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OPTIMAL MANAGEMENT OF RENEWABLE RESOURCES,
ESPECIALLY IN FISHERIES.

by

Shuen-Fong Cheng

A thesis submitted to the Faculty of
Graduate Studies in partial fulfilment
of the requirements for the degree of
Doctor of Philosophy

Carleton University
Department of Mathematics
Ottawa, Ontario

1979

The undersigned recommend to the Faculty
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ABSTRACT

This thesis studies the optimal management problems of renewable resources. The concept of optimal resource management used in this thesis is based on the notion of profit maximization, that is, maximization of the present value of net economic revenues. Our model is a deterministic dynamic model in which time is taken to be a discrete variable.

In our model, we consider that recruitment is dependent on stock. Under favourable conditions, fish (or natural animal) populations have a tendency to increase in numbers. It is, however, possible to remove some fish (or animals) and maintain a constant population size. We divide the population into groups by age or stage in the life cycle. Mathematically speaking, our problems are optimal control problems. By means of the discrete maximum principle, we solve the problems analytically. In the case of non-selective harvest, under certain conditions, it is shown that the optimal harvest sequence is convergent to a constant ("steady" harvesting). In the case of selective harvest, under certain conditions, it is shown that, after a finite adjustment period, the optimal harvest becomes a constant.

To my late Grandmother

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I wish to express my sincere thanks to Dr. Edward Hughes for suggesting the problem and for his invaluable guidance throughout the development of this thesis. I am also grateful to Carleton University for providing me with financial aid during my studies here for the past years.

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Introduction

In real life, we may face a lot of management problems of renewable resources.

In practical policy making there are many other considerations besides net present value. This represents only one of many possible goals. If, for example, there are fishermen whose very survival depends upon an overexploited fishery, or fishing companies holding large investments in boats and gear, would the increased revenues to be obtained from building up the fishery justify a scheme of temporary compensation?

Another question closely related to the above concerns a single company (or public agency) that owns complete property rights in a certain fishery. It can sell its rights to another company, or it can operate the fishery itself. If it sells, what is a fair price? If it operates the fishery itself, what management scheme will it adopt [6]?

All these questions can be treated by considering not the maximization of physical yield, but rather the maximization of economic benefits. Specifically, a rational management policy, whether private or public, will attempt to maximize the total "present value" of profits, present and future, from the resource in question [6].

This thesis studies the optimal management problems of natural renewable resources. Using a modification of Hughes' model [11] (where recruitment is assumed to be independent of stock)

we consider that recruitment is dependent on stock. In our model the parent stock of the k th generation (P_k) gives rise to a determined number of young (Y_k), which ultimately provide a certain number of "recruits" (R_k). Some of the population may be harvested; the remainder form the parent stock (P_{k+1}) of the next generation. The model we deal with is a deterministic one in which time is taken to be a discrete variable and the population is divided into groups by age or stage in the life cycle.

In chapter 1 we introduce the discrete maximum principle [24] which is the main tool for solving our optimization problems. Then we build up the general non-selective harvesting model.

In chapter 2 we deal with a fishery problem by means of a linear model. Using a matrix formulation to represent population dynamics, we apply the discrete maximum principle to the model successfully. And then, we analyse the optimal solution mathematically. We show that the linear model is unrealistic as a description of a natural stock, when it requires the population to grow without bound.

However, it seems not so realistic to assume that the birth and survival rates do not change with time (i.e., the Leslie population matrix L [15, 16] is constant). We find that solution and analysis are still possible when the population matrix changes with time. Suppose the population matrices appear periodically. We find that some cases of this periodic problem reduce to the ordinary non-periodic cases treated in §2.1, but some do not. We only solve a problem with a complete cycle of the form $L_1 L_0$. The general problem may be treated

similarly. Moreover, in chapter 2, we extend our results (obtained in the non-selective single species linear model) to a non-selective multi-species linear model.

In chapter 3 we consider a simple one-dimensional model in which the reproduction function is concave. It is shown that, after a finite adjustment period, the optimal control becomes a constant. This result is similar to Clark's result [7] where the fishing costs are ignored.

The linear model is unrealistic as a description of a natural stock if it requires the population to grow without bound. In chapter 4 we assume that the population matrix L changes (so that the average birth rate to an individual in every age class becomes less) when the population of the stock reaches a certain size. This is a non-linear problem. We have found it necessary to restrict our attention to problems satisfying certain assumptions. The general case is considerably more difficult and may not admit an analytic solution.

In chapters 2, 3 and 4 harvesting is non-selective and so all the results can be applied to fisheries management. However, in the last chapter, we consider an n -dimensional selective harvesting model. This model may be applied to the following harvesting problems: whales, seals [22], forestry [5, 8], etc. In this chapter, the main result (theorems 5.4 and 5.5) is actually an extension of the main result (theorem 3.4) of chapter 3. Also, we apply these results to the age structure problem of a single species.

Some numerical examples are given in chapters 2, 3 and 5.

Chapter 1

The discrete maximum principle and the non-selective harvesting model

§1.1 A discrete-time maximum principle.

The theory of optimal control has for the most part been a development of the past two decades. Its roots are two-fold; it developed in the West through the theory of dynamic programming, beginning with Bellman's principle of optimality [1]. Its development in the Soviet Union at about the same time was from a different theoretical approach, namely the maximum principle of Pontryagin et al. [24], which is basically an extension of the calculus of variations to a more general class of problems.

Despite the fact that its theoretical development has been recent, optimal control has found immediate application to a broad range of problems. This is a result of the extremely general nature of the optimal control problem.

Essentially an optimal control problem consists of:

- (a) a set of differential or difference equations that represent a system that is to be controlled,
- (b) a set of constraints on the variables of the system,
- (c) a set of boundary conditions on the variables,
- (d) and a cost functional, or performance index, which is to be maximized.

We define a discrete-time dynamical system as a system possessing the following:

(i) An ordered subset $\hat{T}$ of positive integers called the time set, i.e., $\hat{T} = \{k\} = \{0, 1, 2, \dots, N\}$.

