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**A Mechanism Free Plane Quadrilateral
Element with Rotational Degrees of Freedom
and the Associated Facet Shell Element**

by

Omer Aziz

B.Eng. Carleton University, Ottawa, Ontario

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfillment of
the requirements for the degree of

Master of Engineering

**Faculty of Engineering
Carleton University
Ottawa, Ontario
August, 1988**

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ISBN 0-315-46284-1

The undersigned recommend to the Faculty of Graduate Studies and Research acceptance of the thesis *A Mechanism Free Plane Quadrilateral Element with Rotational Degrees of Freedom and the Associated Facet Shell Element*, submitted by Omer Aziz, B. Eng; in partial fulfillment of the requirements for the degree of Master of Engineering.

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Dedicated to the memory of Leslie Barber

Abstract

The idea of having a planar finite element with rotational degrees of freedom has been tried and tested almost since the early days of the finite element method. Only recently have successful elements of this type been formulated. The existing elements, however, either fail the patch test or contain spurious mechanism(s). In other cases, the spurious mechanisms have been suppressed but this has led to a decrease in accuracy.

In this thesis, the best of the existing elements of this class was implemented. Extensive tests were conducted to study its accuracy and efficiency. Due to the relatively complex nature of the formulation of this element, the need for a simpler, yet equally accurate, element was identified. Consequently, a new four node quadrilateral element with rotational degrees of freedom was developed. The element, referred to as RQD4, is mechanism free and passes the patch test. It, however, tends to show some locking problem in distorted meshes for a slender cantilever beam. It was determined that the application of reduced integration and substitute shape functions will not remedy this problem. In general, its accuracy is comparable to, or in some cases better than, the existing elements.

RQD4 was then combined with an improved discrete Kirchhoff plate bending element to produce a facet shell element, called QFSE. The two elements were extensively tested and their performance was found satisfactory.

Acknowledgements

I wish to express my deepest gratitude to Dr. A.G. Razaqpur for his supervision and guidance during the course of this research.

Thanks go to my friend Ahmed Abdel-Akher for his inspiring discussions and suggestions.

I would also like to thank my parents, my brother Qamar and my sisters Shabana and Yasmin for the love and support they have always given me.

To my friend Kim for all her help and understanding.

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List of Symbols

A	Area
$\{A\}$	Matrix of coefficients
$[B]$	The strain-displacement matrix
C_n	Continuity of degree n
CST	Constant strain triangle
dof	degree of freedom
$[D_b]$	Flexural rigidity matrix of a plate
$[D_p]$	Matrix of elastic stiffnesses
E	Young's modulus
$\{F_a\}$	Forces on node a
G	Shear modulus
I_{xx}, I_{yy}, I_{xy}	Moments of inertia
$[I]$	Identity matrix
$[J]$	Jacobian matrix
$[K]$	Element stiffness matrix
L	Length
L_i	Triangular coordinate of node i
LST	Linear strain triangle
M	Bending moment
N	Membrane force
$[N]$	Matrix of shape functions
P	Force
QUAD n	n -noded quadrilateral isoparametric element
$\{q\}$	Equivalent nodal vector
t	thickness
$[T], [T']$	Transformation matrices
U	Strain energy

u, v, w	Displacement components
u_n	normal displacement
u_t	tangential displacement
x, y, z	Cartesian coordinates
β_x	Normal rotation about the x -axis
β_y	Normal rotation about the y -axis
δ	Virtual operator (i.e. δu is virtual displacement)
$\{\Delta\}$	Nodal displacements of a structure
$\{\delta\}$	Nodal displacements of an element
$\{\epsilon\}$	Strains
θ_x	Rotation about the x -axis
θ_y	Rotation about the y -axis
$\{\kappa\}$	Vector of curvatures
λ, μ	Constants
ν	Poisson's ratio
ξ, η	Natural coordinates
Π	A functional (i.e. Potential energy)
ρ	Constant
$\{\sigma\}$	Stresses
Ω	True rotation
ω	Rotational degree of freedom

1.1 General

In the finite element method (FEM), the eight-noded serendipity element, shown in Figure 1.1, has become quite popular. It is an isoparametric element and is extensively used for solving plane stress (or strain) problems. Its popularity is due to its easy formulation, relatively good accuracy and its ability to model curved boundaries. Most of the commercial FEM computer programs contain it in their library. However, this element, called QUAD8 here, has some drawbacks.

Figure 1.1: Element QUAD8

1.2 The Problem

The first drawback of QUAD8 is that it only has two degrees of freedom (dof's) per node, namely u and v , which are the planar displacements parallel to the Cartesian axes x and y , respectively. Thus a beam element, which has u , v and θ dof's at each node, where θ is a rotation normal to the plane of the member, cannot be connected to QUAD8. This type of connection would be necessary in the analysis of shear walls with large openings. Of course, transition elements exist that facilitate this type of connection but from the programming viewpoint they are not convenient.

Its second drawback is that computationally it is relatively expensive because of the presence of midside nodes. The following formula, presented by *MacNeal and Harder* (1988), approximates computation time, T :

$$T = aN_e^2 N_c^3 \quad (1.1)$$

where N_e is the number of elements in the mesh, N_c is the sum of the dof's at the corner and two adjacent sides of an element and a is a parameter independent of the element type. It may be seen from Table 1.1, which shows relative computer running times for various elements, that QUAD8 can be eight times slower than a four-noded element with three dof's per node. This is an important consideration, particularly in nonlinear problems where computation time is a crucial factor in terms of expense.

It is for these reasons that many researchers have tried to develop an element with four corner nodes and three dof's per node - u , v and θ . Obviously, if this element is to be competitive with QUAD8, then it must pass the patch test (presented in Chapter 3), have no spurious mechanisms, and rival QUAD8 in terms of accuracy. It also must not fall prey to a school of thought that a planar element is not able to have a rotational degree of freedom. This thought was put forth by *Irons* (*Irons and Ahmad* 1980) which virtually led to the demise of this class of

elements. Irons argument will be presented in Chapter 3, together with how the element discussed and developed here circumvents his objections.

1.3 Objective and Scope

The objective of this thesis is to develop an accurate and efficient four node plane quadrilateral element with three degrees of freedom per node, namely two displacements and a rotation (drilling freedom). The element is to be free of any major defects, such as failing the patch test and containing spurious mechanisms. Once this objective has been realized, a facet shell element will be developed by combining the foregoing plane element with an accurate plate bending element.

In each case extensive tests will be performed to ascertain the accuracy and efficiency of the elements developed. Elements with constant thickness and material properties will be dealt with, but the scope of the work will be limited to static linear analysis.

In pursuing this objective, a complete review of available literature pertaining to this topic will be undertaken. The available elements will be critically discussed and the performance of the elements to be developed will be compared to that of the more important and popular existing elements. The criteria of performance will be based on accuracy and economy of solution.

Table 1.1: Relative computation time for different elements

Element	N_c	Relative time
<p>8 nodes $\frac{2 \text{ dof's}}{\text{node}}$</p>	6	8.000
<p>4 nodes $\frac{2 \text{ dof's}}{\text{node}}$</p>	2	0.296
<p>4 nodes $\frac{3 \text{ dof's}}{\text{node}}$</p>	3	1.000
<p> $\left(\frac{\partial u}{\partial x}\right), \left(\frac{\partial u}{\partial y}\right), \left(\frac{\partial v}{\partial x}\right), \left(\frac{\partial v}{\partial y}\right)$ 4 nodes $\frac{6 \text{ dof's}}{\text{node}}$ </p>	6	8.000

2.1 General

The idea of having a drilling or rotational dof in a plane finite element has been explored since the mid 1960's. A summary of the work done is presented here by discussing a number of elements. For each element its dof's and the method of formulating its stiffness matrix are briefly reviewed. Work of some authors which is directly related to this thesis are reviewed in more detail in later chapters.

2.2 Previous Work

Tocher and Hartz (1967) introduced a three node plane triangular element with six dof's per node, Fig. 2.1a. The dof's used were: u , $\frac{\partial u}{\partial x}$, v , $\frac{\partial v}{\partial y}$, $\frac{1}{2}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})(\equiv \epsilon_{xy})$, $\frac{1}{2}(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x})(\equiv \theta)$. The set of polynomial terms used to define the inplane displacements u and v in the direction of the x and y Cartesian axes, respectively, were:

$$f = \langle 1 \ x \ y \ x^2 \ xy \ y^2 \ x^3 \ xy^2 \ y^3 \rangle$$

The missing x^2y term causes a linear variation of $\frac{\partial u}{\partial y}$ (or $\frac{\partial v}{\partial x}$). They noted that this feature was essential for inter-element compatibility. The stiffness matrix was obtained by dividing each element into 3 subtriangles. The resulting 54×54 stiffness matrix of the combined subtriangles was condensed to an 18×18 matrix. This element was able to model plane stress problems with applied moments. It

was also found to be more accurate than the constant stress triangular element (CST).

Holand and Bergan (1968) commented on this element. They pointed out that the element subdivision was cumbersome and unnecessary. They also noted that only displacement compatibility was necessary for convergence, and that the linear variation of u (or v) with respect to y was an undue constraint. They proposed their own derivation which used the same six dof's per node, Fig. 2.1b, but was based on a full cubic displacement field (10 terms). The two extra coefficients created by having 20 unknowns were solved for by using an internal node with two dof's (u and v). These internal dof's were then condensed out to yield an 18×18 stiffness matrix. No further details were provided as to the accuracy of this element.

MacLeod (1969) pointed out the need for a rotational dof to model shear walls with large openings. He proposed having two rectangular elements alternately placed in the mesh, Fig. 2.1c. Both elements had four nodes and the u and v freedoms at each node, but $\frac{\partial v}{\partial x}$ or $-\frac{\partial u}{\partial y}$ at alternating nodes. The displacement field was defined as:

$$\begin{aligned} u &= a_1 + a_2x + a_3y + a_4xy + a_5y^2 + a_6xy^2 \\ v &= a_7 + a_8x + a_9y + a_{10}xy + a_{11}y^2 + a_{12}x^2y \end{aligned}$$

Proof of boundary compatibility using these displacement functions was provided. It was shown that displacements parallel to an element boundary were linear and those perpendicular to the boundary were quadratic. The 12×12 stiffness matrices for both elements were explicitly derived. He noted that the solutions monotonically approached the theoretical values although the theoretical value was not necessarily the asymptotic value.

Sisodiya (1971) derived a quadrilateral element, called QLC3, for the analysis of box girder bridges, Fig. 2.1d. It had three dof's (u, v, θ) per node, where θ is

described below. The displacement functions assumed were as follows:

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} 1 & \xi & \eta & \xi\eta & 0 & 0 & 0 & 0 \\ 1 & \xi & \eta & \xi^2 & \xi\eta & \xi^3 & \xi^2\eta & \xi^3\eta \end{bmatrix} \{A\}$$

where $\{A\}$ are polynomial constants and ξ and η are natural coordinates. The rotation θ was defined as $\frac{\partial v}{\partial x_\xi}$ where x_ξ is the distance along a line of constant η . This element is coordinate variant and was designed to model problems that display beam-type behaviour. Its performance in problems where beam action was not predominant was also deemed as satisfactory. More details about the behaviour of this element will be shown in Chapter 7. Other elements similar to QLC3 are reviewed by *Moore* (1975).

Olson and Bearden (1979) introduced a new triangular element, Fig. 2.1e, with three dof's per node (u, v, θ). The six dof's per node proposed by *Tocher and Hartz* were used and the resulting 18×18 stiffness matrix was condensed to a 9×9 matrix using nine constraints. The displacement parallel to an edge was restricted to vary only linearly along that edge. This created two constraints per edge. The interior angle at each vertex was forced to remain unchanged. Both of these requirements created a total of nine constraints, enabling the reduction of the stiffness matrix from 18×18 to 9×9 . An extensive set of tests was performed, and it was realized that stress smoothing was necessary. They noted that although the element was compatible, it would not converge to the correct results. The reason it would not converge was because the corner constraints violated the constant stress modes. These authors also noted that the idea of this element was first presented by *Tinawi* (1971).

The utilization of a rotational degree of freedom was not confined to displacement based elements. *Robinson* (1980) introduced a stress based element with moment dof, Fig. 2.1f. The element had two forces and a moment at each node. Boundary stress distributions were replaced with nodal forces using the standard beam functions for loading normal to the boundary and a linear distribution for

tangential loading. No details about the element performance were supplied.

Mohr (1981) noted the computational inefficiency of *Tocher* and *Hartz*' element and consequently introduced a simple rectangular element. He also noted that *Olson* and *Bearden*'s element was less accurate than the linear strain triangle (LST). His idea was to transform global dof's that included rotations to an element with just translational dof's (u and v), Fig. 2.1g. The nodal rotations were obtained by employing a finite difference approximation of $\phi = \frac{1}{2}(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x})$. This element also did not behave better than the LST.

Mohr (1982) addressed the problem again but from a different point of view, Fig. 2.1h. He listed some drawbacks of his previous work, such as the restriction to rectangular shapes and no appreciable increase in solution accuracy. He noted that *Olson* and *Bearden* formulated a 9×9 stiffness matrix by condensing out nine of the 18 dof's in a nine node triangular element. *Mohr* claimed this method must be avoided because it precluded convergence. He introduced a method of nested interpolations. He used three triangular coordinates and expressed the six freedoms at the three midside nodes in terms of these coordinates. He came to the unfortunate realization that additional constraints were necessary since the three drilling freedoms could not uniquely define the six midside freedoms. The first set of constraints involved prescribing quadratic normal displacements and linear tangential displacements. It was clear that more constraints were necessary to define a unique solution. After suggesting a less stringent alternative, he decided it was still best to apply the vertex constraint suggested by *Olson* and *Bearden*. He showed that this element converged but not faster than the LST. It was also shown that accuracy decreased if the drilling freedom was loaded. Unlike the LST, this element gave correct stresses both at the centroid and at the nodes.

Allman (1984) presented a formulation for a displacement based compatible element incorporating vertex rotations, Fig. 2.1i. He noted that all previous displacement based formulations imposed severe restrictions on the displacements in

order to achieve compatibility. He proposed quadratic displacement fields normal to the element sides, instead of previously used cubic functions. The tangential displacements, i.e. displacements parallel to the sides, were assumed to be linear. The displacements of an element boundary were defined as:

$$u_n = a_1 + a_2 s + a_3 s^2$$

$$u_t = a_4 + a_5 s$$

where s is a local coordinate running from one end of a side to the other, u_n is the normal displacement and u_t is the tangential displacement. The five coefficients were determined by the six rotations and translations at the nodes situated on the given side. This problem is mathematically 'overdetermined'. This overdeterminacy was circumvented by applying only five boundary conditions.

The five boundary conditions consisted of the four nodal displacements along a given side plus the difference in the normal displacement derivatives evaluated at the two nodes. As a result, typical displacement functions were given by:

$$u_n = (1 - \frac{s}{l_{12}})u_{n1} + (\frac{s}{l_{12}})u_{n2} + \frac{1}{2}s(1 - \frac{s}{l_{12}})(\omega_1 - \omega_2)$$

$$u_t = (1 - \frac{s}{l_{12}})u_{t1} + (\frac{s}{l_{12}})u_{t2}$$

where ω_i ($i = 1, 2$) is the normal rotation to node i and l_{12} is the length of the element side between nodes 1 and 2. ω_i is not the rotation in the elasticity sense and we shall discuss it in detail later. It was noted that if the nodal translations were equal to zero and the nodal rotations were held constant ($\omega_2 = \omega_1 = \theta_0$), then a spurious mechanism would form. This mechanism could easily be restrained if any node is prevented from rotating. This element represents all rigid body and constant stress states exactly, thus assuring convergence. Test results have shown the element to be competitive with the LST while using fewer dof's per mesh.

Bergan and Felippa (1985) presented an extensive derivation of a triangular element (with rotational dof's) based on the free formulation of *Bergan and Nygård*

(1981), Fig. 2.1j. They noted that Tocher and Hartz' element employed C_1 continuity (displacements and their derivatives) which was unnecessary since only C_0 continuity is required for convergence. The stiffness matrix was derived on the basis that it pass the individual element test (equivalent to passing the patch test). They derived a 9×9 stiffness matrix as follows:

$$[K]_{9 \times 9} = \begin{bmatrix} [K_b]_{6 \times 6} & 0 \\ 0 & \alpha [K_h]_{3 \times 3} \end{bmatrix}$$

where $[K_b]$ is the stiffness matrix corresponding to the basic modes (three rigid body and three constant), $[K_h]$ is the stiffness matrix corresponding to higher order modes, and α is a 'fine tuning' multiplier. This element passed the patch test by virtue of its derivation. It also seemed to behave an order of magnitude better than the CST.

Cook (1986) provided a simplified method using kinematic transformation to generate Allman's element. A transformation matrix $[T]$ was derived which related the dof's of the LST, $\{\delta_{LST}\}$, to the dof's of Allman's triangle (AT), $\{\delta_{AT}\}$:

$$\{\delta_{LST}\}_{12 \times 1} = [T]_{12 \times 9} \{\delta_{AT}\}_{9 \times 1}$$

The stiffness matrix of AT could be determined, based on the principle of virtual work, as follows:

$$[K_{AT}] = [T]_{9 \times 12}^T [K_{LST}]_{12 \times 12} [T]_{12 \times 9}$$

The extension of the idea to quadrilateral elements, using QUAD8 as the parent element, was also mentioned. He stated that the stresses could be calculated by recovering the original displacements. In a test he reported, element AQ (the quadrilateral version of AT) outperformed AT.

MacNeal (1986) provided a detailed history of the elements used in the commercial program MSC/NASTRAN. He pointed out the necessity for rotational dof's in shell structures to resolve a locking problem when non-rectangular elements with

only translational dof's are used. He used Allman's idea and independently derived an element identical to Cook's. This new element solved the element locking problem. A penalty function was used to eliminate the spurious mechanism.

Razaqpur and Nofal (1987) provided the procedure used by Cook to derive the AQ element, Fig. 2.1k. The transformation matrix was identical to Cook's but was based on beam displacement functions and not Allman's functions. An extensive set of planar tests were performed to verify Cook's findings of reliable performance. This element (which they called RQUAD4) was highly recommended. Further details will be given concerning this element in a later chapter in this thesis.

MacNeal (1987) presented an analysis of several four noded membrane elements in the search for a defect free four node element. He noted that AQ (by the same token RQUAD4) had two spurious mechanisms. The first was the mode caused by equal rotations. The second was an 'hourglass' mode caused by the use of two point integration. A Poisson's ratio locking problem was also noted. A new element, named Q4S, was introduced which was based on the AQ formulation with two refinements. The first refinement involved subjecting the element to a least square linear fit to ensure that the strains included three constant terms, four linear terms and no higher order terms. The second refinement was to add auxiliary strain terms in such a way that stress equilibrium conditions for a traction free region were satisfied. This element passed all of the MacNeal-Harder test problems.

MacNeal and Harder (1988) presented a detailed formulation of the new Q4S element, Fig. 2.1k. Penalty function options were introduced to suppress the two spurious modes, but this led to a decrease in accuracy. They concluded that this element was attractive for a wide variety of applications, particularly when computational costs are significant.

