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Models and Permutation Groups

by

LISA LAJEUNESSE

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfillment of
the requirements for the degree of

Master of Science

Department of Mathematics and Statistics

Carleton University

Ottawa, Ontario

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Abstract

This thesis examines various $\aleph_0$ -categorical structures and their automorphism groups. Several ideas and theorems from Model Theory are explored to help describe these structures and their automorphism groups. Particular attention is given to homogeneous structures such as the random graph and the rationals with order. Finally, we look at the problem of finding a dense subgroup of the automorphism group of a structure. This is a subgroup which has the same orbits on n -tuples for each n as the automorphism group.

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I would like to thank my supervisor Brian Mortimer for providing the right balance of help and support throughout the research and writing of this thesis. It was also thanks to him that I found Peter Cameron's book *Oligomorphic Permutation Groups* which first sparked my interest in Model Theory and became my constant companion throughout this thesis.

I also thank the Department of Mathematics and Statistics at Carleton University for providing a pleasant and supportive environment for doing graduate work. It was my observation of the Department's success with their undergraduate students which first inspired me to return to studying mathematics.

I would also like to thank John Poland for helping me reach the decision to return to school to study mathematics. Also, I thank him for his review of the text of this thesis and his suggestions prior to submission.

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Introduction

In this thesis, we look at $\aleph_0$ -categorical structures and their automorphism groups. These are structures which are the unique countable models of their theory. Various ideas from Model Theory are explored to help describe these automorphism groups. This thesis was motivated by a question in Cameron [1], page 86, concerning when we can find, for different automorphism groups, subgroups on a small number of generators which give the same orbits on n -tuples as the original automorphism group.

The ideas from model theory which appear in the thesis include types of an n -tuple, Skolem Theory and indiscernibles, and Stability Theory. A section on stability is included in Chapter 3 because it does seem to enrich our understanding of the examples which are extensively discussed in this thesis, such as the random graph and the structure of the rationals with order.

In Chapter 1, a model-theoretic characterisation of $\aleph_0$ -categorical structures is given. As well, the Chapter concludes with a discussion of non- $\aleph_0$ -categorical theories and their various countable models. The proofs of the theorems in this chapter are my own, though the ideas and techniques are quite commonly found in the standard proofs of these theorems. The one exception is perhaps the notion of n -trees which is used in the proof of Theorem 1.1 and which I have not seen elsewhere.

Chapter 2 gives an equivalent characterisation of $\aleph_0$ -categorical structures, this time in terms of the automorphism group of the structure. Some examples are introduced and discussed. The chapter includes a section on homogeneous structures. In particular, the relationship between homogeneity and symmetry is explained with reference to Fraïssé's Theorem. The proof of Theorem 2.1 is again standard but my own, the overview of the proof of Fraïssé's Theorem and the proof of Theorem 2.3 come from Cameron [1].

Chapter 3 further explores some ideas in model theory which pertain to the composition of a structure, specifically Skolem Theory and stability, in the hope that this might give some general insights into automorphism groups. The proof of Theorem 3.2 comes from Hodges [3]. The proof of Theorem 3.3 is my own.

Chapter 4 introduces a topology on automorphism groups and constructs dense subgroups on a limited number of generators. In particular, the various possibilities for the setwise stabiliser of a set of indiscernibles, which are given in Chapter 3 and Appendix B are discussed. The example of a dense subgroup of $Sym(\mathbb{Z})$ is found elsewhere though the proof that it is such is my own. The other examples and proofs from this chapter are my own.

The source of my historical notes is Hodges [3].

Chapter 1

$\aleph_0$ -categorical Theories

In this thesis, we will be discussing highly symmetric infinite objects called structures. Insight into these objects comes from the branch of mathematical logic called model theory, which classifies and describes mathematical objects by examining their theories and in particular gives a fairly good characterisation of those of high symmetry with some indication of their inherent structure. We will also be examining the automorphism groups of these structures, and again, model theory will give us some description of these groups.

Those with little or no background in mathematical logic may wish to consult Appendix A, which gives a number of preliminary definitions.

If we begin with a structure, we can obtain its theory in a suitable language by taking all true statements about the structure which can be formed in the language according to the rules of first order logic. We note that such a theory is complete. Because we are describing a specific structure, for every logical sentence, exactly one of it and its negation is included in the theory, according to whether or not the sentence is true or false in relation to that structure. The theory is consistent, since the structure we began with is necessarily a model for it.

The theories we will be discussing will normally be complete. By definition, complete theories are consistent and by the Completeness Theorem have models. By a well-known theorem of model theory, the upward-downward Lowenheim-Skolem theorem, any theory over a countable language with an infinite model has models in all infinite cardinalities. We confine our discussion for the most part to countably infinite models.

We say a theory is *categorical* if all models of the theory are isomorphic. By the Lowenheim-Skolem theorem, only the theories of finite structures can be categorical. In the case of a finite structure, all of its properties can be exhaustively listed by its theory, including how many elements there are in its universe, so the theory of a finite structure is always categorical.

We cannot hope that the theories of infinite structures are categorical so we say instead that a theory is $\aleph_0$ -categorical if all countable models of the theory are isomorphic. Likewise, a structure is $\aleph_0$ -categorical if its theory is $\aleph_0$ -categorical.

In this chapter, we will give a comprehensive characterisation of what makes a theory, or a structure, $\aleph_0$ -categorical. An example of an infinite structure which is $\aleph_0$ -categorical is a complete graph on a countably infinite set of vertices. Since its theory contains the statement that any two vertices are adjacent, any countable model of this theory must be a complete graph on a countable set of vertices and any two complete countable graphs are isomorphic.

We will see shortly an example of a theory which is not $\aleph_0$ -categorical. It

may seem surprising that one can have two countable models of the same complete theory which are non-isomorphic. It is the limitation of first-order logic to finite statements which creates this ambiguity in a theory.

To understand this, one can imagine a structure whose universe contains an n -tuple $\bar{a}$ which is dissimilar from other n -tuples in the structure, but to distinguish it from all dissimilar n -tuples would require a listing of an infinite number of properties. The theory of such a structure cannot assert the existence of $\bar{a}$ because the theory can only make finite assertions, and any sentence which attempts to claim $\bar{a}$'s existence by listing a finite subset of its properties, will also describe some other n -tuple in the structure and be asserting its existence as well. Now because the theory cannot assert the existence of an n -tuple with exactly the same properties as $\bar{a}$, there will be models of the theory which do not contain an n -tuple similar to $\bar{a}$ (this is a consequence of the Omitting Types Theorem which is described later). There can be no isomorphism between these two models, since any isomorphism must map $\bar{a}$ to an n -tuple in the new model with precisely the same properties and no such n -tuple exists.

We will explain these ideas more precisely in the next section by introducing the idea of the type of a tuple or an n -type, but first we illustrate with an example. Suppose we have the structure of the natural numbers with order, and our language permits us to name each finite natural number by a constant. The theory of this structure will tell us facts about the structure such as:

Each number has an immediate successor;

There is a smallest number, but no largest number;

Every number but one has an immediate predecessor;

etc.

An alternate countable model of this theory is the natural numbers, named by the constants as before, to which we adjoin a copy of the integers (with their natural ordering) as infinite ordinals dominating the naturals. Note that the statements from the theory listed above are still true in this new model yet there is no isomorphism between the two models. So the theory is not $\aleph_0$ -categorical. Now if we consider one of these infinite ordinals, it is fundamentally distinct from all of the finite natural numbers in our original model in that it has an infinite number of predecessors, but to distinguish it requires listing an infinite number of properties. If we say that it is greater than the integer 1000, and it is not the integer 5000, and add finitely many other properties, there will still remain finite numbers from which we have failed to distinguish it. And it is for exactly this reason that the original theory cannot assert that such a number does not exist, since such an assertion would have to say something like *there is no number greater than n* , where n is some very large number, but since there are natural numbers arbitrarily large, such a statement cannot be in the theory.

1.1 Types

In a language L , if a logical formula has n free variables and we are given an n -tuple in an L -structure, that n -tuple may or may not satisfy the given formula. For example, consider the structure of the rationals with order, $(\mathbb{Q}, <)$. The formula $x < y < z$ has three free variables. It is satisfied by the triple $(1, 2, 173/3)$, but not by the triple $(2, 1, 3)$. The two formulas $\alpha : x < y < z$ and $\beta : y < x < z$ cannot both be satisfied by the same triple. In this case we say that the set $\{\alpha, \beta\}$ is inconsistent, since modulo the theory of $(\mathbb{Q}, <)$ both α and β cannot be satisfied by the same triple.

Given a theory T , an n -type, Σ , is a set of formulas with n free variables which are consistent modulo the theory. This means that if $\alpha_1, \alpha_2, \dots, \alpha_m$ is any finite subset of Σ , then the statement β where

$$\beta = \exists \bar{x} : \alpha_1(\bar{x}) \wedge \alpha_2(\bar{x}) \wedge \dots \wedge \alpha_m(\bar{x})$$

($\bar{x}$ is an n -tuple) is consistent with T (when T is complete then $T \models \beta$). A complete n -type, Σ , is an n -type which is a maximal set, meaning that for each formula θ in n free variables, either $\theta \in \Sigma$ or $\neg\theta \in \Sigma$.

We can think of a complete n -type as being a complete description of an n -tuple which has the potential to exist in some model of the theory because its existence is consistent with the theory. Not only does it have the potential to exist, but the fact that it is consistent with the theory, ensures, by the Compactness

Theorem, that there is some model of the theory in which an n -tuple appears which represents the n -type.

To obtain examples of complete n -types, we can begin with a particular n -tuple, $\bar{a}$, in a structure A which is a model of the theory T . We include in the n -type $\Sigma_{\bar{a}}$, all formulas which the tuple satisfies. This set must be consistent modulo T because it is realised in a model of T by $\bar{a}$, and it must be complete, since for every formula, either it or its negation will be included, according to whether or not the formula is satisfied by the n -tuple $\bar{a}$.

In this case, the n -type $\Sigma_{\bar{a}}$ matches nicely with the tuple $\bar{a}$ and we can say that the type $\Sigma_{\bar{a}}$ is *realised* in the structure A by the tuple $\bar{a}$. However, not all complete types of a theory must be realised in a given model of the theory. When it so happens that a complete type is not realised in some model of the theory we say that the model *omits* the n -type. As we shall see below, it is precisely when some model of a theory omits one of its complete types that a theory is not $\aleph_0$ -categorical.

It is possible to distinguish between complete n -types which are always realised in any model of a given theory, and those which are not. A complete n -type, Σ , is called *principal* if there exists some formula $\sigma \in \Sigma$ such that for every $\theta \in \Sigma$, $T \models \sigma \rightarrow \theta$. We also say that the formula σ is a *support* for the principal n -type Σ .

We can think of principal n -types as being n -types which we can describe

in a finite way. A support for a principal n -type acts as a finite description of it since all other properties of the type are derived modulo the theory from the support formula. As an example, if we think of the real numbers with order, and assume that our language permits us to name each rational number, each rational is a principal 1-type, since we can describe it finitely by specifying which rational it is. On the other hand, a non-rational number needs an infinite number of formulas to describe it, since it requires an infinite number of statements to pin down its position on the real line in relation to the rationals. In fact a Dedekind cut is essentially a non-principal 1-type for the theory of the rational numbers with order.

In the next section we shall prove a theorem which explains the connection between principal types and when a theory is $\aleph_0$ -categorical.

1.2 $\aleph_0$ -Categorical Theories

The following theorem is known as the Ryll-Nardzewski Theorem, though it was proven independently by Engeler, Ryll-Nardzewski and Svenonius in 1959. It summarizes the observations of the last section in a characterization of an $\aleph_0$ -categorical theory.

THEOREM 1.1 *Let T be a complete theory over a countable language with infinite models. Then the following are equivalent:*

- a) *The theory T is $\aleph_0$ -categorical.*
- b) *For each n , there are finitely many complete n -types in T .*
- c) *For each n , every n -type of the theory, T , is principal.*
- d) *For each n , every n -type is realised in every model A of T .*

In Chapter 2 we will add a fifth condition which refers only to the automorphism group of a model of the theory T . This condition is that the automorphism group induce finitely many orbits on the n -tuples of the universe for each particular n .

Before proving Theorem 1.1, we digress by describing an intuitive way of looking at n -types. When our language is countable, for a given n we can construct the n -tree corresponding to the theory T as follows: Let $(\theta_1, \theta_2, \theta_3, \dots)$ be an enumeration of all possible formulas in n free variables. Now construct a binary tree with each of the formulas θ_i associated with an edge at the corresponding depth i in the tree. For each node at depth i the tree can branch in two possible directions with edges labelled θ_i and $\neg\theta_i$. Define the node M at depth m , to be the set of formulas $\{\alpha_1, \alpha_2, \dots, \alpha_m\}$ where $\alpha_1, \alpha_2, \dots, \alpha_m$ are the edges of the unique path from the root to M so, in particular, $\alpha_i = \theta_i$ or $\neg\theta_i$. A node is considered consistent if the set of formulas it represents is a valid n -type. Only consistent nodes are included in the tree.

We observe that the description above does indeed give a tree because if $\{\alpha_1, \alpha_2, \dots, \alpha_{m+1}\}$ is a consistent node then, so is $\{\alpha_1, \alpha_2, \dots, \alpha_m\}$. Also, each branch of the tree must continue indefinitely, since if $\{\alpha_1, \alpha_2, \dots, \alpha_m\}$ is consistent, then one of $\{\alpha_1, \alpha_2, \dots, \alpha_m, \theta_{m+1}\}$ or $\{\alpha_1, \alpha_2, \dots, \alpha_m, \neg\theta_{m+1}\}$ must be consistent as well. Note that the union of the nodes of any infinite path from the root is a complete n -type.

If beyond some node $M = \{\alpha_1, \alpha_2, \dots, \alpha_m\}$ there is a unique path, then we have that for any $\theta_k, k > m$, exactly one of θ_k and $\neg\theta_k$ is a consequence of $T \cup \{\alpha_1, \alpha_2, \dots, \alpha_m\}$. The node $M = \{\alpha_1, \alpha_2, \dots, \alpha_m\}$ represents the complete n -type given by the set of formulas which is the union of all nodes beyond M . So this n -type is a principal n -type, and the conjunction $\sigma = \alpha_1 \wedge \dots \wedge \alpha_m$ is its support. See Figure 1 for an example of principal types in an n -tree.