(ii) A set of states $\{y\} = Y \subseteq R^n$ called the state space, where y is a n -dimensional state-vector.

(iii) A set of controllable inputs $\{s\} = S \subseteq R^r$ called the control space, where s is the r -dimensional control vector. The r components of s are the control variables.

(iv) A first-order vector difference equation which describes the evolution of the state of the system in time, i.e.,

$$(1.1) \quad y_{k+1} = f(y_k, s_k, k) \quad k = 0, 1, \dots, N-1$$

where y_k and s_k are the values of the state vector and control vector at time k , and f is a vector-valued function of dimension n .

The control vector s_k is subject to the constraints

$$(1.2) \quad s_k \in U \quad k = 0, 1, \dots, N.$$

Let Q be a hyperplane in R^n . We assume that $f(y, s, k)$ is continuously differentiable with respect to $y \in Q$ for $k = 0, 1, \dots, N$ and $s \in U$.

Suppose we have a discrete dynamical system described by the difference equation (1.1). In addition we have an initial state $y_0 = z$, and both z and the terminal time N are fixed. Finally we assume that we have a specified cost functional of the form

$$(1.3) \quad J_N = \sum_{k=0}^{N-1} \phi(y_k, s_k, k)$$

which we would like to maximize.

The optimal control problem is to determine the control sequence $\{s_k^*: k = 0, 1, \dots, N-1\}$ and the corresponding trajectory $\{y_k^*: k = 0, 1, \dots, N\}$ such that

$$(1.4) \quad \begin{aligned} y_0^* &= z \\ y_{k+1}^* &= f(y_k^*, s_k^*, k), \quad k = 0, 1, \dots, N-1 \\ s_k^* &\in U \end{aligned}$$

and such that the net revenue (1.3) has its maximum value at $\{s_k\} = \{s_k^*\}$ and $\{y_k\} = \{y_k^*\}$ among all $\{s_k\}$ and $\{y_k\}$ satisfying (1.4).

Theorem 1.1 (The discrete maximum principle). Let $\{y_k^*\}$ be the trajectory of the system described by (1.1) corresponding to the control sequence $\{s_k^*\}$, with $y_0^* = z$. Then if $\{s_k^*\}$ maximizes the net revenue (1.3), it is necessary that there exist a sequence of n -vectors $\{\lambda_k: k = 0, 1, \dots, N\}$ called the Lagrange-multiplier vectors, such that:

(1) The scalar function

$$(1.5) \quad H_k(y_k^*, s_k, \lambda_{k+1}, k) = \phi(y_k^*, s_k, k) + \lambda_{k+1}^T f(y_k^*, s_k, k)$$

called the Hamiltonian is maximized as a function of s_k at $s_k = s_k^*$ for all $k = 0, \dots, N-1$.

(2) The dynamics of y_k^*, λ_{k+1} are determined by the equations:

$$(1.6) \quad y_{k+1}^* = f(y_k^*, s_k^*, k)$$

$$(1.7) \quad \lambda_k = \left. \frac{\partial H_k}{\partial y_k} \right|_{y_k = y_k^*}, \quad y_k \notin Q$$

$$(1.8) \quad \lambda_{N+1} = 0, \quad y_0 \text{ given.}$$

For the proof of theorem 1.1 see [8, 24].

In the following we shall consider an infinite horizon optimization problem:

$$(1.9) \quad \max J = \sum_{k=0}^{\infty} \phi(y_k, s_k, k)$$

subject to (1.4). In order to solve problem (1.9), we first solve the (finite) N -horizon problem (1.3) and then let N tend to ∞ . That is, we shall use the solution for the N -horizon case to approach the solution for the ∞ -horizon case.

§1.2 The non-selective harvesting model [11].

We use a matrix formulation to represent population dynamics. In any given time period (or "year") the population is described by the number of individuals in each age-class. Thus if the lifespan (after recruitment) is n periods, the population is a (column) vector

$y = [y^0, y^1, \dots, y^{n-1}]^T$; the superscript indicates age at the most recent "birthday". We denote by y_k the population-vector in the k -th year. Let $f_0(y)$ indicate an amount of 0-year-old fish as a function of the previous population-vector y (assuming no "harvest"). Note that we are assuming $f_0(y)$ to be known exactly, rather than being given by a probability distribution. Then

$$(1.10) \quad y_{k+1}^0 = f_0(y_k)$$

Let A be a matrix of the form

$$(1.11) \quad A = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ p_0 & 0 & 0 & \dots & 0 & 0 \\ 0 & p_1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & p_{n-2} & 0 \end{bmatrix}$$

where p_i , $0 \leq i \leq n-2$, is the surviving fraction, in the current year, of last year's i -year-olds. Then in the absence of harvesting, the population dynamics are given by the difference equation

$$(1.12) \quad y_{k+1} = [f_0(y_k), 0, \dots, 0]^T + Ay_k$$

Now we consider harvesting: Let y_0 be an initial population-vector of the stock. Suppose a unit effort results in a total catch

of Dy_0 , where D is a constant between 0 and 1 (note this amounts to assuming that harvesting has an equal impact on all age-classes) then an effort t produces a catch of $(1 - (1 - D)^t)y_0$ from the initial population y_0 , and reduces the population $(1 - D)^t y_0$. The dynamics of fishing effort is an assumption and not a transparent conclusion. Other formulations may occur. We assume that the cost of an effort t is proportional to t , with constant of proportionality $c_0 > 0$. If t_k is the fishing effort applying to the stock in year k , we define

$$s_k = (1 - D)^{t_k}$$

We may call s_k the fishing control variable in year k . Clearly the effort t_k in year k produces a catch of $(1 - s_k)y_k$ from population y_k , and reduces the population to $s_k y_k$. It is easy to see

$$t_k = \frac{1}{\ln(1 - D)} \ln s_k$$

Let

$$c = - \frac{c_0}{\ln(1 - D)}$$

We may call c the cost of raising the proportional catch rate by one unit or simply the cost coefficient. Then, the fishing cost in year k is

$$c_0 t_k = -c \ln s_k$$

Finally we define a row-vector u whose components u_i are the average weight of an individual of age i , times the unit price obtainable for fish of age i . Thus u_i is the value of a single individual of age i , and the vector u describes both the growth of the fish and the change (if any) in its unit value that occurs with age.

