 0.1 percent or less. They are well within the tolerable errors for external rate sensing. The notch filter has had the desired effect.

Figure 10-20: Case 1 – Transient Time Response (first 0.5 seconds) of the Spin-axis Tilt Angles (ϕ_x, ϕ_y) to a Step Command of 0.1 degrees About the y-axis, Initial tilt of $(\phi_x, \phi_y) = (0, 0)$ deg, Spin Rate at the Classic Tuned Condition.

Figure 10-21: Case 1 – Steady State Time Response (last 0.1 seconds) of the Transverse-axes Control Torques (T_{c_x}, T_{c_y}) to a Step Command of 0.1 degrees about the y-axis, Initial tilt of $(\phi_x, \phi_y) = (0, 0)$ deg, Spin Rate at the Classic Tuned Condition.

Figure 10-22: Case 2 – Transient Time Response (first 0.5 seconds) of the Spin-axis Tilt Angles (ϕ_x, ϕ_y) to a Step Command of 0.1 degrees about the y-axis, Initial tilt of $(\phi_x, \phi_y) = (0, 0)$ deg, Spin Rate at the Classic Tuned Condition.

Figure 10-23: Case 2 – Steady State Time Response (last 0.1 seconds) of the Transverse-axes Control Torques (T_{c_x}, T_{c_y}) to a Step Command of 0.1 Degrees about the y-axis, Initial Tilt of $(\phi_x, \phi_y) = (0, 0)$ deg, Spin Rate at the Classic Tuned Condition.

10.3 The Effect of Model Variation and Uncertainty

To this point in the discussion on GyroWheel control, the system dynamics have been considered tuned and the control designs have been examined using the standard classical tools of root locus and frequency response techniques. Stability margins have been examined in relation to this fixed plant. In fact, there are a number of known parameters whose variation will affect the transverse-axes dynamics. The essential parameters are:

- a) Spin rate
- b) Torquer scale factor
- c) Tilt sensor scale factor

The torquer scale factor and tilt sensor scale factor are considered relatively constant system characteristics, while the spin rate is a controlled variable in the spin rate control system. Through the linearization of the overall tuned-rotor instrument dynamics, the spin dynamics becomes decoupled from the transverse-axis dynamics. Spin rate is then considered a parameter within the transverse-axis dynamics.

Of these parameters, the spin rate is known to operate with the greatest variation. The spin rate varies from 1300 to 1700 rpm, a ± 15 percent variation from the nominal 1500-rpm spin rate. The variation in spin rate directly affects the nutation mode, which is stabilized by the nutation damper. The anticipated worst-case variation in torquer scale factor is illustrated in Figure 11-1. It is likely that the engineering model design of the torquer will have an almost constant torque scale factor over the tilt range of the transverse-axes.

A formal analysis of robust stability and performance is not performed in this study. This is left for future work. However, methods for coping with the variation in spin rate and its effect on the transverse axes dynamics are discussed in the following section.

10.4 Implementation Issues

The transverse-axes control system is divided into three serial elements: precession controller, nutation damper and notch filter. The precession term controls the low frequency behaviour of the transverse-axes and is insensitive to the anticipated changes in spin rate. However, the nutation damper and notch filter provide phase compensation or amplitude suppression at frequencies related directly to the spin rate. Of these two filters, the effectiveness of the notch filter is particularly sensitive to spin rate variations as discussed in Section 10.2.3.

To address this issue it is necessary to implement the transverse-axes controller in a digital form. In this form, there are at least two approaches, which could be applied to address the variable spin rate issue:

a) **Digital Control System With Fixed Update Period**

The precession controller, nutation damper and notch filter parameters can be varied as a function of the measured spin rate. This adds some complexity to the control system implementation.

b) **Digital Control System With Update Slaved to Spin Rate**

The precession controller, nutation damper and notch filter parameters can be chosen to remain fixed. However, if the controller update is triggered n times for every motor shaft revolution, then the filter cut-off and notch frequencies will change proportionally with the spin rate. This approach is a simple yet effective method of implementing the control system given the present form of the tilt sensor. The tilt sensor outputs a measurement 4 times per revolution of the rotor.

If the update of the transverse-axis control system is triggered after each tilt measurement, the filter cut-off and notch frequencies will vary with spin rate.

This can be illustrated with a simple example. Consider the following lag filter:

$$y(s) = \frac{\omega}{s + \omega} u(s)$$

Using the zero-order hold equivalent for an update period of T , the discrete form of this transfer function is

$$y(z) = \frac{(1 - e^{-\omega T})}{(z - e^{-\omega T})} u(z)$$

For a filter cut-off frequency of 10 radians/second and an update period of 0.01 seconds, the response of the filter to a unit step input is shown in Figure 10-24. Figure 10-24 shows the response of this simple system to a step input for the nominal update frequency of 100 Hz, an update frequency 50 percent less (50 Hz) and an update frequency 50 percent more (150 Hz). When the update frequency drops from 100 Hz to 50 Hz, the time constant of the response doubles, which is equivalent to a halving of the effective cut-off frequency. A doubling of the update frequency results in a halving of the response time constant or a doubling of the effective cut-off frequency.

Given the present tilt sensor concept, the fixed controller with the update period slaved to the spin frequency is the more desirable approach of these two implementations,. This approach was not tested and is left as a topic for future work.

Figure 10-24: Effect of Update Period on the Response of Simple Lag Filter Implemented Using the Zero Order Hold Equivalent, Cut-off Frequency Set at 10 radians/second Using 0.01 sec Update Period, Cut-off Frequency Set at 10 radian/second.

10.5 Characterization of GyroWheel Non-Linear Dynamic Stiffness

The approach for external rate sensing proposed in Chapter 8 relies on estimates of each of the factors influencing the control torques. Part of this knowledge is an accurate characterization of the control torque required to counter precession and hold the de-tuned system at a constant tilt. This stiffness, termed dynamic stiffness (K_{dyn}) in this study has been found for the linear model and is given in equation (6-84). However, simulation of the open loop GyroWheel in Section 9.1 showed that the precession frequency decreases gradually as the tilt of the system is increased. Given that the linear dynamic stiffness is a function of the precession frequency as shown in equation (10-1), this indicates that the non-linear dynamic stiffness will also decrease as tilt increases.

The objective in this study was to characterize this stiffness as a function of tilt angle and spin rate. Three options were considered:

a) Determine Precession Frequency over a range of tilts and spin rates

This method involves initializing the GyroWheel tilt and spin rate at a particular value in the non-linear simulation model and then observing the resulting precession and its frequency. The dynamic stiffness can then be determined from the precession frequency. These simulation tests would be performed with the spin axis control system active. The disadvantage of such an approach is that with the very low precession periods (20000 seconds), a tremendous amount of simulation time would be required to properly characterize the system.

b) Apply a range of transverse-axes torques and spin rates

The transverse-axes angles at which the system settles for the applied torque can be used to determine dynamic stiffness. These simulation tests would be performed with the spin axis control system active but no transverse-axes control system present. The disadvantage here is similar to option a). The presence of the very low precession frequency means that it would take a long time to reach the steady state angle. In addition, with no damping in the gimbals, the system would continue to oscillate at the precession frequency about this angle. This would make it difficult to judge the final angle without performing the simulation for a number of precession periods. Although adding significant gimbal damping would make the system settle more quickly, it has the undesirable effect of introducing changes to the dynamic stiffness as found in the linear analysis of Section 6.4.

c) With the transverse-axes controller active, measure the torque required to hold the transverse-axes at a series of commanded angles.

The system can be commanded to a number of case-referenced angles and the control torque required to hold the system at this angle recorded. Stiffness curves as a function of tilt angle and spin rate can then be generated. The advantage of this approach is that each of the required simulations can be performed over a period of a few seconds rather than tens of thousands of seconds as in options a) and b) above. The disadvantage of this approach is that there is the potential for the transverse-axes control system to introduce additional effects, which alter the torque required to hold the transverse-axes at a particular tilt. This issue has been addressed in Section 10.2 through the introduction of notch filters. A second disadvantage is the fact that the spin rate must also be commanded at a number of values other than the tuned speed. For each of these spin rates, the notch filter

parameters in the simulation must be adjusted to account for the tuned-rotor instrument spin rate.

Of these methods, the third option is the most desirable. The effort required to adjust the notch filter and nutation damper parameters was small compared with the inconvenience of running simulations for many days to characterize the system using options a) or b).

Using option c), simulations were performed for the following conditions:

- a) Step commands of the following tilt angles were made to the transverse—axes control system:

$$\phi_{ref,i} = [0.1 \ 0.2 \ 0.5 \ 1.0 \ 1.5 \ 2.0 \ 2.5 \ 3.0 \ 3.5 \ 4.0 \ 4.5 \ 5.0] \text{ deg}$$

For each tilt case, the transverse axes control torques (T_{c_x}, T_{c_y}) were determined from the simulation response.

- b) For each of the tilt cases, the tests were performed with motor spin control active. They were then repeated with rotor spin control enabled.
- c) The spin control bandwidth (ω_{spin}) was set at each of the following values to determine what effect it had on the dynamic stiffness characteristic:

$$\omega_{spin} = [1 \ 3 \ 10 \ 30 \ 100 \ 300] \text{ Hz}$$

- d) Simulations were performed for five seconds to ensure that the system settled.

- e) Like the analysis of the control torques in Section 10.2, the resulting case referenced torques were passed through a low-pass filter to remove the flutter component.
- f) The lag nutation damper design was used.

10.5.1 Dynamic Stiffness Using Motor Spin Control

Figure 10-25 and Figure 10-26 show the relationship between the control torques $(T_{\epsilon_x}, T_{\epsilon_y})$ and the case-referenced tilt angle ϕ_x for the four spin control bandwidth cases. Figure 10-27 and Figure 10-28 show the deviation of the torques T_{ϵ_x} and T_{ϵ_y} from that predicted by the linear model. The following observations can be made:

- a) The curve characterizing the control torque T_{ϵ_x} versus the tilt angle θ_{ϵ_x} appears to be a cubic relationship.
- b) The torque T_{ϵ_x} required to hold the rotor at 5 degrees tilt varies between $1.0 \times 10^{-4} \text{ N} \cdot \text{m}$ with a 1 Hz spin control bandwidth and $2.5 \times 10^{-2} \text{ N} \cdot \text{m}$ with 300 Hz spin control bandwidth. The latter value is approximately 1000 times larger than that expected with the linear model.
- c) An equally large torque about the case-referenced y-axis is required to hold the rotor at a tilt about the x-axis. At five degrees tilt, the required torque T_{ϵ_y} ranges from $5.0 \times 10^{-4} \text{ N} \cdot \text{m}$ for 1 Hz bandwidth and $1.15 \times 10^{-2} \text{ N} \cdot \text{m}$ with 30 Hz bandwidth.

- d) The curves can also be used to illustrate why as reported in Section 9.2.1, the direction of precession switches at approximately 0.2 degrees when using the non-linear dynamics model. In Figure 10-29, the stiffness curve is shown over the first 0.5 degrees tilt for the 30 Hz spin control bandwidth case. It can be seen how the T_{z_i} versus ϕ_x curve crosses the zero torque axis just beyond 0.2 degrees. It is for this reason that the spin axis precesses in a clockwise direction when the tilt is less than 0.2 degrees, but in a counter-clockwise direction when the tilt is greater than this value.

These results further illustrate why motor spin control is inappropriate for tuned-rotor instruments with large tilt angles (> 1 degree). If the spin control bandwidth is set at suitable levels of 10 Hz or higher, significant torques are required to hold the spin axis at non-zero tilts. This would have an adverse effect on the power budget of the GyroWheel. In addition, with such wide variation in torque versus tilt, it will be harder to accurately characterize this relationship during hardware calibration.

Figure 10-25: Simulated Steady State Transverse x-axis Control Torque (T_{c_x}) as a Function of Case-Referenced x-axis Tilt (ϕ_x), Motor Spin Control with Bandwidth varied from 1 to 300 Hz

Figure 10-26: Simulated Steady State Transverse y-axis Control Torque (T_{cy}) as a Function of Case-Referenced x-axis Tilt (ϕ_x), Motor Spin Control with Bandwidth varied from 1 to 300 Hz.

Figure 10-27: Difference Between Simulated Steady State Transverse x-axis Control Torque (T_{c_i}) and Predicted Linear Control Torque (T_{c_i}) as a Function of Case-Referenced x-axis Tilt (ϕ_x), Motor Spin Control with Bandwidth varied from 1 to 300 Hz.

Figure 10-28: Difference Between Simulated Steady State Transverse y-axis Control Torque (T_{cy}) and Predicted Linear Control Torque (T_{cy}) as a function of case-reference x-axis tilt (ϕ_x), Motor Spin Control with Bandwidth varied from 1 to 300 Hz.

Figure 10-29: Simulated Steady State Transverse x and y-axis Control Torques (T_{c_x}, T_{c_y}) as a function of small case-reference x-axis tilts (ϕ_x) with Motor Spin Control, Spin Control Bandwidth varied from 1 to 300 Hz

10.5.2 Dynamic Stiffness Using Rotor Spin Control

Figure 10-30 and Figure 10-31 show the control torques $T_{\dot{\epsilon}_x}$ and $T_{\dot{\epsilon}_y}$ respectively as a function of the tilt angle ϕ_x for varying spin control bandwidths. The following observations can be made:

- a) The spin control bandwidth has little or no effect on the torque versus tilt curves. This is likely due to the fact that with no spin or transverse-axes controllers active, the rotor has little observable flutter oscillation in its spin rate compared to the motor shaft. Therefore, increasing the rotor spin control bandwidth to suppress the effect of gimbal flutter torque has little effect.
- b) The torque $T_{\dot{\epsilon}_x}$ obtained from the non-linear model match the value predicted by the linear model for tilt angles of up to 0.5 degrees. Beyond this tilt, the error increases significantly. At 2.5 degrees, the torque reaches a local maximum. Beyond this value, the torque actually decreases with tilt, reaching zero at 4.2 degrees. For further increases in tilt beyond this value, the control torque $T_{\dot{\epsilon}_x}$ becomes a larger negative value. This explains why the direction of precession appears to switch directions at approximately 4 degrees in Figure 9-21 when the transverse axis control system is not active and rotor control is used.
- c) Although the control torque $T_{\dot{\epsilon}_y}$ takes on non-zero values with increasing tilt, the magnitude of this torque is small compared to the critical level of $5.0 \times 10^{-7} \text{ N} \cdot \text{m}$ identified in the external rate sensing analysis in Section 8.
- d) The difference between the torque $T_{\dot{\epsilon}_x}(\phi_x)$ found with the non-linear simulation and that determined with the linear model is plotted in Figure 10-32. The

deviations with respect to the linear model are much smaller than that observed when motor spin control is used.

- e) As given in equation (6-44), the relation developed to relate the complex tilt angle to the complex torque required to hold the spin axis at this tilt is the following:

$$T_c = \omega_p \cdot h \phi$$

This relationship still appears to hold for the non-linear dynamics when rotor spin control is applied. For example, the torque required to hold the transverse-axes dynamics at 5 degrees is $-1.305 \times 10^{-5} \text{ N} \cdot \text{m}$. For a rotor angular momentum of $1.005 \text{ N} \cdot \text{m} \cdot \text{sec}$, the expected frequency is $-2.35 \times 10^{-5} \text{ Hz}$ or a precession period of 42,500 seconds. This is very close to the 42,600 second period observed in the free responses shown in Figure 9-7.

Figure 10-30: Simulated Steady State Transverse x-axis Control Torque (T_{cx}) as a Function of Case-Referenced x-axis tilt (ϕ_x), Rotor Spin Control with Bandwidth varied from 1 to 300 Hz

Figure 10-31: Simulated Steady State Transverse y-axis Control Torque (T_{cy}) as a Function of Case-referenced x-axis tilt (ϕ_x), Rotor Spin Control with Bandwidth varied from 1 to 300 Hz.

Figure 10-32: Difference Between Simulated Steady State Transverse x-axis Control Torque (T_{c_x}) and Predicted Linear Control Torque (T_{c_x}) as a function of Case-referenced x-axis tilt (ϕ_x), Motor Spin Control with Bandwidth varied from 1 to 300 Hz.

10.5.3 General Non-Linear Dynamic Stiffness Function

To derive a more general non-linear stiffness function, the effect of spin rate was also examined. To determine the dependency of dynamic stiffness on both tilt angle and spin rate, simulations were performed with the following conditions:

- a) Based on the results of the previous two sections, only rotor spin control was considered. The bandwidth of the rotor spin control system was set at 30 Hz, with control parameters set accordingly.
- b) Step commands of the following tilt angles were made to the transverse-axes control system:

$$\phi_{ref_i} = [0.1 \ 0.2 \ 0.5 \ 1.0 \ 1.5 \ 2.0 \ 2.5 \ 3.0 \ 3.5 \ 4.0 \ 4.5 \ 5.0] \text{ deg}$$

For each tilt case, the transverse axes control torques (T_{c_i}, T_{c_y}) were determined from the simulation response.

- c) The simulations were repeated for each of the following spin rates

$$\dot{\phi}_{ref_z} = [1300 \ 1350 \ 1400 \ 1450 \ 1499.6 \ 1550 \ 1600 \ 1650 \ 1700] \text{ rpm}$$

- d) Simulations were performed for five seconds to ensure that the system settled.
- e) Like the analysis of the control torques in Section 10.2, the case referenced torques responses generated from simulation were passed through a low-pass filter to remove the flutter component.

- f) The lag nutation damper design was used, with the notch filter and nutation damper parameters adjusted for each of the 9 spin rate cases.

To better illustrate the effect of the non-linear dynamics, the difference between the expected linear torque and that obtained from the non-linear dynamics model was plotted as a function of tilt angle and spin rate and is shown in Figure 10-33. The result is a family of curves for each of the spin rates. Each of these curves can be fitted using a cubic polynomial of the form:

$$T_z = k_0(\omega_s) - k_1(\omega_s)\phi + k_2(\omega_s)\phi^2 - k_3(\omega_s)\phi^3 \quad (10-37)$$

Each of the coefficients k_i is a function of the spin rate ω_s and can be determined using a least squares fit. The resulting coefficients are plotted as a function of spin rate in Figure 10-34. The magnitude of the k_0 coefficient is sufficiently small that it can be ignored. As a result, each of the non-linear torque profiles shown in Figure 10-33 can be characterized by the equation:

$$T_z = [k_1(\omega_s) - k_2(\omega_s)\phi + k_3(\omega_s)\phi^2]\phi$$

Therefore, the non-linear portion of the dynamic stiffness is given by the following equation:

$$K_{dyn} = k_1(\omega_s) + k_2(\omega_s)\phi + k_3(\omega_s)\phi^2 \quad (10-38)$$

Re-examining Figure 10-34, the coefficients k_1 , k_2 and k_3 exhibit a near linear profile. Upon closer examination, they are best fitted as parabolas of the following form:

$$k_i(\omega_s) = k_{i0} + k_{i1}\omega_s + k_{i2}\omega_s^2 \quad (10-39)$$

Alternatively, $k_i(\omega_s)$ can be expressed in terms of the spin rate and tuned speed where

$$k_i(\omega_s) = k_{i0} + k_{i1}(\omega_s - \omega_{s_{tune}}) + k_{i2}(\omega_s - \omega_{s_{tune}})^2 \quad (10-40)$$

Based on the curves shown in Figure 10-34, the values of the coefficients k_{i0} , k_{i1} and k_{i2} are given below:

$$\begin{bmatrix} k_{10} & k_{11} & k_{12} \\ k_{20} & k_{21} & k_{22} \\ k_{30} & k_{31} & k_{32} \end{bmatrix} = \begin{bmatrix} 1.264 \times 10^{-13} \frac{\text{N} \cdot \text{m}}{\text{rad}} & 1.543 \times 10^{-9} \frac{\text{Nm} \cdot \text{sec}}{\text{rad}} & -1.156 \times 10^{-6} \frac{\text{Nm} \cdot \text{sec}^2}{\text{rad}} \\ -2.881 \times 10^{-11} \frac{\text{N} \cdot \text{m}}{\text{rad}} & -2.456 \times 10^{-9} \frac{\text{Nm} \cdot \text{sec}}{\text{rad}} & 5.978 \times 10^{-6} \frac{\text{Nm} \cdot \text{sec}^2}{\text{rad}} \\ 3.195 \times 10^{-8} \frac{\text{N} \cdot \text{m}}{\text{rad}} & 2.277 \times 10^{-8} \frac{\text{Nm} \cdot \text{sec}}{\text{rad}} & -7.752 \times 10^{-3} \frac{\text{Nm} \cdot \text{sec}^2}{\text{rad}} \end{bmatrix}$$

The overall dynamic stiffness is a combination of the linear stiffness given in equation (6-44) and the non-linear contribution given in equation (10-38). The resulting dynamic stiffness is

$$\begin{aligned} K_{dyn}(\phi, \omega_s) = & (K - J\omega_s^2) + \frac{K^2}{2\omega_s^2(I_{r_z} - 2I_{r_y} - I_{g_t})} - \frac{K \cdot J}{I_{r_z} - 2I_{r_y} - I_{g_t}} \\ & + \left[k_{10} + k_{11}(\omega_s - \omega_{s_{tune}}) + k_{12}(\omega_s - \omega_{s_{tune}})^2 \right] \\ & + \left[k_{20} + k_{21}(\omega_s - \omega_{s_{tune}}) + k_{22}(\omega_s - \omega_{s_{tune}})^2 \right] \phi \\ & + \left[k_{30} + k_{31}(\omega_s - \omega_{s_{tune}}) + k_{32}(\omega_s - \omega_{s_{tune}})^2 \right] \phi^2 \end{aligned} \quad (10-41)$$

The expression for stiffness is given in terms of the complex case-referenced tilt angle ϕ , rather than the two individual tilt angles ϕ_x and ϕ_y . If the torques $(T_{\dot{x}}, T_{\dot{y}})$ were

required to hold the tilt at the angles (ϕ_x, ϕ_y) , then the following procedure should be applied.

