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Abstract

The Boundary Integral Equation (BIE) method is now generally recognized to be a powerful numerical tool for complex problems in solid mechanics, particularly when cracks are involved. In this study, a brief review of Linear Elastic Fracture Mechanics (LEFM) and the BIE theory are presented.

Stress intensity factors K_I for edge-cracked discs subjected to a temperature gradient are obtained for a wide range of crack and disc geometries using the BIE method in two-dimensions. These results could be superimposed with solutions for other types of loads, such as centrifugal body force, to obtain K_I values for components under thermal and inertia loads.

Much attention is focussed on rotating disc components with a quarter-elliptical corner crack at the bore, which require three-dimensional treatments. The variations of K_I along the crack fronts are evaluated using the BIE method, and results for a number of crack and disc sizes are presented.

These K_I solutions are useful in the structural integrity assessments of cracked disc components.

The undersigned recommend to the Faculty of Graduate Studies and Research acceptance of the thesis;

**BOUNDARY INTEGRAL EQUATION STRESS ANALYSIS
OF CRACKED ROTATING DISCS**

submitted by Zhenhong Jia, B.Eng., in partial fulfilment of the requirements for the degree Master of Mechanical Engineering.

Thesis Supervisor

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Abstract

The Boundary Integral Equation (BIE) method is now generally recognized to be a powerful numerical tool for complex problems in solid mechanics, particularly when cracks are involved. In this study, a brief review of Linear Elastic Fracture Mechanics (LEFM) and the BIE theory are presented.

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These K_I solutions are useful in the structural integrity assessments of cracked disc components.

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Nomenclature

a	-	Characteristic crack length
a	-	Major semi-axis of an ellipse
a_c	-	Critical crack length
ΔA	-	Increase in crack area (Section 4.2)
b	-	Minor semi-axis of an ellipse (Chapter 6)
b	-	Length of the edge-crack in a disc (Tables 6.1 to 6.3)
$C_{ij}(P)$	-	Limiting value of the surface integral of $T_{ij}(P, Q)$
$d(b, c)$	-	c th node of b th boundary element
E	-	Young's modulus
f_i	-	Body-force vector
G	-	Energy release rate defined by Equation (2.7)
G_c	-	Critical value of G
G_I	-	Energy release rate for mode I
$g_i(z)$	-	Prescribed displacements
H	-	Material Modulus defined in Section (2.1)
$h_i(z)$	-	Prescribed tractions
$J(\xi)$	-	Jacobian of transformation
K	-	Stress intensity factor
K_I, K_{II}, K_{III}	-	Stress intensity factors of mode I, II, III
K_{Ic}	-	Critical value of K_I
K_I^*	-	Value of K_I obtained from Equation (4.2), used for extrapolation
$\bar{K}_I$	-	Weighted root-mean-square value of K_I
$K_{I,ref}$	-	Value of K_I at a reference point.
k_0	-	A constant defined in Equation (3.17)
δl	-	Increment of crack length (Section 4.2)
M	-	Total number of boundary elements

N	-	Residual life of a cracked component
$N(\xi)$	-	Shape function
n	-	Unit outer normal
n_i	-	Components of n
p	-	An interior point
P	-	A boundary point
P^a	-	a th nodal point on the boundary
Q	-	General boundary point
R	-	Crack resistance defined in Equation (2.8)
R_1, R_2	-	Inner and outer radius of a disc
r	-	Polar coordinate defined in Figure (2.3)
r_b, r_c	-	Distances of point b and c to the crack front
r_p^*	-	Size of plastic zone
r_{ref}	-	Distance of a reference point to the crack front
$r(s)$	-	Distance of a point on crack face to the crack front
$r^*(s)$	-	$r(s)/r_{ref}$
t_i	-	Traction vector
t	-	Thickness of a rotating disc defined in Figure (6.1)
T_i	-	Kelvin's point load solution for traction
U	-	Strain energy release rate
U_i	-	Kelvin's point load solution for displacement
u	-	Displacement in x direction (Chapter 2)
u_i	-	Displacement vector
ΔU	-	Change of strain energy in an elastic body
v	-	Displacement in y direction
v_b, v_c	-	Displacements of point b, c on crack surface
v_{ref}	-	Displacement of a reference point
$v(s)$	-	Displacement of a point on crack face
$v^*(s)$	-	$v(s)/v_{ref}$
W	-	Surface energy increase (Chapter 2)

Chapter 1

INTRODUCTION

Residual life estimate and structural integrity studies on cracked rotating disc components have long been hampered by the lack of appropriate stress intensity factor K_I solutions. The stress intensity factor is the characterizing parameter which embodies the effect of the stress field and crack geometry in the vicinity of the flaw. According to the theory of Linear Elastic Fracture Mechanics (LEFM), the crack growth rate (da/dN) is a function of the cyclic change in the crack tip stress intensity factor (ΔK). Mathematically, the relationship may take the form.

$$\frac{da}{dN} = f(\Delta K, \text{material, environment, history}) \quad (1.1)$$

where a denotes the crack size; N is the number of cycle and f is an empirically determined function.

For a given material, the crack growth rate da/dN in a component is uniquely determined by the value of the stress intensity factor K and its variations. It can be easily seen that for a complete crack analysis a wide range of K_I solutions are necessary.

The evaluation of stress intensity factors for two dimensional (2 D) problems is relatively simple and a great number of solutions have hitherto been obtained for various structures and loading conditions. However, stress intensity factor results due to the presence of temperature gradient are seldom reported in the fracture mechanics literature. This is probably because of the complexity of thermal stress problems, for temperature distribution must be known *a priori*. With the advent of such powerful numerical techniques as the Finite Element (FE) method and the Boundary Integral Equation (BIE) method,

Alternatively, (2.5) can be rearranged in the form

$$\frac{\pi a \sigma^2}{H} = 2\gamma \quad (2.6)$$

The left hand side of (2.6) has been designated the “energy release rate” or the “crack driving force” G

$$G = \frac{\partial U}{\partial a} = \frac{\pi a \sigma^2}{H} \quad (2.7)$$

The right hand side of (2.6) represents the surface energy increase that would occur owing to a hypothetical crack extension and is hence termed the “crack resistance” R

$$R = \frac{\partial W}{\partial a} = 2\gamma \quad (2.8)$$

It follows that G must be at least equal to R so that the crack growth is possible. Namely

$$G = \frac{\pi a \sigma^2}{H} \geq \frac{\pi a \sigma_c^2}{H} = G_c = R \quad (2.9)$$

where the subscripts c refer to the critical values.

Equation (2.9) can be regarded as the basic condition for crack propagation in Griffith's Criterion. G_c is called the fracture toughness and can be treated as a material property to be determined through a fracture test.

2.1.2 The Stress Intensity Factor (SIF) Approach

The Griffith's criterion is based on energy concepts and it poses some difficulties when directly applied to engineering problems. Noting the inter-relationship between stress, strain and energy, Irwin [1957, 1958] developed another approach to fracture problems. By considering the analyses due to Westergaard [1939], Irwin noted that for a crack subjected to loading which is normal to the crack surface, the stress field in the vicinity of the crack tip could be expressed in the following form

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{K_I \cos \frac{\theta}{2}}{\sqrt{2\pi r}} \begin{bmatrix} 1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \\ 1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \\ \sin \frac{\theta}{2} \cos \frac{3\theta}{2} \end{bmatrix} + O(r^0) \quad (2.10)$$

the solution of LEFM problems involving thermal effects is no longer a formidable task, particularly in 2-D. As part of this study, two-dimensional stress intensity factors for discs with a single edge crack under a temperature gradient are obtained using the BIE method. They are presented in Chapter 5 for a wide range of disc crack geometries.

In reality, the two-dimensional representation of practical engineering problems is no more than a simplification for ease of analysis. To provide more reliable solutions, three-dimensional (3-D) fracture mechanics analysis is necessary. Cracks found in rotating disc components are often found to be located at the internal bore of the disc or at the surface of a through-bolt hole. In the case of the former, the cracks usually initiate from the corner of the bore, where the stresses are the greatest in magnitude. These cracks are very well represented by a quarter of an ellipse.

In Chapter 6, a range of corner cracked rotating discs are considered, the crack shapes are modelled by a quarter of an ellipse with the aspect ratio $b/a = 0.25, 0.5$ and 0.75 . The stress intensity factors for a total of 31 distinct cracked disc configurations are presented, most of which have not been reported in the literature previously.

But first, the basic theory of LEFM and BIE are reviewed, in the next two chapters. This is followed by a brief description of the application of BIE to LEFM.

Four computer programs are used in this study, namely: BIE2D and CFLOW for 2-D elastic stress and potential flow analyses respectively; BIE3D3 and BIE3D-DRAW for 3-D stress analysis. The program BIE3D-DRAW enables the input data file to be examined graphically before it is being used for running with BIE3D3, for the 3-D stress analysis.

Chapter 2

REVIEW OF LINEAR ELASTIC FRACTURE MECHANICS (LEFM)

In many structural problems the conventional strength theories based on tensile yielding or buckling strength together with the use of "safety factors" are sufficient to provide satisfactory designs. However, due to the existence of cracks, catastrophic failure may still occur under "safe" service loadings, indicating that these traditional design concepts are not adequate for such flawed components. In this respect, Fracture Mechanics has now been established as a very useful and important design tool.

2.1 Basic Fracture Theories

2.1.1 The Griffith Criterion

The origins of fracture mechanics may perhaps be traced to the pioneering works of Inglis [1913] and Griffith [1921]. Inglis studied the behavior of the elastic stress field around elliptical holes in a flat sheet. He found that as the radius of curvature of an elliptical hole is reduced to zero, that is, the hole becoming a crack, the stresses at its tip are theoretically infinite. Inglis' theory however, was unable to explain the reason for crack propagation. His conclusion that the stresses at a crack tip become infinite seemed to suggest that a cracked component cannot sustain any load. This, apparently contradicts the physical reality.

To rationalize Inglis' theory, Griffith applied energy conservation principles to the case of a centrally cracked large glass plate, shown schematically in Figure 2.1. The strain energy released in forming a crack of length a in a homogeneous body of unit thickness is

given by

$$U = - \frac{\pi \sigma^2}{2H} a^2 \quad (2.1)$$

The corresponding increase in surface energy is

$$W = 2a\gamma \quad (2.2)$$

where

$$H = \begin{cases} E & \text{for plane stress} \\ E/(1 - \nu^2) & \text{for plane strain} \end{cases}$$

σ is the external loading stress

E is Young's Modulus

ν is Poisson's ratio

γ is the specific surface energy

To provide ease of physical interpretation, it is helpful to plot U and W versus the crack size a (Figure 2.2)

According to Griffith the crack becomes unstable at a critical length a_c when

$$\frac{\partial}{\partial a} (W + U) = 0 \quad (2.3)$$

Substituting (2.1) and (2.2) into (2.3) gives

$$\sigma_c = \sqrt{\frac{2H\gamma}{\pi a}} \quad (2.4)$$

or

$$a_c = \frac{2H\gamma}{\pi \sigma^2} \quad (2.5)$$

Equations (2.4) and (2.5) give the critical value of loading (or crack size) under a certain geometrical (loading) condition

In order to test his premise, Griffith carried out a series of experiments on hard glass and the results turned out to be fairly satisfactory. Nevertheless, for ductile materials, (2.4) and (2.5) were less successful in predicting the critical loads

Alternatively, (2.5) can be rearranged in the form:

$$\frac{\pi a \sigma^2}{H} = 2\gamma \quad (2.6)$$

The left hand side of (2.6) has been designated the "energy release rate" or the "crack driving force" G :

$$G = -\frac{\partial U}{\partial a} = \frac{\pi a \sigma^2}{H} \quad (2.7)$$

The right hand side of (2.6) represents the surface energy increase that would occur owing to infinitesimal crack extension and is hence termed the "crack resistance" R :

$$R = \frac{\partial W}{\partial a} = 2\gamma \quad (2.8)$$

It follows that G must be at least equal to R so that the crack growth is possible. Namely:

$$G = \frac{\pi a \sigma^2}{H} \geq \frac{\pi a \sigma_c^2}{H} = G_c = R \quad (2.9)$$

where the subscripts c refer to the critical values.

Equation (2.9) can be regarded as the basic condition for crack propagation in Griffith's Criterion. G_c is called the fracture toughness and can be treated as a material property to be determined through a fracture test.

2.1.2 The Stress Intensity Factor (SIF) Approach

The Griffith's criterion is based on energy concepts and it poses some difficulties when directly applied to engineering problems. Noting the inter relationship between stress, strain and energy, Irwin [1957,1958] developed another approach to fracture problems. By considering the analyses due to Westergaard [1939], Irwin noted that for a crack subjected to loading which is normal to the crack surface, the stress field in the vicinity of the crack tip could be expressed in the following form:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{K_I \cos \frac{\theta}{2}}{\sqrt{2\pi r}} \begin{bmatrix} 1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \\ 1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \\ \sin \frac{\theta}{2} \cos \frac{3\theta}{2} \end{bmatrix} + O(r^0) \quad (2.10)$$

and

$$\sigma_{zz} = \begin{cases} 0 & \text{for plane stress} \\ (\sigma_{xx} + \sigma_{yy})\nu & \text{for plane strain} \end{cases} \quad (2.11)$$

where K_I is called the stress intensity factor of mode I . The subscript I denotes the opening tendency of the two crack faces; r and θ are polar coordinates defined in Figure 2.3; $O(r^\circ)$ represents terms proportional to non-negative powers of r and are insignificant to the crack tip solutions.

The corresponding displacement field is

$$\begin{bmatrix} u \\ v \end{bmatrix} = \frac{K_I}{4\mu} \sqrt{\frac{2r}{\pi}} (\kappa - 1 + 2 \sin^2 \frac{\theta}{2}) \begin{bmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \end{bmatrix} + O(r^\circ) \quad (2.12)$$

where:

u and v are displacements in x and y direction respectively;

μ is the shear modulus; and

$$\kappa = \begin{cases} 3 - 4\nu & \text{for plane strain} \\ (3 - \nu)/(1 + \nu) & \text{for plane stress} \end{cases}$$

Generally a crack in a solid may be stressed in three different modes which are shown schematically in Figure 2.4. They are:

Mode I (opening): In-plane loading is symmetric with respect to the crack plane;

Mode II (sliding): In-plane loading is skew-symmetric with respect to the crack plane;

Mode III (tearing): Loading is parallel to the leading edge of the crack.

