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RELATIVISTIC BOUND STATE EQUATIONS FOR MESONS IN LIGHT-FRONT MOMENTUM SPACE

by

Valér Drágán

A thesis submitted to the
Faculty of Graduate Studies and Research
in partial fulfillment of the requirements
for the degree of
Master of Science

Ottawa-Carleton Institute for Physics
Department of Physics, Carleton University
Ottawa, Ontario, Canada
December, 1995

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RELATIVISTIC BOUND STATE EQUATIONS FOR MESONS IN
LIGHT-FRONT MOMENTUM SPACE

submitted by Valér Drágán, B.Sc.,

in partial fulfillment of the requirements
for the degree of Master of Science

Thesis Supervisor

Chair, Department of Physics

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Date: 12. 1. 96

Abstract

An attempt has been made to relate the field theoretical Light-Front Tamm-Dancoff equation to relativistic bound state equations for mesons in the constituent quark picture. These bound state integral equations are formulated in momentum space using light-front variables. As an application, an analytical solution with a simple interaction kernel of the linear+Coulomb type is given utilizing hyperspherical harmonics. The mass spectra of charmonium and the Υ family have been successfully reproduced. The wavefunction for the pion has also been derived and used to calculate the decay constant and charge form factor of the pion.

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Introduction

Systems of objects bound together by a fundamental force have always provided excellent opportunities to investigate the nature of that force both experimentally and theoretically. For example, the weakest force of all, gravitation, has been studied by observing the earth-moon system or the solar system composed of planets orbiting around the Sun. Stronger interactions, like the electromagnetic force, produce bound states at the much smaller distance scale of atoms and molecules. The strongest known force—namely *the strong force*—involves even smaller distance scales and so requires higher energies to reveal its characteristics. The importance of a deeper understanding of strongly bound elementary particles is further emphasized by the fact that quarks, which at present are believed to be the most elementary particles interacting via the strong force, only exist in bound states. No free quarks have ever been observed in spite of extensive searches. As a result, highly developed theoretical and experimental tools, that worked extremely well for the electromagnetic and weak interactions, are quite difficult to implement for the study of the strong force. Perturbation theory, for instance, starts with free particles as asymptotic states and treats the interaction energy as a small perturbation to the free Hamiltonian. This formalism is especially suitable for experimentalists who have designed very accurate and sophisticated experiments to check the calculations of perturbation theory. Since electromagnetically interacting free particles (e.g. electrons) are readily available in nature, this situation has led to the development of an extremely successful theory, Quantum Electrodynamics.

Unfortunately, all features of the strong interaction cannot be easily investigated the same way, because the participants of scattering experiments are themselves composite bound objects, unlike the electron. Hence, there is a need to put some more effort into understanding bound states of quarks. Scattering experiments are nevertheless useful in finding structure functions and form factors which describe hadrons as composite objects. The analysis of such functions has led to the observation of Bjorken scaling that has a physical interpretation in the parton model.

The parton model identifies the structure functions with momentum distribution functions describing hadrons as a bunch of pointlike particles called partons, each parton carrying some fraction of the total momentum. Thus inelastic lepton-hadron scattering is viewed as an incoherent sum of elastic scatterings of the lepton on one particular parton inside the hadron.

Another model for the composite structure of hadrons—the quark model— has been proposed on theoretical grounds as a classification scheme for the observed mesons and baryons. Mesons are thought of as composed of quark-antiquark pairs, and baryons as bound states of three quarks. The apparent contradiction between the large number of partons observed and the small number of constituent quarks carrying the quantum number of a hadron is resolved in Quantum Chromodynamics by the distinction between the ‘valence’ quarks and the ‘sea’ quarks. In QCD, a baryon, for instance, is viewed as having three ‘valence’ quarks constantly radiating gluons which create quark-antiquark pairs inside the baryon. These $q\bar{q}$ pairs make up the ‘sea’ of quarks. The gluons, which are the mediators of the strong interaction, together with all the quarks can be identified with the partons.

Therefore, the effective potential in the constituent quark model is the sum of the interaction between valence quarks and contributions from the quark-gluon sea. It is very difficult to derive this potential from first principles in QCD. And what is more, the quark masses that enter the QCD Lagrangian are not the same as constituent quark masses. They are parameters of the chiral symmetry breaking and are usually referred to as current-algebra quark masses [1]. The relationship between constituent and current quarks has been established by Melosh [2] by constructing a unitary current-constituent transformation for free quarks. For interacting quarks, a generalized Melosh transformation has been derived by Berestetsky and Terent'ev [3] using Light-front Dynamics [4],[5].

Light-front Dynamics is based on a description of a Hamiltonian approach to relativistic many-body systems formulated by Dirac [4]. States in this formalism are defined on the light-cone instead of the usual $t = \text{const.}$ surface, and evolve along the $t = z$ direction instead of evolving in time. There are many advantages of this

choice as will be discussed explicitly in Section 1.1. It is possible to describe many-particle systems in analogy with nonrelativistic techniques familiar from Schrödinger theory. Another reason to choose light-front variables is the recent prospering of the Light-front Tamm-Dancoff method (LFTD)[6],[7],[8] which is a promising candidate as a nonperturbative field-theoretical approach to relativistic bound states. Section 1.2 will outline the important features of this method.

We would like to shed some light on the relationship between LFTD and phenomenological descriptions based on the constituent quark model. Equations in the constituent quark model are usually written in spatial coordinates in the Equal-time formalism. A different approach will be followed in this thesis by writing the equation for mesonic wavefunctions in momentum space using the light-front formalism. It will be shown that the LFTD equation has exactly the same structure. Jaus [9] has derived the same equation from the quasipotential formalism [10] as well, which is summarized in Section 2.2.

After establishing the basic equation for the bound state of two quarks, we will move on to solve for the wavefunctions and mass eigenvalues of mesons in Chapter 3. The solution is based on a very elegant analytical technique developed by Lévy [11] employing a mapping to a higher dimensional space which allows the expansion of the wavefunction in terms of hyperspherical harmonics. For the interaction kernel we have chosen a Coulombic potential together with a linear confining potential. Our result for the pion wavefunction is then used in formulas for the calculation of electroweak properties of the pion, namely its decay constant and charge form factor. These properties have been calculated by Jaus [9] and by Cardarelli et al.[12] as well, using light-front variables. Jaus has used an ad-hoc gaussian wavefunction, while Cardarelli et al. have worked in the Isgur & Godfrey [13] potential model with wavefunctions expanded in a harmonic oscillator basis. This model is based on a linear+Coulomb potential including hyperfine and spin-orbit interactions. A brief description of it is given in Section 2.1. The pion decay constant and form factor obtained by the above authors [9],[12] are compared to our results in Section 3.2 and in the Conclusion.

Chapter 1

The Light-front Tamm-Dancoff Method

1.1 Front-form Dynamics

Any dynamical theory that describes particle systems should satisfy the principles of special relativity as confirmed by an overwhelming number of experiments. That is, all inertial observers should experience the same physical laws and be able to write them in the same form. Inertial reference frames, in general, are related by inhomogeneous Lorentz transformations consisting of space-time translations (b^μ) and homogeneous Lorentz transformations (a^μ_ν):

$$x'^\mu = a^\mu_\nu x^\nu + b^\mu \quad (1.1)$$

where x and x' are the space-time coordinates of the same event in two different coordinate systems. The quantities representing the transformations (a, b) , form a group which is referred to as the Poincaré group. This group includes the discontinuous time reversal and parity transformations besides the continuous Lorentz transformations. In the following, by Poincaré group we mean the proper subgroup of the inhomogeneous Lorentz transformations which involves continuous transfor-

mations only. In quantum mechanical systems, the Poincaré group is represented by a set of unitary operators $U(a, b)$ which relate the x -frame state vector $|\Psi\rangle$ to the x' -frame state vector $|\Psi'\rangle$:

$$|\Psi'\rangle = U(a, b) |\Psi\rangle \quad (1.2)$$

Since a and b are continuous, they can be built up from infinitesimal transformations:

$$a_{\mu\nu} = g_{\mu\nu} + \epsilon_{\mu\nu} , \quad b_\mu = \epsilon_\mu \quad (1.3)$$

where $g_{\mu\nu}$ is the metric and the ϵ 's are infinitesimals. The generators of (1.3) are the four-momentum P_μ and the angular momentum tensor $M_{\mu\nu}$ which contains the angular momenta J_i and the boosts K_i :

$$M_{ij} = -M_{ji} = \epsilon_{ijk} J_k \quad M_{0i} = -M_{i0} = K_i \quad (1.4)$$

In spatial coordinate representation with $\hbar = c = 1$, they can be written as

$$P_\mu = i\partial_\mu \quad M_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu) \quad (1.5)$$

Thus $U(a, b)$ corresponding to the infinitesimal transformation (1.3) can be represented by the following form:

$$U(a, b) = 1 + i\epsilon^\mu P_\mu - \frac{i}{2}\epsilon^{\mu\nu} M_{\mu\nu} \quad (1.6)$$

Two successive Poincaré transformations, (a, b) followed by (a', b') , are equivalent to a single transformation (a'', b'') with

$$a'' = a'a \quad , \quad b'' = a'b + b' \quad (1.7)$$

Therefore the U 's have to satisfy the multiplication law

$$U(a'', b'') = U(a', b') U(a, b) \quad (1.8)$$

This implies that the generators (1.5) obey the following commutation relations:

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(g_{\nu\rho} M_{\mu\sigma} - g_{\mu\rho} M_{\nu\sigma} + g_{\mu\sigma} M_{\nu\rho} - g_{\nu\sigma} M_{\mu\rho}) \quad (1.9)$$

$$[P_\mu, P_\nu] = 0 \quad (1.10)$$

$$[P_\mu, M_{\rho\sigma}] = i(g_{\mu\rho} P_\sigma - g_{\mu\sigma} P_\rho) \quad (1.11)$$

It was first pointed out by Dirac [4] that there is no unique way of including interactions in those generators. The only requirement is that a system of interacting particles still has to constitute a representation of the Poincaré group. Therefore, whichever way interaction terms are added to the generators of the corresponding noninteracting system, the above commutation relations must be satisfied. In non-relativistic systems, the Poincaré group is replaced by the Galilean group in the limit of $c \rightarrow \infty$, and it is possible to add an interaction to the generator of time translations (the Hamiltonian) only. All commutation relations can be satisfied this way. In the relativistic case, however, the commutation relations of the Poincaré algebra can only be satisfied by adding interaction terms to more than one generator. It is a matter of choice which generators constitute the dynamical—or interacting—subset and which are kinematical. Thus, in fact, an infinite number of dynamics may be defined, each differing in the way the kinematical subset of generators is separated from the dynamical subset. The kinematical generators form a subgroup of the Poincaré group, usually called the stability group, consisting of generators of Lorentz transformations that leave some 3-dimensional hypersurface of Minkowski-space invariant. The more symmetries this invariant surface has, the larger the stability group, making the dynamical subset smaller. Moreover, causality suggests that only spacelike directions should be contained in the invariant surface so that world lines cannot cross it twice. This is necessary for a unique definition of physical

states. Initiated by Dirac's idea [4] and guided by the above mentioned considerations, Leutwyler and Stern [5] derived five distinct classes of possible kinematic surfaces that can be constructed. These are the following:

- 1) the time instant $x^0 = \text{const.}$;
- 2) one sheet of a hyperboloid $x^2 = a^2 > 0, x_0 > 0$;
- 3) the light-cone $x^0 + x^3 = 0$;
- 4) one sheet of the hyperboloid $x_0^2 - x_1^2 - x_2^2 = a^2 > 0, x_0 > 0$;
- 5) one sheet of the hyperboloid $x_0^2 - x_3^2 = a^2 > 0, x_0 > 0$.

In the most familiar choice, one adds the interaction to the Hamiltonian P_0 which evolves the system in time x_0 . This case is called Instant-form (or Equal-time) Dynamics, for the kinematic subgroup leaves physical quantities defined at an instant of time invariant. In this case the members of the stability group are the generators of space-translations (P_i) and angular momenta J_i which generate rotations in space. There are four dynamical generators which evolve the system away from the surface $x^0 = \text{const}$: the rotationless boosts K_i and the Hamiltonian P_0 .