- i) Determine the magnitude of the tilt angle ϕ

$$\phi = \sqrt{\phi_x^2 + \phi_y^2}$$

- ii) Obtain the torque T_c

$$T_c = K_{\phi m}(\omega_s, \phi) \phi$$

- iii) Determine the individual case-referenced torques

$$T_{c_x} = \frac{\phi_x}{\phi} T_c$$

$$T_{c_y} = \frac{\phi_y}{\phi} T_c$$

Figure 10-33: Difference Between Steady State Transverse x-axis Control Torque (T_{c_x}) Obtained with Simulation and the Predicted Linear Control Torque (T_{c_x}) as a function of case-reference x-axis tilt (ϕ_x) and rotor spin rate, Rotor Spin Control with Bandwidth at 30 Hz

Figure 10-34: Non-Linear Dynamic Stiffness Coefficients as a Function of Spin Rate

10.6 Summary

Through dynamic simulation and linear analysis of the complete linear model, it has been demonstrated that the transverse-axes control system can interact with the twice spin frequency dynamics to alter the dynamic stiffness of the GyroWheel. Introduction of a notch filter at twice the spin frequency allows the GyroWheel dynamic stiffness to be independent of the transverse-axes control design. It was recommended that the transverse-axes control system should be implemented as a digital controller with the update period slaved to the spin rate. This addresses the requirement to vary the notch filter frequency and nutation damper parameters as a function of spin rate.

Based on both transverse-axes control and external rate sensing considerations, the lag design appears to be the best of the three nutation-damper design options. It can accommodate the inclusion of the notch filter in a superior manner to the other two design options. The peak transient control torque is smallest with the lag nutation damper, which should aid external rate sensing. Finally, with the inclusion of the notch filter, the measured control torques when holding at a 0.1 degree tilt angle match that predicted by the open loop dynamics.

The GyroWheel transverse-axes control system described in this chapter has been used to characterize the non-linear dynamic stiffness of the system. An expression for dynamic stiffness as a function of spin rate and spin-axis tilt was developed for the purposes of external rate sensing.

Finally, the SCISCO transfer function has again proved valuable in providing insight in analysis and controller design, which would not otherwise be seen with more traditional methods.

Chapter 11

GyroWheel External Rate Sensing Design

The external rate sensing capability in a tuned-rotor gyroscope relies on the observation that the torque required to hold the gyro rotor at null is proportional to the external rates of the body on which the gyro sits. By measuring the torque applied by the rebalance coils in a tuned rotor gyro, external rates can be obtained. In Chapter 9, it was demonstrated that if a tuned-rotor instrument is operated as a steerable momentum wheel, external rates could still be obtained. The estimated external rates now become a function not only of the torque required to track the momentum demand, but also the commanded spin rate, tilt angles and rate of change of these tilt angles.

The following chapter addresses the issues involved in implementing an external rate-sensing algorithm in a tuned-rotor instrument such as the GyroWheel. An algorithm for estimating the external rates is then proposed. The resulting performance of the GyroWheel external rate sensing capability is demonstrated with two examples.

11.1 Implementation of External Rate Sensing in a Tuned-Rotor Instrument

As first derived in equation (8-21), the external rates can be determined from the two equations presented below:

$$\begin{aligned}\omega_{\dot{x}_1} &= \frac{1}{h_z} \left\{ -K_{\tau q} i_{\omega \dot{x}_1} - 2c'_z \omega_s \phi_{ref_1} - \tilde{K}_{dyn} \dot{\phi}_{ref_1} - (I_{r_1} + I_{s_1}) \omega_s \dot{\phi}_{ref_1} - c'_s \dot{\phi}_{ref_1} \right\} \\ \omega_{\dot{y}_1} &= \frac{1}{h_z} \left\{ K_{\tau q} i_{\omega \dot{y}_1} - 2c'_z \omega_s \phi_{ref_1} - \tilde{K}_{dyn} \dot{\phi}_{ref_1} - (I_{r_1} + I_{s_1}) \omega_s \dot{\phi}_{ref_1} - c'_s \dot{\phi}_{ref_1} \right\}\end{aligned}\quad (11-1)$$

The external rate estimate is dependent on three types of terms: system parameters, measured signals and GyroWheel command inputs. Therefore, to implement the external rate-sensing concept, the following aspects must be considered: characterization of GyroWheel system model, measurement and filtering of GyroWheel dynamic signals and command shaping and latency.

11.1.1 Characterization of System Model

There are four essential parameters, which must be characterized for the purposes of the external rate-sensing algorithm:

- i) Gimbal and rotor inertias $(I_{r_1}, I_{r_2}, I_{s_x}, I_{s_y})$
- ii) Torquer Coil Sensitivity $(K_{\tau q})$
- iii) Gimbal Damping Factor (C_{dyn})
- iv) Dynamic Stiffness (K_{dyn})

Issues related to characterizing these parameters are described below.

a) **Rotor and Gimbal Inertias, $(I_{r_x}, I_{r_y}, I_{g_x}, I_{g_y})$**

The accuracy of estimated gimbal and rotor inertias is a function of the machining precision of each of these two elements. The inertias can be determined based on the material properties and the engineering CAD models used to machine the rotor and gimbal. Once the gimbal or rotor has been machined, there is not a more accurate method of physically measuring the inertia.

b) **Torque scale factor, $T_{\tau q}$**

Based on testing of the first GyroWheel prototype [46], Figure 11-1 shows the torque scale factor as a function of tilt. The torque scale factor decreases as the tilt of the spin-axis increases. Modifications of the magnetic design will reduce the variation of $K_{\tau q}$ over the operating range, but accurate calibration of this characteristic is still required. In addition, the permanent magnet material characteristics are very sensitive to temperature, with accurate temperature control required to ensure that $K_{\tau q}$ is maintained at the calibrated level over the operating range. As described in Section 9.2, the required accuracy of the torque scale factor is 106 parts per million (ppm).

c) **Case-referenced dynamic stiffness, K_{dyn}**

Simulations of the non-linear GyroWheel Dynamics with the complete control system show that the dynamic stiffness is a function of two terms:

- i) A linear stiffness, which is a function of spin rate, flexure stiffness and gimbal inertial characteristics.
- ii) A non-linear correction term which is a function of the terms given in i) as well as spin-axis tilt. This term becomes more dominant as the tilt of the GyroWheel increases.

This relation is summarized in equation (10-41). The functional form of this relation will be used in the rate sensing algorithm, although testing of the device may reveal that some of the coefficients are slightly different from that presented in this document.

Figure 11-1: Torque Scale Factor as a Function of Tilt as Measured in First GyroWheel Prototype

d) **Gimbal damping factor, c_g**

In Chapter 6, it was shown how gimbal damping effectively changes the case-referenced stiffness of the system, introducing a cross-axis stiffness term. Unlike the gimbal spring terms, it cannot be offset through the application of the tuning condition. For this reason, it is desirable to eliminate or more realistically, minimize the damping in the system. When testing the gimbal assembly damping in ambient conditions (1 atmosphere pressure), the damping introduced by the gimbal is quite noticeable. However this is due primarily to the outer gimbal element and the resulting windage, which occurs when the gimbal is perturbed and the flutter occurs. Repeating such a test in vacuum reveals this damping factor to be orders of magnitude smaller. For this reason gimbal damping is assumed to be a negligible factor in the estimation of external rates.

11.1.2 GyroWheel Command Shaping and Latency

The external rate-sensing estimate relies on the following commanded values:

- i) Spin-Axis Tilt Angles $(\phi_{ref_x}, \phi_{ref_y})$
- ii) Spin-Axis Tilt Rate $(\dot{\phi}_{ref_x}, \dot{\phi}_{ref_y})$

Both of these terms are based on the demand of the ACS on the GyroWheel for a desired rate of change of momentum $\dot{h}$. This term is in turn integrated to obtain $h(t)$. From this momentum, the required transverse-axes angles and spin rate can be determined

There are two aspects of these commanded values, which must be considered in ensuring an accurate external rate estimate:

a) Command Update From Spacecraft Attitude Control Computer

Typically, the commands from the attitude control system computer to an actuator sub-system such as a gimballed momentum wheel are sent at a frequency of roughly 1 Hz. With a discrete update period, the command profile of the transverse-axes angles will appear as a series of step commands. In the case of the GyroWheel system, the update period has been chosen as 0.5 Hz. At the initiation of each step command, a short spike will occur in the transient control torque as the transverse-axes angles are accelerated to the new set point. As the change in this set point increases, the transient torque spike will become greater. This spike will have an adverse effect on estimating the external rates.

Based on the performance requirements listed in Section 2.2.3, the maximum rate of change of a transverse-axis angle is 0.01 deg/sec. For a 1 second update period, this translates into an angle step change of 0.01 degrees. Figure 11-2 shows the simulated response of the transverse-axes angles and control torques when the transverse-axis angle about the x-axis is commanded from zero to 0.01 degrees. The peak torque in this simulation is $5.5 \times 10^{-3} \text{ N} \cdot \text{m}$ on the y-axis. Based on the discussion of sub-system requirements for a 1 Nm-sec GyroWheel given in Section 8.2.2, torques of the magnitude of $5.0 \times 10^{-7} \text{ N} \cdot \text{m}$ are significant in terms of sensing external rates because they correspond to potential sensed rate errors of 0.1 deg/hr. Therefore a transient torque of $5.5 \times 10^{-3} \text{ N} \cdot \text{m}$ can be considered very large, having a negative effect on the external rate estimate.

Ideally, it would be better to ramp the position command between the successive angular set points with these commands internally generated within the tuned-rotor instrument computer. To illustrate this idea, Figure 11-3 shows a GyroWheel tilt demand on the x-axis during a manoeuvre of SCI-SAT [47]. By reducing the update period from 1 to 0.1 seconds, the maximum step change in angle has reduced from 0.01 degrees to 0.001

degrees. In fact, the update period can likely be reduced as low as 0.01 seconds, thereby reducing the transient torques by two orders of magnitude.

b) Generation of the Transverse-axes Angular Rates

The simplest way to command the GyroWheel is to provide commands for spin rate and spin-axis tilt. The tilt rate commands are implicitly contained in the tilt angle demand and can be obtained for external rate estimation by simply differentiating the position command locally in the tuned-rotor instrument control computer. This could be done by taking the previous and present transverse-axes angle commands where

$$\dot{\phi}_{ref}(k) = \frac{\phi_{ref}(k) - \phi_{ref}(k-1)}{\Delta T}$$

The disadvantage of such an approach is that the rate is constant over the interval and often has a large step change every new ACS command update. The approach of interpolating between update periods does not improve matters because the rate command is already constant over the interval according to the relation above.

For these reasons, the rate of change of the transverse-axes commanded angles must be sent as a command to the tuned-rotor instrument controller from the ACS. Like the position command, the present and previous commands can be used to interpolate rates at intermediate intervals between ACS updates. The requirement for the ACS to send the rate of change of transverse-axes angles provides no extra computational burden. As described in Chapter 2, the attitude of the spacecraft is adjusted through control of the rate of change of momentum when using momentum management. This is in turn integrated to provide a momentum demand to the gimballed momentum wheel. From the rate of change of momentum, the rate-of-change of transverse-axes angles can be obtained directly.

Figure 11-2: Linear Time Response of GyroWheel Transverse-axes Control System to 0.01 degree Step Command for x-axis Tilt Angle

Figure 11-3: Portion of the Spin-axis tilt Command (ϕ_{xref}) for Sci-Sat Simulation

Example

11.1.3 Filtering of Measured GyroWheel Dynamic Signals

The proposed concept of extracting the external rates of the base body from measured GyroWheel dynamic signals, command inputs and system models is based on the steady state dynamic characteristics of the GyroWheel and its control system. However, while the GyroWheel is functioning as both a steerable momentum wheel and an external rate sensor, the following dynamic effects must be accounted for when estimating the external rates:

a) Transient dynamics

Every time the tilt or spin rate command changes, transient torques will arise as the spin and transverse-axes control systems attempt to track their respective references. As described in Section 11.1.2, these transient torques will occur over a time frame of the order of the dominant spin and transverse-axes time constants.

b) $2\omega_s$ Mode

At any non-zero tilt, with the tilt being varied with respect to time or held constant, a periodic term at the flutter frequency of $2\omega_s$ will be seen in the tilt, tilt rate and torquer current responses. This periodic term will be the dominant component in the tilt rate and torquer current responses.

The external rate-sensing estimate relies on the direct measurement of only two terms: torquer coil current (i_{coil_x}, i_{coil_y}) and spin rate (ω_s) . In addition it requires the commanded tilt angles $(\phi_{ref_x}, \phi_{ref_y})$ and tilt rates $(\dot{\phi}_{ref_x}, \dot{\phi}_{ref_y})$. By definition, the commands will show no effect of either transient or flutter modes. The spin rate is controlled with a high bandwidth loop and shows very little flutter in its response as shown in Chapter 9.

Only the measured current contains significant transient and flutter terms. In the previous section, the magnitude of the transient torques for a relatively small angular change of 0.01 degrees was shown to be $5.5 \times 10^{-3} \text{ Nm}$. The command can be smoothed somewhat through linear interpolation between successive commands, but the effects of this type of transient torque must be further removed from the measured current signals using filtering before the external rates are calculated.

Similarly, the $2\omega_z$ torques were shown to be significant in simulations presented in Chapter 10. Figure 10-16 shows the transverse-axes control torque responses for the case where the GyroWheel is commanded with a 0.1 deg step from an initial zero tilt, with the system operating at the classic tuned speed. Examining the steady state response, the amplitude of the flutter torque was 4.0×10^{-5} Nm. After applying a fourth order Butterworth lowpass filter on this torque signal, the steady-state magnitude of T_{ϵ_i} is 5.84×10^{-7} Nm as shown in Figure 10-21. Evidently a filter such as that used to interpret the steady state control torque for this simulation should be applied to the current in the actual transverse-axes control design before it is processed in the external rate-sensing algorithm.

Given the issues related to both transient and $2\omega_z$ torques, the following two filters were set up in the transverse-axes control system and external rate-sensing algorithm within the GyroWheel prototype simulation:

- a) **Filter the torquer current with 4th order Butterworth filter with cut-off frequency at 10 Hz.**

The cut-off frequency on the current filter is chosen to be slightly below the 50 Hz flutter frequency. This filter can be implemented in analog electronics and can be thought of as an anti-aliasing filter. If this filter were implemented in a digital GyroWheel controller, it would require an update frequency of at least 100Hz with 500 Hz being more desirable. It could be considered “over-design” to implement this high an update frequency when the sample frequency of the GyroWheel rate sensing estimate by the spacecraft ACS is likely to be of the order of 1-2 Hz. By filtering the current, the high frequency components can be removed and the resulting signal sampled at a lower frequency by the digital GyroWheel controller.

- b) **Filter the overall external rate estimate with a second 4th order Butterworth lowpass filter at a cut-off frequency at 0.5-1 Hz.**

The second filter is used to remove any of the transient torque effects, which are introduced when the external rate is calculated using equation (11-1). As the bandwidth for the transverse-axes control system is set at 5 Hz, a low-pass filter at 0.5 Hz should provide significant attenuation of the transient torque effects while not compromising the desired bandwidth of the external rate sensing capability at 0.1 Hz.

11.2 Rate Estimation Algorithm

The following algorithm is based on the expression for estimating the external rates presented in equation (8-21). The effect of gimbal damping has been ignored for reasons discussed above.

- a) **Filter the Torquer Coil Currents (Implemented Transverse-axes Control System)**

The current in the x and y torquer coils must be passed through a 4th order Butterworth lowpass filter with a cut-off frequency of 10 Hz.

- b) **Determine the Pre-Filtered external rate estimate**

The pre-filtered external rate estimate is calculated based on the measured current in the x and y torquer coils, the commanded tilt on the x and y axes and the corresponding rate of change of tilt on the x and y axes where:

$$\begin{aligned}
\omega_{b_x} &= \frac{1}{(h - \gamma_s \omega_s)} \left(-K_{sq} i_{coil_x} + K_{dyn} \phi_{ref_x} - (h + \gamma_s \omega_s) \dot{\phi}_{ref_x} \right) \\
\omega_{b_y} &= \frac{1}{(h - \gamma_s \omega_s)} \left(K_{sq} i_{coil_y} - K_{dyn} \phi_{ref_y} - (h + \gamma_s \omega_s) \dot{\phi}_{ref_y} \right)
\end{aligned} \tag{11-2}$$

K_{dyn} is found using the approximation describing the non-linear dynamic stiffness of the device given in (10-41).

c) Filter the External Rate Estimate

A 4th order Butterworth lowpass filter is applied to both ω_{b_x} and ω_{b_y} estimates to lessen the effect of transient control torques. This cut-off frequency is set at a value between the minimum rate sensing bandwidth and the transverse-axes control system and spin control system bandwidths. A cut-off frequency of 0.5 Hz was used for the simulations presented in the following section.

11.3 Simulations of GyroWheel External Rate Sensing

To demonstrate the performance of the rate-sensing concept, two examples are presented in the following section. In each simulation study, it is assumed that the GyroWheel is positioned on a spacecraft body whose attitude is changing with time. The GyroWheel is being commanded to vary its momentum by tracking a time-varying spin-axis tilt demand. The external rate-sensing algorithm has been implemented in the simulation in order to extract the rates of the spacecraft body. In the first simulation study, a generic spin-axis tilt command is provided which varies sinusoidally with time. In the second study the momentum demand for a SCI-SAT [47] maneuver is used. In each study, the merits of the proposed rate sensing approach in the GyroWheel are demonstrated.

11.3.1 System Configuration For Simulations

The baseline 1 Nm-sec GyroWheel design was used in the system simulation model described in Appendix D. The system parameters are described in Table 6-1. Based on the recommendations in Chapter 10, the lag design option was used for the transverse-axes control system.

11.3.2 GyroWheel Commands and Spacecraft External Rates

A brief explanation is provided below of the rates specified for the spacecraft body and the tilt and spin rate demands placed on the GyroWheel system.

11.3.2.1 GyroWheel Commands

a) Generic Momentum Demand

As a first test of the rate sensing capability, the following sinusoidal GyroWheel tilt input was applied:

$$\begin{aligned}\phi_{ref_x} &= a \sin(\omega_c t) \\ \phi_{ref_y} &= 0\end{aligned}$$

The forcing frequency ω_c was chosen to be 0.002 Hz with the amplitude a set at 1.5 degrees. The spin rate command was held constant at the classic tuned frequency in order to simplify this first examination of the rate-sensing concept. The commands were applied for a period of 3000 seconds, the approximate period of the external rate signal.

b) SCI-SAT Momentum Demand

To use a more realistic GyroWheel command, an example from a SCI-SAT simulation [47] was chosen. SCI-SAT uses a 1 Nm-sec GyroWheel, operating in the 1300-1700 rpm range. A portion of the ACS generated GyroWheel commands were chosen where the most severe rates of change of spin rate and transverse-axes angles occurred. This corresponds to the 3000-second interval shown in Figure 11-4. The update period for the position commands was 1 second. With this coarse update period, there are significant step changes in the transverse-axes angles, which would result in large control torque transients. As noted in Section 11.1.2, the step changes in tilt angle command are as large as 0.01 degrees, which result in transient transverse axes torques of as large as 0.006 Nm. Within the GyroWheel computer function in the simulation model, the position commands were interpolated between successive inputs, to produce updated commands every 0.01 sec.

