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FLEXURE OF BEAMS AND PLATES
RESTING ON WINKLER MEDIA:
A COMPARISON OF ANALYTICAL TECHNIQUES

by

© A.K.M. Ghulam Sarwar, B.Sc. Engg.

A thesis submitted to the Faculty of Graduate Studies
and Research in partial fulfillment of the requirements
for the degree of

MASTER OF ENGINEERING

Department of Civil Engineering

Faculty of Engineering

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Ottawa, Canada

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The undersigned recommend to the Faculty of Graduate Studies
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MASTER OF ENGINEERING.

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ABSTRACT

Flexural behaviour of beams and circular plates resting on elastic soil media are examined using an energy approach and the finite element method. The soil is idealized as a Winkler medium. Analytical solutions are developed for the interaction problems to take into account only the elastic behaviour of each system.

Beams with free ends which are subjected to concentrated and uniformly distributed loads and concentrated moments are analyzed. For the finite element analysis, a cubic polynomial displacement function having two generalized components of displacements at each node is considered. In the energy approach, Fourier series and polynomial functions are used to define the deflected shape of the beam. These displacement functions are either modified or left unmodified while developing the analyses. A modified displacement function is the one which satisfies the static boundary conditions existing at the free ends of a beam. To check the accuracy of both the finite element and the various energy solutions, a conversational programme is developed to analyze the exact solution. Numerical results are developed to illustrate the influence of relative rigidity on beam-deflections, flexural moments and shear forces. The influence of boundary conditions on the energy solutions is also examined.

This thesis also examines the axisymmetric flexure of circular plates resting on a Winkler medium. Flexural behaviour of a plate is represented by the Poisson-Kirchhoff thin-plate theory. For the finite

element analysis, circular plates are modelled as an assemblage of annular ring elements together with a solid central element. For the annular element, a cubic polynomial displacement function having two generalized components of displacements is used. A special formulation is made for the central element. In the energy method, separate solutions are developed for the cases of complete circular plates, annular plates and circular plates with a central rigid pedestal; the displacement functions for the plates are expressed in terms of even-order polynomials consisting of 6, 8 and 7 terms respectively. In each case, the displacement function is modified to satisfy the static and kinematic boundary conditions of the problem. From the energy and the finite element analyses, numerical results are developed to illustrate the effect of relative rigidity on plate deflections.

The finite element analysis for the circular plate is extended to include the analysis of a cylindrical storage tank. A typical problem is solved to illustrate the effect of edge restraint of the circular plate on its deflection and radial moment.

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LIST OF SYMBOLS

Symbol	Meaning
[A]	Matrix relating nodal displacements with generalized coordinates
$A_1, A_2, \dots, A_{10}$	Constants of integration or constant coefficients
a	Radius of circular plate
$a_1, a_2, \dots, a_n$	Constant coefficients in Fourier series
B	Width of beam
b	Radius of annular region or of a pedestal
c	Distance of a load from left end of beam
{C}	Vector containing coefficients of displacement function
[C]	4x4 matrix, associated with annular foundation element stiffness
[C]	3x3 matrix, associated with central foundation element stiffness
[D]	Constitutive relationship matrix
D	Flexural rigidity of a circular plate
$D_2, D_4, \dots$	Constant coefficients
d_1	Distance of a load from origin
E	Modulus of elasticity
e_1	Distance of load from origin
F_1, F_2, F_3, F_4	Special trigonometric functions
{F}	Vector obtained from potential energy due to load
H	Level of liquid measured from base
h	Thickness of shell wall or b/a for circular plate

Symbol	Meaning
I	Moment of inertia of beam-section
k	Modulus of subgrade reaction
$[k_B]$	Stiffness matrix of beam element
$[k_F]$	Stiffness matrix due to foundation element
$[k_P]$	Stiffness matrix of annular plate element
$[k_R]$	Stiffness matrix for a beam or plate element
$[k_{FC}]$	Stiffness matrix for central foundation element
$[k_{PC}]$	Stiffness matrix for central solid element
l	Length of beam, beam element or shell element
$[M]$	Matrix of constants in energy solutions
M_0	Moment in a beam
M_r	Radial moment in circular plate
M_{xx}	Bending stress in the direction of axis of symmetry
$M_{\theta\theta}$	Bending stress in the circumferential direction
N_x	Membrane stress in the direction of axis of symmetry
$N_{\theta\theta}$	Membrane stress in the circumferential direction
P	Concentrated load on a beam
Q_r	Shear force in circular plate
$\{Q\}$	Nodal load vector
$\{q\}$	Nodal displacement vector
q, q_0	Distributed load
R_1, R_2	Special terms associated with Z functions
r_1, r_2	Inner and outer radius of annular element
s	Slope of beam at centre

Symbol	Meaning
S_{mn}, T_m, T_{mn}	Special terms used in summation formulae
t	Thickness of circular plate
u'	Displacement along the length of an element
U_B	Strain energy due to bending of beam
U_F	Strain energy due to deformation of foundation
U_L	Potential energy due to external loads
U_P	Strain energy due to bending of a circular plate
U_R	General name for strain energy due to bending of beam or plate
V, V_0	Shear forces in a beam
v'	Displacement perpendicular to length of shell element
w	Displacement of a circular plate
y	Displacement of a beam
y_0	Displacement of beam at origin
Z_1, Z_2, Z_3, Z_4	Special functions for circular plate
$\{\alpha\}$	Vector of generalized displacement components
β'	Rotation of shell element
γ	Unit weight of liquid
$\{\delta\}$	Vector of nodal displacements
ξ	Variable non-dimensional distance
λ_a	Relative rigidity of circular plate-soil system
λ_l	Relative rigidity of beam-soil system
ν	Poisson's ratio

CHAPTER 1

INTRODUCTION

1.1 GENERAL

Beams^{*} and plates^{*} supported on deformable elastic media constitute problems of importance to the field of structural mechanics (Hetenyi, 1966; Selvadurai, 1979a). Solutions to such problems are of particular importance in the analysis and design of structural foundations^{*} resting on soil and rock media (Selvadurai, 1979a). Examples of beams resting on soil media include the behaviour of rails and railroad ties, grillage beams and distributor beams in floor systems.

In addition to flexible beams, circular plates are extensively used as the bases of storage tanks, nuclear reactors and chimneys. In order to determine the optimum dimensions of such structural foundations, it is necessary to examine their flexural behaviour under anticipated loads. Since both the beams and plates are continuously supported on the soil, it is reasonable to assume that the nature of the supporting soil would greatly affect their flexural behaviour. To allow for this structure^{*}-foundation^{**}-interaction, a unified method of analysis is required.

* beam, plate, structural foundation or structure refer to the structural unit transmitting loads directly to the supporting soil medium.

** foundation refers to the supporting soil or the soil medium.

Since the fundamental laws of stresses and strains for soils are quite complex, it becomes necessary to utilize mathematically simple idealizations of soil behaviour to make possible the analytical study of soil-structure interaction problems. One such elementary model commonly used to represent the behaviour of a soil medium is the Winkler (Spring) Model. This model appears to have been first utilized by E. Winkler in 1867. The exact analytical solution for a prismatic beam resting on a Winkler medium was obtained by Hetenyi (1946). Similarly, the exact solution for symmetrically loaded infinite plate resting on a Winkler medium was given by Hertz (1884). The exact analyses both for a beam and a circular plate involve the solution of fourth order differential equations which require rigorous analytical treatments when applied to most practical situations. Formal solutions of such equations related to beams are available in terms of special functions which can be expressed in exponential forms. Similar solutions for circular plates contain certain special functions, the arguments of which include parameters of the soil-plate system.

The main objective of this thesis is to examine the flexural behaviour of beams and circular plates resting on a Winkler medium using numerical techniques such as the energy method and the finite element method. The analysis procedures developed are applicable to the elastic behaviour of the structure-foundation system. Fig. 1.1 shows a typical beam with free edges resting on a Winkler medium and subjected to concentrated and distributed loads and concentrated moments. The beam may

be prismatic or its cross-section may change along the length. The energy method is so developed that only prismatic beams could be examined. The distribution of loading may be arbitrary.

Fig. 1.2 illustrates the types of circular plates investigated in this thesis. The plate may be a complete, an annular or containing a pedestal as shown in Fig. 1.2. It is assumed that the plates have free edges and are subjected to axisymmetric loads such as uniform pressures and a concentrated load.

The finite element formulation of the circular plate has further been extended to include the analysis of an axisymmetric storage tank (Fig. 1.3) problem. In this development, only axisymmetric loads are considered.

1.2 WINKLER'S MODEL

The Winkler's Model (sometimes referred to as the Winkler-Zimmermann Model) was first proposed by E. Winkler in 1867. It assumes that the deflection (y) at any point on the surface of an idealized elastic foundation is directly proportional to the pressure (p) at that point, and completely independent of pressures or deflections occurring at other, immediately neighbouring points along the length of the beam or the foundation. The response of such a model when subjected to a uniform load is illustrated in Fig. 1.4a. The corresponding relationship governing p and y is given by

$$p = ky \quad (1.1)$$

where k is a constant. The constant k is described as the modulus of

subgrade reaction and has dimensions ML^{-3} . A model of such supporting medium can best be represented by a bed of closely spaced, independent elastic spring elements (Fig. 1.4b), which have a stiffness k . Each of the springs will then be deformed only by the pressure directly applied on it, while neighbouring springs remain unaffected.

Winkler's model formed the basis of H. Zimmermann's classical work on the analysis of railroad track published in 1888. Though the early investigators thought chiefly of soil as the supporting medium, it was later realized that there are other situations where Winkler's assumptions are much more rigorously satisfied. Hetenyi (1946) quotes two such important fields of application. One of these is concerned with the flexure of a network of beams, which is characteristic in the construction of floor-systems for ships, buildings and floating bridges. The other deals with thin shells of revolution and include application to problems encountered in the analysis of boilers, pressure vessels and containers as well as large-span modern reinforced concrete halls and domes. Hertz's (1884) analysis of a floating circular plate is also based on this assumption. While Winkler's theory holds rigorously for most of the problems mentioned here, its application to soil-structure interaction problems should be regarded only as a practical approximation (Hetenyi, 1946). In spite of its simplicity, this theory is widely used for the analysis of soil-structure interaction problems. In a review paper, Hetenyi (1966) catalogued various works related to interaction analysis showing that by far the largest number of investigations in this field is based on

Winkler's hypothesis. Selvadurai (1979a) summarizes a number of model tests and field observations related to rigid plates, flexible beams and large scale structural-foundations and concludes that for beams or plates with a high relative rigidity[†], Winkler's model provides a satisfactory representation of the deformational characteristics of the soil medium.

In practice, the constant k is determined from standard plate-loading tests (Terzaghi, 1956). It has been observed that k is not a unique property of a soil medium but various factors such as size and shape of the test-plate as well as the depth at which a test is carried out affect the results. On the basis of experimental evidence, Terzaghi (1956) establishes procedures to account for these factors. An excellent summary regarding the evaluation of constant k from plate-loading tests is given by Selvadurai (1979a).

In situations where such a test cannot be made, recourse may be made to approximate results derived for the estimation of k for various types of soils. Such tables are given by Terzaghi (1956) and Miner and Seastone (1955).

1.3 LITERATURE REVIEW

In this section, a brief review of previous investigations relating to the analysis of beams and plates resting on a Winkler medium will be given. However, for the sake of clarity, the discussion for the beam and plate will be treated separately.

[†] A factor expressed in terms of the properties of the beam (or plate) and the supporting soil medium

1.3.1 Beamson Elastic Foundation

Winkler's hypothesis leads, in the case of a deformed beam, to a linear ordinary differential equation of fourth order and solutions have been obtained for a variety of loading and boundary conditions. These are available in several comprehensive books on the subject (Hayashi, 1921; Hetenyi, 1946; Vlasov and Leontiev, 1960). The fact that exact solutions for the same beam problem can be expressed in various mathematical forms has been realized by Hetenyi (1959). By equating equivalent expressions obtained through: (a) formal solution of differential equation, (b) by superposition on the infinite beam and (c) by expansion into characteristic functions, he obtained certain summation formulae.

However, in its purely analytical form, Hetenyi's (1946) presentation is not very useful to practising engineers. Therefore, various attempts have been made to present it in a more usable form. One such way is to tabulate the influence coefficients for beam-deflections, bending moments and shear forces due to a concentrated load or a concentrated moment. Dodge (1964) has presented such tabulated results. Miranda and Nair (1966) also tabulated certain special functions which were used to define the deflected shape of a beam resting on a Winkler medium.

Hetenyi (1946) observed that the solution of a beam problem can be expressed as a series of eigenfunctions corresponding to the normal modes of vibration of the beam in question. He utilized such functions to obtain exact solutions for hinged-end beams resting on a Winkler medium.

These characteristic functions were used by Hendry (1958) and later by Iyenger and Anantharamu (1965) to examine problems of beams with free ends. It should be noted that the use of eigenfunctions for non-hinged-ended beams must be regarded as only an approximation (Hetenyi, 1966) because, for such beams, these eigenfunctions are not the same in the presence or absence of an axial force.

One drawback of the analytical method is that only prismatic beams resting on a Winkler medium could be treated with relative ease. Iyenger (1965) generalized and improved the analytical method to include non-prismatic beams by using the concept of transfer matrices. His solution gives a complete analysis for non-prismatic beams having arbitrary end conditions and subjected to arbitrary distribution of lateral loads. In the above analysis, the variation of beam-rigidities is incorporated in a stepwise manner. Standard functions used in the method have also been tabulated by Iyenger (1965).

Iwanczewska and Lewandowski (1968) have presented a wide range of general slope deflection equations for beams resting on a Winkler medium. It has enabled one to use the displacement method (see e.g. Willems and Lucas, 1968) of analysis for this problem.

Finite element techniques have also been used for the analysis of beams resting on Winkler media. Just, Starzewski and Ronan (1971) have developed a finite element analysis using a fourth order displacement function to represent the elastic curve. They also developed an approximate Winkler solution by using a finite element technique where the

supporting medium was represented by a discrete number of springs located at the edges of discretized beam elements. Their main observation was that while a good agreement existed between the finite element and exact solution for contact stresses, a considerable difference was noticed in the results for the shear forces.

Miyahara and Ergatoudis (1976) developed a finite element analysis for a skeletal structure where they used line elements for the foundation beams. To make the superstructure beam-foundation system into a compatible unit, they considered 12 degrees of freedom for the beam element. The soil medium was considered to be of Winkler type which offers resistance to shear, torsional and normal stresses. It was shown that this scheme gave solutions different from the equivalent problem in which the soil medium was represented by isolated spring elements.

Kramrisch and Rogers (1961) presented an approximate Winkler solution for beams and combined footings conforming to certain limitations regarding continuity, rigidity and variations in loadings and column spacings. The restrictions placed upon the method in respect of column spacing and loads were utilized to obtain the maximum negative moment under the columns by means of an approximate formula using a straight line expression instead of the trigonometric function of the classical theory. The evaluation of the contact pressures, positive moments and shears were simplified by the assumption of a straight line distribution of contact pressure with a maximum value under the columns and a minimum value at

the centre of each bay. Just et al (1971) observed that within the scope of its application, Kramrisch and Rogers' (1961) method gives results which are in very good agreement with the classical solution.

Baker (1957) has developed an approximate Winkler method known as the Soil-Line-Method for the analysis of structural foundations (beams and rectangular plates only) in a framed structure. The simplification introduced is due to the assumed linear variation of soil pressure along the length of main beams. Baker (1957) has, however, demonstrated that for a beam the Soil-Line-Method gives results in close agreement with the exact Winkler solution.

1.3.2 Plates on Elastic Foundation

Analytical solutions relating to thin plates on a Winkler medium are available for only a limited case of plate geometry, loading and boundary conditions. A complete account of these problems is given by Timoshenko and Woinowsky-Krieger (1959), Hetenyi (1966) and Selvadurai (1979a). The analysis of a floating elastic plate of infinite extent was presented by Hertz (1884). Hertz solved the differential equation governing the plate deflections (Eqn. (4.4), Chapter 4) for the special case of an infinite plate with a concentrated load. (It may be noted that the situation of a floating plate exactly fulfills Winkler's assumption; the constant k being equivalent to the unit weight of the liquid.) An extensive coverage of circular axisymmetrically deformed plates on a Winkler medium can be found in a classical book by Schleicher (1926).

Schleicher used Bessel functions to solve the plate differential equation. Some problems of practical interest were also solved by Hetenyi (1946).

Thin rectangular plates resting on a Winkler medium were analyzed by Vint and Elgood (1935) using the classical Rayleigh-Ritz procedure. They also conducted experiments on a steel plate bearing on independently acting springs to verify their theoretical predictions. A satisfactory agreement between the experimental and theoretical results of the plate deflections was obtained. Murphy (1937) used the Ritz procedure to solve the plate problem and corroborated his results by testing steel plates on a rubber subgrade which was 3" thick. The coefficient of subgrade reaction was obtained by pressing rigid punches into the subgrade (rubber). The experimental results for displacements of the plate were in reasonable agreement with the theoretical predictions based on the Winkler assumption (i.e. comparisons were made with theoretical results given by Vint and Elgood, 1935). It should be pointed out that the subgrade used by Murphy (1937) is far too thin to represent an elastic continuum type foundation. Pickett et al (1951) examined the rectangular plates on the Winkler foundation using a Fourier Series approach. Allen and Severn (1960, 1961, 1963) employed the method of finite difference to obtain solutions for plates having complicated boundary conditions.

The finite element method has also been used for the analysis of thin circular and rectangular plates resting on a Winkler medium. The stiffness matrix for a thin rectangular plate-bending element was derived by Melosh (1961) and by Zienkiewicz and Cheung (1964). Severn (1966) used

the stiffness matrix derived by Zienkiewicz and Cheung (1964) in the finite element analysis of foundation mats resting on a Winkler medium. Sewarn (1966), however, considered the effect of spring couplings of foundation in addition to the vertical resistance of springs. He compared his results with those of Vint and Elgood (1935) for the case when spring coupling was not considered. A satisfactory agreement was observed between the plate deflections obtained by both these methods.

Smith (1967) developed various finite element analyses whereby beams, circular, rectangular or irregular plates with arbitrarily variable elastic and sectional properties bearing on an elastic foundation can be analyzed with equal facility. The idealized soil-media may be a uniform or non-uniform Winkler, or coupled or semi-infinite elastic medium with arbitrary anisotropy and inhomogeneity. Non-linear (large deflection) elastic analysis in any of the structural foundation types described above could be tackled by using an iterative procedure. In Smith's (1967) work, the finite element method using rectangular elements had been applied to problems of moderately thick (though a very poor approximation compared with E. Reissner's work of 1955) plates for the first time. An explicit stiffness matrix suitable for the analysis of circular plates was also derived. Smith (1967) also applied the method of finite difference for the analysis of axisymmetrically loaded circular plates resting on Winkler media. He conducted some experiments on axisymmetrically loaded circular plates resting on sand subgrades. A satisfactory agreement between the actual and predicted values of plate deflections was observed.

E. Reissner (1955) analyzed thin, infinite elastic plates resting on a Winkler medium under a variety of loading conditions. The thick plate theory developed by Reissner (1955) includes the effects of shear deformation and normal pressure on the deflections of a plate (these effects are neglected in the classical Poisson-Kirchhoff thin plate theory). According to Reissner, a plate could be analyzed following the thin plate theory so long as the value of $2a/h$ is approximately equal or greater than unity, where the quantities a and h refer to the radius of load surfaces and thickness of the plate, respectively. When the parameter $2a/h$ is between 1 and 2, the effect of shear deformation becomes significant. Also, the non-linear behaviour of a plate (effect of the stretching of the middle surface of a plate on deflection) becomes prominent when the deflection exceeds $h/2$.

Frederick (1955) presented the governing differential equations and their solution for the symmetrical and non-symmetrical bending of thick circular plates resting on a Winkler medium. In this development, Reissner's (1955) thick plate theory was adapted. Thick rectangular plates on a Winkler medium were also analyzed by Frederick (1962) using the thick plate theory. Frederick (1957) demonstrated that the solution using Reissner's (1955) theory gives higher maximum deflection, while moments and shear forces give lower values when compared with the classical thin plate solution.

Problems related to moderately large deflections of uniformly loaded circular and rectangular plates resting on a Winkler medium were

examined by Sinha (1963). He used a variational approach to derive the governing non-linear differential equations for the plate deflections. It may be noted that Sinha's (1963) solution is an approximate one, because he disregarded the strain energy component due to second invariant of the middle surface strains of the plate.

The finite element analysis of large deflection analysis of rectangular plates on a Winkler medium was carried out by Yang (1970). In this work, the non-linear behaviour of the system was approximated by a step-by-step linear incremental procedure. He compared his results with the results obtained by Sinha (1963). A satisfactory agreement was obtained between the results obtained from these two methods.

Results concerning the interaction of the superstructure, the structural foundation and a Winkler medium are also available in the literature. Lee and Harrison (1970) developed analytical methods for the analysis of combined footings and rectangular plates on Winkler media which take into account the effect of superstructure. Cheung and Zienkiewicz (1965) developed a finite element analysis to solve the problem of a square storage tank. However, their analysis was capable of taking into account the Winkler's as well as elastic-halfspace idealizations of the soil medium. Cheung and Zienkiewicz (1965) compared their results (rectangular plate on a Winkler medium) with that of Vint and Elgood (1935). A satisfactory agreement was noted.

Davies (1961) examined the problem of cylindrical storage tanks resting on a Winkler medium. In this work, he determined the moments at the continuous connection between the circular plate and the cylindrical element. The rotations at the edge of the plate due to uniform couples and vertical forces at the edge itself were calculated using simplified expressions.

1.4 OBJECTIVES

The main objectives of this thesis can be summarized as follows:

- 1) To develop analyses using an energy approach and the finite element technique for the problem of a beam resting on a Winkler medium.
- 2) To develop a conversational computer program to examine the behaviour of a beam resting on a Winkler medium using the exact analysis.
- 3) To develop the energy and finite element methods of analysis for the axisymmetrically loaded circular plates resting on Winkler media. The following types of plates are considered:
 - i) a complete circular plate
 - ii) an annular plate
 - iii) a circular plate with a central rigid pedestal.
- 4) To develop the finite element analysis for the axisymmetric interaction of an axisymmetric tank and a Winkler medium by taking into account the effect of the combined shell-circular plate and foundation interaction.