In contrast, a non-principal n -type is represented by a path in the tree for which no initial path segment uniquely determines the rest of the path. In other words, from any node M , there is a node, L , at greater depth such that L branches in both directions θ_l and $\neg\theta_l$. To understand why this is, if a path P represents a principal n -type, Σ , it has a support $\sigma \in \Sigma$. Since each formula appears at some level in the tree, we can assume σ appears before some node M , so that $M = \{\alpha_1, \alpha_2, \dots, \alpha_{m-1}, \sigma\}$. Now since $T \models \sigma \rightarrow \alpha_i \forall \alpha_i \in \Sigma$, then at any further node, only the first of $T \cup \{\alpha_1, \alpha_2, \dots, \alpha_{m-1}, \sigma, \dots, \alpha_i\}$ and $T \cup \{\alpha_1, \alpha_2, \dots, \alpha_{m-1}, \sigma, \dots, \neg\alpha_i\}$ is a consistent set. Therefore, each node beyond M can only branch in one of the

Figure 1 Principal Types in an n -tree

two directions. The bold path in Figure 2 is an example of a non-principal type in an n -tree.

Proof. (of Theorem 1.1)

a) $\Rightarrow$ b)

Assume T is $\aleph_0$ -categorical and for some n there are an infinite number of n -types. Then in the n -tree we have constructed above, there are an infinite number of paths. By a repeated application of the pigeon-hole principle, we can construct an infinite path by iteratively choosing between (at most) two branches at each node, the branch which contains an infinite number of sub-branches. This path

Figure 2 Example of a Non-principal Type

gives a non-principal n -type, Σ , since from any node, there is always an infinite number of paths further down which branch from the path. By a well-known theorem of model theory, the Omitting Types Theorem¹, there is a countable model of T which omits the n -type Σ because it is non-principal. We will call this model A . But there is also a model of T which realises the n -type Σ . To obtain this model, expand the language L by adding n new constants $c_1, c_2, \dots, c_n$

¹ See Chang & Keisler [2], Theorem 2.2.9, p.79, for a proof.

to obtain L' , which is countable as well. For every formula $\alpha \in \Sigma$, add the sentence $\alpha(c_1, c_2, \dots, c_n)$ to T to obtain T' . Since Σ is a complete n -type of T , each finite subset of T' has a model, and therefore by the Compactness Theorem, T' has a model B' . Note that we can choose B' to be countable since the language L' is countable. The reduct of B' to the language L gives a model, B , of the original theory T which realises the type Σ . This type is realised by the tuple which is the image in B of the constants $c_1, \dots, c_n$ under the model B' .

Now the two models A and B cannot be isomorphic, since there is an n -tuple in B whose type is not realised by any n -tuple in A , and any isomorphism must map an n -tuple, to another n -tuple of the same type.

b) $\Rightarrow$ c)

Clearly, if there are finitely many n -types for a given n , then each path in the n -tree must be uniquely determined beyond some node at a given depth. Therefore each path (or each n -type) must be principal.

c) $\Rightarrow$ d)

Let Σ be a principal n -type with a support σ . By the definition of an n -type, the sentence β given by $\exists \bar{a} : \sigma(\bar{a})$, is consistent with the theory T , and since T is complete, $T \models \beta$. Thus, the n -type, Σ , is realised in any model of T by the n -tuple whose existence is asserted by β . Thus, any principal n -type is realised in any model of T , so if all n -types are principal, then all n -types are realised in any model of T .

d) $\Rightarrow$ a)

If all n -types are realised in any two models A and B of T , then we can apply what is called a back and forth procedure to construct an isomorphism between the two models. Before we do this, we establish two points. First, if $\bar{a} = (a_1, \dots, a_n)$ and $\bar{b} = (b_1, \dots, b_n)$ are n -tuples in A and B which represent the same n -type Σ , then the finite mapping $\theta_n : A|_{\bar{a}} \mapsto B$ given by $\theta_n(a_i) = b_i$ is a finite isomorphism. If this were not the case, there would be some atomic formula α for which, without loss of generality, $\alpha(\bar{a})$ and $\neg\alpha(\bar{b})$. But then $\bar{a}$ and $\bar{b}$ would not represent the same n -type.

The second point is that two n -tuples, $\bar{a}$ and $\bar{b}$, in the models A and B respectively, which represent the same n -type Σ , can be extended to $n + 1$ -tuples which represent the same $n + 1$ -type Σ' in the following way: if we adjoin to the tuple $\bar{a}$ any new element c of the universe $|A|$, we can find a d in $|B|$ such that the $n + 1$ -tuples $(\bar{a}, c)$ and $(\bar{b}, d)$ give the same $n + 1$ -type. (Note: If we choose c distinct from the elements in $\bar{a}$, then d must be distinct from the elements in $\bar{b}$, since our language includes equality and this property will be included in the complete $n + 1$ -type). We find this d as follows:

By the Omitting Types Theorem, if Σ is a non-principal n -type, then there is some model of T which omits Σ . Since all n -types are realised in all models of T , we conclude that all n -types are principal. Now to find the desired extension of the n -tuples $\bar{a}$ and $\bar{b}$, let β be a support for the $n + 1$ -type corresponding to $(\bar{a}, c)$.

Then the statement $\exists x : \beta(\bar{a}, x)$, is contained in the n -type for $\bar{a}$ and therefore in the n -type for $\bar{b}$ since these types are the same. This implies that $\exists y : \beta(\bar{b}, y)$ is satisfied in the model B . The element d assured by this existential statement will provide an extension to an $n + 1$ -tuple $(\bar{b}, d)$ which is of the same $n + 1$ -type as $(\bar{a}, c)$. This tells us that the mapping $\theta_n : A|_{\bar{a}} \mapsto B$ defined on $a_1, \dots, a_n$ can be extended as a finite isomorphism to $\theta_{n+1} : A|_{(\bar{a}, c)} \mapsto B$ with $\theta_{n+1}(c) = d$.

We now apply the back and forth procedure as follows: We enumerate the universes of A and B , then at each odd stage of our process, $2i - 1$, we choose the first element of our enumeration of $|A|$ which we have not yet used. We add this element c to our previous $2i - 2$ -tuple to obtain a $2i - 1$ -tuple in A , we extend to an isomorphism θ_{2i-1} as explained above, by finding an element d in $|B|$ which matches the given $2i - 1$ -tuple in $|A|$. At the even stage, $2i$, we choose the first element, d' , of our enumeration of $|B|$ not yet appearing in the $2i - 1$ tuple in B and obtain a $2i$ -tuple. We then find a matching c' in the universe of A which enables us to extend the isomorphism θ_{2i-1} to an isomorphism θ_{2i} , with $\theta_{2i}(c') = d'$. Then the union: $\bigcup_{i < \omega} \theta_i$ will give us an isomorphism from A to B . The back and forth argument ensures that this isomorphism has A as its domain and B as its range.

Thus the theory T is $\aleph_0$ -categorical. $\square$

1.3 Non- $\aleph_0$ -Categorical Theories

What happens when a theory is not $\aleph_0$ -categorical? When the theory is sufficiently out of focus to describe more than one structure, what different types of models can result? When a theory is not $\aleph_0$ -categorical, there are further distinctions involving n -types which we can make to give us insights into the sorts of models the theory describes. In some sense, we can talk about the existence of smallest possible and largest possible models, and the number of models between these two.

1.3.1 Smallest Possible Models

We explained in the previous section that if Σ is a principal type of the theory T , then it must be realised in every model of T . If we could find a model A of T in which only principal types are realised, then in some sense this model would be as small as possible. (*Note:* Though we conclude that if there are finitely many types realised in a structure then all types are principal, it is not true that if all types that are realised are principal, then there are only finitely many of them. There can be an infinite number of principal types realised when there is a non-principal type in the theory).

We also know that every formula consistent with T must be realised in any model of T . Thus, if a model A exists in which only principal types are realised, then any formula β consistent with T must be satisfied by a tuple a which realises a principal type Σ and therefore, satisfies its support σ . But since $\beta \in \Sigma$, then

$T \models \sigma \rightarrow \beta$ (by the definition of support). Also, σ has the property that for all formula θ either $\sigma \rightarrow \theta$ or $\sigma \rightarrow \neg\theta$ modulo the theory T . Formulas with this property are called *complete* in T . Thus, the existence of an A which only realises principal types, implies that the theory T has the property that any formula consistent with T is the consequence of a formula complete in T . Such a theory is called *atomic*.

The converse can also be shown¹, that if T is atomic, then a model A of T exists, for which all realised types are principal. We also call the model A *atomic*, and such a model has the following properties:

1) *N*-type characterisation

All n -types realised in the model A are principal.

2) Uniqueness

Any two countable atomic models are isomorphic. Actually, we proved this already in Theorem 1.1 when we proved that if all n -types were realised in every model of T (and therefore all n -types were principal) then T was $\aleph_0$ -categorical. The proof hinged on the fact that all realised types were principal, which is what we have in an atomic model. In the $\aleph_0$ -categorical case all models are atomic and therefore there is only one model up to isomorphism.

3) Smallness

The countable atomic model A is elementarily embedded in every model of

¹ Chang & Keisler [2], Theorem 2.3.2, p.93-94

T . A model with this property is called *prime*. Thus prime models and countably atomic models coincide. Recall that an *elementary embedding* is an embedding which preserves all formulas. Thus, if we define a partial ordering on Ω the models of T by $A \preceq B$ iff A can be elementarily embedded in B , then A is a minimum element of Ω .

It is important to note that atomic models do not necessarily exist for an arbitrary theory. The following example appears in Chang & Keisler [2], page 94.

Let the language L have countably many unary relations P_i . Let the axioms of T be all sentences of the form $\exists x (P_{i_1}(x) \wedge P_{i_2}(x) \wedge \dots \wedge P_{i_n}(x) \wedge \neg P_{j_1}(x) \wedge \neg P_{j_2}(x) \wedge \dots \wedge \neg P_{j_m}(x))$ where the i 's and j 's are arbitrary but distinct.

This theory has the following interesting properties:

- 1) No formula is complete.
- 2) No type is principal.
- 3) Every formula corresponding to an axiom with the existential quantifier removed is consistent with the theory.
- 4) There exists a distinct 1-type for every subset S of the natural numbers.

Define $P_i(x) \in \Sigma_S$ if $i \in S$ and $\neg P_i(x) \in \Sigma_S$ otherwise.

From (4), there are uncountably many 1-types from which we can conclude further that there are uncountably many n -types for each n .

It is clear from what we have said above that this theory is not atomic, and so there is no atomic model of T . As we move now to largest possible models, we

will find that T also does not have a largest possible countable model.

1.3.2 Largest Possible Models

What would be the corresponding properties that we would like a largest possible countable model B to have?

1) N -type characterisation

All n -types are realised in the model B .

2) Bigness

All countable models of T can be elementarily embedded in B .

3) Uniqueness

The model B is unique up to isomorphism, subject to the above two properties.

Such a model is called *countably saturated*. Again, countably saturated models do not necessarily exist for any theory T . The condition for such a model to exist is that there be countably many complete n -types for each n . It is clear that we cannot have a countable model which realises an uncountable number of n -types for a particular n . From this we see that the example given above of a non-atomic theory does not have a countably saturated model.

Theories which have countably saturated models have countable atomic models. If a theory does not have a countable atomic model, then for some n , some consistent formula cannot be realised by a principal n -type. This means that

wherever this consistent formula appears in the n -tree, beyond it there are no principal paths so that all paths continue to branch indefinitely. Then there must be an uncountable number of paths in the n -tree, since beyond the appearance of the formula in the tree, no paths are principal, so each path must keep branching indefinitely, giving $2^{\aleph_0}$ paths.

An interesting consequence of this is that if T does not have a countably saturated model, then we can construct an uncountable number of non-isomorphic countable models, each realising a different collection of the uncountably many non-principal n -types of the theory. The converse, that theories which have countable atomic models must have countably saturated models is not necessarily true.

One final interesting fact which we will not prove, is that non- $\aleph_0$ -categorical theories always have at least 3 distinct countable models.¹

1.4 $\aleph_0$ -Categorical Structures

We have described when a theory is $\aleph_0$ -categorical; how can we determine whether or not a structure is $\aleph_0$ -categorical? One way is to consider the theory of the structure and determine (or at least find some upper bound) for the number of n -types. In practise, however, it is easier to work with a structure than with its theory. Particularly if the structure isn't $\aleph_0$ -categorical, then its theory represents many models, and we may not have a clear picture of what sort of objects it describes. A method which distinguishes between $\aleph_0$ -categorical structures and

¹ Chang & Keisler [2], Theorem 2.3.15, p.104.

non- $\aleph_0$ -categorical structures by referring only to the structures and not to their theories would be preferable. We don't have to look very far to find such a method.

The proof of Theorem 1.1 gives us the following:

The structure A is $\aleph_0$ -categorical $\implies$

There are finitely many n -types for each $n \implies$

All n -types are realised in the structure A .

We conclude that if for some n , there are infinitely many n -types realised in the structure A , then A is not $\aleph_0$ -categorical. Is the converse true? If we can show that for each n there are finitely many n -types realised in the structure, can we conclude that it is $\aleph_0$ -categorical?

The answer is yes, which we explain as follows: If the theory of A is not $\aleph_0$ -categorical, then for some n , there are infinitely many n -types. If we consider the n -tree of types for this particular n , recall that it must have an infinite path which represents a non-principal n -type. Therefore there is no node we can find, beyond which it is the unique path in the tree. From this we must have that there are an infinite number of paths branching off from this path at different nodes, each distinguishable from each of the others by some formula. For each of these infinite paths, choose a node in its path which occurs after the point at which it branches from the original path. Each of these nodes are terminal points of some finite path in the tree, and each finite path represents a finite conjunction of formulas, which if consistent with the theory, must be realised in every model of the theory

(remember, all finite, not necessarily complete, types are realised in every model of a theory). Thus, each of these finite types are realised in A , and because each is distinguishable from all other types in the infinite set by some formula, each represents a different n -type in the theory of A . Thus, if there are infinitely many n -types for some n , then there must be infinitely many distinct n -types realised in every model of the theory. From this we can conclude the converse to our original statement, and we have the following condition on $\aleph_0$ -categorical structures:

A is an $\aleph_0$ -categorical structure if and only if for every n there are finitely many n -types realised in A .

Chapter 2

Automorphism Groups and Homogeneous Structures

2.1 Automorphism Groups

Given any structure, A , we define an *automorphism*, g , of the structure to be any permutation on the universe of A which preserves the given relations of the structure. In other words, g is an automorphism of the structure if and only if, for each R , an n -ary relation on the structure and $(a_1, \dots, a_n) \in |A|^n$, $(a_1, \dots, a_n) \in R \iff (a_1^g, \dots, a_n^g) \in R$. The set of all such permutations forms a group under composition which we call the automorphism group of the structure, $\text{Aut}(A)$.