If an effort t_k is applied to the stock in year k , then the net return on the year's harvest, starting with population y_k , is

$$(1.13) \quad (1 - s_k)uy_k + c \ln s_k$$

Discounting future revenues by a factor α ($0 < \alpha < 1$), we find the total present value of the resource to be

$$(1.14) \quad J = \sum_{k=0}^{\infty} \alpha^k [(1 - s_k)u \cdot y_k + c \ln s_k]$$

where y_0 is the initial population, and the subsequent populations are determined by

$$(1.15) \quad y_{k+1} = [f_0(s_k y_k), 0, \dots, 0]^T + s_k A y_k$$

Our problem, therefore, is to maximize J subject to (1.15), and the additional constraints $0 < s_k \leq 1$, which are required by the definition of s_k , i.e.,

$$(1.16) \quad \max J = \sum_{k=0}^{\infty} \alpha^k [(1 - s_k)uy_k + c \ln s_k]$$

subject to

$$(1.17) \quad y_{k+1} = [f_0(s_k y_k), 0, \dots, 0]^T + s_k A y_k$$

$$(1.18) \quad 0 < s_k \leq 1, y_0 \text{ given.}$$

Chapter 2

The non-selective linear model

§2.1 The single species linear model.

If $y_k = [y_k^0, y_k^1, \dots, y_k^{n-1}]^T$ is a population vector giving the population numbers in the various age-classes in the k th year and if f_i , $i = 0, 1, \dots, n-1$ represents the average birth rate to an individual in class i , then the population number in age class 0 in the $(k+1)$ th year is

$$(2.1) \quad y_{k+1}^0 = y_{k+1}^0 = \sum_{i=0}^{n-1} f_i y_k^i$$

Thus, in the absence of harvesting, the population dynamics are given by

$$y_{k+1} = L y_k$$

where L is the Leslie growth matrix [19, 16]

$$(2.2) \quad L = \begin{bmatrix} f_0 & f_1 & \dots & f_{n-2} & f_{n-1} \\ p_0 & 0 & \dots & 0 & 0 \\ 0 & p_1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & p_{n-2} & 0 \end{bmatrix}$$

Considering harvesting, our optimization problem becomes

$$(2.3) \quad \max J(\{s_k\}) = \sum_{k=0}^{\infty} \alpha^k [(1 - s_k) u y_k + c \ln s_k]$$

subject to

$$(2.4) \quad y_{k+1} = s_k L y_k, \quad 0 < s_k \leq 1, \quad y_0 \text{ given.}$$

In order to apply the discrete maximum principle, we let N be a positive integer and consider the problem:

$$(2.5) \quad \max J_N = \sum_{k=0}^N \alpha^k [(1 - s_k) u y_k + c \ln s_k]$$

subject to (2.4). Suppose that $\{s_k\}$ is an optimal sequence. According to the discrete maximum principle, there exists a sequence $\{\lambda_k\}$ of Lagrange multipliers such that

$$(2.6) \quad H_k = \alpha^k [(1 - s_k) u y_k + c \ln s_k] + \lambda_{k+1}^T s_k L y_k$$

is maximized as a function of s_k at $s_k = s_k^*$, $k = 0, 1, \dots, N$, and

$$(2.7) \quad \lambda_k^T = s_k \lambda_{k+1}^T L + \alpha^k (1 - s_k) u$$

$$(2.8) \quad -\alpha^k u y_k + \frac{\alpha^k c}{s_k} + \lambda_{k+1}^T L y_k \begin{cases} = 0 & \text{if } s_k \in (0, 1) \\ \geq 0 & \text{if } s_k = 1 \end{cases}$$

$$(2.9) \quad y_{k+1} = s_k L y_k$$

and

$$(2.10) \quad \lambda_{N+1} = 0, y_0 \text{ given.}$$

Multiplying (2.7) by y_k , we have

$$(2.11) \quad \lambda_k^T y_k = s_k \lambda_{k+1}^T L y_k + \alpha^k (1 - s_k) u y_k$$

(2.8) implies that

$$(2.12) \quad -\alpha^k s_k u y_k + \alpha^k c + s_k \lambda_{k+1}^T L y_k = 0, \text{ if } s_k < 1.$$

Combining (2.11) and (2.12), we have

$$(2.13) \quad \lambda_k^T y_k = \alpha^k u y_k - \alpha^k c$$

and similarly

$$(2.14) \quad \lambda_{k+1}^T y_{k+1} = \alpha^{k+1} u y_{k+1} - \alpha^{k+1} c$$

Multiplying (2.9) by λ_{k+1}^T ,

$$\lambda_{k+1}^T y_{k+1} = s_k \lambda_{k+1}^T L y_k$$

From (2.12),

$$(2.15) \quad s_k \lambda_{k+1}^T L y_k = \alpha^k s_k u y_k - \alpha^k c$$

and so

$$(2.16) \quad \lambda_{k+1}^T y_{k+1} = \alpha^k s_k u y_k - \alpha^k c$$

Thus (2.14), (2.16) and (2.9) imply

$$(2.17) \quad s_k (u y_k - \alpha u L y_k) = c - \alpha c$$

Hence, we have

$$(2.18) \quad s_k = (c - \alpha c)(u y_k - \alpha u L y_k)^{-1}$$

if $u y_k - \alpha u L y_k \neq 0$. We have, therefore, the following theorem.

Theorem 2.1. If $\{s_k\}$ is an optimal sequence which maximizes problem (2.5), then it is necessary that

$$(2.19) \quad s_k = (c - \alpha c)(u y_k - \alpha u L y_k)^{-1}$$

for all k such that $s_k \in (0,1)$, if $u y_k - \alpha u L y_k \neq 0$, $k = 0,1,\dots,N-1$.