Fig. 1.1

Beams on Winkler Media

Fig. 1.2: Circular Plates Resting on Winkler Media
 (a) Complete (b) Annular (c) Plate with
 Infinitely Rigid Pedestal

Fig. 1.3: Storage Tank

Fig. 1.4: A Winkler Medium

(a) Response of the Medium

(b) Equivalent Representation for (a)

CHAPTER 2

THE THEORETICAL FORMULATIONS

2.1 GENERAL

The general description presented in this section illustrates how the finite element technique and energy methods could be used for the analysis of a structure-foundation-interaction problem related to idealized soil. As outlined previously, both these analytical techniques utilize the Winkler idealization of the soil medium.

2.2 FINITE ELEMENT ANALYSIS

In applying this method to a structure-foundation-interaction problem, a displacement formulation is considered. This follows directly from the consideration that for such a system the displacements but not the stresses are primary unknowns. In any finite element analysis, the obvious first step is to discretize the body or continuum being analyzed. In this process, the given body is subdivided into an equivalent system of elements. The assemblage of such units represents the original body. For the present interaction analysis, two types of elements will be obtained from discretization. The properties of the elements of beam or plate are described by modulus of elasticity (E) and Poisson's ratio (ν) of the constituent material as well as by its thickness. The elements of soil are characterized by modulus of sub-

grade reaction (k) only. The displacement field associated with an element can be represented by polynomial functions. The number of unknown constants characterizing the polynomial function is identically equal to nodal degrees of freedom. The same displacement function is used for both the soil and other structural elements of beam or plate, thus the displacements of two systems would be completely compatible.

Stiffness matrix (K_R) associated with a beam or plate element could be combined with the corresponding soil element stiffness matrix (K_F) term by term to obtain what may be called as total stiffness matrix (K_E) of an element. The distributed forces applied to the body are converted into equivalent concentrated forces at the nodes. The equilibrium relation between the element stiffness matrix (K_E), nodal force vector (Q) and nodal displacement vector (q) can be expressed as a set of simultaneous linear algebraic equations

$$[K_E]\{q\} = \{Q\} \quad (2.1)$$

The stiffness matrices and nodal load vectors can be obtained using a variational principle (Desai and Abel, 1972; Zienkiewicz, 1967; Melosh, 1963). For that purpose, the assumed displacement function is first expressed in terms of nodal displacements. This in turn is used to calculate the total energy functional Π of the system. Each component of the total energy functional will be functions of nodal displacements of the element under consideration. The total energy of the soil

medium-structural foundation-load system is composed of three components:

- i) the strain energy of the idealized soil medium (U_F)
- ii) the strain energy of the flexure of beam or structural foundation (U_R)
- iii) the potential energy of externally applied loads (U_L).

Hence,

$$\Pi = U_R + U_F + U_L \quad (2.2)$$

The theorem of minimum potential energy states that of all kinematically admissible states of deformation those which satisfy equilibrium give a minimum value of the total potential energy, i.e. for the total potential energy to be minimum

$$\partial \Pi = 0$$

or

$$\frac{\partial U_R}{\partial q_i} dq_i + \frac{\partial U_F}{\partial q_i} dq_i + \frac{\partial U_L}{\partial q_i} dq_i = 0 \quad (2.3)$$

This condition generates the equilibrium condition as given by Eqn.

(2.1). In other words, the element of the stiffness matrix in the i th row and j th column for the beam or plate (K_R) and that of the foundation (K_F) and the i th term of the load vector (Q) can be obtained from the following:

$$(K_{ij})_R = \frac{\partial^2 U_R}{\partial q_i \partial q_j}$$

$$(K_{ij})_F = \frac{\partial^2 U_F}{\partial q_i \partial q_j}$$

(2.4a-c)

$$Q_i = - \frac{\partial U_L}{\partial q_i}$$

whence

$$[K_E] = [K_R] + [K_F] \quad (2.5)$$

Once the element stiffness matrices are determined in this manner, the overall or global stiffness matrix is obtained by assembling the stiffness matrices for the individual elements. Similarly, the overall or global load vector is obtained from the element nodal load vectors. Basis for an assembly technique is that the nodal interconnections require the displacement at a node to be the same for all elements adjacent to that node. The overall equilibrium between the global stiffness matrix (K), the global load vector (F) and global displacement vector (θ) is again expressed as a set of simultaneous equations

$$[K]\{\theta\} = \{F\} \quad (2.6)$$

It is important to note that the global stiffness matrix (K) is symmetric about the leading diagonal and is banded. Symmetry is a direct consequence of the reciprocal theorem and the banded nature could

be ensured by systematic numbering of nodes. These properties must be utilized for maximum economy of computer storage space.

From the solution of system of equations given by Eqn. (2.6), nodal displacements will be known. This is achieved by Gaussian Elimination technique with Payne and Irons' method providing the means so that problems involving any mechanical and kinematic boundary conditions may be treated.

The flexural moments and shearing forces in a beam or a circular plate is obtained by using the standard techniques of finite element analysis. Computed values at a node obtained from the two connecting elements are averaged.

2.3 ENERGY METHOD

In the energy method, a deflection function is first assumed that represents the deflected shape of the entire body. The decision as to what function to select for a particular problem can be based upon an intuitive idea of the true deformation of the system. From an analytical standpoint, trigonometric or polynomial functions are normally the most convenient forms to be used. However, any such representation for the deflected shape must satisfy the kinematic boundary constraints particular to the problem. The accuracy of the method can be improved by selecting a deflected shape which satisfies both the mechanical and kinematic boundary conditions of the problem. Although the accuracy is

usually improved by increasing the number of independent terms, the computation becomes correspondingly more complex. The number of terms and hence the number of unknown functions in the displacement function can be taken such that a reasonable degree of accuracy in displacements is obtained.

In applying the energy method to the analysis of soil-structure interaction problems, the same displacement function is used for both the structure and its foundation. Like finite element analysis, a consideration of total potential energy functional would be necessary to formulate a problem. For such a system, total energy functional can be computed as a function of the constants characterizing the assumed deflected shape. The arbitrary constants are uniquely determined by the set of linearly independent algebraic equations generated from the minimization of the total potential energy functional. Thus, if there are n number of undetermined constants in a displacement function, n number of simultaneous equations will be formed. These equations are sometimes best expressed in the matrix form as follows:

$$[M]\{C\} = \{F\} \quad (2.7)$$

where

$[M]$ is a matrix of constants arising from the minimization of the strain energy of the system,

$\{C\}$ is a vector containing the coefficients of displacement function,

$\{F\}$ is a vector arising from the minimization of the work done by external loads.

Equation (2.7) is solved to obtain the values of C_i ($i=1, \dots, n$).

One of the simple ways to solve this set of equations is to use the Gaussian elimination scheme with back substitution. As the values of C 's are known, the displacement field of the entire body is completely defined. Moment and shear force at any location of the body are obtained by differentiating the deflection line.

2.4 SUMMARY OF ANALYSIS TECHNIQUES

Of the two methods of analysis that have been discussed, both have advantages and disadvantages. Since the degree of polynomial or number of unknowns representing the deflected shape that could be taken in finite element method depends on the nodal degrees of freedom, therefore, to improve the accuracy of results, the structure has to be subdivided into a large number of elements. The piecewise deflected shape of various elements taken together in order represents the overall deflected shape of the entire body. Essentially, the method is a product of modern digital electronic computers with a large storage capacity. The adoption of large numbers of elements increases the necessary storage space in the computer and also computation time is increased. However, this method has the capability of accommodating factors such as variation of properties of soil medium and of beam or other structural components.

On the other hand, the energy method would offer a solution in which a higher number of unknowns could be used in the deflection function and only one such function could represent the deflected shape of the entire body. Consequently, a single set of equations need to be deduced and solved to define the displacement field completely. The disadvantage of this method is that it can not readily accommodate variation in the properties of the soil and the structural foundation. The basic theory underlying the energy method can, however, be employed for such cases by combining the energies of the various sections where properties become discontinuous. The size of the coefficient matrix (M) would still remain the same provided the deflected shape is represented by a continuous function. In general, it can be said that the energy method is a powerful tool being as versatile as the finite element method which can profitably be used for the analysis of soil-structure interaction problems.

CHAPTER 3

ANALYSES OF BEAMS

3.1 GENERAL

Problems related to beams on elastic foundations have been investigated by many researchers (Hayashi, 1921; Hetenyi, 1946; Vlasov and Leontiev; 1960, Iyenger, 1965). Comprehensive accounts of such investigations are given by Hetenyi (1966) and Selvadurai (1979a). Beams are generally defined by a characteristic parameter λ , which incorporates its flexural rigidity as well as the elasticity of the idealized soil medium. For a beam of constant width B , having modulus of elasticity E and moment of inertia I , the term λ is defined as

$$\lambda = \left(\frac{Bk}{4EI} \right)^{1/4} \quad (3.1)$$

where k is the modulus of foundation. It is seen that λ has the dimension of length⁻¹. The term λl which is an absolute number is frequently referred to as the relative rigidity. From the mathematical point of view, some of the beams may be considered to be infinitely long while others may be considered to have a finite length.

From practical considerations, beams are classified (Hetenyi, 1946) into three groups as follows:

- i) Short beams $\lambda \ell < \pi/4$
- ii) Medium beams $\pi/4 < \lambda \ell < \pi$
- iii) Long beams $\lambda \ell > \pi$

Short beams are very rigid while medium and long beams are flexible. Various cases of loadings may occur on such beams which are either point loads and concentrated moments or distributed loads. Complicated cases of loading can be solved from superposition of the results from these elementary solutions. Herein, the analysis will be restricted to beams of finite length subjected to concentrated loads, uniformly distributed loads and concentrated moments.

3.2 EXACT ANALYSIS

Consider a straight beam of constant width B and length ℓ , supported along its entire length by a Winkler type of foundation. The beam is subjected to a uniformly distributed load as shown in Fig. 3.1.

According to Winkler's assumption, the reaction p at any point along the length of the beam is

$$p = Bky \quad (3.2)$$

where k (ML^{-3}) is the modulus of the foundation and y is the deflection at the point being considered. Considering the equilibrium of an element (Fig. 3.2), it can be shown that

$$\frac{dV}{dx} = Bky - q \quad (3.3)$$

From the Bernoulli-Euler beam theory, it is known that

$$\frac{dM}{dx} = V ; - EI \frac{d^2 y}{dx^2} = M \quad (3.4)$$

The equation governing the flexure of a beam resting on a Winkler medium can be obtained by combining Eqns. (3.3) and (3.4). Thus,

$$\frac{EI d^4 y}{dx^4} + Bky - q = 0 \quad (3.5)$$

This is the differential equation for the deflection curve of a beam supported on a Winkler type of foundation. The general solution for Eqn. (3.5) has been given by Hetenyi (1946) and takes the form

$$y = e^{\lambda x} (A_1 \cos \lambda x + A_2 \sin \lambda x) + e^{-\lambda x} (A_3 \cos \lambda x + A_4 \sin \lambda x) + y_p \quad (3.6)$$

where A_1 , A_2 , A_3 and A_4 are constants of integration and y_p is the particular solution corresponding to a given loading q .

An explicit solution for the arbitrary constants (A_i , $i=1,4$) can be obtained by incorporating the boundary conditions at the ends of the beam. Thus, the above solution (Eqn. 3.6) is particularly suited to beams of infinite and semi-infinite length, because it is possible in such cases to use the regularity condition ($y=0$ at $x=\infty$) to eliminate two of the constants of integration (Miranda and Nair, 1966). Hence A_1 and A_2 can easily be excluded. For finite beams, the

solution presents computational difficulties due to discontinuity in loadings. However, it is possible to present the general solution to a finite beam problem by employing a technique known as the "method of initial parameters" which reduces the computation work to a great extent. The details of this method are given by Hetenyi (1946) and Selvadurai (1979a). Final expressions for the deflection (y), rotation (θ), flexural moment (M) and shearing forces (V) in a beam can be written as follows:

$$\begin{aligned}
 y &= y_o F_1(\lambda x) + \frac{1}{\lambda} \theta_o F_2(\lambda x) - \frac{1}{\lambda^2 EI} M_o F_3(\lambda x) - \frac{V_o}{\lambda^3 EI} F_4(\lambda x) \\
 &\quad + \frac{1}{\lambda^3 EI} PF_4[\lambda(x-u_p)] + \frac{1}{\lambda^3 EI} \int_c^x q F_4[\lambda(x-u)] du - \frac{1}{\lambda^2 EI} MF_3[\lambda(x-u_M)] \\
 \theta &= -4\lambda y_o F_4(\lambda x) + \theta_o F_1(\lambda x) - \frac{1}{\lambda EI} M_o F_2(\lambda x) - \frac{1}{\lambda^2 EI} V_o F_3(\lambda x) \\
 &\quad + \frac{1}{\lambda^2 EI} PF_3[\lambda(x-u_p)] + \frac{1}{\lambda^2 EI} \int_c^x q F_3[\lambda(x-u)] du - \frac{1}{\lambda EI} MF_2[\lambda(x-u_M)] \\
 M &= \frac{Bk}{\lambda^2} y_o F_3(\lambda x) + \frac{Bk}{\lambda^3} \theta_o F_4(\lambda x) + M_o F_1(\lambda x) + \frac{1}{\lambda} V_o F_2(\lambda x) \\
 &\quad - \frac{1}{\lambda} PF_2[\lambda(x-u_p)] - \frac{1}{\lambda} \int_c^x q F_2[\lambda(x-u)] du + MF_1[\lambda(x-u_M)] \\
 V &= \frac{Bk}{\lambda} y_o F_2(\lambda x) + \frac{Bk}{\lambda^2} \theta_o F_3(\lambda x) - 4\lambda M_o F_4(\lambda x) + V_o F_1(\lambda x) \\
 &\quad - PF_1[\lambda(x-u_p)] - \int_c^x q F_1[\lambda(x-u)] du - 4\lambda MF_4[\lambda(x-u_M)]
 \end{aligned}$$

(3.7a-d)

In the above expressions, y_0 , θ_0 , M_0 and V_0 are respectively the values of deflection, rotation, flexural moment and shear force at one end of the beam, while u_p , u_M and u are the distances of concentrated load, concentrated moment and distributed load respectively corresponding to Fig. (3.3). Also

$$F_1(\lambda x) = \cosh \lambda x \cos \lambda x$$

$$F_2(\lambda x) = \frac{1}{2} [\cosh \lambda x \sin \lambda x + \sinh \lambda x \cos \lambda x]$$

$$F_3(\lambda x) = \frac{1}{2} \sinh \lambda x \sin \lambda x$$

$$F_4(\lambda x) = \frac{1}{4} (\cosh \lambda x \sin \lambda x - \sinh \lambda x \cos \lambda x)$$

and the value of λ being as defined by Eqn. (3.1).

In using Eqns. (3.7a-d) all terms containing the concentrated load P are to be disregarded, provided $x \leq u_p$. The same technique applies for the concentrated moment M when $x \leq u_M$ and for the distributed load q when $x \leq c$. For all the integrals containing the q term, d becomes the upper limit when $x > d$. It should also be noted that the notation $F_i(\lambda[x-u])$ refers to $F_i(\lambda\xi)$ evaluated at $\xi=x-u$.

To apply Eqn. (3.7a-d) for the analysis of a beam similar to the one shown in Fig. (3.3) in which the ends are free, it is observed that flexural moment (M) and shear force (V) are zero at end $x=l$. With this condition, the two unknown initial parameters y_0 and θ_0 can be determined from Eqn. (3.7a) and (3.7b). The set of equations given by Eqn. (3.7a-d) is now completely defined.

3.3 ENERGY METHOD OF ANALYSIS

Using the energy method, two separate solutions have been developed considering the displacement functions for the beam in the following forms:

- i) Fourier series
- ii) Polynomial

Both types of analysis are again approached in two ways. In the first case, a displacement function is assumed and solutions are developed without regard to mechanical boundary conditions at the free ends of a beam. In the second case, the assumed deflection function is modified so as to satisfy the mechanical boundary conditions at the free ends.

In order to simplify the analysis, each loading is divided into symmetrical and antisymmetrical loading components as shown in Fig. 3.4. Complete solution for any loading is obtained by superposition of results from the corresponding symmetrical and antisymmetrical components. The sequence of steps needed in the analysis is necessarily the same as those outlined in Chapter 2 (Section 2.3). In the following, complete details concerning one such analysis will be outlined while for others, only the final results will be presented.

Also, the following description is given for a beam of unit width and modulus of foundation k . For a beam of width B , the effective value of k could be obtained as $k_{\text{eff}} = k.B$

3.3.1 Solutions by Fourier Series

In literature on bending of beams without elastic supports, frequent use is made of the fact that any elastic line which passes through the points, $x=0, y=0$ and $x=l, y=0$ can be represented in the form of a trigonometric series as

$$y = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} \quad (3.8)$$

With suitable modifications, this series can be applied to the analysis of beams on elastic foundations.

3.3.1.1 Symmetric loads

Considering the nature of the deflected shape of the beam for the type of loading (Fig. 3.5), the displacement function can be taken as

$$y = y_0 + \sum_{n=1,3,\dots}^{\infty} a_n \sin \frac{n\pi x}{l} \quad (3.9)$$

The constant y_0 is eliminated by considering the equilibrium of the beam due to vertical forces. For a beam with two concentrated loads (Fig. 3.5a), this condition would require

$$P - k \int_0^l y dx = 0 \quad (3.10)$$

Substituting the value of y from Eqn. (3.9) into Eqn. (3.10), and upon integration, it is found that

$$y_0 = \frac{P}{kl} - \frac{2}{\pi} \sum_{n=1,3}^{\infty} \frac{a_n}{n} \quad (3.11)$$

Therefore, the displacement function for the case of two symmetrically placed concentrated loads can be written as

$$y = \frac{P}{k\ell} + \sum_{n=1,3}^{\infty} a_n \left(\sin \frac{n\pi x}{\ell} - \frac{2}{n\pi} \right) \quad (3.12)$$

Having obtained the displacement function, one can proceed to calculate the components of total potential energy functional of the system. The strain energy of flexure of the beam (U_B) is given by

$$U_B = \frac{EI}{2} \int_0^{\ell} \left(\frac{d^2 y}{dx^2} \right)^2 dx = \frac{\pi^4 EI}{4\ell^3} \sum_{n=1,3}^{\infty} n^4 a_n^2 \quad (3.13)$$

The corresponding strain energy of the deformation in the foundation (U_F) is given by

$$U_F = \frac{k}{2} \int_0^{\ell} y^2 dx = k\ell \left[\sum_{n=1}^{\infty} \frac{a_n^2}{4} - \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{a_m}{m} \sum_{n=1}^{\infty} \frac{a_n}{n} \right] \quad (3.14)$$

and the potential energy of the loads (U_L) is given by

$$U_L = -P[y]_{x=c} = -P \left[\frac{P}{k\ell} + \sum_{n=1}^{\infty} a_n \left(\sin \frac{n\pi c}{\ell} - \frac{2}{n\pi} \right) \right] \quad (3.15)$$

Minimization of the total potential energy functional requires

$$\frac{\partial U_B}{\partial a_n} da_n + \frac{\partial U_F}{\partial a_n} da_n + \frac{\partial U_L}{\partial a_n} da_n = 0$$

The above condition gives the following set of simultaneous equations for a_m :

$$a_m \left(m^4 \pi^4 \frac{EI}{2\ell^3} + \frac{k\ell}{2} \right) - \frac{4k\ell}{m\pi} \sum_{n=1,3}^{\infty} \frac{a_n}{n} = P \left(\sin \frac{m\pi c}{\ell} - \frac{2}{m\pi} \right) \quad (3.16)$$

With the use of Eqn. (3.16), one can determine as many 'a' terms as necessary for the required accuracy of the deflection line. The result (3.16) can be expanded in the form:

$$a_1 \left(1^4 \pi^4 \frac{EI}{2\ell^3} + \frac{k\ell}{2} \right) - \frac{4k\ell}{\pi^2} \left(\frac{a_1}{1} + \frac{a_3}{3} + \frac{a_5}{5} + \dots \right) = P \left(\sin \frac{\pi c}{\ell} - \frac{2}{\pi} \right)$$

$$a_3 \left(3^4 \pi^4 \frac{EI}{2\ell^3} + \frac{k\ell}{2} \right) - \frac{4k\ell}{\pi^2} \left(\frac{a_1}{3} + \frac{a_3}{3} + \frac{a_5}{5} + \dots \right) = P \left(\sin \frac{3\pi c}{\ell} - \frac{2}{3\pi} \right)$$
(3.17)

From this set of simultaneous equations, coefficients a_1, a_3, a_5 etc. can be obtained. Thus, the displacement field of the beam given by Eqn. (3.12) is uniquely determined by the 'a' terms obtained above.