Therefore, the stress field near the tip of a crack under a combined loading may be given as:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{K_I \cos \frac{\theta}{2}}{\sqrt{2\pi r}} \begin{bmatrix} 1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \\ 1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \\ \sin \frac{\theta}{2} \cos \frac{3\theta}{2} \end{bmatrix} + \frac{K_{II} \sin \frac{\theta}{2}}{\sqrt{2\pi r}} \begin{bmatrix} -(2 + \cos \frac{\theta}{2} \cos \frac{3\theta}{2}) \\ \cos \frac{\theta}{2} \cos \frac{3\theta}{2} \\ \cot \frac{\theta}{2} - \cos \frac{\theta}{2} \sin \frac{3\theta}{2} \end{bmatrix} + O(r^\circ) \quad (2.13)$$

and

$$\begin{bmatrix} \sigma_{zz} \\ \sigma_{yz} \end{bmatrix} = \frac{K_{III}}{\sqrt{2\pi r}} \begin{bmatrix} -\sin \frac{\theta}{2} \\ \cos \frac{\theta}{2} \end{bmatrix} + O(r^\circ) \quad (2.14)$$

K_{II} and K_{III} being the stress intensity factors of mode *II* and mode *III* respectively.

It is evident that the stresses resulting from complex loading modes are simply a superposition of those from each individual loading modes. In most practical engineering problems, only mode *I* is involved. Thus for the rest of this study, only this mode will be considered.

From equation (2.10) it could be seen that the stress field near a crack tip is solely determined by the stress intensity factor K_I . The magnitude of K_I depends on the geometry and boundary conditions. In general, it may be expressed in the form:

$$K_I = \sigma Y \sqrt{a} \quad (2.15)$$

where

σ is the nominal applied stress;

Y is the compliance function (non-dimensional geometric function);

a is the characteristic length of the crack.

As K_I is the measure for all the stresses and strains near a crack tip, it is readily seen that K_I is the characterizing parameter governing crack propagation. This indicates that there exists a critical value K_{Ic} such that crack extension would occur when

$$K_I \geq K_{Ic} \quad (2.16)$$

K_{Ic} may be regarded as a material property and termed the "plane strain fracture toughness". Using the virtual work principle, Irwin [1957] has shown that the relationship between G_I and K_I is simply

$$G_I = \frac{K_I^2}{H} \quad (2.17)$$

where H was defined previously in equation (2.1).

It is also obvious that the same relationship holds for the critical values, namely

$$G_{Ic} = \frac{K_{Ic}^2}{H} \quad (2.18)$$

2.2 Limitations of LEFM

From the equations presented in the previous section, it can be seen that the stresses at a crack tip are theoretically infinite. In practice, materials cannot sustain such stresses as plastic deformation around the crack tip will occur.

A gross estimate of the size of a crack tip plastic zone can be obtained by determining the distance r_p^* from the crack tip in which the elastic stress σ_{yy} is larger than the yield stress σ_{ys} (Figure 2.5). Substituting σ_{ys} for σ_{yy} into equation (2.10) and taking the plane $\theta = 0$ result in

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r_p^*}} = \sigma_{ys} \quad \text{or} \quad r_p^* = \frac{K_I^2}{2\pi\sigma_{ys}^2} \quad (2.19)$$

Irwin [1960] proposed a different correction for plastic zone. According to his expression, the diameter of the plastic zone is twice as large as the one defined in equation (2.19), and the stress distribution is also changed accordingly (Figure 2.6). Much work has been done on the estimation of the plastic zone size (see, e.g. Dugdale [1960] and Barenblatt [1962]) and similar results have been obtained.

Another approach worth mentioning is the concept of Crack Opening Displacement (COD) introduced by Wells [1962]. COD is a measure of crack tip deformation and fracture takes place when a certain critical value of COD is reached. The COD approach allows the existence of crack tip plastic zone, thus problems beyond the range of LEFM may be handled.

It should however be emphasized that for problems with relatively small crack tip plastic zone, the principles of LEFM, such as K_I concept, can be applied without significant errors being introduced, as evidenced by the enormous amount of work already published in the literature. Where there are large amounts of plastic deformation, Elastic Plastic Fracture Mechanics (EPFM) must be employed.

2.3 Evaluation of Stress Intensity Factor K_I

Analytical methods of evaluating the K_I based on various stress functions have been

developed for a number of simple problems. These include Westergaard's stress function method, Muskhelishvili's stress function, Stress concentration method, Integral transforms and dislocation models. Such methods have the advantage of leading to explicit expressions of K_I and therefore can be used conveniently. However, when applying to practical problems with complex geometries and boundary conditions, the analytical methods are generally incapable of providing solutions. Recourse to experimental or numerical method must therefore be made. Photoelasticity and compliance methods fall into the former category. In the case of the latter, the most commonly used techniques are the Finite Element (FE) method (see, e.g. Zienkiewicz [1977]) and the Boundary Integral Equation (BIE) method (see, e.g. Cruse [1975], Tan and Fenner [1979]). In this study, the BIE method is employed. Various techniques for obtaining the value of K_I by the BIE will be discussed in Chapter 4, following the review of the BIE method for elastostatics in the next chapter.

Chapter 3

REVIEW OF THE BOUNDARY INTEGRAL EQUATION (BIE) METHOD

The Boundary Integral Equation (BIE) method, as a numerical technique for solving boundary value problems, was first formulated by Rizzo [1967] in plane elastostatics. This work was subsequently extended by Cruse [1969, 1973] to the region of three-dimensional problems. The numerical strategies adopted in these early works were relatively unsophisticated and the advantage of the BIE method over other numerical techniques such as the Finite Element (FE) method was not always clear.

Significant developments of the BIE method were thereafter achieved by a number of authors, such as Lachat and Watson [1975, 1976], Rizzo and Shippy [1977] and Tan and Fenner [1978, 1979]. Their main contribution lies in the relatively advanced numerical formulation and implementation, leading to high accuracy and superior efficiency when used to solve practical problems.

The formulation of the BIE may be classified into two different categories which are termed the indirect method and the direct method respectively. In the former approach, the integral equations are obtained in terms of fictitious density functions which have no physical significance but once obtained will provide the displacements and tractions. On the other hand, when using the direct method, the integral equations are formulated in terms of the physical quantities directly. In this study, only this latter approach will be used and discussed. The numerical strategy adopted follows that originally proposed by Lachat and Watson [1976]. Cartesian tensor notations will be used throughout, where repeated suffices imply summation, and partial differentiation with respect to the coordinates is indicated by a comma suffix. Both two-dimensional and three-dimensional treatments will be discussed in what follows.

3.1 Formulation of the BIE

The stresses in a linear elastic continuum R made of a homogeneous, isotropic material in static equilibrium obey the equation:

$$\sigma_{ij,j} + f_i = 0 \quad \text{in } R \quad (3.1)$$

$$i, j = \begin{cases} 1, 2 & \text{in two-dimensions (2-D)} \\ 1, 2, 3 & \text{in three-dimensions (3-D)} \end{cases}$$

where σ_{ij} is a second-order stress tensor; f_i is a body force vector (force per unit volume). Also, the strain-displacement relationship is given by

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad (3.2)$$

where ϵ_{ij} is the strain tensor and u_i is the displacement component in the x_i -direction.

Considering possible thermal expansion, the generalized Hook's Law becomes

$$\sigma_{ij} = 2\mu(\epsilon_{ij} + \frac{\nu'}{1-2\nu'}\epsilon_{kk}\delta_{ij}) - \gamma\Theta\delta_{ij} \quad (3.3)$$

In (3.3), μ is the shear modulus, and ν' is defined by

$$\nu' = \begin{cases} \nu & \text{for 3-D} \\ \nu^* & \text{for 2-D} \end{cases} \quad (3.4a)$$

where

$$\nu^* = \begin{cases} \nu & \text{for plane strain} \\ \nu/(1+\nu) & \text{for plane stress} \end{cases} \quad (3.4b)$$

δ_{ij} is the well known Kronecker Delta; Θ is the temperature distribution; and γ is a material parameter defined as:

$$\gamma = \begin{cases} 2\mu\alpha(\frac{1+\nu}{1-2\nu}) & \text{for 3-D and plane strain} \\ 2\mu\alpha(\frac{1+\nu}{1-\nu}) & \text{for plane stress} \end{cases}$$

with ν and α being the Poisson's ratio and coefficient of thermal expansion respectively.

Substituting (3.3) into (3.1) and making use of (3.2) lead to the Cauchy-Navier equation of thermo-elastic equilibrium in terms of displacements:

$$\mu(u_{i,jj} + \frac{1}{1-2\nu'}u_{j,ji}) + f_i - \gamma\Theta_{,i} = 0 \quad (3.5)$$

To solve (3.5) for a particular elastic body R bounded by S (Figure 3.1), boundary conditions must be appropriately applied. These may be:

$$u_i(\underline{x}) = g_i(\underline{x}) \quad \text{on } S_0 \quad (3.6a)$$

$$t_i(\underline{x}) = \sigma_{ij}n_j = h_i(\underline{x}) \quad \text{on } S_1 \quad (3.6b)$$

where $u_i(\underline{x})$ and $t_i(\underline{x})$ are displacement and traction components respectively on S ; $g_i(\underline{x})$ and $h_i(\underline{x})$ are prescribed functions and n_j is the unit outward normal vector on S .

The simplest conceivable singular solutions to (3.5), also known as the "fundamental solutions", correspond to that for the Kelvin's problem of a point load in an infinite body. The displacements and tractions for this problem are given by the following tensor fields:

$$U_{ij}(p, Q) = \frac{1}{16\pi\mu(1-\nu)r} [(3-4\nu)\delta_{ij} + r_{,i}r_{,j}] \quad (3.7)$$

$$T_{ij}(p, Q) = -\frac{1-2\nu}{8\pi(1-\nu)r^2} \left[\frac{\partial r}{\partial n} (\delta_{ij} + \frac{3}{1-2\nu} r_{,i}r_{,j}) - n_j r_{,i} + n_i r_{,j} \right] \quad (3.8)$$

for 3-D, and

$$U_{ij}(p, Q) = \frac{1}{8\pi\mu(1-\nu^*)} [(3-4\nu^*) \ln \frac{1}{r} \delta_{ij} + r_{,i}r_{,j}] \quad (3.9)$$

$$T_{ij}(p, Q) = -\frac{1-2\nu^*}{4\pi(1-\nu^*)r} \left[\frac{\partial r}{\partial n} (\delta_{ij} + \frac{2}{1-2\nu^*} r_{,i}r_{,j}) - n_j r_{,i} + n_i r_{,j} \right] \quad (3.10)$$

for 2-D

In the above four equations, $U_{ij}(p, Q)$ and $T_{ij}(p, Q)$ correspond to the displacement and traction respectively, in the x_i -direction at point Q due to the application of a unit load in the x_j direction at point p ; r is the distance vector from p to Q defined by

$$r = \sqrt{(x_{Qi} - x_{pi})(x_{Qi} - x_{pi})} \quad (3.11a)$$

and its derivative

$$\frac{\partial r}{\partial n} = n_i \frac{\partial r}{\partial x_{Qi}} = \frac{n_i}{r} (x_{Qi} - x_{pi}) \quad (3.11b)$$

Substituting the fundamental solution into Betti's reciprocal work theorem (Sokolnikoff [1956]):

$$\int_V \sigma_{ij} \epsilon_{ij}^* dV = \int_V \sigma_{ij}^* \epsilon_{ij} dV \quad (3.12)$$

and making use of the divergence theorem

$$\int_V a_{i,j} dV = \int_S a_i n_j dS \quad (3.13)$$

yield the Somigliana's identity which gives the displacements $u_i(p)$ at an interior point p in terms of the boundary data. In 3-D it is given by:

$$\begin{aligned} u_i(p) = & \int_S t_j(Q) U_{ij}(p, Q) dS - \int_S u_j(Q) T_{ij}(p, Q) dS \\ & + \int_S \mu_0 \gamma \left[\frac{1}{r} (n_i - r_{,i} \frac{\partial r}{\partial n}) \Theta(Q) - r_{,i} \frac{\partial \Theta(Q)}{\partial n} \right] dS \\ & + \int_R f_j U_{ij}(p, Q) dV \end{aligned} \quad (3.14)$$

while in 2-D, it is:

$$\begin{aligned} u_i(p) = & \int_S U_{ij}(p, Q) [t_j(Q) - n_j \gamma \Theta(Q)] dS \\ & - \int_S T_{ij}(p, Q) u_j(Q) dS \\ & - \mu_0 \gamma \int_S [(2 \ln r - 1) n_i + 2 r_{,i} \frac{\partial r}{\partial n}] \Theta(Q) dS \\ & + \mu_0 \gamma \int_S r r_{,i} (2 \ln r - 1) \frac{\partial \Theta(Q)}{\partial n} dS \\ & + \int_R f_j U_{ij}(p, Q) dV \end{aligned} \quad (3.15)$$

where μ_0 is also a material parameter given by

$$\mu_0 = \frac{1 - 2\nu'}{16\pi\mu(1 - \nu')}$$

Note that, in (3.14) and (3.15) there are volume integrals involving the body force f_j , and a direct evaluation of which would diminish the advantage of the BIE as a boundary solution technique. Fortunately, for most commonly encountered body forces such as

gravitational and centrifugal forces, procedures exist, which can transform the volume integrals into surface integrals. Suppose that f_j can be expressed as

$$f_j = -\phi_{,j} \quad (3.16)$$

where ϕ is a scalar potential, such that

$$\phi_{,jj} = k_0, \quad \text{a constant} \quad (3.17)$$

Then it can be shown (see, Appendix I) that

$$\begin{aligned} \int_R f_j U_{ij}(p, Q) dV &= \mu_0 \int_S \left[\frac{1}{r} (n_i - r_{,i} \frac{\partial r}{\partial n}) \phi(Q) \right. \\ &\quad \left. - r_{,i} \frac{\partial \phi(Q)}{\partial n} \right] dS + \mu_0 \int_S k_0 n_i r dS \\ &\quad - \int_S n_j \phi(Q) U_{ij}(p, Q) dS \end{aligned} \quad (3.18)$$

for 3-D, and

$$\begin{aligned} \int_R f_j U_{ij}(p, Q) dV &= \mu_0 \int_S r r_{,i} (2 \ln r - 1) \frac{\partial \phi(Q)}{\partial n} dS \\ &\quad - \mu_0 \int_S [n_i (2 \ln r - 1) + 2 r_{,i} \frac{\partial r}{\partial n}] \phi(Q) dS \\ &\quad - \mu_0 \int_S k_0 n_i r^2 (\ln r - 1) dS \\ &\quad - \int_S n_j \phi(Q) U_{ij}(p, Q) dS \end{aligned} \quad (3.19)$$

for 2-D.