The dynamics based on the light-cone as invariant surface is referred to as Front-form Dynamics. It is the most advantageous form in that the kinematic subgroup is the largest, containing seven generators. If we define the light-front coordinate system as introduced by Dirac [4]:

$$(x^+, x^1, x^2, x^-) = \left(\frac{x^0 + x^3}{\sqrt{2}}, x^1, x^2, \frac{x^0 - x^3}{\sqrt{2}} \right) \quad (1.12)$$

then the seven kinematic generators can be written as:

$$P^+, \bar{P}_\perp, J_3, K_3, \frac{K_2 - J_1}{\sqrt{2}}, \frac{K_1 + J_2}{\sqrt{2}} \quad (1.13)$$

Here transverse angular momenta are mixed with transverse boosts and neither J_1 nor J_2 in itself is kinematic as opposed to the Instant-form where all components

of the angular momentum are kinematic. This fact, that J_3 is kinematic while $\vec{J}_\perp$ are dynamic, is responsible for the loss of manifest rotational invariance. As a result, angular momentum cannot be separated from the dynamical part of the problem, and states cannot be expressed as angular momentum eigenstates of J^2 and J_3 . The three dynamical generators that evolve the system away from the initial surface $x^+ = 0$ can be written as

$$P^-, \frac{K_1 - J_2}{\sqrt{2}}, \frac{K_2 + J_1}{\sqrt{2}} \quad (1.14)$$

where P^- is the light-front Hamiltonian being conjugate to the light-front time x^+ .

Beside having the largest stability group, a major advantage of Front-form Dynamics becomes apparent in quantum field theory: interacting quantum fields defined on the light-front generate Fock-states from the physical vacuum. It is an implication of the Einstein-relation :

$$P^2 = 2P^+P^- - \vec{P}_\perp^2 = M^2 \quad (1.15)$$

Both P^+ and P^- must be positive for positive energies, so by momentum conservation, pair creation from the vacuum is impossible.

Another useful application of light-front coordinates can be found in techniques employing infinite momentum frames [14] [15] where the coordinate system is boosted longitudinally to the speed of light. Thus motion of the particles is essentially along the $x^0 = x^3$ direction. It must be emphasized, however, that the use of light-front coordinates is not equivalent to boosting the system to infinite momentum frames. The simplicity of the vacuum is present in any frame employing light-front coordinates, not just in infinite momentum frames.

1.2 The Tamm-Dancoff Method

In the 1950's Tamm and Dancoff [16] independently proposed an approximation technique in an attempt to treat strong interaction processes involving hadrons. They tried to solve a relativistic Schrödinger equation for the deuteron in a truncated Fock-space. The nonrelativistic limit of this equation was shown to be just the nonrelativistic Schrödinger equation with a Yukawa potential. The method was formulated in the usual Equal-time form.

In solving Tamm-Dancoff type equations, one starts by expanding the state in terms of eigenstates of the free Hamiltonian. These states are also eigenstates of the number operators of the free fields (Fock states). The next step is to project the equation onto each member of the Fock space. As a result, one obtains an infinite set of coupled integral equations for the amplitudes of finding the system in a particular Fock state.

The basic idea of the Tamm-Dancoff method is that the essential physics is assumed to be describable in a truncated Fock space, where only a few Fock states are considered important. Adding more excitations would only refine the theory. Thus the infinite set of integral equations is truncated to a finite set.

In solving these coupled integral equations, vacuum creation and annihilation terms are brought together since the Hamiltonian contains terms which create (or annihilate) particle-antiparticle pairs from (or into) the vacuum. Thus disconnected, divergent vacuum diagrams result, just like in normal perturbation theory for scattering problems. However, in perturbation theory, the contribution of these divergent terms factor out and they do not contribute to transitions. Hence they are easily thrown away as being spurious. On the other hand, in Tamm-Dancoff type equations, the divergent vacuum terms are part of the kernels of integral equations and it makes them hard to dispose of. For details on the attempts to renormalize Tamm-Dancoff theory in the Equal-time formalism, see the papers by Dyson [17] and by Dalitz and Dyson [18]. It was found by those authors that the renormalization of the divergent self-energies and vacuum terms appearing in the integral

equations resulted in the appearance of further nonphysical singularities. Due to these difficulties, the method was soon abandoned.

In the 1970's, interest in the Tamm-Dancoff method was stimulated again by the use of infinite momentum frames. Weinberg [14] has shown that all diagrams that caused divergences in old-fashioned perturbation theory are eliminated in such frames. Later it was realized that divergent vacuum diagrams do not even appear in Front-form dynamics, independently of the reference frame.

Although the most difficult problem with the original Tamm-Dancoff proposal is thus removed in light-front coordinate systems, there are still ultraviolet divergences related to large transverse momenta. Furthermore, for the exchange of massless particles, p^+ can be zero. This infrared problem manifests itself in two ways. Very often kernels are of the convolution type, that is they depend on $\vec{p} - \vec{p}'$ where p' runs from 0 to ∞ , so they are divergent both for small magnitudes of $\vec{p}$ and $\vec{p}'$ and for $\vec{p} = \vec{p}'$. This second case is usually referred to as collinear singularity, since it occurs when the angle between $\vec{p}$ and $\vec{p}'$ is zero.

The ultraviolet divergences can be removed by a renormalization scheme proposed by Perry and Harindranath [6] in 1+1 dimension and extended to 3+1 dimensions by Wilson et al.[7] and by Wort [8]. This involves the assumption that the bare masses appearing in the integral equations are Fock space sector dependent. Wort's method, in a nutshell, consists of the following steps. First, the contribution of 'instantaneous' terms in the Hamiltonian to the highest sector integral equation was omitted. These terms only appear in light-front form, they are not present in the equal-time formalism. They are higher order in the coupling constant than the sector order and analogous to 'seagull' diagrams in QED. All instantaneous terms appearing in one sector would be needed to cancel other terms from higher sectors, so these terms were just an artifact of the truncation of Fock space. The consequence of omitting them is that the highest sector equation has become an algebraic equation. Then, the use of a threshold condition allows the identification of fermion and boson mass counterterms which represent an infinite renormalization. These counterterms follow from the condition that at threshold, since the

highest sector amplitude contains a pole corresponding to the scattering states, its coefficient must vanish. The next step is to solve for the highest sector amplitude, and then eliminate it from lower sectors. Wort only considered the first two sectors in the first Tamm-Dancoff approximation, but the method is easily generalized to including an arbitrary number of Fock states.

In 1+1 dimensions, the above procedure eliminates all divergences. In 3+1 dimensions, however, there are further divergences remaining in the mass counterterms at high transverse momenta. These are removed by introducing an upper cutoff Λ in transverse momenta, and using coupling constant renormalization. We will not discuss the details here, only mention that an important consequence of such renormalization is the so-called triviality constraint. It is a constraint on the largest possible value of the renormalized coupling constant for any given finite cutoff. As the cutoff is increased, the maximum allowed value of the renormalized coupling decreases in order to prevent the bare coupling from becoming imaginary. The cutoff can only go to infinity if the coupling went to zero. For asymptotically free theories, this does not present a problem since the coupling constant does vanish at large momentum scales. The infrared problem (i.e. when $p^+ \rightarrow 0$) is dealt with using a lower cutoff in the light-front longitudinal momentum (p^+).

Let us turn to an example to illustrate how the Tamm-Dancoff method works. We will mainly follow Wort's arguments [8], who considered a two-fermion system with an interaction mediated by a pseudoscalar boson. In the first Tamm-Dancoff approximation, the state $|\Psi(P)\rangle$ is expanded in the truncated Fock space where only the two-fermion state and the two-fermion/one-boson state are retained:

$$|\Psi(P)\rangle = \int d^3p_1 d^3p_2 \Delta_\delta^0(\Lambda, p_1, p_2) C_0(p_1, p_2) B^\dagger(p_1) b^\dagger(p_2) |0\rangle \\ + \int d^3p_1 d^3p_2 d^3p_3 \Delta_\delta^1(\Lambda, p_1, p_2, p_3) C_1(p_1, p_2, p_3) B^\dagger(p_1) b^\dagger(p_2) a^\dagger(p_3) |0\rangle \quad (1.16)$$

where the cutoff functions are defined by

$$\begin{aligned}\Delta_\delta^0(\Lambda, p_1, p_2) &= \theta(p_1^+) \theta(p_2^+) \delta(P^+ - p_1^+ - p_2^+) \delta^2(\vec{P}_\perp - \vec{p}_{1\perp} - \vec{p}_{2\perp}) \\ &\times \theta\left(\Lambda - \frac{m_1^2 + p_{1\perp}^2}{2p_1^+} - \frac{m_2^2 + p_{2\perp}^2}{2p_2^+}\right)\end{aligned}\quad (1.17)$$

$$\begin{aligned}\Delta_\delta^1(\Lambda, p_1, p_2, p_3) &= \theta(p_1^+) \theta(p_2^+) \theta(p_3^+) \\ &\times \delta(P^+ - p_1^+ - p_2^+ - p_3^+) \delta^2(\vec{P}_\perp - \vec{p}_{1\perp} - \vec{p}_{2\perp} - \vec{p}_{3\perp}) \\ &\times \theta\left(\Lambda - \frac{m_1^2 + p_{1\perp}^2}{2p_1^+} - \frac{m_2^2 + p_{2\perp}^2}{2p_2^+} - \frac{m_3^2 + p_{3\perp}^2}{2p_3^+}\right)\end{aligned}\quad (1.18)$$

and sum over spins is suppressed. Now projecting the Einstein equation

$$(2P^+P^- - P_\perp^2)|\Psi\rangle = M^2|\Psi\rangle \quad (1.19)$$

onto noninteracting two-fermion and two-fermion/one-boson states

$$\begin{aligned}|p_B, p_b\rangle &= B^\dagger(p_B) b^\dagger(p_b) |0\rangle \\ |p_B, p_b, p_a\rangle &= B^\dagger(p_B) b^\dagger(p_b) a^\dagger(p_a) |0\rangle\end{aligned}$$

one obtains two integral equations for C_0 and C_1 , where C_0 is the probability amplitude to find the state $|\Psi\rangle$ in the two-fermion state, and C_1 is the amplitude for the state to be in the two-fermion/one-boson state. Introducing the momentum fraction variables

$$x = \frac{p_B^+}{P^+} \quad z = \frac{q^+}{P^+} \quad (1.20)$$

and carrying out the renormalization process briefly discussed above, after eliminating the highest sector amplitude C_1 , one gets the following equation for the lowest sector amplitude C_0 :

$$\begin{aligned}
\left[M^2 - \frac{m_B^2}{x} - \frac{m_b^2}{1-x} - \frac{p_{B\perp}^2}{x(1-x)} + \tilde{\lambda}^2 \Delta \right] C_0(x, \vec{p}_{B\perp}) = \\
\frac{2\tilde{\lambda}^2}{(2\pi)^3} m_B m_b \int d^2 q \int dz \frac{1}{(x-z)[x(1-x)z(1-z)]^{\frac{1}{2}}} \\
\times \left[\frac{\Theta(z-x)}{D_a} - \frac{\Theta(x-z)}{D_b} \right] C_0(z, \vec{q}_\perp)
\end{aligned} \tag{1.21}$$

where

$$D_a = M^2 - \frac{m_B^2}{x} - \frac{m_b^2}{1-z} - \frac{m_a^2}{z-x} - \frac{z p_{B\perp}^2}{x(z-x)} - \frac{(1-x)q_\perp^2}{(z-x)(1-z)} + \frac{2\vec{p}_{B\perp} \cdot \vec{q}_\perp}{z-x} \tag{1.22}$$

D_b is similar with $x \rightarrow 1-x$, $m_B \leftrightarrow m_b$, $z \rightarrow 1-z$. $\tilde{\lambda}$ is the renormalized coupling constant, Δ is the mass correction term, m_B and m_b are renormalized fermion masses and m_a is the renormalized boson mass. The amplitude C_0 is the momentum-space wavefunction for the bound state of two fermions. This equation was derived in the Yukawa model where the exchange particle is a massive pseudoscalar boson.

