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PERTURBATION THEORY IN NORMED RINGS

by

KRISHAN L. BHATLA

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH IN PARTIAL
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ABSTRACT

Suppose A is a Banach semi-prime algebra with identity 1 in which a non-trivial socle S is defined. Barnes [3] defined an element t of A to be Fredholm if t is invertible modulo S . In the first chapter, we prove results on perturbation of a Fredholm element analogous to those for a Fredholm operator. Suppose t is Fredholm and if $\|u\|$ is small or u is an element of a two-sided ideal I with $S \subseteq I \subseteq k(h(S))$, then $t + u$ is Fredholm. Also then index of $t + u$ is the same as that of t . Similar results for semi-Fredholm elements - the elements which are left (right) invertible modulo the socle - are obtained.

In the second chapter, we obtain a condition which subsumes the classical conditions mentioned above for $t - u$ to be Fredholm when t is Fredholm. If t is Fredholm having a generalized inverse s and if $1 - us$ (or equivalently $1 - su$) is a Fredholm element, then $t - u$ is Fredholm with index of $t - u$ equal to index of t . This implies that the set F of Fredholm elements of A is an open semi-group and index is a continuous function on F .

The Fredholm region $\Phi(t)$ of an element t , consisting of all complex numbers λ for which $\lambda - t$ is Fredholm, is an open set and the mapping $\lambda \rightarrow i(\lambda - t)$ is constant on each connected component of $\Phi(t)$. This and some properties of Fredholm spectrum $\sigma_\Phi(t) = \mathbb{C} - \Phi(t)$ of t , analogous to those of spectrum of t , are obtained.

If t is an element with a generalized inverse s , we obtain conditions so that $t - u$ has a generalized inverse. When t is a Fredholm element, the conditions become simply the requirement that $1 - us$ (or equivalently $1 - su$) is a Fredholm element.

We call a g -invertible element t to be Saphar type if the right annihilator of At is contained in $t^n A$ for all n and obtain conditions for $t - u$ to be Saphar type when t is Saphar type. If u commutes with t , the conditions reduce to the requirement that $\|u\|$ is small.

We prove some consequences of this. Further, if V is an open set in complex plane and $\lambda - t$ has a holomorphic generalized inverse on V , then $\lambda - t$ is Saphar type for each λ in V . $S(n, p; t)$, consisting of complex numbers λ for which $\lambda - t$ is Saphar type with $\alpha(\lambda - t) = n$ and $\beta(\lambda - t) = p$, is an open region in complex plane.

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NOTATIONS AND TERMINOLOGY

The following notations and terminology will be used.

$B(X)$	The set of all bounded linear operators on X .
$F(X)$	The set of all bounded operators on X which have finite rank.
$K(X)$	The set of all compact operators on X .
$N(T)$	Null space of T .
$R(T)$	Range of T .
$\theta(I)$	Order of I .
$R(B)$	Right annihilator of B .
$L(B)$	Left annihilator of B .
$\alpha(t)$	Order of $R(At)$.
$\beta(t)$	Order of $L(tA)$.
$i(t)$	Index of t .
F	The set of Fredholm elements of A .
Φ_l	The set of elements t of A such that t is g -invertible with $\alpha(t)$ finite.
Φ_r	The set of elements t of A such that t is g -invertible with $\beta(t)$ finite.
$\Phi(t)$	Fredholm region of t .
$\sigma_\Phi(t)$	Fredholm spectrum of t .
$\Phi^B(t)$	Fredholm region of t considered as an element of B .
$\sigma^B_\Phi(t)$	Fredholm spectrum of t considered as an element of B .
$\{t\}''$	Double commutant of $\{t\}$.

- $P(t)$ The set of all u in $\{t\}^*$ such that $t-u$ is Saphar type.
 $S(t)$ Saphar region of t .
 $S(n,p;t)$ $\{\lambda \in \mathbb{C} : \lambda - t \text{ is Saphar type with } \alpha(\lambda - t) = n, \beta(\lambda - t) = p\}$.
 $k(h(S))$ The intersection of all primitive ideals containing S .

INTRODUCTION

The aim of this dissertation is to study perturbation theory in normed rings. Various authors have attempted to study generalized Fredholm theory. Among others, Barnes [3], starting with a semi-prime ring A with identity in which a non-zero socle is defined, defined a Fredholm element as one that is invertible modulo the socle. In the same paper, he proved some basic results analogous to those of Fredholm operators. Earlier, in [2], he proved some basic properties of the order of an ideal, a concept that replaces the dimension of a vector space. In [3], he defined the index of a Fredholm element t of A as the difference in the orders of left annihilator of tA and right annihilator of At and proved the logarithmic property $i(uv) = i(u) + i(v)$ for Fredholm elements u, v .

Throughout this study, the concept of a generalized inverse plays a crucial role. Some basic properties of a generalized inverse can be found in [11]. In the first chapter, we prove results on perturbation of a Fredholm element analogous to the classical theorems on perturbation of a Fredholm operator by an operator of small norm or by a compact operator. If t is a Fredholm element and $\|u\|$ is small, then $t + u$ is Fredholm. Moreover, $i(t+u) = i(t)$, $\alpha(t+u) \leq \alpha(t)$ and $\beta(t+u) \leq \beta(t)$. Also, if t is Fredholm and u is in a two-sided ideal I , where $S \subseteq I \subseteq k(h(S))$, then $t + u$ is Fredholm and $i(t+u) = i(t)$.

We define a set of semi-Fredholm elements $\phi_l [\phi_r]$ as consisting of those elements t of A that are g -invertible with $\alpha(t) [\beta(t)]$ finite. These elements are characterized as those which are left (right) invertible modulo the socle. Then we prove perturbation results for these semi-Fredholm elements similar to those for Fredholm elements.

We begin the second chapter with a result that the set F of Fredholm elements of a ring A with involution is closed for involution. In the second section, we prove some results on products of Fredholm elements and of elements of $\phi_l (\phi_r)$. If the product of two elements is in F , then either both are inside F or both are outside F . This and some more results have been proved. Now, the product of two g -invertible elements is not, in general, g -invertible. We study some conditions for the g -invertibility of the product.

Then, in the third section of the second chapter, we come back to the perturbation of a Fredholm element. If t is a Fredholm element having a g_2 -inverse s and if $1 - us [1 - su]$ is Fredholm, then $t - u$ is Fredholm and

$$i(t-u) = i(t) + i(1-us) [i(t-u) = i(t) + i(1-su)].$$

Now, if $\|u\| < \|s\|^{-1}$ or u is in I where $S \subseteq I \subseteq k(h(S))$, then $1 - us$ is Fredholm. Thus, this result contains the results on perturbation of a Fredholm element proved in the first chapter as special cases. Moreover, we deduce that F is an open semi-group and index is a continuous function on F . We define Fredholm region $\phi(t)$ of t to consist of all

complex λ such that $\lambda - t$ is Fredholm and prove that $\Phi(t)$ is an open subset of the complex plane. Also the mapping $\lambda \rightarrow i(\lambda - t)$ is locally constant and therefore is constant on each connected component of $\Phi(t)$. Moreover, $\Phi(t+b) = \Phi(t)$, where b is in I with $S \subseteq I \subseteq k(h(S))$.

We call the complement of $\Phi(t)$ in the complex plane the Fredholm spectrum of t and denote it by $\sigma_\Phi(t)$. The remaining portion of the section is devoted to a number of properties of Fredholm spectrum analogous to those of spectrum. In particular, an analogue of the spectral mapping theorem for polynomials and the continuity properties of the function $x \rightarrow \sigma_\Phi(x)$ have been discussed.

In the third chapter, we study perturbation theory and generalized inverses. In the first section, the main result gives some conditions for $t - u$ to be g -invertible when t is g -invertible. Suppose t is g -invertible and s is a g_1 -inverse of t . If $I - us$ is Fredholm having a g_1 -inverse v and $(1-ts)vu(1-st)$ is g -invertible, then $t - u$ is g -invertible. Even though this result seems very special, it contains some important results as special cases. In particular, it contains the results on perturbation of a Fredholm element and of an element of $\Phi_I(\Phi_I)$ given previously. In one corollary, we can actually construct a g_2 -inverse $s(1-us)^{-1} = (1-su)^{-1}s$ of $t - u$.

Now, P. Saphar [9,10] studied operators T which are g -invertible and satisfy the condition " $N(T) \subseteq R(T^n)$ for all n ". We show that this last condition is equivalent to the algebraic condition " $R[B(X)T] \subseteq T^n B(X)$ for all n " for any g -invertible operator T . Thus,

we call a g -invertible element t to be Saphar type if $R(At) \subseteq t^n A$ for all n .

We study conditions for $t - u$ to be Saphar type when t is Saphar type. We prove that if t is Saphar type and if $\|u\|$ is small along with $uI \subseteq I$, where $I = \bigcap_{n=1}^{\infty} t^n A$, then $t - u$ is Saphar type. With the conditions of this result, we see that $R[A(t-u)]$ is module isomorphic to $R(At)$ and $(t-u)A$ is module isomorphic to tA . We note that the condition $uI \subseteq I$ is satisfied by any u which commutes with t . Then, letting $P(t)$ denote the set of all u in the double commutant $\{t\}''$ of $\{t\}$ for which $t - u$ is Saphar type, we see that $P(t)$ is an open subset of $\{t\}''$. Moreover, the mappings $u \rightarrow$ isomorphism class of $R[A(t-u)]$, $u \rightarrow$ isomorphism class of $(t-u)A$ and $u \rightarrow \bigcap_{n=1}^{\infty} (t-u)^n A$ are locally constant on $P(t)$.

Although Saphar type elements seem to be very special, we have a property which shows their importance. We prove that if V is an open set in $\mathbb{C}$ and $\lambda - t$ has a holomorphic g_1 -inverse on V , then $\lambda - t$ is Saphar type for each λ in V . We also show that if A is a Banach algebra and t is in A such that $\bigcap_{n=1}^{\infty} t^n A$ is closed, then there exists a neighbourhood V of the origin in $\mathbb{C}$ such that

$$R[A(\lambda - t)] \subseteq I \subseteq \bigcap_{n=1}^{\infty} (\lambda - t)^n A \quad \text{for all } \lambda \text{ in } V - \{0\}.$$

If we consider the open set $\Phi(t) \cap S(t)$, we see that $\alpha(\lambda - t)$ and $\beta(\lambda - t)$ are constant locally and therefore on each connected component of $\Phi(t) \cap S(t)$. For any pair n, p of positive integers, we define

$S(n, p; t)$ to consist of all complex λ in $\Phi(t) \cap S(t)$ for which
 $\alpha(\lambda - t) = n$ and $\beta(\lambda - t) = p$. Then, $S(n, p; t)$ is an open set and the
 mapping $\lambda \rightarrow \bigcap_{n=1}^{\infty} (\lambda - t)^n A$ is locally constant on $S(n, p; t)$. If A is
 a B^* -algebra, then $S(p, n; t^*)$ equals the complex conjugate of
 $S(n, p; t)$.

CHAPTER 1

PERTURBATION OF A FREDHOLM ELEMENT - I

§1.1 PRELIMINARIES

Let A be a semi-prime ring, i.e., a ring having no nilpotent one-sided ideal. Also suppose A contains non-zero minimal one-sided ideals. Then by Schur's Lemma ([8], Lemma 2.1.5, p. 45), each minimal right (left) ideal in A is generated by an idempotent, called a minimal idempotent. Moreover if $I = eA$ is a minimal ideal, e being a minimal idempotent, then eAe is a division ring with identity e . The algebraic sum of right (left) minimal ideals of A is called the right (left) socle of A . Also, by ([8], Lemma 2.1.11, p. 46), if the right (left) socle of A exists, then it is a two-sided ideal. Moreover, in our case, since A is a semi-prime ring containing non-zero minimal one-sided ideals, by ([8], Lemma 2.1.12, pp. 46-47), the right and left socle exist and are equal. The resulting two-sided ideal is called the socle of A and we will denote it by S unless otherwise specified. For the above basic properties of minimal idempotents, see ([8], p. 45-47).

Now suppose A has identity 1 and let t be an element of A . Barnes [3] defined:

Definition 1.1.1: t is called right (left) Fredholm if t is right (left) invertible modulo the socle S . Also t is called a Fredholm element if t is both right Fredholm and left Fredholm, i.e., if t is

invertible modulo the socle S . Another definition by Barnes [2] is:

Definition 1.1.2: A right (left) ideal I of A has a finite order if and only if I can be written as a sum of a finite number of minimal right (left) ideals of A . The order of I , denoted by $\theta(I)$, is defined to be the smallest number of minimal right (left) ideals of A whose sum is I . We say that the zero ideal has finite order zero.

In case I is a two-sided ideal, by [2], Corollary to Theorem 2.2, the order of I considered as a right ideal is the same as the order of I considered as a left ideal so that there is no ambiguity about the definition of $\theta(I)$ in this case.

The following notation will be used for right (left) annihilator of a subset B of A .

$$R(B) = \{x \in A : Bx = 0\}$$

$$L(B) = \{x \in A : xB = 0\}$$

We will now state some results of Barnes [2,3] for future reference.

Proposition 1.1.3: Assume that I is a right ideal of finite order n .

If $e_1, e_2, \dots, e_m$ are minimal idempotents in A such that $e_1A + e_2A + \dots + e_mA \subseteq I$, and this sum is direct, then $m \leq n$. A similar statement holds for left ideals of finite order.

Proof: See [2], Theorem 2.1, p. 497.

Proposition 1.1.4: Suppose I is a non-zero right (left) ideal of finite order n . Then any maximal orthogonal set of minimal idempotents in I contains n elements and if $\{e_1, \dots, e_n\}$ is such a set, then $I = eA$ ($I = Ae$) where $e = e_1 + \dots + e_n$.

Proof: See [2], Theorem 2.2, p. 497.

Remark 1.1.5: If I is any right (left) ideal of finite order n and $M = \{e_1, \dots, e_n\}$ is a maximal orthogonal set of minimal idempotents in I , we will call M an orthogonal basis for I . Also note that if I is a right (left) ideal of finite order and J is right (left) ideal contained in I , then J has finite order and $\theta(J) \leq \theta(I)$.

Proposition 1.1.6: An element t of A is right Fredholm (left Fredholm) if and only if there exists an idempotent e in the socle S such that

$$tA = (1-e)A \quad (At = A(1-e))$$

Proof: See [3], Theorem 2.3, p. 87.

Now suppose X is a complex Banach space and let $B(X)$ denote the ring of bounded linear operators on X . Then we have:

Proposition 1.1.7: $B(X)$ is a semi-prime ring.

Proof: It is enough to show that $B(X)$ does not contain any non-zero one-sided ideals with square zero.

Suppose I is a non-zero right ideal in $B(X)$ such that $I^2 = 0$.

Choose a non-zero element T of I . Then $T^2 = 0$, since T^2 is

in I^2 . This implies $R(T) \subseteq N(T)$.

Let B be an invertible element of $B(X)$. Then TB is in I since I is a right ideal. Also, $R(TB) = R(T)$ and $N(TB) = B^{-1}N(T)$. Since $(TB)^2$ is in I^2 , we have $(TB)^2 = 0$. Consequently,

$$R(TB) \subseteq N(TB)$$

$$\text{i.e. } R(T) \subseteq B^{-1}N(T)$$

so that $B R(T) \subseteq N(T)$.

On the other hand, if I is a non-zero left ideal in $B(X)$ such that $I^2 = 0$, we choose a non-zero T in I and any invertible B in $B(X)$.

$$\text{Then, } BT \in I \text{ and } R(BT) = B R(T), N(BT) = N(T).$$

Also, $(BT)^2 = 0$ implies $R(BT) \subseteq N(BT)$, i.e. $BR(T) \subseteq N(T)$.

In both cases, we arrive at $BR(T) \subseteq N(T)$ for each invertible B in $B(X)$. But we can always choose an invertible B in $B(X)$ which does not map $R(T)$ into $N(T)$. For example; let $x_1 \in R(T)$ and $x_2 \in X - N(T)$. Let X_1, X_2 be one-dimensional subspaces spanned by x_1, x_2 respectively. Since $R(T) \subseteq N(T)$, X_1, X_2 are independent.

We can write $X = X_1 \oplus X_2 \oplus X_3$ for some closed subspace X_3 .

Define $B : X \rightarrow X$, the linear transformation which maps $x_1 \rightarrow x_2$, $x_2 \rightarrow x_1$ and is identity mapping on X_3 .

This proves the proposition.

We note that the family of all one-dimensional projections in $B(X)$ is precisely the family of minimal idempotents and the family $F(X)$ of all finite dimensional operators is the socle of $B(X)$. We note the following.

Proposition 1.1.8: Let T be an element of $B(X)$. Then

$$(i) \quad \theta(R[B(X)T]) = \dim. N(T)$$

$$(ii) \quad \theta(L[TB(X)]) = \text{co-dim. } R(T).$$

when the dimensions and orders in (i) and (ii) are finite.

Proof: Suppose $\theta(R[B(X)T]) = n$ and $\dim N(T) = p$. Using Proposition 1.1.4, we can write

$R[B(X)T] = PB(X)$ where $P = P_1 + \dots + P_n$, $\{P_1, \dots, P_n\}$ being a maximal orthogonal set of minimal idempotents in $R[B(X)T]$.

Then, since P is in $R[B(X)T]$,

$$TP = 0,$$

and therefore, $R(P) \subseteq N(T)$.

