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THE THEORY OF POLYKAY
STATISTICS WITH APPLICATIONS
TO SURVEY SAMPLING

by

BRIAN THOMAS COLLINS

DEPARTMENT OF MATHEMATICS
CARLETON UNIVERSITY

FEBRUARY 1973

Brian Thomas Collins

1973

The undersigned recommend to the Faculty
of Graduate Studies acceptance of the thesis

"The Theory of Polykay Statistics with Applications to
Survey Sampling"

submitted by Brian Thomas Collins, B.Sc.,
in partial fulfilment of the requirements for
the degree of Master of Science.

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ABSTRACT

The polykay statistics for without replacement sampling from a finite universe of units are developed. A technique of linearizing products of polykays is presented and the formulae derived by this technique are used in simplifying the derivation of many important results in estimation theory for sample surveys. The theory of polykays for with replacement sampling are presented for the first time here and they are then applied to investigate the properties of replicated sampling. Polykays for bivariate populations are investigated and some new results are presented. These statistics are then used to derive expressions for the bias and the mean square error of some ratio estimators. Finally the polykays for a two-way array are studied and a technique for sampling from a two-way array is presented and compared with simple random sampling.

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(i)

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INTRODUCTION

In developing a theory to study a finite population of units no assumptions are made about the distribution of values itself. Hence it is necessary to study the universe via distribution-free statistics. The usual way to characterize a finite universe is through its moments but this, however, leads to algebraic difficulties. For example, it is well known that the expected value of a sample moment is not in general identical to the equivalent population moment. This means that the algebraic manipulations involved in studying the distribution of moments becomes very unwieldy.

A study of the distribution of higher order moments is essential because one must be concerned with the accuracy and precision of estimators when developing an estimation theory. Further, if there are two possible unbiased estimators of a parameter then one must be able to estimate the variance of each in order to choose the more precise estimator. This implies that one must be able to estimate the higher order moments of the universe in question. The polykays presented in this study will reduce the algebra involved to a considerable degree because they have been so defined that they have desirable properties that may be exploited when taking the expected value of linear combinations of them.

(iii)

In this study all samples will be assumed to be selected without replacement from a finite universe of units unless otherwise stated. The notation used will be an extension of that given by Cochran [2] which has become the standard reference for survey sampling.

Consider a universe U composed of N units,
 $U = \{U_1, U_2, \dots, U_N\}$. The variate values associated with these units are respectively $\{y_1, y_2, \dots, y_N\}$. A simple random sample of size n is selected without replacement from U . The variate values associated with the realized sample are $\{y_1, y_2, \dots, y_n\}$. The population arithmetic mean is $\bar{Y} = \sum_{i=1}^N y_i / N$ and the sample arithmetic mean is $\bar{y} = \sum_{i=1}^n y_i / n$. Evidently $\bar{y}$ is an unbiased estimator of $\bar{Y}$. The r -th population moment is defined as

$$\mu_r = \sum_{i=1}^N (y_i - \bar{Y})^r / N ,$$

and the r -th sample moment as

$$m_r = \sum_{i=1}^n (y_i - \bar{y})^r / n , \quad r = 1, 2, \dots .$$

The population mean square is defined by

$$S^2 = \sum_{i=1}^N (y_i - \bar{Y})^2 / (N - 1) ,$$

and the sample mean square by

(iv)

$$s^2 = \sum_{i=1}^n (y_i - \bar{y})^2 / (n - 1) .$$

The symbol $\sum_{i \neq j}^n$ will denote a summation over the n realized sample values of all terms which have different subscripts where indicated. Thus for example

$$\begin{aligned} \sum_{i \neq j}^n y_i y_j &= y_1 y_2 + y_1 y_3 + \dots + y_1 y_n + y_2 y_1 + y_2 y_3 + \dots + y_n y_{n-1} \\ &= \sum_{i=1}^n \sum_{\substack{j=1 \\ i \neq j}}^n y_i y_j . \end{aligned}$$

It is well known that $\bar{y}$ is an unbiased estimator of $\bar{Y}$ and that the variance of $\bar{y}$ is given by $(\frac{1}{n} - \frac{1}{N})S^2$. However, S^2 is a population parameter which is generally unknown; hence it must be estimated from the sample so that statements about the precision of the estimator $\bar{y}$ can be made. Now s^2 is a random variable which is an unbiased estimator of S^2 so that it is desirable to estimate the variance of s^2 , $V(s^2)$. Thus

$$\begin{aligned} V(s^2) &= E(s^4) - \{E(s^2)\}^2 \\ &= E\{(\sum_{i=1}^n y_i^2 - (\sum_{i=1}^n y_i)^2/n)^2 / (n-1)^2\} - S^4 \\ &= E\{(1 - \frac{2}{n} + \frac{1}{n^2})\sum_{i=1}^n y_i^4 + (1 - \frac{2}{n} + \frac{3}{n^2})\sum_{i \neq j}^n y_i^2 y_j^2 \\ &\quad + (\frac{-2}{n} + \frac{6}{n^2})\sum_{i \neq j}^n y_i^2 y_j y_k + (\frac{-4}{n} + \frac{4}{n^2})\sum_{i \neq j}^n y_i^3 y_j \\ &\quad + \sum_{i \neq j}^n y_i y_j y_k y_\ell\} / (n-1)^2 - S^4 . \end{aligned} \tag{0.1}$$

(v)

Define a_i , $i = 1, 2, \dots, N$ to be an indicator variable taking the value unity if the i -th unit is in the sample and zero otherwise. Then

$$E(\sum_{i=1}^n y_i^4) = E(\sum_{i=1}^N y_i^4 a_i) = \sum_{i=1}^N y_i^4 E(a_i) = \sum_{i=1}^N y_i^4 n/N$$

where $E(a_i) = 1(\frac{n}{N}) + 0(\frac{N-n}{N}) = \frac{n}{N}$. Similarly

$$\begin{aligned} E(\sum_{i \neq j}^n y_i^3 y_j) &= E(\sum_{i \neq j}^N y_i^3 y_j a_i a_j) = \sum_{i \neq j}^N y_i^3 y_j E(a_i a_j) \\ &= \sum_{i \neq j}^N y_i^3 y_j n(n-1)/N(N-1), \end{aligned}$$

$$E(\sum_{i \neq j}^n y_i^2 y_j^2) = n(n-1) \sum_{i \neq j}^N y_i^2 y_j^2 / N(N-1),$$

$$E(\sum_{i \neq j \neq k}^n y_i^2 y_j y_k) = n(n-1)(n-2) \sum_{i \neq j \neq k}^N y_i^2 y_j y_k / N(N-1)(N-2),$$

and

$$E(\sum_{i \neq j \neq k \neq l}^n y_i y_j y_k y_l) = n(n-1)(n-2)(n-3) \sum_{i \neq j \neq k \neq l}^N y_i y_j y_k y_l / N(N-1)(N-2)(N-3).$$

Substitution of these results into (0.1) yields

$$\begin{aligned} V(s^2) &= \frac{1}{(n-1)^2} \left[\frac{(n-1)^2}{n} - \frac{(n-1)}{n(N-1)} \{ (n-2)(n-3) - (n-1) \} - \right. \\ &\quad \left. 4 \frac{(n-1)(n-2)(n-3)}{n(N-1)(N-2)} - 6 \frac{(n-1)(n-2)(n-3)}{n(N-1)(N-2)(N-3)} \right] \mu_4 \end{aligned}$$

(vi)

$$\begin{aligned} & + \frac{1}{(n-1)^2} \left[\frac{(n-1)N}{n(N-1)} \{(n-1)^2 + 2\} + 2 \frac{(n-1)(n-2)(n-3)N}{n(N-1)(N-2)} \right. \\ & \quad \left. + \frac{3(n-1)(n-2)(n-3)N}{n(N-1)(N-2)(N-3)} - \frac{N^2(n-1)^2}{(N-1)^2} \right] \mu_2^2 . \end{aligned} \quad (0.2)$$

Further simplification of (0.2) yields

$$\begin{aligned} V(s^2) &= \left[\frac{N\{(n-1)(N^2 - 5N + 6) - (N-1)(n^2 - 5n + 6)\}}{n(N-1)(N-2)(N-3)(n-1)} \right] \mu_4 \\ &+ \left[\frac{N(N-n)[3n - nN^2 + 3(N-1)^2]}{n(N-1)(N-2)(N-3)(n-1)} \right] \mu_2^2 . \end{aligned}$$

Note that the algebraic manipulations have already become extremely tedious even when deriving such a basic result. A simpler derivation and formula for the variance of s^2 is given in Chapter 2.

In Chapter 1 the polykays will be introduced and their properties will be developed. It will be shown that they are unbiased estimators of the equivalent population polykays and that a large class of them is invariant under translations of the variate values. In Chapter 2 it will be shown that a product of polykays can be rewritten as a linear combination of polykays thereby simplifying the process of evaluating the expectation of a product of polykays. This linearization will be used to derive variances and covariances of sample polykays.

In Chapter 3 the properties of polykeys are developed when the sample is drawn with replacement. These results are presented for the first time in this paper. The application of polykeys to replicated sampling is then described and these results are compared to those derived for without replacement sampling.

In Chapter 4 the bivariate analogues of the univariate polykeys are defined and the theory is developed on an exact parallel to univariate polykey theory. Table A5 giving explicit formulae for the bivariate polykeys and Table A6 giving expressions for products of polykeys are presented for the first time in this study. The bivariate polykeys are then applied to the theory of ratio estimation. The approach used in this paper is different from that found in the literature.

The statistics analogous to polykeys in a two-way array are introduced in Chapter 5 and some comparisons with simple random sampling are given. Finally in Chapter 6 a variety of techniques for calculating the value of polykeys for a data set are described.

REVIEW OF THE LITERATURE

In 1929 R. A. Fisher [6] attacked the problem of studying functions of the higher order moments by considering the population cumulants. The cumulants $\kappa_1, \kappa_2, \kappa_3, \dots$, of a population with moments $\mu_1, \mu_2, \mu_3, \dots$, are defined by the relationship

$$\kappa_1 t + \kappa_2 t^2/2! + \kappa_3 t^3/3! + \dots = \ln\{1 + \mu_1 t + \mu_2 t^2/2! + \dots\}$$

where $\ln$ denotes the natural logarithm. Fisher introduced what he called "K statistics" which are unbiased estimators of the population cumulants and studied the cumulants of their distribution.

In 1940 the multiple K statistics or semiinvariants were introduced by Dressel [5] as unbiased estimators of products of cumulants. However, the set of multiple K statistics introduced by Dressel was not a complete set. This paper escaped notice at the time. A paper by Irwin and Kendall [10] in 1944 presented a technique for deriving the properties of finite population K statistics from the already known properties of the infinite population K statistics.

In 1950 Tukey [23] reintroduced some of the multiple K statistics. He studied only the finite population properties of the K statistics. In a later paper, Tukey [24], 1956, set up the finite population K statistics, which he called "polykays"

on an exact theoretical parallel to those for the infinite population case. The finite population polykays have proved to be extremely useful in the theory of sampling. Tukey [23] has shown that many derivations and formulae themselves can be greatly simplified through the use of polykays.

The K statistics for a bivariate infinite population were developed by Cook [27]. The finite population multivariate K statistics were derived by Robson [21] who applied them to the problem of developing unbiased ratio estimators. Hooke [9] introduced the polykays for a two-way array which are based on a sampling scheme known as "bisampling".

The most powerful tool made available by the use of polykays is the vast number of relationships that exist between different polykays. Several authors, notably David and Kendall [3], Wishart [26], Abdel Aty [1] and Tukey [23] have provided extensive tables of results. Several techniques for the calculation of polykays from a data set are given by Schaeffer and Dwyer [22].

CHAPTER 1

SYMMETRIC FUNCTIONS AND POLYKAYS

1.1. Symmetric Functions and Inheritance on the Average.

The difficulties inherent in the previous developments can be traced primarily to the fact that the sample moments are not in general unbiased estimators of the corresponding population moments. The statistic s^2 and the population parameter S^2 were introduced as alternatives to the second sample moment

$m_2 = \sum_{i=1}^n (y_i - \bar{y})^2/n$ and the second population moment
 $\mu_2 = \sum_{i=1}^N (y_i - \bar{Y})^2/N$. This was done because $E(s^2) = S^2$ while $E(m_2) \neq \mu_2$.

A statistic will be defined as being "inherited on the average" if its expected value taken over all possible samples of a given size n is equal to the equivalent population parameter. Thus s^2 is inherited on the average while m_2 is not. This property is extremely useful in simplifying the algebra involved in taking expectations.

Define the "a-th power sum" to be

$$(a) = \sum_{i=1}^n y_i^a.$$

Define the "generalized symmetric functions" to be

$$[ab \dots e] = \sum \nolimits^n \neq y_i^a y_j^b \dots y_v^e .$$

The power sums are obviously a special case of the augmented symmetric functions for $(a) = [a]$.

The generalized symmetric functions may be written in terms of the power sums. For example consider $[a \ b]$:

$$\begin{aligned} [a \ b] &= \sum \nolimits^n \neq y_i^a y_j^b \\ &= (\sum \nolimits^n y_i^a)(\sum \nolimits^n y_i^b) - \sum \nolimits^n y_i^{a+b} \\ &= (a)(b) - (a + b) . \end{aligned} \tag{1.1}$$

David and Kendall [3] have provided extensive tables of such conversions. A portion of this table is reproduced as TABLE A1 in the Appendices.

The "brackets" are defined by

$$\begin{aligned} \langle a \rangle &= \sum \nolimits^n x_i^a / n , \\ \langle a \ b \rangle &= \sum \nolimits^n \neq y_i^a y_j^b / n(n - 1) , \\ \langle ab \dots e \rangle &= \sum \nolimits^n \neq y_i^a y_j^b \dots y_\ell^e / n^{(\rho)} , \end{aligned} \tag{1.2}$$

where $n^{(\rho)} = n(n - 1) \dots (n - \rho + 1)$, ρ being the number of different subscripts in the numerator of the sum.

The brackets may be written in a more compact form.

Assume that a bracket consists of v different values p_1 having multiplicity π_1 , p_2 having multiplicity π_2 , ..., p_v having multiplicity π_v . Then write $\langle p_1 p_1 \dots p_1 p_2 p_2 \dots p_2 \dots p_v p_v \dots p_v \rangle = \langle p_1^{\pi_1} p_2^{\pi_2} \dots p_v^{\pi_v} \rangle$.

A similar compact form is used for the augmented symmetric functions.

The "weight" of a bracket is defined to be $r = \sum_{i=1}^v p_i \pi_i$. The weight of a product of brackets is the sum of the weights of the individual brackets.

The "order" of a bracket is defined to be $\rho = \sum_{i=1}^v \pi_i$.

From (1.1) it follows that

$$\left(\sum y_i^a \right) \left(\sum y_i^b \right) = \sum y_i^{a+b} + \sum y_i^a y_j^b, \quad \text{so that } n^2 \langle a \times b \rangle = n \langle a + b \rangle + n(n-1) \langle a b \rangle, \text{ and hence}$$

$$\langle a b \rangle = (n \langle a \times b \rangle - \langle a + b \rangle) / (n-1). \quad (1.3)$$

Similarly,

$$\left(\sum y_i^a y_j^b \right) \left(\sum y_i^c \right) = \sum y_i^a y_j^b y_k^c + \sum y_i^{a+c} y_j^b + \sum y_i^a y_j^{b+c}, \quad \text{so that } n^2(n-1) \langle ab \times c \rangle = n(n-1)(n-2) \langle abc \rangle + n(n-1) \langle a+c, b \rangle + n(n-1) \langle a, b+c \rangle,$$

$$\text{and hence } \langle abc \rangle = (n \langle ab \times c \rangle - \langle a+c, b \rangle - \langle a, b+c \rangle) / (n-2). \quad (1.4)$$

From relations (1.3) and (1.4) it is evident that a bracket of high order may be written in terms of brackets of lower order which greatly simplifies the final calculation of such statistics.

Define $\langle ab \dots e \rangle'$ to be the population equivalent of $\langle ab \dots e \rangle$, that is

$$\langle ab \dots e \rangle' = \sum_{i,j,\dots,k}^N y_i^a y_j^b \dots y_k^e / N^{(\rho)} .$$

Lemma 1.1. The bracket $\langle ab \dots e \rangle$ is inherited on the average, i.e.,

$$E\{\langle ab \dots e \rangle\} = \langle ab \dots e \rangle' . \quad (1.5)$$

Proof. Let

$$\begin{aligned} a_i &= 1 \quad \text{if the } i\text{-th unit is in the sample of size } n \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Then

$$\begin{aligned} E\{\langle ab \dots e \rangle\} &= E\left\{\sum_{i,j,\dots,k}^n y_i^a y_j^b \dots y_k^e / n^{(\rho)}\right\} \\ &= E\left\{\sum_{i,j,\dots,k}^N y_i^a y_j^b \dots y_k^e a_i a_j \dots a_k / n^{(\rho)}\right\} \\ &= \sum_{i,j,\dots,k}^N y_i^a y_j^b \dots y_k^e E\{a_i a_j \dots a_k\} / n^{(\rho)} . \end{aligned}$$

There are a total of $\binom{N}{n}$ samples of size n . Any specific set of ρ different units occurs in $\binom{N-\rho}{n-\rho}$ of these samples. Thus

$$E(a_i a_j \dots a_k) = \binom{N-\rho}{n-\rho} / \binom{N}{n}.$$

Hence

$$\begin{aligned} E\{<ab \dots e>\} &= \sum \neq y_i^a y_j^b \dots y_k^e \binom{N-\rho}{n-\rho} / \binom{N}{n} n(\rho) \\ &= \sum \neq y_i^a y_j^b \dots y_k^e / N(\rho) \\ &= <ab \dots e>' . \end{aligned}$$

It follows immediately from Lemma 1.1 that any linear function of the brackets is also inherited on the average.

1.2. $\odot$ - Multiplication and First Order Polykays.