Robinson (1988) reviewed Razaqpur and Nofal's formulation of RQUAD4. He noted that the element was sensitive to aspect ratio. However, the sensitivity is about the same order as QUAD8 and far better than QUAD4. He also noted a

reduction in the element's capability due to the presence of spurious modes. He challenged other researchers to develop a quadrilateral element with rotational dof's, without spurious modes, and with no sensitivity to shape distortion.

Allman (1988) introduced another quadrilateral element (AQ2) based on the assemblage of four Allman triangle (AT) elements, Fig. 2.11. His purpose was to eliminate the spurious mechanism of AQ. He successfully did this by constraining the centroid rotation, ω_0 , to be the weighted average of the rotations of the four triangles. He noted that this created a problem when the strains were determined using the standard strain-displacement relationships. Examination of the spurious mechanism displacement functions showed that the perimeter displacements were all equal to zero, yet there were internal strains. He solved this anomaly by calculating the interior stresses by minimizing the complementary energy of an equilibrium stress field given the perimeter displacements. Thus zero perimeter displacements would give zero interior stresses. He calculated the stresses based on Airy's stress function. Several tests showed this element to be accurate even for very coarse meshes. Its performance is compared to other elements in Chapter 7.

This review has mainly concentrated on displacement based models. There have been other formulations using hybrid models but this type of formulation is not widely used in practice. The elements discussed in this survey do not cover all the attempts made at the development of plane elements with rotational dof's. Reference has been made by some of the authors to the work of other investigators who have attempted to solve this problem. Their work is not, however, readily available in the literature.

2.3 Summary

Table 2.1 summarizes the characteristics of the important elements reviewed in this chapter. The table indicates the problems associated with each element. We observe that each is beset by one or more problems. Some of the elements are not attractive because they fail the patch test. Others, such as Tocher and Hartz' triangular element, have too many degrees of freedom and are therefore computationally inefficient. The recent element of Allman, i.e. AQ2, appears to be defect free, but its formulation is too complex. Hence, the search for a simple, accurate and efficient quadrilateral element is still a worthwhile endeavour.

Figure 2.1: Previous elements with a drilling freedom

Figure 2.1: Previous elements with a drilling freedom (continued)

Table 2.1: Characteristics of some existing plane elements with a drilling freedom

Element	Date	Fails the Patch test	Computationally inefficient	Spurious mechanism	Subject to Irons' paradox
Tocher + Hartz	1967	×	×		×
Holand + Bergan	1968	×	×		×
MacLeod	1969	×			×
Sisodiya (QLC3)	1971	×			×
Olson + Bearden (OB)	1979	×			×
Allman (AT)	1984			×	
Cook (AQ)	1986			×	
Razaqpur + Nofal (RQUAD4)	1987			×	
MacNeal + Harder (Q4S)	1988			×	
Allman (AQ2)	1988				

3:1 General

The finite element method (FEM) is widely used in almost all fields of engineering. The method involves dividing a complex domain into finite subdomains, called finite elements or simply elements. The properties of each element, such as the stiffness matrix and the nodal load vector, are determined based on systematic procedures. These elemental matrices are combined to model the whole domain which results in a set of simultaneous equations. These are solved for after the imposition of the necessary boundary conditions. Element strains and stresses are then calculated from the nodal displacement values obtained from the solution of the simultaneous equations.

The procedure described is typical of displacement-based finite elements, the subject of interest in this thesis. Other finite element procedures, in which the nodal unknowns or degrees of freedom may be stresses or mixed stress and displacements are also possible. However, these are outside the scope of the current work and therefore will not be discussed.

In the following section the basic equations of FEM will be presented for the sake of completeness and reference in subsequent chapters. The discussion will be brief, but the reader will be directed to other sources for a more complete presentation.

3.2 Basic Equations of FEM

In the displacement approach, the principle of minimization of the total potential energy, Π , of the structure can be used to derive the finite element equations. From basic mechanics, for stable equilibrium Π must be a minimum for small variations of admissible displacements. An admissible field is a displacement field that satisfies the kinematic (essential) boundary conditions of the problem. The potential energy Π is given by:

$$\Pi = U + V \quad (3.1)$$

where U is the strain energy of deformation and V is the potential of the applied loads. The strain energy U is given by:

$$U = \frac{1}{2} \int_{Vol} \{\epsilon\}^T [D] \{\epsilon\} dvol \quad (3.2)$$

where $\{\epsilon\}$ is the strain vector, $[D]$ is the elasticity matrix and $dvol$ is the volume of a differential element of the body. On the other hand, the potential of the applied forces is given by:

$$V = - \int_{Vol} \{f\}_b^T \{\delta\} dvol - \int_s \{f\}_s^T \{\delta\} ds \quad (3.3)$$

where $\{f\}_b$ is the vector of body forces per unit volume, $\{f\}_s$ is the vector of surface tractions, $\{\delta\}$ is the displacement field and s is the portion of the boundary where tractions are prescribed. For simplicity, we shall henceforth drop the domain of integration symbols since these are implicitly understood. Substituting Eqs. (3.2) and (3.3) into Eq. (3.1) yields:

$$\Pi = \frac{1}{2} \int \{\epsilon\}^T [D] \{\epsilon\} dvol - \int \{f\}_b^T \{\delta\} dvol - \int \{f\}_s^T \{\delta\} ds \quad (3.4)$$

From theory of elasticity

$$\{\epsilon\} = [L] \{\delta\} \quad (3.5)$$

where $[L]$ is a linear operator matrix.

According to finite element procedures, the displacement field $\{\delta\}$ can be expressed in terms of discrete nodal displacement values $\{\Delta\}$, i.e.

$$\{\delta\} = [N] \{\Delta\} \quad (3.6)$$

where $[N]$ is a matrix of "trial functions" (or "shape functions" when referring to individual finite elements). Substitution of Eq. (3.6) into Eq. (3.5) leads to:

$$\{\epsilon\} = [L][N] \{\Delta\} = [B] \{\Delta\} \quad (3.7)$$

in which the matrix $[B]$ is commonly referred to as the strain-displacement matrix. Substituting for $\{\delta\}$ and $\{\epsilon\}$ from Eqs. (3.6) and (3.7) into Eq. (3.4) yields:

$$\Pi = \frac{1}{2} \int \{\Delta\}^T [B]^T [D] [B] \{\Delta\} dvol - \int \{f\}_b^T [N] \{\Delta\} dvol - \int \{f\}_s^T [N] \{\Delta\} ds \quad (3.8)$$

or

$$\Pi = \frac{1}{2} \{\Delta\}^T \left(\int [B]^T [D] [B] dvol \right) \{\Delta\} - \left(\int \{f\}_b^T [N] dvol + \int \{f\}_s^T [N] ds \right) \{\Delta\} \quad (3.9)$$

which can be written as

$$\Pi = \frac{1}{2} \{\Delta\}^T [K] \{\Delta\} - \{q\} \{\Delta\} \quad (3.10)$$

where

$$[K] = \int [B]^T [D] [B] dvol \quad (3.11)$$

$$\{q\} = \int \{f\}_b^T [N] dvol + \int \{f\}_s^T [N] ds \quad (3.12)$$

In the above $[K]$ is the structural stiffness matrix and $\{q\}$ is the equivalent nodal load vector.

The principal of stationary potential energy states that under equilibrium

$$\delta \Pi = \left\{ \frac{\partial \Pi}{\partial \{\Delta\}} \right\}^T \delta \{\Delta\} = 0 \quad (3.13)$$

for any small admissible variation $\delta \{\Delta\}$. Applying the above to Eq. (3.10), yields

$$\frac{\partial \Pi}{\partial \{\Delta\}} = [K] \{\Delta\} - \{q\} = 0 \quad (3.14)$$

Eq. (3.14) expresses the important equations of equilibrium which could be solved for the nodal displacements $\{\Delta\}$.

In the finite element method the structure stiffness matrix and nodal load vector are obtained by a suitable combination of elements stiffness and nodal load matrices. Furthermore, for other than very simple elements, such as triangular and rectangular elements, the integrations in Eqs. (3.11) and (3.12) are performed using numerical integration, principally Gauss quadrature. To perform this integration, the stiffness and load matrices are expressed in terms of the natural coordinates ξ and η . These coordinates vary between -1 and +1 over each element. Expressed in this fashion, for an element Eq. (3.11) can be typically written as:

$$[K]^e = \int_{-1}^1 \int_{-1}^1 t [B(\xi, \eta)]^T [D] [B(\xi, \eta)] \det[J] d\xi d\eta \quad (3.15)$$

where t is the element thickness, $\det[J]$ is the determinant of the Jacobian matrix and the superscript e signifies an element. The matrix $[J]$ simply relates the derivatives of the shape functions N_i with respect to the Cartesian coordinates to the derivatives with respect to the natural coordinates. Thus

$$\begin{Bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix} \quad (3.16)$$

where

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \quad (3.17)$$

Using Gaussian quadrature, Eq. (3.15) can be written as a summation:

$$[K]^e = \sum_{i=1}^n \sum_{j=1}^n t [B(\xi_i, \eta_j)]^T [D] [B(\xi_i, \eta_j)] w_i w_j \det[J] \quad (3.18)$$

in which w_i and w_j are weighing functions pertaining to sampling points (i, j) , which are specified by the integration method. The number of sampling points, $n \times n$, is selected by the user, although a minimum number of points is needed to ensure accuracy.

3.3 Displacement-based Plane Finite Elements

Development of elements for plane elasticity problems is relatively simple. The stress-strain relationship for an elastic material (neglecting initial stresses and strains) is:

$$\{\sigma\} = [D] \{\epsilon\} \quad (3.19)$$

where

$$[D] = \frac{E}{1 - \nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \quad (3.20)$$

for plane stress.

The finite element representation of strain for an n -noded element is:

$$\epsilon = \sum_{i=1}^n \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 \\ 0 & \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \end{Bmatrix} \quad (3.21)$$

$$= [B] \{\delta\} \quad (3.22)$$

where, as mentioned earlier, N_i are the trial functions.

The trial functions, N_i , chosen for an element must have certain properties. The aim is to ensure monotonic convergence to the exact solution as the element size decreases. The displacement functions must satisfy several requirements including completeness. For completeness the displacement fields should include proper representation of:

1. any rigid body motion ✓
2. constant strain states

The first condition is satisfied if the shape functions sum to unity at any point in the element. The second condition is needed because if the element size is continuously reduced, then it must be able to attain a state of constant strain.

In addition to the two requirements listed, the displacement functions must be smooth and continuous, and compatibility must be satisfied along the boundaries between adjacent elements. The order of continuity depends on the potential energy functional. For plane stress the highest order derivative in the potential energy functional is of order one, thus only C_0 continuity is required between adjacent elements. This means that only the displacements across the element boundaries must be continuous.

3.4 The Patch Test

As mentioned earlier, every element must be able to model any constant stress state correctly (this includes the rigid body modes). Passing the patch test ensures that the constant stress states are modelled correctly for an arbitrary patch of elements and not just a single element. As noted by *Crisfield* (1986), it is almost universally accepted that finite elements should pass the patch test.

To apply the test, displacement fields consistent with the constant stress states are applied to the nodes along the boundary of a patch of elements. The patch must consist of several elements at least one of which must have no nodes on the patch boundary. For an element to pass the test, the calculated displacements inside the patch must be the same as the imposed displacement field. The stresses should also be consistent with the applied displacement field.

The patch test that we shall use in this thesis is that suggested by *MacNeal and Harder* (1985), (Fig. 3.1).

Figure 3.1: MacNeal and Harder suggested patch test

The prescribed displacements for the planar test are as follows:

$$u = 10^{-3}\left(x - \frac{y}{2} + 0.06\right) \quad (3.23)$$

$$v = 10^{-3}\left(-\frac{x}{2} + y - 0.12\right) \quad (3.24)$$

where u and v are displacements in the x and y directions, respectively. The corresponding stresses are:

$$\sigma_x = \sigma_y = 1333$$

$$\tau_{xy} = 400$$

The displacement fields for the bending test are:

$$w = -10^{-3}(x^2 + x(0.12 - y) + (0.12 - y)^2)/2 \quad (3.25)$$

$$\theta_x = 10^{-3}\left(\frac{x}{2} + (0.12 - y)\right) \quad (3.26)$$

$$\theta_y = -10^{-3}\left(-x - \frac{0.12 - y}{2}\right) \quad (3.27)$$

where w is the deflection normal to the plane of the element, θ_x and θ_y are rotations about the x and y axes, respectively

$$\theta_x = \frac{\partial w}{\partial y} \quad (3.28)$$

$$\theta_y = -\frac{\partial w}{\partial x} \quad (3.29)$$

The corresponding bending moments are:

$$M_x = M_y = 1.111 \times 10^{-7}$$

$$M_{xy} = 0.333 \times 10^{-7}$$

3.5 Irons' Paradox

As mentioned earlier, plane elasticity formulations only require C_0 continuity. This means that plane elements need not have rotations (derivatives of the displacements) as degrees of freedom. *Irons and Ahmad* (1980) carried this statement further by suggesting that not only is compatibility of rotations not required in plane elasticity, but it is impossible for an element to have this degree of freedom without introducing serious errors. They presented a proof to reinforce their arguments. The proof was originally presented for triangular elements, but is applied to quadrilateral elements here.

Figures 3.2a show the element displacements due to rotations Ω , defined as the mean rotation of the two sides. Figure 3.2b shows the element displacements for $u_B = 1$. We notice that the rotational dof Ω is equal to ϕ_i . Since we have introduced only a nodal displacement and not rotation, we must correct for the rotation ϕ_i by making it zero. This correction is shown in Fig. 3.2c, which shows that a stimulus at node B causes a response along sides AD and DC . This is inadmissible in principle because a degree of freedom is allowed to cause displacements only along those boundaries on which it is located.

This proves that the rotation Ω cannot be used as a degree of freedom. This does not, however, mean that only displacements must be used as dof's in plane elements. The proof does not rule out the possibility of using other measures of rotation which may not be the true nodal rotation. This idea is pursued further in Chapter 4.

Figure 3.2: (a) Displacements due to Ω (b) Uncorrected response due to $u_B = 1$ (c) Corrected response

4.1 — General

This chapter presents the detailed derivation of those quadrilateral membrane elements with rotational degrees of freedom which use the basic concept of Allman. The objective of the current presentation is twofold: firstly, to describe the basic concept of Allman and to clarify the ambiguity concerning the nature of the rotational or drilling degree of freedom; secondly, to demonstrate the need for improving the existing elements. These will furnish the rationale for the modifications introduced in this thesis, which will be described in the following chapter.

4.2 Allman's Approach

Earlier attempts aimed at the development of plane elements with rotational dof were unsuccessful because they did not model the constant stress states correctly. *Allman* (1984) proposed a new approach which permitted modelling these states exactly.

The salient features of his approach can be described by reference to the three node triangular element illustrated in Fig. 4.1a. The nodal dof's consist of the usual Cartesian components of displacement, u and v , and a so-called "drilling freedom", ω_i , which will be elaborated upon later. It must, however, be pointed out at the outset that ω is not a rotation in the normal elasticity sense of the word. The first step involves the selection of the normal and tangential components of

Figure 4.1: (a) Allman's triangle (b) Typical side 1 - 2

displacement, u_n and u_t , along a typical element side 1 - 2 of length l_{12} where 1 and 2 are the node numbers at the ends of side 1 - 2, Fig. 4.1b. Assume that u_n varies quadratically and u_t varies linearly as

$$u_n = a_1 + a_2 s + a_3 s^2 \quad (4.1)$$

$$u_t = a_4 + a_5 s \quad (4.2)$$

where s is an axis running from end 1 to 2. The five unknown coefficients a_1 to a_5 can be determined using the boundary values at the nodes, i.e. u_1 , v_1 , ω_1 , u_2 , v_2 and ω_2 . However, there are only five unknowns and six boundary conditions, thus, the problem is overdetermined.

To overcome this difficulty, Allman used the following boundary conditions to define the end displacements

$$u_n|_{s=0} = u_{n1}$$

$$\begin{aligned}
u_n|_{s=l_{12}} &= u_{n2} \\
u_t|_{s=0} &= u_{t1} \\
u_t|_{s=l_{12}} &= u_{t2}
\end{aligned} \tag{4.3}$$

where $u_{n1}, \dots$, etc are calculated in terms of the displacements u and v using

$$u_n = u \sin \theta + v \cos \theta \tag{4.4}$$

$$u_t = u \cos \theta - v \sin \theta \tag{4.5}$$

in which θ is the angle between the x and s -axes. The fifth boundary condition was chosen to be

$$\left. \frac{\partial u_n}{\partial s} \right|_{s=l_{12}} - \left. \frac{\partial u_n}{\partial s} \right|_{s=0} = \omega_2 - \omega_1 \tag{4.6}$$

Note that Eq. (4.6) shows the difference between the drilling freedoms. The five boundary conditions in Eqs. (4.3) and (4.6) were used to determine a_1 to a_5 uniquely. This led to:

$$u_n = \left(1 - \frac{s}{l_{12}}\right) u_{n1} + \left(\frac{s}{l_{12}}\right) u_{n2} + \frac{1}{2}s \left(1 - \frac{s}{l_{12}}\right) (\omega_1 - \omega_2) \tag{4.7}$$

$$u_t = \left(1 - \frac{s}{l_{12}}\right) u_{t1} + \left(\frac{s}{l_{12}}\right) u_{t2} \tag{4.8}$$

The application of the same procedure along sides 2-3 and 3-1 will lead to equations similar to Eqs. (4.7) and (4.8) for the normal and tangential displacements along those sides.

Using Eqs. (4.4) and (4.5) in conjunction with Eqs. (4.7) and (4.8), the displacement field inside the element can be written as

$$\begin{aligned}
u &= u_1 L_1 + u_2 L_2 + u_3 L_3 + \frac{1}{2} l_{12} \sin \theta_{12} (\omega_1 - \omega_2) L_1 L_2 \\
&\quad + \frac{1}{2} l_{23} \sin \theta_{23} (\omega_2 - \omega_3) L_2 L_3 \\
&\quad + \frac{1}{2} l_{31} \sin \theta_{31} (\omega_3 - \omega_1) L_3 L_1
\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}
 v = & v_1 L_1 + v_2 L_2 + v_3 L_3 + \frac{1}{2} l_{12} \cos \theta_{12} (\omega_1 - \omega_2) L_1 L_2 \\
 & + \frac{1}{2} l_{23} \cos \theta_{23} (\omega_2 - \omega_3) L_2 L_3 \\
 & + \frac{1}{2} l_{31} \cos \theta_{31} (\omega_3 - \omega_1) L_3 L_1
 \end{aligned} \quad (4.10)$$

where L_1 , L_2 and L_3 are triangular coordinates given by

$$\begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} = \frac{1}{2\Delta} \begin{bmatrix} (x_2 y_3 - x_3 y_2) & (y_2 - y_3) & (x_3 - x_2) \\ (x_3 y_1 - x_1 y_3) & (y_3 - y_1) & (x_1 - x_3) \\ (x_1 y_2 - x_2 y_1) & (y_1 - y_2) & (x_2 - x_1) \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} \quad (4.11)$$

In the above, $x_1, y_1, \dots$, etc. are the Cartesian coordinates of the nodes, and Δ is the area of the element.