THEOREM 2.1 *The structure A is $\aleph_0$ -categorical if and only if $\text{Aut}(A)$ induces finitely many orbits on $|A|^n$ for all n .¹*

Such a group is called *oligomorphic*. This term is Greek and means 'of few shapes'. In this case the shapes are the different n -types or orbits and we consider these to be few because there are finitely many. The proof follows from the fact that the orbits on $|A|^n$ and the n -types of the theory of A correspond exactly when A is $\aleph_0$ -categorical, and in the case when A is not $\aleph_0$ -categorical, the number of

¹ This is essentially an alternate form of Theorem 1.1.

orbits is greater than or equal to the number of n -types which are realised in A .

Proof. (of Theorem 2.1)

To prove the forward implication we assume that A is $\aleph_0$ -categorical. From Theorem 1.1, all n -types are realised in A (of which there are finitely many) and all n -types are principal. In Theorem 1.1 we used this to apply a back and forth procedure to construct an isomorphism between any two models A and B to extend a finite isomorphism between n -tuples $\bar{a}$ and $\bar{b}$ which realised the same n -type in the theory.

If we take a copy of A as our second model B , the above procedure will yield an automorphism which maps any n -tuple to any other n -tuple in A which realises the same n -type. Thus, any two n -tuples which realise the same n -type are in the same n -orbit. We conclude that there are finitely many n -orbits if A is $\aleph_0$ -categorical.

To prove the other direction, clearly two n -tuples realising different types cannot be in the same n -orbit, since they are distinguished by some formula and any automorphism must preserve all formulas. Therefore, the number of n -orbits is greater than or equal to the number of n -types realised in the structure. So if there are finitely many n -orbits for all n , then there are finitely many n -types realised for all n , and therefore, by Theorem 1.1, the structure A is $\aleph_0$ -categorical.

□

2.2 Homogeneous Structures

Some structures have the property that if there exists a finite isomorphism between two n -tuples $\bar{a}$ and $\bar{b}$ in the structure, then this isomorphism can be extended to an automorphism of the structure. Such structures are called *homogeneous*, because if two n -tuples look the same locally (they cannot be distinguished by existing relations amongst their elements) then when we take a step back and look at them within the structure as a whole, they still cannot be distinguished.

In such cases, to count the n -orbits for a given n , it is sufficient to count the isomorphism types on n -tuples. Furthermore, if the structure has finitely many i -ary relations for each i , we can conclude immediately that the structure is $\aleph_0$ -categorical, since if m_i is the number of i -ary relations, then the number of orbits, F_n on an n -tuple is bounded above by the following:

$$F_n \leq \prod_{i=1}^n (2^{m_i})^{\frac{n!}{(n-i)!}}$$

This equation assumes that relations do not hold when some elements of the n -tuple are equal (for example, in graphs this means that loops do not occur at vertices). If one does not have this property in the model, one can use the above formula with added relations with fewer arguments to express the different possibilities, (for example, vertices at which loops occur can be distinguished by a unary relation).

2.3 Examples of Structures

Example 2.1 A Structure with no Relations

Consider the structure $\mathbf{N}$ consisting of the naturals with no relations. Since there are no relations restricting the automorphism group, we find that $\text{Aut}(\mathbf{N}) = \text{Sym}(\mathbf{N})$. $\text{Sym}(\mathbf{N})$ has one orbit on $\mathbf{N}^n$ for every n and so the structure is $\aleph_0$ -categorical. Furthermore, the structure is homogeneous, since there is only one isomorphism type for all n -tuples, corresponding to a single n -type. Any n -tuple can be mapped to any other n -tuple by an automorphism of the structure.

Example 2.2 The Rationals with Order

Define $(\mathbf{Q}, <)$ to be the structure consisting of the rationals with a binary relation, $R(a, b)$ iff $a < b$. Two n -tuples are in the same n -orbit if the position of the i th element in relation to all other elements in the tuple is the same for all i in both tuples. For example, the 5-tuple $(5, 3, 2, 1, 4)$ has the same ordering as $(7/3, -64, -65.6, -101, 1)$ since for both tuples the 4th element is the least, the 3rd element is the second least, etc. Both 5-tuples are in the same 5-orbit since we can find an automorphism of $(\mathbf{Q}, <)$ which maps each of the bounded intervals $[1, 2], [2, 3], [3, 4]$ and $[4, 5]$ by linear interpolations to the intervals $[-101, -65.6], [-65.6, -64]$ etc. and which maps the unbounded intervals $(-\infty, 1]$ and $[5, \infty)$ by translations to $(-\infty, -101]$ and $[7/3, \infty)$.

Note that orbits and orderings of n -tuples coincide. But orderings of n -tuples are just the finite isomorphism types, so this structure is homogeneous, and since

there is only one binary relation, it is $\aleph_0$ -categorical. In fact, we can count the number of isomorphism types as follows: For $n = 1$ there is one isomorphism type (one orbit since a simple translation will map any rational to any other rational). For $n = 2$, the second element in the pair can either be less or greater than the first, giving two isomorphism types. When adding an element to an n -tuple to obtain an $n + 1$ -tuple, we have $n + 1$ possibilities of where in the ordering of the existing n -tuple the new element falls, as the least element, between the least and the second least element, etc. Thus, if F_n is the number of isomorphism types of n -tuples, then there are $(n + 1)F_n$ isomorphism types on $n + 1$ -tuples. Thus, by induction, we have that $F_n = n!$.

Example 2.3 A Non-homogeneous Structure

All graphs are structures with a single binary relation, A , representing adjacency. Construct a graph on an infinite set of vertices as follows: Take an infinite set of subgraphs on four vertices. These subgraphs are of two types of which there is an infinite number of each. Those of the first type are complete graphs on four vertices, K_4 . Those of the second type are complete graphs minus a single edge. See Figure 3.

In each subgraph of the type K_4 or K_4 minus an edge, we can find a triangle. Each of these triangles gives a triple of the same isomorphism type. However, these triples split into one of two different orbits in the automorphism group, depending on whether or not they are contained in a K_4 graph. This is because they realise

Figure 3 A Non-homogeneous Structure

different 3-types. A formula which distinguishes them is the following (here a, b , and c are the free variables, representing the triangle vertices):

$$\exists x : A(a, x) \wedge A(b, x) \wedge A(c, x)$$

Note that this formula describes the existence of something (a vertex, or a path etc.) in the vicinity of the triangle. It is not a property which only involves the three vertices of the triangle. Thus, isomorphisms on n -tuples and n -types do not coincide and a finite isomorphism between triangles cannot necessarily be extended to an automorphism of the whole structure. Thus, this structure is not homogeneous, although it is $\aleph_0$ -categorical.

Example 2.4 The Random Graph

Construct a graph on a countably infinite set of vertices by tossing a coin for each pair of vertices to determine whether or not there is an edge joining them.

It is shown in Cameron [1], page 87, that with a probability of one, the resulting graph G has the following universal property:

** Given any two finite, disjoint subsets, A and B , of the vertices of G , there exists a vertex $x \in G \setminus A \cup B$ such that x is adjacent to all of the vertices in the set A , and not adjacent to any of the vertices in the set B .*

From (*) it is not difficult to prove that:

- a) Every finite graph appears as a subgraph of G .
- b) Every finite graph in G is embedded in every possible finite supergraph.

Using b) and a back and forth argument, it is possible to show that G is homogeneous. In other words, the number of n -types is the same as the isomorphism types on n vertices, which is just the number of labelled graphs on n vertices. Thus, the orbits of G can be represented by the number of labelled finite graphs. Note that the number of orbits in this case is equal to

$$\prod_{i=1}^n (2^{m_i})^{\binom{n}{i}}.$$

This is less than the upper bound given above for the number of orbits on a homogeneous structure because the adjacency relation is symmetric, and therefore for two vertices, a and b , the relation $A(a,b)$ cannot occur independently from $A(b,a)$. Otherwise, the edges between different vertices have been produced by independent coin tosses, so there is nothing else in the theory of the structure to restrict the occurrence of possible n -types.

We can conclude that G is $\aleph_0$ -categorical because it has finitely many orbits on n -tuples for each n . Though property (*) is not expressible in first-order logic, and therefore not in the theory of G , each instance of property b) for specific finite graphs, which is a consequence of property (*), is in the theory, and so the back and forth argument, which relies on b) to execute, establishes that any two countable graphs with property (*) are isomorphic and therefore are described by the same theory. Thus, with probability one, the random procedure described above will, up to isomorphism, produce a specific graph which is called the random graph.

Example 2.5 A non- $\aleph_0$ -categorical Structure

This time we construct a graph on a countably infinite set of vertices which is not $\aleph_0$ -categorical. Construct a tree with a root vertex of degree one, and all subsequent vertices at depth i in the tree branch to i new vertices. See Figure 4.

It is not difficult to establish that such properties as the existence of paths of fixed lengths, or the degree of a vertex (if it is finite), are expressible in first-order logic. Thus, there is no automorphism which will map a vertex at depth i , to a vertex at depth j where $i \neq j$, since these vertices have different degrees, and are therefore distinguished by some formula. Since there are an infinite number of depths in the tree, there must be infinitely many orbits on the vertices of the tree. Thus, this structure cannot be $\aleph_0$ -categorical. What would another non-isomorphic model of the theory of this graph look like?

Figure 4 A Non- R_0 -Categorical Structure

Let's try to imagine what sort of statements are in the theory of this graph.

It would include such statements as:

There exists exactly one vertex of degree one.

There exists exactly one vertex of degree three.

The vertex of degree one is connected by an edge to the vertex of degree three.

etc.

In fact, the theory of this structure would provide a very good description of the graph, it would be able to describe all of the vertices and their degrees, and how they are connected to one another.

However, there are some properties of this graph which cannot be stated in

first-order logic:

Every vertex has finite degree.

Since the degrees of the vertices are unbounded, such a statement would require an infinite number of disjunctions:

For every vertex, v , the degree of v is one or two or three etc.

Such a statement is not permitted in first-order logic. Similarly, the property that the graph is connected cannot be expressed because the paths connecting the vertices are arbitrarily long. So the theory of this graph describes it very clearly and asserts its presence as a subgraph of any model, but cannot preclude the existence of other vertices of infinite degree which are necessarily in a different connected component, since the theory of the original graph also includes statements like:

The vertices of degree 100 are joined by an edge to vertices of degree 99 or of degree 101.

Thus, an alternate model of this theory would include the original graph with a tree with each vertex of infinite degree on a countably infinite number of vertices. In fact, any graph which included the original graph, as well as any countable collection of countably infinite graphs with all vertices of infinite degree and without finite cycles would be a model of the given theory.

Note that in all of these models, the graph we began with occurs as a subgraph. It is not difficult to show that it is elementarily embedded in each of these models

and therefore, it is an atomic model.

2.4 Symmetry in Homogeneous Structures

It seems reasonable to think of homogeneous structures as highly symmetric ones. Since any finite isomorphism between substructures can be extended to an automorphism of the whole structure, we conclude that any finite regions which look the same also look the same globally. For example, in a homogeneous structure without unary relations, each single element is isomorphic to any other single element. Therefore, the automorphism group of the structure must be transitive. This hints that the degree of symmetry in a homogeneous structure depends on the predicates in the language which describes it. Consider that for a structure, A , with an automorphism group that has at most countably many orbits on n -tuples for each n , we can change the language, L , and define new relations on A whereby the structure becomes homogeneous. This is achieved as follows: For each n , enumerate the distinct orbits on n -tuples as O_{n_i} and for each such orbit, create an n -ary relation R_{n_i} which holds for each n -tuple in the O_{n_i} th orbit. This structure M is called the *canonical structure* of the automorphism group of A .

THEOREM 2.2 $Aut(A) = Aut(M)$

Proof.

It is clear that $Aut(A) \subseteq Aut(M)$, since by the definition of M , a one-to-one mapping g preserves the relations of M iff it takes each n -tuple to an n -tuple in

the same n -orbit of $\text{Aut}(A)$, which is trivially satisfied for all automorphisms of $\text{Aut}(A)$.

To show that $\text{Aut}(M) \subseteq \text{Aut}(A)$, let $g \in \text{Aut}(M)$ and assume $g \notin \text{Aut}(A)$. Then there exists a formula ϕ in A and an n -tuple $\bar{a}$ with $\phi(\bar{a})$ but not $\phi(\bar{a}^g)$. Thus, $\bar{a}$ and $\bar{a}^g$ cannot be of the same n -type in A since they are distinguished by the formula ϕ and so cannot be in the same orbit of $\text{Aut}(A)$. But then they must be distinguished by the relations in $\text{Aut}(M)$ which correspond to their respective orbits. This contradicts the fact that $g \in \text{Aut}(M)$. Thus we conclude that $\text{Aut}(A) = \text{Aut}(M)$. $\square$

We further observe that if the structure, A , is $\aleph_0$ -categorical, since each orbit of $\text{Aut}(A)$ corresponds to an n -type which is principal and therefore has a formula which acts as its support, ϕ_n , the canonical structure, M , consists only of relations which are definable in the original structure A (by the formulas ϕ_n).

Since $\aleph_0$ -categorical structures can so easily be regarded as homogeneous structures under a language which is in some sense equivalent to the original language, we further restrict our examination to homogeneous structures over specific languages with finitely many relations. For example, if we consider countable structures over languages with a single binary relation, we find that the homogeneous structures are highly symmetric and in fact, there are relatively few examples of them. This scarcity is explained by a famous theorem of Fraïssé which appeared in 1954. We begin with a definition. Given any structure, A , define the

age of A , $\text{Age}(A)$, to be the class of all finitely generated substructures of A . We can now state Fraissé's theorem:

THEOREM 2.3 *Let C be a countable class of finitely generated structures with the following three properties:*

- 1) *C is closed under taking substructures and under taking isomorphisms.*
- 2) *C satisfies the Joint Embedding Property (JEP): If M and N are structures in C , then there exists a structure R in C which embeds both M and N .*
- 3) *C satisfies the Amalgamation Property (AP): If M and N are structures in C which share an isomorphic substructure, then there exists a structure R in C which embeds both M and N and amalgamates them along their isomorphic substructures.*

Then there exists a countable homogeneous structure H whose age is C and which is unique up to isomorphism.

Note: JEP is only a special case of AP if the class C includes the empty structure.

The proof of this theorem is found in Cameron [1], page 40. The Joint Embedding Property and the Amalgamation Property are used iteratively on a chain of structures in C to construct a countable structure H which satisfies $C \subseteq \text{Age}(H)$. The structures are embedded into larger structures in such a way that all possible one point extensions appear at each substructure. This ensures that a back and forth argument, similar to the one used in Theorem 1.1, can be executed to establish the homogeneity and uniqueness of H . The fact that C is closed under

taking substructures ensures as well that $\text{Age}(H) \subseteq C$.

We also know that the converse of this theorem is true. If H is a homogeneous structure, then $\text{Age}(H)$ is a class of finite structures which satisfies the three properties of the theorem.

The random graph, described in example 2.4, is the countable homogeneous structure whose age consists of all finite graphs. Note that the class of all finite graphs satisfies the three properties given in Fraïssé's Theorem.

