UNIVERSAL CYCLES FOR $(n - 1)$ -PARTITIONS OF AN n -SET

by

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Abstract

The sequence 00010111 has the interesting property that every binary sequence of length 3 appears exactly once within the sequence if we consider the end as looping back to the beginning. Such sequences are known as de Bruijn cycles and have found use in applications ranging from ancient Indian drumming to modern number theory, communications and coding theory.

We may also consider generating such sequences for other combinatorial objects, for example, k -subsets of an n -set, partitions of an n -set or permutations of an n -set. This thesis focuses on finding such cycles (called *universal cycles*) for partitions of an n -set into exactly $n - 1$ parts. This problem will turn out to be equivalent to finding certain Hamiltonian cycles in the complete graph on $n - 1$ vertices, and further reduces to determining when a permutation of a set of certain permutations equals the identity.

We describe the construction of such cycles, give necessary and sufficient conditions for when a Hamiltonian cycle can be translated into a universal cycle and prove a conjecture by Hurlburt that no universal cycles exist if n is even.

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Chapter 1

Introduction

Twenty five hundred years ago Indian rhythmists invented a new word:

“*yamātārājabhānasalagām*” (see [Kak00]). A syllable with an “ā” is called a *guru* (or long beat), otherwise it is a *laghu* (or short beat). They used this expression to help remember all possible triplets of *guru*’s and *laghu*’s. The word begins with *laghu, guru, guru*, follows with *guru, guru, guru*, and so on. If we write a 1 for a *laghu* and a 0 for a *guru* we get the binary string $B=1000101110$. We easily see that every binary sequence of length 3 appears exactly once somewhere in B . Furthermore, notice that the first two syllables match the last two, so we may place the first 8 digits of B around a circle to get a “memory wheel” where every sequence of length 3 appears precisely once somewhere on the circle.

In the 1880s the French engineer Émile Baudot (see [Kak00]) discovered and used a similar memory wheel when designing a new teletype machine, the difference being that his wheel contained all binary sequences of length five. In 1894 Rivière [Riv94] asked if it was possible to construct such memory wheels for any set of all m -ary sequences of length k . Soon after, he was affirmatively answered by Flye-Sainte

Figure 1.1: Indian memory wheel

Marie [Mar94]. In 1946 De Bruijn rediscovered the proof and so such memory wheels are popularly known as *m-ary De Bruijn cycles of order k*, but are also called *full length nonlinear shift register sequences*.

In the early 1990s, Chung et al. [CDG92] and Hurlburt [Hur90] asked when such sequences are possible for other sets of “combinatorial objects”. Here we define a set of combinatorial objects as a set whose elements can be generated in some pre-specified way. Suppose we can represent each object with some finite word over an alphabet $\mathcal{A}$ and we are able to generate some sequence from $\mathcal{A}$ such that each of the combinatorial objects is represented exactly once within the sequence as a subword of k consecutive letters. Chung et al. called such a sequence a *universal cycle*.

Their papers consider cases when the combinatorial objects are permutations of an n -set, partitions of an n -set, and k -subsets of an n -set. There are clearly many other objects one could consider including certain matroids, various designs (e.g. BIBD’s or Steiner triple systems), or finite dimensional subspaces of a vector space over the field $GF(q)$. We may even consider constructing a universal cycle out of the set of universal cycles for a family of combinatorial objects. Many of these are still open questions.

Given a set of combinatorial objects there are many questions one might ask.

- How will we choose an appropriate representation for the objects?
- Is one representation of the objects in some sense “better” than another?
- Does there exist a universal cycle for this set?
- If so, how many are there?
- Given an object, is there some way to determine where it is in the universal cycle?
- Are there higher dimensional analogues in the sense of a matrix with each submatrix representing a combinatorial object (a universal torus)?
- Can we find a sequence that represents each object t times (a t -cover universal cycle)?

In Chapter 2 we discuss the general foundations for universal cycles while chapter 3 deals with the classical case of de Bruijn cycles. Chapter 4 looks at universal cycles for partitions of an n -set, but Chapter 5 is the focus of this thesis where we analyze universal cycles for partitions of an n -set into exactly $n - 1$ parts. In [Hur90], Hurlburt shows the existence of a universal cycle for $(n - 1)$ -partitions of an n -set when n is odd and conjectures that none exist when n is even. We show that the problem of finding universal cycles is equivalent to figuring out when a permutation of permutations equals the identity, and we give necessary and sufficient conditions for this.

We shall make use of the following definitions and conventions throughout this paper. It is assumed that the reader is familiar with elementary graph theory (see [Wil96]) and permutation theory (see [DF91]), however as a reminder we state the following.

A directed path (i.e. a walk) on a directed graph G is called *Hamiltonian* if the path visits each vertex of G exactly once. Similarly a walk is *Eulerian* if it uses each edge of G exactly once. A walk is a *cycle* (or a *circuit*) if it begins and ends on the same vertex. A directed graph is strongly connected if there exists a path between any two vertices.

Theorem 1.1 *A graph G is Eulerian if and only if G is strongly connected and for each vertex v , $\text{indegree}(v) = \text{outdegree}(v)$.*

We let the n -set $\{1, 2, \dots, n\}$ be denoted by $[n]$ where n is a positive integer. We also denote the set $\{0, 1, 2, \dots, n\}$ by $[n]_0$.

A *permutation* of $[n]$ is a bijective function $\pi : [n] \rightarrow [n]$. Having said this, when working with permutations it is usually more convenient to use *cyclic* notation and we do so. We adopt the convention that permutations are multiplied (i.e. composed) from *right to left*. A permutation of the form $(i\ j)$ is called a *transposition* and we have the following fact (see [DF91]):

Theorem 1.2 *Fix n and let us denote the identity permutation on $[n]$ by ϵ . Suppose $\epsilon = \beta_1 \beta_2 \cdots \beta_r$, where every β_i is a transposition. Then r is an even number.*

We also make the following definition.

Definition 1.3 Let σ be a permutation on $[n]$. The *order* of σ is the least r such that $\sigma^r = \epsilon$.

Chapter 2

Universal Cycles

Definition 2.1 Let $\mathcal{F} = \{C_1, C_2, \dots, C_n\}$ be a finite family of combinatorial objects. Let $\mathcal{A}$ be a set of symbols (an *alphabet*) and let $\mathcal{A}^k$ be the Cartesian product of k copies of $\mathcal{A}$, where we only consider k finite. We call members of $\mathcal{A}^k$ *words*.

A *representation scheme* of $\mathcal{F}$ is an equivalence relation R on $\mathcal{A}^k$ together with a one-to-one function f from $\mathcal{F}$ to the set of R -equivalence classes on $\mathcal{A}^k$. The number k is the *rank* of the representation scheme.

By a *representation* of $C \in \mathcal{F}$ we mean the element $f(C)$. We take a word $(a_{i_1}, a_{i_2}, \dots, a_{i_k})$ from the R -equivalence class of $f(C)$ and write $f(C) = (a_{i_1}, a_{i_2}, \dots, a_{i_k})$. We usually omit the commas and write $f(C) = a_{i_1}a_{i_2} \dots a_{i_k}$, or $C = a_{i_1}a_{i_2} \dots a_{i_k}$ if there is no risk of confusion about which representation scheme we are using.

The set $f(\mathcal{F})$ is called a *representation* of $\mathcal{F}$.