The rate of change of the commanded transverse-axes angles was determined by differentiating the transverse-axes angle commands. This produces a constant tilt rate command over each 1-second interval. Interpolating between 1-second intervals to get smaller changes in the tilt rate command does not yield a smoother command profile. Examining Figure 11-4, the jitter on the rate command profile is approximately 5×10^{-5} deg/sec or 0.18 deg/hr. This term will contribute directly to the instantaneous external rate sensing error.

It should be noted that the rate of change of these commands with respect to time represent a worst case and may ultimately be judged an overly demanding test of the rate sensing capability.

Figure 11-4: Time History of Spin-axis tilt Angle and Angular Rate Commands for SCI-SAT External Rate Sensing Example

11.3.2.2 External Rates

The spacecraft rates on the x and y axes of the spacecraft, as expressed in the GyroWheel case frame were chosen to be

$$\begin{bmatrix} \omega_{b_x} \\ \omega_{b_y} \end{bmatrix} = Z \begin{bmatrix} \sin \omega_{orbit} t \\ \cos \omega_{orbit} t \end{bmatrix}$$

ω_{orbit} was chosen to be 0.001 radian/second corresponding to the orbit period of an 800 km altitude circular orbit. The peak magnitude of the sinusoidal rate variation (Z) was chosen to be 1×10^{-5} radian/second (2 degree/hour), which is 20 times larger than the desired accuracy of the rate-sensing algorithm.

11.3.3 Simulation Results

11.3.3.1 Generic Example

Figure 11-5 shows the time response of the spin-axis tilt angles in response to the 1.5-degree sinusoidal command on the x-axis. There is no perceptible tracking error on the x-axis while the y-axis tilt exhibits a peak tracking error of the order of 0.005 degrees. The y-axis tilt response appears to exhibit a beat like frequency in its response over the 3000-second period. This is not a dynamic effect, but is instead due to the plot sample interval, which is set at 0.2 seconds. Consequently, the flutter frequency at roughly 50 Hz is being under-sampled, resulting in an aliased signal. Figure 11-6 shows the tracking error for both x and y axes with the plot interval set at 2 msec over the duration of 0 to 250 seconds, the period of the applied tilt command. The sampling interval of 2 msec is 10 times smaller than the period of the flutter oscillation and avoids the aliasing effect.

Here the tracking error is dominated by the flutter oscillations, which grow to a peak at 125 seconds. This corresponds to the point where the tilt reaches its first maximum of 1.5 degrees. Due to the gyroscopic coupling, both x and y-axes exhibit the same amplitude of tracking error.

Figure 11-7 shows the response of the two transverse-axes control torques. The larger of the two torques occurs on the y-axis. This is due to the commanded rate of change of the x-axis transverse angle

$$\dot{\phi}_{ref_x} = \alpha \omega_z \cos \omega_z t$$

Due to gyroscopic coupling, this will induce a torque about the y-axis. The x-axis torque contains a component at the command frequency but also a low frequency drift of approximately 1.0×10^{-5} Nm which corresponds to the influence of the spacecraft body motion on the GyroWheel. This influence cannot be seen in the y-axis torque due to the larger effect of the control torque required to track the x-axis tilt command.

Like the y-axis tilt shown in Figure 11-5, the plot interval for the torques is 0.2 seconds. Consequently, the flutter torque shown in this figure is in an aliased form. Plotting at a finer interval over the first 250 seconds of the simulation, the control torques are shown in Figure 11-8. The influence of flutter torques can be seen in the x-axis torque response with the flutter reaching a maximum value of $\pm 1.0 \times 10^{-6}$ Nm at 125 seconds, when the x-axis tilt reaches its maximum of 1.5 degrees.

Figure 11-9 shows the estimate of the external rate. The tracking of the spacecraft body precession rates is good on the x-axis, with a slightly more perceptible error on the y-axis. Showing only the tracking error in Figure 11-10, the signals are periodic with the same frequency as the command (ω_z) with amplitudes of ± 0.04 deg/hr and ± 0.01 deg/hr on the x and y-axes respectively. These errors are within the desired accuracy requirement of 0.1 deg/hr.

Figure 11-5: Response of Commanded and Actual Spin-axis Tilt Angles For Commanded Input of $a \sin \omega_z t$ on **x-axis ($a = 1.5 \text{ deg}$, $\omega_z = 0.002 \text{ Hz}$), Applied External Rates of $A \sin \omega_{orbit} t$ and $A \cos \omega_{orbit} t$ on the **x** and **y** Spacecraft Axes ($A = 2.05 \frac{\text{deg}}{\text{hour}}$, $\omega_{orbit} = 0.000185 \text{ Hz}$), Classic Tuned Speed, Spin-Axis Tilt Initially at Null**

Figure 11-6: Detailed Response of Commanded and Actual Spin-axis Tilt Angles Over First 200 seconds For Commanded Input of $a \sin \omega_c t$ on x-axis ($a = 1.5 \text{ deg}$, $\omega_c = 0.002 \text{ Hz}$), Applied External Rates of $A \sin \omega_{orbit} t$ and $A \cos \omega_{orbit} t$ on the x and y Spacecraft Axes ($A = 2.05 \frac{\text{deg}}{\text{hour}}$, $\omega_{orbit} = 0.000185 \text{ Hz}$), Classic Tuned Speed, Spin-Axis Tilt Initially at Null

Figure 11-7: Response of Transverse-axes Control Torques For Commanded Input of $a \sin \omega_c t$ on x-axis ($a = 1.5 \text{ deg}$, $\omega_c = 0.002 \text{ Hz}$), Applied External Rates of $A \sin \omega_{orbit} t$ and $A \cos \omega_{orbit} t$ on the x and y Spacecraft Axes ($A = 2.05 \frac{\text{deg}}{\text{hour}}$, $\omega_{orbit} = 0.000185 \text{ Hz}$), Classic Tuned Speed, Spin Axis Tilt Initially at Null

Figure 11-8: Detailed Response of Transverse-axes Control Torques Over First 200 seconds for Commanded Input of $a \sin \omega_c t$ on x-axis ($a = 1.5 \text{ deg}$, $\omega_c = 0.002 \text{ Hz}$), Applied External Rates of $A \sin \omega_{orbit} t$ and $A \cos \omega_{orbit} t$ on the x and y Spacecraft Axes ($A = 2.05 \frac{\text{deg}}{\text{hour}}$, $\omega_{orbit} = 0.000185 \text{ Hz}$), Classic Tuned Speed, Spin-Axis Tilt Initially at Null

Figure 11-9: Response of External Rate Sensing Estimate Versus Actual Rates for Commanded Input of $a \sin \omega_c t$ on x-axis ($a = 1.5 \text{ deg}$, $\omega_c = 0.002 \text{ Hz}$), Applied External Rates of $A \sin \omega_{orbit} t$ and $A \cos \omega_{orbit} t$ on the x and y Spacecraft Axes ($A = 2.05 \frac{\text{deg}}{\text{hour}}$, $\omega_{orbit} = 0.000185 \text{ Hz}$), Classic Tuned Speed, Spin-Axis Tilt Initially at Null

Figure 11-10: Response of External Rate Sensing Error for Commanded Input of $a \sin \omega_c t$ on x-axis ($a = 1.5 \text{ deg}$, $\omega_c = 0.002 \text{ Hz}$), Applied External Rates of $A \sin \omega_{orbit} t$ and $A \cos \omega_{orbit} t$ on the x and y Spacecraft Axes ($A = 2.05 \frac{\text{deg}}{\text{hour}}$, $\omega_{orbit} = 0.000185 \text{ Hz}$), Classic Tuned Speed, Spin-Axis Tilt Initially at Null

11.3.3.2 SCI-SAT External Rate Sensing Example

Figure 11-11 shows the time response of the spin-axis tilt angles compared against the commands sent from the ACS. There is no perceptible tracking error exhibited in these responses. The transverse-axes control torques are shown in Figure 11-12. The large jump in the control torque T_{c_y} to 3.0×10^{-4} Nm at 1000 seconds corresponds to the point at which the ϕ_{ref_x} command begins ramping to -1.4 degrees over 100 seconds. Similarly the T_{c_x} control torque jumps to 1.0×10^{-4} Nm at 1100 seconds for a period of 200 seconds. Over this interval the ϕ_{ref_y} command is ramped up to 1.1 degrees.

The external rate-sensing estimate is plotted against the actual external rate in Figure 11-13 with the errors shown in Figure 11-14. The following observations can be made:

- a) The estimate appears to track the actual external rate.
- b) At 1000 and 1100 seconds, there are significant errors of up to 3 deg/hr in the external rate estimate. As described above, these points correspond to times where there are significant discontinuities in the x and y-axis tilt commands respectively. The resulting control torque transients cause spikes in the external rate estimate. It should be noted that the commanded spin axis tilt command profile might be too severe. Putting a realistic restriction on the maximum acceleration of the commanded profiles will reduce the magnitude of these spikes to a more tolerable level.
- c) Ignoring the spikes at 1000 and 1100 seconds, there is still a noise component to the estimate with a variation of approximately ± 0.6 deg/hr. The magnitude of this error is larger than the desired value of 0.1 deg/hr.

Unlike the previous example, the rate estimation errors are noise-type errors. This noise could become a negative factor on the spacecraft attitude determination and control system (ACDS), depending on how the external rates are used. A typical application of strap-down tuned-rotor gyroscopes in spacecraft ADCS' is for attitude sensing. Based on an initial attitude reference obtained from an absolute position sensor such as a star tracker, the subsequent attitude can be obtained by integrating the external rate estimates from the tuned rotor gyroscope. Once integrated, the noisy component of the external rate estimate shown in Figure 11-13 and Figure 11-14 will have little effect on an attitude estimate. However if the rates are being used directly in the attitude control system rate control loop, then the noise may have a more significant effect. The two factors contributing to the noise and potential remedies are discussed below:

a) Severe rates of change in the transverse-axes control angle commands

As shown at 1000 and 1100 seconds, large transient errors in the external rate estimate occur because of large step changes in spin-axis tilt rate $(\dot{\phi}_x, \dot{\phi}_y)$ as shown in Figure 11-4. These tilt rates will result in the desired rate of change of GyroWheel momentum $(\dot{h})$ required by the spacecraft ACS. This illustrates the coupled relationship between the actuator and sensor functions within the GyroWheel instrument. As more demanding $\dot{h}$ profiles are required, the external rate sensing capability will degrade. Requirements analysis will be necessary to strike the best balance between the two capabilities.

b) Update interval of the spin-axis tilt rates

As described in Section 11.3.2.1, the spin axis tilt commands were updated at a very frequent interval of 0.01 sec. The result is that the command profile appears as a series of ramp commands repeated at 1-second intervals. However the change in slope at each 1 second interval still produces a transient in the control

torque response, i.e. the 1 second update on the tilt rate still affects the rate estimate. This is seen in Figure 11-15.

Examining the filtered torque response over the short interval of 1500 to 1525 seconds, it can be seen that the torque spikes occur every 1 second. The magnitude of the spikes is as large as 1.0×10^{-5} - 1.5×10^{-5} Nm. For a 1 Nm-sec GyroWheel, this can result in a jitter on the external rate sensing estimate of 2-3 deg/hr. Application of the 0.5 Hz low-pass filter in the external rate estimator only attenuates this to 0.6 deg/hr or less.

One approach to reducing the effect of these transient torques is to decrease the command update interval from 1 second. Alternatively the spacecraft ACS can provide both the transverse-axes angle commands and angular rate commands as described in Section 11.1.2. The GyroWheel can then interpolate the intermediate spin-axis tilt angles and rates between successive ACS updates.

Figure 11-11: Response of Commanded and Actual Spin-axis tilt Angles For SCI-SAT ACS Commands, Applied External Rates of $A \sin \omega_{orbit} t$ and $A \cos \omega_{orbit} t$ on the x and y Spacecraft Axes ($A = 2.05 \frac{\text{deg}}{\text{hour}}$, $\omega_{orbit} = 0.000185 \text{ Hz}$), Classic Tuned Speed, Spin-axis Tilt Initially at Null

Figure 11-12: Response of Transverse-axes Control Torques For SCI-SAT ACS Commands, Applied External Rates of $A \sin \omega_{orbit} t$ and $A \cos \omega_{orbit} t$ on the x and y Spacecraft Axes ($A = 2.05 \frac{\text{deg}}{\text{hour}}$, $\omega_{orbit} = 0.000185 \text{ Hz}$), Classic Tuned Speed, Spin-axis tilt Initially at Null

Figure 11-13: Response of Estimated and Actual External Rates For SCI-SAT ACS Commands, Applied External Rates of $A \sin \omega_{orbit} t$ and $A \cos \omega_{orbit} t$ on the x and y Spacecraft Axes ($A = 2.05 \frac{\text{deg}}{\text{hour}}$, $\omega_{orbit} = 0.000185 \text{ Hz}$), Classic Tuned Speed, Spin-axis tilt Initially at Null

Figure 11-14: Response of External Rate Sensing Error For SCI-SAT ACS Commands, Applied External Rates of $A \sin \omega_{orbit} t$ and $A \cos \omega_{orbit} t$ on the x and y Spacecraft Axes ($A = 2.05 \frac{\text{deg}}{\text{hour}}$, $\omega_{orbit} = 0.000185 \text{ Hz}$), Classic Tuned Speed, Spin-axis Tilt Initially at Null

Figure 11-15: Detailed Response of Filtered (10 Hz Butterworth lowpass) Transverse-axes control Torque For SCI-SAT ACS Commands, Applied External Rates of $A \sin \omega_{orbit} t$ and $A \cos \omega_{orbit} t$ on the x and y spacecraft axes ($A = 2.05 \frac{\text{deg}}{\text{hour}}$, $\omega_{orbit} = 0.000185 \text{ Hz}$), Classic Tuned Speed, Spin-axis Tilt Initially at Null

11.4 Summary

The proposed external rate-sensing concept relies on a good knowledge of the low-frequency dynamics of the tuned-rotor instrument and accurate measurement of the devices transverse-axes control torques. To achieve the desired accuracy of 0.1 deg/hr for the GyroWheel system implementation should focus on

- a) Accurate characterization of the dynamic stiffness
- b) Command update period and shaping of the ACS spin rate and transverse-axes angle and angular rate commands
- c) Filtering of the measured current

Simulation tests of the GyroWheel system reveal that the desired accuracy of 0.1 deg/hr is achievable although more strict requirements on command update period and tilt rate commands may be required.

Chapter 12

Concluding Remarks

The tuned-rotor gyroscope concept was introduced in the early 1960's and developed and refined through the 1970's. Since then, studies have focused on developing a better understanding of its dynamics and effective methods of system control in order to extract improved accuracy from the device. Very recently, the observation has been made that with small modifications to its mechanical configuration, a tuned-rotor instrument can function simultaneously as a steerable momentum wheel and an external rate sensor. This has spawned a number of questions concerning system dynamics and control of this new, more general tuned-rotor instrument. The research in this thesis focused on addressing a number of these issues with the findings summarized below.

12.1 Conclusions

The research into dynamics and control of tuned rotor instruments was divided into two areas. The first focused on developing models and techniques for analysing the dynamics and control systems for a general tuned-rotor instrument. The second applied the findings of this research to the problem of designing a control system prototype for the GyroWheel, the first commercial example of such a system.

12.1.1 Research into Tuned-Rotor Instrument Dynamics and Control

The operating range of a general tuned-rotor instrument differs from the classic device in two ways. The spin rate is varied about the tuned speed and the spin axis is intentionally tilted a significant amount from null. The variation in spin rate and the potential for the non-linear dynamics to have a greater influence at significant tilts were viewed as critical factors influencing both the control system design and development of a more general external rate sensing technique. To address these issues, a detailed three-degree of freedom model of the tuned-rotor instrument was developed using Lagrangian dynamics techniques. This model describes the dynamics in spinning coordinates. A linearized version of the model was developed for small tilts of the spin axis. After transforming the model to case-referenced coordinates, a detailed transfer function model, termed the Complete Model of the transverse-axes dynamics was developed which includes all the effects that occur at twice the spin frequency due to fluttering gimbal in a tuned rotor instrument.

Detailed analysis was performed on the Complete Model of the tuned rotor gyroscope to determine the influence of mechanical parameters on the system modes and dominant low frequency characteristics. The transfer function model was shown to contain four poles, which correspond to those identified by Staley [13]. Analysis of this new complete transfer function model also revealed that two of these modes are fixed, as they coincide with the two transmission zeros of the system. An explicit solution to the linear equations of motion was developed which served as an aid in the validation of the overall non-linear dynamics simulator. The concept of case-referenced dynamic stiffness of the transverse-axes dynamics of a tuned-rotor instrument was introduced for the purposes of external rate sensing algorithm development. Finally, using the Complete Model of the transverse-axes dynamics, there is no zero-frequency mode among the four characteristic modes when the Traditional Tuning Condition is applied. Based on this observation, a

new tuning condition was developed which should eliminate a very low frequency drift observed in some tuned-rotor instrument systems.

There is a class of two-input, two output dynamic systems whose inputs and outputs are coupled in quadrature. Examples of such systems are the tilt of the spin axis of a spinning rigid body and the phase currents in a multi-phase AC regulator. These systems can be described in the more compact form of a single-complex-input, single-complex-output (SCISCO) transfer function with complex coefficients. In this form, analysis of the dynamics is greatly simplified. In this dissertation, a general classical control approach for SCISCO systems was proposed. With some modifications to the rules for root locus and frequency domain techniques, approaches were presented for judging relative stability and designing control systems for SCISCO systems. To illustrate this approach, both root locus and frequency domain methods were applied to the problem of nutation damping in a tuned-rotor gyroscope.

A method for determining the external rates of a base body on which a tuned rotor instrument is mounted was developed. This approach assumes that the spin rate and tilt of the spin axis are varied with time as part of the instruments role as a momentum actuator. The algorithm relies on the measurement of two signals: the current in the two torquer coils of the device and the measured spin rate as well as the commanded tilt angles. It relies on a good characterization of the low frequency dynamics of the tuned-rotor instrument. Through an error analysis, it was demonstrated that such a concept is capable of delivering accuracies of the order of 0.1 deg/hr based on current, spin rate and angular tilt measurement capabilities of existing commercial sensors.

12.1.2 Application of Research Findings to GyroWheel Control System Design

All past treatments of tuned-rotor instrument control assume that a high bandwidth control system on the motor shaft will maintain the spin rate at the tuned speed. The development of the non-linear model of the general tuned-rotor instrument provided the means to examine the influence of spin control methods on the behaviour of the GyroWheel device. Before applying the spin controller, examination of the open loop behaviour revealed that up to 0.1 degrees tilt, the system behaves like a linear system. Beyond this level of tilt, the slowest of the four characteristic modes of the tuned rotor instrument decreases in frequency as the tilt of the spin axis is increased. In addition, beyond a tilt of approximately 4.1 degrees, the direction of precession motion changes. Introduction of traditional motor shaft control of the spin rate revealed that the precession frequency of the system varies by several orders of magnitude, depending on the control bandwidth. Direct spin control of the rotor was shown to be far superior. The resulting precession frequency is independent of bandwidth and of the same magnitude as that observed in the open loop system.

The classical control techniques developed for SCISCO systems were applied to the design of the transverse-axes control system. Three different designs were developed and compared on the basis of stability/performance and their impact on the calibration of the low-frequency system characteristics for external rate sensing. To test these designs, the overall system with transverse-axes control system, spin axis control system and complete non-linear dynamics was simulated. It was shown that the transverse-axes control system interacts with high frequency dynamics behaviour at twice the spin frequency to alter the low frequency behaviour of the system. Introduction of a notch filter at 2 times the spin frequency eliminates this problem. Using this modified transverse-axes control system, the GyroWheel dynamic stiffness was characterized over the spin rate and tilt operating range through a series of simulations. A general

expression defining the dynamic stiffness of the tuned-rotor instrument as a function of tilt and spin rate was then developed.

With the low-frequency dynamics of the tuned-rotor instrument dynamics characterized, the external rate-sensing algorithm was tested. The issues involved in implementing the external rate-sensing algorithm were discussed. The effectiveness of the concept was demonstrated for two separate cases, with the results indicating that a 0.1 deg/hr rate sensing accuracy is achievable.