The solutions for the case of uniformly distributed loads and concentrated moments can be obtained in a similar manner starting with the same displacement function. The constant y_0 is eliminated by applying the result (Eqn. (3.10)). For concentrated moments (Fig. 3.5b), the equilibrium of vertical forces requires that

$$k \int_0^{\ell} y \, dx = 0 \quad (3.18)$$

The corresponding relation for distributed loads (Fig. 3.5c) is

$$q(d-c) - k \int_0^{\ell} y \, dx = 0 \quad (3.19)$$

The resulting expression for displacement function (y) is used to calculate the total energy of each system. The potential energy due to the external loads (U_L) is calculated from the relations:

Concentrated moments (Fig. 3.5b)

$$U_L = - M_o \left[\frac{dy}{dx} \right]_{x=c} \quad (3.20)$$

Distributed loads (Fig. 3.5c)

$$U_L = -q \int_c^d y \, dx \quad (3.21)$$

Total energy of each system is then minimized in the manner described earlier. Final expressions for displacement function (y) and set of simultaneous equations thus derived for both the systems are given below:

Concentrated moments (Fig. 3.5b)

$$y = \sum_{n=1}^{\infty} a_n \left(\sin \frac{n\pi x}{l} - \frac{2}{n\pi} \right) \quad (3.22a-b)$$

$$a_m \left(EI \frac{\pi^4 m^4}{2l^3} + \frac{kl}{2} \right) - \frac{4kl}{m\pi^2} \sum_{n=1,3}^{\infty} \frac{a_n}{n} = \frac{M_o}{l} \pi m \cos \frac{m\pi c}{l}$$

Distributed loads (Fig. 3.5c)

$$y = \frac{q(d-c)}{kl} + \sum_{n=1,3}^{\infty} a_n \left(\sin \frac{n\pi x}{l} - \frac{2}{n\pi} \right) \quad (3.23a-b)$$

$$a_m \left(EI \frac{\pi^4 m^4}{2l^3} + \frac{kl}{2} \right) - \frac{4kl}{m\pi^2} \sum_{n=1,3}^{\infty} \frac{a_n}{n} = ql \left[\cos \frac{m\pi c}{l} - \cos \frac{m\pi d}{l} - 2 \left(\frac{d-c}{l} \right) \right]$$

3.3.1.2 Antisymmetric loads

For antisymmetrical loadings, the beam will have deflection curves as shown in Fig. 3.6. Thus, the general equation for a deflection

curve for this type of loading can be expressed in the form

$$y = S\left(\frac{l}{2} - x\right) + \sum_{n=2,4}^{\infty} a_n \sin \frac{n\pi x}{l} \quad (3.24)$$

where S is the slope of the tangent to the deflection line at the center of the beam (i.e., $(\frac{dy}{dx})_{x=l/2}$).

The slope S can be determined by considering equilibrium of the beam under the application of external and internal moments. Applying this condition for a beam with two equally spaced concentrated loads of opposite sign (Fig. 3.6a), one can write

$$P(l-2c) + k \int_0^l yx dx = 0 \quad (3.25)$$

By substituting Eqn. (3.24) in (3.25) and performing the integration, it is found that

$$S = 12P \frac{(l-2c)}{kl^3} - \frac{12}{l\pi} \sum_{n=2,4}^{\infty} \frac{a_n}{n} \quad (3.26)$$

By substituting this in Eqn. (3.24), the deflection curve can be expressed in the form

$$y = \left[12P \frac{(l-2c)}{kl^3} - \frac{12}{l\pi} \sum_{n=2,4}^{\infty} \frac{a_n}{n} \right] \left[\frac{l}{2} - x \right] + \sum_{n=2,4}^{\infty} a_n \sin \frac{n\pi x}{l} \quad (3.27)$$

The energy components U_B , U_F and U_L can be calculated in the manner described for symmetric concentrated loads. It is found that:

$$U_B = \sum_n a_n^2 \pi^4 \frac{EI}{4\ell^3}$$

$$U_F = \sum_n a_n^2 \frac{k\ell}{4} - \frac{6k\ell}{\pi^2} \sum_n \frac{a_n}{n} \sum_m \frac{a_m}{m}$$

$$U_L = -P \left[12P \frac{(\ell-2c)^3}{k\ell^3} - \frac{12}{\ell\pi} \sum_n \frac{a_n}{n} \right] \left[\frac{\ell}{2} - c \right] - P \sum_n a_n \sin \frac{n\pi c}{\ell}$$

By minimization of the total energy functional, a system of linear equations similar to Eqn. (3.16) can be obtained, i.e.

$$a_m \left(EI \frac{\pi^4 m^4}{2\ell^3} + \frac{k\ell}{2} \right) - \frac{12k\ell}{m\pi^2} \sum_n \frac{a_n}{n} = P \left[\sin \frac{m\pi c}{\ell} - \frac{6}{m\pi} (\ell-2c) \right] \quad (3.28)$$

where $m(2,4,6,\dots,\infty)$ and $n(2,4,6,\dots,\infty)$.

Solutions for concentrated moments and distributed loads can be obtained by adopting the procedures outlined earlier. Finally, one would obtain the following sets of equations for y and a_m :

Concentrated moments (Fig. 3.6b)

$$y = \left(\frac{12M_o}{k\ell^3} - \frac{12}{\ell\pi} \sum_n \frac{a_n}{n} \right) \left(\frac{\ell}{2} - x \right) + \sum_n a_n \sin \frac{n\pi x}{\ell} \quad (3.29a-b)$$

$$a_m \left(EI \frac{\pi^4 m^4}{2\ell^3} + \frac{k\ell}{2} \right) - \frac{12k\ell}{m\pi^2} \sum_n \frac{a_n}{n} = \frac{M_o}{\ell} \left(\frac{12}{m\pi} + m\pi \cos \frac{m\pi c}{\ell} \right)$$

Distributed loads (Fig. 3.6c)

$$y = \left[\frac{6q(d-c)(\ell-c-d)}{k\ell^3} - \frac{12}{\ell\pi} \sum_n \frac{a_n}{n} \right] \left(\frac{\ell}{2} - x \right) + \sum_n a_n \sin \frac{n\pi x}{\ell} \quad (3.30a-b)$$

$$a_m \left(EI \frac{\pi^4 m^4}{2\ell^3} + \frac{k\ell}{2} \right) - \frac{12k\ell}{m\pi^2} \sum_n \frac{a_n}{n} = \frac{q}{\ell m \pi} [6(d^2 - c^2 + c\ell - d\ell) + \ell^2 (\cos \frac{m\pi c}{\ell} - \cos \frac{m\pi d}{\ell})]$$

3.3.2 Solution by Polynomial Functions

Polynomial functions are used to define the deflected shape of a beam resting on a Winkler type of foundation. To apply the series, the centre of the beam is taken as the origin of the reference coordinate system (X and Y). The axial distance is measured to the right as positive. To simplify the algebra, the length of the beam is taken to be equal to 2ℓ .

3.3.2.1 Symmetric loads

For the loads shown in Fig. 3.7, the assumed displacement function is given by

$$y = \ell \left[C_0 + \sum_{n=1,2}^{\infty} C_{2n} \left(\frac{x}{\ell} \right)^{2n} \right] \quad (3.31)$$

This equation is similar in character to the displacement function given by Eqn. (3.9) which was used for the analysis of symmetric loads by the Fourier series. Hence, the same steps as outlined in Section 3.3.1.1 could be applied here. The details of the calculations are omitted and only the final form of results are presented here.

The displacement function (y) is given by

$$y = \frac{P}{2k\ell} + \frac{q(d-c)}{2k} + \ell \sum_n C_{2n} \left[\left(\frac{x}{\ell} \right)^{2n} - \left(\frac{1}{2n+1} \right) \right] \quad (3.32)$$

where constants (C's) are obtained from the set of simultaneous equations

$$\begin{aligned} & \frac{4EI}{\ell} m(2m-1) \sum_{m,n} \frac{n(2n-1)}{2m+2n-3} C_{2n} + 2k\ell^3 \sum_{m,n} C_{2n} \left\{ \frac{1}{2m+2n+1} - \frac{1}{(2m+1)(2n+1)} \right\} \\ & = P\ell \left[a^{2m} - \frac{1}{2m+1} \right] + 2M_0 m b^{2m-1} + \frac{q\ell^2}{2m+1} [d^{2m+1} - c^{2m+1} + c \cdot d] \end{aligned} \quad (3.33)$$

where

$$\begin{aligned} m & (1, 2, \dots, \infty) \\ n & (1, 2, \dots, \infty) \end{aligned} \quad (3.34)$$

3.3.2.2 Antisymmetric loads

For the loads shown in Fig. 3.8, the assumed displacement function (y) is given by

$$y = \ell \sum_{n=1,2}^{\infty} C_{2n-1} \left(\frac{x}{\ell} \right)^{2n-1} \quad (3.35)$$

where C's are arbitrary constants.

Eqn. (3.35) is directly used to calculate energy components U_B , U_F and U_L . Following the procedure described in Section 3.3.1.1, a set of simultaneous equations could be derived. The constants (C's) are to be obtained from the following:

$$\begin{aligned} & \frac{8EI}{\ell} (2m-1)(m-1) \sum_{m,n} \frac{(2n-1)(n-1)}{2m+2n-5} C_{2n-1} + 2k\ell^3 \sum_{m,n} \frac{C_{2n-1}}{2m+2n-1} \\ & = P\ell a^{2m-1} + M_0 (2m-1) b^{2m-2} + q\ell^2 \left(\frac{d^{2m} - c^{2m}}{2m} \right) \end{aligned} \quad (3.36a)$$

where m and n are as given by Eqn. (3.34).

3.3.3 Solutions Based on Boundary Conditions

The energy methods of analysis which have been discussed so far are developed without any consideration to static boundary conditions existing at the free ends of a beam. All the Fourier series solutions automatically satisfy the conditions for flexural moments at the ends of a beam but not for the shear force. On the other hand, none of the polynomial solutions satisfies any of the traction boundary conditions. But, for improved accuracy, a displacement field associated with a particular problem should satisfy all the boundary conditions related to that problem. Considering these factors, two sets of solutions have been developed where displacement functions are chosen to satisfy the static boundary conditions at the edges of a beam. The sequence of steps needed in any such analysis is necessarily the same one as outlined previously. In fact, the displacement functions given by Eqs. (3.12), (3.22a), (3.23a), (3.27), (3.29a), (3.30a), (3.32) and (3.35) are utilized for this purpose. Any displacement function (y) given by a Fourier series solution is to be modified from the relation $EI \left[\frac{d^3 y}{dx^3} \right]_{x=0}^{x=l} = 0$.

Using this condition, one constant coefficient could be eliminated from the displacement function. But, from a polynomial function (y), two constants could be eliminated using the conditions:

$$EI \left[\frac{d^2 y}{dx^2} \right]_{x=\pm l} = 0$$

$$EI \left[\frac{d^3 y}{dx^3} \right]_{x=\pm l} = 0$$

However, since a considerable complexity is involved in the process of elimination of two constants, polynomial solutions are developed from only seven terms in the series. Also, the simultaneous equations derived from such solutions can be expressed conveniently in matrix forms. Complete sets of results for the analysis using both the Fourier series and polynomial displacement functions are given in Appendix A.

3.4 FINITE ELEMENT ANALYSIS

The exact displacement function for a laterally loaded beam is given by

$$y = a_1 + a_2x + a_3x^2 + a_4x^3 \quad (3.37)$$

where a_1 , a_2 , a_3 and a_4 are arbitrary constants. This displacement function is used to develop the finite element analysis of a beam resting on the Winkler type of foundation.

3.4.1 Element Stiffness Matrix

A beam and a foundation element is shown in Fig. 3.9 with nodal degrees of freedom as indicated. Explicit form of a beam element stiffness matrix (K_B) is given by Desai and Abel (1972) and Just et al (1971). Therefore, only stiffness matrix (K_F) due to foundation will be derived here.

From the displacement function (y), the nodal displacements can be written in the matrix form

$$\begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & \ell & \ell^2 & \ell^3 \\ 0 & 1 & 2\ell & 3\ell^2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix} \quad (3.38)$$

The energy due to deformation of a foundation element (U_F) is given by

$$U_F = \frac{k}{2} \int_0^\ell y^2 dx \quad (3.39a)$$

The above equation can also be written in the form

$$U_F = \frac{k}{2} \int_0^\ell \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix}^T \begin{Bmatrix} 1 \\ x \\ x^2 \\ x^3 \end{Bmatrix} [1 \quad x \quad x^2 \quad x^3] \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix} dx \quad (3.39b)$$

Substituting Eqns. (3.38) and (3.39b) into Eqn. (2.4b) and carrying out the necessary mathematical operations, the foundation element stiffness matrix K_F is obtained. Combined stiffness for a beam and foundation element is given by Eqn. (3.40) below. It may be noted that the EI and k terms denote the contributions from the beam and foundation respectively.

$$[K_E] = [K_B] + [K_F] = \begin{bmatrix} \frac{12EI}{\ell^3} + \frac{13k\ell}{35} & \frac{6EI}{\ell^2} + \frac{11k\ell^2}{210} & -\frac{12EI}{\ell^3} + \frac{9k\ell}{70} & \frac{6EI}{\ell^2} - \frac{13k\ell^2}{420} \\ & \frac{4EI}{\ell} + \frac{k\ell^3}{105} & -\frac{6EI}{\ell^2} + \frac{13k\ell^2}{420} & \frac{2EI}{\ell} - \frac{k\ell^3}{140} \\ \text{Symmetric} & & \frac{12EI}{\ell^3} + \frac{13k\ell}{35} & -\frac{6EI}{\ell^2} - \frac{11k\ell^2}{210} \\ & & & \frac{4EI}{\ell} + \frac{k\ell^3}{105} \end{bmatrix} \quad (3.40)$$

3.4.2 Nodal Load Vector

A concentrated load or couple is accounted for by introducing a node at the point of its application. The consistent load vector for a distributed load can be obtained from Eqn. (2.4c). For the case shown in Fig. 3.9, the load vector (Q) could be derived as

$$\{Q\} = \left[\frac{q\ell}{2} \quad \frac{q\ell^2}{12} \quad \frac{q\ell}{2} - \frac{q\ell^2}{12} \right]^T \quad (3.41)$$

3.5 COMPUTER PROGRAMS

A variety of methods have so far been presented for the analysis of a beam on Winkler foundation. The results of all such analyses have been computerized in standard Fortran IV.

A conversational program has been developed following the exact analysis. The program reads the beam and foundation properties as well as the magnitudes and locations of loads. The user may input the data either in Imperial or SI units but should be consistent. Option is provided to supply the properties of the beam in various ways. Typical values of modulus of foundation for different types of soil (Miner and Seastone, 1955) have been incorporated in the program, so that the user may utilize them if such informations are not readily available from the results of soil tests.

The program can handle the analysis for concentrated loads and moments as well as for uniformly distributed loads. Each load is treated separately and displacements, bending moments and shear forces are

calculated at assigned locations. Final results for various loadings are obtained as the superposition of all the separate results from individual analysis. The listings of the program, together with an example, are presented in Appendix B.

3.6 COMPARISON OF RESULTS

In this section, the results obtained from the energy and finite element methods of analysis are compared. The effect of relative rigidity (λl) on the beam-deflections, flexural moments and shear forces are also examined.

For the purpose of comparison of results obtained from various methods of analysis, a beam with the loading as shown in Fig. 3.10 is chosen. The particular problem was examined by Just et al (1971) who used the finite element technique. However, in the analysis shown, no mention is made as to how many elements are used. For the present purpose, 20 elements are used in the finite element analysis whereas the energy solutions are obtained by using 10 terms in the displacement functions.

The error in beam displacements (Table 3.1) indicates that the magnitude of error is very small in all cases. Since the computed values are almost the same, the results from various analyses appear to be on the same curve (Fig. 3.11a). The distribution of flexural moments is shown in Fig. 3.11b. It is seen that the results obtained from both the energy and finite element method of analysis underestimate the values given by Just et al (1971) and those results given by the exact method.

The difference is particularly noticeable at the points below the loads; the maximum being of the order of 4% for the finite element analysis. The comparison for shear force is shown in Fig. 3.11c. Unlike flexural moments, computed values for shearing forces are overestimated by 8% at the centre of the beam.

In Section 3.4 (Chapter 3), the energy solutions are derived in two ways where the displacement functions are either modified to suit the static boundary conditions at the free ends of the beam or these functions are used without any modification. To study the effect of the use of modified displacement functions (in both cases of Fourier series and polynomial) on the accuracy of the results, a problem as shown in Fig. 3.12 is solved. In either case, both the modified and unmodified displacement functions are used taking 10 terms in the displacement functions. The results are compared graphically in Figs. 3.13(a-f). Corresponding results from the exact analysis are also presented in each figure. It is observed that the solution obtained by using a modified displacement function yields slightly better results than that of the corresponding unmodified function. But the difference is not so significant in terms of extra computation involved in the analysis. An excellent agreement in beam deflections in either the case of polynomial or Fourier series solution is observed. But the results for flexural moments and shear forces show slight variations. The error for polynomial solution using unmodified displacement function is comparatively

more, the maximum being of the order of 1.1% for moment (Fig. 3.13e) and 2.6% for shear (Fig. 3.13f).

In any finite element analysis, the results are generally influenced by the number of elements used in the discretization. Likewise, the accuracy of an energy solution is influenced by the number of terms included in the displacement function. The beam with the loading as shown in Fig. 3.12 is used to investigate the accuracy of various approximations. By increasing the number of terms in the displacement functions for energy solutions or number of elements in the finite element analysis, displacement, flexural moment and shear force at the centre of the beam are noted in every case. The results are compared with the corresponding values obtained from exact analysis (Table 3.2).

It is observed that with the use of increased number of terms (or elements) the results are improved. In the case of Fourier series solution, the computed value of displacement shows an error of 0.009% when only 4 terms are used, but the corresponding results for flexural moment and shear force show an error of -0.61 and -3.25 respectively. With the use of 10 terms, a better agreement is obtained. The results for polynomial solution indicate that the series is also convergent with few terms but not at the same rate as the Fourier series. The results obtained from the finite element analysis indicate that while convergence in displacements is obtainable with the use of only 10 elements, accurate results for flexural moments and shear forces would require the use of large number of elements, say 25.

The results so far reported are based on the relative rigidity parameter λl in the range of 4 to 6, which may correspond to combined footing type foundations encountered in practice. For rigid beams with $\lambda l < 3$, better agreement in results is obtained from all the various types of analysis which are mentioned earlier. For higher values of λl , some inaccuracy is developed. To study this factor, a beam with a concentrated load at centre is analyzed for a wide selection of the parameter λl , ranging from 1 to 50. The computed values for displacements and moments at $x=.5l$ and shear forces at $x=.4l$ are presented in a non-dimensional form in Table 3.3.

From the results for displacements, it is observed that all methods yield approximately the same result so long as $\lambda l < 10$, the error increases as λl is increased. The results obtained from energy method using polynomial functions indicate a poor correlation. For $\lambda l = 50$, the central deflection computed by this method shows an error of -48.5%. The result from Fourier series solution is slightly better where an error of about -28% is noted. The corresponding error from finite element analysis is found to be about -10%. It is also observed that results obtained from the finite element analysis are consistent over the range of λl as studied; whereas the energy solutions begin to show inaccuracy when λl exceeds 20. From the results for bending moments and shear forces, it is observed that errors are severe in these cases. Fourier series solution yields acceptable results up to λl equal to 10, but beyond this,

the results do not bear any correlation. In that case, even the finite element method yields poor results.

To illustrate the effect of relative rigidity on beam-displacements, a set of results corresponding to exact analysis is presented in Fig. 3.14. It may be observed that when $\lambda\ell \gg 4$, the displacement at or near the edges of the beam becomes negative. This is justified by the assumed idealized connection between the foundation and structure.

Some problems related to distributed loads have also been examined. If a beam is subjected to a distributed load of uniform intensity along its entire length, the displacement of that beam at every point will be constant. Therefore, the flexural moments and shear forces at every location will be identically equal to zero. Such behaviour will occur regardless of the rigidity of the beam. To study the effect of distributed loads spread over a portion of the beam, the problem as shown in Fig. 3.15a, is examined for a wide range of relative rigidities. In Fig. 3.15b, the displacement profile of the beam obtained from a Fourier series solution has been compared with the corresponding exact one. A little variation is observed between the two sets of curves. It may be of interest to note the variation in the beam displacements as $\lambda\ell$ increases. For $\lambda\ell=10$, the displacements of the beam remain almost limited to the loaded portion. With further increase in $\lambda\ell$, this effect becomes pronounced and at $\lambda\ell=50$, the distribution of displacements below the load approaches near uniform. This is consistent with the Winkler's hypothesis. The condition

that $\lambda l \rightarrow \infty$ indicates a beam of such flexibility that the load could be considered to have been placed directly on the foundation medium. In that case, uniform displacements should occur below the load. Since $\lambda l = 50$ approximates such a situation, the displacement profile slightly different from a uniform one is justified.

The bending moment and shear force in the beam diminish with the increase in λl . Since the displacements approach uniformity at high values of λl , almost no bending moment and shear force should develop. This fact is substantiated in Figs. 3.15c-d.

The effect of the location of the load (ξ) on the beam deflections, bending moments and shear forces have also been studied. In this case, a Fourier series solution consisting of 10 terms has been used. The value of λl is kept constant. For a concentrated load, the variation of displacements is shown in Fig. 3.16a, while corresponding results for flexural moments and shear forces are shown in Figs. 3.16b and 3.16c respectively.

Using the Fourier series solution, a set of curves has also been developed to examine the effect of a concentrated moment on the beam deflections, bending moments and shear forces. In Fig. 3.17, the effect of the location of moment is shown.

Fig. 3.1.

A Beam on Winkler Medium

Fig. 3.2

A Small Element of the Beam

Fig. 3.3

Typical Loads on a Beam

Fig. 3.4

Symmetrical and Antisymmetrical
Load Components

Fig. 3.5

Symmetric Loads and Deflected Shapes

(a)

(b)

(c)

Fig. 3.6

**Fourier Series solution
For Antisymmetrical Loads**

Table 3.2 Convergence of results: Error expressed in percentage with respect to exact values. (Based on Fig. 3.12)

NO. OF TERMS	ENERGY METHODS		NO. OF ELEMENTS	FINITE ELEMENT ANALYSIS
	FOURIER SERIES	POLYNOMIAL		

Displacement

2	-.09	-2.498	10	.002
4	-.009	-0.126	15	0
6	.003	-0.020	20	0
8	.003	0.008	25	0
10	0	0	30	0

Moment

2	-2.588	-20.538	10	3.008
4	-0.610	- 3.245	15	0.746
6	0.456	- 0.599	20	0.738
8	0.164	0.827	25	0.188
10	-0.173	0.329	30	0.091

Shear

2	38.505	-76.745	10	4.963
4	4.290	19.338	15	1.213
6	- 1.421	16.445	20	1.212
8	- 2.533	5.459	25	0.316
10	- 0.365	0.402	30	0.012

Table 3.1. Comparison of Displacements. (Ex. 1, given by Just et al, 1971).