Suppose $\mathcal{F} = \mathcal{A}^k$. If we take R as the usual component wise equality relation and f as the identity function then we say that $\mathcal{F}$ is *self-represented*.

Example. Let $\mathcal{F}$ be the family of binary sequences of length k . Then of course $\mathcal{F}$ can be self-represented. Another representation of a binary sequence (albeit not

particularly useful) can be described as follows. Let $C = b_{k-1}b_{k-2} \dots b_0$ be any binary sequence of length k and $\mathcal{A} = [k-1]_0 \cup \{*\}$. Here “ $*$ ” is just a symbol. Recall that if we consider C as the base 2 representation of an integer, then $C = b_{k-1}2^{k-1} + b_{k-2}2^{k-2} + \dots + b_02^0$.

Define the set $\mathcal{J} = \{j | b_j = 1\}$ and the multiset $\mathcal{X} = \{*, *, \dots, *\}$, where $|\mathcal{X}| = k - |\mathcal{J}|$, i.e. $|\mathcal{X}|$ is the number of zeroes in C .

Consider the multiset $\mathcal{J} \cup \mathcal{X}$. Then any listing of the elements of this set is a word which represents C . For example, if $C = 101101$ then we have $C = 2^5 + 2^3 + 2^2 + 2^0$. Thus, C has a representation $5, 3, 2, 0, *, *$. Notice that the order of the elements in the representation is not important, e.g. $*, 3, 0, 5, *, 2$ represents C as well.

As a rule of thumb we wish to keep the number of calculations needed in converting the representation to the represented object to a minimum. It is also worthwhile to keep the size of the alphabet to a minimum. Therefore, it is usually desirable to self-represent the family of objects however this is not always possible (e.g. if our combinatorial objects are trees with k nodes). Picking an appropriate representation scheme is often a balancing act between simplicity and applicability.

Definition 2.2 Let $\mathcal{F} = \{C_1, C_2, \dots, C_n\}$ be a family of combinatorial objects of rank k over some alphabet $\mathcal{A}$. Let $\alpha = (a_1, a_2, a_3, \dots, a_{n+k-1})$ be a word from $\mathcal{A}$. If for all $i = 1, \dots, n$ we have C_i represented by $a_i a_{i+1} \dots a_{i+k-1}$, then α is called a *universal string* or a *Ustring*. Furthermore, if for $j = 1, \dots, k-1$ we have $a_j = a_{n+j}$ then α is called a *universal cycle* or a *Ucycle*.

Remark. When we display a Ucycle, we shall usually omit the final $k-1$ letters, relying on the context to make clear the Ucycleness.

Representation schemes of families of combinatorial objects fall into two types, *ordered* and *unordered*. Ordered representation schemes are those in which the ordering of the symbols in a word is crucial for recovering the object. The simplest example are De Bruijn sequences. On the other hand, we define an unordered representation as one in which the order of the elements is irrelevant when recovering the object, i.e., any permutation on the symbols in the word also represent the object in question. For instance, see the second representation scheme discussed in the above example.

Definition 2.3 Let $\mathcal{F}$ be a family of combinatorial objects of rank k over an alphabet $\mathcal{A}$ with an ordered representation scheme. We define a directed graph T for this representation scheme, called the *transition graph* in the following way. To each $C \in \mathcal{F}$ we assign a vertex in T and label it with the equivalence class $f(C)$. If $C = a_1a_2 \dots a_k$ then we call the subword $a_1a_2 \dots a_{k-1}$ the *prefix* of C and $a_2a_3 \dots a_k$ the *suffix* of C . We construct a directed edge from vertex v_1 to vertex v_2 if a suffix of v_1 is equivalent to a prefix of v_2 , where we have defined an equivalence relation between words of length $k - 1$ beforehand.

If $\mathcal{F}$ has ordered representations, then a universal string corresponds to a directed Hamiltonian path in T while a universal cycle is a directed Hamiltonian cycle. As we shall see the converse is not always true; a Hamiltonian cycle is not, in general, always a universal cycle. However, note that if each object in $\mathcal{F}$ is represented by a *unique* word then the Hamiltonian cycle is a universal cycle. Unfortunately, the problem of deciding the Hamiltonicity of a graph is, in general, NP-complete. However, it is sometimes possible to have enough structure within T to enable us to reduce the problem to a much simpler one.

Using unordered representations to find a Ustring or Ucycle is often much more

difficult as we cannot construct a transition graph due to the fact that we may not be able to determine *a priori* which ordering of a representation is to be used. Of course, one could make a transition graph with *every* possible ordering of every representation but this would be computationally infeasible except for the most trivial cases. All is not lost however, as the representation may have enough structure to be able to find universal cycles. The majority of work so far on unordered representations has centered on self-represented k -subsets of an n -set and the interested reader is referred to [Hur90, Jac93] for further details.

We may naturally generalize the notion of a universal cycle (and analogously a universal string) in the following way.

Definition 2.4 Let a representation scheme of a family of combinatorial objects have rank k over some alphabet $\mathcal{A}$. A t -cover universal cycle is a word α from $\mathcal{A}$ such that each member of the family is represented by exactly t subwords of α .

A universal cycle as defined earlier is a 1-cover Ucycle and so we can make t -cover Ucycles for arbitrary t by simply repeating the original Ucycle t times. If there is no 1-cover Ucycle for some family of combinatorial objects, is there some t such that a t -cover Ucycle for this family exists?

The theory of universal cycles is young with many open questions. Most families of combinatorial objects have yet to be considered, and those that have are rich with structure which often has an intrinsic interest of its own.

Chapter 3

De Bruijn Cycles

Classically, De Bruijn cycles are Ucycles where we take $\mathcal{F}$ to be the self-represented family of binary sequences of length k . These cycles are also known as full length nonlinear shift-register sequences and have applications in diverse areas such as communication and coding theory, random word generators and even number theory (see [Hur90, Fre82]). We define an m -ary de Bruijn cycle of order k as a Ucycle for the family of all m -ary sequences of length k .

Theorem 3.1 (Flye-Sainte Marie 1894 and De Bruijn 1946) *m -ary De Bruijn cycles of order k exist for all m and k .*

Proof. Fix m and construct a transition graph T_k as described in Chapter 2. To prove existence we must prove that T_k is Hamiltonian. To do this we construct an arc digraph T_k^* as follows. If $(a_1a_2 \dots a_k, a_2a_3 \dots a_{k+1})$ is an arc in T_k then we will construct a vertex in T_k^* and label it as “ $a_2a_3 \dots a_k$ ”. Note that we get the set of all m -ary $(k-1)$ -tuples as the vertices.

Directed arcs in T_k^* are constructed the same way as in T_k . Thus $T_k^* \cong T_{k-1}$. If

$(a_1a_2 \dots a_{k-1}, a_2a_3 \dots a_k)$ is an arc in T_k^* then we label this arc as “ $a_1a_2 \dots a_k$ ”. Hence an Eulerian cycle in T_{k-1} corresponds to a Hamiltonian cycle in T_n .

