The Structure of the Banach Algebra $LUC(G)^*$

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Abstract

Let G be a locally compact group. The ideal structure of the algebra $\text{LUC}(G)^*$, the dual of the commutative C^* -algebra $\text{LUC}(G)$ of bounded left uniform continuous functions on G , has been discussed by many authors recently. Also, the proof that the Banach algebra $L_1(G)^{**}$ determines G revealed interesting structural features, in particular, it shows that the algebra $\text{LUC}(G)^*$, plays an important role in the study of $L_1(G)^{**}$, the second dual of the group algebra $L_1(G)$. The algebra $\text{LUC}(G)^*$ is not only of significance in Banach algebra theory and abstract harmonic analysis, but also in the theory of semigroup compactifications as well as topological dynamics.

A large part of this thesis is focused on the systematic account of the structure of the Banach algebra $\text{LUC}(G)^*$ and also to investigate the extension of Curtis and Figà-Talamanca's result to certain subspaces of $L_\infty(G)$, as obtained by A.T.-M. Lau [11].

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Chapter 1

Introduction

The second dual $L_1(G)^{**}$ of the group algebra of a locally compact group G has been studied in a series of papers recently (see, e.g., [12], [14], [7], [15], [16], [17], [18], [19], [13]). Most complete information has been obtained for the compact case [10]. Extensions to non-compact groups present some difficulties but successes have been achieved, notably in showing that the topological center of $L_1(G)^{**}$ for any locally compact group G is $L_1(G)$ [14], [17]. The proof that the Banach algebra $L_1(G)^{**}$ determines G also revealed interesting structural features, in particular, it showed once more that the algebra $\text{LUC}(G)^*$, the dual of the commutative C^* -algebra $\text{LUC}(G)$ of bounded left uniform continuous functions on G , plays an important role in the study of $L_1(G)^{**}$. In this thesis, we study a topic on which much has already been published, that of the Banach algebra $\text{LUC}(G)^*$, and present an account of some of its structural properties, including, example, a discussion of the continuity of multiplication in $\text{LUC}(G)^*$ (cf. Chapter 3.4), and its representation in an appropriate algebra of operators (cf. Chapter 4).

This algebra has been of interest in abstract harmonic analysis for over fifty years, and is referred to as a “highly legitimate object of study” in Hewitt–Ross’ standard treatise [8]; see (19.23) (b) in [8]. Moreover, the authors (ibid.) point out

that $\text{LUC}(G)^*$ “[presents] grave difficulties [...] because of the great complexity of [its] structure.” Finally, we would like to mention that the algebra $\text{LUC}(G)^*$ is not only of significance in Banach algebra theory and abstract harmonic analysis, but also in the theory of semigroup compactifications (see, e.g., [1]) as well as topological dynamics. Here, the spectrum of $\text{LUC}(G)$ plays a crucial rôle since it is the largest semigroup compactification of G (any other semigroup compactification is a quotient), called the *LUC-compactification* of G , or – in topological dynamics – the *greatest ambit*.

In what follows we give a brief outline of the thesis, describing the content of each chapter and indicating its relevance to the existing literature and current research.

In Chapter 2 we shall give the necessary definitions, notations, and basic facts needed for this thesis. Curtis and Figà-Talamanca have proved in [2], Theorem 3.3, that the dual Banach algebra $\text{LUC}(G)^*$ is isometrically algebra isomorphic to the algebra of bounded linear operators from $L_\infty(G)$ into $L_\infty(G)$ commuting with convolution by $L_1(G)$ -functions from the left. In Chapter 3, we shall investigate the extension of Curtis and Figà-Talamanca’s result to certain subspaces of $L_\infty(G)$ as obtained by A. T.-M. Lau [11]. Furthermore, we shall show that the space of bounded operators from $\text{LUC}(G)$ into $\text{LUC}(G)$ commuting with left translation is isometrically isomorphic to $\text{LUC}(G)^*$.

In Chapter 4, we shall extend the representation theorem of the measure algebra for $M(G)$, obtained by Størmer and Ghahramani, to the case of the Banach algebra $\text{LUC}(G)^*$, with the Arens product, for an arbitrary locally compact group G . We consider the isometric representation of $\text{LUC}(G)^*$, in $\mathcal{B}(\mathcal{B}(L_p(G)))$ and $\mathcal{CB}(\mathcal{B}(\mathcal{H}))$, as presented in [21].

The main contributions of the thesis are the following:

- we give a systematic account of the structure of the Banach algebra $\text{LUC}(G)^*$;
- we correct a mistake in the literature by giving a correct proof of a result by A.

T.-M. Lau [11] whose argument has a gap (cf. our Proposition 3.1);

- we present an alternative description of the topological center of the algebra $\text{LUC}(G)^*$ through a commutation relation (cf. Section 3.4).

Chapter 2

Preliminaries

2.1 Introduction

In this chapter, we shall give the necessary definitions and basic facts that will be used in later chapters. Arens multiplication and topological centers of certain dual algebras are also discussed. Terminology and notations not explained in this chapter will be explained or referenced in the subsequent chapters.

2.2 Convolution

Let G denote a locally compact group with a fixed left Haar measure m . Integration with respect to m will be denoted by $\int_G \dots dx$. By $L_1(G)$ we shall denote the space of all complex-valued measurable functions f on G such that $\int_G |f| dx < \infty$. The convolution product of the functions f and g is defined by

$$f * g(x) = \int_G f(y)g(y^{-1}x)dy, \tag{2.1}$$

whenever the integral makes sense. This turns $L^1(G)$ into a Banach algebra.

The product coincides with the convolution of f and g seen as measures in $M(G)$ (the space of all complex regular Borel measures on G) which are absolutely contin-

uous with respect to the Haar measure m . For a detailed analysis on convolution products, we refer to [8].

2.3 Banach Algebras

We do not review the basics of Banach algebras, but we briefly state the definition and some of the algebraic properties, and give some examples of Banach algebras.

Definition 2.1. A **complex algebra** is a vector space A over the complex field $\mathbb{C}$ in which an associative and distributive multiplication is defined, that is,

$$x(yz) = (xy)z, (x + y)z = xz + yz, x(y + z) = xy + xz$$

for $x, y, z \in A$, and which is related to scalar multiplication so that

$$\alpha(xy) = x(\alpha y) = (\alpha x)y$$

for x and $y \in A$, $\alpha \in \mathbb{C}$.

Definition 2.2. A **Banach algebra** is an algebra A over $\mathbb{C}$ that has a norm $\| \cdot \|$ relative to which A is a Banach space and such that for all a, b in A we have $\|ab\| \leq \|a\|\|b\|$.

If A has an identity e then it is assumed that $\|e\| = 1$. Examples of Banach algebras can be found in any standard functional analysis textbook. But we look at two important examples, namely the group and the measure algebra of a locally compact group G . A topological space (X, τ) is called *locally compact* if every point in X has a compact neighborhood. Equivalently, τ has a basis of relatively compact sets.

Definition 2.3. A **topological group** is a group G equipped with a Hausdorff topology in which the group operations are continuous. That is, the maps $G \times G \ni (x, y) \rightarrow xy$ and $G \ni x \rightarrow x^{-1} \in G$ are continuous.

Let G be a locally compact topological group and let $C_0(G)$ be the space of all continuous functions on G that vanish at infinity. If $\mu, \nu \in M(G)$, define

$$\begin{aligned} L : C_0(G) &\rightarrow \mathbb{C} \quad \text{by} \\ L(f) &= \int_G \int_G f(xy) d\mu(x) d\nu(y) \\ &= \int_G \int_G f(xy) d\nu(y) d\mu(x). \end{aligned}$$

Then L is a linear functional on $C_0(G)$ and

$$|L(f)| \leq \int_G \int_G |f(xy)| d|\mu|(x) d|\nu|(y) \leq \|f\| \|\mu\| \|\nu\|, \quad \forall f \in C_0(G). \quad (2.2)$$

So $L \in C_0(G)^* = M(G)$. Define $\mu * \nu$ by $L(f) = \int_G f d\mu * \nu$ for f in $C_0(G)$. That is,

$$\int_G f d\mu * \nu = \int_G \int_G f(xy) d\mu(x) d\nu(y). \quad (2.3)$$

Note that $\|L\| = \|\mu * \nu\| \leq \|\mu\| \|\nu\|$. It follows that $M(G)$ is a Banach algebra. The product $\mu * \nu$ is called the *convolution* of μ and ν . Equivalently, the *convolution* of μ and ν in $M(G)$ can be defined as follows:

$$\forall \mu, \nu \in M(G), \forall f \in C_0(G) \quad \forall x \in G$$

1. $(\mu * f)(x) = \mu(l_x f)$, where $(l_x f)(y) := f(xy)$ for all $x, y \in G$.
2. $\langle \mu * \nu, f \rangle = \langle \mu, \nu * f \rangle$.

Let $C_b(G)$ denote the space of complex-valued bounded continuous functions on G with the supremum norm. Let $LUC(G)$ ($RUC(G)$) denote the space of bounded left (right) uniformly continuous complex-valued functions on G , i.e., the space of functions $f \in C_b(G)$ for which the left (right) translation map

$$G \ni x \mapsto l_x f \in (C_b(G), \|\cdot\|_\infty), \quad \text{resp.,} \quad G \ni x \mapsto r_x f \in (C_b(G), \|\cdot\|_\infty)$$

is continuous; here, $(l_x f)(y) := f(xy)$ as before and $(r_x f)(y) := f(yx)$ for all $x, y \in G$.