The polykays will be defined to be certain linear combinations of the brackets and hence will be inherited on the average. It is convenient to introduce polykays by defining a symbolic multiplication operator " $\odot$ " with the properties

1. $c_1 \odot c_2 = c_1 c_2$ where c_1 and c_2 are constant,
2. $c \odot <a_1 a_2 \dots a_v> = c <a_1 a_2 \dots a_v>$ where c is constant,

$$3. \langle a_1 a_2 \dots a_v \rangle \odot \langle b_1 b_2 \dots b_p \rangle = \langle a_1 a_2 \dots a_v b_1 b_2 \dots b_p \rangle ,$$

$$4. \langle a_1 a_2 \dots a_v \rangle \odot [\langle b_1 b_2 \dots b_p \rangle + \langle c_1 c_2 \dots c_q \rangle] = \\ = \langle a_1 a_2 \dots a_v b_1 b_2 \dots b_p \rangle + \langle a_1 a_2 \dots a_v c_1 c_2 \dots c_q \rangle .$$

Analogous to the exponential and logarithmic functions, define the $\odot$ - exponential and $\odot$ - logarithmic functions:

$$\odot\text{-exp}\{X\} = 1 + X + (X \odot X)/2! + (X \odot X \odot X)/3! + \dots ,$$

and

$$\odot\text{-log}\{1 + X\} = X - (X \odot X)/2 + (X \odot X \odot X)/3 - \dots .$$

These two functions satisfy the following properties:

1. $\odot\text{-log}\{\odot\text{-exp}\{X\}\} = X ,$
2. $\odot\text{-exp}\{\odot\text{-log}\{X\}\} = X ,$
3. $\odot\text{-log}\{(1 + X_1)(1 + X_2)\} = \odot\text{-log}(1 + X_1) + \odot\text{-log}(1 + X_2) .$

The first property is proved by noting that

$$\exp\{x\} = 1 + x + x^2/2! + x^3/3! + \dots , \text{ and } \log\{x+1\} = x - x^2/2 + x^3/3 - \dots .$$

Thus the coefficient of the r -th term, $X \odot X \odot \dots$ (r times) $\dots \odot X$

in $\odot\text{-log}\{\odot\text{-exp}\{x\}\}$ is identical to the coefficient of x^r in

$\log\{\exp\{x\}\} = x$. The second and third properties are proved in

a similar fashion.

$$\text{Let } M(t) = 1 + \langle 1 \rangle t + \langle 2 \rangle t^2/2! + \langle 3 \rangle t^3/3! + \dots$$

The "first order sample polykays" $K(1), K(2), K(3), \dots$ are defined by

$$\psi(t) = \odot - \log\{M(t)\} , \quad (1.6)$$

where $\psi(t) = K(1)t + K(2)t^2/2! + K(3)t^3/3! + \dots$. Now

$$K(1)t + K(2)t^2/2! + K(3)t^3/3! + \dots$$

$$= \odot - \log\{1 + \langle 1 \rangle t + \langle 2 \rangle t^2/2! + \dots\}$$

$$= (t\langle 1 \rangle + t^2\langle 2 \rangle/2! + t^3\langle 3 \rangle/3! + t^4\langle 4 \rangle/4! + \dots)$$

$$- (t\langle 1 \rangle + t^2\langle 2 \rangle/2! + t^3\langle 3 \rangle/3! + \dots) \odot (t\langle 1 \rangle + t^2\langle 2 \rangle/2! + t^3\langle 3 \rangle/3! + \dots)/2$$

$$+ (t\langle 1 \rangle + t^2\langle 2 \rangle/2! + \dots) \odot (t\langle 1 \rangle + t^2\langle 2 \rangle/2! + \dots) \odot (t\langle 1 \rangle + t^2\langle 2 \rangle/2! + \dots)/3$$

$$- (t\langle 1 \rangle + \dots) \odot (t\langle 1 \rangle + \dots) \odot (t\langle 1 \rangle + \dots) \odot (t\langle 1 \rangle + \dots)/4$$

$$+ \dots$$

$$= t\langle 1 \rangle + t^2\{\langle 2 \rangle - \langle 1 \rangle \odot \langle 1 \rangle\}/2!$$

$$+ t^3\{\langle 3 \rangle - 3\langle 1 \rangle \odot \langle 2 \rangle + 2\langle 1 \rangle \odot \langle 1 \rangle \odot \langle 1 \rangle\}/3!$$

$$+ t^4 \{ \langle 4 \rangle - 4 \langle 1 \rangle \odot \langle 3 \rangle - 3 \langle 2 \rangle \odot \langle 2 \rangle + 12 \langle 1 \rangle \odot \langle 1 \rangle \odot \langle 2 \rangle \\ - 6 \langle 1 \rangle \odot \langle 1 \rangle \odot \langle 1 \rangle \odot \langle 1 \rangle \} / 4! + \dots$$

Equating coefficients of t^i on the left and right hand sides gives

$$K(1) = \langle 1 \rangle ,$$

$$K(2) = \langle 2 \rangle - \langle 1 \ 1 \rangle ,$$

$$K(3) = \langle 3 \rangle - 3 \langle 12 \rangle + 2 \langle 111 \rangle , \quad (1.7)$$

and

$$K(4) = \langle 4 \rangle - 4 \langle 13 \rangle - 3 \langle 22 \rangle + 12 \langle 112 \rangle - 6 \langle 1111 \rangle .$$

The general expression for $K(r)$ in terms of the brackets is given in the following theorem:

Theorem 1.1.

$$K(r) = r! \sum \frac{(-1)^{\rho-1} (\rho - 1)! \langle p_1^{\pi_1} p_2^{\pi_2} \dots p_v^{\pi_v} \rangle}{\pi_1! \pi_2! \dots \pi_v! (p_1!)^{\pi_1} (p_2!)^{\pi_2} \dots (p_v!)^{\pi_v}} \quad (1.8)$$

$$r = 1, 2, \dots$$

where the order is $\rho = \sum \pi_i$ and the weight is $r = \sum p_i \pi_i$, the summation extending over all choices of $p_1, p_2, \dots, p_v, \pi_1, \pi_2, \dots, \pi_v$ such that $\sum p_i \pi_i = r$, i.e., over all partitions of r .

Proof. The ρ -th term in the $\odot$ -log expansion of (1.6) is

$$(-1)^{\rho-1} (t^{<1>} + t^2 \langle 2 \rangle / 2! + \dots) \odot (t^{<1>} + t^2 \langle 2 \rangle / 2! + \dots) \odot \dots \\ \odot (t^{<1>} + t^2 \langle 2 \rangle / 2! + \dots) / \rho . \quad (1.9)$$

An explicit solution of (1.6) in terms of the brackets is found by equating coefficients of t on both sides. The coefficient of $K(r)$ on the right hand side of (1.6) is $t^r / r!$.

Now

$$t^r = (t^{p_1})^{\pi_1} (t^{p_2})^{\pi_2} \dots (t^{p_v})^{\pi_v} \quad (1.10)$$

since $\sum p_i \pi_i = r$. Hence a contribution to the coefficient of t^r on the left hand side of (1.6) is given by the coefficient of (1.10). Further since $\sum \pi_i = \rho$ this contribution comes from (1.9), the ρ -th term of the expansion of $\odot\text{-log}\{M(t)\}$.

Now the coefficient of $(t^{p_1})^{\pi_1} (t^{p_2})^{\pi_2} \dots (t^{p_v})^{\pi_v}$ is

$$\frac{(-1)^{\rho-1} \langle p_1 \rangle \odot \langle p_1 \rangle \odot \dots \odot \langle p_1 \rangle \odot \langle p_2 \rangle \odot \langle p_2 \rangle \odot \dots \odot \langle p_2 \rangle \odot \dots \odot \langle p_v \rangle \odot \langle p_v \rangle \odot \dots \odot \langle p_v \rangle}{\rho \, p_1! p_1! \dots p_1! p_2! p_2! \dots p_2! \dots p_v! p_v! \dots p_v!} .$$

The number of times this term occurs in (1.9) is the number of ways of arranging a total of ρ objects of which π_1 are of the first kind, π_2 are of the second kind, ..., and π_v are of the v -th kind, i.e., $\rho! / \pi_1! \pi_2! \dots \pi_v!$.

By the second property of the $\odot$ -operator,

$$\begin{aligned} & \langle p_1 \rangle \odot \langle p_1 \rangle \odot \dots \odot \langle p_1 \rangle \odot \langle p_2 \rangle \odot \langle p_2 \rangle \odot \dots \odot \langle p_2 \rangle \odot \dots \odot \langle p_v \rangle \odot \langle p_v \rangle \odot \dots \odot \langle p_v \rangle \\ &= \langle p_1^{\pi_1} p_2^{\pi_2} \dots p_v^{\pi_v} \rangle . \end{aligned}$$

Hence one component of the coefficient of t^r is

$$\frac{(-1)^{\rho-1}}{\rho} \left(\frac{\rho!}{\pi_1! \pi_2! \dots \pi_v!} \right) \left(\frac{\langle p_1^{\pi_1} p_2^{\pi_2} \dots p_v^{\pi_v} \rangle}{(p_1!)^{\pi_1} (p_2!)^{\pi_2} \dots (p_v!)^{\pi_v}} \right) .$$

The overall coefficient of t^r is found by summing all terms of this type. The result follows immediately.

To illustrate the application of Theorem 1.1 consider the first order sample polykay $K(5)$. The possible partitions of 5 are 5, 1 4, 2 3, 1² 3, 1 2², 1³ 2, and 1⁵. Then from (1.8) the terms summing to yield $K(5)$ are $5!(-1)^0 0! \langle 5 \rangle / 5!$, $5!(-1)^1 1! \langle 1 \ 4 \rangle / 1!4!$, $5!(-1)^1 1! \langle 2 \ 3 \rangle / 2!3!$, $5!(-1)^2 2! \langle 1^2 \ 3 \rangle / 2!(1!)^2 3!$, $5!(-1)^2 2! \langle 1 \ 2^2 \rangle / 2!1!(2!)^2$, $5!(-1)^3 3! \langle 1^3 \ 2 \rangle / 3!(1!)^3 2!$ and $5!(-1)^4 4! \langle 1^5 \rangle / 5!(1!)^5$. Thus

$$\begin{aligned} K(5) &= \langle 5 \rangle - 5 \langle 1 \ 4 \rangle - 10 \langle 2 \ 3 \rangle + 20 \langle 1^2 \ 3 \rangle + 30 \langle 1 \ 2^2 \rangle \\ &\quad - 60 \langle 1^3 \ 2 \rangle + 24 \langle 1^5 \rangle . \end{aligned}$$

The first order population polykays $K(r)'$ will be defined

to be the same function of $\langle p_1^{\pi_1} p_2^{\pi_2} \dots p_v^{\pi_v} \rangle'$ that $K(r)$ is of $\langle p_1^{\pi_1} p_2^{\pi_2} \dots p_v^{\pi_v} \rangle$. Hence let

$$M'(t) = 1 + \langle 1 \rangle' t + \langle 2 \rangle' t^2/2! + \langle 3 \rangle' t^3/3! + \dots$$

and

$$\psi'(t) = K(1)' t + K(2)' t^2/2! + K(3)' t^3/3! + \dots$$

Define $K(r)'$ through the relationship

$$\psi'(t) = \ominus - \log\{M'(t)\}.$$

Analogous to (1.8)

$$K(r)' = r! \sum \frac{(-1)^{\rho-1} (\rho-1)! \langle p_1^{\pi_1} p_2^{\pi_2} \dots p_v^{\pi_v} \rangle'}{\pi_1! \pi_2! \dots \pi_v! (p_1!)^{\pi_1} (p_2!)^{\pi_2} \dots (p_v!)^{\pi_v}} \quad (1.11)$$

which is proved in an identical fashion.

The sample polykays are a linear function of the brackets and hence will be inherited on the average, i.e.,

$$E\{K(r)\} = K(r)', \quad r = 1, 2, \dots \quad (1.12)$$

The polykays can be expressed in terms of the central sample moments:

$$\begin{aligned}
 K(1) &= m_1 , \\
 K(2) &= (n/(n-1))m_2 = s^2 , \\
 K(3) &= (n^2/(n-1)(n-2))m_3 , \\
 K(4) &= (n^2/(n-1)(n-2)(n-3))[(n+1)m_4 - 3(n-1)m_2^2] .
 \end{aligned} \tag{1.13}$$

The corresponding population formulae are

$$\begin{aligned}
 K(1)' &= \mu_1 , \\
 K(2)' &= (N/(N-1))\mu_2 = S^2 , \\
 K(3)' &= (N^2/(N-1)(N-2))\mu_3 , \\
 K(4)' &= (N^2/(N-1)(N-2)(N-3))[(N+1)\mu_4 - 3(N-1)\mu_2^2] .
 \end{aligned} \tag{1.14}$$

The complexity of $K(4)$ and $K(4)'$ indicates that the polykays are not merely a constant multiple of the moments but are actually a completely different set of statistics.

1.3. Higher-Order Polykays.

Higher order polykays may be defined using the $\odot$ -operator and first order polykays.

Define the "p-th order sample polykays" to be

$$K(a_1 a_2 \dots a_p) = K(a_1) \odot K(a_2) \odot \dots \odot K(a_p) . \tag{1.15}$$

The "weight" of this polykay is defined to be $r = \sum_{i=1}^{\rho} a_i$.
The compact notation used for the brackets is also used for the
higher order polykays. Thus

$$K(p_1^{\pi_1} p_2^{\pi_2} \dots p_v^{\pi_v}) = K(p_1 p_1 \dots p_1 p_2 p_2 \dots p_2 \dots p_v p_v \dots p_v)$$

where p_1 has multiplicity π_1 , p_2 has multiplicity π_2 , ...,
 p_v has multiplicity π_v .

Now from (1.15) it follows that

$$\begin{aligned} K(12) &= K(1) \odot K(2) = \langle 1 \rangle \odot [\langle 2 \rangle - \langle 11 \rangle] \\ &= \langle 12 \rangle - \langle 111 \rangle, \end{aligned} \tag{1.16}$$

$$\begin{aligned} K(22) &= K(2) \odot K(2) = [\langle 2 \rangle - \langle 11 \rangle] \odot [\langle 2 \rangle - \langle 11 \rangle] \\ &= \langle 22 \rangle - 2\langle 112 \rangle + \langle 1111 \rangle, \end{aligned}$$

and

$$\begin{aligned} K(113) &= K(1) \odot K(1) \odot K(3) \\ &= \langle 1 \rangle \odot \langle 1 \rangle \odot [\langle 3 \rangle - 3\langle 12 \rangle + 2\langle 111 \rangle] \\ &= \langle 113 \rangle - 3\langle 112 \rangle + 2\langle 1111 \rangle. \end{aligned}$$

The brackets can now be expressed in terms of polykays
using (1.15). Taking $\odot$ -exponentials in (1.6) yields

$$M(t) = \odot\text{-exp}\{\psi(t)\}$$

which, when expanded, gives

$$\begin{aligned} & 1 + \langle 1 \rangle t + \langle 2 \rangle t^2/2! + \langle 3 \rangle t^3/3! + \dots \\ &= 1 + (K(1)t + K(2)t^2/2! + K(3)t^3/3! + \dots) \\ & \quad + (K(1)t + K(2)t^2/2! + \dots) \odot (K(1)t + K(2)t^2/2! + \dots)/2! \\ & \quad + (K(1)t + \dots) \odot (K(1)t + \dots) \odot (K(1)t + \dots)/3! + \dots \\ &= 1 + t K(1) + t^2(K(2) + K(1) \odot K(1))/2! \\ & \quad + t^3(K(3) + 3 K(1) \odot K(2) + K(1) \odot K(1) \odot K(1))/3! + \dots \end{aligned}$$

Equating coefficients of the t gives

$$\langle 1 \rangle = K(1) ,$$

$$\langle 2 \rangle = K(2) + K(11) ,$$

and

$$\langle 3 \rangle = K(3) + 3K(12) + K(111) .$$

Theorem 1.2.

$$\langle r \rangle = r! \sum \frac{K(p_1^{\pi_1} p_2^{\pi_2} \dots p_v^{\pi_v})}{\pi_1! \pi_2! \dots \pi_v! (p_1!)^{\pi_1} (p_2!)^{\pi_2} \dots (p_v!)^{\pi_v}} \quad (1.17)$$

$$r = 1, 2, 3, \dots ,$$

the summation extending over all partitions of r such that $\sum p_i \pi_i = r$.

The higher order brackets can also be expressed in terms of polykeys by using the $\odot$ operator. For example,

$$\begin{aligned} \langle 12 \rangle &= \langle 1 \rangle \odot \langle 2 \rangle = K(1) \odot [K(2) + K(11)] \\ &= K(12) + K(111) . \end{aligned}$$

Tables expressing polykeys in terms of brackets and brackets in terms of polykeys are provided by Tukey [24]. This table is reproduced as TABLE A2 in the Appendices. A table giving conversions of weight 12 has been published by Aty [1].

The less cumbersome notation $K_a \equiv K(a)$, $K_{ab} \equiv K(ab)$, etc., will be used when it is not ambiguous.

Higher order population polykeys can also be defined. Define the " p -th order population polykeys" to be

$$K(a_1 a_2 \dots a_p)' = K(a_1)' \odot K(a_2)' \odot \dots \odot K(a_p)' . \quad (1.18)$$

Paralleling the results for sample polykeys it follows that

$$\begin{aligned} K(12)' &= K(1)' \odot K(2)' = \langle 1 \rangle' \odot [\langle 2 \rangle' - \langle 11 \rangle'] \\ &= \langle 12 \rangle' - \langle 111 \rangle' . \end{aligned} \quad (1.19)$$

Further analogous to (1.17)

$$\langle r \rangle' = r! \sum \frac{K(p_1^{\pi_1} p_2^{\pi_2} \dots p_v^{\pi_v})}{\pi_1! \pi_2! \dots \pi_v! (p_1!)^{\pi_1} (p_2!)^{\pi_2} \dots (p_v!)^{\pi_v}} \quad (1.20)$$

Hence the expressions for population brackets in terms of population brackets may be derived from TABLE A2 by replacing "K()" by "K()'" and replacing "< >" by "< >'" .

The alternate notation presented earlier for sample polykeys will also be used for population polykeys. Hence define $K'_a \equiv K(a)'$, $K'_{ab} = K(ab)'$, etc.

Obviously the higher order sample polykeys are inherited on the average, because they are linear functions of the brackets. Hence

$$E\{K_{a_1 a_2 \dots a_p}\} = K'_{a_1 a_2 \dots a_p} \quad .$$

The polykeys are appealing because they possess the inheritance on the average property. However any statistic which is a linear function of the brackets is inherited on the average, but the polykeys have other useful properties.

1.4. An Invariance Property.

It will be shown here that a certain class of polykeys are

invariant under translations of the origin of the variate values. This property can be used to greatly simplify the algebra involved in many computations involving polykays. Recall that

$$M(t) = 1 + \langle 1 \rangle t + \langle 2 \rangle t^2/2! + \dots \quad (1.21)$$

Expansion of the brackets on the right side of (1.21) gives

$$\begin{aligned} M(t) &= 1 + t \sum_{i=1}^n y_i/n + t^2 \sum_{i=1}^n y_i^2/n2! + \dots \\ &= (1 + ty_1 + t^2 y_1^2/2! + t^3 y_1^3/3! + \dots)/n \\ &\quad + (1 + ty_2 + t^2 y_2^2/2! + \dots)/n \\ &\quad + \dots + (1 + ty_n + t^2 y_n^2/2! + \dots)/n \\ &= \sum_{i=1}^n e^{ty_i/n} . \end{aligned} \quad (1.22)$$

Now consider a transformation $y_i^* = y_i + \delta$, $i = 1, 2, \dots, N$.

Define

$$\langle ab \dots e \rangle^* = \sum \neq y_i^{*a} y_j^{*b} \dots y_m^{*e} / n^{(\rho)} ,$$

where ρ is the number of different subscripts. Define

$M(t)^* = 1 + \langle 1 \rangle^* t + \langle 2 \rangle^* t^2/2! + \dots$. The first order polykays

$K(r)^*$ for the new variate values are defined by the relation

$$\psi(t)^* = \odot\text{-log}\{M(t)^*\} , \quad (1.23)$$

where

$$\psi(t)^* = K(1)^*t + K(2)^*t^2/2! + \dots .$$

Hence, recasting (1.23) using (1.22),

$$\begin{aligned} \psi(t)^* &= \odot\text{-log}\left\{ \sum_{i=1}^n e^{ty_i^*}/n \right\} \\ &= \odot\text{-log}\left\{ \sum_{i=1}^n e^{t(y_i + \delta)}/n \right\} \\ &= \odot\text{-log}\left\{ e^{t\delta} \left(\sum_{i=1}^n e^{ty_i} \right) / n \right\} \\ &= \odot\text{-log}\{e^{t\delta} M(t)\} \\ &= t\delta + \odot\text{-log}\{M(t)\} \\ &= t\delta + \psi(t) . \end{aligned} \quad (1.24)$$

Equating coefficients of t in (1.24) gives

$$K(r)^* = \begin{cases} K(1) + \delta & \text{if } r = 1 \\ K(r) & \text{otherwise} \end{cases} . \quad (1.25)$$

Thus all first order polykays except $K(1)$ are invariant under a translation of the origin.