Recall the conditions of Theorem 1.1. T_{k-1} is strongly connected since if $a_1a_2 \dots a_{k-1}$ and $b_1b_2 \dots b_{k-1}$ are vertices, then there are always (many) directed paths between them. For example,

$$\begin{aligned} a_1a_2 \dots a_{k-1} &\rightarrow a_2 \dots a_{k-1}0 \rightarrow a_3a_4 \dots a_{k-1}00 \rightarrow \dots \\ &\rightarrow 000 \dots 0 \rightarrow 000 \dots 0b_1 \rightarrow 000 \dots 0b_1b_2 \rightarrow \dots \rightarrow b_1b_2 \dots b_{k-1}. \end{aligned}$$

The second condition is also easy to see since for any vertex $a_1a_2 \dots a_{k-1}$ there are exactly m vertices of the form $xa_2 \dots a_{k-1}$, and each has an arc going to $a_1a_2 \dots a_{k-1}$. Similarly there are m vertices of the form $a_2 \dots a_{k-1}x$ each of which is incident with an arc originating from $a_1a_2 \dots a_{k-1}$. Hence for any vertex the indegree equals the outdegree and so T_{k-1} is Eulerian which implies T_k is Hamiltonian. $\blacksquare$

We give an easy algorithm to construct binary De Bruijn cycles of order k .

Theorem 3.2 (Preferred Ones Algorithm) *Given k , the following algorithm always produces a binary de Bruijn cycle of order k .*

1. Place k initial zeros.
2. Subsequently place ones unless doing so results in the repetition of a binary sequence, in which case place a zero.
3. Stop when we can place neither a one nor a zero.

Similarly there is a preferred zeros algorithm, and even a preferred “same” algorithm. For a good explanation and proofs of correctness, see [Fre82].

Figure 3.1: Transition graph for $n = 3$

It is even possible to count the number of m -ary De Bruijn cycles of order k . We denote this number by $B(m, k)$. De Bruijn and Van Aardenne-Ehrenfest [dBvAE51] found that

$$B(m, k) = \frac{(k!)^{k^{m-1}}}{k^m}.$$

There are many other questions one could ask. For example we may consider finding a De Bruijn torus. That is, a matrix B which has every $m \times n$ k -ary matrix as a submatrix (where we identify opposite sides of B). If we take $m = n = k = 2$, then the following matrix works.

$$\begin{array}{|c|c|c|c|} \hline 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \\ \hline 1 & 0 & 1 & 1 \\ \hline 0 & 1 & 1 & 1 \\ \hline \end{array}$$

We could also ask structural questions, for example, does there exist a symmetric binary de Bruijn cycle, i.e. where the first half of the cycle is the $(0,1)$ -complement of the last half. A good discussion of these questions can be found in [Hur90].

Chapter 4

Partitions of $[n]$

A *partition* of $[n]$ is a collection $\mathcal{C}$ of non-empty disjoint subsets of $[n]$ such that $\bigcup_{C \in \mathcal{C}} C = [n]$. For example a partition of $[7]$ is the collection $\{\{134\}, \{2\}, \{567\}\}$. Instead of using the cumbersome brackets we will denote partitions as, for example, $134|2|567$.

The number of ways of partitioning $[n]$ into k parts is called a Stirling number of the second kind, denoted by $S(n, k)$. We define $S(0, 0) = 1$ and $S(n, k) = 0$ if $k > n$. $S(n, k)$ satisfies the relation

$$S(n, k) = S(n - 1, k - 1) + kS(n - 1, k).$$

The total number of ways of partitioning $[n]$ is $\sum_{k=1}^n S(n, k)$.

For our purposes we represent a partition of $[n]$ by using letters of the English alphabet, $\mathcal{A} = \{a, b, c, \dots\}$, with the understanding that we may adjoin other symbols as needed. A partition is represented by a string of n letters such that the i th and j th letters (from the left) are the same if and only if i and j are contained in the same subset. For example, $134|2|567$ can be represented as $abaaccc$.

Note that the crucial aspect of this representation scheme is not the letters used but the relative positions of the identical letters. Thus, $134|2|567$ can also be represented as $deddaaa$. We call these two representations *equivalent* and in general two representations are equivalent if they are identical up to a permutation of their letters. Here we have an example of a representation scheme where every object has multiple ordered representations. We call this scheme the *CDGH-representation* (for Chung, Diaconis, Graham and Hurlburt).

Since this representation scheme is ordered, we may construct the transition graph T_n in the same way as usual: vertices of T_n are the equivalence class of partitions of $[n]$ and we include a directed edge from vertex x to vertex y if the suffix of the representation of x is equivalent to the prefix of the representation of y .

Theorem 4.1 (Hurlburt 1990) *Let $\mathcal{F}$ be the set of partitions of $[n]$ with the CDGH-representation scheme. Let T_n be the transition graph for the representations $\mathcal{F}$. Then T_n is Hamiltonian for all n .*

Proof. We can reduce this graph to a new graph T_n^* in a similar way as in Theorem 3.1. We let the vertices be the partitions of $[n-1]$ and a directed edge from vertices x to y (in T_n^*) if there exists a partition P of $[n]$ such that x is equivalent to the prefix and y is equivalent to the suffix of P . We label this edge as “ P ”.

Thus we have that T_n is Hamiltonian if T_n^* is Eulerian. To show connectivity of T_n^* , let x_0 be any vertex in T_n^* , and let the letter “ z ” be the last letter appearing in the representation of x_0 , where “ z ” has multiplicity k , i.e.,

$$x_0 = * * \cdots * * \underbrace{zz \cdots zz}_{k \text{ times}},$$

where the asterisks are any letters and $1 \leq k \leq n-1$. Then we let P be the partition of $[n]$ such that x_0 is the prefix and P equivalent to the concatenation of x_0 and

another z . Thus there is an edge from x_0 to the suffix of P , say the vertex x_1 where,

$$x_1 = * \cdots * * \underbrace{zz \cdots zz}_{k+1 \text{ times}}.$$

Therefore, we have a path from any vertex x to the vertex with representation “ $zz \cdots zz$ ”. By a similar argument we see that there is also a path from the vertex “ $zz \cdots zz$ ” to any vertex y . Thus, there is always a path between any two vertices in T_n^* .

Eulerianicity is easy to check since any partition of $[n - 1]$ into k parts has indegree and outdegree precisely $k + 1$. ■

Unfortunately we may now have a problem: a Hamiltonian cycle in T_n may not always translate into a universal cycle! For example if $n = 3$ the following is a Hamiltonian cycle: $123 \rightarrow 12|3 \rightarrow 13|2 \rightarrow 1|2|3 \rightarrow 1|23 \rightarrow 123$. If we now try to translate this into a universal cycle we get the Ustring “ $aaabacc$ ”, which is not a universal cycle (since $c \neq a$). In fact, it is easy to check that for $n = 2, 3$ there are no Ucycles, however for $n = 4$ the following is a Ucycle: “ $abcbccccddcdeec$ ”.

Theorem 4.2 (Hurlburt 1990) *A Ucycle for partitions of $[n]$ always exists if $n \geq 4$ using the CDGH-representation scheme.*

Proof. We consider the transition graph T_n and reduce to the Eulerian graph T_n^* as in the previous theorem. Essentially we show that there always exists an Eulerian subgraph in T_n^* which we call a *breaker*. The inclusion of the Eulerian cycle of the breaker within the Eulerian cycle of the full graph T_n^* enables us to prevent contradictions when checking for Ucycleness.