Knowing u and v as given by Eqs. (4.9) and (4.10), one could proceed to determine the element stiffness matrix using the standard procedure of finite elements. Note, however, that in defining the strains, only the derivatives of u and v are involved. Thus, this problem is not in any way related to micropolar elasticity.

4.2.1 The Drilling Freedom ω versus the True Rotation

According to the theory of elasticity, the true rotation, Ω , is given by

$$\Omega = \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \quad (4.12)$$

which in conjunction with Eqs. (4.9) and (4.10) for u and v , and after some manipulation (see *Allman* (1984)), will lead to

$$\Omega = \Omega_0 + \frac{1}{3} \sum_{i=1}^3 (3L_i - 1) \omega_i \quad (4.13)$$

where Ω_0 is the rotation at the centroid of the element and is equal to

$$\Omega_0 = \frac{1}{3} (\Omega_1 + \Omega_2 + \Omega_3) \quad (4.14)$$

where Ω_1, Ω_2 and Ω_3 are the true nodal rotations. If ω_0 is defined as

$$\omega_0 = \frac{1}{3}(\omega_1 + \omega_2 + \omega_3) \quad (4.15)$$

then it can be shown that

$$\Omega_i - \Omega_0 = \frac{3}{4}(\omega_i - \omega_0) \quad (4.16)$$

where i denotes a given node. For example, for node 2

$$\omega_2 = 2(\Omega_2 - \Omega_0) + \frac{1}{2}(\omega_1 + \omega_3) \quad (4.17)$$

It is therefore obvious that the drilling freedom ω is related to the true rotation but the two are not the same.

Physically it can be shown that if Ω were the actual rotation at a given node, then ω would be the relative rotation of the tangent of the deformed shape from the chord joining the end nodes of a side in this configuration, Fig. 4.2

The presentation thus far has shown Allman's basic concept. The triangular element of Allman developed according to this concept performed better than the constant strain triangle but was inferior to the linear strain triangle. Furthermore, the element had one zero energy mode or mechanism which would be activated if

$$\begin{aligned} u_i &= v_i = 0 \\ \omega_i &= \text{constant} \end{aligned} \quad (4.18)$$

where $i = 1, 2, 3$ are the node numbers of element AT. This can be shown by inserting the conditions of Eq. (4.18) into Eqs. (4.9) and (4.10). Although this mechanism can be easily suppressed by prescribing a zero value for ω at some arbitrary point in the mesh, it is nevertheless a nuisance. More importantly, however, the performance of this element is marginal despite the addition of the drilling freedom.

Figure 4.2: Physical meaning of ω

4.3 Coordinate Transformation Approach

Cook (1986) presented an alternative formulation for obtaining Allman's element using a coordinate transformation approach. He demonstrated that the stiffness matrix of the LST element could be transformed to yield the Allman element, Fig. 4.3. This can be summarized, as follows:

Substituting $s = \frac{l_{12}}{2}$ into Eqs. (4.7) and (4.8) yields

$$u_{n4} = \frac{u_{n1}}{2} + \frac{u_{n2}}{2} + \frac{l_{12}}{8}(\omega_1 - \omega_2) \quad (4.19)$$

$$u_{t4} = \frac{u_{t1}}{2} + \frac{u_{t2}}{2} \quad (4.20)$$

The following kinematic relationships may also be written by reference to Eqs. (4.4) and (4.5)

$$\begin{aligned} u &= u_n \sin \theta + u_t \cos \theta \\ v &= u_n \cos \theta - u_t \sin \theta \end{aligned} \quad (4.21)$$

Combining Eqs. (4.4), (4.5), (4.19), (4.20) and (4.21) yields:

$$u_4 = \frac{1}{2}u_1 + \frac{1}{2}u_2 + \frac{l_{12}}{8}(\omega_1 - \omega_2) \sin \theta_{12} \quad (4.22)$$

where node 4 is the midside node between corner nodes 1 and 2, and θ_{12} is the angle between the x -axis and side 1 - 2.

Figure 4.3: Coordinate transformation

Alternatively, since $\sin \theta_{12} = \frac{y_1 - y_2}{l_{12}}$

$$u_4 = \frac{u_1}{2} + \frac{u_2}{2} + \frac{y_1 - y_2}{8}(\omega_1 - \omega_2) \quad (4.23)$$

After writing a similar expression for v_4 and considering the remaining two sides, the following relation is defined between the midside node displacements of LST

and the corner node displacements of AT

$$\begin{Bmatrix} u_4 \\ v_4 \\ u_5 \\ v_5 \\ u_6 \\ v_6 \end{Bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & A_{12} & \frac{1}{2} & 0 & -A_{12} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -B_{12} & 0 & \frac{1}{2} & B_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & A_{23} & \frac{1}{2} & 0 & -A_{23} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & -B_{23} & 0 & \frac{1}{2} & B_{23} \\ \frac{1}{2} & 0 & -A_{31} & 0 & 0 & 0 & \frac{1}{2} & 0 & A_{31} \\ 0 & \frac{1}{2} & B_{31} & 0 & 0 & 0 & 0 & \frac{1}{2} & -B_{31} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ \omega_1 \\ u_2 \\ v_2 \\ \omega_2 \\ u_3 \\ v_3 \\ \omega_3 \end{Bmatrix} \quad (4.24)$$

where $A_{ij} = \frac{y_i - y_j}{8}$, $B_{ij} = \frac{x_i - x_j}{8}$, x_i and y_i are the Cartesian coordinates of node i . Thus, the complete transformation between LST and AT is

$$\{u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4, u_5, v_5, u_6, v_6\}_{LST} = [T] \{u_1, v_1, \omega_1, u_2, v_2, \omega_2, u_3, v_3, \omega_3\}_{AT} \quad (4.25)$$

or

$$\{\delta\}_{LST} = [T] \{\delta\}_{AT} \quad (4.26)$$

where $[T]$ is a 12×9 transformation matrix, $\{\delta\}$ is the displacement vector and the subscripts LST and AT denote the linear strain triangle and Allman triangle, respectively. The upper half of $[T]$ contains mostly 1's and 0's and its bottom half consists of the matrix in Eq. (4.24).

Using the principle of virtual work, the stiffness matrix, $[K]$, of the two elements can be related by:

$$[K]_{AT} = [T]^T [K]_{LST} [T] \quad (4.27)$$

Cook noted that the stresses could be calculated by recovering $\{\delta\}_{LST}$ from Eq. (4.25) after $\{\delta\}_{AT}$ have been calculated. Thus,

$$\{\sigma\} = [D][B]_{LST} \{\delta\}_{LST} \quad (4.28)$$

He also indicated that the procedure can be extended to quadrilateral elements in which QUAD8 can be used as a parent element. He did not, however, present the details of the derivation although he did show a cantilever beam as a test case in which the quadrilateral element showed better performance than the triangular element.

4.4 Element RQUAD4

Razaqpur and Nofal (1987) presented the full derivation for the quadrilateral version of AT and called this element RQUAD4. They decided to pursue this in view of the improved performance of the quadrilateral counterpart of the Allman triangle as reported by Cook. The fundamental difference of their derivation, however, was that they used the beam displacement functions for u_n , and a linear function for u_t . Coincidentally, both the beam equations and Allman's relationships yield the same values at $s = \frac{1}{2}$. Thus the transformation matrix derived by Razaqpur and Nofal is identical to Cook's AQ element. This derivation is briefly summarized. The transformation matrix $[T]$ is derived such that

$$\{\delta\}_{QUAD8} = [T] \{\delta\}_{RQUAD4} \quad (4.29)$$

where $[T]$ is a 16×12 matrix and is formulated using the same procedure as the

$[T]$ matrix for AT presented in the previous section. Explicitly, $[T]$ is given by:

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & A_{12} & \frac{1}{2} & 0 & -A_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -B_{12} & 0 & \frac{1}{2} & B_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & A_{23} & \frac{1}{2} & 0 & -A_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & -B_{23} & 0 & \frac{1}{2} & B_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & A_{34} & \frac{1}{2} & 0 & -A_{34} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & -B_{34} & 0 & \frac{1}{2} & B_{34} \\ \frac{1}{2} & 0 & -A_{41} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & A_{41} \\ 0 & \frac{1}{2} & B_{41} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & -B_{41} \end{bmatrix} \quad (4.30)$$

where A_{ij} and B_{ij} are as defined earlier, and

$$\begin{aligned} \{u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4, u_5, v_5, u_6, v_6, u_7, v_7, u_8, v_8\}_{QUAD8} = \\ [T] \{u_1, v_1, \omega_1, u_2, v_2, \omega_2, u_3, v_3, \omega_3, u_4, v_4, \omega_4\}_{RQUAD4} \end{aligned} \quad (4.31)$$

It should be noted that Eq. (4.29) is a one way transformation, i.e.

$$\{\delta\}_{RQUAD4} \neq [T]^T \{\delta\}_{QUAD8} \quad (4.32)$$

since

$$[T]^T [T] \neq [I] \quad (4.33)$$

where $[I]$ is the identity matrix. It is possible, however, to express $\{\delta\}_{RQUAD4}$ in terms of $\{\delta\}_{QUAD8}$, but it involves the stiffness matrices of both RQUAD4 and QUAD8, making the relationship very unwieldy.

Razaqpur and Nofal made the formulation more efficient by making it unnecessary to explicitly formulate $[K]_{QUAD8}$. Using (3.15), the stiffness matrix for QUAD8 is

$$[K]_{QUAD8} = \int \int [B]^T [D] [B] t |J| d\xi d\eta \quad (4.34)$$

Pre and post-multiplication of Eq. (4.34) by $[T]^T$ and $[T]$, respectively, gives

$$\begin{aligned} [T]^T [K]_{QUAD8} [T] &= \int \int [T]^T [B]^T [D] [B] [T] t |J| d\xi d\eta \\ &= [K]_{RQUAD4} \end{aligned} \quad (4.35)$$

Thus, letting

$$[B]_{RQUAD4} = [B]_{QUAD8} [T] \quad (4.36)$$

leads to

$$[K]_{RQUAD4} = \int \int [B]_{RQUAD4}^T [D] [B]_{RQUAD4} t |J| d\xi d\eta \quad (4.37)$$

Hence, the strain-displacement matrix of the new element may be found using Eq. (4.36) and then $[K]_{RQUAD4}$ can be calculated using Eq. (4.37).

RQUAD4 suffers from the same mechanism as AT, but again this could be suppressed by fixing any one node from rotating in the whole mesh. Many tests were performed by Razaqpur and Nofal, and they found the performance of RQUAD4 to be very good. They highly recommended its use.

In addition to the tests mentioned above, the author found that RQUAD4 also passes the patch test. It must be noted, however, that the boundary conditions had to be modelled carefully since this patch test (presented in Chapter 3) was not created with a rotational dof in mind. The prescribed displacement functions for the patch test, Eqs. (3.23) and (3.24), show that the edges must displace linearly, thus ω must be set to zero along the boundaries in order to satisfy this requirement. All displacements in the interior of the patch were not prescribed, as required by the test.

It can be shown that RQUAD4 is exempt from the so-called Irons' paradox discussed in Section 3.5.

4.5 Modifications to RQUAD4

MacNeal and Harder (1988) independently derived RQUAD4 and found some deficiencies in the element. Thus, they suggested several modifications to remedy these. The deficiencies they referred to are listed below:

1. The presence of the spurious energy mode caused by $u_i = v_i = 0$, ω_i constant ($i = 1 \cdots 4$).
2. The presence of another mechanism, known as the "hourglass mode" present in rectangular elements, caused by reduced, i.e. 2×2 , integration.
3. The presence of higher order strain modes for non-rectangular elements.

Each of these is described below, and their suggested remedy is presented.

4.5.1 Smoothed Strains

An analysis of RQUAD4's internal dof's would show that it has three rigid body modes, three constant stress states and two spurious mechanisms, leaving four modes to be accounted for. They found that these four 'other states' were linear if the element was rectangular, but higher order Jacobian terms caused them to be nonlinear for non-rectangular shapes. They also found that the mechanism due to two point integration disappears for non-rectangular shapes, making the internal dof count uncertain. They suggested 'smoothing' the raw strains of RQUAD4 such that the four 'other' states would be linear for all element shapes. The 'hourglass' mode would also always be spurious, thus removing the degree of freedom count uncertainty. This smoothing had to be done in such a way that satisfaction of the patch test was preserved. This was achieved by means of the following procedure: Let

$$\{\epsilon_s^{(e)}\} = \sum_{j=1}^n S_{sj} \{\epsilon_j\} \quad (4.38)$$

where $\{\epsilon_g^{(s)}\}$ are the smoothed strains at Gauss point g , S_{gj} are smoothing coefficients, $\{\epsilon_j\}$ are the unsmoothed strains and n is the order of integration (4 for two point and 9 for three point integration). Thus,

$$[B_g^{(s)}] = \sum_{j=1}^n S_{gj} [B_j] \quad (4.39)$$

where $[B_g^{(s)}]$ is the smoothed strain-displacement matrix at Gauss point g and $[B_j]$ is the original unsmoothed strain-displacement matrix at Gauss point j . The smoothed Gauss point stresses, $\{\sigma_g^{(s)}\}$, are

$$\{\sigma_g^{(s)}\} = [D][B_g^{(s)}]\{\delta\} \quad (4.40)$$

where $\{\delta\}$ are the nodal displacements. Using Eqs. (4.39) and (4.40), the nodal point forces, $\{F^{(s)}\}$, may be related to the Gauss point stresses as follows:

$$\begin{aligned} \{F^{(s)}\} &= [K^s]\{\delta\} \\ &= \sum_{g=1}^n J_g [B_g^{(s)}]^T [D][B_g^{(s)}]\{\delta\} \\ &= \sum_{g=1}^n J_g \left[\sum_{j=1}^n S_{gj} [B_j]^T \right] \{\sigma_g^{(s)}\} \\ &= \sum_{j=1}^n [B_j]^T \left[\sum_{g=1}^n J_g S_{gj} \{\sigma_g^{(s)}\} \right] \end{aligned} \quad (4.41)$$

similarly the unsmoothed forces are:

$$\begin{aligned} \{F\} &= [K]\{\delta\} \\ &= \sum_{j=1}^n J_j [B_j]^T \{\sigma_j\} \end{aligned} \quad (4.42)$$

where all the symbols are as defined before. The smoothed and unsmoothed nodal forces are required to be equal for all linear and constant stress states i.e. for $\{\sigma\} = \{\sigma^{(s)}\} = 1, x, y$:

$$J_j \{\sigma_j\} = \sum_{g=1}^n J_g S_{gj} \{\sigma_g^{(s)}\} \quad (4.43)$$

The matrix equivalent of Eq. (4.43) is:

$$[J][X] = [S]^T[J][X] \quad (4.44)$$

where $[J]$ is a matrix with the J_g terms on the diagonal and

$$[X] = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ & \vdots & \\ 1 & x_n & y_n \end{bmatrix} \quad (4.45)$$

where x_n and y_n are the coordinates of Gauss point n . The matrix equivalent of constant stress condition is:

$$\{\sigma\} = \{\sigma^{(s)}\} = [X] \quad (4.46)$$

thus,

$$\{\epsilon\} = \{\epsilon^{(s)}\} = [X] \quad (4.47)$$

This leads to:

$$[X] = [S][X] \quad (4.48)$$

where $[S]$ is the smoothing matrix made up of the S_{gg} terms, $[S] = [I]$ is the trivial solution, where $[I]$ is the identity matrix.

Using the method of least squares to minimize the difference between $\{\epsilon^{(s)}\}$ and $\{\epsilon\}$ over the surface of the element:

$$[S] = [X][X^T J X]^{-1}[X]^T[J] \quad (4.49)$$

Eq. (4.49) may be verified by substituting into Eqs. (4.48) and (4.44)

In the formulation of $[B_g^{(s)}]$, a first pass over the Gauss points is required to assemble $[X]$ (Gauss point coordinates) and $[J]$.

4.5.2 Auxiliary Strain Functions

Smoothing the strains ensured that the four 'other' states will always be linear. The representation of all linear states is still incomplete since there is a total of six possible linear stress states. The consequence of this is that in the linear bending state $\epsilon_z = y$, the required $\epsilon_y = -\nu y$ is not represented. MacNeal and Harder called this a Poisson ratio locking problem.

They resolved this problem by invoking stress equilibrium conditions (Fig. 4.4) over the element:

$$\begin{aligned}\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} &= 0 \\ \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} &= 0\end{aligned}\tag{4.50}$$

These conditions have the effect of reducing the number of independent linear stress states to four, which the element is able to model. They used auxiliary strain-displacement terms, suggested by *Hughes* (1980), to satisfy the stress equilibrium conditions.

The latter procedure involves

$$\{\epsilon_g\} = [B_g^{(s)}] \{\delta\} + [B_{go}] \{\delta_o\}\tag{4.51}$$

where $\{\delta\}$ are the nodal displacements, $[B_{go}]$ are the auxiliary strain function terms and $\{\delta_o\}$ are auxiliary displacements solved by minimizing the strain energy, U . For linear materials

$$U = \frac{1}{2} \sum_{g=1}^n J_g \{\epsilon_g\}^T [D] \{\epsilon_g\}\tag{4.52}$$

Setting $\frac{\partial U}{\partial \{\delta_o\}} = 0$,

$$0 = \sum_g J_g [B_{go}]^T \{\sigma_g\}\tag{4.53}$$

Thus, the final strain-displacement relationship is

$$\{\epsilon_g\} = [B_g] \{\delta\}\tag{4.54}$$

Figure 4.4: Stress equilibrium conditions

where

$$[B_g] = [B_g^{(s)}] - [B_{go}][K_{oo}]^{-1}[K_o] \quad (4.55)$$

$$[K_{oo}] = \sum_{g=1}^n J_g [B_{go}]^T [D] [B_{go}] \quad (4.56)$$

and

$$[K_o] = \sum_{g=1}^n J_g [B_{go}]^T [D] [B_g^{(s)}] \quad (4.57)$$

where $[B_{go}]$ is determined such that it satisfies the stress equilibrium conditions and the patch test.

Satisfaction of the patch test means that $[B_{go}]$ contribute nothing to the average strains of the element, i.e. the nodal forces due to the constant stress states are zero. This means that

$$\sum_{g=1}^n J_g [B_{go}] = 0 \quad (4.58)$$

It can be seen that Eq. (4.58) is satisfied if:

$$[B_{go}] = \begin{bmatrix} X_g & 0 \\ 0 & Y_g \\ Y_g & X_g \end{bmatrix} \quad (4.59)$$

where the particular functions of X_g and Y_g which satisfy all the requirements are

$$X_g = \frac{I_{yy}x_g - I_{xy}y_g}{I_{xx}I_{yy} - I_{xy}^2} \quad (4.60)$$

$$Y_g = \frac{I_{xx}y_g - I_{xy}x_g}{I_{xx}I_{yy} - I_{xy}^2} \quad (4.61)$$

and

$$I_{xx} = \sum_{g=1}^n J_g x_g^2 \quad (4.62)$$

$$I_{yy} = \sum_{g=1}^n J_g y_g^2 \quad (4.63)$$

$$I_{xy} = \sum_{g=1}^n J_g x_g y_g \quad (4.64)$$

assuming the centroid of the element is to be the origin of the coordinate system. Refer to *MacNeal and Harder* (1988) for a more detailed derivation.

