 

A resolution test chart featuring several groups of horizontal and vertical lines of varying thicknesses. Each group is accompanied by a numerical value indicating the resolution level. The values include 1.0, 1.1, 1.25, 1.4, 1.6, 1.8, 2.0, 2.2, 2.5, 2.8, 3.2, 3.6, 4.0, 4.5, 5.0, 5.6, 6.3, 7.1, 8.0, and 9.0. The chart is used to measure the resolving power of imaging systems.

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Some Contributions to L-estimation in Linear Regression and Autoregression Models

by

Soroush Alimoradi, B.Sc., M.Sc.

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfillment of
the requirements for the degree of
Doctor of Philosophy

Department of Mathematics and Statistics
Ottawa-Carleton Institute of Mathematics and Statistics

Carleton University
Ottawa, Ontario
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Soroush Alimoradi, B.Sc., M.Sc.
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Abstract

The theory of linear combinations (L-estimation) of order statistics/sample quantiles for the location model is well-known. This thesis considers the problem of L-estimation in linear and autoregression models.

First, we extend L-estimation methods for the regression parameters in a linear model based on a few selected regression quantiles and deal with the question of *optimum* regression quantiles. We also provide tables of optimum spacings, coefficients of ABLUE and the quantities required to calculate the asymptotic efficiencies of these simplified L-estimators with t and the logistic errors. Second, we consider Bahadur type representation of the regression quantiles and the trimmed LSE under less restrictive conditions than that of Jurečková (1984) and we prove the asymptotic normality of these estimators. Third, we consider the estimation of the regression parameters under some uncertain prior sub-space restriction leading to the combination of two diverse areas, namely, robust estimation and shrinkage estimation via preliminary test approach.

Though, there has been some progress on L-estimation for the linear model, relatively little progress has been made with the most popular time series model in application, namely, the AR(p)-model. We develop the autoregression quantiles (ARQ) approach due to Koul and Saleh (1995) in order to deal with some aspects of L-estimation of the parameters of the p^{th} order autoregression model. First, we

consider the problem of L-estimation of the parameters of the AR(p)-model based on a few selected ARQ's. We also provide tables of optimum spacings, coefficients of ABLUE and the quantities required to calculate the asymptotic efficiencies of these simplified L-estimators with t and the logistic errors. Second, we consider Bahadur type representation of the autoregression quantiles and the trimmed least square estimators (TLES) under the same conditions as we have done for regression model. Finally, we consider the preliminary test approach to shrinkage estimation to improve the robustified L-estimators in a sub-hypothesis problem.

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Chapter 1

Introduction

In the nineteenth century, linear combination of order statistics (the so called L-estimators) were in common use especially in astronomical calculations. Sometimes estimators were calculated by dropping smaller or larger values or both from the set of all observations giving the concept of *trimming of outliers*. L-estimators, naturally, are an integral part of *robust* estimation (i.e. estimators which are not sensitive to small deviation from model distribution) of location parameter. In the case of location model sample mean has minimum variance in the class of unbiased estimators when the errors are normal and has also asymptotic efficiency of one. But this estimator has zero efficiency for the Cauchy distribution. Looking for alternative estimators which may be robust opposed to the sample mean has been apparent since the eighteenth century. There are three classes of estimator in the location model which are commonly used as robust estimators, namely, *maximum likelihood type estimators* (M-estimators), *estimators based on ranks* (R-estimators) and *linear combination of order statistics* (L-estimators). We describe them now.

(i) **M-estimators:** Consider the location model

$$Y_i = \beta_0 + Z_i \quad ; \quad Z_i \in \mathbb{R}, \quad i = 1, \dots, n \quad (1.1)$$

where Z_i 's, $i = 1, \dots, n$ are *i.i.d* with *c.d.f.* F_0 . The M-estimator of β_0 is any solution of minimization problem

$$\sum_i \rho(Y_i - b) = \min ; \quad b \in \mathbb{R} \quad (1.2)$$

where ρ is an arbitrary function, or solve the pseudo-likelihood equations with chosen ψ -functions

$$\sum_i \psi(Y_i - b) = 0. \quad (1.3)$$

If ρ is convex and $\psi = \rho'$ these definitions coincide. Note that if we choose $\rho(Y_i - b) = -\log f_0(Y_i - b)$, it gives the ordinary MLE with $\psi = f'_0/f_0$. The asymptotic properties of this kind of estimators are given by Huber (1981).

(ii) **R-estimators:** A robust estimator of location may be derived from a test statistic that is distribution-free under an appropriate null hypothesis about β_0 . This technique was first proposed by Hodges and Lehmann (1963). A test can be carried out based on one or two samples. In the case of one sample, let $Y_1, \dots, Y_n$ be a random sample from a continuous distribution with *c.d.f.* $F_0(y - \beta_0)$ when $F_0(\cdot)$ is symmetric about zero. Assume that $H_0 : \beta_0 = 0$ vs $H_1 : \beta_0 > 0$ and let $V(Y_1, \dots, Y_n)$ be a test statistic which satisfies the following conditions:

- i) H_0 is rejected when $V(Y_1, \dots, Y_n)$ is large.
- ii) For each $(y_1, \dots, y_n)$, $V(y_1 + h, \dots, y_n + h)$ is a nondecreasing function of h .
- iii) Under H_0 , the distribution of $V(Y_1, \dots, Y_n)$ is symmetric about a value ξ for every $F_0(\cdot)$ which is continuous and symmetric about 0.

Now, we motivate a Hodges-Lehmann estimator of β_0 as follows. The random variables $Y_1 - \beta_0, \dots, Y_n - \beta_0$ are independent with symmetric identical distribution. Thus, it is desirable to find an estimator of β_0 , say $\hat{\beta}_0$, such that $Y_1 - \hat{\beta}_0, \dots, Y_n - \hat{\beta}_0$

is symmetrically distributed about 0. The Hodges-Lehmann estimator of β_0 based on a test statistic $V(Y_1, \dots, Y_n)$ satisfying (i), (ii) and (iii) is given by

$$\hat{\beta}_0 = \frac{\beta_0^* + \beta_0^{**}}{2}, \quad (1.4)$$

where

$$\begin{aligned} \beta_0^* &= \sup\{\beta_0 : V(Y_1 - \beta_0, \dots, Y_n - \beta_0) > \xi\}, \\ \text{and } \beta_0^{**} &= \sup\{\beta_0 : V(Y_1 - \beta_0, \dots, Y_n - \beta_0) < \xi\}. \end{aligned} \quad (1.5)$$

More details about this estimator and other R-estimators based on one sample with asymptotic properties can be seen in Randles and Wolfe (1979), Huber (1981) and Puri and Sen (1985) among others.

(iii) L-estimator: The mathematical theory of L-estimation of location and scale parameters for known distributions began with the work of Ogawa (1951) and Lloyd (1952). Lloyd's (1952) work pertains to *best linear unbiased estimation* (BLUE) based on order statistics in finite samples while Ogawa's (1951) work is based on the asymptotic distributions of several *sample quantiles* in large samples due to Mosteller (1946). This paved the way for the developments regarding the asymptotically best linear unbiased estimation (ABBLUE) of the location and scale parameters from known distributions. These simplified L-estimators are precursors of the so-called Gastwirth's (1966) *robust* estimator of the location parameter of symmetric distributions.

Let $Y_{(1)}, \dots, Y_{(n)}$ denote the order statistics of the sample $Y_1, \dots, Y_n$. Then a sequence of estimators whose elements are linear combinations of order statistics take the form

$$\hat{\beta}_{0n} = n^{-1} \sum_{i=1}^n J\left(\frac{i}{n+1}\right) Y_{(i)} \quad (1.6)$$

for some function $J(\cdot)$ on $(0, 1)$, and are called L-estimator with continuous weight function. Typically, $J(\cdot)$ is chosen to satisfy $J(t) = J(1 - t)$ and $\int_0^1 J(t) dt = 1$.

An asymptotically efficient L-estimator of location parameter β_0 in $F_0(z - \beta_0)$ is obtained by setting

$$J(F_0(z)) = \frac{[\log f_0(z)]''}{I(F_0)},$$

where $I(F_0)$ is the Fisher's information number. Readers may refer to Jung (1955), Chernoff, Gastwirth and Johns (1967), and Stigler (1974) among others. An L-estimator is said to be robust only if for some $\lambda \in (0, 1/2]$, $J(t)$ has the support $[\lambda, 1 - \lambda]$. Examples of such robust estimators are the *quick and dirty* or *inefficient* estimators of Mosteller (1946) and Gastwirth's (1966) L-estimators. Undoubtedly, the most important class of L-estimators is the class of trimmed means which have the weight function

$$J(t) = \frac{1}{1 - 2\lambda}, \quad t \in [\lambda, 1 - \lambda]. \quad (1.7)$$

The idea of such a trimmed estimator is due to Tukey (1960). The asymptotic behavior of this estimator is treated in Stigler (1973). These subjects are again treated in chapter 2.

Analogue of M-estimators in the linear multiple regression model carry over from the location model virtually intact (see for example, Huber (1981)), though the problem of invariance under reparametrization of the design matrix remains a problem. As regards the R-estimator of the regression coefficients, the problem of alignment crops in the estimators of parameters in a sub-hypothesis problem (see Puri and Sen (1985)). As regards L-estimators of parameters in the linear model, the definition of *order statistic/quantile* became the nagging problem. However, Bickel (1973) extended the idea of linear combination of order statistics by ordering the residuals based on a preliminary estimator such as a least-square estimator. These estimators may be computed by iteratively reweighted least squares but are complicated and have the disadvantage of being not invariant to reparametrization of design matrix.

An alternative approach to L-estimators in the linear model is due to the well-

accepted definition of *regression quantiles* and their properties by Koenker and Bassett (1978) which extends the definition of *sample quantiles* in location model to the linear model. The idea begins with the observation that the ordinary sample quantiles/order statistics of the location model, may be regarded as the M-estimations with ρ -function

$$\rho_{\lambda}(x) = \begin{cases} \lambda x, & x \geq 0 \\ (\lambda - 1)x, & x < 0, \end{cases} \quad (1.8)$$

where $\lambda \in (0, 1)$. Then, it is straight forward to see that this ρ -function in the linear model yields a p -dimensional analogue of the sample quantiles. The median case ($\lambda = 1/2$) is the well-known least-absolute error (LAE) regression. This concept seems to provide a reasonable basis for the construction of robust L-estimators of regression parameters. The asymptotic behavior of the p dimensional regression quantiles is strikingly similar to the behavior of the ordinary sample quantiles. Later on Ruppert and Carroll (1980) studied the trimmed least squares estimators (TLSE) based on regression quantiles. They derived the Bahadur-type representation of regression quantiles up to the order $o_P(n^{-1/2})$ as well as the asymptotic distribution of the trimmed LSE when the design matrix satisfies the following conditions on the design elements:

$$x_{i1} = 1 \quad i = 1, \dots, n; \quad \sum_{i=1}^n x_{ij} = 0 \quad j = 2, \dots, p+1.$$

Jurečková (1984) considered the same problem under general conditions on the elements of the design matrix. She also derived the Bahadur-type representation for regression quantiles and linear representation of the trimmed LSE up to the order $O_P(n^{-3/4})$ as well as asymptotic distribution of TLSE, when the error distribution function F is absolutely continuous and its second derivative is bounded. For the design matrix she considered conditions such as the average of the fourth power of design matrix elements should be bounded as n goes to infinity.

The main contribution on L-estimation begins with Chapter 3 on the linear re-

gression model. First, we extend L-estimation methods for the regression parameters in a linear model based on a few selected regression quantiles and deal with the question of *optimum* regression quantiles. We also provide tables of optimum spacings, coefficients of ABLUE and the quantities required to calculate the asymptotic efficiencies of these simplified L-estimators with t and the logistic errors. Second, we consider Bahadur type representation of the regression quantiles and the trimmed LSE under less restrictive conditions than that of Jurečková (1984) and we prove the asymptotic normality of these estimators. Third, we consider the estimation of the regression parameters under some uncertain prior sub-space restriction leading to the combination of two diverse areas, namely, robust estimation and shrinkage estimation via preliminary test approach.

Though, there has been some progress on L-estimation for the linear model, relatively little progress has been made with the most popular time series model in application, namely, the AR(p)-model.

In the final chapter 4, we develop the autoregression quantiles (ARQ) approach due to Koul and Saleh (1995) in order to deal with some aspects of L-estimation of the parameters of the p^{th} order autoregression model. First, we consider the problem of L-estimation of the parameters of the AR(p)-model based on a few selected ARQ's. We also provide tables of optimum spacings, coefficients of ABLUE and the quantities required to calculate the asymptotic efficiencies of these simplified L-estimators with t and the logistic errors. Second, we consider Bahadur type representation of the autoregression quantiles and the trimmed least square estimators (TLES) under the same conditions as we have done for regression model. Finally, we consider the preliminary test approach to shrinkage estimation to improve the robustified L-estimators in a sub-hypothesis problem.

Chapter 2

Location and Scale Model

2.1 Introduction

The object of this chapter is to *review* the developments of the estimation of location-scale parameters of a distribution based on order statistics and *a few selected sample quantiles* leading to the so-called the theory of L-estimation. Section 2.2 contains the preliminary developments of L-estimation theory based on order statistics due to Lloyd (1952). In section 2.3, we present the theory of L-estimation based on *a few sample quantiles* due to Ogawa (1951). This section contains some *new* tables of *optimum spacings* and *related coefficients* for the L-estimation of location and scale parameters for the class of t-distributions. The last section (2.4) contains the trimmed least-squares estimator for location parameter. These ideas are treated in full detail which will be continued in later chapters with linear and autoregressive models.

2.2 Estimation of location and scale parameters based on order statistics

Consider the location model

$$Y_j = \beta_0 + \sigma Z_j, \quad j = 1, \dots, n \quad (2.1)$$

where $Z_1, Z_2, \dots, Z_n$ are i.i.d. r.v's with continuous cdf $F_0(z)$ (and pdf $f_0(z)$), $(\beta_0, \sigma)'$ stands for the location and scale parameters of the distribution. Let $Y_{(\cdot)} = (Y_{(1)}, \dots, Y_{(n)})'$ be the order statistic vector corresponding to $Y = (Y_1, \dots, Y_n)'$. Similarly, let $Z_{(\cdot)} = (Z_{(1)}, \dots, Z_{(n)})'$ be the order statistic vector corresponding to $Z = (Z_1, \dots, Z_n)'$. Clearly,

$$Y_{(\cdot)} = \beta_0 \mathbf{1}_n + \sigma Z_{(\cdot)}, \quad \mathbf{1}_n = (1, \dots, 1)'. \quad (2.2)$$

Further,

$$EY_{(\cdot)} = E(Y_{(1)}, \dots, Y_{(n)})' = \beta_0 \mathbf{1}_n + \sigma EZ_{(\cdot)} \quad (2.3)$$

and

$$\text{cov}(Y_{(\cdot)} Y_{(\cdot)}') = \sigma^2 \text{cov}(Z_{(\cdot)} Z_{(\cdot)}') \quad (2.4)$$

Since F_0 is known, $EZ_{(\cdot)} = \alpha_{(\cdot)}$ and $\text{cov}(Z_{(\cdot)} Z_{(\cdot)}') = V$ are completely known. Thus, Lloyd (1952) proposed the linear (L-) estimation of $(\beta_0, \sigma)'$ based on $Y_{(\cdot)}$ by minimization of the quadratic form

$$(Y_{(\cdot)} - \beta_0 \mathbf{1}_n - \sigma \alpha_{(\cdot)})' V^{-1} (Y_{(\cdot)} - \beta_0 \mathbf{1}_n - \sigma \alpha_{(\cdot)}) \quad (2.5)$$

with respect to β_0 and σ . The minimization process leads to the normal equations

$$\mathbf{1}_n' V^{-1} (Y_{(\cdot)} - \beta_0 \mathbf{1}_n - \sigma \alpha_{(\cdot)}) = 0, \quad (2.6)$$

and

$$\alpha_{(\cdot)}' V^{-1} (Y_{(\cdot)} - \beta_0 \mathbf{1}_n - \sigma \alpha_{(\cdot)}) = 0. \quad (2.7)$$

This results in solving for β_0 and σ from the equations

$$\hat{\beta}_0 d_{11} + \hat{\sigma} d_{12} = M_1 \quad (2.8)$$

$$\hat{\beta}_0 d_{12} + \hat{\sigma} d_{22} = M_2 \quad (2.9)$$

where

$$d_{11} = \mathbf{1}_n' V^{-1} \mathbf{1}_n, \quad d_{22} = \alpha_{(\cdot)}' V^{-1} \alpha_{(\cdot)}, \quad d_{12} = \alpha_{(\cdot)}' V^{-1} \mathbf{1}_n, \\ M_1 = \mathbf{1}_n' V^{-1} Y_{(\cdot)}, \quad M_2 = \alpha_{(\cdot)}' V^{-1} Y_{(\cdot)} \quad \text{and} \quad D = d_{11} d_{22} - d_{12}^2. \quad (2.10)$$

The explicit expression for $\hat{\beta}_0$ and $\hat{\sigma}$ are given by

$$\hat{\beta}_0 = \frac{1}{D} (M_1 d_{22} - M_2 d_{12}) \quad (2.11)$$

and

$$\hat{\sigma} = \frac{1}{D} (M_2 d_{11} - M_1 d_{12}) \quad (2.12)$$

respectively. The covariance matrix for $(\hat{\beta}_0, \hat{\sigma})'$ is then given by

$$\text{Cov}(\hat{\beta}_0, \hat{\sigma}) = \frac{\sigma^2}{D} \begin{pmatrix} d_{22} & -d_{12} \\ -d_{12} & d_{11} \end{pmatrix}. \quad (2.13)$$

The estimators $\hat{\beta}_0$ and $\hat{\sigma}$ are best linear unbiased estimators (BLUEs) of β_0 and σ respectively.

In the fifties and sixties many researchers such as Sarhan, Greenberg, Ogawa, Kulldorff and Saleh investigated properties of the above estimators based on a few order statistics. The studies covered uniform, exponential, double-exponential, logistic and normal distributions to name a few. The results of such studies have been documented in the monograph *Contributions to Order Statistics* by Sarhan and Greenberg (1962), *Order Statistics* by David (1981) and *Order Statistics and Inference* by Balakrishnan and Cohen (1991) among others. The methodology of the BLUE of (β_0, σ) remains the same based on k order statistics except that the observation vector $Y_{(.)}$ of n -dimension is taken as

$$Y_{(.)}^k = (Y_{(n_1)}, \dots, Y_{(n_k)})' \quad (2.14)$$

where $Y_{(n_1)}, \dots, Y_{(n_k)}$ are the selected order statistics (with ranks $n_1, \dots, n_k$) from $Y_{(.)} = (Y_{(1)}, \dots, Y_{(n)})'$ and correspondingly V is changed to $k \times k$ matrix $V^{(k)} = \text{cov}(Z_{(.)}^{(k)} Z_{(.)}^{(k)'})$, D is changed to $D^{(k)}$, and $d_{ij} \rightarrow d_{ij}^{(k)}$, for finite sample situations. The problem of optimum order statistics for a given k ($2 \leq k \leq n$) arose and was solved specially for the exponential distribution by Sarhan and Greenberg (1962), Kulldorff (1964) and Saleh (1967) in complete and type II censored samples. The

relative efficiency (R.E.) of BLUEs' based on $k(2 \leq k \leq n)$ order statistics compared to the BLUEs' based on all observations may be written as

$$R.E.(BLUE_{(k)}; BLUE_{(n)}) = \frac{D^{(k)}}{D} = \frac{d_{11}^{(k)}d_{22}^{(k)} - (d_{12}^{(k)})^2}{d_{11}d_{22} - d_{12}^2}. \quad (2.15)$$

2.3 Estimation of location and scale parameters based on sample quantiles

Mosteller (1946) proposed the use of so-called *inefficient statistics* based on a few *sample quantiles* as *quick and dirty* substitutes for more conventional estimators. Mosteller's idea was quickly extended by Ogawa (1951) and provided the asymptotically best linear unbiased estimators (ABLUEs) of β_0 and σ based on $k(2 \leq k < n)$ selected *sample quantiles*. Now we discuss this area of research which expanded during the fifties and sixties having implications in robust estimation of location parameters. First, we consider various definitions.

Definition 2.3.1 Let Z be a random variable with cdf $F_0(z)$. Then, $Q_0(\lambda)$ is called the *population quantile-function* of order λ ($0 < \lambda < 1$) given by

$$Q_0(\lambda) = \inf\{z : F_0(z) \geq \lambda\} = F_0^{-1}(\lambda) \quad (2.16)$$

Definition 2.3.2 Let Z be a random variable with cdf $F_0(z)$ and pdf $f_0(z)$. The density evaluated at $Q_0(\lambda)$, namely, $f_0(Q_0(\lambda))$ is called the *density-quantile function* and is denoted by $q_0(\lambda)$.

Some quantile-functions and density-quantile function are given below.

Table 2.1: Quantile functions and density quantile functions

<i>Dist.</i>	<i>Densities</i>	$u = Q_0(\lambda)$	$q_0(\lambda)$
<i>Normal</i>	$(2\pi)^{-1/2} \exp(-z^2/2)$	$\Phi^{-1}(\lambda)$	$(2\pi)^{-1/2} \exp\{-\frac{1}{2}[\Phi^{-1}(\lambda)]^2\}$
<i>Cauchy</i>	$\pi^{-1}(1+z^2)^{-1}$	$\tan[\pi(\lambda - \frac{1}{2})]$	$\pi^{-1} \sin^2(\pi\lambda)$
<i>Logistic</i>	$e^{\frac{\pi}{\sqrt{3}}z} / (1 + e^{\frac{\pi}{\sqrt{3}}z})^2$	$\frac{\sqrt{3}}{\pi} \ln\left(\frac{\lambda}{1-\lambda}\right)$	$\lambda(1-\lambda)$
<i>Log Nor.</i>	$(2\pi)^{-1/2} z^{-1} e^{-\frac{1}{2}(\ln z)^2}$	$\exp[\Phi^{-1}(\lambda)]$	$(2\pi)^{\frac{1}{2}} e^{\{\frac{1}{2}(\Phi^{-1}(\lambda))^2 - \Phi^{-1}(\lambda)\}}$
<i>Double Exp.</i>	$\frac{1}{2} e^{- z }$	$\ln(2\lambda), \lambda < \frac{1}{2}$	$\begin{cases} \lambda & \text{if } \lambda < \frac{1}{2} \\ 1 - \lambda & \text{if } \lambda > \frac{1}{2} \end{cases}$

Let $Z_1, Z_2, \dots, Z_n$ be a random sample of size n from the common *cdf* $F_0(z)$. Let $F_{0n}(z)$ defined by

$$F_{0n}(z) = n^{-1}(\#Z_i \leq z) = n^{-1} \sum_{i=1}^n I(Z_i \leq z) \quad (2.17)$$

be the empirical *cdf* corresponding to $F_0(z)$. Then

$$\bar{Q}_{0n}(\lambda) = \inf\{z : F_{0n}(z) \geq \lambda\} \equiv F_{0n}^{-1}(\lambda) \quad (2.18)$$

is the sample-quantile of order λ ($0 < \lambda < 1$). Another popular way is to write

$$\bar{Q}_{0n}(\lambda) = Z_{(j)}, \quad \text{if } \frac{j-1}{n} \leq \lambda < \frac{j}{n} \quad (2.19)$$

for $j = 1, 2, \dots, n$. This is a piecewise constant function over the interval $[0, 1]$ and $Z_{(j)}$ stands for the order statistic with rank j ($1 \leq j \leq n$). More commonly, one writes

$$\begin{aligned} \bar{Q}_{0n}(\lambda) &= Z_{([n\lambda]+1)} \quad \text{if } n\lambda \text{ is not an integer} \\ &= Z_{(n\lambda)} \quad \text{if } n\lambda \text{ is an integer} \end{aligned} \quad (2.20)$$

However, a more general definition of *sample quantile of order* λ for a location model

$$Y_i = \beta_0 + Z_i, \quad j = 1, \dots, n \quad (2.21)$$

is the solution of the following minimization problem:

$$\min_{b \in \mathbb{R}} \left[\sum_{i \in \{i: y_i \geq b\}} \lambda |y_i - b| + \sum_{i \in \{i: y_i < b\}} (1 - \lambda) |y_i - b| \right]. \quad (2.22)$$

This criterion was alluded to by Ferguson (1967) in an exercise and extended by Koenker and Bassett (1978). The case of the median is achieved when $\lambda = 1/2$ which is well-known as the solution of the minimization problem:

$$\min_{b \in \mathbb{R}} \left[\sum_{i=1}^n |y_i - b| \right] \quad (2.23)$$

Sample quantiles in the location model are either identified with a single order statistic or a group of order statistics from the observed sample, for example in the case of the median from an even sample, they are identified with a closed interval between two order statistics and some combination of these two order statistics, say, the average. In fact if F_0 is continuous, then the solution of (2.22) is unique if and only if

$$\lambda - 1 < \sum_{i \neq h} \left\{ 1/2 - 1/2 \text{sign}(y_i - y_h) - \lambda \right\} < \lambda. \quad (2.24)$$

See for example Bassett and Koenker (1978). The expression in the bracket takes the value $-\lambda$ if $y_i > y_h$ and $1 - \lambda$ if $y_i < y_h$ so that (2.24) reduces to the requirement that the number of y_i 's less than y_h be strictly between $n\lambda - 1$ and $n\lambda$. This demands that $n\lambda$ be non-integral. In the case of the median the requirement boils down to n being odd.

It is easy to verify that the sample quantiles satisfy the location and scale equivariance properties. Now we consider the asymptotic distribution of several sample quantiles, a result due to Mosteller (1946).

Theorem 2.3.3 *Let $Y_1, Y_2, \dots, Y_n$ be a random sample corresponding to the random variable with cdf $F_0\left(\frac{y - \beta_0}{\sigma}\right)$ and pdf $\frac{1}{\sigma} f_0\left(\frac{y - \beta_0}{\sigma}\right)$ continuous and positive at $Q(\lambda_i) = \beta_0 + \sigma Q_0(\lambda_i)$, $i = 1, \dots, k$ ($2 \leq k \leq n$) where $0 < \lambda_1 < \lambda_2 < \dots < \lambda_k < 1$ are prefixed real numbers in $(0, 1)$. Define $n_j = n\lambda_j + o(n^{-1/2})$, $j = 1, 2, \dots, k$. Then,*

$$n^{1/2} \left(Y_{[n\lambda_1]+1} - Q(\lambda_1), \dots, Y_{[n\lambda_k]+1} - Q(\lambda_k) \right)' \xrightarrow{D} \mathcal{N}_k(0, \sigma^2 \Omega_{k \times k}) \quad (2.25)$$

where

$$\Omega_{k \times k} = \left(\left(\frac{\lambda_i \wedge \lambda_j - \lambda_i \lambda_j}{q_0(\lambda_i) q_0(\lambda_j)} \right) \right). \quad (2.26)$$

See Sen and Singer (1993) section 4.3 for the proof.

Ogawa (1951) formulated the asymptotically best linear unbiased estimators (ABLUEs) of β_0 and σ based on k ($2 \leq k \leq n$) selected sample quantiles using Mosteller's work. Thus, he minimized the quadratic form

$$(Y_{(\cdot)} - \beta_0 \mathbf{1}_k - \sigma Q_0(\lambda))' \Omega^{-1} (Y_{(\cdot)} - \beta_0 \mathbf{1}_k - \sigma Q_0(\lambda)) \quad (2.27)$$

with respect to β_0 and σ . Here, we have

$$Y_{(\cdot)} = (Y_{[n\lambda_1]+1}, \dots, Y_{[n\lambda_k]+1})', \quad (2.28)$$

$$\mathbf{1}_k = (1, \dots, 1)' \quad \text{a } k \text{ tuple of 1's} \quad (2.29)$$

$$\text{and } Q_0(\lambda) = (Q_0(\lambda_1), \dots, Q_0(\lambda_k))'. \quad (2.30)$$

Thus, ABLUE of $(\beta_0, \sigma)'$ based on $Y_{(\cdot)}$ is given by

$$\hat{\beta}_0 = \frac{1}{\Delta} \{K_2 M_1 - K_3 M_2\} = a_1 Y_{[n\lambda_1]+1} + \dots + a_k Y_{[n\lambda_k]+1}, \quad (2.31)$$

$$\hat{\sigma} = \frac{1}{\Delta} \{K_1 M_2 - K_3 M_1\} = b_1 Y_{[n\lambda_1]+1} + \dots + b_k Y_{[n\lambda_k]+1} \quad (2.32)$$

and

$$a_i = \frac{q_0(\lambda_i) K_2}{\Delta} \left\{ \frac{q_0(\lambda_i) - q_0(\lambda_{i-1})}{\lambda_i - \lambda_{i-1}} - \frac{q_0(\lambda_{i+1}) - q_0(\lambda_i)}{\lambda_{i+1} - \lambda_i} \right\} - \quad (2.33)$$

$$\frac{q_0(\lambda_i) K_3}{\Delta} \left\{ \frac{q_0(\lambda_i) Q_0(\lambda_i) - q_0(\lambda_{i-1}) Q_0(\lambda_{i-1})}{\lambda_i - \lambda_{i-1}} - \frac{q_0(\lambda_{i+1}) Q_0(\lambda_{i+1}) - q_0(\lambda_i) Q_0(\lambda_i)}{\lambda_{i+1} - \lambda_i} \right\},$$

$$b_i = \frac{q_0(\lambda_i) K_1}{\Delta} \left\{ \frac{q_0(\lambda_i) Q_0(\lambda_i) - q_0(\lambda_{i-1}) Q_0(\lambda_{i-1})}{\lambda_i - \lambda_{i-1}} - \frac{q_0(\lambda_{i+1}) Q_0(\lambda_{i+1}) - q_0(\lambda_i) Q_0(\lambda_i)}{\lambda_{i+1} - \lambda_i} \right\} - \quad (2.34)$$

$$-\frac{q_0(\lambda_i) K_3}{\Delta} \left\{ \frac{q_0(\lambda_i) - q_0(\lambda_{i-1})}{\lambda_i - \lambda_{i-1}} - \frac{q_0(\lambda_{i+1}) - q_0(\lambda_i)}{\lambda_{i+1} - \lambda_i} \right\}.$$

$i = 1, 2, \dots, k$, where

$$K_1 = \mathbf{1}_k' \Omega^{-1} \mathbf{1}_k, \quad K_2 = \mathbf{Q}_0(\lambda)' \Omega^{-1} \mathbf{Q}_0(\lambda), \quad \Delta = K_1 K_2 - K_3^2, \quad (2.35)$$

$$K_3 = \mathbf{Q}_0(\lambda)' \Omega^{-1} \mathbf{1}_k, \quad M_1 = \mathbf{1}_k' \Omega^{-1} \mathbf{Y}_{(.)} \text{ and } M_2 = \mathbf{Q}_0(\lambda)' \Omega^{-1} \mathbf{Y}_{(.)}. \quad (2.36)$$

The variance-covariance matrix of the ABLUE is given by

$$\frac{\sigma^2}{n\Delta} \begin{pmatrix} K_2 & -K_3 \\ -K_3 & K_1 \end{pmatrix}. \quad (2.37)$$

The explicit expressions for K_1 , K_2 and K_3 are given by

$$K_1 = \sum_{i=1}^k \frac{[q_0(\lambda_i) - q_0(\lambda_{i-1})]^2}{\lambda_i - \lambda_{i-1}} \quad (2.38)$$

$$K_2 = \sum_{i=1}^k \frac{[q_0(\lambda_i)Q_0(\lambda_i) - q_0(\lambda_{i-1})Q_0(\lambda_{i-1})]^2}{\lambda_i - \lambda_{i-1}} \quad (2.39)$$

$$K_3 = \sum_{i=1}^k \frac{[q_0(\lambda_i) - q_0(\lambda_{i-1})][q_0(\lambda_i)Q_0(\lambda_i) - q_0(\lambda_{i-1})Q_0(\lambda_{i-1})]}{\lambda_i - \lambda_{i-1}}. \quad (2.40)$$

The joint asymptotic relative efficiency (JARE) of the ABLUE of β_0 and σ may be obtained by the expression for the regular cases as

$$JARE[ABLUE : MLE] = \frac{K_1 K_2 - K_3^2}{I_{11} I_{22} - I_{12}^2}, \quad (2.41)$$

where

$$I = \begin{pmatrix} I_{11} & I_{12} \\ I_{12} & I_{22} \end{pmatrix} \quad (2.42)$$

is the information matrix for the location-scale distribution. The ARE of $\hat{\beta}_0$ or $\hat{\sigma}$ are then given by

$$ARE[\hat{\beta}_0 : \text{MLE of } \beta_0] = \frac{(K_1 K_2 - K_3^2) I_{22}}{K_2 (I_{11} I_{22} - I_{12}^2)} \quad (2.43)$$

and

$$ARE[\hat{\sigma} : \text{MLE of } \sigma] = \frac{(K_1 K_2 - K_3^2) I_{11}}{K_1 (I_{11} I_{22} - I_{12}^2)} \quad (2.44)$$

Note that $I_{12} = 0$ for symmetric distributions. Elements of information matrix are considered below for some symmetric distributions.

Distribution	I_{11}	I_{22}	I_{12}
Normal	1	2	0
Cauchy	1/2	1/2	0
Student's(ν d.f)	$\frac{\nu+1}{\nu+3}$	$\frac{2\nu}{\nu+3}$	0
Logistic	$\frac{\pi^2}{9}$	$\frac{3+\pi^2}{9}$	0
Double - Exp	1	1	0

Naturally, the question of *optimum spacing* vector arises and may be obtained by maximizing (2.41), (2.43) and (2.44) by choosing $Q_0(\lambda_1), \dots, Q_0(\lambda_k)$ appropriately. Some numerical results on optimum spacings are given in appendix A for the t-distribution with ν degrees of freedom for $\nu = 1(1)3, 5, 10, 20, 500$ and $k=2(1)10$.

In this section, we have discussed the so-called *inefficient statistics* due to Mosteller's (1946) initiatives based on a few sample quantiles as *quick and dirty* substitutes for more conventional estimators. It can be seen that the estimators of this type are almost as efficient as the maximum likelihood estimators. Gastwirth (1966) and others established some estimators of this type with good efficiency properties for a variety of distributions. Our check with the class of t-distributions provides such evidence of efficiency as well. Although these *quick and dirty* estimators are slightly inefficient in a fixed parametric model, in practice they may actually be preferable to putatively optimal estimators like the sample mean, if there is some uncertainty about the shape of the underlying distribution. These simple estimators may be called *simplified L-estimators*. Recent work has devoted to extending results of this type beyond a fixed number of quantiles. The asymptotic theory of such L-estimators has achieved classical standing through the efforts of Chernoff et al (1967) and Bickel (1968). The most common L-estimator of location is the λ -trimmed mean which is simply the mean of the sample after the proportion λ of largest and smallest observations have been removed. In the next section we discuss this estimator with the

asymptotic theory.

2.4 Trimmed Least-Squared Estimation of Location Parameter And Its Asymptotic Distribution

We have considered the L-estimation of location and scale parameters based on all order statistics or on a few optimum sample quantiles which resulted in a *quick* estimator of β_0 and σ for each distribution under consideration, sacrificing very little on the asymptotic relative efficiency. The idea of derivation was based on the generalized least squares principle.

For the location model

$$Y = \beta_0 \mathbf{1}_n + Z, \quad (2.45)$$

where $Z = (Z_1, \dots, Z_n)'$ have components which are *i.i.d* r.v's from a continuous cdf $F_0(z)$ (with pdf $f_0(z)$), the least squares estimator (LSE) of β_0 is the sample mean $\bar{Y}_n$ given by

$$\bar{Y}_n = (\mathbf{1}_n' \mathbf{1}_n)^{-1} \mathbf{1}_n' Y. \quad (2.46)$$

Tukey (1960) proposed the trimmed estimator of β_0 as

$$\bar{Y}_n(r) = \frac{1}{n - 2r} \{Y_{(r+1)} + \dots + Y_{(n-r)}\}, \quad (2.47)$$

where r is the number of trimmed order statistics from the bottom and the top. This estimator was the integral part of the Huber's (1981) robust estimator of β_0 . We obtain a similar estimator by first defining the $n \times n$ diagonal matrix A whose elements are

$$a_{ii} = \begin{cases} 1, & \text{if } \bar{Q}_n(\lambda_1) < Y_i \leq \bar{Q}_n(\lambda_2) \\ 0, & \text{otherwise,} \end{cases} \quad (2.48)$$

where $\bar{Q}_n(\lambda_1)$ and $\bar{Q}_n(\lambda_2)$ are the sample quantiles of order λ_1 and λ_2 respectively, $(0 < \lambda_1 < 1/2 < \lambda_2 < 1)$ used for trimming the sample. Thus, minimizing

$$(Y - \beta_0 \mathbf{1})' A (Y - \beta_0 \mathbf{1}) \quad (2.49)$$

w.r.t β_0 , we obtain the trimmed least squares estimator (TLSE) as follows:

$$\bar{\beta}_{0n}(\lambda_1, \lambda_2) = (\mathbf{1}'_n A \mathbf{1}_n)^{-1} (\mathbf{1}'_n A Y). \quad (2.50)$$

The objective of this section is to provide an asymptotic linearity (AL) result for this statistic. Before we proceed with the representation of $\bar{\beta}_{0n}(\lambda_1, \lambda_2)$, we present some of the historical results regarding sample quantiles. The sample quantiles are themselves excellent summary data and fascinating random variables so much so that many researchers have devoted their life-time research to their study. One such result is due to Bahadur (1966) giving a representation of sample quantiles, namely,

$$\bar{Q}_n(\lambda) - \beta_0 - Q_0(\lambda) = n^{-1} \sum_{i=1}^n \Psi_F(Z_i) + O(n^{-3/4} \log n) \quad \text{a.s.} \quad (2.51)$$

where

$$\Psi_F(Z_i) = -[q_0(\lambda)]^{-1} \{I(Z_i \leq Q_0(\lambda)) - \lambda\} \quad (2.52)$$

and when F_0 has at least two derivatives in the neighborhood of $Q_0(\lambda)$ and is bounded in that neighborhood and $q_0(\lambda) > 0$. Note that the R.H.S of (2.51) is the sum of *i.i.d.* r.v. Hence, the asymptotic joint distribution of several sample quantiles follow easily than using the density method due to Mosteller (1946). More apt is the strong approximation results due to Csörgő and Révész (1975). First, we recall the definition of Brownian bridge process $\{B(\lambda) : 0 < \lambda < 1\}$; it is a zero-mean normal process with covariance function

$$K_B(\lambda_1, \lambda_2) = \lambda_1 \wedge \lambda_2 - \lambda_1 \lambda_2. \quad (2.53)$$

Theorem 2.4.1 (Csörgő and Révész, 1975) *A Brownian bridge $\{B_n(\lambda) : 0 < \lambda < 1\}$ can be defined for each n such that almost surely*

$$\sup_{0 < \lambda < 1} |n^{1/2} \{\bar{Q}_n(\lambda) - \lambda\} - B_n(\lambda)| = O(n^{-1/2} \log n) \quad (2.54)$$

where $\bar{Q}_n(\lambda)$ is the quantile of order λ from a sample of size n from the uniform distribution.

For the purpose of statistical inference, we can interpret the forgoing result as follows for the location-scale case

$$\frac{n^{1/2}}{\sigma} q_0(\lambda) \{ \bar{Q}_n(\lambda) - \beta_0 - \sigma Q_0(\lambda) \} \sim B(\lambda) \quad (2.55)$$

For instance, see Parzen (1979). Eubank (1981) in his Ph.D thesis developed the results of Ogawa (1951) for the estimation of β_0 and σ based on a few selected sample quantiles using the result in (2.55). Further, he defined optimum density function which generates the optimum spacings for the ABLUE of β_0 and σ .

Now, we return to the TLSE case. In this case, we need only a weaker representation for $\bar{Q}_n(\lambda)$ as given by

$$\bar{Q}_n(\lambda) - \beta_0 - Q_0(\lambda) = -n^{-1} [q_0(\lambda)]^{-1} \sum_{i=1}^n \{ I(Z_i \leq Q_0(\lambda)) - \lambda \} + o_P(1). \quad (2.56)$$

We are going to show this representation under similar condition as we need to represent $\tilde{\beta}_{0n}(\lambda_1, \lambda_2)$ given at (2.50). First, we consider the model (2.45) and the process

$$T_n(t; \lambda) = n^{-1/2} \sum_{i=1}^n \{ I(Y_i - t \leq 0) - \lambda \}. \quad (2.57)$$

Now, defining

$$\beta_0(\lambda) = \beta_0 + Q_0(\lambda), \quad (2.58)$$

we have the process (2.57) at $\beta_0(\lambda)$ given by

$$T_n(\beta_0(\lambda); \lambda) = n^{-1/2} \sum_{i=1}^n \{ I(Z_i \leq Q_0(\lambda)) - \lambda \}. \quad (2.59)$$

As a result, we have the following theorem under the condition F1:

F1 F_0 is continuous and has positive and continuous density function f_0 in the neighborhood of $Q_0(\lambda)$.

Theorem 2.4.2 *Under the condition F1*

$$n^{1/2}(\bar{Q}_n(\lambda) - \beta_0(\lambda)) = -[q_0(\lambda)]^{-1}T_n(\beta_0(\lambda); \lambda) + o_P(1). \quad (2.60)$$

The theorem is proved based on the following two lemmas with new proofs.

Lemma 2.4.3 *Let the condition F1 hold. Then, for each $b > 0$*

$$\sup_{|t| \leq b} |T_n(\beta_0(\lambda) + n^{-1/2}t; \lambda) - T_n(\beta_0(\lambda); \lambda) - tq_0(\lambda)| = o_P(1) \quad (2.61)$$

PROOF:

The expression (2.61) is bounded above by

$$\begin{aligned} & \sup_{|t| \leq b} \left| \left\{ T_n(\beta_0(\lambda) + n^{-1/2}t; \lambda) - E[T_n(\beta_0(\lambda) + n^{-1/2}t; \lambda)] \right\} \right. \\ & \quad \left. - \left\{ T_n(\beta_0(\lambda); \lambda) - E[T_n(\beta_0(\lambda); \lambda)] \right\} \right| \\ & + \sup_{|t| \leq b} \left| E[T_n(\beta_0(\lambda) + n^{-1/2}t; \lambda)] - E[T_n(\beta_0(\lambda); \lambda)] - tq_0(\lambda) \right| \\ & = I_1 + I_2. \end{aligned} \quad (2.62)$$

Rewriting I_1 and I_2 as

$$I_1 = \sup_{|t| \leq b} |T_n^0(t; \lambda) - T_n^0(0; \lambda)| \quad (2.63)$$

where

$$T_n^0(t; \lambda) = T_n(\beta_0(\lambda) + n^{-1/2}t; \lambda) - E[T_n(\beta_0(\lambda) + n^{-1/2}t; \lambda)] \quad (2.64)$$

and

$$I_2 = \sup_{|t| \leq b} n^{1/2} |F_0(Q_0(\lambda) + n^{-1/2}t) - F_0(Q_0(\lambda)) - n^{-1/2}tq_0(\lambda)| \quad (2.65)$$

First, we consider I_2 . Under condition F1 and using the mean-value theorem, we get

$$I_2 \leq b \cdot \sup_{|\Delta(t)| \leq \delta_n} |f_0(Q_0(\lambda) + \Delta(t)) - f_0(Q_0(\lambda))| \quad (2.66)$$

where $\delta_n = n^{-1/2} b$. So, $\forall \epsilon > 0 \exists N^\epsilon$ and $\delta > 0$ such that $\delta_n < \delta \forall n > N^\epsilon$.

Therefore, by F1 we have $I_2 \leq \epsilon$.

Now, we consider the following set

$$\mathcal{N}(b) = \{t \in \mathbb{R} : |t| \leq b\} \quad (2.67)$$

For I_1 , first we will show that $I_1 = o_P(1)$ for each $t \in \mathcal{N}(b)$ where

$$\begin{aligned} I_1 &= \left| T_n^0(t; \lambda) - T_n^0(0; \lambda) \right| \\ &= n^{-1/2} \left| \sum_{i=1}^n I(Z_i \leq Q_0(\lambda) + n^{-1/2}t) - F_0(Q_0(\lambda) + n^{-1/2}t) \right. \\ &\quad \left. - I(Z_i \leq Q_0(\lambda)) + F_0(Q_0(\lambda)) \right| \\ &= n^{-1/2} \left| \sum_{i=1}^n (V_i(t) - \nu_n(t)) \right|, \end{aligned} \quad (2.68)$$

where

$$\begin{aligned} V_i(t) &= I(Q_0(\lambda) < Z_i \leq Q_0(\lambda) + n^{-1/2}t), \\ \nu_n(t) &= E(V_i(t)) = F_0(Q_0(\lambda) + n^{-1/2}t) - F_0(Q_0(\lambda)). \end{aligned} \quad (2.69)$$

Then, for any $\epsilon > 0$

$$\begin{aligned} P\left\{ \left| n^{-1/2} \sum_{i=1}^n (V_i(t) - \nu_n(t)) \right|^2 > \epsilon \right\} &\leq \frac{1}{\epsilon^2} E\left(n^{-1/2} \sum_{i=1}^n (V_i(t) - \nu_n(t)) \right)^2 \\ &= \frac{1}{\epsilon^2} \frac{1}{n} \sum_{i=1}^n E(V_i(t) - \nu_n(t))^2 \\ &= \frac{1}{\epsilon^2} (\nu_n(t) - \nu_n^2(t)). \end{aligned} \quad (2.70)$$

But, $\nu_n(t) \rightarrow 0$ as $n \rightarrow \infty$ by continuity of F_0 . Therefore, $I_1 = o_P(1)$. For the tightness of the process we use the compactness property of the set $\mathcal{N}(b)$. Since the set $\mathcal{N}(b)$ is compact, it suffices to show that for every $\epsilon > 0$, there is a $\delta > 0$ such that for each $s \in \mathcal{N}(b)$,

$$\limsup_n P\left(\sup_{|t-s| \leq \delta} \left| T_n^0(t; \lambda) - T_n^0(s; \lambda) \right| > 2\epsilon \right) < \epsilon. \quad (2.71)$$

Using the definition of $T_n^0(\cdot; \cdot)$ we have

$$\begin{aligned}
 & T_n^0(t; \lambda) - T_n^0(s; \lambda) \\
 &= n^{-1/2} \sum_{i=1}^n \left\{ \left(I(Z_i \leq Q_0(\lambda) + n^{-1/2}t) - I(Z_i \leq Q_0(\lambda) + n^{-1/2}s) \right) \right. \\
 &\quad \left. - \left(F_0(Q_0(\lambda) + n^{-1/2}t) - F_0(Q_0(\lambda) + n^{-1/2}s) \right) \right\} \\
 &= n^{-1/2} \sum_{i=1}^n \left\{ \left(\alpha_i(t) - \alpha_i(s) \right) - \left(\mu_n(t) - \mu_n(s) \right) \right\}, \tag{2.72}
 \end{aligned}$$

where

$$\alpha_i(t) = I(Z_i \leq Q_0(\lambda) + n^{-1/2}t) \quad \text{and} \quad \mu_n(t) = E(\alpha_i(t)) \tag{2.73}$$

Now, using the monotone property of the indicator function with the relationship between s and t , i.e

$$s - \delta \leq t \leq s + \delta, \tag{2.74}$$

we have

$$\begin{aligned}
 & T_n^0(t; \lambda) - T_n^0(s; \lambda) \\
 &\leq n^{-1/2} \sum_{i=1}^n \{ \alpha_i(s + \delta) - \alpha_i(s) - \mu_n(s - \delta) + \mu_n(s) \} \\
 &= n^{-1/2} \sum_{i=1}^n \{ \alpha_i(s + \delta) - \alpha_i(s) - \mu_n(s + \delta) + \mu_n(s) \\
 &\quad + \mu_n(s + \delta) - \mu_n(s - \delta) \} \\
 &= T_n^0(s + \delta; \lambda) - T_n^0(s; \lambda) + n^{1/2} \{ \mu_n(s + \delta) - \mu_n(s - \delta) \} \tag{2.75}
 \end{aligned}$$

Similarly, we have

$$\begin{aligned}
 & T_n^0(s - \delta; \lambda) - T_n^0(s; \lambda) + n^{1/2} \{ \mu_n(s - \delta) - \mu_n(s + \delta) \} \\
 &\leq T_n^0(t; \lambda) - T_n^0(s; \lambda) \leq T_n^0(s + \delta; \lambda) - T_n^0(s; \lambda) \\
 &\quad + n^{1/2} \{ \mu_n(s + \delta) - \mu_n(s - \delta) \} \tag{2.76}
 \end{aligned}$$

Thus,

$$\begin{aligned}
 |T_n^0(t; \lambda) - T_n^0(s; \lambda)| &\leq |T_n^0(s + \delta; \lambda) - T_n^0(s; \lambda)| \\
 &\quad + |T_n^0(s - \delta; \lambda) - T_n^0(s; \lambda)| \\
 &\quad + 2n^{1/2} |\mu_n(s + \delta) - \mu_n(s - \delta)| \\
 &= I_{11} + I_{12} + I_{13} = o_P(1) \quad \text{as } n \rightarrow \infty
 \end{aligned} \tag{2.77}$$

By continuity of F_0 we have $|\mu_n(s + \delta) - \mu_n(s)| \rightarrow 0$ as $n \rightarrow \infty$ and $I_{11} = o_P(1) = I_{12}$ by previous discussion. Hence the lemma.