For example, if the elastic body of constant material density ρ , is rotating about the x_3 axis with angular velocity ω , also a constant, then

$$f_1 = \rho \omega^2 x_1; \quad f_2 = \rho \omega^2 x_2; \quad f_3 = 0 \quad (3.20)$$

$$\phi = -\frac{1}{2} \rho \omega^2 (x_1^2 + x_2^2) \quad (3.21)$$

and

$$k_0 = -2\rho\omega^2 \quad (3.22)$$

Substituting (3.18) and (3.19) into (3.14) and (3.15) respectively and taking the interior point p to a boundary point P by the usual limiting process results in the BIE for thermoelastic problems:

$$\begin{aligned}
 C_{ij}(P)u_j(P) + \int_S u_j(Q)T_{ij}(P, Q)dS \\
 &= \int_S [t_j(Q) - n_j\phi(Q)]U_{ij}(P, Q)dS \\
 &+ \mu_0 \int_S \frac{1}{r} (n_i - r_{,i} \frac{\partial r}{\partial n}) [\phi(Q) + \gamma\Theta(Q)]dS \\
 &- \mu_0 \int_S r_{,i} \frac{\partial}{\partial n} [\phi(Q) + \gamma\Theta(Q)]dS \\
 &+ \mu_0 \int_S k_0 r n_i dS
 \end{aligned} \tag{3.23}$$

for 3-D, and

$$\begin{aligned}
 C_{ij}(P)u_j(P) + \int_S u_j(Q)T_{ij}(P, Q)dS \\
 &= \int_S \{t_j(Q) - n_j[\phi(Q) + \gamma\Theta(Q)]\}U_{ij}(P, Q)dS \\
 &- \mu_0 \int_S [(2 \ln r - 1)n_i + 3r_{,i} \frac{\partial r}{\partial n}] [\phi(Q) + \gamma\Theta(Q)]dS \\
 &+ \mu_0 \int_S r r_{,i} (2 \ln r - 1) \frac{\partial}{\partial n} [\phi(Q) + \gamma\Theta(Q)]dS \\
 &- \mu_0 \int_S k_0 r^2 (\ln r - 1) n_i dS
 \end{aligned} \tag{3.24}$$

for 2-D,

where

$$C_{ij}(P) = \lim_{\epsilon \rightarrow 0} \int_{S_\epsilon} T_{ij}(P, Q)dS \tag{3.25}$$

In (3.25) S_ϵ is the surface which is the part of R contained within a sphere of radius ϵ centered at P . If the tangent plane at P is continuous, then

$$C_{ij}(P) = \frac{1}{2} \delta_{ij} \tag{3.26}$$

Equations (3.23) and (3.24) are the integral constraint equations relating the boundary tractions to the boundary displacements. They may generally be solved only by

numerical means. Once the displacements and the tractions on the boundary S are obtained, the displacements at any desired interior point can be calculated using (3.14) and (3.18) or (3.15) and (3.19), depending on the dimensionality of the problem. The stresses may also be obtained by differentiating the displacements at p , yielding the Somigliana's identity for stresses. The explicit expressions for the identity can be found in Appendix I.

From the above discussion, it is evident that the most important feature of the BIE is that it reduces the problem dimensionality by one: a plane domain is represented by its line boundary and a three-dimensional body requires only that its surface be modelled. This not only reduces the effort needed in mesh data preparation, it also improves the efficiency of the computing work in terms of both storage and CPU time.

3.2 Numerical Treatment of the BIE

There are several modelling strategies for solving the BIE. The main differences between these strategies lie in the way in which the surface of the domain is modelled and the assumption made on the variation of the primary variables.

The simplest method used by Rizzo [1967] and Cruse [1969] involved discretizing the surface of the solution domain by straight-line segments (in 2-D) or flat triangular elements (in 3-D), over each of which, the unknown variables were assumed to be constants. In this present study, the numerical formulation adopted is that originally used by Lachat and Watson [1975, 1976] and Tan and Fenner [1978, 1979]. The surface of a 3-D body is represented by eight-node quadrilateral and/or six-node triangular isoparametric elements as illustrated in Figures (3.2) and (3.3). In 2-D, the boundary of the domain is represented by three-node quadratic isoparametric line elements as shown in Figure 3.4. The representation of the geometry and the unknowns over each element is in terms of quadratic shape functions of the intrinsic coordinates $N(\xi)$, as are used in the Finite Element method (see, e.g. Zienkiewicz [1977]). Thus

$$x_i(\xi) = N^e(\xi)x_i^e \quad (3.27a)$$

$$u_i(\xi) = N^e(\xi)u_i^e \quad (3.27b)$$

$$t_i(\xi) = N^c(\xi)t_i^c \quad (3.27c)$$

In equation (3.27) the repeated superscripts c indicate summation.

where

$$c = \begin{cases} 1, 8 & \text{for quadrilateral elements;} \\ 1, 6 & \text{for triangular elements} \\ 1, 3 & \text{for line elements} \end{cases}$$

and the explicit forms of $N^c(\xi)$ are given in Appendix II.

Substitution of (3.27) into (3.23) and (3.24) results in the BIE of the following form:

$$\begin{aligned} C_{ij}(P^a)u_j(P^a) &+ \sum_{b=1}^M \sum_{c=1}^{8 \text{ or } 6} u_j(P^{d(b,c)}) \int_{S_b} T_{ij}(P^a, Q) N^c(\xi) J(\xi) d\xi \\ &= \sum_{b=1}^M \sum_{c=1}^{8 \text{ or } 6} [t_j(P^{d(b,c)}) - n_j \psi(P^{d(b,c)})] \\ &\quad \times \int_{S_b} U_{ij}(P^a, Q) N^c(\xi) J(\xi) d\xi \\ &\quad + \sum_{b=1}^M \sum_{c=1}^{8 \text{ or } 6} [\psi(P^{d(b,c)}) + \gamma \Theta(P^{d(b,c)})] \\ &\quad \times \int_{S_b} \frac{\mu_0}{r} (n_i - r_{,i} \frac{\partial r}{\partial n}) N^c(\xi) J(\xi) d\xi \\ &\quad - \sum_{b=1}^M \sum_{c=1}^{8 \text{ or } 6} \frac{\partial}{\partial n} [\psi(P^{d(b,c)}) + \gamma \Theta(P^{d(b,c)})] \\ &\quad \times \int_{S_b} \mu_0 r_{,i} N^c(\xi) J(\xi) d\xi \\ &\quad + \sum_{b=1}^M \mu_0 k_0 \int_{S_b} r n_i J(\xi) d\xi \end{aligned} \quad (3.33)$$

for 3 D, and

$$\begin{aligned}
C_{ij}(P^a)u_j(P^a) &+ \sum_{b=1}^M \sum_{c=1}^3 u_j(P^{d(b,c)}) \int_{S_b} T_{ij}(P^a, Q) N^c(\xi) J(\xi) d\xi \\
&= \sum_{b=1}^M \sum_{c=1}^3 \{t_j(P^{d(b,c)}) - n_j[\psi(P^{d(b,c)}) + \gamma\Theta(P^{d(b,c)})]\} \\
&\times \int_{S_b} U_{ij}(P^a, Q) N^c(\xi) J(\xi) d\xi \\
&- \sum_{b=1}^M \sum_{c=1}^3 [\psi(P^{d(b,c)}) + \gamma\Theta(P^{d(b,c)})] \\
&\times \int_{S_b} \mu_0 [(2 \ln r - 1)n_i + 2r_{,i} \frac{\partial r}{\partial n}] N^c(\xi) J(\xi) d\xi \\
&+ \sum_{b=1}^M \sum_{c=1}^3 \frac{\partial}{\partial n} [\psi(P^{d(b,c)}) + \gamma\Theta(P^{d(b,c)})] \\
&\times \int_{S_b} \mu_0 r r_{,i} (2 \ln r - 1) N^c(\xi) J(\xi) d\xi \\
&- \sum_{b=1}^M \mu_0 k_0 \int_{S_b} n_i r^2 \ln(r - 1) J(\xi) d\xi
\end{aligned} \tag{3.34}$$

for 2 D.

In equations (3.33) and (3.34), M is the total number of boundary elements, P^a is the a th node, $d(b, c)$ is the c th node of the b th element, and $J(\xi)$ is the Jacobian of transformation. The same principle applies when evaluating the interior point solutions. If there are q distinct surface nodes, then for 3 D problems, (3.33) represents a system of $3q$ linear algebraic equations to be solved for the unknown nodal values of tractions and displacements; whereas for those of 2 D the number of equations is $2q$. The solution of these equations can be obtained by direct Gaussian elimination.

Chapter 4

APPLICATION OF THE BIE METHOD TO LEFM

Elastic stress analysis of cracked solid components is concerned mainly with determining the stress fields in the vicinity of the crack tips or fronts. Therefore, in the application of the BIE method to LEFM, the stress intensity factor K_I is the most important parameter to be determined.

When sufficient isoparametric elements are being used to model the boundary of a physical problem, satisfactory results may be expected. But these conventional elements shown in Figures 3.2 to 3.4 are unable to represent exactly the singular stress behavior around the crack tip or front. Special quarter-point shifting technique may be introduced, as first proposed by Barsoum [1974] and Henshell and Shaw [1975] independently, to overcome this difficulty.

Most of the techniques for evaluating values of K_I by the BIE method were first developed for use with the FE method. They may be classified into two categories: direct and indirect method. The former method uses the local displacements or stresses to obtain K_I values directly. The latter method, on the other hand, is based on energy principles. In this chapter, the different ways of evaluating the stress intensity factors from numerical data are discussed.

4.1 Direct Method to obtain K_I values

The direct method of evaluating K_I was first proposed by Chan *et al.* [1970] in their finite element fracture analysis. Both the displacements and the stresses near the crack tip could be used, according to Chan *et al.*, to calculate the K_I values for a particular crack.

4.1.1 Displacement Method

The local displacement normal to crack face can be obtained from equation (2.12) by letting the polar angle $\theta = \pi$, such that

$$v = \frac{4K_I}{H} \sqrt{\frac{r}{2\pi}} \quad (4.1)$$

from which

$$K_I = \frac{1}{4} H v \sqrt{\frac{2\pi}{r}} \quad (4.2)$$

where r is the distance from the crack tip or front; v is the displacement of the crack surface in its normal direction. H is the material stiffness, which equals Young's modulus E , for the case of plane stress and equals $E/(1 - \nu^2)$, for plane strain.

It should be pointed out that the true value of K_I cannot be calculated directly from (4.2), because this relationship is valid only for $r = 0$. Therefore, it is necessary to take the limit of this Equation:

$$K_I = \lim_{r \rightarrow 0} \left(\frac{1}{4} H v \sqrt{\frac{2\pi}{r}} \right) \quad (4.3)$$

To obtain the stress intensity factor, according to (4.3), values of K_I , say K_I^* , are calculated using (4.2) at a series of nodal points along a radial line on the crack face, with sufficient refined elements around the crack. The K_I^* curves plotted against r rapidly approach a constant slope with increasing distance r from the crack tip or front. The intercept of the tangent to the constant slope portion of the curve with the K_I^* axis is used as the K_I estimate. This process can be seen clearly in Figure 4.1, which is for a cracked rotating disc problem being analysed in the present study. The resulting value of K_I is 20.078. From the figure 4.1 it is evident that the values from points immediately adjacent to the crack tip are not useable to perform the extrapolation. This is true for almost all the numerical solutions, since they cannot represent the crack tip stress singularity exactly, unless this singularity is implicitly modelled.

4.1.2 Formula Method

The extrapolation method based on displacements requires relatively more nodal

points near the crack front. K_I results from coarser BIE meshes would not establish high accuracy. An alternative method was proposed by Navid *et al.* [1985]. This method involves plotting v in Equation (4.1) against $\sqrt{r}$ for a series of points. The result, at least for points close to the crack front, is a straight line passing through the origin. The gradient of the line is proportional to K_I . Hence

$$K_I = \frac{v_c - v_b}{\sqrt{r_c} - \sqrt{r_b}} \frac{\sqrt{2\pi}}{4} H \quad (4.4)$$

where v_b and v_c are displacements at two adjacent nodal points b and c in the direction normal to the crack face; r_b and r_c are their corresponding distances to the crack front. Figure 4.2 shows this relation for the same problem as in Figure 4.1. It can be seen that the nodal points used in Equation (4.4) may be the third and fourth point from the crack tip. However, in the case of coarser meshes, fewer nodal points are available, the first and second point would generally give satisfactory accuracy. Taking the first two nodal points as point b and c , the value calculated is 20.111, which is less than 0.2 percent deviation from that of displacement method.

4.1.3 Stress Method

Rewriting Equation (2.10) for σ_{yy} and letting $\theta = 0$, leads to another explicit expression for K_I :

$$K_I = \sigma_{yy} \sqrt{2\pi r} \quad (4.5)$$

where σ_{yy} is stress component normal to the crack plane and ahead of the crack tip. r is the distance from the crack tip.

Similar extrapolating process to Equation (4.3) can be employed to obtain the K_I value. The result is, however, less accurate than the other methods discussed for a given mesh (see, e.g. Tan and Fenner [1979]). Figure 4.3 shows the extrapolation process. The K_I value calculated is 22.228, which is about 10 percent higher than those obtained above. The reason for the inferiority of the stress method is to be expected because displacements are primary variables in the BIE method, whereas the stresses are secondary results and therefore are generally less accurate than the displacements.

4.2 Indirect Method

The indirect method was originally developed by Cruse and Meyers [1977] for use with the BIE method in three-dimensions. It involves the use of the crack-face displacements and the strain energy release rate G . This rate is obtained from two BIE solutions which are identical in all respects except for a small increment in crack size, and is given by

$$G = \frac{\Delta U}{\Delta A} \quad (4.6)$$

where ΔU is the change of strain energy due to the increase of the area of crack face ΔA . The value of G thus obtained is related to a weighted root-mean-square value of K_I denoted by $\bar{K}_I^2$:

$$\bar{K}_I^2 = GH = \frac{1}{\Delta A} \int_{\Delta A} K_I^2(s) \delta l(s) ds \quad (4.7)$$

In the above equation δl is the distance between the perturbed and unperturbed crack fronts, and s is the linear distance around the crack front, as shown in Figure 4.4.

The variation of K_I along the crack front $K_I(s)$ can be shown (see, e.g. Cruse and Meyers [1977]) to take the form:

$$\frac{K_I(s)}{K_{Iref}} = \frac{v(s)/v_{ref}}{\sqrt{r(s)/r_{ref}}} = \frac{v^*(s)}{\sqrt{r^*(s)}} \quad (4.8)$$

where K_{Iref} is the stress intensity factor at a particular reference point whose displacement and distance from the crack front are v_{ref} and r_{ref} respectively; $v(s)$ and $r(s)$ are values along the crack front and are obtained in the BIE output directly.

Making use of (4.8), Equation (4.7) becomes

$$GH = \frac{1}{\Delta A} \int_{\Delta A} K_{Iref}^2 \left[\frac{v^*(s)}{\sqrt{r^*(s)}} \right]^2 \delta l(s) ds \quad (4.9)$$

and, on rearranging

$$K_{Iref} = \left\{ \frac{1}{\Delta A} \int_{\Delta A} \left[\frac{v^*(s)}{\sqrt{r^*(s)}} \right]^2 \frac{\delta l(s)}{GH} ds \right\}^{-\frac{1}{2}} \quad (4.10)$$

Once K_{Iref} is known at a particular point, $K_I(s)$ can be calculated along the whole crack front by employing Equation (4.8).

The determination of G needs two BIE runs, so the indirect method is generally more expensive than the direct ones, although a less refined mesh can now be used. Nevertheless, techniques exist which offer considerable economies (see, e.g. Tan and Fenner [1979]). By these techniques, the second solution can be obtained without re-establishing and reducing all the linear equations. This is because only a few coefficients are affected when the crack front is perturbed, thus most of the terms in the final set of linear equations remain unchanged.