There are two important features of the equation. First note that it is a 3-dimensional momentum space integral equation with a kernel that depends on the mass of the bound state. We can rewrite the equation formally by grouping some terms together in order to see its structure more clearly. If the mass correction term is taken to the right hand side and included in the kernel with a delta-function, the equation can be written in the following form:

$$\left[M^2 - M_0^2(m_B, m_b, \vec{p}_{B\perp}) \right] C_0(x, \vec{p}_{B\perp}) = \kappa \int d^2 q_\perp \int dz K(M, x, \vec{p}_{B\perp}; z, \vec{q}_\perp) C_0(z, \vec{q}_\perp) \tag{1.23}$$

where κ is an eigenvalue depending on the coupling constant and on the masses of the constituent fermions. M_0^2 is a function of the momentum and masses of the

constituents :

$$M_0^2 = \frac{m_B^2}{x} + \frac{m_b^2}{1-x} + \frac{p_{B\perp}^2}{x(1-x)} \quad (1.24)$$

It can be rewritten to get

$$M_0^2 = \frac{m_B^2 + p_{B\perp}^2}{x} + \frac{m_b^2 + p_{B\perp}^2}{1-x} \quad (1.25)$$

The reason for choosing the name M_0 will become obvious in Section 2.1 where it is shown that M_0 is the mass operator of the noninteracting two-particle system.

The second feature of the Tamm-Dancoff equation is that the integration over longitudinal momentum is treated distinctively from transverse momentum integration. This is an indication of the loss of manifest rotational invariance. In the previous section we saw that this problem arises in every dynamical theory defined on the light-front. A remedy exists in the form of a momentum dependent 'Melosh'-type rotation [2] [3] that has to be applied to the spins of individual particles. We will indicate in the next chapter how this is done.

The solution of the Tamm-Dancoff bound state equation given by Wort was a numerical one. First the 3-dimensional integral was reduced to a 2-dimensional integral, after Fourier-expanding the amplitude in terms of the azimuthal angle and performing the angular integration. Then the longitudinal (x) and transverse ($q_\perp$) integrations were performed using the Gaussian quadrature technique. Thus the integral equation was transformed into a matrix equation. In this process the kernel was symmetrized in order to be able to apply the so-called 'Lanczos algorithm' which is an algorithm for tridiagonalizing a symmetric matrix. Finding the eigenvalues of a tridiagonal matrix requires much less computer storage, so matrices of much larger size could be diagonalized this way. The bound state mass was being fixed as an input and the eigenvalue to solve for was the coupling constant.

The Yukawa potential model works quite well as long as it is applied to interactions between nucleons or other hadrons. In this thesis, however, we are dealing with

quark-antiquark bound states. The interaction between quarks is governed by the strong force which is not of the Yukawa type. We can think of the Yukawa potential as a weak remnant of the strong colour force acting between quarks, just like the Van der Waals interaction acting between atoms or molecules is a remnant of the much stronger electromagnetic force acting between the constituents of atoms. The best theory so far to describe the strong interaction is Quantum Chromodynamics, in which the strong force is believed to be mediated by massless gluons, in analogy to photons mediating the electromagnetic force in Quantum Electrodynamics. It would be desirable to derive a Tamm-Dancoff equation from the field theoretical interaction Hamiltonian of QCD so that the kernel would correspond to the true interaction between quarks. Up to date, no one has done this, it remains an open research problem. It is very hard to determine which are the diagrams that give rise to confinement and which are insignificant in the Tamm-Dancoff approximation. Thus we do not know the form of the true kernel for the strong interaction from fundamental field theory.

There exist fairly successful phenomenological models that are able to account for many meson states (see references [13], [19]) in the usual Equal-time formalism, although there is no overall satisfactory treatment applicable to all mesons and baryons. One of our motivations was to find a potential model which can be cast into a form that gives rise to an integral equation of the Tamm-Dancoff structure, if written in momentum space in terms of light-front variables. Our aim was to see how such phenomenologically motivated Tamm-Dancoff-like integral equation can be solved in an alternative way to Wort's method. Wort's solution was not rotationally invariant and he had to solve a two dimensional integral equation numerically. We have attempted to find a method that reduces the equation to a one dimensional integral equation and that recovers rotational invariance. However we did not use the Yukawa potential in our kernel, so our solution cannot be compared with that of Wort. Instead, the linear+Coulomb potential model was applied and Chapter 2 will elaborate on its motivation.

Chapter 2

Light-front Formalism for Mesons

2.1 Description of $q\bar{q}$ Bound States

The general equations for the wavefunctions of mesons and baryons using light-front dynamics were first derived by Terent'ev and Berestetsky [3]. Several other authors discussed the same formalism and applied it to specific bound state and scattering problems [20] [21] [22]. Here we present the results of their work relevant to quark-antiquark systems.

Using the Einstein-relation (1.15), the Hamiltonian for a single particle of mass m in the light-front system can be written as

$$p^- = \frac{p_\perp^2 + m^2}{2p^+} \quad (2.1)$$

Since p^+ is a conserved quantity, the Hamiltonian formally has a structure similar to the nonrelativistic Hamiltonian for a particle of mass p^+ . This analogy allows the use of familiar techniques to construct many-particle systems.

First, let us consider a system of two particles α and β , with momentum k_α and k_β respectively. In a nonrelativistic system, one can separate the internal motion from the motion of the system as a whole by defining relative and total momenta, so that the Hamiltonian will contain two terms, one depending on the internal momentum only and the other depending on the total momentum only. The same

technique is applicable to the relativistic light-front Hamiltonian, which is a great advantage of the Front-form Dynamics. So let us define the total momentum to be

$$\begin{aligned}\vec{P}_\perp &= \vec{k}_{\alpha\perp} + \vec{k}_{\beta\perp} \\ P^+ &= k_\alpha^+ + k_\beta^+\end{aligned}\quad (2.2)$$

where $\perp$ denotes the transverse component, and a relative transverse momentum $\vec{p}_\perp$ such that

$$\frac{\vec{p}_\perp}{\mu_{\alpha\beta}} = \frac{\vec{k}_{\alpha\perp}}{\mu_\alpha} - \frac{\vec{k}_{\beta\perp}}{\mu_\beta} \quad (2.3)$$

where $\mu_{\alpha\beta}$ is analogous to the reduced mass of nonrelativistic Hamiltonians:

$$\mu_{\alpha\beta} = \frac{\mu_\alpha \mu_\beta}{\mu_\alpha + \mu_\beta}, \quad \text{with } \mu_\alpha = k_\alpha^+, \quad \mu_\beta = k_\beta^+ \quad (2.4)$$

The third component of the internal momentum should be chosen to get a meaningful orbital angular momentum. Before defining it, we will write the Hamiltonian in terms of $\vec{p}_\perp$ and the momentum fraction used in Tamm-Dancoff theory:

$$x = \frac{k_\alpha^+}{P^+}, \quad 1 - x = \frac{k_\beta^+}{P^+} \quad (2.5)$$

The relative transverse momentum becomes

$$\vec{p}_\perp = (1 - x)\vec{k}_{\alpha\perp} - x\vec{k}_{\beta\perp} \quad (2.6)$$

and the total Hamiltonian of the noninteracting two-particle system is

$$H_0 = \frac{P_\perp^2 + M_0^2}{2P^+} \quad (2.7)$$

$$M_0^2 = \frac{p_\perp^2 + m_\alpha^2}{x} + \frac{p_\perp^2 + m_\beta^2}{1 - x} \quad (2.8)$$

It looks like the one-particle Hamiltonian (2.1) with an effective mass M_0 . It is ap-

parent now that M_0 is equivalent to the M_0 of the Tamm-Dancoff equation (1.25). It only depends on internal variables, so the total motion of the system and the internal motion is separated just like in nonrelativistic Schrödinger theory. It enables us to define a wavefunction of the internal variables ($\psi(\vec{p}_\perp, x)$):

$$\Psi = \Psi_{CM}(\vec{P}_\perp, P^+) \psi(\vec{p}_\perp, x) \quad (2.9)$$

The interaction can be switched on by adding an interaction term $\hat{W}$ to the mass-square operator, so the Hamiltonian for the interacting system becomes

$$H = \frac{P_\perp^2 + M^2}{2P^+} \quad (2.10)$$

$$M^2 = M_0^2 + \hat{W} \quad (2.11)$$

The internal state should then satisfy the mass-square eigenvalue equation:

$$M^2|\psi\rangle = M_B^2|\psi\rangle \quad (2.12)$$

where M_B is the mass of the bound state.

As already indicated in Chapter 1, only one component of the angular momentum operator is kinematic, namely J_3 , so the dynamics on the light-front is not manifestly rotational invariant. Also, the relative orbital angular momentum is quite complicated in terms of the variables x and $\vec{p}_\perp$. Only l_3 has the usual form:

$$l_3(\vec{p}_\perp) = i \epsilon_{3ij} \frac{\partial}{\partial p_i} p_j \quad (2.13)$$

where i and j denote the transverse components which are unchanged by the transformation to light-front coordinates. But l_1 and l_2 involve the transformed components resulting in extremely cumbersome expressions. This difficulty can be remedied by a mapping from the x momentum fraction variable to a new momentum p_3 which will be regarded as the third component of the internal momentum 3-vector

$\vec{p} = (\vec{p}_\perp, p_3)$. This mapping, found by Terent'ev [23], is the following:

$$x = \frac{E_\alpha + p_3}{E_\alpha + E_\beta} \quad \text{where} \quad E_{\alpha,\beta} = \sqrt{m_{\alpha,\beta}^2 + p^2} \quad (2.14)$$

The noninteracting mass operator then becomes intuitively simple:

$$M_0 = E_\alpha + E_\beta \quad (2.15)$$

With the new p_3 component, the operator $\tilde{l}(\vec{p})$ can be written in the usual form:

$$\tilde{l}(\vec{p}) = i\nabla_p \times \vec{p} \quad (2.16)$$

which is just the familiar quantum mechanical operator form of the classical quantity $\vec{x} \times \vec{p}$ in momentum representation.

The spin of a two-particle system is the total angular momentum in the rest frame of the system. It includes the relative orbital angular momentum and the individual spins of each composite particle. Spin—besides mass—is used to classify particles in terms of irreducible representations of the Poincaré group. Just as mass corresponds to the Casimir operator $P_\mu P^\mu$, spin corresponds to the other invariant operator $W_\mu W^\mu$, where W_μ is the Pauli-Lubanski vector:

$$W_\mu = \frac{1}{2} \epsilon_{\nu\rho\sigma\mu} J^{\rho\sigma} P^\nu$$

To define spin, first we have to transform to the rest frame by a Lorentz transformation $L(P)$:

$$W'_\mu = L(P)W_\mu \quad \text{where} \quad L(P)P_\mu = (M, 0, 0, 0) \quad (2.17)$$

The spin is then defined as

$$s_1 = \frac{W'_1}{M} \quad (2.18)$$

$L(P)$ is defined up to an arbitrary rotation. So there are infinitely many possible spin definitions. The canonical spin s_c is the one corresponding to the rotationless Lorentz transformation $L_c(P)$ to the rest frame. In the light-front system we want $L(P)$ to leave the light-front surface $x^+ = \text{const.}$ invariant. These particular Lorentz transformations are denoted by $L_f(P)$. Their generators make up the stability group in the Front-form. $L_f(P)$ transforms light-front vectors according to

$$[L_f(P)A]^+ = \frac{M}{P^+} A^+ \quad (2.19)$$

$$[L_f(P)A]_{\perp} = A_{\perp} - P_{\perp} \frac{A^+}{P^+} \quad (2.20)$$

So the spin appropriate for Front-form Dynamics is

$$s_3 = \frac{W^+}{P^+}, \quad s_{\perp} = \frac{W_{\perp} - P_{\perp} s_3}{M} \quad (2.21)$$

With the definition of W_{μ} , equation (2.21) implies that the angular momentum operator for one particle in the rest frame is related to the front-form spin as follows:

$$J_3 = s_3, \quad J_{1,2} = \frac{P_{1,2} s_3 + M s_{1,2}}{P^+} \quad (2.22)$$

Generalizing it to two-particle systems and expressing in terms of the Tamm-Dancoff variables x and $\vec{p}_{\perp}$, the two-body spin $\vec{\Sigma}$ can be written as

$$\Sigma_3 = l_3(\vec{p}_{\perp}) + s_3^{\alpha} + s_3^{\beta} \quad (2.23)$$

$$\Sigma_{1,2} = l_{1,2}(\vec{p}, x) + \frac{p_{1,2} s_3^{\alpha} + m_{\alpha} s_{1,2}^{\alpha}}{x M_0} + \frac{-p_{1,2} s_3^{\beta} + m_{\beta} s_{1,2}^{\beta}}{(1-x) M_0} \quad (2.24)$$

Thus $\vec{\Sigma}$ is not simply the vector sum of the orbital angular momentum and the individual spins of the particles. Therefore the usual techniques with Clebsch-Gordan coefficients to classify states according to the eigenvalues of the operators M , Σ^2 and Σ_3 cannot be applied.