The next step is to show that the sample quantiles are bounded in probability, which we state below:

Lemma 2.4.4 *Under the condition F1, for each $\lambda \in (0, 1)$,*

$$n^{1/2}(\bar{Q}_n(\lambda) - \beta_0(\lambda)) = O_P(1) \tag{2.78}$$

PROOF:

This Lemma is proved by two different methods. For the first one we assume that $f_0(F_0^{-1}(\lambda)) > 0$ and t be any positive real number. If we take $k_n = n\lambda + o(n^{-1/2})$, we have

$$\begin{aligned}
 &P\{Y_{(k_n)} \leq \beta_0(\lambda) - n^{-1/2}t\} \\
 &= P\{F_n(\beta_0(\lambda) - n^{-1/2}t) \geq n^{-1}k_n\} \\
 &= P\left\{n^{1/2}[F_n(\beta_0(\lambda) - n^{-1/2}t) - F(\beta_0(\lambda) - n^{-1/2}t)]\right. \\
 &\quad \left. \geq n^{1/2}[n^{-1}k_n - F(\beta_0(\lambda) - n^{-1/2}t)]\right\} \\
 &\leq \epsilon.
 \end{aligned} \tag{2.79}$$

Since

$$n^{1/2}[n^{-1}k_n - F(\beta_0(\lambda) - n^{-1/2}t)] = t q_0(\lambda) + o(1), \tag{2.80}$$

by Taylor's expansion while $n^{1/2}[F_n(x) - F(x)]$ is asymptotically normal with mean 0 and finite variance, by taking t large enough we get the result for sufficiently large n . A similar result holds for the right tail as well and the proof of the Lemma is complete.

The second method is based on Jurečková (1984) and is the one which we shall follow in subsequent chapters involving regression and autoregression models. We need the following steps to prove this Lemma

$$i) \quad T_n(\bar{Q}_n(\lambda); \lambda) = o_P(1) \quad (2.81)$$

$$ii) \quad \forall \epsilon > 0, 0 < u < \infty \exists b(=b_u^\epsilon) \text{ and an } N^\epsilon \text{ such that}$$

$$\text{for each } \lambda \in (0, 1)$$

$$P\left(\inf_{|t|>b} |T_n(\beta_0(\lambda) + n^{-1/2}t; \lambda)| > u\right) > 1 - \epsilon \quad (2.82)$$

PROOF of (i)

Follows from Lemma A.2 of Ruppert and Carroll (1980). Lemma A.2 of Ruppert and Carroll (1980) in this case is:

The sequence of solutions $\bar{Q}_n(\lambda)$ of (2.22) satisfies

$$n^{-1/2} \sum_{i=1}^n \{I(y_i - \bar{Q}_n(\lambda) \leq 0) - \lambda\} \rightarrow 0 \quad \text{a.s.} \quad (2.83)$$

PROOF of (ii)

To prove this, we shall follow the Koul and Saleh (1995) approach. By using the non-decreasing property of $T_n(\beta_0(\lambda) + n^{-1/2}t; \lambda)$ with respect to t , we have

$$\inf_{|t|>b} |T_n(\beta_0(\lambda) + n^{-1/2}t; \lambda)|^2 \geq \inf_{|t|=b} |T_n(\beta_0(\lambda) + n^{-1/2}t; \lambda)|^2 \quad (2.84)$$

for each $\lambda \in (0, 1)$. Let us define

$$T_n^*(t; \lambda) = T_n(\beta_0(\lambda); \lambda) + tq_0(\lambda). \quad (2.85)$$

By Lemma 2.4.3 we have

$$T_n(\beta_0(\lambda) + n^{-1/2}t; \lambda) - T_n^*(t; \lambda) = o_P(1). \quad (2.86)$$

Then, for all $\epsilon > 0$ there is an N^ϵ such that

$$\begin{aligned} P\left(\inf_{|t|=b} \left\{ T_n(\beta_0(\lambda) + n^{-1/2}t; \lambda) \right\}^2 > u\right) \\ \geq P\left(\inf_{|t|=b} \left\{ T_n^*(t; \lambda) \right\}^2 > u\right) - \epsilon/3, \quad n > N^\epsilon \end{aligned} \quad (2.87)$$

Thus, it is enough to see the behavior of $T_n^*(t; \lambda)$. Now, using the absolute value property, we have

$$\begin{aligned} P\left(\inf_{|t|=b} \left\{ T_n^*(t; \lambda) \right\}^2 > u\right) \\ \geq P\left(\left| |T_n(\beta_0(\lambda); \lambda)| - b q_0(\lambda) \right| > u^{1/2}\right) \\ \geq P\left(\left| T_n(\beta_0(\lambda); \lambda) \right| \leq -u^{1/2} + b q_0(\lambda)\right) \\ \geq 1 - (2\epsilon/3), \quad \forall n > N^\epsilon \end{aligned} \quad (2.88)$$

where $b > \frac{u^{1/2} + M_\epsilon}{q_0(\lambda)}$. Equation (2.88) results from the fact that $T_n(\beta_0(\lambda); \lambda)$ is a sum of *i.i.d* random variables which by C.L.T converges to a normal distribution as $n \rightarrow \infty$. Let Z be a standard normal variable. Then, $\forall \epsilon > 0 \exists M_\epsilon > 0$ and $N^\epsilon \ni$

$$P\left(|T_n(\beta_0(\lambda); \lambda)| \leq M_\epsilon\right) > 1 - (2\epsilon/3) \quad \forall n > N^\epsilon. \quad (2.89)$$

Now, we apply the results of (2.81) and (2.82) to complete the proof of Lemma 2.4.4.

We observe that for any $b > 0$

$$\begin{aligned} P\left(n^{1/2} \left| \bar{Q}_n(\lambda) - \beta_0(\lambda) \right| > b\right) \\ = P\left(n^{1/2} \left| \bar{Q}_n(\lambda) - \beta_0(\lambda) \right| > b, \left| T_n(\beta_0(\lambda) + n^{-1/2}[n^{1/2}(\bar{Q}_n(\lambda) - \beta_0(\lambda))]; \lambda) \right| < u\right) \\ + P\left(n^{1/2} \left| \bar{Q}_n(\lambda) - \beta_0(\lambda) \right| > b, \left| T_n(\beta_0(\lambda) + n^{-1/2}[n^{1/2}(\bar{Q}_n(\lambda) - \beta_0(\lambda))]; \lambda) \right| \geq u\right) \\ \leq P\left(\inf_{|t|>b} \left| T_n(\beta_0(\lambda) + n^{-1/2}t; \lambda) \right| < u\right) + P\left(\left| T_n(\bar{Q}_n(\lambda); \lambda) \right| \geq u\right) \\ \leq \epsilon/2 + \epsilon/2 \quad \text{by (2.81) and (2.82)} \quad \forall n > N^\epsilon. \end{aligned} \quad (2.90)$$

Now by using Lemmas 2.4.3 and 2.4.4 we get the representation of $\bar{Q}_n(\lambda)$. By Lemma 2.4.3 for every bounded t ,

$$T_n(\beta_0(\lambda) + n^{-1/2}t; \lambda) - T_n(\beta_0(\lambda); \lambda) - tq_0(\lambda) = o_P(1) \quad (2.91)$$

Hence, by applying the Lemma 2.4.4 with $t = n^{1/2}(\bar{Q}_n(\lambda) - \beta_0(\lambda))$, we have

$$\begin{aligned} & T_n(\beta_0(\lambda) + n^{-1/2}[n^{\frac{1}{2}}(\bar{Q}_n(\lambda) - \beta_0(\lambda))]; \lambda) - T_n(\beta_0(\lambda); \lambda) - q_0(\lambda) n^{\frac{1}{2}}(\bar{Q}_n(\lambda) - \beta_0(\lambda)) \\ &= o_P(1). \end{aligned} \quad (2.92)$$

On the other hand

$$T_n(\bar{Q}_n(\lambda); \lambda) - T_n(\beta_0(\lambda); \lambda) - q_0(\lambda) n^{\frac{1}{2}}(\bar{Q}_n(\lambda) - \beta_0(\lambda)) = o_P(1), \quad (2.93)$$

and (2.81) implies

$$\begin{aligned} n^{1/2}(\bar{Q}_n(\lambda) - \beta_0(\lambda)) &= -[q_0(\lambda)]^{-1}T_n(\beta_0(\lambda); \lambda) + o_P(1) \\ &= -n^{-1/2}[q_0(\lambda)]^{-1} \sum_{i=1}^n \{I(Z_i \leq Q_0(\lambda)) - \lambda\} + o_P(1) \end{aligned} \quad (2.94)$$

As we mentioned before, the object of this section is to represent the TLSE of β_0 given in (2.50). Thus, we have

Theorem 2.4.5 *Let F_0 be continuous with continuous and positive density function f_0 in the neighborhood of $Q_0(\lambda_1)$ and $Q_0(\lambda_2)$ respectively. Then*

$$\begin{aligned} i) \quad n^{1/2}(\bar{\beta}_{0n}(\lambda_1, \lambda_2) - \beta_0) \\ = (\lambda_2 - \lambda_1)^{-1} n^{-1/2} \sum_{i=1}^n (\phi(Z_i) - \gamma) + o_P(1) \end{aligned} \quad (2.95)$$

$$ii) \quad n^{1/2}(\bar{\beta}_{0n}(\lambda_1, \lambda_2) - \beta_0) \xrightarrow{D} \mathcal{N}(0, \sigma^2(\lambda_1, \lambda_2)), \quad (2.96)$$

where

$$\begin{aligned} \phi(Z) &= \begin{cases} Q_0(\lambda_1), & \text{if } Z < Q_0(\lambda_1), \\ Z, & \text{if } Q_0(\lambda_1) < Z \leq Q_0(\lambda_2), \\ Q_0(\lambda_2), & \text{if } Z > Q_0(\lambda_2) \end{cases} \\ \gamma &= \lambda_1 Q_0(\lambda_1) + (1 - \lambda_2) Q_0(\lambda_2) \end{aligned} \quad (2.97)$$

and

$$\begin{aligned} \sigma^2(\lambda_1, \lambda_2) &= (\lambda_2 - \lambda_1)^{-2} \left\{ \int_{\lambda_1}^{\lambda_2} (Q_0(u) - \delta_0)^2 du + \lambda_1 (Q_0(\lambda_1) - \delta_0)^2 \right. \\ &\quad \left. + (1 - \lambda_2) (Q_0(\lambda_2) - \delta_0)^2 - [\lambda_1 (Q_0(\lambda_1) - \delta_0) + (1 - \lambda_2) (Q_0(\lambda_2) - \delta_0)]^2 \right\} \end{aligned} \quad (2.98)$$

with

$$\delta_0 = (\lambda_2 - \lambda_1)^{-1} \int_{\lambda_1}^{\lambda_2} Q_0(u) du \quad (2.99)$$

PROOF: We shall prove this Theorem based on the two Lemmas (2.4.6 and 2.4.7). First, we define the following processes

$$\tilde{T}_n(t; \lambda) = n^{-1/2} \sum_{i=1}^n Z_i I(Z_i \leq Q_0(\lambda) + n^{-1/2} t) \quad (2.100)$$

and

$$\begin{aligned} \tilde{T}_n^0(t; \lambda) &= \tilde{T}_n(t; \lambda) - E(\tilde{T}_n(t; \lambda)) \\ &= n^{-1/2} \sum_{i=1}^n \{ Z_i I(Z_i \leq Q_0(\lambda) + n^{-1/2} t) - \mu(Q_0(\lambda) + n^{-1/2} t) \} \end{aligned} \quad (2.101)$$

where

$$\mu(Q_0(\lambda) + n^{-1/2} t) = E[Z_i I(Z_i \leq Q_0(\lambda) + n^{-1/2} t)] \quad (2.102)$$

and

$$\mu'(u) = u f(u). \quad (2.103)$$

Then, we have the Lemma 2.4.6.

Lemma 2.4.6 Under the condition F1 for each $b > 0$,

$$\sup_{|t| \leq b} |\tilde{T}_n(t; \lambda) - \tilde{T}_n(0; \lambda) - t q_0(\lambda) Q_0(\lambda)| = o_P(1). \quad (2.104)$$

PROOF:

$$\begin{aligned}
 & \sup_{|t| \leq b} \left| \tilde{T}_n(t; \lambda) - \tilde{T}_n(0; \lambda) - t q_0(\lambda) Q_0(\lambda) \right| \\
 & \leq \sup_{|t| \leq b} \left| \tilde{T}_n^0(t; \lambda) - \tilde{T}_n^0(0; \lambda) \right| + \sup_{|t| \leq b} \left| E(\tilde{T}_n(t; \lambda)) - E(\tilde{T}_n(0; \lambda)) - t q_0(\lambda) Q_0(\lambda) \right| \\
 & = J_1 + J_2.
 \end{aligned} \tag{2.105}$$

First, note that

$$J_2 = \sup_{|t| \leq b} n^{1/2} \left| \mu(Q_0(\lambda) + n^{-1/2} t) - \mu(Q_0(\lambda)) - n^{-1/2} Q_0(\lambda) q_0(\lambda) t \right|. \tag{2.106}$$

Then by using the mean value theorem, we have

$$J_2 \leq b \cdot Q_0(\lambda) \sup_{|\Delta(t)| \leq \delta_n} \left| f_0(Q_0(\lambda) + \Delta(t)) - f_0(Q_0(\lambda)) \right| + \epsilon_n. \tag{2.107}$$

Hence R.H.S of (2.107) is $o(1)$ as $n \rightarrow \infty$ by condition F1. Now, to prove the Lemma 2.4.6, it is enough to show that $J_1 = o_P(1)$ where

$$J_1 = \sup_{|t| \leq b} \left| \tilde{T}_n^0(t; \lambda) - \tilde{T}_n^0(0; \lambda) \right|. \tag{2.108}$$

Here, it is necessary to show that for each $t \in \mathcal{N}(b)$

$$\left| \tilde{T}_n^0(t; \lambda) - \tilde{T}_n^0(0; \lambda) \right| = o_P(1) \tag{2.109}$$

where

$$\begin{aligned}
 & \left| \tilde{T}_n^0(t; \lambda) - \tilde{T}_n^0(0; \lambda) \right| \\
 & = n^{-1/2} \left| \sum_{i=1}^n Z_i \{ I(Z_i \leq Q_0(\lambda) + n^{-1/2} t) - I(Z_i \leq Q_0(\lambda)) \} \right. \\
 & \quad \left. - \{ \mu(Q_0(\lambda) + n^{-1/2} t) - \mu(Q_0(\lambda)) \} \right| \\
 & = n^{-1/2} \left| \sum_{i=1}^n (\hat{V}_i(t) - \hat{\nu}_n(t)) \right|
 \end{aligned} \tag{2.110}$$

and

$$\hat{V}_i(t) = Z_i I(Q_0(\lambda) < Z_i \leq Q_0(\lambda) + n^{-1/2} t) \text{ and } \hat{\nu}_n(t) = E(\hat{V}_i(t)). \tag{2.111}$$

Now, for each $\epsilon > 0$

$$\begin{aligned}
 & P \left\{ \left(n^{-1/2} \sum_{i=1}^n (\tilde{V}_i(t) - \tilde{v}_n(t)) \right)^2 > \epsilon^2 \right\} \\
 & \leq \frac{1}{\epsilon^2} \frac{1}{n} E \left(\sum_{i=1}^n (\tilde{V}_i(t) - \tilde{v}_n(t)) \right)^2 \\
 & = \frac{1}{\epsilon^2} \frac{1}{n} \sum_{i=1}^n E (\tilde{V}_i(t) - \tilde{v}_n(t))^2 \\
 & = \frac{1}{\epsilon^2} \left\{ E (\tilde{V}_1(t))^2 - (\tilde{v}_n(t))^2 \right\}
 \end{aligned} \tag{2.112}$$

By continuity of f_0 each term inside the above bracket goes to zero as $n \rightarrow \infty$ for each t . For the tightness, we use the same argument as in Lemma 2.4.3. So, it is necessary to show that for all $\epsilon > 0 \exists \delta > 0$ such that for each $s \in \mathcal{N}(b)$

$$\limsup_n P \left(\sup_{|t-s| \leq \delta} \left| \tilde{T}_n^0(t; \lambda) - \tilde{T}_n^0(s; \lambda) \right| > 2\epsilon \right) < \epsilon. \tag{2.113}$$

In this case we have

$$\begin{aligned}
 & \left| \tilde{T}_n^0(t; \lambda) - \tilde{T}_n^0(s; \lambda) \right| \\
 & \leq n^{-1/2} \sum_{i=1}^n \left| Z_i \left\{ I(Z_i \leq Q_0(\lambda) + n^{-1/2} t) - I(Z_i \leq Q_0(\lambda) + n^{-1/2} s) \right\} \right| \\
 & \quad + n^{1/2} \left| \mu(Q_0(\lambda) + n^{-1/2} t) - \mu(Q_0(\lambda) + n^{-1/2} s) \right|.
 \end{aligned} \tag{2.114}$$

By the monotone property of the indicator function with the relation (2.74), the above expression is bounded above by

$$\begin{aligned}
 & n^{-1/2} \sum_{i=1}^n |Z_i| \left\{ I(Z_i \leq Q_0(\lambda) + n^{-1/2} s + n^{-1/2} \delta) \right. \\
 & \quad \left. - I(Z_i \leq Q_0(\lambda) + n^{-1/2} s - n^{-1/2} \delta) \right\} \\
 & \quad + n^{1/2} \left| \mu(Q_0(\lambda) + n^{-1/2} t) - \mu(Q_0(\lambda) + n^{-1/2} s) \right| \\
 & = J_{11} + J_{12}.
 \end{aligned} \tag{2.115}$$

By similar argument as (2.106) and (2.107) $J_{12} = o(1)$. We only need to show that $J_{11} = o_P(1)$. As we see, J_{11} does not depend on t . We use Chebyshev inequality to show that it converges to zero in probability. Let us define

$$\mu^*(x) = E[|Z_i|I(Z_i \leq x)]. \quad (2.116)$$

Then,

$$\begin{aligned} J_{11} = & n^{-1/2} \sum_{i=1}^n \left\{ \left[|Z_i|I(Z_i \leq Q_0(\lambda) + n^{-1/2}s + n^{-1/2}\delta) - \mu^*(Q_0(\lambda) + n^{-1/2}s + n^{-1/2}\delta) \right] \right. \\ & \left. - \left[|Z_i|I(Z_i \leq Q_0(\lambda) + n^{-1/2}s - n^{-1/2}\delta) - \mu^*(Q_0(\lambda) + n^{-1/2}s - n^{-1/2}\delta) \right] \right\} \\ & + n^{1/2} \left\{ \mu^*(Q_0(\lambda) + n^{-1/2}s + n^{-1/2}\delta) - \mu^*(Q_0(\lambda) + n^{-1/2}s - n^{-1/2}\delta) \right\} \quad (2.117) \end{aligned}$$

The second term of R.H.S of J_{11} is less than ϵ for sufficiently small δ by the mean value Theorem. The first part is $o_P(1)$ by the argument similar to the proof of J_1 for fixed t of Lemma 2.4.3. This ends the proof of Lemma 2.4.6.

Before stating the next Lemma we consider the following processes,

$$U_n(t) = n^{-1} \sum_{i=1}^n I(Z_i \leq Q_0(\lambda) + n^{-1/2}t) \quad (2.118)$$

and

$$U_n^0(t) = U_n(t) - E[U_n(t)] = U_n(t) - F_0(Q_0(\lambda) + n^{-1/2}t). \quad (2.119)$$

Then,

Lemma 2.4.7 Assume that the condition F1 hold. Then for every $b > 0$

$$\sup_{|t| \leq b} |U_n(t) - \lambda| \xrightarrow{P} 0 \quad (2.120)$$

PROOF:

$$\sup_{|t| \leq b} |U_n(t) - \lambda|$$

$$\begin{aligned}
&= \sup_{|t| \leq b} \left| n^{-1} \sum_{i=1}^n (I(Z_i \leq Q_0(\lambda) + n^{-1/2}t) - \lambda) \right| \\
&= \sup_{|t| \leq b} n^{-1} \left| \sum_{i=1}^n \left[(I(Z_i \leq Q_0(\lambda) + n^{-1/2}t) - F_0(Q_0(\lambda) + n^{-1/2}t)) \right. \right. \\
&\quad \left. \left. + F_0(Q_0(\lambda) + n^{-1/2}t) - \lambda \right] \right| \\
&\leq \sup_{|t| \leq b} n^{-1} \left| \sum_{i=1}^n \left[(I(Z_i \leq Q_0(\lambda) + n^{-1/2}t) - F_0(Q_0(\lambda) + n^{-1/2}t)) \right] \right| \\
&\quad + \sup_{|t| \leq b} \left| F_0(Q_0(\lambda) + n^{-1/2}t) - \lambda \right| \\
&= I_1 + I_2. \tag{2.121}
\end{aligned}$$

Under the condition F1 we get $I_2 \rightarrow 0$ as $n \rightarrow \infty$. For I_1 , we follow the same argument as in Lemma 2.4.3 along with the weak law of large number.

Now, we change the process $U_n(t)$ to $U_n^*(t)$ as

$$U_n^*(t) = n^{-1} \sum_{i=1}^n I(Q_0(\lambda_1) + n^{-1/2}t < Z_i \leq Q_0(\lambda_2) + n^{-1/2}t). \tag{2.122}$$

Then, if f_0 is continuous in the neighborhood of $Q_0(\lambda_1)$ and $Q_0(\lambda_2)$, Lemma 2.4.7 implies

$$\sup_{|t| \leq b} |U_n^*(t) - (\lambda_2 - \lambda_1)| = o_P(1). \tag{2.123}$$

This result together with the structure of A implies

$$n^{-1} \mathbf{1}' A \mathbf{1} = (\lambda_2 - \lambda_1) + o_P(1) \quad \text{as } n \rightarrow \infty. \tag{2.124}$$

Further, note that

$$\begin{aligned}
n^{-1/2} \mathbf{1}' A \mathbf{Z} &= n^{-1/2} \sum_{i=1}^n Z_i \left\{ I(Z_i \leq Q_0(\lambda_2) + n^{-1/2}[n^{1/2}(\bar{Q}_n(\lambda_2) - \beta_0(\lambda_2))]) \right. \\
&\quad \left. - I(Z_i \leq Q_0(\lambda_1) + n^{-1/2}[n^{1/2}(\bar{Q}_n(\lambda_1) - \beta_0(\lambda_1))]) \right\}. \tag{2.125}
\end{aligned}$$

Now, using Lemma 2.4.6 for the first and second term of the R.H.S of (2.125), we have

$$n^{-1/2} \mathbf{1}' A \mathbf{Z} = T_n(\beta_0(\lambda_1); \lambda_1) Q_0(\lambda_1) - T_n(\beta_0(\lambda_2); \lambda_2) Q_0(\lambda_2)$$

$$+n^{-1/2} \sum_{i=1}^n Z_i I(Q_0(\lambda_1) < Z_i \leq Q_0(\lambda_2)) + o_P(1) \quad (2.126)$$

and by the definition of $T_n(\beta_0(\lambda); \lambda)$ (given in (2.59)) we get

$$n^{-1/2} \mathbf{1}' \mathbf{A} \mathbf{Z} = n^{-1/2} \sum_{i=1}^n (\phi(Z_i) - \gamma) + o_P(1). \quad (2.127)$$

Finally, we have

$$(\mathbf{1}' \mathbf{A} \mathbf{1})^{-1} \mathbf{1}' \mathbf{A} \mathbf{Z} = (\mathbf{1}' \mathbf{A} \mathbf{1})^{-1} \mathbf{1}' \mathbf{A} \mathbf{Y} - \beta_0 \quad (2.128)$$

or

$$(n^{-1} \mathbf{1}' \mathbf{A} \mathbf{1})^{-1} n^{-1/2} \mathbf{1}' \mathbf{A} \mathbf{Z} = n^{1/2} (\tilde{\beta}_{0n}(\lambda_1, \lambda_2) - \beta_0) \quad (2.129)$$

and as a result

$$n^{1/2} (\tilde{\beta}_{0n}(\lambda_1, \lambda_2) - \beta_0) = (\lambda_2 - \lambda_1)^{-1} n^{-1/2} \sum_{i=1}^n (\phi(Z_i) - \gamma) + o_P(1). \quad (2.130)$$

This completes the proof of part (i) of Theorem 2.4.5. The proof of the second part of this theorem follows by the C.L.T property of *i.i.d* random variables.

Theorem 2.4.5 states that TLSE converges in probability to a normal distribution with variance $\sigma^2(\lambda_1, \lambda_2)$. Now, we wish to determine a consistent estimator of $\sigma^2(\lambda_1, \lambda_2)$ which will be helpful for future inference (e.g. testing hypotheses, confidence intervals). Let $Y_1, Y_2, \dots, Y_n$ be a sample from the location model (2.45). We may define an estimate of $\sigma^2(\lambda_1, \lambda_2)$ by

$$\begin{aligned} S_n^2(\lambda_1, \lambda_2) = & (\lambda_2 - \lambda_1)^{-2} \left\{ (n-1)^{-1} \sum_{i=1}^n (Y_i - \delta_n)^2 I(\bar{Q}_n(\lambda_1) < Y_i \leq \bar{Q}_n(\lambda_2)) \right. \\ & + \lambda_1 (\bar{Q}_n(\lambda_1) - \delta_n)^2 + (1 - \lambda_2) (\bar{Q}_n(\lambda_2) - \delta_n)^2 \\ & \left. - [\lambda_1 (\bar{Q}_n(\lambda_1) - \delta_n) + (1 - \lambda_2) (\bar{Q}_n(\lambda_2) - \delta_n)]^2 \right\}, \end{aligned} \quad (2.131)$$

where $\bar{Q}_n(\lambda_i)$, ($i = 1, 2$) is the sample quantile with order λ_i and

$$\delta_n = [n(\lambda_2 - \lambda_1)]^{-1} \sum_{i=1}^n Y_i I(\bar{Q}_n(\lambda_1) < Y_i \leq \bar{Q}_n(\lambda_2)). \quad (2.132)$$

Then,

Theorem 2.4.8 *Let the condition of Theorem 2.4.5 hold. Then*

$$S_n^2(\lambda_1, \lambda_2) \xrightarrow{P} \sigma^2(\lambda_1, \lambda_2) \quad (2.133)$$

PROOF:

To prove this theorem first we shall show that

$$\delta_n \xrightarrow{P} \delta, \quad (2.134)$$

where

$$\delta = (\lambda_2 - \lambda_1)^{-1} \int_{Q(\lambda_1)}^{Q(\lambda_2)} Y dF(y) = \beta_0 + \delta_0 \quad (2.135)$$

and δ_0 defined in (2.99). Equivalently, it is necessary to show that

$$\delta_{0n} \xrightarrow{P} \delta_0, \quad (2.136)$$

where

$$\delta_{0n} = [n(\lambda_2 - \lambda_1)]^{-1} \sum_{i=1}^n Z_i I(\bar{Q}_{0n}(\lambda_1) < Z_i \leq \bar{Q}_{0n}(\lambda_2)). \quad (2.137)$$

We need the following lemma to prove (2.136).

Lemma 2.4.9 *Let $\xi_1, \xi_2, \dots, \xi_n$ be i.i.d r.v's with cdf F and let g be a measurable function such that $g(\cdot) \geq 0$ and $E(g(\xi_1)) < \infty$. Define*

$$W_n(x) = n^{-1} \sum_{i=1}^n [g(\xi_i) I(\xi_i \leq x) - \nu(x)] \quad (2.138)$$

where

$$\nu(x) = E[g(\xi_1) I(\xi_1 \leq x)]. \quad (2.139)$$

Then

$$\sup_{-\infty < x < \infty} |W_n(x)| \rightarrow 0 \quad a.s. \quad (2.140)$$

We shall prove this lemma at the end of this chapter.

Now, we are going to show (2.136)

$$\begin{aligned}
 |\delta_{0n} - \delta_0| &= c \left| n^{-1} \sum_{i=1}^n \left\{ Z_i I(\bar{Q}_{0n}(\lambda_1) < Z_i \leq \bar{Q}_{0n}(\lambda_2)) - \int_{Q_0(\lambda_1)}^{Q_0(\lambda_2)} Z dF_0(z) \right\} \right| \\
 &\leq c \left| n^{-1} \sum_{i=1}^n \left\{ Z_i I(Z_i \leq \bar{Q}_{0n}(\lambda_2)) - \nu(Q_0(\lambda_2)) \right\} \right| \\
 &\quad + c \left| n^{-1} \sum_{i=1}^n \left\{ Z_i I(Z_i \leq \bar{Q}_{0n}(\lambda_1)) - \nu(Q_0(\lambda_1)) \right\} \right| \\
 &= c(I_1 + I_2). \tag{2.141}
 \end{aligned}$$

Thus, we have to show that $I_1 = o_P(1) = I_2$. Let us write the r.v Z_i as

$$Z_i = Z_i^+ - Z_i^- \tag{2.142}$$

and define

$$W_n^\pm(\lambda) = n^{-1} \sum_{i=1}^n Z_i^\pm I(Z_i \leq \bar{Q}_{0n}(\lambda)) - \nu^\pm(Q_0(\lambda)) \tag{2.143}$$

It is necessary to show that $|W_n^+(\lambda)| = o_P(1) = |W_n^-(\lambda)|$, where

$$\begin{aligned}
 |W_n^+(\lambda)| &= \left| n^{-1} \sum_{i=1}^n Z_i^+ I(Z_i \leq \bar{Q}_{0n}(\lambda)) - \nu^+(Q_0(\lambda)) \right| \\
 &\leq \left| n^{-1} \sum_{i=1}^n Z_i^+ I(Z_i \leq \bar{Q}_{0n}(\lambda)) - \nu^+(\bar{Q}_{0n}(\lambda)) \right| \\
 &\quad + \left| \nu^+(\bar{Q}_{0n}(\lambda)) - \nu^+(Q_0(\lambda)) \right|. \tag{2.144}
 \end{aligned}$$

The first term of R.H.S of (2.144) goes to zero a.s. by Lemma 2.4.9. The second term also converges to zero as $n \rightarrow \infty$ by consistency of sample quantile given in Lemma 2.4.4. By the same argument we get that $|W_n^-(\lambda)| \rightarrow 0$ as $n \rightarrow \infty$ and then,

$$|W_n(\lambda)| \rightarrow 0 \quad \text{a.s.} \tag{2.145}$$

Therefore we get that $I_1 = o_P(1) = I_2$. Using similar argument we have

$$\left| n^{-1} \sum_{i=1}^n \left\{ Z_i^2 I(\bar{Q}_{0n}(\lambda_1) < Z_i \leq \bar{Q}_{0n}(\lambda_2)) - \int_{Q_0(\lambda_1)}^{Q_0(\lambda_2)} Z^2 dF_0(z) \right\} \right| = o_P(1). \tag{2.146}$$

Finally, by Lemma 2.4.4 we have

$$\tilde{Q}_{0n}(\lambda) \xrightarrow{P} Q_0(\lambda). \quad (2.147)$$

Combining all the above results we claim the proof of Theorem 2.4.8. ♦

PROOF of Lemma 2.4.9:

Let $0 < M < \infty$ be an arbitrary fixed real number. Then, we may write

$$\begin{aligned} \sup_{-\infty < x < \infty} |W_n(x)| &\leq \sup_{-\infty < x < -M} |W_n(x)| + \sup_{-M \leq x \leq M} |W_n(x)| \\ &\quad + \sup_{M < x < \infty} |W_n(x)| \\ &= I_1 + I_2 + I_3. \end{aligned} \quad (2.148)$$

We have to show that each term of R.H.S of above inequality goes to zero. First, we are going to show that $I_2 \rightarrow 0$ a.s. For this reason we partition the interval $[-M, M]$ into k parts. Then, for each x ,

$$\frac{j}{k}2M \leq x \leq \frac{j+1}{k}2M \quad (2.149)$$

we have

$$\begin{aligned} W_n\left(\frac{j}{k}2M\right) + \nu\left(\frac{j}{k}2M\right) - \nu\left(\frac{j+1}{k}2M\right) &\leq W_n(x) \\ &\leq W_n\left(\frac{j+1}{k}2M\right) + \nu\left(\frac{j+1}{k}2M\right) - \nu\left(\frac{j}{k}2M\right) \end{aligned} \quad (2.150)$$

which implies for each $x \in [\frac{j}{k}2M, \frac{j+1}{k}2M]$,

$$|W_n(x)| \leq 2 \max_{1 \leq j \leq k} \left| W_n\left(\frac{j}{k}2M\right) \right| + \max_{1 \leq j \leq k} \left| \nu\left(\frac{j+1}{k}2M\right) - \nu\left(\frac{j}{k}2M\right) \right|. \quad (2.151)$$

Thus,

$$\begin{aligned} \sup_{-M \leq x \leq M} |W_n(x)| &\leq 2 \max_{1 \leq j \leq k} \left| W_n\left(\frac{j}{k}2M\right) \right| + \max_{1 \leq j \leq k} \left| \nu\left(\frac{j+1}{k}2M\right) - \nu\left(\frac{j}{k}2M\right) \right| \\ &= I_{21} + I_{22}. \end{aligned} \quad (2.152)$$

By S.L.L.N, for each $1 \leq j \leq k$ we have $I_{21} \rightarrow 0$ a.s. We get the same result for the maximum over j since M and k are fixed. For I_{22} , $\forall \epsilon > 0$ there exist $M = (M_\epsilon)$, $k = k_\epsilon(M)$ and $\delta > 0$ such that $\frac{2M}{k} < \delta$. Then,

$$\max_{1 \leq j \leq k} \left| \nu\left(\frac{j+1}{k}2M\right) - \nu\left(\frac{j}{k}2M\right) \right| < \epsilon. \quad (2.153)$$

Now, we consider I_1

$$\begin{aligned} I_1 &= \sup_{-\infty < x < -M} |W_n(x)| \\ &\leq \sup_{-\infty < x < -M} n^{-1} \sum_{i=1}^n g(\xi_i) I(\xi_i \leq x) + \sup_{-\infty < x < -M} \nu(x) \\ &= I_{11} + I_{12}. \end{aligned} \quad (2.154)$$

But, for each $\epsilon > 0 \exists M_\epsilon$ such that

$$I_{12} \leq \int_{-\infty}^{-M} g(\xi) dF(\xi) < \epsilon \quad \forall M > M_\epsilon. \quad (2.155)$$

For, I_{11} we have

$$\begin{aligned} I_{11} &= \sup_{-\infty < x < -M} n^{-1} \sum_{i=1}^n g(\xi_i) I(\xi_i \leq x) \\ &\leq n^{-1} \sum_{i=1}^n g(\xi_i) I(\xi_i \leq -M) \rightarrow \nu(-M) \quad \text{a.s.} \end{aligned} \quad (2.156)$$

From (2.155), we have $\nu(-M) < \epsilon \quad \forall M > M_\epsilon$. Therefore, we get the result for I_1 .

The last one which must be checked is I_3 where,

$$\begin{aligned} I_3 &= \sup_{M < x < \infty} |W_n(x)| \\ &= \sup_{M < x < \infty} \left| n^{-1} \sum_{i=1}^n g(\xi_i) I(\xi_i \leq x) - \nu(x) \right| \\ &= \sup_{M < x < \infty} \left| n^{-1} \sum_{i=1}^n \left\{ g(\xi_i) (1 - I(\xi_i > x)) - E[g(\xi_i) (1 - I(\xi_i > x))] \right\} \right| \\ &\leq \left| n^{-1} \sum_{i=1}^n \{g(\xi_i) - E[g(\xi_i)]\} \right| + \sup_{M < x < \infty} n^{-1} \sum_{i=1}^n g(\xi_i) I(\xi_i > x) \\ &\quad + \sup_{M < x < \infty} E[g(\xi_i) I(\xi_i > x)] \\ &= I_{31} + I_{32} + I_{33}. \end{aligned} \quad (2.157)$$

By S.L.L.N we obtain that $I_{31} \rightarrow 0$ a.s. For I_{32} we have $\forall \epsilon > 0 \exists M_\epsilon$ such that

$$\begin{aligned} I_{32} &= \sup_{M < x < \infty} n^{-1} \sum_{i=1}^n g(\xi_i) I(\xi_i > x) \\ &\leq n^{-1} \sum_{i=1}^n g(\xi_i) I(\xi_i > M) \rightarrow \int_M^\infty g(\xi) dF(\xi) \quad \text{a.s.} \\ &< \epsilon \quad \forall M > M_\epsilon. \end{aligned} \tag{2.158}$$

We also get the same result for I_{33} . i.e, $I_{33} < \epsilon \forall M > M_\epsilon$ which implies that $I_3 \rightarrow 0$ a.s. as $n \rightarrow \infty$ and the proof of the Lemma 2.4.9 is complete.

Chapter 3

Regression Model

3.1 Introduction

Linear combination of order statistics or L-estimators plays an extremely important role in the development of robust methods for location parameters. An important precursor to the trimmed L-estimator is the L-estimator consisting of a few fixed number of order statistics, for example, the median, trimmed, and Gastwirth's estimator. In the last chapter, we have studied L-estimators based on a few selected optimum sample quantiles for some parametric models to evaluate the asymptotic relative efficiencies of such procedures. Hogg (1974) preferred trimmed means and Stigler (1977) concluded that the 10% trimmed mean emerges as the victor in analysing data from 18th and 19th century experiments designed to measure some physical constants. As a first attempt, Bickel (1973) considered a class of one-step L-estimators of regression coefficients in a linear model depending on a "preliminary estimator". Although these estimators have good asymptotic properties, they are computationally complex and apparently not invariant to reparameterization. The object of this chapter is to consider the well accepted definition of *regression quantiles* of Koenker and Bassett (1978) in the linear model and pursue the question of L-estimation based on a few selected *regression quantiles* and trimmed least

squares estimation. The results about the trimmed least square estimator in this chapter are obtained by improving the conditions initially given by Ruppert and Carroll (1980) and Jurečková (1984) and are new.

In section 3.2 we present the definition of regression quantiles and its properties in detail as developed by Koenker and Bassett (1978) and Bassett and Koenker (1978). Other sections that follow are new results. Section 3.3 describes the L-estimation of regression parameters based on few selected regression quantiles for some known error distributions in order to assess the asymptotic relative efficiency (ARE) properties. Section 3.4 contains the trimmed L-estimators while section 3.5 concludes with some improved estimators of regression parameters on the basis of trimmed L-estimators combining the idea of robust estimation and Stein-type estimation using the preliminary test approach of Saleh and Sen (1978-86).

3.2 Regression Quantiles and Their Properties

Consider the linear model

$$Y_{n \times 1} = \beta_0 \mathbf{1}_n + X_{n \times p} \beta_{p \times 1} + Z_{n \times 1}, \quad (3.1)$$

where $Z_{n \times 1} = (Z_1, \dots, Z_n)'$ consists of *i.i.d.* *r.v.*'s with a continuous *cdf* $F_0(z)$ and *pdf* $f_0(z)$, $z \in \mathbb{R}$. It is well-known that the least squares estimator (LSE) of $\beta^* = (\beta_0, \beta')'$ is given by

$$\tilde{\beta}_n^{*L} = (D_n' D_n)^{-1} D_n' Y \quad (3.2)$$

where $D_n = (\mathbf{1}_n \mid X)$ is the design matrix and the quadratic estimator of variance is given by

$$S^2 = \frac{(Y - \tilde{\beta}_0^L \mathbf{1}_n - X \tilde{\beta}_n^L)' (Y - \tilde{\beta}_0^L \mathbf{1}_n - X \tilde{\beta}_n^L)}{n - p - 1}. \quad (3.3)$$

It is further well-known that the LSE of β^* is *not* robust. A natural curiosity is to develop analogues of linear combination of order statistics and simplified quantile estimators for the linear model. Bickel (1973) considered ordering of the residuals based on a *preliminary estimator*. But his method is computationally complicated and needs basically iteratively re-weighted least squares techniques.

However, Koenker and Bassett (1978) extended a well-accepted definition of sample quantiles in linear models which they call *regression quantiles*. We shall now devote our attention to this concept of quantiles, present their properties as in Koenker and Bassett (1978) and apply them to the estimation of regression parameters.

Definition 3.2.1 *regression quantiles of order λ is the solution of the minimization problem*

$$\min_{b \in \mathbb{R}^{p+1}} \left[\sum_{j \in J} \lambda |y_j - d_j' b| + \sum_{j \notin J} (1 - \lambda) |y_j - d_j' b| \right] \quad (3.4)$$

where $J = \{j : y_j - d_j' b \geq 0\}$ and d_j is the j^{th} row of D_n the design matrix. For $p = 0$ and $d_j = 1$ the minimization problem yields b_0 equal to the sample quantile of order λ , while $\lambda = 1/2$ yields b_0 to be the sample median. Thus, the median is the least absolute error estimator (LAE) of β_0 .

Let $\hat{B}(\lambda)$ be the solution sets consisting of $(\hat{\beta}_0(\lambda), \hat{\beta}'_n(\lambda))'$ of the regression quantiles by minimising (3.4). Also, let $\mathcal{H}$ denote the $(p+1)$ -element subsets of $\mathcal{I} = \{1, 2, \dots, n\}$. There are $\binom{n}{p+1}$ possible subsets in $\mathcal{H}$. Elements $h \in \mathcal{H}$ have complements $\bar{h} = \mathcal{I} - h$ and h and $\bar{h}$ partition the vector Y as well as the design matrix D_n . Thus, the notation $Y(h)$ stands for the $(p+1)$ -element vector of y 's from Y , i.e. $\{y_j, j \in h\}$ while $D_n(\bar{h})$ stands for a $(n-p-1) \times (p+1)$ matrix with rows $\{(1, x_j) | j \in \bar{h}\}$. Finally, let

$$\mathbf{H} = \{h \in \mathcal{H} \mid \text{rank } D_n(h) = p+1\}. \quad (3.5)$$

Then, the following theorem gives the form of the solution of (3.4).

Theorem 3.2.2 *If the rank of D_n is $p + 1$, then the set of regression quantiles $\hat{B}(\lambda)$ has at least one element of the form*

$$\hat{\beta}_n^*(\lambda) = D_n(h)^{-1}Y(h) \quad (3.6)$$

for some $h \in \mathcal{H}$. Moreover, $\hat{B}(\lambda)$ is the convex hull of all solutions of the form (3.6).

The proof of Theorem 3.2.2 follows from the linear programming formulation of the minimization problem as given below:

$$\begin{aligned} & \min[\lambda 1_n' r^+ + (1 - \lambda) 1_n' r^-], \quad 1_n = (1, \dots, 1)' \\ & \text{subject to} \quad Y = D_n b + r^+ - r^-, \quad (b, r^+, r^-) \in \mathbb{R}^{p+1} \times \mathbb{R}_+^n \times \mathbb{R}_+^n. \end{aligned} \quad (3.7)$$

For example see Abdelmalek (1974).

Sample quantiles in the location model are identified with a single order statistic in a sample. Theorem 3.2.2 generalized this feature to regression quantiles where *normals* to hyperplanes defined by subset of $(p + 1)$ observations play the role of order statistics.

The following Theorems describe various other properties of regression quantiles (RQs') given in Koenker and Bassett (1978).

Theorem 3.2.3 *If $\hat{\beta}_n^*(\lambda, Y, D_n) \in \hat{B}(\lambda, Y, D_n)$ then the following are elements of the specified transformed problems:*

- (i) $\hat{\beta}_n^*(\lambda, cY, D_n) = c\hat{\beta}_n^*(\lambda, Y, D_n), \quad c \in [0, \infty).$
- (ii) $\hat{\beta}_n^*(1 - \lambda, dY, D_n) = d\hat{\beta}_n^*(\lambda, Y, D_n), \quad d \in (-\infty, 0]. \quad (3.8)$
- (iii) $\hat{\beta}_n^*(\lambda, Y + D_n \gamma, D_n) = \hat{\beta}_n^*(\lambda, Y, D_n) + \gamma, \quad \gamma \in \mathbb{R}^{p+1}.$
- (iv) $\hat{\beta}_n^*(\lambda, Y, D_n C) = C^{-1} \hat{\beta}_n^*(\lambda, Y, D_n), \quad C_{p+1 \times p+1} \text{ non-singular matrix.}$

The following theorem gives the conditions when RQ's are unique.

Theorem 3.2.4 *If the error distribution $F_0(z)$ is continuous, then with probability one: $\hat{\beta}_n^*(\lambda) = D_n(h)^{-1}Y(h)$ is a unique solution if and only if*

$$(\lambda - 1)1'_{p+1} < \sum_{j \neq h} [1/2 - 1/2 \operatorname{sgn}(y_j - d'_j \hat{\beta}_n^*) - \lambda] d'_j D_n(h)^{-1} < \lambda 1'_{p+1} \quad (3.9)$$

where 1_{p+1} is a vector of ones.

The next Theorem states the relationship of the number residual errors $\hat{u}_j = y_j - d'_j \hat{\beta}_n^*(\lambda)$, $j = 1, \dots, n$ that are positive, negative and zero with that of $n\lambda$.

Theorem 3.2.5 *Let $\hat{\mathbf{u}} = (\hat{u}_1, \dots, \hat{u}_n)'$ and $P(\hat{\mathbf{u}})$, $N(\hat{\mathbf{u}})$ and $Z(\hat{\mathbf{u}})$ be the number of positive, negative and zero residuals. Then,*

$$N(\hat{\mathbf{u}}) \leq n\lambda \leq n - P(\hat{\mathbf{u}}) = N(\hat{\mathbf{u}}) + Z(\hat{\mathbf{u}}) \quad (3.10)$$

for every $\hat{\beta}_n^*(\lambda) \in \hat{\mathbf{B}}(\lambda)$. If $\hat{\beta}_n^*(\lambda)$ is unique, i.e., $\hat{\beta}_n^*(\lambda) = \hat{\mathbf{B}}(\lambda)$ then inequalities hold strictly.

The following theorem has the following geometric interpretation: Consider a scatter diagram of sample observations in $\mathbb{R}^2$ with λ^{th} -regression quantiles slicing through the scatter. Now, consider the effect (on the position of the λ -regression quantile) of moving observations up or down in the scatter. The result states that as long as these movements leave observations on the same side of the original line, its position is unaffected.

Theorem 3.2.6 *If $\hat{\beta}_n^*(\lambda) \in \hat{\mathbf{B}}(\lambda, Y, D_n)$, then $\hat{\beta}_n^*(\lambda) \in \hat{\mathbf{B}}(\lambda, D_n \hat{\beta}_n^* + A\hat{\mathbf{u}}, D_n)$ where $\hat{\mathbf{u}} = Y - D_n \hat{\beta}_n^*$ and A is any $n \times n$ diagonal matrix with nonnegative elements.*

Before stating the next Theorem it is necessary to consider the following conditions on the distribution function and the design matrix.

