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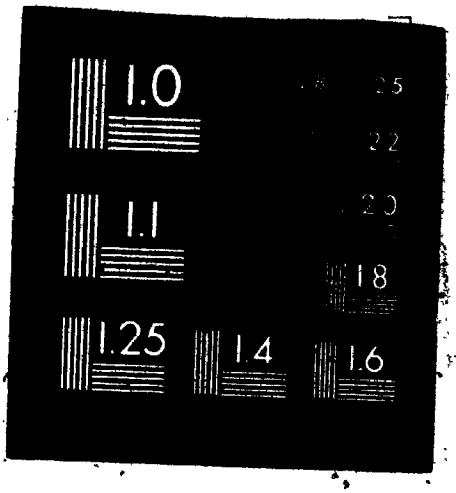

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(Signed) Rilla Nichols

PERMANENT ADDRESS:

18 Walgate Avenue.....

Ottawa, Ontario.....

.....

DATED.....May 20.....1975

Supervisor: C.W.L. Garner

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BAER SUBPLANES OF PROJECTIVE PLANES

BY

Rilla Nichols, B.Sc.

A thesis submitted to the Faculty of
Graduate Studies in partial fulfilment
of the requirements for the degree of

Master of Science

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The undersigned hereby recommend to the Faculty of Graduate Studies acceptance of this thesis, submitted by Rilla Nichols, B.Sc. in partial fulfilment of the requirements for the degree of Master of Science.

D. A. Dawson
.....
CHAIRMAN, DEPARTMENT OF MATHEMATICS

Cyril W. L. Garner
.....
SUPERVISOR

.....
EXTERNAL EXAMINER

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ABSTRACT

The first chapter contains the geometric definitions and concepts which are used in the three main chapters. In the second chapter we discuss four general results involving Baer subplanes. We first consider the definition of Baer subplanes and show the differences in the definition of Baer subplanes of finite and infinite projective planes using an example due to Barlotti [1]. Next we show that a finite projective plane π of square order is uniquely determined by the structure of $\pi - \pi_0$ (denoted π^{π_0}) where π_0 is any Baer subplane of π (see [8] p. 315). We then use a dual affine plane with a certain class of Baer subplanes to define an affine space of dimension greater than two (see [5], [6], [7]). Then this result is used to prove Cofman's result that the Baer subplanes belonging to a derivation set are desarguesian (see [7]). The third and fourth chapters contain two uses of Baer subplanes, namely characterizations and construction of projective planes. We give two characterizations due to Cofman for finite desarguesian planes of square order:

A finite projective plane of square order is desarguesian if and only if:

(1) the vertices of every quadrangle are contained in a unique Baer subplane ([5], [6]);

or (2) the vertices of every quadrangle are fixed by a unique Baer involution ([4]).

Then we describe Ostrom's technique for construction of projective planes, namely derivation. We give a proof that a derivable plane of arbitrary order (finite or infinite) can be used to define a projective plane and we give an example of a derivation of a dual translation plane which leads to the construction of a plane which is neither a translation plane nor a dual translation plane (see [13]).

TABLE OF CONTENTS

I. BASIC CONCEPTS AND DEFINITIONS

An <u>incidence structure</u> is a triple $(\mathcal{P}, \mathcal{B}, \mathcal{I})$ where $\mathcal{P}$, $\mathcal{B}$ and $\mathcal{I}$ are nonempty sets with $\mathcal{P} \cap \mathcal{B} = \emptyset$ and $\mathcal{I} \subset \mathcal{P} \times \mathcal{B}$. The elements of $\mathcal{P}$ are called <u>points</u>; the elements of $\mathcal{B}$ are called <u>blocks</u>; and the elements of $\mathcal{I}$ are <u>flags</u>. Points will usually be denoted by A,B,C,..., and blocks by a,b,c,... . If $(A,a) \in \mathcal{I}$ then we say "A is incident with a" and write "$A \mathcal{I} a$" or "$A \in a$". We shall be interested in the following incidence structures for which blocks are called <u>lines</u>.

A <u>projective plane</u> π is an incidence structure satisfying the following three axioms:

I. Given any two points P, Q of π there is a unique line l of π such that $P,Q \mathcal{I} l$.

II. Given any two lines l, m of π there is a unique point P of π such that $P \mathcal{I} l,m$.

III. There exists a quadrangle in π (i.e. four points no three of which are collinear).

Note that the points incident with a common line are said to be <u>collinear</u> and lines passing through a common point are said to be <u>concurrent</u>.

Clearly axioms I and II are "dual"; that is by replacing the word 'point' in I by 'line' and 'line' by 'point' we get axiom II. It is immediate from axiom III that there exists a quadrilateral (i.e. four lines no three of which are concurrent) and conversely. Thus if we consider

the structure π' dual to a projective plane π we can easily see that π' is also a projective plane. Then given any theorem about a projective plane π, by dualizing the theorem we get a theorem about π'. But π' is a projective plane. Thus the dual of any theorem about a projective plane is still a theorem about a projective plane. This is the <u>Principle of Duality</u> for projective planes.

An <u>affine plane</u> $\mathfrak{A}$ is an incidence structure which satisfies the following axioms:

I Given any two points P, Q of $\mathfrak{A}$, there exists a unique line l of $\mathfrak{A}$ such that P, Q I l.

II Given any line l of $\mathfrak{A}$ and any point P of $\mathfrak{A}$ with P not incident with l (i.e. P $\not\mathrm{I}$ l) there is a unique line m of $\mathfrak{A}$ such that P I m and l and m have no points of $\mathfrak{A}$ in common.

III There is a <u>triangle</u> in $\mathfrak{A}$ (i.e. three noncollinear points of $\mathfrak{A}$).

Define two lines l, m of $\mathfrak{A}$ to be <u>parallel</u> if l = m or if l and m have no points of $\mathfrak{A}$ in common. It is easy to see that parallelism is an equivalence relation on the set of lines of $\mathfrak{A}$.

The incidence structure dual to an affine plane is called a <u>dual affine plane</u>. From axioms I and II of an affine plane, it is clear that a dual affine plane is not an affine plane. The axioms for a dual affine plane $\mathfrak{A}'$ can be obtained by dualizing the axioms of an

affine plane; i.e. they are:

I Given any two lines p,q of $\mathfrak{A}'$ there is a unique point L of $\mathfrak{A}'$ such that L I p,q.

II Given any point L of $\mathfrak{A}'$ and any line p of $\mathfrak{A}'$ with L $\not\mathrm{I}$ p, there is a unique point M of $\mathfrak{A}'$ such that M I p and L and M have no lines of $\mathfrak{A}'$ in common.

III There is a trilateral in $\mathfrak{A}'$ (i.e. three non-concurrent lines).

In this case we have an equivalence relation, parallelism, defined on the points.

We can easily see that the structure π^{l} obtained from a projective plane π by removing some line l of π and all the points of π incident with l is an affine plane. Similarly the structure π^{G} obtained from π by removing some point G of π and all lines of π through G is a dual affine plane. Conversely, given an affine plane $\mathfrak{A}$ we can uniquely embed $\mathfrak{A}$ in a projective plane as follows. Define a structure $\mathfrak{A}^*$ to be $\mathfrak{A}$ together with certain <u>special points</u> and a <u>special line</u> (usually denoted by l_∞) where the 'special points' are the equivalence classes of the lines of $\mathfrak{A}$ under parallelism and the 'special line' is the set of all special points. Then $\mathfrak{A}^*$ is a projective plane and this process is unique. Similarly a dual affine plane $\mathfrak{A}'$ can be uniquely embedded in a projective plane, but here we define <u>special lines</u> as the equivalence classes of parallel points and the

special point as the collection of all special lines. Note that we shall refer to the points and lines of $\mathcal{A}$ or $\mathcal{A}'$ which are not special as affine.

A projective plane π is said to be finite if some line l of π has a finite number of points. Otherwise π is said to be infinite. In particular if some line of a finite projective plane π has n+1 points then π is said to be of order n. Note that there are one-to-one correspondences between the points on any two lines and so the order of a finite projective plane is well-defined. The order of an affine plane or a dual affine plane is the order of the corresponding projective plane. Using the fact that there is a one-to-one correspondence between the points on any line and the lines through any point and using the axioms for a projective plane the following can be shown.

Theorem. For a finite projective plane of order n there are:

(a) n+1 points on each line;

(b) n+1 lines through each point;

(c) n^2+n+1 points in π ; and

(d) n^2+n+1 lines in π .

As an affine plane is obtained by removing one line from a projective plane and one point from each of the remaining lines, the following is an immediate corollary.

<u>Corollary</u>. For a finite affine plane $\mathfrak{A}$ of order n, there are:

(a) n points on each line;

(b) n+1 lines through each point;

(c) n^2 points in $\mathfrak{A}$; and

(d) n^2+n lines in $\mathfrak{A}$.

Similarly the following holds for a dual affine plane.

<u>Corollary</u>. For a finite dual affine plane $\mathfrak{A}'$ of order n, there are:

(a) n+1 points on each line;

(b) n lines through each point;

(c) n^2+n points in $\mathfrak{A}'$; and

(d) n^2 lines in $\mathfrak{A}'$.

A <u>subplane</u> π_0 of a projective plane π is an incidence structure consisting of a subset of the points of π , a subset of the lines of π and a subset of the incidence relation in π such that π_0 satisfies the axioms of a projective plane. An <u>affine subplane</u> $\mathfrak{A}_0$ of an affine plane $\mathfrak{A}$ and a <u>dual affine subplane</u> $\mathfrak{A}_0'$ of a dual affine plane $\mathfrak{A}'$ are defined analagously; but note that the projective extension of $\mathfrak{A}_0$ must contain the special line of $\mathfrak{A}$ as its special line and similarly the projective extension of $\mathfrak{A}_0'$ must contain the special point of $\mathfrak{A}'$ as its special point.

The following is true for subplanes of finite projective planes(see for example [9] p.81).

<u>Theorem</u> (Bruck). If π is a finite projective plane of order n with a proper subplane π_0 of order m then either $n=m^2$ or $n \geq m^2+m$.

The proof of this theorem can be divided into two cases:

(1) every line of π is incident with at least one point of π_0, and every point of π is incident with at least one line of π_0; and

(2) some point of π is not incident with a line of π_0.

The first possibility gives $n=m^2$ (see chapter II § 1) and the second gives $n \geq m^2+m$.

We define a <u>Baer subplane</u> π_0 of a projective plane π of arbitrary order (finite or infinite) to be a subplane of π such that:

(A) every point of π is incident with at least one line of π_0; and

(B) every line of π is incident with at least one point of π_0.

We shall consider the definition of Baer subplanes in more detail in chapter II § 1. By an <u>affine Baer subplane</u> of an affine plane $\mathcal{A}$ we mean an affine subplane $\mathcal{A}_0$ of $\mathcal{A}$ such that the projective extension of $\mathcal{A}_0$ is a Baer subplane of the projective extension of $\mathcal{A}$. A <u>dual affine Baer subplane</u> is defined analagously.

If a projective plane π of arbitrary order has a class of Baer subplanes $\mathcal{B}$ satisfying the condition:

if any two Baer subplanes of $\mathcal{B}$ have three collinear points A,B,C in common then they coincide in all points on the line l joining A,B,C (1.1)

then we say that "segments are well-defined in π "and we call the set of points on l of Baer subplanes in $\mathcal{B}$ containing the three collinear points A,B,C the <u>segment</u> determined by A,B,C.

If l is a line (or G a point) of at least one subplane of $\mathcal{B}$, then let $\mathcal{C}$ be the set of affine (or dual affine) subplanes of π^l (or π^G) whose projective extensions are a subset $\mathcal{B}'$ of $\mathcal{B}$. Then segments are well-defined in π (with respect to subplanes of $\mathcal{B}'$) if and only if in π^l (or π^G) whenever two subplanes of $\mathcal{C}$ have two affine points A,B in common on a line then they coincide in all points on the line AB. We shall refer more often to segments of affine and dual affine planes. If (1.1) is replaced by:

if any two Baer subplanes of $\mathcal{B}$ have three concurrent lines a,b,c in common then they coincide in all their lines through the point of intersection P of a,b,c (1.2)

then we call the set of lines determined by a,b,c through P a <u>segment</u> also and say that "segments are well-defined" in π . A similar result relates the projective case to the affine and dual affine cases.

A projective plane π of arbitrary order (finite or infinite) is said to be <u>derivable</u> if there exists a subset $\mathcal{D}$ of points on some line l_∞ of π such that for any two affine points X,Y of π^{l_∞} with the property that the point of intersection of XY and l_∞ (i.e. $XY \cap l_\infty$) is in $\mathcal{D}$, there exists an affine Baer subplane $\mathcal{A}$ of π^{l_∞} containing X and Y and having precisely the points of $\mathcal{D}$ as its special points. (see Fig. 1). The set $\mathcal{D}$ is called a

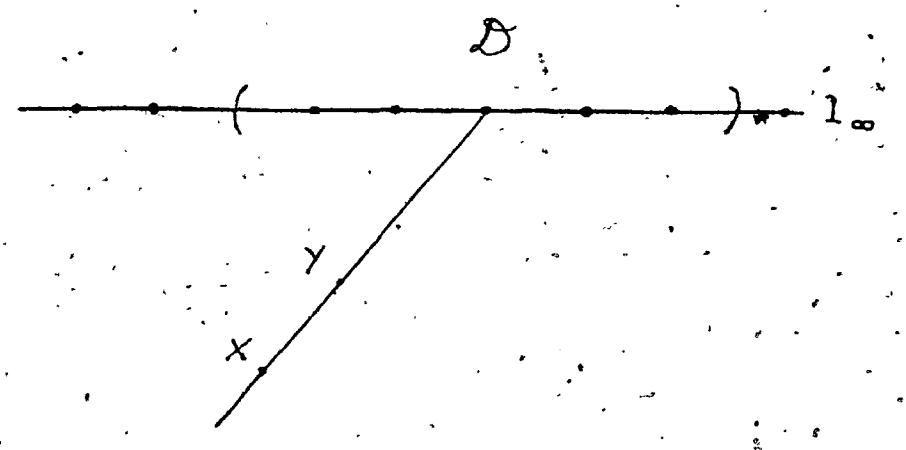

Fig. 1

<u>derivation set</u> for π. The Baer subplanes containing $\mathcal{D}$ are said to <u>belong to</u> $\mathcal{D}$.

An <u>isomorphism</u> between two incidence structures $\mathcal{I}_1, \mathcal{I}_2$ is a bijection of the points of $\mathcal{I}_1$ onto the points of $\mathcal{I}_2$ and a bijection of the blocks of $\mathcal{I}_1$ onto the blocks of $\mathcal{I}_2$ such that the incidence relation is preserved. If parallelism is defined in $\mathcal{I}_1$ and $\mathcal{I}_2$ then the isomorphism

preserves this also. It is easy to show that an isomorphic copy of a projective plane, an affine plane or a dual affine plane is respectively a projective, affine or dual affine plane. If $\mathcal{S}_1 = \mathcal{S}_2 = \mathcal{S}$ then an isomorphism is called a <u>collineation</u> or an <u>automorphism</u>. A point X of $\mathcal{S}$ is said to be <u>fixed by a collineation</u> α if $X^\alpha = X$; similarly a <u>fixed block</u> x is a block x such that $x^\alpha = x$. Note that if a collineation α of a projective plane π is to induce a collineation of the affine plane π^l or of the dual affine plane π^G then l (or G) must be fixed by α.

Denote the set of points and lines fixed by a collineation α of a projective plane (or an affine plane or a dual affine plane) by $F(\alpha)$. If A,B are points of $F(\alpha)$ then $(AB)^\alpha = A^\alpha B^\alpha = AB$ and so the line joining A and B is in $F(\alpha)$; dually the point of intersection of two lines of $F(\alpha)$ is in $F(\alpha)$: such a set is called a <u>closed subset</u> of π. If $F(\alpha)$ satisfies conditions (A) and (B) in the definition of a Baer subplane (i.e. $F(\alpha)$ forms a <u>Baer subset</u>) then α is called a <u>quasiperspectivity</u>. We can show there are only the following three possibilities for the fixed elements of a quasiperspectivity.

(1) The fixed elements form a subplane (i.e. $F(\alpha)$ contains a quadrangle).

(2) $F(\alpha) = \{X \mid X \mathrm{I} l \text{ or } X = V\} \cup \{x \mid V \mathrm{I} x \text{ or } x = l\}$ for some point-line pair V,l with $V \not\mathrm{I} l$. (see Fig.2).

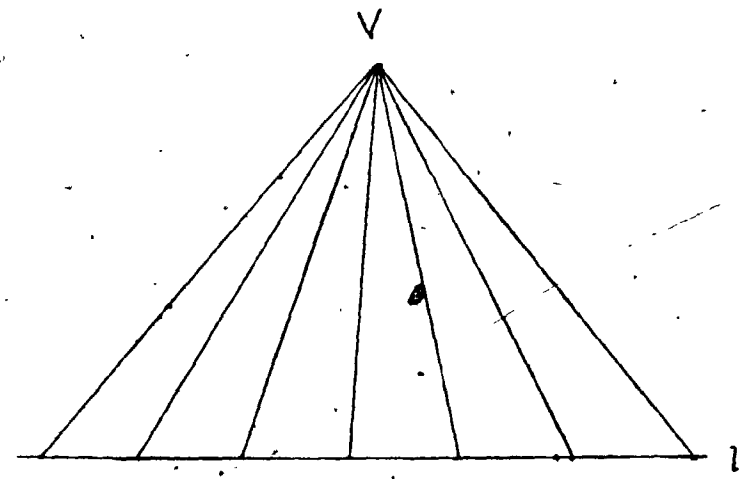

Fig:2

(3) $F(\alpha)$ as in (2) but with $V \mathrm{I} l$ (see Fig.3).

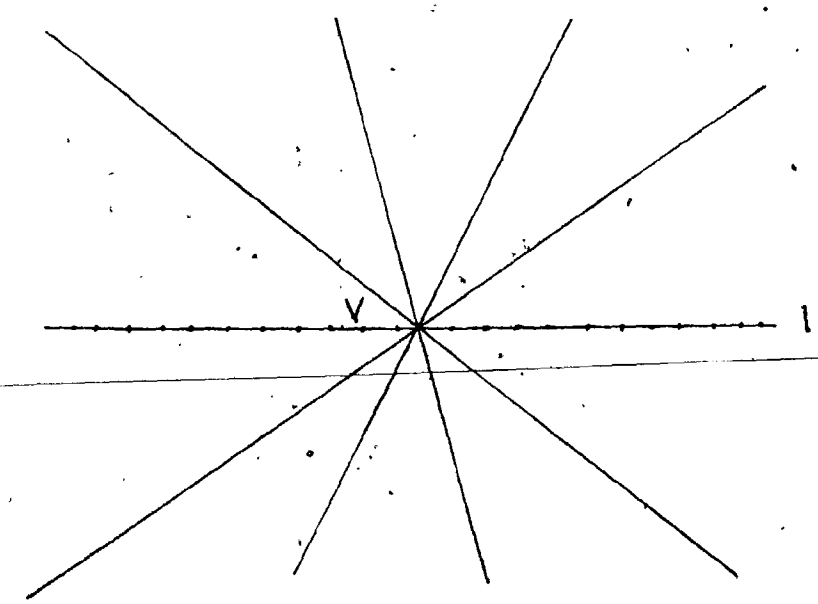

Fig.3

If $F(\alpha)$ is a subplane for a quasiperspectivity α, then α is called a <u>Baer collineation</u>. Note that the fixed subplane of a Baer collineation is a Baer subplane. A

quasiperspectivity α with $F(\alpha)$ as in Fig.2 or Fig.3 is called a (V,l)-<u>perspectivity</u>; if $V \not\perp l$ then α is called an <u>homology</u> and if $V \perp l$, α is called an <u>elation</u>. For (2) and (3) V is called the <u>centre</u> and l the <u>axis</u> of α. A collineation α has a centre if and only if it has an axis. Note that if α is a quasiperspectivity then $F(\alpha)$ is maximal in the sense that if any point or line is adjoined to $F(\alpha)$ the only closed set containing this new set is π.

Let Aut π denote the set of all collineations of a projective plane π. Clearly as Aut π consists of bijections, Aut π is a group under composition. By a <u>collineation group</u> of π we mean a subgroup of Aut π. Let Γ be a collineation group of a projective plane π. Let $\Gamma_{(V,l)}$ denote the subset consisting of all (V,l)-perspectivities of Γ; let $\Gamma_{(m,l)}$ denote the subset consisting of all (X,l)-perspectivities of Γ for some fixed line l and points X incident with some line m; let $\Gamma_{(M,L)}$ denote the subset of all (M,x)-perspectivities of Γ for a fixed point M and all lines x through the point L. If we say that the identity collineation is a (V,l)-perspectivity for all V,l then clearly $\Gamma_{(V,l)}$, $\Gamma_{(m,l)}$ and $\Gamma_{(M,L)}$ are subgroups of Γ (see [9] p.96). Now as Aut π is a group we can define an <u>involution</u> or an <u>involutory collineation</u> to be a collineation of order two. Any involution is a quasiperspectivity. Clearly a Baer involution is a Baer collineation of order two. (For the proof of these statements and other results on collineations see [8] section 3.1 and [9] chapter IV).

A projective plane π is said to be (V,l)-<u>transitive</u> if, given any pair of points X,Y distinct from V and not incident with l but with $V \mathrm{I} XY$ there exists a collineation γ which is a (V,l)-perspectivity of π such that $X^{\gamma} = Y$ (see Fig.4). If for some line l, π is (X,l)-transitive for all

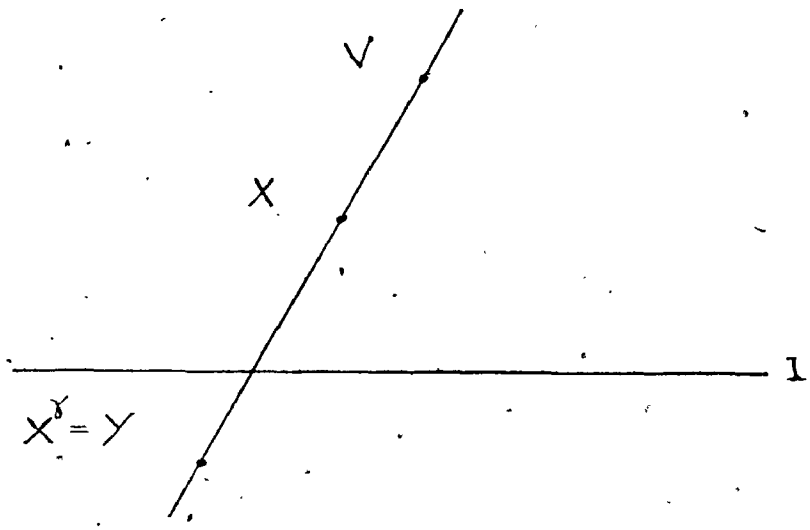

Fig.4

$X \mathrm{I} l$, then l is said to be a <u>translation line</u>. Dually if for some point G, π is (G,x)-transitive for all lines x through G, then G is said to be a <u>translation point</u>. If π has a translation line l or a translation point G, then π is called a <u>translation plane</u> with respect to l or a <u>dual translation plane</u> with respect to G, respectively. Note that if π is a translation plane with respect to some line l, it is natural to consider the affine plane π^{l}, which is also called a <u>translation plane</u>. Elations with axis l in π are called <u>translations</u> in π^{l}.

If every line of a projective plane π is a translation

line then π is called a <u>Moufang plane</u>. In fact it can be shown that if a projective plane π has two distinct translation lines then it is Moufang(see [9] theorem 6.18 p.151).