Consequently, $\dim. R(P) \leq \dim. N(T)$.

Since $P_1, \dots, P_n$ are orthogonal one-dimensional projections, $\dim. R(P) = n$.

Thus, $n \leq p$.

Conversely, suppose $X = N(T) \oplus X_0$ and $N(T) = X_1 \oplus X_2 \oplus \dots \oplus X_p$ where $X_1, X_2, \dots, X_p$ are linearly independent one-dimensional subspaces and X_0 is a closed subspace.

Let $P_1, P_2, \dots, P_p, P$ be the projections onto $X_1, X_2, \dots, X_p, X_0$ respectively. Then $P_1, P_2, \dots, P_p, P$ are orthogonal to each other. Since $R(P_1) \subseteq N(T), \dots, R(P_p) \subseteq N(T)$,

$$TP_1 = 0, \dots, TP_p = 0.$$

Then $P_1, \dots, P_p$ are in $R[B(X)T]$.

This implies $P_1 B(X) \oplus \dots \oplus P_p B(X) \subseteq R[B(X)T]$. Hence, by Proposition 1.1.3, $p \leq \theta(R[B(X)T]) = n$.

(ii) The proof is similar to that of (i).

Thus, for an element t of A , we set

$$\alpha(t) = \theta[R(At)]$$

$$\text{and } \beta(t) = \theta[L(tA)]$$

We give below an algebraic lemma which will be used later.

Lemma 1.1.9: If I is a minimal right (left) ideal in A , then for any element x of A , either xI (Ix) is minimal or $xI = 0$ ($Ix = 0$).

Proof: Let I be a minimal non-zero right ideal in A such that xI is non-zero.

Suppose $J \subseteq xI$ is any right ideal. Then, $H = \{i \in I : xi \in J\}$ is a right ideal contained in I .

Since I is minimal, either $H = I$ or $H = 0$.

In the first case, $J = xH = xI$ and in the second case $J = xH = 0$.

Hence, xI is minimal.

Corollary 1.1.10: Let t be an element of A such that $\theta(tA) < \infty$.

Then

(i) For any element s of A ,

$$\theta(stA) \leq \theta(tA).$$

(ii) For any invertible element s of A ,

$$\theta(stA) = \theta(tA).$$

Proof:

(i) Suppose $\theta(tA) = n$, then $tA = e_1 A + \dots + e_n A$ for some minimal idempotents $e_1, \dots, e_n$ in A . This implies $stA = se_1 A + \dots + se_n A$ where, by the Lemma, each $se_i A$ is either a minimal ideal or zero.

Hence, $\theta(stA) \leq n$.

(ii) If, in (i) above, s is an invertible element, then se_i is non-zero for each i because $se_i = 0$ implies $e_i = 0$ which now contradicts the hypothesis $\theta(tA) = n$. Hence, in this case,

$$\theta(stA) = n.$$

Now let t be a Fredholm element of A . Then there exists an element s such that $1 - st$ and $1 - ts$ belong to the socle S . Now, since $As \subseteq At$, we have

$$R(At) \subseteq R(As) \subseteq (1-st)A.$$

Therefore, since $1 - st$ is in S , $R(At)$ has finite order. Similarly, $L(tA)$ has finite order.

Setting $\alpha(t) = \theta[R(At)]$ and $\beta(t) = \theta[L(tA)]$, we define the index of t by $i(t) = \alpha(t) - \beta(t)$.

Actually, Barnes defined the index by

$$i(t) = \theta[L(tA)] - \theta[R(At)]$$

and proved:

Theorem 1.1.11: Suppose u, v are Fredholm elements in A . Then uv is Fredholm and

$$i(uv) = i(u) + i(v).$$

Proof: See [3], Theorem 3.2, p. 90.

Let $k(h(S))$, the kernel of the hull of S , be the intersection of the primitive ideals of A which contain S . When S is contained in no primitive ideal of A , we take $k(h(S)) = A$. We state the following result of Barnes.

Theorem 1.1.12: Assume that A is normed. Then if u is in $k(h(S))$, the order of $R[A(1-u)]$ is the same as that of $L[(1-u)A]$. If u is in S , the same conclusion holds without the hypothesis that A has a norm.

Proof: See [2], Theorem 3.5, p. 502.

All through this Thesis, we will assume that A is a semi-prime ring with identity 1 and in which a non-trivial socle is defined.

§1.2 BASIC PROPERTIES OF GENERALIZED INVERSES

In this section, we give some properties of a generalized invertible element of a ring. Some people call such an element 'regular'. For basic properties of 'regular' elements, see von-Neumann's work on "Regular Rings" ([11], p. 143).

Definition 1.2.1: Let t be an element of A . An element s of A is called a g_1 -inverse of t if $tst = t$. If such an element s exists, we say that t is g_1 -invertible. If there exists an element s of A such that $tst = t$ and $sts = s$, then s is called g_2 -inverse of t and we say that t is g_2 -invertible.

We note the following simple lemma.

Lemma 1.2.2: Suppose t is g_1 -invertible. If s is a g_1 -inverse of t , then sts is a g_2 -inverse of t and thus t is g_2 -invertible.

Proof: Simple verification.

In view of this Lemma, we call t g -invertible if t is either g_1 -invertible or g_2 -invertible and hence both.

The following simple lemma ([4], Lemma 1, p. 89) will be very useful throughout. We begin with the observation that for any idempotent e of A ,

$$eA = \{x \in A : x = ex\}.$$

Lemma 1.2.3: An element t of A is g -invertible if and only if $tA = eA$ ($At = Ae$) for some idempotent e in A .

Proof: Suppose t is g -invertible and that s is a g_1 -inverse of t .

Then, since $tst = t$, we have $tA = tsA$ and ts is an idempotent.

Conversely, suppose $tA = eA$ where e is an idempotent. Then since t is in eA , $t = et$. Also, since e belongs to tA , $e = ts$ for some s in A . Thus, $t = tst$ and therefore t is g -invertible.

Definition 1.2.4: Two right (left) ideals I, J in A are called complementary if $I \cap J = 0$ and $I \cup J = A$ where $I \cup J$ denotes the smallest ideal containing I and J .

We note the following lemma ([4], Lemma 2, p. 93) without proof:

Lemma 1.2.5: Two right (left) ideals I, J in A are complementary if and only if $I = eA$ and $J = (1-e)A$ ($I = Ae$ and $J = A(1-e)$) for some idempotent e in A .

The following algebraic lemma will be used later.

Lemma 1.2.6: Suppose I, J are two complementary right (left) ideals in A . Then any right (left) ideal J' complementary to I is module isomorphic to J .

Proof: Suppose I, J are complementary right ideals. Then, by Lemma 1.2.5,

$$I = eA \text{ and } J = (1-e)A$$

for some idempotent e in A .

Again, since J' is complementary to I ,

$$I = fA \text{ and } J' = (1-f)A$$

for some idempotent f in A .

Define

$$\phi : (1-e)A \rightarrow (1-f)A$$

by

$$(1-e)x \rightarrow (1-f)x.$$

Suppose

$$(1-f)x_1 = (1-f)x_2.$$

Then

$$x_1 - x_2 = f(x_1 - x_2).$$

This shows that $x_1 - x_2$ is in $fA = eA$.

Consequently, $x_1 - x_2 = e(x_1 - x_2)$.

Thus,

$$(1-e)x_1 = (1-e)x_2.$$

Hence, ϕ is one-one and it also shows, by interchanging the roles of e, f that ϕ is well defined.

Moreover,

$$\begin{aligned} \phi[(1-e)x_1 + (1-e)x_2] &= \phi[(1-e)(x_1 + x_2)] \\ &= (1-f)(x_1 + x_2) \\ &= (1-f)x_1 + (1-f)x_2 \\ &= \phi[(1-e)x_1] + \phi[(1-e)x_2] \end{aligned}$$

and, for any r in A ,

$$\begin{aligned} \phi[(1-e)xr] &= (1-f)xr \\ &= [(1-f)x]r \\ &= \phi[(1-e)x]r \end{aligned}$$

Hence, ϕ is a module isomorphism.

The following theorem describes some basic properties of a g -invertible element.

Theorem 1.2.7: Let t be a g -invertible element of A . If s is a g_2 -inverse of t , then

- (i) $tA = tsA$ where ts is an idempotent and therefore tA is complementary to $(1-ts)A = R(As)$.

Also, then any right ideal complementary to tA is module isomorphic to $(1-ts)A = R(As)$.

- (ii) $R(At) = (1-st)A$ where $1-st$ is an idempotent and therefore $R(At)$ is complementary to $stA = sA$.

Also, any right ideal complementary to $R(At)$ is module isomorphic to $stA = sA$.

- (iii) $At = Ast$ where st is idempotent and therefore At is complementary to $A(1-st) = L(sA)$.

Moreover, any left ideal complementary to At is module isomorphic to $A(1-st) = L(sA)$.

- (iv) $L(tA) = A(1-ts)$ where $1-ts$ is idempotent and therefore $L(tA)$ is complementary to $Ats = As$.

Also, any left ideal complementary to $L(tA)$ is module isomorphic to As .

Proof:

- (i) If s is a g_2 -inverse of t , then
 $t = tst$.

Consequently, $tA = tsA$ and also ts is an idempotent. Now, since $s = sts$, we have

$$s(1-ts) = 0.$$

This implies that $1 - ts$ is in $R(As)$ and therefore $(1-ts)A \subseteq R(As)$.

Also, if $sx = 0$, then

$$\begin{aligned} x &= x - tsx \\ &= (1-ts)x \end{aligned}$$

so that x is in $(1-ts)A$.

Thus, $R(As) \subseteq (1-ts)A$.

Hence, $(1-ts)A = R(As)$.

By Lemma 1.2.5, $tA = tsA$ is complementary to $(1-ts)A = R(As)$.

Finally, by Lemma 1.2.6, any right ideal complementary to tA is module isomorphic to $(1-ts)A = R(As)$.

(ii) If s is a g_2 -inverse of t , then

$$t = tst$$

$$\text{i.e. } t(1-st) = 0.$$

Therefore, $1 - st$ is in $R(At)$ and hence

$$(1-st)A \subseteq R(At).$$

Also, if $tx = 0$, then

$$\begin{aligned} x &= x - stx \\ &= (1-st)x \end{aligned}$$

so that x is in $(1-st)A$ and consequently $R(At) \subseteq (1-st)A$.

Thus, $R(At) = (1-st)A$.

Moreover, by Lemma 1.2.5, $(1-st)A$ is complementary to stA .

Also, since $s = sts$, it follows that $stA = sA$.

Finally, by Lemma 1.2.6, any right ideal complementary to $R(A) = (1-st)A$ is module isomorphic to sA .

(iii) and (iv): The proof is similar to that of (i) and (ii).

§1.3 PERTURBATION OF A FREDHOLM ELEMENT

In this section, we will give some perturbation theorems analogous to the classical perturbation theorems for operators. Throughout we let A be a semi-prime ring with identity 1 in which non-zero socle S is defined. Any additional conditions on A will be assumed whenever needed. We begin with a very useful proposition.

Proposition 1.3.1: Let t be an element of A . Then,

- (i) if t is a Fredholm element, then t is g -invertible.
- (ii) if t is an element of the socle S , then t is g -invertible.

Proof:

(i) Suppose t is a Fredholm element of A . Then, by Proposition 1.1.6, $tA = (1-e)A$ for some idempotent e in S . Since then $1 - e$ is also an idempotent, by Lemma 1.2.3, t is g -invertible.

(ii) Suppose t is an element of the socle S . Then, tA has finite order, say n . Therefore, by Proposition 1.1.4,

$$tA = fA$$

where $f = f_1 + f_2 + \dots + f_n$, $\{f_1, f_2, \dots, f_n\}$ being a maximal orthogonal set of minimal idempotents in tA .

Hence, by Lemma 1.2.3, t is g -invertible.

The following theorem is an analogue of the classical theorem for a Fredholm operator perturbed by a bounded operator of small norm. Assume that A is a complete normed ring.

Theorem 1.3.2: Suppose that t is a Fredholm element of A .

Then, there exists a positive number r such that if b is in A with $\|b\| < r$, then $t + b$ is a Fredholm element and $\alpha(t+b) \leq \alpha(t)$ and $\beta(t+b) \leq \beta(t)$. Also, $i(t+b) = i(t)$.

Proof: Suppose t is a Fredholm element. Then, by Proposition 1.3.1, t is g -invertible. If s is a g_2 -inverse of t , by Theorem 1.2.7, we have

$$(1-st)A = R(At)$$

$$\text{and } A(1-ts) = L(tA).$$

Also, since t is a Fredholm element, $R(At)$ and $L(tA)$ have finite order. Thus, $1 - st$ and $1 - ts$ are in S .

Consider

$$s(t+b) = 1 + sb - (1-st)$$

$$\text{and } (t+b)s = 1 + bs - (1-ts).$$

Take a positive number $r \leq \|s\|^{-1}$. Then, $1 + sb$ and $1 + bs$ are invertible whenever $\|b\| < r$. Thus,

$$(1+sb)^{-1}s(t+b) = 1 - (1+sb)^{-1}(1-st) \quad (I)$$

and

$$(t+b)s(1+bs)^{-1} = 1 - (1-ts)(1+bs)^{-1} \quad (II)$$

where, since S is a two-sided ideal containing $1 - st$ and $1 - ts$, both $(1+sb)^{-1}(1-st)$ and $(1-ts)(1+bs)^{-1}$ belong to S .

Therefore, $t + b$ is both left invertible modulo S and right invertible modulo S .

Hence, $t + b$ is invertible modulo S and therefore is a Fredholm element.

Also,

$$\begin{aligned}
 \alpha(t+b) &= \theta(R[A(t+b)]) \\
 &\leq \theta(R[A(1+sb)^{-1}s(t+b)]) \text{ since } R[A(t+b)] \subseteq R[A(1+sb)^{-1}s(t+b)] \\
 &= \theta(R[A\{1-(1+sb)^{-1}(1-st)\}]) \text{ , using (I).} \\
 &\leq \theta[(1+sb)^{-1}(1-st)A] \text{ since } R[A(1-a)] \subseteq aA \text{ for any } a. \\
 &= \theta[(1-st)A] \text{ using Corollary 1.10.} \\
 &= \theta[R(At)] \text{ since } (1-st)A = R(At). \\
 &= \alpha(t).
 \end{aligned}$$

Similarly, we can show that

$$\beta(t+b) \leq \beta(t).$$

Finally, from (I),

$$\begin{aligned}
 i[(1+sb)^{-1}s(t+b)] &= i[1-(1+sb)^{-1}(1-st)] \\
 &= 0 \text{ , using Theorem 1.1.12}
 \end{aligned}$$

and the fact that $(1+sb)^{-1}(1-st)$ is in S .

Then, by Theorem 1.1.11,

$$i[(1+sb)^{-1}] + i(s) + i(t+b) = 0.$$

Thus ,

$$i(t+b) = -i(s)$$

since $i[(1+sb)^{-1}] = 0$.

Also, since $1 - st$ is in S , using Theorem 1.1.12, we get $i(st) = 0$.

Consequently, by Theorem 1.1.11,

$$i(s) + i(t) = 0.$$

Hence,

$$i(t+b) = i(t).$$

Before coming to the next perturbation result, we note the following algebraic proposition.

Proposition 1.3.3: Let I be a two-sided ideal of A such that $S \subseteq I \subseteq k(h(S))$. Then an element t of A is right (left) invertible modulo S if and only if t is right (left) invertible modulo I .

Proof: We will sketch the proof, first for $k(h(S))$ instead of I and then for I .

By the second isomorphism theorem, there is a one-to-one correspondence between the ideals of A which contain S and the ideals of $R = A/S$. In this correspondence, $k(h(S))$ in A corresponds to $\text{Rad}.R$ in R where the Jacobson radical $\text{Rad}.R$ of R is the intersection of all primitive ideals of R .

Also then, by the same isomorphism theorem, $A/k(h(S))$ is isomorphic to $R/\text{Rad}.R$.

It is enough to show that x in R is right invertible if and only if $[x] = x + \text{Rad}.R$ is right invertible in $R/\text{Rad}.R$. One way, it is obvious. For the other way, suppose $[x]$ is right invertible in $R/\text{Rad}.R$.

Then there is y such that $[x][y] = [1]$. This implies that $1 - xy$ is in $\text{Rad}.R$ and therefore $xy = 1 - (1-xy)$ is right invertible. Hence, x is right invertible.

If we consider I instead of $k(h(S))$, then I in A corresponds to an ideal $I' \subseteq \text{Rad.} R$ in R and A/I is isomorphic to R/I' .

Then, similarly, we see that x in R is right invertible if and only if $[x] = x + I'$ is right invertible in R/I' .

The following is a perturbation theorem analogous to the one for a Fredholm operator perturbed by a compact operator.

Theorem 1.3.4: Suppose t is a Fredholm element of A and let b be a non-zero element of I , where $S \subseteq I \subseteq k(h(S))$. Then, $t + b$ is a Fredholm element. Moreover, $i(t+b) = i(t)$.