A0) F_0 is continuous and has continuous density f_0 in the neighborhood of $Q_0(\lambda_i)$ when $q_0(\lambda_i) > 0$ $i = 1, \dots, k$

$$A1) \quad n^{-1/2} \max_{i,j} |d_{ij}| = o(1).$$

$$A2) \quad \lim_{n \rightarrow \infty} \Sigma_n = \lim_{n \rightarrow \infty} n^{-1} D'_n D_n = \Sigma. \quad \text{p.d.}$$

Then,

Theorem 3.2.7 *Let $(\hat{\beta}_n^*(\lambda_1), \dots, \hat{\beta}_n^*(\lambda_k))$ with $0 < \lambda_1 < \dots < \lambda_k < 1$ for fixed k ($p+1 \leq k \leq n$), be the sequence of unique RQ's from the model (3.1). Define*

$$\beta^*(\lambda) = (\beta_0 + \sigma Q_0(\lambda), \beta_1, \dots, \beta_p). \quad (3.11)$$

Then

$$n^{1/2} [(\hat{\beta}_n^*(\lambda_1) - \beta(\lambda_1)), \dots, (\hat{\beta}_n^*(\lambda_k) - \beta(\lambda_k))] \xrightarrow{D} \mathcal{N}_{k \times p+1}(\mathbf{0}, \Omega \otimes \Sigma^{-1}) \quad (3.12)$$

where

$$\Omega = \left(\left(\frac{\lambda_i \wedge \lambda_j - \lambda_i \lambda_j}{q_0(\lambda_i) q_0(\lambda_j)} \right) \right). \quad (3.13)$$

We recognize that Ω is the covariance matrix of k regression quantile with the spacing $\lambda = (\lambda_1, \dots, \lambda_k)'$. We will also prove this theorem via weighted empirical processes in section 3.4.

Finally, we present the following theorem 3.2.8 which is the basis of studying the problem of estimation of the regression parameters based on a few selected "regression quantiles" in some known error distributions.

Theorem 3.2.8 *Let $a(\lambda) = (a(\lambda_1), \dots, a(\lambda_k))'$ be coefficients such that $\sum_{i=1}^k a(\lambda_i) = 1$ and let A0-A2 hold. Then*

$$\tilde{\beta}_n^*(\lambda) = \sum_{i=1}^k a(\lambda_i) \hat{\beta}_n^*(\lambda_i) \quad (3.14)$$

is invariant to location, scale and reparameterization of design and $n^{1/2}(\tilde{\beta}_n^(\lambda) - \beta^*(\lambda))$ converges in distribution to a $(p+1)$ -variate Gaussian distribution with mean $\mathbf{0}$ and covariance matrix $a(\lambda)' \Omega a(\lambda) \Sigma^{-1}$.*

3.3 L-estimation of the parameters of a linear model based on a few selected regression quantiles with known error distributions.

In this section we consider the simplified L-estimation of the regression and scale parameters of a linear model based on a few selected *regression quantiles*.

Consider the linear model

$$Y = D_n \beta^* + \sigma Z \quad (3.15)$$

where $Z = (Z_1, \dots, Z_n)'$ consists of $i \cdot i \cdot d$. error variables with a continuous *known* *cdf* $F_0(z)$ and *pdf* $f_0(z)$, $z \in \mathbb{R}$ and β^* and σ are the regression and scale parameters respectively. It is well-known that least squares estimation leads to the best linear unbiased estimator (BLUE) of β^* given by (3.2) and the optimum quadratic estimator of σ^2 is given by (3.3). These estimators are not robust. Also the asymptotic efficiency of this BLUE depends on the error distribution.

In this section, we introduce the estimation of $(\beta^*, \sigma)'$ based on a few selected *regression quantiles* (RQ) of Koenker and Bassett (1978) which will relate to simple robust estimators of β^* . This study will also extend the work of Ogawa (1951) on the estimation of location and scale model leading to simplified estimation of the regression and scale parameters of a linear model. Our program is to propose *asymptotically best linear (in regression quantiles) unbiased estimators* (ABBLUE) of β^* and σ based on a few fixed, say k , $(p+2 \leq k < n)$ selected regression quantiles (RQ) when the error distribution is known. We shall also assess their asymptotic relative efficiency properties when the error distribution is the class of *t*-distributions.

For a given integer k ($p+2 \leq k < n$), consider the spacing vector $\lambda = (\lambda_1, \dots, \lambda_k)'$ whose components satisfy the constraints $0 < \lambda_1 < \dots < \lambda_k < 1$. Now, by the

minimisation problem (3.4), we obtain the k RQ's given by the vector

$$\hat{\beta}_{jn}(\lambda) = (\hat{\beta}_{jn}(\lambda_1), \dots, \hat{\beta}_{jn}(\lambda_k))', \quad j = 0, 1, \dots, p \quad (3.16)$$

Then, using Theorem 3.2.7, one obtains that the $k(p+1)$ -dimensional random variable

$$n^{1/2} \left((\hat{\beta}_{0n}(\lambda) - \beta_0 \mathbf{1}_k - \sigma \mathbf{u})', (\hat{\beta}_{1n}(\lambda) - \beta_1 \mathbf{1}_k)', \dots, (\hat{\beta}_{pn}(\lambda) - \beta_p \mathbf{1}_k)' \right)' \quad (3.17)$$

converges in law to the $k(p+1)$ dimensional normal distribution with mean 0 and covariance matrix $\sigma^2(\Sigma^{-1} \otimes \Omega)$ where $\mathbf{u} = (u_1, \dots, u_k)'$ with $u_j = Q_0(\lambda_j)$, $j = 1, \dots, k$, $\mathbf{1} = (1, \dots, 1)'$, a k -tuple of ones, and Ω is given at (3.13).

3.3.1 Joint Estimation of Regression and Scale Parameters

In this subsection, we propose *asymptotically best linear (in regression quantiles) unbiased estimators* (ABLUE) of $(\beta^*, \sigma)'$. ABLUE of $(\beta_0, \beta_1, \dots, \beta_p)'$ and σ can be obtained by minimising the quadratic form using (3.17) as

$$\sum_{i=0}^p \sum_{j=0}^p \sigma_{i+1,j+1} (\hat{\theta}_{in} - \theta_i)' \Omega^{-1} (\hat{\theta}_{jn} - \theta_j), \quad (3.18)$$

where σ_{ij} is the ij^{th} element of covariance matrix Σ and by setting

$$\hat{\theta}_{0n} = \hat{\beta}_{0n}(\lambda), \quad \theta_0 = \beta_0 \mathbf{1}_k + \sigma \mathbf{u}, \quad \hat{\theta}_{jn} = \hat{\beta}_{jn}(\lambda), \quad \theta_j = \beta_j \mathbf{1}_k, \quad (3.19)$$

$j = 1, \dots, p$, to obtain the normal equations:

$$\begin{bmatrix} K_1 \Sigma & m K_3 \\ K_3 m' & K_2 \end{bmatrix} \begin{bmatrix} \tilde{\beta}_n^* \\ \tilde{\sigma}_n \end{bmatrix} = \begin{bmatrix} \Sigma V \\ m' V^* \end{bmatrix}. \quad (3.20)$$

The explicit expressions for $\tilde{\beta}_n^*$ and $\tilde{\sigma}_n$ are given respectively by

$$\tilde{\beta}_n^* = \frac{1}{\Delta} [K_2 V - K_3 V^*], \quad (3.21)$$

$$\text{and } \tilde{\sigma}_n = \frac{1}{\Delta} [K_1 m' V^* - K_3 m' V], \quad (3.22)$$

where

$$V = (V_0, V_1, \dots, V_p)', \quad V_j = \mathbf{1}_k' \Omega^{-1} \dot{\beta}_{jn}(\lambda); \quad j = 0, 1, \dots, p \quad (3.23)$$

$$V^* = (V_0^*, V_1^*, \dots, V_p^*)', \quad V_j^* = \mathbf{u}' \Omega^{-1} \dot{\beta}_{jn}(\lambda); \quad j = 0, 1, \dots, p \quad (3.24)$$

$$K_1 = \mathbf{1}_k' \Omega^{-1} \mathbf{1}_k, \quad K_2 = \mathbf{u}' \Omega^{-1} \mathbf{u}, \quad K_3 = \mathbf{1}_k' \Omega^{-1} \mathbf{u}, \quad (3.25)$$

$$\Delta = K_1 K_2 - K_3^2, \quad \mathbf{m} = (1, \bar{x}_1, \dots, \bar{x}_p)'. \quad (3.26)$$

The asymptotic covariance matrix of $(\hat{\beta}_n^*, \hat{\sigma}_n)'$ is given by

$$\frac{\sigma^2}{n\Delta} \begin{bmatrix} K_2 \Sigma^{-1} & -K_3 \Sigma^{-1} \mathbf{m} \\ -K_3 \mathbf{m}' \Sigma^{-1} & K_1 \end{bmatrix}. \quad (3.27)$$

Also, the explicit forms of K_1 , K_2 and K_3 are given by

$$K_1 = \sum_{i=1}^{k+1} \frac{[q_0(\lambda_i) - q_0(\lambda_{i-1})]^2}{\lambda_i - \lambda_{i-1}}, \quad (3.28)$$

$$K_2 = \sum_{i=1}^{k+1} \frac{[q_0(\lambda_i)Q_0(\lambda_i) - q_0(\lambda_{i-1})Q_0(\lambda_{i-1})]^2}{\lambda_i - \lambda_{i-1}}, \quad (3.29)$$

$$K_3 = \sum_{i=1}^{k+1} \frac{[q_0(\lambda_i) - q_0(\lambda_{i-1})][q_0(\lambda_i)Q_0(\lambda_i) - q_0(\lambda_{i-1})Q_0(\lambda_{i-1})]}{\lambda_i - \lambda_{i-1}} \quad (3.30)$$

with $\lambda_0 = 0$ and $\lambda_{k+1} = 1$. If F_0 is symmetric and $\lambda_i = \lambda_{k-i+1}$, $i = 1, \dots, k$, then, $K_3 = 0$ and the ABLUE of β^* and σ simplify to

$$\hat{\beta}_n^* = \frac{V}{K_1} \quad \text{and} \quad \hat{\sigma}_n = \frac{\mathbf{m}' V^*}{K_2}, \quad (3.31)$$

with variance and covariance given by

$$\text{Var}(\hat{\beta}_n^*) = \frac{\sigma^2}{n K_1} \Sigma^{-1}, \quad \text{Var}(\hat{\sigma}_n) = \frac{\sigma^2}{n K_2} \quad (3.32)$$

and

$$\text{Cov}(\hat{\beta}_n^*, \hat{\sigma}_n) = 0. \quad (3.33)$$

Now, the joint asymptotic relative efficiency (JARE) of the ABLUE $(\hat{\beta}_n^*, \hat{\sigma}_n)^t$ relative to the MLE $(\hat{\beta}_n^{**}, \hat{\sigma}_n)^t$ may be computed as

$$JARE[ABLUE : MLE] = \frac{K_1^p \Delta}{I_{11}^p (I_{11} I_{22} - I_{12}^2)} \quad (3.34)$$

and

$$\begin{aligned} ARE[\hat{\beta}_n^* : MLE] &= \frac{I_{22}^{p+1} \Delta^{p+1}}{K_2^{p+1} (I_{11} I_{22} - I_{12}^2)^{p+1}} \\ \text{and } ARE(\hat{\sigma}_n : MLE) &= \frac{I_{11} \Delta}{K_1 (I_{11} I_{22} - I_{12}^2)} \end{aligned} \quad (3.35)$$

where I_{11} , I_{22} and I_{12} are the elements of the information matrix for a location and scale distribution. Further, for $K_3 = 0$, and $I_{12} = 0$, we obtain (note that $I_{12} = 0$ for symmetric distributions)

$$JARE[ABLUE : MLE] = \frac{K_1^{p+1} K_2}{I_{11}^{p+1} I_{22}} \quad (3.36)$$

and

$$ARE[\hat{\beta}_n^* : MLE] = K_1^{p+1} I_{11}^{-(p+1)} \quad \text{and} \quad ARE[\hat{\sigma}_n : MLE] = K_2 I_{22}^{-1}. \quad (3.37)$$

Another method of estimating (β_0, σ) and β would be to consider two marginal distributions namely, the marginal of

$$\begin{aligned} (i) \quad & n^{1/2} (\hat{\beta}_{0n}(\lambda) - \beta_0 \mathbf{1}_k - \sigma \mathbf{u})', \\ \text{and } (ii) \quad & n^{1/2} \left((\hat{\beta}_{1n}(\lambda) - \beta_1 \mathbf{1}_k)', \dots, (\hat{\beta}_{pn}(\lambda) - \beta_p \mathbf{1}_k)' \right)'. \end{aligned} \quad (3.38)$$

The first one yields the estimators

$$\begin{aligned} \tilde{\beta}_{0n} &= \frac{1}{\Delta} \{K_2 V_0 - K_3 V_0^*\}, \\ \text{and } \tilde{\sigma}_n &= \frac{1}{\Delta} \{K_1 V_0^* - K_3 V_0\}, \end{aligned} \quad (3.39)$$

and the second one yields the estimators of β as

$$\tilde{\beta}_n = \frac{1}{K_1} \mathbf{V}, \quad \mathbf{V} = (V_1, \dots, V_p)'. \quad (3.40)$$

The estimate of σ is $\tilde{\sigma}_n$ which is non-negative but, (3.22) may not be non-negative. The ARE of these estimators relative to the corresponding marginal likelihood estimators may be determined easily.

Now, we consider the explicit expressions for the coefficients in (3.21) and (3.39). They are

$$a = \frac{1}{\Delta} \{K_2 \mathbf{1}'_k \Omega^{-1} - K_3 \mathbf{u}' \Omega^{-1}\} = (a_1, \dots, a_k)' \quad (3.41)$$

$$b = \frac{1}{\Delta} \{K_1 \mathbf{u}' \Omega^{-1} - K_3 \mathbf{1}'_k \Omega^{-1}\} = (b_1, \dots, b_k)' \quad (3.42)$$

where, for $i = 1, 2, \dots, k$,

$$a_i = \frac{q_0(\lambda_i) K_2}{\Delta} \left\{ \frac{q_0(\lambda_i) - q_0(\lambda_{i-1})}{\lambda_i - \lambda_{i-1}} - \frac{q_0(\lambda_{i+1}) - q_0(\lambda_i)}{\lambda_{i+1} - \lambda_i} \right\} - \quad (3.43)$$

$$\frac{q_0(\lambda_i) K_3}{\Delta} \left\{ \frac{q_0(\lambda_i) Q_0(\lambda_i) - q_0(\lambda_{i-1}) Q_0(\lambda_{i-1})}{\lambda_i - \lambda_{i-1}} - \frac{q_0(\lambda_{i+1}) Q_0(\lambda_{i+1}) - q_0(\lambda_i) Q_0(\lambda_i)}{\lambda_{i+1} - \lambda_i} \right\}$$

$$b_i = \frac{q_0(\lambda_i) K_1}{\Delta} \left\{ \frac{q_0(\lambda_i) Q_0(\lambda_i) - q_0(\lambda_{i-1}) Q_0(\lambda_{i-1})}{\lambda_i - \lambda_{i-1}} - \frac{q_0(\lambda_{i+1}) Q_0(\lambda_{i+1}) - q_0(\lambda_i) Q_0(\lambda_i)}{\lambda_{i+1} - \lambda_i} \right\} \\ - \frac{q_0(\lambda_i) K_3}{\Delta} \left\{ \frac{q_0(\lambda_i) - q_0(\lambda_{i-1})}{\lambda_i - \lambda_{i-1}} - \frac{q_0(\lambda_{i+1}) - q_0(\lambda_i)}{\lambda_{i+1} - \lambda_i} \right\}. \quad (3.44)$$

Using these coefficients, we may write the estimators (3.21) and (3.22) as follows:

$$\begin{bmatrix} \tilde{\beta}_{0n} \\ \tilde{\beta}_{1n} \\ \vdots \\ \tilde{\beta}_{pn} \\ \tilde{\sigma}_n \end{bmatrix} = \begin{bmatrix} a_1 \hat{\beta}_{0n}(\lambda_1) + \dots + a_k \hat{\beta}_{0n}(\lambda_k) \\ a_1 \hat{\beta}_{1n}(\lambda_1) + \dots + a_k \hat{\beta}_{1n}(\lambda_k) \\ \vdots \\ a_1 \hat{\beta}_{pn}(\lambda_1) + \dots + a_k \hat{\beta}_{pn}(\lambda_k) \\ \sum_{j=0}^p \bar{d}_j (b_1 \hat{\beta}_{jn}(\lambda_1) + \dots + b_k \hat{\beta}_{jn}(\lambda_k)) \end{bmatrix}, \quad \bar{d}_0 = 1. \quad (3.45)$$

Note that the coefficients for the estimators of the components of β^* are the same satisfying the condition $\sum_{i=1}^k a_i = 1$ as in Theorem 4.3 of Koenker and Bassett (1978) while the coefficients of the estimator of σ satisfying $\sum_{j=1}^k b_j = 0$. It also involves the vector $\mathbf{m}' = (1, \bar{x}_1, \dots, \bar{x}_p)$. Unless $\mathbf{m}' = (1, 0, 0, \dots, 0)$ one may not be

able to guarantee $\bar{\sigma}_n > 0$. This suggests the course of estimation using (3.39) and (3.40), where there is no such anomaly.

In order to obtain the ABLUE of $(\beta'', \sigma)'$ optimally, we maximise $K_1^p \Delta$ with respect to $\lambda_1, \lambda_2, \dots, \lambda_k$ satisfying $0 < \lambda_1 < \lambda_2 < \dots < \lambda_k < 1$. Once we have obtained the optimum spacing vector $\lambda^0 = (\lambda_1^0, \dots, \lambda_k^0)'$, we can determine the optimal coefficients $a^0 = (a_1^0, \dots, a_k^0)'$ and $b^0 = (b_1^0, \dots, b_k^0)'$. For the class of t-distribution with chosen degrees of freedom, we have tabled in Appendix A the optimum spacings, coefficients and the values of $K_1^p \Delta$ for $k = 3(1)10$ and $\nu = 1(1)3, 5, 10, 20, 500$, which covers the Cauchy distribution as well as the normal distribution. We have also considered the Logistic distribution. To illustrate the methodology, we consider the Cauchy distribution. Here, we note that

$$\Omega = \left(\left(\frac{\pi^2(\lambda_i \wedge \lambda_j - \lambda_i \lambda_j)}{\sin^2(\pi \lambda_i) \sin^2(\pi \lambda_j)} \right) \right), \quad (3.46)$$

and

$$Q_0(\lambda_j) = \tan[\pi(\lambda_j - 1/2)]; \quad j = 1, \dots, k. \quad (3.47)$$

The explicit forms of K_1 , K_2 and K_3 are

$$\begin{aligned} K_1 &= \sum_{i=1}^{k+1} \frac{[\sin^2(\pi \lambda_i) - \sin^2(\pi \lambda_{i-1})]^2}{\pi^2(\lambda_i - \lambda_{i-1})}, \\ K_2 &= \sum_{i=1}^{k+1} \frac{[\sin(2\pi \lambda_i) - \sin(2\pi \lambda_{i-1})]^2}{\pi^2(\lambda_i - \lambda_{i-1})}, \\ K_3 &= \sum_{i=1}^{k+1} \frac{[\sin^2(\pi \lambda_i) - \sin^2(\pi \lambda_{i-1})][\sin(2\pi \lambda_i) - \sin(2\pi \lambda_{i-1})]}{2\pi(\lambda_i - \lambda_{i-1})}. \end{aligned} \quad (3.48)$$

The optimum spacings for the estimation of $(\beta'', \sigma)'$ may be obtained by maximising $2^{p+2} K_1 \Delta$ in this case since $I_{11} = I_{22} = 1/2$ and $I_{12} = 0$. We know that optimum spacings for the location-scale parameters are given by

$$\lambda^0 = \left(\frac{1}{k+1}, \frac{2}{k+1}, \dots, \frac{k}{k+1} \right) \quad (3.49)$$

[see for example, Balmer et al (1974) and Cane (1974)]. Using these spacings we obtain the optimum coefficients as

$$\begin{aligned} a_i &= -\frac{4}{k+1} \sin^4 \frac{\pi i}{k+1} \cos \frac{2\pi i}{k+1}, \quad i = 1, \dots, k \\ \text{and } b_i &= -\frac{4}{k+1} \sin^2 \frac{\pi i}{k+1} \cos \frac{2\pi i}{k+1}, \quad i = 1, \dots, k. \end{aligned} \quad (3.50)$$

These coefficients may be used to obtain the ABLUE of β^* and σ respectively. The JARE of the estimator for this case will be $\left(\frac{k+1}{\pi} \sin \frac{\pi}{k+1}\right)^{2p+4}$.

Now, we present some numerical examples using the computational algorithms of Koenker and d'Orey (1987, 1994) for computing the RQs' for two sets of data.

Example 3.3.1 Consider the simple regression model $y = \beta_0 + \beta_1 x + z$, with the following data:

$X :$	78	78	78	78	78	77	76	76	76	76	76	75	77	76
	75	73	71	70	68	68	66	66	65	63	62	60	60	
$Y :$	17.3	17.6	15.0	18.1	18.7	17.9	18.4	18.1	16.3	19.4	17.6	19.5	12.7	17.0
	16.1	17.3	18.4	17.3	16.1	15.9	14.6	16.5	15.8	19.5	15.3	17.4	17.6	

Then, the different fitted lines are as follows,

$LSE :$	$14.72224 + 0.0329 x$
$Regression \text{ Median } (RQ(.5)) :$	$9.9999 + 0.1 x$
$Regression \text{ 1}^{st} \text{ Quantile } (RQ(.25)) :$	$9.1 + 0.099999 x$
$Regression \text{ 3}^{rd} \text{ Quantile } (RQ(.75)) :$	$15.72499 + 0.03125 x$

Now, we represent the solutions for $k = 7$ optimum regression quantiles for the

normal distribution. They are

	λ	$\hat{\beta}_0$	$\hat{\beta}_1$
1	.0171	26.0000019	-0.1727273
2	.0857	16.46249771	-0.01874998
3	.2411	12.499999	0.0499999
4	.50	9.9999294	0.1000009
5	.7589	14.599999	0.049999
6	.9143	19.984619141	-0.00769
7	.9829	19.49999	0.000000089

Based on these regression quantiles, we obtain the estimates of β_0 and β_1 as (14.0088, 0.04139). The following figure shows the scatter plot of data with different fitted lines.

Example 3.3.2 *In this example we consider the regression model*

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + z$$

with the following data,

$X_1 :$	4.3	4.5	4.3	6.1	5.6	5.6	6.1	5.5	5.0	5.6	5.2	4.8	3.8	3.4
	3.6	3.9	5.1	5.9	4.9	4.6	4.8	4.9	5.1	5.4	6.5	6.8	6.2	
$X_2 :$	62	68	74	71	78	85	69	76	83	70	77	84	63	70
	77	63	70	77	63	70	77	56	63	70	49	56	63	
$X_3 :$	78	78	78	78	78	77	76	76	76	76	76	75	77	76
	75	73	71	70	68	68	66	66	65	63	62	60	60	
$Y :$	17.3	17.6	15.0	18.1	18.7	17.9	18.4	18.1	16.3	19.4	17.6	19.5	12.7	17.0
	16.1	17.3	18.4	17.3	16.1	15.9	14.6	16.5	15.8	19.5	15.3	17.4	17.6	

Then, the LSE fit and Regression Median fit is given,

$$LSE : 5.344783 + 0.9234785 x_1 + 0.04712587 x_2 + 0.05217684 x_3$$

$$RM : 5.48703623 + 0.84914058 x_1 - 0.01677862 x_2 + 0.11797349 x_3$$

In this case we obtain the solutions for $k = 7$ optimum regression quantiles for the normal distribution. The results are:

	λ	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$
1	0.0211	-5.79232454	1.84005880	0.07403532	0.08877762
2	0.1009	-7.45990467	1.93643999	0.07792763	0.10249341
3	0.2642	3.67410445	0.86474842	0.02345317	0.09597122
4	0.5000	5.48703623	0.84914058	-0.01677862	0.11797349
5	0.7358	9.57583427	0.45074245	0.05087731	0.03373820
6	0.8991	12.98698711	0.79262042	0.07539532	-0.04833032
7	0.9789	11.04361057	1.01503742	0.06348020	-0.02330833

The estimates of β_0 , β_1 , β_2 , and β_3 based on these regression quantiles are given respectively by (5.065928, 0.941837, 0.0356317, 0.0665379).

We provide here a sample table of optimum spacings, coefficients and related quantiles for ARE computation where error distribution of the linear model ($p=1$) is Cauchy.

k	3	4	5	6	7	8	9	10
λ_1	0.2500	0.1837	0.1535	0.1321	0.1147	0.1013	0.0911	0.0827
a_1	0.0000	-.0769	-.0730	-.0593	-.0446	-.0333	-.0254	-.0197
λ_2	0.5000	0.4099	0.3465	0.2941	0.2500	0.2170	0.1926	0.1732
a_2	1.0000	0.5769	0.2664	0.0892	0.0000	-.0350	-.0453	-.0454
λ_3	0.7500	0.5901	0.5000	0.4364	0.3853	0.3431	0.3074	0.2765
a_3	0.0000	0.5769	0.6132	0.4701	0.3147	0.1860	0.0948	0.0354
λ_4		0.8163	0.6535	0.5636	0.5000	0.4501	0.4089	0.3736
a_4		-.0769	0.2664	0.4701	0.4599	0.3823	0.2934	0.2117
λ_5			0.8465	0.7059	0.6147	0.5499	0.5000	0.4591
a_5			-.0730	0.0892	0.3147	0.3823	0.3650	0.3180
λ_6				0.8679	0.7500	0.6569	0.5911	0.5409
a_6				-.0593	0.0000	0.1860	0.2934	0.3180
λ_7					0.8853	0.7830	0.6926	0.6264
a_7					-.0446	-.0350	0.0948	0.2117
λ_8						0.8987	0.8074	0.7235
a_8						-.0333	-.0453	0.0354
λ_9							0.9089	0.8268
a_9							-.0254	-.0454
λ_{10}								0.9173
a_{10}								-.0197
K_1	0.4053	0.4469	0.4631	0.4712	0.4774	0.4822	0.4855	0.4880
K_2	0.4053	0.4244	0.4457	0.4619	0.4713	0.4770	0.4813	0.4846
K_3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Δ	0.0666	0.0848	0.0956	0.1025	0.1074	0.1109	0.1135	0.1154

In this case, as an example, if we choose $k = 7$ the ARE is

$$ARE = K_1 I_{11}^{-1} = (0.4774) \times 2 = 0.9548.$$

3.3.2 Estimation of Conditional Quantile Function

In this subsection, we consider the estimation of a conditional quantile function defined by

$$Q(\lambda) = t'_0 \beta^* + \sigma Q_0(\lambda), \quad 0 < \lambda < 1 \quad (3.51)$$

based on a few, say k , ($p+2 \leq k < n$) selected RQ's where t_0 is a prefixed vector in advance. To achieve this, we simply substitute the ABLUE of $(\beta'', \sigma)'$ in (3.51) from (3.21) and (3.22) and obtain

$$\tilde{Q}_n(\lambda) = t_0' \tilde{\beta}_n'' + \tilde{\sigma}_n Q_0(\lambda), \quad \text{for each } \lambda \in (0, 1). \quad (3.52)$$

Then the asymptotic variance is given by

$$\text{Var}(\tilde{Q}_n(\lambda)) = \frac{\sigma^2}{n\Delta} \{K_2 t_0' \Sigma^{-1} t_0 - 2K_3 t_0' \Sigma^{-1} m Q_0(\lambda) + K_1 Q_0^2(\lambda)\}. \quad (3.53)$$

Similarly, the MLE of $Q(\lambda)$, say, $\tilde{Q}_n(\lambda)$ has the asymptotic variance given by

$$\frac{\sigma^2}{n|I|} \{I_{22} t_0' \Sigma^{-1} t_0 - 2I_{12} t_0' \Sigma^{-1} m Q_0(\lambda) + I_{11} Q_0^2(\lambda)\}. \quad (3.54)$$

The ARE of $\tilde{Q}_n(\lambda)$ relative to $\tilde{Q}_n(\lambda)$ is given by (3.54) divided by (3.53). The ARE expression lies between

$$Ch_{\min}(K^* I^{*-1}) \leq ARE[\tilde{Q}_n(\lambda) : \tilde{Q}_n(\lambda)] \leq Ch_{\max}(K^* I^{*-1}), \quad (3.55)$$

where

$$K^* = \begin{pmatrix} K_1 \Sigma & m K_3 \\ K_3 m' & K_2 \end{pmatrix} \quad \text{and} \quad I^* = \begin{pmatrix} I_{11} \Sigma & m I_{12} \\ I_{12} m' & I_{22} \end{pmatrix}, \quad (3.56)$$

and $Ch_{\min}(A)$ and $Ch_{\max}(A)$ are a minimum and maximum characteristic root of A . Now, consider two quantities namely (i) $\text{tr}[K^* I^{*-1}]$ and (ii) $|K^* I^{*-1}|^{1/p+2}$. Then, $\frac{1}{p+2} \text{tr}[K^* I^{*-1}]$ will provide us with the average ARE of $\tilde{Q}_n(\lambda)$ relative to $\tilde{Q}_n(\lambda)$ which lies between $Ch_{\min}(K^* I^{*-1})$ and $Ch_{\max}(K^* I^{*-1})$ while $|K^* I^{*-1}|^{1/p+2}$ gives the geometric mean of the characteristics roots of $K^* I^{*-1}$. Hence, we determine the average ARE as

$$\frac{1}{p+2} \text{tr}[K^* I^{*-1}] = [(p+1)K_1 I_{22} - 2K_3 I_{12} + K_2 I_{11}]/(p+2). \quad (3.57)$$

On the other hand,

$$|K^* I^{*-1}| = \frac{K_1^p \Delta}{I_{11}^p (I_{11} I_{22} - I_{12}^2)}. \quad (3.58)$$

The optimization of (3.57) or (3.58) will provide the required optimum spacings for the determination of the regression quantiles required to estimate $(\beta'', \sigma)'$ and thereby $Q(\lambda)$. It is clear that (3.57) and (3.58) do not depend on the design matrix D_n . Alternatively, if $f_0(z)$ is symmetric then, the $(p+2)$ characteristic roots of $(K^* I^{*-1})$ are

$$K_1 I_{11}^{-1}, \dots, K_1 I_{11}^{-1}, K_1 I_{11}^{-1}, \\ (K_1 I_{11}^{-1} + K_2 I_{22}^{-1})/2 \pm 1/2 \left((K_1 I_{11}^{-1} - K_2 I_{22}^{-1})^2 + 4K_3 I_{11}^{-1} I_{22}^{-1} \right)^{1/2}. \quad (3.59)$$

Thus, we maximise

$$(K_1 I_{11}^{-1} + K_2 I_{22}^{-1})/2 + 1/2 \left((K_1 I_{11}^{-1} - K_2 I_{22}^{-1})^2 + 4K_3 I_{11}^{-1} I_{22}^{-1} \right)^{1/2}.$$

w.r.t $Q_0(\lambda)$ to obtain the optimum spacing vector which is independent of D_n .

3.3.3 Estimating functions of parameters

Let $g(\beta^*, \sigma)$ be a function of the regression and scale parameters of the linear model. We consider the estimation of $g(\beta^*, \sigma)$ based on k , $(p+2 \leq k \leq n)$ regression quantiles $\hat{\beta}_{j,n}(\lambda) = (\hat{\beta}_{j,n}(\lambda_1), \dots, \hat{\beta}_{j,n}(\lambda_k))'$, $j = 0, 1, \dots, p$ as in (3.18) and then estimate $g(\beta^*, \sigma)$ by the substitution method as $g(\tilde{\beta}^*, \tilde{\sigma})$. To study the properties of this estimator, we use the local linearization theorem as in Rao (1973). This allows us to obtain the asymptotic variance of $g(\tilde{\beta}^*, \tilde{\sigma})$.

Now, assume that $g(\beta^*, \sigma)$ is defined on $\mathbb{R}^{p+1} \times \mathbb{R}^+$ possessing continuous partial derivatives of order at least two at each point of $\mathbb{R}^{p+1} \times \mathbb{R}^+$. Then, using Rao (1973), the asymptotic variance of $g(\tilde{\beta}^*, \tilde{\sigma})$ may be written as

$$Var(g(\tilde{\beta}^*, \tilde{\sigma})) = \frac{\sigma^2}{n} \omega' K^{*-1} \omega, \quad \omega = (\omega_0, \omega_1, \dots, \omega_{p+1})', \quad (3.60)$$

where

$$\omega_i = \frac{\partial g(\beta^*, \sigma)}{\partial \beta_i}, \quad i = 0, 1, \dots, p; \quad \omega_{p+1} = \frac{\partial g(\beta^*, \sigma)}{\partial \sigma} \quad (3.61)$$

and K^{*-1} is the covariance factor in section 3.3.1.

Similarly, let $g(\tilde{\beta}^*, \tilde{\sigma})$ be the MLE of $g(\beta^*, \sigma)$ with asymptotic variance

$$\text{Var}[g(\tilde{\beta}^*, \tilde{\sigma})] = \frac{\sigma^2}{n} \omega' I^{*-1} \omega. \quad (3.62)$$

The ARE of $g(\tilde{\beta}^*, \tilde{\sigma})$ relative to $g(\beta^*, \sigma)$ is given by

$$\text{ARE}[g(\tilde{\beta}^*, \tilde{\sigma}) : g(\beta^*, \sigma)] = \frac{\omega' I^{*-1} \omega}{\omega' K^{*-1} \omega}. \quad (3.63)$$

To obtain the optimum spacing vector for the estimation of $g(\beta^*, \sigma)$, one has to maximize (3.63) with respect to $Q_0(\lambda_1), \dots, Q_0(\lambda_k)$ for fixed vector ω . However, the function $g(\beta^*, \sigma)$ is nonlinear in general and the vector ω depends on the parameters $(\beta^{**}, \sigma)'$ which is unknown, therefore, we may apply the Courant-Fisher Theorem (Rao, 1973) and find that

$$Ch_{\min}(K^* I^{*-1}) \leq \text{ARE} \leq Ch_{\max}(K^* I^{*-1}). \quad (3.64)$$

Now, we follow section 3.3.2, for optimization problem to determine the optimum spacings. In this way one is able to estimate functions of $(\beta^{**}, \sigma)'$ such as, (i) conditional survival function and (ii) conditional hazard function .

3.4 Trimmed Least-Squared Estimation of Regression Parameters And Its Asymptotic Distribution

In the previous section we considered the L-estimation of regression and scale parameters based on a few selected regression quantiles. These estimators resulted in *quick and robust estimator* of β and σ against outliers for each distribution under consideration while sacrificing very little on the asymptotic relative efficiency. The basic idea of derivation was based on the generalized least squares principle. In

this section we are going to consider another L-estimator namely, the *trimmed least squares estimator* (TLSE) of β .

Consider the regression model

$$Y = D_n\beta + Z, \quad \text{where } \beta = (\beta_0, \beta_1, \dots, \beta_p)' \in \mathbb{R}^{p+1} \quad (3.65)$$

where $Z = (Z_1, \dots, Z_n)'$ has components which are i.i.d r.v's from a continuous cdf $F_0(z)$ (with pdf $f_0(z)$). The least squares estimator (LSE) of β is given by

$$\tilde{\beta}_n^L = (D_n' D_n)^{-1} D_n' Y \quad (3.66)$$

which is obtained by minimizing

$$(Y - D_n\beta)'(Y - D_n\beta) \quad (3.67)$$

with respect to β . For the trimmed least squares estimation (TLSE) of β , we first define an $n \times n$ diagonal matrix A whose elements are

$$a_{ii} = \begin{cases} 0, & \text{if } Y_i \leq d_i \hat{\beta}_n(\lambda_1) \text{ or } Y_i > d_i \hat{\beta}_n(\lambda_2) \\ 1, & \text{otherwise} \end{cases} \quad (3.68)$$

where $\hat{\beta}_n(\lambda_1)$ and $\hat{\beta}_n(\lambda_2)$ are the regression quantiles of orders λ_1 and λ_2 respectively for $0 < \lambda_1 < 1/2 < \lambda_2 < 1$. We obtain the TLSE of β by minimising

$$(Y - D_n\beta)' A (Y - D_n\beta) \quad (3.69)$$

with respect to β obtaining

$$\tilde{\beta}_n(\lambda_1, \lambda_2) = (D_n' A D_n)^{-1} D_n' A Y. \quad (3.70)$$

Ruppert and Carroll (1980) studied the TLSE under the conditions

1. F_0 has continuous density f_0 that is positive on the support of F_0 .
2. The first column of design matrix is $\mathbf{1}$, i.e $d_{i1} = 1$ $i = 1, \dots, n$ and

$$\sum_{i=1}^n d_{ij} = 0, \text{ for } j = 2, \dots, p+1.$$

3. $\lim_{n \rightarrow \infty} (\max_{i,j} (n^{-1/2} |d_{ij}|)) = 0.$
 4. There exists positive definite Σ such that

$$\lim_{n \rightarrow \infty} n^{-1} (D_n' D_n) = \Sigma.$$

Jurečková (1984) stated the representation and asymptotic properties of $\hat{\beta}_n(\lambda_1, \lambda_2)$ under the following conditions:

- i) F_0 is absolutely continuous with density function f_0 such that $f_0(\cdot) > 0$ and $f_0'(\cdot)$ is bounded in a neighborhood of $F_0^{-1}(\lambda)$.
 ii) $\max_{i,j} n^{-1/4} |d_{ij}| = O(1)$, as $n \rightarrow \infty$.
 iii) $\lim_{n \rightarrow \infty} n^{-1} (D_n' D_n) = \Sigma$, where Σ is a positive definite $(p+1) \times (p+1)$ matrix.
 iv) $\frac{1}{n} \sum_{i=1}^n d_{ij}^4 = O(1)$, as $n \rightarrow \infty$, $j = 1, \dots, p+1$.
 v) $\lim_{n \rightarrow \infty} \bar{d}_{nj} = \bar{d}_j$, $j = 1, \dots, p+1$.

We are going to obtain the same results as in Ruppert and Carroll (1980) and Jurečková (1984) under weaker conditions on the design matrix D_n and distribution function F_0 . For this reason we use the weighted empirical approach of Koul (1992) providing new proofs for the theorems and lemmas that follows later on. Before we state the main theorem for TSLE we need an asymptotic linearity result for $\hat{\beta}_n(\lambda)$. First, we consider the conditions A0, A1 and A2 from section 3.2 and define

$$\beta(\lambda) = \beta + Q_0(\lambda) e_1, \quad e_1 = (1, 0, \dots, 0). \quad (3.71)$$

Now, consider the weighted empirical process

$$T_n(t; \lambda) = n^{-1/2} \sum_{i=1}^n d_i \{I(y_i - d_i' t \leq 0) - \lambda\} \quad (3.72)$$

which is basic for Theorem 3.4.1 given below.

Theorem 3.4.1 *Under the conditions A0, A1 and A2 for each $\lambda \in (0, 1)$ we have*

$$n^{1/2} (\hat{\beta}_n(\lambda) - \beta(\lambda)) = -\{q_0(\lambda) \Sigma_n\}^{-1} T_n(\beta(\lambda); \lambda) + o_P(1) \quad (3.73)$$

where

$$T_n(\beta(\lambda); \lambda) = n^{-1/2} \sum_{i=1}^n d_i \{I(Z_i \leq Q_0(\lambda)) - \lambda\}. \quad (3.74)$$

We prove this Theorem with the aid of two Lemmas.

Lemma 3.4.2 *Let the conditions of Theorem 3.4.1 hold. Then for each $0 < \lambda < 1$ and $b > 0$*

$$\sup_{\|t\| \leq b} \|T_n(\beta(\lambda) + n^{-1/2}t; \lambda) - T_n(\beta(\lambda); \lambda) - q_0(\lambda)\Sigma_n t\| = o_P(1). \quad (3.75)$$

PROOF:

This lemma is a weaker version of Theorem 2.3.1 of Koul (1992) where he proves the above result uniformly in λ when A_0 is replaced by assuming f_0 to be uniformly continuous. Denote by $T_n(t; \lambda) = (T_{n0}(t; \lambda), \dots, T_{np}(t; \lambda))$. So, we prove that

$$\sup_{\|t\| \leq b} \left| T_{nj}(\beta(\lambda) + n^{-1/2}t; \lambda) - T_{nj}(\beta(\lambda); \lambda) - q_0(\lambda) \frac{1}{n} \sum_{i=1}^n d_{ij} d'_i t \right| = o_P(1). \quad (3.76)$$

by considering the centered $T_n(\beta(\lambda) + n^{-1/2}t; \lambda)$

$$T_{nj}^0(t; \lambda) = T_{nj}(\beta(\lambda) + n^{-1/2}t; \lambda) - E[T_{nj}(\beta(\lambda) + n^{-1/2}t; \lambda)] \quad (3.77)$$

and the compact set

$$\mathcal{N}(b) = \{t \in \mathbb{R}^{p+1} \mid \|t\| \leq b\}, \quad 0 < b < \infty. \quad (3.78)$$

Then, the expression (3.76) is bounded above by

$$\begin{aligned} & \sup_{\|t\| \leq b} |T_{nj}^0(t; \lambda) - T_{nj}^0(0; \lambda)| \\ & + \sup_{\|t\| \leq b} |E[T_{nj}(\beta(\lambda) + n^{-1/2}t; \lambda)] - E[T_{nj}(\beta(\lambda); \lambda)] - q_0(\lambda) \frac{1}{n} \sum_{i=1}^n d_{ij} d'_i t| \\ & = I_1 + I_2. \end{aligned} \quad (3.79)$$

First, we consider I_2 where

$$I_2 = \sup_{\|t\| \leq b} \left| n^{-1/2} \sum_{i=1}^n d_{ij} \{F_0(\zeta_{ij}(\lambda) + n^{-1/2}d'_i t) - F_0(Q_0(\lambda)) - q_0(\lambda)n^{-1/2}d'_i t\} \right|$$

$$\leq n^{-1/2} \sum_{i=1}^n |d_{ij}| \sup_{\|t\| \leq b} \left| F_0(Q_0(\lambda) + n^{-1/2} \mathbf{d}_i' t) - F_0(Q_0(\lambda)) - q_0(\lambda) n^{-1/2} \mathbf{d}_i' t \right|. \quad (3.80)$$

By the mean value theorem $\exists \Delta_n(t) \in (0, n^{-1/2} \mathbf{d}_i' t)$ such that

$$I_2 \leq n^{-1/2} \sum_{i=1}^n |d_{ij}| \sup_{\|t\| \leq b} |n^{-1/2} \mathbf{d}_i' t| \left| f_0(Q_0(\lambda) + \Delta_n(t)) - f_0(Q_0(\lambda)) \right| \quad (3.81)$$

and by taking $\delta_n = n^{-1/2} \|\mathbf{d}_i\| b$ we have

$$\begin{aligned} I_2 &\leq n^{-1} \sum_{i=1}^n |d_{ij}| \|\mathbf{d}_i\| b \sup_{\Delta_n(t) \leq \delta_n} \left| f_0(Q_0(\lambda) + \Delta_n(t)) - f_0(Q_0(\lambda)) \right| \\ &= o(1) \text{ as } n \rightarrow \infty \text{ and conditions A0 and A2} \end{aligned} \quad (3.82)$$

Now, we prove that $I_1 = o_P(1)$ where

$$I_1 = \sup_{\|t\| \leq b} \left| T_{nj}^0(t; \lambda) - T_{nj}^0(0; \lambda) \right|. \quad (3.83)$$

First, we prove it for each $t \in \mathcal{N}(b)$

$$\begin{aligned} &\left| T_{nj}^0(t; \lambda) - T_{nj}^0(0; \lambda) \right| \\ &= \left| n^{-1/2} \sum_{i=1}^n d_{ij} \left\{ I(Z_i \leq Q_0(\lambda) + n^{-1/2} \mathbf{d}_i' t) - I(Z_i \leq Q_0(\lambda)) \right. \right. \\ &\quad \left. \left. - F_0(Q_0(\lambda) + n^{-1/2} \mathbf{d}_i' t) + F_0(Q_0(\lambda)) \right\} \right| \\ &= \left| n^{-1/2} \sum_{i=1}^n d_{ij} \{ V_i(t) - \nu_{ni}(t) \} \right| \end{aligned} \quad (3.84)$$

where

$$\begin{aligned} V_i(t) &= I(Q_0(\lambda) < Z_i \leq (Q_0(\lambda) + n^{-1/2} \mathbf{d}_i' t)), \\ \nu_{ni}(t) &= F_0(Q_0(\lambda) + n^{-1/2} \mathbf{d}_i' t) - F_0(Q_0(\lambda)) \end{aligned} \quad (3.85)$$

Now, we apply the Chebyshev inequality to get the result.

$$P\left(\left| n^{-1/2} \sum_{i=1}^n d_{ij} \{ V_i(t) - \nu_{ni}(t) \} \right|^2 > \epsilon^2 \right)$$

$$\begin{aligned}
&\leq \frac{1}{\epsilon^2} \frac{1}{n} E \left(\sum_{i=1}^n d_{ij} \{V_i(\mathbf{t}) - \nu_{ni}(\mathbf{t})\} \right)^2 \\
&= \frac{1}{\epsilon^2} \frac{1}{n} \sum_i \sum_k d_{ij} d_{kj} E(V_i(\mathbf{t}) - \nu_{ni}(\mathbf{t})) (V_k(\mathbf{t}) - \nu_{nk}(\mathbf{t})) \\
&= \frac{1}{\epsilon^2} \frac{1}{n} \sum_{i=1}^n d_{ij}^2 E(V_i(\mathbf{t}) - \nu_{ni}(\mathbf{t}))^2 \\
&\leq \epsilon \quad \text{as } n \rightarrow \infty \text{ by A0 and A2,}
\end{aligned} \tag{3.86}$$

which implies $I_1 = o_P(1)$ for fixed $\mathbf{t}$. For tightness, we use the compactness property of set $\mathcal{N}(b)$. Since, $\mathcal{N}(b)$ is a compact set it suffices to show that for each $\mathbf{s} \in \mathcal{N}(b)$ and for every $\epsilon > 0$, there is a $\delta > 0$ such that

$$\limsup_n P \left(\sup_{\|\mathbf{t}-\mathbf{s}\| \leq \delta} |T_{nj}^0(\mathbf{t}; \lambda) - T_{nj}^0(\mathbf{s}; \lambda)| > 2\epsilon \right) < \epsilon. \tag{3.87}$$

We need the following relations to prove (3.87)

$$\mathbf{d}'_i \mathbf{t} = \mathbf{d}'_i \mathbf{s} + (\mathbf{d}'_i \mathbf{t} - \mathbf{d}'_i \mathbf{s}). \tag{3.88}$$

So that

$$\mathbf{d}'_i \mathbf{t} - \mathbf{d}'_i \mathbf{s} = \mathbf{d}'_i (\mathbf{t} - \mathbf{s})$$

and

$$\begin{aligned}
|\mathbf{d}'_i \mathbf{t} - \mathbf{d}'_i \mathbf{s}| &= |\mathbf{d}'_i (\mathbf{t} - \mathbf{s})| \\
&\leq \|\mathbf{d}_i\| \|\mathbf{t} - \mathbf{s}\| \\
&\leq \|\mathbf{d}_i\| \delta.
\end{aligned} \tag{3.89}$$

Thus,

$$\mathbf{d}'_i \mathbf{s} - \|\mathbf{d}_i\| \delta \leq \mathbf{d}'_i \mathbf{t} \leq \mathbf{d}'_i \mathbf{s} + \|\mathbf{d}_i\| \delta. \tag{3.90}$$

Therefore, we have

$$|T_{nj}^0(\mathbf{t}; \lambda) - T_{nj}^0(\mathbf{s}; \lambda)| =$$

$$\begin{aligned}
& \left| n^{-1/2} \sum_{i=1}^n d_{ij} \left\{ \left[I(Z_i \leq Q_0(\lambda) + n^{-1/2} \mathbf{d}'_i \mathbf{t}) - I(Z_i \leq Q_0(\lambda) + n^{-1/2} \mathbf{d}'_i \mathbf{s}) \right] \right. \right. \\
& \quad \left. \left. - \left[F_0(Q_0(\lambda) + n^{-1/2} \mathbf{d}'_i \mathbf{t}) - F_0(Q_0(\lambda) + n^{-1/2} \mathbf{d}'_i \mathbf{s}) \right] \right\} \right| \\
& \leq n^{-1/2} \sum_{i=1}^n |d_{ij}| \left| I(Z_i \leq Q_0(\lambda) + n^{-1/2} \mathbf{d}'_i \mathbf{t}) \right. \\
& \quad \left. - I(Z_i \leq Q_0(\lambda) + n^{-1/2} \mathbf{d}'_i \mathbf{s}) \right| \\
& \quad + n^{-1/2} \sum_{i=1}^n |d_{ij}| \left| F_0(Q_0(\lambda) + n^{-1/2} \mathbf{d}'_i \mathbf{t}) - F_0(Q_0(\lambda) + n^{-1/2} \mathbf{d}'_i \mathbf{s}) \right|. \quad (3.91)
\end{aligned}$$

Now, using the monotone property of the indicator function with the relation (3.90), we can write the first term of (3.91) under the absolute sign as

$$\begin{aligned}
& \left[I(Z_i \leq Q_0(\lambda) + n^{-1/2} \mathbf{d}'_i \mathbf{s} - n^{-1/2} \|\mathbf{d}_i\| \delta) - I(Z_i \leq Q_0(\lambda) + n^{-1/2} \mathbf{d}'_i \mathbf{s}) \right] \\
& \leq \left[I(Z_i \leq Q_0(\lambda) + n^{-1/2} \mathbf{d}'_i \mathbf{t}) - I(Z_i \leq Q_0(\lambda) + n^{-1/2} \mathbf{d}'_i \mathbf{s}) \right] \quad (3.92) \\
& \leq \left[I(Z_i \leq Q_0(\lambda) + n^{-1/2} \mathbf{d}'_i \mathbf{s} + n^{-1/2} \|\mathbf{d}_i\| \delta) - I(Z_i \leq Q_0(\lambda) + n^{-1/2} \mathbf{d}'_i \mathbf{s}) \right]
\end{aligned}$$

so that

$$\begin{aligned}
& \left| I(Z_i \leq Q_0(\lambda) + n^{-1/2} \mathbf{d}'_i \mathbf{t}) - I(Z_i \leq Q_0(\lambda) + n^{-1/2} \mathbf{d}'_i \mathbf{s}) \right| \leq \\
& \leq \left| I(Z_i \leq Q_0(\lambda) + n^{-1/2} \mathbf{d}'_i \mathbf{s} + n^{-1/2} \|\mathbf{d}_i\| \delta) - \right. \\
& \quad \left. I(Z_i \leq Q_0(\lambda) + n^{-1/2} \mathbf{d}'_i \mathbf{s} - n^{-1/2} \|\mathbf{d}_i\| \delta) \right|, \quad (3.93)
\end{aligned}$$

and for the second term, we have

$$\begin{aligned}
& \left| F_0(Q_0(\lambda) + n^{-1/2} \mathbf{d}'_i \mathbf{t}) - F_0(Q_0(\lambda) + n^{-1/2} \mathbf{d}'_i \mathbf{s}) \right| \quad (3.94) \\
& \leq F_0(Q_0(\lambda) + n^{-1/2} \mathbf{d}'_i \mathbf{s} + n^{-1/2} \|\mathbf{d}_i\| \delta) - F_0(Q_0(\lambda) + n^{-1/2} \mathbf{d}'_i \mathbf{s} - n^{-1/2} \|\mathbf{d}_i\| \delta).
\end{aligned}$$

If we substitute (3.93) and (3.94) in (3.91) then add and subtract the corresponding mean, we have

$$\left| T_{ij}^0(\mathbf{t}; \lambda) - T_{ij}^0(\mathbf{s}; \lambda) \right| \leq$$

$$\begin{aligned}
& n^{-1/2} \sum_{i=1}^n |d_{ij}| \left\{ \left[I(Z_i \leq Q_0(\lambda) + n^{-1/2} \mathbf{d}_i' \mathbf{s} + n^{-1/2} \|\mathbf{d}_i\| \delta) \right. \right. \\
& \quad \left. \left. - F_0(Q_0(\lambda) + n^{-1/2} \mathbf{d}_i' \mathbf{s} + n^{-1/2} \|\mathbf{d}_i\| \delta) \right] \right. \\
& \quad \left. - \left[I(Z_i \leq Q_0(\lambda) + n^{-1/2} \mathbf{d}_i' \mathbf{s} - n^{-1/2} \|\mathbf{d}_i\| \delta) \right] \right. \\
& \quad \left. \left. - F_0(Q_0(\lambda) + n^{-1/2} \mathbf{d}_i' \mathbf{s} - n^{-1/2} \|\mathbf{d}_i\| \delta) \right] \right\} \\
& + 2n^{-1/2} \sum_{i=1}^n |d_{ij}| \left\{ F_0(Q_0(\lambda) + n^{-1/2} \mathbf{d}_i' \mathbf{s} + n^{-1/2} \|\mathbf{d}_i\| \delta) \right. \\
& \quad \left. - F_0(Q_0(\lambda) + n^{-1/2} \mathbf{d}_i' \mathbf{s} - n^{-1/2} \|\mathbf{d}_i\| \delta) \right\}. \quad (3.95)
\end{aligned}$$

As we see, the R.H.S of (3.95) (designating it as I_{11} and I_{12} respectively), does not depend on t anymore and $\mathbf{s}$ is fixed. Now we apply the same argument as we mentioned in the proof of I_1 for fixed t to obtain $I_{11} = o_P(1)$. By condition A0 and A2, $I_{12} < \epsilon$ for sufficiently small δ which completes the proof of Lemma 3.4.2.