Definition 2.4. Let A be a Banach algebra. A net $(\chi_\alpha)_{\alpha \in I}$ of elements in A is called an approximate identity for A if the relations

$$a = \lim_{\alpha} \chi_\alpha a = \lim_{\alpha} a \chi_\alpha \quad \text{hold for every } a \in A.$$

2.4 Arens Multiplication and Topological Centers of Certain Dual Algebras

Let A denote a complex Banach algebra, A^* the dual space of A (as a Banach space) and A^{**} the second dual space of A . The canonical embedding of A into A^{**} is denoted by $\pi : A \rightarrow A^{**}$. In 1951, Arens introduced two natural Banach algebra products on A^{**} each of which is an extension of the original product in A as canonically embedded in A^{**} . Subsequently, these products have become known as the *Arens products*. We denote by $\odot$ the first (= left) and by $\diamond$ the second (= right) *Arens product*. We denote by $*$ the product in A . Both *Arens products* are defined in three steps. For $a, b \in A$, $f \in A^*$ and $m, n \in A^{**}$, the elements $f \odot a$, $m \odot f$ in A^* and $m \odot n$ in A^{**} are defined as follows:

$$\langle f \odot a, b \rangle = \langle f, a * b \rangle$$

$$\langle n \odot f, a \rangle = \langle n, f \odot a \rangle$$

$$\langle m \odot n, f \rangle = \langle m, n \odot f \rangle.$$

The basic properties of this multiplication are as follows. For n fixed in A^{**} , the mapping $m \mapsto m \odot n$ is w^* - w^* -continuous. For m fixed in A^{**} , the mapping $n \mapsto m \odot n$ is in general not w^* - w^* -continuous unless $m \in A$. Hence the *topological center* of A^{**} with respect to this multiplication is defined as follows:

$$Z_t^1(A^{**}) = \{m \in A^{**} : n \mapsto m \odot n \text{ is } w^* - w^* - \text{continuous on } A^{**}\}.$$

The second Arens multiplication is defined as follows: For $a, b \in A$, $f \in A^*$ and $m, n \in A^{**}$, the elements $a \diamond f$, $f \diamond m$ in A^* and $m \diamond n$ in A^{**} are defined by the equalities:

$$\langle a \diamond f, b \rangle = \langle f, b * a \rangle$$

$$\langle f \diamond m, a \rangle = \langle m, a \diamond f \rangle$$

$$\langle m \diamond n, f \rangle = \langle n, f \diamond m \rangle.$$

For fixed m in A^{**} , the mapping $n \mapsto m \diamond n$ is w^* - w^* -continuous on A^{**} . For fixed n in A^{**} , the mapping $m \mapsto m \diamond n$ is in general not w^* - w^* -continuous on A^{**} unless $n \in A$. Hence the topological center of A^{**} with respect to this multiplication is defined as follows:

$$Z_t^2(A^{**}) = \{n \in A^{**} : m \mapsto m \diamond n \text{ is } w^* - w^* - \text{continuous on } A^{**}\}.$$

If for each $m, n \in A^{**}$, the equality $m \odot n = m \diamond n$ holds, then the algebra A is said to be *Arens regular*. As a well-known example, we note that all C^* -algebras are Arens regular.

It is easy to see that we have:

$$\begin{aligned} A \text{ Arens regular} &\Leftrightarrow Z_t^1 = A^{**} \\ &\Leftrightarrow Z_t^2 = A^{**} \\ &\Leftrightarrow Z_t^1 = Z_t^2 = A^{**}. \end{aligned}$$

We refer to [3] and [22] for a discussion of basic properties of Arens multiplication.

Remark 2.1. For any Banach algebra A , the algebraic center of A^{**} endowed with the first (respectively, second) Arens product, is included in both topological centers. Accordingly, the algebraic center of the algebra $\text{LUC}(G)^*$ is included in its topological center. If A is commutative, then both algebraic centers of A^{**} (with respect to the first and second Arens product) coincide and are equal to the first and

second topological center. This together with the clearly topological definition of the "topological center" justifies the terminology.

2.5 The Algebra $L_1(G)$ and its Second Dual

Recall that the convolution of functions f and g is defined through

$$f * g(x) = \int_G f(y)g(y^{-1}x)dy, \quad (2.4)$$

whenever the integral makes sense. We note that, with the notation introduced in Section 2.4, $h \odot f = \tilde{f} * h \in \text{LUC}(G)$ for all $h \in L_\infty(G)$ and $f \in L_1(G)$; here, $\text{LUC}(G)$ denotes the space of all complex-valued bounded left uniformly continuous functions on the group G , and $\tilde{f} := \frac{1}{\Delta} \check{f}$, where Δ is the modular function on G , and $\check{f}(x) := f(x^{-1})$ for all $x \in G$. Whenever we consider $L_1(G)^{**}$ as a Banach algebra, we regard it as equipped with the first Arens product. We recall that the first topological center of $L_1(G)^{**}$ is defined to be the set of all functionals $m \in L_1(G)^{**}$ which satisfy $m \odot n = m \diamond n$ for all $n \in L_1(G)^{**}$. Equivalently, the topological center consists of all functionals $m \in L_1(G)^{**}$ such that left multiplication by m is w^* - w^* -continuous on $L_1(G)^{**}$. A detailed analysis of topological centers in the general context of biduals of group algebras can be found, for example, in [17].

2.6 The Space $\text{LUC}(G)$ and its Dual

There is a notion of topological center for the Banach algebra $\text{LUC}(G)^*$. First let us recall the natural construction of the product in $\text{LUC}(G)^*$.

For $m, n \in \text{LUC}(G)^*$, $f \in \text{LUC}(G)$ and $x \in G$ one sets:

- (i) $(n \cdot f)(x) := \langle n, l_x f \rangle;$
- (ii) $\langle m \cdot n, f \rangle := \langle m, n \cdot f \rangle.$

Note that $n \cdot f$ indeed defines an element of $\text{LUC}(G)$; this is precisely the property of being *left introverted* shared by $\text{LUC}(G)$. The product in (ii) makes $\text{LUC}(G)^*$ a Banach algebra, and $\text{LUC}(G)$ becomes a left $\text{LUC}(G)^*$ -module with the action introduced in (i). We refer to [1] for a (much) more detailed discussion.

In analogy to the case of $L_1(G)^{**}$, one defines the topological center of $\text{LUC}(G)^*$ to be the set of elements $m \in \text{LUC}(G)^*$ such that left multiplication by m is w^* - w^* -continuous on $\text{LUC}(G)^*$. We recall that we denote by $M(G) = C_0(G)^*$ the algebra of all complex (finite) Radon measures on G . We shall consider $M(G)$ as a subspace of $\text{LUC}(G)^*$, due to the canonical embedding via integration: $\langle \mu, f \rangle = \int_G f d\mu$, for $\mu \in M(G)$, $f \in \text{LUC}(G)$.

Chapter 3

Structure Results for $\text{LUC}(G)^*$ and Related Algebras

3.1 Introduction

The principal aim of this chapter is to provide some structural properties for $\text{LUC}(G)^*$ and related algebras. We shall also include some new findings and give an alternate description of the topological center of $\text{LUC}(G)^*$. The next section gives a number of lemmata and definitions which are needed to prove the main theorems of this chapter. Most of these are quite well-known. For the sake of completeness we shall present the proofs. The main theorems are then presented in the third section.

3.2 Some Technical Lemmata And Definitions

Let T be a bounded linear operator from $L_\infty(G)$ into $L_\infty(G)$ which commutes with left (respectively, right) translations, if T is in addition w^* - w^* -continuous, then, by the classical Wendel Theorem, there exists a complex Radon measure μ on G such

that

$$T(f) = f * \mu \quad (\text{respectively, } T(f) = \mu * f),$$

for all f in $L_\infty(G)$. In [13], Wendel considers the action of $M(G)$ on $L_1(G)$, but we will consider the dual action of $M(G)$ on $L_\infty(G)$ in this thesis.

Definition 3.1. *A linear operator $T : L_\infty(G) \rightarrow L_\infty(G)$ is said to **commute with convolution from the left** if $T(\varphi * f) = \varphi * T(f)$ for each $\varphi \in L_1(G)$ and each $f \in L_\infty(G)$.*

If $g \in L_1(G)$, $x \in G$, the maps $l_x, r_x, l_g, r_g : L_\infty(G) \rightarrow L_\infty(G)$ are defined by $l_x f(y) = f(xy)$, $r_x f(y) = f(yx)$, $l_g(f) = (\frac{1}{\Delta} \check{g}) * f$, and $r_g(f) = f * \check{g}$ for each $f \in L_\infty(G)$, $y \in G$ (where $\check{g}(x) = g(x^{-1})$ and Δ is the modular function on G ; note that in [8], the notation f^* is used instead of $\check{f}$). Given $\varphi \in L_1(G)$ and $f \in L_\infty(G)$, the functions $\varphi * f$ and $f * \check{\varphi}$ on G defined by

$$(\varphi * f)(x) = \int_G \varphi(y) f(y^{-1}x) dy \quad \text{and} \quad (f * \check{\varphi})(x) = \int_G f(y) \check{\varphi}(y^{-1}x) dy$$

belong to $C_b(G)$. In fact, $\varphi * f \in \text{LUC}(G)$ and $f * \check{\varphi} \in \text{RUC}(G)$ ([8], Theorem 20.16).

Lemma 3.1. (a) *The mapping $g \rightarrow \frac{1}{\Delta} \check{g}$ is a linear isometry and homeomorphism of $L_1(G)$ onto itself.*

(b) *The mapping $f \rightarrow \check{f}$ is a linear isometry and w^* -homeomorphism of $L_\infty(G)$ onto itself.*

(c) *For any $g, \varphi \in L_1(G)$, $f \in L_\infty(G)$, we have*

$$\langle f * \check{g}, \varphi \rangle = \langle f, \varphi * g \rangle = \langle (\frac{1}{\Delta} \check{\varphi}) * f, g \rangle.$$

(d) *For any $g, \varphi \in L_1(G)$, $(\frac{1}{\Delta})(\varphi * g)^\vee = (\frac{1}{\Delta} \check{g}) * (\frac{1}{\Delta} \check{\varphi})$.*

(e) *For any $f \in L_\infty(G)$, $\varphi \in L_1(G)$, $\check{f} * \check{\varphi} = (\varphi * f)^\vee$.*

Proof. (a) By ([8], Theorem 20.2), we see that $g \rightarrow \frac{1}{\Delta} \check{g}$ is a linear isometry of $L_1(G)$ onto itself. Since it is inverse to itself, the remaining assertion is immediate.

(b) Let $T : L_1(G) \rightarrow L_1(G)$ be the map in (a).

Consider its adjoint $T^* : L_\infty(G) \rightarrow L_\infty(G)$. If $f \in L_\infty(G)$ and $g \in L_1(G)$, then

$$\langle T^* f, g \rangle = \langle f, \frac{1}{\Delta} \check{g} \rangle = \langle \check{f}, g \rangle \text{ by ([8], Theorem 20.2).}$$

Hence $T^* f = \check{f}$. Now the claim follows from part (a).