Theorem 2.2. The sequence of optimal controls $\{s_k\}$ for problem (2.5) is given by

$$(2.20) \quad s_k = \begin{cases} 1 & \text{if } c - \alpha c \geq uy_k - \alpha uLy_k \\ (c - \alpha c)(uy_k - \alpha uLy_k)^{-1} & \text{if } c - \alpha c \leq uy_k - \alpha uLy_k \end{cases}$$

$k = 0, 1, \dots, N-1$ and $s_N = c(uy_N)^{-1}$.

Proof. The Hamiltonian $H_k(y_k, \lambda_{k+1}, s_k, k)$ is a concave function of s_k for fixed y_k and λ_{k+1} . If $c - \alpha c \geq uy_k - \alpha uLy_k > 0$, then the Hamiltonian will take its maximum value at some point $s_k = (c - \alpha c)(uy_k - \alpha uLy_k)^{-1} \geq 1$. Since H_k is a concave continuous function of s_k and $H_k \rightarrow -\infty$ as $s_k \rightarrow 0$, we see that H_k is an increasing function of $s_k \in (0, 1]$. However, the optimal control s_k has to lie in the interval $(0, 1]$, and so the best choice for the optimal control is $s_k = 1$ if $c - \alpha c \geq uy_k - \alpha uLy_k > 0$, for $k = 0, 1, \dots, N-1$.

If $uy_k - \alpha uLy_k \leq 0$, then equation (2.17) does not hold. This implies that there is no $s_k \in (0, 1)$ satisfying the necessary condition (2.7), (2.8) and (2.9). Since H_k is a continuous concave function of $s_k \in (0, 1]$ for fixed y_k and λ_{k+1} , and $\ln s$ tends to $-\infty$ as s tends to 0, we see that H_k is an increasing function of $s_k \in (0, 1]$. Hence H_k will take its maximum value at $s_k = 1$ in this case.

At last, if $uy_k - \alpha uLy_k > c - \alpha c$, then

$s_k = (c - \alpha c)(uy_k - \alpha uLy_k)^{-1} \in (0, 1]$ is optimal by theorem 2.1.

Note that verifying the necessary condition does not, by itself, constitute a sufficiency proof.

For each admissible control sequence $s = \{s_k^*\}$, $J_N(s) \rightarrow J(s)$, as $N \rightarrow \infty$. Denote $\hat{J}_N = \max_s J_N(s)$ and $\hat{J} = \max_s J(s)$. Then $\hat{J}_N = J_N(\{s_k\})$, where $\{s_k\}$ is given by (2.20). We show that $\hat{J} = J(\{s_k\})$ where each s_k is given by (2.20). Let $s^* = \{s_k^*\}$ be any admissible control sequence, then

$$J(s^*) = \lim_{N \rightarrow \infty} J_N(s^*) \leq \lim_{N \rightarrow \infty} \hat{J}_N$$

$$= \lim_{N \rightarrow \infty} J_N(\{s_k\}) = J(\{s_k\})$$

Hence $\hat{J} = J(\{s_k\})$ where $\{s_k\}$ is given by (2.20). Note that, if $|\theta| < \alpha^{-1}$ for all eigenvalues θ of L , the series (2.3) converges.

Theorem 2.3. The sequence of optimal controls $\{s_k\}$ for infinite horizon case (i.e., $N = \infty$), is given by (2.20).

In order to give some economic interpretation to our main results we shall follow Clark's discussion [7 and 8 pp 103-104]. The Hamiltonian expression in year k

$$H_k(y_k, s_k, \lambda_{k+1}, k) = \alpha^k [(1 - s_k) u y_k + c \ln s_k] \\ + \lambda_{k+1}^T s_k L y_k$$

is the sum of two value flows:

$$\alpha^k [(1 - s_k) u y_k + c \ln s_k]$$

is the flow of the net return on the k th year's harvest (discounted by α^k), and $s_k Ly_k$ is the flow of "investment" in capital

$$y_{k+1} = s_k Ly_k$$

to express this investment flow in the value terms, it must be multiplied by the vector λ_{k+1} of shadow price of capital. The interpretation of Lagrange-multiplier vectors as vectors of shadow price is well established in economic analysis [8, p. 252]. The term "shadow price" refers to the fact that the asset's value is not its direct sale value but the value imputed from its future productivity [8, p. 104]. We emphasize here that the "investment" being made implies building up the capital (fish) stock, i.e., not catching a fish this year is equivalent to "investing" for future years; whereas "disinvestment" means harvesting.

Let us call $\lambda_{k+1}^T y_{k+1}$ the value of the capital asset y_{k+1} (in year $k+1$). Then the Hamiltonian $H_k(y_k, s_k, \lambda_{k+1}, k)$ in year k represents the sum of the net return (discounted by α^k) on the year's harvest and the value of the capital asset $y_{k+1} = s_k Ly_k$. The discrete maximum principle then asserts that an optimal control s_k must maximize this sum H_k (with y_k and λ_{k+1} fixed).

We refer to $uy - c$ as the net stock-value of a population y . If an optimal control s_k ($0 < s_k \leq 1$) is applied to the stock in year k , by (2.14), we see that the value of the capital asset $y_{k+1} = s_k Ly_k$ in year $k+1$ is equal to the net stock-value of y_{k+1} discounted by α^{k+1} . For convenience, we refer to $\alpha(uy - c)$ as the discounted net

stock-value of y .

If the net stock-value of y_k in year k is non-positive, one should harvest no stock that year, because no profit can be made by harvesting. In fact, $uy_k - c \leq 0$ implies that

$$(1 - s_k)uy_k + c \ln s_k \leq [(1 - s_k) + \ln s_k]c \leq 0$$

for all $s_k \in (0, 1]$. Hence we assume that the net stock-value of y_k is positive, i.e., $uy_k - c > 0$.