The next Lemma will show that the regression quantiles are bounded in probability, i.e

Lemma 3.4.3 *Under the condition of Theorem 3.4.1*

$$n^{1/2}(\hat{\beta}_n(\lambda) - \beta(\lambda)) = O_P(1). \quad (3.96)$$

We need the following steps to prove this Lemma

i)

$$\begin{aligned}
T_{nj}(\hat{\beta}_n(\lambda); \lambda) &= n^{-1/2} \sum_{i=1}^n d_{ij} \{ I(y_i - \mathbf{d}_i' \hat{\beta}_n(\lambda) \leq 0) - \lambda \} \\
&= o_P(1)
\end{aligned} \quad (3.97)$$

ii)

$$T_{nj}(\beta(\lambda); \lambda) = O_P(1) \quad (3.98)$$

iii) $\forall \epsilon > 0, 0 < u < \infty \exists b(=b'_u)$ and an N^ϵ such that for each $\lambda \in (0, 1)$

$$P\left(\inf_{\|\mathbf{t}\| > b} \|T_n(\beta(\lambda) + n^{-1/2} \mathbf{t}; \lambda)\| > u\right) > 1 - \epsilon \quad (3.99)$$

PROOF of (i):

We prove this step via Koul and Saleh (1995). By assumption A0, F_0 is continuous hence we obtain $\hat{\beta}_n(\lambda)$ as a unique solution of the minimization problem. Now use of Theorem 3.3 of Koenker and Bassett (1978) yields after writing $sgn(x) \equiv 1 - 2I(x \leq 0) + I(x = 0)$, for each $\lambda \in (0, 1)$ with probability one (w.p.1)

$$(\lambda - 1)\mathbf{1}'_{p+1} < \sum_{i \in h^c} \mathbf{d}'_i \{I(y_i - \mathbf{d}'_i \hat{\beta}_n(\lambda) \leq 0) - \lambda\} \mathbf{D}_n^{-1}(h) \\ + 1/2 \sum_{i \in h^c} \mathbf{d}'_i I(y_i - \mathbf{d}'_i \hat{\beta}_n(\lambda) = 0) \mathbf{D}_n^{-1}(h) < \lambda \mathbf{1}'_{p+1}, \quad (3.100)$$

where $\mathbf{D}_n(h)$ is defined in section 3.2. Let $\mathbf{w}'_n(\lambda)$ denote the vector inside the inequalities, i.e

$$(\lambda - 1)\mathbf{1}'_{p+1} < \mathbf{w}'_n(\lambda) < \lambda \mathbf{1}'_{p+1}. \quad (3.101)$$

Also, for each $i \in h$ by Theorem 3.2.1 we have

$$\hat{\beta}_n(\lambda) = \mathbf{d}'_i \mathbf{D}_n^{-1}(h) \mathbf{Y}(h) = (0, \dots, \overset{i^{th}}{1}, \dots, 0) \mathbf{Y}(h) = y_i \quad (3.102)$$

which implies that w.p.1

$$I(y_i - \mathbf{d}'_i \hat{\beta}_n(\lambda) = 0) = 0 \quad \forall i \in h^c. \quad (3.103)$$

Then, w.p.1 we have

$$\left[\sum_{i=1}^n \mathbf{d}'_i \{I(y_i - \mathbf{d}'_i \hat{\beta}_n(\lambda) \leq 0) - \lambda\} \right. \\ \left. - \sum_{i \in h} \mathbf{d}'_i \{I(y_i - \mathbf{d}'_i \hat{\beta}_n(\lambda) \leq 0) - \lambda\} \right] \mathbf{D}_n^{-1}(h) = \mathbf{w}'_n(\lambda). \quad (3.104)$$

Note that (3.103) implies w.p.1

$$I(y_i - \mathbf{d}'_i \hat{\beta}_n(\lambda) \leq 0) = 1 \quad \forall i \in h. \quad (3.105)$$

Therefore by the definition of $\mathbf{T}_n(\hat{\beta}_n(\lambda); \lambda)$, we have

$$\mathbf{T}_n(\hat{\beta}_n(\lambda); \lambda) - n^{-1/2} \sum_{i \in h} \mathbf{d}'_i (1 - \lambda) = n^{-1/2} \mathbf{D}'_n(h) \mathbf{w}_n(\lambda) \quad (3.106)$$

which implies that

$$\|T_n(\hat{\beta}_n(\lambda); \lambda)\| \leq n^{-1/2} \sum_{i \in h} \|d_i\| (1 - \lambda) + n^{-1/2} \|D'_n(h)w_n(\lambda)\|. \quad (3.107)$$

From (3.101), we further obtain

$$\|D'_n(h)w_n(\lambda)\| \leq \|D'_n(h)1_{p+1}\| \quad (3.108)$$

where

$$D'_n(h)1_{p+1} = \left(\sum_{j=0}^p d_{0j}, \dots, \sum_{j=0}^p d_{pj} \right). \quad (3.109)$$

But, each component is bounded above by $\max_i \|d_i\| \sqrt{p+1}$ which implies

$$n^{-1/2} \|D'_n(h)1_{p+1}\| \leq n^{-1/2} \max_i \|d_i\| (p+1). \quad (3.110)$$

Also we have

$$n^{-1/2} \sum_{i \in h} \|d_i\| (1 - \lambda) \leq n^{-1/2} \max_i \|d_i\| (p+1). \quad (3.111)$$

Thus, we get w.p.1

$$\|T_n(\hat{\beta}_n(\lambda); \lambda)\| \leq 2(p+1)n^{-1/2} \max_i \|d_i\| \quad (3.112)$$

and the R.H.S converges to zero in probability by A1.

Now, we prove step(ii),

PROOF of (ii):

To show (3.98), we look at the asymptotic distribution of $T_{nj}(\beta(\lambda); \lambda)$. Since, Z'_i s are *i.i.d* we use the Hájek-Šidak C.L.T, Theorem 3.3.6 of Sen and Singer (1993). For this reason we check the condition which is satisfied by conditions A1 and A2, i.e

$$\frac{n^{-1} \max_i d_{ij}^2}{n^{-1} \sum_{i=1}^n d_{ij}^2} \rightarrow 0 \text{ as } n \rightarrow \infty \quad (3.113)$$

which implies that for each $\lambda \in (0, 1)$

$$\frac{n^{-1/2} \sum_{i=1}^n d_{ij} \{I(Z_i \leq Q_0(\lambda)) - \lambda\}}{[\lambda(1-\lambda)n^{-1} \sum_{i=1}^n d_{ij}^2]^{1/2}} \xrightarrow{p} \mathcal{N}(0, 1). \quad (3.114)$$

Therefore, this property implies that for every $\epsilon > 0$ there exists an M_ϵ such that

$$P(T_{nj}(\beta(\lambda); \lambda) \leq M_\epsilon) > 1 - 2\epsilon/3. \quad (3.115)$$

We apply this result to prove step (iii).

PROOF of (iii):

We prove this part via Koul and Saleh (1995). First, we define the following process for simplicity

$$T_n^*(t; \lambda) = T_n(\beta(\lambda) + n^{-1/2}t; \lambda). \quad (3.116)$$

Now, every $t \in \mathbb{R}^{p+1}$ with $\|t\| > b$ could be written as $t = r\theta$, $|r| > b$ and $\|\theta\| = 1$.

Also note that $T_n^{**}(r\theta; \lambda) \cdot \theta$ is nondecreasing function of r , $\forall \theta \in \mathbb{R}^{p+1}$. Therefore, Cauchy-Schwarz inequality implies for each $\lambda \in (0, 1)$

$$\inf_{\|t\| > b} \|T_n^*(t; \lambda)\|^2 \geq \inf_{|r| > b, \|\theta\|=1} \{T_n^{**}(r\theta; \lambda) \cdot \theta\}^2 \geq \inf_{|r|=b, \|\theta\|=1} \{T_n^{**}(r\theta; \lambda) \cdot \theta\}^2. \quad (3.117)$$

Now, let

$$\hat{T}_n^*(r, \theta; \lambda) = T_n^{**}(0; \lambda) \cdot \theta + q_0(\lambda) r \theta' \Sigma_n \theta \quad (3.118)$$

and

$$\begin{aligned} k_n &= \inf\{\theta' \Sigma_n \theta, \|\theta\| = 1\} \\ k &= \lim_n k_n = \inf\{\theta' \Sigma \theta, \|\theta\| = 1\} \end{aligned} \quad (3.119)$$

$$A_n^\epsilon = [k(1 - \epsilon) \leq k_n \leq (1 + \epsilon)k].$$

Since Σ is positive definite we have $k > 0$. This with the result of step (ii), implies that for every $\epsilon > 0$ there exists an M_ϵ , N_1^ϵ and N_2^ϵ such that

$$P(A_n^\epsilon) = P(|k_n - k| < \epsilon k) \geq 1 - \epsilon/3, \quad \forall n \geq N_1^\epsilon \quad (3.120)$$

and

$$P\left(\sup_{\|\theta\|=1} |T_n^{**}(0; \lambda) \cdot \theta| \leq M_\epsilon\right) \geq 1 - \epsilon/3, \quad \forall n \geq N_2^\epsilon. \quad (3.121)$$

Also, by Lemma 3.4.2 $\exists$ an N_3^ϵ such that

$$\begin{aligned} & P\left(\inf_{|r|=b, \|\theta\|=1} \left\{T_n''(r\theta; \lambda) \cdot \theta\right\}^2 > u\right) \\ & \geq P\left(\inf_{|r|=b, \|\theta\|=1} \left\{\tilde{T}_n^*(r, \theta; \lambda)\right\}^2 > u\right) - \epsilon/3, \quad n \geq N_3^\epsilon. \end{aligned} \quad (3.122)$$

Now, we use the fact that $||d| - |c|| \leq |d + c|$, $d, c \in \mathbb{R}$. Then, we have

$$\begin{aligned} & P\left(\inf_{|r|=b, \|\theta\|=1} \left\{\tilde{T}_n^*(r, \theta; \lambda)\right\}^2 > u\right) \\ & \geq P\left(|T_n''(0; \lambda) \cdot \theta| - q_0(\lambda) b \theta' \Sigma_n \theta| > u^{1/2} \quad \forall \|\theta\| = 1\right) \\ & \geq P\left(|T_n''(0; \lambda) \cdot \theta| \leq -u^{1/2} + b k_n q_0(\lambda) \quad \forall \|\theta\| = 1\right) \\ & \geq P\left(\sup_{\|\theta\|=1} |T_n''(0; \lambda) \cdot \theta| \leq -u^{1/2} + b k(1 - \epsilon) q_0(\lambda); A_n^\epsilon\right) \\ & \geq 1 - (2\epsilon/3), \quad \forall n \geq n^\epsilon = N_1^\epsilon \vee N_2^\epsilon \vee N_3^\epsilon \end{aligned} \quad (3.123)$$

by (3.120) and (3.121), as long as $b \geq \frac{(M_\epsilon + u^{1/2})}{k(1 - \epsilon)q_0(\lambda)}$, which completes the proof of step (iii).

Now, we use the results (3.97) and (3.99), to prove the Lemma similar to Jurečková (1984) by defining

$$t = n^{1/2}(\hat{\beta}_n(\lambda) - \beta(\lambda)). \quad (3.124)$$

Thus, for all $\epsilon > 0$, $0 < u < \infty \exists b(= b_\epsilon) > 0$ and N^ϵ so that

$$\begin{aligned} & P(n^{1/2} \|\hat{\beta}_n(\lambda) - \beta(\lambda)\| > b) \\ & = P\left(n^{1/2} \|\hat{\beta}_n(\lambda) - \beta(\lambda)\| > b, \|T_n(\beta(\lambda) + n^{-1/2}t; \lambda)\| < u\right) \\ & \quad + P\left(n^{1/2} \|\hat{\beta}_n(\lambda) - \beta(\lambda)\| > b, \|T_n(\beta(\lambda) + n^{-1/2}t; \lambda)\| \geq u\right) \\ & \leq P\left(\inf_{\|t\| > b} \|T_n(\beta(\lambda) + n^{-1/2}t; \lambda)\| < u\right) + P(\|T_n(\hat{\beta}_n(\lambda); \lambda)\| \geq u) \\ & = \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned} \quad (3.125)$$

Therefore, the proof of Lemma 3.4.2 is complete.

PROOF of Theorem 3.4.1

It follows from Lemma 3.4.2 that

$$T_n(\beta(\lambda) + n^{-1/2}t; \lambda) - T_n(\beta(\lambda); \lambda) - \Sigma_n t q_0(\lambda) = o_P(1) \quad (3.126)$$

for every sequence of random vector t , such that $\|t\| = O_P(1)$. Hence, by substituting t with $n^{1/2}(\hat{\beta}_n(\lambda) - \beta(\lambda))$ we have

$$\begin{aligned} T_n(\beta(\lambda) + n^{-1/2}[n^{1/2}(\hat{\beta}_n(\lambda) - \beta(\lambda))]; \lambda) - T_n(\beta(\lambda); \lambda) \\ - q_0(\lambda) \Sigma_n [n^{1/2}(\hat{\beta}_n(\lambda) - \beta(\lambda))] = o_P(1). \end{aligned} \quad (3.127)$$

On the other hand

$$T_n(\hat{\beta}_n(\lambda); \lambda) - T_n(\beta(\lambda); \lambda) - q_0(\lambda) \Sigma_n [n^{1/2}(\hat{\beta}_n(\lambda) - \beta(\lambda))] = o_P(1). \quad (3.128)$$

But, by (3.97) we have $T_n(\hat{\beta}_n(\lambda); \lambda) = o_P(1)$. Therefore, the proof of Theorem 3.4.1 is complete. ♣

The following Corollaries give the asymptotic distribution of RQ's.

Corollary 3.4.4 *Assume the conditions of theorem 3.4.1 satisfied. Then,*

$$n^{1/2}(\hat{\beta}_n(\lambda) - \beta(\lambda)) \xrightarrow{D} \mathcal{N}_{p+1}\left(0, \frac{\lambda(1-\lambda)}{q_0^2(\lambda)} \Sigma^{-1}\right) \quad (3.129)$$

Corollary 3.4.5 *Let the conditions A0-A2 hold. Then, for each $\lambda_1, \dots, \lambda_k$ such that $0 < \lambda_1 < \dots < \lambda_k < 1$ the asymptotic joint distribution of the regression quantiles is given by*

$$[n^{1/2}(\hat{\beta}_n(\lambda_1) - \beta(\lambda_1)), \dots, n^{1/2}(\hat{\beta}_n(\lambda_k) - \beta(\lambda_k))] \xrightarrow{D} \mathcal{N}_{p+1 \times k}(0, \Omega \otimes \Sigma^{-1}) \quad (3.130)$$

where $\Omega = [\omega_{ij}]$ and $\omega_{ij} = \frac{\lambda_i - \lambda_j}{q_0(\lambda_i)q_0(\lambda_j)}$, $(i < j)$.

PROOF

By Theorem 3.4.1 we write

$$\begin{aligned} & \left[n^{1/2}(\hat{\beta}_n(\lambda_1) - \beta(\lambda_1)), \dots, n^{1/2}(\hat{\beta}_n(\lambda_k) - \beta(\lambda_k)) \right] = \\ & - \left\{ \frac{1}{q_0(\lambda_1)}, \dots, \frac{1}{q_0(\lambda_k)} \right\} \otimes \Sigma_n^{-1} \left[T_n(\beta(\lambda_1); \lambda_1), \dots, T_n(\beta(\lambda_k); \lambda_k) \right] \end{aligned} \quad (3.131)$$

and then by corollary 3.4.4 and Cramer-Wold device we get the result. ♠

Now, we state the main Theorem of TLSE similar to Jurečková (1984).

Theorem 3.4.6 *Assume that the conditions A0, A1 and A2 hold. Then, for each λ_1 and λ_2 such that $0 < \lambda_1 < 1/2 < \lambda_2 < 1$*

$$\begin{aligned} \text{i)} \quad & n^{1/2}(\tilde{\beta}_n(\lambda_1, \lambda_2) - \beta) = \\ & n^{-1/2} \{ (\lambda_2 - \lambda_1) \Sigma_n \}^{-1} \sum_{i=1}^n \mathbf{d}_i (\phi(Z_i) - \gamma) + o_P(1) \end{aligned} \quad (3.132)$$

$$\text{ii)} \quad n^{1/2}(\tilde{\beta}_n(\lambda_1, \lambda_2) - \beta) \xrightarrow{D} \mathcal{N}_{p+1}(\mathbf{0}, \sigma^2(\lambda_1, \lambda_2) \Sigma^{-1}) \quad (3.133)$$

where $\phi(Z_i)$, γ and $\sigma^2(\lambda_1, \lambda_2)$ are as defined in (2.97) and (2.98) respectively.

PROOF

This Theorem will be proved also with the aid of two Lemmas. First, we define the following processes

$$\hat{T}_n(t; \lambda) = n^{-1/2} \sum_{i=1}^n \mathbf{d}_i Z_i I(Z_i \leq Q_0(\lambda) + n^{-1/2} \mathbf{d}_i' t) \quad (3.134)$$

and

$$\begin{aligned} \hat{T}_n^0(t; \lambda) &= \hat{T}_n(t; \lambda) - E(\hat{T}_n(t; \lambda)) \\ &= n^{-1/2} \sum_{i=1}^n \mathbf{d}_i \left\{ Z_i I(Z_i \leq Q_0(\lambda) + n^{-1/2} \mathbf{d}_i' t) \right. \\ & \quad \left. - \mu(Q_0(\lambda) + n^{-1/2} \mathbf{d}_i' t) \right\}, \end{aligned} \quad (3.135)$$

where $\mu(\cdot)$ is defined in (2.103). Then,

Lemma 3.4.7 *Under the condition of Theorem 3.4.6 for each $b > 0$*

$$\sup_{\|t\| \leq b} \left\| \hat{T}_n(t; \lambda) - \hat{T}_n(0; \lambda) - q_0(\lambda)Q_0(\lambda)\Sigma_n t \right\| = o_P(1). \quad (3.136)$$

PROOF

Consider $\hat{T}_{nj}(t; \lambda)$, $j = 0, \dots, p$ as a j^{th} component of $\hat{T}_n(t; \lambda)$. Then, we have

$$\begin{aligned} & \sup_{\|t\| \leq b} \left| \hat{T}_{nj}(t; \lambda) - \hat{T}_{nj}(0; \lambda) - q_0(\lambda)Q_0(\lambda)\Sigma_n t \right| \\ & \leq \sup_{\|t\| \leq b} \left| \hat{T}_{nj}^0(t; \lambda) - \hat{T}_{nj}^0(0; \lambda) \right| \\ & \quad + \sup_{\|t\| \leq b} \left| E(\hat{T}_{nj}(t; \lambda)) - E(\hat{T}_{nj}(0; \lambda)) - q_0(\lambda)Q_0(\lambda)\Sigma_n t \right| \\ & = J_1 + J_2. \end{aligned} \quad (3.137)$$

First, note that

$$J_2 \leq n^{-1/2} \sum_{i=1}^n |d_{ij}| \sup_{\|t\| \leq b} \left| \mu(Q_0(\lambda) + n^{-1/2} d_i' t) - \mu(Q_0(\lambda)) - q_0(\lambda)Q_0(\lambda)n^{-1/2} d_i' t \right|. \quad (3.138)$$

Then, by the mean value Theorem there is a $\Delta(t) \in (0, n^{-1/2} d_i' t)$ such that

$$\begin{aligned} J_2 & \leq b n^{-1} \sum_{i=1}^n |d_{ij}| \|d_i\| \sup_{\|t\| \leq b} \left| (Q_0(\lambda) + \Delta(t)) f_0(Q_0(\lambda) + \Delta(t)) \right. \\ & \quad \left. - Q_0(\lambda) f_0(Q_0(\lambda)) \right| \\ & = o(1). \end{aligned} \quad (3.139)$$

Since, $\|t\| \leq b$ implies that $\Delta(t) \leq \delta_n = n^{-1/2} \|d_i\| b$ then, by conditions A0 and A2 we get the result. To prove Lemma 3.4.7 it is enough to show that $J_1 = o_P(1)$ where

$$J_1 = \sup_{\|t\| \leq b} \left| \hat{T}_{nj}^0(t; \lambda) - \hat{T}_{nj}^0(0; \lambda) \right|. \quad (3.140)$$

First, we consider for each $t \in \mathcal{N}(b)$

$$\left| \hat{T}_{nj}^0(t; \lambda) - \hat{T}_{nj}^0(0; \lambda) \right|$$

$$\begin{aligned}
&= \left| n^{-1/2} \sum_{i=1}^n d_{ij} \left\{ Z_i I(Z_i \leq (Q_0(\lambda) + n^{-1/2} \mathbf{d}'_i \mathbf{t}) - \mu(Q_0(\lambda) + n^{-1/2} \mathbf{d}'_i \mathbf{t}) \right. \right. \\
&\quad \left. \left. - Z_i I(Z_i \leq Q_0(\lambda)) + \mu(Q_0(\lambda)) \right\} \right| \\
&= \left| n^{-1/2} \sum_{i=1}^n d_{ij} \left\{ \hat{V}_i(\mathbf{t}) - \hat{\nu}_{ni}(\mathbf{t}) \right\} \right| \tag{3.141}
\end{aligned}$$

where

$$\begin{aligned}
\hat{V}_i(\mathbf{t}) &= Z_i I(Q_0(\lambda) < Z_i \leq Q_0(\lambda) + n^{-1/2} \mathbf{d}'_i \mathbf{t}) \\
\hat{\nu}_{ni}(\mathbf{t}) &= \mu(Q_0(\lambda) + n^{-1/2} \mathbf{d}'_i \mathbf{t}) - \mu(Q_0(\lambda)). \tag{3.142}
\end{aligned}$$

Now, we apply Chebyshev inequality to show that (3.141) goes to zero in probability.

$$\begin{aligned}
&P\left(\left| n^{-1/2} \sum_{i=1}^n d_{ij} \left\{ \hat{V}_i(\mathbf{t}) - \hat{\nu}_{ni}(\mathbf{t}) \right\} \right|^2 > \epsilon^2\right) \\
&\leq \frac{1}{\epsilon^2} \frac{1}{n} E\left(\sum_{i=1}^n d_{ij} \left\{ \hat{V}_i(\mathbf{t}) - \hat{\nu}_{ni}(\mathbf{t}) \right\}\right)^2 \\
&\quad \frac{1}{\epsilon^2} \frac{1}{n} \sum_{i=1}^n d_{ij}^2 E\left(\hat{V}_i(\mathbf{t}) - \hat{\nu}_{ni}(\mathbf{t})\right)^2 \\
&\leq \epsilon \quad \text{as } n \rightarrow \infty \text{ by condition A0 and A2.} \tag{3.143}
\end{aligned}$$

Therefore, $J_1 = o_P(1)$ for fixed $\mathbf{t}$. For the tightness, we follow similar argument as in expression (3.87) and use the compactness property of $\mathcal{N}(b)$. So, it is necessary to show that

$$\forall \epsilon > 0 \exists \delta > 0 \ni \forall \mathbf{s} \in \mathcal{N}(b)$$

$$\limsup_n P\left(\sup_{\|\mathbf{t}-\mathbf{s}\| \leq b} \left| \hat{T}_{nj}^0(\mathbf{t}; \lambda) - \hat{T}_{nj}^0(\mathbf{s}; \lambda) \right| > 2\epsilon\right) < \epsilon. \tag{3.144}$$

In this case we have

$$\begin{aligned}
&\left| \hat{T}_{nj}^0(\mathbf{t}; \lambda) - \hat{T}_{nj}^0(\mathbf{s}; \lambda) \right| \leq \\
&\quad n^{-1/2} \sum_{i=1}^n |d_{ij} Z_i| \left| I(Z_i \leq Q_0(\lambda) + n^{-1/2} \mathbf{d}'_i \mathbf{t}) - I(Z_i \leq Q_0(\lambda) + n^{-1/2} \mathbf{d}'_i \mathbf{s}) \right| \\
&\quad + n^{-1/2} \sum_{i=1}^n |d_{ij}| \left| \mu(Q_0(\lambda) + n^{-1/2} \mathbf{d}'_i \mathbf{t}) - \mu(Q_0(\lambda) + n^{-1/2} \mathbf{d}'_i \mathbf{s}) \right| \tag{3.145}
\end{aligned}$$

By inequality (3.93) the above expression is bounded above by

$$\begin{aligned}
 & n^{-1/2} \sum_{i=1}^n |d_{ij} Z_i| \left\{ I(Z_i \leq Q_0(\lambda) + n^{-1/2} d'_i s + n^{-1/2} \|d_i\| \delta) \right. \\
 & \quad \left. - I(Z_i \leq Q_0(\lambda) + n^{-1/2} d'_i s - n^{-1/2} \|d_i\| \delta) \right\} \\
 & + n^{-1/2} \sum_{i=1}^n |d_{ij}| \left| \mu(Q_0(\lambda) + n^{-1/2} d'_i t) - \mu(Q_0(\lambda) + n^{-1/2} d'_i s) \right| \\
 & = J_{11} + J_{12}.
 \end{aligned} \tag{3.146}$$

By the mean value Theorem and the same argument as in (3.139) we have

$$\sup_{\|t-s\| \leq \delta} J_{12} = o(1). \tag{3.147}$$

To show $\sup_{\|t-s\| \leq \delta} J_{11} = o_P(1)$ we note that J_{11} does not depend on t and so it is enough to show that $J_{11} = o_P(1)$. For this we center J_{11} to obtain

$$\begin{aligned}
 J_{11} &= n^{-1/2} \sum_{i=1}^n |d_{ij}| \left\{ |Z_i| I(Z_i \leq Q_0(\lambda) + n^{-1/2} d'_i s + n^{-1/2} \|d_i\| \delta) \right. \\
 & \quad \left. - \mu^*(Q_0(\lambda) + n^{-1/2} d'_i s + n^{-1/2} \|d_i\| \delta) \right\} \\
 & \quad - \left\{ I(Z_i \leq Q_0(\lambda) + n^{-1/2} d'_i s - n^{-1/2} \|d_i\| \delta) \right. \\
 & \quad \left. - \mu^*(Q_0(\lambda) + n^{-1/2} d'_i s - n^{-1/2} \|d_i\| \delta) \right\} \Big\} \\
 & + n^{-1/2} \sum_{i=1}^n |d_{ij}| \left\{ \mu^*(Q_0(\lambda) + n^{-1/2} d'_i s + n^{-1/2} \|d_i\| \delta) \right. \\
 & \quad \left. - \mu^*(Q_0(\lambda) + n^{-1/2} d'_i s - n^{-1/2} \|d_i\| \delta) \right\}.
 \end{aligned} \tag{3.148}$$

By the same argument as we did for fixed t in J_1 , the first term of R.H.S of (3.148) goes to zero in probability. The second term is also less than ϵ by continuity of μ^* for sufficient small δ . Therefore, the proof of lemma is complete.

Before we state the next lemma it is necessary to introduce the following processes, namely

$$U_{jk}(t) = n^{-1} \sum_{i=1}^n d_{ij} d_{ik} I(Z_i \leq Q_0(\lambda) + n^{-1/2} d'_i t) \tag{3.149}$$

and

$$U_{jk}^0(t) = U_{jk}(t) - E[U_{jk}(t)]. \quad (3.150)$$

Then,

Lemma 3.4.8 *Under the conditions of Theorem 3.4.6, for $0 < b < \infty$ and $\lambda \in (0, 1)$*

$$\sup_{\|t\| \leq b} |U_{jk}(t) - \lambda \sigma_{jk}| \xrightarrow{P} 0 \quad (3.151)$$

where σ_{jk} is the $j \times k^{\text{th}}$ element of Σ .

PROOF

First, we prove this lemma for each t . Then,

$$\begin{aligned} & |U_{jk}(t) - \lambda \sigma_{jk}| \\ &= \left| n^{-1} \sum_{i=1}^n d_{ij} d_{ik} I(Z_i \leq Q_0(\lambda) + n^{-1/2} \mathbf{d}'_i t) - \lambda \sigma_{jk} \right| \\ &= \left| n^{-1} \sum_{i=1}^n d_{ij} d_{ik} \{ I(Z_i \leq Q_0(\lambda) + n^{-1/2} \mathbf{d}'_i t) - F_0(Q_0(\lambda) + n^{-1/2} \mathbf{d}'_i t) \right. \\ &\quad \left. + F_0(Q_0(\lambda) + n^{-1/2} \mathbf{d}'_i t) \} - \lambda \sigma_{jk} F_0(Q_0(\lambda)) \right| \\ &\leq \left| n^{-1} \sum_{i=1}^n d_{ij} d_{ik} \{ I(Z_i \leq Q_0(\lambda) + n^{-1/2} \mathbf{d}'_i t) - F_0(Q_0(\lambda) + n^{-1/2} \mathbf{d}'_i t) \} \right| \\ &\quad + \left| n^{-1} \sum_{i=1}^n d_{ij} d_{ik} F_0(Q_0(\lambda) + n^{-1/2} \mathbf{d}'_i t) - \lambda \sigma_{jk} F_0(Q_0(\lambda)) \right| \\ &= I_1 + I_2. \end{aligned} \quad (3.152)$$

First we show that $I_1 = o_P(1)$ that is to say that for every $\epsilon > 0$, we have

$$\begin{aligned} & P \left\{ \left| n^{-1} \sum_{i=1}^n d_{ij} d_{ik} \{ I(Z_i \leq Q_0(\lambda) + n^{-1/2} \mathbf{d}'_i t) - F_0(Q_0(\lambda) + n^{-1/2} \mathbf{d}'_i t) \} \right|^2 > \epsilon^2 \right\} \\ &\leq \frac{1}{\epsilon^2} \frac{1}{n^2} E \left\{ \sum_{i=1}^n d_{ij} d_{ik} \{ I(Z_i \leq Q_0(\lambda) + n^{-1/2} \mathbf{d}'_i t) - F_0(Q_0(\lambda) + n^{-1/2} \mathbf{d}'_i t) \} \right\}^2 \\ &= \frac{1}{\epsilon^2} \frac{1}{n^2} \sum_{i=1}^n d_{ij}^2 d_{ik}^2 E \{ I(Z_i \leq Q_0(\lambda) + n^{-1/2} \mathbf{d}'_i t) - F_0(Q_0(\lambda) + n^{-1/2} \mathbf{d}'_i t) \}^2 \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\epsilon^2} \frac{1}{n^2} \sum_{i=1}^n d_{ij}^2 d_{ik}^2 \\
&\leq \frac{1}{\epsilon^2} \left(n^{-1/2} \max_{i,j} |d_{ij}| \right)^2 \cdot n^{-1} \sum_{i=1}^n d_{ik}^2 \\
&\leq \epsilon \text{ as } n \rightarrow \infty \text{ by A1 and A2.}
\end{aligned} \tag{3.153}$$

Continuity of F_0 together with C-S inequality and condition A2 implies that $I_2 = o(1)$. Therefore, we have the result for fixed t . For tightness of I_1 we follow the proof of the Lemma 3.4.2. Next we consider I_2 . For this we have

$$\begin{aligned}
\sup_{\|t\| \leq b} I_2 &= \sup_{\|t\| \leq b} \left| n^{-1} \sum_{i=1}^n d_{ij} d_{ik} F_0(Q_0(\lambda)) + n^{-1/2} d'_i t \right) - \sigma_{jk} F_0(Q_0(\lambda)) \right| \\
&\leq \sup_{\|t\| \leq b} \left| n^{-1} \sum_{i=1}^n d_{ij} d_{ik} \left[F_0(Q_0(\lambda) + n^{-1/2} d'_i t) - F_0(Q_0(\lambda)) \right] \right| \\
&\quad + F_0(Q_0(\lambda)) \left| n^{-1} \sum_{i=1}^n d_{ij} d_{ik} - \sigma_{jk} \right| \\
&\leq n^{-1} \sum_{i=1}^n |d_{ij} d_{ik}| \sup_{\|t\| \leq b} \left| F_0(Q_0(\lambda) + n^{-1/2} d'_i t) - F_0(Q_0(\lambda)) \right| + o(1) \\
&\leq \left(n^{-1} \sum_{i=1}^n d_{ij}^2 \right)^{1/2} \left(n^{-1} \sum_{i=1}^n d_{ik}^2 \right)^{1/2} \cdot o(1) + o(1) \\
&\leq \epsilon \text{ as } n \rightarrow \infty \text{ by C-S inequality and A0 and A2.}
\end{aligned} \tag{3.154}$$

Now we change the process $U_{jk}(\cdot)$ to

$$U_{jk}^*(t) = n^{-1} \sum_{i=1}^n d_{ij} d_{ik} I(Q_0(\lambda_1) + n^{-1/2} d'_i t < Z_i \leq Q_0(\lambda_2) + n^{-1/2} d'_i t) \tag{3.155}$$

for each $\lambda_1, \lambda_2 \ni 0 < \lambda_1 < \lambda_2 < 1$. Then, we have

$$\sup_{\|t\| \leq b} \left| U_{jk}^*(t) - (\lambda_2 - \lambda_1) \sigma_{jk} \right| = o_P(1). \tag{3.156}$$

We apply (3.156) to prove Theorem 3.4.6.

PROOF of Theorem 3.4.6 (i)

By using the result of Lemma 3.4.8 in the matrix form we have

$$n^{-1} D'_n A D_n = n^{-1} \sum_{i=1}^n d_i d'_i I(Q_0(\lambda_1) + n^{-1/2} d'_i t < Z_i \leq Q_0(\lambda_2) + n^{-1/2} d'_i t)$$

$$= (\lambda_2 - \lambda_1)\Sigma + o_P(1) \quad \text{as } n \rightarrow \infty. \quad (3.157)$$

By the structure of matrix A ,

$$\begin{aligned} n^{-1/2} D'_n A Z &= n^{-1/2} \sum_{i=1}^n d_i \left\{ I(Z_i \leq Q_0(\lambda_2) + d'_i(\hat{\beta}_n(\lambda_2) - \beta(\lambda_2))) \right. \\ &\quad \left. - I(Z_i \leq Q_0(\lambda_1) + d'_i(\hat{\beta}_n(\lambda_1) - \beta(\lambda_1))) \right\} Z_i \\ &= \tilde{T}_n(t_2; \lambda_2) - \tilde{T}_n(t_1; \lambda_1) \end{aligned} \quad (3.158)$$

where

$$t_i = n^{1/2}(\hat{\beta}_n(\lambda_i) - \beta(\lambda_i)) \quad i = 1, 2. \quad (3.159)$$

But,

$$\begin{aligned} \tilde{T}_n(t_2; \lambda_2) - \tilde{T}_n(t_1; \lambda_1) &= [\tilde{T}_n(t_2; \lambda_2) - \tilde{T}_n(0; \lambda_2) - q_0(\lambda_2)Q_0(\lambda_2)\Sigma_n t_2] \\ &\quad - [\tilde{T}_n(t_1; \lambda_1) - \tilde{T}_n(0; \lambda_1) - q_0(\lambda_1)Q_0(\lambda_1)\Sigma_n t_1] \\ &\quad + [\tilde{T}_n(0; \lambda_2) - \tilde{T}_n(0; \lambda_1) + q_0(\lambda_2)Q_0(\lambda_2)\Sigma_n t_2 \\ &\quad - q_0(\lambda_1)Q_0(\lambda_1)\Sigma_n t_1]. \end{aligned} \quad (3.160)$$

By Lemma 3.4.7 the first and second terms of (3.160) are $o_P(1)$. Also, by substituting t_i with its representation given in the Theorem 3.4.1 we have

$$\begin{aligned} n^{-1/2} D'_n A Z &= T_n(\beta(\lambda_1); \lambda_1)Q_0(\lambda_1) - T_n(\beta(\lambda_2); \lambda_2)Q_0(\lambda_2) \\ &\quad + n^{-1/2} \sum_{i=1}^n d_i Z_i \{ I(Q_0(\lambda_1) < Z_i \leq Q_0(\lambda_2)) \} + o_P(1) \end{aligned} \quad (3.161)$$

and finally by the definition of $T_n(\beta(\lambda); \lambda)$ and using the fact that $I(Z_i \leq a) = 1 - I(Z_i > a)$ we have

$$\begin{aligned} n^{-1/2} D'_n A Z &= n^{-1/2} \sum_{i=1}^n d_i \left\{ I(Z_i \leq Q_0(\lambda_1))Q_0(\lambda_1) + Z_i I(Q_0(\lambda_1) < Z_i \leq Q_0(\lambda_2)) \right. \\ &\quad \left. + I(Z_i > Q_0(\lambda_2))Q_0(\lambda_2) - \lambda_1 Q_0(\lambda_1) - (1 - \lambda_2)Q_0(\lambda_2) \right\} + o_P(1) \\ &= n^{-1/2} \sum_{i=1}^n d_i \{ \phi(Z_i) - \gamma \} + o_P(1) \end{aligned} \quad (3.162)$$

where $\phi(Z_i)$ and γ are given in the (2.97). Finally we have

$$(D'_n A D_n)^{-1} D'_n A Z = (D'_n A D_n)^{-1} D'_n A Y - \beta \quad (3.163)$$

or

$$(n^{-1} D'_n A D_n)^{-1} n^{-1/2} D'_n A Z = n^{1/2} (\bar{\beta}_n(\lambda_1, \lambda_2) - \beta). \quad (3.164)$$

As a result

$$n^{1/2} (\bar{\beta}_n(\lambda_1, \lambda_2) - \beta) = ((\lambda_2 - \lambda_1) \Sigma_n)^{-1} n^{-1/2} \sum_{i=1}^n d_i \{ \phi(Z_i) - \gamma \} + o_P(1). \quad (3.165)$$

Therefore, the proof of the first part of Theorem 3.4.6 is complete.

PROOF of Theorem 3.4.6 (ii)

The result of the second part is given by using the result of part (i), considering F_0 symmetric with $\lambda_1 = 1 - \lambda_2$. In this case we have $\gamma = 0$ and $E(\phi(Z_i)) = 0$. Thus, by similar argument as (3.114) we have

$$\frac{n^{-1/2} \sum_{i=1}^n d_{ij} \phi(Z_i)}{[\sigma^2(\lambda_1, \lambda_2) n^{-1} \sum_{i=1}^n d_{ij}^2]^{1/2}} \xrightarrow{D} \mathcal{N}(0, 1) \quad (3.166)$$

and so W.L.G. the proof of the second part is complete. ♠

As we saw in the result of theorem 3.4.6, TLSE converges to normal distribution with variance $\sigma^2(\lambda_1, \lambda_2) \Sigma^{-1}$. Here we define a consistent estimator of $\sigma^2(\lambda_1, \lambda_2)$. Let $Y_1, Y_2, \dots, Y_n$ be sample from regression model (3.65). Then the estimation of $\sigma^2(\lambda_1, \lambda_2)$ is given by

$$\begin{aligned} S_n^2(\lambda_1, \lambda_2) = & (\lambda_2 - \lambda_1)^{-2} \left\{ (n-1)^{-1} \sum_{i=1}^n (Y_i - \delta_n)^2 I(\hat{\beta}_n(\lambda_1) < Y_i \leq \hat{\beta}_n(\lambda_2)) \right. \\ & + \lambda_1 (Q_n(\lambda_1) - \delta_n)^2 + (1 - \lambda_2) (Q_n(\lambda_2) - \delta_n)^2 \\ & \left. - [\lambda_1 (Q_n(\lambda_1) - \delta_n) + (1 - \lambda_2) (Q_n(\lambda_2) - \delta_n)]^2 \right\}, \end{aligned} \quad (3.167)$$

where

$$\begin{aligned} \delta_n &= (\lambda_2 - \lambda_1)^{-1} n^{-1} \sum_{i=1}^n Y_i I(d'_i \hat{\beta}_n(\lambda_1) < Y_i \leq d'_i \hat{\beta}_n(\lambda_2)) \\ &= c n^{-1} \sum_{i=1}^n Y_i I(Q_0(\lambda_1) + n^{-1/2} d'_i t_1 < Z_i \leq Q_0(\lambda_2) + n^{-1/2} d'_i t_2) \end{aligned} \quad (3.168)$$

and $\mathbf{t}_i = n^{1/2}(\hat{\beta}_n(\lambda_i) - \beta(\lambda_i))$. The following theorem states the consistency of $S_n^2(\lambda_1, \lambda_2)$.

Theorem 3.4.9 *Let the conditions of Theorem 3.4.6 hold. Then,*

$$S_n^2(\lambda_1, \lambda_2) \xrightarrow{P} \sigma^2(\lambda_1, \lambda_2). \quad (3.169)$$

PROOF:

To prove this theorem first we shall show that

$$\delta_n \xrightarrow{P} \delta \quad (3.170)$$

where

$$\delta = (\lambda_2 - \lambda_1)^{-1} \int_{Q_0(\lambda_1)}^{Q_0(\lambda_2)} Y dF(y) = \mathbf{d}'\boldsymbol{\beta} + \delta_0 \quad (3.171)$$

and δ_0 is given in (2.99). Equivalently, it is necessary to show that

$$\delta_{0n} \xrightarrow{P} \delta_0 \quad (3.172)$$

where

$$\delta_{0n} = c n^{-1} \sum_{i=1}^n Z_i I(Q_0(\lambda_1) + n^{-1/2} \mathbf{d}'_i \mathbf{t}_1 < Z_i \leq Q_0(\lambda_2) + n^{-1/2} \mathbf{d}'_i \mathbf{t}_2). \quad (3.173)$$

For this reason we see

$$\begin{aligned} |\delta_{0n} - \delta_0| &= c \left| n^{-1} \sum_{i=1}^n Z_i I(Q_0(\lambda_1) + n^{-1/2} \mathbf{d}'_i \mathbf{t}_1 < Z_i \leq Q_0(\lambda_2) + n^{-1/2} \mathbf{d}'_i \mathbf{t}_2) \right. \\ &\quad \left. - \int_{Q_0(\lambda_1)}^{Q_0(\lambda_2)} Z dF_0(z) \right| \\ &\leq c \left| n^{-1} \sum_{i=1}^n Z_i I(Z_i \leq Q_0(\lambda_2) + n^{-1/2} \mathbf{d}'_i \mathbf{t}_2) - \nu(Q_0(\lambda_2)) \right| \\ &\quad + c \left| n^{-1} \sum_{i=1}^n Z_i I(Z_i \leq Q_0(\lambda_1) + n^{-1/2} \mathbf{d}'_i \mathbf{t}_1) - \nu(Q_0(\lambda_1)) \right| \\ &= c(I_1 + I_2) \end{aligned} \quad (3.174)$$

where

$$\nu(u) = \int_{-\infty}^u Z dF_0(z). \quad (3.175)$$

So, we have to show that $I_1 = o_P(1) = I_2$ where

$$\begin{aligned} I_1 &\leq c \left| n^{-1} \sum_{i=1}^n Z_i I(Z_i \leq Q_0(\lambda_1) + n^{-1/2} d'_i t_1) - \nu(Q_0(\lambda_1) + n^{-1/2} d'_i t_1) \right| \\ &\quad + c \left| \nu(Q_0(\lambda_1) + n^{-1/2} d'_i t_1) - \nu(Q_0(\lambda_1)) \right| \\ &= I_{11} + I_{12}. \end{aligned} \quad (3.176)$$

In this case we may write $I_{11} = I_{11}^+ - I_{11}^-$ where

$$I_{11}^\pm = c \left| n^{-1} \sum_{i=1}^n Z_i^\pm I(Z_i \leq Q_0(\lambda_1) + n^{-1/2} d'_i t_1) - \nu^\pm(Q_0(\lambda_1) + n^{-1/2} d'_i t_1) \right|. \quad (3.177)$$

Then Lemma 2.4.9 implies that $I_{11}^\pm = o_P(1)$. Also, by continuity of $\nu(\cdot)$ we have $I_{12} = o(1)$. Therefore, $I_1 = o_P(1)$ and similarly $I_2 = o_P(1)$.