If a projective plane π is (V,l)-transitive for all V and l in π then π is said to be <u>desarguesian</u>. Any finite Moufang plane is desarguesian (see [9] corollary to theorem 6.20 p.153). Also the dual of a desarguesian plane is a desarguesian plane (see [8] section 1.4 p.26).

Note that an affine plane or a dual affine plane is desarguesian, Moufang, translation or dual translation if the corresponding projective plane is desarguesian, Moufang, translation or dual translation respectively.

Now we shall consider the method found in [9] chapter V, of coordinatizing a projective plane.

Let π be a projective plane and let R be a set of symbols of cardinality the same as the order of π such that 0,1 are distinct elements of R. Let ∞ be a symbol not in R. We must now determine what properties R must satisfy if it is to coordinatize π .Let l_∞, l_1, l_2 be any trilateral of π .and let I be any point of π , not incident with l_∞, l_1 or l_2. Let $X=l_2\cap l_\infty$, $Y=l_1\cap l_\infty$, $O=l_1\cap l_2$. Now we shall coordinatize π with respect to the quadrangle O,X,Y,I using the elements of the set $R\cup\{\infty\}$.

Let $A=XI\cap l_1$, $B=YI\cap l_2$, $J=AB\cap l_\infty$ (see Fig. 5). Arbitrarily assign the elements of R to the points of $l_1\setminus\{Y\}$ but such that 0 is assigned to O and 1 is assigned to A. The

coordinates assigned to each point of π and to each line of π are given by Table I and Table II.

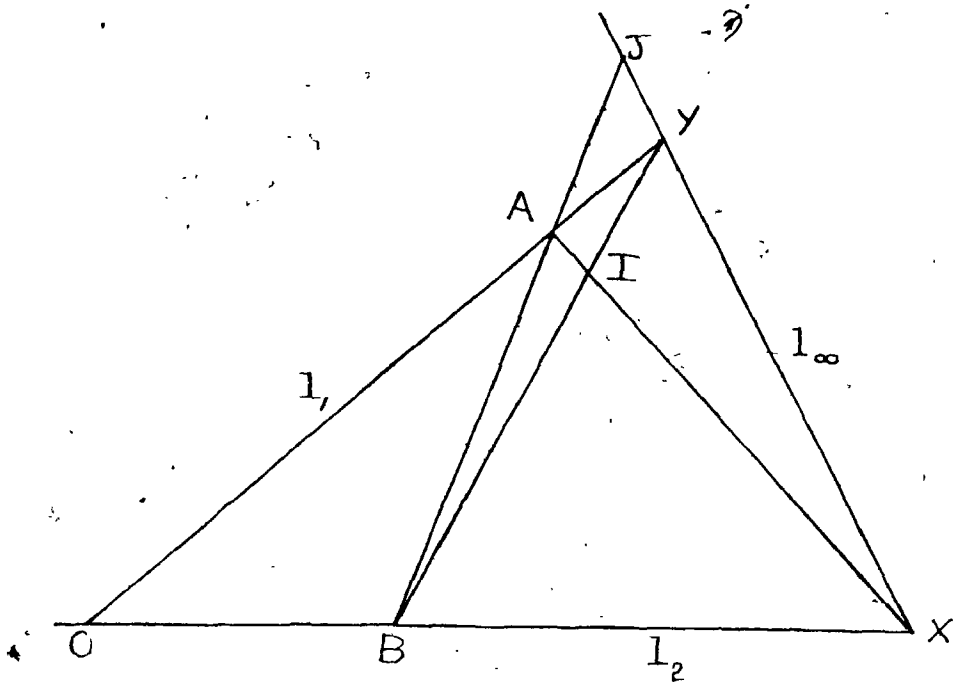

Fig.5

Table I. COORDINATES OF THE POINT P

POINT	COORDINATES	
$P \mathrm{I}\, l_1, P \neq Y$	$(0,y)$	if y is assigned to P.
$P \mathrm{I}\, l_2$	$(y,0)$	if JP intersects l_1 in $(0,y)$.
$P \not\mathrm{I}\, l_1, l_2, l_\infty$	(x,y)	if YP intersect l_2 in $(x,0)$ and XP intersects l_1 in $(0,y)$.
$P \mathrm{I}\, l_\infty, P \neq Y$	(m)	if PB intersects l_1 in $(0,m)$.
$P = Y$	(∞)	

(See Fig. 6-9.)

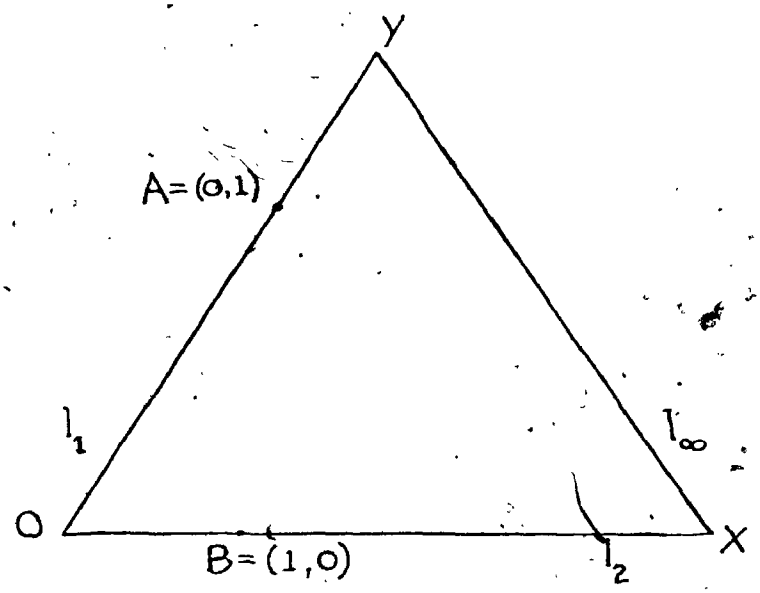

Fig.6

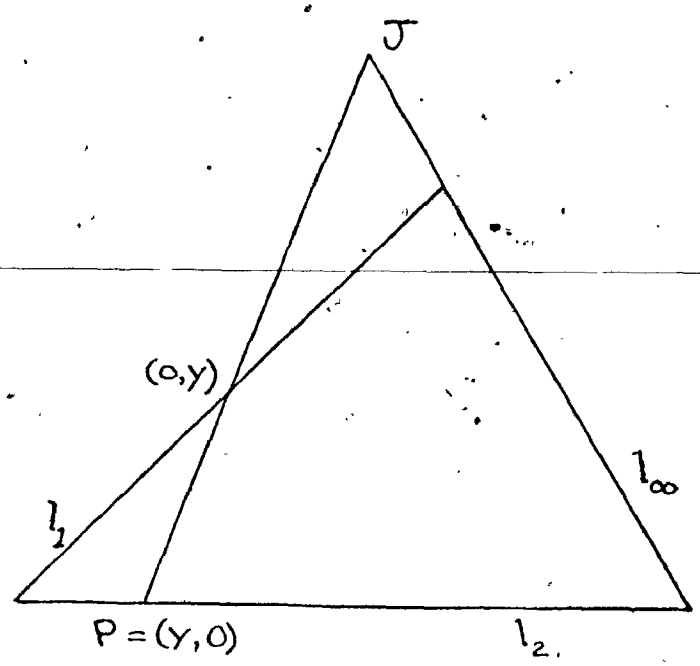

Fig.7

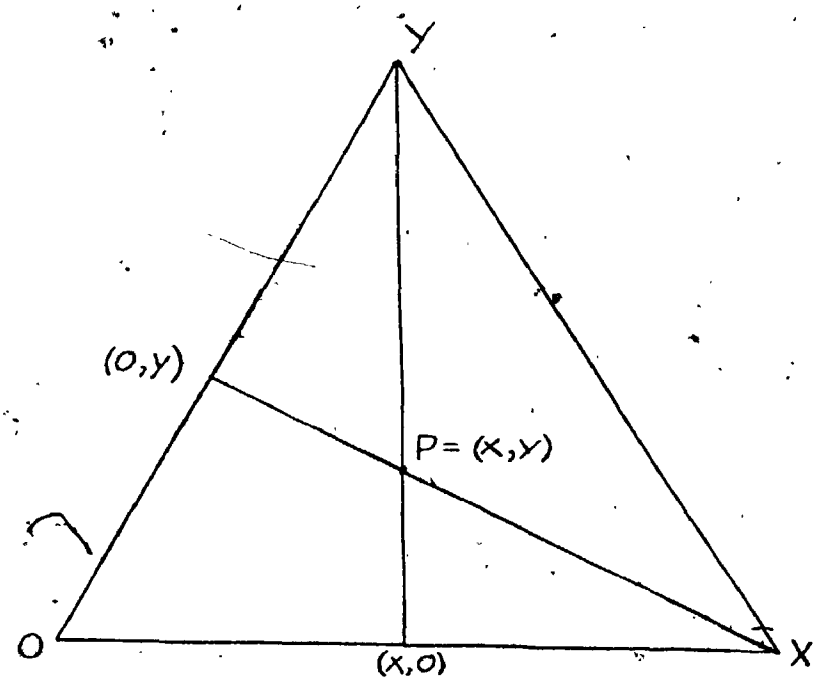

Fig.8

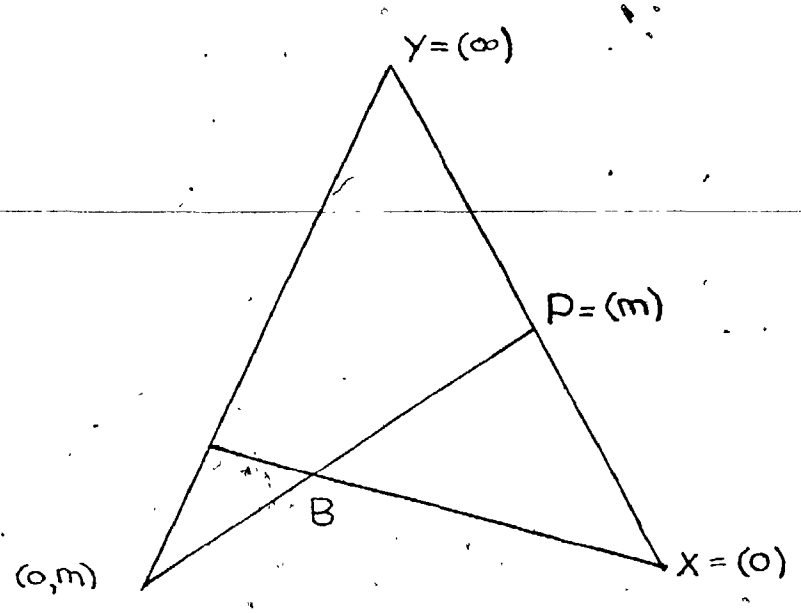

Fig.9

Table II. COORDINATES OF THE LINE l

LINE	COORDINATES	
$l \neq l_\infty$, $Y \not\in l$	$[m,k]$	if $l \cap l_\infty$ is the point (m) and $l \cap l_1$ is the point $(0,k)$.
$l \neq l_\infty$, $Y \in l$	$[k]$	if $l \cap l_2$ is the point $(k,0)$.
$l = l_\infty$	$[\infty]$	

(See Fig.10-11.)

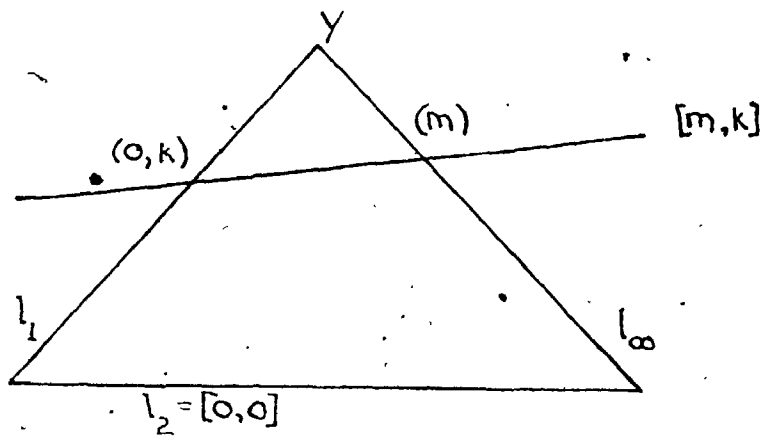

Fig.10

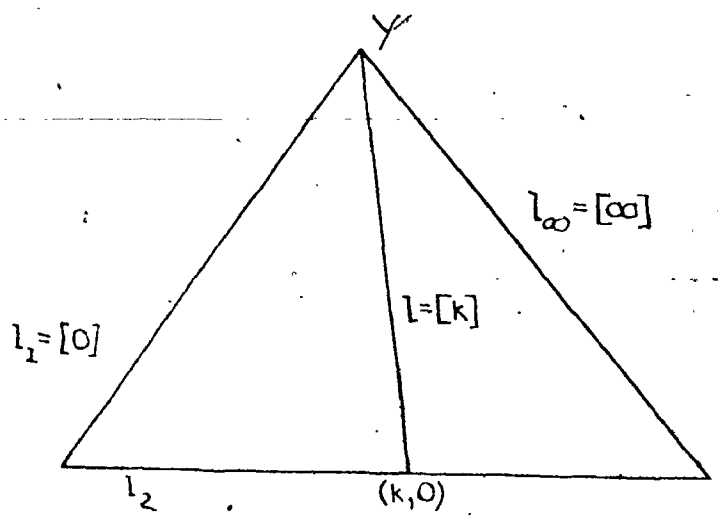

Fig.11

A <u>ternary operation</u> on a set S is a rule which assigns to any three ordered elements a,b,c of S a unique element k of S. A set S together with a ternary operation T is called a <u>ternary ring</u>.

For a projective plane π coordinatized by a set R as above, we define the following ternary operation T:

$$T(a,b,c)=k \iff (b,c) I [a,k].$$

Note that with the above notation (0,k) is the unique point of intersection of l_1 and the line joining (a) to (b,c). Thus k is uniquely determined by T(a,b,c); i.e. T is a ternary operation.

It can be shown (see [9] chapter V) that the incidences in π are given as follows:

$(x,y) I [m,k] \iff T(m,x,y)=k$ for all $m,x,y,k \in R$;

$(x,y) I [k] \iff x=k$ for all $x,y,k \in R$;

$(x) I [m,k] \iff x=m$ for all $m,x,k \in R$;

$(x) I [\infty]$ for all $x \in R$;

$(\infty) I [k]$ for all $k \in R$.

We can define two binary operations $+, \cdot$ on R, $R^*=R \setminus \{0\}$ respectively by

$$a+b=T(1,a,b), \text{ for all } a,b \in R$$

$$a \cdot b=T(a,b,0), \text{ for all } a,b \in R.$$

Note that since T is a ternary operation a+b and a·b are uniquely determined by a and b, thus + and · are binary operations.

A nonempty set G together with a binary operation ·

is a <u>loop</u> if

(i) there exists a unique $x \in G$ such that $a \cdot x = b$ for any $a, b \in G$, $a \neq 0$;

(ii) there exists a unique $y \in G$ such that $y \cdot a = b$ for any $a, b \in G$, $a \neq 0$;

(iii) there exists an element $e \in G$ such that $e \cdot x = x \cdot e = x$, for all $x \in G$.

It can be shown that if R is a ternary ring coordinatizing a projective plane with $+, \cdot$ defined on R and R^* as above then $(R,+)$ and $(R^*, \cdot)$ are loops. If R coordinatizes a projective plane then (R,T) is called a <u>planar ternary ring</u> (see [9] chapter V for further properties of R).

If (R,T) is a planar ternary ring then T is said to be <u>linear</u> if $T(m,x,y) = mx+y$. In general T need not be linear (see for example the Hughes plane in [8], section 5.4 and [9], chapter IX, section 6).

There is a correspondence between the algebraic structure of R and certain geometric properties of π, in particular with the existence of perspectivities (see [9] chapter VI). Some of these will be considered after the following definitions.

A planar ternary ring (R,T) is a <u>quasifield</u> (or left quasifield) if:

(i) T is linear,

(ii) $+$ is associative and

(iii) $a(b+c) = ab+ac$ for all a, b, c R.

(iii) is called the <u>left distributive law</u>. A planar ternary ring (R,T) is called a <u>right quasifield</u> if it satisfies (i), (ii) and (iii)' where

(iii)' $(a+b)c=ac+bc$ for all $a,b,c \in R$.

(iii)' is called the <u>right distributive law</u>.

It can be shown that algebraically a right quasifield R is a set together with two binary operations $+,\cdot$ such that

(1) $(R,+)$ is an abelian group;

(2) $(R^*,\cdot)$ is a loop;

(3) $(x+y)z=xz+yz$ for all $x,y,z \in R$;

(4) $x0=0$ for all $x \in R$ where 0 is the identity of $+$;

(5) $xa=xb+c$ has a unique solution x in R for any $a,b,c \in R$, $a \neq b$.

(For a left quasifield replace (3), (4), (5) by (3)', (4)', (5)' where

(3)' $x(y+z)=xy+xz$ for all $x,y,z \in R$;

(4)' $0x=0$ for all $x \in R$ where 0 is the identity of $+$;

(5)' $ax=bx+c$ has a unique solution $x \in R$ for any $a,b,c \in R$, $a \neq b$.)

A <u>semifield</u> is a left quasifield which also satisfies the right distributive law.

The <u>kernel</u> K of a right quasifield Q is the set of all $k \in K$ such that:

(i) $k(x+y)=kx+ky$

(ii) $k(xy)=(kx)y$

for all x,y in Q. It can be shown that the kernel K of a right quasifield Q is a skewfield and moreover that Q is

a left vector space over K. (See [9] chapter VII for the analagous results for a (left) quasifield).

Now geometrically a projective plane π can be coordinatized by a quasifield if and only if π is a translation plane with respect to l_∞; π can be coordinatized by a right quasifield if and only if π is a dual translation plane with respect to (∞); π can be coordinatized by a semifield if and only if π is a translation plane with respect to l_∞ and a dual translation plane with respect to (∞); π can be coordinatized by a skewfield if and only if π is desarguesian (see [9] p. 154).

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, I)$ be an incidence structure whose blocks are called lines. <u>The subspaces</u> of $\mathcal{S}$ are those subsets $\mathcal{S}$ of $\mathcal{P}$ such that if P,Q are two distinct points of $\mathcal{S}$ then there is a line joining P,Q, in $\mathcal{S}$. A <u>hyperplane</u> $\mathcal{H}$ of $\mathcal{S}$ is a maximal subspace, i.e. $\mathcal{P}$ is the only possible subspace containing $\mathcal{H}$ properly.

<u>A projective space</u> (or a <u>projective geometry</u>) is the system of subspaces of an incidence structure $\mathcal{S}$ which satisfies the following axioms:

I Given any two points P,Q of $\mathcal{P}$ there is a unique line l of $\mathcal{L}$ such that P,Q I l.

II Any line l of $\mathcal{L}$ is incident with at least three points of $\mathcal{P}$.

III If two distinct lines l,m of $\mathcal{L}$ have a point P of $\mathcal{P}$ in common; and if Q,R are points different from

P on l, and if S,T are points different from P on m, then the lines QS and RT have a common point (see Fig.12).

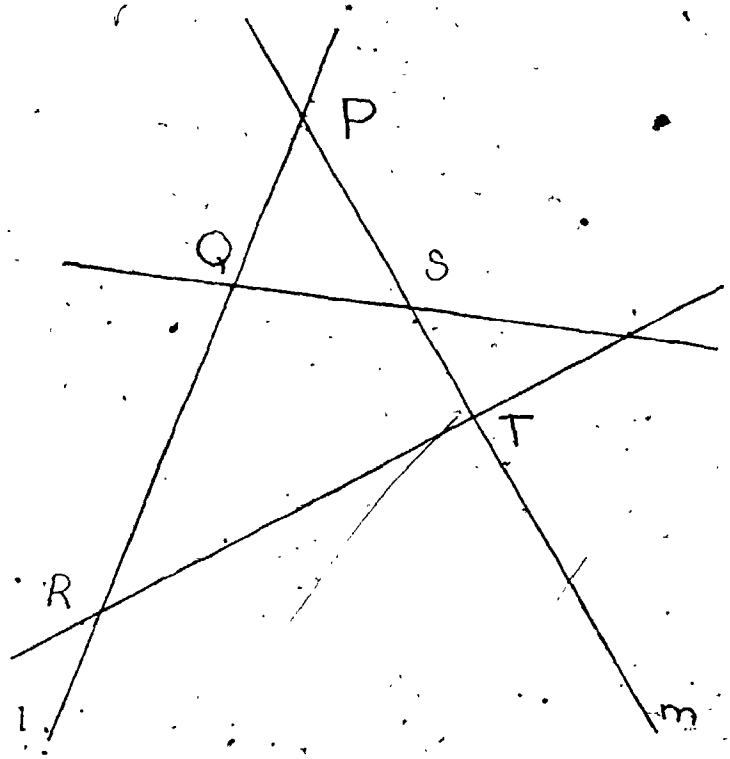

Fig.12

If π is a projective space with at least two nonintersecting lines then π can be thought of as the collection of all subspaces of a right or left vector space V over some skewfield K. Denote such a projective space by $\mathcal{P}(V)$. Incidence in $\mathcal{P}(V)$ is given by inclusion. We are only interested in finite-dimensional projective spaces where the <u>dimension</u> of $\mathcal{P}(V)$ is n if the vector space dimension of V over K is n+1. If U is a vector subspace of V of dimension m+1 then U is a <u>(projective)</u>

<u>subspace</u> of $\mathcal{P}(V)$ of dimension m (written d(U)=m). Subspaces of $\mathcal{P}(V)$ of dimension 0,1,2,n-1 are respectively <u>points</u>, <u>lines</u>, <u>planes</u> and <u>hyperplanes</u> of $\mathcal{P}(V)$.

The following can be proved.

1. Planes of $\mathcal{P}(V)$ are projective planes.
2. (Grassman's Identity) $d(U+W)+d(U\cap W)=d(U)+d(W)$ where $U+W$ is the subspace of $\mathcal{P}(V)$ containing U and W; and $U\cap W$ is the subspace consisting of all points of the intersection of U and W.
3. The planes of $\mathcal{P}(V)$ are desarguesian .

(See [8] section 1.4.1 .)

According to Lenz [11] , an <u>affine space</u> (or an <u>affine geometry</u>) is a system $\mathcal{A}$ of subspaces of an incidence structure whose blocks are called lines, such that $\mathcal{A}$ satisfies the following:

I There is an equivalence relation called parallelism defined between the blocks; two blocks or lines are said to be parallel if they lie in a common plane and have no points in common; a line is parallel to itself.

II For any point-line pair P,b there exists a unique line c in the plane determined by P and b such that c is incident with P and parallel to b.

III For any four distinct points A,B,C,D such that the line AB is parallel to the line CD and for any point $P \mathrm{I} AC$ either $P \mathrm{I} CD$ or AB

and PD have a point in common (see Fig. 13).