(c) For $f \in L_\infty(G)$, $g, \varphi \in L_1(G)$, we have

$$\begin{aligned} \langle f * \check{g}, \varphi \rangle &= \int_G (f * \check{g})(x) \varphi(x) dx \\ &= \int_G \int_G f(y) \check{g}(y^{-1}x) dy \varphi(x) dx && ([8], \text{Corollary 20.14 (iii)}) \\ &= \int_G \int_G g(x^{-1}y) \varphi(x) dx f(y) dy && (\text{Fubini's Theorem}) \\ &= \int_G (\varphi * g)(y) f(y) dy && ([8], \text{Theorem 20.10 (i)}) \\ &= \langle f, \varphi * g \rangle. \end{aligned}$$

On the other hand,

$$\begin{aligned} \langle (\frac{1}{\Delta} \check{\varphi}) * f, g \rangle &= \int_G ((\frac{1}{\Delta} \check{\varphi}) * f)(x) g(x) dx \\ &= \int_G \int_G \Delta(y^{-1}) \varphi(y^{-1}) f(y^{-1}x) dy g(x) dx && ([8], \text{Theorem 20.2}) \\ &= \int_G \int_G \varphi(y) f(yx) dy g(x) dx \\ &= \int_G \int_G f(yx) g(x) dx \varphi(y) dy && (\text{Fubini's Theorem}) \\ &= \int_G \int_G f(x) g(y^{-1}x) dx \varphi(y) dy \\ &= \int_G (f * \check{g})(y) \varphi(y) dy && ([8], \text{Corollary 20.14 (i)}) \\ &= \langle f * \check{g}, \varphi \rangle. \end{aligned}$$

(d) The map $g \rightarrow \frac{1}{\Delta} \check{g}$ is an involution of the group algebra $L_1(G)$ (with convolution as multiplication) by [8], Theorems 20.22 (iv) and 20.23 (iv).

(e) Let $f \in L_\infty(G)$, $\varphi \in L_1(G)$ and $x \in G$, then

$$\begin{aligned} (f \check{*} \check{\varphi})(x) &= \int_G \check{f}(y) \check{\varphi}(y^{-1}x) dy \\ &= \int_G f(y^{-1}) \varphi(x^{-1}y) dy \\ &= \int_G f(y^{-1}x^{-1}) \varphi(y) dy \\ &= (\varphi * f)^\sim(x). \end{aligned}$$

□

Lemma 3.2. Let $m \in L_\infty(G)^*$, $f \in L_\infty(G)$, $\varphi, g \in L_1(G)$.

Then we have the following:

$$f \odot g = (\frac{1}{\Delta} \check{g}) * f \text{ and } \iota(\varphi) \odot f = f * \check{\varphi}, \quad (3.1)$$

where $\iota : L_1(G) \rightarrow L_1(G)^{**}$ is the canonical embedding.

Proof. Using the definition of the first Arens product on $L_1(G)^{**}$, we have for any $h \in L_1(G)$ that

$$\langle f \odot g, h \rangle = \langle f, g * h \rangle = \langle (\frac{1}{\Delta} \check{g}) * f, h \rangle$$

by Lemma 3.1 (c). Hence $f \odot g = (\frac{1}{\Delta} \check{g}) * f$.

Also, if $h \in L^1(G)$, $\langle \iota(\varphi) \odot f, h \rangle = \langle \iota(\varphi), f \odot h \rangle = \langle (\frac{1}{\Delta} \check{h}) * f, \varphi \rangle = \langle f * \check{\varphi}, h \rangle$

by Lemma 3.1 (c) again. Hence $\iota(\varphi) \odot f = f * \check{\varphi}$. □

Definition 3.2. For each $m \in L_\infty(G)^*$, define a mapping $m_L : L_\infty(G) \rightarrow L_\infty(G)$ by $\langle m_L(f), g \rangle = \langle (\frac{1}{\Delta} \check{g}) * f, m \rangle$ for any $f \in L_\infty(G)$, $g \in L_1(G)$. The operator m_L is called the **topological left introversion** of m .

Similarly, the **topological right introversion** $m_R : L_\infty(G) \rightarrow L_\infty(G)$ is defined by $m_R(f)g = \langle f * \check{g}, m \rangle$ for any $f \in L_\infty(G)$, $g \in L_1(G)$.

Lemma 3.3. (a) $m_L : L_\infty(G) \rightarrow L_\infty(G)$ is linear, bounded, $\|m_L(f)\|_\infty \leq \|m\| \|f\|_\infty$ and $m_L(f) = m \odot f$ for any $f \in L_\infty(G)$.

(b) If $n \in L_\infty(G)^*$, $f \in L_\infty(G)$, $\varphi, g \in L_1(G)$ then $\iota(\varphi)_L = r_\varphi$, and $n_L(g * f) = g * n_L(f)$, where r_φ is explained in Definition 3.1 (for the last assertion, see also [26], Lemma 4.3 (B)).

(c) If $m_\alpha \rightarrow m$ in the norm topology of $L_\infty(G)^*$, then $(m_\alpha)_L \rightarrow m_L$ in the uniform operator topology.

(d) If $m_\alpha \rightarrow m$ in the weak* topology of $L_\infty(G)^*$, then for any $f \in L_\infty(G)$, we have $(m_\alpha)_L(f) \rightarrow m_L(f)$ in the weak* topology of $L_\infty(G)$.

Proof. (a) We have for all $g \in L_1(G)$:

$$|\langle m_L(f), g \rangle| = |\langle (\frac{1}{\Delta} \check{g}) * f, m \rangle| \leq \|m\| \|g\|_1 \|f\|_\infty \quad (\text{by the definition of } m_L).$$

Hence $\|m_L(f)\|_\infty \leq \|m\| \|f\|_\infty$. Also, for all $g \in L_1(G)$:

$$\begin{aligned} \langle m_L(f), g \rangle &= \langle (\frac{1}{\Delta} \check{g}) * f, m \rangle = \langle f \odot g, m \rangle \quad (\text{Lemma 3.2}) \\ &= \langle m \odot f, g \rangle. \end{aligned}$$

Thus $m_L(f) = m \odot f$.

(b) Recall $r_g(f) = f * \check{g}$ from Definition 3.1 for $g \in L_1(G)$, $f \in L_\infty$. Now, if $\varphi \in L_1(G)$,

$$\begin{aligned} (\iota(\varphi))_L(f) &= \iota(\varphi) \odot f \quad (\text{by (a) above}) \\ &= f * \check{\varphi} \quad (\text{by Lemma 3.2}) \\ &= r_\varphi(f). \end{aligned}$$

Let $n \in L_\infty(G)^*$, $f \in L_\infty(G)$ and $\varphi, g \in L_1(G)$. Then

$$\langle n_L(g * f), \varphi \rangle = \langle n, \tilde{\varphi} * (g * f) \rangle,$$

but, using Lemma 3.1 (c),

$$\begin{aligned} \langle g * n_L(f), \varphi \rangle &= \langle n_L(f), \tilde{g} * \varphi \rangle \\ &= \langle n, (\tilde{g} * \varphi)^\sim * f \rangle \\ &= \langle n, \tilde{\varphi} * (g * f) \rangle. \end{aligned}$$

Hence $n_L(g * f) = g * n_L(f)$.

(c) Assume $m_\alpha \rightarrow m$ in the norm topology of $L_\infty(G)^*$.

Since $(m_\alpha)_L - m_L = (m_\alpha - m)_L$, $\|((m_\alpha)_L - m_L)f\| \leq \|m_\alpha - m\| \|f\|_\infty \forall f \in L_\infty(G)$ by (a). Hence $\|(m_\alpha)_L - m_L\| \leq \|m_\alpha - m\| \rightarrow 0$.

(d) Let $m_\alpha \rightarrow m$ in the weak* topology of $L_\infty(G)^*$. Then for any $f \in L_\infty(G)$, $g \in L_1(G)$,

$$\langle (m_\alpha)_L f, g \rangle - \langle m_L f, g \rangle = \langle m_\alpha, (\frac{1}{\Delta} \check{g}) * f \rangle - \langle m, (\frac{1}{\Delta} \check{g}) * f \rangle \rightarrow 0.$$

□

Definition 3.3. Let X be any nonvoid set, and let $\mathcal{B}(X)$ be the set of all bounded complex-valued functions defined on X . The symbol $\mathcal{B}^r(X)$ will denote the set of all bounded real-valued functions in $\mathcal{B}(X)$. Let $\mathfrak{S}(X)$ be a linear subspace of $\mathcal{B}^r(X)$. A **mean** M on $\mathfrak{S}(X)$ is a real linear functional on $\mathfrak{S}(X)$ having the property that

$$(i) \quad \inf\{f(x) : x \in X\} \leq M(f) \leq \sup\{f(x) : x \in X\} \text{ for all } f \in \mathfrak{S}(X).$$

Note that if M is a real linear functional on $\mathfrak{S}(X)$ and if $\mathfrak{S}(X)$ contains the constant functions, then property (i) holds if and only if $M(f) \geq 0$ whenever $f \in \mathfrak{S}(X)$ with $f \geq 0$, and $M(1) = 1$.

Definition 3.4. A closed linear subspace X of $L_\infty(G)$ is said to be **topologically left introverted** if for any mean m on $L_\infty(G)$, $m_L(X) \subset X$.

Remark 3.1: By our definition of m_L , we can say that X is a topologically left introverted closed subspace of $L_\infty(G)$ if, for each $m \in L_\infty(G)^*$ and $f \in X$, the functional on $L_1(G)$ defined by

$$\varphi \rightarrow \langle \frac{1}{\Delta} \check{\varphi} * f, m \rangle \text{ is also an element in } X.$$

Definition 3.5. If X is a closed linear subspace of $L_\infty(G)$, then X is said to be **topologically left invariant** if $\varphi * X \subset X$ for each $\varphi \in L_1(G)$.

For the rest of this section we let X be a topologically left *introverted* and left *invariant* closed subspace of $L_\infty(G)$, and then study a result obtained by A. T.-M. Lau stating that there exists a closed subspace Y of $LUC(G)$ which is also topologically left introverted, and the Banach algebra Y^* with the Arens product is isometric and algebra isomorphic to the algebra of bounded linear operators from X into X commuting with convolution from the left.