Proof: Suppose t is a Fredholm element, i.e. t is invertible modulo S . Then, by Proposition 1.3.3, t is invertible modulo I . Thus, there is an element s of A such that both $1 - st$ and $1 - ts$ are in I . Then, for b in I ,

$$s(t+b) = 1 + sb - (1-st)$$

$$\text{and } (t+b)s = 1 + bs - (1-ts)$$

where both $sb - (1-st)$ and $bs - (1-ts)$ are in I , I being a two-sided ideal. Thus, $t + b$ is invertible modulo I and therefore invertible modulo S . Hence, $t + b$ is a Fredholm element. Moreover, by Theorem 1.1.12,

$$i(ts) = i[1 - (1-ts)] = 0.$$

Therefore, by Theorem 1.1.11,

$$i(t) + i(s) = 0.$$

Again, using Theorems 1.1.12 and 1.1.11, we have

$$i[s(t+b)] = i[1 + \{(sb - (1-st))\}]$$

$$= 0$$

and $i(s) + i(t+b) = 0$.

Hence, $i(t+b) = i(t)$.

An immediate corollary follows.

Corollary 1.3.5: Suppose t is a Fredholm element of A . If b is a non-zero element of S , then $t + b$ is a Fredholm element. Moreover, $i(t+b) = i(t)$.

Remark 1.3.6: If $A = B(X)$, the ring of bounded linear operators on a complex Banach space X . Then, by [6], Theorem 1, p. 804,

$$F(X) \subseteq K(X) \subseteq k(h(F(X)))$$

where $K(X)$, $F(X)$ denote the ideals of compact operators, bounded operators of finite rank respectively. Also $k(h(S))$ is precisely the ideal of inessential operators.

Moreover, Kaplansky [7] defined an element a of a Banach algebra A to be completely continuous if the maps $a \rightarrow ax$ and $a \rightarrow xa$ are completely continuous linear operators. The set of all completely continuous elements of A obviously forms a two-sided ideal. Moreover, if A is a semi-simple Banach algebra, the set of all completely continuous elements of A is contained in $k(h(S))$ (see [2], p. 495). In view of this, Theorem 1.3.4 is an analogue of the theorem on perturbation of Fredholm operator by a compact operator.

§1.4 PERTURBATION OF AN ELEMENT OF Φ_ℓ (Φ_r)

In this section, we define a sort of semi-Fredholm element and give perturbation results for the same, similar to those for a Fredholm element given in the previous section. Throughout, we let A be a semi-prime ring with identity 1 and in which non-zero socle is defined.

We begin with a definition.

Definition 1.4.1: Define

$$\Phi_\ell = \{t \in A : t \text{ is } q\text{-invertible and } \alpha(t) < \infty\}$$

$$\Phi_r = \{t \in A : t \text{ is } q\text{-invertible and } \beta(t) < \infty\}$$

We have the following useful lemma similar to one of Atkinson [1] for operators.

Lemma 1.4.2: For elements t, u of A ,

- (i) if $ut - t$ is invertible, then t is invertible
- (ii) if $ut - t$ is q -invertible, then t is q -invertible
- (iii) if $ut - t$ is in Φ_ℓ , then t is in Φ_ℓ
- (iv) if $ut - t$ is in Φ_r , then so is t .
- (v) suppose A is normed. If $(ut - t)A$ is a closed ideal, then so is tA .

Proof:

- (i) Suppose $ut - t$ is invertible. Let $(ut - t)^{-1} = v$. Then, $(ut - t)v = 1$ and $v(ut - t) = 1$. This implies $t(utv - v) = 1$ and $(vtu - v)t = 1$. Thus, t is both right invertible and left invertible, therefore invertible.

(ii) Suppose $tut - t$ is g -invertible. If r is a q_1 -inverse of $tut - t$, we have $(tut-t)r(tut-t) = tut - t$. This can be rearranged into

$$t(u-r+rtu+utr-utrtu)t = t$$

Hence t is g -invertible.

(iii) Suppose $tut - t$ is in Φ_ℓ . Then, $tut - t$ is g -invertible and $\theta(R[A(tut-t)]) < \infty$. By (ii) then, t is g -invertible. Also,

$$R(at) \subseteq R[A(tut-t)]$$

Then $\theta[R(at)] \leq \theta[R[A(tut-t)]] < \infty$

Hence, t is in Φ_ℓ .

(iv) The proof is similar to that of (iii).

(v) Assume that $(tut-t)A$ is closed. Suppose $tx_n \rightarrow y$, y being in A . Then, since multiplication is continuous,

$$tutx_n \rightarrow tuy$$

Thus, $tutx_n - tx_n \rightarrow tuy - y$

i.e. $(tut-t)x_n \rightarrow (tu-1)y$

Since $(tut-t)A$ is closed, $(tu-1)y$ is in $(tut-t)A$ and consequently,

$(tu-1)y = (tut-t)x$, for some x in A . This implies that $y = t(uy-utx+x)$ is in tA . Hence, tA is closed.

The following is another very useful lemma.

Lemma 1.4.3: Let t, s be elements of A . If st is a Fredholm element, then both s and t are g -invertible.

Proof: Suppose st is a Fredholm element, i.e. st is invertible modulo S . Then there is an element u in A such that $1 - ust$ and $1 - stu$ are both in S . This implies that $t - tust = t(1 - ust)$ and $s - stus = (1 - stu)s$ are both in S , since S is a two-sided ideal. Then, by Proposition 1.3.1, $t - tust$ and $s - stus$ are both q -invertible. Hence, by Lemma 1.4.2, both s and t are q -invertible.

The following theorem gives a characterization of elements of Φ_ℓ , Φ_r . First, we note that if s is in S , then $1 - s$ is Fredholm since we can now write $1 - 1(1-s) = s$ and $1 - (1-s)1 = s$ and s belongs to S .

Theorem 1.4.4: Let t be an element of A . Then t belongs to Φ_ℓ if and only if t is left invertible modulo S . Similarly, t is in Φ_r if and only if t is right invertible modulo S .

Proof: Suppose t is an element of Φ_ℓ . Then t is q -invertible and $R(At)$ has finite order. If s is a g_1 -inverse of t , we have $t - tst = 0$. Also, since $R(At)$ has finite order, $R(At) \subseteq S$. Then, $t(1-st) = 0$ implies that $1 - st$ is in $R(At) \subseteq S$. Thus t is left invertible modulo S .

Conversely, suppose t is left invertible modulo S . Then there is an s in A such that $1 - st$ belongs to S . Also, then $st = 1 - (1-st)$ is Fredholm and this implies, by Lemma 1.4.3, that t is q -invertible. Moreover, $R(At) \subseteq R(Ast)$. Also, since st is Fredholm, $R(Ast)$ has finite order and therefore $R(At)$ has finite order. Hence, t is in Φ_ℓ .

The proof of the corresponding result for t in Φ_r is similar.

The following is a theorem on perturbation of an element of ϕ_ℓ (or ϕ_r) by an element of small norm.

Theorem 1.4.5: Suppose A is a complete normed ring and let $t \in \phi_\ell$. Then there exists a positive number r such that if b is an element of A with $\|b\| < r$, then $t + b$ is in ϕ_ℓ . Moreover, $\alpha(t+b) \leq \alpha(t)$.

A similar result holds for ϕ_r , the inequality being $\beta(t+b) \leq \beta(t)$.

Proof: Suppose t is an element of ϕ_ℓ . Then t is g -invertible and $R(At)$ has finite order. If s is a g_1 -inverse of t , then $t = tst$. This implies $t(1-st) = 0$. Thus, $1-st$ is in $R(At)$. Also, since $R(At)$ has finite order, $R(At) \subseteq S$. Therefore $1-st$ belongs to the socle S . We write

$$s(t+b) = 1 + sb - (1-st).$$

Choose any positive number $r \leq \|s\|^{-1}$ so that $1 + sb$ is invertible whenever $\|b\| < r$. Thus,

$$(1+sb)^{-1}s(t+b) = 1 - (1+sb)^{-1}(1-st)$$

where $(1+sb)^{-1}(1-st)$ belongs to S , since S is a two-sided ideal containing $1-st$. Hence, $t+b$ is left invertible modulo S . Thus, by Theorem 1.4.4, $t+b$ is an element of ϕ_ℓ . Moreover,

$$\begin{aligned} \alpha(t+b) &= \theta(R[A(t+b)]) \\ &\leq \theta(R[A(1+sb)^{-1}s(t+b)]) \text{ since } R[A(t+b)] \subseteq R[A(1+sb)^{-1}s(t+b)], \\ &= \theta(R[A(1-(1+sb)^{-1}(1-st))]), \text{ using } (*). \\ &\leq \theta[(1+sb)^{-1}(1-st)A] \text{ since } R[A(1-a)] \subseteq aA. \\ &= \theta[(1-st)A], \text{ using Corollary 1.1.10.} \\ &= \theta[R(At)], \text{ since } (1-st)A = R(At) \text{ by Theorem 1.2.7.} \\ &= \alpha(t). \end{aligned}$$

The proof of the corresponding result for ϕ_r is similar.

Remark 1.4.6: As a consequence of the above theorem, we see that ϕ_l and ϕ_r are open subsets of A .

The following is a theorem on the perturbation of an element of $\phi_l(\phi_r)$ by an element of I where $S \subseteq I \subseteq k(h(S))$.

Theorem 1.4.7: Let I be a two-sided ideal of A such that $S \subseteq I \subseteq k(h(S))$.

Suppose v is a non-zero element of I . Then,

- (i) $t + v$ belongs to ϕ_l whenever t is in ϕ_l .
- (ii) $t + v$ is in ϕ_r whenever t is in ϕ_r .

Proof:

(i) Suppose t is an element of ϕ_l . Then, by Theorem 1.4.4, t is left invertible modulo S . This implies, using Proposition 1.3.3, that t is left invertible modulo I . Therefore, there exists an element s in A such that $1 - st$ belongs to I .

We write

$$s(t+v) = 1 + [sv - (1-st)]$$

and observe that $sv - (1-st)$ belongs to I since I is a two-sided ideal.

Hence, $t + v$ is left invertible modulo S . Thus, by Theorem 1.4.4, $t + v$ is in ϕ_l .

- (ii) The proof is similar to that of (i).

The following is an immediate corollary.

Corollary 1.4.8: Suppose v is an element of S . Then,

- (i) $t + v$ belongs to Φ_ℓ whenever t is in Φ_ℓ
- (ii) $t + v$ is in Φ_r whenever t is in Φ_r .

CHAPTER 2

PERTURBATION OF A FREDHOLM ELEMENT - II

In this chapter, we give some results on products of Fredholm elements, elements of ϕ_l (ϕ_r) and of g -invertible elements. Moreover, we will study the perturbation of a Fredholm element in some details. As in Chapter 1, we assume that A is a semi-prime ring with identity 1 and in which non-zero socle S is defined.

§2.1 RINGS WITH INVOLUTION

In this section, we let A be a ring with involution $*$. We will prove that if t is a Fredholm element of A , then so is t^* and index satisfies $i(t^*) = -i(t)$. In the following, for a subset B of A , B^* denotes the set of x^* for all x in B . We begin with a lemma.

Lemma 2.1.1:

- (i) I is a right (left) ideal in A if and only if I^* is a left (right) ideal in A .
- (ii) I is a minimal right (left) ideal in A if and only if I^* is a minimal left (right) ideal in A .
- (iii) The socle S of A satisfies $S^* = S$.
- (iv) For any right (left) ideal I having finite order, we have $\theta(I^*) = \theta(I)$.
- (v) For an element t of A ,

$$R(At^*) = [L(tA)]^*$$

and $L(t^*A) = [R(At)]^*$

Proof:

(i) Direct from definition of involution.

(ii) Suppose I is a minimal right (left) ideal in A . Then I^* is a left (right) ideal in A . Let J be a left (right) ideal in A such that $0 \subseteq J \subseteq I^*$. Then $0 \subseteq J^* \subseteq I^{**} = I$. Since I is minimal, either $J^* = I$ or $J^* = 0$. Then either $J = I^*$ or $J = 0$. Hence, I^* is minimal.

(iii) Suppose e is a minimal idempotent in A and $I = eA$ is a minimal right ideal. Then, by (ii), $I^* = A^*e^* = Ae^*$ is a minimal left ideal.

Therefore, e^* is a minimal idempotent. Now, let x be an element of S .

Then, $x = e_1x_1 + e_2x_2 + \dots + e_nx_n$ for some minimal idempotents

$e_1, e_2, \dots, e_n$ and some $x_1, x_2, \dots, x_n$ in A . This implies that

$x^* = x_1^*e_1^* + x_2^*e_2^* + \dots + x_n^*e_n^*$ is also in S since $e_1^*, e_2^*, \dots, e_n^*$

are minimal idempotents. Then, $x = x^{**}$ is in S^* . Therefore, $S \subseteq S^*$.

Conversely, let x^* be an element of S^* where x is in S . Then, from above, x^* is in S . Thus, $S^* \subseteq S$.

(iv) Suppose I is a left ideal having finite order n . Let

$I = Ae_1 + Ae_2 + \dots + Ae_n$, where $e_1, e_2, \dots, e_n$ are minimal idempotents.

Then,

$$\begin{aligned} I^* &= (Ae_1 + Ae_2 + \dots + Ae_n)^* \\ &= e_1^*A + e_2^*A + \dots + e_n^*A \end{aligned}$$

where $e_1^*, e_2^*, \dots, e_n^*$ are also minimal idempotents, as shown in (iii).

Also, if I^* can be written as a sum of m ($< n$) minimal ideals,

$$\text{i.e. } I^* = f_1 A + f_2 A + \dots + f_m A$$

then $I = Af_1^* + Af_2^* + \dots + Af_m^*$ is a sum of m minimal ideals and

therefore contradicts the hypothesis that $\theta(I) = n$. Hence, $\theta(I^*) = n$.

(v) Let x be an element of $R(At^*)$. Then, $t^*x = 0$. Consequently, $x^*t = x^*t^{**} = 0$. Therefore, x^* is in $L(tA)$ and hence $x = x^{**}$ belongs to $[L(tA)]^*$. Conversely, suppose x^* is an element of $[L(tA)]^*$ where x is in $L(tA)$. Then, $xt = 0$ implying that $t^*x^* = 0$. Thus, x^* belongs to $R(At^*)$. Hence, $R(At^*) = [L(tA)]^*$. Similarly, we see that $L(t^*A) = [R(At)]^*$.

The following theorem shows that the set of Fredholm elements of A is closed for involution.

Theorem 2.1.2: If t is a Fredholm element of A , then t^* is also a Fredholm element. Moreover, $\alpha(t^*) = \beta(t)$, $\beta(t^*) = \alpha(t)$ and $i(t^*) = -i(t)$.

Proof: Suppose t is a Fredholm element of A . Then there exists s in A such that $1 - st$ and $1 - ts$ both belong to S . By Lemma 2.1.1 (iii),

$$\text{both } 1 - t^*s^* = (1-st)^*$$

$$\text{and } 1 - s^*t^* = (1-ts)^*$$

belong to S . Thus, t^* is invertible modulo S and therefore is a Fredholm element. Moreover, by Lemma 2.1.1 (v),

$$\begin{aligned} \alpha(t^*) &= \theta[R(At^*)] \\ &= \theta([L(tA)]^*) \\ &= \theta[L(tA)] \text{ using Lemma 2.1.1 (iv).} \\ &= \beta(t). \end{aligned}$$

Similarly, we have

$$\beta(t^*) = \alpha(t).$$

Hence,

$$\begin{aligned} i(t^*) &= \alpha(t^*) - \beta(t^*) \\ &= \beta(t) - \alpha(t) \\ &= -i(t). \end{aligned}$$

§2.2 PRODUCTS OF FREDHOLM ELEMENTS AND PRODUCTS OF g-INVERTIBLE ELEMENTS

If s, t are two Fredholm elements of A , then it is obvious that st is Fredholm. In this section, we give a converse of this and some more results on products of Fredholm elements and on products of elements of $\Phi_g(\Phi_r)$. Finally, we will study conditions for product of two g -invertible elements to be g -invertible.

Theorem 2.2.1: Suppose s, t are elements of A such that st is a Fredholm element. Then s is a Fredholm element if and only if t is a Fredholm element.

Proof: Suppose t is a Fredholm element, then there exists v in A such that $1 - vt$ and $1 - tv$ are both in S . By symmetry, we see that v is also a Fredholm element.

Let $1 - vt = x_1$ and $1 - tv = x_2$ where x_1, x_2 are in S . Then $tv = 1 - x_2$ and therefore $stv = s - sx_2$. Thus, $s = stv + sx_2$. Since, by hypothesis, st is Fredholm and also v is Fredholm, stv is Fredholm. Also, sx_2 is in S . Then, by Corollary 1.3.5, $stv + sx_2$ is Fredholm, i.e. s is a Fredholm element.

Conversely, if s is Fredholm, then there is u in A such that $1 - us = x_3$, $1 - su = x_4$ are both in S . Then, we can write $t = ust + x_3$ and show that t is Fredholm.

The following seemingly stronger theorem follows easily from Theorem 2.2.1.

Theorem 2.2.2: Suppose s, t are elements of A such that st is Fredholm. If $\alpha(s)$ is finite, then both s and t are Fredholm elements.

Proof: Suppose st is a Fredholm element. Then, by Lemma 1.4.3, s is g -invertible. If u is a g_1 -inverse of s , then by Theorem 1.2.7,

$$(1-us)A = R(As)$$

$$\text{and } A(1-su) = L(sA).$$

Now, $L(sA) \subseteq L(stA)$ and, since st is Fredholm, $L(stA)$ has finite order. Thus, $L(sA)$ has finite order. Also, $R(As)$ has finite order by hypothesis. Consequently, both $1-us$ and $1-su$ belong to the socle S . Hence, s is a Fredholm element. Also then, by Theorem 2.2.1, t is a Fredholm element.