This is achieved by including within the breaker the partition of $[n]$ into one part (i.e. $[n]$ itself). The representation is the block “ $aa \dots a$ ”. Notice that since this

Figure 4.1: Eulerian cycle with breaker in T_n^*

block of a 's has length n , we can never infer whether a letter to the left of the block equals (or does not equal) a letter to the right. So if we construct a Ustring with this breaker included, as we pass the block " $aa \dots a$ " we can "adjust" the letters to the right so that they match up with the beginning of the Ustring to form a Ucycle. We require Eulerianicity to guarantee that the edges leading up to and from the block " $aa \dots a$ " are not used elsewhere. Figure 4.1 may be helpful in visualizing this idea.

The following is a breaker cycle for all $n \geq 4$ where $x_i \neq x_j$ and $y_i \neq y_j$ if $i \neq j$.

$$x_1 x_2 x_3 \dots x_{n-1} \underbrace{z z \dots z}_n y_1 y_2 y_3 \dots y_{n-1}.$$

Chapter 5

$(n - 1)$ -Partitions of $[n]$

5.1 Introduction

A k -partition of $[n]$ (for $n > 1$) is a collection of exactly k non-empty disjoint subsets of $[n]$ such that their union is $[n]$. We attempt to construct Ucycles using the family $\mathcal{F}$ of k -partitions as our combinatorial objects. We shall use the same CDGH-representation scheme for k -partitions as we did for general partitions in Chapter 4. If $k = n$ then a Ustring is any listing of n distinct letters, and if $k = 1$ then a Ucycle is any listing of n identical letters. This chapter deals with the case $k = n - 1$. Unless otherwise noted, all calculations in this chapter are done modulo n (but we also emphasize this on occasion).

We first note that a partition of $[n]$ into $n - 1$ subsets must be of the form $|1|2|\cdots|i,j|\cdots|n|$ for some $1 \leq i < j \leq n$. Thus, each $(n - 1)$ -partition corresponds to the choosing of exactly 2 elements from $[n]$, so that $|\mathcal{F}| = C(n, 2)$. For now we shall denote any $(n - 1)$ -partition by $|i, j|$, where i and j are the pair contained in the same subset.

We take the first $n - 1$ letters of the English alphabet as $\mathcal{A}$ (if $n > 27$ then we can add extra symbols as needed). Thus, $|i, j|$ is represented by a string of n letters where all letters are distinct except the i th and j th which are the same. For example, suppose $n = 7$. Then $|5, 7|$ is represented as $abcdefe$. Notice that in any Ustring the letter which comes after the second e must be a as otherwise we would get a block of n symbols with more than two equal letters. Hence, in any Ucycle for $n = 7$, $|5, 7|$ must be followed by $|4, 6|$, which in turn must be followed in order by $|3, 5|$, $|2, 4|$ and $|1, 3|$. At this point as we shift one letter to the right, we have a block of 6 distinct letters, thus the next letter can be any one of these six, i.e., $|1, 3|$ can be followed by $|i, 7|$ for any i which has not been previously chosen and where $1 \leq i \leq n - 1$.

It is easy to see that the above construction rule applies for any n . That is, suppose we construct a transition graph T , where the vertices are labeled by $|i, j|$. Then for every $k = 1, \dots, n - 1$ we are forced to have the path $|k, n| \rightarrow |k - 1, n - 1| \rightarrow \dots \rightarrow |1, n - k + 1|$. Let us contract this path into a single vertex and relabel it as the number “ $n - k$ ” with the understanding that in any Hamiltonian cycle in T , touching this vertex represents traversing the entire path. We call the vertex of T labeled by i , the *path* i .

As noted above, for all $k = 1, 2, \dots, n - 1$ and $m = 2, \dots, n - 1$ we have a directed path from $|1, m|$ to $|k, n|$. Thus after contracting all our forced paths, T reduces to the complete digraph on $n - 1$ vertices where each vertex has a unique label in the set $[n - 1]$. Furthermore, since any Hamiltonian cycle will by definition visit each vertex exactly once we may ignore loops and take T as the complete simple graph, K_{n-1} .

Any of the $(n - 1)!$ Hamiltonian cycles in T corresponds at least to a universal string. The question remains how many of these Ustrings are in fact Ucycles? Hurlburt [Hur90] proved that if n is odd, then a Ucycle always exists and guessed that

if n is even, none exist. The goal of this chapter is to prove this conjecture and attempt to show (with only partial success) exactly when any given Hamiltonian cycle corresponds to a universal cycle.

We begin by examining closely the construction of a universal string. Suppose we are given a Hamiltonian cycle as a sequence of vertices of K_{n-1} , i.e. we describe the cycle as $k_1, k_2, \dots, k_{n-1}$, where this sequence is some permutation of $1, 2, \dots, n-1$. It is useful here to consider the “ n -complement” sequence $l_1, l_2, \dots, l_{n-1}$ where $l_i = n - k_i$. We call l_i the *complement path* of k_i .

To construct the Ustring we begin by placing $n-1$ distinct letters, e.g. $a_1 a_2 \dots a_{n-1}$. Next, a_n is chosen to be the same letter as a_{l_1} . Then we are forced by the complement path l_1 to take $a_{n+1} = a_1, a_{n+2} = a_2, \dots, a_{n+l_1-1} = a_{l_1-1}$. Following this, the sequence $a_{l_1+1}, a_{l_1+2}, \dots, a_{n+l_1-1}$ is a sequence of $n-1$ distinct letters, and we follow the above procedure using l_2 and so on, until we have constructed the complement path l_{n-1} at which point the construction of the Ustring is complete. We may then check for Ucycleness.

Remark. We call a subsequence of $n-1$ distinct letters an *end state*.

Example. Let $n = 7$ and take the Hamiltonian cycle 2,6,3,1,4,5. The 7-complement is 5,1,4,6,3,2. Our alphabet is the set $\{a, b, c, d, e, f\}$. We begin by writing the sequence $abcdef$. Our first complement path is 5 so we take the 5th letter in our sequence (in this case e) and write it next. Following this we are forced to take a, b, c, d (in this order). So our Ustring so far looks like $abcdefeabcd$ and our new end state is the subsequence $feabcd$. The second complement path is 1, so we take the first letter of this subsequence (in this case f) and place it as the next letter. We immediately have a new end state so we continue onto the next complement path which is

4. By continuing this construction, we get the Ustring $abcdefeabdfceabdbfcea fbdec$. In this case since $abcdef \neq a fbdec$, our Hamiltonian cycle does not result in a Ucycle. The reader may wish to try the Hamiltonian cycle 1,2,3,4,5,6 to verify that this produces the Ucycle $abcdef fabcdedfabcaedfab cdef$.

5.2 The σ Permutations

The key to understanding which Hamiltonian cycles produce Ucycles and which do not lies in understanding what happens to the end states in the construction of the Ucycle. In particular notice that each path k_i corresponds to a permutation of the $n - 1$ distinct letters into the next end state of $n - 1$ distinct letters. Thus we can think of each k_i not only as a subpath in a Hamiltonian cycle but also as a certain permutation and the Hamiltonian cycle $k_1, \dots, k_{n-1}$ as a product of permutations. For clarity, we denote the permutation corresponding to k_i as σ_{k_i} , but we use these terms interchangeably.