The moments of inertia (I_{xx} , I_{yy} and I_{xy}) are assembled along with $[X]$ and $[J]$ during the first pass over the Gauss points. The final strain-displacement matrix $[B_g]$ is then determined using Eq. (4.54). Thus, the stiffness matrix of Q4S is:

$$[K]_{Q4S} = \int \int [B_g]^T [D] [B_g] t |J| d\xi d\eta \quad (4.65)$$

These modifications were implemented in this thesis and incorporated into the computer program. Hence, Q4S is in the element library of the program.

4.5.3 Some Remarks on Q4S

The element Q4S is a modified version of RQAD4. MacNeal and Harder noted that every modification they introduced ensured that its ability to pass the patch test remained intact.

When Q4S was incorporated into the computer program in this thesis, every effort was made to verify its correct implementation. Several planar tests presented by MacNeal and Harder in their paper were successfully duplicated. The writer has not been successful, however, to make Q4S pass the patch test. *Harder* (1988) stated that it indeed passes the patch test and it was probably incorrectly implemented in the present work. To further check the implementation, the stiffness matrix of a single Q4S element of arbitrary shape was calculated manually and the calculations are shown in Appendix A. Fortunately, the computer results were identical to those obtained by manual calculations. Although this discrepancy has still not been resolved, it will be assumed that Q4S passes the patch test.

4.5.4 Penalty Functions for Controlling the Spurious Energy Modes

MacNeal and Harder (1988) employed penalty functions to control the two spurious

modes of element Q4S. The penalty function corresponding to the 'hourglass' mode is not included here since this mechanism does not exist if three point integration is used. The second penalty function was introduced to avert the other spurious mode. It involves a procedure in which the corner rotations are 'tied' to the rotation at the centroid. Specifically, let

$$\Theta = \frac{1}{4} \sum_{i=1}^4 (\theta_i - \theta_0) \quad (4.66)$$

where θ_0 is the rotation at the centroid

$$\theta_0 = \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)_0 \quad (4.67)$$

The energy due to the penalty function is:

$$P = VG\gamma\Theta^2 \quad (4.68)$$

where V is the element volume, G is the shear modulus and γ is a dimensionless constant. They noted that Θ remains zero for the rigid and constant strain states. The increment to the stiffness matrix caused by P is:

$$[K_{ij}] = \frac{\partial^2 P}{\partial u_i \partial u_j} \quad (4.69)$$

The constant γ is left as a variable for the user to specify. They recommended that $\gamma = 10^{-6}$ is suitable for most problems, except for 'nonlinear almost flat shell problems' where they recommended a higher value. They also found a decrease in accuracy if the penalty functions were used, i.e. a 3 percent decrease in accuracy for the tip deflection of a cantilever beam for the regular mesh and a 9 percent decrease for the irregular mesh (see Section 7.2.2). Deciding on a value for γ for each problem may be as undesirable as using the element with the mechanism. Due to the ambiguity of defining γ and the decrease in accuracy caused by it, penalty functions are not used in this study. Note, however, that they did mention that the penalty functions did not affect the ability of Q4S to pass the patch test.

In Chapter 7, a series of tests are presented in which the performance of this element is compared to other elements of the same family.

4.6 Summary

This chapter has presented the formulation for quadrilateral elements with rotational degrees of freedom. It has also been shown - by clarifying the meaning of the rotational dof - that these elements do not fall prey to what we call the Irons' paradox. This fact and their good performance in other tests, including the patch test, have shown RQUAD4 and Q4S to be competitive elements. In particular, the modifications introduced by *MacNeal and Harder* (1988) have made Q4S invariant to axis orientation. They have also eliminated the so-called Poisson ratio locking problem. Unfortunately, the modifications have made the straightforward original formulation of the element more complex.

The main problem, yet to be resolved, is the spurious mechanism. It was shown that the penalty function approach may not be the best answer. Alternative simple approaches for improving RQUAD4's accuracy, as well as eliminating its mechanisms, have been adopted in this study, and are presented in the next chapter.

5 Refining and Improving RQUAD4

5.1 General

This chapter covers three principle topics. First, the implementation of the element RQUAD4 is further simplified and facilitated by deriving its shape functions and consistent load vector in explicit form. Secondly, several attempts at improving the accuracy of RQUAD4 are presented. These include modifications - such as reduced integration for shear, substitute shape functions and bubble functions - which have been used in the past to improve the performance of many elements. The third part deals with the fundamental problem remaining in both RQUAD4 and Q4S; namely, the spurious mechanism. The problem is to eliminate the mechanism, while still satisfying the patch test. The attempt to eliminate the mechanism led to the derivation of a modified RQUAD4 element which will be called RQD4. This element behaves like RQUAD4 in every way, but has no mechanism.

It must be mentioned that the attempts at improving the behaviour of RQUAD4 presented in this chapter precede the publication of MacNeal and Harder's work presented in the previous chapter.

5.2 Explicit Shape Functions

Calculating the explicit shape functions for RQUAD4 has several benefits. Many programs introduce new elements requiring only the shape functions and nodal dof's of new elements. It also assists in the identification of the order of polynomials involved in the displacement functions. Finally, the derivation of the consistent

load vector is much easier with the availability of explicit shape functions. To derive these functions, one may proceed as follows.

Referring to Eq. (4.36)

$$[B]_{RQUAD4} = [B]_{QUAD8}[T]$$

$$= \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & N_6 & 0 & N_7 & 0 & N_8 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & N_6 & 0 & N_7 & 0 & N_8 \end{bmatrix} [T] \quad (5.1)$$

where $N_1 \dots N_8$ are the shape functions of QUAD8. Since

$$[B]_{RQUAD4} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} [N]_{RQUAD4} \quad (5.2)$$

it can be concluded from Eqs. (5.1) and (5.2) that

$$[N]_{RQUAD4} = [N^*] = [N]_{QUAD8}[T] \quad (5.3)$$

Thus, the shape functions of RQUAD4, $[N^*]$, can be written in explicit form:

$$\left. \begin{aligned} N_1^* &= (1 - \xi - \eta + \xi\eta)/4 \\ N_2^* &= (1 + \xi - \eta - \xi\eta)/4 \\ N_3^* &= (1 + \xi + \eta + \xi\eta)/4 \\ N_4^* &= (1 - \xi + \eta - \xi\eta)/4 \end{aligned} \right\} [N]_{QUAD4}$$

$$\begin{aligned} N_{5u}^* &= \frac{(y_1 - y_2)}{16}(1 - \eta - \xi^2 + \xi^2\eta) - \frac{(y_4 - y_1)}{16}(1 - \xi - \eta^2 + \xi\eta^2) \\ N_{5v}^* &= -\frac{(x_1 - x_2)}{16}(1 - \eta - \xi^2 + \xi^2\eta) + \frac{(x_4 - x_1)}{16}(1 - \xi - \eta^2 + \xi\eta^2) \\ N_{6u}^* &= -\frac{(y_1 - y_2)}{16}(1 - \eta - \xi^2 + \xi^2\eta) + \frac{(y_2 - y_3)}{16}(1 + \xi - \eta^2 - \xi\eta^2) \\ N_{6v}^* &= \frac{(x_1 - x_2)}{16}(1 - \eta - \xi^2 + \xi^2\eta) - \frac{(x_2 - x_3)}{16}(1 + \xi - \eta^2 - \xi\eta^2) \end{aligned} \quad (5.4)$$

$$\begin{aligned}
N_{7u}^* &= -\frac{(y_2 - y_3)}{16}(1 + \xi - \eta^2 - \xi\eta^2) + \frac{(y_3 - y_4)}{16}(1 + \eta - \xi^2 - \xi^2\eta) \\
N_{7v}^* &= \frac{(x_2 - x_3)}{16}(1 + \xi - \eta^2 - \xi\eta^2) - \frac{(x_3 - x_4)}{16}(1 + \eta - \xi^2 - \xi^2\eta) \\
N_{8u}^* &= -\frac{(y_3 - y_4)}{16}(1 + \eta - \xi^2 - \xi^2\eta) + \frac{(y_4 - y_1)}{16}(1 - \xi - \eta^2 + \xi\eta^2) \\
N_{8v}^* &= \frac{(x_3 - x_4)}{16}(1 + \eta - \xi^2 - \xi^2\eta) - \frac{(x_4 - x_1)}{16}(1 - \xi - \eta^2 + \xi\eta^2)
\end{aligned}$$

Note that the subscript u (v) implies that the particular shape function is for the displacement u (v), where:

$$u(\xi, \eta) = N_1^*u_1 + N_2^*u_2 + N_3^*u_3 + N_4^*u_4 + N_{5u}^*\omega_1 + N_{6u}^*\omega_2 + N_{7u}^*\omega_3 + N_{8u}^*\omega_4 \quad (5.5)$$

$$v(\xi, \eta) = N_1^*v_1 + N_2^*v_2 + N_3^*v_3 + N_4^*v_4 + N_{6v}^*\omega_1 + N_{7v}^*\omega_2 + N_{8v}^*\omega_3 + N_{5v}^*\omega_4 \quad (5.6)$$

It may be recalled that ξ and η are natural coordinates varying between -1 and +1.

It can be seen from Eqs. (5.5) and (5.6) that RQUAD4 becomes the four noded isoparametric element QUAD4 if all rotations ω_i are set equal to zero. Plotting these shape functions (Fig. 5.1) further demonstrates that the rotational dof, ω , represents the rotation of the tangent to the deformed side from the chord, and not the total rotation of the corners. It may also be seen from Eqs. (5.5) and (5.6) that RQUAD4 is like a hierarchical element because it starts with the displacement functions of QUAD4 and adds some quadratic terms to them.

5.3 Consistent Load Vector

Consider a uniform load acting along edge 1-2, Fig. 5.2. Using the principle of virtual work:

$$P_i = \int_{s=0}^{l_{12}} N_i p_0 ds \quad (i = 1, 4) \quad (5.7)$$

where the integration is carried over the loaded edge, l_{12} . P_1 and P_2 would be the same as the nodal loads obtained for QUAD4 since $N_1^* \cdots N_4^* = [N]_{QUAD4}$, thus

Figure 5.1: Shape functions of RQUAD4

Figure 5.2: Element with uniform load acting on edge 1 – 2

$$P_1 = P_2 = \frac{p_0 l_{12}}{2}.$$

Considering the nodal moments:

$$M_1 = \int_{s=0}^{l_{12}} N_{\xi v}^* p_0 ds \quad (5.8)$$

where $N_{\xi v}^*$ is the shape function corresponding to ω_1 . p_0 acts only along edge 1 – 2, thus

$$M_1 = \int_{s=0}^{l_{12}} \frac{x_2 - x_1}{16} (1 - \eta - \xi^2 + \eta^2 \xi) p_0 ds \quad (5.9)$$

Noting that η is constant along edge 1 – 2 ($\eta = -1$) and substituting $ds = \frac{l_{12}}{2} d\xi$ leads to

$$M_1 = p_0 \int_{-1}^1 \frac{l_{12}^2}{16} (1 - \xi^2) d\xi \quad (5.10)$$

where $l_{12} = x_2 - x_1$ in this case. Carrying out the integration,

$$M_1 = \frac{p_0 l_{12}^2}{12} \quad (5.11)$$

The above nodal moment is equal to the corresponding fixed end moment due to a uniform load acting on a beam. The same procedure may be performed for a point and varying load as illustrated in Fig. 5.3. It may be seen that the nodal

Figure 5.3: Consistent nodal loads of RQUAD4 corresponding to common types of loading.

end moments for a varying load are:

$$M_1 = -M_2 = \frac{p_A l_{12}^2}{24} + \frac{p_B l_{12}^2}{24} \quad (5.12)$$

The latter end moments may not seem intuitively correct because they are equal and opposite. The reason for this is that applying $\omega_1 = 1$ causes a parabolic displacement variation normal to side 1 – 2, unlike a beam where the displacement varies as a cubic function.

5.4 Strategies for Improving the Accuracy of RQUAD4

Hinton and Campbell (1974), and *Zienkiewicz* (1977) have shown that QUAD8 suffers from spurious shear stresses in certain applications. One example of this is the uniformly loaded cantilever of Fig. 5.4. Two remedies for this problem in the past have been the application of reduced integration for shear (*Hinton and Campbell* 1974) and substitute shape functions (*Chinniah* 1985). Since QUAD8 is the parent element for RQUAD4, it was felt that these remedies may also improve the behaviour of RQUAD4. Thus, their implementation was carried out as discussed below.

5.4.1 Reduced Integration

An attempt at improving the shear behaviour of QUAD8 was made by using reduced integration for shear. In other words, the shear portion of the stiffness matrix was calculated using one point quadrature, while the normal strains were integrated using three point integration.

Test results after this modification were extremely unsatisfactory (rotations were of the order of 10^{10} !). This indicated that the centroid alone is not a good sampling point for shear.

Figure 5.4: Spurious shear exhibited by QUAD8 in uniformly loaded cantilever

5.4.2 Substitute Shape Functions ($\bar{N}_i$)

A detailed derivation of substitute shape functions is presented by *Chinniah* (1985). Therefore, they are briefly reviewed here.

In the formulation of QUAD8, the following displacement function is assumed:

$$u(\xi, \eta) = a_1 + a_2\xi + a_3\eta + a_4\xi^2 + a_5\xi\eta + a_6\eta^2 + a_7\xi^2\eta + a_8\xi\eta^2 \quad (5.13)$$

This is an incomplete cubic function, and it has been stated that the erratic shear behaviour of this element is due to the presence of terms higher than the quadratic terms. To eliminate these higher order terms, a smoothing function, $\bar{u}$, is defined as a complete quadratic:

$$\bar{u}(\xi, \eta) = b_1 + b_2\xi + b_3\eta + b_4\xi^2 + b_5\xi\eta + b_6\eta^2 \quad (5.14)$$

where the coefficients b_i are determined as functions of a_i using the standard least square procedure:

$$\kappa = \int \int (u - \bar{u})^2 d\xi d\eta \quad (5.15)$$

κ is minimized, thus

$$\frac{\partial \kappa}{\partial b_i} = 0 \quad i = 1 \dots 6 \quad (5.16)$$

The smoothed shape functions resulting from the minimization process are:

$$\begin{aligned} \bar{N}_1 &= -\frac{1}{12}(\xi + \eta) + \frac{1}{4}(-1 + \xi^2 + \xi\eta + \eta^2) \\ \bar{N}_2 &= -\frac{1}{3}\eta + \frac{1}{2}(1 - \xi^2) \\ \bar{N}_3 &= \frac{1}{12}(\xi - \eta) + \frac{1}{4}(-1 + \xi^2 - \xi\eta + \eta^2) \\ \bar{N}_4 &= \frac{1}{3}\xi + \frac{1}{2}(1 - \eta^2) \\ \bar{N}_5 &= \frac{1}{12}(\xi + \eta) + \frac{1}{4}(-1 + \xi^2 + \xi\eta + \eta^2) \end{aligned} \quad (5.17)$$

$$\bar{N}_6 = \frac{1}{3}\eta + \frac{1}{2}(1 - \xi^2)$$

$$\bar{N}_7 = \frac{1}{12}(-\xi + \eta) + \frac{1}{4}(-1 + \xi^2 - \xi\eta + \eta^2)$$

$$\bar{N}_8 = -\frac{1}{3}\xi + \frac{1}{2}(1 - \eta^2)$$

These shape functions are only applied for calculating the shear strain terms:

$$\{\epsilon\} = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix} = [B] \{\delta\} \quad (5.18)$$

It was found that the RQUAD4 element derived from this modified QUAD8 behaved virtually identically to the original RQUAD4 element. Therefore, substitute shape functions were not incorporated into the formulation of RQUAD4.

5.4.3 Bubble Functions

Bubble or nodeless functions are special functions that have zero value on the boundary of an element but are not zero inside the element. In his formulation of the AT element, *Cook* (1986) used a bubble function to improve its accuracy. He found that the introduction of this function gave an increase in accuracy of at best 2%.

To investigate whether bubble functions will also improve the accuracy of RQUAD4, a bubble function was added in this study to QUAD8 before its stiffness matrix was transformed to that of RQUAD4. Briefly, $[K]_{QUAD9}$ was formulated using the bubble function

$$N_9 = (1 - \xi^2)(1 - \eta^2) \quad (5.19)$$

thus,

$$[K]_{QUAD9} \begin{Bmatrix} \{\delta_i\} \\ \{\delta_a\} \end{Bmatrix} = \begin{Bmatrix} \{F_i\} \\ \{F_a\} \end{Bmatrix} \quad (5.20)$$

where $\{\delta_a\}$ are the (u and v) displacements of the internal node (at the centroid) and $\{\delta_i\}$ are the usual nodal displacements of QUAD8. If the internal node is not loaded ($\{F_a\} = 0$), then;

$$[K]_{QUAD8'} = [K]_{QUAD8} - [K_{ia}][K_{aa}]^{-1}[K_{ai}] \quad (5.21)$$

where

$$[K_{ia}] = [K_{ai}]^T = \int \int [B]_{QUAD8}^T [D] [B_a] dvol \quad (5.22)$$

and

$$[K_{aa}] = \int \int [B_a]^T [D] [B_a] dvol \quad (5.23)$$

A condensation algorithm (Cook 1981) was used to reduce the stiffness matrix $[K]_{QUAD8}$ to $[K]_{QUAD8'}$. The stiffness matrix of the modified RQUAD4, called RQUAD4', is then given by:

$$[K]_{RQUAD4'} = [T]^T [K]_{QUAD8'} [T] \quad (5.24)$$

Unfortunately, no great accuracy was noted as a result of the introduction of the bubble functions. It was therefore decided not to retain this modification in the computer program.

5.5 The Mechanism

The remaining deficiency of both RQUAD4 and Q4S is their spurious mechanism. Although it may be suppressed if any node in the mesh is restricted from rotating, this is a nuisance. Moreover, it is undesirable to apply more boundary conditions than the problem physically requires.

MacNeal and Harder (1988) have suggested the use of penalty functions, but this is not particularly appealing because the user must intervene and specify a value for them, which may change from one problem to another.

An element, called RQD4, is derived in this study which has no mechanism and also passes the patch test. It is shown that this new element does not fail a test which RQUAD4 and Q4S do due to their mechanisms. Before this element is derived, however, the significance and ramifications of the presence of a mechanism are further studied.

5.5.1 Why the Mechanism Exists?

Partitioning the strain-displacement matrix of RQUAD4 in the manner shown below will show when and why the mechanism will occur.

$$[B]_{RQUAD4} = \begin{Bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{Bmatrix} \begin{bmatrix} N_1^* & 0 & N_{8u}^* \\ & \dots & \\ 0 & N_1^* & N_{8v}^* \end{bmatrix} \quad (5.25)$$

$$= \begin{bmatrix} [B]_{QUAD4} & [B]_{\theta} \end{bmatrix} \quad (5.26)$$

where $[B]_{\theta}$ and $[B]_{QUAD4}$ are the strain-displacement matrices corresponding to the rotational dof's, and the inplane displacements, respectively.