Define the "number of unit parts" of a bracket or polykay to be the number of 1's which occur in the expression. For example K_2 , K_2 , both have zero unit parts, K_1 , K_{13} both have one unit part while K_{114} has two unit parts.

Theorem 1.3. Any polykay having zero unit parts is invariant under translations of origin. For a polykay with h unit parts a translation of the origin through a distance δ has an effect of order δ^h i.e. the resulting expression is a polynomial in δ of order h .

Proof. Let $K(11 \dots 1a_1a_2 \dots a_\rho)$ where $a_i \neq 1$, $i = 1, 2, \dots, \rho$, be any polykay with h unit parts. Translate the variate values by an amount δ and calculate the polykay $K(11 \dots 1a_1a_2 \dots a_\rho)^*$ for the new population of transformed values. Rewriting $K(11 \dots 1a_1a_2 \dots a_\rho)^*$ using (1.25) gives

$$\begin{aligned} K(11 \dots 1a_1a_2 \dots a_\rho)^* &= K(1) * \odot K(1) * \odot \dots \odot K(1) * \odot K(a_1) * \odot K(a_2) * \odot \dots \odot K(a_\rho)^* \\ &= [K(1) + \delta] \odot [K(1) + \delta] \odot \dots \odot [K(1) + \delta] \odot K(a_1) \odot K(a_2) \odot \dots \odot K(a_\rho) \end{aligned}$$

which is a polynomial in δ of order h . For example

$$\begin{aligned} K(112)^* &= [K(1) + \delta] \odot [K(1) + \delta] \odot K(2) \\ &= [K(11) + 2\delta K(1) + \delta^2] \odot K(2) \\ &= K(112) + 2\delta K(12) + \delta^2 K(2) . \end{aligned}$$

Let $K(a_1 a_2 \dots a_\rho)$ be a polykay with no unit parts. Then

$$\begin{aligned} K(a_1 a_2, \dots a_\rho)^* &= K(a_1)^* \odot K(a_2)^* \odot \dots \odot K(a_\rho)^* \\ &= K(a_1) \odot K(a_2) \odot \dots \odot K(a_\rho) \\ &= K(a_1 a_2 \dots a_\rho) . \end{aligned}$$

Hence any polykay with zero unit parts is invariant under translations of origin as was to be shown.

CHAPTER 2
PRODUCTS OF POLYKAYS

2.1. Some Multiplication Techniques.

Consider the problem of determining the variance of the random variable K_2 , viz.,

$$V(K_2) = EK_2^2 - (EK_2)^2 .$$

Thus not only must one be able to compute the expected value of K_2 but also the expected value of the square of K_2 . Further, to determine the covariance between K_1 and K_2 one must be able to evaluate the expectation of the product K_1K_2 . The polykays, however, have been defined so that one can easily determine expected values of linear combinations of them. The techniques given here will demonstrate how products of polykays may be rewritten as a linear combination of polykays, thereby greatly simplifying the process of determining expected values of such products.

Tukey [25] has presented a method of multiplying polykays which depends upon the multiplication of brackets.

Recall that

$$[ab \dots e] = \sum^n \neq y_i^a y_j^b \dots y_k^e .$$

Consider the product

$$\begin{aligned}
 [a][b] &= (\sum_i^n y_i^a)(\sum_i^n y_i^b) = \sum_i^n y_i^{a+b} + \sum_{i \neq j}^n y_i^a y_j^b \\
 &= [a+b] + [a \ b] .
 \end{aligned}
 \tag{2.1}$$

Now

$$\langle ab \dots e \rangle = [ab \dots e]/n^{(\rho)}
 \tag{2.2}$$

where ρ is the number of terms in the symmetric function.

Substituting (2.2) in (2.1) gives

$$n^2 \langle a \times b \rangle = n \langle a+b \rangle + n(n-1) \langle a \ b \rangle ,$$

and thus

$$\langle a \times b \rangle = \langle a+b \rangle/n + (n-1) \langle a \ b \rangle/n .$$

Similarly

$$[a][b \ c] = [a+b \ c] + [b \ a+c] + [a \ b \ c] ,
 \tag{2.3}$$

and

$$\begin{aligned}
 [ab][cd] &= [a+c \ b+d] + [a+d \ b+c] + [a+c \ b \ d] \\
 &\quad + [a+d \ b \ c] + [a \ b+c \ d] + [a \ b+d \ c] \\
 &\quad + [a \ b \ c \ d] .
 \end{aligned}
 \tag{2.4}$$

In general in determining the product of two symmetric functions $[a_1 a_2 \dots a_p]$ and $[b_1 b_2 \dots b_h]$ one must consider all symmetric functions formed when each unit from the first symmetric function is identical to or different from a unit of the second symmetric function. For example in the product $[ab][cd] = [\sum_{i=1}^n y_i^a y_j^b] [\sum_{k=1}^n y_k^c y_\ell^d]$ there are two possible ways of both units being identical $[\sum y_i^{a+c} y_j^{b+d}] = [a+c \ b+d]$ and $[a+d \ b+c]$. There are four ways of matching exactly one element $[a+c \ b \ d]$, $[a+d \ b \ c]$, $[a \ b+c \ d]$, and $[a \ b+d \ c]$. Finally there is one way in which no units correspond $[a \ b \ c \ d]$. The totality of these terms yields (2.4).

Conversion of (2.3) and (2.4) to brackets gives

$$\langle a \rangle \langle bc \rangle = \langle a+b \ c \rangle / n + \langle a+c \ b \rangle / n + (n-2) \langle abc \rangle / n ,$$

and

$$\begin{aligned} \langle ab \rangle \langle cd \rangle &= [\langle a+c \ b+d \rangle + \langle a+d \ b+c \rangle] / n(n-1) + \\ &[\langle a+c \ b \ d \rangle + \langle a+d \ b \ c \rangle + \langle a \ b+c \ d \rangle \\ &+ \langle a \ b+d \ c \rangle] (n-2) / n(n-1) + \langle a \ b \ c \ d \rangle (n-2)(n-3) / n(n-1) . \end{aligned}$$

Hence the product of any two brackets is a linear function of the brackets. To convert a formula involving symmetric functions to one involving brackets one must introduce coefficients involving

n . Assume a bracket of order a is multiplied with one of order b and that one of the terms in the product is of order m . The coefficient of this term would be $n^{(m)}/n^{(a)}n^{(b)}$. This is easily seen from the definition of the brackets. Tukey [25] provides a table of multiplication factors of brackets of order 4 or less.

The multiplication of polykays can now be accomplished by first writing the individual polykays in terms of brackets and multiplying the brackets using the rules just established and then converting the resulting product back to polykays.

For example,

$$\begin{aligned}
 K_2^2 &= \{ \langle 2 \rangle - \langle 11 \rangle \} \{ \langle 2 \rangle - \langle 11 \rangle \} \\
 &= \langle 2 \rangle^2 - 2 \langle 11 \rangle \langle 2 \rangle + \langle 11 \rangle^2 \\
 &= \{ (n-1) \langle 22 \rangle + \langle 4 \rangle \} / n - 2 \{ (n-2) \langle 112 \rangle + 2 \langle 13 \rangle \} / n \\
 &\quad + \{ (n-2)(n-3) \langle 1^4 \rangle + 4(n-2) \langle 112 \rangle + 2 \langle 22 \rangle \} / n(n-1) \\
 &= \langle 4 \rangle / n + [(n-1)/n + 2/n(n-1)] \langle 22 \rangle \\
 &\quad + \{ -4 \langle 13 \rangle / n + [-2(n-2)/n + 4(n-2)/n(n-1)] \langle 112 \rangle \\
 &\quad + (n-2)(n-3) \langle 1^4 \rangle / n(n-1) \} . \tag{2.5}
 \end{aligned}$$

TABLE A2 in the Appendices gives

$$\langle 4 \rangle = K_4 + 3K_{22} + 4K_{13} + 6K_{112} + K_{1111} ,$$

$$\langle 22 \rangle = K_{22} + 2K_{112} + K_{1111} ,$$

$$\langle 13 \rangle = K_{13} + 3K_{112} + K_{1111} ,$$

$$\langle 112 \rangle = K_{112} + K_{1111} ,$$

and

$$\langle 1111 \rangle = K_{1111} .$$

Substituting into (2.5) gives

$$\begin{aligned} K_2^2 &= K_4/n + K_{22}[3/n + (n-1)n + 2/n(n-1)] \\ &\quad + K_{13}[4/n - 4/n] + K_{112}[6/n + 2((n-1)/n + 2/n(n-1)) \\ &\quad - 12n - 2(n-2)/n + 4(n-2)/n(n-1)] \\ &\quad + K_{1111}[1/n + (n-1)/n + 2/n(n-1) - 4/n - 2(n-2)/n \\ &\quad + 4(n-2)/n(n-1) + (n-2)(n-3)/n(n-1)] \\ &= K_4/n + (n+1)K_{22}/(n-1) . \end{aligned} \tag{2.6}$$

This method of multiplication is evidently tedious and becomes even more complex when higher order products are involved. Fortunately this procedure can be simplified.

Now K_2 is invariant under a translation of the origin

of observation. Hence it is reasonable to suspect that K_2^2 is also invariant; (2.6) shows this to be true. Had this been evident prior to the derivation of (2.6) a good deal of tedious calculation could have been avoided.

Define " $0(1^g)$ " to be a linear combination of brackets or polykeys, each of which have at least g unit parts. For example, in (2.5) the last term is simply $0(1^1)$. Further, the expression $K_{112} + 3K_{113} + K_{11123}$ is $0(1^2)$.

If a bracket has g unit parts then obviously its conversion to the polykey form gives a linear combination of polykeys, each of which contains at least g unit parts. Hence a linear combination of brackets which is $0(1^g)$ when converted to polykeys is again $0(1^g)$.

Consider an identity between polykeys e.g., equation (2.6). Define the "unit order" of each side of the identity to be the greatest number of unit parts in one term. For example, in the identity

$$K_2 K_1^2 = K_4/n^2 + 2K_{13}/n + K_{211} \quad (2.7)$$

the term on the left-hand side has 2 unit parts and the 3 terms on the right-hand side have 0, 1, 2 unit parts respectively. Hence the unit order of the left-hand side is 2 and the unit order of the right-hand side is 2.

Consider the effect that a translation of origin would have on (2.7). Using (1.25) gives

$$K_2(K_1+\delta)(K_1+\delta) = K_4/n^2 + 2(K_1+\delta)K_3/n + K_2\odot[K_1+\delta][K_2+\delta] ,$$

and thus

$$\begin{aligned} K_2K_1^2 + 2\delta K_1K_2 + \delta^2 K_2 &= K_4/n^2 + 2(K_{13} + \delta K_3)/n \\ &+ K_{211} + 2\delta K_{12} + \delta^2 K_2 . \end{aligned}$$

Equating coefficients of δ gives

$$K_2K_1^2 = K_4/n^2 + 2K_{13}/n + K_{211} ,$$

$$2K_1K_2 = 2K_3/n + 2K_{12} \quad \text{and} \quad K_2 = K_2 .$$

More general, consider any polykay identity in which one side is linear in the polykays and assume that all trivial cancellations have been made. For example, in the equation

$$K_2^2 = K_4/n + K_{11} + (n+1)K_{22}/n - K_{11}$$

the terms K_{11} , $-K_{11}$ are trivially cancelled.

If the variate values are transformed by a constant value δ then the linear side of the polykay identity will become a polynomial in δ with terms up to order δ^h where h is the

unit order of the linear side. The coefficient of δ^h on the linear side will be linear in the polykays and will not identically vanish because all trivial cancellations were made earlier. Hence the non-linear side must also involve coefficients of order δ^h . Thus the unit order of the non-linear side must be at least h . This gives (unit order on linear side) $\leq$ (unit order on other side).

Now consider the problem of trying to derive a linear expression for the product of two polykays each of which has zero unit order. Using the result just derived one can say that the unit order of the linear side is zero. Hence on the linear side the coefficient of any polykay of unit order 1 or greater is zero. For example consider again the calculation of K_2^2 . The development is identical to equation (2.5) but at that point the term in brace brackets is written $0(1^1)$, viz.,

$$K_2^2 = \langle 4 \rangle / n + [(n+1)/n + 2/n(n-1)] \langle 22 \rangle + 0(1^1) \quad (2.8)$$

Converting (2.8) to polykays gives

$$K_2^2 = K_4/n + K_{22}(n+1)/(n-1) + 0(1^1) .$$

Due to the fact that K_2^2 has no unit parts the term $0(1^1) = 0$, giving formula (2.6).

Similarly, if the unit order on the nonlinear side is one then upon writing such an expression as a linear function of

the polykays the terms $0(1^2)$ can be deleted because such polykays have coefficient zero.

The reduction in computational effort is even greater when evaluating higher order expressions. This is due to the relatively large number of terms that can be ignored. Wishart [26] has provided tables giving products of polykays of up to weight 6 as a linear combination of polykays. Some formulae relating to polykays of weights 7 and 8 are also provided. A different multiplication technique was employed in developing these tables. The polykays are first written in terms of augmented symmetric functions and then converted to power sums. The product is determined and the power sums in the result are converted back to symmetric functions and thence to polykays. Consider

$$\begin{aligned} K_2 &= \langle 2 \rangle - \langle 11 \rangle = [2]/n - [11]/n^{(2)} \\ &= (2)/n - \{(1)^2 - (2)\}/n^{(2)} \\ &= \{(2) - (1)^2/n\}/(n-1) . \end{aligned}$$

Therefore,

$$K_2^2 = \{n^2(2)^2 - 2n(2)(1) + (1)^4\}/(n^{(2)})^2 .$$

The conversions from "[]" to "()" functions are made using TABLE A1 in the Appendices. From TABLE A1

$$n^2(2) = n^2[4] + n^2[2^2]$$

$$-2n(2)(1)^2 = -2n[4] - 4n[31] - 2n[2^2] - 2n[211] + [1^4]$$

and

$$(1)^4 = [4] + 4[31] + 3[2^2] + 6[211] + [1^4] .$$

Therefore,

$$K_2^2 = \{(n-1)^2[4] - 4(n-1)[31] + (n^2-2n+3)[2^2] - 2(n-3)[21^2] + [1^4]\}/(n^{(2)})^2 .$$

The "[]" are converted to polykeys using TABLE A2 of the Appendices, viz.,

$$(n-1)^2[4]/n^{(2)} = (n-1)K_4 + 4(n-1)K_{31} + 3(n-1)K_{22} + 6(n-1)K_{211} + (n-1)K_{1111} ,$$

$$-4(n-1)[31]/n^{(2)} = -4(n-1)K_{31} - 12(n-1)K_{211} - 4(n-1)K_{1111} ,$$

$$(n^2-2n+3)[22]/n^{(2)} = (n^2-2n+3)K_{22} + (2n^2-4n+6)K_{211} + (n^2-2n+3)K_{1111} ,$$

$$-2(n-3)[211]/n^{(2)} = (-2n^2+10n-12)K_{211} + (-2n^2+10n+12)K_{1111} ,$$

and

$$[1111]/n^{(2)} = (n^2-5n+6)K_{1111} .$$

This gives

$$K_2^2 = \{(n-1)K_4 + n(n+1)K_{22}\}/n^{(2)} = K_4/n + (n+1)K_{22}/(n-1) .$$

This last method proves to be just as tedious as the one initially considered. However the technique used to simplify the Tukey method of multiplication could also be used here. This gives one a method of checking the derivation of any product of polykays by providing a second method of performing the same computation.

A portion of the table of products given by Wishart [26] is presented in TABLE A3 of the Appendices.

It is also important to note that any result in this section has an analogue in terms of the population polykays with n replaced by N . For example, paralleling

$$K_2^2 = K_4/n + (n+1)K_{22}/(n-1) ,$$

there is

$$K_2'^2 = K_4'/N + (N+1)K_{22}'/(N-1) .$$

These population analogues will be useful in simplifying formulae for the variance of polykays.

2.2. Variances and Covariances of Polykeys.

In Section 2.1 it was mentioned that in order to calculate variances and covariances of polykeys one must be able to calculate the expected value of products of polykeys. By using the multiplication formulae of TABLE A3 in the Appendices this problem has been reduced to that of calculating the expected value of a linear combination of polykeys.

Consider the problem of calculating the variance of $K_1 = \bar{y}$. TABLE A3 gives

$$K_1^2 = K_2/n + K_{11}.$$

$$\text{Hence } K_1'^2 = K_2'/N + K_{11}'.$$

The variance of K_1 is given by

$$\begin{aligned} V(K_1) &= EK_1^2 - (EK_1)^2 \\ &= E(K_2/n + K_{11}) - K_1'^2 \\ &= K_2'/n + K_{11}' - (K_2'/N + K_{11}') \\ &= (1/n - 1/N)K_2' \\ &= (1/n - 1/N)S^2, \end{aligned} \tag{2.9}$$

a familiar result to any sample survey statistician. Further, an

unbiased estimator of the variance of K_1 is provided by

$$\begin{aligned}\hat{V}(K_1) &= (1/n - 1/N)K_2 \\ &= (1/n - 1/N)s^2.\end{aligned}$$

In the Introduction the variance of $s^2 = \sum(y_i - \bar{y})^2/(n-1)$ was derived. An alternative and simpler derivation is as follows:

$$\begin{aligned}V(K_2) &= EK_2^2 - (EK_2)^2 \\ &= E(K_4/n + (n+1)K_{22}/(n-1)) - K_2'^2 \\ &= K_4'/n + (n+1)K_{22}'/(n-1) - (K_4'/N + (N+1)K_{22}'/(N-1)) \\ &= (1/n - 1/N)K_4' + (2/(n-1) - 2/(N-1))K_{22}'.\end{aligned}\quad (2.10)$$

This result is significantly less cumbersome than (0.2).

Now from TABLE A3,

$$K_{22}' = (N-1)K_2'^2/(N+1) - (N-1)K_4'/N(N+1). \quad (2.11)$$

In (1.14) conversions between population polykays and moments were given. In particular,

$$K_2' = (N/(N-1))\mu_2$$

and

$$K'_4 = (N^2/(N-1)(N-2)(N-3))[(N+1)\mu_4 - 3(N-1)\mu_2^2] . \quad (2.12)$$

Substituting (2.12) into (2.11) yields

$$K'_{22} = \frac{-N\mu_4}{(N-2)(N-3)} + \left[\frac{N^2}{(N+1)(N-1)} + \frac{3N(N-1)}{(N+1)(N-2)(N-3)} \right] \mu_2^2 .$$

Hence, from (2.10),

$$\begin{aligned} V(K_2) = & \left\{ \left(\frac{1}{n} - \frac{1}{N} \right) \frac{N^2(N+1)}{(N-1)(N-2)(N-3)} - \left(\frac{2}{(n-1)} - \frac{2}{(N-1)} \right) \frac{N}{(N-1)(N-2)} \right\} \mu_4 \\ & + \left\{ \left(\frac{1}{n} - \frac{1}{N} \right) \frac{(-3N^2)}{(N-2)(N-3)} + \left(\frac{2}{(n-1)} - \frac{2}{(N-1)} \right) \left(\frac{N^2}{(N+1)(N-1)} + \right. \right. \\ & \left. \left. + \frac{3N(N-1)}{(N+1)(N-2)(N-3)} \right) \right\} \mu_2^2 . \quad (2.13) \end{aligned}$$

The coefficient of μ_4 in (2.13) simplifies to

$$(N-n)N(nN-n-N-1)/n(N-1)(N-2)(N-3)(n-1)$$

and the coefficient of μ_2^2 in (2.13) reduces to

$$(N-n)N\{3n-nN^2+3(N-1)^2\}/n(N-1)^2(N-2)(N-3)(n-1) .$$

These simplifications make (2.13) identical to the result given in the Introduction.