DIST x/l	EXACT VALUE (Metrex 10^{-3})	ERROR IN P.C. WITH RESPECT TO CORRESPONDING EXACT VALUE GIVEN AT THE 2nd COLUMN				
		FINITE ELEMENT (20 elements)	ENERGY-POLYNOMIAL SOLUTION		ENERGY-FOURIER SERIES	
			10 terms	10 terms*	10 terms*	10 terms
0.0	4.10539	-0.58	-1.08	-0.55	-0.54	-0.56
0.1	6.84542	-0.13	-0.51	-0.14	-0.13	-0.15
0.2	9.14745	0.00	0.49	-0.03	-0.01	0.02
0.3	11.15550	0.11	-0.61	0.05	0.10	0.10
0.4	11.76190	0.07	0.81	0.06	0.08	0.08
0.5	11.78610	0.08	-0.83	0.03	0.07	0.06
0.6	10.56230	0.01	0.82	-0.02	0.02	0.02
0.7	9.09365	0.01	-0.81	-0.02	-0.01	0.003
0.8	6.95958	-0.07	0.55	-0.10	-0.06	-0.07
0.9	4.87365	-0.10	-0.63	-0.13	-0.10	-0.12
1.0	2.55399	-0.34	-1.16	-0.31	-0.30	-0.36

* Displacement functions satisfy the boundary conditions

$$k = 42.2 \text{ MN/m}^2$$

$$EI = 1675 \text{ MNm}^2$$

Fig. 3.10

A Problem Solved by Just Et. Al(1971)

Fig. 311b

Comparison of Bending Moment

Comparison of Shear Force

Fig. 3.11c

$$E = 1.17 \times 10^5 \text{ psi}$$

$$\lambda l = 4.0$$

$$k = 100 \text{ lb/in}^3$$

Fig. 3.12

A Beam on Winkler Medium

Fig. 3.13a

Effect of boundary conditions
on beam-displacements.

Fig. 3.13b Effect of Static Boundary Condition
On Bending Moment

Fig. 3.13c
Effect of boundary conditions
on Shear force (Fourier series
solution using 10 terms)

Fig. 3.13d
Effect of Boundary Conditions on
Displacement(Polynomial Solution
Using 10 Terms).

Fig. 3.13e
Effect of Boundary Conditions on
Bending Moment (Polynomial Solution
Using 10 terms)

Fig. 3.13f
Effect of Boundary Conditions on
Shear Force (Using 10 Terms in the
Polynomial Function)

Table 3.2 Convergence of results: Error expressed in percentage with respect to exact values. (Based on Fig. 3.12)

NO. OF TERMS	ENERGY METHODS		NO. OF ELEMENTS	FINITE ELEMENT ANALYSIS
	FOURIER SERIES	POLYNOMIAL		

Displacement

2	-.09	-2.498	10	.002
4	-.009	-0.126	15	0
6	.003	-0.020	20	0
8	.003	0.008	25	0
10	0	0	30	0

Moment

2	-2.588	-20.538	10	3.008
4	-0.610	- 3.245	15	0.746
6	0.456	- 0.599	20	0.738
8	0.164	0.827	25	0.188
10	-0.173	0.329	30	0.091

Shear

2	38.505	-76.745	10	4.963
4	4.290	19.338	15	1.213
6	- 1.421	16.445	20	1.212
8	- 2.533	5.459	25	0.316
10	- 0.365	0.402	30	0.012

Table 3.3. Effect of Relative Rigidity (λl) .

λl	EXACT METHOD	FINITE ELEMENT	ENERGY METHOD	
			POLYNOMIAL	FOURIER SERIES

Displacements at Centre/(P/k l)

1	.10124 E+01	.10124 E+01	.10124 E+01	.10124 E+01
4	.21599 E+01	.21599 E+01	.21581 E+01	.21395 E+01
10	.50008 E+01	.49995 E+01	.49298 E+01	.49838 E+01
20	.10000 E+02	.99594 E+01	.89970 E+01	.97322 E+01
50	.25000 E+02	.22501 E+02	.12829 E+02	.18140 E+02

Moment at Centre/(P l)

1	.12143 E+00	.12120 E+00	.11614 E+00	.11926 E+00
4	.65866 E-01	.64147 E-01	.57697 E-01	.60799 E-01
10	.25003 E-01	.24109 E-01	.16868 E-01	.19944 E-01
20	.12500 E-01	.10703 E-01	.47885 E-02	.74785 E-02
50	.50000 E-02	.31142 E-02	.28173 E-03	.85088 E-03

Shear (at $x=.4l$)/P

1	.39883 E+00	.39879 E+00	.46558 E+00	.34942 E+00
4	.29330 E+00	.28771 E+00	.36007 E+00	.24386 E+00
10	.99329 E-01	.63924 E-01	.16501 E+00	.49984 E-01
20	-.28159 E-01	-.14746 E-01	.24558 E-01	-.76269 E-01
50	.95565 E-03	.43227 E-03	-.11617 E-02	.21260 E-01

Fig. 3.14

Effect of Relative Rigidity on
Beam-displacements (Exact Analysis)

Fig. 3.15a

Beam with Distributed Load

Fig. 3.15b

Effect of Relative Rigidity λl
on Beam-displacements

(c)

Fig. 3.15(c-d)
Effect of Relative Rigidity on
(c) Moment ; (d) Shear

(d)

Fig. 3.16a
 Variation of Displacement for Different
 Position of Load (Based on Relative
 Rigidity=4).

Fig. 3.16b
Variation of Moment for
Different Position of Load
(Based on Relative Rigidity
Equal to 4)

Fig. 3.16c

Variation of Shear for Different
Position of Load (Based on Relative
Rigidity Equal to 4)

Fig. 3.17a
Variation of Displacement for
Different Load Position (Based
on Relative Rigidity Equal to 4)

Fig. 3.17b
 Variation of Bending Moment
 Due to Position of Applied
 Moment (Based on Relative Rigidity
 Equal to 4)

$$\xi = c/l$$

Fig. 3.17c

Variation of Shear for Different
Position of Load (Based on Relative
Rigidity Equal to 4)

CHAPTER 4

ANALYSES OF CIRCULAR PLATES

4.1 GENERAL

Problems related to circular plates on Winkler type of foundation have been investigated by many researchers (Hertz, 1884; Schleicher, 1926; Hetenyi, 1946; Smith, 1967). A comprehensive account of such investigations is given by Hetenyi (1966) and Selvadurai (1979a). In this work, consideration is given only to axisymmetrically loaded circular plates. It is assumed that the flexural behaviour of the plate can be represented by the classical Poisson-Kirchhoff thin plate theory. The fundamental assumptions of this theory can be outlined as follows:

- 1) The thickness of the plate is sufficiently small as compared to its lateral dimension.
- 2) The deflection of the plate is small in comparison to its thickness.

When considering the small deflection Poisson-Kirchhoff thin plate theory, the elastic strain energy of a circular plate subjected to the axisymmetric deflection $w(r)$ is composed of only the flexural energy of the plate U_p (Timoshenko and Woinowsky-Krieger, 1959), which is given by

$$U_p = \int \int_S \frac{D}{2} \left[\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right)^2 - \frac{2(1-\nu)}{r} \frac{\partial w}{\partial r} \frac{\partial^2 w}{\partial r^2} \right] r dr d\theta \quad (4.1)$$

where D is the flexural rigidity of the plate and S corresponds to the region of the plate.

Assuming that there is no separation at the interface between the plate and the foundation, the elastic strain energy U_F of the foundation subjected to the same displacement field $w(r)$ in the region S is given by

$$U_F = \frac{k}{2} \iint_S w^2(r) r dr d\theta \quad (4.2)$$

where k is the modulus of subgrade reaction.

The potential energy U_L due to an externally applied uniformly distributed load of intensity q in the region of the plate S_1 is given by

$$U_L = -q \iint_{S_1} w(r) r dr d\theta \quad (4.3)$$

4.2 EXACT ANALYSIS

The exact analysis of an axisymmetrically loaded circular plate resting on a Winkler medium involves the solution of a fourth order linear differential equation. This is based on the assumptions of classical thin plate theory outlined previously. For a circular plate of any radius a , thickness t , Poisson's ratio ν and modulus of elasticity E , which is acted upon by a concentrated load of magnitude P at the centre of the plate and distributed load of uniform intensity q units per unit area as shown in Fig. 4.1, the differential equation governing the plate deflection $w(r)$ is given by (Timoshenko and Woinowsky-Krieger, 1959)

$$\nabla^2 \nabla^2 w(r) = \frac{q - kw}{D}$$

or

(4.4)

$$\frac{d^4 w}{dr^4} + \frac{2}{r} \frac{d^3 w}{dr^3} - \frac{1}{r^2} \frac{d^2 w}{dr^2} + \frac{1}{r^3} \frac{dw}{dr} = \frac{q - kw}{D}$$

where

$$\nabla^2 = \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr}$$

D is the flexural rigidity of the plate given by

$$D = \frac{Et^3}{12(1-\nu^2)} \quad (4.5)$$

k is the modulus of subgrade reaction

w is the displacement of the plate with the downward direction as positive.

This is a linear differential equation of fourth order. Along the unloaded parts of the plate where $q=0$, Eqn. (4.4) will take the form

$$\frac{d^4 w}{dr^4} + \frac{2}{r} \frac{d^3 w}{dr^3} - \frac{1}{r^2} \frac{d^2 w}{dr^2} + \frac{1}{r^3} \frac{dw}{dr} = \frac{-kw}{D} \quad (4.5)$$

The general solution of Eqn. (4.5) was obtained by Schleicher (1926) in terms of certain special functions which are termed as Z functions. In a simplified way, his solution can be expressed as:

$$w(r) = C_1 Z_1(2x) + C_2 Z_2(2x) + C_3 Z_3(2x) + C_4 Z_4(2x) \quad (4.6)$$

where C_1, C_2, C_3 and C_4 are arbitrary constants to be evaluated from the boundary conditions of the problem. The argument $2x$ is related to

the radial distance r of the point where displacement w is to be computed and the plate and foundation properties by $2x=r(k/D)^{1/2}$. The Z functions can be represented in the form of a power series as:

$$\begin{aligned} Z_1(2x) &= 1 - \frac{x^4}{2!^2} + \frac{x^8}{4!^2} - \frac{x^{12}}{6!^2} + \dots \\ Z_2(2x) &= \frac{-x^2}{1!^2} + \frac{x^6}{3!^2} - \frac{x^{10}}{5!^2} \\ Z_3(2x) &= \frac{Z_1(2x)}{2x} - \frac{2}{\pi} [R_1 + \log_e \gamma x Z_2(2x)] \\ Z_4(2x) &= \frac{Z_2(2x)}{2} + \frac{2}{\pi} [R_2 + \log_e \gamma x Z_3(2x)] \end{aligned} \quad (4.7)$$

where

$$\begin{aligned} R_1 &= x^2 - \frac{\phi(3)}{3!^2} + \frac{\phi(5)}{5!^2} - \frac{\phi(7)}{7!^2} + \dots \\ R_2 &= \frac{\phi(2)}{2!^2} - x^4 - \frac{\phi(4)}{4!^2} x^8 + \frac{\phi(6)x^{12}}{6!^2} - \end{aligned}$$

$$\phi(n) = \sum_{i=1}^n \left\{ \frac{1}{i} \right\}$$

and $\log_e \gamma = .577216$

Schleicher (1926) tabulated these Z functions and gave various relations among their derivatives. Hertz (1884) obtained a solution for the case of an infinitely extended plate being subjected to a single concentrated load P acting at the origin. Some problems of practical interest were also solved by Hetenyi (1946).

It should be mentioned that the solution of Eqn. (4.4) for the case of uniformly distributed load occupying a finite region of the plate is quite difficult. However, both the energy method and finite element techniques presented in Chapter 2 can be conveniently used for this purpose. Any load placed symmetrically with respect to the axis of symmetry could be examined by these methods without expending a large amount of computational effort.

4.3 ENERGY METHOD

Using the energy method, separate solutions have been developed for each of the following three types of circular plate foundations:

- i) Circular plates without any opening
- ii) Pedestal foundations
- iii) Annular foundations.

4.3.1 Circular Plate Without Opening

An even order polynomial consisting of six terms is used to represent approximately the deflected shape ($w(r)$) of the plate foundation system (Fig. 4.1). This could be written as

$$w(r) = a(C_0 + C_2 \frac{r^2}{a^2} + C_4 \frac{r^4}{a^4} + \dots + C_{10} \frac{r^{10}}{a^{10}}) \quad (4.8)$$

where C_0, C_2, C_4 etc. are arbitrary constants.

The above equation satisfies the kinematic constraint of the deformation of the plate at its centre where slope should be zero. Two

2 2

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2 2
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constants namely C_2 and C_4 can, however, be eliminated from Eqn. (4.8) by invoking the Poisson-Kirchhoff boundary conditions applicable to the free edge of the plate. These conditions require that

$$(M_r)_{r=a} = -D \left(\frac{d^2 w}{dr^2} + \frac{\nu}{r} \frac{dw}{dr} \right)_{r=a} = 0 \quad (4.9)$$

$$(Q_r)_{r=a} = -D \frac{d}{dr} \left(\frac{d^2 w}{dr^2} + \frac{1}{r} \frac{dw}{dr} \right)_{r=a} = 0$$

By making use of Eqn. (4.9), expression for $w(r)$ reduces to the form

$$w(r) = a \left[C_0 + C_6 \left(A_2 \frac{r^2}{a^2} - 4.5 \frac{r^4}{a^4} + \frac{r^6}{a^6} \right) + C_8 \left(B_2 \frac{r^2}{a^2} - \frac{12r^4}{a^4} + \frac{r^8}{a^8} \right) + C_{10} \left(D_2 \frac{r^2}{a^2} - \frac{25r^4}{a^4} + \frac{r^{10}}{a^{10}} \right) \right] \quad (4.10)$$

where

$$A_2 = \frac{12+6\nu}{1+\nu}$$

$$B_2 = \frac{44+20\nu}{1+\nu}$$

and

$$D_2 = \frac{105+45\nu}{1+\nu}$$

Eqn. (4.10) can, however, be written in much simpler form as

$$w(r) = a \left[C_0 + \sum_{m=1}^5 (C_6 A_{2m} + C_8 B_{2m} + C_{10} D_{2m}) \left(\frac{r}{a} \right)^{2m} \right] \quad (4.11)$$

where the terms A_{2m} , B_{2m} and D_{2m} are known for all values of m (i.e. by comparing Eqns. (4.11) and (4.10)).

Having obtained the displacement function (Eqn. (4.11)), components U_p , U_F and U_L of the total potential functional for the system are calculated from the relations (4.1), (4.2) and (4.3), i.e.

$$U_p = \frac{D}{2} \int_0^{2\pi} \int_0^a \left[\left(\frac{d^2 w}{dr^2} + \frac{1}{r} \frac{dw}{dr} \right)^2 - \frac{2(1-\nu)}{r} \frac{dw}{dr} \frac{d^2 w}{dr^2} \right] r dr d\theta \quad (4.12a)$$

$$U_F = \frac{k}{2} \int_0^{2\pi} \int_0^a w^2 r dr d\theta \quad (4.12b)$$

$$U_L = -q \int_0^{2\pi} \int_{d_1}^{e_1} w r dr d\theta \quad (4.12c)$$

The total energy functional thus obtained is minimized with respect to coefficients C_0 , C_6 , C_8 , and C_{10} . The details of calculations are omitted here; only the linear equations generated from the minimization of total energy functional are given in matrix form

$$[M]\{C\} = \{F\} \quad (4.12d)$$

where

$\{C\}$ is the vector of unknown constants C_i

$\{F\}$ is the vector derived from external forces

and $[M]$ is the matrix derived from strain energy of the system; given by

$$\begin{aligned}
 [M] = \pi * \sum_{m=1}^5 \sum_{n=1}^5 & \begin{bmatrix} 1 & \frac{A_{2m}}{m+1} & \frac{B_{2m}}{m+1} & \frac{D_{2m}}{m+1} \\ \text{Symmetrical} & \frac{A_{2m}A_{2n}}{m+n+1} & \frac{A_{2m}B_{2n}}{m+n+1} & \frac{A_{2m}D_{2n}}{m+n+1} \\ & \frac{B_{2m}B_{2n}}{m+n+1} & \frac{B_{2m}D_{2n}}{m+n+1} & \\ & & \frac{D_{2m}D_{2n}}{m+n+1} & \end{bmatrix} \\
 + \frac{8\pi}{R} \sum_{m=1}^5 \sum_{n=1}^5 & \begin{bmatrix} 0 & 0 & 0 & 0 \\ T_{mn}A_{2m}A_{2n} & T_{mn}A_{2m}B_{2n} & T_{mn}A_{2m}D_{2n} & \\ & T_{mn}B_{2m}B_{2n} & T_{mn}B_{2m}D_{2n} & \\ & & T_{mn}D_{2m}D_{2n} & \end{bmatrix} \quad (4.13a)
 \end{aligned}$$

where

$$R = a^4 k/D$$

and the value of T_{mn} for $m=i$ and $n=j$ is given by

$$T_{ij} = 2ij \left(\frac{2ij-i-j+2iv-v+1}{i+j-1} \right)$$

$$\{F\} = \frac{\pi q}{ka} \sum_{m=1}^5 \begin{Bmatrix} e^2 - d^2 \\ (e^{2m+2} - d^{2m+2}) A_{2m}/(m+1) \\ (e^{2m+2} - d^{2m+2}) B_{2m}/(m+1) \\ (e^{2m+2} - d^{2m+2}) D_{2m}/(m+1) \end{Bmatrix} + \frac{1}{ka^4} \begin{Bmatrix} Pa \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (4.13b)$$

where

$$e = e_1/a$$

$$d = d_1/a$$

4.3.2 Circular Plate With a Central Rigid Pedestal

The polynomial function approximating the deflected shape of the plate ($r=b$ to $r=a$) can be considered in two stages. For the outer region of the plate, the displacement function can be assumed in the form

$$w(r) = a(C_0 + C_2 \frac{r^2}{a^2} + C_4 \frac{r^4}{a^4} + \dots + C_{12} \frac{r^{12}}{a^{12}}) \quad (4.14a)$$

For the central region extending up to radius b which is assumed to be infinitely rigid, the displacement function becomes

$$w(r) = a(C_0 + C_2 \frac{b^2}{a^2} + C_4 \frac{b^4}{a^4} + \dots + C_{12} \frac{b^{12}}{a^{12}}) \quad (4.14b)$$

However, to ensure continuity between the inner and outer regions of the plate, the displacement functions must also satisfy that $(\frac{dw(r)}{dr})_{r=b} = 0$. Using this condition, one constant C_2 could be eliminated from both the equations. From the modified displacement functions so obtained, two other constants (C_4 and C_6) could be eliminated by employing the boundary conditions (Eqn. (4.9)) for the free edge of the plate. Finally, the displacement functions for the inner and outer regions of the plate can be reduced in the following forms:

For the inner region of the plate

$$w(r) = a(C_0 + C_8 A' + C_{10} B' + C_{12} D')$$

where A' , B' and D' are known constants.

And for the outer region

$$w(r) = a \left[C_0 + \sum_{m=1}^6 (C_8 A_{2m} + C_{10} B_{2m} + C_{12} D_{2m}) \frac{r^{2m}}{a^{2m}} \right] \quad (4.15)$$

where the values of A_{2m} , B_{2m} and D_{2m} ($m=1,2,\dots,6$) are functions of Poisson's ratio ν of the plate and the ratio b/a .

The components U_p , U_F and U_L are then calculated separately for the inner and outer plate regions using the appropriate displacement functions. Total potential energy functional obtained by summation of these quantities is minimized with respect to constants C_0 , C_8 , C_{10} and C_{12} . The set of linear equations thus derived is given in the form of Eqn. (4.12d):

$$\begin{aligned}
 [M] = & \pi \sum_{m=1}^6 \sum_{n=1}^6 \begin{bmatrix} 1 & A_{2m} T_m & B_{2m} T_m & D_{2m} T_m \\ A_{2m} A_{2n} T_{mn} & A_{2m} B_{2n} T_{mn} & A_{2m} D_{2n} T_{mn} \\ & B_{2m} B_{2n} T_{mn} & B_{2m} D_{2n} T_{mn} \\ \text{Symmetrical} & & D_{2m} D_{2n} T_{mn} \end{bmatrix} \\
 & + \frac{8\pi}{R} \sum_{m=1}^6 \sum_{n=1}^6 \begin{bmatrix} 0 & 0 & 0 & 0 \\ A_{2m} A_{2n} S_{mn} & A_{2m} B_{2n} S_{mn} & A_{2m} D_{2n} S_{mn} \\ & B_{2m} B_{2n} S_{mn} & B_{2m} D_{2n} S_{mn} \\ \text{Symmetrical} & & D_{2m} D_{2n} S_{mn} \end{bmatrix} \quad (4.16a) \\
 & + \frac{\pi b^2}{a^2} \begin{bmatrix} 0 & A' & B' & D' \\ & A'^2 & A'B' & A'D' \\ & & B'^2 & B'D' \\ \text{Symmetrical} & & & D'^2 \end{bmatrix}
 \end{aligned}$$

and the values of T_m , T_{mn} and S_{mn} when $m=i$ and $n=j$ is given by

$$T_i = \frac{1-h^{2i+2}}{i+1}$$

$$T_{ij} = \frac{1-h^{2i+2j+2}}{i+j+1}$$

$$S_{ij} = \frac{ij(2ij-i-j-v+2iv+1)(1-h^{2i+2j+2})}{i+j-1}$$

where $h = b/a$.