Accordingly,

$$\begin{aligned} [K]_{RQUAD4} &= \int \int [B]_{RQUAD4}^T [D] [B]_{RQUAD4} dvol \\ &= \int \int \begin{bmatrix} [B]_{QUAD4}^T \\ [B]_{\theta}^T \end{bmatrix} [D] \begin{bmatrix} [B]_{QUAD4} & [B]_{\theta} \end{bmatrix} dvol \\ &= \int \int \begin{bmatrix} [B]_{QUAD4}^T [D] [B]_{QUAD4} & [B]_{QUAD4}^T [D] [B]_{\theta} \\ [B]_{\theta}^T [D] [B]_{QUAD4} & [B]_{\theta}^T [D] [B]_{\theta} \end{bmatrix} \\ &= \begin{bmatrix} [K]_{QUAD4} & [K]_{QUAD4-\theta} \\ [K]_{\theta-QUAD4} & [K]_{\theta-\theta} \end{bmatrix} \quad (5.27) \end{aligned}$$

Each submatrix in Eq. (5.27) displays the characteristics of an independent stiffness matrix, i.e. the columns in each represent the reactions at all nodes due to a unit displacement corresponding to that column. For example, the first column of $[K]_{QUAD4}$ represents all the forces due to a unit displacement in the direction of dof one, while the first column of $[K]_{\theta-QUAD4}$ shows the moments at all nodes due to the same displacement. These forces and moments are in equilibrium independently. This is obvious because $[K]_{QUAD4}$ is the stiffness matrix of QUAD4 whose columns always add up to zero. For this reason, RQUAD4 fails in some cases. For example, in Figure 5.5a, the sum of the applied moments do not add to zero ($M_1 = 0, M_2 = 0, M_3 = M, M_4 = 0$); and the element fails in this test. This may be remedied by fixing any node from rotating thus allowing it to resist the applied moment. Failure occurs in the loading case of Fig. 5.5b because the applied forces are not in total equilibrium independently from the applied moments.

In summary, it can be stated that there will be no mechanism in a finite element mesh provided three displacement and one rotational restraints are applied to the body. The displacement restraints would be identical to those needed in normal plane elasticity problems for preventing rigid body motions. The rotational restraint can be satisfied by fixing one rotational dof anywhere in the mesh. However, the latter restraint would only be necessary if there is an external moment applied to the body being analyzed. Despite this easy method of suppressing the mechanism, as stated earlier, an element with a mechanism is not appealing to the user who is not familiar with it. Hence an attempt will be made to completely eliminate it. The objective is to rid the element of the mechanism, while still satisfying the patch test and retaining the element's accuracy.

5.5.2 A Strategem for Eliminating the Mechanism

Figure 5.6a shows the deformations corresponding to the zero energy mode ($u_i = v_i = 0, \omega_i = \text{constant}$). It can be seen that the whole element undergoes no dis-

(a)

(b)

Figure 5.5: Two cases where RQUAD4 and Q4S fail

placement since this is the only way that a quadratic variation of u_n can be satisfied under the foregoing nodal displacements. It can also be shown, however, that even an element with cubic displacement functions, Fig. 5.6b, would have the mechanism if the nodes are located at the corners and midpoints of the element.

If the deformation mode shown in Fig. 5.6b is related to the nodal displacements of QUAD8, we will conclude that all the nodal displacements are equal to zero and no strain energy is generated. This zero energy condition can be written as:

$$U = \frac{1}{2} \{\delta\}_{QUAD8}^T [K] \{\delta\}_{QUAD8} = 0 \quad (5.28)$$

where U is the strain energy, $[K]$ is the element stiffness matrix. Knowing that

$$\{\delta\}_{QUAD8} = [T] \{\delta\}_{RQUAD4} \quad (5.29)$$

where $[T]$ is the transformation matrix,

$$U = \frac{1}{2} \{\delta\}_{RQUAD4}^T [T]^T [K] [T] \{\delta\}_{RQUAD4} = 0 \quad (5.30)$$

since $\{\delta\}_{RQUAD4} \neq \{0\}$, it implies

$$\det[K]_{RQUAD4} = 0 \quad (5.31)$$

which proves the presence of a mechanism.

In Fig. 5.6b we notice that the element undergoes displacements everywhere except at the nodes and midside points. Thus it may be appropriate to relate the nodal rotations of RQUAD4 to the displacements normal to the element side, u_n , at points other than midside points. For example, we could choose QUAD12, Fig. 5.7, as the parent element for which u_n varies as a complete cubic. The element would not have a mechanism since the nodes located at the $\frac{1}{3}$ and $\frac{2}{3}$ points of the element sides undergo displacements for the previously spurious mode.

The choice of QUAD12 also renders the rotations ω_i to be the actual nodal rotations in the elasticity sense. Pursuing this idea, a 16×12 transformation

Figure 5.6: Displacement mode corresponding to spurious mechanism
 (a) Quadratic displacement functions (b) Cubic displacement functions

Figure 5.7: Element QUAD12

matrix, $[T']$, can be derived which relates the nodal dof of QUAD12 to those of RQUAD4. For the sake of completeness, $[T']$ is given below:

$$[T'] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ [T'_{a12}] & [T'_{b12}] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & [T'_{a23}] & [T'_{b23}] & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & [T'_{a34}] & [T'_{b34}] & 0 & 0 & 0 & 0 \\ [T'_{b41}] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & [T'_{a41}] & 0 & 0 & 0 \end{bmatrix} \quad (5.32)$$

where

$$[T'_{a,ij}] = \begin{bmatrix} \mu A_{ij}^2 + 9\mu & -\mu A_{ij} B_{ij} & 2\mu A_{ij} \\ -\mu A_{ij} B_{ij} & \mu B_{ij}^2 + 9\mu & -2\mu B_{ij} \\ \mu B_{ij}^2 + \frac{7}{2}\mu & \mu A_{ij} B_{ij} & \mu A_{ij} \\ \mu A_{ij} B_{ij} & \mu A_{ij}^2 + \frac{7}{2}\mu & -\mu B_{ij} \end{bmatrix}$$

and

$$[T'_{b,ij}] = \begin{bmatrix} \mu B_{ij}^2 + \frac{7}{2}\mu & \mu A_{ij} B_{ij} & -\mu A_{ij} \\ \mu A_{ij} B_{ij} & \mu A_{ij}^2 + \frac{7}{2}\mu & \mu B_{ij} \\ \mu A_{ij}^2 + 9\mu & -\mu A_{ij} B_{ij} & -2\mu A_{ij} \\ -\mu A_{ij} B_{ij} & \mu B_{ij}^2 + 9\mu & 2\mu B_{ij} \end{bmatrix}$$

where $\mu = \frac{2}{27}$, A_{ij} and B_{ij} are as defined before.

The element based on this transformation has no mechanism. Unfortunately, it fails the patch test and is plagued by Irons' paradox. On the other hand, it

models beam behaviour very well. For example, a cantilever beam loaded by a point load at its tip can be modelled accurately using a single element.

Based on the above observations, we are faced with the following dichotomy: for plane elasticity elements with rotational degrees of freedom if u_n is cubic, the element will be free of a mechanism but will fail the patch test; if u_n is quadratic, it passes the patch test but has a mechanism. Can there be a happy medium? We shall pursue this in the next section.

5.6 Formulation of a Mechanism Free RQUAD4

Consider the following normal displacement function, u_n , for a typical side 1 – 2, Fig. 5.8

$$u_n|_{s=\frac{l_{12}}{3}} = \left(\frac{20}{27} - \frac{\lambda}{27}\right) u_{n1} + \left(\frac{7}{27} + \frac{\lambda}{27}\right) u_{n2} + \left(\frac{4}{27} - \frac{\lambda}{54}\right) l_{12}\omega_1 - \left(\frac{2}{27} + \frac{\lambda}{54}\right) l_{12}\omega_2 \quad (5.33)$$

and

$$u_n|_{s=\frac{2l_{12}}{3}} = \left(\frac{7}{27} + \frac{\lambda}{27}\right) u_{n1} + \left(\frac{20}{27} - \frac{\lambda}{27}\right) u_{n2} + \left(\frac{2}{27} + \frac{\lambda}{54}\right) l_{12}\omega_1 - \left(\frac{4}{27} - \frac{\lambda}{54}\right) l_{12}\omega_2 \quad (5.34)$$

where λ is a variable such that if $\lambda = 2$, the displacement functions of RQUAD4 are obtained, and if $\lambda = 0$, then the displacement functions of the element discussed in the preceding section are obtained. Therefore, the displacement functions in Eqs. (5.33) and (5.34) cover the two extremes. The tangential functions, u_t , are still assumed to be linear:

$$u_t|_{s=\frac{l_{12}}{3}} = \frac{2}{3}u_{t1} + \frac{1}{3}u_{t2} \quad (5.35)$$

and

$$u_t|_{s=\frac{2l_{12}}{3}} = \frac{1}{3}u_{t1} + \frac{2}{3}u_{t2} \quad (5.36)$$

We will vary λ between 0 and 2, and will subject the corresponding element for each

Figure 5.8: New u_n displacement functions

value to the patch test. Table 5.1 shows various values of λ and the corresponding maximum error of both the displacements and stresses in the patch. It may be

Table 5.1: λ and the corresponding maximum error in the patch test

λ	Max. Error (%)	
	Displacement	Stress
1.5	5.1	7.8
1.6	4.1	6.2
1.7	3.0	4.6
1.8	2.0	3.0
1.85	1.5	2.3
1.9	1.0	1.5
1.95	0.5	0.8
1.96	0.4	0.5
1.97	0.3	0.4
1.98	0.2	0.4
1.99	0.1	0.1

observed that the element corresponding to a value of $\lambda = 1.95$ satisfies the patch test with over 99% accuracy. Thus, $\lambda = 1.95$ will be accepted as a suitable value. The corresponding normal displacement functions are:

$$\begin{aligned}
 u_n|_{s=\frac{l_{12}}{2}} &= \frac{361}{540}u_{n1} + \frac{179}{540}u_{n2} + \frac{121}{1080}l_{12}\omega_1 - \frac{119}{1080}l_{12}\omega_2 \\
 u_n|_{s=\frac{2l_{12}}{3}} &= \frac{179}{540}u_{n1} + \frac{361}{540}u_{n2} + \frac{119}{1080}l_{12}\omega_1 - \frac{121}{1080}l_{12}\omega_2
 \end{aligned} \tag{5.37}$$

The relevant transformation matrix, $[T]$, is:

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ [T_{a12}] & [T_{b12}] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & [T_{a23}] & [T_{b23}] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & [T_{a34}] & [T_{b34}] & 0 & 0 & 0 & 0 & 0 \\ [T_{b41}] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & [T_{a41}] & 0 & 0 & 0 & 0 \end{bmatrix} \quad (5.38)$$

in which

$$[T_{aij}] = \begin{bmatrix} \frac{2}{3} + \frac{A_{ij}^2}{540} & \frac{-A_{ij}B_{ij}}{540} & \frac{121A_{ij}}{1080} \\ \frac{-A_{ij}B_{ij}}{540} & \frac{2}{3} + \frac{B_{ij}^2}{540} & \frac{-121B_{ij}}{1080} \\ \frac{1}{3} - \frac{A_{ij}^2}{540} & \frac{A_{ij}B_{ij}}{540} & \frac{119A_{ij}}{1080} \\ \frac{A_{ij}B_{ij}}{540} & \frac{1}{3} - \frac{B_{ij}^2}{540} & \frac{-119B_{ij}}{1080} \end{bmatrix}$$

and

$$[T_{bij}] = \begin{bmatrix} \frac{1}{3} - \frac{A_{ij}^2}{540} & \frac{A_{ij}B_{ij}}{540} & \frac{-119A_{ij}}{1080} \\ \frac{A_{ij}B_{ij}}{540} & \frac{1}{3} - \frac{B_{ij}^2}{540} & \frac{119B_{ij}}{1080} \\ \frac{2}{3} + \frac{A_{ij}^2}{540} & \frac{-A_{ij}B_{ij}}{540} & \frac{-121A_{ij}}{1080} \\ \frac{-A_{ij}B_{ij}}{540} & \frac{2}{3} + \frac{B_{ij}^2}{540} & \frac{121B_{ij}}{1080} \end{bmatrix}$$

where as defined earlier, $A_{ij} = \frac{v_i - v_j}{l_{ij}}$, $B_{ij} = \frac{z_i - z_j}{l_{ij}}$ and the $[T]$ matrix relates $\{\delta\}_{QUAD12}$ to $\{\delta\}_{RQD4}$ in the following manner:

$$\begin{aligned} \{u_1, v_1, u_2, v_2, u_3, v_3, \dots, v_{12}\}_{QUAD12} = \\ [T] \{u_1, v_1, \omega_1, u_2, \omega_2, \dots, \omega_4\}_{RQD4} \end{aligned} \quad (5.39)$$

The stiffness matrix of the new element, called RQD4, is formulated using the same procedure as the one used to determine the stiffness matrix of RQUAD4. The transformation matrix of Eq. (4.30) is substituted by that in Eq. (5.38), and $[B]_{QUAD8}$ is replaced by $[B]_{QUAD12}$ such that:

$$[B]_{RQD4} = [T][B]_{QUAD12} \quad (5.40)$$

The modification introduced here is somewhat similar to the penalty function approach discussed in Section 4.5.4, except there is no decrease in the accuracy of the element and there is no need for any intervention by the user. These modifications may easily be made to the subroutine that calculates the stiffness matrix of RQUAD4. It remains to be seen whether the zero energy mode has disappeared.

Towards this end, an eigenanalysis was performed to determine all of the deformation modes of a single element. Figure 5.9 shows these modes and it may be seen that the previously spurious mechanism now has a small amount of energy associated with it (as evident from the non-zero eigenvalue pertaining to the specific mode of deformation). Thus the zero energy mode has been removed. The three zero eigenvalues correspond to the usual rigid body motions. We can further check this by performing the tests in which RQUAD4 and Q4S fail due to their mechanisms.

Figure 5.10 shows a cantilever beam loaded with an applied moment at the free end. This is the pathological case that both RQUAD4 and Q4S are not able to model unless a node is restricted from rotating. Not only is RQD4 able to model this problem but the displacement and rotation at the free end are very accurate; i.e. 91% and 100% accurate, respectively. The rotational dof is assumed to be the same as that of RQUAD4. It was also found that the behaviour of RQD4 was identical to that of RQUAD4 in all of the tests of Chapter 7, with no exception.

Figure 5.9: Eigenmodes of a single RQD4 element

Figure 5.10: Cantilever with applied moment at the free end

5.7 Summary

After further clarifying the behaviour of RQUAD4 by deriving the explicit shape functions, several attempts were made to improve its accuracy. This included the utilization of reduced integration for shear and the application of substitute shape functions for calculating the shear strains. It was concluded that these artifices did not significantly improve the accuracy of the element. Additionally, the problem of the mechanism was addressed and resolved with the derivation of a new element called RQD4. The formulation of RQD4 is not much more complex than the original formulation. A subroutine that calculates the stiffness matrix of RQUAD4 may be modified to calculate $[K]_{RQD4}$ by simply replacing the $[T]$ matrix and the QUAD8 shape functions with those of QUAD12. The behaviour of this element is the same as RQUAD4, but the mechanism is eliminated.

6 A Quadrilateral Facet Shell Element

6.1 General

Due to their importance in civil and mechanical structural applications, shell structures have received a considerable amount of attention by researchers (*Gallagher* 1975). Flat or facet shell finite elements were the first genre of shell elements developed in the early 1960's. Later, more sophisticated elements were developed based on specialized shell theories or three dimensional continuum equations formulated in a discrete form. The culmination of this work has been the development of the semi-loof shell element by *Irons* (1976).

While the more sophisticated elements are more accurate, they are also much more complex and their implementation requires a great amount of CPU time. The latter is a very important consideration in nonlinear problems where reformulation of the stiffness matrix may be necessary at each load step or even each iteration. For this reason, in the past few years there has emerged a renewed interest in the development of facet shell elements.

The basic concept involved in the development of flat shell elements is simple: a membrane element is combined with a plate bending element and the result is a flat shell element with nodal dof's consisting of the nodal dof's of the membrane and the bending elements. Clearly, the accuracy of the shell element is dependent upon the accuracy of the constituent membrane and plate bending elements.

In the past, the most common membrane element used in the formulation of such shell elements has been the constant strain triangle or its quadrilateral counterpart. These elements have the usual degrees of freedom per node, namely

the planar displacements u and v . The use of u and v creates difficulties in the analysis of shells where two adjacent elements become coplanar." In this case an artificial stiffness is added to the stiffness matrix to endow the elements with stiffness corresponding to the drilling dof (Zienkiewicz 1977).

The planar quadrilateral element discussed in the previous chapter does not pose this problem due to the presence of the rotational dof. Hence this element is a good candidate for the development of a quadrilateral flat shell element.

Due to the relatively good accuracy of RQD4, a coarse mesh is generally adequate to obtain sufficiently accurate results for practical applications. In shell problems, however, an equally accurate and reliable bending element will be needed. A search through available literature (Batoz, Bathe and Ho 1980, Chiniah 1985) showed that the quadrilateral plate bending element IDKQ is probably the best available displacement-based quadrilateral element. Hence, it was decided to combine RQD4 with IDKQ in order to develop a flat shell element called Quadrilateral Facet Shell Element (QFSE). Before any results for this element are shown, we will present its full formulation.

6.2 Formulation of Element IDKQ

6.2.1 Basics of Discrete Kirchhoff Elements

The discrete Kirchhoff approach is a good way to obtain a plate bending element that converges to the Kirchhoff solution of thin plates while only enforcing C_0 continuity. It may be considered to be a combination of the Mindlin and Kirchhoff plate theories. Both plate theories will be summarized before the formulation of the discrete Kirchhoff quadrilateral element is reviewed.

Mindlin plate theory accounts for the effect of flexural and transverse shear deformations. It assumes that points on the normal to the undeformed middle

surface remain on a straight line but the line is not necessarily normal to the deformed middle surface. Thus, using small displacement theory, the components of displacement at a point with coordinates x, y, z are given by;

$$\begin{aligned} u &= z\beta_x(x, y) \\ v &= z\beta_y(x, y) \\ w &= w(x, y) \end{aligned} \quad (6.1)$$

where w is the transverse displacement, β_x and β_y are the rotation of the normals to the undeformed middle surface of the plate and are measured in the $x - z$ and $y - z$ planes, respectively, Fig. 6.1.