An unbiased estimator of the variance of s^2 is provided by

$$\hat{V}(s^2) = \hat{V}(K_2) = (1/n - 1/N)K_4 + (2/(n-1) - 2/(N-1))K_{22} .$$

Consider the problem of determining the covariance of the sample mean $\bar{x}$ and the sample variance s^2 , viz.,

$$\begin{aligned} \text{Cov}(\bar{x}, s^2) &= \text{Cov}(K_1, K_2) = E(K_1 K_2) - EK_1 EK_2 \\ &= E(K_3/n + K_{21}) - K_1' K_2' \\ &= K_3'/n + K_{21}' - (K_3'/N + K_{21}') \\ &= (1/n - 1/N)K_3' . \end{aligned}$$

An unbiased estimator of the covariance of $\bar{x}$, and s^2 is

$$\hat{\text{Cov}}(\bar{x}, s^2) = (1/n - 1/N)K_3 .$$

Next consider the problem of evaluating the third central moment of $\bar{x} = K_1$, $E[(K_1 - K_1')^3] = \mu_3(\bar{x})$. Now $\mu_3(\bar{x})$ is obviously invariant under translations of the origin. Hence if it is written as a linear combination of polykays then those polykays must have no unit parts. Hence it is possible to ignore terms that are $O(1^1)$.

Now

$$\begin{aligned}
 E[(K_1 - K'_1)^3] &= E(K_1^3) - 3K'_1 E(K_1^2) + 3K_1'^2 E(K_1) - K_1'^3 \\
 &= E(K_3/n^2 + 3K_{21}/n + K_{111}) - 3K'_1 E(K_2/n + K_{11}) + 2K_1'^3 \\
 &= K'_3/n^2 - 3K'_1 K'_2/n - 3K'_1 K'_{11} + 2K_1'^3 + O(1^1) \\
 &= K'_3/n^2 - 3[K'_3/N + K_{21}] - 3[2K'_{21}/N + K'_{111}]/n \\
 &\quad + 2[K'_3/N^2 + 3K'_{21}/N + K'_{111}] + O(1^1) \\
 &= K'_3/n^2 - 3K'_3/nN + 2K'_3/N^2 + O(1^1) \\
 &= K'_3(1/n - 1/N)(1/n - 2/N) ,
 \end{aligned}$$

where the term $O(1^1)$ is dropped for it is zero.

The techniques presented in this Section can be used to calculate other useful results, e.g., the expected value of higher order sample central moments and covariances of higher order polykays.

By doing these derivations via the polykays an unbiased estimator of the result can always be written down immediately.

CHAPTER 3

POLYKAYS FOR SAMPLING WITH REPLACEMENT

3.1. Cumulants.

Let $y_{[1]}, y_{[2]}, \dots, y_{[n]}$ denote the variate values attached to a sample of n units drawn with replacement from a population of N individuals.

Thus the $y_{[i]}$ are independent and identically distributed random variables. Define

$$\langle a_1 a_2 \dots a_\rho \rangle_R = \sum^n \neq y_{[i]}^{a_1} y_{[j]}^{a_2} \dots y_{[v]}^{a_\rho} / n^{(\rho)} \quad . \quad (3.1)$$

Taking the expected value of (3.1) over the universal set of N^n possible samples of size n , it follows that

$$\begin{aligned} E\{\langle a_1 a_2 \dots a_\rho \rangle_R\} &= E\left\{\sum^n \neq y_{[i]}^{a_1} y_{[j]}^{a_2} \dots y_{[v]}^{a_\rho} / n^{(\rho)}\right\} \\ &= \sum^n \neq E\{y_{[i]}^{a_1}\} E\{y_{[j]}^{a_2}\} \dots E\{y_{[v]}^{a_\rho}\} / n^{(\rho)} \\ &= \sum^n \neq \left(\sum^N y_i^{a_1}\right) \left(\sum^N y_j^{a_2}\right) \dots \left(\sum^N y_i^{a_\rho}\right) / n^{(\rho)} N^\rho \\ &= \left(\sum^N y_i^{a_1}\right) \left(\sum^N y_i^{a_2}\right) \dots \left(\sum^N y_i^{a_\rho}\right) \sum^n \neq 1 / n^{(\rho)} N^\rho \\ &= \langle a_1 \rangle' \langle a_2 \rangle' \dots \langle a_\rho \rangle' \quad . \quad (3.2) \end{aligned}$$

Hence the functions $\langle a_1 a_2 \dots a_p \rangle_R$ are not inherited on the average. Recall that the brackets which are the equivalent statistics for sampling without replacement are inherited on the average. Thus the most useful of all the properties of these functions in the without replacement case is no longer valid in the with replacement case. The left hand side of (3.2) may be written in terms of the $\odot$ -multiplication defined in Section 1.2, so that (3.2) becomes

$$E\{\langle a_1 \rangle_R \odot \langle a_2 \rangle_R \odot \dots \odot \langle a_p \rangle_R\} = \langle a_1 \rangle' \langle a_2 \rangle' \dots \langle a_p \rangle' . \quad (3.3)$$

Define the polykays, $K(p_1 \dots p_h)_R$, to be the same function of $\langle a_1 a_2 \dots a_p \rangle_R$ that $K(p_1 p_2 \dots p_h)$ is of $\langle a_1 a_2 \dots a_p \rangle$ in the without replacement case. Hence,

$$K(1)_R = \langle 1 \rangle_R ,$$

$$K(2)_R = \langle 2 \rangle_R - \langle 11 \rangle_R ,$$

$$K(11)_R = \langle 11 \rangle_R ,$$

and so on.

The general expression for $K(r)_R$ is

$$K(r)_R = r! \sum_{\pi} \frac{(-1)^{p-1} (p-1)! \langle p_1^{\pi_1} p_2^{\pi_2} \dots p_v^{\pi_v} \rangle_R}{\pi_1! \pi_2! \dots \pi_v! (p_1!)^{\pi_1} (p_2!)^{\pi_2} \dots (p_v!)^{\pi_v}} , \quad (3.4)$$

where $\rho = \sum \pi_i$, the summation extending over all partitions of r such that $\sum p_i \pi_i = r$. The population cumulants, $\kappa_1, \kappa_2, \kappa_3, \dots$ are defined by

$$\exp\{\kappa_1 t + \kappa_2 t^2/2! + \dots\} = 1 + \mu_1 t + \mu_2 t^2/2! + \mu_3 t^3/3! + \dots$$

Paralleling the procedure used in proving Theorem 1.1 it can be seen that

$$\kappa_r = r! \sum \frac{(-1)^{\rho-1} (\rho-1)! \langle p_1 \rangle^{\pi_1} \langle p_2 \rangle^{\pi_2} \dots \langle p_v \rangle^{\pi_v}}{\pi_1! \pi_2! \dots \pi_v! (p_1!)^{\pi_1} (p_2!)^{\pi_2} \dots (p_v!)^{\pi_v}} \quad (3.5)$$

where, as before, $\rho = \sum p_i \pi_i$ and the summation is over all p_i and π_i such that $\sum p_i \pi_i = r$.

Hence,

$$\begin{aligned} E\{K(r)_R\} &= E\left\{r! \sum \frac{(-1)^{\rho-1} (\rho-1)! \langle p_1^{\pi_1} p_2^{\pi_2} \dots p_v^{\pi_v} \rangle}{\pi_1! \pi_2! \dots \pi_v! (p_1!)^{\pi_1} (p_2!)^{\pi_2} \dots (p_v!)^{\pi_v}}\right\} \\ &= r! \sum \frac{(-1)^{\rho-1} (\rho-1)! \langle p_1 \rangle^{\pi_1} \langle p_2 \rangle^{\pi_2} \dots \langle p_v \rangle^{\pi_v}}{\pi_1! \pi_2! \dots \pi_v! (p_1!)^{\pi_1} (p_2!)^{\pi_2} \dots (p_v!)^{\pi_v}} \\ &= \kappa_r \end{aligned} \quad (3.6)$$

Further, using (3.3),

$$\begin{aligned} E\{K(ab \dots e)_R\} &= E\{K(a)_R \odot K(b)_R \odot \dots \odot K(e)_R\} \\ &= \kappa_a \kappa_b \dots \kappa_e \quad . \end{aligned} \quad (3.7)$$

The multiplication formulae given in TABLE A3 of the Appendices obviously are valid as well for the polykeys relating to sampling with replacement. For example,

$$K(1)_R^2 = K(2)_R/n + K(11)_R$$

so that the variance, $\text{Var}(K(1)_R)$ of $K(1)_R$ may be developed:

$$\begin{aligned} \text{Var}(K(1)_R) &= E\{K(1)_R^2\} - E\{K(1)_R\}^2 \\ &= E\{K(2)_R/n + K(11)_R\} - \kappa_1^2 \\ &= \kappa_2/n + \kappa_1^2 - \kappa_1^2 \\ &= \kappa_2/n \quad . \end{aligned} \quad (3.8)$$

Since $E\{K(1)_R\} = \kappa_1 = \bar{Y}$ and $E(K_1) = \bar{Y}$ both $K(1)_R$ and K_1 are unbiased estimators of $\bar{Y}$. The more efficient estimator is that with the smaller variance. Now

$$\begin{aligned} \kappa_2' &= \langle 2 \rangle' - \langle 1 \rangle'^2 \\ &= \langle 2 \rangle' - (\langle 11 \rangle'(N-1)/N + \langle 2 \rangle'/N) \\ &= (\langle 2 \rangle' - \langle 11 \rangle')(N-1)/N \\ &= K_2' (N-1)/N \quad . \end{aligned}$$

Hence,

$$\begin{aligned}\text{Var}\{K(1)_R\} - \text{Var}\{K_1\} &= K_2'(N - 1)/nN - K_2'(N - n)/nN \\ &= K_2'(n - 1)/nN \\ &\geq 0 .\end{aligned}\tag{3.9}$$

Thus, for $n > 1$ sampling without replacement yields a more precise estimator of the population mean, an intuitively obvious result.

3.2. Replicated Sampling.

Let a simple random sample of size n be selected without replacement from a population of N primary sampling units. Let t be an estimator of a parameter T , such that $E(t) = T$. It is possible that the estimator t is based upon a complex subsampling scheme so that variance estimation is very difficult or impossible as in the case of systematic sampling. In either of these cases an unbiased estimator of the variance of t may be developed using the following scheme known as "replicated sampling", due to Mahalanobis [16] and [17].

Draw a simple random sample of n units without replacement and using the variate values associated with these units the statistic t is calculated and denote the random value obtained

by t_1 . The n units are replaced and a second sample of n units is selected without replacement from the same population. The statistic t is again calculated and the random value realized is denoted by t_2 . The procedure is repeated h times. It follows that $t_1, t_2, t_3, \dots, t_h$ are independent and identically distributed random variables.

The replicated sampling estimator of T is $\hat{t} = \sum_{i=1}^h t_i/h$. Evidently $E(\hat{t}) = T$, so that $\hat{t}$ is an unbiased estimator of T . Consider the problem of estimating the variance of $\hat{t}$. Each sample i of size n gives rise to a statistic t_i . There are $\binom{N}{n}$ possible samples. Hence the replicated sampling scheme just described is formally equivalent to the random selection with replacement of a sample of size h from a population of $\binom{N}{n}$ individuals. Thus from (3.8),

$$V(\hat{t}) = V\left(\sum_{i=1}^h t_i/h\right) = V(K(1)_R) = \mu_2/h \quad (3.10)$$

An unbiased estimator of this exact variance is given by

$$K(2)_R/h = (\langle 2 \rangle_R - \langle 1 \rangle_R^2)/h = \sum_{i=1}^h (t_i - \hat{t})^2/h^2 \quad (3.11)$$

Raj [19] gives an expression for the variance of $K(2)_R$ using a direct summation technique. However the variance can be derived using polykeys:

$$\begin{aligned}
 V(K(2)_R) &= E(K(2)_R^2) - E(K(2)_R)^2 \\
 &= E(K(4)_R/h + (h+1)K(22)_R/(h-1)) - \kappa_2^2 \\
 &= \kappa_4/h + (h+1)\kappa_2^2/(h-1) - \kappa_2^2 \\
 &= \kappa_4/h + 2\kappa_2^2/(h-1) , \tag{3.12}
 \end{aligned}$$

and an unbiased estimator of (3.12) is given by

$$\hat{V}(K(2)_R) = K(4)_R/h + 2K(22)_R/(h-1) .$$

This derivation of the variance of $K(2)_R$ is simpler than that given by Raj. Substituting $\kappa_4 = \mu_4 - 3\mu_2^2$ and $\kappa_2 = \mu_2$ in (3.12) gives

$$\begin{aligned}
 V(K(2)_R) &= (\mu_4 - 3\mu_2^2)/h + 2\mu_2^2/(h-1) \\
 &= \mu_4/h - (h-3)\mu_2^2/(h-1) ,
 \end{aligned}$$

which is identical to the result given by Raj. Hence the variance of (3.11) is

$$V(K(2)_R/h) = \mu_4/h^3 - (h-3)\mu_2^2/h^2(h-1) .$$

Thus using replicated sampling one can readily develop an unbiased estimator of the variance of any estimator t , and the variance of the estimator of variance. It is obvious that one

could apply this technique to higher order moments of t . Further the technique of replicated sampling can be applied to any sample from any type of population including the infinite case.

3.3. Comparison With Sampling Without Replacement.

It has been shown that without replacement sampling yields a more precise estimate of a population mean than with replacement sampling. The order of magnitude of the quantitative difference of the variance is $1/N$. Thus, for large populations the two sampling procedures are effectively equivalent.

Recall that

$$\langle a \rangle' \odot \langle b \rangle' = \langle a \rangle' \langle b \rangle' (N - 1)/N + \langle a + b \rangle'/N .$$

Hence as $N \rightarrow \infty$

$$\langle a \rangle' \odot \langle b \rangle' \rightarrow \langle a \rangle' \langle b \rangle' .$$

Similarly one can show that as $N \rightarrow \infty$

$$\langle a \rangle' \odot \langle b \rangle' \odot \dots \odot \langle e \rangle' \rightarrow \langle a \rangle' \langle b \rangle' \dots \langle e \rangle' .$$

This result coupled with (1.11) and (3.5) shows that

$$K(r)' \rightarrow \kappa_r \quad \text{as } N \rightarrow \infty .$$

The definition of $K(ab \dots e)'$, (1.15), shows that

$$K(ab \dots e) \rightarrow \kappa_a \kappa_b \dots \kappa_e \quad \text{as } N \rightarrow \infty \quad (3.13)$$

or equivalently

$$E\{K(ab \dots e)\} \rightarrow E\{K(ab \dots e)_R\} \quad \text{as } N \rightarrow \infty .$$

Hence the expected value of the polykeys for the two sampling schemes become identical as N goes to infinity.

CHAPTER 4

BIVARIATE POLYKEYS

4.1. Bivariate Symmetric Functions and Bivariate Polykeys.

Consider a population in which the i -th unit has two variate values, x_i and y_i associated with it. Bivariate analogues of all the preceding univariate statistics and parameters can be established.

Assume that a simple random sample of size n is taken without replacement from a population of N units. Define the "bivariate power sum"

$$((a_1 a_2)) = \sum_{i=1}^n x_i^{a_1} y_i^{a_2} . \quad (4.1)$$

Define the "bivariate augmented symmetric function"

$$[(a_1 a_2)(b_1 b_2) \dots (e_1 e_2)] = \sum_{i=1}^n x_i^{a_1} y_i^{a_2} x_j^{b_1} y_j^{b_2} \dots x_v^{e_1} y_v^{e_2} . \quad (4.2)$$

The bivariate power sums are obviously a special case of the bivariate augmented symmetric functions, i.e.,

$$[(a_1 a_2)] = ((a_1 a_2)) .$$

Further, any bivariate symmetric function may be expressed in terms of products of bivariate power sums. For example, consider

$$\begin{aligned}
 ((a_1 a_2))((b_1 b_2)) &= (\sum_i^n x_i^{a_1} y_i^{a_2}) (\sum_j^n x_j^{b_1} y_j^{b_2}) \\
 &= \sum_i^n x_i^{a_1+b_1} y_i^{a_2+b_2} + \sum_i^n x_i^{a_1} y_i^{a_2} x_j^{b_1} y_j^{b_2} \\
 &= ((a_1+b_1, a_2+b_2)) + [(a_1 a_2)(b_1 b_2)] .
 \end{aligned}$$

Therefore

$$[(a_1 a_2)(b_1 b_2)] = ((a_1 a_2))((b_1 b_2)) - ((a_1+b_1, a_2+b_2)) .$$

Tables of conversions between bivariate symmetric functions and bivariate power sums have been provided by David, Kendall, and Barton [4]. A portion of these tables has been reproduced in TABLE A4 of the Appendices.

Define the sample "bivariate brackets",

$$\langle (a_1 a_2)(b_1 b_2) \dots (e_1 e_2) \rangle = [(a_1 a_2)(b_1 b_2) \dots (e_1 e_2)] / n^{(\rho)} , \quad (4.3)$$

where ρ is the number of different subscripts in the numerator.

The equivalent population parameters may be defined as follows. Let

$$\begin{aligned}
 ((a_1 a_2))' &= \sum_i^N x_i^{a_1} y_i^{a_2} , \\
 [(a_1 a_2)(b_1 b_2) \dots (e_1 e_2)]' &= \sum_i^N x_i^{a_1} y_i^{a_2} x_j^{b_1} y_j^{b_2} \dots x_v^{e_1} y_v^{e_2} ,
 \end{aligned}$$

and

$$\langle (a_1 a_2)(b_1 b_2) \dots (e_1 e_2) \rangle' = [(a_1 a_2)(b_1 b_2) \dots (e_1 e_2)]' / N^{(\rho)} . \quad (4.4)$$

As in the univariate case a compact notation is now introduced. Assume a bracket contains v different terms, say $(p_1 q_1)$ with multiplicity π_1 , $(p_2 q_2)$ with multiplicity $\pi_2, \dots$, and $(p_v q_v)$ with multiplicity π_v . Write

$$\begin{aligned} & \langle (p_1 q_1)(p_1 q_1) \dots (p_1 q_1)(p_2 q_2)(p_2 q_2) \dots (p_2 q_2) \dots (p_v q_v)(p_v q_v) \dots (p_v q_v) \rangle \\ &= \langle (p_1 q_1)^{\pi_1} (p_2 q_2)^{\pi_2} \dots (p_v q_v)^{\pi_v} \rangle . \end{aligned}$$

The "first variate weight" is defined as $r_1 = \sum \pi_i p_i$, the "second variate weight" as $r_2 = \sum \pi_i q_i$ and the "weight" as $r = r_1 + r_2$.