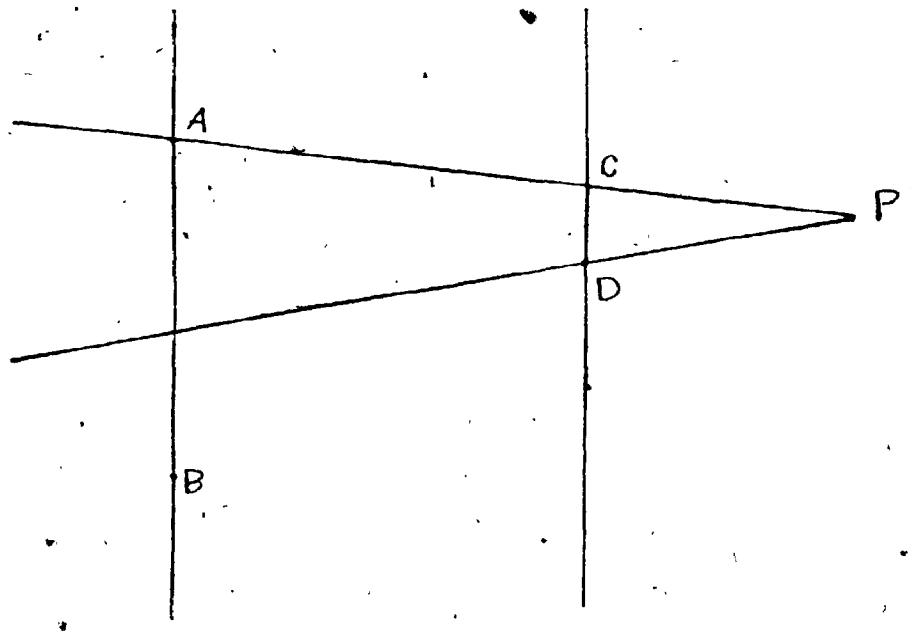

Fig. 13

An affine space $\mathcal{A}$ of dimension greater than two (i.e. $\mathcal{A}$ has at least two planes) corresponds to a projective space with a hyperplane $\mathcal{H}$ and all its points removed. (Note that obtaining an affine plane from a projective plane is a special case of this). Again points, lines and planes of $\mathcal{A}$ are subspaces of $\mathcal{P}(V)\setminus\mathcal{H}$ of dimension 0, 1 and 2 respectively. The planes of $\mathcal{A}$ are affine. If the dimension of $\mathcal{A}$ is greater than two, then the planes of $\mathcal{A}$ are desarguesian (see [8] section 1.1.1). An affine space is of dimension three if and only if some plane either intersects another plane in a unique line or the planes have no points in common (i.e. they are parallel).

A <u>collineation</u> of a projective or affine space is a bijection of its point set which maps lines onto lines. From the definition of subspaces it follows that a collineation maps a subspace onto a subspace. Analagously to the plane perspectivities, perspectivities of $\mathcal{P}(V)$ can be defined. A collineation α of $\mathcal{P}(V)$ (or $\mathcal{A}$) is a <u>perspectivity</u> if there is a point P such that all subspaces containing P are fixed and if there is a hyperplane $\mathcal{H}$ such that every point of $\mathcal{H}$ is fixed. P is called the <u>centre</u> and $\mathcal{H}$ the <u>axis</u> of the collineation α. If P is not incident with $\mathcal{H}$, α is an <u>homology</u> and if P is incident with $\mathcal{H}$, α is an <u>elation</u>.

II. SOME RESULTS ON BAER SUBPLANES

This chapter consists of four general results involving Baer subplanes of projective planes. The first section deals with the definition of Baer subplanes. In particular we show that in finite planes condition (A) holds if and only if condition (B) holds. We then give an example due to Barlotti [1] that shows the independence of (A) and (B) in the infinite case. The second section shows that in some ways the relationship between a finite projective plane π and the substructure π^{π_0} and π^1 are analagous. From the first chapter, π^1 is an affine plane and π is uniquely determined up to isomorphism by the structure of π^1. We shall show that in the finite case a similar result is true for π^{π_0}: that is, π is uniquely determined up to isomorphism by the structure of π^{π_0}. π^{π_0} is a special case of an elliptic semi-plane (see [8] p.315). The result of section 2 is mentioned in [8] (p.317) but here will be proved using only elementary properties of projective planes.

The third section of this chapter contains a method of defining an affine space of dimension $d>2$ using a dual affine plane with a 'sufficiently large' class of Baer subplanes with segments well-defined such that the Baer subplanes are the duals of some of the planes of the affine space. This is a generalization of results due to Cofman [5], [6], [7]. It is used in § 4 and chapter III § 1.

The fourth section of this chapter is the dualization of a result of Cofman [7]. It shows that the Baer subplanes belonging to a derivation set of a projective plane of arbitrary order (finite or infinite) are desarguesian.

§1. Definition of Infinite Baer Subplanes

In the first chapter, a Baer subplane π_0 of a projective plane π was defined to be a subplane π_0 of π satisfying:

(A) every line of π is incident with at least one point of π_0, and

(B) every point of π is incident with at least one line of π_0.

Suppose that π is a finite projective plane of order n and π_0 is a subplane of order m which satisfies condition (A); then $n=m^2$. For if P is any point of $\pi \setminus \pi_0$ on some line l of π_0, then each of the n lines different from l through P has exactly one point of π_0 and each point of π_0 not on l together with P determines a unique line of $\pi \setminus \pi_0$. Thus n must equal the number of points of π_0 not on l, that is $n=m^2$. Conversely if $n=m^2$ then $n^2+n+1=m^2+m+1+(m^2+m+1)(n-m)$, that is the number of lines of π is equal to the sum of the number of lines of π_0 and the number of lines of $\pi \setminus \pi_0$ through points of π_0. Thus every line of π is incident with at least one point of

π_0. Therefore for a finite projective plane π of order n and a subplane π_0 of order m condition (A) holds if and only if $n=m^2$. Dually condition (B) holds if and only if $n=m^2$. Thus in defining a Baer subplane of a finite projective plane we need only require one of (A) or (B) or that the subplane is of order m in a plane of order m^2.

The definition of Baer subplanes of infinite projective planes is quite different, for there exist subplanes of infinite projective planes satisfying one of conditions (A) or (B) but not the other as well as Baer subplanes which satisfy both conditions. We shall now give an example due to Barlotti [1] of an infinite projective plane π with a subplane π_0 satisfying condition (A) but not condition (B). The following construction of the plane π can be found in Bruck and Bose [2].

Let Σ be a projective space of dimension four based on an infinite countable skewfield F. (Note that in [2] Σ can be any projective space of even dimension. The proof of the more general case is the same as this specific one). Let Σ' be a fixed projective subspace of Σ of dimension three. An <u>anomalous spread</u> S of Σ' is a set of lines of Σ' such that every point of Σ' is contained in exactly one line of S but some plane of Σ' does not contain any line of S; for a proof of the existence of such spreads see [3]. Note that the fact that F is infinite is used here. Let S be an anomalous spread of Σ' (see Fig. 14). Construct the system $\pi = \pi(\Sigma, \Sigma', S)$ as follows. The

points of π are of two types: type (a) points of π are the points of $\Sigma \setminus \Sigma'$; and type (b) points of π are the lines of S. The lines of π are also of two types: type (a) lines of π are planes of Σ which intersect Σ' in a line of S and are not contained in Σ'; and one type (b) line of π, namely Σ'. The incidence relation in π is induced by the incidence in Σ.

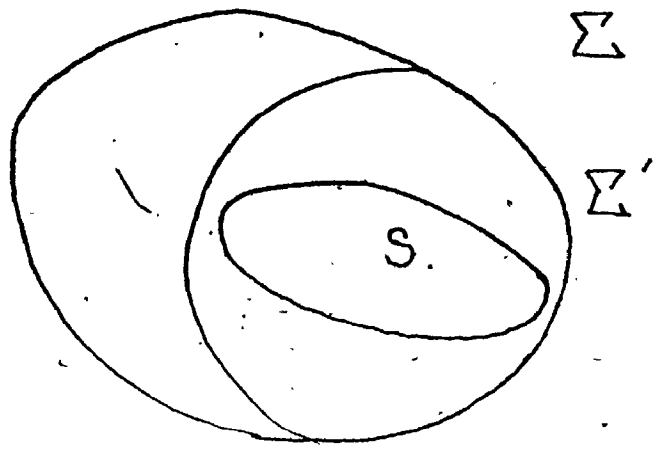

Fig. 14

<u>Theorem 2.1.1</u>. $\pi = \pi(\Sigma, \Sigma', S)$ is a projective plane.

Proof. Let P be a type (a) point of π and J a type (b) point. Then P is a point of $\Sigma \setminus \Sigma'$ and J is a line of S, (see Fig. 15).

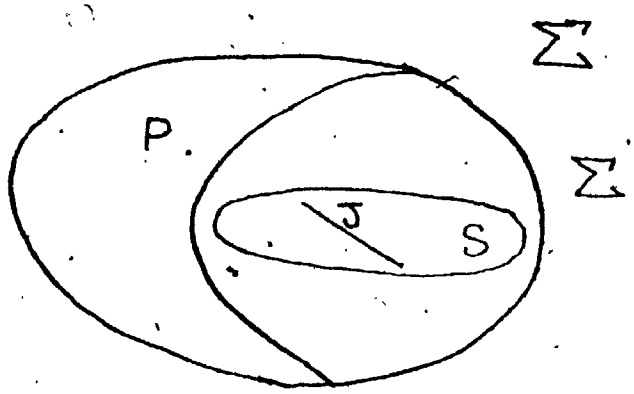

Fig. 15

Thus there is a unique plane λ of Σ containing P and J. Now P is a point of λ and P is not in Σ'. Therefore λ is not a plane of Σ' and since a plane of Σ not in a given three-dimensional projective space of Σ intersects the three-space in a line, $\lambda \cap \Sigma' = J$. Thus λ is a type (a) line of π and since P is not in Σ', λ is clearly the unique line of π containing P and J.

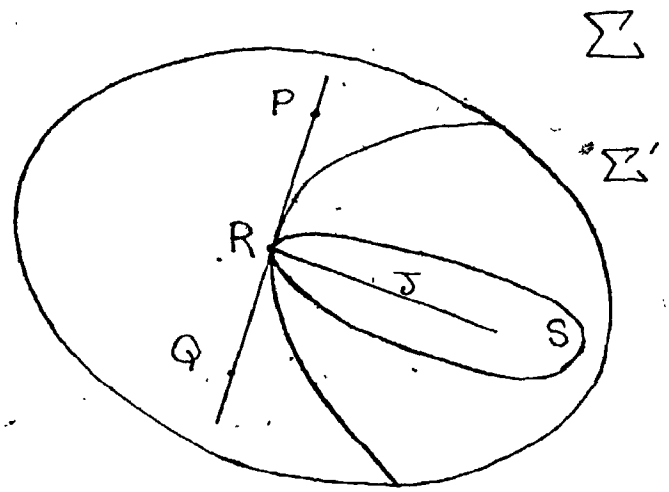

Fig. 16

Now suppose P and Q are two distinct type (a) points of π (see Fig. 16). That is, P and Q are points of $\Sigma \sim \Sigma'$ and hence the line PQ of Σ is not in Σ' and thus PQ intersects Σ' in a unique point R. As S is a spread of Σ', R is on a unique line J of S. If there exists a line l of π containing P and Q then clearly l is a type (a) line and thus the plane λ of Σ which corresponds to l must intersect Σ' in a unique line of S. As P and Q are in λ we have the line PQ in λ and

hence the point R is in λ . Thus $\lambda \cap \Sigma' = J$. On the other hand we have already shown that points P and J will be in a unique type (a) line of π which as a plane of Σ must thus contain the point R. Thus the line PR is in the plane but Q is incident with PR and thus Q is in any plane containing P and J. Thus P and Q are contained in a unique line of π.

If J and K are two type (b) points of π then clearly the only line of π containing them both is Σ'.

From the definition of type (a) lines, any type (a) line intersects Σ' in a unique type (b) point of π. Let l_1 and l_2 be two distinct type (a) lines of π. Then $l_i \cap \Sigma' = J_i$ for i=1,2 where the J_i are type (b) points of π. Suppose first that $J_1 = J_2 = J$. (see Fig. 17).

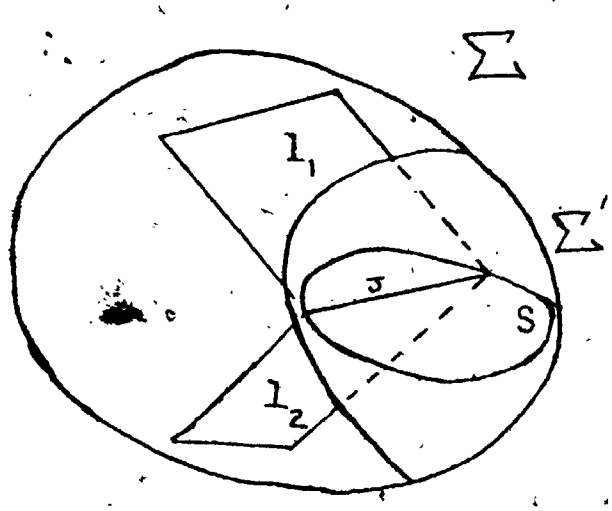

Fig. 17

Then, since from the first part of this proof a type (a)

point of π and a type (b) point of π determine a unique line of π, l_1 and l_2 can have no type (a) point of π in common. Any type (a) line of π has a unique type (b) point of π; thus l_1 and l_2 intersect in the unique point J of π. Now suppose $J_1 \neq J_2$ (see Fig.18). Then J_1 and J_2 have no points of Σ in common

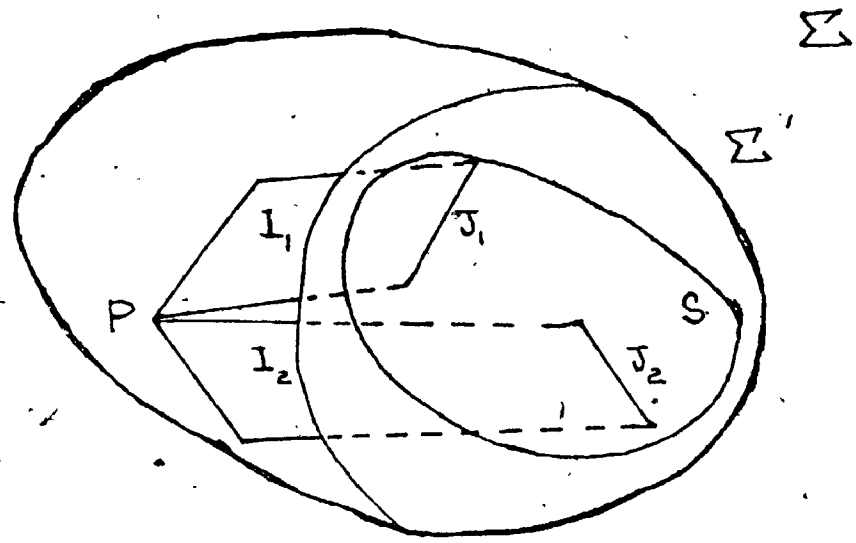

Fig.18

for J_1 and J_2 are lines of S and hence of Σ' and any point of Σ is incident with a unique line of S. Thus $l_1 \cap l_2$ and Σ' have no points of Σ in common. But l_1 and l_2 are planes of Σ and hence have at least one point P in common. P is not in Σ' and so P is a point of π. Since we already know that any two points of π lie on a unique common line of π it is immediate that two lines of π intersect in a unique common point of π.

Since Σ is defined over an infinite skewfield the existence of a quadrangle is immediate. Thus π is a projective plane.

Now consider the system π_o given as follows. Since S is an anomalous spread of Σ' there is at least one plane α of Σ' which does not contain any line of S. Let π_o' be a plane of Σ which is not contained in Σ' and which intersects Σ' in a line p of α. Let the points of π_o be the points of π_o' not in Σ' and the type (b) points of π represented by lines of S incident with p. The lines of π_o are the lines of π with at least two points of π_o. Clearly π_o is a subplane of π.

We shall now show that π_o satisfies condition (A), that is, any line of π is incident with at least one point of π_o. Let a type (a) line l of π be represented by a plane λ of Σ. Then $l \cap \pi_o' = \lambda \cap \pi_o'$ contains at least one point Q of Σ. If Q is incident with p then the plane λ of Σ representing l contains the line q of S through Q (since Q is incident with the line p of Σ' and thus is a point of Σ' and so incident with a unique line q of S and Q is a point of λ which is a type (a) line of π). The line q is a line of S through a point of p and hence is a type (b) point of π_o incident with the line l. If Q is not incident with p then as $\pi_o' \cap \Sigma' = p$, Q is a point of $\Sigma \setminus \Sigma'$, that is, Q is a type (a) point of π_o incident with l. The only type (b) line of π is Σ' and Σ' is a line of π_o since p is not a line of S. Hence more than two type (b) points of π are incident with p and so p is in Σ'.

Thus π_0 is a subplane of π satisfying condition (A).

However, π_0 is not a Baer subplane of π for it does not satisfy condition (B) as can be seen from the following. Let Ω be the hyperplane of Σ containing α and π_0'. Let A be a point of Ω such that A is not in α, π_0' or Σ'. (Such an A exists since Ω is not equal to Σ'). Thus A is a type (a) point of π. Now A is not incident with any line of π_0; for suppose A is incident with the line l of π_0. Then l corresponds to a plane λ of Σ containing A and a line J of S. Also $\lambda \cap \pi_0'$ is a line of Σ. But a plane containing A and a line of π_0' is in Ω. Thus λ and J are in Ω. Since J is in Σ' and Ω, J must be in α (for $\Sigma' \cap \Omega = \alpha$). But S is an anomalous spread and by choice α was a plane containing no lines of S. This gives a contradiction and so condition (B) is not satisfied. By duality condition (B) does not imply condition (A) in infinite projective planes.

§ 2. Uniqueness of π^{π_0}

Suppose $\pi = (\mathcal{P}, \mathcal{L}, I)$ is a finite projective plane of order $n = m^2$ and $\pi_0 = (\mathcal{P}_0, \mathcal{L}_0, I_0)$ is a Baer subplane of order m of π. Consider the structure $\mathcal{S} = \pi^{\pi_0} = (\mathcal{P} \setminus \mathcal{P}_0, \mathcal{L} \setminus \mathcal{L}_0, I \setminus I_0')$ $= (\overline{\mathcal{P}}, \overline{\mathcal{L}}, \overline{I})$ where $I_0' = \{(P,b) \in I \mid P \in \mathcal{P}_0 \text{ and/or } b \in \mathcal{L}_0\}$. Clearly $|\overline{\mathcal{P}}| = |\mathcal{P}| - |\mathcal{P}_0| = n^2+n+1-(n+\sqrt{n}+1) = n^2-\sqrt{n}$ and dually $|\overline{\mathcal{L}}| = n^2-\sqrt{n}$. For any point P of $\mathcal{S}$, P is not in π_0 which implies, as π_0 is a Baer subplane, there is exactly one line of π_0 through P, and so there are n lines of $\mathcal{S}$ through P. Dually there are n points of $\mathcal{S}$ on any line of $\mathcal{S}$. Define two lines of $\mathcal{S}$ to be

"parallel" if they are equal or if they have no point of $\mathcal{S}$ in common; and if l is parallel to m, we write $l \| m$. Define two points of $\mathcal{S}$ to be "parallel" if they are equal or if they do not lie on a common line of $\mathcal{S}$; again write $A \| B$. Clearly parallelism between lines and between points is reflexive and symmetric; that it is also transitive follows immediately from the fact that each line and point of $\mathcal{S}$ is incident with exactly one point and line respectively of π_0. Thus parallelism defines two equivalence relations in $\mathcal{S}$, one on the set of lines and one on the set of points.

Following the example of affine and dual affine planes we will show that $\mathcal{S}$ can be uniquely extended to a projective plane. We embed $\mathcal{S}$ in a projective plane by the addition of ideal elements defined in terms of the equivalence classes under parallelism. Define an <u>ideal point</u> to be a class of parallel lines and an <u>ideal line</u> to be a class of parallel points. If $\mathcal{X}$ is an ideal point and the lines $l, m \in \mathcal{X}$ then $l \| m$ and so $l \cap m$ is a point of π_0. Then as there are $\sqrt{n}+1$ lines of π_0 through $l \cap m$, the number of lines of $\mathcal{S}$ in $\mathcal{X}$ is $n+1-(\sqrt{n}+1)=n-\sqrt{n}$. Dually the number of points of $\mathcal{S}$ in an ideal line $\mathcal{L}$ is $n-\sqrt{n}$. Let $\mathcal{P}^*$ be the set of all ideal points and $\mathcal{L}^*$ the set of all ideal lines. Define $\underline{\mathcal{S}} = (\mathcal{P} \cup \mathcal{P}^*, \mathcal{L} \cup \mathcal{L}^*, \mathcal{I} \cup \mathcal{I}^*) = (\underline{\mathcal{P}}, \underline{\mathcal{L}}, \underline{\mathcal{I}})$ where $\mathcal{I}^*$ is defined as follows:

$\mathcal{I}^*$(i) If $P \in \mathcal{P}^*, \mathcal{L} \in \mathcal{L}^*$ then $P \mathcal{I}^* \mathcal{L} \iff P \in \mathcal{L}$.

I* (ii) If $\mathfrak{X} \in \mathcal{P}^*$, $l \in \mathcal{L}$ then $\mathfrak{X} \, I \, l \Leftrightarrow l \in \mathfrak{X}$.

I* (iii) If $\mathfrak{X} \in \mathcal{P}^*$, $\mathfrak{L} \in \mathcal{L}^*$ then $\mathfrak{X} I \mathfrak{L} \Leftrightarrow$ for all $l \in \mathfrak{X}$ and $P \in \mathfrak{L}$, $P \not I l$.

I* (i) and I*(ii) are natural definitions for incidence; that I*(iii) is the only possible definition for incidence of an ideal point and an ideal line if $\mathcal{S}$ is to be embedded in a projective plane is shown by the following lemma.

<u>Lemma 2.2.1</u> Using the above notation, if $\mathcal{S}$ is a projective plane then $\mathfrak{X} I \mathfrak{L}$ if and only if for all $P \in \mathfrak{L}$ there does not exist $l \in \mathfrak{X}$ with $P I l$.

Proof. Suppose $\mathfrak{X}, P I \mathfrak{L}$ and there exists l in $\mathfrak{X}$ with $P I l$. Then we have $\mathfrak{X}$, $P I l$ and $\mathfrak{X}$, $P I \mathfrak{L}$ in a projective plane which is a contradiction as an ideal element cannot also be an ordinary element (that is, an element of $\mathcal{S}$).

Conversely suppose for all P in $\mathfrak{L}$ there does not exist an l in $\mathfrak{X}$ with $P I l$. Since $\mathcal{S}$ is a projective plane P and $\mathfrak{X}$ must determine a unique line for each P in $\mathfrak{L}$. Clearly this must be an ideal line as $\mathfrak{X}$ is incident with an ordinary line l if and only if l is in $\mathfrak{X}$. P is in a unique equivalence class under parallelism and hence on the unique ideal line $\mathfrak{L}$. Thus $\mathfrak{X} I \mathfrak{L}$.

<u>Lemma 2.2.2</u> $\mathcal{S}$ is a projective plane of order n.

Proof. To prove that two points of $\mathcal{P}$ lie on a unique line of $\mathcal{L}$ there are three cases to consider:

Case (i). If P, Q are in $\overline{\mathcal{P}}$ then there are two possibilities; either PQ is a line of $\overline{\mathcal{L}}$ or $P \| Q$. If $PQ \in \overline{\mathcal{L}}$ then as $\overline{\mathcal{L}} \subset \mathcal{L}$ PQ is a line of $\mathcal{L}$ through P and Q which is clearly unique. If $P \| Q$ then since parallelism is an equivalence relation there is a unique ideal line ξ in $\mathcal{L}^*$ with $P, Q \in \xi$.