The bending strain is:

$$\{\epsilon_b\} = z \{\kappa\} \quad (6.2)$$

where $\{\kappa\}$ is the vector of curvatures

$$\{\kappa\} = \begin{bmatrix} \frac{\partial \beta_x}{\partial x} \\ \frac{\partial \beta_y}{\partial y} \\ \frac{\partial \beta_x}{\partial y} + \frac{\partial \beta_y}{\partial x} \end{bmatrix} \quad (6.3)$$

The transverse shear strain, γ , assumed to be constant through the thickness, is

$$\{\gamma\} = \begin{bmatrix} \frac{\partial w}{\partial x} + \beta_x \\ \frac{\partial w}{\partial y} + \beta_y \end{bmatrix} \quad (6.4)$$

The strain energy may then be defined as:

$$U = U_b + U_s \quad (6.5)$$

where

$$U_b = \frac{1}{2} \int_A \{\kappa\}^T [D_b] \{\kappa\} dx dy \quad (6.6)$$

$$U_s = \frac{1}{2} \int_A \{\gamma\}^T [D_s] \{\gamma\} dx dy \quad (6.7)$$

Figure 6.1: Plate under bending; (a) Undeformed shape (b) Deformed shape

and

$$[D_b] = \frac{Eh^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \quad (6.8)$$

$$[D_s] = \frac{Eh\rho}{2(1+\nu)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (6.9)$$

for an isotropic homogeneous plate of constant thickness h , ρ is a correction factor which accounts for the non-uniformity of the transverse shear stress through the plate thickness and the other symbols are as defined before.

The bending moments $\{M\}$ and shear forces $\{Q\}$ are obtained by integration of the stresses through the thickness:

$$\{M\} = \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \{\sigma\} z dz = [D_b] \{\kappa\} \quad (6.10)$$

and

$$\{Q\} = \begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = \rho \int_{-\frac{h}{2}}^{\frac{h}{2}} \{\sigma_s\} dz \rho = [D_s] \{\gamma\} \quad (6.11)$$

The expression for U given by Eq. (6.5) may be used to formulate finite elements where shear effects are important, i.e. for thick or moderately thick plates and multilayered plates. Only C_0 requirements need be fulfilled for an element based on Mindlin plate theory.

Kirchhoff plate theory is applicable to thin plates where the transverse shear effects are considered to be negligible compared to the bending effects. Thus, normals to the undeformed middle surface are assumed to remain straight and normal after deformation, which means that the shear strains $\{\gamma\}$ are assumed to be zero. Hence, according to Eq. (6.4)

$$\beta_x = -\frac{\partial w}{\partial x} \quad (6.12)$$

$$\beta_y = -\frac{\partial w}{\partial y} \quad (6.13)$$

Elements based on Kirchhoff theory have their drawbacks since they require C_1 continuity, which makes them stiff.

Discrete Kirchhoff elements use Mindlin plate theory as a starting point and then the Kirchhoff hypothesis is applied at discrete points, thus circumventing the need for C_1 continuity.

6.2.2 Discrete Kirchhoff Quadrilateral DKQ

The fundamentals of the discrete Kirchhoff approach can be best demonstrated by means of dealing with a specific finite element. Towards this end, we shall present the procedure for the discrete Kirchhoff quadrilateral element, which constitutes the basis for the Improved Discrete Kirchhoff Quadrilateral (IDKQ), implemented in this thesis.

For thin plates, the transverse shear is considered negligible compared to the bending energy, thus the stiffness matrix of the element is only based on

$$U = \frac{1}{2} \int_A \{\kappa\}^T [D_b] \{\kappa\} dx dy$$

The formulation of the DKQ element is based on the assumptions that β_x and β_y vary quadratically over the element. Thus

$$\beta_x = \sum_{i=1}^8 N_i \beta_{xi} \quad (6.14)$$

$$\beta_y = \sum_{i=1}^8 N_i \beta_{yi} \quad (6.15)$$

where β_{xi} and β_{yi} are the nodal values of β at the corners and mid-points, and $N_i(\xi, \eta)$ are the shape functions of the eight-noded serendipity element. The geometry of the DKQ element is shown in Fig. 6.2.

The Kirchhoff assumptions are applied at discrete points in the following manner:

Figure 6.2: Geometry of DKQ element

1.

$$\{\gamma\} = \begin{bmatrix} \beta_x + \frac{\partial w}{\partial x} \\ \beta_y + \frac{\partial w}{\partial y} \end{bmatrix} = 0$$

at corner nodes 1, 2, 3 and 4

2. $\beta_{sk} + \left(\frac{\partial w}{\partial s}\right)_k = 0$ at midnodes ($k = 5, 6, 7, 8$)3. A linear variation of β_n is imposed along the sides

$$\beta_{nk} = \frac{1}{2}(\beta_{ni} + \beta_{nj})$$

where k is the midnode between nodes i and j .4. No interpolation function is assumed for w over the whole element, but a cubic variation along the sides is assumed such that;

$$\left(\frac{\partial w}{\partial s}\right)_k = -\frac{3}{2l_{ij}}w_i - \frac{1}{4}\left(\frac{\partial w}{\partial s}\right)_i + \frac{3}{2l_{ij}}w_j - \frac{1}{4}\left(\frac{\partial w}{\partial s}\right)_j$$

where l_{ij} is the length of side ij .

Note that since $\frac{\partial w}{\partial s}$ and β_s vary quadratically along the sides and that $\gamma_s = \beta_s + \frac{\partial w}{\partial s} = 0$ is applied at three points along the side, the Kirchhoff hypothesis is satisfied along the entire boundary.

It is necessary to relate β_x and β_y to the nodal dof of DKQ, $\{\delta\}$, where

$$\{\delta\}^T = [w_1, \theta_{x1}, \theta_{y1}, w_2, \dots, \theta_{y4}]$$

The following geometric relationships may be utilized for each side:

$$\begin{Bmatrix} \beta_x \\ \beta_y \end{Bmatrix} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{Bmatrix} \beta_n \\ \beta_s \end{Bmatrix} \quad (6.16)$$

and

$$\begin{Bmatrix} \frac{\partial w}{\partial s} \\ \frac{\partial w}{\partial n} \end{Bmatrix} = \begin{bmatrix} c & s \\ s & -c \end{bmatrix} \begin{Bmatrix} \theta_x \\ \theta_y \end{Bmatrix} \quad (6.17)$$

where $c = \cos \gamma_{ij}$ and $s = \sin \gamma_{ij}$ and γ_{ij} is the angle between the x axis and the normal to the side under consideration, Fig. 6.2.

Using Eqs. (6.12)-(6.17) β_x and β_y may be expressed in the following manner

$$\beta_x = \{H_x\}^T(\xi, \eta) \{\delta\} \quad (6.18)$$

$$\beta_y = \{H_y\}^T(\xi, \eta) \{\delta\} \quad (6.19)$$

where $\{H_x\}$ and $\{H_y\}$ are the new shape functions expressed in terms of the eight-node element shape functions, $\{N_i\}$.

6.2.3 The Improved Discrete Kirchhoff Quadrilateral

Chinniah (1985) found that although the performance of the DKQ and its triangular counterpart DKT was good, their convergence rate for displacements was slow. He, therefore, modified the original formulation to derive the IDKQ element.

These modifications include the use of substitute shape functions and applying the discrete Kirchhoff hypothesis at the interior of the element as well as along the boundaries. Since the Kirchhoff hypothesis is applied at the interior (centroid) of the element, the nine-noded Lagrange element is used instead of the eight-noded serendipity element used by *Batoz and Tahar* (1982) for the formulation of DKQ.

The substitute shape functions entail the replacement of the shape functions of the Lagrangian element, $\{N_i\}$, by shape functions derived as the least square polynomial versions of $\{N_i\}$. They are derived as follows: A typical shape function $\{N_i\}$ is of the form

$$\begin{aligned} N_i &= a_1 + a_2\xi + a_3\eta + a_4\xi^2 + a_5\xi\eta + a_6\eta^2 + a_7\xi^2\eta + a_8\xi\eta^2 + a_9\xi^2\eta^2 \\ &= \sum_{j=1}^9 a_j \xi^{m_j} \eta^{n_j} \end{aligned}$$

The substitute shape function is chosen as

$$\bar{N}_i = \sum_{j=1}^8 b_j \xi^{m_j} \eta^{n_j}$$

The unknown coefficients b_i are obtained in terms of the known coefficients a_i by using the least square smoothing procedure. In other words, the function Π is minimized according to:

$$\Pi = \int_A (N_i - \bar{N}_i)^2 d\xi d\eta \quad (6.20)$$

i.e.

$$\frac{\partial \Pi}{\partial b_j} = 0 \quad (j = 1 \dots 8) \quad (6.21)$$

The solution of the eight simultaneous equations implied by Eq. (6.21) leads to the following smoothed shape functions:

$$\begin{aligned} \bar{N}_1 &= -\frac{1}{36} + \frac{1}{12}\xi^2 + \frac{1}{4}\xi\eta + \frac{1}{12}\eta^2 - \frac{1}{4}\xi^2\eta - \frac{1}{4}\xi\eta^2 \\ \bar{N}_2 &= -\frac{1}{36} + \frac{1}{12}\xi^2 - \frac{1}{4}\xi\eta + \frac{1}{12}\eta^2 - \frac{1}{4}\xi^2\eta + \frac{1}{4}\xi\eta^2 \end{aligned}$$

$$\begin{aligned}
\bar{N}_3 &= -\frac{1}{36} + \frac{1}{12}\xi^2 + \frac{1}{4}\xi\eta + \frac{1}{12}\eta^2 + \frac{1}{4}\xi^2\eta + \frac{1}{4}\xi\eta^2 \\
\bar{N}_4 &= -\frac{1}{36} + \frac{1}{12}\xi^2 - \frac{1}{4}\xi\eta + \frac{1}{12}\eta^2 + \frac{1}{4}\xi^2\eta - \frac{1}{4}\xi\eta^2 \\
\bar{N}_5 &= \frac{1}{18} - \frac{1}{2}\eta - \frac{1}{6}\xi^2 + \frac{1}{3}\eta^2 + \frac{1}{2}\xi^2\eta \\
\bar{N}_6 &= \frac{1}{18} + \frac{1}{2}\xi + \frac{1}{3}\xi^2 - \frac{1}{6}\eta^2 - \frac{1}{2}\xi\eta^2 \\
\bar{N}_7 &= \frac{1}{18} + \frac{1}{2}\eta - \frac{1}{6}\xi^2 + \frac{1}{3}\eta^2 - \frac{1}{2}\xi^2\eta \\
\bar{N}_8 &= \frac{1}{18} - \frac{1}{2}\xi + \frac{1}{3}\xi^2 - \frac{1}{6}\eta^2 + \frac{1}{2}\xi\eta^2 \\
\bar{N}_9 &= \frac{8}{9} - \frac{2}{3}\xi^2 - \frac{2}{3}\eta^2
\end{aligned} \tag{6.22}$$

Chinniah found that although using the Lagrangian shape functions improved the convergence rate for displacements, it slowed down the convergence rate for stresses, while $\bar{N}_i$ improved the convergence of both.

Chinniah not only applied the Kirchhoff hypothesis at the corners and mid-points of the element, but also at the centroid:

$$\begin{aligned}
(\beta_x)_{\text{centroid}} &= -\frac{\partial w}{\partial x} \\
(\beta_y)_{\text{centroid}} &= -\frac{\partial w}{\partial y}
\end{aligned} \tag{6.23}$$

Using this additional constraint and those listed in the previous section, he related β_x and β_y to the vector of nodal dof's as follows:

$$\beta_x = \sum_{i=1}^{12} G_i \delta_i = \{G\}^T \{\delta\} \tag{6.24}$$

$$\beta_y = \sum_{i=1}^{12} H_i \delta_i = \{H\}^T \{\delta\} \tag{6.25}$$

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where the new shape functions $\{G\}$ and $\{H\}$ may be expressed in terms of $\bar{N}_i$,

$$\begin{aligned}
 G_1 &= \frac{p_1}{4}\bar{N}_5 + \frac{p_4}{4}\bar{N}_8 + \frac{3}{2\Delta}y_{42}\bar{N}_9 \\
 G_2 &= -\bar{N}_1 + \frac{(s_1-2)}{4}\bar{N}_5 + \frac{(s_4-2)}{4}\bar{N}_8 + \frac{y_{42}}{4\Delta}(x_{21} + x_{41})\bar{N}_9 \\
 G_3 &= -\frac{q_1}{4}\bar{N}_5 - \frac{q_4}{4}\bar{N}_8 + \frac{y_{42}}{4\Delta}(y_{21} + y_{41})\bar{N}_9 \\
 G_4 &= -\frac{p_1}{4}\bar{N}_5 + \frac{p_2}{4}\bar{N}_6 + \frac{3}{2\Delta}y_{13}\bar{N}_9 \\
 G_5 &= -\bar{N}_2 + \frac{(s_1-2)}{4}\bar{N}_5 + \frac{(s_2-2)}{4}\bar{N}_6 + \frac{y_{13}}{4\Delta}(x_{12} + x_{32})\bar{N}_9 \\
 G_6 &= -\frac{q_1}{4}\bar{N}_5 - \frac{q_2}{4}\bar{N}_6 + \frac{y_{13}}{4\Delta}(y_{12} + y_{32})\bar{N}_9 \\
 G_7 &= -\frac{p_2}{4}\bar{N}_6 + \frac{p_3}{4}\bar{N}_7 + \frac{3}{2\Delta}y_{24}\bar{N}_9 \\
 G_8 &= -\bar{N}_3 + \frac{(s_2-2)}{4}\bar{N}_6 + \frac{(s_3-2)}{4}\bar{N}_7 + \frac{y_{24}}{4\Delta}(x_{23} + x_{43})\bar{N}_9 \\
 G_9 &= -\frac{q_2}{4}\bar{N}_6 - \frac{q_3}{4}\bar{N}_7 + \frac{y_{24}}{4\Delta}(y_{23} + y_{43})\bar{N}_9 \\
 G_{10} &= -\frac{p_3}{4}\bar{N}_7 + \frac{p_4}{4}\bar{N}_8 + \frac{3}{2\Delta}y_{31}\bar{N}_9 \\
 G_{11} &= -\bar{N}_4 + \frac{(s_3-2)}{4}\bar{N}_7 + \frac{(s_4-2)}{4}\bar{N}_8 + \frac{y_{31}}{4\Delta}(x_{14} + x_{34})\bar{N}_9 \\
 G_{12} &= -\frac{q_3}{4}\bar{N}_7 - \frac{q_4}{4}\bar{N}_8 + \frac{y_{31}}{4\Delta}(y_{14} + y_{34})\bar{N}_9 \\
 \\
 H_1 &= -\frac{t_1}{4}\bar{N}_5 + \frac{t_4}{4}\bar{N}_8 + \frac{3}{2\Delta}x_{24}\bar{N}_9 \\
 H_2 &= -\frac{q_1}{4}\bar{N}_5 - \frac{q_4}{4}\bar{N}_8 + \frac{x_{24}}{4\Delta}(x_{21} + x_{41})\bar{N}_9 \\
 H_3 &= -\bar{N}_1 + \frac{(r_1-2)}{4}\bar{N}_5 + \frac{(r_4-2)}{4}\bar{N}_8 + \frac{x_{24}}{4\Delta}(y_{21} + y_{41})\bar{N}_9 \\
 H_4 &= \frac{t_1}{4}\bar{N}_5 - \frac{t_2}{4}\bar{N}_6 + \frac{3}{2\Delta}x_{31}\bar{N}_9
 \end{aligned} \tag{6.26}$$

$$\begin{aligned}
H_5 &= -\frac{q_1}{4}\bar{N}_5 - \frac{q_2}{4}\bar{N}_6 + \frac{x_{31}}{4\Delta}(x_{12} + x_{32})\bar{N}_9 \\
H_6 &= -\bar{N}_2 + \frac{(r_1 - 2)}{4}\bar{N}_5 + \frac{(r_2 - 2)}{4}\bar{N}_6 + \frac{x_{31}}{4\Delta}(y_{12} + y_{32})\bar{N}_9 \quad (6.27) \\
H_7 &= \frac{t_2}{4}\bar{N}_6 - \frac{t_3}{4}\bar{N}_7 + \frac{3}{2\Delta}x_{42}\bar{N}_9 \\
H_8 &= -\frac{q_2}{4}\bar{N}_6 - \frac{q_3}{4}\bar{N}_7 + \frac{x_{42}}{4\Delta}(x_{23} + x_{43})\bar{N}_9 \\
H_9 &= -\bar{N}_3 + \frac{(r_2 - 2)}{4}\bar{N}_6 + \frac{(r_3 - 2)}{4}\bar{N}_7 + \frac{x_{42}}{4\Delta}(y_{23} + y_{43})\bar{N}_9 \\
H_{10} &= \frac{t_3}{4}\bar{N}_7 - \frac{t_4}{4}\bar{N}_8 + \frac{3}{2\Delta}x_{13}\bar{N}_9 \\
H_{11} &= -\frac{q_3}{4}\bar{N}_7 - \frac{q_4}{4}\bar{N}_8 + \frac{x_{13}}{4\Delta}(x_{14} + x_{34})\bar{N}_9 \\
H_{12} &= -\bar{N}_4 + \frac{(r_3 - 2)}{4}\bar{N}_7 + \frac{(r_4 - 2)}{4}\bar{N}_8 + \frac{x_{13}}{4\Delta}(y_{14} + y_{34})\bar{N}_9
\end{aligned}$$

The curvature vector $\{\kappa\}$ is given by;

$$\{\kappa\} = [B] \{\delta\} \quad (6.28)$$

where

$$[B] = [\{B_1\} \cdots \{B_{12}\}]$$

and

$$\{B_i\} = \left\{ \begin{array}{c} \frac{\partial G_i}{\partial x} \\ \frac{\partial H_i}{\partial y} \\ \frac{\partial G_i}{\partial y} + \frac{\partial H_i}{\partial x} \end{array} \right\}$$

Thus

$$[K]_{\text{IDKQ}} = \int_A \{B\}^T [D_b] \{B\} dx dy \quad (6.29)$$

$$= \int_{-1}^1 \int_{-1}^1 \{B\}^T [D_b] \{B\} [J] d\xi d\eta \quad (6.30)$$

where all symbols are as previously defined. This may be calculated using a 3×3 integration scheme.

Thus the bending moments $\{M\}$, at any point are;

$$\{M\} = \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = [D_b] \{B(x, y)\} \{\delta\} \quad (6.31)$$

6.3 Performance of IDKQ

Chinniah (1985) showed that this element passed the patch test and behaved favourably in other tests. The author also found that IDKQ passes the bending patch test discussed in Section 3.4. Two additional tests were performed in this study to further investigate IDKQ's behaviour.

The first is a simply supported beam in bending, Fig. 6.3. Table 6.1 shows the central deflection and bending moment at A for several mesh sizes. It can be seen that IDKQ converges to the correct solution. The second test is a flat plate whose left edge is simply supported and whose right edge is free, Fig. 6.4. Figure 6.5 shows the bending moment M_x at several points along the flat plate. The IDKQ values are compared with a boundary element solution, denoted as BE (*Hartmann and Zotemantel* 1986). The two solutions seem comparable in this problem for which a theoretical solution is not available.

$$E = 10^6 \text{ kN/m}^2$$

$$t = 0.1 \text{ m}$$

$$\nu = 0.333$$

Figure 6.3: Simply supported beam modelled by IDKQ elements

Table 6.1: Vertical deflection and M_z at A for simply supported beam

Mesh	δ_A	M_{zA}
2×1	0.818	1.000
4×1	0.968	1.000
8×1	1.005/	1.000
Beam Solution	1.000	1.000

Figure 6.4: Plate analyzed using element IDKQ

Figure 6.5: z-displacements along; (a) line AI (b) line E-E5

Favourable results found by Chinniah combined with satisfactory results obtained here showed IDKQ to be a good quadrilateral plate bending element; thus it was incorporated as the bending portion of the shell element.