Lemma 3.4. *Let (φ_α) be a bounded approximate identity (BAI) in $L_1(G)$.*

- (a) *Then $(\frac{1}{\Delta}\check{\varphi}_\alpha)$ is also a bounded approximate identity in $L_1(G)$.*
- (b) *If $f \in L_\infty(G)$, then $w^* - \lim_\alpha \varphi_\alpha * f = f = w^* - \lim_\alpha f * \check{\varphi}_\alpha$.*

Proof. (a) If $\varphi \in L_1(G)$ then $\psi := \frac{1}{\Delta}\check{\varphi} \in L_1(G)$. Hence, using Lemma 3.1 (d),

$$\begin{aligned} (\frac{1}{\Delta}\check{\varphi}_\alpha) * \varphi &= (\frac{1}{\Delta}\check{\varphi}_\alpha) * (\frac{1}{\Delta}\check{\psi}) \\ &= \frac{1}{\Delta}(\psi * \varphi_\alpha)^\sim \rightarrow \frac{1}{\Delta}\check{\psi} = \varphi \end{aligned}$$

in the norm topology of $L_1(G)$.

(b) Let (φ_α) be a bounded approximate identity in $L_1(G)$. By (a), so is $(\frac{1}{\Delta}\check{\varphi}_\alpha)$. In particular, $(\frac{1}{\Delta}\check{\varphi}_\alpha) * g \rightarrow g$ weakly in $L_1(G)$. Hence, if $f \in L_\infty(G)$,

$$\langle \varphi_\alpha * f, g \rangle = \langle f, (\frac{1}{\Delta}\check{\varphi}_\alpha) * g \rangle \rightarrow \langle f, g \rangle.$$

The equality is due to Lemma 3.1 (c). Consequently, $\varphi_\alpha * f \rightarrow f$ in the weak* topology of $L_\infty(G)$. Similarly, $f * \check{\varphi}_\alpha \rightarrow f$ in the weak* topology of $L_\infty(G)$. $\square$

Lemma 3.5. *Let X be a topologically left invariant and left introverted subspace of $L_\infty(G)$. Let $Y = L_1(G) * X$. Then the following hold.*

(a) *Y is a closed subspace of $LUC(G)$ which is also topologically left invariant and introverted.*

(b) *For each $m \in Y^*$ and $f \in X$ define the map $m_L : X \rightarrow X$ by*

$$\langle m_L(f), \varphi \rangle = \langle (\frac{1}{\Delta}\check{\varphi}) * f, m \rangle,$$

where $\varphi \in L_1(G)$. Then $\|m_L\| = \|m\|$.

(c) For $m, n \in Y^*$ we have $(m \odot n)_L = m_L \circ n_L$, where $\langle m \odot n, g \rangle = \langle m, n_L(g) \rangle$ for each $g \in Y$.

Proof. (a) It follows from [8], Theorem 20.16, and [9], Theorem 32.22 that Y is a closed subspace of $LUC(G)$. Clearly, Y is topologically left invariant. To see that Y is topologically left introverted, let $m \in L_\infty(G)^*$, $\psi \in L_1(G)$, and $f \in X$. Then

$$\langle (\frac{1}{\Delta} \check{\varphi}) * (\psi * f), m \rangle = \langle \psi * h, \varphi \rangle$$

where h is the functional on $L_1(G)$ defined by the map

$$h(\xi) = \langle (\frac{1}{\Delta} \check{\xi}) * f, m \rangle,$$

for each $\xi \in L_1(G)$. Since X is left introverted, $h \in X$. It follows that Y is also left introverted.

(b) We have $\|m_L\| \leq \|m\|$ by Lemma 3.3. To see the reverse inequality, let (φ_α) be an approximate identity of $L_1(G)$ with $\|\varphi_\alpha\| \leq 1$. Then $(\frac{1}{\Delta} \check{\varphi}_\alpha)$ is also an approximate identity of $L_1(G)$ by Lemma 3.4 (a). Let $f \in Y$. Then, by [9], (32.48) (a),

$$\|(\frac{1}{\Delta} \check{\varphi}_\alpha) * f - f\|_\infty \rightarrow 0 \quad \text{since } f \in LUC(G).$$

Hence,

$$\|m_L(f)\|_\infty \geq |\langle m_L(f), \varphi_\alpha \rangle| = |\langle m, (\frac{1}{\Delta} \check{\varphi}_\alpha) * f \rangle| \rightarrow |\langle m, f \rangle|.$$

Consequently, $\|m_L(f)\|_\infty \geq |\langle m, f \rangle|$. Since f is arbitrary, we have $\|m_L\| \geq \|m\|$.

(c) Let $f \in X$ and $\varphi \in L_1(G)$. Then

$$\begin{aligned} \langle (m \odot n)_L(f), \varphi \rangle &= \langle (\frac{1}{\Delta} \check{\varphi}) * f, m \odot n \rangle \\ &= \langle n_L((\frac{1}{\Delta} \check{\varphi}) * f), m \rangle \\ &= \langle (\frac{1}{\Delta} \check{\varphi}) * n_L(f), m \rangle \quad (\text{by Lemma 3.3 (b)}) \\ &= \langle m_L(n_L(f)), \varphi \rangle. \end{aligned}$$

□

Remark 3.2: It is easy to see that an analogous statement also holds for a topologically right invariant and right introverted subspace X of $L_\infty(G)$ with $Y = X * L_1(G)^\vee$, and $m_R : X \rightarrow X$ defined by

$$\langle m_R(f), \check{\varphi} \rangle = \langle f * \check{\varphi}, m \rangle.$$

If $X = L_\infty(G)$, then

$$L_1(G) * X = \text{LUC}(G) \text{ and } X * L_1(G)^\vee = \text{RUC}(G).$$

Let $f \in \text{LUC}(G)$, and $m \in \text{LUC}(G)^*$. Then analogously to the product we defined in Section 2.6, we define a function $m_l(f)$ on G by

$$\langle m_l(f), x \rangle = m(l_x f) \quad \text{for each } x \in G.$$

Then as known ([8], (19.23)), $m_l(f)$ is also a function in $\text{LUC}(G)$. Similarly, we define for each $f \in \text{RUC}(G)$ and each $m \in \text{RUC}(G)^*$ a function $m_r(f)$ in $\text{RUC}(G)$ by

$$m_r(f)(x) = m(r_x f) \quad \text{for each } x \in G.$$

The following Lemma relates m_L with m_l , and m_R with m_r , where m_L and m_R are as introduced in Definition 3.2.

Lemma 3.6. *If $f \in \text{LUC}(G)$, then $m_L(f) = m_l(f)$. Also, if $f \in \text{RUC}(G)$, then $m_R(f) = m_r(f)$.*

Proof. Let $f \in \text{LUC}(G)$. Let φ be a fixed element in $L_1(G)$ with compact support.

If $m = \delta_a$ for some $a \in G$, then

$$\begin{aligned} \langle m_L(f), \varphi \rangle &= ((\frac{1}{\Delta} \check{\varphi}) * f)(a) = \int_G (\frac{1}{\Delta} \check{\varphi})(x) f(x^{-1}a) dx \\ &= \int_G \varphi(x) m(l_x f) dx = \langle m_l(f), \varphi \rangle. \end{aligned}$$

If $m \in \text{LUC}(G)^*$, $m \geq 0$, and $\|m\| = 1$, then there exists a net m_α of convex combinations of point evaluations such that $m_\alpha(h)$ converges to $m(h)$ for each $h \in \text{LUC}(G)$. Then

$$\langle (m_\alpha)_l(f), \varphi \rangle = \langle (m_\alpha)_L(f), \varphi \rangle = \langle (\frac{1}{\Delta} \check{\varphi}) * f, m_\alpha \rangle \longrightarrow \langle (\frac{1}{\Delta} \check{\varphi}) * f, m \rangle = \langle m_L(f), \varphi \rangle.$$

On the other hand, if K is the support of φ , then the set $\{l_x f \mid x \in K\}$ is a norm compact subset of $\text{LUC}(G)$. It follows that the net $m_\alpha(l_x f)$ converges to $m(l_x f)$ uniformly on K . Hence

$$\langle (m_\alpha)_l(f), \varphi \rangle = \int_K \varphi(x) m_\alpha(l_x f) dx$$

converges to

$$\int_K \varphi(x) m(l_x f) dx = \langle m_l(f), \varphi \rangle.$$

Since every linear functional in $\text{LUC}(G)^*$ is a linear combination of positive ones with norm one, this proves the first part of the lemma.

The proof for functions in $\text{RUC}(G)$ is similar. □

3.3 The Main Theorems

If X is a topologically left invariant and left introverted closed subspace of $L_\infty(G)$, then Y^* , where $Y = L_1(G) * X$, becomes a Banach algebra with the Arens product $\odot$ as defined in Lemma 3.5. The following is the main result in [11]; see *ibid.*, Theorem 1.

Theorem 3.1. *Let X be a topologically left invariant and topologically left introverted closed subspace of $L_\infty(G)$. Let T be a bounded linear operator from X into X . Then the following statements are equivalent:*

- (a) *T commutes with convolution from the left;*
- (b) *there exists a bounded linear functional m on $L_1(G) * X$ such that $T = m_L$.*

Consequently, the algebra of bounded linear operators from X into X commuting with convolution from the left is isometrically isomorphic to the Banach algebra $(L_1(G) * X)^*$ with the Arens product.

Proof. For the part (a) $\Rightarrow$ (b), let (φ_α) be a bounded approximate identity in $L_1(G)$. Then, denoting by *iota* the canonical embedding of $L_1(G)$ into its second dual, we set $\varphi'_\alpha := \iota(\varphi_\alpha) \mid_{X^*} \in X^*$. Now, the net $(T^*(\varphi'_\alpha))$, restricted to $Y = L_1(G) * X$, is bounded in Y^* . Let m be a weak*-cluster point of $(T^*(\varphi'_\alpha))$. By passing to a subnet, if necessary, we may assume that $T^*(\varphi'_\alpha)$ converges to m in the weak* topology of Y^* . Let $f \in L_\infty(G)$. Then for each $\varphi \in L_1(G)$ we have:

$$\begin{aligned} \langle m_L(f), \varphi \rangle &= \lim_\alpha \langle (\frac{1}{\Delta} \check{\varphi}) * f, T^*(\varphi'_\alpha) \rangle \\ &= \lim_\alpha \langle T((\frac{1}{\Delta} \check{\varphi}) * f), \varphi_\alpha \rangle \\ &= \lim_\alpha \langle (\frac{1}{\Delta} \check{\varphi}) * T(f), \varphi_\alpha \rangle \\ &= \lim_\alpha \langle T(f), \varphi * \varphi_\alpha \rangle \quad (\text{by Lemma 3.1(c)}) \\ &= \langle T(f), \varphi \rangle \quad ((\varphi_\alpha) \text{ is a BAI}). \end{aligned}$$

The part (b) $\Rightarrow$ (a) follows from Lemma 3.3 (b).