Let some end state be written as $a_1 a_2 \dots a_{n-1}$. Then by the construction method for k_i we get that the Ustring must become $a_1 a_2 \dots a_{n-1} a_{n-k_i} a_1 a_2 \dots a_{n-k_i-1}$. Thus our next end state is the word $b_1 b_2 \dots b_{n-1} = a_{n-k_i+1} \dots a_{n-1} a_{n-k_i} a_1 a_2 \dots a_{n-k_i-1}$. In other words, the indices are permuted and

can be described by the following set of mappings:

$$\begin{array}{rcl}
 1 & \rightarrow & k_i + 1 \\
 2 & \rightarrow & k_i + 2 \\
 & \vdots & \\
 n - k_i - 1 & \rightarrow & n - 1 \\
 n - k_i & \rightarrow & k_i \\
 n - k_i + 1 & \rightarrow & 1 \\
 & \vdots & \\
 n - 1 & \rightarrow & k_i - 1.
 \end{array}$$

Closer inspection reveals that we can describe this permutation as

$$\sigma_{k_i}(j) = \begin{cases} k_i & \text{if } j = n - k_i, \\ k_i + j \pmod{n} & \text{if } j \neq n - k_i. \end{cases}$$

Notice that $k_i + j = 0$ only when $j = n - k_i$. Hence, $\sigma_{k_i}(j) \neq 0$ for any k_i since when $j = n - k_i$ we immediately go back to k_i .

It is usually more convenient to have a permutation written in cycle notation. It is not difficult to see that $\sigma_{k_i} = (k_i, 2k_i, 3k_i, \dots)(\dots)$. That is, we start with k_i and repeatedly add k_i until we encounter the j such that $j k_i \equiv n - k_i \pmod{n}$ (so that $(j + 1) k_i \equiv 0 \pmod{n}$). We then close this bracket and if necessary continue with the first unused member of $[n - 1]$, adding k_i until we get a repetition, etc. For example, if $n = 9$, then $\sigma_4 = (4 \ 8 \ 3 \ 7 \ 2 \ 6 \ 1 \ 5)$ and $\sigma_6 = (6 \ 3)(1 \ 7 \ 4)(2 \ 8 \ 5)$.

Lemma 5.1 $\sigma_{k_i}^{-1} = \sigma_{n-k_i}$ for all $k_i \in [n - 1]$.

Proof. Fix some k_i and let $j \in [n - 1]$. We have two cases to consider (all calculations are done modulo n):

Case i: Suppose $j = n - k_i$. Then,

$$\sigma_{n-k_i}(\sigma_{k_i}(j)) = \sigma_{n-k_i}(k_i) = n - k_i = j.$$

Case ii: Suppose $j \neq n - k_i$. Then,

$$\sigma_{n-k_i}(\sigma_{k_i}(j)) = \sigma_{n-k_i}(k_i + j) = n - k_i + k_i + j = j + n = j.$$

■

5.3 Existence and Enumeration

The following theorems should now be clear.

Theorem 5.2 *Let $k_1, k_2, \dots, k_{n-1}$ be a Hamiltonian cycle in K_{n-1} . Then this cycle forms a Ucycle for $(n - 1)$ -partitions using the CDGH-representation scheme if and only if the product of permutations $\sigma_{k_1}\sigma_{k_2} \cdots \sigma_{k_{n-1}} = \epsilon$, where ϵ is the identity permutation.*

Theorem 5.3 *If $k_1, k_2, \dots, k_{n-1}$ is a Ucycle for $(n - 1)$ -partitions using the CDGH-representation scheme, then any cyclic rotation, $k_j, k_{j+1}, \dots, k_{n-1}, k_1, k_2, \dots, k_{j-1}$ is also a Ucycle. Furthermore, any two Ucycles where one is a cyclic rotation of the other are equivalent (up to a cyclic rotation of the Ucycle and relabelling of the letters).*

Proof. Let $k_1, k_2, \dots, k_{n-1}$ be a Ucycle, i.e., $\sigma_{k_1}\sigma_{k_2} \cdots \sigma_{k_{n-1}} = \epsilon$. Then by left multiplication by respective inverses we get

$$\sigma_{k_j}\sigma_{k_{j+1}} \cdots \sigma_{k_{n-1}} = \sigma_{k_{j-1}}^{-1} \cdots \sigma_{k_1}^{-1}.$$

We may now right multiply the terms on the right-hand side by their respective inverses to get

$$\sigma_{k_j} \sigma_{k_{j+1}} \cdots \sigma_{k_{n-1}} \sigma_{k_1} \sigma_{k_2} \cdots \sigma_{k_{j-1}} = \epsilon.$$

Whence $k_j, k_{j+1}, \dots, k_{n-1}, k_1, k_2, \dots, k_{j-1}$ gives rise to a Ucycle. Furthermore, the construction of a given path is independent of the previously constructed paths (up to equivalence). Thus, if one Ucycle is a cyclic rotation of another they are equivalent.

■

Theorem 5.4 *If $\alpha = k_1, k_2, \dots, k_{n-1}$ lifts to a Ucycle for $(n-1)$ -partitions using the CDGH-representation scheme, then $\alpha^* = k_{n-1}, k_{n-2}, \dots, k_1$ lifts to a Ucycle.*

Proof. The construction of α^* is simply the reverse construction of α . Thus if α is a Ucycle then so is α^* . ■

Theorem 5.5 *Let $k_1, k_2, \dots, k_{n-1}$ be a Hamiltonian cycle in K_{n-1} . Suppose that for all k_i , one of the following (recursive) conditions are true:*

1. *The path $n - k_i$ is adjacent to k_i , or*
2. *The path $n - k_i$ is separated from k_i by a subsequence $k_{i+1}, \dots, k_{n-k_i-1}$ which itself satisfies conditions 1 and 2.*

Then $k_1, k_2, \dots, k_{n-1}$ forms a Ucycle for $(n-1)$ -partitions using the CDGH-representation scheme.

Essentially the two conditions in Theorem 5.5 state that if we can keep “crossing” out adjacent pairs which sum to n (corresponding to the two permutations being inverses) then we end up with the identity permutation and thus a Ucycle. Another way of looking at these conditions is by labeling $n - 1$ points on a circle by $k_1, k_2, \dots, k_{n-1}$

(in order) and connecting the pairs which sum to n with a chord. If there are no intersecting chords, then we have a Ucycle.

Theorem 5.6 *Let $n = 2m + 1$ and let N_n denote the number of Hamiltonian cycles that satisfy the conditions of Theorem 5.5. Then*

$$N_n = \frac{2^m(2m)!}{(m+1)!}.$$

Proof. The number of possibilities of joining $2m$ points on a circle with m non-intersecting chords is the m th Catalan number, C_m (see [CG96]) defined by,

$$C_m = \frac{(2m)!}{m!(m+1)!}.$$

For each possibility, we have $2m$ ways of labelling the endpoints of the first chord with some k and $n - k$, $2m - 2$ ways of labelling the endpoints of the second chord, etc. Thus we have,

$$N_n = 2m(2m-2)(2m-4) \dots (2)C_m = 2^m(m!)C_m = \frac{2^m(2m)!}{(m+1)!}$$

■

Corollary 5.7 *Let M_n be the number of Hamiltonian cycles that satisfy the conditions of Theorem 5.5, no two of which are equivalent up to cyclic rotation. Then*

$$M_n = \frac{2^m(2m-1)!}{(m+1)!}.$$

Proof. For any Hamiltonian cycle, there are $n - 1 = 2m$ cyclic rotations, hence,

$$M_n = \frac{2^m(2m)!}{(2m)(m+1)!} = \frac{2^m(2m-1)!}{(m+1)!}.$$

■

It would be quite tempting to conjecture the sufficiency of Theorem 5.5. Indeed, this would be true up to $n = 7$. Unfortunately there is a collection of counterexamples for $n \geq 9$ which will be discussed in the next section.