We can now explain why there are sometimes so few countable homogeneous structures over a given finite language. In some cases, from conjecturing that the age of a homogeneous structure H contains a certain number of finite structures and does not contain certain others, we can use the amalgamation property on these finite structures to describe all of $\text{Age}(H)$, and then the uniqueness part of Fraïssé's theorem ensures that we have only one homogeneous object corresponding to this case. Since there are a limited number of possible cases to examine when a language is finite, we can hope to describe all homogeneous structures over that language.

For example, if we consider structures with a single binary relation, and we are interested in those which are homogeneous and whose theory includes an axiom so that the relation is symmetric but not reflexive, we are interested in finding all homogeneous graphs. If we then conjecture that the age of one such graph, G , contains a triangle, but does not contain the complete graph on four vertices, K_4 ,

we can conclude by Ramsey's Theorem that the class contains the complement of K_n for each n . If we further conjecture that it contains the two graphs on three vertices other than the triangle and its complement, we find that there is only one class which contains these finite graphs and which satisfies the three conditions of Fraisse's theorem. This uniquely determines the age of G and we can conclude that G is the unique homogeneous object which embeds all finite graphs not embedding K_4 .

Lachlan and Woodrow [4], have used such arguments to establish that the only homogeneous countable graphs are either the disjoint union of m complete graphs on n vertices where at least one of m and n is countably infinite, or its complement; the random graph; or for each $n \geq 1$, the graph R_n , which embeds every finite graph not embedding K_{n+1} or its complement.

Chapter 3

Model Theory

In this chapter we try to understand more about the automorphism groups of $\aleph_0$ -categorical structures. To do this, we examine what model theory tells us about the composition of the structures themselves. For example, the random graph, R , contains many infinite complete or complements of complete subgraphs. We will prove later that for some such subgraph, M , any automorphism of M (which is in fact any permutation of the vertices of M) extends uniquely to an automorphism of the whole graph. In this case, we can regard the automorphism group of the random graph in terms of how it acts on this subset of vertices. This example generalises in some sense to all $\aleph_0$ -categorical structures.

3.1 Indiscernibles which Generate a Structure

We say a set $X \subseteq Y$, *generates* Y over a model A if Y is the smallest set containing X in the universe of A which is closed with respect to taking terms. In other words, if $\bar{y}$ is an n -tuple from Y^n and f is any n -ary function of the language, then $f(\bar{y}) \in Y$.

Clearly, if X generates Y , then the action of an automorphism g on the set X , uniquely determines its action on the superset Y . In fact, if $y \in Y$ then y can

be expressed as a term of elements in X , $y = t(x_1, \dots, x_n)$ so that the image of y under an automorphism, g , of A , is completely determined by the action of g on the finitely many elements, $\{x_1, \dots, x_n\}$, of X .

Consider the existential formula $\psi_x(\bar{y}) = \exists x \phi(\bar{y}, x)$. If there exists a term $t_{\bar{y}} = t(\bar{y})$ such that the sentence $\forall \bar{y} \psi_x(\bar{y}) \rightarrow \phi(\bar{y}, t_{\bar{y}})$ is in the theory, T , then we say that the term $t_{\bar{y}}$ *witnesses* the formula ψ_x .

As an example, consider the structure of the rationals in the interval $[0,1]$ with the relation $\leq$. When we choose a language for this structure, we may or may not choose to include constants, c_0 and c_1 to represent the elements 0 and 1, respectively, of the structure. In either case, our theory of the structure will include the sentence:

$$\exists x \forall y y \leq x$$

which asserts the existence of a maximum element, 1, on the interval. If we have included a constant c_1 in our language, then c_1 acts as a witness for the existential sentence above if our theory includes the axiom:

$$\exists x \forall y y \leq x \rightarrow \forall y y \leq c_1$$

In this case, c_1 must be mapped to the element one by the model.

We say that the theory T has *built-in Skolem* functions and is a *Skolem* theory if every existential formula ψ_x has a witness.

Given a structure, A , we call a subset X of the universe of A an *indiscernible*

set with respect to some simple ordering $(X, <)$, if no formula, ϕ , can distinguish two tuples, $\bar{a}, \bar{b} \in X^n$, similarly ordered by $<$. In other words, if $\bar{a} = (a_1, \dots, a_n)$ and $\bar{b} = (b_1, \dots, b_n)$ with $a_1 < \dots < a_n$ and $b_1 < \dots < b_n$ then $A \models \phi(\bar{a}) \leftrightarrow \phi(\bar{b})$ for all n -ary formulas ϕ . The ordering $(X, <)$ need not be definable in the structure.

We now bring these three concepts together by stating a well-known theorem of Model Theory which tells us that in some cases we can choose an indiscernible subset from the universe of a structure which will be a generating set for the whole structure. Ehrenfeucht and Mostowski proved this theorem in 1956. See Chang & Keisler [2], Theorem 3.3.11, page 149, for a proof.

THEOREM 3.1 (*Ehrenfeucht-Mostowski*) *If T is a Skolem Theory with infinite models, then there exists some model of T , A , with an indiscernible set $(X, <)$ which is order-isomorphic to the rationals, $\mathbb{Q}$, and which generates the whole of the structure A . Furthermore, any automorphism of X with respect to the ordering $<$ extends uniquely to an automorphism of the whole structure.*

We note that since the set X is countable, and our language is countable, then the model which is generated by X over the language must be countable as well. Interpreting this theorem for $\aleph_0$ -categorical structures, we conclude that the unique countable model of the Skolem theory T must contain this set of generating indiscernibles.

We conclude that if an $\aleph_0$ -categorical theory is a Skolem theory, we can represent the action of its automorphism group by its action on a generating set of

indiscernibles. Moreover, the setwise stabiliser of this set of indiscernibles acts on $(X, <)$ in such a way that there exists a one-to-one mapping from $\mathbb{Q}$ to X such that every automorphism of $(\mathbb{Q}, <)$ occurs in the setwise stabiliser. One can list the groups which act on $\mathbb{Q}$ and contain $Aut(\mathbb{Q}, <)$ as a subgroup:

(A) $Aut(\mathbb{Q}, <)$

(B) $Aut(\mathbb{Q}, <)$ together with an automorphism which flips the order. In other words, the relation which must be preserved by all automorphisms is a ternary relation of betweenness.

(C) The set of automorphisms which preserve a cyclic order on the rationals. We can think of this as the cyclic order we obtain when we identify $+\infty$ and $-\infty$ with one another to obtain what we call a point at infinity. So the relation which must be preserved is a ternary relation $C(a, b, c)$ on the rationals where the relation holds if and only if one of the following is true:

i) $a < b < c$

ii) $b < c < a$

iii) $c < a < b$

We can think of an automorphism of this structure, other than an order preserving automorphism of the rationals, as a cycle. It would require, for example, that the images of convergent sequences of rationals also converge (we must extend our notion of convergence to include sequences which converge to the point at infinity). Alternately, there exists some irrational number r for each automorphism

g such that if $x < y$ then $y^g < x^g$ if and only if $x < r < y$.

(D) The above group together with an automorphism which reverses the cyclic order. In this case, the relation which must be preserved is quaternary. $D(a, b, c, d)$ holds if and only if some cyclic reordering of (a, b, c, d) is either strictly increasing or strictly decreasing.

(S) The symmetric group.

Further background on this theorem is contained in Appendix B which uses a topology on automorphism groups that is introduced in Chapter 4.

It is hoped by this comprehensive list, that we might have a good understanding of how the automorphism group of an $\aleph_0$ -categorical structure acts on a set of indiscernibles order isomorphic to $(\mathbb{Q}, <)$ when they generate the structure. As this lists the possibilities for the action of the setwise stabiliser of a set of indiscernibles, in the case of a Skolem Theory, we need only consider in addition automorphisms of the structure which map a generating indiscernible set to sets that we can think of as conjugate or parallel sets of indiscernibles.

Much of the above discussion hinges on the theory T being a Skolem Theory. The theories we examine in this thesis are not Skolem theories. In fact, when a theory is Skolemized, (a procedure whereby functions are added to the language to provide terms which act as witnesses for the existential formulas) the language is expanded and when the language describing a structure expands, the automorphism group of that structure may collapse to a subgroup. For this reason, we may

find that when we Skolemize a theory, we have introduced so much rigidity into the automorphism group of the structure that the theory is no longer $\aleph_0$ -categorical.

Nevertheless, any $\aleph_0$ -categorical structure has an infinite set of indiscernibles, order-isomorphic to $(\mathbb{Q}, <)$. In some cases, though the theory is not a Skolem theory, the action of the group on this set of indiscernibles uniquely describes its action on the whole structure.

3.2 Stability

Sometimes a model of a theory contains an infinite set of tuples which can be simply ordered by some formula of the language. In this sense, the model is similar to the rationals which are simply ordered by $<$.

We say a theory, T , is *unstable* if there exists a model A of T such that there exists an infinite sequence of n -tuples $\{\bar{a}_i\}_{i < \omega}$ from $|A|^n$ and a formula $\phi(\bar{x}, \bar{y})$ on pairs of n -tuples such that

$$\phi(\bar{a}_i, \bar{a}_j) \leftrightarrow i < j.$$

Stable theories are theories for which no such formula exists for any model of the theory.

We see from the above that the theory of the rationals with order is unstable. Another property of the rationals with order is that though they are a countable set, they contain an uncountable number of gaps, namely the non-rational real numbers. This property can also be generalised to give us an alternate definition

which distinguishes stable structures from unstable structures. To formalize these ideas, we return to an expanded notion of n -types.

3.2.1 N -Types Over a Set

Recall that a complete n -type is a maximally consistent set of formulas in n free variables. These formulas are formed from the language L . We now define a complete n -type over a subset X of our universe of A , which we shall call $tp(\bar{x}/X)$. This is a maximal consistent set of formulas, in which the formulas are permitted to refer directly to elements from the set X . In other words, it is the n -type of a tuple in the language L^+ obtained when we expand L to include a constant for each element of X .

For example, in the random graph, if we choose a single vertex for our set X which our expanded language labels as the constant v , then the complete 3-type, formally corresponding to a triangle in the random graph, now splits into a number of distinct 3-types over X , distinguished by the relationship of the three vertices of a specific triangle to the vertex v . So the set X permits us to make finer distinctions between various n -types.

Of course the set X need not be finite. As another example, consider 1-types over the set $X = \mathbb{Q}$ in the structure $(\mathbb{Q}, <)$, with constants c_r added for each rational r . A maximal consistent set of formulas with one free variable over the set $\mathbb{Q}$, would consist of a set of formulas of the sort $c_r < x$ possibly together with

formulas $x < c_s$. The fact that all the formulas of a single type must together form a consistent set means that if the two formulas above are contained in the same 1-type then $r < s$. We can see that the distinct 1-types over the set $\mathbb{Q}$ include each Dedekind cut.

3.2.2 Stone Spaces

We call the set of all complete n -types with respect to a theory T the n th Stone space of T , notated as $S_n(T)$. We can also consider the n th Stone space over a set X in a model A of T , $S_n(X; A)$, which is the set of all complete n -types over the set X . Stone spaces have a topology: we take as a basic closed set, the set of all types containing a given formula ϕ on n free variables, notated as $\|\phi\|$. Note that closed sets are also open, since the complement of a basic closed set is also closed, since it is given by the formula's negation.

For an example of a Stone space, the 1st Stone space of the theory of the rationals with order consists of only one type since all elements satisfy the same formulas. However, as we did in the last example, if we expand our language to include constants for each rational, in other words, if we consider $S_1(\mathbb{Q}; \mathbb{Q})$, the 1st Stone space consists of a type for each real number as well as other types for elements which dominate or are dominated by every rational. The basic open sets of the topology consist of all finite unions and intersections of open and closed intervals on these types with real endpoints or no endpoints at all, since every

formula consists of finite conjunctions and disjunctions of formulas of the type $x < c_r$ or $x > c_r$ or $x = c_r$ and their negations.

The Stone space $S_0(T)$ where T is not a complete theory, consists of all complete theories $\bar{T}$ which contain T . The famous Compactness Theorem of first-order logic is so-named because it is in essence a proof that $S_0(T)$ is a compact topological space: $S_0(T) = \emptyset$ iff T does not have a model. But,

$$S_0(T) = \bigcap_{\phi \in T} \|\phi\|$$

These are closed sets of the topology. By the compactness theorem of logic, some finite subset $T_n = \{\phi_i\}_{i=1}^n$ of T has no model which means that

$$\bigcap_{i=1}^{i=n} \|\phi_i\| = \emptyset$$

This gives us that $S_0(T)$ is a compact topological space.

We say that a theory is λ -stable for some cardinal λ if for each model A of T , and for each n , if X is a subset of the universe of A and $|X| \leq \lambda$ then the size of the Stone space, $|S_n(X; A)| \leq \lambda$.

THEOREM 3.2 *A theory is stable if and only if it is λ -stable for some infinite cardinal λ .*

Proof. We will give a sketch of the proof for one direction of the theorem which appears in Hodges [3], page 308-309. The proof shows how the existence of a formula and an infinite set of tuples in a model A of T which witnesses the instability

of T will enable us, for any cardinal λ , to demonstrate that T is not λ -stable. This will establish that if a theory T is λ -stable for some cardinal λ then it is a stable theory.

Given any cardinal λ , it is possible to construct a set Q of cardinality λ which is dense in a set R of cardinality greater than λ . For the set R , consider all sequences on $\{0, 1\}$ of length $\leq \lambda$ which are not identically 1 from some point onwards. There are $2^\lambda > \lambda$ such sequences. Choose as the subset Q those sequences which are identically 0 from some point onwards in the sequence. This set has cardinality λ and is dense in R .

Assume that the theory T is unstable and so there exists a formula $\phi(\bar{x}, \bar{y})$ and a set of n -tuples $\{\bar{a}_i\}_{i < \omega}$ in a model A of T with $A \models \phi(\bar{a}_i, \bar{a}_j) \leftrightarrow i < j$. Now add to the language of T an n -tuple of constants $\bar{c}_r$ for each $r \in Q$. Then the theory $\bar{T} = T \cup \{\phi(\bar{c}_r, \bar{c}_p) : r < p \in Q\}$ has a model B by the Compactness Theorem. Now we consider n -types over a set X_Q in B where X_Q consists of the images in the model B of the constants occurring in each of the $\bar{c}_r$'s. We then obtain a distinct n -type corresponding to each element of the set R , since Q is dense in R according to the simple ordering $<$ and the formula ϕ mimics this ordering on the n -tuples $\{\bar{c}_r\}_{r \in Q}$. This set X_Q has cardinality λ but the size of the Stone space $S_n(X; A)$ is of cardinality 2^λ since we have a distinct n -type for each element of R and R has cardinality 2^λ . $\square$

To reiterate these ideas, in the same way that the rationals are dense in the

reals, we can always construct a set Q of a particular cardinality, which is dense in a set R of cardinality $2^{|Q|}$. The formula ϕ permits us to construct a larger theory which the Compactness Theorem ensures is consistent, in which ϕ orders a set of tuples of the same cardinality as Q . In a model for this theory, if we take the reduct to our original language we have a model for T . Now we take the elements of the tuples ordered by ϕ as our set X . The number of n -types over this set will be of the same cardinality as R since Q is dense in R and the formula ϕ permits us to distinguish a tuple for each element of R using the tuples for the elements from Q . This gives us that T cannot be λ -stable for any cardinal λ .