As established by Berestetsky and Terent'ev [3], relativistic invariance requires

the operators M and Σ_i to commute. The constraints on the form of M implied by this condition are very hard to establish given the complicated composition rule for spin. There exists, however, a unitary transformation Ω that transforms Σ_i to $\Sigma_i^{(\Omega)}$ which satisfies the simple composition law familiar from nonrelativistic quantum mechanics. For a single free particle this transformation was derived by Melosh [2]. Berestetsky and Terent'ev [3] constructed an analogous transform for a system of interacting particles. For the two-particle case we quote their result:

$$\Omega = U_\alpha(\vec{p}) U_\beta(-\vec{p}) \quad (2.25)$$

where $U_{\alpha(\beta)}$ transforms the light-front spin of particle $\alpha(\beta)$ into the canonical spin. So $U_{\alpha(\beta)}$ is a representation for the Lorentz-transformation $L_c(\vec{p})L_f^{-1}(\vec{p})$. The calculations of Ref.[3] were corrected by Bakker et al.[20], and the correct expression for U_α is

$$U_\alpha(\vec{p}) = \exp \left\{ 2i \epsilon_{3jk} s_j^\alpha \hat{p}_k \arctan \left(\frac{|\vec{p}_\perp|}{m_\alpha + E_\alpha(\vec{p}) + p_3} \right) \right\} \quad (2.26)$$

With this transformation, we have the usual composition rule for the transformed spin $\vec{\Sigma}^{(\Omega)}$:

$$\vec{\Sigma}^{(\Omega)} = \Omega \vec{\Sigma} \Omega^{-1} = \vec{l}(\vec{p}) + \vec{s}^\alpha + \vec{s}^\beta \quad (2.27)$$

$\vec{\Sigma}^{(\Omega)}$ has the same eigenvalue spectrum as $\vec{\Sigma}$ since Ω is a unitary transformation. Also, $\vec{\Sigma}^{(\Omega)}$ satisfies the angular momentum commutation relations, it commutes with all the kinematic generators and with M as well. Therefore it is possible to define wavefunctions which represent simultaneous eigenstates of $\Sigma^{(\Omega)^2}$, $\Sigma_3^{(\Omega)}$ and M . The wavefunction also has to be transformed. If we denote the eigenvalues of $\Sigma^{(\Omega)^2}$ and $\Sigma_3^{(\Omega)}$ by j and j_3 respectively, the internal wavefunction can be written as a product of a momentum-dependent and a spin-dependent part. For our system

of two particles of helicities λ_α and λ_β , the spin-dependent part being

$$R(\lambda_\alpha \lambda_\beta, jj_3) = \langle \lambda_\alpha \lambda_\beta | jj_3 \rangle \quad (2.28)$$

the total internal wavefunction in momentum space has the form:

$$\psi(\vec{p}; \lambda_\alpha \lambda_\beta, jj_3) = \phi(\vec{p}) R(\lambda_\alpha \lambda_\beta, jj_3) \quad (2.29)$$

The Melosh transformation acts on the spin-dependent part, as we have seen, so the Melosh-rotated wavefunction can be written as

$$\psi^{(\Omega)}(\vec{p}; \lambda_\alpha \lambda_\beta, jj_3) = \phi(\vec{p}) R^{(\Omega)}(\lambda_\alpha \lambda_\beta, jj_3) \quad (2.30)$$

where

$$R^{(\Omega)}(\lambda_\alpha \lambda_\beta, jj_3) = \sum_{\lambda'_\alpha \lambda'_\beta} \langle \lambda_\alpha | U_\alpha(\vec{p}) | \lambda'_\alpha \rangle \langle \lambda_\beta | U_\beta(-\vec{p}) | \lambda'_\beta \rangle \langle \lambda'_\alpha \lambda'_\beta | jj_3 \rangle \quad (2.31)$$

The 2×2 irreducible representation of U is given by

$$\langle \lambda | U(\vec{p}) | \lambda' \rangle = \chi_\lambda^\dagger \frac{m + x M_0 - i \vec{\sigma}(\vec{n} \times \vec{p})}{\sqrt{(m + x M_0)^2 + p_\perp^2}} \chi_{\lambda'} \quad (2.32)$$

where χ_λ is the Pauli spinor and $\vec{n} = (0, 0, 1)$. The spin-dependent wavefunctions $R^{(\Omega)}(\lambda_\alpha \lambda_\beta, jj_3)$ for S-state pseudoscalar and vector mesons are given in Appendix A.

With the above structure of the wavefunction (2.30), the mass eigenvalue equation (2.12) will involve the momentum-dependent part $\phi(\vec{p})$ only, as long as the interaction term $\hat{W}$ does not contain spin-spin or spin-orbit interactions. In this thesis, only S-states were calculated so that the spin-orbit term does not affect our results. The spin-spin coupling results in a hyperfine splitting of the energy levels which we neglect. This ignorance can be justified for mesons with large constituent quark masses moving nearly nonrelativistically, but the splitting can be compa-

able to the magnitude of the energy levels for low mass and highly relativistic systems. A proper model for mesons would have to take those effects into account, but constructing such a complete model was beyond the scope of this project. In what follows, we are assuming a spin-independent interaction $\hat{W}(\vec{p}, \vec{p}')$. The mass eigenvalue equation (2.12) in momentum space can then be written as

$$\left[M_B^2 - M_0^2(\vec{p}) \right] \phi(\vec{p}) = \frac{1}{(2\pi)^3} \int d^3 p' \hat{W}(\vec{p}, \vec{p}') \phi(\vec{p}') \quad (2.33)$$

This equation has the same structure as the Tamm-Dancoff equation (1.23). Since the true interaction kernel has yet to be found from fundamental principles, we can only motivate it phenomenologically with the help of our present knowledge of Quantum Chromodynamics. According to QCD, the mediator of the strong interaction between quarks is the gluon carrying a colour charge. By summing all higher order diagrams that contribute to the gluon propagator, it can be shown (see references [1] [24] [25]) that the coupling constant $\alpha_s = \frac{g^2}{4\pi}$ is not really a constant but depends on the momentum exchange Q^2 :

$$\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 + \frac{\alpha_s(\mu^2)}{12\pi} (11N - 2n_f) \log\left(\frac{Q^2}{\mu^2}\right)} \quad (2.34)$$

where N is the number of colours, n_f is the number of quark flavours and μ is a reference momentum. In the denominator, the $-2n_f$ term arises from summing all higher order quark loops and the term involving the factor $11N$ comes from gluon loops. In contrast, the QED mediator—the photon—does not carry electric charge and therefore no self-coupling of photons occur. There are fermion loop corrections only, which result in a running coupling constant $\alpha(Q^2) = \frac{e^2}{4\pi}$ of the form :

$$\alpha(Q^2) = \frac{\alpha(\mu^2)}{1 - \frac{\alpha(\mu^2)}{3\pi} \log\left(\frac{Q^2}{\mu^2}\right)} \quad (2.35)$$

Hence, in QED, the strength of the coupling (i.e. electric charge) decreases at lower Q^2 corresponding to larger distances. Physically speaking, the electric charge is

'shielded' by the polarization effect of virtual fermion-antifermion pairs. In contrast, gluon loops have an 'antishielding' effect, that is at large distances the colour charge becomes stronger and stronger. At short distances, on the other hand, the interaction strength decreases. In the competition between the screening effect of quark loops and the antiscreening of gluon self-couplings, the latter is the winner as long as $11N > 2n_f$ which is the case in QCD with $N = 3$ colours and $n_f = 6$ quark flavours.

This behaviour of the strong coupling constant is responsible for asymptotic freedom in the high Q^2 region, where the perturbative result (2.34) for α_s is valid. It may also play a role in confinement, but it is still not known whether the function $\alpha_s(Q^2)$ can be continued smoothly to the infrared region where perturbation theory fails. Numerical calculations in lattice gauge theory (Wilson [26]) suggest that there is indeed a continuous transition between the strong and weak coupling regimes. The theory is based on the discretization of space-time which provides a natural nonperturbative regularization scheme. Since energies corresponding to wavelengths shorter than the lattice spacing have no meaning, the discrete nature of space-time guarantees an upper bound for momenta. Thus by increasing the lattice spacing, the theory is well suited for studying the infrared region. Calculations in the strong coupling limit (see e.g. Cheng and Li [1]) show that the confining potential grows linearly with distance. The weak coupling behaviour of the potential can be examined by decreasing the lattice spacing, and a Coulomb-like $1/r$ -dependence is obtained. This can also be calculated in a more familiar way, just as the electromagnetic Coulomb potential is derived from Quantum Electrodynamics. The amplitude for one gluon exchange between a quark and an antiquark can be written down the same way as for one photon exchange in QED. There are two differences, however, due to the colour degree of freedom.

One is the inclusion of a 3-component colour vector c in specifying quark states:

$$c = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ for red, } \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ for blue, } \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ for green}$$

which are analogous to the Dirac spinors $u(\lambda)$. The other difference is in the quark-gluon vertex factor. The QED electron-photon vertex factor is simply $-ie\gamma^\mu$ since the photon is electrically neutral. Gluons, on the other hand, do carry colour charge and it complicates the vertex factor which is written as $-ig_s\lambda^\alpha\gamma^\mu$. The 8 Gell-Mann matrices λ^α are the $SU(3)$ analogues to the Pauli matrices of $SU(2)$. They satisfy the commutation relations $[\lambda^\alpha, \lambda^\beta] = 2if^{\alpha\beta\gamma}\lambda^\gamma$, where $f^{\alpha\beta\gamma}$ are called the structure constants of the group $SU(3)$.

Thus the one-gluon-exchange amplitude is identical to the one-photon-exchange amplitude with an extra colour factor

$$f = \frac{1}{4}(c_3^\dagger \lambda^\alpha c_1)(c_4^\dagger \lambda^\alpha c_2)$$

Here 1 and 2 denote the incoming quarks, 3 and 4 denote the outgoing quarks. All observed mesons are in the colour singlet state for which $f = \frac{4}{3}$ [27]. Hence, the effective QCD potential between a quark and an antiquark has the same form as the electromagnetic Coulomb potential $-\frac{\alpha}{r}$ with the addition of the colour factor:

$$V_{q\bar{q}} = -\frac{4}{3} \frac{\alpha_s}{r}$$

The colour factor for the colour octet states would be $f = -\frac{1}{6}$ resulting in a repulsive potential. Therefore $q\bar{q}$ bound states can only exist in the colour singlet state.

Several, fairly successful attempts [13],[19] have been made to calculate meson spectra using the linear+Coulomb potential model. The most comprehensive treatment was carried out by Isgur and Godfrey [13]. They solved a relativistic Schrödinger equation in the Instant form by diagonalizing the Hamiltonian in a harmonic oscillator basis. They had included in their potential a one-gluon-exchange

term described above, a linear confining potential, a hyperfine interaction term arising from spin-spin coupling, the spin-orbit term and finally a term that takes into account $q\bar{q}$ annihilation via gluons. They had built a semiquantitative model of relativistic effects by introducing a smearing function to the coordinate-space potentials, since the momentum dependence of their QCD-motivated potential could not readily be translated into spatial dependence. They obtained an impressively good fit to the spectra of most of the mesons with some discrepancies in the light meson sectors. They also calculated a large number of decay amplitudes with reasonably good agreement with experimental data.

The relative success of the linear+Coulomb model motivated us to consider this type of potential in our interaction kernel. However, our approach is different from that of Ref.[13] in many respects. We have written a fully relativistic equation in momentum space instead of coordinate space. Our equation is formulated in terms of light-front variables as opposed to the Instant form in [13]. We have ignored hyperfine, spin-orbit and annihilation interaction terms. As we will see in Chapter 3, instead of working in the harmonic oscillator basis which is well-suited for the linear potential, we expanded our wavefunctions in terms of hyperspherical harmonics which are eigensolutions of the Coulomb potential.