6.4 Quadrilateral Facet Shell Element QFSE

It was shown in Section 3.2 that the stiffness matrix of an element can be written as

$$[K] = \int \{B\}^T [D] \{B\} dV \quad (6.32)$$

For a thin shell, the strain $\{\epsilon\}$ at any point is given by

$$\{\epsilon\} = \{\epsilon_o\} - z \{\kappa\} \quad (6.33)$$

where $\{\epsilon_o\}$ are the inplane strains of the projection of the same point on the reference plane, $\{\kappa\}$ is the vector of curvatures due to bending and z is the distance of the point from the middle plane. From previous discussions

$$\{\epsilon_o\} = \{B_p\} \{\delta_p\} \quad (6.34)$$

$$\{\kappa\} = \{B_b\} \{\delta_b\} \quad (6.35)$$

where the subscripts p and b stand for inplane and bending components. Hence Eq. (6.33) can be rewritten as;

$$\{\epsilon\} = (\{B_p\} - z \{B_b\}) \begin{Bmatrix} \{\delta_p\} \\ \{\delta_b\} \end{Bmatrix} = \{B\} \{\delta\} \quad (6.36)$$

If we assume the shell to be isotropic and if the reference plane coincides with the middle plane of the shell, then the inplane and bending parts become uncoupled and the $[D]$ matrix will be given by

$$[D] = \begin{bmatrix} [D_p] & \\ & [D_b] \end{bmatrix} \quad (6.37)$$

where $[D_p]$ is the usual elasticity matrix for plane stress problems, as given by Eq. (3.20) and $[D_b]$ is the plate bending constitutive matrix given by Eq. (6.8)

Substituting for $[B]$ and $[D]$ from Eqs. (6.36) and (6.37) into Eq. (6.32) yields:

$$\begin{aligned} [K] &= \int \left\{ \begin{array}{c} \{B_p\}^T \\ -z \{B_b\}^T \end{array} \right\} \left[\begin{array}{c} [D_p] \\ [D_b] \end{array} \right] \left\{ \begin{array}{c} \{B_p\} \\ -z \{B_b\} \end{array} \right\} dV \\ &= \int \left[\begin{array}{cc} \{B_p\}^T [D_p] \{B_p\} & 0 \\ 0 & -z \{B_b\}^T [D_b] \{B_b\} \end{array} \right] dV \end{aligned} \quad (6.38)$$

$$= \left[\begin{array}{cc} [K_p] & 0 \\ 0 & [K_b] \end{array} \right] \quad (6.39)$$

where

$$[K_p] = \int \{B_p\}^T [D_p] \{B_p\} dV \quad (6.40)$$

$$[K_b] = \int \{B_b\}^T [D_b] \{B_b\} dV \quad (6.41)$$

In the above $[K_p]$ and $[K_b]$ are the stiffness matrices of the inplane and bending elements, respectively. The element yields the usual inplane and bending stress resultants

$$\left\{ \begin{array}{c} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{array} \right\}$$

Note that if the shell is anisotropic or if this element is used in nonlinear analysis for materials with different tensile and compressive strength properties, such as concrete, the inplane and bending terms may become coupled and the off-diagonal terms in Eqs. (6.39) will not consequently be zero. In this study, however, we limit our work to isotropic shells and linear elasticity.

The consistent load vector of the shell element is simply the combined inplane forces and moments obtained from the consistent load vectors of the inplane and bending elements. In the next chapter several benchmark problems have been studied to ascertain the performance of this element.

7.1 General

This chapter deals with the testing of the new elements RQD4 and QFSE developed in this thesis. The tests are divided into two parts; membrane problems and shell problems. Many of the tests are based on the proposed standard set of problems that have been compiled by *MacNeal and Harder* (1985). These consist of a library of problems which demonstrate frequently encountered element failure modes. Some additional tests have also been added to further expose any deficiencies of the new elements.

The major problem that these tests are designed to expose is a phenomenon called 'locking', where an element exhibits excessive stiffness for a particular element shape or load type. The membrane tests consist of: an aspect ratio test, cantilever beam problem, pure inplane bending of a square plate, and a square plate subjected to a parabolically varying edge tension. The shell tests consist of the Scordelis-Lo cylindrical shell roof, a hemispherical shell, a folded plate structure and a pinched cylinder. The results of elements RQD4 and QFSE are compared with those of other elements in the literature (Table 2.1) and with theoretical solutions. By theoretical it is meant the closed-form solution based on classical methods of analysis. The membrane elements Q4S and QLC3 are coupled with the bending element IDKQ to form the shell element, but they are still referred to by their membrane namesake.

7.2 Membrane Element Tests

7.2.1 Aspect Ratio Test

This single element test, presented by *Robinson* (1988), is used to check RQD4's sensitivity to aspect ratio, denoted by $AR \approx \frac{L}{2c}$, where L and c are the dimensions of the element, Fig. 7.1. The element is loaded with a parabolically varying end shear, represented in the analysis by two equal nodal loads applied at the top and bottom right-hand corners of the element. The boundary conditions are $u = 0$

Figure 7.1: Aspect ratio test

at $x = y = 0$ and $u = v = 0$ at $x = 0, y = c$. These conditions, however, result in a spurious mode ('hourglass' mechanism) for element Q4S. Therefore, for this element $v = 0$ at $x = y = 0$ had to be applied in addition to the conditions listed above. The elements tested are: RQD4, Q4S, QLC3, QUAD8 (3 pt. integration)

and QUAD4. The QUAD8 results were obtained from Robinson's work referred to earlier.

The quantity of interest in this test is the tip deflection at $x = l, y = 0$. The theoretical value of this displacement for aspect ratios of 1, 2, 3, 4 and 8 are listed in Table 7.1 and were obtained from Robinson's work. It may be seen from Fig. 7.2, which shows normalized tip displacement versus aspect ratio, that RQD4 converges to an accuracy of about 71% at $AR = 8$. This is inferior to QUAD8, which is at 79% at the same aspect ratio. Note that QUAD8 does not converge to a constant value like RQD4. It is not surprising that RQD4's sensitivity is marginally inferior to its parent element, yet vastly superior to QUAD4 which has an accuracy of 4% at $AR = 8$.

Table 7.1: Theoretical displacement v at A for different aspect ratios of the element

Aspect Ratio	1	2	3	4	8
L	0.12	0.24	0.36	0.48	0.96
$v_{\text{theo.}}$	20.541	119.96	377.13	870.93	6790.0

Figure 7.2: Normalized tip deflection vs. aspect ratio for a single element

7.2.2 Cantilever Beam

The next test is of prime interest due to its popularity in the literature. It is a cantilever beam (Fig. 7.3a) presented by *Olson and Bearden* (1979), among many others. This test is important due to its simplicity and that all of the principle element modes (constant and linearly varying strains and curvatures) may be evoked by applying loads at the free end. The beam is fully fixed at the left end, and a parabolically varying shear load is applied at the right end (total load = 40 kips in the current test). Mesh sizes from 1×1 to 16×4 elements were used, along with a 4×1 mesh of distorted elements (Fig. 7.3b). The elements tested are: RQD4, Q4S, AQ2, QLC3, AT, OB, CST and LST. The theoretical tip deflection is given along with the FEM results in Table 7.2. The theoretical

$$\begin{aligned} \nu &= 0.25 \\ t &= 1.0'' \\ E &= 30.0 \times 10^6 \end{aligned}$$

Figure 7.3: Cantilever beam; (a) Regular mesh (b) Irregular mesh

longitudinal stresses at A and B are given in Table 7.3. The results of AQ2, AT, OB, LST and CST were obtained from Olson and Bearden's work. The stress at B was determined for the elements mentioned above, while the stress at A was calculated for RQD4, Q4S, QLC3 and QUAD4. The FEM results were normalized with respect to their theoretical values.

It may be seen from Table 7.2 that RQD4 and Q4S tip deflection results converge rather quickly. RQD4 has an 8% error in the 4×1 mesh while Q4S only has a 2% error. This is true because the modifications introduced in Q4S were partly intended to improve its beam-type behaviour. Table 7.3 shows that RQD4, Q4S and QLC3 give the correct stresses even if just one element is used. Tables 7.4 and 7.5 show the tip deflection and stresses for different types of element shapes. It is believed that certain basic problems, such as shear-locking and Poisson's ratio locking manifest themselves in distorted elements. Furthermore, the distorted element tests determine the sensitivity of the element to changes in shape. It may be observed that Q4S fares better than the other elements in this test.

Table 7.2: Convergence of tip deflection for the cantilever beam

Mesh Size	RQD4	Q4S	AQ2	QLC3	QUAD4	AT	OB	CST	LST
1 × 1	0.712 (12)*	0.757 (12)		0.708 (12)	0.134 (8)				
2 × 1	0.881 (18)	0.937 (18)		0.878 (18)	0.375 (12)				
4 × 1	0.923 (30)	0.982 (30)	0.850 (30)	0.922 (30)	0.681 (20)	0.758 (30)	0.647 (30)	0.255 (20)	
8 × 1	0.933 (54)	0.993 (54)		0.933 (54)	0.857 (36)				
8 × 2	0.977 (81)	0.995 (81)	0.954 (81)	0.970 (81)	0.888 (54)	0.917 (81)	0.838 (81)	0.559 (54)	0.998 (170)
16 × 2	0.982 (153)	0.999 (153)		0.975 (153)	0.953 (102)				
16 × 4			0.987 (255)			0.976 (255)	0.925 (255)	0.875 (170)	0.999 (594)
Exact	[0.3558]								

* Number of degrees of freedom given in parentheses

Table 7.3: Convergence of longitudinal stress at *A* and *B* for the cantilever beam

Mesh Size	RQD4	Q4S	AQ2	QLC3	QUAD4	AT	OB	CST	LST
1×1	1.000 (12)*	1.000 (12)		1.000 (12)	0.143 (8)				
2×1	1.000 (18)	1.000 (18)		1.000 (18)	0.400 (12)				
4×1	1.000 (30)	1.000 (30)	0.878 (30)	1.000 (30)	0.727 (20)	0.657 (30)			
8×1	1.000 (54)	1.000 (54)		1.000 (54)	0.914 (36)				
8×2	1.017 (81)	1.000 (81)	0.973 (81)	1.017 (81)	0.911 (54)	0.872 (81)	0.798 (81)	0.722 (54)	0.985 (170)
16×2	1.017 (153)	1.000 (153)		1.017 (153)	0.976 (102)				
16×4			0.995 (255)			0.993 (255)	0.930 (255)	0.892 (170)	1.000 (594)
Exact	[40×10^3 pt. A] [60×10^3 pt. B]								

* Number of degrees of freedom given in parentheses

Table 7.4: Tip deflection for regular and irregular meshes of the cantilever beam

Element	Regular Mesh	Irregular Mesh
RQD4	0.923	0.938
Q4S	0.982	0.987
QLC3	0.922	0.828
QUAD8	0.985	0.994
QUAD4	0.681	0.590
Exact	[0.3558]	

Table 7.5: Longitudinal stress at A for regular and irregular meshes for the cantilever beam

Element	Regular Mesh	Irregular Mesh
RQD4	1.000	0.956
Q4S	1.000	1.050
QLC3	1.000	0.684
QUAD4	0.727	0.632
Exact	[40×10 ³]	

7.2.3 Slender Cantilever Beam

Figures 7.4 (a), (b) and (c) show a slender cantilever beam divided into meshes of various element shapes as suggested by *MacNeal and Harder* (1985). This example is important because it tests the combination of different element shapes with linearly varying strains. Many elements display the locking phenomenon if

Figure 7.4: Slender cantilever beam (a) Regular elements (b) Trapezoidal elements (c) Parallelogram elements

these shapes are used in a beam-type problem. It may be noted that the aspect

ratio of the elements in this example is greater than in the previous test.

The elements tested are: RQD4, Q4S and QLC3. The theoretical tip deflection is given in Table 7.6 along with the FEM results. It may be seen that QLC3 exhibits locking when either trapezoidal or parallelogram elements are used. Q4S is not affected by the changes in element shapes. Again it should be emphasized that the changes introduced in this element were designed to improve its behaviour in beam-type problems. RQD4 also shows a reduction in accuracy in the trapezoidal and parallelogram case. The drop in accuracy is not, however, as severe as that of QLC3.

Table 7.6: Tip deflection of slender beam normalized with the exact value

Element	Rectangular Elements	Trapezoidal Elements	Parallelogram Elements
RQD4	0.904	0.764	0.842
Q4S	0.994	0.990	0.986
QLC3	0.904	0.492	0.342
Exact	[0.108]		

7.2.4 Pure Inplane Bending of a Square Plate

Figure 7.5 shows a square plate subjected to pure bending, as presented by *Allman* (1984), and other investigators. Only the shaded quarter was modelled due to symmetry and antisymmetry about two axes. Mesh sizes varied from 1×1 to 8×8 . The elements tested were: RQD4, Q4S, AQ2, QLC3, QUAD4 and AT. The results for AT and AQ2 were obtained from *Allman* (1984) and *Allman* (1988).

Figure 7.5: Pure inplane bending of a square plate

The x and y displacements at C normalized with their respective theoretical values, are shown in Tables 7.7 and 7.8. Table 7.9 shows the stress in the x direction at C (σ_{xC}), normalized with respect to its theoretical value. The theoretical values were obtained from Allman's work. Elements RQD4 and Q4S are virtually identical in accuracy in all of the results. Their results are superior to those of QLC3 and AT.

Table 7.7: Convergence of x -displacement at C for the square plate under inplane bending

Mesh Size	RQD4	Q4S	AQ2	QLC3	QUAD4	AT
1×1	0.964 (12)*	1.026 (12)	0.945 (12)	0.976 (12)	0.908 (8)	0.902 (12)
2×2	0.988 (27)	1.008 (27)	0.982 (27)	0.885 (27)	0.975 (18)	0.964 (27)
3×3	0.994 (48)	1.004 (48)		0.904 (48)	0.986 (32)	
4×4	0.996 (75)	1.002 (75)	0.994 (75)	0.920 (75)	0.992 (50)	0.988 (75)
8×8	0.999 (243)	1.001 (243)	0.998 (243)	0.953 (243)		0.996 (243)
Exact	[0.500]					

Table 7.8: Convergence of y -displacement at C for the square plate under inplane bending

Mesh Size	RQD4	Q4S	AQ2	QLC3	QUAD4	AT
1×1	0.956 (12)*	1.034 (12)	0.945 (12)	-0.034 (12)	0.870 (8)	0.883 (12)
2×2	0.986 (27)	1.010 (27)	0.982 (27)	0.431 (27)	0.971 (18)	0.951 (27)
3×3	0.993 (48)	1.005 (48)		0.581 (48)	0.986 (32)	
4×4	0.996 (75)	1.003 (75)	0.994 (75)	0.672 (75)	0.992 (50)	0.985 (75)
8×8	0.999 (243)	1.001 (243)	0.998 (243)	0.825 (243)		0.997 (243)
Exact	[0.325]					

* Number of degrees of freedom given in parentheses

Table 7.9: Convergence of $(\frac{\sigma_x}{\sigma_0})$ at C for the square plate under inplane bending

Mesh Size	RQD4	Q4S	AQ2	QLC3	AT
1×1	0.918 (12)*	0.984 (12)	0.978 (12)	0.921 (12)	0.944 (12)
2×2	1.012 (27)	0.992 (27)	0.991 (27)	0.641 (27)	0.989 (27)
3×3	1.008 (48)	0.995 (48)		0.584 (48)	
4×4	1.006 (75)	0.996 (75)	0.996 (75)	0.559 (75)	0.996 (75)
8×8	1.003 (243)	0.997 (243)	0.998 (243)	0.521 (243)	0.998 (243)
Exact	[1.000]				

* Number of degrees of freedom given in parentheses

7.2.5 Plate Subjected to Parabolic Edge Tension

The next test is a square plate subjected to parabolic normal stress at two opposite edges. *Olson and Bearden* (1979) and *Allman* (1984) have analyzed this problem. Again only one quarter of the plate need be modelled due to symmetry. The theoretical values were obtained from Olson and Bearden's work. The elements tested are: RQD4, Q4S, AQ2, QLC3, AT and OB.

The normalized x displacement at B is given in Table 7.10. The x and y displacements at C , normalized with their respective theoretical values, are given in Tables 7.11 and 7.12. The normalized y displacement at D is given in Table 7.13 and the normal stress at A (σ_{yA}) is shown in Table 7.14. Elements RQD4 and Q4S give good results at all points.

Figure 7.6: Plate subjected to parabolic edge tension

This is most apparent in Table 7.11, where the horizontal displacement at C , predicted by these two elements, is much better than those predicted by the other elements.

Table 7.10: Convergence of x -displacement at B for the plate under parabolic edge tension

Mesh Size	RQD4	Q4S	AQ2	QLC3	AT	OB
1×1	1.135 (12)*	1.316 (12)	0.990 (12)	1.161 (12)	0.249 (12)	0.640 (12)
2×2	1.036 (27)	1.099 (27)	1.006 (27)	1.022 (27)	0.909 (27)	0.691 (27)
4×4	1.002 (75)	1.015 (75)	1.002 (75)	0.997 (75)	0.977 (75)	0.757 (75)
8×8	1.000 (243)	1.004 (243)	1.000 (243)	0.999 (243)	0.995 (243)	
Exact	[-0.1383]					

Table 7.11: Convergence of x -displacement at C for the plate under parabolic edge tension

Mesh Size	RQD4	Q4S	AQ2	QLC3	AT	OB
1×1	-0.104 (12)*	0.468 (12)	0.082 (12)	-2.427 (12)	-0.649 (12)	1.358 (12)
2×2	1.349 (27)	2.172 (27)	0.711 (27)	-0.391 (27)	0.167 (27)	0.883 (27)
4×4	0.926 (75)	1.071 (75)	0.896 (75)	0.551 (75)	0.769 (75)	1.128 (75)
8×8	0.971 (243)	1.017 (243)	0.961 (243)	0.855 (243)	0.930 (243)	
Exact	$[1.623 \times 10^{-2}]$					

* Number of degrees of freedom given in parentheses

Table 7.12: Convergence of y -displacement at C for the plate under parabolic edge tension

Mesh Size	RQD4	Q4S	AQ2	QLC3	AT	OB
1×1	1.341 (12)*	1.193 (12)	1.461 (12)	1.676 (12)	0.604 (12)	1.168 (12)
2×2	1.058 (27)	0.923 (27)	1.159 (27)	1.257 (27)	1.265 (27)	1.041 (27)
4×4	0.998 (75)	0.968 (75)	1.051 (75)	1.074 (75)	1.087 (75)	0.951 (75)
8×8	1.001 (243)	0.990 (243)	1.015 (243)	1.026 (243)	1.028 (243)	
Exact	[-0.1162]					

* Number of degrees of freedom given in parentheses

Table 7.13: Convergence of y -displacement at D for the plate under parabolic edge tension

Mesh Size	RQD4	Q4S	AQ2	QLC3	AT
1×1	1.017 (12)*	1.088 (12)	0.937 (12)	0.934 (12)	0.282 (12)
2×2	0.999 (27)	1.021 (27)	0.983 (27)	0.971 (27)	0.959 (27)
4×4	0.989 (75)	0.993 (75)	0.996 (75)	0.982 (75)	0.988 (75)
8×8	0.997 (243)	0.998 (243)	0.999 (243)	0.995 (243)	0.997 (243)
Exact	[-0.4617]				

* Number of degrees of freedom given in parentheses

Table 7.14: Convergence of σ_y at A for the plate under parabolic edge tension

Mesh Size	RQD4	Q4S	AQ2	QLC3	AT	OB
1×1		1.200 (12)*	1.051 (12)		0.893 (12)	
2×2	0.992 (27)	1.039 (27)	1.033 (27)	0.954 (27)	0.974 (27)	0.882 (27)
4×4	1.001 (75)	1.003 (75)	1.008 (75)	0.991 (75)	0.990 (75)	0.930 (75)
8×8	1.000 (243)	1.001 (243)	1.002 (243)	0.998 (243)	0.995 (243)	
Exact	$[8.59 \times 10^6]$					

* Number of degrees of freedom given in parentheses

7.3 Shell Element Tests

7.3.1 Cylindrical Shell Roof

The Scordelis-Lo cylindrical shell roof is shown in Fig. 7.7 and is probably the most popular thin shell problem in the literature. Among others, it was used by *Olson and Bearden* (1979) to test their element. This singly curved shell problem is important since the membrane and bending deformations both contribute to the vertical displacements. This is especially true for the vertical deflection at the center of the free edge (point B in Fig. 7.7). The shell is loaded by its self weight, and is supported by two end diaphragms ($v = w = 0$) but the sides are free. Using symmetry only one quarter of the roof was modelled and the loads were lumped at the nodes. Several meshes were used including the 8×12 grid shown in Fig. 7.7.