Case (ii) If $P, Q \in \mathcal{P}^*$ then by previous remarks there are n points of $\mathcal{S}$ on each of the $n-\sqrt{n}$ lines of $\mathcal{S}$ in the equivalence classes P and Q. Thus the number of points of $\mathcal{S}$ on lines of $\mathcal{S}$ in P and Q is at most $2n(n-\sqrt{n})$. Since equivalence classes are disjoint and only one point has been removed from each line of $\mathcal{S}$, each line of $\mathcal{S}$ in P intersects each of the $n-\sqrt{n}$ lines of Q in a point of $\mathcal{S}$. Thus the number of points of $\mathcal{S}$ on lines of $\mathcal{S}$ in $P \cup Q$ is $(n-\sqrt{n})(n-\sqrt{n})$. Hence the number of ordinary points on lines in $P \cup Q$ is $2n(n-\sqrt{n})-(n-\sqrt{n})(n-\sqrt{n})=n^2-n$, and so the number of points of $\mathcal{S}$ not on any line of $\mathcal{S}$ in $P \cup Q$ is $n^2-\sqrt{n}-(n^2-n)=n-\sqrt{n}$. Consider any two of these $n-\sqrt{n}$ points, say A, B. Then there are unique lines a, b of π_0 through A, B respectively. Clearly considering P, Q as points of π_0 they must be on both these lines and so $PQ=a=b$ in π_0. Thus $A \| B$. Varying B over all the $n-\sqrt{n}$ points and then varying A over the same set shows that these points form a class of parallel points ξ. Then by I* (iii) $P, Q \mathrm{I} \xi$ and ξ is unique as any ideal line

has $n-\sqrt{n}$ points.

Case (iii). If $P \in \overline{\mathcal{P}}$, $Q \in \mathcal{P}^*$ then there are two possibilities. If there is an l in the equivalence class Q such that $P I l$ then $P, Q I l$ and l is unique by definition of equivalence classes. Suppose there does not exist l in Q such that $P I l$. Now every point of $\mathcal{P}$ is on a unique line of π_0 and so every point of $\overline{\mathcal{P}}$ is in a unique parallel class, say P is in ξ. Consider the point Q' in π_0 which is the point of intersection in π of the lines of $\mathcal{J}$ in Q. Then PQ' is a line of π_0; for if PQ' is a line of $\pi \setminus \pi_0$, PQ' is a line of $\mathcal{J}$ and so PQ' is a line in Q since Q is the set of lines of $\mathcal{J}$ passing through Q' in π. But $PQ' \in Q \cap \overline{\mathcal{L}}$ contradicts the assumption. Thus PQ' is a line of π_0. Now suppose $Q \not I \xi$; that is suppose there exists l in $Q \subseteq \overline{\mathcal{L}}$ and R in $\xi \subseteq \overline{\mathcal{P}}$ with $R I l$. Then $R \in \xi$, $P \in \xi$ and so RP is a line of π_0. From the above we also have PQ' in π_0. Thus as P is in $\overline{\mathcal{P}}$ and so is on a unique line of π_0, $RP = PQ' = RQ' = l$. Thus l is a line of π_0 contradicting that l is in $Q \subseteq \overline{\mathcal{L}}$. Thus $Q I \xi$.

Arguments dual to the above show that two lines of $\underline{\mathcal{J}}$ intersect in a unique point of $\underline{\mathcal{J}}$,

Since π has a Baer subplane, $n \geq 4$. By previous remarks therefore, any point of $\overline{\mathcal{P}}$ has at least four lines of $\overline{\mathcal{L}}$ through it and any line of $\overline{\mathcal{L}}$ has at least four points of $\overline{\mathcal{P}}$ on it. Clearly then as $\overline{\mathcal{P}} \subseteq \underline{\mathcal{P}}$, $\overline{\mathcal{L}} \subseteq \underline{\mathcal{L}}$, $\underline{\mathcal{J}}$ contains a quadrangle.

Finally consider a line $l \in \overline{\mathcal{L}}$. Then l is incident with n points of $\overline{\mathcal{P}}$ and a unique point of $\mathcal{P}^*$. Thus l is incident with $n+1$ points of $\underline{\mathcal{S}}$. Therefore $\underline{\mathcal{S}}$ is a projective plane of order n.

<u>Lemma 2.2.3</u>. $\mathcal{S}^* = (\mathcal{P}^*, \mathcal{L}^*, I^*)$ is a Baer subplane of $\underline{\mathcal{S}}$.

Proof. Two points of $\mathcal{P}^*$ lie on a unique line of $\mathcal{L}^*$ by case (ii) in the proof of lemma 2.2.2. The dual shows that two lines of $\mathcal{L}^*$ pass through a unique point of $\mathcal{P}^*$. By considering orders, every ideal line has $\sqrt{n}+1$ ideal points. Thus if $\mathcal{S}^*$ has a quadrangle it is a Baer subplane of $\underline{\mathcal{S}}$. Note $|\mathcal{L}^*| = |\underline{\mathcal{L}}| - |\overline{\mathcal{L}}| = n+\sqrt{n}+1 > 1$ and $|\mathcal{P}^*| = |\underline{\mathcal{P}}| - |\overline{\mathcal{P}}| = n+\sqrt{n}+1 > 1$. Let Γ be any ideal line. Then there are $n \geq 2$ ideal points not on Γ. Thus as there are at least three points on Γ, $\mathcal{S}^*$ has a quadrangle and so is a Baer subplane of $\underline{\mathcal{S}}$.

<u>Lemma 2.2.4</u>. π^{π_0} is isomorphic to $\pi'^{\pi_0'}$ if and only if there exists an isomorphism α from π onto π' such that $\pi_0^\alpha = \pi_0'$.

Proof. Suppose $\beta : \pi^{\pi_0} \to \pi'^{\pi_0'}$ is an isomorphism. Define $\alpha : \pi \to \pi'$ by

$$X^\alpha = \begin{cases} X^\beta & \text{if } X \in \pi^{\pi_0} \\ Y & \text{if } X \in \pi_0 \end{cases}$$

where if x is any line of π^{π_0} through X, Y is the unique point of π_0' on x^β.

$$x^\alpha = \begin{cases} x^\beta & \text{if } x \in \pi^{\pi_0} \\ y & \text{if } x \in \pi_0 \end{cases}$$

where if X is any point of π^{π_0} incident with x, y is the unique line of π_0' through x^β. Now if X is a point of π_0 and $X I x,y$ with x,y lines of π^{π_0}, then in π^{π_0}, $x \| y$ and so as β is an isomorphism from π^{π_0} onto $\pi'^{\pi_0'}$, $x^\beta \| y^\beta$. Thus $Y = x^\beta \cap y^\beta$ is a point of π_0'. As x,y are lines of π^{π_0}, x^β, y^β are lines of $\pi'^{\pi_0'}$; that is x^β, y^β are not in π_0' so Y is the unique point of π_0' through x^β and y^β. Thus $X^\alpha = Y$ is well-defined. A dual argument shows that α is well-defined on lines. Clearly, by definition of α, $\pi_0^\alpha = \pi_0'$ and $(\pi^{\pi_0})^\alpha = \pi'^{\pi_0'}$. Then as α restricted to π^{π_0} is identically β, we need only show that α is one-to-one from π_0 onto π_0'. Suppose $X^\alpha = Y^\alpha$ are in π_0'. Let x be any line of $\pi'^{\pi_0'}$ with $X^\alpha = Y^\alpha I x$. As $\beta: \pi^{\pi_0} \longrightarrow \pi'^{\pi_0'}$ is an isomorphism there exists $y \in \pi^{\pi_0}$ with $y^\beta = x$. By definition of α the pre-images of X^α and Y^α are on y; but there is a unique point of π_0 on y since y is not in π_0. Thus X=Y and so α is one-to-one. A dual argument shows that α is a one-to-one map of the lines also. Now let Y be a point of π_0' and let y be a line of $\pi'^{\pi_0'}$ such that $Y I y$. Consider $y^{\beta^{-1}}$, the pre-image of y which is in π^{π_0}. There is a unique point, say X, in π_0 with $X I y^{\beta^{-1}}$. Then as $(y^{\beta^{-1}})^\beta = y$, $X^\alpha = Y$. A dual argument shows that α is also a surjective mapping of the lines.

Now we must show that α preserves incidence. Suppose X and y are in π and $X I y$. If X,y are in π^{π_0}, clearly $X^\alpha I y^\alpha$ since X^α, y^α are in $\pi'^{\pi_0'}$ and conversely. If X is a point of π_0 and y is a line of π^{π_0}, then $X I y$ if and only if $X^\alpha I y^\alpha$ by definition of α; and dually, if

X is a point of π^{π_0} and y is a line of π_0, $X I y$ if and only if $X^\alpha I y^\alpha$. If X and y are in π_0 then by lemma 2.2.1 $X I y$ if and only if for all lines l of π^{π_0} with $X I l$ and for all points P of π^{π_0} with $P I y$, $P \not I l$; since β preserves parallelism, this happens if and only if for all l^β, P^β (since β is onto) with $X^\alpha I l^\beta = l^\alpha$ and $P^\beta = P^\alpha I y^\alpha$, $P^\alpha \not I l^\alpha$; that is if and only if $X^\alpha I y^\alpha$.

Conversely if α is an isomorphism from π onto π' such that $\pi_0^\alpha = \pi_0'$, then clearly α induces an isomorphism from π^{π_0} onto $\pi'^{\pi_0'}$.

The above results are summarized in the following theorem.

<u>Theorem 2.2.1</u>. If π is a finite projective plane and π_0 is a Baer subplane of π then π is uniquely determined up to isomorphism by the structure of π^{π_0}.

§3. <u>An Affine Space</u>

In this section an incidence structure $\mathcal{S}$ will be defined using lines and Baer subplanes of a dual affine plane. It will be shown that $\mathcal{S}$ is an affine space of dimension greater than two. This result will then be used in §4 to show that the Baer subplanes belonging to a derivation set of a derivable projective plane are desarguesian. The dual of the result, that is an incidence structure defined using points and Baer subplanes of an affine plane, will be used in chapter III §1.

Suppose $\mathcal{A}$ is a dual affine plane of arbitrary order (finite or infinite) with a class $\mathcal{G}$ of Baer subplanes such that $|\mathcal{G}|>1$. Let G be the special point of $\mathcal{A}$. Let $\mathcal{P}$ be the set of affine lines of the subplanes in $\mathcal{G}$. Suppose further, that the following conditions hold:

(1) any trilateral of $\mathcal{P}$ is in a unique subplane of $\mathcal{G}$;

(2) if two subplanes $\mathcal{A}_1, \mathcal{A}_2$ of $\mathcal{G}$ have two affine lines, say x,y, of $\mathcal{P}$ in common then $\mathcal{A}_1$ and $\mathcal{A}_2$ coincide in all their lines through $x \cap y$; that is, segments are well-defined. Denote the segment determined by x and y by $\overline{x \cap y}$.

Define an incidence structure $\mathcal{S}$ whose points are the lines of $\mathcal{P}$ and whose blocks are the segment determined by two lines of $\mathcal{P}$. The points of $\mathcal{S}$ will be called <u>$\mathcal{S}$-points</u> and will usually be denoted by a,b,... . A block of $\mathcal{S}$ is a subset of lines through some affine point of $\mathcal{A}$. Blocks contained in a point X of $\mathcal{A}$ will usually be denoted by $\overline{X}$ or $\overline{\Pi}_i$. Define two types of planes in $\mathcal{S}$: <u>type $\mathbb{B}$</u> and <u>type $\mathbb{L}$</u>. Planes of type $\mathbb{B}$ are the duals of the Baer subplanes in $\mathcal{G}$, i.e. the points of the subplanes of $\mathcal{G}$ are lines of type $\mathbb{B}$ planes and the lines of planes of $\mathcal{G}$ are the points of the planes of type $\mathbb{B}$. If X is an affine point of $\mathcal{A}$ such that the lines of $\mathcal{A}$ through X coincide with the affine lines of $\mathcal{P}$ through X, then the points and blocks of $\mathcal{S}$ incident with X form a plane of type $\mathbb{L}$.

In order to show that $\mathcal{S}$ is an affine space of

dimension greater than two we need the following lemmas.

<u>Lemma 2.3.1</u>. Planes of type $\mathbb{B}$ and $\mathbb{L}$ are affine planes.

Proof. Planes of type $\mathbb{B}$ are the duals of the planes of $\mathcal{C}$ and hence are affine.

Now consider a plane of type $\mathbb{L}$ determined by the point H of $\mathcal{A}$. Let a,b be any two distinct affine lines through H (note $a,b \in \mathcal{P}$). Let d be any other line of $\mathcal{P}$ ~~such~~ that a,b,d form a trilateral in $\mathcal{A}$. By assumption (1) a, b,d are contained in a unique subplane, $\mathcal{A}_1$ say, of $\mathcal{C}$. Let x be any line through G such that x is not a special line of $\mathcal{A}_1$. Note $H \not\mathrm{I} x$, for if $H \mathrm{I} x$ then as $G \mathrm{I} x$, $GH=x$. But $H=a \cap b$ is a point of $\mathcal{A}_1$. Thus $x=GH$ is a special line of $\mathcal{A}_1$ which is a contradiction. Consider the mapping φ from the affine lines of $\mathcal{A}_1$ into the affine lines of H given by

$$\varphi(y)=(x \cap y)H$$

where y is any affine line of $\mathcal{A}_1$. See Fig.19. Then φ is

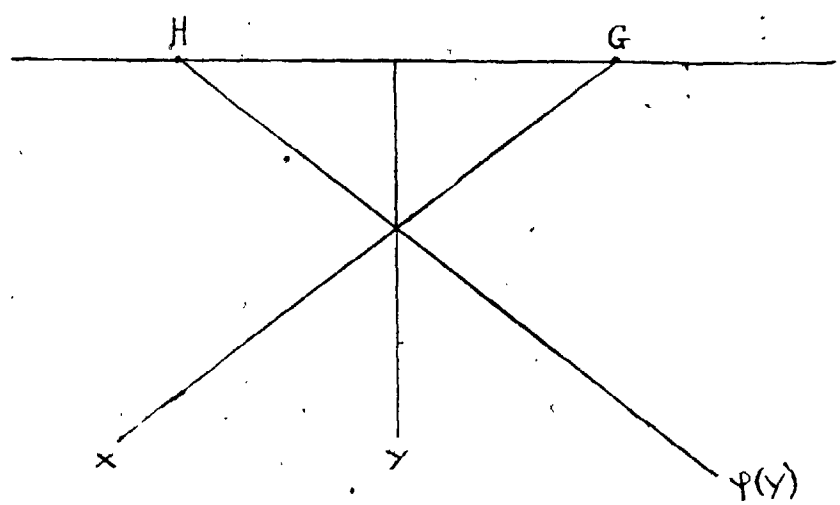

Fig.19

one-to-one; for if $\varphi(y)=\varphi(z)$ for y,z in $\mathcal{A}_1$ then $(x\cap y)H=(x\cap z)H=m$ say. But $x\cap y$, $x\cap z I x,m$ and $x\neq m$ (since $H I m$ and $H \not I x$) which implies $x\cap y=x\cap z$ and so y,z are lines of $\mathcal{A}_1$ and pass through the same point of x. As x is not a special line of $\mathcal{A}_1$, there is a unique line of $\mathcal{A}_1$ through each affine point of x. Thus y=z and so φ is one-to-one.

Now to see that φ is surjective let m be any affine line through H. Let $m\cap x=M$. Then there is a unique line

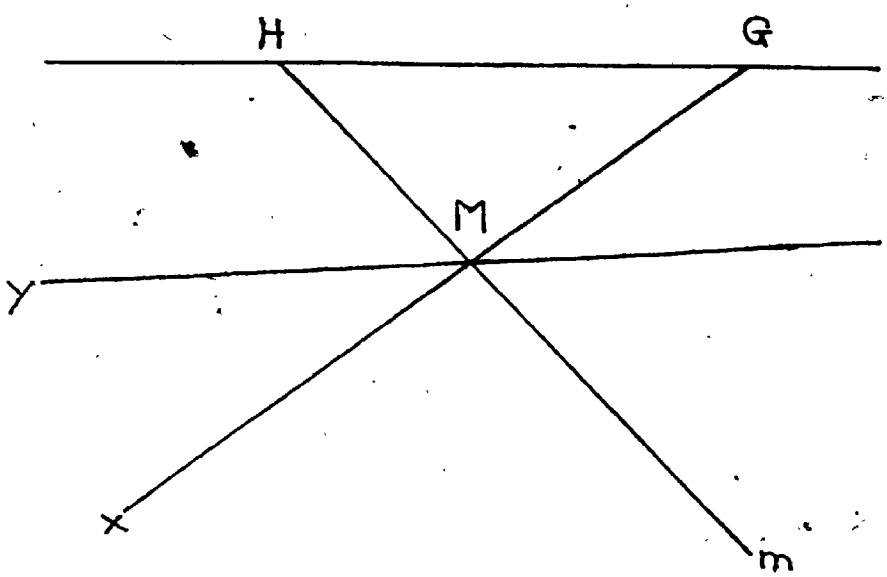

Fig. 20

of $\mathcal{A}_1$ through M, say y (see Fig. 20). Clearly $\varphi(y)=m$ and so φ is surjective.

We must now show that φ preserves incidence; that is, (i) that the segments of $\mathcal{A}_1$ are mapped to the blocks of $\mathcal{J}$ through H and (ii) that the pre-image of any block of $\mathcal{J}$ through H is a segment of $\mathcal{A}_1$. For (i) let u,v be lines of $\mathcal{A}_1$ and let $\varphi(u)=(u\cap x)H=m$, $\varphi(v)=(v\cap x)H=n$. First suppose

at least one of $u \neq m$, $v \neq n$ holds, say $u \neq m$. Then by assumption (1) u, v, m are contained in a unique subplane, $\mathcal{A}_2$ say, of $\mathcal{C}$. Since $u \cap m$ is a point of $\mathcal{A}_2$ we have $x = (u \cap m)G$ is a special line of $\mathcal{A}_2$ and also $v \cap x = n \cap x$ is a point of $\mathcal{A}_2$ which implies $n = (n \cap x)H$ is a line of $\mathcal{A}_2$. Thus we have u, v, m, n, x and $H = m \cap n$ in $\mathcal{A}_2$. By definition $\overline{u \cap v}$ is contained in $\mathcal{A}_2$. Let y be in $\overline{u \cap v}$. We want to show that $\varphi(y)$ is in $\overline{m \cap n}$; i.e. $\varphi(y)$ is in $\mathcal{A}_2$. Now $y \cap x$ is in $\mathcal{A}_2$ and so $\varphi(y) = (y \cap x)H$ is in $\mathcal{A}_2$. Thus by assumption (2) $\varphi(y)$ is in $\overline{m \cap n}$. Now suppose $u = m$ and $v = n$. Then $\overline{u \cap v} = \overline{m \cap n}$ and so if y is in $\overline{u \cap v}$, $\varphi(y) = (y \cap x)H = y$ in $\overline{m \cap n}$ since y in $\overline{u \cap v}$ implies that $H \mathrm{I} y$. Thus blocks of $\mathcal{A}_1$ are taken into blocks through H by φ. Now for (ii) with the same notation as for (i), consider z in $\overline{m \cap n}$. If we first consider $u \neq m$, we know φ is surjective and so there is some w in $\mathcal{A}_1$ with $\varphi(w) = z$. We want to show that w is in $\overline{u \cap v}$. Now $\overline{m \cap n}$ is in $\mathcal{A}_2$ and so z is a line of $\mathcal{A}_2$. As x is the special line of $\mathcal{A}_2$, $z \cap x$ is a point of $\mathcal{A}_2$ and thus $(u \cap v)(z \cap x)$ is a line of $\mathcal{A}_2$. But u, v are lines of $\mathcal{A}_1 \cap \mathcal{A}_2$ and so by assumption (2) $\overline{u \cap v}$ is contained in $\mathcal{A}_1 \cap \mathcal{A}_2$. Thus the line $(u \cap v)(z \cap x)$ is in $\mathcal{A}_1$. But $w = \varphi(z)$ is the unique line of $\mathcal{A}_1$ through $z \cap x$. Thus $w = (u \cap v)(z \cap x)$ is a line of $\mathcal{A}_1$ and so w is in $\overline{u \cap v}$. Finally if $u = m$ and $v = n$ then $\overline{m \cap n}$ is contained in $\mathcal{A}_1$ and so z is in $\mathcal{A}_1$ and $\varphi(z) = z$.

<u>Lemma 2.3.2</u>. Let $\mathcal{A}$ be a dual affine plane with a class $\mathcal{C}$ of Baer subplanes such that any trilateral of $\mathcal{A}$ is in at most one Baer subplane of $\mathcal{C}$. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two Baer

subplanes of $\mathcal{C}$ containing a common affine point L such that the affine lines of $\mathcal{A}_1$ through L coincide with the affine lines of $\mathcal{A}_2$ through L. Then there is a special line x in $\mathcal{A}$ such that the affine points of $\mathcal{A}_1$ on x coincide with the affine points of $\mathcal{A}_2$ on x.

Proof. Let G be the special point of $\mathcal{A}$. Let a be an affine line of $\mathcal{A}_2 \setminus \mathcal{A}_1$ and let A be the unique point of $\mathcal{A}_1$ on a. Consider two cases:

Case (i). Suppose $G \mathrel{I} AL$. Then GL=AL is a special line of $\mathcal{A}_1$ and $\mathcal{A}_2$. Since a,AL are lines of $\mathcal{A}_2$ and $A \mathrel{I} a$ we have $A = a \cap AL$ in $\mathcal{A}_2$. We will now show that $\mathcal{A}_1$ and $\mathcal{A}_2$ coincide in all affine points of AL. Suppose there exists a point

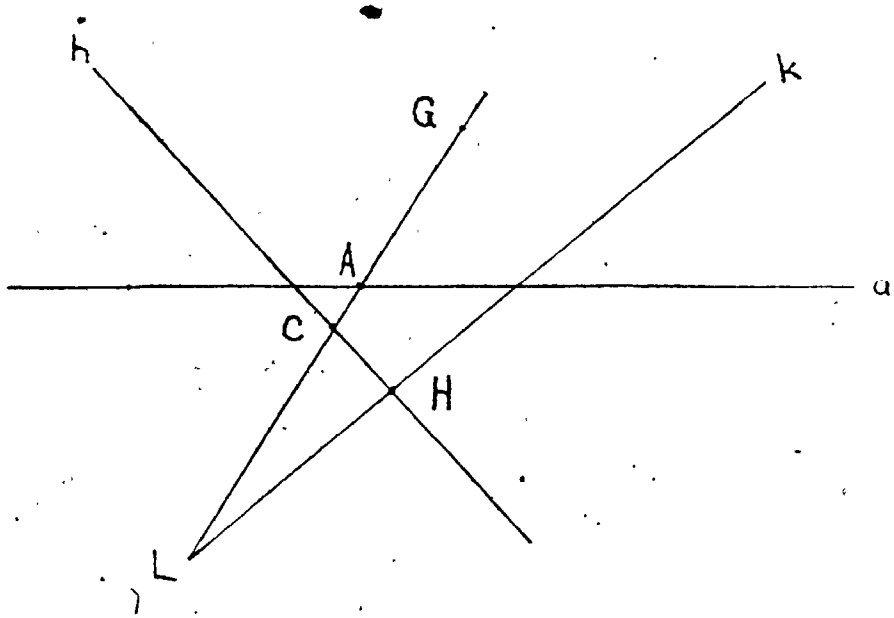

Fig.21

C in $\mathcal{A}_1 \setminus \mathcal{A}_2$ with $C \mathrel{I} AL$. Let h be any line of $\mathcal{A}_1 \setminus \mathcal{A}_2$ through C and let H be the unique point of $\mathcal{A}_2$ on h. Then

H,L are points of $\mathcal{A}_2$ and so HL is a line of $\mathcal{A}_2$, say HL=k (see Fig.21). But $\mathcal{A}_1$ and $\mathcal{A}_2$ coincide in all lines through L; thus k is a line of $\mathcal{A}_1$. Therefore k and h are lines of $\mathcal{A}_1$ and so $H=k\cap h$ is a point of $\mathcal{A}_1$. Let l be any line of $\mathcal{A}_1\cap\mathcal{A}_2$ through L such that $l\neq AL, HL$. (There exist at least three lines of $\mathcal{A}_1\cap\mathcal{A}_2$ through L.) Then AH,HL and l form a trilateral in $\mathcal{A}_1\cap\mathcal{A}_2$; but any trilateral is contained in at most one subplane of $\mathcal{C}$ contradicting the existence of C. Thus the set L_1 of points of $\mathcal{A}_1$ on AL is a subset of the set L_2 of points of $\mathcal{A}_2$ on AL. Similarly L_2 is a subset of L_1. Thus $L_1=L_2$.