Recall that any permutation σ_k , when written in cycle notation, “skips” the number 0. However it is useful to look at it in another way. Let us consider σ_k not as skipping 0, but rather as keeping 0 fixed.

Let ς_k be the permutation defined by

$$\varsigma_k(j) = j + k \pmod{n}.$$

Note that ς is defined on the set $[n - 1]_0$.

Lemma 5.8 *The ς permutations have the following two properties:*

1. For any $k, l \in [n - 1]$,

$$\varsigma_k \varsigma_l = \varsigma_l \varsigma_k$$

2. For any transposition $(s \ t)$ and any $k \in [n - 1]$,

$$(s \ t) \varsigma_k = \varsigma_k(s - k, t - k)$$

Proof. Property 1 is trivial so we show Property 2. Note that the only numbers we need to examine are $s, t, s - k$ and $t - k$, as all other numbers will be fixed. Now on either side we have that $s - k \rightarrow t - k \rightarrow t$ and $t - k \rightarrow s - k \rightarrow s$. Thus any number is mapped to the same number on both sides. ■

Remark. Note that $(s \ t) \varsigma_{n-k} = \varsigma_{n-k}(s + k, t + k)$.

Lemma 5.9 *Let $k \in [n - 1]$. Then,*

$$\sigma_k = \varsigma_k(0, n - k)$$

Proof. Again we need only show that the two numbers in the transposition $(0, n - k)$ are mapped to the same number on both sides. On the right hand side we have that $0 \rightarrow n - k \rightarrow 0$ and $n - k \rightarrow 0 \rightarrow k$. On the left hand side we have that 0 is fixed and $n - k \rightarrow k$. Thus the equation holds. ■

We now take any Hamiltonian cycle $n - k_1, n - k_2, \dots, n - k_{n-1}$, and rewrite it in terms of ς -permutations and transpositions

$$\sigma_{n-k_1} \sigma_{n-k_2} \cdots \sigma_{n-k_{n-1}} = \varsigma_{n-k_1}(0, k_1) \varsigma_{n-k_2}(0, k_2) \cdots \varsigma_{n-k_{n-1}}(0, k_{n-1}). \quad (5.1)$$

We can now use Property 2 of Lemma 5.8 to rearrange the right hand side of (5.1) so that all transpositions are on the right hand side.

Let

$$S_i := \sum_{l=i}^{n-1} k_l \pmod{n}.$$

Then by successive applications of Property 2 of Lemma 5.8, we get

$$(\text{equation (5.1)}) = \varsigma_{n-k_1} \cdots \varsigma_{n-k_{n-1}}(S_2, S_1)(S_3, S_2) \cdots (S_{n-1}, S_{n-2})(0, S_{n-1})$$

Now if n is odd, then $\varsigma_{n-k_1} \cdots \varsigma_{n-k_{n-1}} = \epsilon$ due to the commutivity of the ς_k 's and the fact that for each ς_k , its inverse ς_{n-k} appears exactly once. We therefore have the following.

Theorem 5.10 *If n is odd, then the Hamiltonian cycle $\sigma_{n-k_1} \sigma_{n-k_2} \cdots \sigma_{n-k_{n-1}}$ is a Ucycle for $(n - 1)$ -partitions of $[n]$ if and only if the product of transpositions $(S_2, S_1)(S_3, S_2) \cdots (S_{n-1}, S_{n-2})(0, S_{n-1})$ is the identity.*

Remark. We call the product $(S_2, S_1)(S_3, S_2) \cdots (S_{n-1}, S_{n-2})(0, S_{n-1})$ the *associated product of transpositions*

A note on how to quickly construct the associated product of transpositions given the Hamiltonian cycle $n - k_1, n - k_2, \dots, n - k_{n-1}$. First, suppose n is odd. Then we have $S_1 \equiv \sum_{i=1}^{n-1} k_i \equiv 0 \pmod{n}$. Thus for any $1 < j \leq n - 1$,

$$\sum_{i=j}^{n-1} k_i \equiv - \sum_{i=1}^{j-1} k_i \equiv (n - k_1) + (n - k_2) + \dots + (n - k_{j-1}) \pmod{n}.$$

Hence,

$$S_j \equiv (n - k_1) + \dots + (n - k_{j-1}) \pmod{n}.$$

Let $P_j \equiv \sum_{i=1}^j (n - k_i)$, (i.e., $P_j = S_{j+1}$ for $1 \leq j \leq n - 2$ and $P_{n-1} = S_1$)

Then we take this sequence of partial sums $(\text{mod } n)$, $P_1, P_2, \dots, P_{n-1}$ and intertwine them as

$$(P_1, P_{n-1})(P_2, P_1) \cdots (P_{n-2}, P_{n-3})(P_{n-1}, P_{n-2}).$$

A similar procedure works if $n = 2m$ is even, except we have that $\sum_{i=1}^{n-1} (n - k_i) \equiv m \pmod{n}$. That is, we take the sequence of partial sums $(\text{mod } n)$ and intertwine them as

$$(P_1, m)(P_2, P_1) \cdots (P_{n-2}, P_{n-3})(P_{n-1}, P_{n-2}).$$

Note that $P_{n-1} \equiv m + (n - k_1) + \dots + (n - k_{n-1}) \equiv m + m \equiv 0$.

Example. Let $n = 9$ and take the Hamiltonian cycle $8, 7, 5, 6, 1, 2, 4, 3$. Then the sequence of partial sums $(\text{mod } 9)$ is $8, 6, 2, 8, 0, 2, 6, 0$ and so the product of transpositions is $(8\ 0)(6\ 8)(2\ 6)(8\ 2)(0\ 8)(2\ 0)(6\ 2)(0\ 6)$. Notice that this product is the identity, while the Hamiltonian cycle does not satisfy the conditions of Theorem 5.5. This is our first example of a *star Ucycle*, which is discussed in the next section.

Let $n = 8$ and take the Hamiltonian cycle $7, 5, 6, 1, 2, 4, 3$. Then the sequence of partial sums $(\text{mod } 8)$ is $7, 4, 2, 3, 5, 1, 4$. Adding 4 to each we get the sequence

3,0,6,7,1,5,0 and intertwining them we get the associated product of transpositions $(3\ 4)(0\ 3)(6\ 0)(7\ 6)(1\ 7)(5\ 1)(0\ 5)$.

Theorem 5.11 *If n is even then no Ucycles for $(n - 1)$ -partitions $[n]$ using the CDGH-representation scheme exist.*

Proof. First note that in the product $\varsigma_{n-k_1} \cdots \varsigma_{n-k_{n-1}}$, $\varsigma_{n/2}$ does not cancel with anything (since it is its own inverse and it only appears once). However note that $\varsigma_{n/2} = (0, n/2)(1, 1 + n/2)(2, 2 + n/2) \cdots (n/2 - 1, n - 1)$, which is a product of transpositions and contains each number between 0 and $n - 1$ exactly once.