Figure 7.7: Cylindrical shell roof

The theoretical values, based on shallow shell theory, for the displacements at several points and the longitudinal stress at B were obtained from Olson and Bearden's work. The elements used were: QFSE, Q4S, QLC3 and OB.

Figure 7.8 shows the x -displacement at A . The vertical displacement at B is shown in Fig. 7.9 and the y -displacement at B in Fig. 7.10. Figure 7.11 shows the vertical displacement at C . The longitudinal stress at B is shown in Fig. 7.12. It may be seen that the results of elements QFSE, Q4S and QLC3 are virtually identical. The displacements and stresses determined using the 8×8 mesh are in

error by at most 5%. All the previous results are normalized with respect to their theoretical value. Figure 7.13 shows the vertical displacement of the shell along the circumferential line CB and Fig. 7.14 shows the longitudinal displacements along AD using the results of the 8×8 mesh. The displacements shown in Figs. 7.13 and 7.14 are in good agreement with theory, obtained from graphical results.

Figure 7.8: Convergence of x -displacement at A for cylindrical shell roof

Figure 7.9: Convergence of z-displacement at B for cylindrical shell roof

Figure 7.10: Convergence of y-displacement at B for cylindrical shell roof

Figure 7.11: Convergence of z -displacement at C for cylindrical shell roof

Figure 7.12: Convergence of σ_x at B for cylindrical shell roof

Figure 7.13: z -displacements along line CB for cylindrical shell roof

Figure 7.14: x -displacements along line DA for cylindrical shell roof

7.3.2 Hemispherical Shell

Figure 7.15 shows a hemispherical shell under two concentrated loads as presented by *MacNeal and Harder* (1985). This doubly curved shell problem represents a hemisphere with four point loads alternating in sign at 90 degree intervals at the equator. The hole at the top is to avoid the use of a triangular element near the axis of the revolution. This problem is important since both the membrane

Figure 7.15: Hemispherical shell

and bending stiffnesses contribute to the displacement under the load. Thus, this

example has exposed deficiencies in both the membrane and bending portions of shell elements. The elements tested are: QFSE, Q4S and QLC3.

Figure 7.16 shows the displacement under load plotted versus the total number of degrees of freedom in the mesh. The values were normalized with respect to the theoretical value given by MacNeal and Harder. It is important to mention that the so called theoretical value was not obtained from exact shell analysis. It was estimated from the theoretical solution of a full hemispherical shell without a hole at the pole. It may be seen from Fig. 7.16 that both QFSE and Q4S are converging to a value which is about 7.5% higher than the theoretical value, while QLC3 converges to a value 20% below the expected displacement.

Figure 7.16: Convergence of deflection under point of load for hemispherical shell

7.3.3 Folded Plate Roof

Figure 7.17 shows a non-prismatic folded plate structure. This problem is of special interest since it requires non-rectangular elements. This example was used by *Olson and Bearden* (1979) and is based on experimental work done by *Johnson and Lee* (1968), who also provided the theoretical values.

The structure consists of tapered plates which are six inches wide at one end and two inches at the other. It is supported by end diaphragms and the sides are free. The plate is loaded by three line loads as shown in Fig. 7.17. Only one half of the structure was modelled due to symmetry. The elements used are: QFSE, Q4S, QLC3 and OB.

Figures 7.18-7.20 show the normalized vertical deflections at points *B*, *D* and *F*, respectively. Figures 7.21 and 7.22 show the longitudinal stresses at points *B* and *F*, respectively. These results are also normalized. It may be seen that in all of the results QFSE and Q4S behave better than QLC3 and OB. QFSE and Q4S give identical results in Figs. 7.21 and 7.22. Note, however, that QFSE converges slightly closer to the theoretical displacements than Q4S. Therefore, in this example QFSE behaves as well as, and in some cases better, than Q4S.

Figure 7.17: Folded plate roof

Figure 7.18: Convergence of z-displacement at B for folded plate roof

Figure 7.19: Convergence of z-displacement at D for folded plate roof

Figure 7.20: Convergence z-displacement at F for folded plate roof

Figure 7.21: Convergence of σ_x at B for folded plate roof

Figure 7.22: Convergence of σ_x at F for folded plate roof

7.3.4 Pinched Cylinder

The next application is the pinched cylinder test shown in Fig. 7.23. Many elements experience shear locking and membrane locking due to the differences in in-plane and out-of-plane stiffness. The pinched cylinder is a well known example used to detect these types of locking phenomena (*Schnobrich* 1985).

The cylinder is subjected to two equal and opposite point loads as shown in the figure. Only one eighth of the cylinder needs to be modelled due to symmetry. Boundary conditions of $u = v = 0$ were applied at both ends. The elements used are: QFSE, Q4S and QLC3.

Figure 7.23: Pinched cylinder

Figures 7.24-7.27 show the stresses and displacements at various points and Fig. 7.24 shows the z -deflection under load versus the total number of degrees of freedom. The deflections are normalized with respect to the theoretical values. Figure 7.27 shows the z -displacements along line DC . Figure 7.28 shows the x -displacements along line AD and Fig. 7.29 shows the z -displacements along line BC . All of the elements closely represent the theoretical values.

Figure 7.24: Convergence of z-displacement at C for pinched cylinder

Figure 7.25: Convergence of x-displacement at D for pinched cylinder

Figure 7.26: Convergence of σ_x at C for pinched cylinder

Figure 7.27: Convergence of σ_y at C for pinched cylinder

Figure 7.28: x -displacements along line AD for pinched cylinder

Figure 7.29: z -displacements along line BC for pinched cylinder

Figure 7.30: z -displacements along line DC for pinched cylinder

7.4 Summary

This chapter has shown the performance of RQD4 and QFSE compared to Q4S and other elements in tests designed to expose various deficiencies of finite elements. Table 7.15 summarizes the performance of all the elements tested. The quantity recorded for tests with a point load is the deflection under the load, which is a direct measure of the strain energy. In the tests with distributed loads, the largest displacement component is recorded.

The elements are given a grade in each test according to the performance scale proposed by *MacNeal and Harder* (1985). Their proposed marking scheme is given below:

Grade	Rule
A	$2\% \geq \text{error}$
B	$10\% \geq \text{error} > 2\%$
C	$20\% \geq \text{error} > 10\%$
D	$50\% \geq \text{error} > 20\%$
F	$\text{error} > 50\%$

It may be seen from Table 7.15 that QFSE (or RQD4) passes all of the tests. It does, however, exhibit some locking in the trapezoidal and parallelogram element meshes of the slender beam. It seems that this locking problem has been eliminated by the modifications introduced in Q4S.

Both QFSE and Q4S pass the patch test. The superior performance in these test proves that this new class of elements with a rotational dof ω will be very competitive in modern applications.

Table 7.15: Summary of test results

Test	QFSE	Q4S	AQ2	QLC3	AT	OB
Cantilever Beam						
Regular Mesh	B	A	C	B	D	D
Irregular Mesh	B	A		C		
Slender Cantilever						
Regular Elements	B	A		B		
Trapezoidal Elements	D	A		F		
Parallelogram Elements	C	A		F		
Pure Inplane Bending of a Square Plate (4 × 4)	A	A	A	B	A	
Plate Subjected to Parabolic Edge Tension (4 × 4)	A	A	A	A	A	
Cylindrical Shell Roof (8 × 8)	A	A		A		
Hemispherical Shell (20 × 20)	B	B		D		
Folded Plate (8 × 8)	A	B		C		C
Pinched Cylinder (12 × 12)	A	A		A		

8 Conclusions and Recommendations

8.1 Summary

A detailed survey of quadrilateral membrane elements with rotational degrees of freedom was conducted. It was discovered that many of them are flawed except for the family of elements based on Allman's concept. The latter elements, however, have at least one spurious mechanism. Although this mechanism can be averted by artificially restraining at least one node from rotating, it is believed that this is not particularly desirable.

In the present work a new quadrilateral element with a rotational dof was derived based on Allman's concept. This element is mechanism free and it is much simpler to formulate and implement than the elements mentioned earlier. Extensive numerical tests showed it to be comparable to the best of the existing elements in nearly all situations. In addition, several procedures such as explicit shape function formulation, reduced integration and substitute shape functions, were implemented to improve the efficiency and accuracy of the element. Unfortunately some of these did not lead to increased accuracy.

Due to its good accuracy, the new membrane element was combined with an existing discrete Kirchhoff plate bending element to develop a new facet shell element. The element was subsequently tested for accuracy and efficiency. The conclusion was that in view of its simplicity and accuracy, it should be competitive in nonlinear analysis.

For the sake of completeness, the best of existing elements of this type, namely Q4S, was also implemented in a computer program. The element was tested and

the results were compared with those reported by its authors. While almost all of the tests they reported were duplicated here, the element failed to pass the patch test. This is contrary to the findings of the original authors who reported successful passing of this test. This discrepancy has still not been resolved despite extensive checking of the implementation.

8.2 Conclusions

1. It is possible to have a successful planar quadrilateral element with a rotational degree of freedom. This rotational or drilling degree of freedom cannot, however, be the true rotation if the element is to pass the patch test.
2. Allman's concept permits the formulation of these elements and a family of elements exists, but they are not defect free, they are either too complicated to formulate or they suffer from a spurious mechanism.
3. It is possible to formulate an element which is: simple, mechanism free, and rivals the accuracy of existing elements, as formulated in this study. The current element, called RQD4, gives good results in all cases except a slender cantilever beam with non-rectangular elements. The devices of reduced integration and substitute shape functions will not solve this locking phenomenon.
4. The combination of RQD4 with an improved discrete Kirchhoff plate bending element yielded a simple and accurate shell element as indicated by the good results of an extensive set of tests on shells.
5. Based on the preceding considerations, it can be stated that RQD4 and its associated shell element will be competitive both in linear and nonlinear problems.

8.3 Recommendations for Future Work

1. A study should be undertaken into whether some strain smoothing procedure, similar to that in Q4S, can be utilized to eliminate the locking phenomenon displayed in RQD4 without creating a spurious mechanism.
2. Some additional benchmark tests may be needed to further examine situations in which the locking phenomenon referred to may occur.
3. Work on an improved triangular counterpart of RQD4 is desirable because it will facilitate modelling the geometry of complex structures.
4. It would be desirable to combine RQD4 with a beam element in problems such as shear walls to determine its accuracy and compatibility. This is necessary since the drilling freedom of RQD4 is not the true rotation while the rotational freedom in a beam is.
5. The element has proven to be competitive in linear problems. It is important to solve nonlinear problems where it may offer the greatest advantage over elements such as QUAD8. Note, however, that for geometrically nonlinear problems more work will be needed to fully implement the element.

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Appendix A

Hand Calculation of the Stiffness Matrix of a Single Q4S Element

1. Choose an arbitrary quadrilateral element, Fig A.1

Figure A.1: Single element of arbitrary shape

2. Calculate the values of N_{QUAD4} .

Node	$N(\xi, \eta)$	Gauss Point			
		1	2	3	4
1	$\frac{(1-\xi)(1-\eta)}{4}$	$\frac{2.488034}{4}$	$\frac{2}{12}$	$\frac{2}{12}$	$\frac{0.178633}{4}$
2	$\frac{(1+\xi)(1-\eta)}{4}$	$\frac{2}{12}$	$\frac{0.178633}{4}$	$\frac{2.488034}{4}$	$\frac{2}{12}$
3	$\frac{(1+\xi)(1+\eta)}{4}$	$\frac{0.178633}{4}$	$\frac{2}{12}$	$\frac{2}{12}$	$\frac{2.488034}{4}$
4	$\frac{(1-\xi)(1+\eta)}{4}$	$\frac{2}{12}$	$\frac{2.488034}{4}$	$\frac{0.178633}{4}$	$\frac{2}{12}$

3. Calculate coordinates of Gauss points (needed to assemble $[X]$ matrix)

$$x(\xi, \eta) = \sum_{i=1}^4 (N_i(\xi, \eta)_{QUAD4}) x_i$$

$$y(\xi, \eta) = \sum_{i=1}^4 (N_i(\xi, \eta)_{QUAD4}) y_i$$

$$\begin{aligned} x_{g1} &= \left[(2.488034)(-1) + \left(\frac{2}{3}\right)(1) + (0.178633)(2) + \left(\frac{2}{3}\right)(-2) \right] / 4 \\ &= -0.699359 \end{aligned}$$

$$\begin{aligned} y_{g1} &= \left[(2.488034)(-1) + \left(\frac{2}{3}\right)(-2) + (0.178633)(2) + \left(\frac{2}{3}\right)(1) \right] / 4 \\ &= -0.699359 \end{aligned}$$

similarly:

i	x_{gi}	y_{gi}
2	-1.032692	0.69935867
3	0.6993586	-1.032692
4	1.032692	1.032692

which yields

$$[X] = \begin{bmatrix} 1 & -0.699359 & -0.699359 \\ 1 & -1.032692 & 0.699359 \\ 1 & 0.699359 & -1.032692 \\ 1 & 1.032692 & 1.032692 \end{bmatrix}$$

4. Determine the $[J]$ matrix.

$$J_1 = 1.3839746$$

$$J_2 = 2.25$$

$$J_3 = 2.25$$

$$J_4 = 3.116025$$

5. Determine $[S]$

$$[S] = [X][X^T J X]^{-1} [X]^T [J]$$

$$[S] = \begin{bmatrix} 0.421641 & 0.417618 & 0.417618 & -0.256877 \\ 0.256877 & 0.814516 & -0.185484 & 0.114091 \\ 0.256877 & -0.185484 & 0.814516 & 0.114091 \\ -0.114091 & 0.082382 & 0.082382 & 0.949327 \end{bmatrix}$$

This matched the program's calculation of $[S]$

6. Calculate terms of the auxiliary strain-displacement matrix

Calculate I_{xx} , I_{yy} , I_{xy} (centroid is origin thus, $\bar{x}$ and $\bar{y} = 0$)

$$I_{xx} = \sum_{g=1}^4 J_g x_g^2$$

$$I_{yy} = \sum_{g=1}^4 J_g y_g^2$$

$$I_{xy} = \sum_{g=1}^4 J_g x_g y_g$$

thus,

$$I_{xx} = 7.5$$

$$I_{yy} = 7.5$$

$$I_{xy} = 0.75$$

7. Calculate X_g and Y_g

g	$X_g = \frac{I_{yy}x_g - I_{xy}y_g}{I_{xx}I_{yy} - I_{xy}^2}$	$Y_g = \frac{I_{xx}y_g - I_{xy}x_g}{I_{xx}I_{yy} - I_{xy}^2}$
1	-0.084771	-0.084771
2	-0.148502	0.108098
3	0.108098	-0.148502
4	0.125175	0.125175

8. Calculate $[B_g^{(*)}]$

$$[B_g^{(*)}] = \sum_{j=1}^4 S_{gj} [B_j]$$

i.e.

$$[B_g^{(*)}]_{g=1} = S_{11}[B_1]_{RQD4} + S_{12}[B_2]_{RQD4} + S_{13}[B_3]_{RQD4} + S_{14}[B_4]_{RQD4}$$

These were verified with program's calculations

9. Determine $[K_{oo}]^{-1}$

$$[K_{oo}] = \sum_{g=1}^4 J_g [B_{go}]^T [D] [B_{go}]$$

where $E = 1.0 \times 10^6$, $\nu = 0.25$, $t = 0.001$ thus,

$$[D] = \begin{bmatrix} 1066.6667 & 266.6667 & 0 \\ 266.6667 & 1066.6667 & 0 \\ 0 & 0 & 400.00 \end{bmatrix}$$

Thus,

$$[K_{oo}]^{-1} = \begin{bmatrix} 0.00507298 & 0.00023059 \\ 0.00023059 & 0.00507298 \end{bmatrix}$$

Verified with program's calculations

10. Verified calculation of $[K_o]$

$$[K_o] = \sum_{g=1}^4 J_g [B_{go}]^T [D] [B_g^{(*)}]$$

11. Verified calculation of $[B_g]_{Q4S}$

$$[B_g]_{Q4S} = [B_g^{(s)}] - [B_{go}][K_{oo}]^{-1}[K_o]$$

12. Determine K_{Q4S}

$$[K]_{Q4S} = \sum_{g=1}^4 [B_g]^T [D] [B_g] J_g$$

Calculated to be:

620	218	-53	-314	-46	227	-309	-172	-187	3	0	12
218	620	53	0	3	-12	-172	-309	187	-46	-314	-227
-53	53	109	44	18	-38	27	-27	-33	-18	-44	-38
-314	0	44	450	-162	-323	137	-25	182	-274	187	97
-46	3	18	-162	460	60	49	-189	168	159	-274	-246
227	-12	-38	-323	60	361	-151	49	-188	246	-97	-135
-309	-172	27	137	49	-151	361	148	173	-189	-25	-49
-172	-309	-27	-25	-189	49	148	361	-173	49	137	151
-187	187	-33	182	168	-188	173	-173	410	-168	-182	-188
3	-46	-18	-274	159	246	-189	49	-168	460	-162	-60
0	-314	-44	187	-274	-97	-25	137	-182	-162	450	323
12	-227	-38	97	-246	-135	-49	151	-188	-60	323	361

Verified to be equal to the stiffness matrix calculated by the subroutine.

END

0|6|0|7|8|9

FIN