The final assertion follows immediately from Lemma 3.5. □

Remark 3.3: The above result in case $X = L_\infty(G)$ is due to Curtis and Figà-Talamanca (see [2], Theorem 3.3); in this situation, note that $\text{LUC}(G) = L_1(G) * L_\infty(G)$ (cf. [9], (32.45) (b)).

The following Proposition is Lemma 2 in [11]; unfortunately, the proof given there has a gap: it is only valid under the additional assumption that the operator T is w^* - w^* -continuous. Nevertheless, as we shall show below through a different argument, the statement is correct.

Proposition 3.1. *Let T be a bounded linear operator from $L_\infty(G)$ into $L_\infty(G)$. Let $f \in L_\infty(G)$.*

(a) If $T(\varphi * f) = \varphi * T(f)$ for each $\varphi \in L_1(G)$, then $T(l_x f) = l_x T(f)$ for each $x \in G$.

(b) If $T(f * \check{\varphi}) = T(f) * \check{\varphi}$ for each $\varphi \in L_1(G)$, then $T(r_x f) = r_x T(f)$ for each $x \in G$.

Proof. Due to Theorem 3.1, there is $m \in \text{LUC}(G)^*$ such that $T = m_L$. Hence, it suffices to show that for all $f \in L_\infty(G)$, $g \in L_1(G)$ and $x \in G$ we have:

$$\langle m_L(l_x f), g \rangle = \langle l_x(m_L f), g \rangle.$$

Since by [8], Remark 20.11 (iii),

$$\varphi * (l_x h) = \Delta(x)(r_x \varphi) * h$$

for all $\varphi \in L_1(G)$ and $h \in L_\infty(G)$, we have:

$$\langle m_L(l_x f), g \rangle = \langle m, (\frac{1}{\Delta} \check{g}) * (l_x f) \rangle = \Delta(x) \langle m, (r_x(\frac{1}{\Delta} \check{g})) * f \rangle.$$

On the other hand, we obtain:

$$\langle l_x(m_L f), g \rangle = \langle m_L f, l_{x^{-1}} g \rangle = \langle m, (\frac{1}{\Delta} (l_{x^{-1}} g)) * f \rangle.$$

Thus, it only remains to show that

$$\Delta(x)(r_x(\frac{1}{\Delta} \check{g})) = (\frac{1}{\Delta} (l_{x^{-1}} g)),$$

which is seen by the following simple calculation: for every $t \in G$ we get that

$$\begin{aligned} \Delta(x)(r_x(\frac{1}{\Delta} \check{g}))(t) &= \Delta(x)\Delta(x^{-1}t^{-1})g(x^{-1}t^{-1}) \\ &= \Delta(t^{-1})(l_{x^{-1}}g)(t^{-1}) \\ &= (\frac{1}{\Delta}(l_{x^{-1}}g))(t). \end{aligned}$$

This finishes the proof.

Part (b) can be proved similarly. □

The following is Theorem 2 in [11]; we provide an alternative proof here.

Theorem 3.2. *Let G be any non-compact, locally compact group. Then there exists a bounded linear operator from $L_\infty(G)$ into $L_\infty(G)$ commuting with convolution from left which is not w^* - w^* -continuous.*

Proof. Denote by $\mathcal{B}_{L^1}(L_\infty)$ the algebra of all bounded linear operators on $L_\infty(G)$ commuting with convolution from the left. By Theorem 3.1, we have

$$\mathcal{B}_{L^1}(L_\infty) \cong \text{LUC}(G)^* \supseteq C_0(G)^\perp \neq (0)$$

since G is non-compact; here, $C_0(G)^\perp$ denotes the annihilator of $C_0(G)$ in $\text{LUC}(G)^*$. (In fact, compactness of G is equivalent to the equality $C_0(G) = \text{LUC}(G)$). Take $m \in C_0(G)^\perp$ such that $\langle m, 1 \rangle \neq 0$ (which is possible by the Hahn–Banach Theorem since $1 \notin C_0(G)$), then consider $m_L \in \mathcal{B}_{L^1}(L_\infty(G))$. Assume m_L was w^* - w^* -continuous. Take $K_\alpha \subseteq G$ compact such that $G = \bigcup_{\alpha \in I} K_\alpha$, then $h_\alpha := \chi_{K_\alpha}$ converges w^* to 1 in $L_\infty(G)$. This implies that $m_L(h_\alpha)$ converges w^* to $m_L(1)$. Note that $m_L(h_\alpha) = 0$ for all α since for an arbitrary $g \in L_1(G)$, we have:

$$\langle m_L(h_\alpha), g \rangle = \langle m, (\frac{1}{\Delta} \check{g}) * h_\alpha \rangle = 0;$$

the last equality follows from the fact that $(\frac{1}{\Delta} \check{g}) * h_\alpha \in C_0(G)$ (see [9], (32.44) (e)).

Now take $\varphi \in L_1(G)$ such that $\int_G (\frac{1}{\Delta} \check{\varphi})(s) ds \neq 0$. Then we obtain:

$$\begin{aligned} 0 = \langle m_L(h_\alpha), \varphi \rangle &\longrightarrow \langle m_L(1), \varphi \rangle \\ &= \langle m, (\frac{1}{\Delta} \check{\varphi}) * 1 \rangle \\ &= (\int_G \frac{1}{\Delta} \check{\varphi}(s) ds) \langle m, 1 \rangle \\ &\neq 0, \quad \text{a contradiction.} \end{aligned}$$

□

The following result is Theorem 3 in [11], where in the proof the statement of our Proposition 3.1 is used. As we have shown, the latter is indeed correct, hence applicable here. Nevertheless, we present a different, more elegant and elementary approach.

Theorem 3.3. *Let G be any locally compact group and let T be any bounded linear operator from $\text{LUC}(G)$ into $\text{LUC}(G)$. Then the following statements are equivalent:*

- (a) *T commutes with convolution from the left;*
- (b) *T commutes with left translations;*
- (c) *there exists a bounded linear functional m on $\text{LUC}(G)$ such that $T = m_L = m_l$.*

Hence, the algebra of bounded linear operators on $\text{LUC}(G)$ commuting with left translations is isometrically isomorphic to the algebra $\text{LUC}(G)^*$.

Proof. That (a) implies (b) is seen as follows. Let (φ_α) be a bounded approximate identity of $L_1(G)$. Then, by [9], (32.48) (a), we know that

$$\|\varphi_\alpha * h - h\|_\infty \rightarrow 0$$

for all $h \in \text{LUC}(G)$. Moreover, according to [8], Remark 20.11 (iii), we have

$$\varphi * (l_x g) = \Delta(x)(r_x \varphi) * g$$

for all $\varphi \in L_1(G)$ and $g \in L_\infty(G)$. Hence, for each $x \in G$ and $f \in \text{LUC}(G)$, we obtain (using only norm-limits):

$$\begin{aligned} T(l_x f) &= \lim_{\alpha} T(\varphi_\alpha * l_x f) = \lim_{\alpha} T(\Delta(x)(r_x \varphi_\alpha) * f) \\ &= \lim_{\alpha} \Delta(x)(r_x \varphi_\alpha) * T(f) = \lim_{\alpha} \varphi_\alpha * (l_x T(f)) \\ &= l_x T(f). \end{aligned}$$

This proves our claim.

If (b) holds, let δ_e denote the functional on $\text{LUC}(G)$ defined by $\delta_e(f) = f(e)$ for each $f \in \text{LUC}(G)$, where e is the identity of G . Let $m = T^*(\delta_e)$. Then for each $f \in \text{LUC}(G)$ we have

$$(m_l f)(x) = \langle l_x f, T^*(\delta_e) \rangle = \langle T(l_x f), \delta_e \rangle = \langle l_x T(f), \delta_e \rangle = T(f)(x).$$

Hence $m_l = T$. By Lemma 3.6, $T = m_L$. This proves (c).

That (c) implies (a) follows from Theorem 3.1.

The final assertion follows from Theorem 3.1, noting that $\text{LUC}(G) = L_1(G) * L_\infty(G) = L_1(G) * \text{LUC}(G)$ (see [9], (32.45) (a), (b)). $\square$

3.4 An Alternative Description of the Topological Center of $\text{LUC}(G)^*$

Proposition 3.2. *The topological center $Z_t(\text{LUC}(G)^*)$ of the Banach algebra $\text{LUC}(G)^*$ is the set of functionals $m \in \text{LUC}(G)^*$ such that:*

(i) *for all $f \in \text{LUC}(G)$ the function $m_r(f)$ defined by $(m_r(f))(x) := \langle m, r_x f \rangle$ ($x \in G$) is again in $\text{LUC}(G)$.*

(ii) *for all $n \in \text{LUC}(G)^*$ and for all $f \in \text{LUC}(G)$ we have: $\langle n, m_r(f) \rangle = \langle m, n_l(f) \rangle$.*

Proof. See [12], Lemma 2. $\square$

The following result characterizes the topological centre of $\text{LUC}(G)^*$ through an actual commutation relation which may be seen as another justification of the terminology "topological center".

Proposition 3.3. *Let $m \in \text{LUC}(G)^*$. We have $m \in Z_t(\text{LUC}(G)^*)$ if and only if*

(i) *for all $f \in \text{LUC}(G)$ the function $m_r(f)$ defined by $(m_r(f))(x) := \langle m, r_x f \rangle$ ($x \in G$) belongs to $\text{LUC}(G)$,*

(ii) for all $n \in \text{LUC}(G)^*$ we have: $m_r n_l = n_l m_r$.

Proof. If condition (i) and condition (ii) are satisfied then for all $m \in \text{LUC}(G)^*$ and $f \in \text{LUC}(G)$ we have:

$$\langle m, n_l f \rangle = (m_r n_l f)(e) = (n_l m_r f)(e) = \langle n, m_r f \rangle,$$

and by Proposition 3.2 we see that $m \in Z_t(\text{LUC}(G)^*)$.

Next, let $m \in Z_t(\text{LUC}(G)^*)$. Then, by Proposition 3.2, we have (i).