The following interesting corollary is immediate from the above theorem.

Corollary 2.2.3: If t^n is a Fredholm element for some positive integer n , then t is Fredholm.

Proof: We write $t^n = t t^{n-1}$ and note that

$$R(At) \subseteq R(At^n).$$

Since t^n is Fredholm, $R(At^n)$ has finite order. Therefore, $R(At)$ has finite order. Hence, by Theorem 2.2.2, t is Fredholm.

We have another result in the above sequence.

Theorem 2.2.4: Suppose A is with involution $*$. Let s, t be elements of A such that st is a Fredholm element. If $\beta(t)$ is finite, then both s and t are Fredholm elements.

Proof: Suppose st is a Fredholm element and $L(tA)$ has finite order. Then, by Theorem 2.1.2, $t^*s^* = (st)^*$ is a Fredholm element and $\alpha(t^*) = \beta(t)$ is finite. Therefore, by Theorem 2.2.2, both t^* and s^* are Fredholm. Again, by Theorem 2.1.2, $t = t^{**}$ and $s = s^{**}$ are Fredholm elements.

Whereas it is obvious that product of two Fredholm elements is a Fredholm element, it is not so obvious that product of two elements of $\Phi_\ell(\Phi_r)$ is in $\Phi_\ell(\Phi_r)$. However, it is true. Before coming to this result, we give the following lemma of Barnes ([3], Lemma 3.3, p. 90) by molding it to our situation. The following lemma due to Barnes will be used in the proof.

Lemma 2.2.5: Assume that N is a right ideal of order n and u is an element of A such that $[R(Au)] \cap N = 0$. Then, $\theta(uN) = n$. A similar statement holds when N is a left ideal.

Proof: See [3], Lemma 1.5, p. 86.

Lemma 2.2.6: Suppose s, t are g -invertible elements of A .

- (i) If $R(As)$ and $R(At)$ have finite order, then $R(Ast)$ has finite order and $\theta[R(Ast)] = \theta[R(At)] + \theta[(tA) \cap R(As)]$.
- (ii) If $L(sA)$ and $L(tA)$ have finite order, then $L(stA)$ has finite order and $\theta[L(stA)] = \theta[L(sA)] + \theta[(As) \cap L(tA)]$.

Proof:

- (i) Suppose u is a g_1 -inverse of t and v is a g_1 -inverse of s . Then, by Theorem 1.2.7,

$$R(At) = eA$$

$$\text{and } At = A(1-e)$$

$$\text{where } e = 1 - ut.$$

$$\text{Also, } R(As) = gA$$

$$\text{and } As = A(1-g)$$

$$\text{where } g = 1 - vs.$$

Define $T : (1-e)A \rightarrow tA$ by $T[(1-e)x] = t(1-e)x$.

Then T is a module homomorphism. Moreover, if $t(1-e)x = 0$, then

$$(1-e)x \text{ is in } R(At) = eA$$

Thus $(1-e)x = e(1-e)x = 0$. Therefore, T is a module isomorphism. Then,

$$K = T^{-1}[(tA) \cap (gA)]$$

is a right ideal contained in $(1-e)A$ and therefore $K \cap eA = 0$.

Then, by Lemma 2.2.5,

$$\theta(K) = \theta[(tA) \cap (gA)]$$

Now, we will show that

$$R(Ast) = eA \oplus K$$

Clearly, $eA = R(At) \subseteq R(Ast)$.

Also, if x is in K , then tx belongs to $tA \cap g(A)$ and therefore to

$gA = R(As)$. Thus, $stx = 0$ implies that x is in $R(Ast)$. Consequently,

$K \subseteq R(Ast)$. Conversely, let x be an element of $R(Ast)$. Then $stx = 0$

and therefore tx is in $R(As)$. Thus, tx belongs to $(tA) \cap R(As)$.

Also, $t(1-e)x = tutx = tx$ is in $(tA) \cap R(As) = (tA) \cap (gA)$. Consequently,

$(1-e)x$ belongs to K . Then $s = ex + (1-e)x$ is in $eA + K$. Hence,

$R(Ast) = eA \oplus K$. From this, we obtain

$$\theta[R(Ast)] = \theta(eA) + \theta(K)$$

$$= \theta(eA) + \theta[(tA) \cap (gA)] , \text{ using (I).}$$

(ii) The proof is similar to that of (i).

The following is the result on product of two elements of Φ_ℓ (Φ_r) mentioned earlier.

Theorem 2.2.7: Let s, t be elements of A .

(i) If s, t are in Φ_ℓ , then st belongs to Φ_ℓ and

$$\alpha(st) \leq \alpha(s) + \alpha(t).$$

(ii) If s, t are in Φ_r , then st belongs to Φ_r and

$$\beta(st) \leq \beta(s) + \beta(t).$$

Proof:

(i) Suppose s, t are elements of Φ_ℓ . Let u be a g_1 -inverse of t and v be a g_1 -inverse of s . Then, by Theorem 1.2.7,

$$(1-ut)A = R(At)$$

$$\text{and } (1-vs)A = R(As).$$

Since $R(At)$ and $R(As)$ have finite order, both $1-ut$, $1-vs$ belong to the socle S . We write

$$uvst = 1 - [(1-ut) + u(1-vs)t]$$

where, since S is a two-sided ideal, $1-ut + u(1-vs)t$ is in S .

Hence, st is left invertible modulo S and therefore, by Theorem 1.4.4, is in Φ_ℓ . Finally, by Lemma 2.2.6,

$$\alpha(st) \leq \alpha(s) + \alpha(t).$$

(ii) If s, t are in Φ_r and u, v are q_1 -inverses of t, s respectively, we write

$$stuv = 1 - [(1-sv) + s(1-tu)v].$$

Then, in a similar way as in (i), st is in Φ_r . Also, the inequality $\beta(st) \leq \beta(s) + \beta(t)$ follows from Lemma 2.2.6.

The following result gives a sort of converse to Theorem 2.2.7.

Theorem 2.2.8: Let s, t be elements of A .

- (i) If st is in Φ_ℓ , then t belongs to Φ_ℓ .
- (ii) If st is in Φ_r , then s is in Φ_r .

Proof:

(i) Suppose st is an element of Φ_ℓ . Then, by Theorem 1.4.4, st is left invertible modulo the socle S . Therefore, there exists an element u of A such that $1 - ust$ is in S . This shows that t is left invertible modulo S and therefore, by Theorem 1.4.4, t is in Φ_ℓ .

(ii) The proof is similar to that of (i).

Remark 2.2.9: We have seen in Lemma 1.4.3 that if st is a Fredholm element, then s and t are both g -invertible. In fact, we can get a little more, i.e. t belongs to Φ_ℓ and s belongs to Φ_r .

Let s, t be two g -invertible elements of A . Now, in general, the product st is not g -invertible; is clear from the following simple example; (see [4], p. 122).

Example 2.2.10: Let X be a Banach space and T belong to $B(X)$ such that $R(T)$ is not closed. Define P and Q in $B(X \oplus X)$ by means of matrices:

$$P = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \quad Q = \begin{bmatrix} 0 & 0 \\ T & I \end{bmatrix}$$

where I is identity operator in $B(X)$. Then, by simple verification, $P^2 = P$ and $Q^2 = Q$. Thus, since P and Q are projections, they are g -invertible. But

$$QP = \begin{bmatrix} 0 & 0 \\ T & 0 \end{bmatrix}$$

does not have closed range and therefore is not g -invertible.

Thus, it is clear that some additional conditions on g -invertible elements s, t are needed for the product st to be g -invertible. In the following theorem, we consider some such conditions.

Theorem 2.2.11: Suppose s, t are g -invertible elements of A . Let $\tilde{s}, \tilde{t}$ be g_2 -inverses of s, t respectively. If $e = t\tilde{t}$ and $f = \tilde{s}s$, we have:

- (i) if u is a g_1 -inverse of st , then tus is a g_1^* -inverse of fe
- (ii) if v is a g_1 -inverse of fe , then $\tilde{t}v\tilde{s}$ is a g_1 -inverse of st .

Proof: Suppose $\tilde{s}, \tilde{t}$ are g_2 -inverses of s, t respectively, then by Theorem 1.2.7, $e = t\tilde{t}$ and $f = \tilde{s}s$ are idempotents such that $tA = eA$ and $R(As) = (1-f)A$.

(i) Suppose u is a g_1 -inverse of st , then

$$\begin{aligned} fe(tus)fe &= \tilde{s}s \tilde{t}t(tus)\tilde{s}s \tilde{t}t \\ &= \tilde{s}s tus \tilde{t}t, \text{ using } \tilde{t}t\tilde{t} = t, \text{ etc.} \\ &= \tilde{s} st \tilde{t}, \text{ using } stust = st \\ &= fe. \end{aligned}$$

Thus, fe is g -invertible and tus is a g_1 -inverse of fe .

(ii) Suppose v is a q_1 -inverse of fe . Then, $fe v fe = fe$ and therefore

$$\tilde{s}s \tilde{t}t v \tilde{s}s \tilde{t}t = \tilde{s}s \tilde{t}t$$

$$\text{Thus } \tilde{s}(st \tilde{t} v \tilde{s} st - st)\tilde{t} = 0$$

$$\text{i.e. } \tilde{s} k \tilde{t} = 0$$

$$\text{where } k = st\tilde{t} v \tilde{s}st - st$$

(*)

Then, $k\tilde{t}$ belongs to $R(\tilde{A}s)$ and therefore

$$k\tilde{t}A \subseteq R(\tilde{A}s)$$

(I)

Also, from (*),

$$kR(At) = 0.$$

(II)

Moreover, from Theorem 1.2.7,

$$\tilde{t}A = \tilde{t}tA \text{ and } R(At) = (1-\tilde{t}t)A$$

are complementary. Thus, from (I), (II),

$$kA \subseteq R(\tilde{A}s).$$

Also, from (*),

$$kA \subseteq sA.$$

Again, from Theorem 1.2.7,

$$R(\tilde{A}s) = (1-s\tilde{s})A \text{ and } sA = s\tilde{s}A$$

are complementary. Therefore,

$$kA = 0.$$

Hence $k = 0$, since $L(A) = 0$.

Thus, from (*),

$$st \tilde{t} v \tilde{s} st = st.$$

Consequently, $\tilde{t} v \tilde{s}$ is a g_1 -inverse of st .

Corollary 2.2.12:

- (i) If u is a g_2 -inverse of st , then tus is a g_2 -inverse of fe .
- (ii) If v is a g_2 -inverse of fe , then $\tilde{t}v\tilde{s}$ is a g_2 -inverse of st .

Proof:

- (i) If u is a g_2 -inverse of st , we have seen in the theorem that tus is a g_1 -inverse of fe . Also, now

$$\begin{aligned} tus fe tus &= tus \tilde{s}s \tilde{t}\tilde{t} tus \\ &= tustus, \text{ since } \tilde{t}\tilde{t}t = t \text{ and } \tilde{s}s s = s \\ &= tus, \text{ since } ust u = u. \end{aligned}$$

Hence, tus is a g_2 -inverse of fe .

- (ii) If v is a g_2 -inverse of fe , we have seen in the theorem that $\tilde{t}v\tilde{s}$ is a g_1 -inverse of st . But now

$$\begin{aligned} \tilde{t}v\tilde{s} st \tilde{t}v\tilde{s} &= \tilde{t}v fe v\tilde{s} \\ &= \tilde{t}v\tilde{s}, \end{aligned}$$

since v is a g_2 -inverse of fe . Hence, $\tilde{t}v\tilde{s}$ is a g_2 -inverse of st .

Remark 2.2.13: It may be noted that if one can find e, f in the above theorem which commute, then fe is an idempotent. In this case, fe is g -invertible, fe being then its own g_2 -inverse.

In general, it is not easy to decide the g -invertibility of the product of two elements. But, one simple case is almost obvious.

Theorem 2.2.14: Let s, t be elements of A such that s is left invertible and t is right invertible. Then st is g -invertible. Moreover, if $\tilde{s}$ is a left inverse of s and $\tilde{t}$ is a right inverse of t , then $\tilde{\tilde{ts}}$ is a g_2 -inverse of st .

Proof: If $\tilde{s}s = 1$ and $t\tilde{t} = 1$, then

$$st \tilde{\tilde{ts}} st = st$$

and $\tilde{\tilde{ts}} st \tilde{\tilde{ts}} = \tilde{\tilde{ts}}.$

§2.3 PERTURBATION OF A FREDHOLM ELEMENT

In this section, we will consider perturbation of a Fredholm element in more detail. We will give a perturbation theorem which contains Theorems 1.3.2 and 1.3.4 as special cases. Further, we will define the Fredholm spectrum of an element and discuss some of its properties. As usual, A will denote a semi-prime ring with identity 1 in which a non-zero socle S is defined.

We begin with a lemma.

Lemma 2.3.1: Let s, t be elements of A . Then,

- (i) $1 - ts$ is g -invertible if and only if $1 - st$ is g -invertible.
- (ii) $1 - ts$ is an element of ϕ_l if and only if $1 - st$ is in ϕ_l .
Moreover, $\alpha(1-st) = \alpha(1-ts)$.
- (iii) $1 - ts$ is in ϕ_r if and only if $1 - st$ is in ϕ_r . Moreover,
 $\beta(1-st) = \beta(1-ts)$.
- (iv) $1 - ts$ is invertible if and only if $1 - st$ is invertible.
- (v) If A is normed, then $(1-ts)A$ is a closed ideal if and only if $(1-st)A$ is closed.

Proof:

- (i) Suppose $1 - ts$ is g -invertible. If w is a g_1 -inverse of $1 - ts$ then

$$(1-ts)w(1-ts) = 1 - ts$$

$$\text{i.e. } w - tsw - wts + ts wts = 1 - ts.$$

Now,

(*)

$$\begin{aligned}
(1-st)(1+swt)(1-st) &= 1 - s(2.1-w+wtst-ts+tsw-tswts)t \\
&= 1 - s[2.1-ts-(1-ts)]t, \text{ using } (*) \\
&= 1 - st.
\end{aligned}$$

Thus, $1 - st$ is g -invertible.

The converse follows by symmetry.

(ii) Suppose $1 - ts$ is an element of ϕ_ℓ , then $1 - ts$ is g -invertible and $R[A(1-ts)]$ has finite order. By (i) then, $1 - st$ is also g -invertible.

Suppose w is a g_1 -inverse of $1 - ts$, then from (i), $1 + swt$ is a g_1 -inverse of $1 - st$. Now,

$$1 - (1+swt)(1-st) = s[1-w(1-ts)]t \quad (*)$$

Also, by Theorem 1.2.7,

$$R[A(1-st)] = [1 - (1+swt)(1-st)]A$$

Then,

$$\begin{aligned}
\theta(R[A(1-st)]) &= \theta([1 - (1+swt)(1-st)]A) \\
&= \theta(s[1-w(1-ts)]tA), \text{ using } (*) \\
&\leq \theta([1-w(1-ts)]tA), \text{ using Corollary 1.1.10} \\
&\leq \theta([1-w(1-ts)]A), \text{ since } [1-w(1-ts)]tA \subseteq [1-w(1-ts)]A \\
&= \theta(R[A(1-ts)]), \text{ since } [1-w(1-ts)]A = R[A(1-ts)]
\end{aligned}$$

The converse follows by symmetry.

(iii) The proof is similar to that of (ii).

(iv) Suppose $1 - ts$ is invertible. Now,

$$\begin{aligned}
(1-st)[1+s(1-ts)^{-1}t] &= 1 - st + (s-sts)(1-ts)^{-1}t \\
&= (1-st) + s(1-ts)(1-ts)^{-1}t \\
&= 1
\end{aligned}$$

and similarly

$$[1+s(1-ts)^{-1}t](1-st) = 1.$$

Hence, $1-st$ is invertible.

(v) Let $(1-ts)A$ be a closed ideal in A .

Suppose $(1-st)x_n \rightarrow y$, y belonging to A . Then, since multiplication in A is continuous,

$$t(1-st)x_n \rightarrow ty$$

$$\text{i.e. } (1-ts)t x_n \rightarrow ty.$$

Since $(1-ts)A$ is closed, ty is in $(1-ts)A$. Thus, $ty = (1-ts)x$, for some x in A . Therefore,

$$\begin{aligned} sty &= s(1-ts)x \\ &= (1-st)sx \end{aligned}$$

$$\begin{aligned} \text{Then } y &= (1-st)y + sty \\ &= (1-st)y + (1-st)sx \end{aligned}$$

and therefore y belongs to $(1-st)A$. Hence, $(1-st)A$ is closed.

Remark: If A is an algebra, the previous lemma can be stated as:

(i) $\lambda - st$ is g -invertible if and only if $\lambda - ts$ is g -invertible, etc. since we can write $\lambda - ts = \lambda [1 - (\frac{1}{\lambda} t)s]$ etc.

We now come to a theorem on perturbation of a Fredholm element.

Theorem 2.3.2: Let t be a Fredholm element of A . If s is a g_2 -inverse of t , then s is Fredholm and $i(s) = -i(t)$. If u is an element of A such that $1-us$ (or $1-su$) is Fredholm, then $t-u$ is Fredholm and $i(t-u) = i(t) + i(1-us)$ (or $i(t-u) = i(t) + i(1-su)$).