Case (ii). Suppose $G\not\in AL$. Then AL is a line of $\mathcal{A}_1$ and so AL is a line of $\mathcal{A}_2$, as $\mathcal{A}_1$ and $\mathcal{A}_2$ coincide in all lines

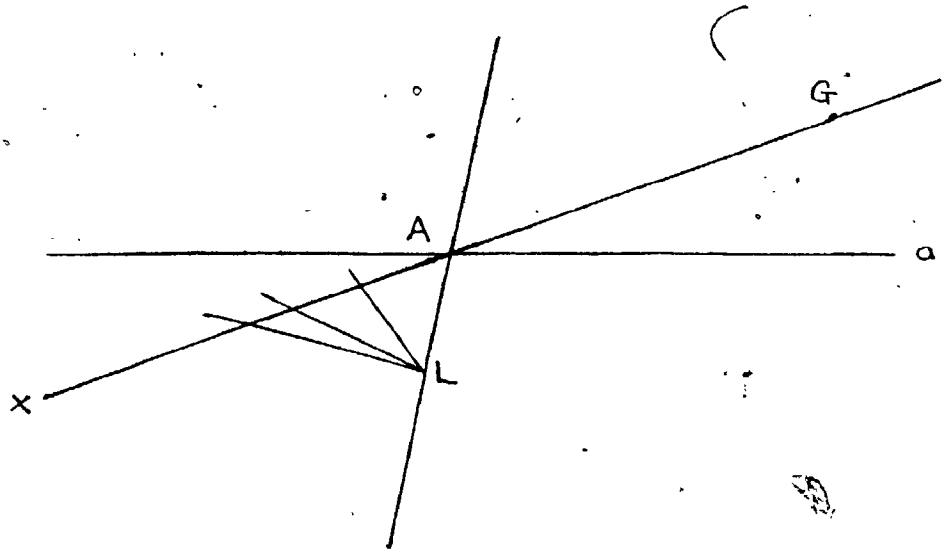

Fig.22

through L. Hence $A=a\cap AL$ is a point of $\mathcal{A}_2$. Let AG=x in $\mathcal{A}_1\cap\mathcal{A}_2$ (see Fig.22). Clearly the affine points of $\mathcal{A}_2$ on x coincide with the affine points of $\mathcal{A}_1$ on x as they are

just the intersections with lines through L. This completes the proof of lemma 2.3.2.

Define parallelism in $\mathcal{J}$ as follows:

> Two distinct blocks of $\mathcal{J}$ are parallel if and only if they are disjoint sets of lines say $\bar{A}$, $\bar{B}$ belonging to a common plane of $\mathcal{J}$ (denoted by $\bar{A} \| \bar{B}$). Each block is parallel to itself.

Note if $\bar{A}, \bar{B}$ are distinct parallel blocks in a type $\mathbb{B}$ plane of $\mathcal{J}$, then $A \neq B$ as point sets of $\mathcal{A}$ and $G \perp AB$ in $\mathcal{A}$. If $\bar{A}, \bar{B}$ are parallel in a type $\mathbb{L}$ plane, say determined by a point L of $\mathcal{A}$, then A=B=L.

<u>Lemma 2.3.3</u>. Parallelism is an equivalence relation.

Proof. Clearly parallelism is reflexive and symmetric. Let $\bar{A}, \bar{B}, \bar{C}$ be three blocks of $\mathcal{J}$ such that $\bar{A} \| \bar{B}$ and $\bar{B} \| \bar{C}$. Then $\bar{A}, \bar{B}$ and $\bar{B}, \bar{C}$ are contained in planes of $\mathcal{J}$, say $\mathcal{J}_1, \mathcal{J}_2$ respectively. Four cases can be distinguished.

Case (i). If $\mathcal{J}_1 = \mathcal{J}_2$ then as the planes of $\mathcal{J}$ are affine by lemma 2.3.1, $\bar{A} \| \bar{C}$.

Case (ii). Suppose $\mathcal{J}_1$ and $\mathcal{J}_2$ are distinct and both of type $\mathbb{L}$, say determined by points L,L' respectively. Then by the remark immediately preceding this lemma $\bar{A} \| \bar{B}$ implies A=B=L and $\bar{B} \| \bar{C}$ implies B=C=L'. Thus L=L' and $\mathcal{J}_1 = \mathcal{J}_2$ which is a contradiction. Thus case (ii) cannot occur.

Case (iii). Suppose $\mathcal{J}_1$ is a plane of type $\mathbb{B}$, say $\mathcal{A}_1$, and $\mathcal{J}_2$ is a plane of type $\mathbb{L}$, say $\mathcal{L}$, determined by the

point L of $\mathcal{A}$. Let x,y be arbitrary $\mathcal{J}$-points of $\bar{A}$ and z an arbitrary $\mathcal{J}$-point of $\bar{C}$. Clearly x,y,z are not collinear as $\mathcal{J}$-points for $\bar{A},\bar{B}$ are in a plane of type $\mathbb{B}$ and $\bar{B},\bar{C}$ are in a plane of type $\mathbb{L}$. Thus x,y,z form a

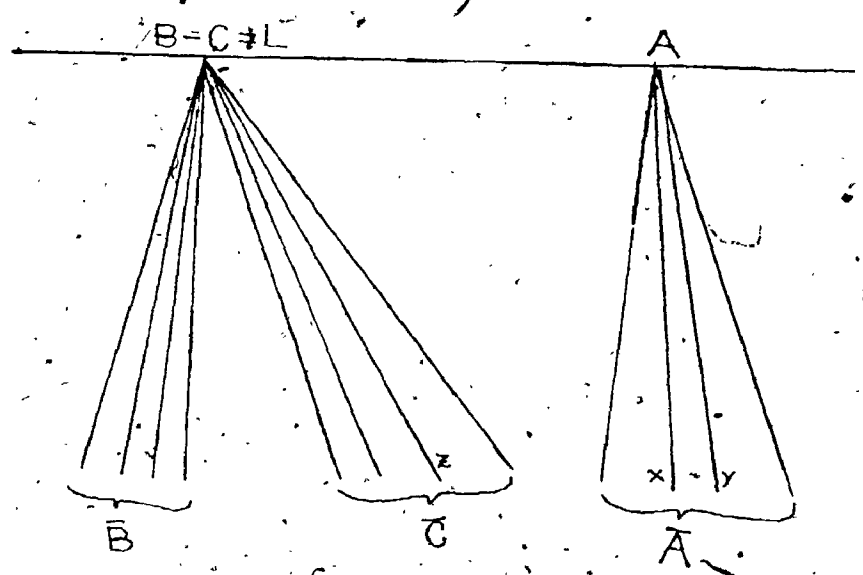

Fig. 23

trilateral in $\mathcal{P}$ (see Fig.23). Then by assumption (1) of this section x,y,z are contained in a unique plane $\mathcal{A}_2$ of $\mathcal{C}$. $\mathcal{A}_2$ intersects $\mathcal{L}$ in a block $\bar{C}_1$ through z (since LA=BA is the improper line of $\mathcal{A}_2$). Note $\bar{C}_1 \parallel \bar{B}$, for if w is in $\bar{C}_1 \cap \bar{B}$ then $w \mathrm{I} \bar{B} \subseteq \mathcal{A}_1$ and so x,y,w are in $\mathcal{A}_1 \cap \mathcal{A}_2$. Thus by assumption (1) $\mathcal{A}_1 = \mathcal{A}_2$ and so z is in $\mathcal{A}_1$ but this contradicts $\bar{B} \parallel \bar{C}$ as $\bar{B} \subseteq \mathcal{A}_1$ and z is in $\bar{C}$. Thus $\bar{C}=\bar{C}_1$ must be a parallel to $\bar{B}$ through z in the plane of type $\mathbb{L}$ carried by L=B=C. Thus $\bar{A},\bar{C}$ are contained in $\mathcal{A}_2$ and $\bar{A} \parallel \bar{C}$ as the special line of A is the same as the special

line of C, i.e. $G \mathrm{I} AC$.

Case (iv). Suppose $\mathcal{P}_1, \mathcal{P}_2$ are both planes of type $\mathbb{B}$, say $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively. $\bar{A} \parallel B$ and $\bar{B} \parallel \bar{C}$ implies that $G \mathrm{I} AB, BC$ and hence $AB = BC$. By lemma 2.3.2 $\mathcal{A}_1$ and $\mathcal{A}_2$ coincide in all their affine points on some special line x in $\mathcal{A}_1 \cap \mathcal{A}_2$. Suppose first that x is the special line of $\bar{B}$. Now $C \not\mathrm{I} x$, as $\bar{B} \parallel \bar{C}$, and C is in $\mathcal{A}_2$; thus C is in $\mathcal{A}_1$ and so $\mathcal{A}_1$ intersects the lines through C in a block $\bar{C}_1$. Segments through a common point are disjoint or equal. $\bar{C}$ is not contained in $\mathcal{A}_1$; thus $\bar{C} \parallel \bar{C}_1$, in a plane of type $\mathbb{L}$ carried by C. Clearly $\bar{A} \parallel \bar{C}_1$ as $G \mathrm{I} AC$, giving $\bar{A} \parallel \bar{C}$ by case (iii). Now suppose $y \neq x$ is the special line of $\bar{B}$. Let a,b be two distinct affine lines in $\bar{A}$ (see Fig.24). The points $x \cap a$,

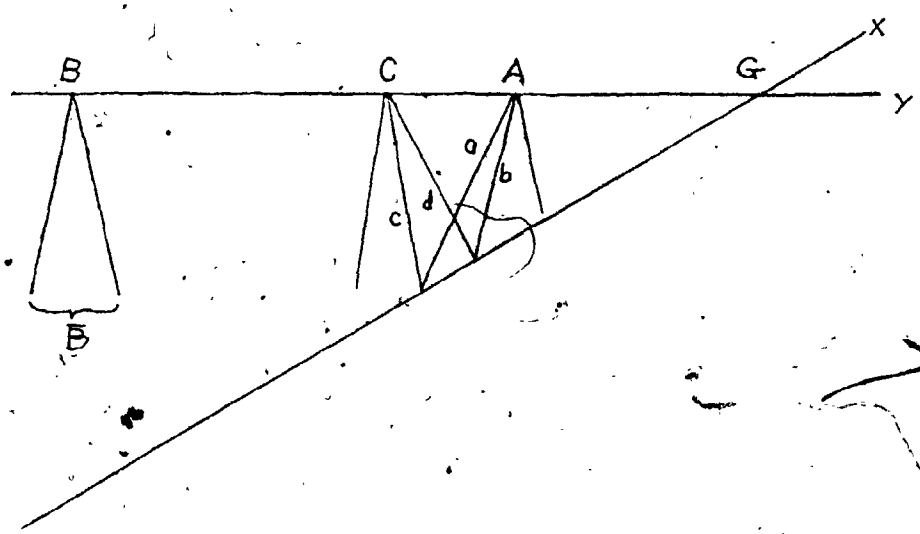

Fig.24

$x \cap b$ are in $\mathcal{A}_1$ and so as $\mathcal{A}_1$ and $\mathcal{A}_2$ coincide in all their points on x, $x \cap a$ and $x \cap b$ are in $\mathcal{A}_2$. Hence $\mathcal{A}_2$ contains the lines $c=(x \cap a)C$, $d=(x \cap b)C$. The trilateral a,b,c is contained in a unique subplane $\mathcal{A}_3$ of $\mathcal{G}$ by (1). $\mathcal{A}_3$ contains the special lines $y=(a \cap b)G$ and $z=(a \cap c)G$ and so also the line $d=(x \cap b)(c \cap y)$. Thus $\overline{c \cap d}=\bar{C}$ and $\overline{a \cap b}=\bar{A}$. Thus $\mathcal{A}_3$ contains $\bar{A}$ and $\bar{C}$ and $\bar{A} \parallel \bar{C}$ in $\mathcal{A}_3$ since both $\bar{A}$ and $\bar{C}$ have y as a special line. This completes the proof that parallelism is an equivalence relation.

<u>Lemma 2.3.4</u> Two points of $\mathcal{J}$ determine a unique block of $\mathcal{J}$.

Proof. Let x and y be any two distinct $\mathcal{J}$-points. By (2) $\overline{x \cap y}$ is the unique block of $\mathcal{J}$ determined by x and y.

<u>Lemma 2.3.5</u> Three points of $\mathcal{J}$ not on a common block of $\mathcal{J}$ are contained in a unique plane of $\mathcal{J}$.

Proof. Let a,b,c be three points of $\mathcal{J}$ not on a common block of $\mathcal{J}$, that is a,b,c are lines of $\mathcal{P}$ with c not in $\overline{a \cap b}$. If a,b,c form a trilateral in $\mathcal{P}$ then by (1) a,b,c are contained in a unique plane of type $\mathbb{B}$. Clearly they do not have a point in common and thus are not in a plane of type $\mathbb{L}$. If a,b,c have a point of $\mathcal{A}$ in common, say $a \cap b=b \cap c=L$, then a,b,c are contained in the unique plane of type $\mathbb{L}$ determined by L. As $L \sqsupset a,b,c$ but c is not in $\overline{a \cap b}$, a,b,c are not in a plane of type $\mathbb{B}$.

<u>Corollary 2.3.1</u> A point of $\mathcal{J}$, and a nonincident block of

$\mathcal{J}$ determine a unique plane of $\mathcal{J}$.

We are now ready to prove the following theorem.

<u>Theorem 2.3.1</u>. The incidence structure $\mathcal{J}$ defined above is an affine space of dimension greater than two.

Proof. We shall use the result due to Lenz [11] quoted in chapter I p.23 to prove this theorem. Note first that from lemma 2.3.3 there is an equivalence relation called parallelism between the blocks of $\mathcal{J}$. Suppose we have a point-block pair P,b; then if $P \mathrm{I} b$, b is clearly the unique block of $\mathcal{J}$ through P parallel to b. If $P \not\mathrm{I} b$ then by corollary 2.3.1 P and b are in a unique plane of $\mathcal{J}$. By lemma 2.3.1 the planes of $\mathcal{J}$ are affine planes and so there is a unique parallel to b incident with P. Finally given four distinct points A,B,C,D of $\mathcal{J}$ with $\overline{AB} \| \overline{CD}$, A,B,C,D are in a unique plane of $\mathcal{J}$ and as these are affine planes for any point $P \mathrm{I} \overline{AC}$ either $P \mathrm{I} \overline{CD}$ or $\overline{AB}$ and $\overline{PD}$ have a point in common. Thus by the result due to Lenz [11] $\mathcal{J}$ is an affine space. Since there is more than one subplane in $\mathcal{C}$, $\mathcal{J}$ consists of more than one plane and hence $\mathcal{J}$ is of dimension greater than two.

§4. <u>Baer Subplanes Belonging to a Derivation Set</u>

Suppose π is a derivable projective plane of finite or infinite order with a derivation set $\mathcal{D}$ contained on some line l. Denote by $\mathcal{B}$ the set of Baer subplanes of π , belonging to $\mathcal{D}$. Let G be any point in $\mathcal{D}$ and consider the

dual affine plane $\mathcal{A} := \pi^G$. Since G is a point of $\mathcal{D}$, all the subplanes in $\mathcal{B}$ are dual affine subplanes of $\mathcal{A}$. Denote the set of subplanes of $\mathcal{B}$ considered as subplanes of $\mathcal{A}$ by $\mathcal{C}$. Let $\mathcal{P}$ be the set of lines different from l through the points of $\mathcal{D}\setminus\{G\}$, i.e. $\mathcal{P}$ is the set of lines in the subplanes of $\mathcal{C}$. Now we shall show that the planes dual to the planes of $\mathcal{C}$ are some of the affine hyperplanes contained in a three-dimensional affine space. Hence these planes and their duals in $\mathcal{C}$ are desarguesian. This implies Cofman's result (see [7]) that the subplanes of $\mathcal{B}$ are desarguesian. First we prove the following lemmas.

<u>Lemma 2.4.1</u> Any trilateral in $\mathcal{P}$ is contained in exactly one subplane of $\mathcal{C}$.

Proof. Let a,b,c be any trilateral in $\mathcal{P}$. Then as l is not a line of $\mathcal{P}$ at least two of the vertices of the trilateral are not in $\mathcal{D}$; say $a\cap b=C$ and $a\cap c=B$ are not in $\mathcal{D}$ (see Fig. 25). Now $BC\cap l$ is a point of $\mathcal{D}$ and so as $\mathcal{D}$ is a derivation

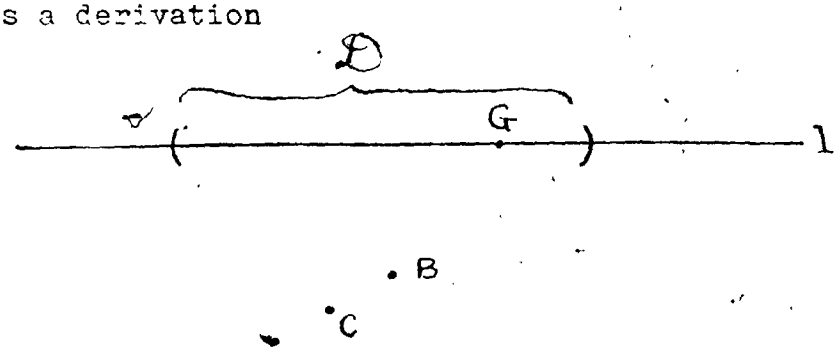

Fig. 25

set B and C are contained in a unique subplane of $\mathcal{B}$ and so in a unique plane of $\mathcal{G}$, say $\mathcal{A}_1$. Then as a,b,c are lines of $\mathcal{P}$ we have $a\cap l$, $b\cap l$, $c\cap l$ are points of $\mathcal{D}\setminus\{G\}\subseteq\mathcal{A}_1$ and so $a=B(a\cap l)$, $b=C(b\cap l)$, $c=B(c\cap l)$ are lines of $\mathcal{A}_1$.

<u>Lemma 2.4.2</u>. If two planes $\mathcal{A}_1$, $\mathcal{A}_2$ of $\mathcal{G}$ have two affine lines a,b in common then they have all their lines through $a\cap b$ in common.

Proof. The proof can be divided into two cases.
Case (i). If $a\cap b$ is not a point of $\mathcal{D}$ then all lines through $a\cap b$ in any subplane of $\mathcal{G}$ are the lines joining $a\cap b$ to the points of $\mathcal{D}\setminus\{G\}$ (see Fig.26). Hence all lines through $a\cap b$ in $\mathcal{A}_1$ and $\mathcal{A}_2$ are the same.

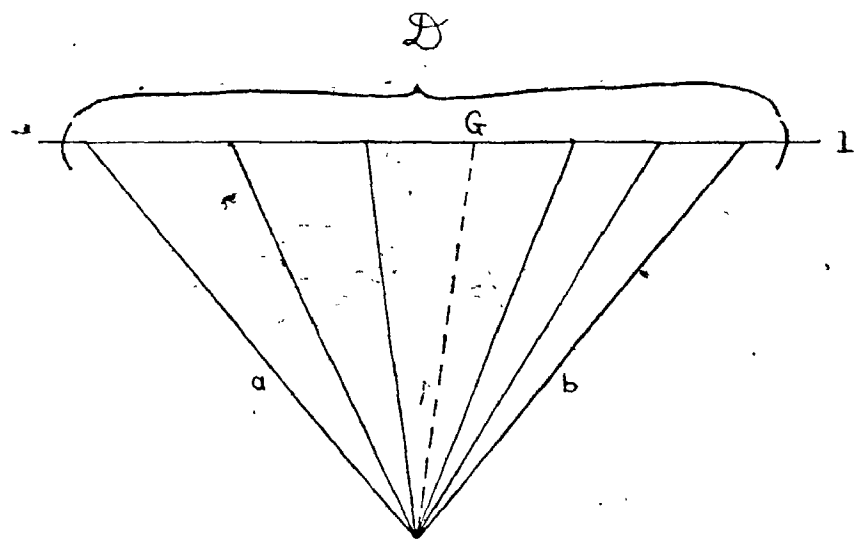

Fig.26

Case (ii). If $a\cap b$ is a point of $\mathcal{D}$, then suppose there exists a line c with $a\cap b\,I\,c$ and c in $\mathcal{A}_1\setminus\mathcal{A}_2$. Consider

the subplanes π_1, π_2 in $\mathcal{B}$ from which $\mathfrak{A}_1$ and $\mathfrak{A}_2$ respectively arise. Let H be any point of π_1 distinct from $a \cap b$ with $H I c$ (see Fig.27). H is not a point of π_2, for H in π_2, $a \cap b$ in π_2 and H, $a \cap b I c$ implies that c is a line of π_2

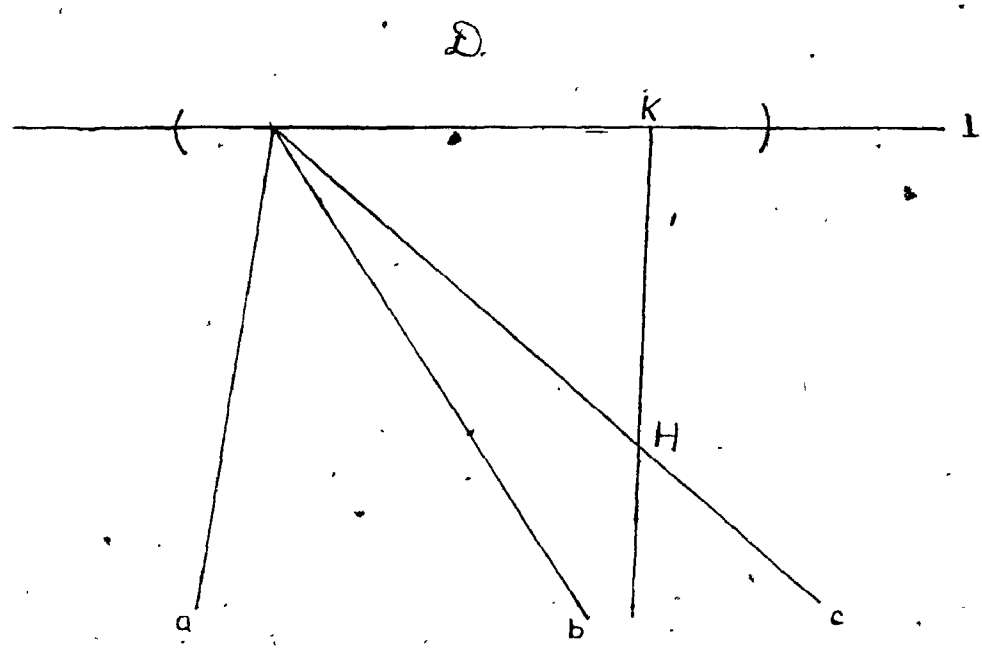

Fig.27

contradicting the assumption that c is not a line of $\mathfrak{A}_2$. Then as π_2 is a Baer subplane there is exactly one line of π_2 through H, say h. As h is in π_2 we have $K = h \cap l$ in $\mathcal{D}$. But $\mathcal{D} \subseteq \pi_1$ and so K is a point of π_1 and as H is also a point of π_1, $HK = h$ is a line of π_1. Thus h,a,b are in π_1 and π_2 which implies that $h \cap a$ and $h \cap b$ are in $\mathfrak{A}_1$ and $\mathfrak{A}_2$. As π_1, π_2 are Baer subplanes belonging to a derivation set $\mathcal{D}$ they are uniquely determined by any two of their affine points; thus we have $\pi_1 = \pi_2$. Then $\mathfrak{A}_1$ and $\mathfrak{A}_2$ coincide contradicting the existence of c. Hence $\mathfrak{A}_1$ and $\mathfrak{A}_2$ must

have all their lines through $a \cap b$ in common.