Also,

$$\{F\} = \frac{\pi q_0 b^2}{ka^3} \begin{Bmatrix} 1 \\ A' \\ B' \\ D' \end{Bmatrix} + \frac{P}{ka^3} \begin{Bmatrix} 1 \\ A' \\ B' \\ D' \end{Bmatrix} + \frac{\pi q}{ka} \sum_{m=1}^6 \begin{Bmatrix} e^2 - d^2 \\ (e^{2m+2} - d^{2m+2}) A_{2m}/(m+1) \\ (e^{2m+2} - d^{2m+2}) B_{2m}/(m+1) \\ (e^{2m+2} - d^{2m+2}) D_{2m}/(m+1) \end{Bmatrix} \quad (4.16b)$$

where

$$d = d_1/a$$

$$e = e_1/a$$

4.3.3 Annular Plate Foundations

This type of foundation is represented in Fig. 4.3. The radius of the circular plate is denoted by a , while the radius of central annular region is denoted by b . To approximate the deflected shape of the plate, the following displacement function is chosen:

$$w(r) = a(C_0 + C_2 \frac{r^2}{a^2} + C_4 \frac{r^4}{a^4} + \dots + C_{14} \frac{r^{14}}{a^{14}}) \quad (4.17)$$

where $C_0, C_2, \dots, C_{14}$ are constants. However, four of the constants namely C_2, C_4, C_6 and C_8 could be eliminated from this expression by invoking the Poisson-Kirchhoff boundary conditions applicable to the free inner and outer edges of the plate. The appropriate conditions are

$$(M_r)_{r=a} = (M_r)_{r=b} = (Q_r)_{r=a} = (Q_r)_{r=b} = 0 \quad (4.18)$$

The displacement function so modified can be written in the form

$$w(r) = a[C_0 + \sum_{m=1}^7 (C_{10} A_{2m} + C_{12} B_{2m} + C_{14} D_{2m}) \frac{r^{2m}}{a^{2m}}] \quad (4.19)$$

where all A, B and D's are functions of Poisson's ratio ν of the plate and the ratio b/a .

Having obtained the displacement function, the total potential energy functional of the system is calculated following the method outlined previously. The result so obtained is then minimized with respect to coefficients C_0, C_{10}, C_{12} , and C_{14} from which 4 linear equations are derived. These are given in the matrix form as follows:

$$\begin{aligned}
 [M] = & \pi \sum_{m=1}^7 \sum_{n=1}^7 \left[\begin{array}{cccc}
 1-h^2 & A_{2m} T_m & B_{2m} T_m & D_{2m} T_m \\
 & A_{2m} A_{2n} T_{mn} & A_{2m} B_{2n} T_{mn} & A_{2m} D_{2n} T_{mn} \\
 \text{Symmetrical} & & B_{2m} B_{2n} T_{mn} & B_{2m} D_{2n} T_{mn} \\
 & & & D_{2m} D_{2n} T_{mn}
 \end{array} \right] \\
 + \frac{8\pi}{R} \sum_{m=1}^7 \sum_{n=1}^7 & \left[\begin{array}{cccc}
 0 & 0 & 0 & 0 \\
 & A_{2m} A_{2n} S_{mn} & A_{2m} B_{2n} S_{mn} & A_{2m} D_{2n} S_{mn} \\
 \text{Symmetrical} & & B_{2m} B_{2n} S_{mn} & B_{2m} D_{2n} S_{mn} \\
 & & & D_{2m} D_{2n} S_{mn}
 \end{array} \right] \quad (4.20a)
 \end{aligned}$$

where

$$h = b/a$$

$$R = a^4 k/D$$

and the values of T_m , T_{mn} , S_{mn} when $m=i$ and $n=j$ are given by:

$$T_i = \frac{1-h^{2i+2}}{i+1}$$

$$T_{ij} = \frac{1-h^{2i+2j+2}}{i+j+1}$$

$$S_{ij} = \frac{ij(2ij-i-j+2iv+1)(1-h^{2i+2j+2})}{i+j-1}$$

Also, that

$$\{F\} = \frac{\pi q}{ka} \sum_{m=1}^5 \begin{Bmatrix} e^2 - d^2 \\ (e^{2m+2} - d^{2m+2}) A_{2m}/(m+1) \\ (e^{2m+2} - d^{2m+2}) B_{2m}/(m+1) \\ (e^{2m+2} - d^{2m+2}) D_{2m}/(m+1) \end{Bmatrix} \quad (4.20b)$$

where

$$e = e_1/a$$

$$d = d_1/a$$

4.4 FINITE ELEMENT ANALYSIS

For the finite element analysis, circular plates are modelled as an assemblage of annular ring elements. The axisymmetric nature of the problem reduces the computation essentially to that of one-dimensional line element. Accordingly, a cubic displacement model having two generalized components of displacements at each node is considered in the formulation of element stiffness matrices. The overall analysis follows the general description as outlined in Chapter 2. In this section, only the derivation of element stiffness matrices and load vectors will be briefly presented.

4.4.1 Plate Element Stiffness

The assumed displacement function for an annular ring element is given by

$$w(r) = a_1 + a_2 r + a_3 r^2 + a_4 r^3 \quad (4.21)$$

where a_1 , a_2 , a_3 and a_4 are generalized coordinates. Nodal displacements associated with an element are as shown in Fig. 4.4. These nodal displacements can be related to the generalized components of displacements as

$$\{\alpha\} = [A^{-1}]\{\delta\} \quad (4.22)$$

where

$\{\alpha\}$ is a column vector containing a_1 , a_2 , a_3 and a_4

$[A]$ is a 4x4 matrix, formed by the coordinates of nodes

$\{\delta\}$ is a column vector containing nodal displacements δ_1 , δ_2 , δ_3 and δ_4

Expression (4.1) for the elastic strain energy due to the bending of the plate can be written in the matrix form as

$$U_p = \frac{D}{2} \int_0^{2\pi} \int_{r_1}^{r_2} \{\alpha\}^T \begin{bmatrix} 1 \\ r \\ r^2 \\ r^3 \end{bmatrix} \begin{bmatrix} d^2/dr^2 \\ 1/r \cdot d/dr \end{bmatrix}^T \begin{bmatrix} 1 & v \\ v & 1 \end{bmatrix} \begin{bmatrix} d^2/dr^2 \\ 1/r \cdot d/dr \end{bmatrix} \begin{bmatrix} 1 & r & r^2 & r^3 \end{bmatrix} r dr d\theta \quad (4.23)$$

Using Eqn. (4.22), the generalized coordinate vector α can be eliminated from Eqn. (4.23) which is then integrated to obtain U_p .

Element stiffness matrix K_p is derived from the relation

$$(K_p)_{i,j} = \frac{\partial^2 U_p}{\partial \delta_i \partial \delta_j}$$

One could then obtain

$$[K_p] = 2\pi D[A^{-1}]^T[B][A]^{-1} \quad (4.24a)$$

where

$$[A] = \begin{bmatrix} 1 & r_1 & r_1^2 & r_1^3 \\ 0 & 1 & 2r_1 & 3r_1^2 \\ 1 & r_2 & r_2^2 & r_2^3 \\ 0 & 1 & 2r_2 & 3r_2^2 \end{bmatrix} \quad (4.24b)$$

$$[B] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \log \frac{r_2}{r_1} & 2(1+\nu)(r_2-r_1) & 1.5(1+2\nu)(r_2^2-r_1^2) \\ & 4(1+\nu)(r_2^2-r_1^2) & 6(1+\nu)(r_2^3-r_1^3) \\ \text{Symmetrical} & & (11.25+9\nu)(r_2^4-r_1^4) \end{bmatrix} \quad (4.24c)$$

4.4.2 Stiffness Matrix Due to Foundation

The elastic strain energy (U_F) due to the deformation of foundation (Eqn. (4.2)) can be written in the form

$$U_F = \frac{k}{2} \int_0^{2\pi} \int_{r_1}^{r_2} \{\alpha\}^T \begin{Bmatrix} 1 \\ r \\ r^2 \\ r^3 \end{Bmatrix} [1 \ r \ r^2 \ r^3] \{\alpha\} r dr d\theta \quad (4.25)$$

Substituting Eqn. (4.22) in Eqn. (4.25) and integrating the latter, U_F can be determined in terms of nodal displacements. Stiffness matrix K_F is obtained by making use of the relation

$$(K_F)_{ij} = \frac{\partial^2 U_F}{\partial \delta_i \partial \delta_j}$$

Final results for stiffness matrix (K_F) due to a foundation element can be expressed as

$$[K_F] = 2\pi k [A^{-1}]^T [C] [A^{-1}] \quad (4.26a)$$

where matrix $[A]$ is given by Eqn. (4.24b) and $[C]$ is shown below:

$$[C] = \begin{bmatrix} \frac{r_2^2 - r_1^2}{2} & \frac{r_2^3 - r_1^3}{3} & \frac{r_2^4 - r_1^4}{4} & \frac{r_2^5 - r_1^5}{5} \\ & \frac{r_2^4 - r_1^4}{4} & \frac{r_2^5 - r_1^5}{5} & \frac{r_2^6 - r_1^6}{6} \\ & & \frac{r_2^6 - r_1^6}{6} & \frac{r_2^7 - r_1^7}{7} \\ \text{Symmetrical} & & & \frac{r_2^8 - r_1^8}{8} \end{bmatrix} \quad (4.26b)$$

4.4.3 Consistent Load Vector

As discussed previously, consistent load vector is calculated by minimizing the potential energy (U_L) due to the load. Assuming that the plate element (Fig. 4.4) is subjected to a distributed load of uniform intensity q units per unit area, it can be shown that

$$U_L = -q \int_0^{2\pi} \int_{r_1}^{r_2} w(r) r dr d\theta \quad (4.27)$$

This can also be written in the form

$$U_L = -q \int_0^{2\pi} \int_{r_1}^{r_2} [a_1 \ a_2 \ a_3 \ a_4] \begin{Bmatrix} 1 \\ r \\ r^2 \\ r^3 \end{Bmatrix} r dr d\theta \quad (4.28)$$

By making use of Eqn. (4.22), the vector containing a_1, a_2, a_3 and a_4 can be eliminated from Eqn. (4.28). Upon integrating the same and carrying out the operations indicated by Eqn. (2.4c), the load vector Q could be obtained as follows:

$$\{Q\} = 2\pi q[A^{-1}]^T \begin{Bmatrix} (r_2^2 - r_1^2)/2 \\ (r_2^3 - r_1^3)/3 \\ (r_2^4 - r_1^4)/4 \\ (r_2^5 - r_1^5)/5 \end{Bmatrix} \quad (4.29)$$

4.4.4 Special Central Element

It is seen that the matrix B in Eqn. (4.24c) contains a term $\log(r_2/r_1)$. Thus, when $r_1=0$, this term becomes infinite, indicating that the stiffness matrix for the annular plate element could not be directly applied for a central solid element. To alleviate this problem, a special element with three degrees of freedom is introduced to model the central region of the plate (Fig. 4.5). Since the magnitude of rotation at the centre of the plate is zero (i.e. $(\frac{dw(r)}{dr})_{r=0}=0$), the degree of freedom corresponding to this rotation could be omitted. From the same consideration, it follows that the displacement function be so chosen that the rigid body rotation term is absent. Thus, the assumed displacement function for this element is given by

$$w(r) = a_1 + a_2 r^2 + a_3 r^3 \quad (4.30)$$

where a_1 , a_2 and a_3 are arbitrary constants.

Stiffness matrix K_{PC} for the plate, K_{FC} for the foundation and load vector Q_C are derived in the same way as discussed in Section 2.2.

The following results will then be obtained:

$$[K_{PC}] = 2\pi D [A_1^{-1}]^T [B_1] [A_1^{-1}]$$

$$[K_{FC}] = 2\pi k [A_1^{-1}]^T [C_1] [A_1^{-1}] \quad (4.31a-c)$$

$$\{Q_C\} = 2\pi q [A_1^{-1}]^T \begin{Bmatrix} r_2^2/2 \\ r_2^4/4 \\ r_2^5/5 \end{Bmatrix}$$

where

$$[A_1] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & r_2^2 & r_2^3 \\ 0 & 2r_2 & 3r_2^2 \end{bmatrix}$$

$$[B_1] = \begin{bmatrix} 0 & 0 & 0 \\ 4(1+\nu)r_2^2 & 6(1+\nu)r_2^3 \\ \text{Symmetrical} & (11.25+9\nu)r_2^4 \end{bmatrix}$$

and

$$[C_1] = \begin{bmatrix} \frac{r_2^2}{2} & \frac{r_2^4}{4} & \frac{r_2^5}{5} \\ \frac{r_2^6}{6} & \frac{r_2^7}{7} \\ \text{Symmetrical} & \frac{r_2^8}{8} \end{bmatrix}$$

4.5 STORAGE TANK PROBLEM

The analyses of circular plates which have been discussed so far are related to those with free edge boundary conditions. However, in practical situations where they form an integral part of a superstructure, the edges are usually restrained. The effect of edge restraint is investigated by analyzing the behaviour of an axisymmetric storage tank problem (Fig. 4.6). A description of the finite element modelling for the problem is outlined below.

4.5.1 Finite Element Model

Finite element analysis involves the modelling of the circular plate and the wall of the tank or the shell. As before, the circular plate is subdivided into annular ring elements and special element at the centre of the plate whose stiffnesses have already been derived. The wall of the tank is assumed to form a thin shell and is modelled as an assemblage of conical elements (Fig. 4.6). It is assumed that the behaviour of each element could be represented by membrane theory. The derivation of stiffness matrix for such an element is given below.

4.5.1.1 Element stiffness matrix

A typical axisymmetric conical shell element is shown in Fig. 4.7a. A detailed description for the element stiffness matrix is given by Bowes and Russell (1975). Therefore, only a summarized form of the derivation is presented here.

The degrees of freedom of the element are shown in Fig. 4.7b.

The displacements of the element at any typical position located by x' are u' , v' and β' which correspond to the displacements in the direction of X' axis, Y' axis and a clockwise rotation respectively. Polynomial expressions for u' and v' are given by

$$u' = a_1 + a_2 x'$$

$$v' = a_3 + a_4 x' + a_5 x'^2 + a_6 x'^3$$

from which it follows that

(4.32a-c)

$$\beta' = a_4 + 2a_5 x' + 3a_6 x'^2$$

The generalized coordinates can be related with the nodal displacement vectors as

$$\begin{Bmatrix} \delta'_1 \\ \delta'_2 \\ \delta'_3 \\ \delta'_4 \\ \delta'_5 \\ \delta'_6 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & \ell & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \ell & \ell^2 & \ell^3 \\ 0 & 0 & 0 & 1 & 2\ell & 3\ell^2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{Bmatrix} \quad (4.33)$$

[A]

or

$$\{\delta'\} = [A]\{\alpha\}$$

The stiffness matrix K_s is given by:

$$[K_s] = \begin{bmatrix} \lambda^T & 0 \\ 0 & \lambda^T \end{bmatrix} [A^{-1}]^T [B^T][D][B] d_{vol} [A^{-1}] \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \quad (4.34)$$

where

$$[B] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{\sin\theta}{y} & \frac{x'\sin\theta}{y} & \frac{\cos\theta}{y} & \frac{x'\cos\theta}{y} & \frac{x'^2\cos\theta}{y} & \frac{x'^3\cos\theta}{y} \\ 0 & 0 & 0 & 0 & 2 & 6x' \\ 0 & 0 & 0 & \frac{\sin\theta}{y} & \frac{2x'\sin\theta}{y} & \frac{3x'^2\sin\theta}{y} \end{bmatrix} \quad (4.35a)$$

$$[D] = \frac{Et}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 & 0 \\ \nu & 1 & 0 & 0 \\ 0 & 0 & t^2/12 & \nu t^2/12 \\ 0 & 0 & \nu t^2/12 & t^2/12 \end{bmatrix} \quad (4.35b)$$

$$[\lambda] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (4.35c)$$

Eqn. (4.34) is integrated numerically by the use of a six-point Gaussian Quadrature Formula (Kuo, 1960).

4.5.1.2 Consistent load vector

Consistent load vector due to the pressure exerted by the liquid inside the storage tank is derived following the method outlined in Chapter 2. Referring to Fig. 4.7b, the work done by the load can be

written as

$$U_L = - \int_0^{\ell} 2\pi\gamma(H-x)y v' dx' \quad (4.36a)$$

where γ is the unit weight of the liquid.

Eqn. (4.36a) can be written as:

$$U_L = -2\pi\gamma \int_0^{\ell} (y_1 + x' \sin\theta)(H - x' \cos\theta) \begin{bmatrix} 0 & 0 & 1 & x' & x'^2 & x'^3 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{Bmatrix} dx' \quad (4.36b)$$

From this expression, a_i 's ($i=1,6$) are eliminated first by making use of Eqn. (4.22), then integration is performed. Following the operations indicated by Eqn. (2.4c), the load vector Q is obtained as follows:

$$\{Q\} = \frac{\pi\gamma\ell}{60} \begin{bmatrix} \lambda^T & 0 \\ 0 & \lambda^T \end{bmatrix} [A^{-1}]^T \begin{Bmatrix} 0 \\ 0 \\ 120T_1 + 60T_2 - 40T_3 \\ \ell(60T_1 + 40T_2 - 30T_3) \\ \ell^2(40T_1 + 30T_2 - 24T_3) \\ \ell^3(30T_1 + 24T_2 - 20T_3) \end{Bmatrix} \quad (4.37)$$

where

$$T_1 = x_1 H$$

$$T_2 = l(H\sin\theta - y\cos\theta)$$

$$T_3 = \frac{l^2}{2} \sin 2\theta$$

4.5.1.3 Assembly Procedure

In assembling the total stiffness matrix and load vector from the foundation, plate and shell element stiffnesses and load vectors, a mismatch of the directions of vertical degrees of freedom is observed (Fig. 4.8). This is overcome by using a new transformation matrix in Eqns. (4.24) and (4.26). By doing so, a downward vertical deflection is obtained as positive. The matrix λ in Eqn. (4.26c) is now given by:

$$\lambda = \begin{bmatrix} -\cos\theta & \sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (4.38)$$

Also, the horizontal degree of freedom at the junction of shell and the plate is omitted during the assembling procedure.

4.5.1.4 Stresses

For a given liquid pressure in the storage tank, the nodal displacements can be found in the manner outlined previously in Chapter 2. By separating the appropriate components of displacements, the stresses in the wall of the tank can be determined. The vector of stresses (S) is given by:

$$\{S\} = \begin{Bmatrix} N_x \\ N_\theta \\ M_{xx} \\ M_{\theta\theta} \end{Bmatrix} = [D][B][\lambda^{-1}] \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \{\delta\} \quad (4.39)$$

where the stresses N_x , M_{xx} respectively refer to the membrane and bending stresses in the x-directions while N_θ and $M_{\theta\theta}$ represent the equivalent components in the circumferential direction. The matrices A, B, D and λ are given by Eqns. (4.33), (4.35a), (4.35b) and (4.38) respectively.

True stresses are found from the expressions (Bowes and Russell, 1975):

$$\begin{aligned} S_1 &= 6M/t^2 \\ S_2 &= N/t \end{aligned} \quad (4.40)$$

where M and N are as described above and t indicates the thickness of the shell element.

4.6 COMPARISON OF RESULTS

In this section, the results obtained from the analysis of circular plates are presented. Throughout the presentation of results, the quantity λa representing the relative rigidity of the plate-foundation system is defined in the following manner:

$$\lambda a = a (k/D)^{\frac{1}{4}} \quad (4.41)$$

where

- k is the modulus of subgrade reaction
- D is the flexural rigidity of the plate as given by Eqn. (4.5)
- and a is the radius of the plate.

The results obtained from the finite element and energy methods of analysis are compared with the analytical solution for the infinite plate problem given by Hertz (1884). This problem is concerned with a thin elastic plate resting on a Winkler medium and subjected to a concentrated load at the centre. A graphical representation of Hertz's analysis based on $\lambda a = 5$ is presented by Timoshenko and Woinowsky-Krieger (1959). This is reproduced in Fig. 4.9 where corresponding results from finite element and energy methods of analysis are also plotted. It is observed that the finite element analysis yields almost exactly the same results as given by Hertz (1884). On the other hand, the energy method underestimates the central deflection by about 20%. However, a close agreement is observed for points located away from the point of application of the load.

Some problems related to distributed loads have been examined. If a distributed load of uniform intensity acts over the entire region of a plate, the displacements of the plate will be uniform at every point. Such behaviour will occur regardless of the rigidity (λa) of the plate-foundation system. To study the effect of distributed loads spread over a portion of the plate, the problem as shown in Fig. 4.10a is examined for a wide range of relative rigidities. In Fig. 4.10b, the variation in central displacements with respect to various λa is plotted. It is observed that both the methods yield nearly the same result as long as $\lambda a < 5$. Portion of the curve with λa lying in the range .1 to .6 is

characterized by its flat slope, indicating that the central displacement is nearly constant. In fact, at λa equal to 0.1, the displacement is uniform all over the plate and at 0.6, the displacements differ only slightly from that of a uniform distribution. This fact is substantiated in Fig. 4.11 where the curves showing the variation of displacements have been plotted in a much exaggerated scale. It may be noted that as $\lambda a \rightarrow 0$, the plate becomes infinitely rigid. So a uniform or nearly uniform distribution of displacement of the plate at low values of λa (i.e. $\lambda a = .1$ to .6) is justified.

Referring back to Fig. 4.10 again, it may be observed that with the increase of λa (i.e. $\lambda a > .6$), the central displacement of the plate begins to increase. This increase continues so long as $\lambda a \leq 5$, beyond which results obtained from both the methods show a decrease. A discrepancy is also observed between the two curves indicating that the energy method exhibits a value different from the corresponding value obtained from the finite element analysis.

The distribution of displacement of a plate for λa equal to 40 is shown in Fig. 4.12. It is observed that the displacement is limited to the loaded region of the plate. Finite element analysis shows almost a constant displacement of magnitude q/k all over the loaded region. This is consistent with the theoretical predictions because the situation that λa equals 40 indicates a plate of infinite flexibility relative to the stiffness of the soil such that the load may be assumed to be transmitted

directly to the soil. The discrepancy in the results obtained from the energy method of analysis may be attributed to the nature of displacement function used in the formulation. Selvadurai (1979b) developed the energy method of analysis to examine the axisymmetric interaction of a circular plate resting on an elastic-halfspace. Since the displacements induced in a plate-elastic halfspace system can be represented by a continuous function at all ranges of relative rigidities, the derived results were found to be consistent. For the present work, the assumed displacement function also represents a continuous curve. However, this function is likely to be inadequate in representing the discontinuous step-function type displacement profile obtained for high values of λa . An alternative to the power series approximation would be to use a Fourier series representation of displacement function (Selvadurai, 1979c) i.e.

$$w(r) = \sum_{n=0}^m C_n \cos\left(\frac{n\pi r}{a}\right).$$

This would better simulate the discontinuous behaviour obtained as the relative rigidity (λa) becomes high.

Results from the analysis of a concentrated load placed at the centre of the plate (Fig. 4.13) reveal similar results as discussed in connection with distributed loads. The reason that the energy method yields results which are lower than the corresponding ones from finite element analysis may be attributed to the use of a limited number of terms in the displacement function.