The numerical accuracy of the indirect method is generally accepted to be better than the other methods, but this is obtained at greater cost, as mentioned previously.

Chapter 5

STRESS INTENSITY FACTORS FOR AN INTERNAL EDGE CRACK IN CIRCULAR DISCS

The problems of cracks emanating from a hollow circular elastic body such as a disc or cylinder have been investigated by several researchers over the past number of years (see, e.g. Emery and Segedin [1972], Delale and Erdogan [1982] and Tan [1983]). Loading conditions studied include internal pressure, external tension and centrifugal body force. However, to the author's knowledge, thus far no results involving thermal loading exist in the open literature.

In this chapter, two-dimensional BIE analyses will be carried out to obtain the K_I values for single-edge crack in the bore of a disc-like hollow body subjected to a radial temperature gradient. A wide range of geometries will be considered.

5.1 Review of Previous Work

Real cracks are usually three-dimensional by nature. But for many engineering problems, the two-dimensional assumption is made to simplify the solution process and to provide suitable upper bound estimates.

Winne *et al.* [1958] carried out a series of experiments on large, bored and notched rotating discs and found out that the discs burst at average tangential stresses lower than the material yield strength. Using the Griffith-Irwin theory, they calculated the fracture toughness G_c from the rotating experiments for different materials. Emery and Segedin [1972] studied the stress intensity for external cracks in hollow cylinders subjected to various loading conditions by a technique similar to the finite difference approximations. Labbens *et al.* [1976] used the weight function method to obtain some stress intensity

factor values for inner edge cracks in cylinders under different mechanical loadings. Tracy [1979] suggested a modified mapping collocation method to calculate the stress intensity factors for a ring subjected to uniform tension at the outer boundary. Since the above discussed methods are different approximations, their accuracies are questionable, and the results obtained are only applicable to a very limited amount of geometries and loading conditions.

Recently, Delale and Erdogan [1982] developed a new formulation for the evaluation of K_I values in a radially edge cracked cylinder or disc. A wide range of geometries as well as loading conditions were studied and the numerical results, according to these authors, are highly accurate. However, the mathematical manipulation involved in their formulation is rather sophisticated and the solution procedure includes the evaluation of infinite series which may have convergence difficulties.

Alternatively, the well known Finite Element method and the Boundary Integral Equation method can be used to solve the cracked disc/cylinder problems easily. Owen and Griffiths [1973] and Buchalet and Bamford [1976] have presented some results of K_I for cracked rotating discs and pressure vessels. The FE method was used in their investigation. Tan [1983] employed the BIE method to analyse the edge cracks in rotating discs. The K_I values for a wide range of single edge cracks occurring at the bore of the discs were presented.

It is clear from the above review that the cracked disc/cylinder problem has attracted great attention due to its importance in the engineering practice. But it is surprising to note that a very important aspect of this problem, namely, the effect of thermal loads, has been totally ignored. It will now be considered in this chapter.

5.2 Test Run

In the present investigation, the two dimensional BIE computer program BIE2D is used. In order to assess the veracity of this program, a test run on a simple thermal stress problem was designed. The problem considered was a thick walled disc with different

temperatures Θ_1 and Θ_2 at its inner and outer circumferential surfaces respectively (Figure 5.1). Assuming the Young's modulus E and the coefficient of thermal expansion α are independent of the temperature, then the hoop stress σ_ϕ in the disc can be expressed analytically as

$$\sigma_\phi = E\alpha(\Theta_1 - \Theta_2) \frac{1 - \ln(\frac{R_2}{r}) - \frac{R_2^2}{R_2^2 - R_1^2} (1 + \frac{R_2^2}{r^2}) \ln(\frac{R_2}{R_1})}{2 \ln(\frac{R_2}{R_1}) \eta} \quad (5.1)$$

where

$$\eta = \begin{cases} 1 & \text{for plane stress} \\ 1 - \nu & \text{for plane strain} \end{cases}$$

Taking advantage of the axisymmetry of the problem domain, only one-quarter segment was modelled (Figure 5.2). Where $R_1 = 1$, $R_2 = 8$. The other parameters used in this test run are:

$$\Theta_1 = 100; \quad \Theta_2 = 0$$

$$E = 1000; \quad \alpha = 0.0001$$

A twelve-element BIE mesh was designed (Figure 5.3) to model the segment shown in Figure 5.2. In the input data of BIE2D, the temperature distribution along the radial direction Θ_r must be included. This may be easily done using the steady-state heat conduction theory which gives:

$$\Theta_r = \Theta_2 + \frac{(\Theta_1 - \Theta_2) \ln(\frac{R_2}{r})}{\ln(\frac{R_2}{R_1})} \quad (5.2)$$

Applying appropriate boundary conditions, namely, displacements of the straight boundaries are zero in their outer normal directions and tractions at the circular surfaces are zero, the stress σ_ϕ was obtained for each nodal point as shown in Figure 5.4 together with the exact solution. It can be seen that the BIE results are very accurate indeed even for a relatively coarse mesh.

5.3 Present Numerical Investigation

A disc with a single-edge crack is shown in Figure 5.5. In very much the same way as the test problem, the disc (or cylinder) is subjected to pure thermal loading: temperature

at its inner and outer surface are Θ_1 and Θ_2 respectively, both are constants. The range of geometries considered was identical to that treated by Tan [1983], namely, the radius ratio R_2/R_1 ranging from 2 to 8, and the relative crack length a/W varying from 0.1 to 0.7, with $W = R_2 - R_1$. Poisson's ratio ν was taken to be 0.3, and in each case, the problem was treated under conditions of plane stress as well as plane strain.

5.3.1 Determination of Temperature Distribution

Generally thermal stresses arise whenever a non-uniform temperature field exists in a homogeneous body. Consider, for example, a cracked vessel containing a high temperature fluid, being located in a lower temperature environment. The steady-state temperature variation in the vessel cannot generally be determined analytically. A numerical means is usually adopted, and for the two-dimensional case, this is not a formidable task.

The BIE method is employed in this study to determine the temperature field in the cracked disc for the range of geometries considered. A BIE computer program CFLOW is used. This program was written for two-dimensional analysis of harmonic problems. Detailed discussion about the program itself will not be presented here.

It is seen from Figure 5.5 that for the cracked disc, there is a symmetry about x axis, therefore only half of the physical domain need to be modelled (Figure 5.6). The mixed boundary conditions involved both boundary temperature and temperature gradient. The temperature designated by Θ equals Θ_1 at the bore and the crack surface, and it equals Θ_2 at the outer surface. The temperature gradient $d\Theta/dn$, on the straight boundary, where Θ is to be determined, is zero. The boundary of the half disc is discretized into line elements (Figure 5.7). Temperature and its gradient over each element are treated to be constants in CFLOW. The resulting discrete values of temperature and its gradient may not be used directly in the input of BIE2D unless special care is taken when designing the CFLOW mesh. This is mainly due to the difference between the elements used in the two programs: three-node isoparametric elements are used in BIE2D, whereas CFLOW uses one-node constant elements. In order to make full use of results from CFLOW, a least square fitting was carried out to each CFLOW output of boundary temperature and its

gradient before applying to BIE2D, so that a continuous temperature field over the entire boundary was obtained.

5.3.2 Stress/Strain Analysis

Once the temperature and its gradient distributions are known over the entire boundary, the other boundary conditions for the stress analysis are the traction displacement constraint conditions. They are shown in Figure 5.8, where the displacements at the straight boundary (excluding the crack surface) are zero in y direction, and the remaining part of the boundary are traction free.

The boundary of the numerical solution domain is represented by a series of three-node isoparametric elements (Figure 5.9). The total number of elements used range from 28 to 30, depending on the radius ratio R_2/R_1 and the crack size a/W . In order to introduce the square root singular behavior at the vicinity of the crack tip, the well known quarter point shifting technique was applied. This involves moving the two nodal points adjacent to the crack tip to such positions that their distances from the tip are one quarter of the lengths of their elements (see, Figure 5.9).

As primary results, the stress intensity factors K_I were obtained using the displacement method described in Subsection 4.4.1. Loading conditions considered are that with higher temperature Θ_1 at the inner bore and lower temperature Θ_2 at the outer surface. However, solutions from such condition can be used for the opposite temperature gradient, namely, $\Theta_1 < \Theta_2$, but the signs of the values must be reversed. The normalized values of K_I are presented in Table 5.1 for both the plane stress and plane strain conditions as well as in Figure 5.10 for plane stress only. The normalizing factor is $\sigma_0\sqrt{\pi a}$, where

$$\sigma_0 = E\alpha(\Theta_1 - \Theta_2) \frac{1 - \frac{2R_2^2}{R_2^2 - R_1^2} \ln\left(\frac{R_2}{R_1}\right)}{2 \ln\left(\frac{R_2}{R_1}\right)\eta} \quad (5.3)$$

which is from Equation (5.1) by letting $r = R_1$ and hence is the hoop stress at the bore of an uncracked disc subjected to the same thermal load conditions at the curved boundaries.

The hoop stress distribution along the x -direction and the circumferential strain variation over the outer surfaces are also presented for a number of selected cases in

Figure 5.12 to 5.23. The normalizing factor for the former is σ_0 defined in (5.3) and for the strain is ϵ_0 given by

$$\epsilon_0 = \alpha(\Theta_1 - \Theta_2) \frac{1 - \frac{2R_2^2}{R_2^2 - R_1^2} \ln\left(\frac{R_2}{R_1}\right)}{2 \ln\left(\frac{R_2}{R_1}\right) \eta} \quad (5.4)$$

representing the hoop strain over the outer surface of an uncracked disc.

5.4 Results and Discussion

5.4.1 Stress Intensity Factors

The normalized results of K_I values are presented both quantitatively (Table 5.1) and graphically (Figure 5.10). The difference between the results from the cases of plane stress and those of plane strain are generally less than 1 percent, therefore only one set of results (plane stress) are plotted, as shown in Figure 5.10.

It should be noted that for a positive temperature gradient $\Theta_1 - \Theta_2 > 0$, K_I takes a negative value; the normalizing factor $\sigma_0 \sqrt{\pi a}$ is however also negative. A negative K_I seems to contradict its definition with the term "opening mode", since a minus K_I indicates the crack closing. Fracture would thus not happen at all. However, the physical interpretation may be sensed when different load conditions such as centrifugal body force, external tension and/or internal pressure are acting together with the thermal loads. In such cases, the value of K_I is simply the algebraic sum of those for each of the individual load cases. Hence the effect of a negative K_I is to reduce the total value of the stress intensity factor.

The fact that there was no previous work done in the pure thermal loading cases for this disc problem makes it difficult to establish the numerical accuracy of the results. However, the good agreement of BIE results with the known exact solution for simple problems and the author's experience of using the BIE2D program suggest that the computed values of K_I are within 5 percent of the true values.

For the sake of completeness, the K_I results for single-edge cracks in rotating discs, obtained by Tan [1983] are also shown in Table 5.2 and Figure 5.11.

5.4.2 Hoop stresses along $-x$ direction

The existence of a crack results in a different stress field in a component from that in an unflawed one. The critical area in a cracked disc, as far as fracture is concerned, is at the vicinity of the crack tip. There the stress field is solely characterized by the stress intensity factor K_I . The stresses remote from the crack tip, on the other hand, may not be characterized by K_I . Although also of interest, they are generally of less significance and are presented here for the sake of completeness.

Six typical cases are chosen, the hoop stress distribution along $-x$ direction ($\phi = 180^\circ$, Figure 5.5) are presented. In Figures 5.12 to 5.14, the results for discs with $R_2/R_1 = 2$, and $a/W = 0.1, 0.4, 0.7$ are presented while Figures 5.15 to 5.17 show those for discs with $R_2/R_1 = 8$ and $a/W = 0.1, 0.4, 0.7$.

With no loss of generality, only plane stress type solutions are presented and discussed. The analytical solution for an uncracked disc is plotted in each case for comparison. All the results are normalized with respect to the stress at the inner bore of an uncracked disc.

As expected, the re-distribution of σ_ϕ increases with the relative crack length a/W , the most severe cases are those with $a/W = 0.7$ (Figure 5.14 and 5.17). Hoop stresses at the bore surface are generally higher than that of an uncracked disc, whilst at the outer surface, they are lower (absolute value greater). A further examination to the effect of radius ratio on the hoop stress re-distribution reveals that for a particular crack length a/W , the stress variation caused by the crack is more severe for a thick-walled disc than that for a thin-walled one. This trend may be seen in Figures 5.14 and 5.17.

5.4.3 Strains at the Outer Circumferential Periphery

Hoop strains are measurable quantities and hence are of physical importance such as in monitoring crack growth. The hoop strains ϵ_ϕ along the outer surface of a disc subjected to thermal loads are shown in Figures 5.18 to 5.23, for the same six cases as mentioned in the last subsection. The normalizing factor ϵ_0 is the hoop strain at the outer

periphery of an unflawed disc.

Figure 5.18 shows the variation of ϵ_ϕ with angle ϕ for the case with $R_2/R_1 = 2$, $a/W = 0.1$. The $\epsilon_\phi - \phi$ curve is fairly close to a horizontal straight line. But careful examination reveals that the ϵ_ϕ reaches a maximum $\epsilon_{\phi \max} = 1.05\epsilon_0$ at $\phi = 0^\circ$ and decreases to minimum $\epsilon_{\phi \min}$ of $0.96\epsilon_0$ at $\phi = 30^\circ$. With the increase in the relative crack length a/W , the deviation of the $\epsilon_\phi - \phi$ curve from a straight line becomes more evident, but their general trends are similar to the one shown in Figure 5.18.

5.5 Concluding Remarks

In many engineering fracture problems, thermal loading is absent. But in rotating disc problems, this is not often so. Unlike other fracture problems, a thermal fracture analysis requires knowledge of the entire temperature field before the stress analysis can be carried out. So the solution of a thermal type stress problem consists of two steps:

- 1) Solution for the temperature and its gradient distribution.
- 2) Using the results from the first step as part of the boundary conditions, carry out the stress analysis.

It is evident that the BIE method is a useful tool for solving these types of problems.

Chapter 6

ROTATING DISCS WITH A QUARTER-ELLIPTICAL CORNER CRACK

In practice, cracks in rotating discs are rarely straight-fronted but are instead, usually part through the disc thickness. To provide accurate solutions for this problem, three dimensional treatment is necessary. In the present chapter, three dimensional BIE analysis will be carried out to study the stress intensity behavior of a corner quarter-elliptical crack in a rotating disc (Figure 6.1). K_I values for a range of crack sizes and disc configurations will be presented.

6.1 Review of Previous Work

Application of fracture mechanics to the structural integrity assessments of cracked spinning disc components was hampered by the lack of three dimensional solutions of the stress intensity factor K_I . In recent years, the advances made in such numerical techniques as the Finite Element method and the Boundary Integral Equation method have made it possible to obtain accurate stress intensity factors for three dimensional crack problems, but there has been relatively little attention paid to the rotating disc problem.