First we consider the case

$$c - \alpha c > uy_k - \alpha uy_k$$

which is equivalent to

$$(uy_k - c)/(uy_k - c) > \frac{1}{\alpha}.$$

We refer to the ratio $(uy_k - c)/(uy_k - c)$ as the growth rate of the net stock-value corresponding to y_k . Then the above inequality implies that the growth rate of the net stock-value of y_k is greater than the rate of increase of money $\frac{1}{\alpha}$. This means that the net stock-value is growing more rapidly than the growth of money, and so we should harvest no stock (in year k), i.e., $s_k = 1$. On the other hand, suppose

$$c - \alpha c \leq uy_k - \alpha uy_k$$

which is equivalent to

$$(uLy_k - c)/(uy_k - c) \leq \frac{1}{\alpha}$$

Hence, the growth rate of the net stock-value corresponding to y_k is less than or equal to the rate of increase of money. Harvest the stock is suggested. Our optimal harvest policy will, in this case, reduce the population y_k to such a level $s_k y_k$ that the growth rate of the net stock-value (corresponding to the reduced population $s_k y_k$) is equal to the rate of increase of money, i.e.

$$(s_k uLy_k - c)/(s_k uy_k - c) = \frac{1}{\alpha}$$

which is exactly (2.19).

In the finite N-horizon case, as we do not make any investment to the stock in year N, we harvest in such a way that the net return in year N is maximized. Hence

$$\left. \frac{d}{ds} [(1-s)uy_N + c \ln s] \right|_{s=s_N} = 0$$

or

$$s_N = c(uy_N)^{-1}.$$

This gives us some economic interpretation for theorem 2.2 (the

finite N -horizon case). Similar interpretation can be made for theorem 2.3 (the infinite horizon case).

In the following we shall investigate the convergence of the sequences of the optimal controls and the optimal population vectors.

A population matrix L defined by (2.2) is primitive if some power L^k of L is positive (i.e., all entries > 0). A sufficient condition for a population matrix L to be primitive is that the survival rates p_i , $i = 0, 1, \dots, n-2$, the last birth rate f_{n-1} and birth rates f_i in the middle age groups are non-zero (i.e., two successive age groups, not including the first, have positive birth rates f_i) [10]. The Perron-Frobenius theorem [13] states that a primitive positive matrix like L has a unique positive eigenvalue ξ_1 such that:

(a) Corresponding to ξ_1 there is a positive eigenvector v_1 ,

(b) If ξ_i is another eigenvalue of L then
 $|\xi_i| < \xi_1$.

Transform L into the Jordan-Weierstrass canonical form by some matrix H [4, p. 88].

$$(2.21) \quad H^{-1}LH = \begin{bmatrix} M_1 & 0 & 0 & \dots \\ 0 & M_2 & 0 & \dots \\ 0 & 0 & M_3 & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix} \equiv \begin{bmatrix} M_1 & 0 \\ 0 & K \end{bmatrix}$$

Each M_i has one of the forms

$$[\xi_i], \quad \begin{bmatrix} \xi_i & 1 \\ 0 & \xi_i \end{bmatrix}, \quad \begin{bmatrix} \xi_i & 1 & 0 \\ 0 & \xi_i & 1 \\ 0 & 0 & \xi_i \end{bmatrix}, \text{ etc.}$$

where ξ_i is an eigenvalue of L . Since ξ_1 is simple, $M_1 = \xi_1$. From (2.21), we have

$$(2.22) \quad L = H \begin{bmatrix} \xi_1 & 0 \\ 0 & K \end{bmatrix} H^{-1}$$

and so,

$$\xi_1^{-1}L = H \begin{bmatrix} 1 & 0 \\ 0 & E \end{bmatrix} H^{-1}$$

and hence,

$$(2.23) \quad (\xi_1^{-1}L)^m = H \begin{bmatrix} 1 & 0 \\ 0 & E^m \end{bmatrix} H^{-1}$$

where $E = \xi_1^{-1}K$. By property (b), we see that all the eigenvalues of E are less than 1 in absolute value. Hence $E^m \rightarrow 0$, as $m \rightarrow \infty$ [13, theorem 9.2.1]. We have, from (2.23),

$$(\xi_1^{-1}L)^m \rightarrow H \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{bmatrix} H^{-1}, \text{ as } m \rightarrow \infty.$$

Since ξ_1 is simple, the eigencolumn belonging to ξ_1 is $v_1 = H_{\cdot 1}$, the first column of H ; the eigenrow belonging to ξ_1 is $w_1 = (H^{-1})_1$, the first row of H^{-1} [4, page 90]. This implies that

$$H \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{bmatrix} H^{-1} = H_{\cdot 1} (H^{-1})_1 = v_1 w_1$$

with $w_1 v_1 = 1$, and so

$$(\xi_1^{-1}L)^m \rightarrow v_1 w_1, \text{ as } m \rightarrow \infty.$$

We have, therefore, the following lemma.

Lemma 2.4. The sequence $(\xi_1^{-1}L)^m$ converges to $v_1 w_1$, where v_1 and w_1 are respectively eigencolumn and eigenrow of ξ_1 , normalized so that $w_1 v_1 = 1$.

Lemma 2.5. The sequence $y_k / \|y_k\|$ converges to a multiple of v_1 .

Lemma 2.5 is well-known in population models [10, p. 88].

Lemma 2.5 shows that, the iterates $y_k = s_{k-1} L y_{k-1}$,

$k = 1, 2, \dots$ have the property that their components in v_1 tend, as $k \rightarrow \infty$, to dominate all other components.

Let V be the 1-dimensional subspace spanned by v_1 . Then, for any starting value $y_0 \in V$, it is clear that all the iterates $y_k \in V$, $k = 0, 1, \dots$. If $y_0 \in V$, therefore, the problem is 1-dimensional and can be solved easily. Let $y_0 \in V$ and denote

$$(2.24) \quad y_k = x_k v_1 \quad k = 0, 1, \dots$$

Since $y_{k+1} = s_k L y_k = s_k x_k \xi_1 v_1$, we have

$$(2.25) \quad x_{k+1} = s_k x_k \xi_1 \quad k = 0, 1, \dots$$

Substituting (2.24) into (2.20), the optimal sequence, in this case, is

$$(2.26) \quad s_k = \begin{cases} 1 & \text{if } c - \alpha c \geq uv_1 x_k (1 - \alpha \xi_1) \\ \frac{c - \alpha c}{uv_1 (1 - \alpha \xi_1) x_k} & \text{if } c - \alpha c < uv_1 x_k (1 - \alpha \xi_1) \end{cases}$$

$$k = 0, 1, \dots$$

Case (I): $1 < \xi_1 < \alpha^{-1}$.