3.3 The Random Graph

The random graph is unstable. Choose tuples $\{\bar{a}_i\}_{i < \omega}$ and the formula $\phi(\bar{a}_i, \bar{a}_j)$ as follows: Let $\bar{a}_1$ be any pair of vertices (x_1, y_1) which are not adjacent. Assume we have chosen distinct pairs of vertices (x_i, y_i) for the tuple $\bar{a}_i$ for each $i < k$. For the tuple $\bar{a}_k$, choose a vertex x_k such that it is not adjacent to any of the vertices, $y_i, i < k$. This is possible by the universal property of the random graph. Choose a vertex y_k which is adjacent to all of the vertices x_i for $i < k$ and non-adjacent to x_k . Let $\bar{a}_k = (x_k, y_k)$. Now the formula $\phi(\bar{a}_i, \bar{a}_j)$ given by x_i is adjacent to y_j is true if and only if $i < j$. The formula ϕ and the tuples $\{\bar{a}_i\}_{i < \omega}$ witness that the theory of the random graph is unstable.

By the discussion in section 3.2, we should be able to find a set of vertices

X of countable cardinality such that the number of distinct n -types over X is uncountable. If we distinguish any infinite set of vertices for our set X , we find that the number of 1-types that could be realised by a vertex v is uncountable, since the edge relationships between v and each vertex in X are independent. Each possible 1-type corresponds to a subset of X , where we distinguish which vertices in X are adjacent to the vertex v . (Note that all of these types cannot be realised in the random graph since it is countable). Thus the Stone space has cardinality $2^{|X|}$ whence T cannot be $\aleph_0$ -stable.

We now prove the theorem introduced at the beginning of the chapter, which appears as a problem in Hodges [2], page 356:

THEOREM 3.3 *The random graph contains an infinite set of vertices, such that any permutation of these vertices extends uniquely to a permutation of the random graph.*

Proof. We begin by constructing a countably infinite graph around an infinite subset of vertices, R_0 , so that any permutation of the vertices in R_0 extends uniquely to an automorphism of the whole graph. We then prove that the graph we have constructed is in fact the random graph.

We build a countable graph as follows: We begin with a countably infinite set of vertices, R_0 , with no edges. Assume we have defined a series of countably infinite sets of vertices, $R_0, R_1, \dots, R_{k-1}$ and all edge relationships within the union

of vertices $\bigcup_{i \leq k-1} R_i$ to give the graph G_{k-1} .

Vertices:

At this stage, we add a countable set of new vertices R_k . The vertices of R_k are expressed as certain distinct subsets of $R_0 \cup \dots \cup R_{k-1}$ where the subset defining a new vertex consists of the vertices in G_{k-1} to which the new vertex is adjacent. Only those finite subsets of the vertices of G_{k-1} with a certain distribution across the sets $R_0, \dots, R_{k-1}$ form vertices in R_k :

$$R_k = \{X \subset R_0 \cup \dots \cup R_{k-1} : |X \cap R_i| = k - i \ \forall i < k\}. (*)$$

Note that each X is a finite set so that, inductively, R_k is countable since $R_0 \cup \dots \cup R_{k-1}$ is countable.

Edge Relationships:

If X is a new vertex in R_k , we set $X \text{ adj } Y$ with $Y \in R_0 \cup \dots \cup R_{k-1}$ if and only if $Y \in X$. If both X and Y are new vertices in R_k , we set them non-adjacent.

We let G_k be the graph on the union of vertices $\bigcup_{i \leq k} R_i$ with the given edge relationships. At each stage, G_k is a countable graph.

We now let

$$G = \bigcup_{i < \omega} G_i.$$

G is the countable union of countable graphs, and so G is also countable.

Note that two vertices X_i and Y_j from R_i and R_j respectively, with $i < j$, are adjacent if and only if $X_i \in Y_j$.

We now claim that any permutation, g , of the vertex set R_0 extends uniquely to an automorphism of G . To show that it extends, we see that if we consider g 's induced action on R_1 (as the one-element subsets of R_0), and extend g to g_1 accordingly, this gives an automorphism of $R_0 \cup R_1$. Also, g_1 's action, restricted to R_1 is a bijection. We continue by induction, assuming g has been extended to an automorphism, g_{k-1} of the graph G_{k-1} with the property that its action on each vertex set R_i with $i \leq k-1$ is a bijection.

If $X \in R_k$, then $X \subset R_0 \cup \dots \cup R_{k-1}$. So we can extend g_{k-1} to g_k by considering the induced action of g_{k-1} on the subset X in R_k . Thus, $X^{g_k} = Y$, where $Y = \{v_1^{g_{k-1}}, \dots, v_n^{g_{k-1}}\}$ when $X = \{v_1, \dots, v_n\}$. Note that $Y \in R_k$ since Y has the same distribution (*) over $R_0, \dots, R_{k-1}$ because g_{k-1} is a bijective mapping on R_i for all $i \leq k-1$. Also g_k is a bijection of R_k to itself, since R_k consists of exactly one vertex for each finite subset of $R_0 \cup \dots \cup R_{k-1}$ of the given distribution (*), and because g_{k-1} is a bijection of each of the R_{k-1} 's.

Also, g_k is an automorphism of G_k since any bijection of the R_i 's onto themselves is an admissible action restricted to R_i , since R_i has no edges. Secondly, the vertex $X \in R_k$ which is adjacent to precisely the vertices in $R_0 \cup \dots \cup R_{k-1}$ which it contains, is mapped to the vertex Y which is adjacent to precisely the images of the elements of X under the automorphism g_{k-1} .

So $\bar{g} = \lim_{k \rightarrow \infty} g_k$ is an automorphism of the graph G which extends the action of g on R_0 . Furthermore, $\bar{g}$ is the unique extension of g to an automorphism of

the graph G . Any automorphism which setwise stabilizes R_0 , must also setwise stabilize R_k for any k , since the vertices of R_k are the only vertices of the graph G which are adjacent to precisely k vertices from R_0 . Once this is established, any automorphism whose action has been determined on the vertices of $R_0 \cup \dots \cup R_{k-1}$ must be uniquely extended to R_k since the only possible image of a vertex X adjacent to the vertices $\{v_1, \dots, v_n\}$, must be the vertex Y which is adjacent to precisely the images of the vertices $\{v_1, \dots, v_n\}$.

Finally, we must show that the graph G is the random graph. We shall do this, by establishing that G has the universal property: If A and B are finite subsets of vertices of G , then there exists a vertex X of G such that X is adjacent to all of the vertices in A and none of the vertices in B .

Let A and B be a given pair of finite sets of vertices. Choose an integer k such that the subsets A and B are both contained in $\bigcup_{i \leq k} R_i$. Let $n = |A| + k$. We now choose a vertex $X \in R_n$ such that $A \subset X$ and $B \cap X = \emptyset$. This will ensure that X is the desired vertex.

In order that $X \in R_n$, we must choose a subset of $\bigcup_{i \leq n} R_i$ with the distribution (*) $|X \cap R_i| = n - i$. We know that we can choose X to contain A since

$$|A \cap R_i| \leq |A| = n - k < n - i \quad \forall i \leq k.$$

Since all occurrences of elements of A are in R_i for $i \leq k$, then the number of vertices we choose from R_i for X is always greater than the maximum number of vertices from A which can be contained in R_i . Then from each R_i we choose the correct

number of vertices for X , being sure to include any vertices of A which occur, and to not include any vertices from B . Since each set of vertices R_i is infinite, this can be done. The X which we have obtained has the desired property.

Thus, G satisfies the universal property and is countable. Therefore, G is the random graph, and we have found an infinite set of vertices of the random graph such that any permutation of the vertices extends uniquely to an automorphism of the graph. $\square$

Note that the above construction and proof executes just as well if instead of beginning with an infinite set of vertices with no edges for R_0 , we begin with any infinite, countable graph. Then we have that any automorphism of this initial countable graph extends uniquely to an automorphism of the whole structure.

3.4 Describing an Automorphism

Continuing with the example of the random graph, we note that its theory is not a Skolem theory. Though there are existential formulas, for example, the formula which asserts the existence of a vertex connected to all three vertices of an arbitrary triple, there are no terms (in fact, the language has no terms nor even any definable functions) which witness this formula. Nevertheless, we have shown that there is an infinite set of vertices X for which any permutation of X extends uniquely to an automorphism of the whole graph. The set, X , is also a set of indiscernibles for the random graph. We can define an order $<$, on X (remember

that the order does not need to be definable in the language) such that $(X, <)$ is dense and therefore order-isomorphic to $(\mathbb{Q}, <)$. The automorphism group of the random graph acts upon this set as a supergroup of $Aut(X, <)$, in fact, it acts as the symmetric group on X . Furthermore, the action of an automorphism group element upon this set uniquely determines its action on the whole structure.

Elements of the universe are fixed under automorphisms in different ways. For example, consider again the structure of the rational interval $[0, 1]$ with ordering. Whether or not our language includes constants for the values 0 and 1, it is clear that these points must be fixed by any automorphism of the structure. This is because they are the unique potential witnesses for the formulas asserting the existence of maximum and minimum elements. If instead we had a partial ordering on a structure, one can imagine a formula asserting the existence of maximal elements, but because there are many potential witnesses for this formula, the automorphism group of the structure may map maximal elements to other maximal elements, and thus not fix them. It may be that an automorphism's action on an infinite number of indiscernibles must be specified, before the image of a maximal element is determined. For this reason, the partial ordering example has more in common with what happens in the random graph.

In the case of the random graph, it is an easy consequence of its universal property and homogeneity that the image of a vertex under an automorphism, g , cannot be uniquely determined by g 's action on a finite subset of vertices. However,

from the proof of Theorem 3.3, if we limit ourselves to consider automorphisms from the setwise stabiliser of the subgraph R_0 , then the image of a vertex from any one of the R_i 's is uniquely determined once the automorphism's action on finitely many of the vertices of R_0 is specified.

This distinguishes a theory such as that of the random graph from a Skolem Theory. In a Skolem Theory, because the indiscernibles generate the structure, every element of the structure is a term on a finite number of indiscernible elements. Thus, every element of the structure is fixed by any automorphism, once that automorphism's action on a finite subset of indiscernibles is defined. In Chapter 4, we will be interested in finding automorphisms which map an arbitrary n -tuple to any other n -tuple in the same n -orbit. If every element in the structure is a term on a finite subset of the generating set of indiscernibles, we can translate this problem into a problem of finding an automorphism which maps an arbitrary n -tuple of indiscernibles to any other n -tuple of indiscernibles.

Chapter 4

Dense Subgroups

In order for the automorphism group, G , of a structure, to act on a countably infinite set Γ and have finitely many orbits on Γ^n for each n , it must be very large. In fact, it can be shown that $|G| = 2^{\aleph_0}$. As it is the automorphism group of an $\aleph_0$ -categorical structure, Cameron [1], page 84, uses an argument of Jim Kister's to prove further that G will have a free subgroup on $2^{\aleph_0}$ generators. Can we find relatively small subgroups of G which have the same orbits on n -tuples? The answer is yes, in fact, in some cases we can find subgroups with the same orbits on as few as 2 generators. Subgroups which have the same orbits on n -tuples are called dense subgroups, due to the following topology:

Given that G acts on a countably infinite set Γ , without loss of generality we can assume that G acts on $\mathbb{N}$. Define a metric on G as follows:

Given $g, h \in G$ let

$$d(g, h) = 1/2^n$$

where $g(i) = h(i)$ and $g^{-1}(i) = h^{-1}(i)$ for $i \leq n$ and $g(n+1) \neq h(n+1)$ or $g^{-1}(n+1) \neq h^{-1}(n+1)$. Then the metric, d , defines a topology on G and $Sym(\mathbb{N})$ is a complete metric space.

THEOREM 4.1 *H is a dense subgroup of G if and only if H and G have the same orbits on n -tuples for each n .*

Proof. If H is a subgroup which is dense in G , then if $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$ are in the same G -orbit, we can define $m = \max\{a_1, \dots, a_n\}$. We know there exists $g \in G$ such that $(a_1, \dots, a_n)^g = (b_1, \dots, b_n)$. Since H is dense in G , there exists $h \in H$, such that $d(g, h) \leq 1/2^m$, so that $a_i^g = a_i^h$ for all i since $a_i \leq m$. Thus, $(a_1, \dots, a_n)$ is in the same H -orbit as $(b_1, \dots, b_n)$.

Now assume that H and G have the same orbits on n -tuples for each n . We must show that if $g \in G$ we can find an $h \in H$ such that $d(g, h) < 1/2^m$ for any m . We consider the $m+1$ -tuple $(1, \dots, m, m+1)$, and take the maximum of the inverse images of these elements $\{1^{g^{-1}}, \dots, m+1^{g^{-1}}\}$ with $m+1$ to give an integer d . Then we can find an $h \in H$ such that $(1^h, \dots, d^h) = (1^g, \dots, d^g)$ since H and G have the same orbits on n -tuples for each n . This ensures that both $(1^h, \dots, m+1^h) = (1^g, \dots, m+1^g)$ and that $(1^{h^{-1}}, \dots, m+1^{h^{-1}}) = (1^{g^{-1}}, \dots, m+1^{g^{-1}})$. Then by the definition of our metric, $d(h, g) \leq 1/2^{m+1}$. $\square$

4.1 A Dense subgroup of $Sym(\mathbb{Z})$

Consider the symmetric group acting on a countably infinite set which we label using the integers $\mathbb{Z}$. Recall from Chapter 2 that this is the automorphism group of a countably infinite set without relations.

THEOREM 4.2 *The symmetric group on the integers, $Sym(\mathbb{Z})$, has a dense sub-*

group on two generators.

Proof. Define a subgroup H to be generated by the following two elements:

$$h(n) = n + 1$$

$$g(n) = \begin{cases} n & \text{if } n \leq 0 \\ n - 2 & \text{if } n \text{ is odd and } n \geq 3 \\ 2 & \text{if } n = 1 \\ n + 2 & \text{if } n \text{ is even and } n \geq 2 \end{cases}$$

Note that h is a simple translation of the integers; g fixes all non-positive integers, and cycles the positive integers as $(\dots, 7, 5, 3, 1, 2, 4, 6, 8, \dots)$.