Cursory survey of the fracture mechanics literature shows that only a few authors have dealt with 3 D cracks in turbine discs to date. Cruise and Besuner [1975] are probably the first to have reported on the numerical study of this problem. Using the BIE method, these authors outlined the fracture mechanics methodology for estimating the stress intensity factor K_I of a corner quarter elliptical crack at a through bolt hole in a spinning disc. However, the numerical modelling domain used in their analysis was a rectangular solid which did not account for the influence of the curved hole geometry. Thus good accuracy may not be expected if one applies the results directly to a rotating disc.

Kobayashi *et al* [1976] presented a different approach for evaluating the stress intensity factors of a corner crack at the bore of a rotating disc by using the alternating method. This method consists of two major parts, the first of which involves the derivation of quarter infinite solid with various hoop stress loadings prescribed on the crack surface. The second part applies curvature correction factors obtained from two dimensional analogs of pressurized edge cracks at a bore in an infinite solid. However, careful study reveals that the alternating method employed by these authors suffers from several inadequacies which result in considerable errors (see, e.g. Tan [1983]).

Tan [1983, 1985] presented K_I values for a range of cracked rotating discs, again using the BIE method. Loading conditions included centrifugal body force and radial tensile stress on the external circumferential surface which simulate the effect of blades or buckets at the periphery of a spinning disc. Tan's results are perhaps the first reliable set to the three dimensional cracked rotating disc problem to have appeared in the literature. Another powerful numerical technique, the Finite Element method, has also been used to solve the 3-D cracked disc problems recently by Pagotto [1985]. Pagotto [1985] developed K_I solutions for elliptical corner cracks at the inner bore of a disc subjected to a range of polynomial loading conditions. Twelve different crack configurations were considered in his investigation. No other 3-D results for this problem appear to be available in the open literature.

6.2 Test Runs

Although the program employed in this study BIE3D3 has been extensively and successfully used over the past number of years, it was felt necessary to conduct test runs on simple problems to gain familiarity with its use and to establish its veracity. Two such test problems were treated, the first one being studied is that of an uncracked rotating disc shown in Figure 6.2, and the second is that of a centrally cracked plate under uniform tension shown in Figure 6.3.

Unlike most 3-D finite element computer packages, in which the mesh design is usually

a very tedious task, hence a mesh generating sub program being important, the present BIE program did not make use of a mesh generator. Instead the mesh data were prepared by the user as part of the input file. In general, for an eight-node quadrilateral or a six node triangular element, only the coordinates of the corner nodes need be presented as the program may generate the mid point coordinates automatically as required. In order to ensure that the correct geometric data are being read in, a computer program BIE3D DRAW was employed. In this way, the heavy cost incurred by using an erroneous data set may be avoided.

The disc, as shown in Figure 6.2, is taken to rotate at angular velocity $\omega = 1$; its material density and Young's modulus are taken as $\rho = 1$ and $E = 1000$; Poisson's ratio is $\nu = 0$, inner and outer radius are $R_1 = 1$, $R_2 = 2$; thickness in axial direction is $t = 2$. It is understood that units chosen for the parameters above can be totally arbitrary as long as they represent a consistent set. The value of Poisson's ratio was purposely chosen to be zero so that the 2-D and 3-D analytical solutions for the rotating disc problem are identical, as given by:

$$\sigma_{\phi} = \frac{3}{8} \rho \omega^2 (R_1^2 + R_2^2 + \frac{R_1^2 R_2^2}{r^2} - \frac{r^2}{3}) \quad (6.1)$$

The BIE results are as shown in Figure 6.5, together with the exact solution. The maximum error was found to be less than 0.5 percent with the mesh employed. It should be remarked that the stress solutions are merely secondary results in a BIE analysis, it can therefore be expected that the displacements, as the primary results, would have a better accuracy than 0.5 percent for this particular test problem.

The second test problem is shown in Figure 6.3; the width of the plate $W = 10$, and the crack length $a = 1$; material parameters are $E = 1000$ and $\nu = 0.3$. Again the units are a consistent set. Several 2-D plane strain approximate solutions of K_I exist for this problem. Among them three expressions are widely used:

$$K_I = \sigma \sqrt{\pi a} [1 + 0.256(\frac{a}{W}) - 1.152(\frac{a}{W})^2 + 12.20(\frac{a}{W})^3] \quad (6.3)$$

is derived by Isida,

$$K_I = \sigma \sqrt{\pi a} \sqrt{\sec(\frac{\pi a}{W})} \quad (6.4)$$

by Feddersen, and

$$K_I = \sigma \sqrt{\pi a} \frac{1}{\sqrt{1 - (\frac{2a}{W})^2}} \quad (6.5)$$

by Dixon.

In the above three equations σ is the remote tensile stress. For the present test problem, K_I value calculated from the above expressions are 1.819 (Isida), 1.818 (Feddersen) and 1.809 (Dixon).

The numerical results were obtained from a 73 element BIE mesh, which represents one eighth of the whole plate (Figure 6.6). It was found that there is a variation of K_I through the plate thickness. At the mid point m (see Figure 6.3), $K_I = 1.903$, which is 4.7 percent higher than that of Feddersen's, whereas for the surface point s , $K_I = 1.877$, 3.3 percent higher. No attempt is made here to investigate the varying trend of K_I with plate thickness, but from experience it is believed that the 3-D numerical results are no more than 5 percent off the true value.

6.3 Present Numerical Investigation

A rotating disc with a corner quarter-elliptical crack at its inner bore is shown in Figure 6.1. The cracked disc geometries considered in the present study are as follows:

$$R_2/R_1 = 6 \text{ and } 8;$$

$$t/R_1 = 2 \text{ and } 4;$$

$$a/t = 0.25, 0.5 \text{ and } 0.75;$$

$$b/a = 0.25, 0.5 \text{ and } 0.75;$$

where R_1 and R_2 are inner and outer radius respectively; t is the disc axial thickness; a and b are the semi-major and semi-minor axes of the ellipse representing the crack periphery. Although in total, there are 36 cases to be treated, five of these cases have been studied by Tan [1983, 1985], using the BIE method. They are: $R_2/R_1 = 8$, $b/a = 0.75$, $t/R_1 = 2$, $a/t = 0.5, 0.75$ and $t/R_1 = 4$, $a/t = 0.25, 0.5, 0.75$. Hence only 31 different

cracked discs were considered in the present investigation. The loading treated is that of centrifugal body force due to a constant angular velocity ω of the disc about its central axis. Poisson's ratio was taken to be 0.3 throughout.

As there is a plane of symmetry, only one-half of the disc was modelled. Six different meshes are shown in Figures 6.7 to 6.12, representing various radius ratios, elliptical aspect ratios as well as crack depths. Total number of nodal points ranges from 301 to 306, depending on the crack shape and the disc geometry. The square root r singular behavior of stress near the crack front was introduced by the widely used quarter-point shifting technique which can be seen in Figure 6.13.

For the cracked disc considered, the boundary conditions are as shown in Figure 6.14. The "cut" cross-section (shaded area) was restrained from moving in its normal direction (y -direction) because of the property of symmetry. To preclude rigid body motion, a number of nodal constraints were also applied in both x - and z -direction.

Once the mesh data and boundary conditions were correctly read in, BIE3D3 would calculate and then output the displacements and stresses at each nodal point. The solutions at any interior point, if desired, may also be generated, but they are of no interest in this study. K_I values were evaluated using the displacement method described in Chapter 4 (Equation 4.3). Plane strain conditions were assumed for all points along the crack front except where it intersects the free surfaces; there, plane stress assumption was applied.

6.4 Results and Discussion

6.4.1 Stress Intensity Factors

The variation of the stress intensity factor K_I along the crack front in the rotating discs analysed are presented in a non-dimensional form for various crack and disc geometries in Figures 6.15 to 6.26. In these figures, the abscissas θ is as defined in Figure 6.1 and the normalizing factor $\sigma_0\sqrt{\pi b}$ is the stress intensity factor for an infinite plate containing a central crack of length b subjected to remote tension σ_0 . The magnitude of stress σ_0 is

given by

$$\sigma_0 = \frac{3+\nu}{4} \rho \omega^2 (R_2^2 + \frac{1-\nu}{3+\nu} R_1^2) \quad (6.5)$$

which is the hoop stress at the inner surface of an uncracked disc rotating at the same speed as the cracked one. As the normalizing factor for a given crack size is a constant, the K_I curves represent the absolute trends of the stress intensity factor along the crack front.

Figures 6.15 to 6.18 show the stress intensity factors for those cracks with elliptical aspect ratio $b/a = 0.25$. For all the twelve curves, the trends are qualitatively very similar to each other. The effect of the relative thickness t/R_1 on the stress intensity factors may be seen in Figure 6.15 to 6.16. For $a/t = 0.25$, the values of K_I when $t/R_1 = 2$ are generally about 13 percent higher than the corresponding values for the case with $t/R_1 = 4$. Increases in the value K_I due to doubling t/R_1 are about 16 percent for $a/t = 0.5$ and 17 percent for $a/t = 0.75$. Figures 6.17 to 6.18 show the effect of changing the radius ratio R_2/R_1 from 6 to 8. A careful comparison reveals that for the same size of crack and disc thickness, the magnitude of the normalized stress intensity factor $K_I/\sigma_0\sqrt{\pi b}$ for disc with $R_2/R_1 = 8$ and one with $R_2/R_1 = 6$ deviates only marginally from each other typically by less than 1 percent. It should however be noted that the normalizing factor $\sigma_0\sqrt{\pi b}$ is higher for the bigger R_2/R_1 ratio than for the latter.

The variation of K_I along the fronts of the elliptical cracks with $b/a = 0.5$ are as shown in Figures 6.19 to 6.22. As can be seen, the trends for these geometries are somewhat different from those seen in Figures 6.15 to 6.18. Generally, K_I decreases from $\theta = 0^\circ$ to a minimum at θ between 10° to 15° thereafter it increases fairly constantly until reaches the maximum value at the flat surface. In Figures 6.19 and 6.20 where $R_2/R_1 = 6$ the drop in the stress intensity factor $K_I/\sigma_0\sqrt{\pi b}$ due to the change of thickness t/R_1 from 2 to 4 is about 14 percent for the three a/t ratios. Figures 6.21 and 6.22 show the corresponding results for the discs with $R_2/R_1 = 8$. As in the four cases with $b/a = 0.25$ (Figures 6.15 to 6.18) the effect of changing R_2/R_1 from 6 to 8 is not significant, the differences due to this change are generally less than 2 percent. Figures 6.23 to 6.26 show the $K_I/\sigma_0\sqrt{\pi b}$ value for the cracks with aspect ratio $b/a = 0.75$. Similar trends to those with $b/a = 0.5$

(Figures 6.19 to 6.22) may be noticed. Decreases of $K_I/\sigma_0\sqrt{\pi b}$ value due to the doubling of the disc thickness t/R_1 are about 15 percent. As the crack sizes in these four cases are relatively large, the influence of the radius ratio R_2/R_1 changing becomes more evident as is seen in Figures 6.24 and 6.26.

6.4.2 A comparison with 2-D solutions

As mentioned previously in Chapter 5, two-dimensional solutions may generally be considered as the upper bound of three-dimensional problems with similar geometries. Therefore, it is interesting to compare the 2-D results of stress intensity factors obtained by Tan [1983] (Table 5.2 and Figure 5.11) with the present 3-D results. Noting that the 3-D equivalence of the relative crack length is $b/(R_2 - R_1)$, the entire range of 3-D results and their 2-D counterparts are listed in Table 6.1 to 6.3, where the 3-D values are taken to be the maximum values along the crack front for the corresponding crack sizes.

It can be seen that the 3-D values approach the 2-D solution as the crack penetration a/t increases as to be expected. Maximum $K_I(3D)/K_I(2D)$ ratio was found to be 93.9 percent for the last case in Table 6.1, which is for $b/a = 0.25$ and $a/t = 0.75$. Since the crack shape is very close to a straight fronted one and three-quarters of the disc thickness has been penetrated by the crack, the closeness to the 2-D solution can again be expected. In general however, for the same depth of crack penetration, the maximum stress intensity factor for the cases treated at the front of a corner elliptical crack was found to be less than 25 percent lower than that for a 2-D edge crack in the disc.

6.4.3 Accuracy of the K_I Results

The accuracy of the K_I results presented is difficult to ascertain as no exact solutions are available. A mesh refinement exercise could be carried out to verify the convergence of solutions to their true values. But this can be very expensive indeed. Moreover, experience with the use of the BIE technique has shown this to be unnecessary as the results are most likely to be sufficiently accurate for engineering use. Comparisons with solutions obtained by other techniques can also provide the level of confidence with which to use the present

results.

For the problems studied, the only results reported for the rotating disc are those of Pagotto's [1985], four cases of which are identical to those treated above. They are: $R_2/R_1 = 8$; $t/R_1 = 2$; $b/a = 0.25, 0.5$; $a/t = 0.25, 0.5$. The four sets of K_I results are plotted separately, together with Pagotto's in Figures 6.27 to 6.30. Several significant features may be seen from these figures. Despite the evident deviation between the two sets of solutions, where the FE (Pagotto's) results are higher than the BIE ones, the trends of K_I along the crack fronts agree very well. The deviations of K_I along the crack front between the two solutions are quite consistent, although not negligible. For the aspect ratio $b/a = 0.25$, overall deviation between the two groups of results are as large as 15 percent, whilst for $b/a = 0.5$, the differences are about 10 percent. It should however be pointed out that no mesh refinement exercise was performed either by Pagotto in his FE work, and that the FE computer package (ASAS) have always tended to overestimate the true K_I values (Pagotto [1985]).

6.4.4 Strains Over the Outer Surface

As strains are measurable quantities, it may perhaps be also interesting to present the circumferential strain distribution along the outer surface of a cracked rotating disc. They are shown in Figures 6.31 to 6.40, for ten selected disc geometries, all of which are for the greatest crack penetration treated, given the disc size and crack shape. The hoop strains are normalized with respect to ϵ_0 , where

$$\epsilon_0 = \frac{3 + \nu}{4E} \rho \omega^2 (R_1^2 + \frac{1 - \nu}{3 + \nu} R_2^2) \quad (6.6)$$

Equation (6.6) gives the hoop strain at the outer surface of a uncracked rotating disc under the condition of plane stress. Solutions at the two free surfaces ($z = 0$, $z = t$) are both plotted.

Two common features are observed for all the ten figures:

- 1) For each curve, there is a drop of strain at $\phi = 135^\circ$, thereafter, it rises rapidly with the angle ϕ .

- 2) Strains at the cracked side of the disc ($z = 0$) are always greater than or equal to that at the other side.

The location of the maximum strain is found to be at $\phi = 180^\circ$ for most of the cases except where the absolute crack size becomes relatively large, as shown in Figures 6.36, 6.38, 6.39 and 6.40. The minimum outer-circumferential strain is found to be generally located at $\phi = 0^\circ$, on which plane the crack lies, although in some instances (see, e.g. Figures 6.31, 6.33, 6.34 and 6.37) it occurred at $\phi = 135^\circ$. As to be expected, the deviations of strains from ϵ_0 become greater as the crack size increased.