However, in most practical situations, the relative rigidity (λa) of a circular plate foundation seldom exceeds a value of 5, a value in

the range 1 to 3 being typical (viz. for a concrete raft $a=50$ ft; $t=5$ ft; $E=3 \times 10^6$ psi; $\nu=.15$; $k=250$ lb per cu. inch; $\lambda a=4.64$). For such rigid plates, a useful approximation of displacements and flexural moments could be obtained from a finite element analysis using a coarse element subdivision. Comparable results could also be obtained from an energy solution using only two or three terms in the displacement function. This is examined by solving the problem shown in Fig. 4.14 where λa of the system is equal to 3. The results are variously compared in tabular and graphical forms as shown in Table 4.1 and Figs. 4.15 to 4.17 respectively.

Table 4.1 illustrates that while only 4 finite element subdivisions are needed to obtain a desired accuracy in the displacements, reasonable values for flexural moments could only be obtained by increasing the mesh size to 12 elements. The results from the energy method of analysis reveal that with three terms in the displacement function, computed displacements tend to reach a limiting value but radial and tangential moments show a variation of 2.25 and $-.84\%$ respectively with the use of even 4 terms. Further comparisons showing the effect in the use of varying number of elements or terms (as applicable) on the plate displacement and radial moment distribution are presented in Figs. 4.15 to 4.17. From Fig. 4.15a, it is observed that the energy method shows a lower value for end-displacement than the corresponding value obtained from the finite element analysis. Fig. 4.15b illustrates that the maximum difference between the two curves for radial moments occurs at points beyond the loaded region.

The results obtained from the energy method of analysis of annular and pedestal type of circular plate foundations are represented in Figs. 4.18 to 4.20. The variation in displacements for an annular, pedestal and a solid circular plate with no openings are compared in Fig. 4.18. Identical plate dimensions and loading conditions are used for all these cases. The central region of the plate extending radially up to b has been considered to be of infinite and zero stiffness for the pedestal and annular type of foundation, respectively. It is observed that the maximum central and differential deflection occurs in the case of the annular type of foundation, while both of these are lower for the pedestal type. The variations of radial moments are compared in Fig. 4.19.

The effect of increased annularity or increased size in the infinitely rigid portion at plate-centre on the plate deflections are compared in Fig. 4.20. While the difference between the two curves for the pedestal type of foundation (Fig. 4.20b) is observed to be small, a marked difference is observed in the two similar curves for the annular type of foundation (Fig. 4.20a).

The results obtained from the analysis of a cylindrical storage tank are presented in Figs. 4.21, 4.22 and 4.23. If such a tank is filled with a liquid, uniform pressures will be developed at the bottom (i.e. on the circular plate). In addition, uniform couples will be generated at the continuous connection between the wall and the circular plate. It is of interest to observe the variations in plate displacements and radial moments due to these forces. Effect of rigidity of the wall

of the tank is illustrated in Fig. 4.21. The quantity N is the ratio between the moduli of elasticity of the materials of the tank and the circular plate (it is assumed that Poisson's ratio for both the materials is the same). The curves for $N=0$ correspond to the case of the circular plate loaded by a uniform pressure as that at the bottom of the tank where the restraining effect due to the wall is absent. It is observed that as N increases, central deflection of the plate decreases but the edge displacement shows an increase (Fig. 4.21a). From Fig. 4.21b, it is observed that the factor N has a great influence on the distribution of radial moments of the plate. For $N=50$, negative moments are produced near the wall-plate junction.

Effect of varying thickness of the circular plate on its displacements and radial moments is shown in Figs. 4.22a-b. It is observed that as a/t decreases, the distribution of displacement tends to become uniform. The radial moments show an increase with increased plate thickness.

Effect of varying properties of plate and soil on the plate displacements and radial moments is shown in Fig. 4.23. The quantity M is defined in the same figure. A small value of M corresponds to a rigid plate. It is observed that as M increases, central displacement of the plate increases. But, the case is reversed for radial moments. For $M=1$, the maximum moment in the plate is found to occur at a distance from the origin.

Fig. 4.2: Circular Plate With Central Pedestal of Infinite Rigidity

Fig. 4.1 A Complete Circular Plate

Fig. 4.3 An Annular Plate

Fig. 4.4 An Annular Ring Element

Fig. 4.5: Special Central Element

Fig. 4.6
A Storage Tank

Fig. 4.7a
A Conical Element

Fig. 4.7b
A Conical Element With
Degrees of Freedom

Fig. 4.8
Degrees of Freedom
at Tank-Plate Junction

Fig. 4.9

Comparison of Displacements
With Hertz's (1884) Analysis

Fig. 4.10b
Variation of Central Displacement
with respect to Relative Rigidity - λa

Fig. 4.11

Variation of Displacement for
Rigid and Infinitely Rigid Plates
(Based on $b/a = 0.6$)

Fig. 4.12

**Variation of Displacement
for Infinitely Flexible Plate
(Relative Rigidity Equal to 40)**

Fig. 4.13

Variation of Central
Displacement with respect
to Relative Rigidity

Table 4.1. The Variation in Percent in Computed Values (at a radial distance $a/2$) with respect to results from a finite element analysis (Fig. 4.14) using 80 elements.

FINITE ELEMENT ANALYSIS

NO. OF ELEMENTS	4	6	8	10	12
Displacement	-0.020	-0.011	-0.007	-0.001	0
Radial Moment	-5.300	-3.061	-1.710	-1.207	-0.900
Tangential Moment	-0.859	-0.495	-0.330	-0.263	-0.190

ENERGY METHOD

NO. OF TERMS	2	3	4
Displacement	1.682	-0.103	0.001
Radial Moment	38.722	-1.171	2.249
Tangential Moment	-16.397	-1.506	-0.838

Relative Rigidity = 3.0

Fig. 4.14

A Circular Plate Resting on a Winkler Medium

Fig. 4.15: Comparison of Energy and Finite Element Methods of Analysis

Fig. 4.16

Effect Due to Number of Terms
in the Displacement Function
(Energy Method of Analysis)

Fig. 4.17: Effect due to Mesh Sizes
(Finite Element Analysis)

Fig. 4.18

Variation of Displacement for Different Types of Circular plates Resting on Winkler Medium.

Fig. 4.19

Variation of Radial Moment
for Different Types of Circular
Plates

Fig. 4.20
Variation in Displacement

Fig. 4.21 Effect of Rigidity Ratio (N) on the Response of a Cylindrical Tank

(a) On Displacement (b) On Radial Moment
 (Based on Fig.4.6; $M=.005$, $h/t=0.25$, $a/t=20.0$,
 $b/a=0.8$, $H/a=2.0$ and Poisson's Ratio=0.3)

Fig.4.22: Effect of Plate Thickness (a/t)
 (a) On Displacement; (b) On Radial Moment
 (Based on $M = 0.005, a/h = 100, E_s/E = 1.0,$
 $b/a = 0.8, H/a = 2.0$ and Poisson's Ratio $= 0.3$)

$$\text{Modular Ratio}(M) = 12k(1-\nu^2) a/E$$

Fig. 4.23: Effect of Modular Ratio(M)

- (a) Variation in Displacement
 (b) Variation in Radial Moment
 (Based on Fig.4.6; $E_s/E=1.0$,
 $h/t=0.25$, $b/a=0.8$, $H/a=2.0$,
 $\nu=0.3$)

CHAPTER 5

CONCLUSIONS AND RECOMMENDATIONS FOR FUTURE STUDIES

5.1 CONCLUSIONS

5.1.1 Beams

Numerical results are developed using the energy and the finite element methods of analysis for beams resting on Winkler media to illustrate the effects of relative rigidity on deflections, flexural moments and shear forces. The convergence of both the finite elements and the various energy solutions to the exact analysis are examined. Effects of boundary conditions on the energy solutions are also investigated. From the various analyses of beams, it is possible to arrive at the following general conclusions:

- a) In the context of analyses of beams having low relative rigidities ($\lambda l \leq 4$), both the finite element technique and the energy methods of analysis yield satisfactory results which are comparable with the exact analytical solution. The agreement with beam-deflections derived from various methods is good (within 0.5%). The results for flexural moments and shear forces show only slight variations (of the order of -4% and -8% respectively). For a beam having λl equal to 5 (which may be described as a moderately flexible beam), the maximum error in the deflections, flexural moments and

shear forces obtained by any one of the methods of analyses is of the order of 0.01%, -1.7% and -4.60%, respectively.

- b) The effect of increased relative rigidity of the beam-soil system is to increase the deflection of the beam at or near the point of application of the load. The flexural moments and shear forces are, however, decreased.

The energy and finite element solutions respond to the increased effect of relative rigidity favourably well, provided that $\lambda l < 10$. Errors in the computed values for the beam-deflections, flexural moments and shear forces are introduced as λl increases. The results obtained from the polynomial solutions indicate a very poor correlation with those obtained from the exact analysis. The available results seem to suggest that both the polynomial and Fourier series solutions may be used to examine beam deflections and moments as long as λl does not exceed 20. For beams having $\lambda l > 20$, finite element method can be used only for the approximate determination of displacements.

- c) In the case of energy methods, the solution obtained by using a modified displacement function which satisfies the static boundary conditions at the free ends of a beam (in either the Fourier series or polynomial displacement function) yields slightly better results than those obtained from the use of corresponding unmodified function. The difference between any two sets of results is, however, not significant in terms of extra computations needed in the analysis. The

available results seem to suggest that a structure-foundation interaction analysis could be based on displacement functions which satisfy only the kinematic boundary conditions.

- d) The Fourier series solution indicates that accurate results for displacements can be obtained even when only 4 terms are used in the displacement function. However, in order that flexural moments and shear forces be within 5% of the exact value, at least 10 terms should be used. Polynomial solution also converges to the exact one but the rate of convergence is rather slow. In the case of finite element analysis, while accurate results for displacements are obtained by using 10 elements, useful results for flexural moments and shear forces would necessitate the use of a more refined mesh (say 25 elements).

5.1.2 Circular Plates

From the finite element and energy methods of analyses of various types of circular plates resting on Winkler media, numerical results are developed to illustrate the effect of relative rigidity of the plate-foundation system on the overall plate deflections, central deflections and radial moments. The main observations can be summarized as follows:

- a) The effect of increased relative rigidity of a circular plate-soil system is to increase the central deflection of the plate.

- b) For rigid and moderately flexible plates having $\lambda a < 5$, the results obtained from the finite element and energy methods of analyses are found to be in good agreement. However, as λa increases beyond 5, central deflection of the plate computed by the energy method is found to be lower than the corresponding results obtained from the finite element analysis. Comparison with an existing analytical solution demonstrates that the finite element analysis is accurate (practically no error in displacements is observed).
- c) In the energy method of analysis of a circular plate, the chosen displacement function which is an even order polynomial consisting of six terms, is found to be ineffective in representing the deflected shape at high values of relative rigidity (say $\lambda a > 40$). This discrepancy is not observed in the case of finite element method of analysis.

5.2. RECOMMENDATIONS FOR FUTURE STUDIES

The present study has indicated the need for further studies in the following areas.

5.2.1 Beams

- (a) The energy solutions developed in the thesis consider only prismatic beams having free ends. The analyses could be extended to include non-prismatic beams under any support conditions (viz. end clamped or hinged).

- b) In the present finite element analysis, non-prismatic beams with a stepwise variation of rigidity only can be examined. In practical situations, the sectional and elastic properties of a beam may vary in a different manner. It is recommended that a more general finite element analysis be carried out so that a beam having any variation in shape and rigidity could be examined.
- c) In the energy and finite element methods of analyses, only concentrated and uniformly distributed loads and concentrated moments are examined. The future analyses may be extended to include any arbitrary distributed loads and moments.
- d) The various analyses for beams developed in this thesis are applicable only in the elastic range. But, in practical situations, inelastic behaviour may be anticipated due to excessive differential settlement. It is, therefore, recommended to extend the present study to inelastic range.

5.2.2 Circular Plates

- a) In the energy method, the chosen even order polynomial function consisting of six terms is found to be ineffective in representing the deflected shape of a circular plate at high values of relative rigidity. A Fourier series representation of the deflected shape of the plate is likely to be effective in such a case (Selvadurai, 1979c). It is recommended that an energy method of analysis be developed for circular plates using the Fourier series.

- b) The present study considers only axisymmetrically loaded circular plates. It is recommended that analyses be developed to include non-symmetric loads as well.
- c) The present analyses are developed on the basis of Poisson-Kirchhoff thin-plate small-deflection theory. However, in practical situations, the plate may be thick enough to include the effect of normal pressure and shear deformation on its deflections. It is recommended that an analysis be performed using thick plate theory.

REFERENCES

- Allen, D. and Severn, R. (1960,1961,1963). The Stresses in Foundation Rafts, Proceedings of the Institution of Civil Engineers, London, Vol. 15 (1960); Vol. 20 (1961); Vol. 25 (1963).
- Baker, A. (1957). "Raft Foundations", Concrete Publications Ltd., London.
- Bickley, W.G. (1941). Formulae for Numerical Differentiation, Math. Ga3, Vol. 25, pp. 19-27.
- Bowes, W.H. and Russell, L.T. (1975). "Stress Analysis by the Finite Element Method for Practicing Engineers", D.C. Heath and Company, Massachusetts, U.S.A.
- Cheung, Y.K. and Zienkiewicz, O.C. (1965). Plates and Tanks on Elastic Foundations: An Application of the Finite Element Method, Int. Journal of Solids and Structures, Vol. 1.
- Davies, J.D. (1961). Cylindrical Concrete Tanks on Elastic Foundations, Civil Engineering, London, Vol. 56, pp. 61-64.
- Desai, C.S. and Abel, J.F. (1972). "Introduction to the Finite Element Method", Van Nostrand Reinhold Co., New York.
- Dodge, A. (1964). Influence Functions for Beams on Elastic Foundations, Proc. A.S.C.E., Journal of the Structural Division, Vol. 90, ST4, pp. 63-101.
- Frederick, D. (1955). On Some Problems in Bending of Thick Circular Plates on an Elastic Foundation, A.S.M.E. Ann. Meet., Chicago, Ill., Pap. 55-A-36, pp. 6.
- Frederick, D. (1957). Thick Rectangular Plates on an Elastic Foundation, Trans. A.S.C.E., Vol. 122, pp. 1069-1089.
- Hayashi, K. (1921). "Theory of Beams on Elastic Foundation" (in German), Springer-Verlag, Berlin.
- Hendry, A. (1958). A New Method for the Analysis of Beams on Elastic Foundations, Civil Engineering and Public Works Review, Vol. 53.
- Hertz, H. (1884). On the Equilibrium of Floating Elastic Plates (in German), Wiedemanns Ann., Vol. 22, p. 449.

- Hetenyi, M. (1946). "Beams on Elastic Foundation", University of Michigan Press.
- Hetenyi, M. (1959). A Comparison of Various Solutions for Beams on Elastic Foundations, Acta Tech., Acad. Sci. Hungaricae, Budapest, Vol. 26, 1/2, pp. 21-28.
- Hetenyi, M. (1966). Beams and Plates on Elastic Foundations and Related Problems, Applied Mechanics Review, Vol. 19, 2, pp. 95-102.
- Iwanczewska, A. and Lewandowski, J. (1968). "Obliczanie Konstrukcji na Sprezystym Podlozu", Arkady, Warsaw.
- Iyenger, S.R. and Anantharamu, S. (1965). Influence Lines for Beams on Elastic Foundations, Proc. A.S.C.E., Journal of the Structural Division, Vol. 91, ST3, pp. 45-56.
- Iyenger, S.R. (1965). Matrix Analysis of Finite Beams on Elastic Foundations, Journal Inst. of Engineers, India.
- Just, D.J., Starzewski, K. and Ronan, P.B. (1971). Finite Element Method of Analysis of Structures Resting on Elastic Foundations, Proc. Symposium on Foundation Structure Interaction, University of Birmingham, England.
- Kramrisch, F. and Rogers, P. (1961). Simplified Design of Combined Footings, Proc. A.S.C.E. Journal of the S.M. and F.E. Division, Vol. 87, SM5, pp. 45-68.
- Kuo, S.S. (1966). Numerical Methods and Computers, Don Mills, Addison-Wesley, Ontario.
- Lee, I.K. and Harrison, H.B. (1970). Structure and Foundation Interaction Theory, Proc. A.S.C.E. Journal of the Structural Division, Vol. 90, ST2, pp. 177-196.
- Livesley, R.K. (1953). Some Notes on the Mathematical Theory of a Loaded Elastic Plate Resting on an Elastic Foundation, Quart. J. Mech. Appl. Math., Vol. 6, Part 1, pp. 32-34.
- Melosh, R. (1961). A Stiffness Matrix for the Analysis of Thin Plates in Bending, Journal of the Aerospace Sciences, Vol. 28.
- Melosh, R. (1963). Basis for Derivation of Matrices for the Direct Stiffness Method, A.I.A.A. Journal, 1.
- Miner, D. F. and Seastone, J.B. (1955). "Handbook of Engineering Materials", Wiley, New York.

- Miranda, C. and Nair, K. (1966). Finite Beams on Elastic Foundation, Proc. A.S.C.E., Journal of the Structural Division, Vol. 92, ST2, pp. 131-140.
- Miyahara, F. and Ergatoudis, J.G. (1976). Matrix Analysis of Structure-Foundation Interaction, Proc. A.S.C.E., Journal of the Structural Division, Vol. 102, ST1, pp. 251-261.
- Murphy, G. (1937). Stresses and Deflections in Loaded Rectangular Plates on Elastic Foundations, Iowa State College Eng. Expt. Sta. Bull. 135.
- Payne and Irons, referred to as private communication in The Finite Element Method in Engineering Science by Zienkiewicz (1971).
- Pickett, G., Raville, M.E., Jones, W.C. and McCormick, F.J. (1951). Deflections, Moments and Reactive Pressures for Concrete Pavements, Kansas State Coll. Bull. No. 65, p. 73.
- Reissner, E. (1955). Stresses in Elastic Plates over Flexible Subgrades, Proc. A.S.C.E., Vol. 81, Separ. No. 690, p. 29.
- Schleicher, F. (1926). "Circular Plates on Elastic Foundations" (in German), Springer Verlag, Berlin.
- Selvadurai, A.P.S. (1979a). "Elastic Analysis of Soil-Foundation Interaction", Developments in Geotechnical Engineering, Elsevier Scientific Publishing Co., New York.
- Selvadurai, A.P.S. (1979b). The Interaction Between a Uniformly Loaded Circular Plate and an Isotropic Elastic Halfspace: A Variational Approach, Jour. Structural Mechanics, Vol. 7, 3, pp. 231-246.
- Selvadurai, A.P.S. (1979c). A Ritz-Galerkin Approach for the Analysis of Contact Problems (unpublished work).
- Severn, R.T. (1966). The Solution of Foundation Mat Problems by Finite Element Method, Structural Engineer, Vol. 44, 6, pp. 223-228.
- Sinha, S.N. (1963). Large Deflections of Plates on Elastic Foundations, Proc. A.S.C.E., Journal Eng. Mech. Division, Vol. 89, EM1, pp. 1-24.
- Smith, I.M. (1967). "The Behaviour of Flexible Slabs on Idealized and Actual Foundations", Ph.D. Thesis, University of Glasgow, England.

- Terzaghi, K. (1956). Evaluation of Coefficients of Subgrade Reactions, Geotechnique, Vol. 5, 4, p. 297.
- Timoshenko, S.P. and Woinowsky-Krieger, S. (1959). "Theory of Plates and Shells", 2nd Ed., McGraw-Hill Book Co., New York.
- Vint, J. and Elgood, W. (1935). Deformation of a Loaded Bloom Plate, Philosophical Magazine, Vol. 19, Series 7.
- Vlasov, V.Z. and Leontiev, N.N. (1960). "Beams, Plates and Shells on Elastic Foundations" (in Russian), Gosudarstvennoe Izdatelstve Fiziko-Matem, Literaturny, Moskava, p. 491.
- Willems, N. and Lucas, W.M. (1968). "Matrix Analysis for Structural Engineers", Prentice-Hall Inc., Englewood Cliffs, N.J.
- Winkler, E. (1867). "Die Lehre von der Elastizität und Festigkeit", Prag., p. 182.
- Yang, H.T.Y. (1970). Flexible Plate Finite Element on Elastic Foundation, Proc. A.S.C.E., Journal of the Structural Division, Vol. 96, ST10, pp. 2083-3000.
- Zienkiewicz, O.C. (1971). "The Finite Element Method in Engineering Science", 2nd Ed., McGraw-Hill, New York.
- Zienkiewicz, O.C. and Cheung, Y.K. (1964). The Finite Element Method for Analysis of Elastic Isotropic and Orthotropic Slabs, Proc. Instn. of Civil Engrs., Vol. 28, pp. 471-488.
- Zimmermann, H. (1884). "Die Berechnung des Eisenbahnoerbaues", Berlin, 2nd Ed., Berlin, 1930.