By Theorem 4.1, in order for $H = \langle h, g \rangle$ to be dense in $Sym(\mathbb{Z})$, for any two n -tuples $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$, there must be an element of H which will map a_i to b_i for $i \leq n$. Without loss of generality, assume that the elements of the n -tuple $(b_1, \dots, b_n)$ are strictly increasing. We can assume this because any mapping which maps $(a_1, \dots, a_n)$ to $(b_1, \dots, b_n)$ will map $(a_{\sigma(1)}, \dots, a_{\sigma(n)})$ to $(b_{\sigma(1)}, \dots, b_{\sigma(n)})$ for any permutation σ of $\{1, 2, \dots, n\}$.

Step 1

If some of the a_i 's are non-positive, map all of them to the positive integers by a sufficient power of the translation h , say h^s . Use a suitable power of the element g , say g^q , which cycles the positive integers, to map $h^s(a_1)$ to the integer 1. Note that the image of a_1 at this stage is the minimum of the images of the a_i 's. Apply h^{-1} to map a_1 to 0. Define the composition of these elements of H to be f_1 , ($f_1 = h^{-1}g^qh^s$).

Inductive Assumption

Assume at step k that we have defined an automorphism from H , f_k , so that the first k elements of $(a_1, \dots, a_n)$ are properly spaced, that is:

$$(1) f_k(a_i) - f_k(a_{i-1}) = b_i - b_{i-1} \text{ for } i \leq k$$

$$(2) f_k(a_k) = 0$$

$$(3) f_k(a_i) > 0 \text{ for } i > k$$

Note from (1) and (2) that for $i \leq k$, $f_k(a_i) \leq 0$.

Step $k+1$

Apply some power, r_1 , of the element g so that

$$\min_{i > k} \{g^{r_1} f_k(a_i)\} > b_{k+1} - b_k - 1.$$

This is done so that we can translate $g^{r_1} f_k(a_k) = 0$ to $b_k - b_{k+1} + 1$ by the translation $h^{b_k - b_{k+1} + 1}$ while still leaving the images of the a_i 's for $i > k$ positive. This ensures that condition (3) is still satisfied. Note that neither the application of g nor negative powers of h will disturb the spacing of the images of the a_i 's with $i \leq k$. So (1) is still satisfied. Since $f_k(a_{k+1}) > 0$, we can apply some power, r_2 , of the element g so that $f_k(a_{k+1})$ is mapped to the element 1. Note that the image of a_{k+1} is now the minimum of the a_i 's for $i \geq k+1$ and also the spacing of the image of the elements a_k and a_{k+1} matches the spacing of b_k and b_{k+1} . Finally apply h^{-1} to map the image of a_{k+1} , which is $g^{r_2} h^{b_k - b_{k+1} + 1} g^{r_1} f_k(a_{k+1})$, to 0. Note now that (2) is satisfied as well. Define $f_{k+1} = h^{-1} g^{r_2} h^{b_k - b_{k+1} + 1} g^{r_1} f_k$.

Final step

When $k = n$, we have that $f_n(a_i) - f_n(a_{i-1}) = b_i - b_{i-1} \forall i \leq n$, and $f_n(a_n) = 0$. We now translate by h^{b_n} to map $f_n(a_n)$ to b_n .

Define $f = h^{b_n} f_n$. Now f is an element of H which maps the tuple $(a_1, \dots, a_n)$ to $(b_1, \dots, b_n)$.

So we have found a subgroup, H , of $Sym(\mathbb{Z})$ that is dense and which is generated by two elements. $\square$

4.2 A Transitive Subgroup of $Aut(\mathbb{Q}, <)$

Our goal is to find a dense subgroup of $Aut(\mathbb{Q}, <)$ on two generators; we begin by constructing a subgroup on two generators which is transitive.

THEOREM 4.3 *$Aut(\mathbb{Q}, <)$ has a transitive subgroup on two generators.*

Proof. To obtain the transitive subgroup, we take one of the generators to be a simple translation, the other generator compresses each even, interval $[2k-2, 2k-1]$, $k \geq 1$, to a width of $\frac{k}{k+1}$. See Figure 5.

The definitions for $f, g \in Aut(\mathbb{Q}, <)$ are as follows:

$$f(s) = s + 1$$

$$g(s) = \begin{cases} s & \text{if } s \leq 0 \\ q + \sum_{k=1}^q \frac{k}{k+1} + r - \frac{\min(1,r)}{q+2} & \text{if } 0 < s \leq 2q + r \text{ and } 0 \leq r < 2 \end{cases}$$

Clearly, f is an automorphism of $(\mathbb{Q}, <)$. To verify that g is an automorphism, we assume $p > s$ and show that $g(p) > g(s)$.

Figure 5 The Automorphism g

Case 1 : s is non-positive.

Since g is the identity on all negative rationals we need only verify that positive rationals are mapped to positive rationals. If $p > 0$ and $p = 2q + r$, then

$$\begin{aligned} g(p) &= q + \sum_{k=1}^q \frac{k}{k+1} + r - \frac{\min(1, r)}{q+2} \\ &\geq q + r - \frac{\min(1, r)}{q+2} \\ &\geq q + r \frac{q+1}{q+2} > 0 \end{aligned}$$

since either q or r is strictly positive.

Case 2 : $p > s > 0$.

Let $p = 2q + r$ and $s = 2q' + r'$.

Subcase 1): If $q = q'$ then $r > r'$.

Then

$$\begin{aligned} g(p) - g(s) &= q + \sum_{k=1}^q \frac{k}{k+1} + r - \frac{\min(1, r)}{q+2} \\ &\quad - \left(q + \sum_{k=1}^q \frac{k}{k+1} + r' - \frac{\min(1, r')}{q+2} \right) \\ &= r - \frac{\min(1, r)}{q+2} - r' + \frac{\min(1, r')}{q+2} \end{aligned}$$

If $r > r' \geq 1$ then

$$g(p) - g(s) = r - r' > 0.$$

If $r' < 1$ then

$$g(p) - g(s) \geq r \frac{q+1}{q+2} - r' \frac{q+1}{q+2} > 0.$$

Subcase ii): If $q \neq q'$ then $q > q'$.

Then

$$\begin{aligned} g(p) - g(s) &= q + \sum_{k=1}^q \frac{k}{k+1} + r - \frac{\min(1, r)}{q+2} \\ &\quad - (q' + \sum_{k=1}^{q'} \frac{k}{k+1} + r' - \frac{\min(1, r')}{q'+2}) \\ &= q - q' + r - r' + \sum_{k=q'+1}^q \frac{k}{k+1} - \frac{\min(1, r)}{q+2} + \frac{\min(1, r')}{q'+2} \\ &= r - r' + \sum_{k=q'+1}^q (1 + \frac{k}{k+1}) - \frac{\min(1, r)}{q+2} + \frac{\min(1, r')}{q'+2} \end{aligned}$$

Case a: If $r > r'$ we continue with

$$\begin{aligned} g(p) - g(s) &\geq \sum_{k=q'+1}^q (1 + \frac{k}{k+1}) - \frac{1}{q+2} \\ &\geq 1 - \frac{1}{q+2} > 0. \end{aligned}$$

Case b: If $r < r' \leq 1$ then

$$\begin{aligned} g(p) - g(s) &= r - r' + \sum_{k=q'+1}^q (1 + \frac{k}{k+1}) - \frac{r}{q+2} + \frac{r'}{q'+2} \\ &\geq -1 + 1 + \frac{q'+1}{q'+2} - \frac{r}{q'+2} + \frac{r'}{q'+2} \\ &= \frac{q'+1}{q'+2} + \frac{r' - r}{q'+2} \\ &> 0. \end{aligned}$$

Case c: If $r < 1 < r' < 2$ then

$$\begin{aligned}
 g(p) - g(s) &= r - r' + \sum_{k=q'+1}^q \left(1 + \frac{k}{k+1}\right) - \frac{r}{q+2} + \frac{1}{q'+2} \\
 &\geq 1 + \frac{q'+1}{q'+2} + \frac{1}{q'+2} + r - r' - \frac{r}{q+2} \\
 &= 2 - r' + r - \frac{r}{q+2} \\
 &> 0.
 \end{aligned}$$

Case d: If $1 \leq r < r' < 2$ then

$$\begin{aligned}
 g(p) - g(s) &= r - r' + \sum_{k=q'+1}^q \left(1 + \frac{k}{k+1}\right) - \frac{1}{q+2} + \frac{1}{q'+2} \\
 &\geq -1 + 1 + \frac{q'+1}{q'+2} - \frac{1}{q+2} + \frac{1}{q'+2} \\
 &= 1 - \frac{1}{q+2} \\
 &> 0.
 \end{aligned}$$

This completes the proof that g is an element of $\text{Aut}(\mathbb{Q}, <)$.

Now we must show that $H = \langle f, g \rangle$ is transitive. We begin by showing that for any pair (s, m) where s is a rational number and m is a positive integer, there exists an $h_{s,m} \in H$ such that $h_{s,m}(s) = s + 1/m$.

Let m and s be chosen. Define

$$t = m - 2 + \sum_{k=1}^{m-1} \frac{k}{k+1}.$$

Now given t and s , we can find an s' to satisfy $0 \leq s' - t < 1$ and $s' - s = d$, an integer. We now show that the commutator $[f^2, g](s') = s' + 1/m$. At this point, conjugation of the commutator by f^d to translate s to s' will map s to $s + 1/m$.

Since $0 \leq s' - t < 1$, we have that $s' = t + r'$ where $0 \leq r' < 1$.

Then

$$\begin{aligned} s' &= m - 2 + \sum_{k=1}^{m-1} \frac{k}{k+1} + r' \\ &= m - 2 + \sum_{k=1}^{m-2} \frac{k}{k+1} + r' + 1 - \frac{1}{m} \\ &= g(2(m-2) + r) \end{aligned}$$

where $r = r' + 1$ and $1 \leq r < 2$.

So,

$$\begin{aligned} g^{-1}(s') &= 2(m-2) + r \\ f^{-2}g^{-1}(s') &= 2(m-3) + r \\ gf^{-2}g^{-1}(s') &= m - 3 + \sum_{k=1}^{m-3} \frac{k}{k+1} + r - \frac{\min(1, r)}{m-1} \\ &= m - 3 + \sum_{k=1}^{m-3} \frac{k}{k+1} + r - \frac{1}{m-1} \\ f^2gf^{-2}g^{-1}(s') &= m - 3 + \sum_{k=1}^{m-2} \frac{k}{k+1} - \frac{m-2}{m-1} + r - \frac{1}{m-1} + 2 \\ &= m - 2 + \sum_{k=1}^{m-2} \frac{k}{k+1} + r \\ &= m - 2 + \sum_{k=1}^{m-2} \frac{k}{k+1} + r' + 1 \\ &= s' + 1/m \end{aligned}$$

Thus, we can define $h_{s,m} = f^{-d}f^2gf^{-2}g^{-1}f^d$, with $h_{s,m}(s) = s + 1/m$. By repeated iteration of such elements $h_{s,m}$, we can map s to any p where $p = s + n/m$. (Note: We can assume $p \geq s$ by conjugation with some power of f). $\square$

4.3 A Dense subgroup of $Aut(\mathbb{Q}, <)$

THEOREM 4.4 *$Aut(\mathbb{Q}, <)$ has a dense subgroup on two generators.*

Without giving an explicit definition, we will describe a two-generator, dense subgroup of $Aut(\mathbb{Q}, <)$ as a modification of the transitive subgroup described in Section 4.2. If we recall the dense subgroup of $Sym(\mathbb{Z})$, the proof that it was dense relied on an element of the subgroup which fixed an arbitrarily large subset (the negative integers), where we could iteratively “park” each a_i of a given n -tuple, $\bar{a}$, after having mapped it to its desired position relative to the others.

Similarly, if we had arbitrarily large regions of $\mathbb{Q}$ which were fixed by the commutator $f^2gf^{-2}g^{-1}$, we could iteratively map each a_i of a given n -tuple to its desired value modulo 1, then translate it to such a region while using an appropriate commutator to map subsequent a_i ’s to their desired positions.

To give an intuitive description of how the commutator works, the automorphisms g and g^{-1} contract and expand different positive intervals by a scale of $\frac{k-1}{k}$ and its inverse, respectively, as k ranges over the integers $k > 1$. These intervals are separated by intervals of width one upon which no scaling is performed. Figure 6 gives the path of a rational s' as it is mapped by the commutator. This point begins in a position greater than the interval of width $\frac{k-1}{k}$ being expanded by g^{-1} , the expansion is performed, and then the point is translated to the other side of that interval, before the inverse contraction is performed. Finally it is translated

Figure 6 Action of the Commutator $[f^2, g]$

back to within the vicinity of s' . Each particular point only experiences the effects of the expansions and contractions of those intervals which lie on its left. Because the image of the point s' is on different sides of the specific interval of width $\frac{k-1}{k}$ when the expansion and contraction are performed, but on the same side of all other intervals being expanded and contracted, the net effect after the commutator is applied is that only the expansion of the interval of width $\frac{k-1}{k}$ affects the final position of the point. So the rational s' is increased by a value of $1/k$ by the commutator.

If the intervals which are not scaled by the automorphism g had widths of arbitrarily large size, then an arbitrarily large subinterval would remain between the same pair of expanded and contracted intervals when translated by f^{-2} and so would be fixed by the commutator. If we consider all possible integer pairs (k, m) , of which there are countably many, we can modify the automorphism g to give an automorphism $\hat{g}$, so that for each pair (k, m) we have an interval which is scaled by

f^{-1} between two intervals of width m which are fixed by the commutator $[f^2, \hat{g}]$.

In the process of mapping an n -tuple and leaving most of its elements fixed while moving another by $1/k$, it may be necessary to have an element on one side of a contracted interval, and the other elements of the tuple on the other side in a region which is fixed by the commutator. This introduces a technical problem, since it is only possible if we begin with a tuple $\bar{a}$, and wish to map to a tuple $\bar{b}$ which has elements separated by some minimum distance; a distance of two would suffice since the intervals for expansion have width no greater than one, and the commutator involves f^2 which translates by two.

Repeated application of $\hat{g}^{-1}$ which expands the rational line together with translations by f , will give an automorphism r_1 which introduces a distance of two between the consecutive b_i 's. A similar element r_2 can be found so that $r_2(\bar{a}) = (r_2(a_1), \dots, r_2(a_n))$ has consecutive entries separated by at least as much as the corresponding entries in $r_1(\bar{b})$. We can now map the tuple $\bar{a}$ to $r_1(\bar{a})$, then use the strategy given below to map $r_1(\bar{a})$ to $r_2(\bar{b})$ and use the inverse of r_2 to finally map to $\bar{b}$.