If $n \in Z_t(\text{LUC}(G)^*)$, $f \in \text{LUC}(G)$ and $x \in G$ then

$$\langle \delta_x \cdot n, f \rangle = (n \cdot f)(x) = (n_l f)(x) = \langle n, l_x f \rangle. \quad (3.2)$$

Also we have:

$$(\delta_x \cdot n)_l f = r_x(n_l f), \quad (3.3)$$

since for all $t \in G$ we obtain:

$$\begin{aligned} ((\delta_x \cdot n)_l f)(t) &= \langle \delta_x \cdot n, l_t f \rangle \\ &= \langle n, l_x(l_t f) \rangle \quad (\text{by equation (3.2)}) \\ &= \langle n, l_{tx} f \rangle \\ &= (n_l f)(tx) \\ &= (r_x(n_l f))(t). \end{aligned}$$

Now let us check condition (ii). For any $n \in Z_t(\text{LUC}(G)^*)$, $f \in \text{LUC}(G)$, $x \in G$ (noting that $m_r f \in \text{LUC}(G)$), we have:

$$\begin{aligned} (m_r n_l f)(x) &= \langle m, r_x(n_l f) \rangle \\ &= \langle m, (\delta_x \cdot n)_l f \rangle \quad (\text{by equation (3.3)}) \\ &= \langle \delta_x \cdot n, m_r f \rangle \quad (\text{since } m \in Z_t(\text{LUC}(G)^*) \text{ and by Proposition 3.2}) \\ &= \langle n, l_x(m_r f) \rangle \quad (\text{by equation (3.2)}) \\ &= (n_l m_r f)(x). \end{aligned}$$

This finishes the proof. □

We remark that the topological center of $\text{LUC}(G)^*$ can actually be described explicitly: it is a deep result due to A. T.-M. Lau ([12], Theorem 1) that $Z_t(\text{LUC}(G)^*)$ equals the measure algebra $M(G)$, for any locally compact group G (see also [17] for a different proof).

Chapter 4

Isometric Representation of

$\text{LUC}(G)^*$ on $\mathcal{B}(L_2(G))$

4.1 Introduction

The material in this chapter is taken from [21]. In this chapter we will construct an isometric representation of the Banach algebra $\text{LUC}(G)^*$, with the Arens product, as completely bounded $\mathcal{R}(G)$ -bimodule homomorphisms on $\mathcal{B}(L_2(G))$, for an arbitrary locally compact group G . This will be an extension of previous work by F. Ghahramani and E. Størmer who considered the case of the measure algebra $M(G)$; the corresponding image algebra turns out to be the normal part of the image of $\text{LUC}(G)^*$ under our representation.

4.1.1 Completely Bounded Operators

Definition 4.1. *Let E and F be operator spaces, i.e., closed subspaces of $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{K})$, respectively, for some Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$. Consider a bounded linear*

operator $\Phi : E \longrightarrow F$. Then we define the ***n*-th amplification**

$$\begin{aligned} \Phi^{(n)} : M_n(E) &\longrightarrow M_n(F) \\ [a_{ij}] &\longmapsto [\Phi(a_{ij})]. \end{aligned}$$

We say that Φ is **completely bounded (c.b.)** if

$$\|\Phi\|_{cb} := \sup_n \|\Phi^{(n)}\| < \infty.$$

$\|\cdot\|_{cb}$ is a norm which turns

$$\mathcal{CB}(E, F) := \{\Phi : E \longrightarrow F \mid \Phi \text{ c.b.}\}$$

into a Banach space. If $E = F$, then this is a Banach algebra. We will be concerned with the case $E = F = \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, and we will simply write $\mathcal{CB}(\mathcal{B}(\mathcal{H}))$ instead of $\mathcal{CB}(\mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{H}))$.

We will only need basic facts from the theory of completely bounded maps and operator spaces, which we will recall below. For a rich account of the theory, we refer the reader to [4] and [23].

4.1.2 The Construction

The GNS construction provides an isometric representation of a C*-algebra in $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. For duals or preduals of von Neumann algebras which are at the same time Banach algebras, there is no analogue of such a representation in the theory of operator algebras. Viewing these as operator spaces, they do of course have a completely isometric embedding into some $\mathcal{B}(\mathcal{H})$ – but this embedding does not, in general, respect the algebra structure.

Examples of Banach algebras that are preduals of von Neumann algebras or even duals of C*-algebras, naturally arise in abstract harmonic analysis. We just mention the group algebra $L_1(G)$ or the measure $M(G)$ for a locally compact group G . The

idea of a “GNS construction” for duals of C^* -algebras takes us again back to the theory of operator spaces. Roughly speaking, passing to the dual necessitates adding one level of non-commutativity for the representation space. Hence, instead of the von Neumann algebra $\mathcal{B}(\mathcal{H})$, one should look for a representation in the algebra $\mathcal{CB}(\mathcal{B}(\mathcal{H}))$ of completely bounded maps on $\mathcal{B}(\mathcal{H})$, which is both an operator space and a Banach algebra.

In 1980, E. Størmer ([24], Proposition 4.6) gave the construction of an isometric representation of the algebra $M(G)$ as operators on the space $\mathcal{B}(L_2(G))$, for a locally compact abelian group G . F. Ghahramani ([6], Theorem 1) showed the same result for a general locally compact group G .

In this chapter, the left regular representation in $\mathcal{B}(L_2(G))$ is replaced by its non-commutative analogue, the “left conjugated” representation: instead of taking left translates of functions in $L_2(G)$, one takes the conjugation $L_{t^{-1}}TL_t$ of an operator $T \in \mathcal{B}(L_2(G))$ by left translation. If $T = M_f$ is just the multiplication operator by a function $f \in L_\infty(G)$, this corresponds to the usual commutative situation of shifting f .

We shall extend the representation theorem of Størmer and Ghahramani to the case of the Banach algebra $\text{LUC}(G)^*$, with the Arens product, for an arbitrary locally compact group G . We place ourselves in the framework of operator spaces, noting that the representation takes values in the set of completely bounded $\mathcal{R}(G)$ -bimodule homomorphisms on $\mathcal{B}(L_2(G))$, where $\mathcal{R}(G)$ denotes the von Neumann algebra generated by the right regular representation on $L_2(G)$.

Our image algebras $\tilde{\Gamma}_l(\text{LUC}(G)^*)$ and $\Gamma_l(M(G))$ provide natural w^* -closed subalgebras of the algebras $\mathcal{CB}_{\mathcal{R}(G)}(\mathcal{B}(\mathcal{H}))$ and $\mathcal{CB}_{\mathcal{R}(G)}^\sigma(\mathcal{B}(\mathcal{H}))$, respectively. Here, $\mathcal{CB}_{\mathcal{R}(G)}(\mathcal{B}(\mathcal{H}))$ denotes the algebra of completely bounded $\mathcal{R}(G)$ -bimodule maps on $\mathcal{B}(\mathcal{H})$, and $\mathcal{CB}_{\mathcal{R}(G)}^\sigma(\mathcal{B}(\mathcal{H}))$ denotes its normal (i.e., w^* - w^* -continuous) part.

The representation of $\text{LUC}(G)^*$ as specific completely bounded operators on $\mathcal{B}(\mathcal{H})$ adds to our understanding of the structure of the algebra $\text{LUC}(G)^*$, in particular its relation to the measure algebra $M(G)$. It is interesting to observe that multiplication in $\mathcal{CB}(\mathcal{B}(\mathcal{H}))$ is w^* - w^* -continuous only in the left variable exactly as is the case for the Arens product in $\text{LUC}(G)^*$. This is in striking contrast to the situation for von Neumann algebras, where the multiplicative structure does not show such an asymmetry. On the other hand, w^* - w^* -continuity in the right variable holds precisely for elements in $\mathcal{CB}^\sigma(\mathcal{B}(\mathcal{H}))$, the normal part, and it will become apparent from our structural analysis of the representations that the latter corresponds to the topological center of $\text{LUC}(G)^*$. Our interest in the algebra $\mathcal{CB}(\mathcal{B}(\mathcal{H}))$ stems from the fact that it provides a common representation theoretical framework for algebras such as $M(G)$ and $\text{LUC}(G)^*$ – as we shall see below – and their dual objects, $M_{cb}A(G)$ (the completely bounded multipliers of the Fourier algebra $A(G)$) and $\text{UCB}(\widehat{G})^*$; for the latter algebras, we refer to [20].

We let L_t be the operator of left translation on $L_2(G)$, where $t \in G$, and, more generally, on $L_p(G)$, for $1 < p < \infty$. The operator of right translation on $L_p(G)$ will be denoted by R_t ($t \in G$). If $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is a C^* -algebra, we let $\mathcal{CB}_{\mathcal{A}}(\mathcal{B}(\mathcal{H}))$ be the space consisting of all completely bounded $\mathcal{A}$ -bimodule homomorphisms on $\mathcal{B}(\mathcal{H})$, and $\mathcal{CB}_{\mathcal{A}}^\sigma(\mathcal{B}(\mathcal{H}))$ denote the normal (i.e., w^* - w^* -continuous) ones.

Proposition 4.1. *On $\mathcal{CB}(\mathcal{B}(\mathcal{H}))$, consider the locally convex topology generated by the linear functionals $\Phi \mapsto \langle \Phi^n(T_n)\xi_n, \eta_n \rangle$, where $\Phi \in \mathcal{CB}(\mathcal{B}(\mathcal{H}))$, $T_n \in \mathcal{B}(\mathcal{H}^n)$, $\xi_n, \eta_n \in \mathcal{H}^n$, and $\Phi^{(n)}$ denotes the n -fold amplification. This provides the formula*

$$\|\Phi\|_{cb} = \sup\{|\langle \Phi^{(n)}(T_n)\xi_n, \eta_n \rangle| : n \in \mathbb{N}, T_n \in \text{Ball}(\mathcal{B}(\mathcal{H}^n)), \xi_n, \eta_n \in \text{Ball}(\mathcal{H}^n)\}$$

for the completely bounded norm. Furthermore, on $\text{Ball}(\mathcal{CB}(\mathcal{B}(\mathcal{H})))$, the w^* -topology is just the $\sigma(\mathcal{CB}(\mathcal{B}(\mathcal{H})), \mathcal{B}(\mathcal{H}) \otimes \mathcal{H} \otimes \mathcal{H})$ -topology.

Proof. See [21], Proposition 2.1. □

Remark 4.1. Noting that we have $\mathcal{CB}^\sigma(\mathcal{B}(\mathcal{H})) = \mathcal{CB}(\mathcal{K}(\mathcal{H}), \mathcal{B}(\mathcal{H}))$, the w^* -topology is given by the $\sigma(\mathcal{CB}(\mathcal{K}(\mathcal{H}), \mathcal{B}(\mathcal{H})), \mathcal{K}(\mathcal{H}) \otimes \mathcal{H} \otimes \mathcal{H})$ -topology.