Proof: Suppose t is a Fredholm element of A . Then, by Proposition 1.3.1, t is g -invertible. Also, $R(At)$ and $L(tA)$ have finite order. If s is a g_2 -inverse of t , then by Theorem 1.2.7, $(1-st)A = R(At)$ and $A(1-ts) = L(tA)$. Since $R(At)$ and $L(tA)$ have finite order, both $1-st$, $1-ts$ belong to the socle S . Thus, s is Fredholm.

Since both t , s are Fredholm, st and also tst are Fredholm.

Now, $t = tst$. This implies, using Theorem 1.1.11, that

$$i(t) = i(t) + i(s) + i(t). \text{ Therefore}$$

$$i(s) = -i(t)$$

Now suppose u is an element of A such that $1-us$ is Fredholm.

Again, by Theorem 1.1.11,

$$s - sus = s(1-us) \text{ is Fredholm and } i(s-sus) = i(s) + i(1-us).$$

Now, since s is a g_2 -inverse of t ,

$$sts - sus = s - sus$$

i.e., $s(t-u)s = s - sus$ is Fredholm.

Therefore, since $s(t-u)s$ and s are Fredholm, by Theorem 2.2.1 applied twice, we see that $t-u$ is Fredholm. Also

$$i[s(t-u)s] = i(s-sus)$$

and therefore

$$i(s) + i(t-u) + i(s) = i(s) + i(1-us)$$

Thus,

$$i(t-u) = -i(s) + i(1-us)$$

$$= i(t) + i(1-us), \text{ using } (*).$$

Similarly, the result follows when $1-su$ is Fredholm.

The following corollaries are immediate.

Corollary 2.3.3: Let t be a Fredholm element of A and s be a g_2 -inverse of t . If u is an element of A such that $1 - us$ (or $1 - su$) is invertible, then $t - u$ is Fredholm and $i(t-u) = i(t)$.

Proof: Since $1 - us$ is invertible, it is Fredholm and $i(1-us) = 0$.

Remark 2.3.4: Let A be a complete normed ring. Then " $1 - us$ is Fredholm" contains the conditions of Theorems 1.3.2 and 1.3.4. If $\|u\| < \|s\|^{-1}$, then $1 - us$ is invertible and therefore Fredholm. Also, if u is in a two-sided ideal I , where $S \subseteq I \subseteq k(h(S))$, then us is in I and therefore $1 - us$ is Fredholm. Hence, as mentioned earlier, Theorem 2.3.2 contains the results of Theorems 1.3.2 and 1.3.4 as special cases.

Corollary 2.3.5: Let A be a complete normed ring and F denote the set of Fredholm elements of A . Then F is an open subset of A .

Proof: Let t_0 be an element of F . We write

$$t = t_0 + (t - t_0).$$

If s_0 is a g_2 -inverse of t_0 , by Corollary 2.3.4, t is Fredholm whenever $\|t - t_0\| < \|s_0\|^{-1}$. Hence, F is an open set in A .

The set F of Fredholm elements in A is thus an open semi-group and it is an immediate consequence of Corollary 2.3.3 that index is a continuous function on F . We record this in:

Corollary 2.3.6: The index function i is a continuous function on F .

Now suppose A is an algebra and t is an element of A . We define the Fredholm region of t .

Definition 2.3.7: The Fredholm region of t , denoted by $\Phi(t)$, is the set of all complex numbers λ such that $\lambda 1 - t$ is Fredholm.

We will now denote $\lambda 1$ by λ .

The following important result is a consequence of Theorem 2.3.2.

Theorem 2.3.8: Suppose A is a Banach algebra and let t be an element of A . Then,

- (i) $\Phi(t)$ is an open subset of $\mathbb{C}$.
- (ii) The mapping $\lambda \rightarrow i(\lambda - t)$ is locally constant on $\Phi(t)$. Moreover, on each connected component of $\Phi(t)$, the index of $\lambda - t$ is constant.

Proof:

- (i) Suppose λ_0 is any element of $\Phi(t)$, then $\lambda_0 - t$ is Fredholm.

We write

$$\lambda - t = (\lambda_0 - t) - (\lambda_0 - \lambda).$$

Then, by Corollary 2.3.4, whenever $\|\lambda - \lambda_0\| < \|s_0\|^{-1}$ where s_0 is a g_2 -inverse of $\lambda_0 - t$, we have that $\lambda - t$ is Fredholm and $i(\lambda - t) = i(\lambda_0 - t)$.

Thus, there exists an open neighbourhood V of λ_0 such that $V \subseteq \Phi(t)$. Hence, $\Phi(t)$ is an open set.

- (ii) We have shown in (i) that for each λ_0 in $\Phi(t)$, there exists a neighbourhood V of λ_0 such that $i(\lambda - t) = i(\lambda_0 - t)$ for all λ in $V \subseteq \Phi(t)$. Thus, the mapping $\lambda \rightarrow i(\lambda - t)$ is locally constant on $\Phi(t)$.

Now, let D be a connected component of $\Phi(t)$. Let λ_1, λ_2 be any two points of D . Suppose C is a smooth curve joining λ_1 to λ_2 lying entirely in D .

Suppose λ is a point on C , then as shown in (i), there exists $\epsilon > 0$ such that whenever $|\mu - \lambda| < \epsilon$, μ belongs to $\Phi(t)$ and $i(\mu - t) = i(\lambda - t)$.

Thus, we obtain an open cover of C and using the fact that C is compact, we get a finite open cover of C .

Since each of the sets of this open cover overlaps with at least one other and $i(\mu - t)$ is constant on each one, we obtain

$$i(\lambda_1 - t) = i(\lambda_2 - t).$$

This completes the proof.

Throughout the rest of this section, we continue to assume that A is a semi-prime algebra. Let t belong to A and b be any element of I where I is a two-sided ideal such that $S \subseteq I \subseteq k(h(S))$. Then:

Proposition 2.3.9:

$$\Phi(t+b) = \Phi(t)$$

and $i(\lambda - t - b) = i(\lambda - t)$ for all λ in $\Phi(t)$.

Proof: If λ is in $\Phi(t)$, then $\lambda - t$ is Fredholm. Therefore, by Theorem 1.3.4, $\lambda - t - b$ is Fredholm. Thus λ is in $\Phi(t+b)$.

Also, the converse follows by the same argument. Moreover, by the same Theorem 1.3.4, $i(\lambda - t - b) = i(\lambda - t)$ for all λ in $\Phi(t)$.

We define the Fredholm spectrum of t , denoted by $\sigma_\Phi(t)$, by

Definition 2.3.10:

$$\sigma_{\phi}(t) = \mathbb{C} - \phi(t).$$

Then we have

Corollary 2.3.11: With the notation of Proposition 2.3.9,

$$\sigma_{\phi}(t+b) = \sigma_{\phi}(t)$$

Let A be a semi-prime Banach algebra and suppose I is any closed two-sided proper ideal in A such that $S \subseteq I \subseteq k(h(S))$. Let π denote the canonical mapping $A \rightarrow A/I$, and let $[t]$ denote the image $\pi(t)$ of t . We will give some properties of the Fredholm spectrum of t . In the following, $\sigma([t])$ and $\rho([t])$ will denote, as usual, the spectrum and resolvent set of $[t]$ respectively.

Proposition 2.3.12:

$$\sigma_{\phi}(t) = \sigma([t])$$

Proof: It is easy to see that $\phi(t) = \sigma([t])$. In fact, λ is in $\phi(t)$ if and only if $\lambda - t$ is Fredholm. The latter is equivalent to saying that $[\lambda - t]$ is invertible in A/I , i.e. $\lambda - [t]$ is invertible in A/I , since $[\lambda - t] = \lambda[1] - [t] = \lambda - [t]$.

The following corollary is immediate. It may be noted that when A is a normed algebra and I is a proper closed two-sided ideal in A , then A/I , with the usual quotient norm, is a normed algebra. Also, when A is a Banach algebra, so is A/I .

Corollary 2.3.13: $\sigma_\phi(t)$ is a non-empty, closed and bounded subset of $\mathbb{C}$.

Proof: Since $\sigma([t])$ is a non-empty, compact subset of $\mathbb{C}$, the result follows.

Suppose $p(z) = \sum_{k=0}^n \alpha_k z^k$ is a complex polynomial, then $p(t) = \sum_{k=0}^n \alpha_k t^k$, where $t^0 = 1$, is an element of A . We have the following analogue of the spectral mapping theorem for polynomials.

Theorem 2.3.14: $p(\sigma_\phi(t)) = \sigma_\phi(p(t))$. Moreover, if μ belongs to $p(\phi(t))$ and $\lambda_1, \lambda_2, \dots, \lambda_n$ are the zeros of $p(z) - \mu$ counted according to their multiplicity, then

$$i[\mu - p(t)] = \sum_{k=1}^n i(\lambda_k - t).$$

Proof: Suppose λ is a point of $\sigma_\phi(t)$. Since λ is a zero of $p(\lambda) - p(t)$, we have

$$p(\lambda) - p(t) = (\lambda - t)q(t)$$

where $q(t)$ is a polynomial.

Then, $p(\lambda) - p(t) = (\lambda - t)q(t) = q(t)(\lambda - t)$. (*)

Now, if $p(\lambda)$ is in $\phi(p(t))$, then $p(\lambda) - p(t)$ is Fredholm and is therefore invertible modulo the socle S .

Thus, from (*), $\lambda - t$ is both right and left invertible modulo S . Hence $\lambda - t$ is Fredholm which contradicts the supposition.

Conversely, suppose μ is in $\sigma_\phi(p(t))$. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the complex zeros of $\mu - p(z)$. Then

$$\mu - p(z) = \alpha(\lambda_1 - z)(\lambda_2 - z) \dots (\lambda_n - z), \quad \alpha \neq 0$$

Thus, $\mu - p(t) = \alpha(\lambda_1 - t)(\lambda_2 - t) \dots (\lambda_n - t)$.

Now at least one λ_1 should be in $\sigma_\phi(t)$, because otherwise, from the last equation, $\mu - p(t)$ would be Fredholm.

Thus, $\mu = p(\lambda_1)$ is in $p(\sigma_\phi(t))$.

Finally, suppose μ is in $p(\phi(t))$, i.e. $\mu = p(\lambda)$, λ in $\phi(t)$.

Now, as above, we can write

$$p(\lambda) - p(t) = (\lambda - t)q(t) = q(t)(\lambda - t).$$

Also since $\lambda - t$ is Fredholm, $p(\lambda) - p(t)$ is Fredholm. Now if $\lambda_1, \lambda_2, \dots, \lambda_n$ are zeros of $\mu - p(z)$ counted according to their multiplicity, we can write

$$\mu - p(t) = \alpha(\lambda_1 - t)(\lambda_2 - t) \dots (\lambda_n - t).$$

Then, by Theorem 1.1.11,

$$i[\mu - p(t)] = \sum_{k=1}^n i(\lambda_k - t).$$

Using Lemma 2.3.1, we have:

Proposition 2.3.15: For s, t in $\hat{A}$, we have:

$$\sigma_\phi(st) \cup \{0\} = \sigma_\phi(ts) \cup \{0\}.$$

Proof: Suppose $\lambda \neq 0$ is in $\phi(st)$. Then $\lambda - st$ is Fredholm. Thus, by Lemma 2.3.1 (ii)-(iii), $\lambda - ts$ is Fredholm. Hence, λ is in $\phi(ts)$.

The converse follows by symmetry.

Proposition 2.3.16: Suppose t is a Fredholm element of A . If s is a g_1 -inverse of t , then

$$\sigma_\phi(s) = \{1/\lambda : \lambda \in \phi(t)\}$$

Proof: Suppose t is Fredholm. Then, by Lemma 1.3.1, t is g -invertible.

If s is a g_1 -inverse of t , then $1 - st$ is in S .

Let $1 - st = f$. We write

$$\begin{aligned} \frac{1}{\lambda} - s &= -\frac{1}{\lambda} s(\lambda - t) + \frac{1}{\lambda} (1 - st) \\ &= -\frac{1}{\lambda} s(\lambda - t) + \frac{1}{\lambda} f \end{aligned} \quad (*)$$

If $\frac{1}{\lambda} - s$ is Fredholm, then by Theorem 1.3.4, $-\frac{1}{\lambda} s(\lambda - t)$ is Fredholm.

Thus, since s is Fredholm, by Theorem 2.2.1, $\lambda - t$ is Fredholm.

Conversely, if $\lambda - t$ is Fredholm and since s is also Fredholm, then $-\frac{1}{\lambda} s(\lambda - t)$ is Fredholm. Hence, by Theorem 1.3.4, $-\frac{1}{\lambda} s(\lambda - t) + \frac{1}{\lambda} f$ is Fredholm, i.e. $\frac{1}{\lambda} - s$ is Fredholm.

If A is a $*$ -algebra, we have:

Proposition 2.3.17: If t is an element of A , then $\sigma_\phi(t^*) = \text{complex conjugate of } \sigma_\phi(t)$.

Proof: We will show that

$$\phi(t^*) = \text{complex conjugate of } \phi(t).$$

Suppose λ is in $\phi(t)$, then $\lambda - t$ is Fredholm. Thus there exists some s in A such that both $1 - s(\lambda - t)$ and $1 - (\lambda - t)s$ are in the socle S .

Now since, by Lemma 2.1.1, S is a star ideal, $[1-s(\lambda-t)]^*$ and $[1-(\lambda-t)s]^*$ belong to S . Therefore, $1 - (\bar{\lambda}-t^*)s^*$ and $1 - s^*(\bar{\lambda}-t^*)$ are in S . Hence, $\bar{\lambda} - t^*$ is Fredholm, i.e. $\bar{\lambda}$ is in $\Phi(t^*)$. Similarly, we get the other way around.

Let B be a semi-prime closed subalgebra of A which contains the identity 1 of A . Assume that S_B, S_A denote the socle of B, A respectively. Then $S_B = S_A \cap B$. Suppose t belongs to B . Let $\sigma_\Phi^B(t), \sigma_\Phi^A(t)$ denote the Fredholm spectrum of t considered as an element of B, A respectively. Then, we have:

Proposition 2.3.18:

$$\sigma_\Phi^A(t) \subseteq \sigma_\Phi^B(t).$$

Moreover, if the socle of A is contained in B , we have

$$\text{bdy } \sigma_\Phi^B(t) \subseteq \text{bdy } \sigma_\Phi^A(t)$$

where bdy denotes topological boundary.

Proof: We will show that

$$\Phi^B(t) \subseteq \Phi^A(t)$$

where $\Phi^B(t), \Phi^A(t)$ denote the Fredholm region of t considered as an element of B, A respectively.

Suppose λ is in $\Phi^B(t)$. Then $\lambda - t$, as an element of B , is invertible modulo S_B . Thus there exists an element s of B such that $1 - s(\lambda - t)$ and $1 - (\lambda - t)s$ belong to S_B .

Since $S_B \subseteq S_A$ and $B \subseteq A$, this implies that λ is in $\phi^A(t)$.

In the case where $S_A \subseteq B$, we have $S_B = S_A = S$, say. Then B/S is a subalgebra of A/S and the canonical mapping $B \rightarrow B/S$ is a restriction of the canonical mapping $A \rightarrow A/S$. In this case, $\pi(t) = [t]$ is in $B/S \subseteq A/S$. Then,

$$\begin{aligned} \text{bdy } \sigma_{\phi}^B(t) &= \text{bdy } \sigma^{B/S}([t]) \\ &\subseteq \text{bdy } \sigma^{A/S}([t]) \\ &= \text{bdy } \sigma_{\phi}^A(t), \end{aligned}$$

using ([8], Theorem 1.6.12, p. 33) and where $\sigma^{B/S}([t])$, $\sigma^{A/S}([t])$ denote the spectrum of $[t]$ considered as an element of B/S , A/S respectively.

Finally, we have the following continuity properties of the function $x \rightarrow \sigma_{\phi}(x)$. Let A be a Banach algebra.

Proposition 2.3.19: Suppose x is an element of A and V is a neighbourhood of zero in $\mathbb{C}$. Then there exists $\delta > 0$ such that $\sigma_{\phi}(y) \subseteq \sigma_{\phi}(x) + V$ whenever $\|y - x\| < \delta$.

Proof: Let $U = \sigma_{\phi}(x) + V$ and suppose the theorem is false. Then, for each n , there exists x_n in A and λ_n in $\mathbb{C}$ such that $\|x_n - x\| < 1/n$, $\lambda_n \in \sigma_{\phi}(x_n)$ and $\lambda_n \notin U$. This implies that

$$\begin{aligned} |\lambda_n| &\leq \|x_n\| \\ &\leq \|x\| + 1/n \\ &\leq \|x\| + 1 \end{aligned}$$

Then, by passing to a subsequence if necessary, we get a complex number λ such that $\lim \lambda_n = \lambda$. Since U is an open set and the sequence $\{\lambda_n\}$ is outside U , $\lambda \notin U$ and therefore $\lambda \notin \sigma_\phi(x)$. Thus, $\lambda - x$ is Fredholm.

Since, by Corollary 2.3.5, the set F of all Fredholm elements of A is open and $\lim(\lambda_n - x_n) = \lambda - x$, $\lambda_n - x_n$ is Fredholm for large n . This is a contradiction to the fact that $\lambda_n \in \sigma_\phi(x_n)$ for all n .