What follows is a strategy for mapping an increasing n -tuple x to an increasing n -tuple y where consecutive elements in the tuple $\bar{y}$ are separated by a distance of at least two and the distances between consecutive elements in $\bar{x}$ exceed the distances between the corresponding consecutive elements in $\bar{y}$.

Step 1

Apply the inverse of the commutator $f^2 \tilde{q} f^{-2} \tilde{q}^{-1}$ conjugated by some power of the translation f iteratively (we shall call this automorphism h_1) to decrease the element x_1 to some rational so that $h_1(x_1) = y_1 \equiv 0$ modulo 1 and such that the other elements are left fixed. To do this, it is necessary to choose a sufficiently large m as the width of the fixed interval.

Step j

Assume we have reached the j th stage where we have found some h_j so that

- (1) $h_j(x_1) = y_1 \equiv 0$ modulo 1
- (2) $h_j(x_i) = h_j(x_{i-1}) = y_i = y_{i-1}$ for $i \leq j$
- (3) $h_j(x_i) = h_j(x_{i-1}) > y_i = y_{i-1}$ for $i > j$.

Condition (1) ensures that by the end of the induction we can translate by a power of f to map the image of x_1 to y_1 . Condition (2) ensures that the other x_i 's will also be mapped to the appropriate y_i 's. Condition (3) is used throughout the induction to ensure that conditions (1) - (3) can be maintained by positioning the tuple $h_j(\bar{x})$ in such a way that only $h_j(x_{j+1})$ is being modified by the commutator.

We wish to find an h which maps $h_j(x_{j+1})$ so that $h \circ h_j(x_{j+1}) = h \circ h_j(x_j) = y_{j+1} = y_j$ and so that h leaves the other elements of the tuple fixed. Then this automorphism $h \circ h_j$, which we shall call h_{j+1} , affects only the image of x_{j+1} by decreasing it to the appropriate distance from $h_j(x_j)$. It thus preserves the equalities in (1) and (2) for $i \leq j$ and since it is decreasing the element $h_j(x_{j+1})$

by precisely the necessary amount, the inequality in (3) becomes the equality in (2) for $i = j + 1$. The only other equation which is affected is the inequality in (3) for $i = j + 2$ which is preserved, since the element being subtracted on the left-hand side of the equation is decreased by h . Thus, if we can find such an element h , we will have that h_{j+1} will satisfy the conditions of the inductive assumption.

To find h , we must decrease the element $h_j(x_{j+1})$ by some rational n/k . Given this integer k and the integer m so that $m > h_j(x_n) - h_j(x_1)$, we can use the translation f to position the tuple around an interval of width $\frac{k-1}{k}$ which is expanded by $\hat{g}$ and leaves intervals of width m fixed on either side. We position the tuple in such a way that the only element of the tuple affected by the inverse of the commutator $f^2 \hat{g} f^{-2} \hat{g}^{-1}$ will be $h_j(x_{j+1})$. If we repeat this n times, we have the desired mapping.

Final Step

Since we finish with $h_n(x_1) - y_1 \equiv 0$ modulo 1, we can use a power of the translation f to map $h_n(x_1)$ to y_1 . Thus, we will have mapped $\bar{x}$ to $\bar{y}$ using the automorphisms f and $\hat{g}$.

4.4 More on Dense Subgroups

We cannot find an integer, n , sufficiently large so that the automorphism group of an arbitrary $\aleph_0$ -categorical structure has a dense subgroup on n generators. The following example is of an automorphism group such that any dense subgroup

requires more than n generators and is found in Cameron [1], page 86, where it is credited to Dugald Macpherson. Consider a finite group, G , requiring more than n generators. We can find a faithful action of G on a finite set, X , and give this set the canonical structure induced by the group's action. We can then add an infinite set of elements with no further structure. Though this structure is $\aleph_0$ -categorical, its automorphism group does not contain a dense subgroup on two generators, since a dense subgroup must induce G on X , and therefore requires at least as many generators as G .

What we would like to do is find an embedding of an arbitrary $\aleph_0$ -categorical structure, A , into another $\aleph_0$ -categorical structure, B , such that $Aut(A)$ can also be embedded in $Aut(B)$ in a natural way, permitting it to act on the image of A as it did before, and such that $Aut(B)$ contains a dense subgroup on two generators.

We can almost do this. But in order to embed A in the desired structure, we must first expand our language L to include a predicate, E_2 , to correspond to a binary relation, e_2 , which is definable in A by the formula $\phi(x, y) = (x = y) \vee (x \neq y)$. Any definable relation which is added to a structure, makes no appreciable difference to that structure. In this case, in particular, this relation is tautologically satisfied by any pair (x, y) in any structure, as it does not rely on the relations of the particular structure A for its definition. We shall call this expanded language L^+ and the expanded structure A^+ .

LEMMA 4.5 *For any $\aleph_0$ -categorical structure, A , there exists a structure, A_{int}*

which embeds the expanded structure A^+ , such that $\text{Aut}(A_{wr})$ is equal to the wreath product $\text{Aut}(A)wr\text{Sym}(\mathbb{Z})$.

Proof. We define the structure A_{wr} as follows:

Let Γ be the universe of A , and define the universe of A_{wr} to be $\Gamma \times \mathbb{Z}$.

Define a set of relations R_{wr} on the universe $\Gamma \times \mathbb{Z}$ from the set of relations R on Γ by

$$R_{wr} = \{r'_n \mid r_n \in R, \text{ an } n\text{-ary relation and } \forall i \in \mathbb{Z}, ((a_1, i), \dots, (a_n, i)) \in r'_n \text{ iff } (a_1, \dots, a_n) \in r_n \text{ in } A\} \cup \{e'_2 \mid ((a_1, i), (a_2, i)) \in e'_2 \forall a_1, a_2 \in \Gamma, i \in \mathbb{Z}\}.$$

Clearly the mapping f from Γ to $\Gamma \times \mathbb{Z}$ defined as $f(a) = (a, 1)$ and which takes each n -ary relation r_n to r'_n and the binary relation e_2 to the equivalence relation e'_2 is an embedding of A^+ into A_{wr} .

Now, we will prove that $\text{Aut}(A_{wr})$ acts on $\Gamma \times \mathbb{Z}$ as the wreath product $\text{Aut}(A)wr\text{Sym}(\mathbb{Z})$. We shall call the subsets $\Gamma \times \{i\}$ of $\Gamma \times \mathbb{Z}$ its strands and we note that the wreath product's action on the universe preserves the integrity of the strands by setwise permuting them.

Let $g \in \text{Aut}(A)wr\text{Sym}(\mathbb{Z})$. Assume $g = ((\dots, g_{-2}, g_{-1}, g_0, g_1, g_2, \dots), k)$ where $g_i \in \text{Aut}(A)$ and acts on the strand $\Gamma \times \{i\}$ followed by the action of $k \in \text{Sym}(\mathbb{Z})$ which permutes the strands of $\Gamma \times \mathbb{Z}$. So $g(a, i) = (a^{g_i}, i^k)$. The only n -tuples contained in relations of $\text{Aut}(A_{wr})$ are of the form $\bar{a} = ((a_1, i), \dots, (a_n, i)) \in r'_n$ with $(a_1, \dots, a_n) \in r_n$ in the structure A . Then $g(\bar{a}) = ((a_1^{g_1}, i^k), \dots, (a_n^{g_n}, i^k))$ is also in r'_n since g_i is an automorphism of A , which ensures that $(a_1^{g_1}, \dots, a_n^{g_n})$ is in r_n . Note

that the relation e'_2 is also preserved in the same way. So $Aut(A)wrSym(\mathbb{Z}) \subseteq Aut(A_{wr})$.

Because of the equivalence relation e'_2 , any automorphism, g , of $Aut(A_{wr})$ must permute the strands $\Gamma \times \{i\}$ without breaking them apart. In other words, if $g(a, i) = (a^g, j)$, then any other element (b, i) of $\Gamma \times \{i\}$ must be mapped by g to an element in $\Gamma \times \{j\}$. Thus, $Aut(A_{wr}) \subseteq Sym(\Gamma)wrSym(\mathbb{Z})$ and we know that we can write $g \in Aut(A_{wr})$ in the form $g = ((\dots, g_{-2}, g_{-1}, g_0, g_1, g_2, \dots), k)$ with each g_i representing a permutation of the set Γ .

We observe however that each of the g_i 's acting on the strand $\Gamma \times \{i\}$ must be in $Aut(A)$. Otherwise, if $g_i \notin Aut(A)$ then there exists an n -tuple $(a_1, \dots, a_n)$ and an n -ary relation r_n of A with, without loss of generality, $(a_1, \dots, a_n) \in r_n$ and $(a_1^{g_i}, \dots, a_n^{g_i}) \notin r_n$. But clearly, $((a_1, i), \dots, (a_n, i)) \in r'_n$ but $((a_1^{g_i}, i^k), \dots, (a_n^{g_i}, i^k)) \notin r'_n$ by the definition of r'_n . So we conclude that $Aut(A)wrSym(\mathbb{Z}) = Aut(A_{wr})$. $\square$

LEMMA 4.6 *$Aut(A)$ is oligomorphic if and only if $Aut(A)wrSym(\mathbb{Z})$ is oligomorphic.*

Proof. Let f_n be the number of orbits on n -tuples in $Aut(A)$ for each n and let f_n^{wr} be the number of orbits on n -tuples in $Aut(A)wrSym(\mathbb{Z})$. If $Aut(A)wrSym(\mathbb{Z})$ is oligomorphic, then the fact that $Aut(A)$ is oligomorphic follows from the fact that $f_n \leq f_n^{wr}$ for all n , since if $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$ are in different orbits in $Aut(A)$, then $((a_1, i), \dots, (a_n, i))$ and $((b_1, i), \dots, (b_n, i))$ are in different orbits in

$Aut(A)wrSym(\mathbb{Z})$.

Assume that $Aut(A)$ is oligomorphic so that f_n is finite for each n . Since the wreath product setwise permutes the strands $\Gamma \times \{i\}$, we can think of each strand as a bin, and different n -tuples from $\Gamma \times \mathbb{Z}$ can be distinguished by how many elements are drawn from the same bin and at which entries they appear in the tuple. For example, if we consider $\mathbb{Q} \times \mathbb{Z}$ acted upon by $Aut(\mathbb{Q}, <)wrSym(\mathbb{Z})$, then the triple $((0.2, 1), (0.3, 1), (0.1, 2))$ is in the same 3-orbit as $((0.1, 7), (0.4, 7), (0.1, 5))$ since for both triples, the first and second entries are both drawn from the same strand $\mathbb{Q} \times \{1\}$ and $\mathbb{Q} \times \{7\}$ respectively, and as pairs in $(\mathbb{Q}, <)$, $(0.2, 0.3)$ and $(0.1, 0.4)$ are in the same orbit. Likewise, different distributions of the i_k 's in the tuple correspond to distinct orbits. In other words, the tuple $\bar{b} = ((b_1, j_1), \dots, (b_n, j_n))$ is in the same orbit as $\bar{a} = ((a_1, i_1), \dots, (a_n, i_n))$ if and only if for each pair k and m with $1 \leq k, m \leq n$, $i_k = i_m \iff j_k = j_m$ and furthermore, if $i_{s_1} = \dots = i_{s_p}$ then $(a_{s_1}, \dots, a_{s_p})$ is in the same p -orbit in $Aut(A)$ as $(b_{s_1}, \dots, b_{s_p})$.

So to count the number of orbits f_n^{wr} , we must consider each partition $p = (k_1, \dots, k_s)$ of the integer n with $k_1 \leq \dots \leq k_s$. Each partition represents a distribution of the elements (a_r, i_r) of the tuple $\bar{a}$ where for each k_i there are k_i elements of a drawn from the same strand of $\Gamma \times \mathbb{Z}$. We must also consider the ordering in the n -tuple of the elements in matched strands. This gives us

$$\binom{n}{k_1 \dots k_s}$$

possibilities.

However, if some of the k_i 's are equal, we don't wish to distinguish between their ordering in the tuple. So each partition of n , $(k_1, \dots, k_s)$, gives a partition of s , $(m_1, \dots, m_p)$ which describes how many of the k_i 's are equal. In other words, $k_1 = \dots = k_{m_1}, k_{m_1+1} = \dots = k_{m_1+m_2}, \dots, k_{m_1+\dots+m_{p-1}+1} = \dots = k_s$. Then we wish to divide $\binom{n}{k_1, \dots, k_s}$ further by $m_1! \dots m_p!$

Now, each element k_i of the partition corresponds to a k_i -tuple drawn from the same strand. The number of possible orbits on this tuple in $Aut(A)$ is f_{k_i} . If P_n is the set of all possible partitions of the integer n , we have the final relation between f_n and f_n^{wr} :

$$f_n^{wr} = \sum_{(k_1, \dots, k_s) \in P_n} \frac{\binom{n}{k_1, \dots, k_s}}{m_1! \dots m_p!} f_{k_1} \dots f_{k_s}$$

Thus, if f_n is finite for each n , then f_n^{wr} is finite for each n . $\square$

LEMMA 4.7 *$Aut(A)wrSym(\mathbb{Z})$ has a dense subgroup on two generators.*

If we recall the dense subgroup of $Sym(\mathbb{Z})$ on two generators from Section 4.1, one generator, h , was a translation, the second generator, g , fixed the negative integers and cycled the positive integers. We shall define our two generators H and G of a dense subgroup of $Aut(A)wrSym(\mathbb{Z})$ by using these two generators for the top group's action on the strands of $\Gamma \times \mathbb{Z}$.

So our element H is defined to be

$$H = (\dots, 1_{A_{-1}}, 1_{A_0}, 1_{A_1}, \dots, h)$$

where 1_{A_i} from the base group acts as the identity on the strand $\Gamma \times \{i\}$ which is followed by the action of h from the top group, which maps the strand $\Gamma \times \{i\}$ to the strand $\Gamma \times \{i+1\}$.