Definition 4.2. A family of functions $(f_i)_{i \in I} \subseteq \text{LUC}(G)$ will be referred to as **equi left uniformly continuous** (in short: **equi-LUC**), if for every $\varepsilon > 0$ there is a neighborhood U of e , such that for all $i \in I$ and for all $x \in U$ one has $\|l_x f_i - f_i\|_\infty < \varepsilon$.

In a similar manner, using right translations, we define **equi right uniformly continuous** (in short: **equi-RUC**) families of functions.

Lemma 4.1. Let $\xi \in L_p(G), \eta \in L_q(G)$, where $\frac{1}{p} + \frac{1}{q} = 1$. Then the families of functions $(\gamma_l)_{T \in \text{Ball}(\mathcal{B}(L_p(G)))}$ and $(\gamma_r)_{T \in \text{Ball}(\mathcal{B}(L_q(G)))}$ are equi-LUC, where $\gamma_l(t) := \langle L_{t^{-1}} T L_t \xi, \eta \rangle$ and $\gamma_r(t) := \langle R_t T R_{t^{-1}} \xi, \eta \rangle$.

Proof. Considering γ_l , for $x \in G$ and $T \in \text{Ball}(\mathcal{B}(L_p(G)))$, we have

$$\begin{aligned} \|l_x \gamma_l - \gamma_l\|_\infty &= \sup_{t \in G} |\langle T L_{xt} \xi, L_{xt} \eta \rangle - \langle T L_t \xi, L_t \eta \rangle| \\ &= \sup_{t \in G} |\langle T L_{xt} \xi, L_{xt} \eta \rangle - \langle T L_t \xi, L_{xt} \eta \rangle + \langle T L_t \xi, L_{xt} \eta \rangle - \langle T L_t \xi, L_t \eta \rangle| \\ &\leq \sup_{t \in G} \|T L_{xt} \xi - T L_t \xi\|_p \|L_{xt} \eta\|_q + \sup_{t \in G} \|T L_t \xi\|_p \|L_{xt} \eta - L_t \eta\|_q \\ &\leq \|T\| \|\eta\|_q \sup_{t \in G} \|L_{xt} \xi - L_t \xi\|_p + \|T\| \|\xi\|_p \sup_{t \in G} \|L_{xt} \eta - L_t \eta\|_q \\ &\leq \|\eta\|_q \|L_x \xi - \xi\|_p + \|\xi\|_p \|L_x \eta - \eta\|_q. \end{aligned}$$

The first claim now follows from the strong continuity of left translation; see [5], Proposition 2.41. For the proof concerning γ_r , we refer to [21], Lemma 2.4. □

Lemma 4.2. Let $T \in \mathcal{K}(L_p(G))$ be a compact operator, $\xi \in L_p(G), \eta \in L_q(G)$. Then the functions γ_l and γ_r , as defined in Lemma 4.1, are elements of $C_0(G)$.

Proof. We prove the left case only, for the proof of the right case we refer to [21], Lemma 2.5. Now, if $T = x \otimes y$ (where $x \in L_p(G), y \in L_q(G)$) is an operator of rank

1, we obtain:

$$\langle L_{t-1}TL_t\xi, \eta \rangle = \langle x, L_t\eta \rangle \langle y, L_t\xi \rangle \in C_0(G).$$

Now, approximating $T \in \mathcal{K}(L_p(G))$ by finite ranks operators, we see that γ_l is a $\|\cdot\|_\infty$ -limit of $C_0(G)$ -functions, which yields the result. $\square$

4.2 Representation of the Algebra $\text{LUC}(G)^*$

Theorem 4.1. *Consider the mapping*

$$\Gamma_l : M(G) \hookrightarrow \mathcal{CB}^\sigma(\mathcal{B}(\mathcal{H})), \quad \mu \mapsto \Gamma_l(\mu),$$

where for $T \in \mathcal{B}(\mathcal{H})$, $\xi, \eta \in \mathcal{H}$ we set: $\langle \Gamma_l(\mu)T\xi, \eta \rangle := \int_G \langle L_{t-1}TL_t\xi, \eta \rangle d\mu(t)$. This representation admits a w^* -continuous extension

$$\tilde{\Gamma}_l : \text{LUC}(G)^* \hookrightarrow \mathcal{CB}(\mathcal{B}(\mathcal{H})), \quad m \mapsto \tilde{\Gamma}_l(m),$$

where for $T \in \mathcal{B}(\mathcal{H})$, $\xi, \eta \in \mathcal{H}$ we set: $\langle \tilde{\Gamma}_l(m)T\xi, \eta \rangle := m_t(\langle L_{t-1}TL_t\xi, \eta \rangle)$. Then Γ_l as well as $\tilde{\Gamma}_l$ are linear isometric algebra homomorphisms. They are further w^* -homeomorphisms onto their (w^* -closed) image algebras. The latter consist of $\mathcal{R}(\mathcal{G})$ -bimodule homomorphisms. More precisely, $\tilde{\Gamma}_l(\text{LUC}(G)^*)$ and $\Gamma_l(M(G))$ are w^* -closed subalgebras of the algebras $\mathcal{CB}_{\mathcal{R}(\mathcal{G})}(\mathcal{B}(\mathcal{H}))$ and $\mathcal{CB}_{\mathcal{R}(\mathcal{G})}^\sigma(\mathcal{B}(\mathcal{H}))$, respectively.

Proof. We prove the above theorem in five steps. We prove

1. the complete boundedness of the mapping $\tilde{\Gamma}_l(m)$, $m \in \text{LUC}(G)^*$;
2. the normality of $\Gamma_l(\mu)$, for $\mu \in M(G)$;
3. the algebra homomorphism property for $\tilde{\Gamma}_l$;
4. the isometry $\|\tilde{\Gamma}_l(m)\|_{cb} = \|m\|$ for $m \in \text{LUC}(G)^*$;

5. the $\sigma(\text{LUC}(G)^*, \text{LUC}(G))$ - $\sigma(\mathcal{CB}(\mathcal{B}(\mathcal{H})), \mathcal{CB}(\mathcal{B}(\mathcal{H}))_*)$ -continuity of $\tilde{\Gamma}_l$ (similarly for Γ_l).

1.) By Lemma 4.1, $\tilde{\Gamma}_l$ is well-defined, taking values in $\mathcal{B}(\mathcal{B}(\mathcal{H}))$. We now apply Proposition 4.1 and obtain for arbitrary $m \in \text{LUC}(G)^*$:

$$\begin{aligned} \|\tilde{\Gamma}_l(m)\| &\leq \|\tilde{\Gamma}_l(m)\|_{cb} = \sup_{n \in \mathbb{N}, \|T_n\|=\|\xi_n\|=\|\eta_n\|=1} |\langle \tilde{\Gamma}_l(m)^{(n)}(T_n)\xi_n, \eta_n \rangle| \\ &= \sup |m_t(\langle T_n(L_t\xi_n), L_t\eta_n \rangle)| \\ &\leq \|m\| \sup |\langle T_n(L_t\xi_n), L_t\eta_n \rangle| \\ &\leq \|m\|(< \infty). \end{aligned}$$

2.) We have to prove that

$$\Gamma_l(\mu)(T_\alpha) \longrightarrow 0 \text{ (WOT)}$$

if $T_\alpha \in \text{Ball}(\mathcal{B}(\mathcal{H}))$ with $T_\alpha \longrightarrow 0$ (WOT). Let $\xi, \eta \in \mathcal{H}$. Then we have:

$$\langle \Gamma_l(\mu)(T_\alpha)\xi, \eta \rangle = \int_G \langle L_{t^{-1}}T_\alpha L_t \xi, \eta \rangle d\mu(t),$$

where by Lemma 4.1, $\langle L_{t^{-1}}T_\alpha L_t \xi, \eta \rangle$ is equi-LUC (and norm-bounded). It further converges pointwise to 0 by the assumption. We thus deduce that it converges to 0 uniformly on compact subsets of G . Now the regularity of μ allows us to conclude the proof.

3.) Let $m, n \in \text{LUC}(G)^*$, $T \in \mathcal{B}(\mathcal{H})$, $\xi, \eta \in \mathcal{H}$. We claim that

$$\langle \tilde{\Gamma}_l(m \cdot n)T\xi, \eta \rangle = \langle \tilde{\Gamma}_l(m)[\tilde{\Gamma}_l(n)T]\xi, \eta \rangle. \quad (4.1)$$

Using the definition of $\tilde{\Gamma}_l$ we get for the left-hand side of equation (4.1) the expression: $m(n_t \cdot \langle T(L_t\xi), L_t\eta \rangle)$, and for the right-hand side we obtain: $m_x(\langle (\tilde{\Gamma}_l(n)T)(L_t\xi), L_x\eta \rangle)$.

We thus have to prove the equation

$$(n_t \cdot \langle T(L_t\xi), L_t\eta \rangle)(x) = \langle (\tilde{\Gamma}_l(n)T)(L_x\xi), L_x\eta \rangle.$$

The left-hand side turns out to be:

$$n_t(l_x[\langle T(L_t\xi), L_t\eta \rangle]) = n_t(\langle T(L_{xt}\xi), L_{xt}\eta \rangle),$$

whereas the right-hand side is

$$n_t(\langle T[L_t(L_x\xi)], L_t(L_x\eta) \rangle) = n_t(\langle T(L_{xt}\xi), L_{xt}\eta \rangle).$$

This shows the desired equality.