We have tried to make a more direct connection with QCD by working in momentum space. As we have seen, the field theoretical approach in the Light-Front Tamm-Dancoff theory resulted in integral equations of the same structure as (2.33). In order to strengthen this connection, the quasipotential formalism will be discussed in the next section, which is yet another semi-field-theoretic approach leading to the same equation.

2.2 The Quasipotential Formalism

A systematic development of the quasipotential formalism was given by Jaus and Woolcock [10] who applied it to pion exchange effects in two-nucleon systems. It was also shown by Jaus [9] that the mass eigenvalue equation (2.33) can be obtained

from the quasipotential formalism as well.

Let us start with the Bethe-Salpeter equation [28], which is a relativistic two-particle equation based on the Feynman-diagram approach to scattering. The infinite sum of all Feynman diagrams for two-particle scattering can be represented by the following iterative integral equation:

$$\mathcal{M}(p, p''; P) = \mathcal{U}(p, p''; P) + \int \frac{d^4 p'}{(2\pi)^4} \mathcal{U}(p, p'; P) G(p'; P) \mathcal{M}(p', p''; P) \quad (2.36)$$

Its diagrammatic meaning is illustrated in Figure 2.1. $\mathcal{M}$ is the scattering amplitude

Figure 2.1: Diagrammatic representation of the Bethe-Salpeter Equation (2.36)

as a function of the initial and final relative 4-momenta, p and p'' respectively. $\mathcal{U}$ is the interaction kernel representing the infinite sum of all irreducible Feynman diagrams, and G is the two-particle propagator. Hence, the equation combines all irreducible diagrams with all reducible ones and sums them to get the full scattering amplitude. However, the full kernel would be just as hard to determine as the full amplitude itself, so the kernel is usually approximated by the first few terms of its perturbation expansion. When the important physics is described by boson exchange, the so-called ladder sum gives a good approximation in which $\mathcal{U}$ represents the one-boson exchange. If we write the Bethe-Salpeter equation in operator form as an iterative sum, then a geometric series is obtained which can be formally summed:

$$\begin{aligned} \mathcal{M} &= \mathcal{U} + \mathcal{U}G\mathcal{U} + \mathcal{U}G\mathcal{U}G\mathcal{U} + \dots + (\mathcal{U}G)^n \mathcal{U} + \dots \\ &= (1 - \mathcal{U}G)^{-1} \mathcal{U} \end{aligned} \quad (2.37)$$

Thus the scattering amplitude $\mathcal{M}$ has a pole when there exists an operator Γ such that $\Gamma = \mathcal{U}G\Gamma$ or explicitly:

$$\Gamma(p; P) = \int \frac{d^4 p'}{(2\pi)^4} \mathcal{U}(p, p'; P) G(p'; P) \Gamma(p'; P) \quad (2.38)$$

The function $\Gamma(p; P)$ is interpreted as the vertex function describing the bound state. The integration is over all 4 components of momentum and the interaction contains a relative energy dependence. This is equivalent to a relative time dependence of the Fourier transformed kernel. In general one does not know what boundary condition to use for relative times. In order to be able to describe the bound state by a time-independent effective potential, equation (2.38) has to be reduced to a 3-dimensional equation. A number of covariant prescriptions for reduction were proposed in the literature. A good summary and comparison of these methods can be found in Woloshyn and Jackson [29]. All of them are based on the introduction of a new 3-dimensional propagator g by replacing G with $g + (G - g)$ so that the bound state equation in operator form becomes

$$\Gamma = W g \Gamma \quad \text{with} \quad W = \mathcal{U} + \mathcal{U} (G - g) W \quad (2.39)$$

We will follow Jaus [9] and select the most widely used g , known as the Blankenbecler-Sugar [30] propagator, which puts the constituent quarks on their mass shell, thus fixing the time component of the momentum:

$$g(\vec{p}; P) = 2\pi i \int \frac{ds}{P^2 - s} \delta^+(k_\alpha^2 - m_\alpha^2) \delta^+(k_\beta^2 - m_\beta^2) (\not{k}_\alpha + m_\alpha)(-\not{k}_\beta + m_\beta) \quad (2.40)$$

where $s = (k_\alpha + k_\beta)^2$ and $\delta^+(k^2 - m^2) \equiv \theta(k^+) \delta(k^2 - m^2)$. The integral in terms of the momentum fraction x and $\vec{p}_\perp$:

$$g(\vec{p}_\perp, x; P) = 2\pi i \delta^+(k_\beta^2 - m_\beta^2) \theta(x) \theta(1 - x) \frac{\Lambda^+(k_\alpha) \Lambda^-(k_\beta)}{x (P^2 - M_0^2)} \quad (2.41)$$

where the spin projection operators $\Lambda^+(k_\alpha)$ and $\Lambda^-(k_\beta)$ are given by (A.4). The

spinor basis is given in Appendix A. Transforming from dx to dp_3 with a Jacobian $\frac{x(1-x)M_0}{E_\alpha E_\beta}$, and redefining both the vertex function and the kernel with spinors attached:

$$\hat{\Gamma}(\vec{p}, \lambda_\alpha \lambda_\beta) = \left(\frac{M_0}{2E_\alpha E_\beta} \right)^{\frac{1}{2}} \bar{u}(k_\alpha \lambda_\alpha) \Gamma v(k_\beta \lambda_\beta) \quad (2.42)$$

$$\begin{aligned} \hat{W}(\vec{p}, \vec{p}'; \lambda_\alpha \lambda_\beta, \lambda'_\alpha \lambda'_\beta) &= \left(\frac{M_0}{2E_\alpha E_\beta} \right)^{\frac{1}{2}} \left(\frac{M'_0}{2E'_\alpha E'_\beta} \right)^{\frac{1}{2}} \\ &\times \bar{u}(k'_\alpha \lambda'_\alpha) \bar{v}(k_\beta \lambda_\beta) W u(k_\alpha \lambda_\alpha) v(k'_\beta \lambda'_\beta) \end{aligned} \quad (2.43)$$

the quasipotential bound state equation can be written as

$$\hat{\Gamma}(\vec{p}, \lambda_\alpha \lambda_\beta) = \int \frac{d^3 p'}{(2\pi)^3} \hat{W}(\vec{p}, \vec{p}'; \lambda_\alpha \lambda_\beta, \lambda'_\alpha \lambda'_\beta) \frac{\hat{\Gamma}(\vec{p}', \lambda'_\alpha \lambda'_\beta)}{P^2 - M'^2_0} \quad (2.44)$$

This equation can be identified with (2.33) if we define the wavefunction as

$$\psi(\vec{p}, \lambda_\alpha \lambda_\beta) = \frac{\hat{\Gamma}(\vec{p}, \lambda_\alpha \lambda_\beta)}{P^2 - M^2_0} \quad (2.45)$$

since $P^2 = M^2_B$.

There are two alternative approaches one can follow to solve this equation. One is by motivating the 4-dimensional interaction kernel from its Feynman diagram representation and reduce it to a 3-dimensional one by putting the constituent particles on their mass shell. The second approach is to postulate some 3-dimensional effective potential in coordinate space, and use its momentum space Fourier transform directly in (2.33). To illustrate the first method, let us write down the kernel for a hypothetical interaction including a one-boson-exchange term and another term resembling the 4-dimensional Fourier transform of the linear potential:

$$\mathcal{U} = \frac{c_1}{(k_\alpha - k'_\alpha)^2} + \frac{c_2}{(k_\alpha - k'_\alpha)^4} \quad (2.46)$$

In terms of the light-front Tamm-Dancoff variables $\vec{p}_\perp$ and x it has a form typical in Tamm-Dancoff equations (see (1.22) for comparison) if we put both particles α

and β on their mass shell obtaining the 3-dimensional reduced kernel:

$$W = \frac{c_1}{D} + \frac{c_2}{D^2} \quad (2.47)$$

$$\text{with } D = (x - z) \left[\frac{p_1^2 + m_\beta^2}{1 - x} - \frac{p_1'^2 + m_\beta^2}{1 - z} - \frac{p_1^2 + m_\alpha^2}{x} - \frac{p_1'^2 + m_\alpha^2}{z} + \frac{2(\vec{p}_1 - \vec{p}_1')^2}{x - z} \right]$$

Now if we map from the x momentum fraction to the new p_3 component defined in (2.14) we find

$$W = \frac{c_1}{(\vec{p} - \vec{p}')^2} + \frac{c_2}{(\vec{p} - \vec{p}')^4} \quad (2.48)$$

The significance of this result is that the 3-dimensional reduced kernel W is just the 3-dimensional Fourier transform of the linear+Coulomb potential which will be calculated explicitly in Chapter 3.

Having established a strong connection between many different approaches to the two-particle bound state problem, in the next chapter we will find solutions to equation (2.33) and try to relate our results to experimental data and to theoretical calculations done by others.

Chapter 3

Solution of The Mass Eigenvalue Equation and Applications

3.1 The Method of Solution and Results for S-state Mesons

As an application of the light-front formalism developed for mesons, we will find solutions to the mass eigenvalue equation (2.33) using an interaction kernel motivated by the linear+Coulomb potential model. The solution is based on a general technique for solving momentum-space integral equations. This technique was developed by Lévy [11] who solved both the nonrelativistic Schrödinger equation and the relativistic Dirac equation with the Coulomb potential (hydrogen atom) and also with the Yukawa potential. He also investigated a more general case, when the momentum-space potential is expanded in a series of powers of the modulus of the momentum. Since the Fourier transforms of central potentials of the form $V(r) = r^\nu$ are all functions of the modulus of the momentum, the method is applicable to a broad range of models.

We will apply this method to obtain wavefunctions and corresponding mass eigenvalues for mesonic states. As it was shown in previous chapters, the equation to be solved in Tamm-Dancoff theory is the Einstein equation (1.19) which is equivalent

to a mass-square eigenvalue equation :

$$(M_0^2 + \hat{W}) |\psi\rangle = M_B^2 |\psi\rangle \quad (3.1)$$

On the other hand, phenomenological potential models correspond to a mass eigenvalue equation which is linear in mass:

$$(M_0 + V) |\psi\rangle = M_B |\psi\rangle \quad (3.2)$$

where the Coulomb+linear potential is used for V :

$$V(r) = -\frac{a}{r} + br \quad (3.3)$$

Let us rearrange terms in (3.2) and then operate on both sides by $(M_B + M_0)$ from the left. We get

$$(M_B^2 - M_0^2) |\psi\rangle = (M_B + M_0) V |\psi\rangle \quad (3.4)$$

Thus (3.1) and (3.2) are equivalent provided that the operators $\hat{W}$ and V are related as follows:

$$\hat{W} = (M_B + M_0) V \quad (3.5)$$

If we denote the 3-dimensional Fourier transform of the potential $V(\vec{r})$ by $U(\vec{q})$:

$$U(\vec{q}) = \int d^3r V(\vec{r}) e^{i\vec{q}\cdot\vec{r}} \quad (3.6)$$

then equation (3.4) can be written in momentum space as:

$$[M_B^2 - M_0^2(p)] \phi(\vec{p}) = [M_B + M_0(p)] \int \frac{d^3p'}{(2\pi)^3} U(\vec{p} - \vec{p}') \phi(\vec{p}') \quad (3.7)$$

and the condition (3.5) for the kernels becomes

$$\hat{W}(\vec{p}, \vec{p}') = [M_B + M_0(p)] U(\vec{p} - \vec{p}') \quad (3.8)$$

It is apparent that the equation quadratic in mass has a kernel depending on the bound state mass itself, while the kernel of the equation linear in mass is mass-independent. Although this fact would not present any serious difficulty in applying Lévy's method to the quadratic equation, we have chosen to solve the equation linear in mass, for our particular kernel was formulated to better suit this equation. Had we derived the kernel from field theory it would suit the quadratic equation better, that is $\hat{W}(\vec{p}, \vec{p}')$ would look algebraically simpler than $U(\vec{p}, \vec{p}')$. This is indeed the case for the Tamm-Dancoff equation (1.21) or (1.23) which has a kernel derived from the field theoretical Yukawa Lagrangian. If we were to convert (1.23) into an equation linear in mass, its kernel would become more complicated:

$$K(\vec{p}, \vec{p}') \rightarrow \frac{1}{M_B + M_0(p)} K(\vec{p}, \vec{p}') \quad (3.9)$$

Lévy's method is applicable in both cases.