By similar argument we have

$$\begin{aligned} &\left| n^{-1} \sum_{i=1}^n Z_i^2 I(Q_0(\lambda_1) + n^{-1/2} d'_i t_1 < Z_i \leq Q_0(\lambda_2) + n^{-1/2} d'_i t_2) \right. \\ &\quad \left. - \int_{Q_0(\lambda_1)}^{Q_0(\lambda_2)} Z^2 dF_0(z) \right| = o_P(1). \end{aligned} \quad (3.178)$$

Also, by same argument as (2.147) we have

$$Q_{0n}(\lambda) \xrightarrow{P} Q_0(\lambda). \quad (3.179)$$

Combining the above results we obtain the proof of Theorem 3.4.9.

3.5 Trimmed Estimation of Regression Parameters Under Uncertain Prior Information.

In this section we discuss the estimation of β when there exists some uncertain prior information on β in the form of a hypothesis. Consider the usual linear model

(3.65). We are interested in the robust estimation of β when it is suspected but not sure that the null hypothesis

$$H_0 : \mathbf{R}\beta = \mathbf{r} \quad (3.180)$$

may hold, where $\mathbf{R}$ is a $q \times p+1$ matrix of rank $q(\leq p+1)$. Saleh and Han (1990) considered the problem based on least-squares estimator of β while Saleh and Shiraishi (1989) dealt with R and M estimators. In this chapter, we consider (λ_1, λ_2) -trimmed estimator for the linear model (3.65) as discussed in section 3.4 and the modification there of. For the model (3.65), we denote the (λ_1, λ_2) -trimmed estimator of β by $\tilde{\beta}_n(\lambda_1, \lambda_2)$ and is called the *unrestricted trimmed L-estimator* (UTLE) after trimming off λ_1 and λ_2 proportion of observations using (3.68). When H_0 is true, the corresponding trimmed estimator of β is denoted by $\tilde{\beta}_n(\lambda_1, \lambda_2)$ and is called the *restricted trimmed L-estimator* (RTLE). It may be pointed out that $\tilde{\beta}_n(\lambda_1, \lambda_2)$ has smaller asymptotic dispersion than $\tilde{\beta}_n(\lambda_1, \lambda_2)$ under $H_0 : \mathbf{R}\beta = \mathbf{r}$, although, $\tilde{\beta}_n(\lambda_1, \lambda_2)$ may be biased and even inconsistent. However, when H_0 is uncertain, we consider a preliminary test (PT) on H_0 and consider $\tilde{\beta}_n(\lambda_1, \lambda_2)$ or $\tilde{\beta}_n(\lambda_1, \lambda_2)$ according as H_0 is accepted or rejected using a reasonable test statistic. We denote this estimator by $\tilde{\beta}_n^{PT}(\lambda_1, \lambda_2)$ and is called the *preliminary test trimmed L-estimator* (PTTLE). Further, we consider some Stein-type estimators incorporating $\tilde{\beta}_n(\lambda_1, \lambda_2)$, $\tilde{\beta}_n(\lambda_1, \lambda_2)$ and the test statistic. The usual *shrinkage trimmed L-estimator* (STLE) will be denoted by $\tilde{\beta}_n^S(\lambda_1, \lambda_2)$ and the *positive-rule shrinkage trimmed L-estimator* (PRSTLE) denoted by $\tilde{\beta}_n^{S+}(\lambda_1, \lambda_2)$.

In subsection 3.5.1, five trimmed estimators of β are introduced. In subsection 3.5.2, the asymptotic distributional quadratic risks(ADQR) of these estimators under local alternatives are investigated. In subsection 3.5.3, the main results on relative dominance of these trimmed estimators are obtained.

3.5.1 Five Trimmed Estimators

This subsection contains the proposed five estimators of β stemming out of the TLSE of β . In the Theorem 3.4.6 (ii), we showed that

$$\sqrt{n}(\bar{\beta}_n(\lambda_1, \lambda_2) - \beta) \xrightarrow{D} N_p(0, \sigma^2(\lambda_1, \lambda_2)\Sigma^{-1}). \quad (3.181)$$

Now, by analogy of LSE of β subject to $R\beta = r$, we propose the RTLE as

$$\tilde{\beta}_n(\lambda_1, \lambda_2) = \bar{\beta}_n(\lambda_1, \lambda_2) - \Sigma_n^{-1}R'(R\Sigma_n^{-1}R')^{-1}(R\bar{\beta}_n(\lambda_1, \lambda_2) - r). \quad (3.182)$$

To obtain the PTTLE of β , we consider the following test statistic, L_n given by

$$L_n = \frac{[R\bar{\beta}_n(\lambda_1, \lambda_2) - r]'[R\Sigma_n^{-1}R']^{-1}[R\bar{\beta}_n(\lambda_1, \lambda_2) - r]}{qS_n^2(\lambda_1, \lambda_2)} \quad (3.183)$$

where $S_n^2(\lambda_1, \lambda_2)$ given in the (3.167). By Theorem 3.4.9, $S_n^2(\lambda_1, \lambda_2) \xrightarrow{P} \sigma^2(\lambda_1, \lambda_2)$.

Hence, as $n \rightarrow \infty$, L_n converges in law to the central chi-squared distribution with q degrees of freedom (df) under H_0 . Hence, we define the PTTLE as

$$\tilde{\beta}_n^{PT}(\lambda_1, \lambda_2) = \bar{\beta}_n(\lambda_1, \lambda_2) - \Sigma_n^{-1}R'(R\Sigma_n^{-1}R')^{-1}(R\bar{\beta}_n(\lambda_1, \lambda_2) - r)I(L_n \leq \chi_{q,\alpha}^2) \quad (3.184)$$

where $\chi_{q,\alpha}^2$ is the upper $100\alpha\%$ point of the central chi-square distribution with q df and $I(A)$ is the indicator function of set A . As for the STLE and PRSTLE we propose the following:

$$\tilde{\beta}_n^S(\lambda_1, \lambda_2) = \bar{\beta}_n(\lambda_1, \lambda_2) - cL_n^{-1}\Sigma_n^{-1}R'(R\Sigma_n^{-1}R')^{-1}(R\bar{\beta}_n(\lambda_1, \lambda_2) - r) \quad (3.185)$$

and

$$\tilde{\beta}_n^{S+}(\lambda_1, \lambda_2) = \tilde{\beta}_n(\lambda_1, \lambda_2) + [1 - cL_n^{-1}]I(L_n > c)(\bar{\beta}_n(\lambda_1, \lambda_2) - \tilde{\beta}_n(\lambda_1, \lambda_2)) \quad (3.186)$$

respectively where $0 < c < 2(q-2)$. Note that as $L_n \rightarrow \infty$, $\tilde{\beta}_n^S(\lambda_1, \lambda_2)\{\tilde{\beta}_n^{S+}(\lambda_1, \lambda_2)\} \rightarrow \bar{\beta}_n(\lambda_1, \lambda_2)$ and as $L_n \rightarrow c$, $\tilde{\beta}_n^S(\lambda_1, \lambda_2)\{\tilde{\beta}_n^{S+}(\lambda_1, \lambda_2)\} \rightarrow \tilde{\beta}_n(\lambda_1, \lambda_2)$, which is similar to the property of PTTLE but (3.185) and (3.186) are smooth version of PTTLE.

3.5.2 Asymptotic Distributional Risks

We consider the asymptotic distributional risks under local alternatives, noting that the test based on L_n is consistent for fixed β such that $R\beta \neq r$ and the PTTLE, STLE and PRSTLE are asymptotically equivalent to UTLE under fixed alternatives. Hence, we study the asymptotic risks under local alternatives:

$$H_a : R\beta = r + n^{-\frac{1}{2}}\delta, \quad \delta = (\delta_1, \dots, \delta_q)'. \quad (3.187)$$

Then we have the following theorem.

Theorem 3.5.1 *Under $\{H_a\}$ and conditions of theorem 3.4.6, the distribution of $L_n, \bar{\beta}_n(\lambda_1, \lambda_2), \tilde{\beta}_n(\lambda_1, \lambda_2)$, and $\tilde{\beta}_n^{PT}(\lambda_1, \lambda_2)$ are given by*

$$(i) \quad \lim P\{L_n \leq \mathbf{x}\} = H_q(\mathbf{x}, \Delta), \quad \Delta = \frac{\delta'(\mathbf{R}\Sigma^{-1}\mathbf{R}')^{-1}\delta}{\sigma^2(\lambda_1, \lambda_2)} \quad (3.188)$$

$$(ii) \quad \lim P\{\sqrt{n}(\bar{\beta}_n(\lambda_1, \lambda_2) - \beta) \leq \mathbf{x}\} = G_{p+1}(\mathbf{x}; \mathbf{0}, \sigma^2(\lambda_1, \lambda_2)\Sigma^{-1}) \quad (3.189)$$

$$(iii) \quad \lim P\{\sqrt{n}(\tilde{\beta}_n(\lambda_1, \lambda_2) - \beta) \leq \mathbf{x}\} \\ = G_{p+1}(\mathbf{x} + \Sigma^{-1}\mathbf{R}'(\mathbf{R}\Sigma^{-1}\mathbf{R}')^{-1}\delta; \mathbf{0}, \sigma^2(\lambda_1, \lambda_2)\mathbf{V}) \quad (3.190)$$

$$(iv) \quad \lim P\{\sqrt{n}(\tilde{\beta}_n^{PT}(\lambda_1, \lambda_2) - \beta) \leq \mathbf{x}\} = \\ H_q(\chi_{q,\alpha}^2; \Delta)G_{p+1}\{\mathbf{x} + \Sigma^{-1}\mathbf{R}'(\mathbf{R}\Sigma^{-1}\mathbf{R}')^{-1}\delta; \mathbf{0}, \sigma^2(\lambda_1, \lambda_2)\mathbf{V}\} \\ + \int_{E(\delta)} G_{p+1}\{\mathbf{x} - \Sigma^{-1}\mathbf{R}'(\mathbf{R}\Sigma^{-1}\mathbf{R}')^{-1}\mathbf{z}; \mathbf{0}, \sigma^2(\lambda_1, \lambda_2)\mathbf{V}\} \\ dG_{p+1}(\mathbf{z}; \mathbf{0}, \sigma^2(\lambda_1, \lambda_2)(\mathbf{R}\Sigma^{-1}\mathbf{R}')) \quad (3.191)$$

where

$$\mathbf{V} = \Sigma^{-1} - \Sigma^{-1}\mathbf{R}'(\mathbf{R}\Sigma^{-1}\mathbf{R}')^{-1}\mathbf{R}\Sigma^{-1} \quad (3.192)$$

and $H_q(\cdot; \Delta)$ is the cdf of a non-central chi-square distribution with q df and non-centrality parameter Δ , $G_{p+1}(\mathbf{x}; \mu, \Sigma)$ is the cdf of a $p+1$ -dimensional normal

distribution with mean μ and covariance matrix Σ and

$$E(\delta) = \{z : \frac{(z + \delta)'(\mathbf{R}\Sigma^{-1}\mathbf{R}')^{-1}(z + \delta)}{\sigma^2(\lambda_1, \lambda_2)} \geq \chi_{q,\alpha}^2\}. \quad (3.193)$$

Furthermore,

$$(v) \quad \sqrt{n}(\tilde{\beta}_n^S(\lambda_1, \lambda_2) - \beta) \stackrel{D}{=} \mathbf{U} - c\sigma^2(\lambda_1, \lambda_2) \frac{\Sigma^{-1}\mathbf{R}'(\mathbf{R}\Sigma^{-1}\mathbf{R}')^{-1}(\mathbf{R}\mathbf{U} + \delta)}{(\mathbf{R}\mathbf{U} + \delta)'(\mathbf{R}\Sigma^{-1}\mathbf{R}')^{-1}(\mathbf{R}\mathbf{U} + \delta)} \quad (3.194)$$

where

$$\mathbf{U} \stackrel{D}{=} \mathcal{N}_{p+1}(\mathbf{0}, \sigma^2(\lambda_1, \lambda_2)\Sigma^{-1}). \quad (3.195)$$

Similarly,

$$(vi) \quad \begin{aligned} \sqrt{n}(\tilde{\beta}_n^{S+}(\lambda_1, \lambda_2) - \beta) &\stackrel{D}{=} \mathbf{U} - \Sigma^{-1}\mathbf{R}'(\mathbf{R}\Sigma^{-1}\mathbf{R}')^{-1}(\mathbf{R}\mathbf{U} + \delta) \\ &\quad + \{1 - c\sigma^2(\lambda_1, \lambda_2)[(\mathbf{R}\mathbf{U} + \delta)'(\mathbf{R}\Sigma^{-1}\mathbf{R}')^{-1}(\mathbf{R}\mathbf{U} + \delta)]^{-1}\} \\ &\quad I\{(\mathbf{R}\mathbf{U} + \delta)'(\mathbf{R}\Sigma^{-1}\mathbf{R}')^{-1}(\mathbf{R}\mathbf{U} + \delta) \geq c\} \\ &\quad \Sigma^{-1}\mathbf{R}'(\mathbf{R}\Sigma^{-1}\mathbf{R}')^{-1}(\mathbf{R}\mathbf{U} + \delta). \end{aligned} \quad (3.196)$$

Assume that, for an estimator $\tilde{\beta}_n^*(\lambda_1, \lambda_2)$ of β ,

$$G^*(\mathbf{x}) = \lim P\{\sqrt{n}(\tilde{\beta}_n^*(\lambda_1, \lambda_2) - \beta) \leq \mathbf{x}\}. \quad (3.197)$$

Then, we define the asymptotic distributional quadratic risk (ADQR) of $\tilde{\beta}_n^*(\lambda_1, \lambda_2)$ by

$$\mathfrak{R}(\tilde{\beta}_n^*(\lambda_1, \lambda_2); \mathbf{W}) = \int_{R^p} \mathbf{x}'\mathbf{W}\mathbf{x}dG^*(\mathbf{x}) = \text{tr}(\Sigma^*\mathbf{W}). \quad (3.198)$$

where

$$\Sigma^* = \int \mathbf{x}\mathbf{x}'dG^*(\mathbf{x}) \quad (3.199)$$

and $\mathbf{W}$ is a positive definite matrix associated with the quadratic loss function

$$L(\tilde{\beta}_n^*(\lambda_1, \lambda_2); \beta) = n(\tilde{\beta}_n^*(\lambda_1, \lambda_2) - \beta)' \mathbf{W} (\tilde{\beta}_n^*(\lambda_1, \lambda_2) - \beta). \quad (3.200)$$

Theorem 3.5.2 *Under the assumption of theorem 3.5.1, the ADQR's of the estimators are given by*

- (i) $\mathfrak{R}(\tilde{\beta}_n(\lambda_1, \lambda_2); \mathbf{W}) = \sigma^2(\lambda_1, \lambda_2) \text{tr} \Sigma^{-1} \mathbf{W},$
- (ii) $\mathfrak{R}(\tilde{\beta}_n(\lambda_1, \lambda_2); \mathbf{W}) = \sigma^2(\lambda_1, \lambda_2) [\text{tr}(\Sigma^{-1} \mathbf{W}) - \text{tr}(\mathbf{B})] + \delta'(\mathbf{R} \Sigma^{-1} \mathbf{R}')^{-1} \mathbf{B} \delta$
- (iii) $\mathfrak{R}(\tilde{\beta}_n^{PT}(\lambda_1, \lambda_2); \mathbf{W}) = \sigma^2(\lambda_1, \lambda_2) \text{tr}(\Sigma^{-1} \mathbf{W}) - \sigma^2(\lambda_1, \lambda_2) \text{tr}(\mathbf{B}) H_{q+2}(\chi_{q,\alpha}^2; \Delta)$
 $+ \delta'(\mathbf{R} \Sigma^{-1} \mathbf{R}')^{-1} \mathbf{B} \delta \{2H_{q+2}(\chi_{q,\alpha}^2; \Delta) - H_{q+4}(\chi_{q,\alpha}^2; \Delta)\},$
- (iv) $\mathfrak{R}(\tilde{\beta}_n^S(\lambda_1, \lambda_2); \mathbf{W}) = \sigma^2(\lambda_1, \lambda_2) \text{tr}(\Sigma^{-1} \mathbf{W}) - \sigma^2(\lambda_1, \lambda_2) c \text{tr}(\mathbf{B}) \{2E[\chi_{q+2}^{-2}(\Delta)]$
 $- c E[\chi_{q+2}^{-4}(\Delta)]\} + c(c+4) \{ \delta'(\mathbf{R} \Sigma^{-1} \mathbf{R}')^{-1} \mathbf{B} \delta E[\chi_{q+4}^{-4}(\Delta)] \},$
- (v) $\mathfrak{R}(\tilde{\beta}_n^{S+}(\lambda_1, \lambda_2); \mathbf{W}) = \mathfrak{R}(\tilde{\beta}_n^S; \mathbf{W}) - \sigma^2(\lambda_1, \lambda_2) \text{tr}(\mathbf{B})$
 $E \left[(1 - c \chi_{q+2}^{-2}(\Delta))^2 I(\chi_{q+2}^2(\Delta) \leq c) \right]$
 $+ \delta'(\mathbf{R} \Sigma^{-1} \mathbf{R}')^{-1} \mathbf{B} \delta \left\{ 2E \left[(1 - c \chi_{q+2}^{-2}(\Delta)) I(\chi_{q+2}^2 \leq c) \right] \right.$
 $\left. - E \left[(1 - c \chi_{q+4}^{-2}(\Delta)) I(\chi_{q+4}^2(\Delta) \leq c) \right] \right\}$

where

$$\mathbf{B} = \mathbf{R} \Sigma^{-1} \mathbf{W} \Sigma^{-1} \mathbf{R}' (\mathbf{R} \Sigma^{-1} \mathbf{R}')^{-1}. \quad (3.201)$$

PROOF:

(i) and (ii) follow by straight forward computations using Theorem 3.5.1 (ii) and (iii). Using Theorem 3.5.1 (iv) – (v), (iii), (iv) and (v) are given by the same argument as in Judge and Bock (1978).

3.5.3 Comparison of ADQR

First, we compare the $\tilde{\beta}_n(\lambda_1, \lambda_2)$ with $\bar{\beta}_n(\lambda_1, \lambda_2)$ by the ratio of the ADQR of

$\tilde{\beta}_n(\lambda_1, \lambda_2)$ to that of $\tilde{\beta}_n(\lambda_1, \lambda_2)$.

$$ARE[\tilde{\beta}_n(\lambda_1, \lambda_2); \tilde{\beta}_n(\lambda_1, \lambda_2)] = \left\{ 1 - \frac{tr(\mathbf{B})}{tr(\Sigma^{-1}\mathbf{W})} + \frac{\delta'(\mathbf{R}\Sigma^{-1}\mathbf{R}')^{-1}\mathbf{B}\delta}{\sigma^2(\lambda_1, \lambda_2)tr(\Sigma^{-1}\mathbf{W})} \right\}^{-1}. \quad (3.202)$$

Hence, $ARE[\tilde{\beta}_n(\lambda_1, \lambda_2); \tilde{\beta}_n(\lambda_1, \lambda_2)] \lesssim 1$ according as

$$\delta'(\mathbf{R}\Sigma^{-1}\mathbf{R}')^{-1}\mathbf{B}\delta \lesssim \sigma^2(\lambda_1, \lambda_2)tr(\Sigma^{-1}\mathbf{W}). \quad (3.203)$$

Specially, if $\delta = 0$, ie, under H_0 , $ARE(\tilde{\beta}_n(\lambda_1, \lambda_2); \tilde{\beta}_n(\lambda_1, \lambda_2)) \geq 1$. Let $ch_{\min}(\mathbf{B})$ and $ch_{\max}(\mathbf{B})$ be the smallest and largest characteristic root of $\mathbf{B}$, then using the relation

$$ch_{\min}(\mathbf{B})\Delta \leq \frac{\delta'(\mathbf{R}\Sigma^{-1}\mathbf{R}')^{-1}\mathbf{B}\delta}{\sigma^2(\lambda_1, \lambda_2)} \leq ch_{\max}(\mathbf{B})\Delta \quad (3.204)$$

$$\begin{aligned} \left\{ 1 - \frac{tr(\mathbf{B})}{tr(\Sigma^{-1}\mathbf{W})} + \frac{ch_{\max}(\mathbf{B})\Delta}{tr(\Sigma^{-1}\mathbf{W})} \right\}^{-1} &\leq ARE(\tilde{\beta}_n(\lambda_1, \lambda_2), \tilde{\beta}_n(\lambda_1, \lambda_2)) \\ &\leq \left\{ 1 - \frac{tr(\mathbf{B})}{tr(\Sigma^{-1}\mathbf{W})} + \frac{ch_{\min}(\mathbf{B})\Delta}{tr(\Sigma^{-1}\mathbf{W})} \right\}^{-1}. \end{aligned} \quad (3.205)$$

Both sides of inequality (3.205) are decreasing in Δ . Next, we investigate the ARE of $\tilde{\beta}_n^{PT}(\lambda_1, \lambda_2)$ relative to $\tilde{\beta}_n(\lambda_1, \lambda_2)$.

$$ARE(\tilde{\beta}_n^{PT}(\lambda_1, \lambda_2), \tilde{\beta}_n(\lambda_1, \lambda_2)) = [1 + h(\delta)]^{-1} \quad (3.206)$$

where

$$\begin{aligned} h(\delta) &= \sigma^{-2}(\lambda_1, \lambda_2) \left\{ \delta'(\mathbf{R}\Sigma^{-1}\mathbf{R}')^{-1}\mathbf{B}\delta \right\} \left\{ \frac{2H_{q+2}(\chi_{q,\alpha}^2, \Delta) - H_{q+4}(\chi_{q,\alpha}^2, \Delta)}{tr(\Sigma^{-1}\mathbf{W})} \right\} \\ &\quad - \left\{ \frac{tr(\mathbf{B})H_{q+2}(\chi_{q,\alpha}^2, \Delta)}{tr(\Sigma^{-1}\mathbf{W})} \right\}. \end{aligned} \quad (3.207)$$

Hence we get $ARE(\tilde{\beta}_n^{PT}(\lambda_1, \lambda_2), \tilde{\beta}_n(\lambda_1, \lambda_2)) \lesssim 1$ according to

$$\frac{\delta'(\mathbf{R}\Sigma^{-1}\mathbf{R}')^{-1}\mathbf{B}\delta}{\sigma^2(\lambda_1, \lambda_2)} \lesssim \frac{tr(\mathbf{B})H_{q+2}(\chi_{q,\alpha}^2, \Delta)}{2H_{q+2}(\chi_{q,\alpha}^2, \Delta) - H_{q+4}(\chi_{q,\alpha}^2, \Delta)}. \quad (3.208)$$

Also let us define

$$h_{\max}(\Delta) = ch_{\max}(\mathbf{B})\Delta \left\{ \frac{2H_{q+2}(\chi_{q,\alpha}^2, \Delta) - H_{q+4}(\chi_{q,\alpha}^2, \Delta)}{\text{tr}(\Sigma^{-1}\mathbf{W})} - \frac{\text{tr}(\mathbf{B})H_{q+2}(\chi_{q,\alpha}^2; \Delta)}{\text{tr}(\Sigma^{-1}\mathbf{W})} \right\} \quad (3.209)$$

$$h_{\min}(\Delta) = ch_{\min}(\mathbf{B})\Delta \left\{ \frac{2H_{q+2}(\chi_{q,\alpha}^2, \Delta) - H_{q+4}(\chi_{q,\alpha}^2, \Delta)}{\text{tr}(\Sigma^{-1}\mathbf{W})} - \frac{\text{tr}(\mathbf{B})H_{q+2}(\chi_{q,\alpha}^2; \Delta)}{\text{tr}(\Sigma^{-1}\mathbf{W})} \right\} \quad (3.210)$$

Then we obtain the inequality

$$[1 + h_{\max}(\Delta)]^{-1} \leq ARE(\hat{\beta}_n^{PT}(\lambda_1, \lambda_2), \bar{\beta}_n(\lambda_1, \lambda_2)) \leq [1 + h_{\min}(\Delta)]^{-1}. \quad (3.211)$$

We compare $\hat{\beta}_n(\lambda_1, \lambda_2)$ with $\bar{\beta}_n(\lambda_1, \lambda_2)$. The difference of the risks is given by

$$\begin{aligned} \mathfrak{R}(\bar{\beta}_n(\lambda_1, \lambda_2); \mathbf{W}) - \mathfrak{R}(\hat{\beta}_n^S(\lambda_1, \lambda_2); \mathbf{W}) = \\ \sigma^2(\lambda_1, \lambda_2) c \text{tr}(\mathbf{B}) \left\{ 2E[\chi_{q+2}^{-2}(\Delta)] - c E[\chi_{q+2}^{-4}(\Delta)] \right\} \\ - c(c+4) \left\{ \delta'(\mathbf{R}\Sigma^{-1}\mathbf{R}')^{-1} \mathbf{B} \delta E[\chi_{q+4}^{-4}(\Delta)] \right\}. \end{aligned} \quad (3.212)$$

Using the relation (3.204), the R.H.S of (3.212) is bounded from below by

$$\begin{aligned} \sigma^2(\lambda_1, \lambda_2) c \text{tr}(\mathbf{B}) \left\{ 2E[\chi_{q+2}^{-2}(\Delta)] \right. \\ \left. - c E[\chi_{q+2}^{-4}(\Delta)] \right\} - c(c+4) ch_{\max}(\mathbf{B}) \Delta E[\chi_{q+4}^{-4}(\Delta)]. \end{aligned} \quad (3.213)$$

We have also the relation

$$E[\chi_{q+2}^{-2}(\Delta)] - (q-2)E[\chi_{q+2}^{-4}(\Delta)] = \Delta E[\chi_{q+4}^{-4}(\Delta)]. \quad (3.214)$$

We may rewrite (3.213) as

$$\sigma^2(\lambda_1, \lambda_2) c \text{tr}(\mathbf{B}) \left\{ (2-g)E[\chi_{q+2}^{-2}(\Delta)] + \{(q-2)g - c\}E[\chi_{q+2}^{-4}(\Delta)] \right\}, \quad (3.215)$$

where $g = (c + 4)h$ and $h = ch_{\max}(\mathbf{B})/tr(\mathbf{B})$. Thus, for (3.212) to be non-negative for all δ it suffices to choose c such that

$$0 \leq g \leq 2 \quad \text{and} \quad (q - 2)g \geq c \geq 0 \quad (3.216)$$

Note that $g \leq 2$ is equivalent to $c \leq 2(h^{-1} - 2)$. This implies that $tr(\mathbf{B}) > 2ch_{\max}(\mathbf{B})$ is the minimum requirement for (3.212) to be non-negative. Also, from non-negativeness of (3.215) at $\delta = 0$, it follows that c has to be less than or equal to $2(q - 2)(\Rightarrow q \geq 3)$. Hence, we get

Theorem 3.5.3 *A sufficient condition for the asymptotic dominance of the STLE over UTLE [i.e. $\mathfrak{R}(\hat{\beta}_n(\lambda_1, \lambda_2); \mathbf{W}) \geq \mathfrak{R}(\hat{\beta}_n^S(\lambda_1, \lambda_2); \mathbf{W})$ for all $\delta \in R^q$ and $\mathbf{W}(p, d)$] is that the shrinkage factor c is positive and it satisfies the following inequality*

$$2E[\chi_{q+2}^{-2}(\Delta)] - c E[\chi_{q+2}^{-4}(\Delta)] - (c - 4) h \Delta E[\chi_{q+4}^{-4}(\Delta)] \geq 0 \quad \forall \Delta \geq 0 \quad (3.217)$$

which, in turn, requires that $q \geq 3$, $0 < c < 2(q - 2)$

and $tr(\mathbf{B}) > 2ch_{\max}(\mathbf{B})$.

By the same argument as in the proof of Theorem 4.3 of Sen and Saleh (1987) we get

Theorem 3.5.4 *Under the sufficient condition of Theorem 3.5.3 the PTTLE fails to dominate the STLE. Also, if for α , the level of significance of the preliminary test, we have*

$$H_{q+2}(\chi_{q,\alpha}^2; 0) \geq r[2(q - 2) - r] / [q(q - 2)]$$

$$r = (q - 2) \wedge [2/(h - 4)]$$

then STLE fails to dominate the PTTLE.

Finally, we compare $\hat{\beta}_n^{S+}(\lambda_1, \lambda_2)$ with $\hat{\beta}_n^S(\lambda_1, \lambda_2)$, the difference of the risks is given by

$$\mathfrak{R}(\hat{\beta}_n^{S+}(\lambda_1, \lambda_2); \mathbf{W}) - \mathfrak{R}(\hat{\beta}_n^S(\lambda_1, \lambda_2); \mathbf{W}) =$$

$$\begin{aligned}
& - \left\{ \sigma^2(\lambda_1, \lambda_2) \text{tr}(\mathbf{B}) E \left[(1 - c \chi_{q+2}^{-2}(\Delta))^2 I(\chi_{q+2}^2(\Delta) \leq c) \right] \right. \\
& \quad + \delta'(\mathbf{R}\Sigma^{-1}\mathbf{R})^{-1}\mathbf{B}\delta \left(E \left[(1 - c \chi_{q+4}^{-2}(\Delta)) I(\chi_{q+4}^2(\Delta) \leq c) \right] \right. \\
& \quad \left. \left. - 2E \left[(1 - c \chi_{q+2}^{-2}(\Delta)) I(\chi_{q+2}^2 \leq c) \right] \right) \right\}. \tag{3.218}
\end{aligned}$$

By using the relation

$$E[\chi_{q+2}^{-2}(\Delta)] - E[\chi_{q+4}^{-2}(\Delta)] = 2E[\chi_{q+2}^{-4}(\Delta)] \tag{3.219}$$

The R.H.S of (3.218) is always negative so

Theorem 3.5.5 *PSTLE dominates STLE.*

In general the dominance picture may be stated as

$$\Re(\hat{\beta}_n(\lambda_1, \lambda_2); \mathbf{W}) < \Re(\hat{\beta}_n^{S+}(\lambda_1, \lambda_2); \mathbf{W}) < \Re(\hat{\beta}_n^S(\lambda_1, \lambda_2); \mathbf{W}) < \Re(\hat{\beta}_n(\lambda_1, \lambda_2); \mathbf{W}) \tag{3.220}$$

while $\Re(\hat{\beta}_n^{PT}(\lambda_1, \lambda_2); \mathbf{W})$ may sit between $\Re(\hat{\beta}_n(\lambda_1, \lambda_2); \mathbf{W})$ and $\Re(\hat{\beta}_n^S(\lambda_1, \lambda_2); \mathbf{W})$ depending on the size of the test.

3.5.4 Conclusion

In this section, we proposed several alternative estimators of the regression parameters based on trimmed least squares in the linear model. It is observed that the performance of the preliminary test estimators near the null-hypothesis is generally better than other estimators depending on the size of the test while the shrinkage estimators having uniform dominance and never depend on the size of the test, as such are preferable. The positive rule is preferred over the usual shrinkage estimator. However, we notice that the shrinkage estimator depends on the rank of the matrix $\mathbf{R}$ which is q and the requirement for the application is that $q \geq 3$ while the

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preliminary test estimator does not depend on the size of q . Thus for $q \leq 2$, one is forced to use the preliminary test estimator. For optimum choice of the size of α , one may benefit using preliminary test estimator while shrinkage estimators are always preferred provided $q \geq 3$.

the following graphs represent the risk of the five estimators discussed in this section.

Figure 3.2: Risk of the five estimators

Chapter 4

Autoregression Model

4.1 Introduction

The purpose of this chapter is to present *the autoregression quantile* (ARQ) approach to some aspects of L- estimation of the parameters for the p^{th} order autoregression, AR(p)-models. This approach is similar to the previous chapter in connection with the linear models and thereby extends the statistical methodology to the most applied models with dependent observations. For the AR(p)-models with Gaussian errors, Saleh (1992) discusses various improved estimators of the parameters based on maximum likelihood estimators (MLEs) and Koul and Saleh (1993) considers the rank theory version of the same problem when the errors are *i.i.d r.v's* from some unknown *d.f* F_0 . In this chapter, we are particularly concerned with the extension of the idea of L-estimation based on a few selected ARQ's and the *trimmed least squares estimators* in the linear model to the AR(p)-models using the ideas of Koenker and Bassett (1978) and Jurečková (1984) for the linear models. Though the results are parallel, the analysis is complex to put the matter in rigorous terms.

The material of this chapter is presented in three sections. Section 4.2 contains the introduction to the AR(p)-model along with its basic properties and the least

squares estimator of the parameters. It also contains the definition of autoregression quantile (ARQ) introduced by Koul and Saleh (1995). The basic idea of the developments for the AR(p) model on autoregression quantiles (ARQ) are based on randomly weighted empirical process due to Koul and Ossainder (1994). Here, we present the asymptotic properties and the asymptotic representation of ARQ's. Section 4.3 contains the discussion of L-estimation of parameters of an autoregression model based on a few selected autoregression quantiles. In section 4.4 we present the *trimmed least squares estimators* of the autoregression parameters while section 4.5 contains estimation of the autoregression parameters under uncertain prior information.

4.2 Autoregression Quantiles

Consider the autoregression model

$$X_i = \rho_0 + \rho_1 X_{i-1} + \cdots + \rho_p X_{i-p} + Z_i \quad i = 0, \pm 1, \pm 2, \cdots \quad (4.1)$$

where $p(\geq 1)$ is a fixed integer, $\boldsymbol{\rho} := (\rho_0, \rho_1, \dots, \rho_p)' \in \mathbb{R}^{p+1}$ is the vector of parameters in (4.1) belonging to the $p+1$ dimensional Euclidean space, and Z_i 's $i = 0, \pm 1, \pm 2, \dots$ are i.i.d random variables with continuous distribution function F_0 .

We assume further the following:

- a1) $E(Z_i) = 0$ $E(Z_i^2) < \infty$
- a2) all roots of the equation $x^p - \rho_1 x^{p-1} - \cdots - \rho_p = 0$ are inside the unit circle for all $x \in \mathbb{C}$.

For a given sample, let us rewrite the model (4.1) in the form

$$X_i = \mathbf{X}_{i-1}' \boldsymbol{\rho} + Z_i, \quad i = 0, \pm 1, \pm 2, \cdots \quad (4.2)$$

where $\mathbf{X}_{i-1} = (1, X_{i-1}, \dots, X_{i-p})'$ and let $\mathbf{X}_n$ be the $n \times (p+1)$ design matrix with the i^{th} row being $\mathbf{X}_{i-1}'$, $(1 \leq i \leq n)$. Since the first element of $\mathbf{X}_{i-1}$ is one,

we define the components of $\mathbf{X}_{i-1}$ as a X_{i-j} , $j = 0, 1, \dots, p$. In the analysis of the autoregression model, under the conditions a1 and a2, the AR(p) model is stationary, causal and invertible. Details may be found in Brockwell and Davis (1991). The invertibility property of $\{X_i\}$ together with continuity of F_0 implies that the rows and columns of $\mathbf{X}_n$ are linearly independent w.p.1. This allows us to consider the use of the inverse of $\mathbf{X}_n$ in the sequel. The causal property with Theorem IV.2.3 of Hannan (1970) implies that the process $\{X_i\}$ is ergodic. We need the ergodic property to prove some of the Theorems in this chapter. The following conditions of the design matrix $\mathbf{X}_n$ are implied by assumption a1 and a2.

$$B1 \quad n^{-1/2} \max_i |X_{i-j}| = o_P(1).$$

$$B2 \quad \{X_{i-j}, 1 \leq i \leq n\} \text{ is square integrable for each } n \geq 1.$$

$$B3 \quad n^{-1} \mathbf{X}'_n \mathbf{X}_n = \boldsymbol{\Sigma}_n \xrightarrow{P} \boldsymbol{\Sigma} \quad p \cdot d.$$

Before stating the results we introduce the necessary definitions and assumptions as follows:

Let $(\Omega, \mathcal{F}, P)$ be a probability space and (Z_i, X_{i-j}, ξ_{ni}) , $1 \leq i \leq n$ be an array of trivariate r.v's defined on $(\Omega, \mathcal{F})$ such that $\{Z_i\}$ are *i.i.d.* F_0 and Z_i is independent of (X_{i-j}, ξ_{ni}) , $1 \leq i \leq n$. Furthermore, let $\{\mathcal{F}_i\}$ be an array of sub- σ fields such that $\mathcal{F}_i \subset \mathcal{F}_{i+1}$, $1 \leq i \leq n$; (X_{1-j}, ξ_{n1}) is $\mathcal{F}_1$ -measurable, the r.v's $\{Z_1, \dots, Z_{l-1}; (X_{i-j}, \xi_{ni}), 1 \leq i \leq l\}$ are $\mathcal{F}_l$ -measurable, $2 \leq l \leq n$; and Z_l is independent of $\mathcal{F}_l$, $1 \leq l \leq n$.

Now, consider the estimation of $\boldsymbol{\rho}$. It is well-known that the least squares estimator of $\boldsymbol{\rho}$ is

$$\hat{\boldsymbol{\rho}}^L = (\mathbf{X}'_n \mathbf{X}_n)^{-1} \mathbf{X}'_n \mathbf{X} \quad (4.3)$$

where $\mathbf{X} = (X_1, \dots, X_n)'$ and

$$n^{1/2}(\hat{\boldsymbol{\rho}}^L - \boldsymbol{\rho}) \xrightarrow{D} \mathcal{N}_{p+1}(\mathbf{0}, \boldsymbol{\Sigma}) \quad (4.4)$$

where $\boldsymbol{\Sigma}$ is the covariance matrix.

We define a λ -autoregression quantile (ARQ) for the model (4.1) following Koul and Saleh (1995). An λ -autoregression quantile $\hat{\rho}_n(\lambda) \in \hat{\mathcal{B}}_n(\lambda)$, $0 < \lambda < 1$, where $\hat{\mathcal{B}}_n(\lambda)$ is the set of all solutions of the minimization problem:

$$\sum g_\lambda(X_i - \mathbf{X}'_{i-1}\mathbf{b}) = \min \quad (4.5)$$

with respect to $\mathbf{b} \in \mathbb{R}^{p+1}$, where $g_\lambda(u) = \lambda u I(u > 0) - (1 - \lambda)u I(u \leq 0)$, $u \in \mathbb{R}$ and $\lambda \in (0, 1)$. Note that $\hat{\rho}_n(1/2)$ is the well-known least absolute deviation estimator (LAE) of ρ , i.e. it is the autoregression median. In this case also, as in KB (1978), $\hat{\mathcal{B}}_n(\lambda)$ is the convex hull of all solutions having the form

$$\mathbf{b}_h = \mathcal{X}_h^{-1} \mathbf{X}_h \quad (4.6)$$

where h is a subset of $\{1, 2, \dots, n\}$ of size $p + 1$ and $\mathcal{X}_h$ is sub-design matrix with rows $\mathbf{X}'_{i-1}$, $i \in h$.

We may write the above minimization problem to the following linear programming problem:

$$\begin{aligned} & \min[\lambda \mathbf{1}'_n \mathbf{r}^+ + (1 - \lambda) \mathbf{1}'_n \mathbf{r}^-], \quad \mathbf{1}_n = (1, \dots, 1)', \\ & \text{subject to} \quad \mathbf{X} = \mathcal{X}_n \mathbf{b} + \mathbf{r}^+ - \mathbf{r}^-, \quad (\mathbf{b}, \mathbf{r}^+, \mathbf{r}^-) \in \mathbb{R}^{p+1} \times \mathbb{R}_+^n \times \mathbb{R}_+^n. \end{aligned} \quad (4.7)$$

To state the results, we may rewrite the condition F1 from chapter 2:

F1 F_0 is continuous and has positive and continuous density function f_0 in the neighborhood of $Q_0(\lambda)$ for each $\lambda \in (0, 1)$.

Now we define for each $\lambda \in (0, 1)$

$$\rho(\lambda) = \rho + Q_0(\lambda) \mathbf{e}_1, \quad \mathbf{e}_1 = (1, 0, \dots, 0)'$$

and the randomly weighted process

$$\mathbf{T}_n(\mathbf{t}; \lambda) = n^{-1/2} \sum_{i=1}^n \mathbf{X}_{i-1} \{I(X_i - \mathbf{X}'_{i-1} \mathbf{t} \leq 0) - \lambda\}. \quad (4.8)$$

Then, we have

$$T_n(\boldsymbol{\rho}(\lambda) + n^{-1/2}\mathbf{t}; \lambda) = n^{-1/2} \sum_{i=1}^n \mathbf{X}_{i-1} \{I(Z_i \leq Q_0(\lambda) + n^{-1/2} \mathbf{X}'_{i-1} \mathbf{t}) - \lambda\}. \quad (4.9)$$

Also, we set $n^{-1/2} \mathbf{X}'_{i-1} \mathbf{t} \equiv \xi_{ni}$ as given in the second paragraph of this section.

Now, we state the following results which are the weaker version of the same results given by Koul and Saleh (1995).

Theorem 4.2.1 *Under the conditions a1, a2 and F1 for each $\lambda \in (0, 1)$*

$$n^{1/2}(\hat{\boldsymbol{\rho}}_n(\lambda) - \boldsymbol{\rho}(\lambda)) = -(\boldsymbol{\Sigma}_n q_0(\lambda))^{-1} T_n(\boldsymbol{\rho}(\lambda); \lambda) + o_P(1) \quad (4.10)$$

Lemma 4.2.2 *Under the conditions of Theorem 4.2.1 for each $\lambda \in (0, 1)$*

$$\sup_{\|\mathbf{t}\| \leq b} \|T_n(\boldsymbol{\rho}(\lambda) + n^{-1/2}\mathbf{t}; \lambda) - T_n(\boldsymbol{\rho}(\lambda); \lambda) - q_0(\lambda) \boldsymbol{\Sigma}_n \mathbf{t}\| = o_P(1) \quad (4.11)$$

The next Lemma will show that the autoregression quantiles are bounded in probability, i.e

Lemma 4.2.3 *Under the condition of Theorem 4.2.1, for each fixed λ .*

$$n^{1/2}(\hat{\boldsymbol{\rho}}_n(\lambda) - \boldsymbol{\rho}(\lambda)) = O_P(1). \quad (4.12)$$

The following corollaries give the asymptotic distribution of ARQs.

Corollary 4.2.4 *Assume that the conditions of Theorem 4.2.1 hold. Then,*

$$n^{1/2}(\hat{\boldsymbol{\rho}}_n(\lambda) - \boldsymbol{\rho}(\lambda)) \xrightarrow{D} \mathcal{N}_{p+1}\left(\mathbf{0}, \frac{\lambda(1-\lambda)}{q_0^2(\lambda)} \boldsymbol{\Sigma}^{-1}\right). \quad (4.13)$$

Corollary 4.2.5 *Let F_0 be continuous with continuous density f_0 in the neighborhood of $Q_0(\lambda_i)$ with $q_0(\lambda_i) > 0$, $i = 1, \dots, k$ and let conditions a1 and a2 hold. Then, for each $\lambda_1, \dots, \lambda_k$ such that $0 < \lambda_1 < \lambda_2 < \dots < \lambda_k < 1$*

$$\left(n^{1/2}(\hat{\boldsymbol{\rho}}(\lambda_1) - \boldsymbol{\rho}(\lambda_1)), \dots, n^{1/2}(\hat{\boldsymbol{\rho}}(\lambda_k) - \boldsymbol{\rho}(\lambda_k))\right) \xrightarrow{D} \mathcal{N}_{p+1 \times k}(\mathbf{0}, \boldsymbol{\Omega} \otimes \boldsymbol{\Sigma}^{-1}) \quad (4.14)$$

where $\boldsymbol{\Omega} = [\omega_{ij}]$ and $\omega_{ij} = \frac{\lambda_i - \lambda_j}{q_0(\lambda_i)q_0(\lambda_j)}$ ($i < j$).

4.3 L-estimation of the parameters of a autoregression model based on a few selected autoregression quantile

In this section we consider the L-estimation of the autoregression and scale parameters of the autoregressive model (4.1) based on the asymptotic normality of autoregression quantiles.

Consider the linear model

$$X = \mathcal{X}_n \rho + \sigma Z \quad (4.15)$$

where $Z = (Z_1, \dots, Z_n)'$ consists of $i.i.d.$ error variables with a continuous cdf $F_0(z)$ and positive pdf $f_0(z)$, $z \in \mathbb{R}$. It is well-known that the least squares method leads to the estimator of ρ given by

$$\hat{\rho}_n^L = (\mathcal{X}_n' \mathcal{X}_n)^{-1} \mathcal{X}_n' Y, \quad (4.16)$$

and an estimator of σ^2 is given by

$$S^2 = \frac{(\mathbf{X} - \mathcal{X}_n \hat{\rho}_n^L)' (\mathbf{X} - \mathcal{X}_n \hat{\rho}_n^L)}{(n - p)}. \quad (4.17)$$

In this section, we introduce the estimation of $(\rho', \sigma)'$ based on a few selected *autoregression quantiles* (ARQ). This study will extend the work of Ogawa (1951) on the estimation of location and scale model leading to simplified estimation of the autoregression and scale parameters of the model (4.15). Our aim is to propose asymptotically best linear (in ARQ) unbiased estimators (ABLUEs) of ρ and σ based on a few fixed, say k , ($p + 2 \leq k < n$), selected autoregression quantiles (ARQ) under the conditions a1 and a2.

For a given integer k ($p+2 \leq k < n$), consider the spacing vector $\lambda = (\lambda_1, \dots, \lambda_k)'$ whose components satisfy the constraint $0 < \lambda_1 < \dots < \lambda_k < 1$. Now, by the

minimization problem in the autoregression case similar to (3.4), we obtain the k ARQs given by the vector

$$\hat{\rho}_{jn}(\lambda) = (\hat{\rho}_{jn}(\lambda_1), \dots, \hat{\rho}_{jn}(\lambda_k))', \quad j = 0, 1, \dots, p. \quad (4.18)$$

Then, using the Corollary 4.2.5, one finds that the $k(p+1)$ -dimensional random variable

$$n^{1/2} \left((\hat{\rho}_{0n}(\lambda) - \rho_0 \mathbf{1}_k - \sigma \mathbf{u})', (\hat{\rho}_{1n}(\lambda) - \rho_1 \mathbf{1}_k)', \dots, (\hat{\rho}_{pn}(\lambda) - \rho_p \mathbf{1}_k)' \right)' \quad (4.19)$$

converges in law to the $k(p+1)$ dimensional normal distribution with mean $\mathbf{0}$ and covariance matrix $\sigma^2 (\Sigma^{-1} \otimes \Omega)$ where $\mathbf{u} = (u_1, \dots, u_k)'$, $u_j = Q_0(\lambda_j) = 1, \dots, k$ and $\mathbf{1} = (1, \dots, 1)'$ a k -tuple of ones, and Ω is given in (4.14).