APPENDIX A

ANALYSIS OF BEAM (DISPLACEMENT FUNCTION
TO SATISFY THE BOUNDARY CONDITION)

A.1 FOURIER SERIES SOLUTION

A.1.1 Symmetric Loads (Fig. A.1)

Displacement function (y) is given by

$$y = \frac{P}{k\ell} + \frac{q(d-c)}{k\ell} + \sum a_n \left(\sin \frac{n\pi x}{\ell} - n^3 \sin \frac{\pi x}{\ell} + \frac{2n^3}{\pi} - \frac{2}{n\pi} \right) \quad (A.1)$$

where a's are obtained from

$$\begin{aligned} & a_m \left(\frac{EI\pi^4 m^4}{2\ell^3} + \frac{k\ell}{2} \right) + \frac{EI\pi^4 m^3}{2\ell^3} \sum_n a_n n^3 \\ & + \frac{4k\ell}{\pi^2} \sum_{m,n} a_n \left(\frac{m^3}{n} + \frac{n^3}{m} + \frac{m^3 n^3 \pi^2}{8} - m^3 n^3 - \frac{1}{mn} \right) \\ & = P \left[\sin \frac{m\pi a}{\ell} - m^3 \sin \frac{\pi a}{\ell} + \frac{2m^3}{\pi} - \frac{2}{m\pi} \right] \\ & + \frac{M_0 m\pi}{\ell} \left[\cos \frac{m\pi b}{\ell} - m^2 \cos \frac{\pi b}{\ell} \right] \\ & + \frac{q\ell}{\pi} \left[\frac{1}{m} \left(\cos \frac{m\pi c}{\ell} - \cos \frac{m\pi d}{\ell} \right) + m^3 \left(\cos \frac{\pi d}{\ell} - \cos \frac{\pi c}{\ell} \right) + \frac{d-c}{\ell} \left(2m^3 - \frac{2}{m} \right) \right] \end{aligned} \quad (A.2)$$

where

$$m \ (3, 5, 7, \dots \infty)$$

$$n \ (3, 5, 7, \dots \infty)$$

A.1.2 Antisymmetric Loads (Fig. A.2)

The displacement function is given by

$$y = \left[\frac{6P(\ell-2a)+12M_0+6q(d-c)(\ell-d-c)}{k\ell^3} \right] \left[\frac{\ell}{2} - x \right] + \frac{12}{\ell\pi} \sum_n a_n \left(\frac{n^3}{16} - \frac{1}{n} \right) \left(\frac{\ell}{2} - x \right) + \sum_n a_n \left(\sin \frac{n\pi x}{\ell} - \frac{n^3}{8} \sin \frac{2\pi x}{\ell} \right) \quad (A.3)$$

where a 's are obtained from the set of equations

$$\begin{aligned} a_m \left(\frac{EI\pi^4 m^4}{2\ell^3} + \frac{k\ell}{2} \right) + \frac{3}{4} \frac{k\ell}{\pi} \sum_n a_n \left(\frac{m^3}{n} + \frac{n^3}{m} - \frac{16}{mn} + \frac{m^3 n^3}{16} + \frac{m^3 n^3 \pi^2}{96} \right) \\ + \frac{EI\pi^4 m^3}{8\ell^3} \sum_n a_n n^3 \\ = P \left[\frac{12}{\ell\pi} \left(\frac{m^3}{16} - \frac{1}{m} \right) \left(\frac{\ell}{2} - a \right) - \frac{1}{8} m^3 \sin \frac{2\pi a}{\ell} + \sin \frac{m\pi a}{\ell} \right] \\ + M_0 \left[\frac{12}{\ell\pi} \left(\frac{1}{m} - \frac{m^3}{16} \right) - \frac{\pi}{4\ell} m^3 \cos \frac{2\pi b}{\ell} + \frac{m\pi}{\ell} \cos \frac{m\pi b}{\ell} \right] \\ + q \left[\frac{6}{\ell\pi} \left(\frac{m^3}{16} - \frac{1}{m} \right) (\ell d - \ell c + c^2 - d^2) + \frac{m^3 \ell}{16\pi} \left(\cos \frac{2\pi d}{\ell} - \cos \frac{2\pi c}{\ell} \right) \right. \\ \left. - \frac{\ell}{m\pi} \left(\cos \frac{m\pi d}{\ell} - \cos \frac{m\pi c}{\ell} \right) \right] \end{aligned} \quad (A.4)$$

where

$$m (4, 6, 8, \dots \infty)$$

$$n (4, 6, 8, \dots \infty)$$

A.2 POLYNOMIAL SOLUTION

A.2.1 Symmetric Loads (Fig. A.3)

Displacement function is given by

$$y = C_0 \ell + \ell \sum_{i=2,4}^{14} (C_6 A_i + C_8 B_i + C_{10} D_i + C_{12} E_i + C_{14} F_i) \left(\frac{x}{\ell}\right)^i$$

All values of A_i , B_i , ..., F_i are zero except the following:

$$\begin{array}{ccccc} A_2 = 15 & B_2 = 56 & D_2 = 135 & E_2 = 264 & F_2 = 455 \\ A_4 = -5 & B_6 = -14 & D_8 = -30 & E_{10} = -30 & F_{12} = -91 \\ A_6 = 1 & B_8 = 1 & D_{10} = 1 & E_{12} = 1 & F_{14} = 1 \end{array}$$

The constants C_0 , C_6 , C_8 , C_{10} , C_{12} , and C_{14} are obtained from

$$[M]\{C\} = \{F\}$$

where

$$[M] = \sum_i \sum_j \begin{bmatrix} T_2 & \frac{T_2 A_i}{i} & \frac{T_2 B_i}{i} & \frac{T_2 D_i}{i} & \frac{T_2 E_i}{i} & \frac{T_2 F_i}{i} \\ & T_{ij} A_i A_j & T_{ij} A_i B_j & T_{ij} A_i D_j & T_{ij} A_i E_j & T_{ij} A_i F_j \\ & & T_{ij} B_i B_j & T_{ij} B_i D_j & T_{ij} B_i E_j & T_{ij} B_i F_j \\ & & & T_{ij} D_i D_j & T_{ij} D_i E_j & T_{ij} D_i F_j \\ & & & & T_{ij} E_i E_j & T_{ij} E_i F_i \\ & & & & & T_{ij} F_i F_j \end{bmatrix}$$

Symmetrical

where

$$T_2 = 2k\ell^3$$

$$T_{ij} = \frac{T_2}{i+j+3} + \frac{2EI}{\ell} ij \frac{(i-1)(j-1)}{i+j-3}$$

Also,

$$F = P\ell \sum_i \begin{Bmatrix} 1 \\ A_i a^i \\ B_i a^i \\ D_i a^i \\ E_i a^i \\ F_i a^i \end{Bmatrix} + q\ell^2 \sum_i \begin{Bmatrix} d^2 - c^2 \\ (d^i - c^i) A_i / i \\ (d^i - c^i) B_i / i \\ (d^i - c^i) D_i / i \\ (d^i - c^i) E_i / i \\ (d^i - c^i) F_i / i \end{Bmatrix} + M_0 \sum_i \begin{Bmatrix} 0 \\ b^{i-1} \cdot i \cdot A_i \\ b^{i-1} \cdot i \cdot B_i \\ b^{i-1} \cdot i \cdot D_i \\ b^{i-1} \cdot i \cdot E_i \\ b^{i-1} \cdot i \cdot F_i \end{Bmatrix}$$

where

$$\{C\}^T = [C_0 \ C_6 \ C_8 \ C_{10} \ C_{12} \ C_{14}]$$

$$i \ (2, 4, 6, \dots, 14)$$

$$j \ (2, 4, 6, \dots, 14)$$

A.2.2 Antisymmetric Loads (Polynomial)

Based on Fig. A.4, displacement function (y) is given by:

$$\frac{y}{\ell} = C_1 \frac{x}{\ell} + \sum_{i=3,5}^{15} (C_7 A_i + C_9 B_i + C_{11} D_i + C_{13} E_i + C_{15} F_i) \left(\frac{x}{\ell}\right)^i$$

where $C_1, C_7, C_9, C_{11}, C_{13}, C_{15}$ are arbitrary constants. Only non-zero values of A_i, B_i ($i=3,5,\dots,15$) etc. are:

$$\begin{array}{lllll} A_3 = 7 & B_3 = 24 & D_3 = 55 & E_3 = 104 & F_3 = 175 \\ A_5 = -4.2 & B_7 = -10.8 & D_9 = -22 & E_{11} = -39 & F_{13} = -63 \\ A_7 = 1 & B_9 = 1 & D_{11} = .1 & E_{13} = 1 & F_{15} = 1 \end{array}$$

The constants $C_1, C_7, C_9, \dots, C_{15}$ are obtained from

$$[M]\{C\} = \{F\}$$

where

$$[M] = \sum_i \sum_j \begin{bmatrix} \frac{T_2}{3} & \frac{T_2 A_i}{i} & \frac{T_2 B_i}{i} & \frac{T_2 D_i}{i} & \frac{T_2 E_i}{i} & \frac{T_2 F_i}{i} \\ & T_{ij} A_i A_j & T_{ij} A_i B_j & T_{ij} A_i D_j & T_{ij} A_i E_j & T_{ij} A_i F_j \\ & & T_{ij} B_i B_j & T_{ij} B_i D_j & T_{ij} B_i E_j & T_{ij} B_i F_j \\ & & & T_{ij} D_i D_j & T_{ij} D_i E_j & T_{ij} D_i F_j \\ & & & & T_{ij} E_i E_j & T_{ij} E_i F_j \\ & & & & & T_{ij} F_i F_j \end{bmatrix}$$

Symmetrical

where

$$T_2 = 2k\ell^3$$

$$T_{ij} = 2k\ell^3 + \frac{2EI}{\ell} \frac{(i-1)(j-1)ij}{i+j-3}$$

$$\{F\} = Pl \sum_i \begin{bmatrix} 1 \\ A_i a_i^i \\ B_i a_i^i \\ D_i a_i^i \\ E_i a_i^i \\ F_i a_i^i \end{bmatrix} + q\ell^2 \sum_i \begin{bmatrix} d^2 - c^2 \\ (d^i - c^i) A_i / i \\ (d^i - c^i) B_i / i \\ (d^i - c^i) D_i / i \\ (d^i - c^i) E_i / i \\ (d^i - c^i) F_i / i \end{bmatrix} + M_0 \sum_i \begin{bmatrix} 1 \\ b^{i-1} \cdot i \cdot A_i \\ b^{i-1} \cdot i \cdot B_i \\ b^{i-1} \cdot i \cdot D_i \\ b^{i-1} \cdot i \cdot E_i \\ b^{i-1} \cdot i \cdot F_i \end{bmatrix}$$

where

$$\{C\}^T = [C_1 \ C_7 \ C_9 \ C_{14} \ C_{13} \ C_{15}]$$

$$i \ (3, 5, 7, \dots, 15)$$

$$j \ (3, 5, 7, \dots, 15)$$

Fig. A.1

Fig. A.2

Fig. A.3

Fig. A.4

```

1.000 C      * * * * *
2.000 C      *
3.000 C      * =====
4.000 C      * PROGRAM FOR THE ANALYSIS OF A BEAM
5.000 C      * RESTING ON WINKLER'S FOUNDATION
6.000 C      * =====
7.000 C      *
8.000 C      * PROGRAMMED BY
9.000 C      * A.K.M. GHULAM SARWAR
10.000 C     * THESIS SUPERVISORS
11.000 C     * PROF. J.L.HUMAR  PROF. A.P.S. SELVADURAI
12.000 C     *
13.000 C     * CARLETON UNIVERSITY
14.000 C     * OTTAWA,CANADA
15.000 C     * JUNE,1979
16.000 C     *
17.000 C     * * * * *
18.000 DIMENSION DISP(50),XM(50),SH(50)
19.000 COMMON/BLOK1/IR,IW,IN,ITOT,JA(80),IA(48),IFI(6),APL(6),
20.000 +IND(6),JBND(8,6),IBND(4,6),FBND(4,6)
21.000 DATA II,IY,ID/'I','Y','D'/
22.000 IR=5
23.000 IW=6
24.000 IN=30
25.000 DO 11 I=1,4
26.000 DO 11 J=1,6
27.000 FBND(I,J)=0.
28.000 11 CONTINUE
29.000 DO 21 I=1,8
30.000 DO 21 J=1,6
31.000 JBND(I,J)=0
32.000 21 CONTINUE
33.000 WRITE(6,700)
34.000 ITOT=1
35.000 CALL REED
36.000 JI=IA(1)
37.000 WRITE(6,699)
38.000 IF(JI.NE.II) GO TO 1
39.000 WRITE(6,701)
40.000 READ(5,991) IF
41.000 WRITE(6,702)
42.000 READ(5,991) IL
43.000 GO TO 2
44.000 1 WRITE(6,703)
45.000 READ(5,991) IF
46.000 WRITE(6,704)
47.000 READ(5,991) IL
48.000 2 WRITE(6,706)

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```
49.000      CALL REED
50.000      XL=AFL(1)
51.000      WRITE(6,708)
52.000      CALL REED
53.000      B=AFL(1)
54.000      WRITE(6,710)
55.000      CALL REED
56.000      JD=IA(1)
57.000      IF(JD.NE.ID) GO TO 4
58.000      WRITE(6,712)
59.000      CALL REED
60.000      D=AFL(1)
61.000      WRITE(6,714)
62.000      CALL REED
63.000      UW=AFL(1)
64.000      QS=UW*B*D
65.000      XI=B*D**3/12.
66.000      GO TO 3
67.000      4 WRITE(6,716)
68.000      CALL REED
69.000      XI=AFL(1)
70.000      WRITE(6,718)
71.000      CALL REED
72.000      QS=AFL(1)
73.000      3 WRITE(6,719)
74.000      CALL REED
75.000      E=AFL(1)
76.000      WRITE(6,720)
77.000      CALL REED
78.000      JY=IA(1)
79.000      IF(JY.NE.IY) GO TO 6
80.000      IF(JI.NE.II) GO TO 5
81.000      WRITE(6,728)
82.000      WRITE(6,730)
83.000      WRITE(6,732)
84.000      GO TO 6
85.000      5 WRITE(6,722)
86.000      WRITE(6,724)
87.000      WRITE(6,726)
88.000      6 WRITE(6,734)
89.000      CALL REED
90.000      XK=AFL(1)
91.000      XK=XK*B
92.000      PP=XL*(XK/(4.*E*XI))**.25
93.000      XLAM=PP/XL
94.000      XLAM=(XK/(4.*E*XI))**.25
95.000      C
96.000      WRITE(6,736)
```

```

97.000      CALL REED
98.000      NSEG=IFI(1)
99.000      ND=NSEG+1
100.000     WRITE(6,333)
101.000     DO 50 I=1,ND
102.000     DISP(I)=QS/XK
103.000     XM(I)=0.
104.000     SH(I)=0.
105.000 50  CONTINUE
106.000     CHP=COSH(PP)
107.000     SHP=SINH(PP)
108.000     SP=SIN(PP)
109.000     CP=COS(PP)
110.000     F2=.5*(CHP*SP+SHP*CP)
111.000     F3=.5*SHP*SP
112.000     F4=.25*(CHP*SP-SHP*CP)
113.000     D=F3*F3-F2*F4
114.000 C
115.000     WRITE(6,738)
116.000     CALL REED
117.000     JY=IA(1)
118.000     IF(JY.NE.IY) GO TO 14
119.000     WRITE(6,740)
120.000     ITOT=2
121.000 12  CALL REED
122.000     A=AFL(1)
123.000     P=AFL(2)
124.000     A=A/XL
125.000     IF(P.EQ.0.) GO TO 14
126.000     CALL CONCL(PP,F2,F3,F4,D,ND,XLAM,XK,P,A,DISP,XM,SH)
127.000     GO TO 12
128.000 14  WRITE(6,742)
129.000     ITOT=1
130.000     CALL REED
131.000     JY=IA(1)
132.000     IF(JY.NE.IY) GO TO 18
133.000     ITOT=2
134.000     WRITE(6,744)
135.000 16  CALL REED
136.000     A=AFL(1)
137.000     CM=AFL(2)
138.000     IF(CM.EQ.0.) GO TO 18
139.000     A=A/XL
140.000     CALL COUPLE(PP,F2,F3,F4,D,ND,XLAM,XK,CM,A,DISP,XM,SH)
141.000     GO TO 16
142.000 C
143.000 C
144.000 C