As the first step towards solving the equation linear in mass,

$$[M_B - M_0(p)] \phi(\vec{p}) = \int \frac{d^3 p'}{(2\pi)^3} U(\vec{p} - \vec{p}') \phi(\vec{p}') , \quad (3.10)$$

the Fourier transform of the potential must be found. Let us start with the Coulomb term:

$$U_c(\vec{q}) = -a \int d^3 r \frac{e^{i\vec{q} \cdot \vec{r}}}{r} \quad (3.11)$$

This can only be evaluated with the introduction of a convergence factor $e^{-\mu r}$ in the integrand and by taking the limit of the final result as $\mu \rightarrow 0$. The result is

$$U_c(\vec{q}) = -\frac{4\pi a}{q^2} \quad (3.12)$$

Note that $U_c(\vec{q})$ is singular as $q \rightarrow 0$. Similarly, for the linear term:

$$U_L(\vec{q}) = \lim_{\mu \rightarrow 0} b \int d^3r r e^{-\mu r} e^{i\vec{q} \cdot \vec{r}} \quad (3.13)$$

In this case we have to treat the infrared limit ($q \rightarrow 0$) with some care. We will see later that the infrared singularity of the Coulomb term ($\sim \frac{1}{q^2}$) does not lead to divergent integrals while any other kernel of the form $\frac{1}{q^n}$ with $n > 2$ results in divergent integrals. For the linear potential with $q \neq 0$:

$$U_L(\vec{q}) = -\frac{8\pi b}{q^4} \quad (3.14)$$

To include the proper infrared limit, U_L should really be written as

$$U_L(\vec{q}) = \lim_{\mu \rightarrow 0} 8\pi b \frac{3\mu^2 - q^2}{(q^2 + \mu^2)^3} \quad (3.15)$$

and its limit is (3.14) only when $q \neq 0$. Having all components of the mass eigenvalue equation derived in momentum space, we make one more simplification. Let us consider quarks and antiquarks of equal mass only for algebraic simplicity. Then the mass operator becomes

$$M_0(p) = 2\sqrt{m^2 + p^2} \quad (3.16)$$

and the equation to be solved in the 3-dimensional $\vec{p}$ -space:

$$\left[M_B - 2\sqrt{m^2 + p^2} \right] \phi(\vec{p}) = - \int \frac{d^3p'}{(2\pi)^3} \left[\frac{4\pi a}{(\vec{p} - \vec{p}')^2} + \frac{8\pi b}{(\vec{p} - \vec{p}')^4} \right] \phi(\vec{p}') \quad (3.17)$$

with the caution that whenever integrals with the infrared singularity are evaluated, we have to revert to (3.15) to get the correct limit.

The idea of Lévy (initiated by Fock [31]) that we exploit here is that transforming from the 3-dimensional momentum space to a 4-dimensional hyperspace allows us to express the solution as a series of hyperspherical harmonics. The transformation

can be accomplished by setting

$$p = m \tan \frac{\psi}{2} \quad \text{and} \quad p' = m \tan \frac{\psi'}{2} \quad (3.18)$$

where m is the quark (or antiquark) mass. This mapping is equivalent to transforming the momentum space into a 3-dimensional hypersphere of radius m embedded in a 4-dimensional hyperspace. This hyperspace is characterized by angular polar coordinates ψ, θ and φ , where θ and φ are the usual polar and azimuthal angles in the original $\vec{p}$ -space. Lévy generalized this idea to an arbitrary number of dimensions. We will not discuss the general case here, since only the transformation from 3 to 4 dimensions is relevant to our calculation.

Equation (3.10) in this new space has the following form:

$$\begin{aligned} \left[M_B - 2m \sec \frac{\psi}{2} \right] \phi(\hat{u}) = & - \int \frac{d\Omega'}{(2\pi)^3} \frac{\pi a m \cos^2 \frac{\psi}{2}}{\cos^4 \frac{\psi'}{2} (1 - \cos \Theta)} \phi(\hat{v}) \\ & - \int \frac{d\Omega'}{(2\pi)^3} \frac{4\pi b \cos^4 \frac{\psi}{2}}{m \cos^2 \frac{\psi'}{2} (1 - \cos \Theta)^2} \phi(\hat{v}) \end{aligned} \quad (3.19)$$

where $\hat{u} = (\psi, \theta, \varphi)$ is a unit vector along $\vec{p}$ and $\hat{v} = (\psi', \theta', \varphi')$ is a unit vector along $\vec{p}'$. The angle between $\vec{p}$ and $\vec{p}'$ is Θ , so that

$$\cos \Theta = \cos \psi \cos \psi' + \sin \psi \sin \psi' \cos \theta' \quad (3.20)$$

since θ and φ can be chosen to be zero. The solid angle element is

$$d\Omega' = d\psi' \sin^2 \psi' \sin \theta' d\theta' d\varphi' \quad (3.21)$$

The integrand can be simplified if we define a new function

$$G(\hat{u}) \equiv \frac{\phi(\hat{u})}{\cos^4 \frac{\psi}{2}} \quad (3.22)$$

and divide both sides by $2m \cos^2 \frac{\psi}{2}$ to obtain a dimensionless equation:

$$\begin{aligned} \frac{M_B}{2m} \cos^2 \frac{\psi}{2} G(\hat{u}) - \cos \frac{\psi}{2} G(\hat{u}) &= -\frac{\pi a}{2(2\pi)^3} \int d\Omega' \frac{G(\hat{v})}{1 - \cos \Theta} \\ &\quad - \frac{2\pi b}{m^2(2\pi)^3} \int d\Omega' \frac{\cos^2 \frac{\psi}{2} \cos^2 \frac{\psi'}{2} G(\hat{v})}{(1 - \cos \Theta)^2} \end{aligned} \quad (3.23)$$

Now $G(\hat{u})$ can be expanded in a series of hyperspherical harmonics:

$$G(\hat{u}) = \sum_{nlm} a_n \mathcal{Y}_{nlm}(\hat{u}) \quad (3.24)$$

The normalized hyperspherical harmonics are given in Sundaresan and Watson [32]:

$$\mathcal{Y}_{nlm}(\hat{u}) = C_n^l(\psi) Y_{lm}(\vartheta, \varphi) \quad (3.25)$$

$$\text{where } C_n^l(\psi) = \left[\frac{2^{2l+1}(n+1)(n-l)!}{\pi(n+l+1)!} \right]^{\frac{1}{2}} l! \sin^l \psi C_{n-l}^{l+1}(\cos \psi) \quad (3.26)$$

$Y_{lm}(\theta, \varphi)$ are the familiar spherical harmonics and $C_N^L(\cos \psi)$ are the Gegenbauer functions. Detailed discussion of the Gegenbauer polynomials can be found in Magnus and Oberhettinger [33] and several useful recursion relations for $C_n^l(\psi)$ were derived by Sundaresan and Watson [32]. Appendix B contains a relevant summary of their formulas.

Substituting the series (3.24) in equation (3.23), we obtain an equation for the expansion coefficients a_n :

$$\begin{aligned} a_n \frac{M_B}{4m} (1 + \cos \psi) \mathcal{Y}_{nlm}(\hat{u}) &= a_n \left[\cos \frac{\psi}{2} \mathcal{Y}_{nlm}(\hat{u}) - \frac{a}{16\pi^2} \int d\Omega' \frac{\mathcal{Y}_{nlm}(\hat{v})}{1 - \cos \Theta} \right. \\ &\quad \left. - \frac{b(1 + \cos \psi)}{16\pi^2 m^2} \int d\Omega' \frac{(1 + \cos \psi') \mathcal{Y}_{nlm}(\hat{v})}{(1 - \cos \Theta)^2} \right] \end{aligned} \quad (3.27)$$

In order to evaluate the integrals, we make use of a theorem by Erdélyi [34] concerning eigenvalues from integral equations for hyperspherical harmonics. The theorem

asserts that hyperspherical harmonics satisfy the following integral equation:

$$\mathcal{Y}_n(\hat{u}) = \Lambda_n \int K(\cos \Theta) \mathcal{Y}_n(\hat{v}) d\Omega' \quad (3.28)$$

with an eigenvalue

$$\frac{1}{\Lambda_n} = 2\pi i^n \int_{-1}^{+1} d\beta K(\beta) \int_{-\infty}^{+\infty} d\alpha \frac{J_{n+1}(\alpha)}{\alpha} e^{-i\alpha\beta} \quad (3.29)$$

always convergent, provided that $K(\cos \Theta)$ is an absolute square integrable function of $\cos \Theta$ in the interval $-1 \leq \cos \Theta \leq +1$. For the Coulomb term in (3.27), $K(\beta) = \frac{1}{1-\beta}$ and the integral is convergent:

$$\frac{1}{\Lambda_n^{Coulomb}} = \frac{4\pi^2}{n+1} \quad (3.30)$$

To arrive at this result we made use of the formula (see Abramowitz et al.[35] p486):

$$\int_{-\infty}^{+\infty} d\alpha \frac{J_{n+1}(\alpha)}{\alpha} e^{-i\alpha\beta} = \frac{2i}{n+1} (-i)^{n+1} (1-\beta)^{1/2} C_n^1(\beta) \quad (3.31)$$

For the linear term, however, the blind substitution of $K(\beta) = \frac{1}{(1-\beta)^2}$ in the eigenvalue integral (3.29) would lead to a divergent integral. We must remember that this form of $K(\beta)$ is only valid for $\mu(\equiv \cos \Theta) \neq 1$, that is when the angle between $\vec{p}$ and $\vec{p}'$ does not vanish. This is a manifestation of the collinear singularity which was mentioned earlier in Section 1.2. The proper way to handle this situation is to use equation (3.15) instead of (3.14) as the linear term in the kernel. This corresponds to

$$K(\beta) = \lim_{\mu \rightarrow 0} \frac{1 - \beta - 3\mu^2}{(1 - \beta + \mu^2)^3} \quad (3.32)$$

Substituting this into (3.29) results in

$$\frac{1}{\Lambda_n^{linear}} = -4\pi^2(n+1) \quad (3.33)$$

From now on we will restrict ourselves to S-states ($l = 0$) and drop the indices l and m from the $\mathcal{Y}_{nlm}$'s. Using the recursion relation (B.7) with $l = 0$, equation (3.27) can be written as

$$a_n \frac{M_B}{4m} \left[\mathcal{Y}_n + \frac{1}{2} \mathcal{Y}_{n-1} + \frac{1}{2} \mathcal{Y}_{n+1} \right] = a_n \left[\cos \frac{\psi}{2} - \frac{a}{4(n+1)} + \frac{3b(n+1)}{8m^2} \right] \mathcal{Y}_n \\ + a_n \frac{b(2n+1)}{8m^2} \mathcal{Y}_{n-1} + a_n \frac{b(2n+3)}{8m^2} \mathcal{Y}_{n+1} + a_n \frac{bn}{16m^2} \mathcal{Y}_{n-2} + a_n \frac{b(n+2)}{16m^2} \mathcal{Y}_{n+2} \quad (3.34)$$

The $\mathcal{Y}_{nlm}$'s are normalized so that they form an orthonormal set:

$$\int d\Omega \mathcal{Y}_{n'l'm'}^*(\hat{u}) \mathcal{Y}_{nlm}(\hat{u}) = \delta_{n'n} \delta_{l'l} \delta_{m'm} \quad (3.35)$$

Contraction of equation (3.34) by operating with $\int d\Omega \mathcal{Y}_n^*(\hat{u}) \dots$ on both sides will couple five consecutive coefficients a_{n-2} , a_{n-1} , a_n , a_{n+1} and a_{n+2} , resulting in a pentadiagonal matrix. In addition, the term involving a $\cos \frac{\psi}{2}$ factor will contribute to all matrix elements. The final matrix equation can be written as

$$\lambda a_{n'} = \sum_{n=0}^{\infty} A_{n'n} a_n + B_{n'} a_{n'-2} + C_{n'} a_{n'-1} + D_{n'} a_{n'} + E_{n'} a_{n'+1} + F_{n'} a_{n'+2} \quad (3.36)$$

$$\text{where } A_{n'n} = \int d\psi \sin^2 \psi \cos \frac{\psi}{2} C_{n'}^0(\psi) C_n^0(\psi) \quad (3.37)$$

and

$$\begin{aligned} \lambda &= \frac{M_B}{4m} & C_{n'} &= \frac{b(2n'+1)}{8m^2} - \frac{M_B}{8m} \\ B_{n'} &= \frac{bn'}{16m^2} & D_{n'} &= \frac{3b(n'+1)}{8m^2} - \frac{a}{4(n'+1)} \\ F_{n'} &= \frac{b(n'+2)}{16m^2} & E_{n'} &= \frac{b(2n'+3)}{8m^2} - \frac{M_B}{8m} \end{aligned} \quad (3.38)$$