Note that since $P_{n-1} \equiv 0 \pmod{n}$, we must have at least one number in $[n - 1]$ not appearing in the sequence of partial sums. But this number does appear exactly once in $\varsigma_{n/2}$. Thus, this number is not fixed and so we do not get the identity permutation. ■

Remark. There is an alternate proof of Theorem 5.11 if $n \equiv 0 \pmod{4}$. If $n \equiv 0 \pmod{4}$, then $\varsigma_{n/2}$ is a product of an even number of transpositions. Thus, the product

$$\varsigma_{n-k_1} \cdots \varsigma_{n-k_{n-1}}(S_2, S_1)(S_3, S_2) \cdots (S_{n-1} S_{n-2})(0, S_{n-1})$$

is a product of an odd number of transpositions which is never the identity.

5.4 Star Ucycles

For $n \geq 9$ we come across certain Hamiltonian cycles whose associated permutation product is the identity while the cycle does not satisfy the conditions of Theorem 5.5. We do not yet fully understand these exceptional cases, however, we endeavour to provide as much information about them as we can.

The Hamiltonian cycle 8,7,5,6,1,2,4,3 is one example. We call this a star Ucycle since if we evenly space 9 points around a circle and join with a chord those pairs which sum to 9, we notice that all 4 chords intersect at the centre of the circle. For $n = 9$, a computer search found that there are exactly 48 exceptional cases, all of which are star Ucycles. Unfortunately not all star formations are Ucycles. However, these 48 essentially reduce to three fundamental cases in the following sense: the other 45 cycles can be obtained from these three by cyclic rotations or inversions. For example from 8,7,5,6,1,2,4,3 we have that 6,1,2,4,3,8,7,5 and 3,4,2,1,6,5,7,8 are star Ucycles. The three fundamental cases for $n = 9$ are:

$$\alpha = 8, 7, 5, 6, 1, 2, 4, 3,$$

$$\beta = 8, 7, 3, 5, 1, 2, 6, 4,$$

$$\gamma = 8, 6, 7, 5, 1, 3, 2, 4.$$

An interesting property of these three cases is that if we take a unit a of $\mathbf{Z}_9$ and multiply each entry in the Hamiltonian cycle by this unit, we get a cyclic rotation of one of the 3 cases. We let $*$ denote the operation of multiplying each term in the sequence by a unit modulo n .

$1 * \alpha = 8, 7, 5, 6, 1, 2, 4, 3$	$1 * \beta = 8, 7, 3, 5, 1, 2, 6, 4$	$1 * \gamma = 8, 6, 7, 5, 1, 3, 2, 4$
$2 * \alpha = 7, 5, 1, 3, 2, 4, 8, 6$	$2 * \beta = 7, 5, 6, 1, 2, 4, 3, 8$	$2 * \gamma = 7, 3, 5, 1, 2, 6, 4, 8$
$4 * \alpha = 5, 1, 2, 6, 4, 8, 7, 3$	$4 * \beta = 5, 1, 3, 2, 4, 8, 6, 7$	$4 * \gamma = 5, 6, 1, 2, 4, 3, 8, 7$
$5 * \alpha = 4, 8, 7, 3, 5, 1, 2, 6$	$5 * \beta = 4, 8, 6, 7, 5, 1, 3, 2$	$5 * \gamma = 4, 3, 8, 7, 5, 6, 1, 2$
$7 * \alpha = 2, 4, 8, 6, 7, 5, 1, 3$	$7 * \beta = 2, 4, 3, 8, 7, 5, 6, 1$	$7 * \gamma = 2, 6, 4, 8, 7, 3, 5, 1$
$8 * \alpha = 1, 2, 4, 3, 8, 7, 5, 6$	$8 * \beta = 1, 2, 6, 4, 8, 7, 3, 5$	$8 * \gamma = 1, 3, 2, 4, 8, 6, 7, 5$

Note that each of α, β and γ falls in an orbit of size 6, however we do not get all of our exceptional permutations. To obtain the rest, let $\eta = 3, 8, 7, 5, 6, 1, 2, 4$. Then we

Figure 5.1: Transposition graph for star Ucycles when $n = 9$

have

$$1 * \eta = 3, 8, 7, 5, 6, 1, 2, 4,$$

$$2 * \eta = 6, 7, 5, 1, 3, 2, 4, 8,$$

$$4 * \eta = 3, 5, 1, 2, 6, 4, 8, 7,$$

$$5 * \eta = 6, 4, 8, 7, 3, 5, 1, 2,$$

$$7 * \eta = 3, 2, 4, 8, 6, 7, 5, 1,$$

$$8 * \eta = 6, 1, 2, 4, 3, 8, 7, 5.$$

We have 24 star Ucycles, and by inverting each, we get the other 24.

We can also consider a graph based on the transpositions in the product (we call it the “transposition graph”). It so happens that if we take the sequence of partial sums modulo 9 for each of the 48 star Ucycles, then each of these sequences happens to have exactly four different numbers. So we construct a graph with four vertices labelled by these four numbers and construct an edge between two vertices if the two labels appear in the same transposition. Interestingly, all 48 of these graphs are isomorphic to the graph in Figure 5.1.

For $n = 11$, our problem does not get easier. We still have star Ucycles (for example, $10, 8, 7, 2, 5, 1, 3, 4, 9, 6$), but we also get two more types. The first can be described as a nested star. An example is $10, 2, 4, 5, 8, 9, 7, 6, 3, 1$. The sequence of entries between 10 and 1 form the identity and they are in a star formation. The second

Figure 5.2: Visualization of star Ucycles for $n = 11$ Figure 5.3: Transposition graphs of star Ucycles for $n = 11$

type can be pictured by again arranging 10 points evenly spaced around a circle. We have two parallel lines while the other 4 pairs form a star. An example of this type is 10,3,2,5,4,8,1,9,6,7. Let us call this type a split star. See figure 5.2 for examples.

We can also draw a transposition graph for these types in the same way as we did for $n = 9$; see Figure 5.3.

In total for $n = 11$, a computer search found that there are 1900 star Ucycles. We summarize with the following table which lists for certain n the total number of Ucycles that satisfy theorem 5.5 (“normal” types), and the number of exceptional cases (found by computer searches, with help from John Dixon whom we graciously thank). The results were found using MATLAB for the lesser n , and GAP for larger n using different computers so a running time comparison would not be appropriate

here.

n	Normal	Exceptional	Total
3	2	0	2
5	16	0	16
7	240	0	240
9	5376	48	5424
11	161280	1900	163180
13	6082560	110688	6193248
15	276756480	???	???

The trend so far is that the ratio of exceptional cases to the total number of Ucycles is increasing. So at some point it may not be appropriate to call these exceptional cases. In any case the reason why these particular exceptions work is still a mystery.

For discussion's sake, let us look at the Ucycle 8,7,3,5,1,2,6,4 (for $n = 9$). It could just be random chance that the permutation $\sigma_2\sigma_6\sigma_4$ is conjugate to $\sigma_7\sigma_3\sigma_5$ in the symmetric group S_9 , however if these exceptional cases arose purely by luck, then we would expect to see other non-star exceptional cases as well. This author believe that the true reason is likely something we have not considered, since it seems unlikely that these star patterns will continue for larger n . We make the following conjecture.