Lemma 4.1 $\langle (a_1 a_2)(b_1 b_2) \dots (e_1 e_2) \rangle$ is inherited on the average, i.e.,

$$E\{\langle (a_1 a_2)(b_1 b_2) \dots (e_1 e_2) \rangle\} = \langle (a_1 a_2)(b_1 b_2) \dots (e_1 e_2) \rangle' . \quad (4.5)$$

The proof of Lemma 4.1 is almost identical to that of Lemma 1.1. Merely replace y_i^a in the proof of Lemma 1.1 by $\begin{smallmatrix} a_1 & a_2 \\ x_i & y_i \end{smallmatrix}$ to prove Lemma 4.1. As an immediate consequence of (4.5) any linear combination of bivariate brackets is also inherited on

the average. Define

$$M(t_1 t_2) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \langle (ij) \rangle t_1^i t_2^j / i! j! \quad . \quad (4.6)$$

Bivariate $\odot$ -multiplication is defined to be an operator with the properties:

1. $c_1 \odot c_2 = c_1 c_2$ if c_1 and c_2 are real numbers.
2. $c \odot \langle (a_1 a_2)(b_1 b_2) \dots (e_1 e_2) \rangle = c \langle (a_1 a_2)(b_1 b_2) \dots (e_1 e_2) \rangle$
where c is any real number.
3. $\langle (a_1 a_2)(b_1 b_2) \dots (e_1 e_2) \rangle \odot \langle (f_1 f_2)(g_1 g_2) \dots (m_1 m_2) \rangle$
 $= \langle (a_1 a_2)(b_1 b_2) \dots (e_1 e_2)(f_1 f_2)(g_1 g_2) \dots (m_1 m_2) \rangle .$
4. $\langle (a_1 a_2) \rangle \odot \{ \langle (b_1 b_2) \rangle + \langle (c_1 c_2) \rangle \} = \langle (a_1 a_2)(b_1 b_2) \rangle + \langle (a_1 a_2)(c_1 c_2) \rangle .$

The "bivariate sample polykeys" are designated by $K((a_1 a_2)(b_1 b_2) \dots (e_1 e_2))$. The first order bivariate sample polykeys are defined through the solution of the identity

$$\emptyset(t_1 t_2) = \odot - \log \{ M(t_1 t_2) \} \quad , \quad (4.7)$$

where $M(t_1 t_2)$ has been defined in (4.6) ,

$$\emptyset(t_1 t_2) = \left\{ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} K((ij)) t_1^i t_2^j / i! j! \right\} - 1$$

and the $\odot$ -log operator is identical to that defined earlier in Section 1.2.

Now from (4.7)

$$\begin{aligned}
 \emptyset(t_1 t_2) &= \odot\text{-log}\{1 + [M(t_1 t_2) - 1]\} \\
 &= [M(t_1 t_2) - 1] + [M(t_1 t_2) - 1] \odot [M(t_1 t_2) - 1]/2 + \dots \\
 &= [<(10)>t_1 + <(01)>t_2 + <(20)>t_1^2/2 + <(11)>t_1 t_2 \\
 &\quad + <(02)>t_2^2/2 + \dots] \\
 &\quad - [<(10)>t_1 + <(01)>t_2 + \dots] \odot [<(10)>t_1 + <(01)>t_2 + \dots]/2 + \dots \\
 &= <(10)>t_1 + <(01)>t_2 + [<(20)> - <(10)^2>]t_1^2/2 \\
 &\quad + [<(11)> - <(10)(01)>]t_1 t_2 + [<(02)> - <(01)^2>]t_2^2/2 + \dots .
 \end{aligned}$$

Equating coefficients of $t_1^i t_2^j$ gives:

$$K((10)) = <(10)> , \quad K((01)) = <(01)> ,$$

$$K((20)) = <(20)> - <(10)^2> , \quad K((02)) = <(02)> - <(01)^2> ,$$

$$K((11)) = <(11)> - <(10)(01)> .$$

The general expression for one part bivariate polykays in terms of bivariate brackets is given in Theorem 4.1.

Theorem 4.1.

$$K((r_1 r_2)) = r_1! r_2! \sum \frac{(-1)^{\rho-1} (\rho-1)! \langle (p_1 q_1)^{\pi_1} (p_2 q_2)^{\pi_2} \dots (p_v q_v)^{\pi_v} \rangle}{\pi_1! \pi_2! \dots \pi_v! (p_1! q_1!)^{\pi_1} (p_2! q_2!)^{\pi_2} \dots (p_v! q_v!)^{\pi_v}}, \quad (4.8)$$

where $\rho = \sum \pi_i$, and the summation extends over all $(p_i q_i)$ and π_i such that $\sum p_i \pi_i = r_1$ and $\sum q_i \pi_i = r_2$.

The proof of Theorem 4.1 is similar to that of Theorem 1.1 and hence is not reproduced here.

The higher order polykays are defined as

$$K((a_1 a_2)(b_1 b_2) \dots (e_1 e_2)) = K((a_1 a_2)) \odot K((b_1 b_2)) \odot \dots \odot K((e_1 e_2)).$$

Thus,

$$\begin{aligned} K((10)(11)) &= K((10)) \odot K((11)) \\ &= \langle (10) \rangle \odot [\langle (11) \rangle - \langle (10)(01) \rangle] \\ &= \langle (10)(11) \rangle - \langle (10) \rangle^2 \langle (01) \rangle. \end{aligned}$$

Bivariate brackets may be written in terms of bivariate polykays by taking $\odot$ -exponentials on both sides of (4.7):

$$\begin{aligned} M(t_1 t_2) &= \odot\text{-exp}\{\emptyset(t_1 t_2)\} \\ &= 1 + [\emptyset(t_1 t_2)] + [\emptyset(t_1 t_2)] \odot [\emptyset(t_1 t_2)]/2 + \dots \end{aligned}$$

$$\begin{aligned}
&= 1 + [t_1 K((10)) + t_2 K((01)) + t_1^2 K((20))/2 \\
&\quad + t_1 t_2 K((11)) + t_2^2 K((02))/2 + \dots] \\
&\quad + [t_1 K((10)) + t_2 K((01)) + \dots] \odot [t_1 K((10)) + t_2 K((01)) + \dots]/2 + \dots \\
&= 1 + t_1 K((10)) + t_2 K((01)) + t_1^2 [K((20)) + K((10)^2)]/2 \\
&\quad + t_2^2 [K((02)) + K((01)^2)]/2 + t_1 t_2 [K((11)) + K((10)(01))] + \dots .
\end{aligned}$$

Equating coefficients of $t_1^i t_2^j$ gives

$$\begin{aligned}
\langle (10) \rangle &= K((10)) , & \langle (01) \rangle &= K((01)) , \\
\langle (20) \rangle &= K((20)) + K((10)^2) , & \langle (02) \rangle &= K((02)) + K((01)^2) , \\
\langle (11) \rangle &= K((11)) + K((10)(01)) .
\end{aligned}$$

In general

$$\langle (r_1 r_2) \rangle = r_1! r_2! \sum \frac{K((p_1 q_1)^{\pi_1} (p_2 q_2)^{\pi_2} \dots (p_v q_v)^{\pi_v})}{\pi_1! \pi_2! \dots \pi_v! (p_1! q_1!)^{\pi_1} (p_2! q_2!)^{\pi_2} \dots (p_v! q_v!)^{\pi_v}} , \quad (4.9)$$

where the summation extends over all p_i , q_i and π_i such that

$$\sum p_i \pi_i = r_1 \quad \text{and} \quad \sum q_i \pi_i = r_2 .$$

The higher order bivariate brackets can be written in terms of polykeys by using symbolic multiplication. Thus

$$\begin{aligned}
 \langle (10)(11) \rangle &= \langle (10) \rangle \odot \langle (11) \rangle \\
 &= K((10)) \odot [K((11)) + K((10)(01))] \\
 &= K((10)(11)) + K((10)^2(01)) \quad .
 \end{aligned}$$

TABLE A5 in the Appendices lists additional conversions between bivariate brackets and bivariate polykeys.

The population bivariate polykeys are designated by $K'((a_1 a_2)(b_1 b_2) \dots (e_1 e_2))$. The one part population polykeys are defined by

$$\emptyset(t_1, t_2) = \odot - \log \{M(t_1 t_2)'\} \quad (4.10)$$

where

$$\emptyset(t_1, t_2)' = \left(\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} K'((ij)) t_1^i t_2^j / i! j! \right) - 1$$

and

$$M(t_1 t_2)' = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \langle (ij) \rangle' t_1^i t_2^j / i! j! \quad .$$

Similar to the derivation of formulae relating sample polykeys to sample brackets it can be shown that the population polykey $K'((a_1 a_2)(b_1 b_2) \dots (e_1 e_2))$ is the same function of the population brackets that the equivalent sample polykey is of the sample brackets. By virtue of Lemma 4.1 it follows that bivariate polykeys are inherited on the average, i.e.,

$$E\{K((a_1 a_2)(b_1 b_2) \dots (e_1 e_2))\} = K'((a_1 a_2)(b_1 b_2) \dots (e_1 e_2)) \quad . \quad (4.11)$$

Analogous to Theorem 1.3 the following invariance property holds for bivariate polykeys.

Theorem 4.2. All bivariate polykeys except those which involve (10) or (01) are invariant under translations of origin.

Proof. (4.6) may be written as

$$\begin{aligned} M(t_1, t_2) &= \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \langle pq \rangle t_1^p t_2^q / p! q! \\ &= \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \left\{ \sum_{i=1}^n x_i^p y_i^q \right\} t_1^p t_2^q / p! q! n \\ &= \sum_{i=1}^n \left\{ \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} x_i^p y_i^q t_1^p t_2^q / p! q! \right\} / n \\ &= \sum_{i=1}^n \left\{ \sum_{p=0}^{\infty} x_i^p t_1^p \left[\sum_{q=0}^{\infty} y_i^q t_2^q / q! \right] / p! \right\} / n \\ &= \sum_{i=1}^n \left\{ \sum_{p=0}^{\infty} x_i^p t_1^p e^{t_2 y_i} / p! \right\} / n \\ &= \sum_{i=1}^n e^{t_1 x_i + t_2 y_i} / n \quad . \end{aligned} \quad (4.12)$$

The proof of the invariance property is now similar to

that given in Section 1.2 relating to the invariance of univariate polykeys.

The results in this Section are direct extensions of the univariate results given in Chapter 1; the parallel will be extended in Section 4.2 where the theory of products of bivariate polykeys is developed.

4.2. Multiplication of Bivariate Polykeys.

It was shown in Chapter 2 that for the univariate case a product of polykeys can be written as a linear combination of polykeys. Similarly a product of bivariate polykeys can be linearized.

Consider the product of two bivariate augmented symmetric functions:

$$\begin{aligned} [(p_1 q_1)][(p_2 q_2)] &= \left(\sum_i^n x_i^{p_1} y_i^{q_1} \right) \left(\sum_i^n x_i^{p_2} y_i^{q_2} \right) \\ &= \sum_i^n x_i^{p_1+p_2} y_i^{q_1+q_2} + \sum_{i \neq j}^n x_i^{p_1} y_i^{q_1} x_j^{p_2} y_j^{q_2} \\ &= [(p_1+p_2 \ q_1+q_2)] + [(p_1 q_1)(p_2 q_2)] . \end{aligned}$$

Hence,

$$\langle (p_1 q_1) \times (p_2 q_2) \rangle = (p_1+p_2 \ q_1+q_2) / n + \langle (p_1 q_1)(p_2 q_2) \rangle (n-1) / n ,$$

which corresponds to the univariate formula

$$\langle p \rangle \langle q \rangle = \langle p + q \rangle / n + \langle p q \rangle (n - 1) / n .$$

Similarly it can be shown that

$$\begin{aligned} \langle (p_1 q_1)(p_2 q_2) \rangle \langle (p_3 q_3) \rangle &= \langle (p_1 + p_3 \quad q_1 + q_3)(p_2 q_2) \rangle / n \\ &+ \langle (p_1 q_1)(p_2 + p_3 \quad q_2 + q_3) \rangle / n \\ &+ \langle (p_1 q_1)(p_2 q_2)(p_3 q_3) \rangle (n - 2) / n , \end{aligned}$$

which corresponds to the univariate product

$$\langle p q \rangle \langle r \rangle = \langle p + r \quad q \rangle / n + \langle p \quad q + r \rangle / n + \langle p q r \rangle (n - 2) / n .$$

These two sets of formulae indicate how one may write a product of bivariate augmented symmetric functions by merely looking at the comparable univariate formula. One merely considers the paired terms inside a univariate bracket to be a vector with two components. Then to replace the univariate formulae with a bivariate one the additions within the bracket is conducted componentwise.

Analogous to Tukey's technique [24] for multiplying univariate polykeys outlined in Section 2.1, bivariate polykeys may be multiplied by first writing them in terms of bivariate brackets, then multiplying the brackets and finally rewriting the brackets in terms

of bivariate polykeys. For example,

$$K((10))K((01)) = \langle (10) \times (01) \rangle = \langle (10)(01) \rangle (n-1)/n + \langle (11) \rangle /n .$$

From TABLE A4 of the Appendices $\langle (10)(01) \rangle = K((10)(01))$ and $K\langle (11) \rangle = K((10)(01)) + K((11))$. Therefore,

$$\begin{aligned} K((10))K((01)) &= K((10)(01))(n-1)/n + [K((10)(01)) + K((11))]/n \\ &= K((11))/n + K((10)(01)) . \end{aligned} \quad (4.13)$$

The univariate analogue is

$$K_1^2 = K_2/n + K_{11} .$$

Products of bivariate polykeys are given in TABLE A6 of the Appendices.

The second technique for multiplying polykeys given in Section 2.1 may be applied to bivariate polykeys. Thus using TABLE A4 of the Appendices

$$\begin{aligned} K((10))K((01)) &= [(10)][(01)]/n^2 \\ &= ((10))((01))/n^2 \\ &= ([(11)] + [(10)(01)])/n^2 \\ &= K((11))/n + K((10)(01)) . \end{aligned}$$

Define the "first variate unit order" of a bivariate polykay to be the number of (10) terms found in the polykay and the "second variate unit order" to be the number of (01) terms found in the polykay. For example, $K((10)^2(02))$ has first variate unit order two and second variate unit order zero.

Paralleling the result given in Section 2.1 one can show that, for any identity in bivariate polykays in which one side is linear and from which all trivial cancellations have been made, (the first variate unit order of the linear side) $\leq$ (the first variate unit order of the other side) and (the second variate unit order of the linear side) $\leq$ (the second variate unit order of the other side).

Define an arbitrary set of bivariate polykays or bivariate brackets to be $O((10)^g)$ if each term of the set has first variate unit order at least g ; define the set to be $O((01)^h)$ if each term has second variate unit order at least h . Consider the product

$$\begin{aligned}
 K((20))K((02)) &= [\langle (20) \rangle - \langle (10)^2 \rangle] [\langle (02) \rangle - \langle (01)^2 \rangle] \\
 &= \langle (20) \times (02) \rangle - \langle (10)^2 \times (02) \rangle - \langle (01)^2 \times (20) \rangle \\
 &\quad + \langle (10)^2 \times (01)^2 \rangle \\
 &= \langle (20)(02) \rangle (n-1)/n + \langle (22) \rangle /n - \langle (10)^2(02) \rangle (n-2)/n \\
 &\quad - 2\langle (12)(10) \rangle /n - \langle (01)^2(20) \rangle (n-2)/n - 2\langle (21)(01) \rangle /n
 \end{aligned}$$

$$\begin{aligned}
 & + \langle (10)^2 (01)^2 \rangle_{(n-2)(n-3)/n(n-1)} + 4 \langle (11)(10)(01) \rangle_{(n-2)/n(n-1)} \\
 & + 2 \langle (11)^2 \rangle_{/n(n-1)} \\
 & = \langle (20)(02) \rangle_{(n-1)/n} + \langle (22) \rangle_{/n} + 2 \langle (11)^2 \rangle_{/n(n-1)} \\
 & + 0 \langle (10)^1 \rangle + 0 \langle (01)^1 \rangle \\
 & = K((20)(02)) + K((22))/n + 2K((11))/(n-1) \\
 & + 0 \langle (10)^1 \rangle + 0 \langle (01)^1 \rangle . \quad (4.14)
 \end{aligned}$$

Since $K((20))$ and $K((02))$ have first and second variate unit orders zero the coefficients of the terms $0 \langle (10)^1 \rangle$ and $0 \langle (01)^1 \rangle$ in (4.14) are zero. Hence (4.14) becomes

$$K((20))K((02)) = K((20)(02)) + K((22))/n + 2K((11)^2)/(n-1) .$$

Similar formulae for products of population polykeys may be developed. The equations are identical except that "K" is replaced by "K'" and n by N . Thus, for example

$$K((20))'K((02))' = K'((20)(02)) + K'((22))/N + 2K'((11)^2)/(N-1) .$$

The formulae cited for products of bivariate polykeys will be used in Section 4.3 where variances and covariances of sample polykeys are considered.

4.3. Variances and Covariances of Bivariate Polykeys.

Paralleling the results of Section 2.2, several results relating to the variances and covariances of sample bivariate polykeys can be developed.

Consider the problem of evaluating the variance of the statistic $K((11))$.

$$\begin{aligned}
 V[K((11))] &= E[K((11))^2] - E[K((11))]^2 \\
 &= E[K((22))/n + nK((11))^2/(n-1) + K((20)(02))/(n-1)] - K'((11))^2 \\
 &= K'((22))/n + nK'((11))^2/(n-1) + K'((20)(02))/(n-1) \\
 &\quad - [K'((22))/N + NK'((11))^2/(N-1) + K'((20)(02))/(N-1)] \\
 &= [\frac{1}{n} - \frac{1}{N}]K'((22)) + [\frac{1}{n-1} - \frac{1}{N-1}][K'((11))^2 + K'((20)(02))] .
 \end{aligned}$$

An unbiased estimator of the variance of $K((11))$ is evidently

$$\hat{V}[K((11))] = [\frac{1}{n} - \frac{1}{N}]K((22)) + [\frac{1}{n-1} - \frac{1}{N-1}][K((11))^2 + K((20)(02))] .$$

It can be shown that

$$K((11)) = \langle (11) \rangle - \langle (10)(01) \rangle$$

$$= \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y}) / (n-1) ,$$

where $\bar{x} = \sum_{i=1}^n x_i / n$ and $\bar{y} = \sum_{i=1}^n y_i / n$. Hence $K((11))$ is the sample covariance of $\bar{x}$ and $\bar{y}$.

Consider the covariance of $\bar{x} = K((10))$ and $\bar{y} = K((01))$,

$$\begin{aligned} \text{Cov}[K((10)), K((01))] &= E[K((10))K((01))] - E[K((10))]E[K((01))] \\ &= E[K((11))/n + K((10)(01))] - K'((10))K'((01)) \\ &= [\frac{1}{n} - \frac{1}{N}]K'((11)) . \end{aligned}$$

An unbiased estimator of $\text{Cov}(\bar{x}, \bar{y})$ is evidently

$$\hat{\text{Cov}}[K((10)), K((01))] = [\frac{1}{n} - \frac{1}{N}]K((11)) .$$

The covariance of $\bar{x} = K((10))$ and $K((11))$ the sample covariance of x and y is

$$\begin{aligned} \text{Cov}[K((10)), K((11))] &= E[K((10))K((11))] - K'((10))K'((11)) \\ &= [\frac{1}{n} - \frac{1}{N}]K'((21)) . \end{aligned}$$

An unbiased estimator is

$$\hat{\text{Cov}}[K((10)), K((11))] = [\frac{1}{n} - \frac{1}{N}]K((21)) .$$

Many fundamental results can be established using the techniques given in this Section. In addition the bivariate polykays can be used to derive still other results in the theory of sampling from a finite universe.

4.4. An Unbiased Product Estimator.

It has been shown that the sample mean $\bar{y}$ is an unbiased estimator of the population mean $\bar{Y}$ in simple random sampling without replacement. If additional concomitant information about the population is available then it may be possible to use this information to obtain a more precise estimator of $\bar{Y}$.

Assume that the population mean $\bar{X}$ for an auxiliary variate is known and further assume that the paired variate values $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ are observed for a simple random sample of n units. Then this information can be used to derive alternate estimators of $\bar{Y}$.