Remark 2.3.20: The above result shows that the Fredholm spectrum $\sigma_\phi(x)$ in a Banach algebra is an upper semi-continuous function of x .

In the following result, we see that commutativity forces $\sigma_\phi(x)$ to be a continuous function of x .

Proposition 2.3.21: Let x be an element of A and let V be a neighbourhood of zero in $\mathcal{C}$. Then there exists $\delta > 0$ such that $\|y - x\| < \delta$ and $xy = yx$ imply $\sigma_\phi(y) \subseteq \sigma_\phi(x) + V$ and $\sigma_\phi(x) \subseteq \sigma_\phi(y) + V$.

Proof: Let I be any closed two-sided ideal in A such that $S \subseteq I \subseteq k(h(S))$. Then A/I , with the usual quotient norm, is a Banach algebra. Let the image of x, y , under the canonical mapping $A \rightarrow A/I$, be denoted by $[x], [y]$ respectively.

Then, by ([8], Theorem 1.6.17, p. 36), there exists $\delta > 0$ such that

$$\|[y] - [x]\| < \delta \quad \text{and} \quad [x][y] = [y][x]$$

imply

$$\sigma([y]) \subseteq \sigma([x]) + V \quad \text{and} \quad \sigma([x]) \subseteq \sigma([y]) + V.$$

Now suppose

$$||y-x|| < \delta \quad \text{and} \quad xy = yx,$$

then

$$||[y]-[x]|| = ||[y-x]|| \\ \leq ||y-x|| < \delta$$

and

$$[x][y] = [y][x]$$

This implies, from above,

$$\sigma([y]) \subseteq \sigma([x]) + V \quad \text{and} \quad \sigma([x]) \subseteq \sigma([y]) + V$$

i.e.

$$\sigma_{\phi}(y) \subseteq \sigma_{\phi}(x) + V \quad \text{and} \quad \sigma_{\phi}(x) \subseteq \sigma_{\phi}(y) + V,$$

using Proposition 2.3.12.

We close this chapter with the following remark.

Remark 2.3.22: Suppose A is a Banach algebra and t is an element of A .

Let $f(\lambda)$ be a complex valued function which is analytic on an open neighbourhood $\mathcal{O}$ of $\sigma_{\phi}(t)$. Then one can define $f(t)$ in the following way.

Let I be a closed two-sided ideal such that $S \subseteq I \subseteq k(h(S))$ and let $[t]$ denote the image of t under the canonical mapping $\pi : A \rightarrow A/I$. Now, since $\sigma_{\phi}(t) = \sigma([t])$, $f(\lambda)$ is analytic on the neighbourhood $\mathcal{O}$ of $\sigma([t])$. Then $f([t])$ can be defined by

$$f([t]) = \frac{1}{2\pi i} \int_{\gamma} f(\lambda) (\lambda - [t])^{-1} d\lambda$$

where γ is a suitable contour contained in $\mathcal{O}$ and containing $\sigma_{\phi}(t)$.

$f(t)$ may then be chosen to be any pre-image of $f([t])$ under the canonical mapping π .

It may be noted that the particular choice of $f(t)$ is arbitrary, any two choices differing by an element of I . It would be interesting to have a better way of defining $f(t)$ in which the choice is not so arbitrary.

CHAPTER 3

PERTURBATION THEORY AND GENERALIZED INVERSES

In this chapter, we will give some results on perturbation theory and generalized inverses. Throughout the chapter, we assume that A is a semi-prime ring with identity 1 and in which non-zero socle, denoted by S , is defined.

§3.1 PERTURBATION OF A g -INVERTIBLE ELEMENT

In this section, we will mainly prove a theorem, analogous to one of Caradus in the context of operators; see ([5], Theorem, p. 73). This theorem contains some important results as special cases. In particular, it contains results proved in Theorems 1.3.2, 1.3.4, 1.4.5, 1.4.7 and 2.3.2.

We begin with a lemma.

Lemma 3.1.1: Let t, v be elements of A and suppose $tv - t$ belongs to the socle S . Then t is g -invertible. Also tA and $R(At)$ are generated by idempotents which differ from tv and $1 - vt$ respectively by elements of the socle.

Proof: Since $tv - t$ is in S , by Proposition 1.3.1, it is g -invertible. Then, by Lemma 1.4.2, t is g -invertible. Also, from the proof of Lemma 1.4.2, if s is a g_1 -inverse of $tv - t$, then $s' = v - vtstv + stv + vts - s$ is a g_1 -inverse of t .

Moreover, by Theorem 1.2.7,

$$tA = ts'A.$$

and $R(At) = (1-s't)A$

where ts' and $1 - s't$ are idempotents.

Also,

$$ts' = tv - (tv-t)stv + (tv-t)s$$

and $1 - s't = 1 - vt + vts(tv-t) - s(tv-t)$

where $(tv-t)stv - (tv-t)s$

and $vts(tv-t) - s(tv-t)$ belong to S .

The following result will be used in the proof of the main theorem.

Proposition 3.1.2: Let t be a g -invertible element of A and suppose u is an element of the socle S ; then $t + u$ is g -invertible.

Proof: If s is a g_1 -inverse of t , then

$$\begin{aligned} (t+u)s(t+u) - (t+u) &= (t+u)(st+su) - (t+u) \\ &= tsu + ust + usu - u, \end{aligned}$$

using $t = tst$.

Moreover, since S is a two-sided ideal and u is in S , $tsu + ust + usu - u$ is an element of S . Therefore, by Proposition 1.3.1, $(t+u)s(t+u) - (t+u)$ is g -invertible. Thus, by Lemma 1.4.2, $t + u$ is g -invertible.

Now we come to the main result of the section.

Theorem 3.1.3: Suppose t is a g -invertible element of A and let s be a g_1 -inverse of t . Let u be an element of A and suppose that $1 - us$

is a Fredholm element having a g_1 -inverse v . Finally, suppose that $(1-ts)vu(1-st)$ is g -invertible. Then $t - u$ is g -invertible.

Proof: Suppose t is g -invertible and s is a g_1 -inverse of t . Further, $1 - us$ is a Fredholm element and therefore, by Proposition 1.3.1, $1 - us$ is g -invertible. Suppose v is a g_1 -inverse of $1 - us$. Then, by Theorem 1.2.7,

$$R[A(1-us)] = [1 - v(1-us)]A$$

$$\text{and } L[(1-us)A] = A[1 - (1-us)v].$$

Now, since $R[A(1-us)]$ and $L[(1-us)A]$ have finite order, the idempotents

$$f_1 = 1 - (1-us)v$$

$$\text{and } f_2 = 1 - v(1-us)$$

belong to the socle S .

Consider

$$\begin{aligned} (t-u) - (t-u)sv(t-u) &= (t-u) - (t-u)sv(t-u) + (1-us)v(t-u) - (1-us)v(t-u) \\ &= [(1-us)-(t-u)s]v(t-u) + [1-(1-us)v](t-u) \\ &= (1-ts)v(t-u) + f_1(t-u) \\ &= (1-ts)v[(1-us)t-u(1-st)] + f_1(t-u) \\ &= (1-ts)[v(1-us)t-vu(1-st)] + f_1(t-u) \\ &= (1-ts)[v(1-us)t-vu(1-st)] + f_1(t-u) \\ &= (1-ts)t - (1-ts)f_2t - (1-ts)vu(1-st) + f_1(t-u) \\ &= -(1-ts)vu(1-st) - (1-ts)f_2t + f_1(t-u), \end{aligned}$$

using $t = tst$. Now $-(1-ts)f_2t + f_1(t-u)$ is in S and, by hypothesis, $-(1-ts)vu(1-st)$ is g -invertible. Thus, by Proposition 3.1.2,

$(t-u) - (t-u)sv(t-u)$ is g -invertible.

Hence, by Lemma 1.4.2, $t - u$ is g -invertible.

Remark 3.1.4: By Lemma 2.3.1, the condition " $1 - us$ is a Fredholm element" is equivalent to " $1 - su$ is a Fredholm element". If we assume the latter and w is a g_1 -inverse of $1 - su$, then $(t-u) - (t-u)ws(t-u)$ reduces to $-(1-ts)uw(1-st)$ + an element of the socle S . Thus we have the following second formulation of the above theorem.

Theorem 3.1.5: Let t be g -invertible and s be a g_1 -inverse of t . Suppose u is an element of A such that $1 - su$ is Fredholm having a g_1 -inverse w . Finally, if $(1-ts)uw(1-st)$ is g -invertible, then $t - u$ is g -invertible.

Although the Theorems 3.1.3, 3.1.5 seem to be rather special, they contain some important results as special cases. Some of them are given below.

Corollary 3.1.6: Suppose t is an element of ϕ_r . If s is a g_1 -inverse of t and $1 - us$ is Fredholm, then $t - u$ is in ϕ_r .

Proof: From Theorem 1.2.7, we have

$$A(1-ts) = L(tA).$$

Thus, since $L(tA)$ has finite order, $1 - ts$ belongs to S . Then, if v is a g_1 -inverse of $1 - us$, $(1-ts)vu(1-st)$ belongs to S . Therefore, by Proposition 1.3.1, $(1-ts)vu(1-st)$ is g -invertible. Hence, by Theorem 3.1.3, $t - u$ is g -invertible. Moreover, from the proof of Theorem 3.1.3,

$$(t-u) - (t-u)sv(t-u) = - (1-ts)vu(1-st) + \text{an element of } S$$

and therefore $(t-u) - (t-u)sv(t-u)$ is in S . Then, by Lemma 3.1.1,

$$(t-u)A = eA$$

where e is an idempotent such that $e = (t-u)sv - f$, f being an element of S .

Moreover,

$$\begin{aligned} 1 - e &= 1 - (t-u)sv + f \\ &= 1 - [ts - 1 + 1 - us]v + f \\ &= 1 + (1-ts)v - (1-us)v + f \\ &= 1 + (1-ts)v - (1 - f_1) + f \quad \text{where } (1-us)v = 1 - f_1, f_1 \text{ in } S \\ &= (1-ts)v + f_1 + f \end{aligned}$$

Thus, since S is a two-sided ideal containing $1 - ts$, f_1 , f , we see that $1 - e$ is in S . Hence,

$$\begin{aligned} L[(t-u)A] &= L(eA) \\ &= A(1-e) \end{aligned}$$

has finite order. Consequently, $t - u$ belongs to Φ_r .

Corollary 3.1.7: Suppose t is an element of Φ_l . If s is a g_1 -inverse of t and $1 - su$ is Fredholm, then $t - u$ is in Φ_l .

Proof: The proof follows using similar calculation as in the previous corollary and using Theorem 3.1.5.

Remark 3.1.8: If A is a complete normed ring and $\|u\| < \|s\|^{-1}$, then $1-us$, $1 - su$ are invertible and therefore Fredholm. Also, if u belongs to I

where $S \subseteq I \subseteq k(h(S))$, then both us , su belong to I and therefore $1 - us$, $1 - su$ are Fredholm. Hence, Corollaries 3.1.6 and 3.1.7 contain the results of Theorems 1.3.2, 1.3.4, 1.4.5 and 1.4.7 as special cases.

Corollary 3.1.9: Let A be a complete normed ring and t be g -invertible with a g_1 -inverse s . If $R(At) \subseteq R(Au)$ or $uA \subseteq tA$ with $\|u\| < \|s\|^{-1}$, then $t - u$ is g -invertible.

Proof: Suppose $R(At) \subseteq R(Au)$ and $\|u\| < \|s\|^{-1}$. Then $1 - us$ is invertible. Take $v = (1 - us)^{-1}$. Also, since $R(At) = (1 - st)A$, the condition $R(At) \subseteq R(Au)$ implies $u(1 - st) = 0$. Thus $(1 - st)vu(1 - st) = 0$. Therefore, by Theorem 3.1.3, $t - u$ is g -invertible. In the case where $uA \subseteq tA$ and $\|u\| < \|s\|^{-1}$, take $w = (1 - su)^{-1}$. Now, $L(tA) \subseteq L(uA)$ and this implies $(1 - ts)u = 0$. Therefore, using Theorem 3.1.5, $t - u$ is g -invertible.

Corollary 3.1.10: Let A be a complete normed ring and t be g -invertible having a g_1 -inverse s . Assume that $R(At) \subseteq tA$. Let I be a right ideal satisfying $R(At) \subseteq I \subseteq tA$ and $tI = I$. Then, if $uI \subseteq I$ and $\|u\| < \|s\|^{-1}$, $t - u$ is g -invertible. Moreover, $(1 - su)^{-1}s$ and $s(1 - us)^{-1}$ are equal defining a g_1 -inverse of $t - u$.

Proof: Since $I = tI$, we have

$$\begin{aligned} tsI &= tstI \\ &= tI, \text{ since } tst = t \\ &= I. \end{aligned}$$

This implies that

$sI \subseteq J$ where $J = \{x \in A : tx \in I\}$ is a right ideal.

Further, we will show that $J \subseteq I$. If $x \in J$, then $tx \in I$. Since $I = tI$, $tx = tx'$ for some x' in I . Thus, $x - x'$ is in $R(At) \subseteq I$. Therefore, $x = (x - x') + x'$ belongs to I . Hence, we have shown that $sI \subseteq I$. Now, since $uI \subseteq I$ and $sI \subseteq I$, using Neumann series expansion, we have

$$(1-us)^{-1}uI \subseteq I \quad (I)$$

Now

$$\begin{aligned} (1-ts)(1-us)^{-1}u(1-st)A &= (1-ts)(1-us)^{-1}uR(At), \text{ since } (1-st)A = R(At) \\ &\subseteq (1-ts)(1-us)^{-1}uI, \text{ since } R(At) \subseteq I \\ &\subseteq (1-ts)I, \text{ using (I)} \\ &\subseteq (1-ts)tA, \text{ since } I \subseteq tA \\ &= 0, \text{ since } t = tst. \end{aligned}$$

Hence, since $L(A) = 0$

$$(1-ts)(1-us)^{-1}u(1-st) = 0 \quad (II)$$

Thus, by Theorem 3.1.3, $t - u$ is g -invertible. Also, since

$s(1-us) = (1-su)s$ we have

$$(1-su)^{-1}s = s(1-us)^{-1}$$

Finally

$$\begin{aligned} (t-u)s(1-us)^{-1}(t-u) &= (ts-us)(1-us)^{-1}(t-u) \\ &= [(1-us) - (1-ts)](1-us)^{-1}(t-u) \\ &= (t-u) - (1-ts)(1-us)^{-1}(t-u) \\ &= (t-u) - (1-ts)(1-us)^{-1}[(1-us)t-u(1-st)] \\ &= t - u \end{aligned}$$

using (II) and $t = tst$. Thus, $s(1-us)^{-1}$ is a g_1 -inverse of $t - u$.

§3.2 PERTURBATION OF SAPHAR TYPE ELEMENTS

Let X be a complex Banach space. P. Saphar [9,10] studied operators T of $B(X)$ which are g -invertible and satisfy the condition $N(T) \subseteq R(T^n)$ for all n . In this section, we will study elements of A that are analogous to such operators. Before we define what we call a Saphar type element, we will express the condition " $N(T) \subseteq R(T^n)$ for all n " in an algebraic form.

We begin with a lemma.

Lemma 3.2.1: Let T be a g -invertible operator in $B(X)$. Let W be an operator in $B(X)$ such that $R(W) \subseteq R(T)$. Then $W = TA$ for some A in $B(X)$.

Proof: Suppose $X = N(T) \oplus X_0$, X_0 being a closed subspace of X . We write $T_1 = T|_{X_0}$. Then $T_1^{-1} : R(T) \rightarrow X_0$ is continuous. Now, since $R(W) \subseteq R(T)$, $T_1^{-1}W$ is defined on whole of X and belongs to $B(X)$.

Define $A = T_1^{-1}W$, then $TA = W$.

Theorem 3.2.2: Suppose T is a g -invertible operator in $B(X)$. Then the following are equivalent.

- (i) $N(T) \subseteq R(T^n)$ for all n .
- (ii) $R[B(X)T] \subseteq T^n B(X)$ for all n .

Proof: Suppose $N(T) \subseteq R(T^n)$ for all n . Let B belong to $R[B(X)T]$, then $TB = 0$. Thus, $R(B) \subseteq N(T)$ and consequently $R(B) \subseteq R(T^n)$ for all n . Moreover, since T is g -invertible and $N(T) \subseteq R(T^n)$, T^n is

g-invertible for all n ; see ([4], p. 149). Then, by Lemma 3.2.1, for each n , we can write $B = T^n W_n$, for some W_n in $B(X)$. Therefore, B is in $T^n B(X)$ and hence $R[B(X)T] \subseteq T^n B(X)$ for all n .

Conversely, suppose $R[B(X)T] \subseteq T^n B(X)$ for all n .

Let x_0 be an element of $N(T)$ with $\|x_0\| = 1$. Then, by the Hahn Banach Theorem, there exists a bounded linear functional x'_0 on X such that $\|x'_0\| = 1$ and $x'_0(x_0) = \|x_0\| = 1$.