12.2 Future Work

Although a number of aspects of tuned-rotor instrument dynamics and its control were addressed in this study, there are still outstanding issues to examine:

- a) Development of robust control methods for SCISCO transfer functions.
- b) Examination of the robustness of the control designs outlined in this study for the GyroWheel, applying the methods in a) to determine the effect of model uncertainty.
- c) Implementation of the transverse-axes controller in digital form with the update triggered by the spin rate. It can then be shown that the nutation damper and notch filter frequencies will vary in the desired way as the spin rate varies.
- d) Investigate appropriate methods of direct rotor control, rather than motor shaft control. This involves designing an actuator, which will drive the rotor spin directly, rather than through the motor input shaft.
- e) As presented in this study, the external rate estimates are made about two of the three spacecraft body axes. However, when the tilt of the tuned-rotor instrument becomes large, the spacecraft rate about its third axis will affect the estimate of

the rate estimates about the other two axes. An approach to cope with this issue must be developed.

- f) Examine the interaction between GyroWheel and spacecraft dynamics.
- g) Determine the bound on the maximum tilt and de-tuned speed for which tuned-rotor instrument can function effectively.

12.3 Summary of Contributions

Contributions have been made in the areas of general control system design and tuned-rotor instrument dynamics and their control:

- a) **Tuned-Rotor Instrument Dynamics**
 - i) A general single-complex-input, single-complex-output transfer function model was developed which includes the twice spin frequency effects that are frequently ignored in past models.
 - ii) A more accurate tuning condition for tuned-rotor instruments has been developed which could eliminate small drifts.
 - iii) Tuned-rotor instruments cannot be tuned with a single spin rate when significant tilts occur. The tuning condition varies as a function of both tilt and spin rate. However, using the GyroWheel as an example, a general functional form to describe the dynamic stiffness of a tuned-rotor instrument was derived. It characterizes dynamic stiffness as a function of the spin rate and spin axis tilt angles.
 - iv) Open-loop tuned-rotor gyroscopes exhibit four characteristic modes of vibration when observed from the stationary case. However, when controlling the transverse-axis tilt, two of these modes can be considered

“fixed modes”. It is for this reason, that they are never observed in closed-loop rebalance control systems.

b) Control System Analysis and Design Techniques for SCISCO Systems

The general classical control techniques for single-input, single-output systems have been generalized for single-complex-input, single-complex-output systems described by a single complex valued transfer function with complex coefficients. This approach applies to a class of systems for which two degrees of freedom can be combined in quadrature, e.g. gyroscopes, spinning rigid bodies, motors, regulators. This approach provides potential for a more intuitive and less time consuming method of examining this particular class of two-input, two-output control systems.

c) Control of Tuned-Rotor Instruments

- i) Rotor rather than motor shaft control is more appropriate for tuned-rotor instruments, particularly when the transverse-axes tilt becomes large.
- ii) External rates of a base body can be estimated from a tuned-rotor instrument even when the spin rate is not at the tuned speed and the tilt is not held at null.
- iii) A dynamic simulation of the non-linear dynamics of the tuned-rotor instrument with control system and external rate sensing has been developed.

Appendix A

Derivation of Dynamic Models

A.1 Relationship Between Gimbal Referenced and Case-Referenced Angles, Rates and Torques

a) Position

As first derived by Cain [17], the relationship between the gimbal angles and case-referenced angles is shown below. This is obtained by first noting that for a given rotor orientation, the elements of the transformation matrices **A** and **A'** given in equations (5-5) and (5-20) must be the same. This must be true in order for either transformation to describe the same orientation of the rotor in its body referenced-frame. Examining the expressions for **A** and **A'**, the case-referenced angles are related to the gimbal angles by the following relationships

$$\begin{aligned}\tan \phi_x &= \frac{S_{\theta_1} C_{\theta_2} C_{\theta_3} - S_{\theta_1} S_{\theta_3}}{C_{\theta_1} C_{\theta_3}} \\ \sin \phi_y &= S_{\theta_1} C_{\theta_3} + S_{\theta_1} C_{\theta_2} S_{\theta_3} \\ \tan \phi_z &= -\frac{C_{\theta_1} S_{\theta_3}}{S_{\theta_1} S_{\theta_2} S_{\theta_3} - C_{\theta_1} C_{\theta_2}}\end{aligned}\tag{A-1}$$

Of interest in the derivation of a suitable transfer function model of the system are the above relationships when the tilt of the spin axis is small. For small tilts of the spin axis, the equations in (A-1) simplify to

$$\begin{aligned}\phi_x &\approx C_{\theta_z} \theta_x - S_{\theta_z} \theta_y \\ \phi_y &\approx S_{\theta_z} \theta_x + C_{\theta_z} \theta_y \\ \phi_z &\approx \theta_z\end{aligned}\tag{A-2}$$

Therefore for small tilts of the spin axis, the input shaft angle is equal to the rotor angle.

b) Rates

The relationship between the gimbal angular rates and the rate of change of the case-referenced angles can be found by noting the following:

$$\bar{\omega}_r = \bar{\omega}'_r\tag{A-3}$$

Substituting the expressions for $\bar{\omega}'_r$ and $\bar{\omega}_r$ given in equations (5-14) and (5-23), the following expression results:

$$\mathbf{J} \begin{bmatrix} \dot{\theta}_x \\ \dot{\theta}_y \\ \dot{\theta}_z \end{bmatrix} = \mathbf{J}' \begin{bmatrix} \dot{\phi}_x \\ \dot{\phi}_y \\ \dot{\phi}_z \end{bmatrix}\tag{A-4}$$

The above expression can be re-arranged into the following form:

$$\begin{bmatrix} \dot{\phi}_x \\ \dot{\phi}_y \\ \dot{\phi}_z \end{bmatrix} = \mathbf{J}'^{-1} \mathbf{J} \begin{bmatrix} \dot{\theta}_x \\ \dot{\theta}_y \\ \dot{\theta}_z \end{bmatrix}\tag{A-5}$$

Substituting the expressions for $\mathbf{J}$ and $\mathbf{J}'$ into (A-5) yields the following expression for the case-referenced angular rates in terms of the gimbal angular rates:

$$\begin{bmatrix} \dot{\phi}_x \\ \dot{\phi}_y \\ \dot{\phi}_z \end{bmatrix} = \frac{1}{C_{\phi_y}} \begin{bmatrix} C_{\phi_z} C_{\phi_x} & -S_{\phi_z} & -S_{\phi_z} S_{\phi_x} - C_{\phi_z} C_{\phi_x} S_{\phi_y} \\ C_{\phi_z} S_{\phi_x} C_{\phi_y} & C_{\phi_z} C_{\phi_x} & C_{\phi_z} C_{\phi_x} S_{\phi_y} - C_{\phi_z} S_{\phi_x} C_{\phi_y} S_{\phi_y} \\ C_{\phi_z} S_{\phi_x} - S_{\phi_z} C_{\phi_x} C_{\phi_y} & S_{\phi_z} S_{\phi_x} & S_{\phi_z} C_{\phi_x} C_{\phi_y} S_{\phi_y} + S_{\phi_z} S_{\phi_x} S_{\phi_y} - C_{\phi_z} C_{\phi_x} C_{\phi_y} \end{bmatrix} \begin{bmatrix} \dot{\theta}_x \\ \dot{\theta}_y \\ \dot{\theta}_z \end{bmatrix} \quad (\text{A-6})$$

Assuming small tilt angles, the above expression simplifies to the following form:

$$\begin{bmatrix} \dot{\phi}_x \\ \dot{\phi}_y \\ \dot{\phi}_z \end{bmatrix} \approx \begin{bmatrix} C_{\phi_z} & -S_{\phi_z} & -S_{\phi_z} \theta_x - C_{\phi_z} \theta_y \\ S_{\phi_z} & C_{\phi_z} & C_{\phi_z} \theta_x - S_{\phi_z} \theta_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_x \\ \dot{\theta}_y \\ \dot{\theta}_z \end{bmatrix} \quad (\text{A-7})$$

The motor rate $\dot{\theta}_z$ and rotor spin rate $\dot{\phi}_z$ are assigned the same constant value of ω_z , i.e.

$$\dot{\theta}_z = \dot{\phi}_z = \omega_z \quad (\text{A-8})$$

The common rotor and motor spin angle then becomes the following:

$$\theta_z = \phi_z = \omega_z t \quad (\text{A-9})$$

Equations (A-2) and (A-7) now can be re-expressed in the following form:

$$\begin{bmatrix} \phi_x \\ \phi_y \end{bmatrix} = \begin{bmatrix} C_{\omega_z t} & -S_{\omega_z t} \\ S_{\omega_z t} & C_{\omega_z t} \end{bmatrix} \begin{bmatrix} \theta_x \\ \theta_y \end{bmatrix} \quad (\text{A-10})$$

$$\begin{bmatrix} \dot{\phi}_x \\ \dot{\phi}_y \end{bmatrix} = \begin{bmatrix} C_{\omega_z t} & -S_{\omega_z t} \\ S_{\omega_z t} & C_{\omega_z t} \end{bmatrix} \begin{bmatrix} \dot{\theta}_x \\ \dot{\theta}_y \end{bmatrix} - \omega_z \begin{bmatrix} S_{\omega_z t} & C_{\omega_z t} \\ -C_{\omega_z t} & S_{\omega_z t} \end{bmatrix} \begin{bmatrix} \theta_x \\ \theta_y \end{bmatrix} \quad (\text{A-11})$$

c) Torques

The case referenced torques consist of:

- i) The two transverse-axes control torques acting from the case about the two orthogonal tilt axes and
- ii) The spin control torque acting about tilted rotor spin axis.

Gimbal torques are defined as torques acting about the three gimbal axes described in Section 5.1.1. Although no control torques act about these axes, gimbal torques can be considered as "generalized torques", representing the equivalent effect of the case-referenced control torques. Gimbal torques are required in order to formulate the equations of motion of the tuned-rotor instrument.

In order to determine the relationship between gimbal torques and case-referenced torques, the principle of virtual work will be employed. Consider the case where the rotor spin axis is tilted at some arbitrary tilt angle with an arbitrary spin angle. If the rotor is constrained from moving and gimbal torques are applied, constraint torques will develop on the rotor. The following relation defines a differential change in attitude of the rotor $\delta\Omega$ expressed in its body frame:

$$\delta\Omega = \delta \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix} \quad (\text{A-12})$$

Noting the relationship $\bar{\omega}_r = \mathbf{J} \begin{bmatrix} \dot{\theta}_x & \dot{\theta}_y & \dot{\theta}_z \end{bmatrix}^T$ defined in Section 5.1.1, the following expression relates the differential change in rotor attitude to differential changes in the gimbal angles:

$$\delta\Omega = \mathbf{J} \delta \begin{bmatrix} \theta_x \\ \theta_y \\ \theta_z \end{bmatrix} \quad (\text{A-13})$$

If the following torques are applied about the gimbal axes,

$$\tau_g = \begin{bmatrix} T_{g_x} \\ T_{g_y} \\ T_{g_z} \end{bmatrix} \quad (\text{A-14})$$

then the resulting constraint torques on the rotor are defined as the following:

$$\mathbf{T}_{rotor} = \begin{bmatrix} T_{rotor_x} \\ T_{rotor_y} \\ T_{rotor_z} \end{bmatrix} \quad (\text{A-15})$$

The following expression describes the virtual work performed in the system:

$$\delta\mathbf{W} = \mathbf{T}_{rotor}^T \delta\Omega - \tau_g^T \delta \begin{bmatrix} \theta_x \\ \theta_y \\ \theta_z \end{bmatrix} \quad (\text{A-16})$$

Substituting equation (A-13) in the above expression results in following expression:

$$\delta\mathbf{W} = (\mathbf{T}_{rotor}^T \mathbf{J} - \tau_g^T) \delta \begin{bmatrix} \theta_x \\ \theta_y \\ \theta_z \end{bmatrix} \quad (\text{A-17})$$

If the rotor is in equilibrium, then the virtual work performed is zero, which is true for any arbitrary gimbal displacement. Therefore the relationship between the constraint torques and gimbal torques is

$$\tau = \mathbf{J}' \mathbf{T}_{rotor} \quad (\text{A-18})$$

Now consider the equivalent problem when case-referenced torques are applied and the rotor is again constrained. The case referenced torques are defined below:

$$\tau' = \begin{bmatrix} T_{\zeta_x} \\ T_{\zeta_y} \\ T_{\zeta_z} \end{bmatrix} \quad (\text{A-19})$$

Using the same rationale as above, the constraint torques on the rotor resulting from the application of case-referenced torques can be found by the relation given below:

$$\tau' = \mathbf{J}'' \mathbf{T}'_{rotor} \quad (\text{A-20})$$

If the constraint torques on the rotor for a set of applied gimbal torques are equivalent to the reactions on the rotor for a set of case-referenced torques, then the two sets of torques are related by the following expression:

$$\tau = \mathbf{J}' (\mathbf{J}'')^{-1} \tau' \quad (\text{A-21})$$

Expanding out the above expression, the gimbal torques are a function of the gimbal and case-referenced angles and the case-referenced torques as shown below:

$$\begin{aligned}
T_{g_x} &= \frac{1}{C_{\phi_v}} \left\{ C_{\theta_v} C_{\phi_z} T_{c_x} + C_{\theta_v} C_{\phi_v} S_{\phi_z} T_{c_y} - (C_{\theta_v} S_{\phi_v} C_{\phi_z} - S_{\theta_v} C_{\phi_v}) T_{c_z} \right\} \\
T_{g_y} &= \frac{1}{C_{\phi_v}} \left\{ -S_{\phi_z} T_{c_x} + C_{\phi_v} C_{\phi_z} T_{c_y} - S_{\phi_v} S_{\phi_z} T_{c_z} \right\} \\
T_{g_z} &= \frac{1}{C_{\phi_v}} \left\{ -(C_{\theta_v} S_{\theta_v} C_{\phi_z} + S_{\theta_v} S_{\phi_z}) T_{c_x} - (C_{\theta_v} S_{\theta_v} C_{\phi_v} S_{\phi_z} - S_{\theta_v} S_{\phi_v} C_{\phi_z}) T_{c_y} \right. \\
&\quad \left. - (C_{\theta_v} S_{\phi_v} S_{\phi_z} C_{\phi_z} + S_{\theta_v} S_{\phi_v} S_{\phi_z} + C_{\theta_v} C_{\phi_v} C_{\phi_z}) T_{c_z} \right\}
\end{aligned} \tag{A-22}$$

For the situation where the tilt angles are small, the expressions describing the gimbal torques as a function of the case-referenced torques simplify to the following form:

$$\begin{aligned}
T_{g_x} &\approx C_{\phi_z} T_{c_x} - S_{\phi_z} T_{c_y} \\
T_{g_y} &\approx -S_{\phi_z} T_{c_x} + C_{\phi_z} T_{c_y} \\
T_{g_z} &\approx T_{c_z}
\end{aligned} \tag{A-23}$$

Finally, assuming that the spin rate is constant at a value of ω_s , the equations in (A-23) further simplify to the following form:

$$\begin{aligned}
T_{g_x} &= C_{\omega_s} T_{c_x} + S_{\omega_s} T_{c_y} \\
T_{g_y} &= -S_{\omega_s} T_{c_x} + C_{\omega_s} T_{c_y} \\
T_{g_z} &= T_{c_z} = 0
\end{aligned} \tag{A-24}$$

A.2 Partial Derivatives Used in the Derivation of the 3-Body Model Equations of Motion

The following partial derivatives were used during the determination of the equations of motion using Lagrange's equation.

$$\frac{\partial \bar{\omega}_m}{\partial \theta_z} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (\text{A-25})$$

$$\frac{\partial \bar{\omega}_m}{\partial \theta_y} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (\text{A-26})$$

$$\frac{\partial \bar{\omega}_m}{\partial \dot{\theta}_z} = \begin{bmatrix} -S_{\theta_z} \omega_{b_x} + C_{\theta_z} \omega_{b_y} \\ C_{\theta_z} \omega_{b_x} - S_{\theta_z} \omega_{b_y} \\ 0 \end{bmatrix} \quad (\text{A-27})$$

$$\frac{\partial \bar{\omega}_m}{\partial \theta_x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (\text{A-28})$$

$$\frac{\partial \bar{\omega}_m}{\partial \theta_y} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (\text{A-29})$$

$$\frac{\partial \bar{\omega}_m}{\partial \dot{\theta}_z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (\text{A-30})$$

$$\frac{\partial \bar{\omega}_g}{\partial \theta_x} = \begin{bmatrix} 0 \\ -S_{\theta_x} \left(-S_{\theta_z} \omega_{b_x} + C_{\theta_z} \omega_{b_y} \right) + C_{\theta_x} \left(\omega_{b_z} - \dot{\theta}_z \right) \\ -C_{\theta_x} \left(-S_{\theta_z} \omega_{b_x} + C_{\theta_z} \omega_{b_y} \right) - S_{\theta_x} \left(\omega_{b_z} - \dot{\theta}_z \right) \end{bmatrix} \quad (\text{A-31})$$

$$\frac{\partial \bar{\omega}_g}{\partial \theta_y} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (\text{A-32})$$

$$\frac{\partial \bar{\omega}_g}{\partial \dot{\theta}_z} = \begin{bmatrix} -S_{\theta_z} \omega_{b_x} - C_{\theta_z} \omega_{b_y} \\ -C_{\theta_x} \left(C_{\theta_z} \omega_{b_x} + S_{\theta_z} \omega_{b_y} \right) \\ S_{\theta_x} \left(C_{\theta_z} \omega_{b_x} + S_{\theta_z} \omega_{b_y} \right) \end{bmatrix} \quad (\text{A-33})$$

$$\frac{\partial \bar{\omega}_g}{\partial \theta_x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad (\text{A-34})$$

$$\frac{\partial \bar{\omega}_g}{\partial \theta_y} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (\text{A-35})$$

$$\frac{\partial \bar{\omega}_g}{\partial \theta_z} = \begin{bmatrix} 0 \\ S_{\theta_z} \\ C_{\theta_z} \end{bmatrix} \quad (\text{A-36})$$

$$\frac{\partial \bar{\omega}_r}{\partial \theta_x} = \begin{bmatrix} -C_{\theta_z} S_{\theta_z} (S_{\theta_z} \omega_{b_x} - C_{\theta_z} \omega_{b_y}) + S_{\theta_z} S_{\theta_z} (\omega_{b_z} - \dot{\theta}_z) \\ -S_{\theta_z} (-S_{\theta_z} \omega_{b_x} + C_{\theta_z} \omega_{b_y}) + C_{\theta_z} (\omega_{b_z} - \dot{\theta}_z) \\ C_{\theta_z} C_{\theta_z} (S_{\theta_z} \omega_{b_x} - C_{\theta_z} \omega_{b_y}) + S_{\theta_z} C_{\theta_z} (\omega_{b_z} - \dot{\theta}_z) \end{bmatrix} \quad (\text{A-37})$$

$$\frac{\partial \bar{\omega}_r}{\partial \theta_y} = \begin{bmatrix} -S_{\theta_z} \dot{\theta}_x - (S_{\theta_z} C_{\theta_z} - S_{\theta_z} C_{\theta_z} S_{\theta_z}) \omega_{b_x} - (S_{\theta_z} S_{\theta_z} - S_{\theta_z} C_{\theta_z} C_{\theta_z}) \omega_{b_y} - C_{\theta_z} C_{\theta_z} (\omega_{b_z} + \dot{\theta}_z) \\ 0 \\ C_{\theta_z} \dot{\theta}_x - (C_{\theta_z} C_{\theta_z} - S_{\theta_z} S_{\theta_z} S_{\theta_z}) \omega_{b_x} - (C_{\theta_z} S_{\theta_z} + S_{\theta_z} S_{\theta_z} C_{\theta_z}) \omega_{b_y} - C_{\theta_z} S_{\theta_z} (\omega_{b_z} + \dot{\theta}_z) \end{bmatrix} \quad (\text{A-38})$$

$$\frac{\partial \bar{\omega}_r}{\partial \theta_z} = \begin{bmatrix} -(C_{\theta_z} S_{\theta_z} - S_{\theta_z} S_{\theta_z} C_{\theta_z}) \omega_{b_x} - (C_{\theta_z} C_{\theta_z} - S_{\theta_z} S_{\theta_z} S_{\theta_z}) \omega_{b_y} \\ -C_{\theta_z} C_{\theta_z} \omega_{b_x} - C_{\theta_z} S_{\theta_z} \omega_{b_y} \\ -(S_{\theta_z} S_{\theta_z} - S_{\theta_z} C_{\theta_z} C_{\theta_z}) \omega_{b_x} - (S_{\theta_z} C_{\theta_z} + S_{\theta_z} C_{\theta_z} S_{\theta_z}) \omega_{b_y} \end{bmatrix} \quad (\text{A-39})$$

$$\frac{\partial \bar{\omega}_r}{\partial \theta_x} = \begin{bmatrix} C_{\theta_z} \\ 0 \\ S_{\theta_z} \end{bmatrix} \quad (\text{A-40})$$

$$\frac{\partial \bar{\omega}_r}{\partial \theta_y} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad (\text{A-41})$$

$$\frac{\partial \bar{\omega}_r}{\partial \dot{\theta}_z} = \begin{bmatrix} -C_{\theta_z} S_{\theta_z} \\ S_{\theta_z} \\ C_{\theta_z} C_{\theta_z} \end{bmatrix} \quad (\text{A-42})$$