```

```

145.000 18 WRITE(6,746)
146.000     ITOT=1
147.000     CALL REED
148.000     JY=IA(1)
149.000     IF(JY.NE.IY) GO TO 22
150.000     WRITE(6,748)
151.000     ITOT=3
152.000 20 CALL REED
153.000     A=AFL(1)
154.000     B=AFL(2)
155.000     Q=AFL(3)
156.000     IF(Q.EQ.0.) GO TO 22
157.000     A=A/XL
158.000     B=B/XL
159.000     CALL UDL(PP,F2,F3,F4,D,ND,XLAM,XR,Q,A,B,DISP,XM,SH)
160.000     GO TO 20
161.000 22 CONTINUE
162.000     XN=NSEG
163.000     SEG=XL/XN
164.000     X=-SEG
165.000     WRITE(6,333)
166.000     WRITE(6,802) PP
167.000     WRITE(6,803)
168.000     WRITE(6,999)
169.000     WRITE(6,805) IL,IL,IF,IL,IF
170.000     DO 777 IK=1,ND
171.000     X=X+SEG
172.000     WRITE(6,806) X,DISP(IK),XM(IK),SH(IK)
173.000 777 CONTINUE
174.000 802 FORMAT(//////////19X,'THE BEAM HAS LAMBDA-L=',F6.2//)
175.000 803 FORMAT(10X,'DISTANCE'3X,'DISPLACEMENT',5X,'B. MOMENT'
176.000     +3X,'SHEAR FORCE')
177.000 805 FORMAT(11X,'(',A3,')',8X,'(',A3,')',9X,'(',A3,'-',
178.000     +A3,')',6X,'(',A3,')'//)
179.000 806 FORMAT(6X,E12.6,3X,E12.6,2X,E12.6,2X,E12.6/)
180.000 999 FORMAT(10X,8(1H-),3X,12(1H-),5X,9(1H-),3X,11(1H-))
181.000 333 FORMAT(1H1,4X,30(1X,1H*)//
182.000     16X,1H*,57X,1H*//6X,1H*,3X,51(1H=),3X,1H*//
183.000     26X,1H*,15X,'AN ANALYSIS OF A FINITE BEAM',12X,
184.000     31H*//6X,1H*,16X,'RESTING ON A WINKLER',
185.000     4' MEDIUM',13X,1H*//
186.000     56X,1H*,3X,51(1H=),3X,1H*//
187.000     66X,1H*,22X,'PROGRAMMED BY',21X,1H*//
188.000     76X,1H*,19X,'A.K.M. GHULAM SARWAR',18X,1H*//
189.000     86X,1H*,19X,'THESIS SUPERVISORS',19X,1H*//
190.000     86X,1H*10X,'PROF. J.L.HUMAR'6X,'PROF. A.P.S. SELVADURAI',3
191.000     X,1H*//
191.000     96X,1H*,18X,'CARLETON UNIVERSITY',19X,1H*//
192.000     36X,1H*,22X,'OTTAWA, CANADA',21X,1H*//

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193.000      16X,1H*,23X,'AUGUST, 1979',22X,1H*//
194.000      36X,1H*,57X,1H*//5X,30(1X,1H*)
195.000      700 FORMAT(/1X,'IN  WHAT UNITS DO YOU WANT TO INPUT?'//
196.000      +1X,'      (IMPERIAL OR METRIC)? I/M'//)
197.000 C
198.000      699 FORMAT(/1X,'USE CONSISTENT UNITS.'//)
199.000 C
200.000      701 FORMAT(/1X,'INPUT THE UNIT OF FORCE- MAX. THREE CHARACTERS'//
201.000      +1X,'FOR  EXAMPLE: TON,KIP,LBS'//)
202.000 C
203.000      702 FORMAT(/1X,'INPUT THE UNIT OF LENGTH-MAX. THREE CHARACTERS'//
204.000      +1X,'FOR  EXAMPLE: FT,IN'//)
205.000 C
206.000      703 FORMAT(/1X,'INPUT THE UNIT OF FORCE- MAX. THREE CHARACTERS'//
207.000      +1X,'FOR  EXAMPLE :KG,KN,N'//)
208.000 C
209.000      704 FORMAT(/1X,'INPUT THE UNIT OF LENGTH-MAX. THREE CHARACTERS'//
210.000      +1X,'FOR  EXAMPLE: M,CM,MM'//)
211.000 C
212.000      706 FORMAT(/1X,'INPUT THE LENGTH OF BEAM'//)
213.000 C
214.000      708 FORMAT(/1X,'SUPPLY THE WIDTH OF THE BEAM'//)
215.000 C
216.000      710 FORMAT(/1X,'DO YOU WANT TO SUPPLY THE BEAM-DEPTH OR '
217.000      +1X,'MOMENT OF INERTIA? D/I'//)
218.000 C
219.000      712 FORMAT(/1X,'DEPTH  OF THE BEAM?'//)
220.000 C
221.000      714 FORMAT(/1X,'INPUT THE UNIT-WT.  OF THE BEAM MATERIAL'//)
222.000 C
223.000      716 FORMAT(/1X,'MOMENT OF INERTIA OF THE BEAM?'//)
224.000 C
225.000      718 FORMAT(/1X,'INPUT SELF-WEIGHT OF THE BEAM PER UNIT LENGTH'//)
226.000 C
227.000      719 FORMAT(/1X,'INPUT THE MODULUS OF ELASTICITY OF THE BEAM'//)
228.000 C
229.000      720 FORMAT(/1X,'DO YOU WANT A LIST OF STANDARD VALUES FOR'//
230.000      +1X,'THE MODULUS OF SUBGRADE REACTION?. Y/N'//)
231.000 C
232.000      734 FORMAT(/1X,'INPUT THE MODULUS OF SUBGRADE REACTION FOR THE,'
233.000      +1X,'SOIL'//)
234.000 C
235.000      736 FORMAT(/1X,'INPUT THE NUMBER OF BEAM SEGMENTS FOR ANALYSIS'//)
236.000 C
237.000      738 FORMAT(/1X,'DO YOU WANT TO ANALYSE FOR CONCENTRATED LOADS. Y/N?'//)
238.000 C
239.000      740 FORMAT(/1X,'FOR EACH LOAD SUPPLY THE DISTANCE FROM LEFT AND'//
240.000      +1X,'THE MAGNITUDE SEPERATED BY A BLANK.'//)

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241.000      *+1X,'HIT CARRIAGE RETURN WHEN ALL CONC. LOAD DATA SUPPLIED'//)
242.000 C
243.000      742 FORMAT(/1X,'DO YOU WANT TO ANALYSE FOR CONC. MOMENTS? Y/N'//)
244.000 C
245.000      744 FORMAT(/1X,'FOR EACH MOMENT, SUPPLY THE DISTANCE FROM LEFT END'//
246.000      +1X,'OF THE BEAM AND MAGNITUDE SEPERATED BY A BLANK.'//
247.000      +1X,'HIT CARRIAGE RETURN WHEN ALL MOMENT DATA SUPPLIED.'//)
248.000 C
249.000      746 FORMAT(/1X,'DO YOU WANT TO ANALYSE FOR DISTRIBUTED LOADS? Y/N'//)
250.000 C
251.000      748 FORMAT(/1X,'FOR EACH DISTRIBUTED LOAD, SUPPLY THE DISTANCE'//
252.000      +1X,'TO THE LEFT,DISTANCE TO THE RIGHT AND THE MAGNITUDE'//
253.000      +1X,'SEPERATED BY BLANKS.'//)
254.000 C
255.000 C
256.000      722 FORMAT(//
257.000      +' **TABLE FOR K VALUE OF SOIL AFTER MINER AND SEASTONE (1955)',
258.000      +'**'/
259.000      +20X,'(KN PER CUBIC METRE*1000)'/
260.000      +2X,61(1H-)//
261.000      +1X,'MAJOR DIVISIONS',6X,'SOIL GROUPS AND TYPICAL DESCRIPTION',
262.000      +4X,'RANGE'/
263.000      +1X,65(1H-)/
264.000      +22X,'WELL-GRADED GRAVEL AND GRAVEL-SAND'/
265.000      +22X,'MIXTURES-LITTLE OR NO FINES',10X,'135-190'/
266.000      +22X,44(1H-)/
267.000      +22X,'WELL-GRADED GRAVEL-SAND-CLAY'/
268.000      +22X,44(1H-)/
269.000      +1X,'GRAVEL AND',11X,'MIXTURES-EXCELLENT BINDER',12X,'110-190'/
270.000      +1X,'GRAVELLY SOILS',7X,'POORLY GRADED GRAVEL AND GRAVEL SAND',/
271.000      +22X,'MIXTURES-LITTLE OR NO FINES',11X,'80-135'/
272.000      +22X,44(1H-)/
273.000      +22X,'GRAVEL WITH FINES,VERY SILTY GRAVEL,'/
274.000      +22X,'CLAYEY GRAVEL,POORLY GRADED GRAVEL-')
275.000      724 FORMAT(/
276.000      +22X,'SAND-CLAY MIXTURES',19X,'70-135'/
277.000      +1X,66(1H-)/
278.000      +22X,'WELL-GRADED SANDS AND GRAVELLY SAND-'/
279.000      +22X,'LITTLE OR NO FINES',20X,'70-155'/
280.000      +22X,44(1H-)/
281.000      +22X,'WELL GRADED SAND-CLAY MIXTURES-'/
282.000      +1X,'SANDS AND',12X,'EXCELLENT BINDER',22X,'70-155'/
283.000      +22X,44(1H-)/
284.000      +1X,'SANDY SOILS',10X,'POORLY GRADED SANDS- LITTLE OR NO'/

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285.000 +22X, 'FINES', 34X, '55-90' /
 286.000 +22X, 44(1H-) /
 287.000 +22X, 'SANDS WITH FINES, VERY SILTY SANDS, ' /
 288.000 +22X, 'CLAYEY SANDS, POORLY GRADED SAND' /
 289.000 +22X, 'AND CLAY MIXTURES', 22X, '50-90' /
 290.000 +1X, 66(1H-) /
 291.000 +22X, 'INORGANIC SILTS AND VERY FINE SANDS, ' /
 292.000 +22X, 'ROCK FLOUR, SILTY OR CLAYEY FINE' /
 293.000 +1X, 'FINE GRAINED', 9X, 'SANDS WITH SLIGHT PLASTICITY', 11X, '40-80')
 294.000 726 FORMAT(/1X, 'SOILS HAVING', 9X, 44(1H-) /
 295.000 +1X, 'LOW TO MEDIUM', 8X, 'INORGANIC CLAYS OF LOW TO MEDIUM', /
 296.000 +1X, 'COMPRESSIBILITY', 6X, 'PLASTICITY- SANDY, SILTY OR'
 297.000 + 'LEAN CLAYS', 4X, '30-60' /
 298.000 +22X, 44(1H-) /
 299.000 +22X, 'ORGANIC SILTS AND SILT-CLAYS OF LOW' /
 300.000 +22X, 'PLASTICITY', 29X, '25-50' /
 301.000 +1X, 66(1H-) /
 302.000 +22X, 'MICACEOUS OR DIATOMACEOUS FINE SANDY' /
 303.000 +22X, 'OR SILTY SOILS, ELASTIC SILTS', 11X, '10-50' /
 304.000 +22X, 44(1H-) /
 305.000 +1X, 'FINE GRAINED', /
 306.000 +1X, 'SOILS HAVING', 9X, 'INORGANIC CLAYS OF HIGH PLASTICITY, ' /
 307.000 +1X, 'MEDIUM TO HIGH', 7X, 'FAT CLAYS', 30X, '10-40' /
 308.000 +1X, 'COMPRESSIBILITY', 6X, 44(1H-) /
 309.000 +22X, 'ORGANIC CLAYS OF MEDIUM TO HIGH' /
 310.000 +22X, 'PLASTICITY', 29X, '10-30' /
 311.000 +1X, 66(1H-))
 312.000 728 FORMAT(/
 313.000 + ' **TABLE FOR K VALUE OF SOIL AFTER MINER AND SEASTONE (1955)',
 314.000 + '***' /
 315.000 +22X, ' (LBS PER CUBIC INCH)' /
 316.000 +2X, 61(1H-) /
 317.000 +1X, 'MAJOR DIVISIONS', 6X, 'SOIL GROUPS AND TYPICAL DESCRIPTION',
 318.000 +4X, 'RANGE' /
 319.000 +1X, 65(1H-) /
 320.000 +22X, 'WELL-GRADED GRAVEL AND GRAVEL-SAND' /
 321.000 +22X, 'MIXTURES-LITTLE OR NO FINES', 10X, '500-700' /
 322.000 +22X, 44(1H-) /
 323.000 +22X, 'WELL-GRADED GRAVEL-SAND-CLAY 400-700' /
 324.000 +22X, 44(1H-) /
 325.000 +1X, 'GRAVEL AND', 11X, 'MIXTURES-EXCELLENT BINDER' /
 326.000 +1X, 'GRAVELLY SOILS', 7X, 'POORLY GRADED GRAVEL AND GRAVEL SAND', /
 327.000 +22X, 'MIXTURES-LITTLE OR NO FINES', 10X, '295-500' /
 328.000 +22X, 44(1H-) /
 329.000 +22X, 'GRAVEL WITH FINES, VERY SILTY GRAVEL, ' /
 330.000 +22X, 'CLAYEY GRAVEL, POORLY GRADED GRAVEL-')
 331.000 730 FORMAT(/
 332.000 +22X, 'SAND-CLAY MIXTURES', 19X, '257-500' /


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333.000 +1X,65(1H-)/
334.000 +22X,'WELL-GRADED SANDS AND GRAVELLY SAND-'/
335.000 +22X,'LITTLE OR NO FINES',19X,'257-570'/
336.000 +22X,44(1H-)/
337.000 +22X,'WELL GRADED SAND-CLAY MIXTURES-'/
338.000 +1X,'SANDS AND',12X,'EXCELLENT BINDER',21X,'257-570'/
339.000 +22X,44(1H-)/
340.000 +1X,'SANDY SOILS',10X,'POORLY GRADED SANDS- LITTLE OR NO'/
341.000 +22X,'FINES',32X,'200-330'/
342.000 +22X,44(1H-)/
343.000 +22X,'SANDS WITH FINES,VERY SILTY SANDS, '/
344.000 +22X,'CLAYEY SANDS,POORLY GRADED SAND' /
345.000 +22X,'AND CLAY MIXTURES',20X,'185-330' /
346.000 +1X,65(1H-)/
347.000 +22X,'INORGANIC SILTS AND VERY FINE SANDS, /
348.000 +22X,'ROCK FLOUR,SILTY OR CLAYEY FINE' /
349.000 +1X,'FINE GRAINED',9X,'SANDS WITH SLIGHT PLASTICITY',9X,'150-295')
732 FORMAT(/1X,'SOILS HAVING',9X,44(1H-)/
351.000 +1X,'LOW TO MEDIUM',8X,'INORGANIC CLAYS OF LOW TO MEDIUM',/
352.000 +1X,'COMPRESSIBILITY',6X,'PLASTICITY- SANDY,SILTY OR'
353.000 +'LEAN CLAYS',1X,'110-220'/
354.000 +22X,44(1H-)/
355.000 +22X,'ORGANIC SILTS AND SILT-CLAYS OF LOW'/
356.000 +22X,'PLASTICITY',28X,'92-185'/
357.000 +1X,65(1H-)/
358.000 +22X,'MICACEOUS OR DIATOMACEOUS FINE SANDY'/
359.000 +22X,'OR SILTY SOILS,ELASTIC SILTS',10X,'35-185'/
360.000 +22X,44(1H-)/
361.000 +1X,'FINE GRAINED',/
362.000 +1X,'SOILS HAVING',9X,'INORGANIC CLAYS OF HIGH PLASTICITY, '/
363.000 +1X,'MEDIUM TO HIGH',7X,'FAT CLAYS',29X,'35-150'/
364.000 +1X,'COMPRESSIBILITY',6X,44(1H-)/
365.000 +22X,'ORGANIC CLAYS OF MEDIUM TO HIGH' /
366.000 +22X,'PLASTICITY',28X,'35-110' /
367.000 +1X,65(1H-))
368.000 991 FORMAT(A3)
369.000 END
370.000 SUBROUTINE CONCL(PP,F2,F3,F4,D,NDIV,XLAM,XK,P,A,DISP,XM,SH)
371.000 DIMENSION DISP(50),XM(50),SH(50)
372.000 B=1.-A
373.000 PPB=PP*B
374.000 CHB=COSH(PPB)
375.000 SHB=SINH(PPB)
376.000 SB=SIN(PPB)
377.000 CB=COS(PPB)
378.000 F1B=CHB*CB
379.000 F2B=.5*(CHB*SB+SHB*CB)
380.000 C1=(F2B*F3-F1B*F4)/D

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381.000 C2=(F1B*F3-F2B*F2)/D
382.000 DO 86 I=1,NDIV
383.000 X=(I-1.)/(NDIV-1.)
384.000 PPX=PP*X
385.000 CHX=COSH(PPX)
386.000 SHX=SINH(PPX)
387.000 CX=COS(PPX)
388.000 SX=SIN(PPX)
389.000 F1X=CHX*CX
390.000 F2X=.5*(CHX*SX+SHX*CX)
391.000 F3X=.5*SHX*SX
392.000 F4X=.25*(CHX*SX-SHX*CX)
393.000 DISP(I)=DISP(I)+(C1*F1X+C2*F2X)*P*XLAM/XK
394.000 XM(I)=XM(I)+(C1*F3X+C2*F4X)*P/XLAM
395.000 SH(I)=SH(I)+(C1*F2X+C2*F3X)*P
396.000 IF(X.LE.A) GO TO 86
397.000 XA=X-A
398.000 PPXA=PP*XA
399.000 CHXA=COSH(PPXA)
400.000 SHXA=SINH(PPXA)
401.000 CXA=COS(PPXA)
402.000 SXA=SIN(PPXA)
403.000 F1XA=CHXA*CXA
404.000 F2XA=.5*(CHXA*SXA+SHXA*CXA)
405.000 F4XA=.25*(CHXA*SXA-SHXA*CXA)
406.000 DISP(I)=DISP(I)+4.*F4XA*P*XLAM/XK
407.000 XM(I)=XM(I)-F2XA*P/XLAM
408.000 SH(I)=SH(I)-F1XA*P
409.000 86 CONTINUE
410.000 RETURN
411.000 END
412.000 SUBROUTINE COUPLE(PP,F2,F3,F4,D,NDIV,XLAM,XK,CM,A,DISP,XM,SH)
413.000 DIMENSION DISP(50),XM(50),SH(50)
414.000 B=I.-A
415.000 PPB=PP*B
416.000 CHB=COSH(PPB)
417.000 SHB=SINH(PPB)
418.000 CB=COS(PPB)
419.000 SB=SIN(PPB)
420.000 F1B=CHB*CB
421.000 F4B=.25*(CHB*SB-SHB*CB)
422.000 C1=(F1B*F3+4.*F4B*F4)/D
423.000 C2=(F1B*F2+4.*F4B*F3)/D
424.000 DO 88 I=1,NDIV
425.000 X=(I-1.)/(NDIV-1.)
426.000 PPX=PP*X
427.000 CHX=COSH(PPX)
428.000 SHX=SINH(PPX)

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429.000 SX=SIN (PPX)
430.000 CX=COS (PPX)
431.000 F1X=CHX*CX
432.000 F2X=.5*(CHX*SX+SHX*CX)
433.000 F3X=.5*SHX*SX
434.000 F4X=.25*(CHX*SX-SHX*CX)
435.000 DISP(I)=DISP(I)-(C1*F1X-C2*F2X)*CM*XLAM*XLAM/XK
436.000 XM(I)=XM(I)-(C1*F3X-C2*F4X)*CM
437.000 SH(I)=SH(I)-(C1*F2X-C2*F3X)*XLAM*CM
438.000 IF(X.LE.A) GO TO 88
439.000 XA=X-A
440.000 PPXA=PP*XA
441.000 CHXA=COSH (PPXA)
442.000 SHXA=SINH (PPXA)
443.000 CXA=COS (PPXA)
444.000 SXA=SIN (PPXA)
445.000 F1XA=CHXA*CXA
446.000 F3XA=.5*SHXA*SXA
447.000 F4XA=.25*(CHXA*SXA-SHXA*CXA)
448.000 DISP(I)=DISP(I)-4.*F3XA*CM*XLAM*XLAM/XK
449.000 XM(I)=XM(I)+F1XA*CM
450.000 SH(I)=SH(I)-4.*F4XA*XLAM*CM
451.000 88 CONTINUE
452.000 RETURN
453.000 END
454.000 SUBROUTINE UDL(PP,F2,F3,F4,D,NDIV,XLAM,XK,Q,A,B,DISP,XM,SH)
455.000 DIMENSION DISP(50),XM(50),SH(50)
456.000 PPA=PP*(1.-A)
457.000 CHA=COSH (PPA)
458.000 SHA=SINH (PPA)
459.000 CA=COS (PPA)
460.000 SA=SIN (PPA)
461.000 PPB=PP*(1.-B)
462.000 CHB=COSH (PPB)
463.000 SHB=SINH (PPB)
464.000 CB=COS (PPB)
465.000 SB=SIN (PPB)
466.000 F2A=.5*(CHA*SA+SHA*CA)
467.000 F2B=.5*(CHB*SB+SHB*CB)
468.000 F3A=.5*SHA*SA
469.000 F3B=.5*SHB*SB
470.000 C1=((F3A-F3B)*F3-(F2A-F2B)*F4)/D
471.000 C2=((F2A-F2B)*F3-(F3A-F3B)*F2)/D
472.000 DO 9 I=1,NDIV
473.000 X=(I-1.)/(NDIV-1.)
474.000 XA=X-A
475.000 XB=X-B
476.000 PPX=PP*X

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477.000 CHX=COSH(PPX)
478.000 SHX=SINH(PPX)
479.000 SX=SIN(PPX)
480.000 CX=COS(PPX)
481.000 PPA=PP*XA
482.000 PPB=PP*XB
483.000 CHXA=COSH(PPA)
484.000 SHXA=SINH(PPA)
485.000 CXA=COS(PPA)
486.000 SXA=SIN(PPA)
487.000 CHXB=COSH(PPB)
488.000 SHXB=SINH(PPB)
489.000 CXB=COS(PPB)
490.000 SXB=SIN(PPB)
491.000 FLXA=CHXA*CXA
492.000 FLXB=CHXB*CXB
493.000 F2XA=.5*(CHXA*SXA+SHXA*CXA)
494.000 F2XB=.5*(CHXB*SXB+SHXB*CXB)
495.000 F3XA=.5*SHXA*SXA
496.000 F3XB=.5*SHXB*SXB
497.000 FLX=CHX*CX
498.000 F2X=.5*(CHX*SX+SHX*CX)
499.000 F3X=.5*SHX*SX
500.000 F4X=.25*(CHX*SX-SHX*CX)
501.000 DISP(I)=DISP(I)+(C1*FLX+C2*F2X)*Q/XR
502.000 XM(I)=XM(I)+(C1*F3X+C2*F4X)*Q/(XLAM*XLAM)
503.000 SH(I)=SH(I)+(C1*F2X+C2*F3X)*Q/XLAM
504.000 IF(X.LT.A) GO TO 9
505.000 DISP(I)=DISP(I)+Q/XR-FLXA*Q/XR
506.000 XM(I)=XM(I)-F3XA*Q/(XLAM*XLAM)
507.000 SH(I)=SH(I)-F2XA*Q/XLAM
508.000 IF(X.LT.B) GO TO 9
509.000 DISP(I)=DISP(I)-Q/XR+FLXB*Q/XR
510.000 XM(I)=XM(I)+F3XB*Q/(XLAM*XLAM)
511.000 SH(I)=SH(I)+F2XB*Q/XLAM
512.000 9 CONTINUE
513.000 RETURN
514.000 END

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FLAG EXBEAM
FLAG VERSION D04
WITH>AD,M6

IN WHAT UNITS DO YOU WANT TO INPUT?

(IMPERIAL OR METRIC)? I/M

?I

USE CONSISTENT UNITS.

INPUT THE UNIT OF FORCE- MAX. THREE CHARACTERS

FOR EXAMPLE: TON,KIP,LBS

?LB

INPUT THE UNIT OF LENGTH-MAX. THREE CHARACTERS

FOR EXAMPLE: FT,IN

?IN

INPUT THE LENGTH OF BEAM

?120

SUPPLY THE WIDTH OF THE BEAM

?10

DO YOU WANT TO SUPPLY THE BEAM-DEPTH OR MOMENT OF INERTIA? D/I

?I

MOMENT OF INERTIA OF THE BEAM?

?426.7

INPUT SELF-WEIGHT OF THE BEAM PER UNIT LENGTH

?0.0

INPUT THE MODULUS OF ELASTICITY OF THE BEAM

?1500000

DO YOU WANT A LIST OF STANDARD VALUES FOR

THE MODULUS OF SUBGRADE REACTION?. Y/N

?Y

TABLE FOR K VALUE OF SOIL AFTER MINER AND SEASTONE (1955)
(LBS PER CUBIC INCH)

MAJOR DIVISIONS	SOIL GROUPS AND TYPICAL DESCRIPTION	RANGE
GRAVEL AND GRAVELLY SOILS	WELL-GRADED GRAVEL AND GRAVEL-SAND MIXTURES-LITTLE OR NO FINES	500-700
	WELL-GRADED GRAVEL-SAND-CLAY	400-700
	MIXTURES-EXCELLENT BINDER POORLY GRADED GRAVEL AND GRAVEL SAND MIXTURES-LITTLE OR NO FINES	295-500
	GRAVEL WITH FINES, VERY SILTY GRAVEL, CLAYEY GRAVEL, POORLY GRADED GRAVEL- SAND-CLAY MIXTURES	257-500
SANDS AND SANDY SOILS	WELL-GRADED SANDS AND GRAVELLY SAND- LITTLE OR NO FINES	257-570
	WELL GRADED SAND-CLAY MIXTURES- EXCELLENT BINDER	257-570
	POORLY GRADED SANDS- LITTLE OR NO FINES	200-330
	SANDS WITH FINES, VERY SILTY SANDS, CLAYEY SANDS, POORLY GRADED SAND AND CLAY MIXTURES	185-330
FINE GRAINED	INORGANIC SILTS AND VERY FINE SANDS, ROCK FLOUR, SILTY OR CLAYEY FINE SANDS WITH SLIGHT PLASTICITY	150-295
SOILS HAVING LOW TO MEDIUM COMPRESSIBILITY	INORGANIC CLAYS OF LOW TO MEDIUM PLASTICITY- SANDY, SILTY OR LEAN CLAYS	110-220
	ORGANIC SILTS AND SILT-CLAYS OF LOW PLASTICITY	92-185
	MICACEOUS OR DIATOMACEOUS FINE SANDY OR SILTY SOILS, ELASTIC SILTS	35-185
FINE GRAINED SOILS HAVING MEDIUM TO HIGH COMPRESSIBILITY	INORGANIC CLAYS OF HIGH PLASTICITY, FAT CLAYS	35-150
	ORGANIC CLAYS OF MEDIUM TO HIGH PLASTICITY	35-110

INPUT THE MODULUS OF SUBGRADE REACTION FOR THE, SOIL

INPUT THE NUMBER OF BEAM SEGMENTS FOR ANALYSIS

?10

DO YOU WANT TO ANALYSE FOR CONCENTRATED LOADS. Y/N?

?Y

FOR EACH LOAD SUPPLY THE DISTANCE FROM LEFT AND

THE MAGNITUDE SEPERATED BY A BLANK.

HIT CARRIAGE RETURN WHEN ALL CONC. LOAD DATA SUPPLIED

?30 5000

?

DO YOU WANT TO ANALYSE FOR CONC. MOMENTS? Y/N

?N

DO YOU WANT TO ANALYSE FOR DISTRIBUTED LOADS? Y/N

?Y

FOR EACH DISTRIBUTED LOAD, SUPPLY THE DISTANCE

TO THE LEFT, DISTANCE TO THE RIGHT AND THE MAGNITUDE

SEPERATED BY BLANKS.

?52 100 100

?

THE BEAM HAS LAMBDA-L= 3.57

DISTANCE	DISPLACEMENT	B. MOMENT	SHEAR FORCE
-----	-----	-----	-----
(IN)	(IN)	(LB -IN)	(LB)
.000000E+00	.303469E-01	.000000E+00	.000000E+00
.120000E+02	.412103E-01	.489479E+04	.859324E+03
.240000E+02	.507500E-01	.216296E+05	.196822E+04
.360000E+02	.554316E-01	.228620E+05	-.174407E+04
.480000E+02	.549525E-01	.993334E+04	-.412805E+03
.600000E+02	.519690E-01	.957469E+04	.744314E+02
.720000E+02	.468193E-01	.105313E+05	.644191E+02
.840000E+02	.393180E-01	.105152E+05	-.971116E+02
.960000E+02	.295090E-01	.736640E+04	-.466966E+03
.108000E+03	.181269E-01	.147738E+04	-.293493E+03
.120000E+03	.631635E-02	-.100044E-10	.909495E-12

STOP

END

31-12-80

FIN

2

6