Define B in $B(X)$ by

$$B(x) = x'_0(x)x_0$$

Then $TB(x) = x'_0(x)T(x_0)$

$$= 0, \text{ since } x_0 \text{ is in } N(T).$$

Thus, $TB = 0$ and therefore B is in $R[B(X)T]$. Then, by hypothesis, B is in $T^n B(X)$ for each n . Hence, for each n , we can write

$$B = T^n B_n, \quad \text{for some } B_n \text{ in } B(X).$$

Thus, $x_0 = B(x_0)$

$$= T^n B_n(x_0)$$

and therefore x_0 is in $R(T^n)$. Hence, $N(T) \subseteq R(T^n)$ for all n .

Thus we define:

Definition 3.2.3: We call an element t of A to be a Saphar type element if t is g-invertible and $R(At) \subseteq t^n A$ for all n .

Before we prove the main theorem on perturbation of a Saphar type element, analogous to one of Saphar ([9], Theorem 1) in the context of

operators, we express another condition in algebraic form. This is done in the following theorem.

Theorem 3.2.4: Suppose X is a complex Banach space and let T be a q -invertible operator in $B(X)$ such that $N(T) \subseteq R(T^n)$ for all n . Set $M = \bigcap_{n=1}^{\infty} R(T^n)$ and $I = \bigcap_{n=1}^{\infty} T^n B(X)$. Then $UM \subseteq M$ if and only if $UI \subseteq I$, for any U in $B(X)$.

Proof: Suppose $UM \subseteq M$.

Let B be an element of I , then B is in $T^n B(X)$ for each n . Therefore, for each n , we can write $B = T^n B_n$, for some B_n in $B(X)$. Then, $Bx = T^n B_n x$ belongs to $R(T^n)$ for each n . Thus Bx is in M and therefore, since $UM \subseteq M$, UBx is in M .

Consequently, $R(UB) \subseteq R(T^n)$ for each n . Then, by Lemma 3.2.1, for each n , we can write $UB = T^n W_n$, for some W_n in $B(X)$. Thus, UB belongs to $T^n B(X)$ for each n , i.e. UB is in I . Hence, $UI \subseteq I$.

Conversely, suppose $UI \subseteq I$. Let $x_0 \in M = \bigcap_{n=1}^{\infty} R(T^n)$, $\|x_0\| = 1$. Then, for each n ,

$$x_0 = T^n x_n, \text{ for some } x_n \text{ in } X. \quad (*)$$

By Hahn Banach Theorem, there exists a bounded linear functional x'_0 on X such that $x'_0(x_0) = \|x_0\| = 1$ and $\|x'_0\| = 1$.

Define B in $B(X)$ by

$$Bx = x'_0(x)x_0, \quad x \in X.$$

Then, $Bx = x'_0(x)T^n x_n$, using $(*)$.

Since T is g -invertible and $N(T) \subseteq R(T^n)$ for all n , by ([4], p. 149), T^n is g -invertible. If $\hat{T}_n$ is a g_1 -inverse of T^n , we obtain

$$\begin{aligned} Bx &= x'_0(x) T^n x_n \doteq x'_0(x) T^n \hat{T}_n T^n x_n \\ &= x'_0(x) T^n \hat{T}_n x'_0, \text{ using } (*) \\ &= T^n \hat{T}_n x'_0(x) x_0 \\ &= T^n \hat{T}_n Bx \end{aligned}$$

Thus, $B = T^n \hat{T}_n B$ and consequently B belongs to $T^n B(X)$ for each n .

Then, B is in I and therefore, by hypothesis, UB is in I . Thus UB is in $T^n B(X)$ for each n and we can write

$$UB = T^n B_n, \quad B_n \text{ being in } B(X).$$

This shows that $R(UB) \subseteq R(T^n)$ for each n and therefore

$$R(UB) \subseteq M.$$

Hence, $Ux_0 = UBx_0$ is in M .

This completes the proof.

The following is a useful result.

Proposition 3.2.5: Let t be a g -invertible element of A such that $R(At) \subseteq t^n A$ for all n , then t^n is g -invertible for all n .

Proof: We will prove the result by induction. For $n = 1$, the result is true by hypothesis.

Suppose t^n is g -invertible for all $n \leq k$. Then, by Theorem 1.2.7, $R(At) = (1-f)A$ and $t^k A = eA$ when e, f are idempotents.

Now, by hypothesis, $R(At) \subseteq t^k A$. Thus, $(1-f)A \subseteq eA$ and therefore,

since $1-f$ is in eA ,

$$1-f = e(1-f)$$

Then $f = 1 - e(1-f)$

and we get

$$fe = e - e(1-f)e$$

$$= efe, \text{ since } e = e^2.$$

Thus, $fe = f^2 e$, since $f = f^2$

$$= fe fe.$$

Now, since fe is idempotent and therefore is g -invertible, by Theorem

2.2.11, $t^{k+1} = t t^k$ is g -invertible.

This completes the proof.

In the following theorem, we obtain conditions for $t - u$ to be Saphar type when t is Saphar type. When u commutes with t , the conditions reduce to the requirement that $\|u\|$ is small.

Theorem 3.2.6: Let A be a complete normed ring and t be a Saphar type element of A with a g_2 -inverse s . Let $I = \bigcap_{n=1}^{\infty} t^n A$ and suppose that u is an element of A such that $uI \subseteq I$ and $\|u\| < \|s\|^{-1}$. Then $t - u$ is an element of Saphar type with

$$R[A(t-u)] \subseteq I \subseteq \bigcap_{n=1}^{\infty} (t-u)^n A.$$

Moreover, $s(1-us)^{-1} = (1-su)^{-1}s$ is a g_2 -inverse of $t - u$.

Proof: Since t is Saphar type, we have

$$R(At) \subseteq I \subseteq tA.$$

One can also easily check that $tI = I$. Thus, by Corollary 3.1.10, $t - u$

is g -invertible and $s(1-us)^{-1}$ is a g_1 -inverse of $t-u$.

We can verify that $s(1-us)^{-1}$ is, in fact, a g_2 -inverse of $t-u$; for

$$\begin{aligned} s(1-us)^{-1}(t-u)s(1-us)^{-1} &= (1-su)^{-1}s(t-u)s(1-us)^{-1} \\ &= (1-su)^{-1}(sts-sus)(1-us)^{-1} \\ &= (1-su)^{-1}(s-sus)(1-us)^{-1} \\ &= (1-su)^{-1}(1-su)s(1-us)^{-1} \\ &= s(1-us)^{-1}. \end{aligned}$$

We will now show that $t-u$ is a Saphar type element. We begin by showing that

$$(t-u)I = I.$$

First, we see that

$$(t-u)I = t(1-su)I.$$

Suppose $(t-u)x$ is an element of $(t-u)I$. Then

$$\begin{aligned} (t-u)x &= tx - ux \\ &= tx - ty, \text{ for some } y \text{ in } A \text{ since } uI \subseteq I \subseteq tA \\ &= tx - tsty \\ &= tx - tsux \\ &= t(1-su)x \end{aligned}$$

and therefore $(t-u)x$ is in $t(1-su)I$. Also, similarly, any element $t(1-su)x$ in $t(1-su)I$ is in $(t-u)I$.

Moreover, since $uI \subseteq I$ by hypothesis and $sI \subseteq I$ as shown in Corollary 3.1.10,

$$(1-su)I \subseteq I.$$

Also, using Neumann series expansion,

$$(1-su)^{-1}I \subseteq I.$$

and therefore

(*)

$$I \subseteq (1-su)I.$$

Thus

$$(1-su)I = I.$$

Hence

$$\begin{aligned} (t-u)I &= t(1-su)I \\ &= tI \\ &= I. \end{aligned}$$

Then, by induction, we get

$$I = (t-u)^n I, \quad \text{for all } n,$$

and therefore

$$I \subseteq (t-u)^n A, \quad \text{for all } n.$$

Thus

$$I \subseteq \bigcap_{n=1}^{\infty} (t-u)^n A.$$

Finally, since $(1-su)^{-1}s$ is a g_2 -inverse of $t-u$, by Theorem 1.2.7,

$$\begin{aligned} R[A(t-u)] &= [1 + (1-su)^{-1}s(t-u)]A \\ &= (1-su)^{-1}[(1-su) - s(t-u)]A \\ &= (1-su)^{-1}(1-st)A \\ &= (1-su)^{-1}R(At), \text{ since } s \text{ is a } g_2\text{-inverse of } t \\ &\subseteq (1-su)^{-1}I, \text{ since } R(At) \subseteq I \\ &\subseteq I, \text{ using } (*). \end{aligned}$$

Hence,

$$R[A(t-u)] \subseteq I \subseteq \bigcap_{n=1}^{\infty} (t-u)^n A.$$

Now, using Theorem 1.2.7, we deduce the following result from Theorem 3.2.6.

Theorem 3.2.7: With the notations of Theorem 3.2.6, we have the following:

- (i) $R[A(t-u)]$ is complementary to sA and is therefore module isomorphic to $R(At)$.
- (ii) $(t-u)A$ is complementary to $R(As)$ and is therefore module isomorphic to tA .
- (iii), (iv) similar results about left annihilator, etc.

Proof:

- (i) From Theorem 3.2.6, $s(1-us)^{-1}$ is a g_2 -inverse of $t-u$. Then, by Theorem 1.2.7, $R[A(t-u)]$ is complementary to $s(1-us)^{-1}A = sA$. Also, $R(At) = (1-st)A$ is complementary to sA and therefore, by the same Theorem 1.2.7, $R[A(t-u)]$ is module isomorphic to $R(At)$.
- (ii) From Theorem 3.2.6, $(1-su)^{-1}s$ is a g_2 -inverse of $t-u$. Therefore, by Theorem 1.2.7, $(t-u)A$ is complementary to $R[A(1-su)^{-1}s] = R(As)$. Also, $tA = tsA$ is complementary to $(1-ts)A = R(As)$. Thus, $(t-u)A$ is, by Theorem 1.2.7, module isomorphic to tA .
- (iii), (iv) The proof is similar to that of (i), (ii).

Remark 3.2.8: We may note that in Theorems 3.2.6 and 3.2.7 the condition $uI \subseteq I^2$ is satisfied by any element u which commutes with t .

Suppose $tu = ut$.

Then $tuI = utI$

$= uI$, since $tI = I^2$.

By induction, it follows that

$$t^n u I = u I, \quad \text{for all } n.$$

Thus,

$$u I = t^n u I \subseteq t^n A, \quad \text{for all } n.$$

Hence,

$$u I \subseteq \bigcap_{n=1}^{\infty} t^n A = I.$$

As a consequence of the previous two theorems, we have the following result. Assume that A is a complete normed ring.

Theorem 3.2.9: Let t be an element of A .

Define $P(t) = \{u \in A : u \in \{t\}'' \text{ and } t - u \text{ is Saphar type}\}$

where $\{t\}''$ denotes the double commutant of $\{t\}$, i.e.

$$\{t\}'' = \{u \in A : ts = st \Rightarrow us = su\}.$$

Then $P(t)$ is an open subset of $\{t\}''$. Moreover, the mappings

$u \rightarrow$ the isomorphism class of $R[A(t-u)]$

$u \rightarrow$ the isomorphism class of $(t-u)A$

and

$$u \rightarrow \bigcap_{n=1}^{\infty} (t-u)^n A$$

are locally constant on $P(t)$.

Proof: Let u_0 be any element of $P(t)$. Then, for any u in $\{t\}''$,

$$u_0 t = t u_0, \quad u t = t u \quad \text{and} \quad u u_0 = u_0 u$$

and thus

$$(u - u_0)(t - u_0) = (t - u_0)(u - u_0).$$

We write

$$I_0 = \bigcap_{n=1}^{\infty} (t-u_0)^n A.$$

Then, by Remark 3.2.8, we have

$$(u-u_0)I_0 \subseteq I_0.$$

Now

$$t - u = (t-u_0) - (u-u_0)$$

where $t - u_0$ is Saphar type.

Thus, by Theorem 3.2.6, $t - u$ is Saphar type whenever $\|u-u_0\|$ is sufficiently small, i.e. u belongs to $P(t)$ whenever $\|u-u_0\|$ is sufficiently small. Hence, $P(t)$ is an open subset of $\{t\}^n$.

Also, by Theorem 3.2.7, whenever $\|u-u_0\|$ is small, $R[A(t-u)]$ is module isomorphic to $R[A(t-u_0)]$ and $(t-u)A$ is module isomorphic to $(t-u_0)A$.

Thus there is an open neighbourhood G of u_0 in $P(t)$ such that the mappings

$$u \rightarrow \text{isomorphism class of } R[A(t-u)]$$

$$\text{and } u \rightarrow \text{isomorphism class of } (t-u)A$$

are constant on G .

To study the third mapping, let s_0 be a g_1 -inverse of $t - u_0$. Then, using the above argument and Theorem 3.2.6, there exists an open subset G of $P(t)$ such that for each u in G ,

$$h(u) = [1 - s_0(u-u_0)]^{-1} s_0$$

is a g_1 -inverse of $t - u$.

Also, for u in G with $\|u - u_0\|$ sufficiently small, we can obtain

$$\begin{aligned}
 \|h(u)\| &= \|[1 - s_0(u - u_0)]^{-1} s_0\| \\
 &\leq \|[1 - s_0(u - u_0)]^{-1}\| \|s_0\| \\
 &\leq [1 + \|s_0(u - u_0)\| + \|s_0(u - u_0)\|^2 + \dots] \|s_0\| \\
 &< [1 + \frac{1}{2} + \frac{1}{2^2} + \dots] \|s_0\|,
 \end{aligned}$$

taking $\|u - u_0\|$ so small that

$$\|s_0(u - u_0)\| < \frac{1}{2}$$

$$< 2\|s_0\|$$

Therefore,

$$\|h(u)\|^{-1} > \frac{1}{2}\|s_0\|^{-1}.$$

Thus, in some open neighbourhood of u_0 in $P(t)$, we have

$$\|u - u_0\| < \|h(u)\|^{-1} \quad \text{and} \quad \|u - u_0\| < \|h(u_0)\|^{-1}.$$

Consequently, by Theorem 3.2.6,

$$\bigcap_{n=1}^{\infty} (t - u_0)^n A \subseteq \bigcap_{n=1}^{\infty} (t - u)^n A$$

and

$$\bigcap_{n=1}^{\infty} (t - u)^n A \subseteq \bigcap_{n=1}^{\infty} (t - u_0)^n A$$

in an open neighbourhood of u_0 in $P(t)$.

Hence,

$$\bigcap_{n=1}^{\infty} (t - u)^n A = \bigcap_{n=1}^{\infty} (t - u_0)^n A$$

in an open neighbourhood of u_0 in $P(t)$.

Let us now suppose that A is a complex Banach algebra and t is an element of A . Define

$$S(t) = \{\lambda \in \mathbb{C} : \lambda - t \text{ is Saphar type}\},$$

then the following is immediate from the above theorem.

Corollary 3.2.10: $S(t)$ is an open subset of $\mathbb{C}$. Also, the mappings

$$\lambda \rightarrow \text{the isomorphism class of } R[A(\lambda-t)]$$

$$\lambda \rightarrow \text{the isomorphism class of } (\lambda-t)A$$

$$\text{and } \lambda \rightarrow \bigcap_{n=1}^{\infty} (\lambda-t)^n A$$

are locally constant on $S(t)$.

Remark: As in Theorem 2.3.8, one can deduce that the above mappings are constant on each connected component of $S(t)$.

We will now continue to assume that A is a complex Banach algebra. Although Saphar type elements seem to be rather special, we will give below a property which brings out their importance. Let t be an element of A . We will prove that if V is an open set in $\mathbb{C}$ and $\lambda - t$ has a holomorphic g_1 -inverse on V , then $\lambda - t$ is Saphar type for all λ in V . We begin with the following lemma.

Lemma 3.2.11: Let t be an element of A . Suppose V is an open subset of $\mathbb{C}$ and x_0 is a fixed element of A . Assume that $f : V \rightarrow A$ is a holomorphic A -valued function and that $(\lambda-t)f(\lambda) = x_0$ for all λ in V . Then, $f(\lambda)$ belongs to $\bigcap_{n=1}^{\infty} (\lambda-t)^n A$ for all λ in V .

Proof: Let λ_1 be a point of V . Since f is holomorphic on V , in some neighbourhood of λ_1 we can write

$$f(\lambda) = \sum_{n=0}^{\infty} a_n (\lambda - \lambda_1)^n, \text{ with } a_n \text{ in } A.$$

Therefore, $\star$

$$\begin{aligned} (\lambda_1 - t)f(\lambda) &= [(\lambda - t) - (\lambda - \lambda_1)]f(\lambda) \\ &= x_0 - (\lambda - \lambda_1)f(\lambda) \end{aligned}$$

Thus

$$(\lambda_1 - t) \sum_{n=0}^{\infty} a_n (\lambda - \lambda_1)^n = x_0 - (\lambda - \lambda_1) \sum_{n=0}^{\infty} a_n (\lambda - \lambda_1)^n.$$

Equating the corresponding coefficients, we get

$$(\lambda_1 - t)a_0 = x_0$$

$$(\lambda_1 - t)a_1 = -a_0$$

$$(\lambda_1 - t)a_2 = -a_1$$

$$\dots\dots\dots$$

Then

$$a_0 = -(\lambda_1 - t)a_1 = (\lambda_1 - t)^2 a_2 = \dots$$

$$a_1 = -(\lambda_1 - t)a_2 = (\lambda_1 - t)^2 a_3 = \dots$$

$$\dots\dots\dots$$

Hence $a_0, a_1, \dots$ all belong to $\bigcap_{n=1}^{\infty} (\lambda_1 - t)^n A$. In particular,

$$f(\lambda_1) = a_0 \text{ belongs to } \bigcap_{n=1}^{\infty} (\lambda_1 - t)^n A.$$

Hence, $f(\lambda)$ belongs to $\bigcap_{n=1}^{\infty} (\lambda - t)^n A$ for all λ in V .