A.3 Non-Linear Control Torques in 3-Body Dynamics Model

The three non-linear control torque terms for the three equations of motion of the general tuned rotor instrument are given below:

$$\begin{aligned}
f_{\tilde{u}_i}(\mathbf{x}, \dot{\mathbf{x}}, \tilde{\omega}_b, \dot{\tilde{\omega}}_b) = & - \left[\frac{1}{2}(I_{r_z} - I_{r_x})S_{\theta_z}S_{\gamma_{\theta_z}}S_{\theta_z} + I_zC_{\theta_z} \right] \dot{\omega}_{b_z} \\
& - \left[\frac{1}{2}(I_{r_z} - I_{r_x})S_{\theta_z}S_{\gamma_{\theta_z}}C_{\theta_z} - I_zS_{\theta_z} \right] \dot{\omega}_{b_x} \\
& - \left[\frac{1}{2}(I_{r_z} - I_{r_x})C_{\theta_z}S_{\gamma_{\theta_z}} \right] \dot{\omega}_{b_y} \\
& - \left[(I_{r_z} - I_{r_x})S_{\gamma_{\theta_z}} \right] \dot{\theta}_x \dot{\theta}_y \\
& - \left[(I_{r_z} - I_{r_x})C_{\gamma_{\theta_z}} - I_{r_y} \right] C_{\theta_z} \dot{\theta}_y \dot{\theta}_z \\
& - \left[\frac{1}{2}I_zS_{\gamma_{\theta_z}} \right] \dot{\theta}_z^2 \\
& - \left[(I_{r_z} - I_{r_x})(S_{\gamma_{\theta_z}}C_{\theta_z} - S_{\theta_z}C_{\gamma_{\theta_z}}S_{\theta_z}) \right] \dot{\theta}_y \omega_{b_x} \\
& - \left[-(I_{r_z} - I_{r_x})S_{\theta_z}S_{\gamma_{\theta_z}}C_{\theta_z} - I_zS_{\theta_z} - I_zC_{\gamma_{\theta_z}}S_{\theta_z} \right] \dot{\theta}_z \omega_{b_x} \\
& + \left[\frac{1}{4}(I_{r_z} - I_{r_x})C_{\theta_z}S_{\gamma_{\theta_z}}S_{\gamma_{\theta_z}} + \frac{1}{2}I_zS_{\gamma_{\theta_z}}S_{\theta_z}^2 \right] \omega_{b_x}^2 \\
& - \left[\frac{1}{2}(I_{r_z} - I_{r_x})C_{\theta_z}S_{\gamma_{\theta_z}}C_{\gamma_{\theta_z}} - \frac{1}{2}I_zS_{\gamma_{\theta_z}}S_{\gamma_{\theta_z}} \right] \omega_{b_x} \omega_{b_y} \\
& - \left[\frac{1}{2}(I_{r_z} - I_{r_x})S_{\theta_z}S_{\gamma_{\theta_z}}C_{\theta_z} - I_zC_{\gamma_{\theta_z}}S_{\theta_z} \right] \omega_{b_x} \omega_{b_z} \\
& - \left[(I_{r_z} - I_{r_x})(S_{\gamma_{\theta_z}}S_{\theta_z} - S_{\theta_z}C_{\gamma_{\theta_z}}C_{\theta_z}) \right] \dot{\theta}_y \omega_{b_y} \\
& - \left[(I_{r_z} - I_{r_x})S_{\theta_z}S_{\gamma_{\theta_z}}S_{\theta_z} + I_zC_{\theta_z} + I_zC_{\gamma_{\theta_z}}C_{\theta_z} \right] \dot{\theta}_z \omega_{b_y} \\
& - \left[\frac{1}{4}(I_{r_z} - I_{r_x})C_{\theta_z}S_{\gamma_{\theta_z}}S_{\gamma_{\theta_z}} - \frac{1}{2}I_zS_{\gamma_{\theta_z}}C_{\theta_z}^2 \right] \omega_{b_y}^2 \\
& - \left[\frac{1}{2}(I_{r_z} - I_{r_x})S_{\theta_z}S_{\gamma_{\theta_z}}S_{\theta_z} - I_zC_{\gamma_{\theta_z}}C_{\theta_z} \right] \omega_{b_y} \omega_{b_z} \\
& - \left[I_zS_{\gamma_{\theta_z}} \right] \dot{\theta}_z \omega_{b_z} \\
& - \left[\frac{1}{2}I_zS_{\gamma_{\theta_z}} \right] \omega_{b_z}^2
\end{aligned} \tag{A-43}$$

$$\begin{aligned}
f_{\tilde{u}_z}(\mathbf{x}, \dot{\mathbf{x}}, \tilde{\omega}_z, \dot{\tilde{\omega}}_z) = & I_r C_{\theta_z} S_{\theta_z} \dot{\omega}_{b_z} - I_r C_{\theta_z} C_{\theta_z} \dot{\omega}_{b_z} - I_r S_{\theta_z} \dot{\omega}_{b_z} \\
& - \left[\frac{1}{2} (I_r - I_r) S_{2\theta_z} \right] \dot{\theta}_x^2 \\
& - \left[(I_r - I_r) C_{2\theta_z} - I_r \right] C_{\theta_z} \dot{\theta}_x \dot{\theta}_z \\
& - \left[\frac{1}{2} (I_r - I_r) C_{\theta_z}^2 S_{2\theta_z} \right] \dot{\theta}_z^2 \\
& - \left[(I_r - I_r) (S_{2\theta_z} C_{\theta_z} + S_{\theta_z} C_{2\theta_z} S_{\theta_z}) - I_r S_{\theta_z} S_{\theta_z} \right] \dot{\theta}_x \omega_{b_z} \\
& - \left[(I_r - I_r) \left(C_{\theta_z} C_{2\theta_z} C_{\theta_z} - \frac{1}{2} S_{2\theta_z} S_{2\theta_z} S_{\theta_z} \right) - I_r C_{\theta_z} C_{\theta_z} \right] \dot{\theta}_z \omega_{b_z} \\
& - \left[\frac{1}{2} (I_r - I_r) (S_{\theta_z} C_{2\theta_z} S_{2\theta_z} - S_{2\theta_z} (C_{\theta_z}^2 - S_{\theta_z}^2 S_{\theta_z}^2)) \right] \omega_{b_z}^2 \\
& - \left[(I_r - I_r) \left(\frac{1}{2} S_{\theta_z}^2 S_{2\theta_z} S_{2\theta_z} - S_{\theta_z} C_{2\theta_z} C_{2\theta_z} + \frac{1}{2} S_{2\theta_z} C_{2\theta_z} \right) \right] \omega_{b_z} \omega_{b_z} \\
& - \left[\frac{1}{2} (I_r - I_r) (C_{\theta_z} C_{2\theta_z} C_{\theta_z} - S_{2\theta_z} S_{2\theta_z} S_{\theta_z}) \right] \omega_{b_z} \omega_{b_z} \\
& - \left[(I_r - I_r) (S_{2\theta_z} S_{\theta_z} - S_{\theta_z} C_{2\theta_z} C_{\theta_z}) - I_r S_{\theta_z} C_{\theta_z} \right] \dot{\theta}_x \omega_{b_z} \\
& - \left[(I_r - I_r) \left(\frac{1}{2} S_{2\theta_z} S_{2\theta_z} C_{\theta_z} - C_{\theta_z} C_{2\theta_z} S_{\theta_z} \right) - I_r C_{\theta_z} S_{\theta_z} \right] \dot{\theta}_z \omega_{b_z} \\
& - \left[\frac{1}{2} (I_r - I_r) (S_{\theta_z} C_{2\theta_z} S_{2\theta_z} - S_{2\theta_z} (S_{\theta_z}^2 - S_{\theta_z}^2 C_{\theta_z}^2)) \right] \omega_{b_z}^2 \\
& - \left[(I_r - I_r) \left(\frac{1}{2} S_{2\theta_z} S_{2\theta_z} C_{\theta_z} + C_{\theta_z} C_{2\theta_z} S_{\theta_z} \right) \right] \omega_{b_z} \omega_{b_z} \\
& - \left[(I_r - I_r) C_{2\theta_z} - I_r \right] C_{\theta_z} \dot{\theta}_x \omega_{b_z} \\
& - \left[(I_r - I_r) C_{\theta_z}^2 S_{2\theta_z} \right] \dot{\theta}_z \omega_{b_z} \\
& - \left[\frac{1}{2} (I_r - I_r) C_{\theta_z}^2 S_{2\theta_z} \right] \omega_{b_z}^2
\end{aligned} \tag{A-44}$$

$$\begin{aligned}
f_{n_1}(\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}, \dot{\ddot{\mathbf{x}}}) = & \left[\frac{1}{2}(I_z - I_x)C_{\theta_1}S_{2\theta_1}S_{\theta_2} - I_2S_{2\theta_1}C_{\theta_2} \right] \dot{\omega}_{b_z} \\
& - \left[\frac{1}{2}(I_z - I_x)C_{\theta_1}S_{2\theta_1}S_{\theta_2} - \frac{I_2}{2}S_{2\theta_1}C_{\theta_2} \right] \dot{\omega}_{b_y} \\
& - [I_z] \dot{\omega}_{b_z} \\
& - \left[\frac{1}{2}(I_z - I_x)S_{\theta_1}S_{2\theta_1} \right] \dot{\theta}_x^2 \\
& - \left[(I_z - I_x)C_{\theta_1}C_{2\theta_1} + I_2C_{\theta_1} \right] \dot{\theta}_x \dot{\theta}_y \\
& - [I_2S_{2\theta_1}] \dot{\theta}_x \dot{\theta}_z \\
& - \left[(I_z - I_x)C_{\theta_1}^2S_{2\theta_1} \right] \dot{\theta}_y \dot{\theta}_z \\
& - \left[\frac{1}{2}(I_z - I_x)S_{\theta_1}S_{2\theta_1}C_{\theta_2} + I_2C_{2\theta_1}S_{\theta_2} \right] \dot{\theta}_x \omega_{b_z} \\
& - \left[(I_z - I_x)(C_{2\theta_1}C_{\theta_2} - S_{\theta_1}S_{2\theta_1}S_{\theta_2})C_{\theta_1} \right] \dot{\theta}_y \omega_{b_z} \\
& - \left[\frac{1}{2}(I_z - I_x)C_{\theta_1}S_{2\theta_1}S_{\theta_2} - \frac{1}{2}I_2S_{2\theta_1}C_{\theta_2} \right] \dot{\theta}_z \omega_{b_z} \\
& - \left[\frac{1}{2}(I_z - I_x)S_{\theta_1}S_{2\theta_1}S_{\theta_2} + I_2C_{2\theta_1}C_{\theta_2} \right] \dot{\theta}_x \omega_{b_y} \\
& - \left[(I_z - I_x)(C_{2\theta_1}S_{\theta_2} - S_{\theta_1}S_{2\theta_1}C_{\theta_2})C_{\theta_1} \right] \dot{\theta}_y \omega_{b_y} \\
& - \left[\frac{1}{2}(I_z - I_x)C_{\theta_1}S_{2\theta_1}C_{\theta_2} + \frac{1}{2}I_2S_{2\theta_1}S_{\theta_2} \right] \dot{\theta}_z \omega_{b_y} \\
& + [I_2S_{2\theta_1}] \dot{\theta}_x \omega_{b_z} \\
& + \left[(I_z - I_x)C_{\theta_1}^2S_{2\theta_1} \right] \dot{\theta}_y \omega_{b_z}
\end{aligned} \tag{A-45}$$

Appendix B

Linear Analysis of Tuned-Rotor Instrument Dynamics

B.1 Poles and Zeros of Complete Transfer Function

The following section illustrates how the poles and zeros of the Complete Transfer Function $G_z(s)$ can be determined in a straightforward manner by first determining the poles and zeros of the equivalent gimbal-referenced transfer function $\mathbf{G}_{d1}(s)$. From equation (5-115), $\mathbf{G}_{d1}(s)$ is given by the following expression:

$$\mathbf{G}_{d1}(s) = \frac{\bar{n}_g(s)}{I_{\zeta}' \mathbf{d}(s)} \quad (\text{B.1})$$

The poles of $\mathbf{G}_{d1}(s)$ are determined first. The denominator polynomial $\mathbf{d}(s)$ is defined in equation (5-116) and can be expressed in terms of factors of $s^2 - \omega_{\zeta}^2$ as shown below:

$$\begin{aligned}
 \mathbf{d}(s) = & \left[1 - \gamma^2 \right] (s^2 - \omega_s^2)^2 \\
 & - \left[\omega_n (\omega_n - 2\omega_s) + 2(\alpha - \gamma\beta) - 4\gamma\omega_s (\omega_n - \omega_s) + 4\gamma^2\omega_s^2 \right] (s^2 - \omega_s^2) \\
 & - \left[2\alpha (\omega_n - 2\omega_s) + \alpha^2 - \beta^2 + 4\alpha\gamma\omega_s^2 \right]
 \end{aligned} \quad (\text{B-2})$$

Treating the denominator $\mathbf{d}(s)$ as a quadratic equation in terms of $s^2 - \omega_s^2$, the roots of the polynomial are:

$$\begin{aligned}
 s^2 - \omega_s^2 = & - \frac{\omega_n (\omega_n - 2\omega_s) + 2(\alpha - \gamma\beta) - 4\gamma\omega_s (\omega_n - \omega_s)}{2(1 - \gamma^2)} \\
 & \pm \sqrt{1 - \frac{4(\alpha - \gamma\beta)^2}{\omega_n^2 (\omega_n - 2\omega_s)^2} - \frac{16\gamma^2\omega_s^2 (\omega_n - \omega_s)^2}{\omega_n^2 (\omega_n - 2\omega_s)^2}} \\
 = & \frac{\omega_n (\omega_n - 2\omega_s)}{2(1 - \gamma^2)} \pm \sqrt{\frac{4(\alpha - \gamma\beta)}{\omega_n (\omega_n - 2\omega_s)} - \frac{8\gamma\omega_s (\omega_n - 2\omega_s)}{\omega_n (\omega_n - 2\omega_s)}} \\
 & \pm \sqrt{\frac{16\gamma\omega_s (\alpha - \gamma\beta)(\omega_n - \omega_s)}{\omega_n^2 (\omega_n - 2\omega_s)^2} - \frac{8\alpha\omega_s (1 - \gamma^2)}{\omega_n (\omega_n - 2\omega_s)}} \\
 & \pm \sqrt{\frac{4(\alpha^2 - \beta^2)(1 - \gamma^2)}{\omega_n^2 (\omega_s - 2\omega_s)^2} - \frac{16\alpha\gamma\omega_s^2 (1 - \gamma^2)}{\omega_n^2 (\omega_s - 2\omega_s)^2}}
 \end{aligned} \quad (\text{B-3})$$

It is now assumed that for terms within the square root sign,

$$1 - \gamma^2 \approx 1 \quad (\text{B-4})$$

The above expression can be simplified by noting that the gimbal inertias (I_{g_1}, I_{g_2}) are roughly two orders of magnitude smaller than the rotor inertias (I_r, I_r). As a result, the expressions within the square root are all small as compared with unity. A binomial expansion can be applied by noting the following:

$$\sqrt{1+x} = 1 - \frac{x}{2} - \frac{x^2}{8} + \dots \quad (\text{B-5})$$

Employing (B-5), the expression in the square root term can be expanded out to include second order terms. The resulting expressions for the roots $s^2 - \omega_s^2$ of the denominator polynomial $\mathbf{d}(s)$ take the form:

$$\begin{aligned} s^2 - \omega_s^2 \approx \frac{1}{(1-\gamma^2)} & \left\{ \frac{-\omega_n}{2}(\omega_n - 2\omega_s) - (\alpha - \gamma\beta) - 2\gamma\omega_s(\omega_n - \omega_s) \right. \\ & \pm \left[\frac{\omega_n}{2}(\omega_n - 2\omega_s) + (\alpha - \gamma\beta) - 2\gamma\omega_s(\omega_n - \omega_s) \right. \\ & \left. \left. - \frac{2\alpha\omega_s}{\omega_n} - \frac{(\alpha^2 - \beta^2)}{\omega_n(\omega_n - 2\omega_s)} - \frac{4\alpha\gamma\omega_s^2}{\omega_n(\omega_n - 2\omega_s)} \right] \right\} \end{aligned} \quad (\text{B-6})$$

The term $\frac{1}{1-\gamma^2}$ can be modified by noting that γ^2 is very small compared with 1.