6.5 Final Remarks

Three-dimensional BIE stress analysis is, in absolute terms, expensive to carry out, although it is still less costly when compared to other numerical techniques. The CPU time required for each 3-D run on the Honeywell Level 66 DPS/C3 Dual Processor Mainframe for the rotating disc problem was about 150 minutes. Average charge, with 50 percent discount on the CP-6 academic account, was about \$280 per run. The relatively high cost and long CPU time have discouraged the use of too fine a mesh for the 3-D analysis, although it must be said that the number of element employed in each case was, from experience, deemed sufficient to yield results with very satisfactory accuracy. It should also be pointed out that the K_I results at the two free surfaces should be used with caution, primarily because there are doubts concerning to meaning of stress intensity factor at the intersection of the crack front and the free surface. A number of authors (see, e.g. Hartranft and Sih [1970], Tan and Fenner [1979]) have pointed out that the stress singularity at the free surface of a 3-D body is one weaker than $r^{-1/2}$. Accordingly, the stress intensity factor at the free surface should theoretically be zero, although numerically only an abrupt drop in K_I could be detected if an extremely refined mesh is employed. No effort was made to investigate this feature in the present study.

Chapter 7

CONCLUSIONS

The basic concepts and principles of Linear Elastic Fracture Mechanics (LEFM) have been briefly reviewed. Attention was mainly focussed on the opening mode of fracture with the application and limitations of the LEFM also described. The development of the Boundary Integral Equation (BIE) method, mainly in the scope of the direct BIE formulation for linear elastic problems, were introduced in Chapter 3. The basic mathematical derivations of the BIE for both two- and three-dimensions were presented, together with a brief discussion of the numerical implementation. No attempt was made to discuss the programing aspects of the BIE as all the computer packages used in the present work were provided by Dr. C.L. Tan of Carleton University. The various techniques to obtain the stress intensity factors K_I using the BIE method were also discussed in Chapter 4. Of the schemes, the direct displacement method which utilises the crack face displacement data, offers a quick, easy and accurate means of obtaining K_I at the crack front.

The BIE method for elastic fracture analysis was successfully applied to the problem of a cracked rotating disc. As K_I solutions for edge cracks in rotating discs under rotational loading have previously been reported in the literature, attention in the present work was focussed on the effect of thermal loads on K_I for edge-cracked disc components. This apparently, has never been done before. To obtain solutions for the thermal stress problem, two separate computer runs were performed. The first run calculated the temperature field over the entire domain and its output was used as part of the input data for the second run, which produced results of stress analysis.

Finally, a range of three-dimensional corner cracked rotating disc geometries were investigated. The stress intensity factors K_I for each the crack geometry were presented and discussed. Most of the results obtained in this study were new, as no similar results, to

the author's knowledge, have been published before. The very few solutions which could be compared directly with were those obtained by Pagotto [1985] using the finite element method; they generally showed good qualitative agreement with each other, although the numerical value deviated by up to 15 percent in some cases.

The 2-D and 3-D results for the stress intensity factors presented are very useful indeed for the structural integrity assessments of cracked rotating disc components. More work could be done to obtain K_I for other forms of loading, such as blade loads and thermal loads in the 3-D study. This was not carried out primarily because of time constraint. An efficient way of achieving this work would be to obtain K_I solution for polynomial component loads and using the superposition principle; this is particularly so if the computer program used can treat multiple loading cases.

APPENDICES

Appendix I: Derivation of BIE (Courtesy of Dr. C.L. Tan)

This appendix presents the derivation of three-dimensional BIE and some important expressions of both 3-D and 2-D BIE.

(a) Derivation of 3-D BIE

The equilibrium equation with body force in a linear elastic body can be expressed as:

$$\sigma_{ij,j} = -f_i \quad (I.1)$$

The Hook's law with thermal effect is given by:

$$\sigma_{ij} = \lambda \delta_{ij} u_{k,k} + \mu (u_{i,j} + u_{j,i}) - \gamma \Theta \delta_{ij} \quad (I.2)$$

ie.

$$\sigma_{ij} = \lambda \delta_{ij} \epsilon_{k,k} + 2\mu \epsilon_{ij} - \gamma \Theta \delta_{ij} \quad (I.3)$$

where

$$\lambda = \frac{2\mu\nu}{1-2\nu}; \quad \gamma = \frac{\alpha E}{1-2\nu}$$

Recalling Betti's theorem for elasticity, namely

$$\int_V \sigma_{ij} \epsilon_{ij}^E dV = \int_V \sigma_{ij}^E \epsilon_{ij} dV \quad (I.4)$$

and the divergence theorem:

$$\int_V a_{i,i} dV = \int_S a_i n_i dS \quad (I.5)$$

But noting that

$$\epsilon_{ij} = \epsilon_{ij}^E + \epsilon_{ij}^T \quad (I.6)$$

where ϵ_{ij}^E and ϵ_{ij}^T are strain tensors due to elastic force and thermal force respectively, then

$$\int_V \sigma_{ij} \epsilon_{ij}^{E*} dV = \int_V \sigma_{ij}^* \epsilon_{ij} dV - \int_V \sigma_{ij}^* \epsilon_{ij}^T dV \quad (I.7)$$

For pure strain without rotation, strain-displacement relation is simply

$$\epsilon_{ij} = u_{i,j} = u_{j,i} \quad (I.8)$$

therefore, (I.7) becomes:

$$\begin{aligned} \int_V (\sigma_{ij} u_i^{E*})_{,j} dV - \int_V \sigma_{ij,j} u_i^{E*} dV \\ = \int_V (\sigma_{ij}^* u_i)_{,j} dV - \int_V \sigma_{ij,j}^* u_i dV \\ - \int_V \sigma_{ij}^* \epsilon_{ij}^T dV \end{aligned} \quad (I.9)$$

Replacing $\sigma_{ij,j}$ by f_i as in (I.1) and applying (I.5):

$$\begin{aligned} \int_S \sigma_{ij} u_i^{E*} n_j dS + \int_V f_i u_i^{E*} dV \\ = \int_S \sigma_{ij}^* u_i n_j dS + \int_V f_i^* u_i dV \\ - \int_V \sigma_{ij}^* \epsilon_{ij}^T dV \end{aligned} \quad (I.10)$$

One of the known solutions which can be used as the pair of asterisked components is given by Kelvin's point load solution:

$$u_i^{E*} = U_{ij} e_j \quad (I.11a)$$

$$t_i^* = \sigma_{ij}^* n_j = T_{ij} e_j \quad (I.11b)$$

$$f_i = 0 \quad (I.11c)$$

where U_{ij} and T_{ij} were given in equations (3.7) and (3.8), e_j is the base vector.

Since the fundamental solutions are singular at point p on which the load applies, equation (I.10) should be evaluated in the region V with V_e deleted from it. Where V_e is

(b) Somigliana's Identities

(i) three-dimensional

Interior point displacements

$$\begin{aligned}
u_i(p) = & - \int_S u_j(Q) T_{ij}(p, Q) dS \\
& + \int_S [t_j(Q) - n_j \phi(Q)] U_{ij}(p, Q) dS \\
& + \mu_0 \int_S \frac{1}{r} (n_i - r_{,i} \frac{\partial r}{\partial n}) [\psi(Q) + \gamma \Theta(Q)] dS \\
& - \mu_0 \int_S r_{,i} \frac{\partial}{\partial n} [\psi(Q) + \gamma \Theta(Q)] dS \\
& + \mu_0 \int_S k_0 r n_i dS
\end{aligned} \tag{I.38}$$

Interior point stresses:

$$\begin{aligned}
\sigma_{ij}(p) = & - \gamma \Theta(p) \delta_{ij} - \int_S S_{kij}(p, Q) u_k(Q) dS \\
& + \int_S D_{kij}(p, Q) [t_k(Q) - \phi(Q) n_k(Q)] dS \\
& + \int_S S_{ij}^*(p, Q) [\psi(Q) + \gamma \Theta(Q)] dS \\
& + \int_S V_{ij}^*(p, Q) \frac{\partial}{\partial n} [\psi(Q) + \gamma \Theta(Q)] dS \\
& - \int_S k_0 W_{ij}^*(p, Q) dS
\end{aligned} \tag{I.39}$$

where $\Theta(p)$ is the temperature at the interior point:

$$\begin{aligned}
S_{kij}(p, Q) = & \frac{E}{8\pi(1-\nu^2)} \frac{1}{r^3} \left\{ 3 \frac{\partial r}{\partial n} [(1-2\nu) \delta_{ij} r_{,k} \right. \\
& + \nu (\delta_{ki} r_{,j} + \delta_{kj} r_{,i}) - 5 r_{,i} r_{,j} r_{,k}] \\
& + n_i [3\nu r_{,j} r_{,k} + (1-2\nu) \delta_{jk}] \\
& + n_j [3\nu r_{,i} r_{,k} + (1-2\nu) \delta_{ki}] \\
& \left. + n_k [3(1-2\nu) r_{,i} r_{,j} - (1-4\nu) \delta_{ij}] \right\} \\
D_{kij}(p, Q) = & \frac{(1-2\nu)}{8\pi(1-\nu)} \frac{1}{r^2} (\delta_{ki} r_{,j} + \delta_{kj} r_{,i} \\
& - \delta_{ij} r_{,k} + \frac{3}{1-2\nu} r_{,i} r_{,j} r_{,k})
\end{aligned}$$

Therefore, (I.13) can be written as:

$$\begin{aligned} \int_S t_{ij} U_{ij} e_j dS + \int_V f_i U_{ij} e_j dV \\ = \int_S u_i T_{ij} e_j dS + u_j(p) e_j \\ - \int_V \sigma_{ij}^* \epsilon_{ij}^T dV \end{aligned} \quad (I.19)$$

Multiplying both sides by e_i , (I.19) becomes

$$\begin{aligned} u_i(p) = \int_S t_{ij} U_{ij} dS - \int_S u_j T_{ij} dS \\ + \int_V f_j U_{ij} dV + \int_V \sigma_{ij}^* \epsilon_{ij}^T e_i dV \end{aligned} \quad (I.20)$$

But

$$\begin{aligned} \sigma_{ij}^* \epsilon_{ij}^T e_i &= \alpha \Theta \delta_{ij} \sigma_{ij}^* e_i \\ &= \alpha \Theta \delta_{ij} [\lambda \delta_{ij} u_{m,m}^* + \mu (u_{i,j}^* + u_{j,i}^*)] e_i \\ &= \alpha \Theta (3\lambda U_{ii,j} + 2\mu U_{ii,i}) \\ &= 2\mu \alpha \frac{1+\nu}{1-2\nu} U_{ii,j} \Theta = \gamma U_{ii,j} \Theta \end{aligned} \quad (I.21)$$

Thus (I.20) becomes

$$\begin{aligned} u_i(p) = \int_S t_{ij}(Q) U_{ij}(p, Q) dS - \int_S u_j(Q) T_{ij}(p, Q) dS \\ + \int_V f_j U_{ij}(p, Q) dV + \int_V \gamma U_{ii,j}(p, Q) \Theta dV. \end{aligned} \quad (I.22)$$

Taking $p \rightarrow P$ (P is a boundary point), (I.22) takes the form:

$$\begin{aligned} C_{ij} u_j^*(P) = \int_S t_{ij}(Q) U_{ij}(P, Q) dS - \int_S u_j(Q) T_{ij}(P, Q) dS \\ + \int_V f_j U_{ij}(P, Q) dV + \int_V \gamma U_{ii,j}(P, Q) \Theta dV \end{aligned} \quad (I.23)$$

The last volume integral in (I.23) can be transformed into a boundary integral. Noted that

$$\gamma U_{ii,j}(P, Q) dV = \frac{\alpha}{4\pi} \frac{1+\nu}{1-\nu} \int_V \left(\frac{1}{r}\right)_{,j} \Theta dV \quad (I.25)$$

But

$$\nabla^2 r = \frac{2}{r} \quad (I.26)$$

hence

$$\int_V \gamma U_{ij,j}(P, Q) \Theta dV = \frac{\alpha}{4\pi} \frac{1+\nu}{1-\nu} \int_V \left(\frac{1}{2} \nabla^2 r \right)_{,i} \Theta dV \quad (I.27)$$

In steady state heat conduction

$$\nabla^2 \Theta = 0 \quad (I.28)$$

thus

$$\begin{aligned} \int_V \left(\frac{1}{r} \right)_{,i} \Theta dV &= \frac{1}{2} \int_V (\nabla^2 r_{,i} \Theta - r_{,i} \nabla^2 \Theta) dV \\ &= \frac{1}{2} \int_V (\Theta_{r,i,j} - r_{,i} \Theta_{,j})_{,j} dV \\ &= \frac{1}{2} \int_S (\Theta_{r,i,j} - r_{,i} \Theta_{,j}) n_j dS \\ &= \frac{1}{2} \int_S \left[\frac{\Theta}{r} (\delta_{ij} - r_{,i} r_{,j}) n_j - r_{,i} \frac{\partial \Theta}{\partial n} \right] dS \\ &= \frac{1}{2} \int_S \left[\frac{\Theta}{r} (n_i - r_{,i} \frac{\partial r}{\partial n}) - r_{,i} \frac{\partial \Theta}{\partial n} \right] dS \end{aligned} \quad (I.29)$$

Therefore

$$\int_V \gamma U_{ij,j}(P, Q) \Theta dV = \frac{\alpha(1+\nu)}{8\pi(1-\nu)} \int_S \left[\frac{\Theta}{r} (n_i - r_{,i} \frac{\partial r}{\partial n}) - r_{,i} \frac{\partial \Theta}{\partial n} \right] dS \quad (I.30)$$

The other volume integral in (I.23) may be transformed into a boundary integral under certain conditions. Assume that:

$$f_j = -\phi_{,j} \quad \text{where } \phi \text{ is a scalar,} \quad (I.31)$$

such that

$$\phi_{,jj} = k_0, \quad \text{a constant} \quad (I.32)$$

then by divergence theorem

$$\begin{aligned} \int_V f_j U_{ij} dV &= - \int_V \phi_{,j} U_{ij} dV \\ &= - \int_V (\phi U_{ij})_{,j} dV + \int_V \phi U_{ij,j} dV \\ &= - \int_V \phi U_{ij,j} dV + \int_S \phi U_{ij} n_j dS \end{aligned} \quad (I.33)$$