4.) Let $(e_j)_{j \in J}$ be a bounded approximate identity (BAI) in $L_1(G)$; then, by [5], Proposition 2.42, we have:

$$\forall b \in \text{LUC}(G) : \|e_j * b - b\|_\infty \longrightarrow 0.$$

We write $M_{\check{b}}$, where $b \in \text{LUC}(G)$, for the corresponding multiplication operator in $\mathcal{B}(L_2(G))$. For $m \in \text{LUC}(G)^*$, $b \in \text{LUC}(G)$ we then obtain:

$$\begin{aligned} \langle \tilde{\Gamma}_l(m) (M_{\check{b}}) \sqrt{e_j}, \sqrt{e_j} \rangle &= m_t(\langle L_{t^{-1}} M_{\check{b}} L_t \sqrt{e_j}, \sqrt{e_j} \rangle) \quad (\text{by the definition of } \tilde{\Gamma}_l) \\ &= m_t(\langle M_{L_{t^{-1}}\check{b}} \sqrt{e_j}, \sqrt{e_j} \rangle) \\ &= \langle m_t, \int_G \check{b}(t^{-1}s) e_j(s) ds \rangle \\ &= \langle m_t, \int_G b(s^{-1}t) e_j(s) ds \rangle \\ &= \langle m, e_j * b \rangle \\ &\longrightarrow \langle m, b \rangle. \end{aligned}$$

Hence, for $m \in \text{LUC}(G)^*$, we have:

$$\begin{aligned} \|m\| &= \sup_{b \in \text{Ball}(\text{LUC}(G))} | \langle m, b \rangle | \\ &= \sup_j \lim | \langle \tilde{\Gamma}_l(m) (M_{\check{b}}) \sqrt{e_j}, \sqrt{e_j} \rangle | \\ &\leq \| \tilde{\Gamma}_l(m) \|. \end{aligned}$$

But we have already proved in part 1.) that

$$\| \tilde{\Gamma}_l(m) \| \leq \| \tilde{\Gamma}_l(m) \|_{cb} \leq \|m\|,$$

whence

$$\|\tilde{\Gamma}_l(m)\|_{cb} = \|\tilde{\Gamma}_l(m)\| = \|m\|.$$

5.) Let $(m_\alpha) \subseteq \text{Ball}(\text{LUC}(G)^*)$ be such that $m_\alpha \rightarrow 0$ (w^*). We claim that

$$\tilde{\Gamma}_l(m_\alpha) \rightarrow 0 \text{ in the } \sigma(\mathcal{CB}(\mathcal{B}(\mathcal{H})), \mathcal{CB}(\mathcal{B}(\mathcal{H}))_*)\text{-topology.}$$

Since $\|\tilde{\Gamma}_l(m_\alpha)\| \leq 1$, we have to show that $\lim_\alpha \langle \tilde{\Gamma}_l(m_\alpha), T \otimes \xi \otimes \eta \rangle \rightarrow 0$ for some fixed $T \in \mathcal{B}(\mathcal{H})$, $\xi, \eta \in \mathcal{H}$ (due to Proposition 4.1). But by the definition of $\tilde{\Gamma}_l$,

$$\lim_\alpha \langle \tilde{\Gamma}_l(m_\alpha), T \otimes \xi \otimes \eta \rangle = \lim_\alpha (m_\alpha)_t(\langle L_{t^{-1}} T L_t \xi, \eta \rangle) \rightarrow 0.$$

Analogously, by Remark 4.1 and Lemma 4.2, we conclude that Γ_l is w^* - w^* -continuous. Now, $\tilde{\Gamma}_l$ being a w^* - w^* -continuous isometry, $\text{Ball}(\text{ran}(\tilde{\Gamma}_l))$ is w^* -compact, hence $\text{ran}(\tilde{\Gamma}_l)$ is w^* -closed. We infer from the closed range theorem that $(\tilde{\Gamma}_l)^{-1}$ is w^* - w^* -continuous. A similar argument applies to Γ_l .

Finally, the bimodule homomorphism property with respect to $\mathcal{R}(\mathcal{G})$ shared by the operators in $\tilde{\Gamma}_l(\text{LUC}(G)^*)$ and $\tilde{\Gamma}_l(\text{M}(G))$ follows at once by the construction of the representations. This completes the proof. $\square$

Let us briefly note that we can in fact represent $\text{LUC}(G)^*$ isometrically as a subalgebra of $\mathcal{B}(\mathcal{B}(L_p(G)))$, for any $p \in (1, \infty)$.

Proposition 4.2. *Consider the mapping $\Phi : \text{LUC}(G)^* \hookrightarrow \mathcal{B}(\mathcal{B}(L_p(G)))$, $m \mapsto \Phi(m)$, where*

$$\langle \Phi(m) T \xi, \eta \rangle := m_t(\langle L_{t^{-1}} T L_t \xi, \eta \rangle),$$

for $T \in \mathcal{B}(L_p(G))$, $\xi \in L_p(G)$, $\eta \in L_q(G)$.

(Here, $\langle \cdot, \cdot \rangle$ denotes the duality of $L_p(G)$ and $L_q(G)$, where $\frac{1}{p} + \frac{1}{q} = 1$). Then Φ defines a representation which enjoys the same properties as the one constructed in Theorem 4.1.

Proof. Following the proof of Theorem 4.1, we remark that the family $\gamma_l(t) = \langle L_{t^{-1}} T L_t \xi, \eta \rangle$ is equi-LUC (cf. Lemma 4.1). That Φ is an isometry is proved by noting that for $m \in \text{LUC}(G)^*$, $b \in \text{LUC}(G)$ we have:

$$\langle \Phi(m)(M_b) \sqrt[4]{e_j}, \sqrt[4]{e_j} \rangle = \langle m, e_j * b \rangle,$$

where (e_j) is the standard BAI in $L_1(G)$. □

4.3 The Image Algebras

We recall the Tomiyama–Takesaki decomposition of

$$\mathcal{CB}(\mathcal{B}(\mathcal{H})) = \mathcal{CB}^\sigma(\mathcal{B}(\mathcal{H})) \oplus \mathcal{CB}^s(\mathcal{B}(\mathcal{H}))$$

into the normal (i.e., w^* - w^* -continuous) and the singular part – where the latter can be described as follows:

$$\mathcal{CB}^s(\mathcal{B}(\mathcal{H})) = \{\Phi \in \mathcal{CB}(\mathcal{B}(\mathcal{H})) : \Phi|_{\mathcal{K}(\mathcal{H})} \equiv 0\};$$

here, $\mathcal{K}(\mathcal{H})$ denotes the C^* -algebra of compact operators on $\mathcal{H}$.

There is an analogous decomposition in abstract harmonic analysis ([7], Lemma 1.1):

$$\text{LUC}(G)^* = \text{M}(G) \oplus_{l_1} C_0(G)^\perp,$$

where $C_0(G)^\perp$ denotes the annihilator of $C_0(G)$ in $\text{LUC}(G)^*$.

Proposition 4.3. $\Gamma_l(\text{M}(G)) = \tilde{\Gamma}_l(\text{LUC}(G)^*) \cap \mathcal{CB}^\sigma(\mathcal{B}(\mathcal{H}))$.

Proof. The inclusion “ $\subseteq$ ” follows from the proof of Theorem 4.1, Step 2. The reverse inclusion is proved as follows. – Fix $\Phi \in \tilde{\Gamma}_l(\text{LUC}(G)^*) \cap \mathcal{CB}^\sigma(\mathcal{B}(\mathcal{H}))$. We have $\Phi = \tilde{\Gamma}_l(m)$, for some $m \in \text{LUC}(G)^*$. Now decompose $m = \mu + n$, where $\mu \in \text{M}(G)$ and $n \in C_0(G)^\perp$. We will show that $n = 0$. Note that $\tilde{\Gamma}_l(n) = \tilde{\Gamma}_l(m) - \tilde{\Gamma}_l(\mu)$ is normal,

so we only have to show that $\tilde{\Gamma}_l(n)$ vanishes on $\mathcal{K}(\mathcal{H})$. But for $T \in \mathcal{K}(\mathcal{H})$, $\xi, \nu \in \mathcal{H}$ we have

$$\langle \tilde{\Gamma}_l(n)T\xi, \eta \rangle = n_t(\langle L_{t^{-1}}TL_t\xi, \eta \rangle) = 0,$$

since by Lemma 4.2, the function $\langle L_{t^{-1}}TL_t\xi, \eta \rangle$ is an element of $C_0(G)$. $\square$

4.4 LUC(G) as a Non-Commutative Coefficient Space

Proposition 4.4. *The following equation holds:*

$$\text{LUC}(G) = \{ \langle L_{t^{-1}}TL_t, \xi \otimes \eta \rangle : T \in \mathcal{B}(L_2(G)), \xi, \eta \in L_2(G) \}.$$

Proof. From Lemma 4.1, we obtain the inclusion “ $\supseteq$ ”, so we only have to show the reverse inclusion. By [9], (32.45) (a), (b), we have the following:

$$\text{LUC}(G) = L_1(G) * L_\infty = L_1(G) * \text{LUC}(G).$$

Let $f \in \text{LUC}(G)$. Then there exist functions $g \in L_1(G)$ and $b \in \text{LUC}(G)$ such that $f = g * b$. Furthermore, there are $g_1, g_2 \in L_2(G)$ with $g = g_1 \overline{g_2}$. Hence, we obtain for all $t \in G$:

$$\begin{aligned} \langle L_{t^{-1}}M_{\check{b}}L_t g_1, g_2 \rangle &= \langle M_{l_{t^{-1}}\check{b}} g_1, \overline{g_2} \rangle \\ &= \int_G \check{b}(t^{-1}s) g_1(s) g_2(s) ds \\ &= \int_G b(s^{-1}t) g(s) ds \\ &= (g * b)(t) \\ &= f(t). \end{aligned}$$

$\square$

Chapter 5

Conclusion and Results

As stated at the beginning of chapter 1, the main objectives of our research have been:

1. to present some basic facts about Banach algebras, as well as a discussion on Arens multiplication and topological centers;
2. to study the Banach algebra $\text{LUC}(G)^*$, and present an account of some of its structural properties;
3. to investigate the extension of Curtis and Figà-Talamanca's result to certain subspaces of $L_\infty(G)$, as obtained by A.T.-M. Lau;
4. to extend the representation theorem of the measure algebra for $M(G)$, obtained by Størmer and Ghahramani, to the case of the Banach algebra $\text{LUC}(G)^*$, with the Arens product for an arbitrary locally compact group G , where we considered the isometric representation of $\text{LUC}(G)^*$, in $\mathcal{B}(\mathcal{B}(L_p(G)))$ and $\mathcal{CB}(\mathcal{B}(\mathcal{H}))$.

The most important novelties in this thesis are with respect to items 2 and 3. In Chapter 3 and, particularly in section 3.3, we were successful in giving a systematic account of the structure of the Banach algebra $\text{LUC}(G)^*$. In addition, we also corrected a mistake in the literature by giving a correct proof of a result by A. T.-M. Lau [11] whose argument had a gap (cf. our Proposition 3.1). Finally, an

alternative description of the topological center of the algebra $\text{LUC}(G)^*$ through a commutation relation (cf. Section 3.4) was also presented.

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