4.3.1 Joint Estimation of Autoregression and Scale Parameters

In this subsection, we propose *asymptotically best linear (in ARQ) unbiased estimators* (ABLUEs) of $(\rho', \sigma)'$. ABLUE of $(\rho_0, \rho_1, \dots, \rho_p)'$ and σ may be obtained by minimizing the quadratic form using (4.19) as

$$\sum_{i=0}^p \sum_{j=0}^p S_{i+1,j+1} (\hat{\theta}_{in} - \theta_i)' \Omega^{-1} (\hat{\theta}_{jn} - \theta_j), \quad (4.20)$$

where S_{ij} is the ij^{th} element of the matrix Σ_n , and by setting

$$\hat{\theta}_{0n} = \hat{\rho}_0(\lambda), \quad \theta_0 = \rho_0 \mathbf{1}_k + \sigma \mathbf{u}, \quad \hat{\theta}_{jn} = \hat{\rho}_{jn}(\lambda), \quad \theta_j = \rho_j \mathbf{1}_k, \quad (4.21)$$

for $j = 0, 1, \dots, p$. The normal equations are then given by

$$\begin{bmatrix} K_1 \Sigma_n & \mathbf{m} K_3 \\ K_3 \mathbf{m}' & K_2 \end{bmatrix} \begin{bmatrix} \hat{\rho}_n \\ \hat{\sigma}_n \end{bmatrix} = \begin{bmatrix} \Sigma_n \mathbf{V} \\ \mathbf{m}' \mathbf{V}^* \end{bmatrix}; \quad \mathbf{m} = (1, \bar{X}_1, \dots, \bar{X}_p)', \quad (4.22)$$

where

$$\bar{X}_j = n^{-1} \sum_{i=1}^n X_{i-j}. \quad (4.23)$$

The explicit expressions for $\hat{\rho}_n$ and $\hat{\sigma}_n$ are given by

$$\hat{\rho}_n = \frac{1}{\Delta} [K_2 V - K_3 V^*] \quad (4.24)$$

$$\text{and } \hat{\sigma}_n = \frac{1}{\Delta} [K_1 m' V^* - K_3 m' V] \quad (4.25)$$

respectively, where

$$V = (V_0, V_1, \dots, V_p)', \quad V_j = \mathbf{1}_k' \Omega^{-1} \hat{\rho}_{jn}(\lambda), \quad (j = 0, 1, \dots, p) \quad (4.26)$$

$$V^* = (V_0^*, V_1^*, \dots, V_p^*)', \quad V_j^* = \mathbf{u}' \Omega^{-1} \hat{\rho}_{jn}(\lambda), \quad (j = 0, 1, \dots, p) \quad (4.27)$$

$$K_1 = \mathbf{1}_k' \Omega^{-1} \mathbf{1}_k, \quad K_2 = \mathbf{u}' \Omega^{-1} \mathbf{u}, \quad K_3 = \mathbf{1}_k' \Omega^{-1} \mathbf{u}, \quad (4.28)$$

$$\text{and } \Delta = K_1 K_2 - K_3^2. \quad (4.29)$$

The asymptotic covariance matrix of $(\hat{\rho}_n', \hat{\sigma}_n)'$ is given by

$$\frac{\sigma^2}{n\Delta} \begin{bmatrix} K_2 \Sigma^{-1} & -K_3 \Sigma^{-1} \mathbf{m} \\ -K_3 \mathbf{m}' \Sigma^{-1} & K_1 \end{bmatrix}. \quad (4.30)$$

Also, the explicit forms of K_1 , K_2 and K_3 are respectively given by

$$K_1 = \sum_{i=1}^{k+1} \frac{[q_0(\lambda_i) - q_0(\lambda_{i-1})]^2}{\lambda_i - \lambda_{i-1}}, \quad (4.31)$$

$$K_2 = \sum_{i=1}^{k+1} \frac{[q_0(\lambda_i) Q_0(\lambda_i) - q_0(\lambda_{i-1}) Q_0(\lambda_{i-1})]^2}{\lambda_i - \lambda_{i-1}}, \quad (4.32)$$

$$K_3 = \sum_{i=1}^{k+1} \frac{[q_0(\lambda_i) - q_0(\lambda_{i-1})] [q_0(\lambda_i) Q_0(\lambda_i) - q_0(\lambda_{i-1}) Q_0(\lambda_{i-1})]}{\lambda_i - \lambda_{i-1}} \quad (4.33)$$

with $\lambda_0 = 0$ and $\lambda_{k+1} = 1$. If F_0 is symmetric and $\lambda_i = \lambda_{k-i+1}$, $i = 1, \dots, k$, then

$K_3 = 0$ and the ABLUE of ρ and σ simplify to

$$\hat{\rho}_n = \frac{V}{K_1} \quad \text{and} \quad \hat{\sigma}_n = \frac{m' V^*}{K_2} \quad (4.34)$$

with variance and covariance given by

$$\text{Var}(\tilde{\rho}_n) = \frac{\sigma^2}{n K_1} \Sigma^{-1}, \quad \text{Var}(\tilde{\sigma}_n) = \frac{\sigma^2}{n K_2} \quad (4.35)$$

and

$$\text{Cov}(\tilde{\rho}_n, \tilde{\sigma}_n) = 0. \quad (4.36)$$

In general the joint asymptotic relative efficiency (JARE) of the ABLUE $(\tilde{\rho}'_n, \tilde{\sigma}_n)'$ relative to the LSE $(\hat{\rho}'_n^L, \hat{\sigma}_n^L)'$ may be computed as

$$\text{JARE}[\text{ABLUE} : \text{LSE}] = \frac{K_1^p \Delta}{I_{11}^p (I_{11} I_{22} - I_{12}^2)} \quad (4.37)$$

and

$$\begin{aligned} \text{ARE}[\tilde{\rho}_n, \text{LSE}] &= \frac{I_{22}^{p+1} \Delta^{p+1}}{K_2^{p+1} (I_{11} I_{22} - I_{12}^2)^{p+1}} \\ \text{and } \text{ARE}[\tilde{\sigma}_n, \text{LSE}] &= \frac{I_{11} \Delta}{K_1 (I_{11} I_{22} - I_{12}^2)} \end{aligned} \quad (4.38)$$

where $I = (I_{ij})$ is the information matrix of the location and scale distribution. Another method of estimating (ρ_0, σ) and ρ would be to consider two marginal distributions, namely, the marginals of

$$(i) \quad n^{1/2} (\hat{\rho}_{0n}(\lambda) - \rho_0 \mathbf{1}_k - \sigma \mathbf{u})' \quad (4.39)$$

$$(ii) \quad n^{1/2} \left((\hat{\rho}_{1n}(\lambda) - \rho_1 \mathbf{1}_k)', \dots, (\hat{\rho}_{pn}(\lambda) - \rho_p \mathbf{1}_k)' \right)'$$

The first one yields the estimators

$$\begin{aligned} \tilde{\rho}_{0n} &= \frac{1}{\Delta} \{K_2 V_0 - K_3 V_0^*\}, \\ \tilde{\sigma}_n &= \frac{1}{\Delta} \{K_1 V_0^* - K_3 V_0\} \end{aligned} \quad (4.40)$$

and the second one yields the estimators of ρ as

$$\dot{\rho}_n = \frac{1}{K_1} V, \quad V = (V_1, \dots, V_p)'. \quad (4.41)$$

The estimator $\hat{\sigma}_n$ of σ is non-negative but (4.25) may not be non-negative. The ARE of these estimators relative to the corresponding marginal likelihood estimators may be determined easily.

Now consider the explicit expressions for the coefficients in (4.24) and (4.40). They are

$$\mathbf{a} = \frac{1}{\Delta} \{K_2 \mathbf{1}'_k \Omega^{-1} - K_3 \mathbf{u}' \Omega^{-1}\} = (a_1, \dots, a_k)' \quad (4.42)$$

$$\mathbf{b} = \frac{1}{\Delta} \{K_1 \mathbf{u}' \Omega^{-1} - K_3 \mathbf{1}'_k \Omega^{-1}\} = (b_1, \dots, b_k)' \quad (4.43)$$

where for $i = 1, 2, \dots, k$ we have

$$a_i = \frac{q_0(\lambda_i) K_2}{\Delta} \left\{ \frac{q_0(\lambda_i) - q_0(\lambda_{i-1})}{\lambda_i - \lambda_{i-1}} - \frac{q_0(\lambda_{i+1}) - q_0(\lambda_i)}{\lambda_{i+1} - \lambda_i} \right\} - \quad (4.44)$$

$$\frac{q_0(\lambda_i) K_3}{\Delta} \left\{ \frac{q_0(\lambda_i) Q_0(\lambda_i) - q_0(\lambda_{i-1}) Q_0(\lambda_{i-1})}{\lambda_i - \lambda_{i-1}} - \frac{q_0(\lambda_{i+1}) Q_0(\lambda_{i+1}) - q_0(\lambda_i) Q_0(\lambda_i)}{\lambda_{i+1} - \lambda_i} \right\}$$

$$b_i = \frac{q_0(\lambda_i) K_1}{\Delta} \left\{ \frac{q_0(\lambda_i) Q_0(\lambda_i) - q_0(\lambda_{i-1}) Q_0(\lambda_{i-1})}{\lambda_i - \lambda_{i-1}} - \frac{q_0(\lambda_{i+1}) Q_0(\lambda_{i+1}) - q_0(\lambda_i) Q_0(\lambda_i)}{\lambda_{i+1} - \lambda_i} \right\} - \frac{q_0(\lambda_i) K_3}{\Delta} \left\{ \frac{q_0(\lambda_i) - q_0(\lambda_{i-1})}{\lambda_i - \lambda_{i-1}} - \frac{q_0(\lambda_{i+1}) - q_0(\lambda_i)}{\lambda_{i+1} - \lambda_i} \right\}. \quad (4.45)$$

Using these coefficients, we may write the estimators (4.24) and (4.25) as follows:

$$\begin{bmatrix} \hat{\rho}_{0n} \\ \hat{\rho}_{1n} \\ \vdots \\ \hat{\rho}_{pn} \\ \tilde{\sigma}_n \end{bmatrix} = \begin{bmatrix} a_1 \hat{\rho}_{0n}(\lambda_1) + \dots + a_k \hat{\rho}_{0n}(\lambda_k) \\ a_1 \hat{\rho}_{1n}(\lambda_1) + \dots + a_k \hat{\rho}_{1n}(\lambda_k) \\ \vdots \\ a_1 \hat{\rho}_{pn}(\lambda_1) + \dots + a_k \hat{\rho}_{pn}(\lambda_k) \\ \sum_{j=0}^p \bar{X}_j (b_1 \hat{\rho}_{jn}(\lambda_1) + \dots + b_k \hat{\rho}_{jn}(\lambda_k)) \end{bmatrix}. \quad (4.46)$$

Once we know the optimum spacing vector $\boldsymbol{\lambda}^0 = (\lambda_1^0, \dots, \lambda_k^0)$, we can determine $\hat{\rho}_{jn}(\lambda_i^0)$, $j = 0, 1, \dots, p$ for each $i = 1, \dots, k$, and the coefficients $(a_1, \dots, a_k)'$ and $(b_1, \dots, b_k)'$.

Tables in appendix A provide some optimum spacings when the errors are distributed as t-distribution and logistic distribution.

Now we present a numerical example using the computational algorithms of Koenker and d'Orey (1987, 1994) extended to the autoregression case for computing ARQs'.

Example 4.3.1 Consider the following 100 observations from autoregressive model of order one ($AR(1)$) given in example 7.3 by Fuller (1976). He generated these data with independent normal $(0,1)$ errors.

0.874	0.850	2.345	2.501	1.657	1.649	2.498	1.330	1.307	3.404
2.445	2.805	1.639	1.240	1.116	0.448	0.377	-0.488	-0.96	-0.579
-1.674	-0.366	-0.922	-1.174	-1.585	-0.613	0.110	0.113	-0.308	0.723
-0.257	1.051	0.803	0.116	-1.454	0.296	1.501	0.880	-0.672	0.436
0.930	1.168	1.999	1.376	1.613	2.030	0.616	0.667	0.707	1.029
-0.366	-1.420	-0.183	-0.044	-0.391	-0.095	-0.971	0.371	-1.622	-2.941
-2.814	-1.784	-2.471	-3.508	-2.979	-0.779	0.869	1.786	0.123	0.093
-0.731	-1.253	-2.213	-0.252	0.403	-0.995	-0.948	0.046	0.091	0.254
2.750	1.673	2.286	1.220	-0.256	0.252	0.325	-0.338	0.378	0.127
-2.006	-2.380	-2.024	-1.085	1.037	-0.467	-0.794	-0.493	-0.157	0.659

In this case the model is

$$X_t = \rho_0 + \rho_1 X_{t-1} + Z_t.$$

The LSE of the parameters (ρ_0, ρ_1) is $(0.0259, 0.7434)$ and the predicted values based

on this estimator are

0.026	0.676	0.658	1.769	1.885	1.258	1.252	1.883	1.015	0.998
2.556	1.844	2.111	1.244	0.948	0.856	0.359	0.306	-0.337	-0.688
-0.405	-1.219	-0.246	-0.660	-0.847	-1.227	-0.430	0.108	0.110	-0.203
0.563	-0.165	0.807	0.623	0.112	-1.055	0.246	1.142	0.680	-0.474
0.350	0.717	0.894	1.512	1.049	1.225	1.535	0.481	0.522	0.551
0.791	-0.246	-1.030	-0.110	-0.007	-0.265	-0.045	-0.696	0.302	-1.180
-2.160	-2.066	-1.300	-1.811	-2.582	-2.189	-0.553	0.672	1.354	0.117
0.095	-0.518	-0.906	-1.619	-0.161	0.325	-0.714	-0.679	0.060	0.094
0.215	2.070	1.270	1.725	0.933	-0.164	0.213	0.268	0.225	0.307
0.120	-1.465	-1.743	-1.479	-0.781	0.797	-0.321	-0.561	0.311	0.094

The autoregression median is given by (0.0807, 0.6872) with the following predictions:

0.081	0.681	0.665	1.692	1.799	1.219	1.214	1.797	0.995	0.979
2.420	1.761	2.008	1.207	0.933	0.848	0.389	0.310	0.255	-0.579
-0.317	-1.070	-0.171	-0.553	-0.726	-1.077	-0.341	0.156	0.158	-0.131
0.578	-0.096	0.803	0.633	0.160	-0.918	0.284	1.112	0.685	-0.381
0.380	0.720	0.883	1.454	1.026	1.189	1.476	0.501	0.539	0.567
0.788	-0.171	-0.895	-0.045	0.050	-0.188	0.015	-0.587	0.336	-1.031
-1.940	-1.853	-1.145	-1.617	-2.330	-1.966	-0.455	0.678	1.308	0.165
0.145	-0.422	-0.780	-1.440	-0.092	0.358	-0.603	-0.571	0.112	0.143
0.255	1.971	1.230	1.652	0.919	-0.095	0.254	0.304	-0.152	0.310
0.168	-1.298	-1.555	-1.310	-0.665	0.793	-0.240	-0.465	-0.258	0.027

Plot 4.1 shows these two results comparing with the original data as follows:

plot 4.1: AR(1) model data and predicted

plot 4.1: cont.

Example 4.3.2 *Gastwirths' Estimator:*

The Gastwirth estimator is the linear combination of $1/3$, $1/2$ and $2/3$ autoregression quantiles with the coefficient .3, .4 and .3 respectively. The estimator of parameters based on this method is (0.0396, 0.696) with the following prediction values

0.040	0.648	0.631	1.672	1.780	1.193	1.187	1.778	0.965	0.949
2.409	1.741	1.992	1.180	0.903	0.816	0.351	0.302	-0.300	-0.629
-0.363	-1.126	-0.215	-0.602	-0.778	-1.133	-0.387	0.116	0.118	-0.175
0.543	-0.139	0.771	0.598	0.120	-0.972	0.246	1.084	0.652	-0.428
0.343	0.687	0.853	1.431	0.997	1.162	1.452	0.468	0.504	0.532
0.756	-0.215	-0.949	-0.088	0.009	-0.233	-0.027	-0.636	0.298	-1.089
-2.007	-1.919	-1.202	-1.680	-2.402	-2.034	-0.503	0.644	1.283	0.125
0.104	-0.469	-0.832	-1.501	-0.136	0.320	-0.653	-0.620	0.072	0.103
0.216	1.954	1.204	1.631	0.889	-0.139	0.215	0.266	-0.196	0.303
0.128	-1.357	-1.617	-1.369	-0.716	0.761	-0.285	-0.513	-0.304	-0.070

Graphical display of these results together with autoregression median and original data is given in plot 4.2.

plot 4.2: AR(1) model data and predicted

plot 4.2: cont.

Example 4.3.3 *Trimmed LSE and Estimation based on 7 RQs.*

Estimate the parameters based on a few selected autoregression quantiles and trimmed least squares estimator are two L-estimators which we shall estimate (ρ_0, ρ_1) based on these two methods. The 10% TLSE of the parameters is (.0138, .7273) and the corresponding predicted values are

0.014	0.649	0.632	1.719	1.833	1.219	1.213	1.831	0.981	0.964
2.490	1.792	2.054	1.206	0.916	0.825	0.340	0.288	-0.341	-0.684
-0.407	-1.204	-0.252	-0.657	-0.840	-1.212	-0.432	0.091	0.096	-0.210
0.540	-0.173	0.778	0.598	0.098	-1.044	0.229	1.105	0.654	-0.475
0.331	0.690	0.863	1.468	1.015	1.187	1.490	0.462	0.499	0.528
0.762	-0.252	-1.019	-0.119	-0.018	-0.271	-0.055	-0.692	0.284	-1.166
-2.125	-2.033	-1.284	-1.783	-2.538	-2.153	-0.553	0.646	1.313	0.103
0.081	-0.518	-0.898	-1.596	-0.169	0.307	-0.710	-0.676	0.047	0.080
0.199	2.014	1.231	1.676	0.901	-0.172	0.197	0.250	-0.232	0.289
0.106	-1.445	-1.717	-1.458	-0.775	0.768	-0.326	-0.564	-0.345	-0.100

The estimation of parameters based on 7 optimal autoregression quantile is (0.0204, .731) with the following prediction values

0.020	0.659	0.642	1.735	1.849	1.232	1.226	1.816	0.993	0.976
2.509	1.808	2.071	1.219	0.927	0.836	0.348	0.296	-0.336	-0.681
-0.403	-1.203	-0.247	-0.654	-0.838	-1.211	-0.428	0.101	0.103	-0.205
0.549	-0.167	0.789	0.607	0.105	-1.042	0.237	1.118	0.664	-0.471
0.339	0.700	0.874	1.482	1.026	1.200	1.504	0.471	0.508	0.537
0.773	-0.247	-1.018	-0.113	-0.012	-0.265	-0.049	-0.689	0.292	-1.165
-2.129	-2.037	-1.284	-1.786	-2.544	-2.157	-0.549	0.656	1.326	0.110
0.088	-0.514	-0.896	-1.597	-0.164	0.315	-0.707	-0.673	0.054	0.087
0.206	2.031	1.243	1.691	0.912	-0.167	0.205	0.258	-0.227	0.297
0.113	-1.446	-1.719	-1.459	-0.773	0.778	-0.321	-0.560	-0.340	-0.094

Plot 4.3 shows these two results as well as the original data.

plot 4.3: AR(1) model data and predicted

plot 4.3: cont.

4.4 Trimmed Least Squares Estimation of AR(p) Parameters

In Section 4.3, we studied the L-estimators of the autoregression parameters under known error distribution based on a few selected autoregression quantiles. In this section, we shall consider another L-estimator of ρ when F_0 is unknown. It is the *Trimmed Least Squares Estimator* (TLSE) of the autoregression parameters. To obtain the TLSE of ρ , we consider the autoregression model (4.2) and define an $n \times n$ diagonal matrix A whose elements are

$$a_{ii} = \begin{cases} 0, & \text{if } X_i \leq X'_{i-1}\hat{\rho}_n(\lambda_1) \text{ or } X_i > X'_{i-1}\hat{\rho}_n(\lambda_2) \\ 1, & \text{otherwise} \end{cases} \quad (4.47)$$

where $\hat{\rho}_n(\lambda_1)$ and $\hat{\rho}_n(\lambda_2)$ are the autoregression quantiles of orders λ_1 and λ_2 respectively for $0 < \lambda_1 < 1/2 < \lambda_2 < 1$. We obtain the TLSE of ρ by minimizing

$$(X - X_n\rho)'A(X - X_n\rho) \quad (4.48)$$

with respect to ρ which yields

$$\bar{\rho}_n(\lambda_1, \lambda_2) = (X'_nAX_n)^{-1}X'_nAX. \quad (4.49)$$

We are going to study the asymptotic property of the above estimator under the same condition as in Section 4.2 and via asymptotic linearity. The following main theorem of this section represents the TLSE of ρ .

Theorem 4.4.1 *Let F_0 be continuous with positive and continuous density f_0 in the neighborhood of $Q_0(\lambda_1)$ and $Q_0(\lambda_2)$. Then, if conditions a1 and a2 hold, for each λ_1 and λ_2 such that $0 < \lambda_1 < 1/2 < \lambda_2 < 1$*

$$i) \quad n^{1/2}(\bar{\rho}_n(\lambda_1, \lambda_2) - \rho) =$$

$$n^{-1/2}\{(\lambda_2 - \lambda_1)\Sigma_n\}^{-1} \sum_{i=1}^n \mathbf{X}_{i-1}(\phi(Z_i) - \gamma) + o_P(1) \quad (4.50)$$

$$ii) \quad n^{1/2}(\bar{\rho}_n(\lambda_1, \lambda_2) - \rho) \xrightarrow{D} \mathcal{N}_{r+1}(\mathbf{0}, \sigma^2(\lambda_1, \lambda_2)\Sigma^{-1}) \quad (4.51)$$

where $\phi(Z_i)$, γ and $\sigma^2(\lambda_1, \lambda_2)$ are defined in (2.97) and (2.98) respectively.

PROOF

This Theorem will be proved with the aid of two Lemmas. First, we define the following processes

$$\tilde{T}_n(\mathbf{t}; \lambda) = n^{-1/2} \sum_{i=1}^n \mathbf{X}_{i-1} Z_i I(Z_i \leq Q_0(\lambda) + n^{-1/2} \mathbf{X}'_{i-1} \mathbf{t}) \quad (4.52)$$

and

$$\tilde{m}_n(\mathbf{t}; \lambda) = n^{-1/2} \sum_{i=1}^n \mathbf{X}_{i-1} \mu(Q_0(\lambda) + n^{-1/2} \mathbf{X}'_{i-1} \mathbf{t}) \quad (4.53)$$

where

$$\mu(x) = E(Z_i I(Z_i \leq x) | \mathcal{F}_i). \quad (4.54)$$

Then, we define the following centered process

$$\begin{aligned} \tilde{T}_n^0(\mathbf{t}; \lambda) &= \tilde{T}_n(\mathbf{t}; \lambda) - \tilde{m}_n(\mathbf{t}; \lambda) \\ &= n^{-1/2} \sum_{i=1}^n \mathbf{X}_{i-1} \left\{ Z_i I(Z_i \leq Q_0(\lambda) + n^{-1/2} \mathbf{X}'_{i-1} \mathbf{t}) \right. \\ &\quad \left. - \mu(Q_0(\lambda) + n^{-1/2} \mathbf{X}'_{i-1} \mathbf{t}) \right\}. \end{aligned} \quad (4.55)$$

Lemma 4.4.2 *Under the condition of Theorem 4.4.1 for each $0 < b < \infty$*

$$\sup_{\|\mathbf{t}\| \leq b} \left\| \tilde{T}_n(\mathbf{t}; \lambda) - \tilde{T}_n(\mathbf{0}; \lambda) - q_0(\lambda) Q_0(\lambda) \Sigma_n \mathbf{t} \right\| = o_P(1). \quad (4.56)$$

PROOF

Consider $\tilde{T}_{nj}(\mathbf{t}; \lambda)$ as the j^{th} component of $\tilde{T}_n(\mathbf{t}; \lambda)$. Then, we have

$$\sup_{\|\mathbf{t}\| \leq b} \left| \tilde{T}_{nj}(\mathbf{t}; \lambda) - \tilde{T}_{nj}(\mathbf{0}; \lambda) - q_0(\lambda) Q_0(\lambda) \Sigma_n \mathbf{t} \right|$$

$$\begin{aligned}
&\leq \sup_{\|\mathbf{t}\| \leq b} \left| \hat{T}_{nj}^0(\mathbf{t}; \lambda) - \hat{T}_{nj}^0(\mathbf{0}; \lambda) \right| \\
&\quad + \sup_{\|\mathbf{t}\| \leq b} \left| \hat{m}_n(\mathbf{t}; \lambda) - \hat{m}_n(\mathbf{0}; \lambda) - q_0(\lambda)Q_0(\lambda)\Sigma_n \mathbf{t} \right| \\
&= J_1 + J_2.
\end{aligned} \tag{4.57}$$

First, note that

$$J_2 \leq n^{-1/2} \sum_{i=1}^n |X_{i-j}| \sup_{\|\mathbf{t}\| \leq b} \left| \mu(Q_0(\lambda) + n^{-1/2}\mathbf{X}'_{i-1}\mathbf{t}) - \mu(Q_0(\lambda)) - q_0(\lambda)Q_0(\lambda)n^{-1/2}\mathbf{X}'_{i-1}\mathbf{t} \right|. \tag{4.58}$$

Then, by mean value Theorem there is a $\Delta(\mathbf{t}) \in (0, n^{-1/2}\mathbf{X}'_{i-1}\mathbf{t})$ such that if $\mathbf{t} \in \mathcal{N}(b)$, $|\Delta(\mathbf{t})| \leq n^{-1/2}\|\mathbf{X}_{i-1}\| b = \delta_n$. Therefore,

$$\begin{aligned}
J_2 &\leq b n^{-1} \sum_{i=1}^n |X_{i-j}| \|\mathbf{X}_{i-1}\| \sup_{\|\mathbf{t}\| \leq b} \left| (Q_0(\lambda) + \Delta(\mathbf{t}))f_0(Q_0(\lambda) + \Delta(\mathbf{t})) \right. \\
&\quad \left. - Q_0(\lambda)f_0(Q_0(\lambda)) \right| \\
&\leq b n^{-1} \sum_{i=1}^n |X_{i-j}| \|\mathbf{X}_{i-1}\| \left\{ \sup_{|\Delta(\mathbf{t})| \leq \delta_n} \left| Q_0(\lambda)(f_0(Q_0(\lambda) + \Delta(\mathbf{t})) - f_0(Q_0(\lambda))) \right| \right. \\
&\quad \left. + \sup_{|\Delta(\mathbf{t})| \leq \delta_n} \left| \Delta(\mathbf{t})f_0(Q_0(\lambda) + \Delta(\mathbf{t})) \right| \right\} \\
&= b O_P(1) \{K \cdot o(1) + o_P(1) O(1)\} \\
&= o_P(1) \quad \text{as } n \rightarrow \infty
\end{aligned} \tag{4.59}$$

by F1 and C-S inequality. To prove Lemma 4.4.2 it is enough to show that $J_1 = o_P(1)$ where

$$J_1 = \sup_{\|\mathbf{t}\| \leq b} \left| \hat{T}_{nj}^0(\mathbf{t}; \lambda) - \hat{T}_{nj}^0(\mathbf{0}; \lambda) \right|. \tag{4.60}$$

First, let $\mathbf{t}$ be fixed.

$$\begin{aligned}
&\left| \hat{T}_{nj}^0(\mathbf{t}; \lambda) - \hat{T}_{nj}^0(\mathbf{0}; \lambda) \right| \\
&= \left| n^{-1/2} \sum_{i=1}^n X_{i-j} \left\{ Z_i I(Q_0(\lambda) < Z_i \leq Q_0(\lambda) + n^{-1/2}\mathbf{X}'_{i-1}\mathbf{t}) \right. \right. \\
&\quad \left. \left. - \mu(Q_0(\lambda) + n^{-1/2}\mathbf{X}'_{i-1}\mathbf{t}) + \mu(Q_0(\lambda)) \right\} \right|
\end{aligned}$$

$$= \left| n^{-1/2} \sum_{i=1}^n X_{i-j} \{ \hat{V}_i(\mathbf{t}) - \hat{\nu}_{ni}(\mathbf{t}) \} \right| \quad (4.61)$$

where

$$\begin{aligned} \hat{V}_i(\mathbf{t}) &= Z_i I(Q_0(\lambda) < Z_i \leq Q_0(\lambda) + n^{-1/2} \mathbf{X}'_{i-1} \mathbf{t}) \\ \hat{\nu}_{ni}(\mathbf{t}) &= \mu(Q_0(\lambda) + n^{-1/2} \mathbf{X}'_{i-1} \mathbf{t}) - \mu(Q_0(\lambda)). \end{aligned} \quad (4.62)$$

Therefore,

$$\begin{aligned} & P \left\{ \left| n^{-1/2} \sum_{i=1}^n X_{i-j} \{ \hat{V}_i(\mathbf{t}) - \hat{\nu}_{ni}(\mathbf{t}) \} \right|^2 > \epsilon^2 \right\} \\ & \leq \frac{1}{\epsilon^2} \frac{1}{n} E \left(\sum_{i=1}^n X_{i-j} \{ \hat{V}_i(\mathbf{t}) - \hat{\nu}_{ni}(\mathbf{t}) \} \right)^2 \\ & = \frac{1}{\epsilon^2} \frac{1}{n} E \left(\sum_{i=1}^n X_{i-j}^2 \{ \hat{V}_i(\mathbf{t}) - \hat{\nu}_{ni}(\mathbf{t}) \}^2 \right) + 2E \left(\sum_{\substack{i,j=1 \\ i < k}}^n X_{i-j} X_{k-j} \{ \hat{V}_i(\mathbf{t}) - \hat{\nu}_{ni}(\mathbf{t}) \} \{ \hat{V}_k(\mathbf{t}) - \hat{\nu}_{nk}(\mathbf{t}) \} \right) \\ & = \frac{1}{\epsilon^2} \frac{1}{n} E \left(\sum_{i=1}^n X_{i-j}^2 \{ E(\hat{V}_i(\mathbf{t}) | \mathcal{F}_i)^2 - \hat{\nu}_{ni}(\mathbf{t})^2 \} \right) \\ & = \frac{1}{\epsilon^2} \frac{1}{n} \sum_{i=1}^n E(X_{i-j}^2) \left(\left\{ \int_{Q_0(\lambda)}^{Q_0(\lambda) + n^{-1/2} \mathbf{X}'_{i-1} \mathbf{t}} Z^2 dF_0(Z) \right\} - \left\{ \int_{Q_0(\lambda)}^{Q_0(\lambda) + n^{-1/2} \mathbf{X}'_{i-1} \mathbf{t}} Z dF_0(Z) \right\}^2 \right) \quad (4.63) \end{aligned}$$

Now, the first term of R.H.S of (4.63) may be written as

$$\begin{aligned} & \frac{1}{\epsilon^2} \frac{1}{n} \sum_{i=1}^n E(X_{i-j}^2) \left\{ \int_{Q_0(\lambda)}^{Q_0(\lambda) + n^{-1/2} \mathbf{X}'_{i-1} \mathbf{t}} Z^2 dF_0(Z) I(n^{-1/2} \mathbf{X}'_{i-1} \mathbf{t} > \delta) \right. \\ & \quad \left. - \int_{Q_0(\lambda)}^{Q_0(\lambda) + n^{-1/2} \mathbf{X}'_{i-1} \mathbf{t}} Z^2 dF_0(Z) I(n^{-1/2} \mathbf{X}'_{i-1} \mathbf{t} \leq \delta) \right\} \\ & \leq \epsilon \quad \text{by a1.} \end{aligned} \quad (4.64)$$

By similar argument we get the same result for the second term. Then, we have

$J_1 = o_P(1)$ for fixed $\mathbf{t}$. For the tightness, we use the compactness property of $\mathcal{N}(b)$.

So, it is necessary to show that

$$\forall \epsilon > 0 \exists \delta > 0 \ni \forall \mathbf{s} \in \mathcal{N}(b)$$

$$\limsup_n P \left(\sup_{\|\mathbf{t} - \mathbf{s}\| \leq \delta} \left| \hat{T}_{nj}^0(\mathbf{t}; \lambda) - \hat{T}_{nj}^0(\mathbf{s}; \lambda) \right| > 2\epsilon \right) < \epsilon. \quad (4.65)$$

In this case, we have

$$\begin{aligned}
 & \left| \tilde{T}_{nj}^0(\mathbf{t}; \lambda) - \tilde{T}_{nj}^0(\mathbf{s}; \lambda) \right| \leq \\
 & n^{-1/2} \sum_{i=1}^n |X_{i-1}| \left| I(Z_i \leq Q_0(\lambda) + n^{-1/2} \mathbf{X}'_{i-1} \mathbf{t}) - I(Z_i \leq Q_0(\lambda) + n^{-1/2} \mathbf{X}'_{i-1} \mathbf{s}) \right| \\
 & + n^{-1/2} \sum_{i=1}^n |X_{i-1}| \left| \mu(Q_0(\lambda) + n^{-1/2} \mathbf{X}'_{i-1} \mathbf{t}) - \mu(Q_0(\lambda) + n^{-1/2} \mathbf{X}'_{i-1} \mathbf{s}) \right|. \quad (4.66)
 \end{aligned}$$

By the inequality (3.93) the above expression is bounded above by

$$\begin{aligned}
 & n^{-1/2} \sum_{i=1}^n |X_{i-1}| \left\{ I(Z_i \leq Q_0(\lambda) + n^{-1/2} \mathbf{X}'_{i-1} \mathbf{s} + n^{-1/2} \|\mathbf{X}_{i-1}\| \delta) \right. \\
 & \quad \left. - I(Z_i \leq Q_0(\lambda) + n^{-1/2} \mathbf{X}'_{i-1} \mathbf{s} - n^{-1/2} \|\mathbf{X}_{i-1}\| \delta) \right\} \\
 & + n^{-1/2} \sum_{i=1}^n |X_{i-1}| \left| \mu(Q_0(\lambda) + n^{-1/2} \mathbf{X}'_{i-1} \mathbf{t}) - \mu(Q_0(\lambda) + n^{-1/2} \mathbf{X}'_{i-1} \mathbf{s}) \right| \\
 & = J_{11} + J_{12}. \quad (4.67)
 \end{aligned}$$

By mean value Theorem, for sufficiently small δ and as $n \rightarrow \infty$ we see that

$\sup_{\|\mathbf{t}-\mathbf{s}\| \leq \delta} J_{12} = o_P(1)$. As we see, J_{11} does not depend on $\mathbf{t}$ and so it is enough to show that $J_{11} = o_P(1)$ where

$$\begin{aligned}
 J_{11} &= n^{-1/2} \sum_{i=1}^n |X_{i-1}| \left\{ \left[|Z_i| I(Z_i \leq Q_0(\lambda) + n^{-1/2} \mathbf{X}'_{i-1} \mathbf{s} + n^{-1/2} \|\mathbf{X}_{i-1}\| \delta) \right. \right. \\
 & \quad \left. \left. - \mu^*(Q_0(\lambda) + n^{-1/2} \mathbf{X}'_{i-1} \mathbf{s} + n^{-1/2} \|\mathbf{X}_{i-1}\| \delta) \right] \right. \\
 & \quad \left. - \left[|Z_i| I(Z_i \leq Q_0(\lambda) + n^{-1/2} \mathbf{X}'_{i-1} \mathbf{s} - n^{-1/2} \|\mathbf{X}_{i-1}\| \delta) \right. \right. \\
 & \quad \left. \left. - \mu^*(Q_0(\lambda) + n^{-1/2} \mathbf{X}'_{i-1} \mathbf{s} - n^{-1/2} \|\mathbf{X}_{i-1}\| \delta) \right] \right\} \\
 & + n^{-1/2} \sum_{i=1}^n |X_{i-1}| \left\{ \mu^*(Q_0(\lambda) + n^{-1/2} \mathbf{X}'_{i-1} \mathbf{s} + n^{-1/2} \|\mathbf{X}_{i-1}\| \delta) \right. \\
 & \quad \left. - \mu^*(Q_0(\lambda) + n^{-1/2} \mathbf{X}'_{i-1} \mathbf{s} - n^{-1/2} \|\mathbf{X}_{i-1}\| \delta) \right\} \\
 & = J_{111} + J_{112} \quad \text{respectively.} \quad (4.68)
 \end{aligned}$$

where

$$\mu^*(x) = E(|Z_i|I(Z_i \leq x)) \quad \text{and} \quad \mu^{*'}(x) = |x|f_0(x). \quad (4.69)$$

Again, by the mean value Theorem we obtain $J_{112} = o_P(1)$. For J_{111} , we use the same idea of proving $J_1 = o_P(1)$ for fixed $\mathbf{t}$. Therefore, the proof of Lemma 4.4.1 is complete.

Before we state the next Lemma, it is necessary to introduce the following processes, namely

$$U_{jk}(\mathbf{t}) = n^{-1} \sum_{i=1}^n X_{i-j} X_{i-k} I(Z_i \leq Q_0(\lambda) + n^{-1/2} \mathbf{X}'_{i-1} \mathbf{t}). \quad (4.70)$$

Then we have following lemma.

Lemma 4.4.3 *Under the conditions of Theorem 4.4.1, for each $0 < b < \infty$ and $\lambda \in (0, 1)$*

$$\sup_{\|\mathbf{t}\| \leq b} |U_{jk}(\mathbf{t}) - \lambda \sigma_{jk}| \xrightarrow{P} 0 \quad (4.71)$$

where σ_{jk} is the $j \times k^{\text{th}}$ element of Σ .

PROOF

By definition of $U_{jk}(\mathbf{t})$ we have

$$\begin{aligned} & \sup_{\|\mathbf{t}\| \leq b} |U_{jk}(\mathbf{t}) - \lambda \sigma_{jk}| \\ &= \sup_{\|\mathbf{t}\| \leq b} \left| n^{-1} \sum_{i=1}^n X_{i-j} X_{i-k} I(Z_i \leq Q_0(\lambda) + n^{-1/2} \mathbf{X}'_{i-1} \mathbf{t}) - \sigma_{jk} \lambda \right| \\ &\leq \sup_{\|\mathbf{t}\| \leq b} \left| n^{-1} \sum_{i=1}^n X_{i-j} X_{i-k} \{I(Z_i \leq Q_0(\lambda) + n^{-1/2} \mathbf{X}'_{i-1} \mathbf{t}) - F_0(Q_0(\lambda) + n^{-1/2} \mathbf{X}'_{i-1} \mathbf{t})\} \right| \\ &\quad + \sup_{\|\mathbf{t}\| \leq b} \left| n^{-1} \sum_{i=1}^n X_{i-j} X_{i-k} F_0(Q_0(\lambda) + n^{-1/2} \mathbf{X}'_{i-1} \mathbf{t}) - \sigma_{jk} \lambda \right| \\ &= I_1 + I_2. \end{aligned} \quad (4.72)$$

So, we have to show that $I_1 = o_P(1) = I_2$. First, consider I_1 for fixed $\mathbf{t} \in \mathcal{N}(b)$.

$$I_1 \leq \left| n^{-1} \sum_{i=1}^n X_{i-j} X_{i-k} \{I(Z_i \leq Q_0(\lambda) + n^{-1/2} \mathbf{X}'_{i-1} \mathbf{t}) - F_0(Q_0(\lambda) + n^{-1/2} \mathbf{X}'_{i-1} \mathbf{t})\} \right|$$

$$\begin{aligned}
& -I(Z_i \leq Q_0(\lambda)) + F_0(Q_0(\lambda)) \Big| \\
& + \Big| n^{-1} \sum_{i=1}^n X_{i-j} X_{i-k} \{I(Z_i \leq Q_0(\lambda)) - F_0(Q_0(\lambda))\} \Big| \\
& = I_{11} + I_{12}.
\end{aligned} \tag{4.73}$$

Let us consider I_{12} first. I_{12} in the absolute value sign is the sum of stationary and ergodic random variables with mean zero and we have

$$E\left(\left| X_{i-j} X_{i-k} \{I(Z_i \leq Q_0(\lambda)) - F_0(Q_0(\lambda))\} \right| \right) \leq E |X_{i-j} X_{i-k}| < \infty. \tag{4.74}$$

Then, by Theorem IV.2.2 of Hannan (1970) we have

$$n^{-1} \sum_{i=1}^n X_{i-j} X_{i-k} \{I(Z_i \leq Q_0(\lambda)) - F_0(Q_0(\lambda))\} \rightarrow 0 \quad \text{a.s.} \tag{4.75}$$

For I_{11} , by the C-S inequality

$$\begin{aligned}
I_{11}^2 & \leq n^{-1} \sum_{i=1}^n X_{i-j}^2 \cdot n^{-1} \sum_{i=1}^n X_{i-k}^2 \left\{ I(Z_i \leq Q_0(\lambda) + n^{-1/2} \mathbf{X}'_{i-1} \mathbf{t}) \right. \\
& \quad \left. - F_0(Q_0(\lambda) + n^{-1/2} \mathbf{X}'_{i-1} \mathbf{t}) - I(Z_i \leq Q_0(\lambda)) + F_0(Q_0(\lambda)) \right\}^2 \\
& = O_P(1) \cdot o_P(1).
\end{aligned} \tag{4.76}$$

The first term of R.H.S of (4.76) is $O_P(1)$ by B3. The second term is $o_P(1)$ by Theorem 2.23 of Hall and Heyde (1980). But we have to check the conditions of this theorem. For this reason let us define

$$\begin{aligned}
U_n^2 & = \sum_{i=1}^n (n^{-1/2} X_{i-j})^2 \left\{ I(Q_0(\lambda) < Z_i \leq Q_0(\lambda) + n^{-1/2} \mathbf{X}'_{i-1} \mathbf{t}) \right. \\
& \quad \left. - F_0(Q_0(\lambda) + n^{-1/2} \mathbf{X}'_{i-1} \mathbf{t}) + F_0(Q_0(\lambda)) \right\}^2 \\
& = \sum_{i=1}^n W_{ni}^2
\end{aligned} \tag{4.77}$$

and

$$V_n^2 = \sum_{i=1}^n E(W_{ni}^2 | \mathcal{F}_i). \tag{4.78}$$

The first condition is

$$\sup_n P(V_n^2 > \eta) \rightarrow 0 \text{ as } \eta \rightarrow \infty. \quad (4.79)$$

But by definition of V_n^2 we have

$$\begin{aligned} V_n^2 &= n^{-1} \sum_{i=1}^n X_{i-j}^2 \left[\left\{ F_0(Q_0(\lambda) + n^{-1/2} \mathbf{X}'_{i-1} \mathbf{t}) - F_0(Q_0(\lambda)) \right\} \right. \\ &\quad \left. \times \left(1 - \left\{ F_0(Q_0(\lambda) + n^{-1/2} \mathbf{X}'_{i-1} \mathbf{t}) - F_0(Q_0(\lambda)) \right\} \right) \right] \\ &\leq n^{-1} \sum_{i=1}^n X_{i-j}^2 \left\{ F_0(Q_0(\lambda) + n^{-1/2} \mathbf{X}'_{i-1} \mathbf{t}) - F_0(Q_0(\lambda)) \right\} \\ &= n^{-1} \sum_{i=1}^n X_{i-j}^2 (n^{-1/2} \mathbf{X}'_{i-1} \mathbf{t} f_0(c)) \\ &\leq n^{-1} \sum_{i=1}^n X_{i-j}^2 n^{-1/2} \|\mathbf{X}_{i-1}\| b f_0(c) \\ &= O_P(1) \cdot o_P(1) \cdot b f_0(c) \\ &= o_P(1). \end{aligned} \quad (4.80)$$

Therefore, the first condition holds. The second one is the conditional Lindeberg condition, i.e.

$$\forall \epsilon > 0 \quad \sum_{i=1}^n E\{W_{ni}^2 I(|W_{ni}| > \epsilon) \mid \mathcal{F}_i\} \xrightarrow{P} 0 \quad (4.81)$$

This condition also holds by the definition of W_{ni} . Then, by Theorem 2.23 of Hall and Heyde (1980) we have

$$|U_n^2 - V_n^2| \xrightarrow{P} 0 \quad (4.82)$$

which implies that $I_1 = o_P(1)$. For tightness, we use the compactness property of $\mathcal{N}(b)$ and nondecreasing property of the indicator function. Then we change the process with respect to $\mathbf{t}$ to a random variable with fixed $\mathbf{s}$ and we use a similar method as we did for fixed $\mathbf{t}$ in this lemma. As a result we get $I_1 = o_P(1)$.

Now, consider I_2 ,

$$I_2 = \sup_{\|\mathbf{t}\| \leq b} \left| n^{-1} \sum_{i=1}^n X_{i-j} X_{i-k} F_0(Q_0(\lambda) + n^{-1/2} \mathbf{X}'_{i-1} \mathbf{t}) - \sigma_{j,k} \lambda \right|$$

$$\begin{aligned}
&\leq \sup_{\|t\| \leq b} \left| n^{-1} \sum_{i=1}^n X_{i-j} X_{i-k} \{ F_0(Q_0(\lambda) + n^{-1/2} \mathbf{X}'_{i-1} \mathbf{t}) - F_0(Q_0(\lambda)) \} \right| \\
&\quad + \left| n^{-1} \sum_{i=1}^n X_{i-j} X_{i-k} - \sigma_{jk} \right| F_0(Q_0(\lambda)) \\
&= o_P(1) + o_P(1) = o_P(1) \quad \text{by C-S inequality and B3.} \tag{4.83}
\end{aligned}$$

Therefore, the proof of Lemma 4.4.3 is complete.