Lemma 2.4.2 implies that any two lines of $\mathcal{P}$ determine a segment as defined in chapter I p.7.

Now define an incidence structure $\mathcal{J}$ whose points are the lines of $\mathcal{P}$ and whose blocks are the segments determined by any two lines of $\mathcal{P}$. Define two types of planes in $\mathcal{J}$: planes of type $\mathbb{B}$ which are the duals of the dual affine subplanes of $\mathcal{G}$ and planes of type $\mathbb{L}$ which consist of the affine lines through any point of $\mathcal{D} \setminus \{G\}$. Clearly lemmas 2.4.1 and 2.4.2 yield conditions (1) and (2) in §3 respectively (see p.42). Thus by theorem 2.3.1 $\mathcal{J}$ is an affine space of dimension greater than two. The planes of type $\mathbb{B}$ are therefore desarguesian and so are their duals in $\mathcal{G}$ and the projective extensions of the planes in $\mathcal{G}$. Hence we have proved the following theorem.

<u>Theorem 2.4.1</u>. Any Baer subplane belonging to a derivation set of a derivable projective plane is desarguesian.

III. USES OF BAER SUBPLANES I

In this chapter we consider the first use of Baer subplanes, namely the characterization of finite projective planes of square order. We shall give characterizations for desarguesian planes although results are known for non-desarguesian planes (see [10] and [16]).

The first section of this chapter uses the dual of theorem 2.3.1 to prove that a projective plane of arbitrary order (finite or infinite) with a 'sufficiently large' class of Baer subplanes and with segments well-defined is Moufang (see [6]). As a corollary we obtain the result of Cofman ([5] and [6]) that a projective plane of square order is desarguesian if and only if the vertices of every quadrangle are contained in a unique Baer subplane.

The second section contains another result due to Cofman ([4]). It is similar to the first characterization of square order desarguesian planes but involves Baer involutions.

§1. Characterization I

In this section we give a characterization for finite desarguesian planes of square order. We first prove the following theorem for planes of arbitrary order (see [6]).

Theorem 3.1.1. Let π be a projective plane with a class $\mathcal{B}$ of Baer subplanes satisfying the following conditions:

(a) any quadrangle of π is contained in exactly one subplane of $\mathcal{B}$;

(b) any two Baer subplanes of $\mathcal{B}$ through three distinct collinear points A,B,C of π coincide in all their points on the line containing A,B,C.

Then π is Moufang and the subplanes of $\mathcal{B}$ are desarguesian.

Proof. Let l be any line of π and consider the affine plane $\mathcal{A} := \pi^{l}$. Let $\mathcal{P}$ be the set of affine points of $\mathcal{A}$. Denote by $\mathcal{C}$ the set of subplanes of $\mathcal{B}$ having l as a line. Let A,B,C be any triangle of points of $\mathcal{P}$. Then AB and AC intersect l in distinct points D and E respectively (see Fig. 28).

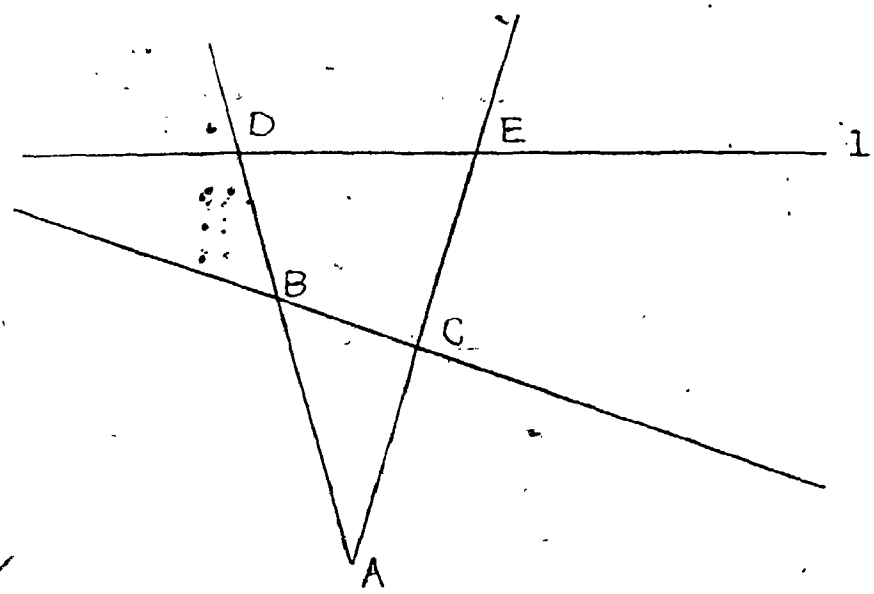

Fig. 28

The triangle A,B,C is in a subplane of $\mathcal{C}$ if and only if the quadrangle B,C,D,E is in a subplane of $\mathcal{B}$; for if A,B,

C are in the subplane $\mathcal{A}_0$ in $\mathcal{C}$ then as $\mathcal{A}_0 = \pi_0^1$ for some π_0 in $\mathcal{B}$, $D = AD \cap l$ and $E = AC \cap l$ are in π_0. Thus B,C,D,E are in π_0. Conversely if B,C,D,E is a quadrangle of some subplane π_0 in $\mathcal{B}$, then as $DE = l$, π_0^1 is a subplane of $\mathcal{C}$ containing the points B,C and $A = BD \cap CE$; that is A,B,C are the vertices of a triangle of π_0^1. Thus by condition (a) of the theorem any triangle of points of $\mathcal{P}$ is in exactly one subplane of $\mathcal{C}$.

If two subplanes $\mathcal{A}_1$, $\mathcal{A}_2$ of $\mathcal{C}$ coincide in two affine points A,B on some line l of $\mathcal{A}$ then $\mathcal{A}_1$ and $\mathcal{A}_2$ also have the same special point, $AB \cap l$. Thus by condition (b) of the theorem $\mathcal{A}_1$ and $\mathcal{A}_2$ coincide in all their points on the line AB; that is, two points A,B determine a unique segment $\overline{AB}$ (see chapter I p.7).

Consider the incidence structure $\mathcal{J}$ whose "points" are the points of $\mathcal{P}$ and whose "blocks" are the segments determined by any two points of $\mathcal{P}$. Define two types of planes in $\mathcal{J}$: type $\mathbb{B}$ planes and type $\mathbb{L}$ planes. Type $\mathbb{B}$ planes are the subplanes of $\mathcal{C}$ and type $\mathbb{L}$ planes correspond to the affine points of some line of π^1. Then by the dual of theorem 2.3.1 $\mathcal{J}$ is an affine space of dimension greater than two. Since the dimension of $\mathcal{J}$ is greater than two, $\mathcal{J}$ is desarguesian.

Let Γ be the group of translations of $\mathcal{J}$. Since $\mathcal{J}$ is desarguesian, Γ is transitive on the points of $\mathcal{J}$ and so as the point sets of $\mathcal{J}$ and π^1 are the same, Γ is transitive on the points of π^1. Since Γ is a group of

translations of $\mathcal{J}$, any plane of $\mathcal{J}$ is parallel to its image under the action of any element of Γ. Now from the proof of theorem 2.3.1 type $\mathbb{L}$ planes are parallel to type $\mathbb{L}$ planes only and type $\mathbb{L}$ planes correspond to the lines of π^1; thus, lines of π^1 are taken to lines of π^1 under Γ, that is, Γ induces a group of collineations of π^1. Clearly type $\mathbb{L}$ planes are parallel if and only if the corresponding lines of π^1 are parallel. Thus Γ is a group of translations of π^1 which is transitive on the points of π^1. Therefore, π^1 is a translation plane.

The choice of 1 was arbitrary; thus π is a Moufang plane. The Baer subplanes of $\mathcal{C}$ are some of the planes of $\mathcal{J}$ and thus are desarguesian. Then the corresponding subplanes of $\mathcal{B}$ are also desarguesian.

Clearly finite planes satisfying theorem 3.1.1 are desarguesian. When the paper by J. Cofman (pp. 60-61 of [16]) appeared in April 1973, it was an open question whether infinite planes satisfying theorem 3.1.1. were desarguesian; no new results are known to the author.

<u>Corollary 3.1.1</u>. A finite projective plane π of square order n is desarguesian if and only if it has a class of Baer subplanes satisfying conditions (a) and (b) of theorem 3.1.1.

Proof. If a finite projective plane π of square order n satisfies the conditions of the theorem it is desarguesian (see Chapter I p. 13).

Suppose conversely that π is a finite desarguesian plane, say of order q^2. Then any quadrangle of π is contained in a unique subplane of π; for as π is desarguesian if we coordinatize π with respect to any quadrangle A,B,C,D then the ternary ring obtained is the unique field K of order q^2, i.e. $K=GF(q^2)$. Now $K_0=GF(q)$ is a subfield of $K=GF(q^2)$ and so K_0 coordinatizes a subplane π_0 of order q containing A,B,C,D. If A,B,C,D are in any other subplane π_1 of order q then π_1 is coordinatized by a subfield K_1 of order q of K. But $K=GF(q^2)$ has a unique subfield of order q, GF(q); thus $K_0=K_1$ and π_0 and π_1 coincide. Thus π satisfies condition (a) of theorem 3.1.1.

Now given any three collinear points A,B,C of π there are

$$N=q^4(q^2-1)/2$$

quadrangles A,B,D,E in π with C incident with DE and

$$N_0 = q^2(q-1)/2$$

quadrangles A,B,D,E in π_0 with C incident with DE. Since π satisfies condition (a) any Baer subplane is uniquely determined by one of its quadrangles. Thus any subplane containing A,B,C is uniquely determined by one of the above quadrangles. The number of subplanes with such quadrangles is therefore

$$\frac{N}{N_0}=\frac{q^4(q^2-1)/2}{q^2(q-1)/2} = q^2(q+1). \qquad (3.1)$$

Suppose A,B,C lie on some line l in a Baer subplane π_0 of π. Let Γ be the group of perspectivities in π with axis l. If α is in Γ then $\pi_0^\alpha = \pi_0$ if and only if the centre P of α is in π_0 and for some nonfixed point X of π_0, X^α is in π_0. (This is clear as we know we can construct the image of any point Y of π_0 using only P,X,X^α and l. Conversely if $\pi_0^\alpha = \pi_0$, α induces a collineation of π_0 and as l is a line of π_0, α has a centre in π_0 which is clearly the same as the centre in π). Thus as π_0 is desarguesian the order of the stabilizer of π_0 in Γ is $1+(q+1)(q-1)+q^2(q-2)=q^2(q-1)$; that is, it consists of all perspectivities with axis l of the desarguesian plane π_0 of order q. Similarly as π is desarguesian, the order of Γ is $q^4(q^2-1)$. Thus the length of the orbit of π_0 under Γ is

$$\frac{q^4(q^2-1)}{q^2(q-1)} = q^2(q+1).$$

The points of l are fixed by any element of Γ. Thus the $q^2(q+1)$ distinct images of π_0 under Γ contain the same q+1 points on l, including the points A,B,C. Now A,B,C are in exactly $q^2(q+1)$ Baer subplanes by (3.1). Hence the images of π_0 under Γ must be all the Baer subplanes containing A,B,C. Therefore π satisfies condition (b) of the theorem and the proof is completed.

§3. Characterization II

This section is concerned with the existence of Baer involutions of finite projective planes. We shall show that a finite projective plane of square order n is desarguesian if and only if the vertices of every quadrangle are fixed by a unique Baer involution.

Suppose π is a finite desarguesian projective plane of order q^2, and so coordinatized by $GF(q^2)$. By corollary 3.1.1 the vertices of every quadrangle of π are contained in a unique Baer subplane of π. Thus we need to show that every Baer subplane of π is fixed by a unique involution. Since π is desarguesian we can coordinatize with respect to any quadrangle and hence we can have any Baer subplane coordinatized by $GF(q)$. Clearly the automorphism of $GF(q^2)$ given by $\alpha: x \to x^q$ will induce a Baer involution in π; for $x^q = x$ if and only if $x^{q-1} = 1$, that is, the set of fixed elements of α is $GF(q)$, since the multiplicative group of $GF(q)$ is the unique subgroup of order $q-1$ in the multiplicative group of $GF(q^2)$. Note that α is an involution as $x^{q^2-1} = 1$ for all nonzero x in $GF(q^2)$. Thus any Baer subplane is fixed by at least one Baer involution.

Now let Γ be the subgroup of Autπ fixing a Baer subplane pointwise. Then Γ fixes a quadrangle pointwise and hence Γ is isomorphic to a subgroup Γ' of Aut $GF(q^2)$ (see for example [9] theorem 2.12).

Clearly the set of fixed elements of Γ' is $GF(q)$. But if $|\Gamma'| = n$, then $GF(q^2)$ is a vector space of dimension n over $GF(q)$ (see for example [9] result 1.6). Thus the order of Γ is 2 and so there is a unique Baer collineation α fixing a Baer subplane pointwise. Clearly α is an involution.

In order to show that a finite projective plane π with the vertices of every quadrangle fixed by a unique Baer involution is desarguesian we need the following lemmas:

Let Γ be the group generated by all Baer involutions of π. Suppose the order of π is n.

<u>Lemma 3.2.1.</u> Γ contains nontrivial elations.

Proof: Let α be a Baer involution with fixed subplane π_0. If N and N_0 are the number of distinct quadrangles in π and π_0 respectively, then $N=(n^2+n+1)(n^2+n)\,n^2(n-1)^2/4!$ and $N_0=(n+\sqrt{n}+1)(n+\sqrt{n})n(\sqrt{n}-1)^2/4!$ Each quadrangle is fixed by a unique Baer involution of Γ. Hence

$$N/N_0=(n+\sqrt{n}+1)(n+1)\,n\sqrt{n}$$

is the number of Baer involutions in Γ. Clearly N/N_0 is even, but the number of involutions in a finite group is odd. Thus as any involution is a quasiperspectivity there must be a nontrivial involutory perspectivity in Γ. If n is even then every nontrivial involutory perspectivity of π is an elation (see [8] section 4.1.9). Suppose n is odd and let α be a nontrivial involutory perspectivity in Γ. Then α is an homology (see [8] Section 4.1.9).

Let the centre of α be A and the axis a. Let B,C be two distinct points on a and D a point not on AB, AC or BC. By assumption there

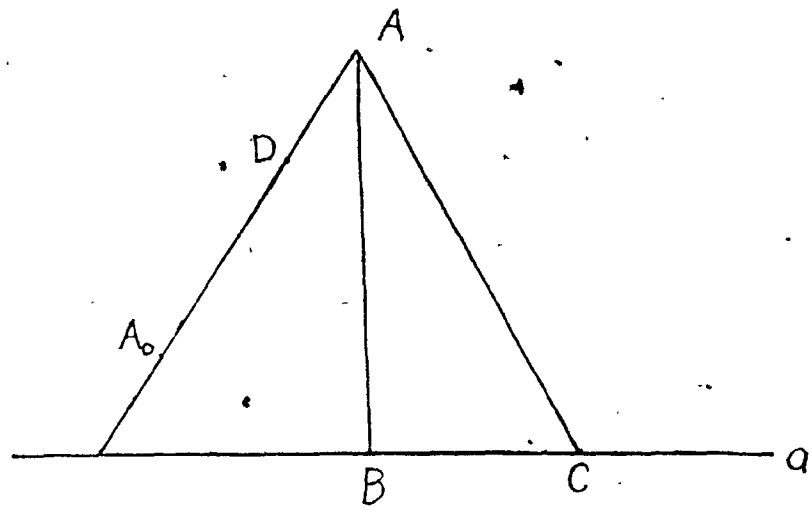

Fig 29

exists a unique Baer involution σ fixing A,B,C,D. Let π_0 be the fixed subplane of σ and let A_0 be a point on AD but not in π_0 (see Fig. 29). Then the unique Baer involution σ_0 fixing A_0,B,C,D cannot fix A by the uniqueness of σ. Thus $\sigma_0^{-1}\alpha\sigma_0$ is an (A^{σ_0},a)-homology (see [9] corollary 2 to lemma 4.11). Note that A^{σ_0} is not incident with a for if A^{σ_0} is incident with a then as A is incident with A_0D, $A^{\sigma_0}=a^{\sigma_0}\cap A^{\sigma_0}D^{\sigma_0}=a\cap A_0D$ which is a contradiction as $a\cap A_0D$ is a fixed point under σ_0. Both α and $\sigma_0^{-1}\alpha\sigma_0$ are involutory homologies with distinct centres and the same axis. Thus $\alpha\sigma_0^{-1}\alpha\sigma_0$ is an elation (see [8] section 3.1.8(b)).

<u>Lemma 3.2.2</u>. No elements of π are fixed by Γ; that is, $F(\Gamma)=\emptyset$.

Proof. Let X be any point of π. Consider a quadrangle containing X as a vertex, say X,Y,Z,U. Let σ be the unique Baer involution fixing X,Y,Z,U with fixed subplane π_0. Let X_0 be any point not in π_0 forming a quadrangle with Y,Z,U. By assumption there exists a Baer involution σ_0 fixing X_0,Y,Z,U. By the uniqueness of σ, σ_0 cannot fix X. A dual proof shows that no line of π is fixed by Γ.

<u>Lemma 3.2.3</u>. For every point A of π which is the centre of a nontrivial elation in Γ, the group Γ contains elations with centre A and at least two distinct axes through A.

Proof. Let α be a nontrivial elation of Γ with centre A and axis a. Let B≠A be a point of a and let C,D be distinct points not on a such that A,B,C,D is a quadrangle. Let σ be

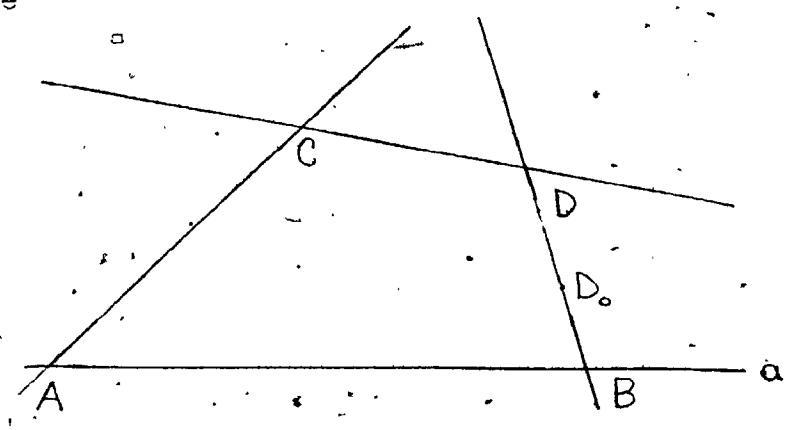

Fig. 30

the unique Baer involution fixing A,B,C,D. Let π_o be the subplane of π fixed by σ. Let D_o be a point of BD not in π_o and let σ_o be the unique Baer involution fixing A,C,D,D_o (see Fig. 30). As σ_o does not fix $B=a\cap DD_o$, σ_o does not fix a. Thus $\sigma_o^{-1}\alpha\sigma_o$ is an elation with centre A and axis $a\sigma_o \neq a$.

The dual of this result follows immediately:

<u>Lemma 3.2.4.</u> For every line a of π which is the axis of a nontrivial elation in Γ, the group Γ contains elations with axis a and at least two distinct centres on a.

Let $E=E(\Gamma)$ be the set of points and lines of π which are respectively centres and axes of nontrivial elations in Γ.

<u>Lemma 3.2.5.</u> There exist at least two nontrivial (A,a)- elations for every flag (A,a) in E.

Proof. Let α be a nontrivial elation with centre A and axis a. Such an elation exists by [9] lemma 4.15. Let B be a point not on a and let $B^{\alpha}=B'$. Let C,D be two distinct points of a different from A. Then Γ contains a unique Baer involution σ fixing B,B',C,D. Choose a point B_o on BB' such that $B_o^{\sigma} \neq B_o$ (see Fig. 31).

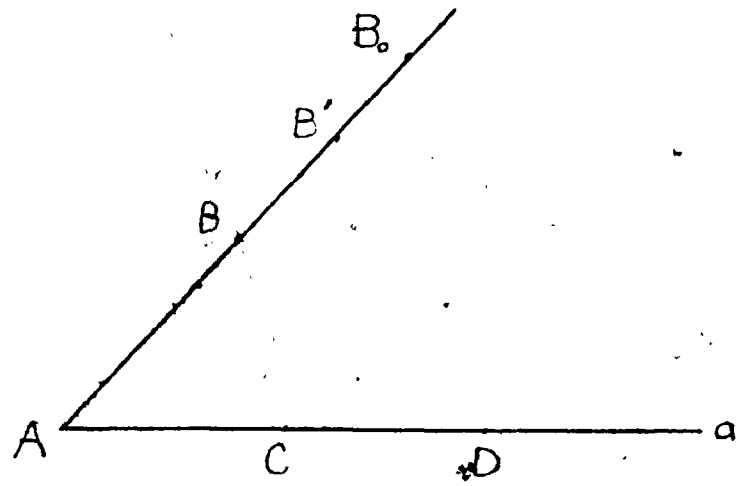

Fig. 31.

The unique Baer involution σ_0 fixing C, D, B, B_0 does not fix B', by the uniqueness of σ. Thus $\sigma^{-1}\alpha\sigma_0 = \alpha_0$ is an elation with centre A and axis a. Note that $\alpha_0 \neq \alpha$ as $B^{\alpha} = B'$ and $B^{\alpha_0} = B'^{\sigma} \neq B'$.

<u>Lemma 3.2.6</u> E is a desarguesian subplane of π and Γ contains all elations of E.

Proof. This result of Piper can be found in [8] (4.3.20(b) and 4.3.22(a)). Since results from various authors and papers are needed to prove this, it seems advisable to omit the proof here.

We are now ready to prove the main theorem of this section.

<u>Theorem 3.2.1</u> A finite projective plane π of square order n is desarguesian if and only if the vertices of every quadrangle are fixed by a unique Baer involution.

Proof. We have already shown that the vertices of every quadrangle of a desarguesian projective plane are fixed by a unique Baer involution.