Therefore, $\frac{1}{1-\gamma^2}$ can then be expressed in term of a binomial expansion, expanding out to only first order terms. This term takes the following form:

$$\frac{1}{1-\gamma^2} \approx 1 + \gamma^2 \quad (\text{B-7})$$

Substituting this approximation in the solution for $s^2 - \omega_s^2$ yields

$$\begin{aligned}
s^2 + \omega_s^2 &\approx \frac{-\omega_n}{2}(\omega_n - 2\omega_s) - (\alpha - \gamma\beta) - 2\gamma\omega_s(\omega_n - \omega_s) - \frac{\gamma^2\omega_n(\omega_n - 2\omega_s)}{2} \\
&= \frac{\omega_n}{2}(\omega_n - 2\omega_s) - (\alpha - \gamma\beta) + 2\gamma\omega_s(\omega_n - \omega_s) - \frac{2\alpha\omega_s}{\omega_n} \\
&\quad - \frac{(\alpha^2 - \beta^2)}{\omega_n(\omega_n - 2\omega_s)} + \frac{\gamma^2\omega_n(\omega_n - 2\omega_s)}{2} - \frac{4\alpha\gamma\omega_s^2}{\omega_n(\omega_n - 2\omega_s)} \quad (B-8)
\end{aligned}$$

Therefore, the two roots s^2 of the fourth order denominator polynomial $\mathbf{d}(s)$ are

$$\begin{aligned}
s^2 &= -\omega_s^2 - \frac{2\alpha\omega_s}{\omega_n} - \frac{(\alpha^2 - \beta^2)}{\omega_n(\omega_n - 2\omega_s)} - \frac{4\alpha\gamma\omega_s^2}{\omega_n(\omega_n - 2\omega_s)} \\
s^2 &= -(\omega_n - \omega_s)^2 - 2(\alpha - \gamma\beta) - 4\gamma\omega_s(\omega_n - \omega_s) - \frac{2\alpha\omega_s}{\omega_n} - \frac{(\alpha^2 - \beta^2)}{\omega_n(\omega_n - 2\omega_s)} \\
&\quad - \gamma^2\omega_n(\omega_n - 2\omega_s) - \frac{4\alpha\gamma\omega_s^2}{\omega_n(\omega_n - 2\omega_s)} \quad (B-9)
\end{aligned}$$

Taking the square roots of each of the terms in (B-9) yields the following expressions for the four roots of the gimbal referenced denominator polynomial $\mathbf{d}(s)$:

$$\begin{aligned}
s &= \pm j\omega_s \sqrt{1 - \frac{2\alpha}{\omega_s\omega_n} - \frac{(\alpha^2 - \beta^2)}{\omega_s^2\omega_n(\omega_n - 2\omega_s)} - \frac{4\alpha\gamma}{\omega_n(\omega_n - 2\omega_s)}} \\
s &= \pm j(\omega_n - \omega_s) \sqrt{1 + \frac{2(\alpha - \gamma\beta)}{(\omega_n - \omega_s)^2} + \frac{4\gamma\omega_s}{(\omega_n - \omega_s)} - \frac{2\alpha\omega_s}{\omega_n(\omega_n - \omega_s)^2}} \\
&\quad - \frac{(\alpha^2 - \beta^2)}{\omega_n(\omega_n - \omega_s)^2(\omega_n - 2\omega_s)} - \frac{\gamma^2\omega_n(\omega_n - 2\omega_s)}{(\omega_n - \omega_s)^2} \\
&\quad - \frac{2\alpha\gamma\omega_s^2}{\omega_n(\omega_n - 2\omega_s)(\omega_n - \omega_s)^2} \quad (B-10)
\end{aligned}$$

The expressions within the square roots can again be simplified by employing the same binomial expansion given in (B-5). Including terms up to second order, the expressions in (B-10) can be expressed in the following form:

$$\begin{aligned}
 s = & \pm j \left[\omega_s + \frac{\alpha}{\omega_n} + \frac{(\alpha^2 - \beta^2)}{2\omega_s\omega_n(\omega_n - 2\omega_s)} + \frac{2\alpha\gamma\omega_s}{\omega_n(\omega_n - 2\omega_s)} \right] \\
 s = & \pm j \left[(\omega_n - \omega_s) + \frac{(\alpha - \gamma\beta)}{(\omega_n - \omega_s)} + 2\gamma\omega_s - \frac{\alpha\omega_s}{\omega_n(\omega_n - \omega_s)} \right. \\
 & \left. - \frac{(\alpha^2 - \beta^2)}{2\omega_n(\omega_n - \omega_s)(\omega_n - 2\omega_s)} - \frac{\gamma^2\omega_n(\omega_n - 2\omega_s)}{2(\omega_n - \omega_s)^2} - \frac{2\alpha\gamma\omega_s^2}{\omega_n(\omega_n - 2\omega_s)(\omega_n - \omega_s)} \right]
 \end{aligned} \tag{B-11}$$

Finally, the poles of the case-referenced transfer function denominator $d(s)$ can be found by noting that $d(s) = \mathbf{d}(s - j\omega_s)$. Therefore the four poles of $d(s)$ can be found by substituting $s - j\omega_s$ for s in the equation (B-11) above. Solving for s yields the following four case-referenced poles of the system:

$$\begin{aligned}
 p_1 = & j \left[2\omega_s + \frac{\alpha}{\omega_n} + \frac{(\alpha^2 - \beta^2)}{2\omega_s\omega_n(\omega_n - 2\omega_s)} + \frac{2\alpha\gamma\omega_s}{\omega_n(\omega_n - 2\omega_s)} \right] \\
 p_2 = & j \left[\omega_n + \frac{(\alpha - \gamma\beta)}{(\omega_n - \omega_s)} + 2\gamma\omega_s - \frac{\alpha\omega_s}{\omega_n(\omega_n - \omega_s)} \right. \\
 & \left. - \frac{(\alpha^2 - \beta^2)}{2\omega_n(\omega_n - \omega_s)(\omega_n - 2\omega_s)} + \frac{\gamma^2\omega_n(\omega_n - 2\omega_s)}{2(\omega_n - \omega_s)^2} - \frac{2\alpha\gamma\omega_s^2}{\omega_n(\omega_n - 2\omega_s)(\omega_n - \omega_s)} \right] \\
 p_3 = & j \left[(2\omega_s - \omega_n) + \frac{(\alpha - \gamma\beta)}{(\omega_n - \omega_s)} + 2\gamma\omega_s + \frac{\alpha\omega_s}{\omega_n(\omega_n - \omega_s)} \right. \\
 & \left. - \frac{(\alpha^2 - \beta^2)}{2\omega_n(\omega_n - \omega_s)(\omega_n - 2\omega_s)} - \frac{\gamma^2\omega_n(\omega_n - 2\omega_s)}{2(\omega_n - \omega_s)^2} + \frac{2\alpha\gamma\omega_s^2}{\omega_n(\omega_n - 2\omega_s)(\omega_n - \omega_s)} \right] \\
 p_4 = & j \left[-\frac{\alpha}{\omega_n} - \frac{(\alpha^2 - \beta^2)}{2\omega_s\omega_n(\omega_n - 2\omega_s)} - \frac{2\alpha\gamma\omega_s}{\omega_n(\omega_n - 2\omega_s)} \right]
 \end{aligned} \tag{B-12}$$

When the system is tuned according to the Traditional tuning condition ($\alpha = 0$), the expressions for the four poles of the system simplify to the following:

$$\begin{aligned}
 p_1 &= j \left[2\omega_s - \frac{\beta^2}{2\omega_s\omega_n(\omega_n - 2\omega_s)} \right] \\
 p_2 &= j \left[\omega_n - \frac{\gamma\beta}{(\omega_n - \omega_s)} + 2\gamma\omega_s - \frac{\beta^2}{2\omega_n(\omega_n - \omega_s)(\omega_n - 2\omega_s)} \right. \\
 &\quad \left. - \frac{\gamma^2\omega_n(\omega_n - 2\omega_s)}{2(\omega_n - \omega_s)} \right] \\
 p_3 &= j \left[(2\omega_s - \omega_n) - \frac{\gamma\beta}{(\omega_n - \omega_s)} - 2\gamma\omega_s - \frac{\beta^2}{2\omega_n(\omega_n - \omega_s)(\omega_n - 2\omega_s)} \right. \\
 &\quad \left. - \frac{\gamma^2\omega_n(\omega_n - 2\omega_s)}{2(\omega_n - \omega_s)} \right] \\
 p_4 &= j \left[\frac{\beta^2}{2\omega_s\omega_n(\omega_n - 2\omega_s)} \right]
 \end{aligned} \tag{B-13}$$

A similar approach can be used to determine the zeros of the gimbal referenced transfer function $\mathbf{G}_{d1}(s)$. This numerator polynomial $\bar{n}_z$ can be expressed in terms of the factor $(s - j\omega_s)$ where

$$\bar{n}_z(s) = \left\{ (s - j\omega_s)^2 - j[\omega_n + 2\gamma\omega_s](s - j\omega_s) - \alpha \right\} \tag{B-14}$$

The roots of the equation are:

$$s - j\omega_s = -\frac{j}{2}[\omega_n + 2\gamma\omega_s] \pm \frac{j\omega_n}{2} \sqrt{1 + \frac{4\gamma\omega_s}{\omega_n} + \frac{4\gamma^2\omega_s^2}{\omega_n^2} - \frac{4\alpha}{\omega_n^2}} \tag{B-15}$$

Like the analysis of the denominator polynomial, the terms within the square root can be expressed in terms of a binomial expansion by noting that all terms except the first are small with respect to unity. Expanding only to the first order terms, the expression in (B-15) simplifies to the following:

$$s - j\omega_s = j \left[-\frac{\omega_n}{2} - \gamma\omega_s \pm \left(\frac{\omega_n}{2} - \gamma\omega_s + \frac{\gamma^2\omega_s^2}{\omega_n} - \frac{\alpha}{\omega_n} \right) \right] \quad (\text{B-16})$$

Therefore, the two roots of the gimbal referenced numerator polynomial are:

$$s = j \left[\omega_s - \frac{\omega_n}{2} - \gamma\omega_s \pm \left(\frac{\omega_n}{2} - \gamma\omega_s + \frac{\gamma^2\omega_s^2}{\omega_n} - \frac{\alpha}{\omega_n} \right) \right] \quad (\text{B-17})$$

These two roots are the zeros of the polynomial $\bar{n}_2(s)$. The zeros of the case-referenced transfer function numerator $\bar{n}_2(s - j\omega_s)$ are found by substituting $s - j\omega_s$ for s in (B-17)

The resulting two case-referenced zeros are:

$$\begin{aligned} z_1 &= j \left[2\omega_s + \frac{\gamma^2\omega_s^2}{\omega_n} - \frac{\alpha}{\omega_n} \right] \\ z_2 &= j \left[(2\omega_s - \omega_n) - 2\gamma\omega_s - \frac{\gamma^2\omega_s^2}{\omega_n} - \frac{\alpha}{\omega_n} \right] \end{aligned} \quad (\text{B-18})$$

B.2 Closed-Form Solution to Transverse-Axes Dynamics

As outlined in Section 6.2, the approach taken to determine the solution to the transverse-axes dynamics equations of motion was to first find the solution in gimbal-referenced coordinates, i.e. in spinning coordinates. Then the solution can be transformed to a form

expressed in terms of non-spinning or case-referenced coordinates. The transverse-axes dynamics equations of equation (6-29) are presented below for completeness:

$$\dot{\mathbf{x}}_g = \mathbf{A}_g \mathbf{x}_g - \mathbf{B}_g \mathbf{u}_g \quad (\text{B.19})$$

The gimbal referenced state vector $\mathbf{x}_g$, control input $\mathbf{u}_g$, state matrix $\mathbf{A}_g$ and control matrix $\mathbf{B}_g$ are defined below:

$$\mathbf{x}_g = [\bar{\theta}_x \quad \theta_y \quad \dot{\theta}_x \quad \dot{\theta}_y]^{-T} \quad (\text{B.20})$$

$$\mathbf{u}_g = [T_{z_x} \quad T_{z_y}]^{-T} \quad (\text{B.21})$$

$$\mathbf{A}_g = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \bar{k}_1 & 0 & 0 & \bar{c}_1 \\ 0 & \bar{k}_2 & \bar{c}_2 & 0 \end{bmatrix} \quad (\text{B.22})$$

$$\mathbf{B}_g = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{\cos \omega_s t}{I_{r_x} + I_{g_x}} & \frac{\sin \omega_s t}{I_{r_x} - I_{g_x}} \\ \frac{-\sin \omega_s t}{I_{r_y}} & \frac{\cos \omega_s t}{I_{r_y}} \end{bmatrix} \quad (\text{B.23})$$

$$\bar{k}_1 = \frac{(I_{r_x} - I_{r_z} - I_{g_z} - 2J)\omega_s^2 - K_x}{I_{r_x} + I_{g_x}} \quad (\text{B.24})$$

$$\bar{k}_2 = \frac{(I_{r_x} - I_{r_z})\omega_s^2 - K_x}{I_{r_y}} \quad (\text{B.25})$$

$$\bar{c}_1 = \frac{(I_{r_x} + I_{r_y} - I_{r_z})\omega_s}{I_{r_x} + I_{g_x}} \quad (\text{B.26})$$

$$\bar{c}_z = \frac{-(I_{r_x} + I_{r_y} - I_{r_z})\omega_s}{I_{r_y}} \quad (\text{B-27})$$

The response of the gimbal-referenced state vector $\mathbf{x}_g(t)$ in terms of the initial state vector $\mathbf{x}_g(t_0)$ and constant applied case-referenced torques $\mathbf{u}_g$ is desired. For the state-space model given in (B.19), the general solution is given by the following expression:

$$\mathbf{x}_g(t) = e^{\mathbf{A}_g t} \mathbf{x}_g(t_0) + \int_0^t e^{\mathbf{A}_g(t-\tau)} \mathbf{B}_g \mathbf{u}_g(\tau) d\tau \quad (\text{B-28})$$

The system's state transition matrix $e^{\mathbf{A}_g t}$ is determined using the following expression:

$$e^{\mathbf{A}_g t} = \mathbf{U} e^{\mathbf{A}_t t} \mathbf{U}^{-1} \quad (\text{B-29})$$

$$e^{\mathbf{A}_t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 & 0 \\ 0 & 0 & e^{\lambda_3 t} & 0 \\ 0 & 0 & 0 & e^{\lambda_4 t} \end{bmatrix} \quad (\text{B-30})$$

$\mathbf{U}$ is the matrix of eigenvectors of the system with state matrix $\mathbf{A}_g$. For the state matrix of the form given in equation (B.22), the eigenvalues can be expressed in the following form:

$$\lambda_{1,2} = \frac{\sqrt{(\bar{k}_1 + \bar{k}_2 + \bar{c}_1 \cdot \bar{c}_2) + \sqrt{(\bar{k}_1 + \bar{k}_2 - \bar{c}_1 \cdot \bar{c}_2)^2 - 4\bar{k}_1 \cdot \bar{k}_2}}}{2} \quad (\text{B-31})$$

$$\lambda_{3,4} = \pm \sqrt{\frac{(\bar{k}_1 + \bar{k}_2 + \bar{c}_1 \cdot \bar{c}_2) - \sqrt{(\bar{k}_1 + \bar{k}_2 + \bar{c}_1 \cdot \bar{c}_2)^2 - 4\bar{k}_1 \cdot \bar{k}_2}}{2}} \quad (\text{B-32})$$

Therefore $e^{\Lambda t}$ can be expressed in the following form:

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 & 0 \\ 0 & e^{-\lambda_1 t} & 0 & 0 \\ 0 & 0 & e^{\lambda_3 t} & 0 \\ 0 & 0 & 0 & e^{-\lambda_3 t} \end{bmatrix} \quad (\text{B-33})$$

The matrix of eigenvectors $\mathbf{U}$ and its inverse $\mathbf{V} = \mathbf{U}^{-1}$ can be determined using the symbolic algebra capability in MAPLE [48]. Common and offsetting terms can be eliminated manually, with the resulting matrices being:

$$\mathbf{U} = \begin{bmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ U_{21} & U_{22} & U_{23} & U_{24} \\ U_{31} & U_{32} & U_{33} & U_{34} \\ U_{41} & U_{42} & U_{43} & U_{44} \end{bmatrix} \quad (\text{B-34})$$

$$\mathbf{U}^{-1} = \mathbf{V} = \begin{bmatrix} V_{11} & V_{12} & V_{13} & V_{14} \\ V_{21} & V_{22} & V_{23} & V_{24} \\ V_{31} & V_{32} & V_{33} & V_{34} \\ V_{41} & V_{42} & V_{43} & V_{44} \end{bmatrix} \quad (\text{B-35})$$

The individual elements $U_{i,j}$ and $V_{i,j}$ are defined in the following expressions:

$$U_{11} = \frac{\lambda_1^2 - \bar{k}_2 - \bar{c}_1 \cdot \bar{c}_2}{\bar{k}_1 \cdot \bar{c}_2}$$

$$U_{12} = U_{11}$$

$$U_{13} = \frac{\lambda_3^2 - \bar{k}_2 - \bar{c}_1 \cdot \bar{c}_2}{\bar{k}_1 \cdot \bar{c}_2}$$

$$U_{14} = U_{13}$$

$$U_{21} = \frac{-\lambda_1(\lambda_1^2 - \bar{k}_1 - \bar{k}_2 - \bar{c}_1 \cdot \bar{c}_2)}{\bar{k}_1 \cdot \bar{k}_2}$$

$$U_{22} = -U_{21}$$

$$U_{23} = \frac{-\lambda_3(\lambda_3^2 - \bar{k}_1 - \bar{k}_2 - \bar{c}_1 \cdot \bar{c}_2)}{\bar{k}_1 \cdot \bar{k}_2}$$

$$U_{24} = -U_{23}$$

$$U_{31} = \frac{\lambda_1(\lambda_1^2 - \bar{k}_2 - \bar{c}_1 \cdot \bar{c}_2)}{\bar{k}_1 \cdot \bar{c}_2}$$

$$U_{32} = U_{31}$$

$$U_{33} = \frac{\lambda_3(\lambda_3^2 - \bar{k}_2 - \bar{c}_1 \cdot \bar{c}_2)}{\bar{k}_1 \cdot \bar{c}_2}$$

$$U_{34} = U_{33}$$

$$U_{41} = U_{42} = U_{43} = U_{44} = 1$$

$$V_{11} = \frac{-\bar{k}_1 \cdot \bar{c}_2}{2(\lambda_3^2 - \lambda_1^2)}$$

$$V_{12} = \frac{\bar{k}_2(\lambda_3^2 - \bar{k}_2 - \bar{c}_1 \cdot \bar{c}_2)}{2\lambda_1(\lambda_3^2 - \lambda_1^2)}$$

$$V_{13} = \frac{\bar{c}_2(\lambda_3^2 - \bar{k}_1 - \bar{k}_2 - \bar{c}_1 \cdot \bar{c}_2)}{2\lambda_1(\lambda_3^2 - \lambda_1^2)}$$

$$V_{14} = \frac{\lambda_3^2 - \bar{k}_2 - \bar{c}_1 \cdot \bar{c}_2}{2(\lambda_3^2 - \lambda_1^2)}$$

$$\begin{aligned}
V_{21} &= V_{11} \\
V_{22} &= -V_{12} \\
V_{23} &= -V_{13} \\
V_{24} &= V_{14} \\
V_{31} &= -V_{11} \\
V_{32} &= \frac{-\bar{k}_2(\lambda_1^2 - \bar{k}_2 - \bar{c}_1 \cdot \bar{c}_2)}{2\lambda_3(\lambda_3^2 - \lambda_1^2)} \\
V_{33} &= \frac{-\bar{c}_2(\lambda_1^2 - \bar{k}_1 - \bar{k}_2 - \bar{c}_1 \cdot \bar{c}_2)}{2\lambda_3(\lambda_3^2 - \lambda_1^2)} \\
V_{34} &= \frac{-(\lambda_1^2 - \bar{k}_2 - \bar{c}_1 \cdot \bar{c}_2)}{2(\lambda_3^2 - \lambda_1^2)} \\
V_{41} &= V_{31} \\
V_{42} &= -V_{32} \\
V_{43} &= -V_{33} \\
V_{44} &= V_{34}
\end{aligned} \tag{B-36}$$

The state transition matrix can now be expressed in the following form:

$$e^{At} = \begin{bmatrix} \sum_{k=1}^4 U_{1k} V_{k1} e^{\lambda_k t} & \sum_{k=1}^4 U_{1k} V_{k2} e^{\lambda_k t} & \sum_{k=1}^4 U_{1k} V_{k3} e^{\lambda_k t} & \sum_{k=1}^4 U_{1k} V_{k4} e^{\lambda_k t} \\ \sum_{k=1}^4 U_{2k} V_{k1} e^{\lambda_k t} & \sum_{k=1}^4 U_{2k} V_{k2} e^{\lambda_k t} & \sum_{k=1}^4 U_{2k} V_{k3} e^{\lambda_k t} & \sum_{k=1}^4 U_{2k} V_{k4} e^{\lambda_k t} \\ \sum_{k=1}^4 U_{3k} V_{k1} e^{\lambda_k t} & \sum_{k=1}^4 U_{3k} V_{k2} e^{\lambda_k t} & \sum_{k=1}^4 U_{3k} V_{k3} e^{\lambda_k t} & \sum_{k=1}^4 U_{3k} V_{k4} e^{\lambda_k t} \\ \sum_{k=1}^4 U_{4k} V_{k1} e^{\lambda_k t} & \sum_{k=1}^4 U_{4k} V_{k2} e^{\lambda_k t} & \sum_{k=1}^4 U_{4k} V_{k3} e^{\lambda_k t} & \sum_{k=1}^4 U_{4k} V_{k4} e^{\lambda_k t} \end{bmatrix} \tag{B-37}$$

To evaluate the response of the gimbal angles, the term $\mathbf{B}_g \mathbf{u}_g(t)$ is expressed in the following form:

$$\mathbf{B}_g \mathbf{u}_g(t) = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{I_{r_x} + I_{g_x}} \left\{ T_{c_x} \left[\frac{e^{j\omega_f t} + e^{-j\omega_f t}}{2} \right] + T_{c_y} \left[\frac{e^{j\omega_f t} - e^{-j\omega_f t}}{2j} \right] \right\} \\ \frac{1}{I_{r_y}} \left\{ -T_{c_x} \left[\frac{e^{j\omega_f t} - e^{-j\omega_f t}}{2j} \right] + T_{c_y} \left[\frac{e^{j\omega_f t} + e^{-j\omega_f t}}{2} \right] \right\} \end{bmatrix} \quad (\text{B-38})$$