Now we define

$$U_{jk}^*(\mathbf{t}) = n^{-1} \sum_{i=1}^n X_{i-j} X_{i-k} I(Q_0(\lambda_1) + n^{-1/2} \mathbf{X}'_{i-1} \mathbf{t} < Z_i \leq Q_0(\lambda_2) + n^{-1/2} \mathbf{X}'_{i-1} \mathbf{t}) \tag{4.84}$$

for each $\lambda_1, \lambda_2 \ni 0 < \lambda_1 < \lambda_2 < 1$. Then, we have

$$\sup_{\|t\| \leq b} |U_{jk}^*(\mathbf{t}) - (\lambda_2 - \lambda_1) \sigma_{jk}| = o_P(1). \tag{4.85}$$

We apply this result to prove Theorem 4.4.1

PROOF of Theorem 4.4.1

(i): By using the result of Lemma 4.4.3 in the matrix form we have

$$\begin{aligned}
n^{-1} \mathbf{X}'_n \mathbf{A} \mathbf{X}_n &= n^{-1} \sum_{i=1}^n \mathbf{X}_{i-1} \mathbf{X}'_{i-1} I(Q_0(\lambda_1) + n^{-1/2} \mathbf{X}'_{i-1} \mathbf{t} < Z_i \\
&\leq Q_0(\lambda_2) + n^{-1/2} \mathbf{X}'_{i-1} \mathbf{t}) \\
&= (\lambda_2 - \lambda_1) \boldsymbol{\Sigma} + o_P(1) \quad \text{as } n \rightarrow \infty. \tag{4.86}
\end{aligned}$$

By the structure of the matrix $\mathbf{A}$,

$$\begin{aligned}
n^{-1/2} \mathbf{X}'_n \mathbf{A} \mathbf{Z} &= n^{-1/2} \sum_{i=1}^n \mathbf{X}_{i-1} \left\{ I(Z_i \leq Q_0(\lambda_2) + \mathbf{X}'_{i-1} (\hat{\boldsymbol{\rho}}_n(\lambda_2) - \boldsymbol{\rho}(\lambda_2))) \right. \\
&\quad \left. - I(Z_i \leq Q_0(\lambda_1) + \mathbf{X}'_{i-1} (\hat{\boldsymbol{\rho}}_n(\lambda_1) - \boldsymbol{\rho}(\lambda_1))) \right\} Z_i \\
&= \hat{\mathbf{T}}_n(\mathbf{t}_2; \lambda_2) - \hat{\mathbf{T}}_n(\mathbf{t}_1; \lambda_1) \tag{4.87}
\end{aligned}$$

where

$$\mathbf{t}_i = n^{1/2} (\hat{\boldsymbol{\rho}}_n(\lambda_i) - \boldsymbol{\rho}(\lambda_i)) \quad i = 1, 2. \tag{4.88}$$

But

$$\begin{aligned}
 \hat{\mathbf{T}}_n(\mathbf{t}_2; \lambda_2) - \hat{\mathbf{T}}_n(\mathbf{t}_1; \lambda_1) &= [\hat{\mathbf{T}}_n(\mathbf{t}_2; \lambda_2) - \hat{\mathbf{T}}_n(\mathbf{0}; \lambda_2) - q_0(\lambda_2)Q_0(\lambda_2)\boldsymbol{\Sigma}_n\mathbf{t}_2] \\
 &\quad - [\hat{\mathbf{T}}_n(\mathbf{t}_1; \lambda_1) - \hat{\mathbf{T}}_n(\mathbf{0}; \lambda_1) - q_0(\lambda_1)Q_0(\lambda_1)\boldsymbol{\Sigma}_n\mathbf{t}_1] \\
 &\quad + [\hat{\mathbf{T}}_n(\mathbf{0}; \lambda_2) - \hat{\mathbf{T}}_n(\mathbf{0}; \lambda_1) + q_0(\lambda_2)Q_0(\lambda_2)\boldsymbol{\Sigma}_n\mathbf{t}_2 \\
 &\quad \quad - q_0(\lambda_1)Q_0(\lambda_1)\boldsymbol{\Sigma}_n\mathbf{t}_1]. \quad (4.89)
 \end{aligned}$$

By Lemma 4.2.2 the first and second term of (4.89) are $o_P(1)$. Also, by substituting $\mathbf{t}_i$ with its representation given in Theorem 4.2.1 we have

$$\begin{aligned}
 n^{-1/2}\mathbf{X}'_n\mathbf{A}\mathbf{Z} &= \mathbf{T}_n(\boldsymbol{\rho}(\lambda_1); \lambda_1)Q_0(\lambda_1) - \mathbf{T}_n(\boldsymbol{\rho}(\lambda_2); \lambda_2)Q_0(\lambda_2) \\
 &\quad + n^{-1/2}\sum_{i=1}^n \mathbf{X}_{i-1}Z_i\{I(Q_0(\lambda_1) < Z_i \leq Q_0(\lambda_2))\} + o_P(1) \quad (4.90)
 \end{aligned}$$

and then by definition of $\mathbf{T}_n(\boldsymbol{\rho}(\lambda); \lambda)$ and using the fact $I(Z_i \leq a) = 1 - I(Z_i > a)$ we have

$$\begin{aligned}
 n^{-1/2}\mathbf{X}'_n\mathbf{A}\mathbf{Z} &= n^{-1/2}\sum_{i=1}^n \mathbf{X}'_{i-1}\left\{I(Z_i \leq Q_0(\lambda_1))Q_0(\lambda_1) + Z_i I(Q_0(\lambda_1) < Z_i \leq Q_0(\lambda_2))\right. \\
 &\quad \left.+ I(Z_i > Q_0(\lambda_2))Q_0(\lambda_2) - \lambda_1 Q_0(\lambda_1) - (1 - \lambda_2)Q_0(\lambda_2)\right\} + o_P(1) \\
 &= n^{-1/2}\sum_{i=1}^n \mathbf{X}_{i-1}\{\phi(Z_i) - \gamma\} + o_P(1) \quad (4.91)
 \end{aligned}$$

where $\phi(Z_i)$ and γ are given in (2.97). Finally, we have

$$(\mathbf{X}'_n\mathbf{A}\mathbf{X}_n)^{-1}\mathbf{X}'_n\mathbf{A}\mathbf{Z} = (\mathbf{X}'_n\mathbf{A}\mathbf{X}_n)^{-1}\mathbf{X}'_n\mathbf{A}\mathbf{X} - \boldsymbol{\rho} \quad (4.92)$$

or

$$(n^{-1}\mathbf{X}'_n\mathbf{A}\mathbf{X}_n)^{-1}n^{-1/2}\mathbf{X}'_n\mathbf{A}\mathbf{Z} = n^{1/2}(\bar{\boldsymbol{\rho}}_n(\lambda_1, \lambda_2) - \boldsymbol{\rho}) \quad (4.93)$$

and as a result

$$n^{1/2}(\bar{\boldsymbol{\rho}}_n(\lambda_1, \lambda_2) - \boldsymbol{\rho}) = ((\lambda_2 - \lambda_1)\boldsymbol{\Sigma}_n)^{-1}n^{-1/2}\sum_{i=1}^n \mathbf{X}_{i-1}\{\phi(Z_i) - \gamma\} + o_P(1). \quad (4.94)$$

Therefore, the proof of the first part of Theorem 4.4.1 is complete.

PROOF of Theorem 4.4.1

(ii): The result of the second part is obtained by using the result of part (i), taking F_0 symmetric with $\lambda_1 = 1 - \lambda_2$ and Corollary 3.1 of Hall and Heyde (1980). For this reason we check the necessary conditions. First, we have to check that

$$\sum_{i=1}^n E \left[n^{-1} \left\| \mathbf{X}_{i-1} \phi(Z_i) \right\|^2 I \left(n^{-1/2} \left\| \mathbf{X}_{i-1} \phi(Z_i) \right\| > \epsilon \right) \mid \mathcal{F}_i \right] \rightarrow 0 \quad (4.95)$$

But

$$\begin{aligned} |\phi(Z_i)| &\leq |I(Z_i \leq Q_0(\lambda_1))| |Q_0(\lambda_1)| + |I(Z_i \geq Q_0(\lambda_2))| |Q_0(\lambda_2)| \\ &\quad + |Z_i I(Q_0(\lambda_1) < Z_i \leq Q_0(\lambda_2))| \end{aligned} \quad (4.96)$$

and because λ_1 and λ_2 are fixed, there are $M_1, M_2 > 0$ such that $|Q_0(\lambda_1)| \leq M_1$ and $|Q_0(\lambda_2)| \leq M_2$. Thus, by taking $M = \max(M_1, M_2)/4$ we have

$$|\phi(Z_i)| \leq M \quad (4.97)$$

and then

$$\begin{aligned} n^{-1} \sum_{i=1}^n E \left[\left\| \mathbf{X}_{i-1} \phi(Z_i) \right\|^2 I \left(\left\| \mathbf{X}_{i-1} \phi(Z_i) \right\| > n^{1/2} \epsilon \right) \mid \mathcal{F}_i \right] \\ \leq M n^{-1} \sum_{i=1}^n E \left[\left\| \mathbf{X}_{i-1} \right\|^2 I \left(\left\| \mathbf{X}_{i-1} \right\| > n^{1/2} \epsilon^* \right) \mid \mathcal{F}_i \right] \end{aligned} \quad (4.98)$$

where $\epsilon^* = \epsilon/M$. Therefore, the first condition is satisfied. For the second condition, we have to show that

$$\sum_{i=1}^n E \left[n^{-1} \left\| \mathbf{X}_{i-1} \phi(Z_i) \right\|^2 \mid \mathcal{F}_i \right] \rightarrow C < \infty. \quad (4.99)$$

In this case, we have

$$\begin{aligned} n^{-1} \sum_{i=1}^n E \left[\left\| \mathbf{X}_{i-1} \phi(Z_i) \right\|^2 \mid \mathcal{F}_i \right] \\ = n^{-1} \sum_{i=1}^n \left\| \mathbf{X}_{i-1} \right\|^2 E \left(\phi^2(Z_i) \mid \mathcal{F}_i \right) \\ = \sigma^2(\lambda_1, \lambda_2) E \left[\left\| \mathbf{X}_0 \right\|^2 \right] \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (4.100)$$

Therefore, the proof of Theorem is complete and we obtain

$$n^{1/2}(\bar{\rho}_n(\lambda_1, \lambda_2) - \rho) \xrightarrow{\mathcal{D}} \mathcal{N}_{p+1}(\mathbf{0}, \sigma^2(\lambda_1, \lambda_2)\Sigma^{-1}) \quad \spadesuit \quad (4.101)$$

4.5 Estimation of Parameters Under Uncertain Prior Information.

In the last two sections we estimated the autoregression parameter ρ using L-estimation based on a few selected autoregression quantiles as well as a trimmed least squares estimator. The estimators could be improved under some prior information on ρ in the form of a subhypothesis. In this section we are going to improve these estimators based on some uncertain prior information. We shall obtain the result for a general autoregression quantile for any fixed λ , which could be replaced by any of the L-estimators defined in the last sections.

Consider the autoregression model (4.2)

$$X_i = \mathbf{X}'_{i-1}\rho + Z_i, \quad \rho \in \mathbb{R}^{p+1} \quad i = 0, \pm 1, \pm 2, \dots \quad (4.102)$$

and let $1 \leq p_1 \leq p+1$ and $p_2 = (p+1) - p_1$. Partition ρ into two subsets namely, $\rho' = (\rho'_1, \rho'_2)$ where $\rho'_i \in \mathbb{R}^{p_i}$ $i = 1, 2$. We are primarily interested in the L-estimation of ρ_1 when it is suspected but not sure that

$$H_0 : \rho_2 = \mathbf{0} \quad (4.103)$$

holds. For this we partition the matrix Σ_n as

$$\Sigma_n = \begin{bmatrix} \Sigma_{n11} & \Sigma_{n12} \\ \Sigma_{n21} & \Sigma_{n22} \end{bmatrix} \quad (4.104)$$

where Σ_{n11} , Σ_{n12} and Σ_{n22} are $p_1 \times p_1$, $p_1 \times p_2$ and $p_2 \times p_2$ matrices respectively, with $\Sigma_{n12} = \Sigma'_{n21}$. Now, write $\hat{\rho}_n(\lambda)$, the autoregression quantile of ρ for fixed

$\lambda \in (0, 1)$, as $\hat{\rho}'_n(\lambda) = (\hat{\rho}'_{1n}(\lambda), \hat{\rho}'_{2n}(\lambda))$. In Corollary 4.2.4, we showed that

$$n^{1/2}(\hat{\rho}_n(\lambda) - \rho(\lambda)) \xrightarrow{D} \mathcal{N}_{p+1}\left(0, \frac{\lambda(1-\lambda)}{q_0^2(\lambda)} \Sigma^{-1}\right). \quad (4.105)$$

Let $\tilde{\rho}_{1n}(\lambda)$ be the *restricted L-estimator* (RLE) of ρ_1 while $\hat{\rho}_{1n}(\lambda)$ be the *unrestricted L-estimator* (ULE) of ρ_1 . This RLE performs better than ULE when $\rho_2 = 0$ holds. We are going to consider the *preliminary test L-estimator* (PTLE) and the *Shrinkage L-estimator* (SLE) of ρ_1 when ρ_2 is suspected to be near 0. For this reason we need to introduce a test statistic to test $H_0 : \rho_2 = 0$ as

$$\mathcal{L}_n = n d^2 \hat{\rho}'_{2n}(\lambda) \Sigma_{n22.1}^{-1} \hat{\rho}_{2n}(\lambda) \quad (4.106)$$

where

$$\Sigma_{n22.1}^{-1} = \Sigma_{n22} - \Sigma_{n21} \Sigma_{n11}^{-1} \Sigma_{n12} \quad \text{and} \quad d^2 = \frac{\lambda(1-\lambda)}{q_0(\lambda)}. \quad (4.107)$$

Then under H_0 , $\mathcal{L}_n$ has asymptotically central chi-square distribution with p_2 degrees of freedom (df). The PTLE of ρ_1 is defined by

$$\hat{\rho}_{1n}^{PT}(\lambda) = \tilde{\rho}_{1n}(\lambda) + I(\mathcal{L}_n \geq \chi_{p_2, \alpha}^2) (\hat{\rho}_{1n}(\lambda) - \tilde{\rho}_{1n}(\lambda)), \quad (4.108)$$

where $\chi_{p_2, \alpha}^2$ is the upper 100 α % point of the central chi-squared distribution with p_2 df. The SLE is defined following Berger, Bock, Brown, Casella and Gleser (1977) as

$$\hat{\rho}_{1n}^S(\lambda) = \tilde{\rho}_{1n}(\lambda) + (\mathbf{I}_{p_1} - k c_n n^{-1} \mathcal{L}_n^{-1} \mathbf{W}^{-1} \Sigma_{n11.2}) (\hat{\rho}_{1n}(\lambda) - \tilde{\rho}_{1n}(\lambda)) \quad (4.109)$$

where $c_n = ch_{p_1}(n \mathbf{W} \Sigma_{n11.2}^{-1})$ = smallest characteristic root of $n \mathbf{W} \Sigma_{n11.2}^{-1}$, and k is a positive shrinkage factor. If one chooses $\mathbf{W} = n^{-1} \Sigma_{n11.2}$, then we get the simpler form of (4.109)

$$\hat{\rho}_{1n}^S(\lambda) = \tilde{\rho}_{1n}(\lambda) + (1 - k \mathcal{L}_n^{-1}) (\hat{\rho}_{1n}(\lambda) - \tilde{\rho}_{1n}(\lambda)), \quad 0 < k < 2(p_2 - 2). \quad (4.110)$$

We improve the SLE by defining the *positive-rule shrinkage L-estimator* (PRSLE) due to Sclove, Morris and Radhakrishnan (1972)

$$\hat{\rho}_{1n}^{S+}(\lambda) = \tilde{\rho}_{1n}(\lambda) + (1 - k \mathcal{L}_n^{-1}) I(\mathcal{L}_n > k) (\hat{\rho}_{1n}(\lambda) - \tilde{\rho}_{1n}(\lambda)). \quad (4.111)$$

Note that $\hat{\rho}_{1n}^{PT}(\lambda)$ and $\hat{\rho}_{1n}^{S+}(\lambda)$ are convex combinations of $\hat{\rho}_{1n}(\lambda)$ and $\hat{\rho}_{1n}^S(\lambda)$ while $\hat{\rho}_{1n}^S(\lambda)$ is not. We shall discuss the asymptotic relative optimality of these estimators in the next subsection in terms of asymptotic distribution risk (ADR).

4.5.1 ADR and ADRE

We follow Saleh and Sen (1986) and Sen and Saleh (1987) to obtain the expressions for ADR of the above estimators. We use the sequence of alternatives

$$K_{(n)} : \rho_{2(n)} = n^{-1/2}\xi, \quad \xi \in \mathbb{R}^{p_2}. \quad (4.112)$$

Let $H_m(\cdot; \Delta)$ represent the cdf of a noncentral chi-square random variable with m df and noncentrality parameter Δ and

$$E(\chi_m^{-2r}(\Delta)) = \int_0^\infty x^{-r} dH_m(x; \Delta), \quad r \geq 1. \quad (4.113)$$

For any estimator ρ_{1n}^* of ρ_1 such that

$$n^{1/2}(\rho_{1n}^* - \rho_1) \xrightarrow{D} \mathcal{N}_{p_1}(\mathbf{0}, V^*) \quad (4.114)$$

under $K_{(n)}$, we define the ADR with respect to the loss

$$n(\rho_{1n}^* - \rho_1)' W (\rho_{1n}^* - \rho_1) \quad (4.115)$$

to be

$$R(\rho_{1n}^* : W) \equiv \text{tr}(W V^*) \quad (4.116)$$

where W is a p.d matrix and tr stands for trace operator. The ADR of four estimators are given in the following theorem.

Theorem 4.5.1 *Under the conditions a1 and a2, the ADR's of the four estimators are given by*

$$R(\hat{\rho}_{1n}(\lambda) : W) = d^2 \text{tr}(W \Sigma_{11.2}^{-1}), \quad (4.117)$$

$$R(\hat{\rho}_{1n}(\lambda) : W) = d^2 \text{tr}(W \Sigma_{11}^{-1}) + \xi' M \xi, \quad (4.118)$$

$$\begin{aligned} R(\hat{\rho}_{1n}^{PT}(\lambda) : W) = & d^2 \left\{ \text{tr}(W \Sigma_{11.2}^{-1}) [1 - H(\chi_{p_2, \alpha}^2; \Delta)] \right. \\ & \left. + \text{tr}(W \Sigma_{11}^{-1}) H_{p_2+2}(\chi_{p_2, \alpha}^2; \Delta) \right\} \\ & + (\xi' M \xi) [2H_{p_2+2}(\chi_{p_2, \alpha}^2; \Delta) - H_{p_2+4}(\chi_{p_2, \alpha}^2; \Delta)], \end{aligned} \quad (4.119)$$

$$\begin{aligned} R(\hat{\rho}_{1n}^S(\lambda) : W) = & d^2 \left\{ \text{tr}(W \Sigma_{11.2}^{-1}) - k \text{tr}(M \Sigma_{22.1}^{-1}) [2E(\chi_{p_2+2}^{-2}(\Delta)) \right. \\ & \left. - k E(\chi_{p_2+2}^{-4}(\Delta))] \right\} \\ & + k(k+4)(\xi' M \xi) E(\chi_{p_2+4}^{-4}(\Delta)), \end{aligned} \quad (4.120)$$

where

$$M = \Sigma_{21} \Sigma_{11}^{-1} W \Sigma_{11}^{-1} \Sigma_{12}. \quad (4.121)$$

We shall now discuss the *asymptotic distribution risk efficiency* (ADRE) results. Note that the risk of $\hat{\rho}_{1n}(\lambda)$ does not depend on ξ and if we consider the Mahalanobis distance (loss) as

$$d^2 n (\hat{\rho}_{1n}(\lambda) - \rho_1(\lambda))' \Sigma_{11.2} (\hat{\rho}_{1n}(\lambda) - \rho_1(\lambda)) \quad (4.122)$$

then $W = d^2 \Sigma_{11.2}$, and $R(\hat{\rho}_{1n}(\lambda) : W)$ reduces to p_1 . Now let

$$M^* = \Sigma_{21} \Sigma_{11}^{-1} \Sigma_{11.2} \Sigma_{11}^{-1} \Sigma_{12} \quad \text{and} \quad M^0 = \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21} \Sigma_{11}^{-1}. \quad (4.123)$$

Theorem 4.5.2 Under the conditions of Theorem 4.5.1, we have

- (i) $R(\hat{\rho}_{1n}(\lambda) : W) \gtrless R(\hat{\rho}_{1n}(\lambda) : W)$ according as $d^{-2}(\xi' M^* \xi) \gtrless \text{tr}(M^0)$.
- (ii) $R(\hat{\rho}_{1n}(\lambda) : W) \gtrless R(\hat{\rho}_{1n}^{PT}(\lambda) : W)$ according as

$$(\xi' M^* \xi) \gtrless \frac{d^2 \text{tr}(M^0) H_{p_2+2}(\chi_{p_2, \alpha}^2; \Delta)}{2H_{p_2+2}(\chi_{p_2, \alpha}^2; \Delta) - H_{p_2+4}(\chi_{p_2, \alpha}^2; \Delta)}.$$
- (iii) if $k = (p_2 - 2)$ and W is an arbitrary positive definite matrix, then

$$R(\hat{\rho}_{1n}^S(\lambda) : W) \leq R(\hat{\rho}_{1n}(\lambda) : W) \quad \forall W \in \mathcal{W} \text{ and } \xi \in \mathbb{R}^{p_2}, \quad (4.124)$$

where

$$\mathcal{W} = \left\{ \mathbf{W} \text{ p.d.} : \frac{\text{tr}(\mathbf{M} \boldsymbol{\Sigma}_{22.1}^{-1})}{\text{ch}_{\max}(\mathbf{M} \boldsymbol{\Sigma}_{22.1}^{-1})} \geq \frac{p_2 + 2}{2} \right\} \quad (4.125)$$

with $\mathbf{M}$ given by (4.121) and $\text{ch}_{\max}(\mathbf{A}) = \text{maximum of the characteristic roots of } \mathbf{A}$.

PROOF:

The proof of (i) and (ii) is straightforward. We prove (iii). Use the identity

$$E(\chi_{p_2+2}^{-2}(\Delta)) - (p_2 - 2)E(\chi_{p_2+2}^{-4}(\Delta)) = \Delta E(\chi_{p_2+4}^{-4}(\Delta)) \quad (4.126)$$

in Theorem 4.5.1 to obtain

$$\begin{aligned} R(\hat{\rho}_{1n}^S(\lambda) : \mathbf{W}) &= R(\hat{\rho}_{1n}(\lambda) : \mathbf{W}) \\ &\quad - (p_2 - 2)d^2 \text{tr}(\mathbf{M} \boldsymbol{\Sigma}_{22.1}^{-1}) \left\{ (p_2 - 2)E(\chi_{p_2+2}^{-4}(\Delta)) \right. \\ &\quad \left. + \left[1 - \frac{(p_2 + 2)(\boldsymbol{\xi}' \mathbf{M} \boldsymbol{\xi})d^2}{2 \text{tr}(\mathbf{M} \boldsymbol{\Sigma}_{22.1}^{-1})\Delta} \right] 2\Delta E(\chi_{p_2+4}^{-4}(\Delta)) \right\}. \end{aligned} \quad (4.127)$$

From (4.126), it follows that

$$R(\hat{\rho}_{1n}^S(\lambda) : \mathbf{W}) \leq R(\hat{\rho}_{1n}(\lambda) : \mathbf{W}) \quad (4.128)$$

for all $\boldsymbol{\xi} \in \mathbb{R}^{p_2}$ if

$$\frac{\text{tr}(\mathbf{M} \boldsymbol{\Sigma}_{22.1}^{-1})}{\text{ch}_{\max}(\mathbf{M} \boldsymbol{\Sigma}_{22.1}^{-1})} \geq \frac{(p_2 + 2)}{2}, \quad (4.129)$$

that is, if $\mathbf{W} \in \mathcal{W}$.

In particular, if $\mathbf{W} = d^2 \boldsymbol{\Sigma}_{11.2}$, then the risk expression (4.127) is the same with $\mathbf{M}$ replaced by $\mathbf{M}^*$ of (4.123). In this case the condition (4.125) reduces to

$$\mathcal{W} = \left\{ \mathbf{W} \text{ p.d.} : \frac{\text{tr}(\mathbf{M}^0)}{\text{ch}_{\max}(\mathbf{M}^0)} \geq \frac{p_2 + 2}{2} \right\}. \quad (4.130)$$

since

$$\text{tr}(\mathbf{M}^* \boldsymbol{\Sigma}_{22.1}^{-1}) = \text{tr}(\mathbf{M}^0).$$

We note that $\text{tr}(\mathbf{M}^0) = \text{tr}(\mathbf{I}_{p_1} - \boldsymbol{\Sigma}_{11.2} \boldsymbol{\Sigma}_{11}^{-1}) \leq \min(p_1, p_2) = p^* > 3$ since $p_2 > 3$. Also, if the rank of $\mathbf{M}^0 \leq 2$, then (4.129) does not hold with $\mathbf{M} = \mathbf{M}^*$. However,

if $p_2 \geq 3$ and $2\text{tr}(\mathbf{M}^0) \geq ch_{\max}(\mathbf{M}^0)$, then (4.129) holds. Thus, for Mahalanobis loss, (iii) holds. Fortunately, the condition (4.129) is verifiable for a given Σ_n for any AR(p) model. For an arbitrary $\mathbf{W}$ not belonging to $\mathcal{W}$, (iii) may not hold for all $\mathbf{M}$. In that case one may use the SLE given in (4.110).

Now we assume that k is an arbitrary positive integer and that $\mathbf{W} = d^{-2}\Sigma_{11,2}$. The following theorem gives a sufficient condition on k for which SLE dominates ULE under ADR.

Theorem 4.5.3 *A sufficient condition for the asymptotic dominance of SLE over ULE [i.e., $R(\hat{\rho}_{1n}(\lambda) : \mathbf{W}) \geq R(\hat{\rho}_{1n}^S(\lambda) : \mathbf{W}) \forall \xi \in \mathbb{R}^{p_2}$] is that the shrinkage factor k is positive and satisfies the inequality*

$$2E(\chi_{p_2+2}^{-2}(\Delta)) - kE(\chi_{p_2+2}^{-4}(\Delta)) - (k+4)h\Delta E(\chi_{p_2+2}^{-4}(\Delta)) \geq 0 \quad \forall \Delta, \quad (4.131)$$

which in turn, requires that

$$p_2 \geq 3, \quad 0 < k < 2(p_2 - 2) \quad \text{and} \quad h(k+4) \leq 2,$$

where $h = ch_{\max}(\mathbf{M}^0)/\text{tr}(\mathbf{M}^0)$ so that $0 < h < 1$.

PROOF:

Details of the proof are similar to Sen and Saleh (1987) and are left out.

Similarly, one has the following theorem.

Theorem 4.5.4 *Under the conditions of Theorem 4.5.3, the PTLE fails to dominate SLE. Also, if for an α ($0 < \alpha < 1$), the level of significance of the PT, we have*

$$H_{p_2+2}(\chi_{p_2,\alpha}^2; 0) \geq q\{2(p_2 - 2) - q\}/p_2(p_2 - 2), \quad (4.132)$$

$$q = (p_2 - 2) \wedge (2/h - 4).$$

then the SLE fails to dominate the PTLE.

However, one may verify that under H_0 the ordering of the ADR of the four estimators is as follows:

$$R(\tilde{\rho}_{1n}(\lambda) : W) \leq R(\tilde{\rho}_{1n}^{FT}(\lambda) : W) \leq R(\tilde{\rho}_{1n}^S(\lambda) : W) \leq R(\dot{\rho}_{1n}(\lambda) : W). \quad (4.133)$$

The general behavior of these risks is similar to those discussed in Sen and Saleh (1987) in connection with linear regression models.

Appendix A

t-dist. (location and scale parameters)

Table A.1: Optimal Spacing for the location and scale parameters along with the coefficients for the ABLUE of t-distribution.

<i>D.F.</i> = 1									
<i>k</i>	2	3	4	5	6	7	8	9	10
λ_1	0.3333	0.2500	0.2000	0.1667	0.1429	0.1250	0.1111	0.1000	0.0909
a_1	0.5000	0.0000	-.0854	-.0833	-.0671	-.0518	-.0398	-.0309	-.0243
b_1	-.8660	-.5000	-.2629	-.1443	-.0841	-.0518	-.0334	-.0225	-.0156
λ_2	0.6667	0.5000	0.4000	0.3333	0.2857	0.2500	0.2222	0.2000	0.1818
a_2	0.5000	1.0000	0.5854	0.2500	0.0777	0.0000	-.0319	-.0427	-.0442
b_2	0.8660	0.0000	-.4253	-.4330	-.3405	-.2500	-.1808	-.1314	-.0967
λ_3		0.7500	0.6000	0.5000	0.4286	0.3750	0.3333	0.3000	0.2727
a_3		0.0000	0.5854	0.6667	0.4893	0.3018	0.1667	0.0809	0.0296
b_3		0.5000	0.4253	0.0000	-.2357	-.3018	-.2887	-.2490	-.2056
λ_4			0.8000	0.6667	0.5714	0.5000	0.4444	0.4000	0.3636
a_4			-.0854	0.2500	0.4893	0.5000	0.4050	0.2927	0.1970
b_4			0.2629	0.4330	0.2357	0.0000	-.1474	-.2127	-.2274
λ_5				0.8333	0.7143	0.6250	0.5556	0.5000	0.4545
a_5				-.0833	0.0777	0.3018	0.4050	0.4000	0.3418
b_5				0.1443	0.3405	0.3018	0.1474	0.0000	-.1001
λ_6					0.8571	0.7500	0.6667	0.6000	0.5455
a_6					-.0671	0.0000	0.1667	0.2927	0.3418
b_6					0.0841	0.2500	0.2887	0.2127	0.1001
λ_7						0.8750	0.7778	0.7000	0.6364
a_7						-.0518	-.0319	0.0809	0.1970
b_7						0.0518	0.1808	0.2490	0.2274
λ_8							0.8889	0.8000	0.7273
a_8							-.0398	-.0427	0.0296
b_8							0.0334	0.1314	0.2056
λ_9								0.9000	0.8182
a_9								-.0309	-.0442
b_9								0.0225	0.0967
λ_{10}									0.9091
a_{10}									-.0243
b_{10}									0.0156
K_1	0.3420	0.4053	0.4376	0.4559	0.4673	0.4748	0.4800	0.4838	0.4866
K_2	0.3420	0.4053	0.4376	0.4559	0.4673	0.4748	0.4800	0.4838	0.4866
K_3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Δ	0.1169	0.1643	0.1915	0.2079	0.2184	0.2255	0.2304	0.2340	0.2367

Table A.2: Optimal Spacing for the location and scale parameters along with the coefficients for the ABLUE of t-distribution.

<i>D.F.</i> = 2									
<i>k</i>	2	3	4	5	6	7	8	9	10
λ_1	0.2688	0.1964	0.1487	0.1183	0.0968	0.0812	0.0693	0.0601	0.0527
a_1	0.5000	0.1091	-.0037	-.0312	-.0355	-.0329	-.0286	-.0244	-.0206
b_1	-.6778	-.4627	-.2674	-.1662	-.1078	-.0731	-.0513	-.0371	-.0275
λ_2	0.7312	0.5000	0.3634	0.2854	0.2327	0.1951	0.1669	0.1451	0.1278
a_2	0.5000	0.7819	0.5037	0.2578	0.1216	0.0523	0.0168	-.0014	-.0106
b_2	0.6778	0.0000	-.3154	-.3306	-.2713	-.2109	-.1625	-.1259	-.0986
λ_3		0.8036	0.6366	0.5000	0.4028	0.3349	0.2850	0.2471	0.2172
a_3		0.1091	0.5037	0.5467	0.4139	0.2718	0.1690	0.1023	0.0600
b_3		0.4627	0.3154	0.0000	-.1760	-.2264	-.2191	-.1933	-.1647
λ_4			0.8513	0.7146	0.5972	0.5000	0.4243	0.3661	0.3206
a_4			-.0037	0.2578	0.4139	0.4175	0.3428	0.2548	0.1809
b_4			0.2674	0.3306	0.1760	0.0000	-.1119	-.1608	-.1717
λ_5				0.8817	0.7673	0.6651	0.5757	0.5000	0.4379
a_5				-.0312	0.1216	0.2718	0.3428	0.3373	0.2902
b_5				0.1662	0.2713	0.2264	0.1119	0.0000	-.0773
λ_6					0.9032	0.8049	0.7150	0.6339	0.5621
a_6					-.0355	0.0523	0.1690	0.2548	0.2902
b_6					0.1078	0.2109	0.2191	0.1608	0.0773
λ_7						0.9188	0.8331	0.7529	0.6794
a_7						-.0329	0.0168	0.1023	0.1809
b_7						0.0731	0.1625	0.1933	0.1717
λ_8							0.9307	0.8549	0.7828
a_8							-.0286	-.0014	0.0600
b_8							0.0513	0.1259	0.1647
λ_9								0.9399	0.8722
a_9								-.0244	-.0106
b_9								0.0371	0.0986
λ_{10}									0.9473
a_{10}									-.0206
b_{10}									0.0275
K_1	0.4519	0.5248	0.5452	0.5609	0.5702	0.5767	0.5813	0.5847	0.5872
K_2	0.5317	0.6161	0.6842	0.7174	0.7389	0.7528	0.7625	0.7695	0.7746
K_3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Δ	0.2403	0.3233	0.3730	0.4024	0.4214	0.4342	0.4433	0.4499	0.4549

Table A.3: Optimal Spacing for the location and scale parameters along with the coefficients for the ABLUE of t-distribution.

<i>D.F.</i> = 3									
<i>k</i>	2	3	4	5	6	7	8	9	10
λ_1	0.2352	0.1672	0.1212	0.0939	0.0750	0.0616	0.0516	0.0440	0.0381
a_1	0.5000	0.1467	0.0322	-.0018	-.0134	-.0165	-.0165	-.0153	-.0138
b_1	-.6069	-.4358	-.2554	-.1657	-.1115	-.0787	-.0573	-.0429	-.0329
λ_2	0.7648	0.5000	0.3405	0.2579	0.2035	0.1662	0.1390	0.1184	0.1024
a_2	0.5000	0.7066	0.4678	0.2520	0.1311	0.0687	0.0349	0.0161	0.0053
b_2	0.6069	0.0000	-.2844	-.2974	-.2444	-.1921	-.1504	-.1187	-.0948
λ_3		0.8328	0.6595	0.5000	0.3867	0.3114	0.2579	0.2184	0.1881
a_3		0.1467	0.4678	0.4995	0.3823	0.2557	0.1648	0.1062	0.0686
b_3		0.4358	0.2844	0.0000	-.1593	-.2033	-.1957	-.1728	-.1481
λ_4			0.8788	0.7421	0.6133	0.5000	0.4118	0.3461	0.2961
a_4			0.0322	0.2520	0.3823	0.3843	0.3168	0.2372	0.1713
b_4			0.2554	0.2974	0.1593	0.0000	-.1017	-.1450	-.1537
λ_5				0.9061	0.7965	0.6886	0.5882	0.5000	0.4277
a_5				-.0018	0.1311	0.2557	0.3168	0.3117	0.2686
b_5				0.1657	0.2444	0.2033	0.1017	0.0000	-.0705
λ_6					0.9250	0.8338	0.7421	0.6539	0.5723
a_6					-.0134	0.0687	0.1648	0.2372	0.2686
b_6					0.1115	0.1921	0.1957	0.1450	0.0705
λ_7						0.9384	0.8610	0.7816	0.7039
a_7						-.0165	0.0349	0.1062	0.1713
b_7						0.0787	0.1504	0.1728	0.1537
λ_8							0.9484	0.8816	0.8119
a_8							-.0165	0.0161	0.0686
b_8							0.0573	0.1187	0.1481
λ_9								0.9560	0.8976
a_9								-.0153	0.0053
b_9								0.0429	0.0948
λ_{10}									0.9619
a_{10}									-.0138
b_{10}									0.0329
K_1	0.5081	0.5940	0.6124	0.6292	0.6379	0.6442	0.6486	0.6518	0.6542
K_2	0.6511	0.7457	0.8395	0.8831	0.9134	0.9327	0.9465	0.9563	0.9637
K_3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Δ	0.3308	0.4430	0.5141	0.5556	0.5826	0.6009	0.6138	0.6233	0.6305

Table A.4: Optimal Spacing for the location and scale parameters along with the coefficients for the ABLUE of t-distribution.

<i>D.F.</i> = 5									
<i>k</i>	2	3	4	5	6	7	8	9	10
λ_1	0.2012	0.1380	0.0936	0.0699	0.0537	0.0428	0.0349	0.0290	0.0246
a_1	0.5000	0.1770	0.0623	0.0248	0.0082	0.0008	-.0026	-.0041	-.0046
b_1	-.5467	-.4089	-.2386	-.1594	-.1092	-.0789	-.0587	-.0449	-.0352
λ_2	0.7988	0.5000	0.3134	0.2275	0.1715	0.1352	0.1094	0.0907	0.0764
a_2	0.5000	0.6459	0.4377	0.2450	0.1351	0.0800	0.0483	-.0298	-.0184
b_2	0.5467	0.0000	-.2619	-.2724	-.2221	-.1753	-.1379	-.1099	-.0886
λ_3		0.8620	0.6866	0.5000	0.3669	0.2840	0.2269	0.1863	0.1561
a_3		0.1770	0.4377	0.4604	0.3559	0.2408	0.1590	0.1072	0.0737
b_3		0.4089	0.2619	0.0000	-.1484	-.1872	-.1783	-.1566	-.1341
λ_4			0.9064	0.7725	0.6331	0.5000	0.3963	0.3220	0.2674
a_4			0.0623	0.2450	0.3559	0.3568	0.2952	0.2217	0.1617
b_4			0.2386	0.2724	0.1484	0.0000	-.0956	-.1346	-.1409
λ_5				0.9301	0.8285	0.7160	0.6037	0.5000	0.4149
a_5				0.0248	0.1359	0.2408	0.2952	0.2908	0.2508
b_5				0.1594	0.2221	0.1872	0.0956	0.0000	-.0667
λ_6					0.9463	0.8648	0.7731	0.6780	0.5851
a_6					0.0082	0.0800	0.1590	0.2217	0.2508
b_6					0.1092	0.1753	0.1783	0.1346	0.0667
λ_7						0.9572	0.8906	0.8137	0.7326
a_7						0.0008	0.0483	0.1072	0.1617
b_7						0.0789	0.1379	0.1566	0.1409
λ_8							0.9651	0.9093	0.8439
a_8							-.0026	0.0298	0.0737
b_8							0.0587	0.1099	0.1341
λ_9								0.9710	0.9236
a_9								-.0041	0.0184
b_9								0.0449	0.0886
λ_{10}									0.9754
a_{10}									-.0046
b_{10}									0.0352
K_1	0.5663	0.6702	0.6890	0.7097	0.7189	0.7261	0.7307	0.7342	0.7368
K_2	0.7926	0.8977	1.0267	1.0827	1.1253	1.1521	1.1717	1.1858	1.1965
K_3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Δ	0.4488	0.6016	0.7074	0.7683	0.8090	0.8365	0.8562	0.8707	0.8816

Table A.5: Optimal Spacing for the location and scale parameters along with the coefficients for the ABLUE of t-distribution.

<i>D.F.</i> = 10									
<i>k</i>	2	3	4	5	6	7	8	9	10
λ_1	0.1702	0.1122	0.0699	0.0499	0.0363	0.0277	0.0217	0.0174	0.0143
a_1	0.5000	0.2002	0.0852	0.0458	0.0255	0.0153	0.0094	0.0059	0.0037
b_1	-.4994	-.3860	-.2217	-.1511	-.1038	-.0758	-.0568	-.0439	-.0346
λ_2	0.8298	0.5000	0.2849	0.1974	0.1406	0.1062	0.0824	0.0658	0.0536
a_2	0.5000	0.5995	0.4148	0.2393	0.1388	0.0879	0.0580	0.0399	0.0282
b_2	0.4994	0.0000	-.2459	-.2543	-.2048	-.1616	-.1270	-.1014	-.0820
λ_3		0.8878	0.7151	0.5000	0.3447	0.2552	0.1951	0.1543	0.1248
a_3		0.2002	0.4148	0.4297	0.3357	0.2290	0.1538	0.1072	0.0769
b_3		0.3860	0.2459	0.0000	-.1420	-.1766	-.1659	-.1447	-.1233
λ_4			0.9301	0.8026	0.6553	0.5000	0.3785	0.2957	0.2367
a_4			0.0852	0.2393	0.3357	0.3356	0.2788	0.2094	0.1537
b_4			0.2217	0.2543	0.1420	0.0000	-.0925	-.1286	-.1327
λ_5				0.9501	0.8594	0.7448	0.6218	0.5000	0.4001
a_5				0.0458	0.1388	0.2290	0.2788	0.2751	0.2375
b_5				0.1511	0.2048	0.1766	0.0925	0.0000	-.0652
λ_6					0.9637	0.8938	0.8049	0.7043	0.5999
a_6					0.0255	0.0879	0.1538	0.2094	0.2375
b_6					0.1038	0.1616	0.1659	0.1286	0.0652
λ_7						0.9723	0.9176	0.8457	0.7633
a_7						0.0153	0.0580	0.1072	0.1537
b_7						0.0758	0.1270	0.1447	0.1327
λ_8							0.9783	0.9342	0.8752
a_8							0.0094	0.0399	0.0769
b_8							0.0568	0.1014	0.1233
λ_9								0.9826	0.9464
a_9								0.0059	0.0282
b_9								0.0439	0.0820
λ_{10}									0.9857
a_{10}									0.0037
b_{10}									0.0346
K_1	0.6222	0.7473	0.7687	0.7962	0.8071	0.8166	0.8222	0.8267	0.8298
K_2	0.9455	1.0613	1.2342	1.3035	1.3627	1.3984	1.4259	1.4455	1.4606
K_3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Δ	0.5883	0.7924	0.9487	1.0379	1.0998	1.1418	1.1724	1.1949	1.2121

Table A.6: Optimal Spacing for the location and scale parameters along with the coefficients for the ABLUE of t-distribution.

<i>D.F.</i> = 20									
<i>k</i>	2	3	4	5	6	7	8	9	10
λ_1	0.1525	0.0981	0.0575	0.0398	0.0277	0.0206	0.0156	0.0122	0.0097
a_1	0.5000	0.2122	0.0967	0.0566	0.0343	0.0227	0.0155	0.0110	0.0080
b_1	-.4750	-.3739	-.2119	-.1460	-.1000	-.0734	-.0550	-.0426	-.0336
λ_2	0.8475	0.5000	0.2668	0.1795	0.1226	0.0900	0.0676	0.0526	0.0417
a_2	0.5000	0.5757	0.4033	0.2366	0.1402	0.0921	0.0629	0.0450	0.0331
b_2	0.4750	0.0000	-.2380	-.2453	-.1958	-.1544	-.1210	-.0966	-.0780
λ_3		0.9019	0.7332	0.5000	0.3299	0.2371	0.1756	0.1353	0.1066
a_3		0.2122	0.4033	0.4135	0.3254	0.2230	0.1512	0.1073	0.0786
b_3		0.3739	0.2380	0.0000	-.1392	-.1717	-.1598	-.1386	-.1177
λ_4			0.9425	0.8205	0.6701	0.5000	0.3664	0.2786	0.2172
a_4			0.0967	0.2366	0.3254	0.3244	0.2704	0.2032	0.1496
b_4			0.2119	0.2453	0.1392	0.0000	-.0915	-.1261	-.1288
λ_5				0.9602	0.8774	0.7629	0.6336	0.5000	0.3898
a_5				0.0566	0.1402	0.2230	0.2704	0.2670	0.2307
b_5				0.1460	0.1958	0.1717	0.0915	0.0000	-.0650
λ_6					0.9723	0.9100	0.8244	0.7214	0.6102
a_6					0.0343	0.0921	0.1512	0.2032	0.2307
b_6					0.1000	0.1544	0.1598	0.1261	0.0650
λ_7						0.9794	0.9324	0.8647	0.7828
a_7						0.0227	0.0629	0.1073	0.1496
b_7						0.0734	0.1210	0.1386	0.1288
λ_8							0.9844	0.9474	0.8934
a_8							0.0155	0.0450	0.0786
b_8							0.0550	0.0966	0.1177
λ_9								0.9878	0.9583
a_9								0.0110	0.0331
b_9								0.0426	0.0780
λ_{10}									0.9903
a_{10}									0.0080
b_{10}									0.0336
K_1	0.6559	0.7958	0.8197	0.8531	0.8655	0.8772	0.8838	0.8893	0.8931
K_2	1.0459	1.1659	1.3735	1.4513	1.5234	1.5655	1.5992	1.6229	1.6416
K_3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Δ	0.6860	0.9278	1.1258	1.2381	1.3186	1.3732	1.4134	1.4432	1.4661

Table A.7: Optimal Spacing for the location and scale parameters along with the coefficients for the ABLUE of t-distribution.

<i>D.F.</i> = 500									
<i>k</i>	2	3	4	5	6	7	8	9	10
λ_1	0.1342	0.0839	0.0454	0.0303	0.0200	0.0143	0.0104	0.0079	0.0061
a_1	0.5000	0.2239	0.1079	0.0671	0.0428	0.0298	0.0213	0.0159	0.0121
b_1	-.4512	-.3619	-.2017	-.1408	-.0959	-.0707	-.0528	-.0410	-.0323
λ_2	0.8658	0.5000	0.2462	0.1602	0.1038	0.0735	0.0530	0.0399	0.0306
a_2	0.5000	0.5521	0.3921	0.2344	0.1419	0.0964	0.0678	0.0501	0.0379
b_2	0.4512	0.0000	-.2303	-.2365	-.1867	-.1472	-.1149	-.0917	-.0739
λ_3		0.9161	0.7538	0.5000	0.3122	0.2168	0.1543	0.1151	0.0877
a_3		0.2239	0.3921	0.3970	0.3153	0.2173	0.1488	0.1077	0.0804
b_3		0.3619	0.2303	0.0000	-.1368	-.1670	-.1538	-.1327	-.1121
λ_4			0.9546	0.8398	0.6878	0.5000	0.3516	0.2588	0.1953
a_4			0.1079	0.2344	0.3153	0.3129	0.2621	0.1970	0.1456
b_4			0.2017	0.2365	0.1368	0.0000	-.0910	-.1239	-.1252
λ_5				0.9697	0.8962	0.7832	0.6484	0.5000	0.3773
a_5				0.0671	0.1419	0.2173	0.2621	0.2586	0.2240
b_5				0.1408	0.1867	0.1670	0.0910	0.0000	-.0651
λ_6					0.9800	0.9265	0.8457	0.7412	0.6227
a_6					0.0428	0.0964	0.1488	0.1970	0.2240
b_6					0.0959	0.1472	0.1538	0.1239	0.0651
λ_7						0.9857	0.9470	0.8849	0.8047
a_7						0.0298	0.0678	0.1077	0.1456
b_7						0.0707	0.1149	0.1327	0.1252
λ_8							0.9896	0.9601	0.9123
a_8							0.0213	0.0501	0.0804
b_8							0.0528	0.0917	0.1121
λ_9								0.9921	0.9694
a_9								0.0159	0.0379
b_9								0.0410	0.0739
λ_{10}									0.9939
a_{10}									0.0121
b_{10}									0.0323
K_1	0.6935	0.8511	0.8789	0.9204	0.9351	0.9500	0.9582	0.9653	0.9701
K_2	1.1639	1.2891	1.5407	1.6282	1.7179	1.7679	1.8103	1.8393	1.8629
K_3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Δ	0.8072	1.0972	1.3541	1.4985	1.6064	1.6796	1.7346	1.7754	1.8072

Appendix A

t-Dist. (Regression Parameters, $p=1$)

Table A.1: Optimal Spacing for the regression and scale parameters along with the coefficients for the ABLUE of t-distribution with $p=1$.

$D.F. = 1$								
k	3	4	5	6	7	8	9	10
λ_1	0.2500	0.1837	0.1535	0.1321	0.1147	0.1013	0.0911	0.0827
a_1	0.0000	-.0769	-.0730	-.0593	-.0446	-.0333	-.0254	-.0197
λ_2	0.5000	0.4099	0.3465	0.2941	0.2500	0.2170	0.1926	0.1732
a_2	1.0000	0.5769	0.2664	0.0892	0.0000	-.0350	-.0453	-.0454
λ_3	0.7500	0.5901	0.5000	0.4364	0.3853	0.3431	0.3074	0.2765
a_3	0.0000	0.5769	0.6132	0.4701	0.3147	0.1860	0.0918	0.0354
λ_4		0.8163	0.6535	0.5636	0.5000	0.4501	0.4089	0.3736
a_4		-.0769	0.2664	0.4701	0.4599	0.3823	0.2934	0.2117
λ_5			0.8465	0.7059	0.6147	0.5499	0.5000	0.4591
a_5			-.0730	0.0892	0.3147	0.3823	0.3650	0.3180
λ_6				0.8679	0.7500	0.6569	0.5911	0.5409
a_6				-.0593	0.0000	0.1860	0.2934	0.3180
λ_7					0.8853	0.7830	0.6926	0.6264
a_7					-.0446	-.0350	0.0918	0.2117
λ_8						0.8987	0.8074	0.7235
a_8						-.0333	-.0453	0.0354
λ_9							0.9089	0.8268
a_9							-.0254	-.0454
λ_{10}								0.9173
a_{10}								-.0197
K_1	0.4053	0.4469	0.4631	0.4712	0.4774	0.4822	0.4855	0.4880
K_2	0.4053	0.4244	0.4457	0.4619	0.4713	0.4770	0.4813	0.4846
K_3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Δ	0.0666	0.0848	0.0956	0.1025	0.1074	0.1109	0.1135	0.1154

Table A.2: Optimal Spacing for the regression and scale parameters along with the coefficients for the ABLUE of t-distribution with $p=1$.