Suppose that π is a projective plane such that the vertices of every quadrangle are fixed by a unique Baer involution; we shall show that $E=\pi$. Suppose E is a proper subplane of π. (Note that E need not be a Baer subplane). Let A,B,C be three noncollinear points of E. Through each of these points there are at least two lines of $\pi \setminus E$. Thus there is a trilateral a,b,c in $\pi \setminus E$ with $A \in a$, $B \in b$ and $C \in c$. Let $a \cap b = C_1$, $a \cap c = B_1$ and $b \cap c = A_1$. Choose a point D different from C, C_1 on CC_1. Then A_1, B_1, C_1, D are the vertices of a quadrangle and so are fixed by an involution σ with fixed subplane π_σ. The line CC_1 contains at

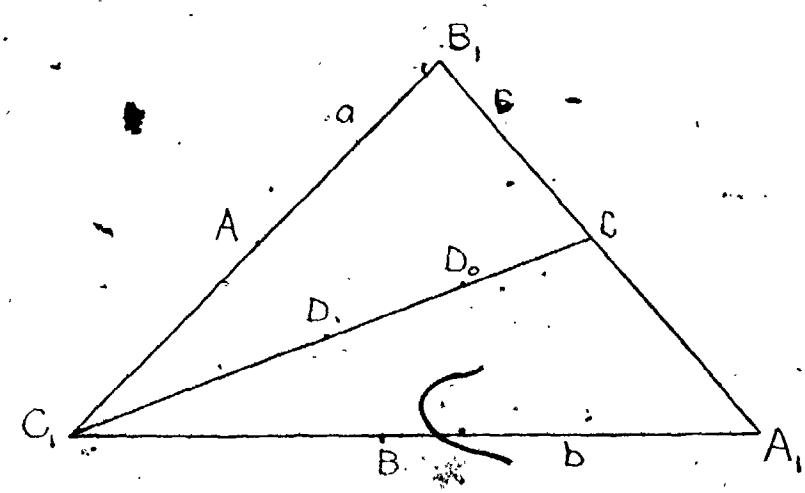

Fig. 32

least one point D_0 of π which is not in π_0 (see Fig. 32). Let σ_0 be the unique Baer involution fixing A_1, B_1, C_1, D_0. Now D_0 is not fixed by σ; thus σ and σ_0 are distinct. But clearly both σ and σ_0 fix the lines a,b,c. As a,b,c are not in E each has at most one point of E; in fact each has exactly one point namely A,B,C respectively. Both σ and σ_0 must fix A, B, C, C_1 for otherwise A^σ, B^σ and C^σ are in E by definition of E and as σ fixes a,b,c $A^\sigma, B^\sigma, C^\sigma$ are incident with a,b,c respectively contradicting the uniqueness of the point of E on a,b,c. Note for example that A^σ is in E since A in E implies there is a nontrivial elation in Γ with centre A, say α. Then α^σ is a nontrivial elation with centre A^σ which implies that A^σ is in E. Similarly for B^σ and C^σ. However σ and σ_0 are distinct Baer involutions and thus cannot fix the same quadrangle. Thus we have a contradiction and therefore $\pi = E$; thus by lemma 3.2.6 π is desarguesian.

The following is an immediate corollary from theorem 3.2.1 and lemma 3.2.6.

<u>Corollary 3.2.1</u> The group of collineations of π generated by all Baer involutions contains all elations of π.

IV Uses of Baer Subplanes II

This chapter deals with Ostrom's method for construction of projective planes. It is known as derivation and is applied to the derivable projective planes defined in chapter I. The proof that finite (square order) derivable planes can be used to define new projective planes can be found in [8] (section 5.4), [9] (chapter X) and [14]. In the first section of this chapter we show that derivable planes of arbitrary order (finite or infinite) can be used to define new planes. Although infinite derivable planes have been defined (see for example [7]) no proof of this result seems to have been published.

In the second section (following [8] section 5.4.7 and [15]) we show that certain dual translation planes can be derived and that the derived planes are not translation planes and are not dual translation planes.

§1. Derivation

'Derivation' is a construction technique due to Ostrom. It requires a projective plane π with a 'sufficiently large' class of Baer subplanes. We replace some of the lines of π with Baer subplanes to obtain a new incidence structure $\overline{\pi}$, which in general is not isomorphic to π.

A projective plane π is said to be <u>derivable</u> if there exists a subset $\mathcal{D}$ of points on some line l_∞

of π such that for any two affine points X,Y of the affine plane π^{l_∞} with $XY \cap l_\infty$ in $\mathcal{D}$ there exists a Baer subplane $\mathcal{B}$ of π^{l_∞} containing X and Y and such that the special points of $\mathcal{B}$ are precisely the points of $\mathcal{D}$. $\mathcal{D}$ is said to be a "derivation set" for π and the Baer subplanes containing the points of $\mathcal{D}$ as their special points are said to "belong to $\mathcal{D}$".

A derivable projective plane π can be used in the following manner to construct a new projective plane $\overline{\pi}$. Let $\mathcal{A}$ denote the affine plane π^{l_∞}. Define an incidence structure $\overline{\mathcal{A}}$ whose points are the affine points of $\mathcal{A}$ and whose lines are of two types: type 1 lines are lines of $\mathcal{A}$ whose special points are not in $\mathcal{D}$, and type 2 lines are the Baer subplanes of $\mathcal{A}$ belonging to $\mathcal{D}$.

We now show that $\overline{\mathcal{A}}$ is an affine plane. Let X and Y be any two points of $\overline{\mathcal{A}}$. If the point $XY \cap l_\infty$ is not in $\mathcal{D}$ then the line XY of $\mathcal{A}$ is the unique type 1 line of $\overline{\mathcal{A}}$ containing X and Y. As the special point of XY is not in $\mathcal{D}$, XY cannot be a line of any Baer subplane of $\mathcal{A}$ belonging to $\mathcal{D}$ and so X and Y are not contained in any type 2 line of $\overline{\mathcal{A}}$. Thus X and Y determine a unique line of $\overline{\mathcal{A}}$. If the point $XY \cap l_\infty$ is in $\mathcal{D}$ then X and Y determine at least one Baer subplane belonging to $\mathcal{D}$. Suppose that X and Y are in $\mathcal{B}_1$ and $\mathcal{B}_2$ where $\mathcal{B}_i$ (i=1,2) are Baer subplanes belonging to $\mathcal{D}$. Then X and Y are distinct affine points in $\mathcal{B}_1$ and $\mathcal{B}_2$ and $\mathcal{D}$ is contained in $\mathcal{B}_1$ and $\mathcal{B}_2$ which implies that $\mathcal{B}_1$ and $\mathcal{B}_2$ coincide in all their

lines through X and Y (since these lines are the uniquely determined joins of X and Y respectively to the points of $\mathcal{D}$). Clearly then by considering the intersections of these lines, $\mathcal{B}_1$ and $\mathcal{B}_2$ coincide in all their points and so $\mathcal{B}_1 = \mathcal{B}_2$. Thus any two points X and Y of $\bar{\mathcal{A}}$ determine a unique line of $\bar{\mathcal{A}}$.

Now suppose we are given an affine line l of $\bar{\mathcal{A}}$ and an affine point P of $\bar{\mathcal{A}}$ such that P is not incident with l. Then we must show that there exists a unique line m of $\bar{\mathcal{A}}$ such that P is incident with m and l is parallel to m.

First suppose that l is a type 1 line of $\bar{\mathcal{A}}$. Then l is a line of $\mathcal{A}$ and so as $\mathcal{A}$ is an affine plane there is a unique line m of type 1 in $\bar{\mathcal{A}}$ such that P is incident with m and l is parallel to m, i.e. in π, $l \cap m$ is a point of $l_\infty \setminus \mathcal{D}$. Suppose there is type 2 line $\mathcal{B}$ of $\bar{\mathcal{A}}$ through P and parallel to l. But $\mathcal{B}$ is a Baer subplane of π and l is a line of π which implies that there exists a point C of $\mathcal{B}$ such that C is incident with l. Clearly C is not in $\mathcal{D}$ as the point $l \cap l_\infty$ is not in $\mathcal{D}$ as l is a type 1 line of $\bar{\mathcal{A}}$. Thus C is an affine point of $\mathcal{A}$, and hence of $\bar{\mathcal{A}}$, in both l and $\mathcal{B}$ contradicting that l and $\mathcal{B}$ are parallel. Thus for any type 1 line l of $\bar{\mathcal{A}}$ and a nonincident point P of $\bar{\mathcal{A}}$ there is a unique line of $\bar{\mathcal{A}}$ through P, parallel to l and it is also of type 1.

Before proving the existence of a parallel to a type 2 line, we prove the following lemma.

<u>Lemma 4.1.1</u>. Let l be any affine line of $\mathcal{A}$ such that $l \cap l_\infty$ is a point of $\mathcal{D}$. Then the points and segments (see chapter I p.7) of l are respectively the points and lines of an affine plane.

Proof: Let H be any point of $\mathcal{D} \setminus \{l \cap l_\infty\}$. From lemma 2.3.1 the affine lines and the segments in H are respectively the points and lines of an affine plane. Define a mapping φ from the lines through H to the points of l by

$$\varphi(x) = x \cap l$$

for any affine line x through H. We wish to prove φ is an isomorphism. Since two lines intersect in a unique point φ is one-to-one and since two points determine a line φ is surjective. Thus we need only show that φ preserves incidence. Note that since $l \cap l_\infty$ is a point of $\mathcal{D}$, any two affine points of l determine a unique Baer subplane belonging to $\mathcal{D}$. Let x and y be any two lines through H (see Fig.33).

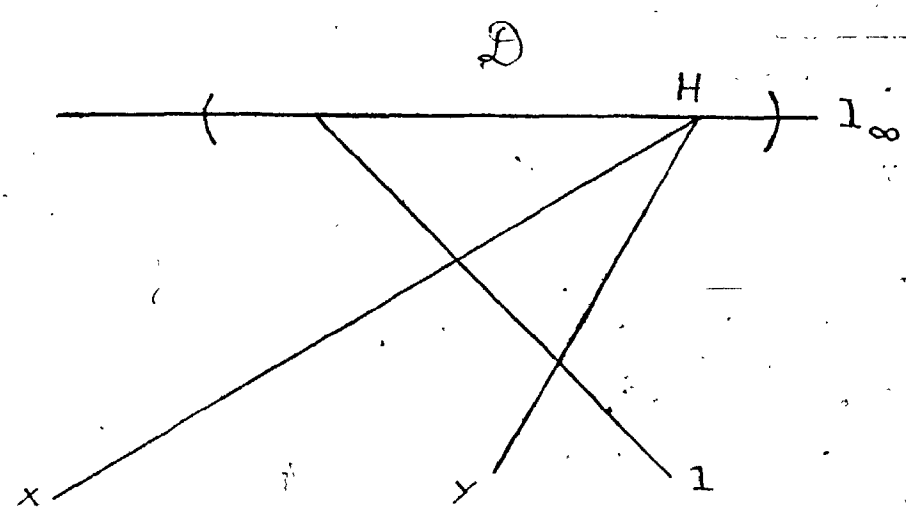

Fig.33

Then $\varphi(x)$ and $\varphi(y)$ determine a Baer subplane $\mathcal{A}_0$ of $\mathcal{A}$ belonging to $\mathcal{D}$. Since H is a point of $\mathcal{D}$, x and y are lines of $\mathcal{A}_0$. If z is an element of $\overline{x \cap y}$ (that is, if z is in the segment through H determined by x and y) then z is in $\mathcal{A}_0$ by lemma 2.3.2 (or from the definition of the segment determined by x and y chapter I p.7). Then as l is in $\mathcal{A}_0$, $\varphi(z)=z \cap l$ is in $\mathcal{A}_0$ and hence $\varphi(z)$ is in $\overline{\varphi(x) \cap \varphi(y)}$. Conversely, suppose $X=\varphi(x)$ and $Y=\varphi(y)$ for x,y through H and let Z be any point of $\overline{X \cap Y}$. Then the points of $\overline{X \cap Y}$ are contained in the unique subplane $\mathcal{A}_1$ belonging to $\mathcal{D}$ determined by X and Y. Since H is in $\mathcal{D}$ the lines x=XH, y=YH and ZH are in $\mathcal{A}_1$. Thus ZH is in $\overline{XH \cap YH}$. But clearly $z=\varphi(ZH)$. Thus φ is an isomorphism which preserves incidence and so the points and segments of l are the points and lines of an affine plane.

Now returning to the proof that $\overline{\mathcal{A}}$ is an affine plane, there remains the case that we have a type 2 line $\mathcal{A}_0$ of $\overline{\mathcal{A}}$ and a point P not in $\mathcal{A}_0$ (see Fig.34). Since P is an

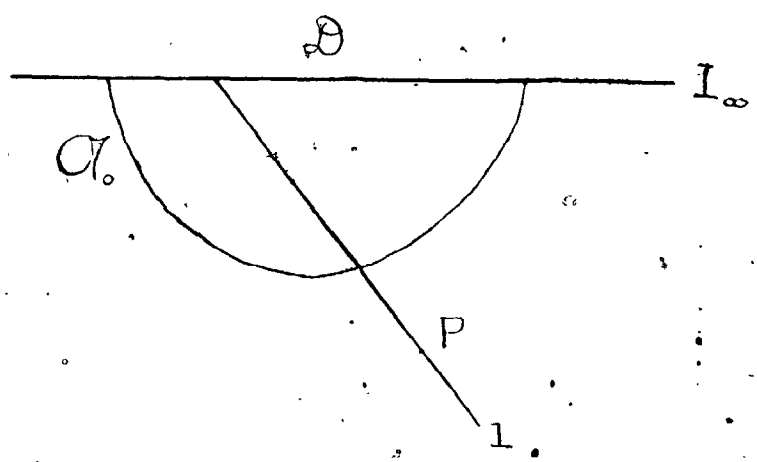

Fig.34

affine point, i.e. $P \not\in l_\infty$, and since $\mathcal{A}_0$ is a Baer subplane there is some line l of $\mathcal{A}_0$ through P. By lemma 4.1.1 the points of l and the segments determined by the points of l are respectively the points and lines of an affine plane, say $\mathcal{L}$. Now l is a line of $\mathcal{A}_0$; thus the points of $l \cap \mathcal{A}_0$ form a segment of l corresponding to a line of $\mathcal{L}$ and P is a point of $\mathcal{L}$ which is not contained in the line $l \cap \mathcal{A}_0$. Thus there exists a unique line of $\mathcal{L}$ through P having no points in common with $l \cap \mathcal{A}_0$. This unique parallel is of the form $l \cap \mathcal{A}_1$ for some Baer subplane $\mathcal{A}_1$ belonging to $\mathcal{D}$ where P is in $\mathcal{A}_1$. Clearly $\mathcal{A}_0$ and $\mathcal{A}_1$ have no points in common on l. If there exists an affine point Q in $\mathcal{A}_0 \cap \mathcal{A}_1$ then $Q \neq P$ (since Q is not incident with l). Also $PQ = RQ$ for some R in $\mathcal{D}$ and so PQ is a line of $\mathcal{A}_0 \cap \mathcal{A}_1$ (see Fig.35). But then $P = PQ \cap l$ is in $\mathcal{A}_0$ which is a contradiction. Hence $\mathcal{A}_0$ and $\mathcal{A}_1$ have no points in common. Thus $\mathcal{A}_1$ is the required parallel.

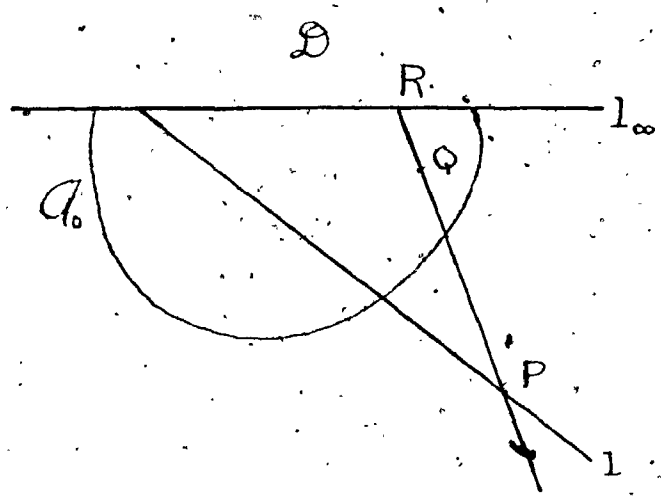

Fig.35

Clearly $\mathcal{O}$ is unique. Obviously the order of π is at least four and so there is a triangle in $\bar{A}$. Thus we have proved the following.

<u>Theorem 4.1.1.</u> $\bar{A}$ is an affine plane.

As the point sets of A and $\bar{A}$ are the same, the following is an immediate corollary.

<u>Corollary 4.1.1.</u> If A is finite of order m^2 so is $\bar{A}$.

Now $\bar{A}$ is an affine plane and thus can be uniquely embedded in a projective plane $\bar{\pi}$. Denote the special line of $\bar{A}$ by $\bar{l}_\infty$. Then $\bar{l}_\infty$ is the collection of equivalence classes of lines of $\bar{A}$ under the relation of parallelism. Let $\mathcal{D}'$ be the complement of $\mathcal{D}$ on l_∞. Type 1 lines of $\bar{A}$ are lines of A through $\mathcal{D}'$ and thus there is a one-to-one correspondence between equivalence classes of type 1 lines of $\bar{A}$ and the points of $\mathcal{D}'$ in A. Denote the set of equivalence classes of type 1 lines of $\bar{A}$ under parallelism by $\mathcal{D}'$ also. Denote the set consisting of the remaining points of $\bar{l}_\infty$ by $\bar{\mathcal{D}}$. $\bar{\mathcal{D}}'$ is the set of equivalence classes of parallel Baer subplanes (i.e. Baer subplanes with no affine points in common) belonging to $\mathcal{D}$ in A and thus there is no natural correspondence between the points of $\bar{\mathcal{D}}$ and any points of π. $\bar{\pi}$ (or $\bar{A} = \bar{\pi}^{\bar{l}_\infty}$) is called the "derived plane of π (or A) with respect to the derivation set $\mathcal{D}$".

We shall now prove a few general results about

derivation.

<u>Lemma 4.1.2.</u> Let π be a finite derivable plane of order n^2 with derivation set $\mathcal{D}$ contained on some line l_∞. If k is an affine line of π^{l_∞} such that $k \cap l_\infty$ is a point of $\mathcal{D}$, then the points and segments of k form a Baer subplane of $\bar{\pi}$ belonging to $\bar{\mathcal{D}}$.

Proof. It follows immediately from lemma 4.1.1 that in $\bar{\mathcal{A}} = \bar{\pi}^{\bar{l}_\infty}$, k determines an affine subplane of order n^2. Thus k is an affine Baer subplane $\bar{k}$ of $\bar{\mathcal{A}}$. Since the parallel lines of $\bar{k}$ when considered as lines of $\bar{\mathcal{A}}$ are of type 2, the special points of $\bar{k}$ must be $\bar{\mathcal{D}}$.

Note that if the derivation set $\mathcal{D}$ is fixed by some collineation α of a derivable projective plane π then the Baer subplanes belonging to $\mathcal{D}$ will be permuted among themselves; i.e. type 2 lines of $\bar{\pi}$ will be taken to type 2 lines of $\bar{\pi}$ by α. Thus we have the following.

<u>Lemma 4.1.3.</u> If α is a collineation of a derivable projective plane π such that the derivation set is fixed by α, then α induces a collineation of the derived plane $\bar{\pi}$.

<u>Theorem 4.1.2.</u> If π is a finite derivable projective plane of order n^2 with derivation set $\mathcal{D}$ contained on l_∞ then the groups of translations of $\mathcal{A} = \pi^{l_\infty}$ and $\bar{\mathcal{A}} = \bar{\pi}^{\bar{l}_\infty}$ are isomorphic.

Proof. By lemma 4.1.3 translations of $\mathcal{A}$ induce collineations of $\bar{\mathcal{A}}$. As the point sets of $\mathcal{A}$ and $\bar{\mathcal{A}}$ are

identical α fixes no affine points of $\bar{\mathcal{A}}$. We now need only show that any type 2 line of $\bar{\mathcal{A}}$ is parallel to its image under α, since a type 1 line of $\bar{\mathcal{A}}$ is a line of $\mathcal{A}$ and hence is parallel to its image under α.

First consider the following situation. Let P and Q be affine points of $\mathcal{A}$ such that $B=PQ\cap l_\infty$ is in $\mathcal{D}$. Let A,C be two distinct points of $\mathcal{D}$ different from B. Let $QC\cap PA=S$ and $SB\cap QA=T$ (see Fig. 36). Clearly P,Q,S,T

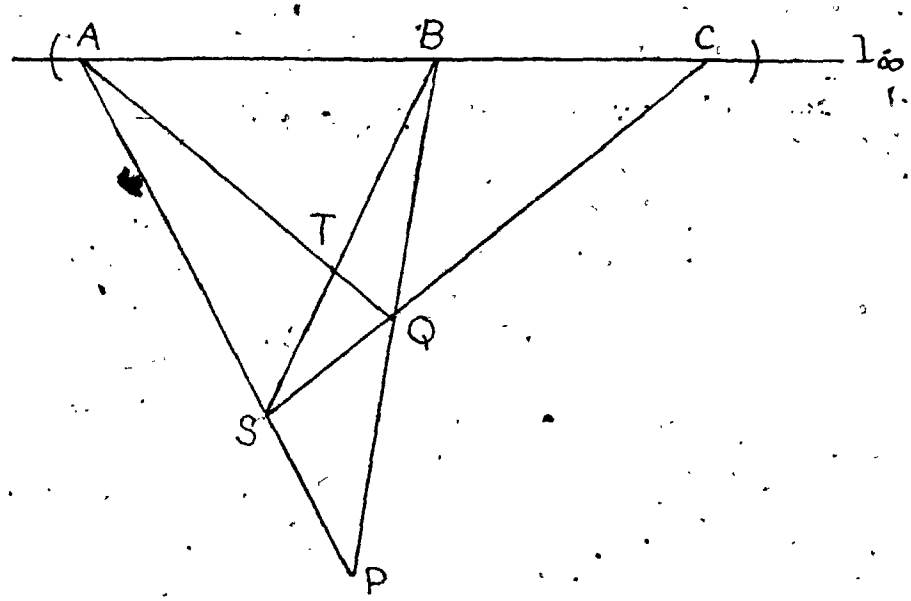

Fig. 36

are all contained in some Baer subplane $\mathcal{B}$ belonging to $\mathcal{D}$. Suppose under the translation α of $\mathcal{A}$ $P^\alpha=Q$; then as $S=SB\cap PA$ we have $S^\alpha=(SB)^\alpha\cap(PA)^\alpha=SB\cap QA=T$ (note that the centre of α must be $l_\infty\cap PP^\alpha$ which is B). Now any Baer subplane belonging to $\mathcal{D}$ is uniquely determined by any two

of its points and $\mathcal{B} \cap \mathcal{B}^{\alpha} \supseteq \{Q, T\}$. Thus $\mathcal{B} = \mathcal{B}^{\alpha}$ if P^{α} is in $\mathcal{B}$ for any P in $\mathcal{B}$.

Now suppose l and m are any type 2 lines of $\bar{\mathcal{A}}$ such that $l^{\alpha} = m$. Suppose l and m correspond respectively to the Baer subplanes $\mathcal{L}$ and $\mathcal{M}$ belonging to $\mathcal{D}$ in $\mathcal{A}$. Then $\mathcal{L}^{\alpha} = \mathcal{M}$. Suppose $\mathcal{L} \neq \mathcal{M}$ but there exists Q in $\mathcal{L} \cap \mathcal{M}$. As $\mathcal{L}^{\alpha} = \mathcal{M}$ there exists a point P in $\mathcal{L}$ such that $P^{\alpha} = Q$. But Q is also in $\mathcal{L}$. Thus by the above comment $\mathcal{L} = \mathcal{M}$ which is a contradiction. Thus if the point set of some type 2 line l of $\bar{\mathcal{A}}$ is not disjoint from l^{α}, l is fixed; that is, l is parallel to l^{α} for any line l of $\bar{\mathcal{A}}$. Thus α induces a translation of $\bar{\mathcal{A}}$.

From lemma 4.1.2, $\bar{\mathcal{A}}$ is derivable with derivation set $\bar{\mathcal{D}}$. It is immediate that $\bar{\bar{\mathcal{A}}} \simeq \mathcal{A}$. Thus the translation groups of $\mathcal{A}$ and $\bar{\mathcal{A}}$ are isomorphic.

§2. An Example

From theorem 4.1.2, we can obtain non-translation planes by deriving a plane which is not a translation plane. Following Ostrom ([13]) and [8] (5.4.7) we shall construct a class of 'semi-translation planes' which are neither translation planes nor dual translation planes.

Lemma 4.2.1. Let π be a finite dual translation plane of order q^2 whose coordinatizing right quasifield Q is of dimension two over its kernel K. π is derivable.