Theorem 2.6. $(1 < \xi_1 < \alpha^{-1})$. If the initial population $y_0 \in V$,

then the optimal sequence of controls is of the form

$\{1, 1, \dots, 1, s_I, \xi_1^{-1}, \xi_1^{-1}, \dots\}$ for some non-negative integer

I , where $s_I = (c - \alpha c)(uy_I - \alpha uLy_I)^{-1}$. Moreover, we have

$$y_k = \xi_1 (c - \alpha c) [uv_1 (1 - \alpha \xi_1)]^{k-I} v_1, \text{ for } k > I.$$

Proof. For convenience, we denote $a = uv_1 (1 - \alpha \xi_1)$. Using the notations defined by (4.24), we have

$$uy_k - \alpha uLy_k = ax_k, \quad k = 0, 1, \dots$$

Suppose $ax_0 \leq c - \alpha c$. Then the optimal control $s_0 = 1$, and so $y_1 = s_0 Ly_0 = \xi_1 y_0 > y_0$. This implies that $ax_1 > ax_0$. Let I be the first integer such that $ax_I > (c - \alpha c)$ and $ax_{I-1} \leq c - \alpha c$. Then we take $s_I = (c - \alpha c)a^{-1}x_I^{-1}$. We have, $y_{I+1} = s_I Ly_I = \xi_1 a^{-1}(c - \alpha c)v_1 = x_{I+1}v_1$ and so $ax_{I+1} = \xi_1(c - \alpha c) > c - \alpha c$. This implies that $s_{I+1} = \xi_1^{-1}$. By mathematical induction, we can show that $s_k = \xi_1^{-1}$ and $y_k = [(c - \alpha c)\xi_1^{-1}]v_1$, for all $k \geq I$.

On the other hand, if $ax_0 > c - \alpha c$, then we take $I = 0$. Repeating the above argument, we have the required result. This completes the proof.

Denote

$$(2.27) \quad y_* = \xi_1^{-1}(c - \alpha c)[uv_1(1 - \alpha\xi_1)]^{-1}v_1$$

Then

$$(2.28) \quad (c - \alpha c)[uy_* - \alpha uLy_*]^{-1} = \xi_1^{-1}$$

From theorem 2.6, we see that there exists an optimal equilibrium population $\xi_1^{-1}y_*$ in the eigenspace V . When the initial population vector $y_0 \in V$ differs from $\xi_1^{-1}y_*$, the optimal harvest policy is either to "invest" (when $y_0 \leq \xi_1^{-1}y_*$) or to "disinvest" (when $y_0 > \xi_1^{-1}y_*$), where investment implies building up the capital (fish) stock. It is easy to show that $y_0 < \xi_1^{-1}y_*$ implies that

$$(uLy_0 - c)/(uy_0 - c) > \frac{1}{\alpha}.$$

Hence, when $y_0 \leq \xi_1^{-1} y_*$, the net stock-value of the asset (i.e., the fish population) y_0 is growing at a rate greater than the growth rate $\frac{1}{\alpha}$ of money. Thus the asset y_0 should be retained and expanded. Conversely, when $y_0 > \xi_1^{-1} y_*$, the asset should be harvested to the optimal equilibrium level $\xi_1^{-1} y_*$. In this case, our optimal harvest policy consists of a most-rapid approach to $\xi_1^{-1} y_*$.

In the following we study the convergence of the optimal sequence $\{s_k, y_k\}$. Define

$$(2.29) \quad \phi(y) = \frac{c - \alpha c}{uy - \alpha uLy} Ly \quad y \neq 0$$

Then $\phi(ry) = \phi(y)$ for all $y \neq 0$ and $r > 0$, i.e., ϕ is constant along rays through the origin.

Lemma 2.7. $(1 < \xi_1 < \alpha^{-1})$. Let $\{y_k\}$ be an optimal sequence of population vectors. Then

$$\phi(y_k) \rightarrow y_*, \text{ as } k \rightarrow \infty$$

Proof. By lemma 2.5, we have

$$(2.30). \quad y_k / \|y_k\| \rightarrow rv_1, \text{ as } k \rightarrow \infty$$

for some $r > 0$. Then

$$\phi(y_k) = \phi(y_k / \|y_k\|) \rightarrow \phi(rv_1) = \phi(v_1) = y_*$$

as $k \rightarrow \infty$, by continuity of $\phi(y)$.

From lemma 2.7 it is easy to see that the optimal sequence $\{y_k\}$ is bounded. In fact, (2.30) implies that

$$u(y_k / \|y_k\|) - \alpha uL(y_k / \|y_k\|) + r(uv_1 - \alpha uLv_1) > 0$$

and so

$$uy_k - \alpha uLy_k > 0 \quad \text{for large } k.$$

If $0 < uy_k - \alpha uLy_k \leq \epsilon - \alpha\epsilon$, then the optimal control $s_k = 1$ by theorem 2.2, and so, by (2.4),

$$(2.31) \quad y_{k+1} = Ly_k \leq \phi(y_k) = (c - \alpha c)(uy_k - \alpha uLy_k)^{-1}Ly_k$$

On the other hand, if $uy_k - \alpha uLy_k > c - \alpha c$, then

$$(2.32) \quad y_{k+1} = (c - \alpha c)(uy_k - \alpha uLy_k)^{-1}Ly_k = \phi(y_k)$$

Thus, from (2.31) and (2.32), we have

$$(2.33) \quad y_{k+1} \leq \phi(y_k) \rightarrow y_*, \text{ as } k \rightarrow \infty.$$

Theorem 2.8 ($1 < \xi_1 < \alpha^{-1}$). Let $\{y_k\}$ be an optimal sequence of population vectors and $\{s_k\}$ the corresponding optimal control sequence for problem (2.3). Then

$$y_k \rightarrow y_* \text{ and } s_k \rightarrow \xi_1^{-1}, \text{ as } k \rightarrow \infty$$