Conjecture. Let n be odd and let $k_1, k_2, \dots, k_{n-1}$ be a Hamiltonian cycle which does *not* satisfy the conditions of Theorem 5.5. Let $S_1, S_2, \dots, S_{n-1}$ be the sequence of partial sums mod n . Then $k_1, k_2, \dots, k_{n-1}$ is a Ucycle if and only if exactly $(n-1)/2$ different numbers appear in $S_1, S_2, \dots, S_{n-1}$.

5.5 Further Explorations

One of the questions in Chapter 1 asked the following: given a combinatorial object, where is it in a given Ustring. In fact this is easy for $(n - 1)$ -partitions of $[n]$.

Theorem 5.12 *Suppose we are given a Hamiltonian cycle $k_1, k_2, \dots, k_{n-1}$ and the corresponding Ustring is the word $x_1 x_2 \dots x_{C(n,2)+n-1}$. Then the partition $|s, t|$, $s < t$, is the subword of length n starting at x_q where*

$$q = 1 + n - t + k_1 + k_2 + \dots + k_{i-1}$$

where $k_i = t - s$ and if $t - s = k_1$ then take $q = 1 + n - t$. (Note this is not done modulo n .)

Proof. Let v be the number such that $n - v = t$. First note that by definition of the paths in the Hamiltonian cycle, $t - s = k_i$ for one of the $n - k_i$ and so $|s, t|$ lies in this path. Now given an end state $y_1 y_2 \dots y_{n-1}$ we would construct the path k_i as usual. In other words, the partition $|n - k_i, n|$ starts at y_1 , $|n - k_i - 1, n - 1|$ starts at $y_2 (= y_{1+1})$ etc. Thus the partition $|s, t|$ starts at $y_{1+v} = y_{1+n-t}$.

Now given the Ustring $x_1 x_2 \dots x_{C(n,2)+n-1}$ and the path k_i at what x_j does the i th end state begin? This is easy. The first end state begins at x_1 , the second begins at x_{k_1+1} , the third begins at $x_{k_1+k_2+1}$ etc. It is easy to see then that the i th end state begins at index $j = 1 + \sum_{h=1}^{i-1} k_h$.

Combining these two pieces we see that from x_1 , the partition $|s, t|$ starts at x_q where,

$$q = 1 + n - t + k_1 + k_2 + \dots + k_{i-1}$$

where $k_i = t - s$ (and if $t - s = k_1$, then take $q = 1 + n - t$). ■

It is easy to generalize the discussions in the preceding sections to talk about t -cover Ustrings and Ucycles. In this case a t -cover Ustring is clearly equivalent to a path in K_{n-1}^* which visits each vertex exactly t times. Here K_{n-1}^* denotes the complete graph on $n - 1$ vertices where we also include a loop at each vertex. This path is a Ucycle if and only if the associated product of transpositions is the identity.

Theorem 5.13 *If n is even then, there always exists a 2-cover Ucycle for $(n - 1)$ -partitions.*

Proof. Take the path $n - 1, 1, n - 1, 1, n - 2, 2, n - 2, 2 \dots, n - n/2, n/2$. ■

Suppose we are given a Hamiltonian cycle α . For what t can we create a t -cover Ucycle by simply taking as our path $\alpha\alpha \dots \alpha$?

Theorem 5.14 *$t = k\phi$, where ϕ is the order of the associated product of transpositions of α and k is any positive integer.*

Proof. This is obvious. ■

Suppose we expand our alphabet to include more than $n - 1$ letters. Brett Stevens([Ste04]) conjectures that if we take n letters, then we can always construct a 1-cover Ucycle for any given Hamiltonian cycle by strategic placing of the extra letter(s). This author is somewhat dubious of this claim in its full generality, but it might be true in many cases.

A crucial property of $(n - 1)$ -partitions is that every partition has 1 subset of size 2 and $n - 2$ singleton subsets. So perhaps the next logical place to study is not k -partitions in general but the family of partitions where each partition has the same size subsets as every other. For example, if we consider the family of partitions where

Figure 5.4: Transition graph for $[5, 2]$ and $[5, 3]$.

each partition has one subset of size k and the other of size $n - k$, and we use the same CDGH-representation scheme for partitions as usual, then in fact this is one way of representing k -subsets of an n -set. Let us denote this family by $[n, k]$.

Example. The following is a Ucycle for $[5, 3]$: $aabbbaabab$. We note two interesting facts about this example. First, this Ucycle doubles as a Ucycle for $[5, 2]$. It is not difficult to see this property holds in general for the CDGH-representation scheme; a Ucycle for $[n, k]$ is a Ucycle for $[n, n - k]$.

In [SBE⁺02], Stevens et al. show that if we self-represent the set of k -subsets of $[n]$, then one cannot construct Ucycles for $k = n - 2$. Therefore the CDGH-representation scheme is advantageous in that we *do* have a Ucycle for $k = n - 2$, for one k and n , however, as we show below, this is not usually the case. Figure 5.4 shows the transition graph constructed as usual, except we have labelled the nodes with the partitions of $[5]$, but constructed the edges by our representation scheme.

We also give a final example; the following is a Ucycle for $[7, 3]$ and $[7, 4]$ using the CDGH-representation scheme:

$$aaabbbbbaaababbbbaabaabbaabababababb.$$

Theorem 5.15 *Ucycles (and Ustrings) for $[n, 2]$ with the CDGH-representation scheme only exist for $n = 2, 3$ and 5 .*

Proof. A Ucycle for $n = 2$ is aa , a Ucycle for $n = 3$ is aab and we have already given a Ucycle for $n = 5$. Now suppose n is none of these three values. Then the partition represented by $aa \dots abb$ (where there are $n - 2$ a 's) must be followed by $aa \dots abba$, which in turn must be followed by $aa \dots abbaa$, and so forth. However it is easy to see that eventually we just wind up back to $aa \dots abb$, thereby missing the partition $ababaa \dots a$ (for example). Thus, we may never construct a Ucycle or even a Ustring for such n . ■

Following our examples of Ucycles for $[7, 3]$ and $[5, 2]$, we propose the following.

Conjecture. A Ucycle for $[n, \frac{n-1}{2}]$ always exists for odd $n \geq 3$ using the CDGH-representation scheme.

The connectivity of the transition graph is at least plausible here since within the representations we have $C(n - 1, \frac{n-1}{2})$ suffixes (and prefixes) with exactly $\frac{n-1}{2}$ a 's and $\frac{n-1}{2}$ b 's, enabling us to write in as the next letter either a or b .

5.6 Conclusion

In [Hur90], Hurlburt shows that the Hamiltonian cycle $1, 2, \dots, n - 1$ is a universal cycle for n odd. We have shown that this permutation of $[n - 1]$ is a simple consequence of Theorem 5.5 and have given methods of determining when any permutation of $[n - 1]$ is a universal cycle. We also proved the conjecture by Hurlburt that no Ucycles for $(n - 1)$ -partitions exist if n is even.

In our attempt to enumerate all Ucycles, we came across a small class of permutations that do not satisfy Theorem 5.5, yet also lift to a Ucycle. These “star” Ucycles continue to be mysterious, yet surely must have a simple property which explains their Ucycleness. Unfortunately, finding this property eludes our efforts and we encourage other young researchers to take up this problem which seems to be quite mathematically interesting.

The theory of universal cycles for k -partitions is still in its infancy. For $2 \leq k \leq n - 2$ we are still stumped due to the fact that there is no known reduction to a Hamiltonian graph.

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