The estimator

$$y_p = \bar{x} \bar{y} / \bar{X} = K((10))K((01))/K'((01)) , \quad (4.15)$$

where $\bar{x} = \sum_{i=1}^n x_i / n$ and $\bar{X} = \sum_{i=1}^N x_i / N$ is called a product estimator of $\bar{Y}$. Assume that x and y are negatively correlated variables. Then if the sample $\bar{x}$ is smaller than $\bar{X}$ then it is reasonable to assume that the $\bar{y}$ values is likely to be larger than $\bar{Y}$. Hence

multiplying $\bar{y}$ by $\bar{x}/\bar{X}$ would tend to make y_p closer to $\bar{Y}$ than $\bar{y}$. An identical type of improvement in precision would occur if $\bar{x}$ were larger than $\bar{X}$. Hence if x and y are negatively correlated then $\bar{y}_p$ is intuitively appealing as an estimator of $\bar{Y}$. In general y_p is biased for $\bar{Y}$ since

$$\begin{aligned} E\{\bar{x} \bar{y}/\bar{X}\} - \bar{Y} &= \{E K((10))K((01)) - K'((10))K'((01))\}/K'((10)) \\ &= \{E[K(11)]/n - K((10)(01))\} - K'((11))/N + K'((10)(01))/K'((10)) \\ &= (\frac{1}{n} - \frac{1}{N})K'((11))/K'((10)) \end{aligned} \quad (4.16)$$

which vanishes only if $n = N$ or if $K'((11)) = 0$, i.e., if the covariance between x and y is zero.

Hence an unbiased estimator of $\bar{Y}$ is

$$y_p^* = \bar{x} \bar{y}/\bar{X} - (\frac{1}{n} - \frac{1}{N})K((11))/\bar{X}$$

which is derived by adding to (4.15) an estimator of the bias given by (4.16). Further,

$$\begin{aligned} y_p^* &= \{K((10))K((01)) - (\frac{1}{n} - \frac{1}{N})K((11))\}/\bar{X} \\ &= \{K((11))/n + K((10)(01)) - (\frac{1}{n} - \frac{1}{N})K((11))\}/\bar{X} \\ &= \{K((10)(01)) + K((11))/N\}/\bar{X} . \end{aligned} \quad (4.17)$$

The variance of (4.17) is

$$\begin{aligned}
 V\{y_p^*\} &= V\{K((10)(01)) + K((11))/N\}/\bar{X}^2 \\
 &= E\{(K((10)(01)) + K((11))/N)^2\}/\bar{X}^2 - E\{(K((10)(01)) + K((11))/N)/\bar{X}\}^2 \\
 &= E\{K((10)(01))^2 + 2K((10)(01))K((11))/N + K((11))^2/N^2\}/\bar{X}^2 - \bar{Y}^2 \\
 &= E\{K((11))^2/n^{(2)} + K((20)(02))/n^{(2)} + K((20)(01))^2/n \\
 &\quad + K((10)^2(02))/n + 2K((11)(10)(01))/n + K((10)^2(01))^2 \\
 &\quad + 2[K((21)(01))/n + K((12)(10))/n - K((11))^2/n^{(2)} \\
 &\quad - K((20)(02))/n^{(2)} + K((10)(01)(11))/N + [K((22))/n \\
 &\quad + K((11))^2/n/(n-1) + K((20)(02))/(n-1)]/N^2\}/\bar{X}^2 \\
 &\quad - [K'((10))^2 K'((01))^2]/\bar{X}^2 . \tag{4.18}
 \end{aligned}$$

Simplification of (4.18) gives

$$\begin{aligned}
 V\{y_p^*\} &= (N-n)\{K'((22)) + N[(N-1) + (n-1)]/(n-1)[K'((20)(02)) + K'((11))^2]\} \\
 &\quad + 2N[K'((21)(01)) + K'((12)(10))] + N^2[K'((20)(01))^2 + K'((10)^2(02))] \\
 &\quad + 2N^2 K'((11)(10)(01))/N^3 \bar{X}^2 , \tag{4.19}
 \end{aligned}$$

which is identical to a result given by Robson [21] who used a somewhat different argument.

The variance of the sample mean of the y variate, $\bar{y}$, is $\text{Var } \bar{y} = (\frac{1}{n} - \frac{1}{N})S^2$. To compare the variances of the competing

estimators $\bar{y}$ and y_p^* write the variance of $\bar{y}$ in terms of bivariate polykeys as

$$\begin{aligned} V(\bar{y}) &= E[K((10))^2] - E[K((01))]^2 \\ &= (N-n)K'((02))/nN \\ &= (N-n)\{K'((22))/N^2 + 2K'((12)(10))/n + K'((20)(02))/N \\ &\quad + K'((10)^2(02))/n\bar{N}^2\}. \end{aligned}$$

Subtracting the variance of the "y-only estimator" , $\bar{y}$ from the variance of the unbiased product estimator y_p^* gives

$$\begin{aligned} V(y_p^*) - V(\bar{y}) &= (N-n)\{N(N-1)K'((20)(02)) + N[(N-1) + (n-1)]K'((11)^2)/(n-1) \\ &\quad + 2N K'((21)(01)) + N^2 K'((20)(01))^2 \\ &\quad + 2N^2 K'((11)(10)(01))\}/N^3\bar{N}^2. \end{aligned}$$

Hence y_p^* is a more precise estimator of $\bar{Y}$ than $\bar{y}$ if and only if

$$\begin{aligned} &N(N-1)K'((20)(02))/(n-1) + N[(N-1) + (n-1)]K'((11)^2)/(n-1) \\ &+ 2N K'((21)(01)) + N^2 K'((20)(01))^2 + 2N^2 K'((11)(10)(01)) < 0. \end{aligned}$$

The estimator y_p given by (4.15) is biased. Consequently

its mean square error is employed in investigating its merits as an estimator of $\bar{Y}$. It may readily be shown that

$$\begin{aligned}
 \text{MSE}(y_p) &= V(y_p) + E^2(y_p - \bar{Y}) \\
 &= \left\{ \left[\frac{1}{n} + \frac{1}{N} - \frac{2}{nN} \right] K'((22)) \right. \\
 &\quad + \left[\frac{1}{n} - \frac{1}{N} \right] [K'((20)(01))^2 + K'((02)(10))^2 + 2K'((11)(10)(01))] \\
 &\quad + \frac{2}{n} \left[\frac{1}{n} - \frac{1}{N} \right] [K'((21)(01)) + K'((12)(10)) + K'((11)^2)] \\
 &\quad \left. + \left[\frac{1}{n^2} - \frac{1}{N^2} \right] K'((20)(02)) \right\} / \bar{X}^2. \tag{4.20}
 \end{aligned}$$

A numerical example involving the competing estimators $\bar{y}$, y_p^* , y_p and two further estimators is provided in Section 4.5.

4.5. An Unbiased Ratio Estimator.

In Section 4.4 it was shown that concomitant information about the x_i values could be used to provide a more precise estimator of $\bar{Y}$. The estimators in Section 4.4 are obviously not the only ones exploiting such information. Mickey [18] has defined a "ratio-type estimator" to be an estimator which is invariant if a scale change is performed on the x-variate and which would undergo the same scale change as the y-variable. Thus both y_p and y_p^* of Section 4.4 are ratio-type estimators.

Robson [21] has used trivariate polykeys to develop an unbiased ratio-type estimator. It is, however, possible to develop the estimator using the bivariate polykeys of the paired values $(x_i, r_i = y_i/x_i)$.

A ratio-type estimator is the mean-of-ratios estimator

$$y_R = \bar{X} \bar{r} = K'((10))K((01)) , \quad (4.21)$$

where $\bar{r} = \sum_{i=1}^n r_i/n$ and, as before, it is assumed that $\bar{X}$ is known.

Now (4.21) is biased for $\bar{Y}$ since

$$\begin{aligned} E(y_R) - \bar{Y} &= \langle(10)\rangle' \langle(01)\rangle' - \bar{Y} \\ &= (N-1)\langle(10)(01)\rangle'/N + \langle(11)\rangle'/N - \bar{Y} . \end{aligned}$$

But

$$\langle(11)\rangle' = \sum_{i=1}^N x_i r_i / N = \sum_{i=1}^N y_i / N = \bar{Y} .$$

Thus the bias in $\bar{Y}$ is

$$E(y_R) - \bar{Y} = (N-1)\{\langle(10)(01)\rangle' - \bar{Y}\}/N \quad (4.22)$$

which is not in general equal to zero. In fact the estimator y_R is not consistent, i.e., does not approach $\bar{Y}$ as n approaches N .

An unbiased estimator of the bias term (4.22) is

$$\begin{aligned}
 & (N-1) \{ \langle (10)(01) \rangle - \bar{y} \} / N \\
 &= (N-1) \{ [n \langle (10) \rangle \langle (01) \rangle - \langle (11) \rangle] / (n-1) - \bar{y} \} / N \\
 &= (N-1) \{ [\bar{x} \bar{r} - \bar{y}] / (n-1) - \bar{y} \} / N \\
 &= (N-1)n \{ \bar{x} \bar{r} - \bar{y} \} / N(n-1) .
 \end{aligned}$$

Therefore an unbiased ratio-type estimator of $\bar{y}$ is given by

$$y_R^* = \bar{X} \bar{r} - (N-1)n \{ \bar{x} \bar{r} - \bar{y} \} / N(n-1) . \quad (4.23)$$

Now (4.23) can be recast using the vector of variate values (x_i, r_i) as

$$\begin{aligned}
 y_R^* &= K'((10))K((01)) - (N-1) \{ \langle (10)(01) \rangle - \bar{y} \} / N \\
 &= K'((10))K((01)) - (N-1)K((11)) / N .
 \end{aligned}$$

The variance of the estimator (4.23) may be derived in exactly the same manner that the variance of y_p^* was developed. Thus

$$\begin{aligned}
 V(y_R^*) &= E\{[K'((10))K((01)) + (N-1)K((11))/N]^2\} - \bar{Y}^2 \\
 &= E\{K'((10))^2 K((01))^2 + 2(N-1)K'((10))K((01))K((11))/N \\
 &\quad + (N-1)^2 K((11))^2/N^2\} - \langle(11)\rangle'^2 \\
 &= K'((10))^2 E[K((02))/n + K((01))^2] \\
 &\quad + 2(N-1)K'((10))E[K((12))/n + K((11)(01))]/N \\
 &\quad + (N-1)^2 E[K((22))/n + K((11))^2 n/(n-1) + K((20)(02))/(n-1)]/N^2 \\
 &\quad - [K'((10)(01)) + K'((11))]^2 .
 \end{aligned}$$

This may be simplified to yield

$$\begin{aligned}
 V(y_R^*) &= (N-n)K'((22))/nN + (N-1)(N-n)K'((11))/(n-1)N^2 \\
 &\quad + [(N-1)^2 n - N(n-1)^2]K'((20)(02))/(n-1)N^2 n \\
 &\quad + (N-n)K'((10))^2(02))/nN .
 \end{aligned} \tag{4.24}$$

In order to determine whether the unbiased ratio-type estimator y_R^* is preferable to the biased ratio-type estimator y_R the variance of y_R^* must be compared with the mean square error of y_R :

$$\begin{aligned}
 MSE(y_R) &= \bar{X}^2 V[\bar{r}] + [(N-1)K'((11))/N]^2 \\
 &= K'((10))^2 (N-n)K'((02))/nN + (N-1)^2 K'((11))^2/N^2
 \end{aligned}$$

$$\begin{aligned}
 &= (N-n)[K'((22))/N^2 + 2K'((12)(10))/N + K'((20)(02))/N \\
 &\quad + K'((20)(01)^2)/nN + (N-1)^2[K'((22))/N \\
 &\quad + K'((11)^2)N/(N-1) + K'((20)(02))/(N-1)]/N^2 \\
 &= K'((22))[(N-n)/n + (N-1)^2] + 2(N-n)K'((12)(10))/nN^2 \\
 &\quad + K'((20)(02))[(N-n)/n + (N-1)]/N^2 + (N-n)K'((20)(01)^2)/nN \\
 &\quad + K'((11)^2)(N-1)/N .
 \end{aligned} \tag{4.25}$$

The values of $V(\bar{y})$, $MSE(y_p)$, $V(y_p^*)$, $MSE(y_R)$ and $V(y_R^*)$ are compared in a numerical example. TABLE 1 displays a population of $N = 100$ paired values. Corresponding to this population, the values for formula (2.9), (4.18), (4.20), (4.24) and (4.25), i.e. $V(\bar{y})$, $MSE(y_p)$, $V(y_p^*)$, $MSE(y_R)$ and $V(y_R^*)$, were calculated for several sample sizes. The computations were carried out using a Xerox Sigma 9 computer; the results are given in TABLE 2. In this illustration both y_p and y_p^* are more efficient than $\bar{y}$ as estimators of $\bar{Y}$. Further for all sample sizes the mean square error of y_p is less than the variance of y_p^* . Thus in the population studied the creation of an unbiased estimator of $\bar{Y}$ resulted in a greater loss in precision than was gained in accuracy. Both the estimators y_R and y_R^* are less precise than $\bar{y}$ in this example. Further, in comparing y_R to y_R^* it is seen that y_R is the better estimator at $n = 10$ but as n increases y_R^* becomes

TABLE 1: Population of 100 paired (x_i, y_i) values.

x_i	y_i	x_i	y_i	x_i	y_i	x_i	y_i	x_i	y_i
5	5	5	6	5	7	5	8	5	9
5	12	5	15	5	20	5	15	5	18
5	18	5	17	4	15	4	16	4	20
4	9	4	10	4	13	4	13	4	15
6	5	6	6	6	6	6	7	6	7
6	10	6	11	6	12	6	12	6	14
6	15	6	15	6	16	6	19	3	11
3	12	3	11	3	15	7	3	7	6
7	16	7	12	7	13	7	10	7	11
8	16	8	15	8	8	8	10	8	11
8	12	8	9	8	13	9	10	9	8
9	6	9	9	9	5	9	7	9	3
10	4	10	7	10	19	10	10	10	11
11	10	11	9	11	8	11	7	11	6
12	9	12	8	12	8	12	7	13	6
13	8	13	7	13	6	14	6	14	5
14	8	14	6	15	8	15	7	15	6
16	6	16	6	16	7	17	9	17	6
17	7	18	9	18	7	19	8	20	7
20	8	22	9	5	5	6	22	8	8

TABLE 2: Mean square errors of alternate estimators of $\bar{Y}$ for specified sample sizes.

Sample Size	$V(\bar{y})$	$MSE(y_p)$	$V(y_p^*)$	$MSE(y_R)$	$V(y_R^*)$
10	2.202	0.451×10^{-3}	0.483×10^{-3}	0.105×10^2	0.130×10^2
20	0.987	0.206×10^{-3}	0.212×10^{-3}	0.801×10^1	0.454×10^1
30	0.570	0.121×10^{-3}	0.123×10^{-3}	0.719×10^1	0.241×10^1
40	0.367	0.786×10^{-4}	0.795×10^{-4}	0.679×10^1	0.147×10^1
50	0.244	0.525×10^{-4}	0.529×10^{-4}	0.654×10^1	0.950
60	0.163	0.351×10^{-4}	0.352×10^{-4}	0.638×10^1	0.620
70	0.144	0.226×10^{-4}	0.227×10^{-4}	0.626×10^1	0.392
80	0.611×10^{-1}	0.131×10^{-4}	0.132×10^{-4}	0.618×10^1	0.226
90	0.271×10^{-1}	0.587×10^{-5}	0.587×10^{-5}	0.611×10^1	0.995×10^{-1}
100	0.00	0.00	0.00	0.606×10^1	0.000

the more accurate estimator. Thus the choice of the estimator of $\bar{Y}$ depends upon the sample size used.

In general it must be recognized that such a comparison is not strictly valid in that the greater cost realized by collecting information on two variates rather than one must be considered.

4.6. Multivariate Polykeys.

Multivariate brackets and polykeys may be defined for a finite population of units where a sampling unit u_i , $i = 1, 2, \dots, N$, has m associated variate values $x_{i1}, x_{i2}, \dots, x_{im}$. Define the multivariate bracket

$$\begin{aligned} & \langle (a_1 a_2 \dots a_m) (b_1 b_2 \dots b_m) \dots (e_1 e_2 \dots e_m) \rangle \\ &= \sum^n \neq \frac{a_1 a_2 \dots a_m b_1 b_2 \dots b_m e_1 e_2 \dots e_m}{x_{i1} x_{i2} \dots x_{im} x_{j1} x_{j2} \dots x_{jm} \dots x_{v1} x_{v2} \dots x_{vm}} / n^{(\rho)} \end{aligned} \quad (4.26)$$

where ρ denotes the number of different first subscripts. For example, with $m = 3$,

$$\langle (134)(133) \rangle = \sum^n \neq \frac{x_{i1}^2 x_{i2}^3 x_{i3}^4 x_{j1} x_{j2}^3 x_{j3}^3}{x_{i1} x_{i2} x_{i3} x_{j1} x_{j2} x_{j3}} / n^{(2)} .$$

Multivariate $\odot$ -multiplication is defined by

$$\begin{aligned} & \langle (a_1 a_2 \dots a_m) \dots (b_1 b_2 \dots b_m) \rangle \odot \langle (c_1 c_2 \dots c_m) \dots (e_1 e_2 \dots e_m) \rangle \\ &= \langle (a_1 a_2 \dots a_m) \dots (b_1 b_2 \dots b_m) (c_1 c_2 \dots c_m) \dots (e_1 e_2 \dots e_m) \rangle . \end{aligned}$$

Define

$$M(t_1 t_2 \dots t_m) = \sum_{v_1=0}^{\infty} \sum_{v_2=0}^{\infty} \dots \sum_{v_m=0}^{\infty} \langle (v_1 v_2 \dots v_m) \rangle t_1^{v_1} t_2^{v_2} \dots t_m^{v_m} / (v_1! v_2! \dots v_m!) .$$

The one part multivariate polykays are defined by

$$\emptyset(t_1 t_2 \dots t_m) = \odot \log \{M(t_1 t_2 \dots t_m)\} ,$$

where

$$\emptyset(t_1 t_2 \dots t_m) = \sum_{v_1=0}^{\infty} \sum_{v_2=0}^{\infty} \dots \sum_{v_m=0}^{\infty} [K((v_1 v_2 \dots v_m)) t_1^{v_1} t_2^{v_2} \dots t_m^{v_m} / v_1! \dots v_m!] - 1 ,$$

$K((v_1 v_2 \dots v_m))$ denotes a one part multivariate polykay and the $\odot \log$ operator is identical to that defined in Section 1.2. Higher order multivariate polykays are defined by symbolic multiplication.

Let

$$\begin{aligned} & K((a_1 a_2 \dots a_m)(b_1 b_2 \dots b_m) \dots (e_1 e_2 \dots e_m)) \\ &= K((a_1 a_2 \dots a_m)) \odot K((b_1 b_2 \dots b_m)) \odot \dots \odot K((e_1 e_2 \dots e_m)) . \end{aligned}$$

The equivalent population parameters are defined by

$$\begin{aligned} & \langle (a_1 a_2 \dots a_m)(b_1 b_2 \dots b_m) \dots (e_1 e_2 \dots e_m) \rangle' \\ &= \sum_{i_1}^N \dots \sum_{i_m}^N \frac{a_1}{x_{i_1}} \frac{a_2}{x_{i_2}} \dots \frac{a_m}{x_{i_m}} \frac{b_1}{x_{j_1}} \frac{b_2}{x_{j_2}} \dots \frac{b_m}{x_{j_m}} \dots \frac{e_1}{x_{v_1}} \frac{e_2}{x_{v_2}} \dots \frac{e_m}{x_{v_m}} / N^{(\rho)} , \end{aligned}$$

where, again, ρ denotes the number of different first subscripts.