Proof. By lemma 2.7, we have

$$\phi(y_k) \rightarrow y_*, \text{ as } k \rightarrow \infty.$$

We denote

$$z_k = \phi(y_k) - y_* \quad k = 0, 1, \dots$$

Then $z_k \rightarrow 0$ as $k \rightarrow \infty$. Since $(u - \alpha uL)(y)$ is a continuous function

of y , there is some positive integer I such that

$$(2.34) \quad |uz_k - \alpha LZ_k| < (\xi_1 - 1)(c - \alpha c), \quad k \geq I + 1$$

and

$$(2.35) \quad y_{I+1} = \phi(y_I)$$

i.e., $y_{I+1} = s_I Ly_I$ with $s_I = (c - \alpha c)(uy_I - \alpha Ly_I)^{-1} \in (0, 1]$. Then

$$\begin{aligned} uy_{I+1} - \alpha Ly_{I+1} &= uy_* - \alpha Ly_* + (uz_{I+1} - \alpha LZ_{I+1}) \\ &= \xi_1(c - \alpha c) + (uz_{I+1} - \alpha LZ_{I+1}) \\ &> c - \alpha c \end{aligned}$$

by (2.28) and (2.34). Hence, by theorem 2.2,

$$s_{I+1} = (c - \alpha c)(uy_{I+1} - \alpha Ly_{I+1})^{-1} < 1$$

and so

$$y_{I+2} = s_{I+1} Ly_{I+1} = \phi(y_{I+1}).$$

Using mathematical induction, we can show that

$$(2.36) \quad y_{k+1} = \phi(y_k), \quad k \geq 1$$

Thus, by lemma 2.7, we have

$$y_k \rightarrow y_*, \quad \text{as } k \rightarrow \infty.$$

Moreover, as $k \rightarrow \infty$,

$$\begin{aligned} s_k &= (c - \alpha c)(u y_k - \alpha u L y_k)^{-1} \\ &\rightarrow (c - \alpha c)(u y_* - \alpha u L y_*)^{-1} = \xi_1^{-1}. \end{aligned}$$

By theorem 2.8, the optimal sequence of controls $\{s_k\}$ defined by (4.20) is asymptotic to the reciprocal of the eigenvalue ξ_1 of the population matrix L , while the corresponding optimal sequence of populations $\{y_k\}$ is asymptotic to the constant population y_* . As the optimal harvest sequence is asymptotic to a constant, it is a "steady" harvest policy.

Case (II): $\xi_1 < 1$.

It is well known that if $\xi_1 < 1$, then $L^k \rightarrow 0$ as $k \rightarrow \infty$ (all entries tend to 0) [13, theorem 9.2.1.]. This implies that

$$L^k y_0 \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Since $s_k \in (0, 1]$, $k = 0, 1, \dots$, we have

$$||y_k|| = ||s_0 s_1 \dots s_{k-1} L^k y_0|| \leq ||L^k y_0||$$

and so, $||y_k|| \rightarrow 0$ as $k \rightarrow \infty$. Thus

$$y_k \rightarrow 0 \text{ as } k \rightarrow \infty$$

This case is of limited interest since it implies that species will die out even with no harvesting. Thus, such a species would never be observed in nature.

Case (III): $\xi_1 = 1$.

This particular case is also of limited interest, so we will only give a brief discussion. Let y_0 be an initial population vector. By lemma 2.4, we see that $L^k \rightarrow v_1 w_1$ as $k \rightarrow \infty$, since $\xi_1 = 1$. This implies that $||L^k y_0||$ is bounded and so is $||y_k||$. In the absence of harvesting, we have

$$y_k = L^k y_0 \rightarrow (w_1 y_0) v_1 \in V, \text{ as } k \rightarrow \infty.$$

Hence, it is sufficient to consider the space V . It is easy to obtain the following result.

Theorem 2.9 ($\xi_1 = 1$). If the initial population $y_0 \in V$, then the optimal sequence of controls is $\{s_0, 1, 1, \dots\}$ where $s_0 = c(uy)^{-1} \in (0, 1)$ or $s_0 = 1$.

Clearly, if the net stock value of $y_0 \in V$ is non-positive (i.e., $uy_0 - c \leq 0$), we harvest no stock as we cannot get profits from the fishery. We assume $uy_0 - c > 0$. Since $\xi_1 = 1$, we have

$$\frac{uLy_0 - c}{uy_0 - c} = 1 < \frac{1}{\alpha}$$

That is, the growth rate of the net stock-value of y_0 is less than the growth rate of money. Thus we should harvest the stock in such a way that the net return on the first year's harvest is maximized, i.e., $s_0 = c(uy_0)^{-1}$. In fact, the optimal control

$$s_0 = (c - \alpha c)(uy_0 - \alpha uLy_0)^{-1} = c(uy_0)^{-1}.$$

Since $Ly_0 = y_0$ (with $y_0 \in V$), actually, we may treat the stock as a non-renewable resource. The optimal harvest policy, of course, is the "extinction" policy: $\{c(uy_0)^{-1}, 1, 1, \dots\}$, provided that $uy_0 - c > 0$.

Case (IV) $\frac{1}{\alpha} > \alpha^{-1}$.

Let y_0 be any initial population. Suppose we take $s_k = (\alpha \xi_1)^{-1}$, $k \geq 0$. Then the population dynamics are given by

$$y_k = s_{k-1}Ly_{k-1} = \alpha^{-k}(\xi_1^{-1}L)^k y_0$$

$k = 1, 2, \dots$. Hence

$$\begin{aligned} (2.38) \quad J &= \sum_{k=0}^{\infty} \alpha^k [(1 - s_k)uy_k + c \ln s_k] \\ &= \sum_{k=0}^{\infty} [(1 - \alpha^{-1}\xi_1^{-1})(\xi_1^{-1}L)^k y_0 + \alpha^k c \ln (\alpha \xi_1)^{-1}] \end{aligned}$$

By lemma 2.4, we have

$$u(\xi_1^{-1}L)^k y_0 \rightarrow (uv_1)(w_1 y_0), \text{ as } k \rightarrow \infty$$

This implies that

$$(1 - \alpha^{-1}\xi_1^{-1})u(\xi_1^{-1}L)^k y_0 + \alpha^k c \ln(\alpha\xi_1)^{-1}$$

does not converge to zero. Hence the series (2.38) does not converge, i.e., $J = \infty$. In this case our linear model is unrealistic as a description of a natural stock, since it requires the population to grow without bound.