$D.F. = 2$								
k	3	4	5	6	7	8	9	10
λ_1	0.2152	0.1578	0.1204	0.0955	0.0781	0.0654	0.0558	0.0483
a_1	0.1330	0.0087	-.0273	-.0339	-.0317	-.0272	-.0227	-.0188
λ_2	0.5000	0.3837	0.3047	0.2479	0.2057	0.1736	0.1487	0.1290
a_2	0.7341	0.4913	0.2781	0.1437	0.0663	0.0237	0.0009	-.0106
λ_3	0.7848	0.6163	0.5000	0.4156	0.3515	0.3015	0.2615	0.2290
a_3	0.1330	0.4913	0.4983	0.3902	0.2774	0.1870	0.1208	0.0746
λ_4		0.8422	0.6953	0.5844	0.5000	0.4337	0.3803	0.3364
a_4		0.0087	0.2781	0.3902	0.3758	0.3166	0.2501	0.1901
λ_5			0.8796	0.7521	0.6485	0.5663	0.5000	0.4454
a_5			-.0273	0.1437	0.2774	0.3166	0.3018	0.2647
λ_6				0.9045	0.7943	0.6985	0.6197	0.5546
a_6				-.0339	0.0663	0.1870	0.2501	0.2647
λ_7					0.9219	0.8264	0.7385	0.6636
a_7					-.0317	0.0237	0.1208	0.1901
λ_8						0.9346	0.8513	0.7710
a_8						-.0272	0.0009	0.0746
λ_9							0.9442	0.8710
a_9							-.0227	-.0106
λ_{10}								0.9517
a_{10}								-.0188
K_1	0.5318	0.5516	0.5648	0.5735	0.5793	0.5834	0.5864	0.5887
K_2	0.6040	0.6733	0.7103	0.7330	0.7481	0.7586	0.7662	0.7719
K_3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Δ	0.1708	0.2049	0.2266	0.2411	0.2510	0.2582	0.2635	0.2675

Table A.3: Optimal Spacing for the regression and scale parameters along with the coefficients for the ABLUE of t-distribution with $p=1$.

$D.F. = 3$									
k	2	3	4	5	6	7	8	9	10
λ_1	0.1862	0.1354	0.1027	0.0805	0.0648	0.0533	0.0447	0.0381	
a_1	0.1694	0.0505	0.0061	-.0103	-.0155	-.0163	-.0154	-.0138	
λ_2	0.5000	0.3661	0.2813	0.2234	0.1819	0.1512	0.1277	0.1094	
a_2	0.6612	0.4495	0.2688	0.1550	0.0873	0.0473	0.0237	0.0097	
λ_3	0.8138	0.6339	0.5000	0.4035	0.3324	0.2787	0.2372	0.2044	
a_3	0.1694	0.4495	0.4503	0.3553	0.2576	0.1804	0.1242	0.0844	
λ_4		0.8646	0.7187	0.5965	0.5000	0.4242	0.3642	0.3160	
a_4		0.0505	0.2688	0.3553	0.3413	0.2886	0.2301	0.1780	
λ_5			0.8973	0.7766	0.6676	0.5758	0.5000	0.4376	
a_5			0.0061	0.1550	0.2576	0.2886	0.2750	0.2418	
λ_6				0.9195	0.8181	0.7213	0.6358	0.5624	
a_6				-.0103	0.0873	0.1804	0.2301	0.2418	
λ_7					0.9352	0.8488	0.7628	0.6840	
a_7					-.0155	0.0473	0.1242	0.1780	
λ_8						0.9467	0.8723	0.7956	
a_8						-.0163	0.0237	0.0844	
λ_9							0.9553	0.8906	
a_9							-.0154	0.0097	
λ_{10}									0.9619
a_{10}									-.0138
K_1	0.6030	0.6224	0.6340	0.6417	0.6470	0.6507	0.6536	0.6557	
K_2	0.7293	0.8211	0.8734	0.9058	0.9272	0.9420	0.9527	0.9607	
K_3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	
Δ	0.2651	0.3180	0.3511	0.3730	0.3881	0.3989	0.4069	0.4130	

Table A.4: Optimal Spacing for the regression and scale parameters along with the coefficients for the ABLUE of t-distribution with $p=1$.

$D.F. = 5$									
k	2	3	4	5	6	7	8	9	10
λ_1		0.1553	0.1084	0.0799	0.0613	0.0485	0.0393	0.0325	0.0273
a_1		0.1966	0.0824	0.0355	0.0143	0.0041	-.0008	-.0032	-.0043
λ_2		0.5000	0.3438	0.2523	0.1933	0.1529	0.1241	0.1027	0.0864
a_2		0.6068	0.4176	0.2570	0.1574	0.0981	0.0621	0.0398	0.0256
λ_3		0.8447	0.6562	0.5000	0.3877	0.3082	0.2507	0.2078	0.1751
a_3		0.1966	0.4176	0.4150	0.3283	0.2402	0.1719	0.1229	0.0884
λ_4			0.8916	0.7477	0.6123	0.5000	0.4120	0.3438	0.2908
a_4			0.0824	0.2570	0.3283	0.3153	0.2668	0.2133	0.1665
λ_5				0.9201	0.8067	0.6918	0.5880	0.5000	0.4275
a_5				0.0355	0.1574	0.2402	0.2668	0.2544	0.2237
λ_6					0.9387	0.8471	0.7493	0.6562	0.5725
a_6					0.0143	0.0981	0.1719	0.2133	0.2237
λ_7						0.9515	0.8759	0.7922	0.7092
a_7						0.0041	0.0621	0.1229	0.1665
λ_8							0.9607	0.8973	0.8249
a_8							-.0008	0.0398	0.0884
λ_9								0.9675	0.9136
a_9								-.0032	0.0256
λ_{10}									0.9727
a_{10}									-.0043
K_1		0.6799	0.7026	0.7160	0.7242	0.7298	0.7337	0.7366	0.7387
K_2		0.8789	0.9994	1.0690	1.1137	1.1438	1.1650	1.1805	1.1921
K_3		0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Δ		0.4062	0.4933	0.5480	0.5842	0.6092	0.6272	0.6405	0.6506

Table A.5: Optimal Spacing for the regression and scale parameters along with the coefficients for the ABLUE of t-distribution with $p=1$.

$D.F. = 10$									
k	2	3	4	5	6	7	8	9	10
λ_1	0.1270	0.0830	0.0583	0.0429	0.0327	0.0256	0.0206	0.0169	
a_1	0.2165	0.1045	0.0571	0.0332	0.0202	0.0127	0.0082	0.0054	
λ_2	0.5000	0.3190	0.2214	0.1615	0.1226	0.0958	0.0768	0.0628	
a_2	0.5669	0.3951	0.2477	0.1571	0.1034	0.0705	0.0495	0.0356	
λ_3	0.8730	0.6810	0.5000	0.3692	0.2806	0.2192	0.1754	0.1431	
a_3	0.2165	0.3951	0.3904	0.3098	0.2273	0.1645	0.1203	0.0893	
λ_4		0.9170	0.7786	0.6308	0.5000	0.3971	0.3197	0.2615	
a_4		0.1049	0.2477	0.3098	0.2981	0.2523	0.2014	0.1577	
λ_5			0.9417	0.8385	0.7194	0.6029	0.5000	0.4151	
a_5			0.0571	0.1571	0.2273	0.2523	0.2413	0.2120	
λ_6				0.9571	0.8774	0.7808	0.6803	0.5819	
a_6				0.0332	0.1034	0.1645	0.2014	0.2120	
λ_7					0.9673	0.9042	0.8246	0.7385	
a_7					0.0202	0.0705	0.1203	0.1577	
λ_8						0.9744	0.9232	0.8569	
a_8						0.0127	0.0495	0.0893	
λ_9							0.9794	0.9372	
a_9							0.0082	0.0356	
λ_{10}									0.9831
a_{10}									0.0054
K_1	0.7568	0.7854	0.8033	0.8139	0.8210	0.8260	0.8296	0.8323	
K_2	1.0408	1.1981	1.2871	1.3469	1.3875	1.4167	1.4381	1.4544	
K_3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	
Δ	0.5962	0.7390	0.8305	0.8922	0.9353	0.9665	0.9898	1.0076	

Table A.6: Optimal Spacing for the regression and scale parameters along with the coefficients for the ABLUE of t-distribution with $p=1$.

$D.F. = 20$									
k	α	3	4	5	6	7	8	9	10
λ_1		0.1112	0.0691	0.0468	0.0332	0.0245	0.0187	0.0146	0.0117
a_1		0.2266	0.1157	0.0675	0.0423	0.0280	0.0194	0.0139	0.0102
λ_2		0.5000	0.3027	0.2022	0.1422	0.1045	0.0794	0.0619	0.0491
a_2		0.5468	0.3843	0.2434	0.1568	0.1058	0.0742	0.0538	0.0401
λ_3		0.8888	0.6973	0.5000	0.3564	0.2623	0.1989	0.1549	0.1233
a_3		0.2266	0.3843	0.3782	0.3009	0.2212	0.1609	0.1188	0.0896
λ_4			0.9309	0.7978	0.6436	0.5000	0.3867	0.3032	0.2418
a_4			0.1157	0.2434	0.3009	0.2900	0.2455	0.1958	0.1534
λ_5				0.9532	0.8578	0.7377	0.6133	0.5000	0.4063
a_5				0.0675	0.1568	0.2212	0.2455	0.2354	0.2068
λ_6					0.9668	0.8955	0.8011	0.6968	0.5937
a_6					0.0423	0.1058	0.1609	0.1958	0.2068
λ_7						0.9755	0.9206	0.8451	0.7582
a_7						0.0280	0.0742	0.1188	0.1534
λ_8							0.9813	0.9381	0.8767
a_8							0.0194	0.0538	0.0896
λ_9								0.9854	0.9506
a_9								0.0139	0.0401
λ_{10}									0.9883
a_{10}									0.0102
K_1		0.8049	0.8383	0.8602	0.8731	0.8819	0.8880	0.8925	0.8959
K_2		1.1466	1.3317	1.4340	1.5050	1.5535	1.5886	1.6146	1.6344
K_3		0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Δ		0.7429	0.9357	1.0611	1.1473	1.2082	1.2527	1.2861	1.3118

Table A.7: Optimal Spacing for the regression and scale parameters along with the coefficients for the ABUE of t-distribution with $p=1$.

<i>D.F.</i> = 500									
<i>k</i>	2	3	4	5	6	7	8	9	10
λ_1	0.0952	0.0554	0.0357	0.0242	0.0171	0.0126	0.0095	0.0073	
a_1	0.2365	0.1259	0.0773	0.0507	0.0352	0.0254	0.0190	0.0146	
λ_2	0.5000	0.2838	0.1811	0.1215	0.0857	0.0627	0.0472	0.0365	
a_2	0.5270	0.3741	0.2396	0.1569	0.1083	0.0779	0.0580	0.0413	
λ_3	0.9048	0.7162	0.5000	0.3411	0.2411	0.1761	0.1324	0.1021	
a_3	0.2365	0.3741	0.3662	0.2924	0.2154	0.1576	0.1177	0.0899	
λ_4		0.9446	0.8189	0.6589	0.5000	0.3739	0.2834	0.2190	
a_4		0.1259	0.2396	0.2924	0.2882	0.2391	0.1901	0.1493	
λ_5			0.9643	0.8785	0.7589	0.6261	0.5000	0.3953	
a_5			0.0773	0.1569	0.2154	0.2391	0.2299	0.2019	
λ_6				0.9758	0.9143	0.8239	0.7166	0.6017	
a_6				0.0507	0.1083	0.1576	0.1901	0.2019	
λ_7					0.9829	0.9373	0.8676	0.7810	
a_7					0.0352	0.0779	0.1177	0.1493	
λ_8						0.9871	0.9528	0.8979	
a_8						0.0254	0.0580	0.0899	
λ_9							0.9905	0.9635	
a_9							0.0190	0.0413	
λ_{10}									0.9927
a_{10}									0.0146
K_1	0.8597	0.8995	0.9274	0.9436	0.9549	0.9628	0.9687	0.9731	
K_2	1.2705	1.4921	1.6105	1.6966	1.7550	1.7981	1.8302	1.8549	
K_3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	
Δ	0.9389	1.2074	1.3851	1.5106	1.6004	1.6669	1.7173	1.7563	

Appendix A

t-Dist. (Regression Parameters, $p=2$)

Table A.1: Optimal Spacing for the regression and scale parameters along with the coefficients for the ABLUE of t-distribution with $p=2$.

$D.F. = 1$							
k	4	5	6	7	8	9	10
λ_1	0.1676	0.1444	0.1276	0.1098	0.0958	0.0866	0.0791
a_1	-.0701	-.0660	-.0561	-.0414	-.0296	-.0226	-.0177
λ_2	0.4133	0.3556	0.3049	0.2500	0.2105	0.1867	0.1681
a_2	0.5701	0.2778	0.1049	0.0000	-.0372	-.0460	-.0455
λ_3	0.5867	0.5000	0.4409	0.3902	0.3482	0.3133	0.2807
a_3	0.5701	0.5764	0.4512	0.3207	0.1957	0.1041	0.0105
λ_4	0.8324	0.6444	0.5591	0.5000	0.4523	0.4134	0.3791
a_4	-.0701	0.2778	0.4512	0.4414	0.3711	0.2908	0.2179
λ_5		0.8556	0.6951	0.6098	0.5477	0.5000	0.4613
a_5		-.0660	0.1049	0.3207	0.3711	0.3473	0.3047
λ_6			0.8724	0.7500	0.6518	0.5866	0.5387
a_6			-.0561	0.0000	0.1957	0.2908	0.3047
λ_7				0.8902	0.7895	0.6867	0.6206
a_7				-.0414	-.0372	0.1011	0.2179
λ_8					0.9042	0.8133	0.7193
a_8					-.0296	-.0460	0.0405
λ_9						0.9131	0.8319
a_9						-.0226	-.0455
λ_{10}							0.9209
a_{10}							-.0177
K_1	0.4530	0.4675	0.4734	0.4782	0.4832	0.4865	0.4886
K_2	0.4106	0.4355	0.4566	0.4693	0.4744	0.4790	0.4830
K_3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Δ	0.0382	0.0445	0.0484	0.0513	0.0535	0.0551	0.0563

Table A.2: Optimal Spacing for the regression and scale parameters along with the coefficients for the ABLUE of t-distribution with $p=2$.

$D.F. = 2$							
k	4	5	6	7	8	9	10
λ_1	0.1651	0.1194	0.0924	0.0748	0.0622	0.0527	0.0454
a_1	0.0168	-.0254	-.0326	-.0305	-.0261	-.0216	-.0176
λ_2	0.3920	0.3146	0.2576	0.2136	0.1786	0.1510	0.1293
a_2	0.4832	0.2886	0.1564	0.0755	0.0284	0.0026	-.0105
λ_3	0.6080	0.5000	0.4208	0.3599	0.3109	0.2706	0.2370
a_3	0.4832	0.4735	0.3762	0.2776	0.1955	0.1316	0.0840
λ_4	0.8349	0.6854	0.5792	0.5000	0.4378	0.3871	0.3448
a_4	0.0168	0.2886	0.3762	0.3547	0.3022	0.2451	0.1925
λ_5		0.8806	0.7424	0.6401	0.5622	0.5000	0.4487
a_5		-.0254	0.1564	0.2776	0.3022	0.2847	0.2517
λ_6			0.9076	0.7864	0.6891	0.6129	0.5513
a_6			-.0326	0.0755	0.1955	0.2451	0.2517
λ_7				0.9252	0.8214	0.7294	0.6552
a_7				-.0305	0.0284	0.1316	0.1925
λ_8					0.9378	0.8490	0.7630
a_8					-.0261	0.0026	0.0840
λ_9						0.9473	0.8707
a_9						-.0216	-.0105
λ_{10}							0.9546
a_{10}							-.0176
K_1	0.5534	0.5666	0.5752	0.5807	0.5844	0.5872	0.5894
K_2	0.6682	0.7050	0.7274	0.7437	0.7553	0.7636	0.7698
K_3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Δ	0.1132	0.1282	0.1385	0.1456	0.1508	0.1546	0.1576

Table A.3: Optimal Spacing for the regression and scale parameters along with the coefficients for the ABLUE of t-distribution with $p=2$.

<i>D.F.</i> = 3							
<i>k</i>	4	5	6	7	8	9	10
λ_1	0.1472	0.1098	0.0843	0.0665	0.0539	0.0446	0.0375
a_1	0.0640	0.0124	-.0078	-.0146	-.0160	-.0152	-.0137
λ_2	0.3774	0.2953	0.2365	0.1930	0.1599	0.1344	0.1143
a_2	0.4360	0.2770	0.1688	0.0995	0.0561	0.0293	0.0130
λ_3	0.6226	0.5000	0.4107	0.3432	0.2907	0.2489	0.2152
a_3	0.4360	0.4212	0.3389	0.2555	0.1867	0.1337	0.0942
λ_4	0.8528	0.7047	0.5893	0.5000	0.4296	0.3730	0.3266
a_4	0.0640	0.2770	0.3389	0.3192	0.2732	0.2236	0.1786
λ_5		0.8902	0.7635	0.6568	0.5704	0.5000	0.4419
a_5		0.0124	0.1688	0.2555	0.2732	0.2571	0.2279
λ_6			0.9157	0.8070	0.7093	0.6270	0.5581
a_6			-.0078	0.0995	0.1867	0.2236	0.2279
λ_7				0.9335	0.8401	0.7511	0.6735
a_7				-.0146	0.0561	0.1337	0.1786
λ_8					0.9461	0.8656	0.7848
a_8					-.0160	0.0293	0.0942
λ_9						0.9554	0.8857
a_9						-.0152	0.0130
λ_{10}							0.9625
a_{10}							-.0137
K_1	0.6262	0.6363	0.6432	0.6482	0.6517	0.6544	0.6564
K_2	0.8089	0.8660	0.9005	0.9229	0.9385	0.9498	0.9582
K_3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Δ	0.1986	0.2231	0.2397	0.2513	0.2598	0.2661	0.2710

Table A.4: Optimal Spacing for the regression and scale parameters along with the coefficients for the ABLUE of t-distribution with $p=2$.

$D.F. = 5$							
k	4	5	6	7	8	9	10
λ_1	0.1196	0.0883	0.0676	0.0532	0.0429	0.0352	0.0294
a_1	0.0963	0.0444	0.0196	0.0072	0.0009	-.0023	-.0038
λ_2	0.3572	0.2679	0.2080	0.1659	0.1351	0.1120	0.0943
a_2	0.4037	0.2621	0.1691	0.1099	0.0721	0.0476	0.0314
λ_3	0.6428	0.5000	0.3969	0.3215	0.2651	0.2219	0.1882
a_3	0.4037	0.3871	0.3114	0.2365	0.1763	0.1308	0.0972
λ_4	0.8804	0.7321	0.6031	0.5000	0.4190	0.3550	0.3040
a_4	0.0963	0.2621	0.3114	0.2928	0.2507	0.2060	0.1660
λ_5		0.9117	0.7920	0.6785	0.5810	0.5000	0.4331
a_5		0.0444	0.1691	0.2365	0.2507	0.2359	0.2093
λ_6			0.9324	0.8341	0.7349	0.6450	0.5669
a_6			0.0196	0.1099	0.1763	0.2060	0.2093
λ_7				0.9468	0.8649	0.7781	0.6960
a_7				0.0072	0.0721	0.1308	0.1660
λ_8					0.9571	0.8880	0.8118
a_8					0.0009	0.0476	0.0972
λ_9						0.9648	0.9057
a_9						-.0023	0.0314
λ_{10}							0.9706
a_{10}							-.0038
K_1	0.7076	0.7192	0.7265	0.7315	0.7350	0.7376	0.7396
K_2	0.9820	1.0571	1.1050	1.1373	1.1599	1.1763	1.1887
K_3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Δ	0.3479	0.3933	0.4238	0.4452	0.4606	0.4721	0.4809

Table A.5: Optimal Spacing for the regression and scale parameters along with the coefficients for the ABLUE of t-distribution with $p=2$.

$D.F. = 10$							
k	4	5	6	7	8	9	10
λ_1	0.0923	0.0652	0.0481	0.0368	0.0289	0.0233	0.0191
a_1	0.1175	0.0659	0.0392	0.0244	0.0157	0.0103	0.0069
λ_2	0.3341	0.2368	0.1754	0.1344	0.1059	0.0853	0.0700
a_2	0.3825	0.2510	0.1663	0.1131	0.0790	0.0566	0.0413
λ_3	0.6659	0.5000	0.3798	0.2948	0.2340	0.1894	0.1559
a_3	0.3825	0.3662	0.2944	0.2237	0.1679	0.1266	0.0961
λ_4	0.9077	0.7632	0.6202	0.5000	0.4053	0.3323	0.2757
a_4	0.1175	0.2510	0.2944	0.2775	0.2374	0.1946	0.1570
λ_5		0.9348	0.8246	0.7052	0.5947	0.5000	0.4218
a_5		0.0659	0.1663	0.2237	0.2374	0.2238	0.1983
λ_6			0.9519	0.8656	0.7660	0.6677	0.5782
a_6			0.0392	0.1131	0.1679	0.1946	0.1983
λ_7				0.9632	0.8941	0.8106	0.7243
a_7				0.0244	0.0790	0.1266	0.1570
λ_8					0.9711	0.9147	0.8441
a_8					0.0157	0.0566	0.0964
λ_9						0.9767	0.9300
a_9						0.0103	0.0413
λ_{10}							0.9809
a_{10}							0.0069
K_1	0.7911	0.8069	0.8167	0.8232	0.8277	0.8310	0.8335
K_2	1.1771	1.2729	1.3357	1.3788	1.4095	1.4323	1.4495
K_3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Δ	0.5828	0.6687	0.7275	0.7690	0.7993	0.8219	0.8393

Table A.6: Optimal Spacing for the regression and scale parameters along with the coefficients for the ABLUE of t-distribution with $p=2$.

$D.F. = 20$							
k	4	5	6	7	8	9	10
λ_1	0.0771	0.0523	0.0373	0.0277	0.0212	0.0166	0.0133
a_1	0.1273	0.0757	0.0482	0.0323	0.0225	0.0162	0.0120
λ_2	0.3188	0.2170	0.1549	0.1150	0.0880	0.0690	0.0553
a_2	0.3727	0.2461	0.1649	0.1143	0.0817	0.0601	0.0453
λ_3	0.6812	0.5000	0.3678	0.2766	0.2131	0.1679	0.1348
a_3	0.3727	0.3565	0.2869	0.2179	0.1640	0.1244	0.0958
λ_4	0.9229	0.7830	0.6322	0.5000	0.3955	0.3161	0.2560
a_4	0.1273	0.2461	0.2869	0.2710	0.2318	0.1897	0.1529
λ_5		0.9477	0.8451	0.7234	0.6045	0.5000	0.4135
a_5		0.0757	0.1649	0.2179	0.2318	0.2191	0.1940
λ_6			0.9627	0.8850	0.7869	0.6839	0.5865
a_6			0.0482	0.1143	0.1640	0.1897	0.1940
λ_7				0.9723	0.9120	0.8321	0.7440
a_7				0.0323	0.0817	0.1244	0.1529
λ_8					0.9788	0.9310	0.8652
a_8					0.0225	0.0601	0.0958
λ_9						0.9834	0.9447
a_9						0.0162	0.0453
λ_{10}							0.9867
a_{10}							0.0120
K_1	0.8443	0.8639	0.8760	0.8841	0.8898	0.8940	0.8971
K_2	1.3086	1.4192	1.4929	1.5440	0.5808	1.6081	1.6290
K_3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Δ	0.7876	0.9149	1.0036	1.0670	1.1137	1.1489	1.1761

Table A.7: Optimal Spacing for the regression and scale parameters along with the coefficients for the ABLUE of t-distribution with $p=2$.

$D.F. = 500$								
k		4	5	6	7	8	9	10
λ_1		0.0619	0.0400	0.0272	0.0193	0.0142	0.0108	0.0083
a_1		0.1365	0.0847	0.0562	0.0393	0.0285	0.0214	0.0165
λ_2		0.3007	0.1947	0.1327	0.0944	0.0695	0.0527	0.0408
a_2	0.	0.3635	0.2417	0.1639	0.1155	0.0843	0.0633	0.0488
λ_3		0.6993	0.5000	0.3532	0.2551	0.1892	0.1439	0.1118
a_3		0.3635	0.3472	0.2799	0.2126	0.1604	0.1226	0.0952
λ_4		0.9381	0.8053	0.6468	0.5000	0.3834	0.2965	0.2326
a_4		0.1365	0.2417	0.2799	0.2651	0.2267	0.1852	0.1492
λ_5			0.9600	0.8673	0.7449	0.6166	0.5000	0.4031
a_5			0.0847	0.1639	0.2126	0.2267	0.2150	0.1903
λ_6				0.9728	0.9056	0.8108	0.7035	0.5969
a_6				0.0562	0.1155	0.1604	0.1852	0.1903
λ_7					0.9807	0.9305	0.8561	0.7674
a_7					0.0393	0.0843	0.1226	0.1492
λ_8						0.9858	0.9473	0.8882
a_8						0.0285	0.0633	0.0952
λ_9							0.9892	0.9592
a_9							0.0214	0.0488
λ_{10}								0.9917
a_{10}								0.0165
K_1		0.9060	0.9310	0.9465	0.9571	0.9646	0.9701	0.9743
K_2		1.4666	1.5955	1.6837	1.7452	1.7899	1.8234	1.8491
K_3		0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Δ		1.0905	1.2873	1.4279	1.5302	1.6066	1.6649	1.7102

Appendix A

t-Dist. (Regression Parameters, $p=3$)

Table A.1: Optimal Spacing for the regression and scale parameters along with the coefficients for the ABLUE of t-distribution with $p=3$.

$D.F. = 1$						
k	5	6	7	8	9	10
λ_1	0.1377	0.1253	0.1071	0.0915	0.0836	0.0772
a_1	-.0610	-.0543	-.0397	-.0265	-.0208	-.0166
λ_2	0.3623	0.3159	0.2500	0.2026	0.1816	0.1652
a_2	0.2860	0.1218	0.0000	-.0393	-.0459	-.0454
λ_3	0.5000	0.4443	0.3929	0.3507	0.3184	0.2860
a_3	0.5500	0.4325	0.3242	0.2006	0.1115	0.0466
λ_4	0.6377	0.5557	0.5000	0.4533	0.4164	0.3837
a_4	0.2860	0.4325	0.4310	0.3652	0.2875	0.2207
λ_5	0.8623	0.6841	0.6071	0.5467	0.5000	0.4627
a_5	-.0610	0.1218	0.3242	0.3652	0.3356	0.2947
λ_6		0.8747	0.7500	0.6493	0.5836	0.5373
a_6		-.0543	0.0000	0.2006	0.2875	0.2947
λ_7			0.8929	0.7974	0.6816	0.6163
a_7			-.0397	-.0393	0.1115	0.2207
λ_8				0.9085	0.8184	0.7140
a_8				-.0265	-.0459	0.0466
λ_9					0.9164	0.8348
a_9					-.0208	-.0454
λ_{10}						0.9228
a_{10}						-.0166
K_1	0.4703	0.4752	0.4786	0.4841	0.4872	0.4890
K_2	0.4264	0.4506	0.4681	0.4716	0.4766	0.4815
K_3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Δ	0.0209	0.0230	0.0246	0.0259	0.0268	0.0275

Table A.2: Optimal Spacing the regression and scale parameters along with the coefficients for the ABLUE of t-distribution with $p=3$.

<i>D.F.</i> = 2						
<i>k</i>	5	6	7	8	9	10
λ_1	0.1162	0.0884	0.0719	0.0599	0.0506	0.0434
a_1	-.0248	-.0313	-.0293	-.0252	-.0207	-.0168
λ_2	0.3199	0.2641	0.2204	0.1837	0.1531	0.1293
a_2	0.2946	0.1645	0.0829	0.0327	0.0040	-.0105
λ_3	0.5000	0.4238	0.3654	0.3176	0.2771	0.2430
a_3	0.4603	0.3668	0.2760	0.2002	0.1387	0.0908
λ_4	0.6801	0.5762	0.5000	0.4404	0.3913	0.3500
a_4	0.2946	0.3668	0.3408	0.2923	0.2409	0.1929
λ_5	0.8838	0.7359	0.6346	0.5596	0.5000	0.4506
a_5	-.0248	0.1645	0.2760	0.2923	0.2742	0.2435
λ_6		0.9116	0.7796	0.6824	0.6087	0.5494
a_6		-.0313	0.0829	0.2002	0.2409	0.2435
λ_7			0.9281	0.8163	0.7229	0.6500
a_7			-.0293	0.0327	0.1387	0.1929
λ_8				0.9401	0.8469	0.7570
a_8				-.0252	0.0040	0.0908
λ_9					0.9494	0.8707
a_9					-.0207	-.0105
λ_{10}						0.9566
a_{10}						-.0168
K_1	0.5676	0.5766	0.5817	0.5851	0.5877	0.5897
K_2	0.7007	0.7216	0.7390	0.7523	0.7615	0.7680
K_3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Δ	0.0727	0.0797	0.0846	0.0882	0.0908	0.0929

Table A.3: Optimal Spacing the regression and scale parameters along with the coefficients for the ABLUE of t-distribution with $p=3$.

$D.F. = 3$						
k	5	6	7	8	9	10
λ_1	0.1165	0.0872	0.0674	0.0538	0.0440	0.0368
a_1	0.0182	-.0057	-.0139	-.0157	-.0151	-.0136
λ_2	0.3047	0.2457	0.2011	0.1668	0.1398	0.1183
a_2	0.2809	0.1779	0.1083	0.0629	0.0339	0.0157
λ_3	0.5000	0.4150	0.3498	0.2985	0.2571	0.2231
a_3	0.4017	0.3277	0.2528	0.1896	0.1394	0.1008
λ_4	0.6953	0.2850	0.5000	0.4327	0.3783	0.3333
a_4	0.2809	0.3277	0.3057	0.2633	0.2186	0.1779
λ_5	0.8835	0.7543	0.6502	0.5673	0.5000	0.4444
a_5	0.0182	0.1779	0.2528	0.2633	0.2462	0.2192
λ_6		0.9128	0.7989	0.7015	0.6217	0.5556
a_6		-.0057	0.1083	0.1896	0.2186	0.2192
λ_7			0.9326	0.8332	0.7429	0.6667
a_7			-.0139	0.0629	0.1394	0.1779
λ_8				0.9462	0.8602	0.7769
a_8				-.0157	0.0339	0.1008
λ_9					0.9560	0.8817
a_9					-.0151	0.0157
λ_{10}						0.9632
a_{10}						-.0136
K_1	0.6376	0.6441	0.6489	0.6523	0.6549	0.6568
K_2	0.8599	0.8964	0.9194	0.9354	0.9472	0.9560
K_3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Δ	0.1421	0.1543	0.1630	0.1694	0.1742	0.1779

Table A.4: Optimal Spacing the regression and scale parameters along with the coefficients for the ABLUE of t-distribution with $p=3$.

$D.F. = 5$						
k	5	6	7	8	9	10
λ_1	0.0959	0.0734	0.0576	0.0461	0.0376	0.0312
a_1	0.0522	0.0244	0.0100	0.0025	-.0014	-.0034
λ_2	0.2786	0.2190	0.1760	0.1441	0.1198	0.1008
a_2	0.2638	0.1763	0.1183	0.0798	0.0539	0.0364
λ_3	0.5000	0.4025	0.3301	0.2749	0.2320	0.1980
a_3	0.3682	0.2993	0.2325	0.1777	0.1352	0.1027
λ_4	0.7214	0.5975	0.5000	0.4232	0.3620	0.3126
a_4	0.2638	0.2993	0.2783	0.2400	0.2002	0.1643
λ_5	0.9041	0.7810	0.6699	0.5768	0.5000	0.4365
a_5	0.0522	0.1763	0.2325	0.2400	0.2243	0.1999
λ_6		0.9266	0.8240	0.7251	0.6380	0.5635
a_6		0.0244	0.1183	0.1777	0.2002	0.1999
λ_7			0.9424	0.8559	0.7680	0.6874
a_7			0.0100	0.0798	0.1352	0.1643
λ_8				0.9539	0.8802	0.8020
a_8				0.0025	0.0539	0.1027
λ_9					0.9624	0.8992
a_9					-.0014	0.0364
λ_{10}						0.9688
a_{10}						-.0034
K_1	0.7213	0.7280	0.7326	0.7358	0.7383	0.7401
K_2	1.0465	1.0974	1.1315	1.1555	1.1728	1.1858
K_3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Δ	0.2833	0.3082	0.3259	0.3388	0.3484	0.3558

Table A.5: Optimal Spacing the regression and scale parameters along with the coefficients for the ABLUE of t-distribution with $p=3$.

$D.F. = 10$						
k	5	6	7	8	9	10
λ_1	0.0710	0.0527	0.0404	0.0319	0.0257	0.0210
a_1	0.0730	0.0444	0.0281	0.0183	0.0122	0.0083
λ_2	0.2474	0.1856	0.1436	0.1139	0.0923	0.0760
a_2	0.2521	0.1718	0.1197	0.0853	0.0620	0.0460
λ_3	0.5000	0.3861	0.3041	0.2442	0.1995	0.1655
a_3	0.3499	0.2838	0.2201	0.1689	0.1300	0.1009
λ_4	0.7526	0.6139	0.5000	0.4103	0.3402	0.2852
a_4	0.2521	0.2838	0.2643	0.2275	0.1893	0.1554
λ_5	0.9290	0.8144	0.6929	0.5897	0.5000	0.4259
a_5	0.0730	0.1718	0.2201	0.2275	0.2129	0.1895
λ_6		0.9473	0.8564	0.7558	0.6598	0.5741
a_6		0.0444	0.1197	0.1689	0.1893	0.1895
λ_7			0.9596	0.8861	0.8005	0.7148
a_7			0.0281	0.0853	0.1300	0.1554
λ_8				0.9681	0.9077	0.8345
a_8				0.0183	0.0620	0.1009
λ_9					0.9743	0.9240
a_9					0.0122	0.0460
λ_{10}						0.9790
a_{10}						0.0083
K_1	0.8092	0.8183	0.8245	0.8287	0.8318	0.8342
K_2	1.2607	1.3262	1.3712	1.4034	1.4272	1.4453
K_3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Δ	0.5404	0.5948	0.6335	0.6620	0.6834	0.6999

Table A.6: Optimal Spacing the regression and scale parameters along with the coefficients for the ABLUE of t-distribution with $p=3$.

$D.F. = 20$						
k	5	6	7	8	9	10
λ_1	0.0569	0.0408	0.0304	0.0233	0.0184	0.0147
a_1	0.0821	0.0530	0.0359	0.0252	0.0183	0.0136
λ_2	0.2271	0.1642	0.1230	0.0948	0.0748	0.0602
a_2	0.2469	0.1697	0.1199	0.0870	0.0648	0.0493
λ_3	0.5000	0.3744	0.2859	0.2229	0.1773	0.1434
a_3	0.3419	0.2773	0.2148	0.1650	0.1275	0.0996
λ_4	0.7729	0.6256	0.5000	0.4007	0.3242	0.2654
a_4	0.2469	0.2773	0.2589	0.2227	0.1849	0.1516
λ_5	0.9431	0.8358	0.7141	0.5993	0.5000	0.4178
a_5	0.0821	0.1697	0.2148	0.2227	0.2089	0.1859
λ_6		0.9592	0.8770	0.7771	0.6758	0.5822
a_6		0.0530	0.1199	0.1650	0.1849	0.1859
λ_7			0.9696	0.9052	0.8227	0.7346
a_7			0.0359	0.0870	0.1275	0.1516
λ_8				0.9767	0.9252	0.8566
a_8				0.0252	0.0648	0.0996
λ_9					0.9816	0.9398
a_9					0.0183	0.0493
λ_{10}						0.9853
a_{10}						0.0136
K_1	0.8661	0.8777	0.8854	0.8909	0.8948	0.8978
K_2	1.4069	1.4832	1.5362	1.5744	1.6028	1.6244
K_3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Δ	0.7915	0.8800	0.9441	0.9916	1.0276	1.0555

Table A.7: Optimal Spacing the regression and scale parameters along with the coefficients for the ABLUE of t-distribution with $p=3$.

$D.F. = 500$						
k	5	6	7	8	9	10
λ_1	0.0433	0.0296	0.0211	0.0156	0.0118	0.0092
a_1	0.0903	.0605	.0425	0.0311	0.0234	0.0180
λ_2	0.2041	0.1407	0.1009	0.0748	0.0569	0.0443
a_2	0.2425	0.1681	0.1203	0.0887	0.0672	0.0521
λ_3	0.5000	0.3602	0.2642	0.1982	0.1521	0.1190
a_3	0.3344	0.2715	0.2100	0.1615	0.1253	0.0985
λ_4	0.7959	0.6398	0.5000	0.3888	0.3046	0.2416
a_4	0.2425	0.2715	0.2543	0.2187	0.1812	0.1483
λ_5	0.9567	0.8593	0.7358	0.6112	0.5000	0.4076
a_5	0.0903	0.1681	0.2100	0.2187	0.2058	0.1830
λ_6		0.9704	0.8991	0.8018	0.6954	0.5924
a_6		0.0605	0.1203	0.1615	0.1812	0.1830
λ_7			0.9789	0.9252	0.8479	0.7584
a_7			0.0425	0.0887	0.1253	0.1483
λ_8				0.9844	0.9431	0.8810
a_8				0.0311	0.0672	0.0985
λ_9					0.9882	0.9557
a_9					0.0234	0.0521
λ_{10}						0.9908
a_{10}						0.0180
K_1	0.9330	0.9481	0.9584	0.9656	0.9710	0.9750
K_2	1.5834	1.6739	1.7374	1.7835	1.8180	1.8445
K_3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Δ	1.1999	1.3528	1.4657	1.5507	1.6159	1.6669

Appendix A

Logistic Distribution Table

Table A.1: Optimal Spacing for the regression and scale parameters along with the coefficients for the ABLUE of Logistic distribution when Δ is maximized

k	2	3	4	5	6	7	8	9	10
λ_1	0.1873	0.1213	0.0752	0.0517	0.0363	0.0265	0.0199	0.0153	0.0120
a_1	0.5000	0.1801	0.0686	0.0326	0.0164	0.0089	0.0051	0.0031	0.0019
b_1	0.3407	-0.2526	-0.1424	-0.0929	-0.0619	-0.0437	-0.0318	-0.0239	-0.0184
λ_2	0.8127	0.5000	0.3001	0.2086	0.1496	0.1112	0.0849	0.0663	0.0527
a_2	0.5000	0.6397	0.4314	0.2355	0.1271	0.0734	0.0442	0.0277	0.0179
b_2	0.3407	0.0000	-0.1686	-0.1724	-0.1370	-0.1054	-0.0809	-0.0629	-0.0496
λ_3		0.8787	0.6999	0.5000	0.3554	0.2657	0.2042	0.1609	0.1292
a_3		0.1801	0.4314	0.4638	0.3566	0.2357	0.1511	0.0989	0.0663
b_3		0.2526	0.1686	0.0000	-0.0977	-0.1213	-0.1132	-0.0974	-0.0816
λ_4			0.9248	0.7914	0.6446	0.5000	0.3863	0.3050	0.2455
a_4			0.0686	0.2355	0.3566	0.3639	0.2996	0.2206	0.1568
b_4			0.1424	0.1724	0.0977	0.0000	-0.0639	-0.0889	-0.0915
λ_5				0.9483	0.8510	0.7343	0.6137	0.5000	0.4060
a_5				0.0326	0.1271	0.2357	0.2996	0.2993	0.2570
b_5				0.0929	0.1370	0.1213	0.0639	0.0000	-0.0451
λ_6					0.9637	0.8888	0.7958	0.6950	0.5910
a_6					0.0164	0.0734	0.1511	0.2206	0.2570
b_6					0.0619	0.1054	0.1132	0.0889	0.0451
λ_7						0.9735	0.9151	0.8391	0.7545
a_7						0.0089	0.0442	0.0989	0.1568
b_7						0.0437	0.0809	0.0974	0.0915
λ_8							0.9801	0.9337	0.8708
a_8							0.0051	0.0277	0.0663
b_8							0.0318	0.0629	0.0816
λ_9								0.9847	0.9473
a_9								0.0031	0.0179
b_9								0.0239	0.0496
λ_{10}									0.9880
a_{10}									0.0019
b_{10}									0.0184
K_1	0.2474	0.2959	0.3042	0.3142	0.3184	0.3219	0.3241	0.3257	0.3270
K_2	0.8521	0.9697	1.1267	1.1961	1.2518	1.2874	1.3143	1.3339	1.3490
K_3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Δ	0.2108	0.2870	0.3427	0.3758	0.3986	0.4144	0.4259	0.4345	0.4411

Table A.2: Optimal Spacing for the regression and scale parameters along with the coefficients for the ABLUE of Logistic distribution when $K_1\Delta$ is maximized

k	3	4	5	6	7	8	9	10
λ_1	0.1381	0.0895	0.0611	0.0433	0.0317	0.0239	0.0184	0.0144
a_1	0.1984	0.0873	0.0424	0.0222	0.0123	0.0071	0.0043	0.0027
b_1	-.2731	-.1680	-.1092	-.0743	-.0526	-.0386	-.0290	-.0224
λ_2	0.5000	0.3331	0.2348	0.1718	0.1294	0.0998	0.0784	0.0626
a_2	0.6032	0.4127	0.2486	0.1486	0.0907	0.0569	0.0366	0.0242
b_2	0.0000	-.1587	-.1707	-.1452	-.1165	-.0921	-.0730	-.0583
λ_3	0.8619	0.6669	0.5000	0.3787	0.2925	0.2302	0.1841	0.1495
a_3	0.1984	0.4127	0.4179	0.3293	0.2368	0.1656	0.1155	0.0811
b_3	0.2731	0.1587	0.0000	-.0840	-.1110	-.1110	-.1008	-.0879
λ_4		0.9105	0.7652	0.6213	0.5000	0.4041	0.3297	0.2719
a_4		0.0873	0.2486	0.3293	0.3204	0.2704	0.2134	0.1636
b_4		0.1680	0.1707	0.0840	0.0000	-.0522	-.0769	-.0846
λ_5			0.9389	0.8282	0.7075	0.5959	0.5000	0.4206
a_5			0.0424	0.1486	0.2368	0.2704	0.2603	0.2284
b_5			0.1092	0.1452	0.1110	0.0522	0.0000	-.0357
λ_6				0.9567	0.8706	0.7698	0.6703	0.5794
a_6				0.0222	0.0907	0.1656	0.2134	0.2284
b_6				0.0743	0.1165	0.1110	0.0769	0.0357
λ_7					0.9683	0.9002	0.8159	0.7281
a_7					0.0123	0.0569	0.1155	0.1636
b_7					0.0526	0.0921	0.1008	0.0846
λ_8						0.9761	0.9216	0.8505
a_8						0.0071	0.0366	0.0811
b_8						0.0386	0.0730	0.0879
λ_9							0.9816	0.9374
a_9							0.0043	0.0242
b_9							0.0290	0.0583
λ_{10}								0.9856
a_{10}								0.0027
b_{10}								0.0224
K_1	0.3000	0.3108	0.3173	0.3212	0.3238	0.3257	0.3270	0.3281
K_2	0.9505	1.0935	1.1794	1.2366	1.2763	1.3050	1.3263	1.3426
K_3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Δ	0.0855	0.1056	0.1187	0.1276	0.1339	0.1384	0.1419	0.1445

Table A.3: Optimal Spacing for the regression and scale parameters along with the coefficients for the ABLUE of Logistic distribution when $K_1^2 \Delta$ is maximized

k	4	5	6	7	8	9	10
λ_1	0.1000	0.0687	0.0490	0.0361	0.0272	0.0210	0.0165
a_1	0.0999	0.0505	0.0271	0.0153	0.0090	0.0055	0.0035
b_1	-.1842	-.1218	-.0839	-.0599	-.0441	-.0333	-.0258
λ_2	0.3475	0.2515	0.1874	0.1430	0.1112	0.0880	0.0707
a_2	0.4061	0.2546	0.1605	0.1023	0.0663	0.0438	0.0295
b_2	-.1510	-.1675	-.1480	-.1224	-.0992	-.0800	-.0648
λ_3	0.6525	0.5000	0.3889	0.3073	0.2462	0.1998	0.1640
a_3	0.4001	0.3898	0.3124	0.2340	0.1711	0.1242	0.0903
b_3	0.1510	0.0000	-.0758	-.1034	-.1072	-.1007	-.0903
λ_4	0.9000	0.7485	0.6111	0.5000	0.4121	0.3426	0.2870
a_4	0.0999	0.2546	0.3124	0.2967	0.2536	0.2063	0.1639
b_4	0.1842	0.1675	0.0758	0.0000	-.0460	-.0694	-.0787
λ_5		0.9313	0.8126	0.6927	0.5879	0.5000	0.4272
a_5		0.0505	0.1605	0.2340	0.2536	0.2403	0.2128
b_5		0.1218	0.1480	0.1034	0.0460	0.0000	-.0310
λ_6			0.9510	0.8570	0.7538	0.6574	0.5728
a_6			0.0271	0.1023	0.1711	0.2063	0.2128
b_6			0.0839	0.1224	0.1072	0.0694	0.0310
λ_7				0.9639	0.8888	0.8002	0.7130
a_7				0.0153	0.0663	0.1242	0.1639
b_7				0.0599	0.0992	0.1007	0.0787
λ_8					0.9728	0.9120	0.8360
a_8					0.0090	0.0438	0.0903
b_8					0.0441	0.0800	0.0903
λ_9						0.9790	0.9293
a_9						0.0055	0.0295
b_9						0.0333	0.0648
λ_{10}							0.9835
a_{10}							0.0035
b_{10}							0.0258
K_1	0.3131	0.3188	0.3224	0.3248	0.3264	0.3277	0.3286
K_2	1.0741	1.1653	1.2256	1.2675	1.2978	1.3203	1.3375
K_3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Δ	0.0330	0.0378	0.0411	0.0434	0.0451	0.0464	0.0474

Table A.4: Optimal Spacing for the regression and scale parameters along with the coefficients for the ABLUE of Logistic distribution when $K_1^3 \Delta$ is maximized

k	5	6	7	8	9	10
λ_1	0.0754	0.0541	0.0399	0.0302	0.0234	0.0184
a_1	0.0573	0.0315	0.0181	0.0108	0.0067	0.0043
b_1	-.1320	-.0919	-.0661	-.0490	-.0371	-.0288
λ_2	0.2628	0.1989	0.1536	0.1206	0.0961	0.0776
a_2	0.2572	0.1682	0.1106	0.0737	0.0497	0.0341
b_2	-.1640	-.1486	-.1259	-.1040	-.0852	-.0698
λ_3	0.5000	0.3950	0.3168	0.2573	0.2113	0.1751
a_3	0.3709	0.3004	0.2305	0.1733	0.1293	0.0964
b_3	0.0000	-.0703	-.0975	-.1034	-.0993	-.0910
λ_4	0.7372	0.6050	0.5000	0.4170	0.3507	0.2971
a_4	0.2572	0.3004	0.2815	0.2422	0.2004	0.1626
b_4	0.1640	0.0703	0.0000	-.0420	-.0643	-.0742
λ_5	0.9246	0.8011	0.6832	0.5830	0.5000	0.4312
a_5	0.0573	0.1682	0.2305	0.2422	0.2276	0.2026
b_5	0.1320	0.1486	0.0975	0.0420	0.0000	-.0281
λ_6		0.9459	0.8464	0.7427	0.6493	0.5688
a_6		0.0315	0.1106	0.1733	0.2004	0.2026
b_6		0.0919	0.1259	0.1034	0.0643	0.0281
λ_7			0.9601	0.8794	0.7887	0.7029
a_7			0.0181	0.0737	0.1293	0.1626
b_7			0.0661	0.1040	0.0993	0.0742
λ_8				0.9698	0.9039	0.8249
a_8				0.0108	0.0497	0.0964
b_8				0.0490	0.0852	0.0910
λ_9					0.9766	0.9224
a_9					0.0067	0.0341
b_9					0.0371	0.0698
λ_{10}						0.9816
a_{10}						0.0043
b_{10}						0.0288
K_1	0.3198	0.3231	0.3253	0.3269	0.3280	0.3289
K_2	1.1534	1.2163	1.2600	1.2916	1.3152	1.3332
K_3	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Δ	0.0121	0.0133	0.0141	0.0147	0.0152	0.0156

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