Rather than evaluate the response of all four states, only the first two (θ_x, θ_y) are evaluated. The gimbal rates ($\dot{\theta}_x, \dot{\theta}_y$) can be obtained more easily by taking the first derivative with respect to time of the gimbal angles responses. Expressions for the time responses of the first two states are shown below:

$$\begin{aligned} x_{g1} = \theta_x = \sum_{k=1}^4 U_{1k} & \left\{ V_{k1} e^{\lambda_k t} \theta_x(0) - V_{k2} e^{\lambda_k t} \theta_y(0) \right. \\ & - V_{k3} e^{\lambda_k t} \dot{\theta}_x(0) - V_{k4} e^{\lambda_k t} \dot{\theta}_y(0) \\ & - \left\{ \frac{V_{k3}}{I_{r_x} + I_{g_x}} [\omega_s S_{\omega_f t} - \lambda_k C_{\omega_f t} + \lambda_k e^{\lambda_k t}] \right. \\ & \quad \left. + \frac{V_{k4}}{I_{r_y}} [\omega_s C_{\omega_f t} + \lambda_k S_{\omega_f t} - \omega_s e^{\lambda_k t}] \right\} T_{c_x} \\ & - \left\{ \frac{-V_{k3}}{I_{r_x} + I_{g_x}} [\omega_s C_{\omega_f t} + \lambda_k S_{\omega_f t} - \omega_s e^{\lambda_k t}] \right. \\ & \quad \left. + \frac{V_{k4}}{I_{r_y}} [\omega_s S_{\omega_f t} - \lambda_k C_{\omega_f t} - \lambda_k e^{\lambda_k t}] \right\} T_{c_y} \left. \right\} \end{aligned} \quad (\text{B-39})$$

$$\begin{aligned}
x_{g_2} = \theta_y = \sum_{k=1}^4 U_{2k} & \left\{ \begin{aligned} & V_{k1} e^{\lambda_k t} \theta_x(0) + V_{k2} e^{\lambda_k t} \theta_y(0) \\ & + V_{k3} e^{\lambda_k t} \dot{\theta}_x(0) + V_{k4} e^{\lambda_k t} \dot{\theta}_y(0) \\ & + \left[\frac{V_{k3}}{I_{r_1} + I_{g_1}} \left[\omega_s S_{\omega_s t} - \lambda_k C_{\omega_s t} + \lambda_k e^{\lambda_k t} \right] \right. \\ & \quad \left. - \frac{V_{k4}}{I_{r_2}} \left[\omega_s C_{\omega_s t} - \lambda_k S_{\omega_s t} - \omega_s e^{\lambda_k t} \right] \right] T_{c_1} \\ & + \left[\frac{-V_{k3}}{I_{r_1} + I_{g_1}} \left[\omega_s C_{\omega_s t} - \lambda_k S_{\omega_s t} - \omega_s e^{\lambda_k t} \right] \right. \\ & \quad \left. - \frac{V_{k4}}{I_{r_2}} \left[\omega_s S_{\omega_s t} - \lambda_k C_{\omega_s t} + \lambda_k e^{\lambda_k t} \right] \right] T_{c_2} \end{aligned} \right\} \quad (B-40)
\end{aligned}$$

The response of the case-referenced angles describing the tilt of the spin axis can now be found by noting their relation to the gimbal angles derived in Section 5.1.3:

$$\begin{bmatrix} \phi_x \\ \phi_y \end{bmatrix} = \begin{bmatrix} C_{\omega_s t} & -S_{\omega_s t} \\ S_{\omega_s t} & C_{\omega_s t} \end{bmatrix} \begin{bmatrix} \theta_x \\ \theta_y \end{bmatrix} \quad (B-41)$$

Substituting equations (B-36) and (B-41) into (B-40) and performing considerable simplification yields the following expressions for the response of the case-referenced angles ϕ_x and ϕ_y :

$$\begin{aligned}
\phi_x(t) = & \left[\sum_{j=1}^4 S_{1j} \cos \omega_j t \right] \phi_x(0) + \left[\sum_{j=1}^4 S_{2j} \sin \omega_j t \right] \phi_y(0) \\ & + \left[\sum_{j=1}^4 R_{1j} \sin \omega_j t \right] \dot{\phi}_x(0) + \left[\sum_{j=1}^4 R_{2j} \cos \omega_j t \right] \dot{\phi}_y(0) \\ & + \left[D_{10} + \sum_{j=1}^4 D_{1j} \cos \omega_j t - D_{15} \cos 2\omega_s t \right] T_{c_1} \\ & - \left[\sum_{j=1}^4 D_{2j} \sin \omega_j t - D_{25} \sin 2\omega_s t \right] T_{c_2} \quad (B-42)
\end{aligned}$$

$$\begin{aligned}
\phi_y(t) = & \begin{bmatrix} \sum_{j=1}^4 S_{1j} \sin \omega_j t \\ \sum_{j=1}^4 R_{1j} \cos \omega_j t \end{bmatrix} \phi_x(0) - \begin{bmatrix} \sum_{j=1}^4 S_{2j} \cos \omega_j t \\ \sum_{j=1}^4 R_{2j} \sin \omega_j t \end{bmatrix} \phi_y(0) \\
& - \begin{bmatrix} \sum_{j=1}^4 D_{1j} \sin \omega_j t + D_{15} \sin 2\omega_s t \\ D_{20} + \sum_{j=1}^4 D_{2j} \cos \omega_j t + D_{25} \cos 2\omega_s t \end{bmatrix} \begin{bmatrix} T_{c_1} \\ T_{c_2} \end{bmatrix}
\end{aligned} \tag{B-43}$$

The position influence coefficients $S_{i,j}$, rate influence coefficients $R_{i,j}$ and compliance terms $D_{i,j}$ are given by the following expressions:

$$\begin{aligned}
S_{11} &= V_{11}(U_{11} + jU_{21}) \\
S_{12} &= V_{11}(U_{11} - jU_{21}) \\
S_{13} &= V_{31}(U_{13} + jU_{23}) \\
S_{14} &= V_{31}(U_{13} - jU_{23}) \\
S_{21} &= -jV_{12}(U_{11} + jU_{21}) \\
S_{22} &= -jV_{12}(U_{11} - jU_{21}) \\
S_{23} &= -jV_{32}(U_{13} + jU_{23}) \\
S_{24} &= -jV_{32}(U_{13} - jU_{23})
\end{aligned} \tag{B-44}$$

$$\begin{aligned}
R_{11} &= jV_{13}(U_{11} - jU_{21}) \\
R_{12} &= -jV_{13}(U_{11} + jU_{21}) \\
R_{13} &= jV_{33}(U_{13} - jU_{23}) \\
R_{14} &= -jV_{33}(U_{13} + jU_{23}) \\
R_{21} &= V_{14}(U_{11} + jU_{21}) \\
R_{22} &= V_{14}(U_{11} - jU_{21}) \\
R_{23} &= V_{34}(U_{13} + jU_{23}) \\
R_{24} &= V_{34}(U_{13} - jU_{23})
\end{aligned} \tag{B-45}$$

$$D_{10} = \left\langle \begin{bmatrix} U_{11} \left(\frac{-V_{13}\lambda_1}{I_r + I_{g1}} - \frac{V_{14}\omega_s}{I_r} \right) - U_{21} \left(\frac{V_{13}\omega_s}{I_r + I_{g1}} + \frac{V_{14}\lambda_1}{I_r} \right) \right] \frac{1}{\omega_s^2 - \lambda_1^2} \\ + \left[U_{13} \left(\frac{-V_{33}\lambda_3}{I_r + I_{g1}} + \frac{V_{34}\omega_s}{I_r} \right) - U_{23} \left(\frac{V_{33}\omega_s}{I_r + I_{g1}} + \frac{V_{34}\lambda_3}{I_r} \right) \right] \frac{1}{\omega_s^2 + \lambda_3^2} \end{bmatrix} \right\rangle \quad (\text{B-46})$$

$$D_{11} = - \left(\frac{U_{11} + jU_{21}}{\omega_s^2 + \lambda_1^2} \right) \left(\frac{-V_{13}\lambda_1}{I_r - I_{g1}} - \frac{V_{14}\omega_s}{I_r} \right) \quad (\text{B-47})$$

$$D_{12} = - \left(\frac{U_{11} - jU_{21}}{\omega_s^2 + \lambda_1^2} \right) \left(\frac{-V_{13}\lambda_1}{I_r - I_{g1}} - \frac{V_{14}\omega_s}{I_r} \right) \quad (\text{B-48})$$

$$D_{13} = - \left(\frac{U_{13} + jU_{23}}{\omega_s^2 + \lambda_3^2} \right) \left(\frac{-V_{33}\lambda_3}{I_r - I_{g1}} - \frac{V_{34}\omega_s}{I_r} \right) \quad (\text{B-49})$$

$$D_{14} = - \left(\frac{U_{13} - jU_{23}}{\omega_s^2 + \lambda_3^2} \right) \left(\frac{-V_{33}\lambda_3}{I_r - I_{g1}} - \frac{V_{34}\omega_s}{I_r} \right) \quad (\text{B-50})$$

$$D_{15} = \left\langle \begin{bmatrix} U_{11} \left(\frac{-V_{13}\lambda_1}{I_r - I_{g1}} - \frac{V_{14}\omega_s}{I_r} \right) - U_{21} \left(\frac{V_{13}\omega_s}{I_r + I_{g1}} - \frac{V_{14}\lambda_1}{I_r} \right) \right] \frac{1}{\omega_s^2 + \lambda_1^2} \\ - \left[U_{13} \left(\frac{-V_{33}\lambda_3}{I_r - I_{g1}} - \frac{V_{34}\omega_s}{I_r} \right) + U_{23} \left(\frac{V_{33}\omega_s}{I_r + I_{g1}} - \frac{V_{34}\lambda_3}{I_r} \right) \right] \frac{1}{\omega_s^2 + \lambda_3^2} \end{bmatrix} \right\rangle \quad (\text{B-51})$$

$$D_{20} = D_{10} \quad (\text{B-52})$$

$$D_{21} = - \left(\frac{U_{11} + jU_{21}}{\omega_s^2 + \lambda_1^2} \right) \left(\frac{V_{13}\omega_s}{I_r - I_{g1}} + \frac{V_{14}\lambda_1}{I_r} \right) \quad (\text{B-53})$$

$$D_{22} = \left(\frac{U_{11} - jU_{21}}{\omega_s^2 + \lambda_1^2} \right) \left(\frac{V_{13}\omega_s}{I_r + I_{g1}} + \frac{V_{14}\lambda_1}{I_r} \right) \quad (\text{B-54})$$

$$D_{23} = - \left(\frac{U_{13} + jU_{23}}{\omega_s^2 - \lambda_3^2} \right) \left(\frac{V_{33}\omega_s}{I_r + I_{g_s}} - \frac{V_{34}\lambda_3}{I_r} \right) \quad (\text{B-55})$$

$$D_{24} = \left(\frac{U_{13} - jU_{23}}{\omega_s^2 + \lambda_3^2} \right) \left(\frac{V_{33}\omega_s}{I_r - I_{g_s}} - \frac{V_{34}\lambda_3}{I_r} \right) \quad (\text{B-56})$$

$$D_{25} = \left\{ \begin{aligned} & \left[U_{11} \left(\frac{-V_{13}\lambda_1}{I_r - I_{g_s}} - \frac{V_{14}\omega_s}{I_r} \right) + U_{21} \left(\frac{V_{13}\omega_s}{I_r - I_{g_s}} + \frac{V_{14}\lambda_1}{I_r} \right) \right] \frac{1}{\omega_s^2 - \lambda_1^2} \\ & - \left[U_{13} \left(\frac{-V_{33}\lambda_3}{I_r - I_{g_s}} - \frac{V_{34}\omega_s}{I_r} \right) - U_{23} \left(\frac{V_{33}\omega_s}{I_r - I_{g_s}} - \frac{V_{34}\lambda_3}{I_r} \right) \right] \frac{1}{\omega_s^2 - \lambda_3^2} \end{aligned} \right\} \quad (\text{B-57})$$

The four characteristic modes ω_k of the transverse-axis dynamics as viewed from the case frame are:

$$\omega_1 = \omega_s - \frac{\lambda_1}{j}, \omega_2 = \omega_s - \frac{\lambda_2}{j}, \omega_3 = \omega_s - \frac{\lambda_3}{j}, \omega_4 = \omega_s - \frac{\lambda_4}{j}; \quad (\text{B-58})$$

B.3 Alternate Method of Obtaining Complete Model Tuning Condition

As shown in the Section 6.4, the approximate solution used to obtain the tuning condition for the Complete dynamic model in Section 6.4 can still result in small drifts. To obtain a more accurate value of tuned speed, the exact expression for the precession pole is obtained from the gimbal-referenced characteristic polynomial $\mathbf{d}(s)$ of the gyro dynamics given in equation (6-8). It has the form

$$\mathbf{d}(s) = \mathbf{d}_2 (s^2 + \omega_s^2)^2 - \mathbf{d}_1 (s^2 + \omega_s^2) + \mathbf{d}_0 \quad (\text{B-59})$$

where

$$\begin{aligned} \mathbf{d}_2 &= (1 - \gamma^2) \\ \mathbf{d}_1 &= \omega_\pi (\omega_\pi - 2\omega_s) - 2(\alpha - \gamma\beta) - 4\gamma\omega_s (\omega_\pi - \omega_s) + 4\gamma^2\omega_s^2 \\ \mathbf{d}_0 &= 2\alpha(\omega_\pi - 2\omega_s)\omega_s - \alpha^2 - \beta^2 + 4\alpha\gamma\omega_s^2 \end{aligned} \quad (\text{B-60})$$

The four roots of $\mathbf{d}(s)$ are obtained from the following expression:

$$s = \pm \sqrt{-\omega_s^2 - \frac{\mathbf{d}_1}{2\mathbf{d}_2} \pm \frac{1}{2\mathbf{d}_2} \sqrt{\mathbf{d}_1^2 - 4\mathbf{d}_2\mathbf{d}_0}} \quad (\text{B-61})$$

The poles of the case-referenced characteristic polynomial $d(s)$ are found by noting that these roots are the same as the roots of $\mathbf{d}(s - j\omega_s)$. The roots of $\mathbf{d}(s - j\omega_s)$ are simply the roots of $\mathbf{d}(s)$, each shifted by $j\omega_s$, i.e.

$$s = j\omega_s \pm \sqrt{-\omega_s^2 - \frac{\mathbf{d}_1}{2\mathbf{d}_2} \pm \frac{1}{2\mathbf{d}_2} \sqrt{\mathbf{d}_1^2 - 4\mathbf{d}_2\mathbf{d}_0}} \quad (\text{B-62})$$

The following expression gives the value of the precession pole:

$$s' = j\omega_s - j \sqrt{\omega_s^2 - \frac{\mathbf{d}_1}{2\mathbf{d}_2} \pm \frac{1}{2\mathbf{d}_2} \sqrt{\mathbf{d}_1^2 - 4\mathbf{d}_2\mathbf{d}_0}} \quad (\text{B-63})$$

To eliminate the precession and tune the system, the expression within the square root sign must equal ω_s^2 . The numerical “method of iteration” can be used to determine the tuned speed using the following approach

- a) Define the spin rate as $\omega_s = f(\omega_\pi)$ where

$$f(\omega_s) = \sqrt{\omega_s^2 + \frac{\mathbf{d}_1}{2\mathbf{d}_2} + \frac{1}{2\mathbf{d}_2} \sqrt{\mathbf{d}_1^2 - 4\mathbf{d}_2\mathbf{d}_0}}$$

- b) Set the initial spin rate ω_s at the Traditional tuned speed.
- c) For an error tolerance ε , perform the following iterative procedure

While $|\omega_s - \omega_{s_last}| \leq \varepsilon$

Set $\omega_s = f(\omega_s)$

Set $\omega_{s_last} = \omega_s$

Appendix C

SCISCO Control System Design

C.1 Relationship Between $G(-j\omega)$ and $\bar{G}(j\omega)$

Consider the following transfer function:

$$G(s) = \frac{a_0 - a_1s + a_2s^2 - a_3s^3 + a_4s^4 - \dots}{b_0 - b_1s + b_2s^2 - b_3s^3 + b_4s^4 - \dots} \quad (\text{C-1})$$

The frequency responses $G(j\omega)$, $G(-j\omega)$ and $\bar{G}(j\omega)$ take the following form:

$$\begin{aligned} G(j\omega) &= \frac{a_0 + a_1j\omega - a_2\omega^2 - a_3j\omega^3 + a_4\omega^4 - \dots}{b_0 + b_1j\omega - b_2\omega^2 - b_3j\omega^3 + b_4\omega^4 - \dots} \\ G(-j\omega) &= \frac{a_0 - a_1j\omega - a_2\omega^2 - a_3j\omega^3 + a_4\omega^4 - \dots}{b_0 - b_1j\omega - b_2\omega^2 + b_3j\omega^3 + b_4\omega^4 - \dots} \\ \bar{G}(j\omega) &= \frac{\bar{a}_0 + \bar{a}_1j\omega - \bar{a}_2\omega^2 - \bar{a}_3j\omega^3 - \bar{a}_4\omega^4 - \dots}{\bar{b}_0 - \bar{b}_1j\omega - \bar{b}_2\omega^2 - \bar{b}_3j\omega^3 + \bar{b}_4\omega^4 - \dots} \end{aligned} \quad (\text{C-2})$$

Each coefficient a_k and b_k is now expressed in terms of a real and imaginary component such that

$$\begin{aligned}a_j &= a_{j_r} + ja_{j_i} \\ b_k &= b_{k_r} + jb_{k_i}\end{aligned}\tag{C-3}$$

The two transfer functions $G(-j\omega)$ and $\overline{G}(j\omega)$ can now be expressed in the following form:

$$\begin{aligned}G(-j\omega) &= \frac{A + jB}{C + jD} \\ \overline{G}(j\omega) &= \frac{A - jB}{C - jD}\end{aligned}\tag{C-4}$$

where the polynomials A , B , C and D are defined as:

$$\begin{aligned}A &= a_{0_r} - a_{1_i}\omega - a_{2_r}\omega^2 - a_{3_i}\omega^3 + a_{4_r}\omega^4 - \dots \\ B &= a_{0_i} - a_{1_r}\omega - a_{2_i}\omega^2 + a_{3_r}\omega^3 - a_{4_i}\omega^4 + \dots \\ C &= b_{0_r} + b_{1_i}\omega - b_{2_r}\omega^2 - b_{3_i}\omega^3 + b_{4_r}\omega^4 - \dots \\ D &= b_{0_i} - b_{1_r}\omega - b_{2_i}\omega^2 - b_{3_r}\omega^3 + b_{4_i}\omega^4 - \dots\end{aligned}\tag{C-5}$$

Therefore the magnitude of $G(-j\omega)$ and $\overline{G}(j\omega)$ are the same, while the phase angle of $G(-j\omega)$ is the negative of $\overline{G}(j\omega)$.

C.2 Determining Classical Root Locus

Evans [49] first proposed an approach for constructing standard root loci for loop gain functions with real-valued coefficients. Consider the following loop gain function of the SISO system shown in Figure 7-5:

$$P_c(s)P_{\phi n}(s)Q(s) = \frac{K(s-z_1)(s-z_2)\dots(s-z_m)}{(s-p_1)(s-p_2)\dots(s-p_n)} \quad (\text{C-6})$$

For the above system to be physically realizable, $m < n$.

One way to interpret the loop gain transfer function is to represent it graphically in the s -plane as described in [50]. In the transfer function of equation (C-6), the term $(s-z_i)$ represents a vector from z_i to s . Similarly, the term $(s-p_j)$ represents a vector from p_j to s . This is illustrated in Figure C-1 below.

Figure C-1: S-Plane Diagram of vectors formed by $s-p_j$ and $s-z_i$

Noting that each vector has a real and imaginary component, both the zeros and poles can be represented by complex exponentials of the form:

$$\begin{aligned} s-z_i &= A_i e^{j\alpha_i} \\ s-p_j &= B_j e^{j\beta_j} \end{aligned} \quad (\text{C-7})$$

where A_i and B_j are the magnitudes of the vectors $(s - z_i)$ and $(s - p_j)$, and α_i and β_j are the phase angles describing the orientation of these respective vectors.

For the closed-loop system shown in Figure 7-5, the characteristic polynomial is

$$P_c(s)P_{dyn}(s)Q(s) - 1 = 0 \quad (C-8)$$

This relation can be rearranged into the following form:

$$P_c(s)P_{dyn}(s)Q(s) = -1 \quad (C-9)$$

The open-loop transfer function $P_c P_{dyn} Q$ is made up of a number of numerator and denominator polynomial vectors $A_i e^{j\alpha_i}$ and $B_j e^{j\beta_j}$ respectively. Similarly, -1 on the right hand side of the equation can be considered as a unit vector directed along the negative real axis from the origin of the s-plane. Examining Figure C-1, for a value of s to be a closed loop pole of the system, it must satisfy equation (C-8). Graphically, this is equivalent to requiring that the left hand side of (C-8) represent a vector with magnitude of 1 and with phase angle of ± 180 degrees (or $\pm \pi$ radians). This leads to the magnitude and phase angle conditions, which are the basis for root locus methods as first proposed by Evans [49].

a) Magnitude Condition

$$\|P_c P_{dyn} Q\| = K \frac{A_1 \cdots A_m}{B_1 \cdots B_n} = 1 \quad (C-10)$$

b) Angle Condition

$$\text{phase}(P_c P_{dyn} Q) = \sum_{i=1}^m \alpha_i - \sum_{j=1}^n \beta_j = \pm (2k - 1)\pi \quad (C-11)$$

where k is any integer.