But

$$U_{ij,j} = \frac{1-2\nu}{8\pi\mu(1-\nu)} \left(\frac{1}{r} \right)_{,i} = 2\mu_0 \left(\frac{1}{r} \right)_{,i} \quad (I.34)$$

where

$$\mu_0 = \frac{1 - 2\nu}{16\pi\mu(1 - \nu)}$$

and

$$\begin{aligned} \int_V \left(\frac{1}{r}\right)_{,i} \psi dV &= \frac{1}{2} \int_V (\nabla^2 r_{,i} \psi - r_{,i} \nabla^2 \psi) dV \\ &+ \frac{1}{2} \int_V r_{,i} \nabla^2 \psi dV \\ &= \frac{1}{2} \int_V [(r_{,ij} \psi)_{,j} - (r_{,i} \psi_{,j})_{,j}] dV \\ &+ \frac{1}{2} \int_S k_0 r n_i dS \\ &= \frac{1}{2} \int_S (r_{,ij} \psi - r_{,i} \psi_{,j}) n_j dS \\ &+ \frac{1}{2} \int_S k_0 r n_i dS \\ &= \frac{1}{2} \int_S \left[\frac{1}{r} (n_i - r_{,i} \frac{\partial r}{\partial n}) \psi - r_{,i} \frac{\partial \psi}{\partial n} \right] dS \\ &+ \frac{1}{2} \int_S k_0 r n_i dS \end{aligned} \quad (I.35)$$

Therefore

$$\begin{aligned} \int_V f_j U_{ij}(P, Q) dV &= \mu_0 \int_S \left[\frac{1}{r} (n_i - r_{,i} \frac{\partial r}{\partial n}) \psi \right. \\ &\quad \left. - r_{,i} \frac{\partial \psi}{\partial n} \right] dS + \mu_0 \int_S k_0 r n_i dS \\ &\quad - \int_S U_{ij}(P, Q) \psi n_j dS \end{aligned} \quad (I.36)$$

Substituting (I.30) and (I.36) into (I.23) leads to the final form of BIE with body force and thermal effect:

$$\begin{aligned} C_{ij}(P) u_j(P) &+ \int_S u_j(Q) T_{ij}(P, Q) dS \\ &= \int_S [t_j(Q) - n_j \psi(Q)] U_{ij}(P, Q) dS \\ &+ \mu_0 \int_S \frac{1}{r} (n_i - r_{,i} \frac{\partial r}{\partial n}) [\psi(Q) + \gamma \Theta(Q)] dS \\ &- \mu_0 \int_S r_{,i} \frac{\partial}{\partial n} [\psi(Q) + \gamma \Theta(Q)] dS \\ &+ \mu_0 \int_S k_0 r n_i dS \end{aligned} \quad (I.37)$$

(b) Somigliana's Identities

(i) three-dimensional

Interior point displacements

$$\begin{aligned}
u_i(p) = & - \int_S u_j(Q) T_{ij}(p, Q) dS \\
& + \int_S [t_j(Q) - n_j \psi(Q)] U_{ij}(p, Q) dS \\
& + \mu_0 \int_S \frac{1}{r} (n_i - r_{,i} \frac{\partial r}{\partial n}) [\psi(Q) + \gamma \Theta(Q)] dS \\
& - \mu_0 \int_S r_{,i} \frac{\partial}{\partial n} [\psi(Q) + \gamma \Theta(Q)] dS \\
& + \mu_0 \int_S k_0 r_{,i} n_i dS
\end{aligned} \tag{I.38}$$

Interior point stresses:

$$\begin{aligned}
\sigma_{ij}(p) = & - \gamma \Theta(p) \delta_{ij} - \int_S S_{kij}(p, Q) u_k(Q) dS \\
& + \int_S D_{kij}(p, Q) [t_k(Q) - \psi(Q) n_k(Q)] dS \\
& + \int_S S_{ij}^*(p, Q) [\psi(Q) + \gamma \Theta(Q)] dS \\
& + \int_S V_{ij}^*(p, Q) \frac{\partial}{\partial n} [\psi(Q) + \gamma \Theta(Q)] dS \\
& - \int_S k_0 W_{ij}^*(p, Q) dS
\end{aligned} \tag{I.39}$$

where $\Theta(p)$ is the temperature at the interior point;

$$\begin{aligned}
S_{kij}(p, Q) = & \frac{E}{8\pi(1-\nu^2)} \frac{1}{r^3} \left\{ 3 \frac{\partial r}{\partial n} [(1-2\nu) \delta_{ij} r_{,k} \right. \\
& + \nu (\delta_{ki} r_{,j} + \delta_{kj} r_{,i}) - 5 r_{,i} r_{,j} r_{,k}] \\
& + n_i [3\nu r_{,j} r_{,k} + (1-2\nu) \delta_{jk}] \\
& + n_j [3\nu r_{,i} r_{,k} + (1-2\nu) \delta_{ki}] \\
& \left. + n_k [3(1-2\nu) r_{,i} r_{,j} - (1-4\nu) \delta_{ij}] \right\} \\
D_{kij}(p, Q) = & \frac{(1-2\nu)}{8\pi(1-\nu)} \frac{1}{r^2} (\delta_{ki} r_{,j} + \delta_{kj} r_{,i} \\
& - \delta_{ij} r_{,k} + \frac{3}{1-2\nu} r_{,i} r_{,j} r_{,k})
\end{aligned}$$

$$S_{ij}^*(p, Q) = \frac{1-2\nu}{8\pi(1-\nu)} \frac{1}{r^2} \left(\frac{1}{1-2\nu} \frac{\partial r}{\partial n} \delta_{ij} + n_i r_{,j} + n_j r_{,i} - 3r_{,i} r_{,j} \frac{\partial r}{\partial n} \right)$$

$$V_{ij}^*(p, Q) = \frac{1-2\nu}{8\pi(1-\nu)} \frac{1}{r} \left(\frac{1}{1-2\nu} \delta_{ij} - r_{,i} r_{,j} \right)$$

$$W_{ij}^*(p, Q) = \frac{(1-2\nu)}{16\pi(1-\nu)} \left(\frac{2\nu}{1-2\nu} \delta_{ij} \frac{\partial r}{\partial n} + n_i r_{,j} + n_j r_{,i} \right)$$

(ii) Two-dimensional

Interior point displacements:

$$\begin{aligned} u_j(p) = & - \int_S u_j(Q) T_{ij}(p, Q) dS \\ & + \int_S \{t_j(Q) - n_j[\phi(Q) + \gamma\Theta(Q)]\} U_{ij}(p, Q) dS \\ & - \mu_0 \int_S [(2 \ln r - 1)n_i + 2r_{,i} \frac{\partial r}{\partial n}] [\phi(Q) + \gamma\Theta(Q)] dS \\ & + \mu_0 \int_S r r_{,i} (2 \ln r - 1) \frac{\partial}{\partial n} [\phi(Q) + \gamma\Theta(Q)] dS \\ & - \mu_0 \int_S k_0 r^2 (\ln r - 1) n_i dS \end{aligned} \quad (I.40)$$

Interior point stresses:

$$\begin{aligned} \sigma_{ij}(p) = & -\gamma\Theta(p) \delta_{ij} - \int_S S_{kij}(p, Q) u_k(Q) dS \\ & + \int_S D_{kij}(p, Q) [t_k(Q) - \phi(Q) n_k(Q)] dS \\ & + \int_S S_{ij}^*(p, Q) [\phi(Q) + \gamma\Theta(Q)] dS \\ & + \int_S V_{ij}^*(p, Q) \frac{\partial}{\partial n} [\phi(Q) + \gamma\Theta(Q)] dS \\ & + \int_S k_0 W_{ij}^*(p, Q) dS \end{aligned} \quad (I.41)$$

where

$$\begin{aligned}
 S_{kij}(p, Q) = & \frac{E}{4\pi(1-\nu^2)} \frac{1}{r^2} \left\{ 2 \frac{\partial r}{\partial n} [(1-2\nu)\delta_{ij}r_{,k} \right. \\
 & + \nu(\delta_{ki}r_{,j} + \delta_{kj}r_{,i}) - 4r_{,i}r_{,j}r_{,k}] \\
 & + 2\nu(n_i r_{,j}r_{,k} + n_j r_{,i}r_{,k}) \\
 & + (1-2\nu)(2n_k r_{,i}r_{,j} + n_j \delta_{ik} + n_i \delta_{jk}) \\
 & \left. + n_k(1-4\nu)\delta_{ij} \right\}
 \end{aligned}$$

$$\begin{aligned}
 D_{kij}(p, Q) = & \frac{1}{4\pi(1-\nu)r} [(1-2\nu)(\delta_{ki}r_{,j} + \delta_{kj}r_{,i} \\
 & - \delta_{ij}r_{,k}) + 2r_{,i}r_{,j}r_{,k}]
 \end{aligned}$$

$$\begin{aligned}
 S_{ij}^*(p, Q) = & \frac{1}{4\pi(1-\nu)r} \left[\frac{\partial r}{\partial n} \delta_{ij} \right. \\
 & \left. + (1-2\nu)(n_i r_{,j} + n_j r_{,i} - 2r_{,i}r_{,j} \frac{\partial r}{\partial n}) \right]
 \end{aligned}$$

$$V_{ij}^*(p, Q) = -\frac{1}{8\pi(1-\nu)} [(2 \ln r) \delta_{ij} + (1-2\nu)(\delta_{ij} - 2r_{,i}r_{,j})]$$

$$W_{ij}^*(p, Q) = \frac{r(2 \ln r - 1)}{16\pi(1-\nu)} \left[2\nu \delta_{ij} \frac{\partial r}{\partial n} + (1-2\nu)(n_i r_{,j} + n_j r_{,i}) \right]$$

Appendix II: Shape Functions

(a) 3-node quadratic line element:

$$N^1(\xi) = \frac{1}{2}\xi(\xi - 1)$$

$$N^2(\xi) = 1 - \xi^2$$

$$N^3(\xi) = \frac{1}{2}\xi(\xi + 1)$$

(b) 6-node triangular element:

$$N^1(\xi) = (1 - \xi_1 - \xi_2)(1 - 2\xi_1 - 2\xi_2)$$

$$N^2(\xi) = 4\xi_1(1 - \xi_1 - \xi_2)$$

$$N^3(\xi) = \xi_1(2\xi_1 - 1)$$

$$N^4(\xi) = 4\xi_1\xi_2$$

$$N^5(\xi) = \xi_2(2\xi_2 - 1)$$

$$N^6(\xi) = 4\xi_2(1 - \xi_1 - \xi_2)$$

(c) 8-node quadrilateral element:

$$N^1(\xi) = \frac{1}{4}(\xi_1 - 1)(1 - \xi_2)(1 + \xi_1 + \xi_2)$$

$$N^2(\xi) = \frac{1}{2}(1 - \xi_1^2)(1 - \xi_2)$$

$$N^3(\xi) = \frac{1}{4}(\xi_1 + 1)(\xi_2 - 1)(1 - \xi_1 + \xi_2)$$

$$N^4(\xi) = \frac{1}{2}(\xi_1 + 1)(1 - \xi_2^2)$$

$$N^5(\xi) = \frac{1}{4}(\xi_1 + 1)(\xi_2 + 1)(\xi_1 + \xi_2 - 1)$$

$$N^6(\xi) = \frac{1}{2}(\xi_2 + 1)(1 - \xi_1^2)$$

$$N^7(\xi) = \frac{1}{4}(\xi_1 - 1)(\xi_2 + 1)(1 + \xi_1 - \xi_2)$$

$$N^8(\xi) = \frac{1}{2}(1 - \xi_1)(1 - \xi_2^2)$$

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Table 5.2a. $K_I/\sigma_0\sqrt{\pi a}$ for a single-edge crack in a rotating disc (Tan (1983))
(plane stress)

$$\sigma_c = \frac{3+\nu}{4} \sigma_0 \left(R_2^2 + \frac{(1-\nu)}{(3+\nu)} R_1^2 \right)$$

a/W	R_2/R_1						
	2	3	4	5	6	7	8
0.1	1.0413	0.9360	0.8636	0.8058	0.7588	0.7222	0.6921
0.2	1.0337	0.8601	0.7600	0.6929	0.6445	0.6101	0.5823
0.3	1.0504	0.8215	0.7063	0.6396	0.5940	0.5613	0.5361
0.4	1.0816	0.8038	0.6828	0.6163	0.5684	0.5414	0.5198
0.5	1.1277	0.8039	0.6772	0.6077	0.5646	0.5334	0.5133
0.6	1.1928	0.8280	0.6924	0.6210	0.5745	0.5488	0.5287
0.7	1.2882	0.8743	0.7295	0.6509	0.6058	0.5755	0.5523

Table 5.1a. $K_I/\sigma_0\sqrt{\pi a}$ for a single-edge crack in a disc subjected to thermal loading (plane stress)

$$\sigma_c = E\alpha(\theta_1 - \theta_2) \frac{1 - 2R_2^2 \ln(R_2/R_1) / (R_2^2 - R_1^2)}{2 \ln(R_2/R_1)}$$

a/w	R_2/R_1						
	2	3	4	5	6	7	8
0.1	0.8903	0.8037	0.7370	0.6841	0.6416	0.6078	0.5784
0.2	0.8046	0.6587	0.5744	0.5249	0.4895	0.4611	0.4391
0.3	0.7204	0.5563	0.4848	0.4345	0.4076	0.3901	0.3686
0.4	0.6768	0.4887	0.4221	0.3799	0.3588	0.3413	0.3273
0.5	0.6321	0.4427	0.3786	0.3415	0.3225	0.3085	0.2964
0.6	0.6106	0.4240	0.3522	0.3144	0.2962	0.2854	0.2746
0.7	0.5959	0.4096	0.3328	0.2945	0.2764	0.2684	0.2570

Table 5.1b. $K_I/\sigma_0\sqrt{\pi a}$ for a single-edge crack in a disc subjected to thermal loading (plane strain)

$$\sigma_0 = E\alpha(\theta_1 - \theta_2) \frac{1 - 2R_2^2 \ln(R_2/R_1) / (R_2^2 - R_1^2)}{2 \ln(R_2/R_1)(1-\nu)}$$

a/W	R_2/R_1						
	2	3	4	5	6	7	8
0.1	0.8872	0.8030	0.7366	0.6834	0.6412	0.6072	0.5783
0.2	0.8043	0.6577	0.5740	0.5244	0.4895	0.4608	0.4388
0.3	0.7195	0.5560	0.4841	0.4344	0.4073	0.3886	0.3692
0.4	0.6765	0.4875	0.4219	0.3807	0.3587	0.3402	0.3263
0.5	0.6324	0.4413	0.3784	0.3410	0.3225	0.3085	0.2964
0.6	0.6074	0.4241	0.3517	0.3137	0.2966	0.2866	0.2762
0.7	0.5944	0.4109	0.3332	0.2940	0.2768	0.2701	0.2572

Table 5.2a. $K_I/\sigma_0\sqrt{\pi a}$ for a single-edge crack in a rotating disc (Tan (1983))
(plane stress)

$$K_I = \frac{3+\nu}{4} \sigma_0 \sqrt{\pi a} \left[R_2^2 + \frac{(1-\nu)}{(3+\nu)} R_1^2 \right]$$

a/W	R_2/R_1						
	2	3	4	5	6	7	8
0.1	1.0413	0.9360	0.8636	0.8058	0.7588	0.7222	0.6921
0.2	1.0337	0.8601	0.7600	0.6929	0.6445	0.6101	0.5823
0.3	1.0504	0.8215	0.7063	0.6396	0.5940	0.5613	0.5361
0.4	1.0816	0.8038	0.6828	0.6163	0.5684	0.5414	0.5198
0.5	1.1277	0.8039	0.6772	0.6077	0.5646	0.5334	0.5133
0.6	1.1928	0.8280	0.6924	0.6210	0.5745	0.5488	0.5287
0.7	1.2882	0.8743	0.7295	0.6509	0.6058	0.5755	0.5523