The one part population polykeys are defined by

$$\emptyset'(t_1 t_2 \dots t_m) = \ominus \log \{M'(t_1 t_2 \dots t_m)\} ,$$

where

$$\emptyset'(t_1 t_2 \dots t_m) = \sum_{v_1=1}^{\infty} \sum_{v_2=1}^{\infty} \dots \sum_{v_m=1}^{\infty} [K'((v_1 v_2 \dots v_m)) t_1^{v_1} t_2^{v_2} \dots t_m^{v_m} / v_1! v_2! \dots v_m!] - 1 ,$$

and

$$M'(t_1 t_2 \dots t_m) = \sum_{v_1=1}^{\infty} \sum_{v_2=1}^{\infty} \dots \sum_{v_m=1}^{\infty} [\langle (v_1 v_2 \dots v_m) \rangle t_1^{v_1} t_2^{v_2} \dots t_m^{v_m} / v_1! v_2! \dots v_m!] .$$

Analogous to Lemmas 2.1 and 4.1 it can be shown that multivariate brackets and hence multivariate polykeys are inherited on the average.

The multivariate polykeys may be used to develop multivariate ratio estimators. Robson [21] used trivariate polykeys to develop the variance of the ratio estimator y_R . In the same paper Robson studied the trivariate analogue of y_p , i.e., an estimator of $\bar{X}_1$ is given by

$$x^* = \bar{x}_1 \bar{x}_2 \bar{x}_3 / \bar{X}_2 \bar{X}_3 .$$

Robson described a technique for adjusting such estimators to make

them unbiased and derived the variance of such estimators for a large population.

CHAPTER 5

POLYKAYS OF A TWO WAY ARRAY

5.1. Symmetric Functions of a Two Way Array.

Consider a population in which each unit is classified according to two criteria. Let the first criterion divide the population into R classes and let the second criterion divide the same population into C classes. It is assumed that there is exactly one unit in the population which satisfies any pair of criteria. Hence the population consists of exactly RC units. Let y_{ij} denote the variate value under study, where $i = 1, 2, \dots, R$, and $j = 1, 2, \dots, C$.

For example consider a population of R television sets numbered $1, 2, \dots, R$. Within each set there are C component parts numbered $1, 2, \dots, C$ which could be defective. The problem is to estimate the number of defective parts. If a simple random sample of parts were taken then it is likely that a large number of television sets would have to be unpackaged and examined, a time consuming and expensive operation. Thus it is desirable to use some other sampling plan.

A "bisample" of the population is taken by choosing without replacement r first subscripts and c second subscripts. The variate values associated with the bisample are y_{ij} where i is

one of the r first subscripts chosen and j is one of the c second subscripts chosen. For example, if the first subscripts chosen were 1 and 5 and the second subscripts chosen were 2 and 4 then the units sampled would be u_{12} , u_{14} , u_{52} and u_{54} , i.e., in reference to the example television sets number 1 and 5 would be selected and within each one parts 2 and 4 would be examined.

The sample generalized symmetric functions of the 2 way array are defined by

$$\begin{aligned} [\begin{smallmatrix} p \\ - \end{smallmatrix}] &= \sum^{r,c} y_{ij}^p / rc, \\ [\begin{smallmatrix} p & q \\ - & - \end{smallmatrix}] &= \sum^{r,c}_{\neq} y_{ij}^p y_{im}^q / rc(c-1), \\ [\begin{smallmatrix} p \\ q \end{smallmatrix}] &= \sum^{r,c}_{\neq} y_{ij}^p y_{kj}^q / rc(r-1), \\ \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} &= \sum^{r,c} y_{ij}^{p_{11}} y_{im}^{p_{12}} y_{kj}^{p_{21}} y_{km}^{p_{22}} / r^{(2)} c^{(2)}, \end{aligned} \quad (5.1)$$

and so on. Here $\sum^{r,c}_{\neq}$ denotes a summation taken over the variate values of the sample which have different subscripts where indicated, e.g.,

$$\sum^{r,c}_{\neq} y_{ij} y_{im} = \sum_{\substack{i=1 \\ i \neq j}}^r \sum_{\substack{j=1 \\ j \neq m}}^c \sum_{\substack{m=1 \\ m \neq i}}^c y_{ij} y_{im}.$$

Corresponding population parameters may be defined using

prime notation, e.g.,

$$[\begin{smallmatrix} p \\ - \end{smallmatrix} \begin{smallmatrix} q \\ - \end{smallmatrix}]' = \sum^{R,C} y_{ij}^p y_{ik}^q / RC(C-1) .$$

The generalized symmetric functions are inherited on the average. The proof is similar to that of Lemma 2.1 and hence is not reproduced here.

As in the case of univariate and bivariate polykays the polykays of a two way array are defined as linear functions of the generalized symmetric functions and are therefore inherited on the average. The polykays of a two way array are defined to be

$$\begin{aligned} K[\begin{smallmatrix} 1 \\ - \end{smallmatrix} \begin{smallmatrix} - \\ - \end{smallmatrix}] &= [\begin{smallmatrix} 1 \\ - \end{smallmatrix} \begin{smallmatrix} - \\ - \end{smallmatrix}] , \\ K[\begin{smallmatrix} 1 \\ - \end{smallmatrix} \begin{smallmatrix} 1 \\ - \end{smallmatrix}] &= [\begin{smallmatrix} 1 \\ - \end{smallmatrix} \begin{smallmatrix} 1 \\ - \end{smallmatrix}] - [\begin{smallmatrix} 1 \\ - \end{smallmatrix} \begin{smallmatrix} - \\ 1 \end{smallmatrix}] , \\ K[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \begin{smallmatrix} - \\ - \end{smallmatrix}] &= [\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \begin{smallmatrix} - \\ - \end{smallmatrix}] - [\begin{smallmatrix} 1 \\ - \end{smallmatrix} \begin{smallmatrix} - \\ 1 \end{smallmatrix}] , \\ K[\begin{smallmatrix} 1 \\ - \end{smallmatrix} \begin{smallmatrix} - \\ 1 \end{smallmatrix}] &= [\begin{smallmatrix} 2 \\ - \end{smallmatrix} \begin{smallmatrix} - \\ - \end{smallmatrix}] - [\begin{smallmatrix} 1 \\ - \end{smallmatrix} \begin{smallmatrix} 1 \\ - \end{smallmatrix}] - [\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \begin{smallmatrix} - \\ - \end{smallmatrix}] + [\begin{smallmatrix} 1 \\ - \end{smallmatrix} \begin{smallmatrix} - \\ 1 \end{smallmatrix}] . \end{aligned} \tag{5.2}$$

Further definitions of polykays of a two-way array are given in Hooke [9].

5.2. Expectations of Symmetric Functions of a Two-Way Array.

As was the case with both univariate and bivariate polykays the products of polykays of a two way array must be studied before

the variances of such polykeys may be developed. Consider the following product:

$$\begin{aligned}
 \begin{bmatrix} a \\ - \end{bmatrix} \begin{bmatrix} b \\ - \end{bmatrix} &= \sum^{r,c} y_{ij}^a \sum^{r,c} y_{ij}^b / r^2 c^2 \\
 &= \{ \sum^{r,c} y_{ij}^a y_{km}^b + \sum^{r,c} y_{ij}^a y_{im}^b + \sum^{r,c} y_{ij}^a y_{kj}^b + \sum^{r,c} y_{ij}^{a+b} \} / r^2 c^2 \\
 &= \{ (r-1)(c-1) \begin{bmatrix} a \\ - \end{bmatrix} \begin{bmatrix} b \\ - \end{bmatrix} + (c-1) \begin{bmatrix} a \\ - \end{bmatrix} \begin{bmatrix} b \\ 1 \end{bmatrix} + (r-1) \begin{bmatrix} a \\ 1 \end{bmatrix} \begin{bmatrix} b \\ - \end{bmatrix} + \begin{bmatrix} a+b \\ - \end{bmatrix} \} / rc .
 \end{aligned}$$

The products of polykeys of a two way array can be calculated from the products of symmetric functions. For example

$$\begin{aligned}
 K \begin{bmatrix} 1 \\ - \end{bmatrix}^2 &= \begin{bmatrix} 1 \\ - \end{bmatrix}^2 \\
 &= \{ (r-1)(c-1) \begin{bmatrix} 1 \\ - \end{bmatrix} \begin{bmatrix} 1 \\ - \end{bmatrix} + (c-1) \begin{bmatrix} 1 \\ - \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (r-1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ - \end{bmatrix} + \begin{bmatrix} 2 \\ - \end{bmatrix} \} / rc \\
 &= K \begin{bmatrix} 2 \\ - \end{bmatrix} / rc + K \begin{bmatrix} 1 \\ - \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} / r + K \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ - \end{bmatrix} / c + K \begin{bmatrix} 1 \\ 1 \end{bmatrix}^2 . \quad (5.3)
 \end{aligned}$$

Using (5.3) the variance of $K \begin{bmatrix} 1 \\ - \end{bmatrix}$ may be calculated:

$$\begin{aligned}
 V \{ K \begin{bmatrix} 1 \\ - \end{bmatrix} \} &= E \{ K \begin{bmatrix} 1 \\ - \end{bmatrix}^2 \} - E \{ K \begin{bmatrix} 1 \\ - \end{bmatrix} \}^2 \\
 &= E \{ K \begin{bmatrix} 2 \\ - \end{bmatrix} / rc + K \begin{bmatrix} 1 \\ - \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} / r + K \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ - \end{bmatrix} / c + K \begin{bmatrix} 1 \\ 1 \end{bmatrix}^2 \} - K' \begin{bmatrix} 1 \\ - \end{bmatrix}^2 \\
 &= K' \begin{bmatrix} 2 \\ - \end{bmatrix} \{ \frac{1}{rc} - \frac{1}{RC} \} + K' \begin{bmatrix} 1 \\ - \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \{ \frac{1}{r} - \frac{1}{R} \} + K' \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ - \end{bmatrix} \{ \frac{1}{c} - \frac{1}{C} \} . \quad (5.4)
 \end{aligned}$$

An unbiased estimator of (5.4) is given by

$$K \begin{bmatrix} 2 \\ - \\ - \end{bmatrix} \left\{ \frac{1}{rc} - \frac{1}{RC} \right\} + K \begin{bmatrix} 1 & 1 \\ - & - \end{bmatrix} \left\{ \frac{1}{r} - \frac{1}{R} \right\} + K \begin{bmatrix} 1 & - \\ 1 & - \end{bmatrix} \left\{ \frac{1}{c} - \frac{1}{C} \right\} .$$

Now $E\{K \begin{bmatrix} 1 \\ - \\ - \end{bmatrix}\} = K' \begin{bmatrix} 1 \\ - \\ - \end{bmatrix} = \sum^{R,C} y_{ij}/RC = \bar{Y}$, i.e., $K \begin{bmatrix} 1 \\ - \\ - \end{bmatrix}$ is an unbiased estimator of the population mean $\bar{Y}$. In order to compare $K \begin{bmatrix} 1 \\ - \\ - \end{bmatrix}$ with the y-only estimator $\bar{y}$ assume a sample of size rc is taken at random and without replacement from the finite population of RC values. Then from (2.9)

$$V(\bar{y}) = \left(\frac{1}{rc} - \frac{1}{RC} \right) \{ \langle 2 \rangle' - \langle 11 \rangle' \} . \quad (5.5)$$

Further

$$\langle 2 \rangle' = \sum^{RC} y_{ij}^2 = \begin{bmatrix} 2 \\ - \\ - \end{bmatrix}' .$$

Also $\langle 11 \rangle'$ is the sum of products of pairs of variate values which are not associated with the same sampling unit,

$$\langle 11 \rangle' = \sum^{RC} y_{ij} y_{im} + \sum^{RC} y_{ij} y_{kj} + \sum^{R,C} y_{ij} y_{km} .$$

Hence (5.5) becomes

$$\begin{aligned} V(\bar{y}) &= \left(\frac{1}{rc} - \frac{1}{RC} \right) \left\{ \begin{bmatrix} 2 \\ - \\ - \end{bmatrix}' - \begin{bmatrix} 1 & 1 \\ - & - \end{bmatrix}' - \begin{bmatrix} 1 & - \\ 1 & - \end{bmatrix}' - \begin{bmatrix} 1 & - \\ - & 1 \end{bmatrix}' \right\} \\ &= \left(\frac{1}{rc} - \frac{1}{RC} \right) \{ K' \begin{bmatrix} 2 \\ - \\ - \end{bmatrix} + 2K' \begin{bmatrix} 1 & - \\ - & 1 \end{bmatrix} \} . \end{aligned} \quad (5.6)$$

Therefore

$$\begin{aligned}
 v\{K[\begin{smallmatrix} 1 \\ - \end{smallmatrix}]\} - v(\bar{y}) &= (\frac{1}{r} - \frac{1}{R})K'[\begin{smallmatrix} 1 \\ - \end{smallmatrix}] + (\frac{1}{c} - \frac{1}{C})K'[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}] \\
 &\quad - 2(\frac{1}{rc} - \frac{1}{RC})K'[\begin{smallmatrix} 1 \\ - \\ 1 \end{smallmatrix}] . \quad (5.7)
 \end{aligned}$$

The estimator $K[\begin{smallmatrix} 1 \\ - \end{smallmatrix}]$ is more precise than $\bar{y}$ if (5.7) is negative.

Returning to the television sets example let y_{ij} equal unity if the ij -th part is defective and 0 if it is non-defective. Then $\bar{Y}$ is evidently the proportion of defective tubes in the population. An artificial set of data showing the y_{ij} for a population of 20 television sets each having 15 component parts is given in TABLE 3. The variances of the competing estimators $\bar{y}$ and $K[\begin{smallmatrix} 1 \\ - \end{smallmatrix}]$ of $\bar{Y}$ were calculated for several sample sizes and are displayed in TABLE 4. For this data set $\bar{y}$ is a more precise estimator of $\bar{Y}$ than $K[\begin{smallmatrix} 1 \\ - \end{smallmatrix}]$. This comparison, however, does not take into account the fact that if one takes a simple random sample then the expected cost would exceed that of a bisample of the same number of parts.

TABLE 3: Population of Television Sets Showing Defective Components.

		PART NUMBER														
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
T E L E V I S I O N N U M B E R	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	2	0	1	0	1	0	1	0	1	1	0	1	1	1	0	1
	3	0	1	0	1	0	1	0	1	1	0	1	1	1	0	1
	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	5	0	1	0	1	0	1	0	1	1	0	1	1	1	0	1
	6	0	1	0	1	0	1	0	1	1	0	1	1	1	0	1
	7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	8	0	1	0	1	0	1	0	1	1	0	1	1	1	0	1
	9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	10	0	1	0	1	0	1	0	1	1	0	1	1	1	0	1
	11	0	1	0	1	0	1	0	1	1	0	1	1	1	0	1
	12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	13	0	1	0	1	0	1	0	1	1	0	1	1	1	0	1
	14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	15	0	1	0	1	0	1	0	1	1	0	1	1	1	0	1
	16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	18	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	19	0	1	0	1	0	1	0	1	1	0	1	1	1	0	1
	20	0	1	0	1	0	1	0	1	1	0	1	1	1	0	1

TABLE 4: Comparison of Variances Due to Bisampling and Simple Random Sampling for Various Sample Sizes.

SAMPLE r	SIZE c	$V(\bar{y})$	$V\{K[\begin{smallmatrix} 1 \\ - \end{smallmatrix}]\}$
10	5	0.368×10^{-2}	0.155×10^{-1}
10	10	0.147×10^{-2}	0.739×10^{-2}
10	15	0.737×10^{-3}	0.468×10^{-2}
20	5	0.147×10^{-2}	0.104×10^{-1}
20	10	0.368×10^{-3}	0.259×10^{-2}
20	15	0.00	0.00

CHAPTER 6

COMPUTATION WITH POLYKEYS

6.1. Calculations Involving Brackets and Polykeys.

The use of polykeys in deriving formulae has been shown to simplify greatly both the derivation and the resulting formulae. Schaeffer and Dwyer [22] have pointed out that the numerical evaluation of formulae involving polykeys depends upon the ease with which the polykeys themselves can be evaluated. Tukey [23] presents a computational method for the evaluation of polykeys of up to weight 4 . The power sums are calculated first and the brackets are computed systematically using the relationships:

$$\begin{aligned}
 \langle a \rangle &= (a)/n , \\
 \langle ab \rangle &= [(a)\langle b \rangle - \langle a+b \rangle]/(n-1) , \\
 \langle abc \rangle &= [(c)\langle ab \rangle - \langle a+c \ b \rangle - \langle a \ b+c \rangle]/(n-2) , \\
 \langle abcd \rangle &= [(d)\langle abc \rangle - \langle a+d \ b \ c \rangle - \langle a \ b+d \ c \rangle - \langle a \ b \ c+d \rangle]/(n-3) .
 \end{aligned}
 \tag{6.1}$$

These formulae can be extended for use in the calculation of higher order brackets. Tukey [23] suggests that the polykeys be calculated using the recursion relations

$$K(1^r) = \langle 1^r \rangle ,$$

$$K(21^r) = \langle 21^r \rangle - K(1^{r+2}) ,$$

$$K(31^r) = \langle 31^r \rangle - [3K(21^{r+1}) + K(1^{r+3})] ,$$

$$K(2^2 1^r) = \langle 2^2 1^r \rangle - [2K(21^{r+2}) + K(1^{r+4})] ,$$

$$K(41^r) = \langle 41^r \rangle - [4K(31^{r+1}) + 3K(2^2 1^r) + 6K(21^{r+2}) + K(1^{r+4})] .$$

Tukey also presented a computing form for these calculations and an involved set of rules to follow to use the form. The main drawback to Tukey's method is that all polykays up to a given weight must be calculated. If only a subset of the polykays are actually needed then this is a great deal of unnecessary labour. Schaeffer and Dwyer [22] present a modification of this technique in which the brackets are calculated using (6.1) and the polykays are then computed directly from these brackets instead of from a set of previously calculated polykays. Schaeffer and Dwyer [22] have also considered techniques for calculating polykays which are invariant under translations of origin for polykays up to and including weight eight.

The multiplication formulae given in TABLE A3 can be used to check calculations of polykays. For example after having calculated K_4 , K_{22} and K_2 one could check whether the relation

$$K^2 = K_4/n + (n+1)K_{22}/(n-1)$$

holds. If it doesn't hold then there is a mistake in the computation.

6.2. Calculation of Bivariate Polykeys.

The calculation of bivariate polykeys can be carried out in an analogous manner to that for univariate polykeys. The bivariate brackets are calculated using the relations

$$\langle a_1 a_2 \rangle = ((a_1 a_2))/n ,$$

$$\langle a_1 a_2 \rangle \langle b_1 b_2 \rangle = [((a_1 a_2)) \langle b_1 b_2 \rangle - \langle a_1 + b_1, a_2 + b_2 \rangle] / (n-1) ,$$

$$\begin{aligned} \langle a_1 a_2 \rangle \langle b_1 b_2 \rangle \langle c_1 c_2 \rangle &= [((a_1 a_2)) \langle b_1 b_2 \rangle \langle c_1 c_2 \rangle - \langle a_1 + b_1, a_2 + b_2 \rangle \langle c_1 c_2 \rangle \\ &\quad - \langle a_1 a_2 \rangle \langle b_1 + c_1, b_2 + c_2 \rangle] / (n-2) , \end{aligned}$$

$$\begin{aligned} \langle a_1 a_2 \rangle \langle b_1 b_2 \rangle \langle c_1 c_2 \rangle \langle d_1 d_2 \rangle &= [((a_1 a_2)) \langle b_1 b_2 \rangle \langle c_1 c_2 \rangle \langle d_1 d_2 \rangle \\ &\quad - \langle a_1 + b_1, a_2 + b_2 \rangle \langle c_1 c_2 \rangle \langle d_1 d_2 \rangle \\ &\quad - \langle b_1 b_2 \rangle \langle a_1 + c_1, a_2 + c_2 \rangle \langle d_1 d_2 \rangle \\ &\quad - \langle b_1 b_2 \rangle \langle c_1 c_2 \rangle \langle a_1 + d_1, a_2 + d_2 \rangle] / (n-3) . \end{aligned}$$

The bivariate polykeys are then calculated using the formulae given in TABLE A5 of the Appendices. The calculations can then be checked by the multiplication formulae for bivariate polykeys given in TABLE A6 of the Appendices.