C.3 Nyquist Stability Criterion

The basis for the standard Nyquist Stability Criterion for SISO systems is described here based on [50]. The characteristic polynomial of the single-input, single-output system feedback control system shown in Figure 7-5 is given by the expression $1 - P_c(s)P_{dyn}(s)Q(s)$. This expression can be expressed in terms of a loop gain function as shown below:

$$1 - P_c(s)P_{dyn}(s)Q(s) = K \frac{(s - z_1)(s - z_2) \cdots (s - z_m)}{(s - p_1)(s - p_2) \cdots (s - p_n)} \quad (C-12)$$

Unlike the interpretation of the open-loop transfer function in equation (C-6), the poles p_i of this transfer function represent the open loop poles while the z_i are the closed loop poles of the system. Without evaluating the roots of $(1 - P_c P_{dyn} Q)$, all p_i are known, while all z_i are unknown.

The objective is to determine whether the system is stable by finding whether any z_i lie in the right half of the s-plane. To determine system stability, the Nyquist Stability Criterion makes use of the representation of $(s - z_i)$ and $(s - p_i)$ as vectors in the s-plane as previously described in Section C.2. In addition, it is noted that the right half of the s-plane is bounded by the Nyquist contour, which consists of the imaginary axis from $-j\infty$ to $j\infty$ and a semi-circle of radius $R \rightarrow \infty$ as shown in Figure C-2. As s travels around the Nyquist contour, the vector $(s - z_i)$ undergoes no net rotation when the zero z_i lies in the left side of the s-plane. When z_i lies in the right side of the s-plane, the vector $(s - z_i)$ is rotated 360 degrees clockwise. Similarly the vector $(s - p_i)$ undergoes a 360 degree clockwise rotation when p_i is inside the Nyquist contour, but no net rotation when p_i is outside the contour.

Finally, consider the rotation of the overall vector $(1 + P_c P_{dyn} Q)$. If any of the zeros z_i are within the Nyquist contour, they will each contribute 360 degrees clockwise rotation to $(1 + P_c P_{dyn} Q)$. If there are m_z unstable closed-loop poles, they will contribute m_z clockwise rotations. If any open-loop pole p_i is within the Nyquist contour, it will contribute a 360-degree counterclockwise rotation. If there are n_p unstable open-loop poles, they will contribute n_p counterclockwise rotations.

For the closed loop system to be stable, $m_z = 0$. This leads to the principle behind the Nyquist stability criterion.

a) A Form of the Nyquist Stability Criterion

Assume that $(1 - P_c(s)P_{dyn}(s)Q(s))$ has n_p open-loop poles within the Nyquist contour. As s travels once around the contour, the closed loop system will be stable if and only if the vector $(1 - P_c(s)P_{dyn}(s)Q(s))$ encircles the origin n_p times in the counterclockwise direction.

This criterion simplifies further by noting that the number of encirclements of $(1 - P_c(s)P_{dyn}(s)Q(s))$ around the origin is the same as the number of encirclements of the vector $P_c(s)P_{dyn}(s)Q(s)$ around the -1 point on the real axis. This leads to the final form of the Nyquist stability criterion:

a) Nyquist Stability Criterion

A feedback system is stable if and only if the number of counterclockwise encirclements of a plot of the loop gain function $P_c(s)P_{dyn}(s)Q(s)$ about the -1

point is equal to the number of poles of $P_c(s)P_{dyn}(s)Q(s)$ inside the right-half plane.

The remaining issue to address is how to handle open loop poles that lie on the imaginary axis, which is part of the Nyquist contour D. These poles are excluded from the Nyquist contour by placing a semi-circular indentation of infinitesimal radius around the nutation and precession pole. Traversing the Nyquist contour, the vector of the loop gain function will undergo a rotation of 180 degrees as it traverses each of these indentations. This is illustrated in Figure C-3.

Figure C-2: Nyquist Contour Showing $(s - z_i)$ and $(s - p_i)$ Vectors

Figure C-3: Nyquist Contour With Indentations at ω_p and ω_p

C.3.1 Simplified Nyquist Criterion for SISO Systems

In many systems, there are no unstable open-loop poles. This allows system stability to be judged using the simplified Nyquist Criterion.

If the single-input, single-output loop function $G_a G_{dyn} H_1$ has no open-loop unstable poles, the closed loop system is stable if and only if the -1 point lies to the left on the polar plot when moving along this plot in the direction of increasing ω from 0 to ∞ .

C.3.2 Relative Stability for SISO Transfer Functions

Two common methods used to characterize relative stability are gain and phase margin, which are illustrated in Figure 7-18 a). The gain margin M_{dB} and phase margin ϕ_m are defined as

a) Gain Margin

The inverse of the magnitude of the vector from the origin to the intersection of $P_c P_{dyn} Q$ with the negative real axis, i.e.

$$M_{dB} = \frac{1}{0C} \quad (C-13)$$

b) Phase Margin

This is the negative phase shift or clockwise rotation that causes the plot of $P_c P_{dyn} Q$ to pass through the -1 point on the real axis, i.e.

$$\phi_m = \pi + \text{phase}(P_c(j\omega_c)P_{dyn}(j\omega_c)Q(j\omega_c)) \quad (C-14)$$

Appendix D

Tuned-Rotor Instrument Non-Linear Simulator

To test many of the ideas proposed in this thesis, a non-linear dynamic simulator of a general tuned-rotor instrument was developed. It consists not only of the non-linear dynamics of a tuned-rotor instrument, but also the sensor and actuator dynamics as well as the spin and transverse-axes control systems. The following section describes the model implemented in the simulator, the method of numerical integration, the development environment and the simulator inputs and outputs

D.1 Simulation Model

Figure D-1 shows a block diagram of the overall system model. It consists of a number of modules, each representing a particular sub-system of the tuned-rotor instrument. A brief description of each subsystem model is given below:

a) **command_generator**

This module generates the following case-referenced commands:

- i) tilt angle commands $(\phi_{ref_x}, \phi_{ref_y})$
- ii) tilt angular rate commands $(\dot{\phi}_{ref_x}, \dot{\phi}_{ref_y})$

- iii) spin rate commands ($\dot{\phi}_{ref_z}$)

These commands are generated at a selectable update period in one of two forms:

- i) Generic Input Profile: Step, ramp or sinusoidal profiles for tilt angles and spin rate
- ii) Stored ACS Command Profile: This is a time history of the desired tilt angle, tilt rate and spin rate commands, which can be taken from a Spacecraft Attitude Control System.

b) **gyrowheel_computer**

This simple module controls the update period and shaping of the GyroWheel tilt, tilt rate and spin rate commands. Between successive update periods from the **command_generator** module, it interpolates intermediate values of the commands at a specific gyrowheel_computer update period. This capability is implemented to study the effect of command frequency on the external rate sensing capability. It was also introduced to allow for the later introduction of the spin and transverse-axes controllers in discrete form.

c) **controller**

This module performs the spin and transverse-axes control functions. The simple spin control system shown in Figure 9-20 and the transverse-axis control system in Figure 10-17 were implemented as analog (continuous) controllers in the simulation to facilitate comparison with test results from the first GyroWheel prototype that employs an analog control system [51,52]. Both the spin and transverse controllers were converted to state models of the following form:

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$$

$$\mathbf{y} = \mathbf{Cx} + \mathbf{Du}$$

They are numerically integrated in the same manner as the non-linear gyro dynamics.

d) **spin_motor**

The simplified dc-motor model shown in Figure 9-20 is implemented as part of the simulation model.

e) **torquer**

This function converts the transverse-axes controller current demands into control torques using the non-linear torquer scale factor function shown in Figure 11-1.

f) **gyrowheel_dynamics**

The non-linear gimbal referenced equations of motion for a tuned rotor instrument (equation (5-53)) are used to describe the dynamics. The three case-referenced torques $\begin{bmatrix} T_{c_x} & T_{c_y} & T_{c_z} \end{bmatrix}^T$ generated by the **torquer** and **spin_motor** modules are the inputs supplied to the **gyrowheel_dynamics** module. These torques are converted to spinning torques $\begin{bmatrix} T_{s_x} & T_{s_y} & T_{s_z} \end{bmatrix}^T$ using the transformation given in equation (5-29). Using these input torques, equation (5-53) is numerically integrated to obtain the gimbal angles $\begin{bmatrix} \theta_x & \theta_y & \theta_z \end{bmatrix}^T$ and gimbal rates $\begin{bmatrix} \dot{\theta}_x & \dot{\theta}_y & \dot{\theta}_z \end{bmatrix}^T$.

g) **tilt_sensor**

The gimbal angles $[\theta_x \ \theta_y \ \theta_z]^T$ and rates $[\dot{\theta}_x \ \dot{\theta}_y \ \dot{\theta}_z]^T$ produced in **gyrowheel_dynamics** are transformed to case-referenced angles $[\phi_x \ \phi_y \ \phi_z]^T$ and rates $[\dot{\phi}_x \ \dot{\phi}_y \ \dot{\phi}_z]^T$ using the transformations in equations (5-25) and (5-27) respectively.

h) **Current Sensor**

The control torques generated in the torquer module are passed through a 4th order Butterworth lowpass filter implemented in state-space continuous form. This filter is numerically integrated in the same manner as the non-linear dynamics and controllers.

g) **rate_sensing**

The external rate-sensing algorithm presented in Section 11.2 is executed in this module. The resultant external rates are passed through a 4th order Butterworth lowpass filter implemented in discrete form. The update period of the filter is set at the GyroWheel computer update period.

D.2 Numerical Integration

A fourth-order simple Runge-Kutta integration algorithm was used to numerically integrate all continuous dynamic states in the controller.

D.3 Development Environment and Model Implementation

The overall simulation was developed using the WATCOM Fortran 77 development environment. Each of the sub-system modules was coded as a Fortran subroutine. Passage of information between modules was facilitated via a common database implemented in common block form. The sequencing and timing control was performed by an overall executive routine called **nlinsim**.

D.4 Simulation Control and Output

The simulation model and runtime definition parameters were specified in two ASCII data files:

a) **runtime.dat**

The user has control over the following quantities:

- h) Simulation Timing: duration, integration step, data output interval
- ii) Initial States: initial angles and rates can be specified
- iii) Command Options: generic or ACS generated command history
Closed/open-loop control for spin/transverse axes
Command profile types, periods, amplitudes

b) model.dat

User can modify the model parameters for any of the control, sensor, actuator or gyro dynamics models.

A time history of the simulation is generated in the form of an ASCII file **plot.dat**. The time history is in the form of an $m \times (n+1)$ matrix. The first column is a vector of time stamps at the selected interval. The remaining n columns are the corresponding time histories of selected sub-system outputs at each time-stamp value. The commercial software tool MATLAB was used to plot these responses using a customized “script” file.

Figure D-1: Block Diagram of Tuned-Rotor Instrument Dynamic Simulation Model

List of References

- 1 A.K. Ward, "Small Satellites - An Overview," *Small Satellites - A One Day Conference*, Royal Astronautical Society, London, 1991.
- 2 Fleeter, R., Warner, R., "Guidance and Control of Miniature Satellites." *Automatic Control in Aerospace, Selected Papers from the 1989 IFAC Symposium*, 1990.
- 3 Sellers, J.J. Sweeting, M., Hodgart, M.S., "UOSAT and Other European Activities in Small Satellite Attitude Control." *Advances in the Astronautical Sciences*, Vol. 86, 1994, pp. 359-370.
- 4 Ovchinnikov, M.Y., "Small Satellites in Russia and in the World: Current Status," *Space Communications*, 1995.
- 5 Staley, D.A. Tyc, G., Frise, P.R., "System and Method for Spacecraft Attitude Control", U.S. Patent Application No. 60/093,801, filed July 23, 1998.
- 6 Larson, W.J., Wertz, J.R., *Space Mission Analysis and Design*, Microcosm Inc and Kluwer Academic Publishers, 2nd Edition, 1992.
- 7 Siahpush, A., Gleave, J., "A Brief Survey of Attitude Control Systems for Small Satellites Using Momentum Concepts," *Proceedings of the 2nd Annual AIAA Utah State University Conference on Small Satellites*, September 1988

- 8 Lebsock, K.L., "Wheel Attitude Control System Comparison Study," Lockheed Missiles and Space Company, Space Systems Division, Report No. ASCS/2589/6211, February 1970.
- 9 Bender, J.D., Brehove, J.D., "Momentum Wheel Platform Steering System and Method". United States Patent, No. 5,261,631, November 16, 1993.
- 10 Lawrence, A., *Modern Inertial Technology*, Springer Verlag, 1993.
- 11 Lipman, J.S., "Application of the Complex Method to Transform Analysis of Spinning Systems with Rotating Nonsymmetries," *Proceedings of the Joint Automatic Control Conference*, June 1968, pp. 102-108.
- 12 Craig, R.J., "Theory of Operation of an Elastically Supported Tuned Gyroscope," *IEEE Transactions on Aerospace and Electronic Systems*, Vol., AES-8, No. 3, May 1972, pp. 280-288.
- 13 Staley, D.A., Cain, J.S., "Dynamics of a Tuned Rotor Gyroscope," *Canadian Aeronautics and Space Journal*, Volume 43, No. 2, June 1997, pp. 106-110.
- 14 Cain, J.S., "Non-Linear Analysis of a Gyroscopic Rotor," M.Eng. Thesis, Department of Mechanical and Aerospace Engineering, Carleton University, Ottawa, April, 1996.
- 15 Howe, E.W., Savet, P.H., "The Dynamically Tuned Free Rotor Gyro", *Control Engineering*, June, 1964, pp. 67-72.

- 16 Goldstein, H., *Classical Dynamics*, Addison Wesley Publishing Company, Reading, Massachusetts, 1980.
- 17 Thomson, W.T., *Introduction to Space Dynamics*, John Wiley and Sons, New York, 1961.
- 18 Mansour, W.M, Lacchini, C., "Two-Axis Dry Tuned-Rotor Gyroscopes," *Journal of Guidance, Control and Dynamics*, Vol. 16, No. 3, May-June 1993, pp. 417-425.
- 19 Cain, J.S., "Investigation of the Crossed Flexure Pivot and the Dynamics of the Momentum Management System Spacecraft Control Component and the Dynamically Tuned Gyroscope", PhD Thesis, Department of Mechanical and Aerospace Engineering, Carleton University, Ottawa, March 1999.
- 20 Kalley, J.J., "Control System Design for Double Gimbaled Wheels," *Journal of Spacecraft*, Vol. 12, No. 10, October 1975.
- 21 Shamma, M., Michaelis, T., "Double Gimbaled Momentum Wheel Design for Small Satellites," *ESA, Spacecraft Guidance, Navigation and Control*, 1991.
- 22 Kao, M.K., Hung, J.C., "A Pseudo Root-Locus Method for the Design of a Class of Two-Input, Two-Output Systems, *Proceedings of the Southeastern Symposium on System Theory*, 1977.
- 23 Briz, F., Degner, M.W., Lorenz, R.D., "Analysis and Design of Current Regulators Using Complex Vectors", *Proceedings of the IEEE Industry Applications Society, Annual Meeting*, New Orleans, Louisiana, October, 1997.

- 24 Staley, D.A., "Tuned Rotor Gyro Rebalance Loops for a Low Rate Environment," ANCON-R.838, Ancon Space Technology Corporation, June 1987.
- 25 Bird, J., "An Analysis of DREO's Analog Rebalance Loops for the CSG-2 Tuned Rotor Gyroscope." Defense Research Establishment Ottawa, Technical Note 87-15, July 1987.
- 26 Wu, Y.A., "Adaptive Inverse Control of a Two-Axis Dry Tuned Gyro," *Proceedings of the 25th Conference on Decision and Control*, Athens, Greece, December 1986, pp. 1520-1526.
- 27 Shingu, H., Otsuki, M., "Non-interacting Control of Dynamically Tuned Dry Gyro and its Application to Measurement of Two-Axis Angular Accelerations," *IFAC Automatic Control in Aerospace*, 1989, pp. 147-152.
- 28 Spong, M.W., Vidyasagar, M., *Robot Dynamics and Control*, John Wiley and Sons, New York, 1989.
- 29 Dzielski, J., Bergmann, E., Paradiso, J.A., Rowell, D., Wormley, D., "Approach to Control Moment Gyroscope Steering Using Feedback Linearization," *Journal of Guidance, Control and Dynamics*, Volume 14, No. 1, Jan. - Feb., 1991, pp. 96-106.
- 30 Bender, J.D., Brehove, J.D., "Momentum Wheel Platform Steering System and Method", United States Patent, No. 5,261,631, November 16, 1993.

- 31 Wertz, J.R., *Spacecraft Attitude Determination and Control*, Kluwer Academic Publishers, Dordrecht, 1978.
- 32 Hung, J.C., White, H.V., "Wheel Induced Errors in the Use of Dry-Tuned Gyros," *IEEE Transactions on Industrial Electronics*, Vol. IE-30, No. 1, February 1983, pp. 7-9.
- 33 David, N., "Design and Simulation of a Momentum Controller for Odin: A Small Satellite with Versatile and Stringent Pointing Requirements". M.Eng. Thesis, Department of Mechanical and Aerospace Engineering, Carleton University, April 1995.
- 34 Cain, J.S., Staley, D.A., Hartl, D., "Inertial Sensors in a Spacecraft Application," *Canadian Aeronautics and Space Journal*, Volume 43, No. 2, June 1997, pp. 101-105.
- 35 Craig, J.G., "Theory of Errors of a Multi-gimbal, Elastically Supported, Tuned Gyroscope". *IEEE Transactions on Aerospace and Electronic Systems*, Volume AES-8, No. 3, May 1972.
- 36 Kao, M.K., Hung, J.C., Berry, E.H., "A Pseudo Root-Locus Method for the Design of a Class of Two-Input, Two-Output Systems". *Proceedings of the 8th Annual Southeastern Symposium on System Theory*, 1976, pp. 264-268.
- 37 MacFarlane, A.G. J., *Complex Variable Methods for Linear Multivariable Feedback Systems*, Taylor & Francis, London, 1980.

- 38 Maciejowski, J.M., *Multivariable Feedback Design*, Addison Wesley Publishing Company, Reading, Massachusetts, 1989.
- 39 Green, M., Limebeer, D.J.N, *Linear Robust Control*, Prentice Hall, Englewood Cliffs, New Jersey, 1995.
- 40 Doyle, J.C., Francis, B.A., Tannenbaum, A.R., *Feedback Control Theory*, Macmillan Publishing Company, New York, 1992.
- 41 Slotine, J.J., Li, W., *Applied Nonlinear Control*, Prentice Hall, Englewood Cliffs, New Jersey, 1991.
- 42 Lefschetz, S., *Differential Equations: Geometric Theory*, Interscience Publishers Inc, New York, 1957.
- 43 MATLAB, Version 5.3, The MathWorks Inc., 1999.
- 44 Wang, L.C., Han, K.W., Directions of Root-Loci in the Stability Equation Method, *Journal of the Franklin Institute*, Vol. 304, No. 6, 1977, pp. 221-229.
- 45 Staley, D.A., *Orbital Mechanics and Spacecraft Control*, Department of Mechanical and Aerospace Engineering, Carleton University, 1997.
- 46 Ower, C., MMS Laboratory Model: Analysis of Performance Tests, MMS Technical Note, Bristol Aerospace Ltd., April 1998.
- 47 Pradhan, S., (Attitude Control Systems Engineer, Bristol Aerospace Ltd.), Personal Communication, April 1997.

- 48 Student Edition of Matlab, Version 4, The MathWorks Inc., Published by Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1995.
- 49 Evans, W.R., Graphical Analysis of Control Systems, *Transactions of the AIEE*, Vol. 67, pp. 547-551, 1948.
- 50 Van de Vegte, J., *Feedback Control Systems*, Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1986
- 51 Ower, C., MMS Laboratory Model Control System, MMS Technical Note, Bristol Aerospace Ltd, October 1997.
- 52 Ower, C., Updates to MMS Laboratory Unit Control System, Bristol Aerospace Ltd., June 1998.