To define the element G , we first refer to a theorem of Dugald Macpherson which is referenced in Cameron [1], page 85, which states that the automorphism group of an $\aleph_0$ -categorical structure contains a dense subgroup which is free of countable rank. So if $\{f_i\}_{i < \omega}$ are the generators of this dense subgroup for the automorphism group of the structure A , then we can enumerate all finite words on these generators, $\{w_i\}_{i < \omega}$. Finally we take a countable enumeration of all possible pairs $\{w_i\}_{i < \omega} \times N$. Now we can define our element G ,

$$G = (\dots, k_{A_{s-1}}, k_{A_s}, k_{A_{s+1}}, \dots, k_{A_{-2}}, k_{A_{-1}}, 1_{A_0}, 1_{A_1}, 1_{A_2}, \dots, g).$$

The base group acts as the identity on each positive strand, $\Gamma \times \{i\}, i \geq 0$, and the automorphism k_{A_s} acts on the negative strand $\Gamma \times \{s\}$ either as the identity, or as one of the words w_i determined as follows: for each enumerated pair (w_i, m) with w_i a word from the dense subgroup of $\text{Aut}(A)$ and m an arbitrary natural number, we have an action of the base group on a set of negatively indexed strands $\Gamma \times \{p, \dots, p+2m\}$ which is the identity on m of the strands on either side of a strand upon which it acts as the word w_i . So G will have the property that for every pair (w_i, m) there is some negative integer p with

$$G = (\dots, \overbrace{1_{A_p}, \dots, 1_{A_{p+m-1}}}^{m \text{ elements}}, w_{iA_{p+m}}, \overbrace{1_{A_{p+m+1}}, \dots, 1_{A_{p+2m}}}^{m \text{ elements}}, \dots, g)$$

so there is some region where G fixes m strands on either side of a strand upon which it acts according to the word w_i .

Now we claim that the subgroup generated by $\langle H, G \rangle$ is dense. Let $a = ((a_1, i_1), \dots, (a_n, i_n))$ and $\bar{b} = ((b_1, j_1), \dots, (b_n, j_n))$ be any two n -tuples from $\Gamma \times \mathbb{Z}$ which are in the same orbit under $Aut(A)wrSym(\mathbb{Z})$. Then, as explained in Lemma 4.6, the distribution of the i 's and j 's over distinct strands are matched in the sense that $i_m = i_k \iff j_m = j_k$. So there exists a permutation from $Sym(\mathbb{Z})$ which will map i_k to j_k for each $0 \leq k \leq n$. Since the two permutations h and g defined in Section 4.1 form a dense subgroup of $Sym(\mathbb{Z})$, there is some word, w , formed from h and g which will map the tuple $(i_1, \dots, i_n)$ to the tuple $(j_1, \dots, j_n)$. Let W be the corresponding word from $\langle H, G \rangle$ which we obtain from w when we substitute the elements h and g from $Sym(\mathbb{Z})$ by the elements H and G from $Aut(A)wrSym(\mathbb{Z})$. We disregard, for the moment, the action of the base group, and note only that the top group will permute the strands as desired so that we have

$$\bar{a}^W = ((a_1^W, j_1), \dots, (a_n^W, j_n))$$

with the strand positions now matching those from $\bar{b}$.

Now consider a maximal set of the indices $(j_{s_1}, \dots, j_{s_k})$ with $j_{s_1} = \dots = j_{s_k}$. If we refer again to the proof of Lemma 4.6, we know that $(a_{s_1}, \dots, a_{s_k})$ is in the same k -orbit as $(b_{s_1}, \dots, b_{s_k})$ under $Aut(A)$. Because the action of W on elements of $\Gamma \times \mathbb{Z}$ from the same strand has been a combination of identity actions and

actions of the words $\{w_i\}_{i \in \omega}$ from $Aut(A)$, then if we consider this as a k -tuple in our copy of Γ , $(a_{s_1}, \dots, a_{s_k})$, we still have that $(a_{s_1}^W, \dots, a_{s_k}^W)$ is in the same k -orbit of $Aut(A)$. Since, $\{w_i\}_{i \in \omega}$ runs through all possible words from a dense subgroup of $Aut(A)$, there is some word w_s which will map the tuple $(a_{s_1}^W, \dots, a_{s_k}^W)$ to the tuple $(b_{s_1}, \dots, b_{s_k})$. Let $m = \max\{j_1, \dots, j_n\} - \min\{j_1, \dots, j_n\}$. Then if we consider the pair (w_s, m) , we can find a negative integer p for which G acts as w_s on the strand $\Gamma \times \{p + m\}$ and leaves m strands fixed on either side. If we use some power r of the translation H , which acts as the identity on each strand, we can map j_{s_1} to $p + m$, then apply the element G which will leave all elements of the n -tuple $\bar{a}^{WH'}$ fixed, except for those in the strand $\Gamma \times \{p + m\}$. We know this because all elements of the n -tuple $\bar{a}^W$ are at a maximum distance of m strands from one another. Thus, we will have that the elements $(a_{s_1}^W, \dots, a_{s_k}^W)$ are mapped to $(b_{s_1}, \dots, b_{s_k})$ as desired, and if we apply this process iteratively to the maximal sets of equal indices, we eventually map $\bar{a}$ to $\bar{b}$.

Thus, the subgroup $\langle H, G \rangle$ is dense in $Aut(A)wrSym(\mathbb{Z})$. $\square$

THEOREM 4.7 *For any $\aleph_0$ -categorical structure, A , the expanded structure A^+ can be embedded in an $\aleph_0$ -categorical structure with a dense subgroup on two generators.*

Proof. The theorem follows from the above three lemmas. $\square$

4.5 Conclusions for Dense Subgroups

We saw in Chapter 3 how an $\aleph_0$ -categorical model for a theory with built-in Skolem functions has a set of indiscernibles, X , which generate the structure A . In other words, every element of the model is either an indiscernible, or it is a term on a finite subset of the indiscernibles. Thus, the action of an automorphism on the indiscernible set determines its action on all of A . This is explained in greater detail in Appendix B. Furthermore, as we saw with the random graph in Chapter 3, it may happen that an automorphism's action on a set of indiscernibles determines its action on the whole structure, even when the theory is not a Skolem theory.

Indiscernibles can always be found in an $\aleph_0$ -categorical structure, even those which are not Skolem theories, with an ordering $(X, <)$ which is order-isomorphic to $(\mathbb{Q}, <)$. If the automorphisms of the indiscernible set extend uniquely to the whole structure, then the setwise stabilizer of X in $\text{Aut} A$, is isomorphic to one of the groups (A), (B), (C), (D) or (S) as described in Chapter 3. The group (A) is the automorphism group of $(\mathbb{Q}, <)$ for which we described a two-generator, dense subgroup in Section 4.3. The case of the group (S) which is the symmetric group on a countable set, was treated in Section 4.1.

The group (B) is the set of automorphisms which preserve or reverse the order on $\mathbb{Q}$. Since the automorphism $\hat{q}$ which was described in section 4.3, fixes the element 0, it can be modified on the negative rationals (this will not interfere

with the fact that $\langle \tilde{g}, f \rangle$ is a dense subgroup) so that it can be expressed as the square of an automorphism which reverses order. With this substitution of the automorphism g by such a square root, we will have a two-generator subgroup of (B) which is dense.

The group (C) is an expansion of the group (A) which includes automorphisms which cycle the rationals. It is possible that a two-generated dense subgroup exists for (C), but we can certainly say that if we add one of the cycle automorphisms of (C) to our dense subgroup of $Aut(\mathbb{Q}, <)$ we obtain a three-generated dense subgroup of (C). The same can be said for the group (D), which is a supergroup of the group (B) which includes automorphisms which cycle the rationals.

In order to find a dense subgroup of the automorphism group of an arbitrary $\aleph_0$ -categorical structure, we need still to consider if there are automorphisms which do not setwise fix the set of indiscernibles. Then we must add some of these automorphisms which map between conjugate sets of indiscernibles to the appropriate automorphisms of the setwise stabiliser (as outlined above) in order to obtain a dense subgroup of $Aut(A)$.

Appendix A

Definitions

A *structure* is a set of elements Γ (called the *universe* of A or $|A|$) together with a set of n -ary relations. An L -*structure* is a structure over a language L containing an n -ary predicate for each n -ary relation of the structure.

A first-order *theory* over a language L (with n -ary predicates and including equality) is a set of sentences formed from the language according to the rules of first-order logic.

A *complete theory* is a theory for which exactly one of every well-formed sentence of the language and its negation is a consequence of the theory.

A *model* of a theory T is a structure A together with a mapping from each n -ary predicate of the language L to an n -ary relation of the structure A such that the sentences of the theory T are satisfied in the structure.

An *isomorphism* g between two models A and B of a theory T is a one-to-one mapping from the universe of A to the universe of B which satisfies the following. For each n -ary predicate of the language, if R_A and R_B are the corresponding relations assigned the predicate by the models A and B , then for each n tuple $\bar{a} \in |A|^n$ $(a_1, \dots, a_n) \in R_A$ if and only if $(a_1^g, \dots, a_n^g) \in R_B$.

An *atomic* formula is a formula without quantifiers.

An *embedding* from an L -structure A into an L -structure B is an injective map from the universe of A into the universe of B such that atomic formulas are preserved.

An *elementary embedding* is an embedding which preserves all formulas.

The *theory* of a structure is the set of all sentences in first-order logic in an appropriate language (with an n -ary predicate for each n -ary relation) which describe the structure.

Given a structure A with universe Γ , an n -ary *function*, f_n , is a mapping from Γ^n to Γ which assigns to each n -tuple of Γ^n exactly one element of Γ . Note that an n -ary function can be regarded as an $n + 1$ -ary relation, R , with the axiom:

$$\forall \bar{a} \in \Gamma^n \exists x R(\bar{a}, x) \wedge (\forall y R(\bar{a}, y) \rightarrow x = y).$$

A *constant* is a function without arguments.

A *term* is iteratively defined to be a function whose arguments are terms.

Appendix B

Possible Supergroups of $Aut(\mathbb{Q}, <)$

In this Appendix, we look at the setwise stabiliser of a set of indiscernibles which generate a structure, A , in the automorphism group of A . There are a limited number of possibilities for such a stabiliser.

THEOREM A.1 *Let G be the automorphism group on a structure A with universe Γ . Then G is a closed subgroup of $Sym(\Gamma)$.*

Proof. Let Γ be enumerated $\{\gamma_i\}_{i < \omega}$ and assume $\{g_i\}_{i < \omega}$ is a convergent sequence of automorphisms from G with $g = \lim_{i \rightarrow \infty} g_i$. Assume $g \notin G$. Then there exists some n -ary relation of A , say r_n and elements $(\gamma_{i_1}, \dots, \gamma_{i_n})$ so that, without loss of generality, $(\gamma_{i_1}, \dots, \gamma_{i_n}) \in r_n$ and $(\gamma_{i_1}^g, \dots, \gamma_{i_n}^g) \notin r_n$. Let $k = \max\{i_1, \dots, i_n\}$. Since $\{g_i\}_{i < \omega}$ converges to g , there exists an integer m such that $d(g_m, g) < 1/2^k$. So $\gamma_{i_1}^{g_m} = \gamma_{i_1}^g$ for all $i \leq k$. This contradicts the fact that $g_m \in G$. Thus, G is a closed subgroup of $Sym(\Gamma)$. $\square$

THEOREM A.2 *If X is a set which generates a structure A , then the setwise stabiliser of X in $Aut(A)$ acts on X as a closed subgroup of $Sym(X)$.*

Proof. Let $\{g_i\}_{i < \omega}$ be a set of automorphisms from the setwise stabiliser of X in $Aut(A)$ whose action on X converges in $Sym(X)$. We must show that $\{g_i\}_{i < \omega}$

converges in $Sym(\Gamma)$ where Γ is the universe of A . This will give us that $g = \lim_{i \rightarrow \infty} g_i$ is an element of $Aut(A)$ from the setwise stabiliser of X and the action of g on the set X is the limit in $Sym(X)$ of the actions of $\{g_i\}_{i < \omega}$ on X .

To show that $\{g_i\}_{i < \omega}$ converges in $Sym(\Gamma)$, we let $\gamma \in \Gamma$ and show that there exists m such that $\gamma^{g_m} = \gamma^{g_k}$ for all $k \geq m$. Since X generates the structure A , there exist finitely many elements of X , $(x_1, \dots, x_n)$ such that γ is a term on these elements, say $\gamma = t(x_1, \dots, x_n)$. Since $\{g_i\}_{i < \omega}$ converges on the set X , we can find an integer m such that $x_i^{g_m} = x_i^{g_k}$ for all $k \geq m$ and $i \leq n$. Then $\gamma^{g_k} = t(x_1, \dots, x_n)^{g_k} = t(x_1^{g_k}, \dots, x_n^{g_k}) = t(x_1^{g_m}, \dots, x_n^{g_m}) = t(x_1, \dots, x_n)^{g_m} = \gamma^{g_m}$. Therefore, the setwise stabiliser of the set X in $Aut(A)$ acts on X as a closed subgroup of $Sym(X)$. $\square$

We say that a permutation group is *highly homogeneous* if there is an element of the group which maps any n -set to any other n -set. Note that the order need not be preserved.

THEOREM A.3 *A highly homogeneous permutation group of countable degree is a dense subgroup of one of the following:*

- (A) *The automorphism group of $(\mathbb{Q}, <)$.*
- (B) *The supergroup of A which preserves or reverses the order on $\mathbb{Q}$.*
- (C) *If one takes for our universe the set of complex points on the unit circle with rational arguments, that is $\{P \in \mathbb{C} \mid P = e^{i\theta} \text{ where } -\pi < \theta < \pi \text{ and } \theta \text{ is rational}\}$, we consider the set of automorphisms which preserves the cyclic*

order on the circle. That is, we must preserve a ternary relation R , where $(P_1, P_2, P_3) \in R$ if and only if the element P_2 lies between the elements P_1 and P_3 when traversing counterclockwise about the circle.

(D) *The supergroup of C which preserves or reverses the cyclic order described above.*

(S) *The symmetric group.*

We will not prove this theorem; the proof is given in Cameron [1], pages 61-63. But as a consequence, we note that any highly homogeneous, closed subgroup of the symmetric group must in fact be one of these five groups. In Chapter 2, when we first introduced $Aut(\mathbb{Q}, <)$ we noted that it was highly homogeneous, since any two similarly ordered n -tuples are in the same n -orbit.

If we consider a generating set of indiscernibles $(X, <)$ for a structure A which is order-isomorphic to $(\mathbb{Q}, <)$, then since all automorphisms of $(X, <)$ extend to automorphisms of A , by Theorem A.2 we must have that the action of the setwise stabiliser of X in $Aut(A)$ on X is both closed and highly homogeneous and therefore its action on X is described by one of the above groups. Let us call this group, $Aut(X_A)$. Furthermore, there is some bijection between $\mathbb{Q}$ and X which embeds the order automorphisms of $\mathbb{Q}$. It is easy to see such an embedding when $Aut(X_A)$ is the group A, B or S . For an embedding into the group C , choose an order preserving bijection from the rationals $\mathbb{Q}$ to the rationals in the open interval $(-\pi, \pi)$. Then this gives a bijection from $\mathbb{Q}$ to the points on the

unit circle described in C' . Clearly this mapping will embed the automorphisms of $Aut(\mathbb{Q}, <)$. This same bijection shows that D will also embed $Aut(\mathbb{Q})$.

We conclude that the group $Aut(X_A)$ is isomorphic to one of the groups listed in Theorem A.3.

References

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