Proof. Consider the set $\mathcal{D} = \{(\alpha) \mid \alpha \in K\} \cup \{(\infty)\}$. We shall now

show that $\mathcal{D}$ is a derivation set for π. For any a,b,c in $\mathcal{Q}$ with $a \neq 0$ let $\mathcal{B}(a,b,c) = \{(\alpha \cdot a+b, \beta \cdot a+c) \mid \alpha, \beta \in K\}$. Note first that $\mathcal{B}(a,b,c)$ is an affine Baer subplane of π^{1} containing $\mathcal{D}$ as its set of special points; for as a,b,c are fixed and α, β vary over the q elements of K, there are q^2 points in $\mathcal{B}(a,b,c)$. Lines of $\mathcal{B}(a,b,c)$ are lines of π with at least two points of $\mathcal{B}(a,b,c)$; thus two points of $\mathcal{B}(a,b,c)$ are on a unique line of $\mathcal{B}(a,b,c)$. In order to see that $\mathcal{B}(a,b,c)$ is an affine plane we must show that any line of π with two points of $\mathcal{B}(a,b,c)$ contains exactly q points of $\mathcal{B}(a,b,c)$ (see [8] 3.2.4 (b)). Suppose $X=(\alpha_1 a+b, \beta_1 a+c)$ and $Y=(\alpha_2 a+b, \beta_2 a+c)$ are in $\mathcal{B}(a,b,c)$. Then X and Y are on the unique line XY of π. First suppose that (∞) is incident with XY, that is, XY has the form $[k]$ for some k in K. Then we must have $\alpha_1 a+b = \alpha_2 a+c = k$. Now if α' in K is fixed and β is varied over K, there are clearly q points of the form $(\alpha' a+b, \beta \cdot a+c)$. Thus there are q points of $\mathcal{B}(a,b,c)$ on the line XY. Suppose now that $\alpha_1 a+b$ is not equal to $\alpha_2 a+c$. Then X and Y are incident with a line $[\lambda, k]$ in π. Now λ is in K for given α_i, β_i in the field K for i=1,2 as can be seen from the following: there clearly exists λ' in K satisfying

$$\lambda'(\alpha_1 - \alpha_2) = \beta_2 - \beta_1 .$$

Then for any a in $\mathcal{Q}$

$$(\lambda' \cdot \alpha_1 - \lambda' \cdot \alpha_2)a = (\beta_2 - \beta_1)a$$

and as the right distributive law holds

$$(\lambda' \cdot \alpha_1)a - (\lambda' \cdot \alpha_2)a = \beta_2 a - \beta_1 a$$

Also λ' is in K, the kernel of Q, and so

$$\lambda'(\alpha_1 a - \alpha_2 a) = \beta_2 a - \beta_1 a.$$

Then for any b,c in Q

$$\lambda'(\alpha_1 a+b - \alpha_2 a-b) = \beta_2 a+c - \beta_1 a-c$$

and so as -1 is in K

$$\lambda'(\alpha_1 a+b) - \lambda'(\alpha_2 a+b) = \beta_2 a+c - (\beta_1 a+c)$$

Re-arranging we get

$$\lambda'(\alpha_1 a+b) + \beta_1 a+c = \lambda'(\alpha_2 a+b) + \beta_2 a+c = k,$$

say. Thus the unique line of π joining X and Y is $[\lambda', k]$, i.e. $\lambda = \lambda'$ ~~and so λ~~ is in K. We want to show this line contains exactly q points of $\mathcal{B}(a,b,c)$. In fact, any line $[\lambda, k]$ where λ is in K and which contains one point $(\alpha_1 a+b, \beta_1 a+c)$ of $\mathcal{B}(a,b,c)$ contains exactly q points of $\mathcal{B}(a,b,c)$. For, a point $(\alpha a+b, \beta a+c)$ is incident with $[\lambda, k]$ if and only if

$$k = \lambda(\alpha a+b) + \beta a+c$$

that is, if and only if

$$\beta a = k - \lambda(\alpha a+b) - c. \qquad (4.1)$$

But $(\alpha_1 a+b, \beta_1 a+c)$ is incident with $[\lambda, k]$ so that

$$k = \lambda(\alpha_1 a+b) + \beta_1 a+c; \qquad (4.2)$$

Substituting (2) in (1) gives

$$\beta a = \lambda(\alpha_1 a+b) + \beta_1 a - \lambda(\alpha a+b). \qquad (4.3)$$

Now for each α in K there is a unique β in Q satisfying (4.3). But λ is in K; thus

$$\beta a = (\lambda\alpha_1 - \lambda\alpha + \beta_1)a$$

and so β is in K. Therefore there are q points of $\mathcal{B}(a,b,c)$ on $[\lambda, k]$. Thus $\mathcal{B}(a,b,c)$ is an affine Baer subplane of π^{1}_{∞}. Finally as $[\lambda, k]$ is a line of $\mathcal{B}(a,b,c)$

for all λ in K, $\mathcal{D}$ is the set of special points of $\mathcal{B}(a,b,c)$.

Suppose that we are given any two points $X=(x_1,y_1)$, $Y=(x_2,y_2)$ such that $XY\cap l_\infty$ is a point of $\mathcal{D}$. We must show that X and Y are in $\mathcal{B}(a,b,c)$ for some a,b,c in Q. If $XY\cap l_\infty=(\infty)$ then XY must be of the form $[k]$ where $x_1=x_2=k$. We want to show that there exist elements a,b,c in Q such that for some α, β_i in K $(i=1,2)$ the system

$$k=\alpha a+b$$
$$y_1=\beta_1 a+c$$
$$y_2=\beta_2 a+c$$

is solvable. Now $y_1=\beta_1 a+c$ and $y_2=\beta_2 a+c$ are solvable simultaneously if and only if there is a point (a,c) incident with $[\beta_1,y_1]$ and $[\beta_2,y_2]$. As π is a projective plane and the lines $[\beta_1,y_1]$ and $[\beta_2,y_2]$ have a unique point of intersection, Z, and as X and Y are distinct, β_1 is not equal to β_2; thus Z is an affine point of the form (a,c) for some a,c in Q. Then given α,k,a in Q there is clearly a unique b in Q such that $k=\alpha a+b$. Suppose now that $XY\cap l_\infty=(\lambda)$ for some λ in K. Thus X and Y are incident with the line $[\lambda,k]$ for some k in Q where

$$k=\lambda\cdot x_1+y_1=\lambda\cdot x_2+y_2 \qquad (4.4)$$

Then X and Y are contained in $\mathcal{B}(x_1-x_2,x_2,y_1)$ for $x_1=1\cdot(x_1-x_2)-x_2$, $y_1=0\cdot(x_1-x_2)+y_1$, $x_2=0\cdot(x_1-x_2)+x_2$ and $y_2=\lambda\cdot(x_1-x_2)+y_1$ since $\lambda\cdot(x_1-x_2)+y_1=\lambda\cdot x_1+y_1-\lambda\cdot x_2=k-\lambda\cdot x_2=y_2$ by (4.4).

Thus a dual translation plane which is coordinatized by a right quasifield of dimension two over its kernel is derivable.

We shall now show that the derived plane $\overline{\pi}$ is a "semi-translation plane" with respect to $\overline{l}_\infty$; that is if P is a point of $\overline{\mathcal{D}}$ then $\overline{\pi}$ admits a group of elations of order q with centre P and axis $\overline{l}_\infty$ (see [14] for the general definition of a semi-translation plane).

<u>Lemma 4.2.2</u>. $\overline{\pi}$ is a semi-translation plane with respect to $\overline{l}_\infty$.

Proof. Consider the mappings T(e) of the set of points of the affine plane $\mathcal{A}$ into itself given by

$$T(e):(x,y) \longrightarrow (x,y+e)$$

where e is in Q. Clearly the mappings ~~T(e)~~ are bijections and they fix the lines [k]. Also as $(0,k) \longrightarrow (0,k+e)$ we must have $[m,k] \rightarrow [m',k+e]$ for some m′ in Q. Then if (x,y) is incident with [m,k], (x,y+e) is incident with [m′,k+e]; that is $k+e=m'\cdot x+y+e$ or $k=m'\cdot x+y=m\cdot x+y$; thus $m=m'$ and $[m,k] \longrightarrow [m,k+e]$. Thus the mappings T(e) are collineations of $\mathcal{A}$ such that any affine line is parallel to its image, that is T(e) is a translation of $\mathcal{A}$ for each e in Q. Then by theorem 4.1.2 T(e) is a translation of $\overline{\pi}$. Let $\Gamma = \{T(e) \mid e \in Q\}$. Then Γ is a group of order q^2 consisting of translations of $\overline{\pi}$.

Now let π_0 be the subplane of $\overline{\pi}$ corresponding

to the line $[0]$ of π, that is $\pi_0 = \{(0,y) \mid y \in Q\}$. Clearly as any γ in Γ fixes the first coordinate of an affine point, π_0 is fixed as a set under Γ. Now Γ is a group of order q^2 consisting of translations of $\mathcal{A}$ with centre (∞); and as the order of $\mathcal{A}$ is q^2, Γ is transitive on the points of any affine line through (∞), in particular on the points of $[0]$. Hence as the point sets of $\mathcal{A}$ and $\bar{\mathcal{A}}$ are the same Γ is transitive on the points of π_0. Considering Γ as a group of translations in $\bar{\mathcal{A}}$, the centres must be in $\bar{\mathcal{D}}$; for suppose P not in $\bar{\mathcal{D}}$ is the centre of T(e) for some e in Q. Since P is a point of π also, it is of the form (m) for some m in $Q \setminus K$. All lines through P are fixed by T(e) and so $[m,k]=[m,k+e]$. But clearly this is possible if and only if e=0, that is T(e) is the identity. Thus the centres are in $\bar{\mathcal{D}}$. We already know that Γ induces a group of translations in π_0 and we know that no non-identity element of Γ fixes any affine points of $\bar{\mathcal{A}}$ and hence of π_0. Thus as π_0 has q^2 points and the order of Γ is q^2, Γ is the complete group of translations of π_0. That is, for each P in $\bar{\mathcal{D}}$, $\bar{\pi}$ admits a group of elations of order q with centre P and axis $\bar{l}_\infty$. Thus $\bar{\pi}$ is a semi-translation plane.

Before showing that $\bar{\mathcal{A}}$ and $\bar{\pi}$ are not translation planes, we need the following lemmas.

<u>Lemma 4.2 3</u>. Given $a, b, c, \bar{a}, \bar{b}, \bar{c}$ in Q then $\mathcal{B}(a,b,c)$ is parallel to $\mathcal{B}(\bar{a},\bar{b},\bar{c})$ if and only if there exists some γ in K such that $\bar{a} = \gamma a$.

Proof. (x,y) is in $\mathcal{B}(a,b,c)$ and $\mathcal{B}(\bar{a},\bar{b},\bar{c})$ if and only if there exist α, β, $\bar{\alpha}$, $\bar{\beta}$ in K such that

$$x = \alpha a + b = \bar{\alpha}\bar{a} + \bar{b}$$
$$y = \beta a + c = \bar{\beta}\bar{a} + \bar{c}. \qquad (4.5)$$

Suppose first that $\bar{a}$ is not equal to γa for any γ in K, that is $\bar{a}$ and a are linearly independent. Now Q is a vector space of dimension two over K and thus as a, $\bar{a}$ are linearly independent vectors of Q any element of Q can be uniquely represented as a linear combination of a and $\bar{a}$ with coefficients from K. In particular there exist unique $\alpha, \beta, \bar{\alpha}, \bar{\beta}$ in K such that

$$\bar{b} - b = \alpha a - \bar{\alpha}\bar{a}$$
$$\bar{c} - c = \beta a - \bar{\beta}\bar{a}.$$

Clearly these relations uniquely determine x and y and hence $\mathcal{B}(a,b,c)$ and $\mathcal{B}(\bar{a},\bar{b},\bar{c})$ are not parallel if a and $\bar{a}$ are linearly independent.

Suppose conversely that a and $\bar{a}$ are linearly dependent that is for some γ in K, $\bar{a} = \gamma \cdot a$. Suppose that $\mathcal{B}(a,b,c)$ and $\mathcal{B}(\bar{a},\bar{b},\bar{c})$ have some point (x,y) in common. Then from (4.5)

$$\bar{b} = (\alpha - \bar{\alpha}\cdot\gamma)a + b$$
$$\bar{c} = (\beta - \bar{\beta}\cdot\gamma)a + c$$

and so $\mathcal{B}(a,b,c)$ and $\mathcal{B}(\bar{a},\bar{b},\bar{c})$ are identical: for if

(u,v) is in $\mathcal{B}(\bar{a},\bar{b},\bar{c})$ then $(u,v)=(\bar{\alpha}\cdot\bar{a}+\bar{b},\bar{\beta}\cdot\bar{a}+\bar{c})$ for some $\bar{\alpha},\bar{\beta}$ in K. Then

$$\begin{aligned}(u,v)&=(\bar{\alpha}\cdot\bar{a}+\bar{b},\bar{\beta}\cdot\bar{a}+\bar{c})\\&=(\bar{\alpha}\cdot\gamma\cdot a+(\alpha-\bar{\alpha}\cdot\gamma)a+b,\ \bar{\beta}\gamma\cdot a+(\beta-\bar{\beta}\cdot\gamma)a+c)\\&=(\alpha\cdot a+b,\ \beta\cdot a+c),\end{aligned}$$

that is, (u,v) is in $\mathcal{B}(a,b,c)$. Similarly if (u,v) is in $\mathcal{B}(a,b,c)$ then for some α,β in K

$$\begin{aligned}(u,v)&=(\alpha\cdot a+b,\ \beta\cdot a+c)\\&=(\alpha\cdot\gamma^{-1}\cdot\bar{a}+\bar{b}-(\alpha-\bar{\alpha}\cdot\gamma)\cdot\gamma^{-1}\cdot\bar{a},\ \beta\cdot\gamma^{-1}\cdot\bar{a}+\bar{c}-(\beta-\bar{\beta}\cdot\gamma)\cdot\gamma^{-1}\cdot\bar{a})\\&=(\bar{\alpha}\cdot\bar{a}+\bar{b},\ \bar{\beta}\cdot\bar{a}+\bar{c}),\end{aligned}$$

note that γ is in the field K, $\gamma\neq 0$ and so γ^{-1} exists. Thus (u,v) is in $\mathcal{B}(\bar{a},\bar{b},\bar{c})$. Thus $\mathcal{B}(a,b,c)$ equals $\mathcal{B}(\bar{a},\bar{b},\bar{c})$. This completes the proof of the lemma.

<u>Lemma 4.2.4</u> If for some c in $\mathcal{Q}$ there is a translation of π taking $(0,0)$ into $(c,0)$ then $y\cdot(x+c)=y\cdot x+y\cdot c$ for all x,y in $\mathcal{Q}$.

Proof. Let τ be a translation of π taking $(0,0)$ into $(c,0)$. Clearly $(0,0)$ and $(c,0)$ are incident with $\mathcal{B}(c,0,0)$ as $(0,0)=(0\cdot c+0,0\cdot 0+0)$ and $(c,0)=(1\cdot c+0,0\cdot 0+0)$. Now as τ is a translation and $(0,0)$ and its image are incident with $\mathcal{B}(c,0,0)$, τ must fix $\mathcal{B}(c,0,0)$ and its parallels. As the point $(c,0)$ is incident with $\mathcal{B}(a,0,c)$ for any a in $\mathcal{Q}$, the image of $\mathcal{B}(a,0,0)$ under τ is the unique parallel to $\mathcal{B}(a,0,0)$ through $(c,0)$. By lemma 4.2.3 all parallels to $\mathcal{B}(a,0,0)$ are of the form $\mathcal{B}(a,x,y)$ for x,y in $\mathcal{Q}$. Clearly $(c,0)$ is incident with $\mathcal{B}(a,c,0)$ as

$(c,0)=(0\cdot a+c, 0\cdot a+0)$; and hence $\mathcal{B}(a,c,0)$ must be the image of $\mathcal{B}(a,0,0)$ under τ for any a in Q. Now the point $(\bar{\alpha}\cdot a, \bar{\beta}\cdot a)$ for any $\bar{\alpha}, \bar{\beta}$ in K and any a in Q is the unique point of intersection of $\mathcal{B}(a,0,0)$ and $\mathcal{B}(c,d,e)$ where $\bar{\alpha}\cdot a=\bar{\alpha}\cdot c+d$ and $\bar{\beta}\cdot a=\bar{\beta}\cdot c+e$. Thus the image of $(\bar{\alpha}\cdot a, \bar{\beta}\cdot a)$ under τ is the unique point of intersection of $\mathcal{B}(a,c,0)$ and $\mathcal{B}(c,d,e)$. Now clearly $(\bar{\alpha}\cdot a+c, \bar{\beta}\cdot a)$ is incident with $\mathcal{B}(a,c,0)$ and as

$$(\bar{\alpha}\cdot a+c, \bar{\beta}\cdot a)=(\bar{\alpha}\cdot c+d+c, \bar{\beta}\cdot c+e)$$
$$=((\bar{\alpha}+1)\cdot c+d, \bar{\beta}\cdot c+e)$$

we have $(\bar{\alpha}\cdot a+c, \bar{\beta}\cdot a)$ incident with $\mathcal{B}(c,d,e)$ also. Hence the image of $(\alpha\cdot a, \beta\cdot a)$ under τ for any a in Q, α, β in K is $(\alpha\cdot a+c, \beta\cdot a)$ where $\alpha\cdot a=\alpha\cdot c+d$ and $\beta\cdot a=\beta\cdot c+e$. In particular then the image of $(0,b)$ for any b in Q under τ is (c,b). Then as $(0,b)$ is incident with $\mathcal{B}(a,0,b)$ for any a in Q, the image of $\mathcal{B}(a,0,b)$ under τ is the unique parallel to $\mathcal{B}(a,0,b)$ through (c,b), that is $\mathcal{B}(a,c,b)$. In general the image of (x,y) for any x,y in Q under τ is $(x+c,y)$; for (x,y) is incident with $\mathcal{B}(x,0,y)$ which by the above is mapped by τ into $\mathcal{B}(x,c,y)$. Thus the image of (x,y) under τ is incident with $\mathcal{B}(x,c,y)$. Also the line determined by (x,y) and $(x,y)^{\tau}$ is fixed and hence is parallel to $\mathcal{B}(c,0,0)$. Thus $(x,y)^{\tau}$ is incident with $\mathcal{B}(c,x,y)$, since $\mathcal{B}(c,x,y)$ is the unique parallel to $\mathcal{B}(c,0,0)$ through (x,y). Clearly $(x+c,y)$ is the unique point of intersection of $\mathcal{B}(c,x,y)$ and $\mathcal{B}(x,c,y)$ as $(x+c,y)=(1\cdot x+c, 0\cdot x+y)=(1\cdot c+x, 0\cdot c+y)$.

Now the image of $(0,b)$ under τ is (c,b) for any b in $\mathcal{Q}$; thus any line $[m,b]$ through $(0,b)$ is mapped to the unique parallel to $[m,b]$ through (c,b) which is $[m,m\cdot c+b]$. Now all points incident with $[m,b]$ for m in $Q\backslash K$ are of the form $(x,b-m\cdot x)$; thus $(x+c,b-m\cdot x)$ is incident with $[m,m\cdot c+b]$, that is

$$m\cdot c+b=m\cdot(x+c)+b-m\cdot x$$

or

$$m\cdot x+m\cdot c=m\cdot(x+c)$$

for all m,x in $\mathcal{Q}$ since result trivial if m is in K.

<u>Lemma 4.2.5</u> π is not a translation plane.

Proof. If $\bar{A} = \pi^{l_\infty}$ is a translation plane then there exist translations of $\bar{A}$ taking $(0,0)$ to $(c,0)$ for any c in $\mathcal{Q}$ which implies by lemma 4.2.4 that $\mathcal{Q}$ satisfies the left distributive law thus contradicting the assumption that $\mathcal{Q}$ is a right quasifield but not a semifield. Hence $\bar{A}$ is not a translation plane; i.e. π is not a translation plane with respect to l_∞. This implies that π is non-desarguesian.

Suppose π is a translation plane with respect to some line l different from l_∞. Since π is a semi-translation plane with respect to l_∞, there exists some collineation displacing any line not equal to l_∞. In particular there exists a collineation α such that $l^\alpha \neq l$. Then π has two distinct translation lines, is

finite, and so desarguesian (see chapter I p. 13). But this is a contradiction. Thus $\overline{\pi}$ is not a translation plane with respect to any line.

<u>Lemma 4.2.6</u> $\overline{\pi}$ is not a dual translation plane if Q does not satisfy, for any $c \neq 0$ in Q, the partial distributive law

$$x\cdot(y+c)=x\cdot y+x\cdot c$$

for all x,y in Q.

Proof. Suppose $\overline{\pi}$ is a dual translation plane with respect to the point P. Then as by the above lemma $\overline{\pi}$ is not desarguesian P must be fixed by all collineations (dual of the argument in lemma 4.2.5 or see [8] ,4.3.23). The mappings $T(e)$ for e in Q from lemma 4.2.2 guarantee that no affine point of $\overline{\pi}$ is fixed by all collineations of $\overline{\pi}$, thus P is incident with $\overline{l}_\infty$. Consider the collineations $E(\alpha)$ of $\overline{\pi}$ given by

$$E(\alpha):(x,y) \longrightarrow (x,\alpha x+y)$$

for α in K. The image of a line $[m,k]$ under some $E(\alpha)$ is $[m-\alpha,k]$. Then as α is in K, $m-\alpha$ is in K if and only if m is in K. Thus the set $\mathcal{D}$ is fixed by $E(\alpha)$ for α in K and so $E(\alpha)$ induces a collineation of $\overline{\pi}$. Clearly as m varies over Q so does $m-\alpha$; thus no point of $\mathcal{D}'$ is fixed in $\overline{\pi}$. But $\mathcal{D}'$ is also a set of points on $\overline{l}_\infty$ in $\overline{\pi}$. Thus as P is fixed, P must be in $\overline{\mathcal{D}}$. Then $P=\overline{l}_\infty \cap \mathcal{B}(c,0,0)$ for some c in Q. If $\overline{\pi}$ is a dual translation plane with respect to P, $\overline{\pi}$ must admit all elations with centre P

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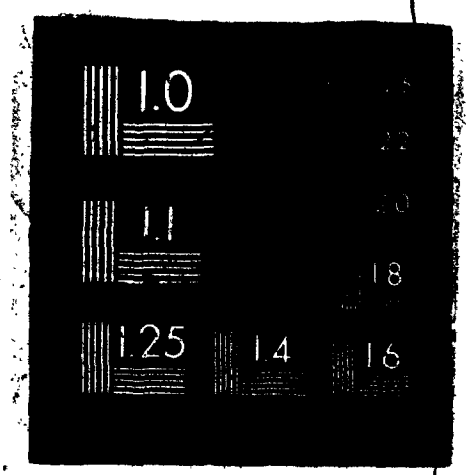

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and axis through P. In particular $\overline{\pi}$ admits all translations with centre P and so admits a translation taking (0,0) into (c,0). But then by lemma 4.2.4, $\overline{\pi}$ satisfies the partial distributive law $x(y+c)=xy+xc$ for all x,y in Q which is a contradiction. Thus $\overline{\pi}$ is not a dual translation plane.

For systems satisfying the conditions of lemma 4.2.6 see [12]. Note that if there is some c in Q satisfying this partial distributive law then from the proof of the lemma c is in K. But the existence of the collineations $E(\alpha)$ implies that the collineation group is transitive on the points of $\overline{\mathcal{D}}$ and so if the partial distributive law is satisfied for some c in K then it is satisfied for all c in K.

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