CHAPTER 7

CONCLUDING REMARKS

This study presents in a concise fashion the basic theory of polykay statistics and applies them in developing a number of important basic results in finite population sampling. The polykays which are functions of certain power sums are so defined that they lead to considerable simplification in the development of basic statistical results that are otherwise extremely tedious and prone to error. Final results which are otherwise completely unwieldy in structure may be presented in a compact form.

It is interesting to note that polykays are avoided in all texts in survey sampling that have come to our attention. This is due, in our opinion, to the lack of a good basic reference that details the essential theory and results that are of primary importance to the survey statistician. The short-hand notation is initially somewhat difficult to adjust to and the properties of the polykays must be memorized. But once these initial stumbling blocks have been overcome the power of this approach becomes apparent. It is our feeling that this study has filled the gap that existed.

Although the application of polykays to survey sampling have been studied, their properties may be utilized in developing certain results in the area of experimental design (see Tukey [25]

and Hooke [9] for example). Recent research by Dwyer and Tracy [28] and Tracy [29] provide results relating to the development of a polykay product without any reference to the symmetric functions.

It has been suggested that polykays are not possibly of such great importance now because of the extensive availability of electronic computers with which large-scale simulations may be made. Such simulations may investigate the sampling properties of any design and estimator. But the nature of the underlying population distribution that must be assumed is often reflected in the conclusions derived from such a study, particularly in a finite population setting where the universe a priori does not follow any of the classically assumed distributions. Thus the polykay statistics will continue to play an important and possibly increasing role in the development of the theory of sampling from a finite universe.

APPENDICES

TABLE A1: Conversions Between Power Sums and Augmented Symmetric Functions

Weight 1 (1) = [1]

Weight 2

	[2]	[1 ²]
(2)	1	-1
(1) ²	1	1

Weight 3

	[3]	[12]	[1 ³]
(3)	1	-1	2
(1)(2)	1	1	-3
(1) ³	1	3	1

Weight 4

	[4]	[13]	[2 ²]	[1 ² 2]	[1 ⁴]
(4)	1	-1	-1	2	-6
(1)(3)	1	1	.	-2	8
(2) ²	1	.	1	-1	3
(1) ² (2)	1	2	1	1	-6
(1) ⁴	1	4	3	6	1

Weight 5

	[5]	[14]	[23]	[1 ² 3]	[12 ²]	[1 ³ 2]	[1 ⁵]
(5)	1	-1	-1	2	2	-6	24
(1)(4)	1	1	.	-2	-1	6	-30
(2)(3)	1	.	1	-1	-2	5	-20
(1) ² (3)	1	2	1	1	.	-3	20
(1)(2) ²	1	1	2	.	1	-3	15
(1) ³ (2)	1	3	4	3	3	1	-10
(1) ⁵	1	5	10	10	15	10	1

TABLE A1 cont'd.

To use Table A1 to express the augmented symmetric functions in terms of the power sums, read down the appropriate column up to and including the diagonal entry, e.g.,

$$[1^2 2] = 2(4) - 2(1)(3) - (2)^2 + (1)^2(2) .$$

Conversely to express the power sums in terms of the augmented symmetric functions read across the appropriate row up to and including the diagonal, e.g.,

$$(2)^2 = [4] + [2^2] .$$

This table is given up to weight 12 in David and Kendall [3].

TABLE A2: Conversions Between Polykeys and Brackets.

To convert a polykey into a bracket read down the column up to and including the diagonal. Then add ones inside the bracket until the weights are homogeneous. For example, the terms in $K(4)$ are $-6\langle 1 \rangle$, $12\langle 2 \rangle$, $-4\langle 3 \rangle$, $-3\langle 22 \rangle$, $\langle 4 \rangle$.

Hence

$$K(4) = -6\langle 1^4 \rangle + 12\langle 1^2 2 \rangle - 4\langle 13 \rangle - 3\langle 22 \rangle + \langle 4 \rangle .$$

Again,

$$K(14) = -6\langle 1^5 \rangle + 12\langle 1^3 2 \rangle - 4\langle 1^2 3 \rangle - 3\langle 122 \rangle + \langle 14 \rangle .$$

To convert brackets to polykeys read across the row up to and including the diagonal inserting ones to yield the correct weight.

This table was first published by Tukey [24].

TABLE A2 cont'd.

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	K(1)	K(2)	K(3)	K(22)	K(4)	K(23)	K(222)	K(5)	K(24)	K(33)	K(223)	K(2222)	K(6)	K(25)	K(34)	K(224)	K(233)	K(7)	K(26)	K(35)	K(44)	K(8)
<1>	1	-1	2	1	-6	-2	-1	24	6	4	2	1	-120	-24	-12	-6	-4	720	120	48	36	-5040
<2>	1	1	-3	-2	12	5	3	-60	-18	-12	-7	-4	360	84	42	24	16	-2520	-480	-192	-144	20160
<3>	1	2	1	.	-4	-1	.	20	4	4	1	.	-120	-20	-14	-4	-4	840	120	64	48	-6720
<22>	1	3	.	1	-3	-3	-3	30	15	9	8	6	-270	-90	-42	-33	-21	2520	630	240	180	-25200
<4>	1	6	4	3	1	.	.	-5	-1	.	.	.	30	5	2	1	.	-210	-30	-10	-12	1680
<23>	1	4	1	3	.	1	.	-10	-4	-6	-2	.	120	30	24	8	10	-1260	-240	-140	-96	13440
<222>	1	3	.	3	.	.	1	.	-3	.	-3	-4	30	30	9	18	9	-630	-300	-90	-72	10080
<5>	1	10	10	15	5	10	.	1	.	.	.	.	-6	-1	.	.	.	42	6	2	.	-336
<24>	1	7	4	9	1	4	3	.	1	.	.	.	-15	-5	-3	-2	.	210	45	15	24	-2520
<33>	1	6	2	9	.	6	.	.	.	1	.	.	-10	.	-4	.	-1	140	10	20	16	-1680
<223>	1	5	1	7	.	2	3	.	.	.	1	.	.	-10	-3	-4	-6	210	120	60	24	-5040
<2222>	1	4	.	6	.	.	4	.	.	.	.	1	.	.	.	-3	.	.	30	.	9	-630
<6>	1	15	20	45	15	60	15	6	15	10	.	.	1	.	.	.	.	-7	-1	.	.	56
<25>	1	11	10	25	5	20	15	1	5	.	10	.	.	1	.	.	.	-21	-6	-3	.	336
<34>	1	9	5	21	1	18	9	.	3	4	3	.	.	.	1	.	.	-35	.	-5	-8	560
<224>	1	8	4	16	1	8	12	.	2	.	4	3	.	.	.	1	.	.	-15	.	-6	420
<233>	1	7	2	15	.	8	9	.	.	1	6	.	.	.	.	.	1	.	-10	-10	.	560
<7>	1	21	35	105	35	210	105	21	105	70	105	.	7	21	35	.	.	1	.	.	.	-8
<26>	1	16	20	60	15	80	60	6	30	10	60	15	1	6	.	15	10	.	1	.	.	-28
<35>	1	13	11	45	5	50	45	1	15	10	45	.	.	3	5	.	10	.	.	1	.	-56
<44>	1	12	18	42	2	48	36	.	12	16	24	9	.	.	8	6	.	.	.	.	1	-35
<8>	1	28	56	210	70	560	420	56	420	280	840	105	28	168	280	210	280	8	28	56	35	1

TABLE A3: Products of Polykeys.

Weight 2 $K(1)^2 = K(2)/n + K(11)$

Weight 3

	K(3)	K(21)	K(1 ³)
K(2)K(1)	1/n	1	
K(1) ³	1/n ²	3/n	1
K(11)K(1)		2/n	1

Weight 4

	K(4)	K(31)	K(22)	K(21 ²)	K(1 ⁴)
K(3)K(1)	1/n	1			
K(2) ²	1/n		(n+1)/(n-1)		
K(2)K(1) ²	1/n ²	2/n	1/n	1	
K(1) ⁴	1/n ³	4/n ²	3/n ²	6/n	1
K(21)K(1)		1/n	1/n	1	
K(2)K(11)		2/n	-2/n ⁽²⁾	1	
K(11)K(11) ²		2/n ²	2/n ²	5/n	1
K(11) ²			2/n ⁽²⁾	4/n	1
K(111)K(1)				3/n	1

This Table is given by Wishart [26].

TABLE A4: Conversions Between Bivariate Power Sums
and Bivariate Augmented Symmetric Functions.

Weight 1

$$[(10)] = ((01))$$

Weight 2

	$[(20)]$	$[(10)^2]$	$[(11)]$	$[(10)(01)]$
$((20))$	1	-1		
$((10))^2$	1	1		
$((11))$			1	-1
$((10))((01))$			1	1

Weight 3

	$[(30)]$	$[(20)(10)]$	$[(10)^3]$	$[(21)]$	$[(20)(01)]$	$[(11)(10)]$	$[(10)^2(01)]$
$((30))$	1	-1	2				
$((20))((10))$	1	1	-3				
$((10))^3$	1	3	1				
$((21))$				1	-1	-1	2
$((20))((01))$				1	1	.	-1
$((11))((10))$				1	.	1	-2
$((10))^2((01))$				1	1	2	1

TABLE A4 cont'd.

Table A4 is a parallel of Table A1. To write the $[(\)(\)]$'s in terms of the $((\))$'s read down the column up to and including the diagonal entry, e.g.,

$$[(20)(01)] = ((20))((01)) - ((21)) .$$

To write the $((\))$ in terms of $[(\)]$ read across the row up to and including the diagonal entry.

Table A4 is symmetric with respect to both x and y . Hence the remaining conversions can be obtained by interchanging all terms inside the bracket, e.g.,

$$[(02)(10)] = ((02))((10)) - ((12)) .$$

TABLE A5: Conversions Between Bivariate Brackets and Bivariate Polykeys.

Weight 1

$$K((10)) = \langle (10) \rangle$$

Weight 2

	$K((10)^2)$	$K((20))$	$K((10)(01))$	$K((11))$
$\langle (10)^2 \rangle$	1	-1		
$\langle (20) \rangle$	1	1		
$\langle (10)(01) \rangle$			1	-1
$\langle (11) \rangle$			1	1

Weight 3

	$K((10)^3)$	$K((20)(10))$	$K((30))$	$K((10)^2(01))$	$K((20)(01))$	$K((11)(10))$	$K((21))$
$\langle (10)^3 \rangle$	1	-1	2				
$\langle (20)(10) \rangle$	1	1	-3				
$\langle (30) \rangle$	1	3	1				
$\langle (10)^2(01) \rangle$				1	-1	-1	2
$\langle (20)(01) \rangle$				1	1	.	-1
$\langle (11)(10) \rangle$				1	.	1	-2
$\langle (21) \rangle$				1	1	2	1

Weight 4

[illegible]

To write $K(())$ in terms of the $\langle () \rangle$'s read down the column up to and including the diagonal entry, e.g.,

$$K((11)(10)) = \langle (11)(10) \rangle - \langle (10)^2(01) \rangle .$$

Other formulae are obtained by symmetry. Thus,

$$K((11)(01)) = \langle (11)(01) \rangle - \langle (01)^2(10) \rangle .$$

To write $\langle () \rangle$ in terms of the $K(())$'s read across the row up to and including the diagonal entry. Thus,

$$\langle (11)(10) \rangle = K((10)^2(01)) + K((11)(10)) .$$

TABLE A6: Products of Bivariate Polykays.

Weight 2

$$K((10))^2 = K((20))/n + K((10)^2)$$

$$K((10))K((01)) = K((11))/n + K((10)(01))$$

Weight 3

$$K((20))K((10)) = K((30))/n + K((20)(10))$$

$$K((10))^3 = K((30))/n^2 + 3K((20)(10))/n + K((10)^3)$$

$$K((10)^2)K((10)) = 2K((20)(10))/n + K((10)^3)$$

$$K((20))K((01)) = K((21))/n + K((20)(01))$$

$$K((10))^2K((01)) = K((21))/n^2 + K((20)(01))/n + 2K((11)(10))/n + K((10)^2(01))$$

$$K((10)^2)K((01)) = 2K((11)(10))/n + K((10)^2(01))$$

$$K((10)(01))K((10)) = K((20)(01))/n + K((10)(11))/n + K((10)^2(01))$$

$$K((11))K((10)) = K((21))/n + K((11)(10))$$

Weight 4

$$K((30))K((10)) = K((40))/n + K((30)(10))$$

$$K((20))^2 = K((40))/n + K((20)^2)(n+1)/(n-1)$$

$$K((20))K((10))^2 = K((40))/n^2 + 2K((30)(10))/n + K((20)^2)/n + K((20)(10)^2)$$

$$K((10))^4 = K((40))/n^3 + 4K((30)(10))/n^2 + 3K((20)^2)/n^2 + 6K((20)(10)^2)/n + K((10)^4)$$

$$K((20)(10))K((10)) = K((30)(10))/n + K((20)^2)/n + K((20)(10)^2)$$

$$K((20))K((10)^2) = 2K((30)(10))/n - 2K((20)^2)/n^{(2)} + K((20)(10)^2)$$

$$K((10))^2K((10)^2) = 2K((30)(10))/n^2 + 2K((20)^2)/n^2 + 5K((20)(10)^2)/n + K((10)^4)$$

$$\begin{aligned}K((10)^2)^2 &= 2K((20)^2)/n^{(2)} + 4K((20)(10)^2)/n + K((10)^4) \\K((10)^3)K((10)) &= 3K((20)(10)^2)/n + K((10)^4) \\K((30))K((01)) &= K((31))/n + K((30)(01)) \\K((21))K((10)) &= K((31))/n + K((21)(10)) \\K((20))K((11)) &= K((31))/n + K((20)(11))(n+1)/(n-1) \\K((20)(10))K((01)) &= K((21)(10))/n + K((20)(11))/n + K((20)(10)(01)) \\K((20)(01)K((10)) &= K((30)(01))/n + K((20)(11))/n + K((20)(10)(01)) \\K((11)(10))K((10)) &= K((21)(10))/n + K((20)(11))/n + K((20)(10)(01)) \\K((20))K((10)(01)) &= K((30)(01))/n + K((21)(10))/n - 2K((20)(11))/n^{(2)} \\&\quad + K((20)(10)(01)) \\K((11))K((10)^2) &= 2K((21)(10))/n - 2K((20)(11))/n^{(2)} + K((11)(10)^2) \\K((10)^2)K((10)(01)) &= 2K((20)(11))/n^{(2)} + 2K((20)(10)(01))/n \\&\quad + 2K((11)(10)^2)/n + K((10)^3(01)) \\K((10)^3)K((01)) &= 3K((10)^2(11))/n + K((10)^3(01)) \\K((10)^2(01))K((10)) &= 2K((20)(10)(01))/n + K((10)^2(11))/n + K((10)^3(01)) \\K((20))K((10))K((01)) &= K((31))/n^2 + K((30)(01))/n + K((21)(10))/n \\&\quad + K((20)(11))/n + K((20)(10)(01)) \\K((11))K((10))^2 &= K((31))/n^2 + 2K((21)(10))/n + K((20)(11))/n + K((20)(10)(01)) \\K((10)^2)K((10))K((01)) &= 2K((21)(10))/n^2 + 2K((20)(11))/n^2 + 2K((20)(10)(01))/n \\&\quad + 3K((10)^2(11))/n + K((10)^3(01)) \\K((10))^3K((01)) &= K((31))/n^3 + K((30)(01))/n^2 + 3K((21)(10))/n^2 + 3K((20)(11))/n^2 \\&\quad + 3K((20)(10)(01))/n + 3K((10)^2(11))/n + K((10)^3(01)) \\K((21))K((01)) &= K((22))/n + K((21)(01)) \\K((20))K((02)) &= K((22))/n + 2K((11)^2)/(n-1) + K((20)(02)) \\K((11))^2 &= K((22))/n + K((11)^2)n/(n-1) + K((20)(02))/(n-1)\end{aligned}$$

$$\begin{aligned}K((20)(01))K((01)) &= K((21)(01))/n + K((20)(02))/n + K((01)^2(20)) \\K((11)(10))K((01)) &= K((12)(10))/n + K((11)^2)/n + K((11)(10)(01)) \\K((20))K((01)^2) &= 2K((01)(21))/n - 2K((11)^2)/n^{(2)} + K((01)^2(20)) \\K((11))K((10)(01)) &= K((21)(01))/n + K((12)(10))/n - K((11)^2)/n^{(2)} \\&\quad - K((20)(02))/n^{(2)} + K((10)(01)(11)) \\K((10)^2)K((01)^2) &= 2K((11)^2)/n^{(2)} + 2K((01)^2(20))/n^{(2)} \\&\quad + 2K((10)^2(02))/n^{(2)} + 4K((11)(10)(01))(n-2)/n^{(2)} \\&\quad + K((10)^2(01)^2) \\K((10)(01))^2 &= K((11)^2)/n^{(2)} + K((20)(02))/n^{(2)} + K((20)(01)^2)/n \\&\quad + K((10)^2(02))/n + 2K((11)(10)(01))/n \\&\quad + K((10)^2(01)^2) \\K((10)^2(01))K((01)) &= 2K((11)(10)(01))/n + K((10)^2(02))/n + K((10)^2(01)^2) \\K((20))K((01))^2 &= K((22))/n^2 + 2K((21)(01))/n + K((20)(02))/n \\&\quad + K((20)(01)^2) \\K((11))K((10))K((01)) &= K((22))/n^2 + K((21)(01))/n + K((12)(10))/n \\&\quad + K((11)^2)/n + K((11)(10)(01)) \\K((10)^2)K((01))^2 &= 2K((12)(10))/n^2 + 2K((11)^2)/n^2 + 4K((11)(10)(01))/n \\&\quad + K((10)^2(02))/n + K((10)^2(01)^2) \\K((10)(01))K((10))K((01)) &= K((21)(01))/n^2 + K((12)(10))/n^2 \\&\quad + K((20)(02))/n^2 + K((11)^2)/n^2 \\&\quad + K((10)^2(02))/n + 3K((11)(10)(01))/n \\&\quad + K((01)^2(20))/n + K((10)^2(01)^2) \\K((10))^2K((01))^2 &= K((22))/n^3 + 2K((21)(01))/n^2 + 2K((12)(10))/n^2 \\&\quad + K((20)(02))/n^2 + 2K((11)^2)/n^2 + 4K((11)(10)(01))/n \\&\quad + K((01)^2(20))/n + K((02)(10)^2)/n + K((10)^2(01)^2)\end{aligned}$$

Other formulae are obtained from symmetry considerations,
e.g.,

$$K((03))K((10)) = 3K((13))/n + K((03)(10))$$

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AMS - Annals of Mathematical Statistics

BIOM - Biometrika

JASA - Journal of the American Statistical Association

PLMS - Proceedings of the London Mathematical Society.