In matrix notation, equation (3.36) is equivalent to

$$T \bar{a} = \lambda \bar{a} \quad (3.39)$$

where the matrix T has the following elements:

$$T = \begin{pmatrix} A_{00} + D_0 & A_{01} + E_0 & A_{02} + F_0 & A_{03} & A_{04} & A_{05} & A_{06} & \dots \\ A_{10} + C_1 & A_{11} + D_1 & A_{12} + E_1 & A_{13} + F_1 & A_{14} & A_{15} & A_{16} & \dots \\ A_{20} + B_2 & A_{21} + C_2 & A_{22} + D_2 & A_{23} + E_2 & A_{24} + F_2 & A_{25} & A_{26} & \dots \\ A_{30} & A_{31} + B_3 & A_{32} + C_3 & A_{33} + D_3 & A_{34} + E_3 & A_{35} + F_3 & A_{36} & \dots \\ A_{40} & A_{41} & A_{42} + B_4 & A_{43} + C_4 & A_{44} + D_4 & A_{45} + E_4 & A_{46} + F_4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and

$$\bar{a}^T = [a_0 \ a_1 \ a_2 \ a_3 \ a_4 \ \dots].$$

Equation (3.39) is not truly an eigenvalue equation since some of the matrix elements themselves depend on the 'eigenvalue' λ . However, it can be converted to a peculiar eigenvalue equation by dividing both sides by λ :

$$T' \bar{a} = 1 \bar{a} \quad \text{with } T' = \frac{1}{\lambda} T \quad (3.40)$$

It is peculiar in the sense that it gives the 'eigenvalue' λ as the solution indirectly only. Some particular λ has to be chosen to be put in matrix T' and then the matrix can be diagonalized. In other words we have to solve the matrix eigenvalue equation

$$T' \bar{a} = \lambda' \bar{a} \quad (3.41)$$

and adjust λ until we get $\lambda' = 1$ in which case equations (3.40) and (3.41) are identical. We have treated the Coulomb coupling strength a and the confinement strength b as fixed parameters and varied the constituent quark mass as well as the bound state mass to adjust λ . Thus several values of M_B were tried until the lowest eigenvalue λ' became equal to 1. The corresponding bound state mass is the mass of the 1S-state. Then by increasing the bound state mass (with m fixed) until the second lowest λ' became equal to 1, we obtained the mass of the 2S-state. This procedure can be continued to obtain all higher S-state masses. For the parameters

a and b we used values that are consistent with those of Eichten and Gottfried [19] and Isgur and Godfrey [13]. Table 3.1 summarizes our results for the masses of the members of two heavy meson families, charmonium and bottomonium. Values of the parameters and constituent quark masses are listed there as well. The experimental mass spectra and the spectra calculated by Isgur and Godfrey (IG) are also given for comparison. The agreement with the IG results is not surprising, since heavy quarkonia are nearly nonrelativistic and hyperfine splitting is negligible.

There is no spectroscopic information about $\psi(4040)$ and $\psi(4160)$ from experiments. This makes the identification of our charmonium 3S-state somewhat ambiguous. Isgur and Godfrey obtained a 2D-state at 4.19 GeV , which is close to $\psi(4160)$, therefore $\psi(4040)$ is most likely the 3S-state. Had we calculated higher l -states, we would have been able to identify all states unambiguously as well.

Wavefunctions were also calculated by finding the eigenvectors that correspond to 1S-state masses. The eigenvector elements are the coefficients a_n of the wavefunction expansion (3.24). Mapping back from the ψ variable to the momentum p , the momentum space wavefunctions were obtained. Figure 3.1 shows $4\pi p^2 |\phi(p)|^2$ as a function of p for a heavy meson (J/ψ) and for a light meson (π).

Parameters: $a = 0.52 \text{ GeV}^2$ $b = 0.18 \text{ GeV}^2$

	M_ψ	$M_\psi \text{ exp.}$	$M_\psi \text{ (IG)}$	M_Υ	$M_\Upsilon \text{ exp.}$	$M_\Upsilon \text{ (IG)}$
1S	3.100	3.097	3.10	9.433	9.460	9.46
2S	3.683	3.686	3.68	10.025	10.023	10.00
3S	4.100	4.040/4.160	4.10	10.369	10.355	10.35
4S	4.448	4.415	4.45	10.640	10.580	10.63
5S				10.880	10.865	10.88
6S				11.100	11.020	11.10

Quark masses: $m_c = 1.4 \text{ GeV}$ $m_b = 4.805 \text{ GeV}$

Table 3.1: Mass spectra of charmonium and bottomonium compared with experimental data and with the IG-model (all masses given in GeV 's).

Since our potential did not take into account the effects of spin-spin coupling and isospin splitting, the bound state mass of the pion does not agree with the

Figure 3.1: J/ψ and π wavefunctions $4\pi p^2 |\phi(p)|^2$ as a function of p in GeV .

data. We have obtained $0.55 GeV$ instead of $0.14 GeV$. Figure 3.1 clearly shows that the pion is much more relativistic than charmonium, its wavefunction having a longer tail into the high momentum region. This was expected from experimental data as well, since hyperfine splitting between the pseudoscalar $\pi(140)$ and the vector $\rho(770)$ is about six times as large as the corresponding splitting between $\eta_c(2980)$ and $J/\psi(3097)$ of the charmonium family. Hyperfine splitting increases as a bound state becomes more relativistic with decreasing constituent masses. This splitting arises from spin-spin coupling which our model ignores, therefore the mass we obtained for the pion is really the unsplit mass in between the singlet and triplet

states. Furthermore, the Coulombic coupling is stronger for the pion: $a = 1.38$ as opposed to the much lower value ($a = 0.52$) for heavy mesons. This is an indication of the running of the strong coupling constant. As an application, the wavefunction for the pion was used to calculate some of the electroweak properties of the pion that can be compared with experimentally obtained values. The results are presented in the next section.

3.2 Pion Decay Constant and Electromagnetic Form Factor

The relativistic constituent-quark model of electroweak properties of mesons using the light-front formalism was investigated by Jaus [9]. He considered the one-loop approximation for the meson vertex in scattering and decay processes and derived formulas for the form factors involved in semileptonic decays and for the decay constant of pseudoscalar mesons. We will outline his results in the next two sections and substitute our wavefunctions in the formulas for the pion decay constant and charge form factor. Jaus had used a Gaussian harmonic oscillator wavefunction with a free parameter in it, which he fitted to the experimental values of decay constants.

Cardarelli et al [12] also calculated electroweak properties for pseudoscalar mesons. Adopting the Isgur & Godfrey model [13] they worked in light-front momentum space therefore their results can readily be compared to ours. They expanded the meson wavefunctions in terms of momentum-space harmonic oscillator functions $R_{n,l}(p^2)$:

$$\phi(p^2) = \sum_{n=0}^{n_{\max}} b_n R_{n,0}(p^2)$$

The coefficients b_n were determined by applying the variational principle to the Isgur & Godfrey Hamiltonian.

3.2.1 Decay Constant

Let us consider the decay of π^- in the one-loop approximation of Figure 3.2 : $\pi^- \rightarrow \mu^- + \bar{\nu}_\mu$. The amplitude is given by

$$\mathcal{M} = \frac{G_F}{\sqrt{2}} P_\mu f_\pi \bar{u}_\mu \gamma^\mu (1 - \gamma_5) v_\nu \quad (3.42)$$

where the decay constant f_π is defined by the hadronic matrix element of the axial vector current as follows:

$$\langle 0 | \bar{u}_\alpha \gamma_\mu \gamma_5 u_d | \vec{P}, 00 \rangle \sqrt{P^+} = i P_\mu f_\pi \quad (3.43)$$

From the diagrammatic representation of Figure 3.2 this matrix element can be written as a trace of Dirac matrices in the following way:

$$i P_\mu f_\pi = i\sqrt{3} \int \frac{d^4 k}{(2\pi)^4} \text{Tr} \left[\Gamma(k_d, k_{\bar{u}}) \frac{k_d + m_d}{k_d^2 - m_d^2 + i\epsilon} \gamma_\mu \gamma_5 \frac{-k_{\bar{u}} + m_{\bar{u}}}{k_{\bar{u}}^2 - m_{\bar{u}}^2 + i\epsilon} \right] \quad (3.44)$$

Γ is the vertex function of Section 2.2 and a factor of $\sqrt{3}$ is needed to take into

Figure 3.2: One-loop Feynman diagram representing the hadronic matrix element in the pion decay amplitude

account the number of colours. The integration is over the relative 4-momentum $k = k_d - k_{\bar{u}}$. The basic assumption made at this point is that the vertex function is independent of the k^- component. Thus the 4-dimensional integral can be reduced to a 3-dimensional one. Converting all momentum variables into the light-front

variables of Chapter 2 and replacing the vertex function with the wavefunction of (2.45), the decay constant can be expressed as

$$f_\pi = \frac{\sqrt{3}}{(2\pi)^{3/2}} \int d^3p \phi(p) \left[\frac{M_0}{2E_d E_{\bar{u}}} \right]^{\frac{1}{2}} \frac{(1-x)m_d + xm_{\bar{u}}}{[M_0^2 - (m_d - m_{\bar{u}})^2]^{1/2}} \quad (3.45)$$

where $M_0 = E_d + E_{\bar{u}}$ with $E = \sqrt{(m^2 + p^2)}$ and $\vec{p}$ is the light-front internal 3-momentum as discussed in Chapter 2. The explicit momentum dependence of the wavefunction with equal mass constituents is given as

$$\phi(p) = \frac{1}{\sqrt{2\pi^2}} \frac{m^4}{(m^2 + p^2)^2} \sum_{n=0}^{\infty} a_n C_n^1 \left(\frac{m^2 - p^2}{m^2 + p^2} \right) \quad (3.46)$$

The pion wavefunction of Figure 3.1 was obtained assuming equal masses for the up and down quarks, $m_{u,d} = 135 \text{ MeV}$. This is somewhat lower than usual in other phenomenological models (see Ref.[13] and [19]). The coefficients a_n were computed by diagonalizing the matrix T' of Equation (3.40). The integral (3.45) resulted in the following value for the decay constant:

$$f_\pi = 90.8 \text{ MeV} \quad (3.47)$$

which compares surprisingly well with the experimental value of $f_\pi = 92.4 \pm 0.2 \text{ MeV}$ of Ref.[36]. This result was obtained for a 550 MeV pion as mentioned earlier. At this mass, the wavefunction of Figure 3.1 had the smoothest possible shape. When the correct pion mass, 140 MeV , was used, the wavefunction had a lot more structure in the low-momentum region and the decay constant had the much higher value of $f_\pi = 120 \text{ MeV}$. This is to be compared to the value of $f_\pi = 127.3 \text{ MeV}$ obtained by Cardarelli et al [12] using the full Isgur & Godfrey potential. They introduced an axial vector coupling constant at the level of constituent quarks which was then adjusted to get the correct experimental value for f_π .