Theorem 3.2.12: Let t belong to A . Suppose V is an open subset of C and that $f : V \rightarrow A$ is holomorphic with

$$(\lambda - t)f(\lambda)(\lambda - t) = \lambda - t, \quad \text{for all } \lambda \text{ in } V.$$

Then $\lambda - t$ is Saphar type for all λ in V .

Proof: Let λ_0 be in V and suppose that x_0 is an element of $R[A(\lambda_0 - t)]$. Then for any λ in V , since $f(\lambda)$ is a g_1 -inverse of $\lambda - t$,

$$(\lambda - t)[1 - f(\lambda)(\lambda - t)]x_0 = 0.$$

Define

$$g : V \rightarrow A$$

by

$$g(\lambda) = [1 - f(\lambda)(\lambda - t)]x_0, \quad \text{for all } \lambda \text{ in } V.$$

Then g is holomorphic and

$$(\lambda - t)g(\lambda) = 0, \quad \text{for all } \lambda \text{ in } V.$$

Thus, by Lemma 3.2.11,

$$g(\lambda) \text{ belongs to } \bigcap_{n=1}^{\infty} (\lambda - t)^n A \text{ for all } \lambda \text{ in } V.$$

Also, x_0 is in

$$R[A(\lambda_0 - t)] = [1 - f(\lambda_0)(\lambda_0 - t)]A$$

and therefore $x_0 = [1 - f(\lambda_0)(\lambda_0 - t)]x_0$, since $1 - f(\lambda_0)(\lambda_0 - t)$ is an idempotent. Thus, $x_0 = g(\lambda_0)$ belongs to $\bigcap_{n=1}^{\infty} (\lambda_0 - t)^n A$. Hence, $R[A(\lambda_0 - t)] \subseteq \bigcap_{n=1}^{\infty} (\lambda_0 - t)^n A$ and therefore $\lambda_0 - t$ is Saphar type.

Using Lemma 3.2.11, we also obtain another interesting result.

First, we have another lemma which will be needed in the proof of that result.

Lemma 3.2.13: Let A be a Banach algebra. Suppose t is an element of A such that $I = \bigcap_{n=1}^{\infty} t^n A$ is a closed right ideal. Then there exists $k > 0$ such that for each x in I , there is an $\hat{x}$ in I satisfying $t\hat{x} = x$ and $\|\hat{x}\| \leq k \|x\|$.

Proof: One can easily check that $tI = I$. Then the continuous linear transformation

$$\begin{aligned} T_t : I &\rightarrow I \\ x &\rightarrow tx \end{aligned}$$

maps complete normed linear space I onto itself. Further,

$N(T_t) = \{x \in I : tx = 0\}$ is a closed subspace of I and the induced mapping

$$\hat{T}_t : I/N(T_t) \rightarrow I$$

maps Banach space $I/N(T_t)$ onto Banach space I and therefore has a bounded inverse.

Then, for any $T_t(y) = x$ in I and denoting the image of y under canonical mapping $I \rightarrow I/N(T_t)$ by $[y]$ etc., we have

$$\|[y]\| = \|\hat{T}_t^{-1}x\| \leq \|\hat{T}_t^{-1}\| \|x\|.$$

Since

$$\|[y]\| = \inf\{\|y+z\| : z \in N(T_t)\},$$

we can choose z in $N(T_t)$ such that

$$\begin{aligned} \|y+z\| &\leq 2\|[y]\| \\ &\leq 2\|\hat{T}_t^{-1}\| \|x\|. \end{aligned}$$

If we write $y + z = \hat{x}$, $2\|\hat{T}_t^{-1}\| = k$, then we have shown that there exists $k > 0$ such that for each x in I , there is an $\hat{x}$ in I satisfying $T_t \hat{x} = x$ and $\|\hat{x}\| \leq k\|x\|$.

This completes the proof.

Theorem 3.2.14: Suppose A is a Banach algebra and t is an element of A such that $I = \bigcap_{n=1}^{\infty} t^n A$ is a closed right ideal. Then there exists a neighbourhood V of the origin in $\mathbb{C}$ such that

$$R[A(\lambda-t)] \subseteq I \subseteq \bigcap_{n=1}^{\infty} (\lambda-t)^n A, \text{ for all } \lambda \text{ in } V - \{0\}.$$

Proof: By Lemma 3.2.13, there exists $k > 0$ such that for each x in I , there is an $\hat{x}$ in I satisfying $t\hat{x} = x$ and $\|\hat{x}\| \leq k\|x\|$.

Now let x_0 be an element of I , then there is an a_0 in I such that $ta_0 = x_0$ and $\|a_0\| \leq k\|x_0\|$. Again, there is an a_1 in I such that $ta_1 = a_0$ and $\|a_1\| \leq k\|a_0\|$ and so on.

Thus, we obtain a sequence $x_0, a_0, a_1, \dots, a_n, \dots$ in I such that $ta_0 = x_0, ta_n = a_{n-1}$ and $\|a_n\| \leq k^n \|a_0\|$.

Set $V = \{\lambda \in \mathbb{C} : |\lambda| < k^{-1}\}$.

Define

$$f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n, \quad \text{for } \lambda \text{ in } V.$$

since
$$\begin{aligned} \|a_n \lambda^n\| &= |\lambda|^n \|a_n\| \\ &\leq |\lambda|^n k^n \|a_0\| \end{aligned}$$

and $|\lambda| k < 1$, the series $\sum_{n=0}^{\infty} a_n \lambda^n$ is convergent and therefore $f : V \rightarrow A$ is a holomorphic map.

Also

$$\begin{aligned} (\lambda-t)f(\lambda) &= (\lambda-t) \sum_{n=0}^{\infty} a_n \lambda^n \\ &= -x_0 + \sum_{n=1}^{\infty} (a_{n-1} - t a_n) \lambda^n \\ &= -x_0, \text{ since } t a_n = a_{n-1}. \end{aligned}$$

Then, by Lemma 3.2.11,

$$f(\lambda) \text{ is in } \bigcap_{n=1}^{\infty} (\lambda-t)^n A, \text{ for any } \lambda \text{ in } V - \{0\}.$$

Consequently, $(\lambda-t)f(\lambda)$ is in $\bigcap_{n=1}^{\infty} (\lambda-t)^n A$ and therefore x_0 is in $\bigcap_{n=1}^{\infty} (\lambda-t)^n A$.

Thus,

$$I \subseteq \bigcap_{n=1}^{\infty} (\lambda-t)^n A.$$

Finally, for any λ in $V - \{0\}$, suppose x is an element of $R[A(\lambda-t)]$.

Then $(\lambda-t)x = 0$ and therefore $x = t(\frac{1}{\lambda} x)$. Thus, x is in tA .

Again, we can write

$$\begin{aligned} x &= t \frac{1}{\lambda} (t \frac{1}{\lambda} x) \\ &= t^2 (\frac{1}{\lambda^2} x) \end{aligned}$$

and therefore x is in $t^2 A$.

Continuing this way, we have shown that x belongs to $t^n A$ for each n . Hence,

$$R[A(\lambda-t)] \subseteq \bigcap_{n=1}^{\infty} t^n A.$$

Barnes ([3], p. 94) gave the following example of a semi-prime algebra with identity in which a non-trivial socle exists. We will give an example of a Saphar type element in this algebra.

Example 3.2.15: Suppose Ω is a compact Hausdorff topological space and X is a Banach space. Let A be the algebra of all continuous functions defined on Ω with values in $B(X)$. The identity in A is the constant function which takes each x in Ω to identity operator in $B(X)$. The norm of $f \in A$ is defined by

$$\|f\| = \sup_{x \in \Omega} \|f(x)\|_1$$

where $\|\cdot\|_1$ is the usual norm on $B(X)$. $e \in A$ is a minimal idempotent of A if and only if there exists x_0 , an isolated point of Ω , and $E \in B(X)$, E a projection on X with one-dimensional range such that

$$e(x_0) = E$$

and $e(x) = 0$ when $x \neq x_0$, $x \in \Omega$.

The socle S of A is the set of all $f \in A$ such that f takes the value zero at all except a finite set of isolated points of Ω and $f(x) \in F(X)$ for all $x \in \Omega$.

Now suppose Ω is a finite discrete topological space and X is any Banach space. In this case, A is the Banach algebra of all functions on Ω with values in $B(X)$.

Let T be a Saphar type operator in $B(X)$, i.e. T is g -invertible and $N(T) \subseteq R(T^n)$ for all n . Then, by Theorem 3.2.2, $R[B(X)T] \subseteq T^n B(X)$ for all n .

Let $h \in A$ be the constant function $h(x) = T$, $x \in \Omega$. We will show that h is a Saphar type element of A .

If U is a g_1 -inverse of T , i.e. $TUT = T$, then $khk = h$ where k is the constant function $k(x) = U$, $x \in \Omega$.

Now to show that $R(Ah) \subseteq h^n A$ for all n , we proceed as follows.

Fix a positive integer n . Suppose $hg = 0$. Then $h(x)g(x) = 0$ for all x and therefore $Tg(x) = 0$ for all x .

Thus, for each x , $g(x) \in R[B(X)T] \subseteq T^n B(X)$. Consequently, for each x ,

$$g(x) = T^n B_n^x, \quad \text{for some } B_n^x \text{ in } B(X).$$

If $f_n(x) = B_n^x$, $x \in \Omega$, then $g = h^n f_n$ and therefore $g \in h^n A$.

This completes the proof. *

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§3.3 SAPHAR TYPE ELEMENTS

In this section, we will consider Saphar type elements which are also Fredholm elements. Suppose A is a Banach algebra and t is an element of A . We have already noted that Fredholm region $\Phi(t)$ of t is an open subset of the complex plane. Also, by Corollary 3.2.10,

$$S(t) = \{\lambda \in \mathbb{C} : \lambda - t \text{ is Saphar type}\}$$

is an open subset of $\mathbb{C}$. Therefore, $\Phi(t) \cap S(t)$ is an open subset of $\mathbb{C}$. Suppose $\Phi(t) \cap S(t)$ is non-empty. Also, it follows from Theorem 3.2.7 that the mappings

$$\lambda \rightarrow \alpha(\lambda - t)$$

$$\text{and } \lambda \rightarrow \beta(\lambda - t)$$

are locally constant on $\Phi(t) \cap S(t)$.

Moreover, as shown in Theorem 2.3.6, it follows that these mappings are constant on each connected component of $\Phi(t) \cap S(t)$. We define the following subset of $\Phi(t) \cap S(t)$.

Definition 3.3.1: For any non-negative integers n and p , we define

$$S(n, p; t) = \{\lambda \in \mathbb{C} : \lambda - t \text{ is Saphar type with}$$

$$\alpha(\lambda - t) = n \text{ and } \beta(\lambda - t) = p\}.$$

We have the following.

Theorem 3.3.2: For non-negative integers n, p , suppose $S(n, p; t)$ is non-empty. Then $S(n, p; t)$ is an open subset of $\mathbb{C}$. Moreover, the mapping

$$\lambda \rightarrow \bigcap_{n=1}^{\infty} (\lambda - t)^n A$$

is locally constant on $S(p, p; t)$.

Proof: Suppose λ_0 is a point of $S(n, p; t)$. Then $\lambda_0 - t$ is Saphar type and $R[A(\lambda_0 - t)]$, $L[(\lambda_0 - t)A]$ have orders n, p respectively.

Writing

$$\lambda - t = (\lambda_0 - t) + (\lambda - \lambda_0)$$

and using Theorems 3.2.6 and 3.2.7, we obtain:

Whenever $|\lambda - \lambda_0|$ is small, $\lambda - t$ is Saphar type and also then $R[A(\lambda - t)]$ is module isomorphic to $R[A(\lambda_0 - t)]$, $L[(\lambda - t)A]$ is module isomorphic to $L[(\lambda_0 - t)A]$.

Consequently, whenever $|\lambda - \lambda_0|$ is small, $\lambda - t$ is Saphar type and

$$\theta(R[A(\lambda - t)]) = \theta(R[A(\lambda_0 - t)]) = n,$$

$$\theta(L[(\lambda - t)A]) = \theta(L[(\lambda_0 - t)A]) = p.$$

Thus there is an open neighbourhood around λ_0 which is contained in $S(n, p; t)$ and therefore $S(n, p; t)$ is open in $\mathbb{C}$.

Again suppose λ_0 is in $S(n, p; t)$. Then from above, there is an open neighbourhood G_0 of λ_0 such that $G_0 \subseteq S(n, p; t)$.

Also, by Theorem 3.2.9, there is an open neighbourhood G_1 of λ_0 such that the mapping

$$\lambda \rightarrow \bigcap_{n=1}^{\infty} (\lambda - t)^n A \quad (*)$$

is constant on G_1 .

Setting $G = G_0 \cap G_1$, we see that $G \subseteq S(n, p; t)$ and the mapping (*) is constant on G .

If A is assumed to be a B^* -algebra, we have the following result.

Theorem 3.3.3: Suppose A is a B^* -algebra and t is an element of A . If t is g -invertible (Saphar type), then t^* is g -invertible (Saphar type). Moreover, for any non-negative integers n and p , we have:

$$\text{complex conjugate of } S(n, p; t) = S(p, n; t^*).$$

Proof: Suppose t is g -invertible and let s be a g_2 -inverse of t , then $tst = t$ and $sts = s$.

This implies that

$$t^* s^* t^* = t^* \quad \text{and} \quad s^* t^* s^* = s^*$$

and therefore s^* is a g_2 -inverse of t^* .

Now, suppose t is Saphar type. Then, from Theorem 3.2.6,

$$r(\lambda) = (1 - \lambda s)^{-1} s$$

is a g_2 -inverse of $t - \lambda$ whenever $|\lambda| < \|s\|^{-1}$.

$$\text{Also, } [r(\lambda)]^* = s^* (1 - \bar{\lambda} s^*)^{-1}$$

and therefore $[r(\lambda)]^*$ is a g_2 -inverse of $t^* - \bar{\lambda}$ whenever $|\bar{\lambda}| < \|s^*\|^{-1}$.

Using the fact that in a B^* -algebra, the condition $\|x^* x\| = \|x\|^2$ implies $\|x^*\| = \|x\|$, we see that $|\bar{\lambda}| < \|s^*\|^{-1}$ is the same as $|\lambda| < \|s\|^{-1}$. Also, the mapping

$$\lambda \rightarrow [r(\lambda)]^* = s^* (1 - \bar{\lambda} s^*)^{-1}$$

is holomorphic.

Thus, $t^* - \bar{\lambda}$ has a holomorphic g_1 -inverse on $V = \{\lambda \in \mathbb{C} : |\lambda| < ||s||^{-1}\}$.

Therefore, by Theorem 3.2.12, $t^* - \bar{\lambda}$ is Saphar type for all λ in

V. In particular, t^* is Saphar type.

Now, suppose λ is in $S(n, p; t)$. Then $t - \lambda$ is Saphar type, $\alpha(t - \lambda) = n$ and $\beta(t - \lambda) = p$.

Therefore, from the above, $t^* - \bar{\lambda} = (t - \lambda)^*$ is Saphar type. Also, from Theorem 2.1.2,

$$\alpha(t^* - \bar{\lambda}) = \alpha[(t - \lambda)^*] = \beta(t - \lambda) = p$$

and similarly

$$\beta(t^* - \bar{\lambda}) = n.$$

Thus, $\bar{\lambda}$ is in $S(p, n; t^*)$.

Hence,

$$\text{complex conjugate of } S(n, p; t) \subseteq S(p, n; t^*).$$

Similarly, we have

$$S(p, n; t^*) \subseteq \text{complex conjugate of } S(n, p; t).$$

This completes the proof.

BIBLIOGRAPHY

1. Atkinson, F.V., On Relatively Regular Operators, Acta. Sci. Math. Szeged, 15(1953), 38-56.
2. Barnes, B.A., Generalized Fredholm Theory for Certain Maps in Regular Representations of an Algebra, Can. J. Math. 20(1968), 495-504.
3. Barnes, B.A., The Fredholm Elements of a Ring, Can. J. Math, 21(1969), 84-95.
4. Caradus, S.R., Generalized Inverses and Operator Theory, Manuscript.
5. Caradus, S.R., Perturbation Theory for Generalized Fredholm Operators, II, Proc. Amer. Math. Soc., 62(1) (1977), 72-76.
6. Kleinecke, D., Almost Finite, Compact and Inessential Operators, Proc. Amer. Math. Soc. 14(1963), 863-868.
7. Kaplansky, I., Dual Rings, Ann. of Math. (2) (1948), 689-701.
8. Rickart, C.E., General Theory of Banach Algebras, Van Nostrand, New York (1960).
9. Saphar, P., Contribution a l'étude des applications linéaires dans un espace de Banach, Bull. Soc. Math., France 92 (1964), 363-384.
10. Saphar, P., Sur les applications lineaires dans un espace de Banach II, Ann. Sci. Ecole Norm. Sup. (3) 92 (1965), 205-240.
11. Von-Neumann, J., Continuous Geometry, Princeton University Press, 1960.

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