Table 5.2b. $K_I/\sigma_0\sqrt{a}$ for a single-edge crack in a rotating disc (Tan (1983))
(plane strain)

$$\sigma_0 = \frac{1}{4} \frac{3-2\nu}{1-\nu} \rho \omega^2 \{ R_2^2 + \frac{(1-2\nu)}{(3-2\nu)} R_1^2 \}$$

a/W	R_2/R_1						
	2	3	4	5	6	7	8
0.1	1.0406	0.9351	0.8620	0.8043	0.7576	0.7217	0.6911
0.2	1.0291	0.8569	0.7568	0.6904	0.6421	0.6079	0.5803
0.3	1.0433	0.8159	0.7014	0.6352	0.5902	0.5578	0.5327
0.4	1.0719	0.7964	0.6758	0.6102	0.5627	0.5359	0.5151
0.5	1.1145	0.7929	0.6677	0.5992	0.5567	0.5301	0.5063
0.6	1.1778	0.8136	0.6801	0.6097	0.5637	0.5289	0.5190
0.7	1.2665	0.8599	0.7174	0.6354	0.5911	0.5615	0.5390

Table 6.1 Comparison of 3-D and 2-D BIE results of Normalized K_I for cracked rotating discs: $b/a=0.25$

R_2/R_1	t/R_1	a/t	$b/(R_2-R_1)$	$\frac{K_I(3D)}{\sigma_0\sqrt{\pi b}}$	$\frac{K_I(2D)}{\sigma_0\sqrt{\pi b}}$	$\frac{K_I(3D)}{K_I(2D)} \%$
6	2	0.25	0.025	0.8565	—	
6	2	0.5	0.05	0.7863	—	
6	2	0.75	0.075	0.7436	—	
6	4	0.25	0.05	0.7544	—	
6	4	0.5	0.10	0.6775	0.7588	89.3
6	4	0.75	0.15	0.6375	0.6950	91.7
8	2	0.25	0.0179	0.8599	—	
8	2	0.5	0.0357	0.7868	—	
8	2	0.75	0.0536	0.7435	—	
8	4	0.25	0.0357	0.7628	—	
8	4	0.5	0.0714	0.6858	—	
8	4	0.75	0.1071	0.6411	0.6830	93.9

Note: the crack length for 2-D is taken to be b here to avoid confusion.

Table 6.2 Comparison of 3-D and 2-D BIE results of Normalized K_I for cracked rotating discs: $b/a=0.5$

R_2/R_1	t/R_1	a/t	$b/(R_2-R_1)$	$\frac{K_I(3D)}{\sigma_0 \sqrt{b}}$	$\frac{K_I(2D)}{\sigma_0 \sqrt{b}}$	$\frac{K_I(3D)}{K_I(2D)}$
6	2	0.25	0.05	0.6963	—	
6	2	0.5	0.10	0.6312	0.7588	83.2
6	2	0.75	0.15	0.6048	0.6954	87.0
6	4	0.25	0.10	0.5942	0.7588	78.3
6	4	0.5	0.20	0.5903	0.6445	82.4
6	4	0.75	0.30	0.5088	0.5940	85.7
8	2	0.25	0.0357	0.6986	—	
8	2	0.5	0.0714	0.6315	—	
8	2	0.75	0.1021	0.6003	0.6830	87.9
8	4	0.25	0.0714	0.6021	—	
8	4	0.5	0.1429	0.5346	0.6350	84.2
8	4	0.75	0.2143	0.5076	0.5781	87.8

Note: the crack length for 2-D is taken to be b here to avoid confusion.

Table 6.3 Comparison of 3-D and 2-D BIE results of Normalized K_I
for cracked rotating discs: $b/a=0.75$

R_2/R_1	t/R_1	a/t	$b/(R_2-R_1)$	$\frac{K_I(3D)}{c_0 \sqrt{b}}$	$\frac{K_I(2D)}{c_0 \sqrt{b}}$	$\frac{K_I(3D)}{K_I(2D)} \%$
6	2	0.25	0.075	0.6284	—	
6	2	0.5	0.15	0.5775	0.6950	83.1
6	2	0.75	0.225	0.5685	0.6251	91.0
6	4	0.25	0.150	0.5347	0.6946	76.9
6	4	0.5	0.30	0.4882	0.5940	82.2
6	4	0.75	0.45	0.4848	0.5620	86.3
8	2	0.25	0.0536	0.6297	—	
8	2	0.5	0.1071	0.5792	0.6830	84.8
8	2	0.75	0.1607	0.5712	0.6230	91.7
8	4	0.25	0.1071	0.5423	0.6830	79.4
8	4	0.5	0.2143	0.4894	0.5781	84.7
8	4	0.75	0.3214	0.4800	0.5311	90.4

Note: the crack length for 2-D is taken to be b here to avoid confusion.

Figure 2.1. A through-cracked glass plate

Figure 2.2. Variation in surface energy and strain energy release rates with crack length

Figure 2.3. The vicinity of a crack tip

Figure 2.4. Three modes of cracking

Figure 2.5. Assumed crack tip plastic zone

Figure 2.6. Irwin's plastic zone correction

Figure 3.1. The physical domain of a boundary value problem

Figure 3.2. An eight-node quadrilateral surface element

Figure 3.3. A six-node triangular surface element

Figure 3.4. A three-node quadratic line element

Figure 4.1. Determination of stress intensity factor by displacement method

Figure 4.2. Crack face displacement vs. square root of r

Figure 4.3. Determination of stress intensity factor by stress method

Figure 4.4. Perturbation of a crack front

Figure 5.1. A disc under temperature gradient

Figure 5.2. Boundary constraint to the test segment

Figure 5.3. BIE mesh design

Figure 5.4. Hoop stress distribution in the radial direction of a disc

Figure 5.5. A disc with an edge crack at its bore

Figure 5.6. The BIE analysing domain of a disc

Figure 5.5. A disc with an edge crack at its bore

Figure 5.6. The BIE analysing domain of a disc

Figure 5.7. A constant element BIE mesh

Figure 5.8. Boundary conditions for a disc under thermal loading

Figure 5.9. An isoparametric element BIE mesh

Figure 5.10. Stress intensity factors for a single-edge crack in a disc subjected to thermal loading ($\bar{p}$ plane stress)

Figure 5.11. Stress intensity factors for a single-edge crack in a rotating disc (plane stress) (courtesy Tan)

Figure 5.12. Hoop stresses along the $\phi=180^\circ$ radial direction of a disc:
 $R_2/R_1=2$, $a/W=0.1$

Figure 5.13. Hoop stresses along the $\phi=180^\circ$ radial direction of a disc:

$$R_2/R_1=2, a/W=0.4$$

Figure 5.14. Hoop stresses along the $\phi=180^\circ$ radial direction of a disc:
 $R_2/R_1=2$, $a/W=0.7$

Figure 5.15. Hoop stresses along the $\phi=180^\circ$ radial direction of a disc:
 $R_2/R_1=8$, $a/W=0.1$

Figure 5.16. Hoop stresses along the $\phi=180^\circ$ radial direction of a disc:

$$R_2/R_1=8, a/W=0.4$$

Figure 5.17. Hoop stresses along the $\phi=180^\circ$ radial direction of a disc:

$$R_2/R_1=8, a/W=0.7$$

Figure 5.18. Hoop strains along the outer surface of a disc:

$$R_2/R_1 = 2, a/h = 0.1$$

Figure 5.19: Hoop strains along the outer surface of a disc:

$$R_2/R_1=2, a/W=0.4$$

Figure 5.20. Hoop strains along the outer surface of a disc:

$$R_2/R_1=2, \quad a/W=0.7$$

Figure 5.21. Hoop strains along the outer surface of a disc:

$$R_2/R_1 = 8, a/W = 0.1$$

Figure 5.22. Hoop strains along the outer surface of a disc:
 $R_2/R_1=8$, $a/W=0.4$

Figure 5.23. Hoop strains along the outer surface of a disc:

$$R_2/R_1=8, a/W=0.7$$

Figure 6.1. A rotating disc with a quarter-elliptical corner crack at its bore

Figure 6.2. An uncracked rotating disc

Figure 6.3. A centrally cracked plate

Figure 6.4. 32 element mesh for a rotating disc

Figure 6.5. Hoop stress distribution in the radial direction of a rotating disc

Figure 6.6. 73 element mesh for a centrally cracked plate

104 elements

306 nodal points

Figure 6.7. BIE mesh for the disc with a corner crack:
 $R_2/R_1=6$, $t/R_1=4$, $b/a=0.25$, $a/t=0.5$

$$R_2/R_1=6, t/R_1=4, b/a=0.25, a/t=0.5$$

104 elements
304 nodal points

Figure 6.8. BIE mesh for the disc with a corner crack:
 $R_2/R_1=8$, $t/R_1=4$, $b/a=0.25$, $a/t=0.75$

103 elements
305 nodal points

Figure 6.9. BIE mesh for the disc with a corner crack:
 $R_2/R_1=6$, $t/R_1=2$, $b/a=0.5$, $a/t=0.5$

103 elements
303 nodal points

Figure 6.10. BIE mesh for the disc with a corner crack:
 $R_2/R_1=6$, $t/R_1=2$, $b/a=0.5$, $a/t=0.75$

103 elements
305 nodal points

Figure 6.11. BIE mesh for the disc with a corner crack:
 $R_2/R_1=6$, $t/R_1=4$, $b/a=0.75$, $a/t=0.25$

103 elements
301 nodal points

Figure 6.12. BIE mesh for the disc with a corner crack:
 $R_2/R_1=6$, $t/R_1=4$, $b/a=0.75$, $a/t=0.75$

Figure 6.13. Mesh idealization near the crack front

Shaded area is constrained in y-direction

Figure 6.14. Boundary conditions of a cracked rotating disc

Figure 6.15. Variation of the stress intensity factor along the front of a quarter-elliptical corner crack in a rotating disc: $R_2/R_1=6$, $t/R_1=2$, $b/a=0.25$

Figure 6.16. Variation of the stress intensity factor along the front of a quarter-elliptical corner crack in a rotating disc: $R_2/R_1=6$, $t/R_1=4$, $b/a=0.25$

Figure 6.17. Variation of the stress intensity factor along the front of a quarter-elliptical corner crack in a rotating disc: $R_2/R_1=8$, $t/R_1=2$, $b/a=0.25$

Figure 6.18. Variation of the stress intensity factor along the front of a quarter-elliptical corner crack in a rotating disc: $R_2/R_1=8$, $t/R_1=4$, $b/a=0.25$

Figure 6.19. Variation of the stress intensity factor along the front of a quarter-elliptical corner crack in a rotating disc: $R_2/R_1=6$, $t/R_1=2$, $b/a=0.5$

Figure 6.20. Variation of the stress intensity factor along the front of a quarter-elliptical corner crack in a rotating disc: $R_2/R_1=6$, $t/R_1=4$, $b/a=0.5$

Figure 6.21. Variation of the stress intensity factor along the front of a quarter-elliptical corner crack in a rotating disc: $R_2/R_1=8$, $t/R_1=2$, $b/a=0.5$

Figure 6.22. Variation of the stress intensity factor along the front of a quarter-elliptical corner crack in a rotating disc: $R_2/R_1=8$, $t/R_1=4$, $b/a=0.5$

Figure 6.23. Variation of the stress intensity factor along the front of a quarter-elliptical corner crack in a rotating disc: $R_2/R_1=6$, $t/R_1=2$, $b/a=0.75$.

Figure 6.24. Variation of the stress intensity factor along the front of a quarter-elliptical corner crack in a rotating disc: $R_2/R_1=6$, $t/R_1=4$, $b/a=0.75$

Figure 6.25. Variation of the stress intensity factor along the front of a quarter-elliptical corner crack in a rotating disc: $R_2/R_1=8$, $t/R_1=2$, $b/a=0.75$ (courtesy Tan)

Figure 6.26. Variation of the stress intensity factor along the front of a quarter-elliptical corner crack in a rotating disc: $R_2/R_1=8$, $t/R_1=4$, $b/a=0.75$ (courtesy Tan)

Figure 6.27. Comparison of BIE with FE results: K_I value of a corner crack in a rotating disc, $R_2/R_1=8$, $t/R_1=2$, $b/a=0.25$, $a/t=0.25$

Figure 6.28. Comparison of BIE with FE results: K_I value of a corner crack in a rotating disc, $R_2/R_1=8$, $t/R_1=2$, $h/a=0.25$, $a/t=0.5$

Figure 6.29. Comparison of BIE with FE results: K_I value of a corner crack in a rotating disc, $R_2/R_1=8$, $t/R_1=2$, $b/a=0.5$, $a/t=0.25$

Figure 6.30. Comparison of BIE with FE results: K_I value of a corner crack in a rotating disc, $R_2/R_1=8$, $t/R_1=2$, $b/a=0.5$, $a/t=0.5$.

Figure 6.31. Hoop strain distribution along the outer surface of a disc:
 $R_2/R_1=6$, $t/R_1=2$, $b/a=0.25$, $a/t=0.75$

Figure 6.32. Hoop strain distribution along the outer surface of a disc:
 $R_2/R_1=6$, $t/R_1=4$, $b/a=0.25$, $a/t=0.75$

Figure 6.33. Hoop strain distribution along the outer surface of a disc:
 $R_2/R_1=8$, $t/R_1=2$, $b/a=0.25$, $a/t=0.75$

Figure 6.34. Hoop strain distribution along the outer surface of a disc $R_2/R_1=8$, $t/R_1=4$, $b/a=0.25$, $a/t=0.75$

Figure 6.35. Hoop strain distribution along the outer surface of a disc:
 $R_2/R_1=6$, $t/R_1=2$, $b/a=0.5$, $a/t=0.75$

Figure 6.36. Hoop strain distribution along the outer surface of a disc:
 $R_2/R_1=6$, $t/R_1=4$, $b/a=0.5$, $a/t=0.75$

Figure 6.37. Hoop strain distribution along the outer surface of a disc:
 $R_2/R_1=8$, $t/R_1=2$, $b/a=0.5$, $a/t=0.75$

Figure 6.38. Hoop strain distribution along the outer surface of a disc:
 $R_2/R_1=8$, $t/R_1=4$, $b/a=0.5$, $a/t=0.75$

Figure 6.39. Hoop strain distribution along the outer surface of a disc:
 $R_2/R_1=6$, $t/R_1=2$, $b/a=0.75$, $a/t=0.75$

Figure 6.40. Hoop strain distribution along the outer surface of a disc:
 $R_2/R_1=6$, $t/R_1=4$, $b/a=0.75$, $\lambda/t=0.75$

END

1 5 0 4 8 6

FIN