3.2.2 Electromagnetic Form Factor

In scattering processes involving the pion, the hadronic vertex can be approximated by the one-loop diagram of Figure 3.3 in view of the constituent quark picture. The

Figure 3.3: One-loop Feynman diagram representing the hadronic vertex contribution to the amplitude for pion-lepton scattering

hadronic matrix element of the vector current for the pion is defined as

$$M_\mu = \langle \vec{P}'', 00 | \bar{q}'' \gamma_\mu q' | \vec{P}', 00 \rangle \sqrt{4P'^+ P''^+} \quad (3.48)$$

Since M_μ is a four vector, its most general structure can be described with form factors as functions of the only independent scalar Q^2 multiplied by the two independent 4-vectors $P' + P''$ and $P' - P''$:

$$M_\mu = F_+(Q^2)(P'_\mu + P''_\mu) + F_-(Q^2)(P'_\mu - P''_\mu) \quad (3.49)$$

In the limit when the mass of the scattering lepton is negligible, the second term is also negligible so that only $F_+(Q^2)$ contributes. This is the electromagnetic or charge form factor that we will calculate. In terms of the usual Instant-form momenta:

$$M_\mu = i \int \frac{d^4 k}{(2\pi)^4} \text{Tr} \left[\Gamma(k_1'', k_2'') \frac{k_1'' + m_1}{k_1''^2 - m_1^2 + i\epsilon} \gamma_\mu \frac{k_1' + m_1}{k_1'^2 - m_1^2 + i\epsilon} \right. \\ \left. \times \Gamma(k_1', k_2') \frac{-k_2' + m_2}{k_2'^2 - m_2^2 + i\epsilon} \right] \quad (3.50)$$

Following Jaus [9], the + component of M_μ can be expressed in terms of light-front variables and thus $F_+(Q^2)$ for equal mass constituents follows as

$$F_+(Q^2) = \int d^3p' \phi(p'') \phi(p') \left[\frac{E''}{E'} \right]^{1/2} \frac{m^2 + \vec{p}_\perp' \cdot \vec{p}_\perp''}{M_0' M_0'' x(1-x)} \quad (3.51)$$

The double-primed momentum can be expressed in terms of the primed momentum and the momentum exchange Q . If the axes are chosen such that $Q^+ = 0$, then $Q^2 = -Q_\perp^2$ and $\vec{p}_\perp'' = \vec{p}_\perp' - (1-x)\vec{Q}_\perp$. x in terms of p_3 is given in Equation (2.14). The integral was evaluated numerically at several different values of Q^2 . The result $F_+(Q^2)$ vs. Q^2 is plotted in Figure 3.4 in comparison with experimental data. It is

Figure 3.4: Pion charge form factor F_+^2 vs. Q^2 . Q^2 is in GeV^2 's on the horizontal axis, and $F_+^2(Q^2)$ is dimensionless on the vertical axis

apparent that the form factor decreases faster with increasing Q^2 than our model predicts. The corresponding curve in Ref.[12] would be almost identical to ours in

the small Q^2 region and their form factor would decrease somewhat slower as Q^2 gets higher and higher. Again, to get agreement with experimental data, Cardarelli et al.[12] introduce new form factors for the constituent quarks.

Conclusion

We have examined relativistic two-particle bound state equations formulated by several different models and theories. The most fundamental of them is the Light-front Tamm-Dancoff method which is a field-theoretical description based on a truncation of Fock space. Applied to strongly interacting systems, this method provides an opportunity to find the relationship between Quantum Chromodynamics—the fundamental field theory of the strong interaction—and the Constituent Quark Model.

We have seized this opportunity and investigated different phenomenological and semi-field-theoretical approaches to the quark-antiquark bound state problem. We have found that the light-front formalism provides a framework in which most models based on the constituent quark picture can be related to the Tamm-Dancoff method in a straightforward manner. We have shown that the equation (2.33) for meson wavefunctions formulated in Front-form Dynamics has an equivalent structure to the Tamm-Dancoff equation.

The connection between fundamental and phenomenological treatments is most apparent in momentum space where the equation for the wavefunction is an integral equation. This creates some mathematical difficulties compared to the more conventional and more familiar differential equations in spatial coordinates. Nevertheless, there exists an elegant and powerful solution method designed for momentum space integral equations by Lévy [11], which is applicable to a broad class of kernels, namely to those depending on the modulus of the momentum. We have applied this technique using a relatively simple interaction kernel of the linear+Coulomb type, motivated by lattice field theory and by successes of the same potential model in the nonrelativistic regime [13],[19],[37].

It must be emphasized that we do not wish to give the impression of creating a complete relativistic model of mesons. Our kernel is too simplistic for that task. The object of this thesis is mainly concerned with treating the integral equations arising from LFTD and solving them accurately. To ascertain how well we succeed in this endeavour, we have included the application to a relativistic model of mesons based

upon phenomenological potentials motivated by QCD. The problem of deriving the kernel in LFTD which gives rise to the relativistic bound states has not been addressed in this thesis.

The potential that we have used does not take spin-spin coupling into account and we have also ignored isospin splitting. These effects are not important in mesons composed of heavy quarks, such as charmonium and bottomonium. Indeed, our results fit the spectra of the ψ and Υ families very well. On the other hand, for light mesons such as the pion, there are some discrepancies between our calculations and experimental data, as expected from our simple model. Still, the fact that we obtained a reasonable wavefunction for the pion and a good result for the decay constant is promising. The Q^2 dependence of the charge form factor is also in qualitative agreement with experimental data, although it does not fall fast enough with increasing Q^2 . This seems to be a common problem in other approaches as well, even in the Isgur and Godfrey model [13] which did include spin-spin interactions. Cardarelli et al.[12] have got around the problem by introducing new form factors at the constituent quark level and adjusting their values to fit the pion form factor to the data.

Also, we have not mentioned spin-orbit coupling, since our calculations were restricted to S-states with zero orbital angular momentum. Lévy's method is readily applicable to higher l -states. The only modification is that higher order Gegenbauer polynomials would be involved in the wavefunction expansion. It would be interesting to see in what circumstances the spin-orbit coupling term becomes important in the interaction, and how it can be incorporated in the momentum space kernel.

We have only treated mesons consisting equal mass (or nearly equal mass) constituents. The sole reason for this was algebraic simplicity. There is no obstacle to performing the same calculations for nonequal mass quarks other than time constraints.

We strongly believe that it is possible to pursue the method of solution presented here further for the actual Tamm-Dancoff kernel. Once that kernel is derived from QCD, our method could be immediately applied to obtain fully relativistic, rota-

tionally invariant wavefunctions for mesons. Based on the promising results of our fairly coarse model, we have a good reason to hope that our approach can fruitfully be developed in the future for LFTD.

Appendix A

Spin-Dependent Wavefunctions

The spin-dependent part of the wavefunction defined in Equation (2.31) and (2.32) was calculated by Jaus [9] in the following spinor basis:

$$\begin{aligned} u(p, \lambda) &= \frac{1}{\sqrt{p^+}} (\not{p} + m) u(\lambda) \\ v(p, \lambda) &= \frac{1}{\sqrt{p^+}} (\not{p} - m) v(\lambda) \end{aligned} \quad (\text{A.1})$$

where

$$u\left(\frac{1}{2}\right) = v\left(-\frac{1}{2}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad u\left(-\frac{1}{2}\right) = v\left(\frac{1}{2}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad (\text{A.2})$$

For the γ matrices the Weyl representation has been used:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (\text{A.3})$$

The spinors are normalized as $\bar{u}(p, \lambda)u(p, \lambda) = -\bar{v}(p, \lambda)v(p, \lambda) = 2m$ and the spin projection operators are given as

$$\begin{aligned}\Lambda^+(p) &= \sum_{\lambda} u(p, \lambda)\bar{u}(p, \lambda) = \not{p} + m \\ \Lambda^-(p) &= -\sum_{\lambda} v(p, \lambda)\bar{v}(p, \lambda) = -\not{p} + m\end{aligned}\quad (\text{A.4})$$

The function $R^{(\Omega)}(\lambda_{\alpha}\lambda_{\beta}, jj_3)$ for pseudoscalar mesons can be written in this basis as

$$R^{(\Omega)}(\lambda_{\alpha}\lambda_{\beta}, 00) = \frac{1}{\sqrt{2}(M_0^2 - (m_{\alpha} - m_{\beta})^2)^{\frac{1}{2}}} \bar{u}(k_{\alpha}\lambda_{\alpha}) \gamma^5 v(k_{\beta}\lambda_{\beta}) \quad (\text{A.5})$$

For vector mesons:

$$\begin{aligned}R^{(\Omega)}(\lambda_{\alpha}\lambda_{\beta}, 1j_3) &= \frac{-1}{\sqrt{2}(M_0^2 - (m_{\alpha} - m_{\beta})^2)^{\frac{1}{2}}} \hat{\epsilon}_{\mu}(j_3) \\ &\times \bar{u}(k_{\alpha}, \lambda_{\alpha}) \left[\gamma^{\mu} - \frac{(k_{\alpha} - k_{\beta})^{\mu}}{M_0 + m_{\alpha} + m_{\beta}} \right] v(k_{\beta}, \lambda_{\beta})\end{aligned}\quad (\text{A.6})$$

where the polarization vectors are given by

$$\begin{aligned}\hat{\epsilon}(0) &= \frac{1}{M_0} \left[\frac{P_{\perp}^2 - M_0^2}{P^+}, P^+, \vec{P}_{\perp} \right] \\ \hat{\epsilon}(\pm 1) &= \left[\frac{2}{P^+} \vec{\epsilon}_{\perp}(\pm 1) \cdot \vec{P}_{\perp}, 0, \vec{\epsilon}_{\perp}(\pm 1) \right], \text{ with } \vec{\epsilon}_{\perp}(\pm 1) = \mp \frac{1}{\sqrt{2}} (1, \pm i)\end{aligned}\quad (\text{A.7})$$

Appendix B

Gegenbauer Polynomials

Hyperspherical harmonics—defined in Equation (3.25)— are extensions of the familiar spherical harmonics $Y_{lm}(\theta, \varphi)$ to 4-dimensional space. The Gegenbauer polynomials are components of the hyperspherical harmonics as written in Equation (3.26). Their generating function is

$$\frac{1}{(1 - 2xr + r^2)^\alpha} = \sum_{n=0}^{\infty} C_n^\alpha(x) r^n \quad |r| < 1, \alpha \neq 0 \quad (\text{B.1})$$

This is a generalization of the generating function for the Legendre polynomials $P_n(x)$ for which $\alpha = 1/2$:

$$C_n^{1/2}(x) = P_n(x) \quad (\text{B.2})$$

Another special case that we used in evaluating the integral (3.31):

$$C_n^1(x) = U_n(x) \quad (\text{B.3})$$

where $U_n(x)$ are the Chebyshev polynomials of the second kind.

Also note that

$$C_0^\alpha(x) = 1 \quad (\text{B.4})$$

$$C_n^1(\cos \theta) = \frac{\sin [(n+1)\theta]}{\sin \theta} \quad (\text{B.5})$$

The recurrence relation between Gegenbauer polynomials

$$(n+1)C_{n+1}^\alpha(x) = 2(n+\alpha)x C_n^\alpha(x) - (n+2\alpha-1)C_{n-1}^\alpha(x) \quad (\text{B.6})$$

implies the recursion relation for C_n^l of Equation (3.26):

$$\cos \psi C_n^l(\psi) = \frac{1}{2} \left[\left(\frac{(n+l+1)(n-l)}{n(n+1)} \right)^{\frac{1}{2}} C_{n-1}^l(\psi) + \left(\frac{(n-l+1)(n+l+2)}{(n+1)(n+2)} \right)^{\frac{1}{2}} C_{n+1}^l(\psi) \right] \quad (\text{B.7})$$

The orthogonality relation (3.35) of the hyperspherical harmonics implies the orthogonality of C_n^α :

$$\int_{-1}^{+1} C_n^\alpha(x) C_m^\alpha(x) (1-x^2)^{\alpha-\frac{1}{2}} dx = \frac{\pi 2^{1-2\alpha} \Gamma(n+2\alpha)}{n! (n+\alpha) \Gamma^2(\alpha)} \delta_{nm} \quad (\text{B.8})$$

The addition theorem for Gegenbauer functions:

$$C_n^\alpha(\cos \Theta) = \frac{\Gamma(2\alpha-1)}{\Gamma^2(\alpha)} \sum_{m=0}^n \frac{2^{2m} \Gamma(n-m+1) \Gamma^2(m+\alpha) (2\alpha+2m-1)}{\Gamma(n+2\alpha+m)} \\ \times \sin^m \psi \sin^m \psi' C_{n-m}^{\alpha+m}(\cos \psi) C_{n-m}^{\alpha+m}(\cos \psi') C_m^{\alpha-1/2}(\cos \theta') \quad (\text{B.9})$$

where $\cos \Theta$ is given by Equation (3.20).

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