From another point of view, we apply theorem 2.3 to this case. For any initial population $y_0 \in V$, we have

$$uy_0 - \alpha Ly_0 = (1 - \alpha\xi_1)uy_0 < 0$$

since $\alpha\xi_1 > 1$. By theorem 2.3, we have $s_0 = 1$ (i.e., no harvesting). Then $y_0 = Ly_0 = \xi_1 y_0$ is in V and again $s_1 = 1$. It is easy to see that $y_k \in V$ and $s_k = 1$ for all k . This means that no harvest is made every year, and the stock will tend to infinity. Of course this is impossible in the real world. In chapter 4 we shall consider a modification of this model to make it more realistic.

Hirshleifer's book [12, p. 88] on Investment Theory has a discussion of the same situation as $\xi_1 > \alpha^{-1}$, in the context of a discussion of a tree-harvesting problem. If such a situation ($J = \infty$)

were to come into existence, it would necessarily set in motion forces that would adjust the rate of increase of money α^{-1} upward so as to make the attainable present value J of the stock finite again. The attainable present value J remains infinite so long as $\xi_1 > \alpha^{-1}$. The owner of the stock (with infinite wealth) could, at existing market prices, buy up all of society's (finite) current income flow. (And still have infinite wealth left!) His attempt to exchange, for consumptive purposes, enormous future flows for large fractions of society's current income would drive up the price of the latter relative to the former - raise the rate of increase of money α^{-1} . The process can only stop when α^{-1} rises high enough to make his wealth finite. Hence, α^{-1} is driven up until $\xi_1 = \alpha^{-1}$. Suppose y_0 is in V . Consider the finite N -horizon case. According to theorem 2.2, if $\xi_1 = \alpha^{-1}$ then

$$s_N = c(uy_N)^{-1} \text{ and } s_k = 1, k = 0, \dots, N-1$$

where $y_N = \xi_1^N y_0$. Then the corresponding present value is

$$J_N = [1 - \alpha^N c(uy_0)^{-1}] uy_0 + \alpha^N c \ln \frac{\alpha^N c}{uy_0}$$

Hence, if $\xi_1 = \alpha^{-1}$, we have

$$J_N \rightarrow uy_0, \text{ as } N \rightarrow \infty,$$

i.e., $J = uy_0$. Thus, if $\xi_1 = \alpha^{-1}$ then the maximum present value J

is finite. However, if $\xi_1 = \alpha^{-1}$ forever, the linear model still fails to be realistic. We note that, in the finite N -horizon case (with $\xi_1 = \alpha^{-1}$ and $y_0 \in V$), $y_N = \xi_1^N y_0$ since $s_k = 1$ for all $k = 0, 1, \dots, N-1$. As N is large enough, y_N will be large enough so that there is not enough room in the sea for such a big population. Thus modification of the model is still suggested.

In the following we shall consider some examples in which the data are of suitable units.

Example 2.1. We consider the fisheries problem with Williamson's matrix [10]

$$L = \begin{bmatrix} 0 & 9 & 12 \\ 1/3 & 0 & 0 \\ 0 & 1/2 & 0 \end{bmatrix}$$

which has the eigenvalue $\xi_1 = 2$. If $\alpha^{-1} < 2$, then our linear model fails to describe the stock. In fact, if we take the control $s_k = (\alpha \xi_1)^{-1} = (2\alpha)^{-1}$, $k = 0, 1, \dots$ then the present value $J = \infty$. However, α is not a parameter of the stock, but of ourselves. If α is small enough, for example $\alpha = 0.4$, we can get fish to market.

In order to check our results, we assume that $\alpha = 0.4$, $c = 0.5$, $u_0 = 1$, $u_1 = 1.1$ and $u_2 = 1.2$. Suppose the initial population vector $y_0 = [21, 7, 25]^T$. Then the optimal sequence of controls s_k converges to 0.5 while the optimal population vectors y_k converges to $y_* = [2.4324, 0.4054, 0.1013]^T$. The results, obtained on a Xerox Sigma 9 computer, are shown in table 1.

Example 2.2 [9]. We consider a problem of harvesting the blue whale, *Balaenoptera musculus*, using vital data from the 1930's before overfishing altered the vital rates. The sex ratio in catches was approximately 1♂:1♀, so that we shall consider the female population only. The time step chosen is two years, which is reasonable biologically in view of a 12 month generation period and 7 months of nursing during which a female cannot become pregnant, although it makes less sense for whaling fleets. The life expectancy is about 40 years, but the vital rates do not change much over age 12 so that all age groups over 12 were collapsed into one giving

the matrix

$$M = \begin{bmatrix} 0 & 0 & 0.19 & 0.44 & 0.50 & 0.50 & 0.45 \\ 0.87 & 0 & \text{---} & \text{---} & \text{---} & \text{---} & 0 \\ 0 & 0.87 & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} \\ & & 0.87 & \text{---} & \text{---} & \text{---} & \text{---} \\ & & & 0.87 & \text{---} & \text{---} & \text{---} \\ & & & & 0.87 & 0 & 0 \\ 0 & \text{---} & \text{---} & 0 & 0.87 & 0.87 \end{bmatrix}$$

M has dominant eigenvalue $\lambda_1 = 1.0986$ with stable age distribution [9]

$$z = [0.221, 0.175, 0.138, 0.11, 0.087, 0.069, 0.20]^T$$

We note that the difference between M and the Leslie growth matrix is that $M_{77} \neq 0$. However, it is easy to check that our results can be applied to M as to L (see §2.4). Suppose

$$(2.44) \quad \alpha = 0.9, c = 1, u = [1, 1.01, 1.02, 1.03, 1.04, 1.05, 1.06]$$

with an initial population vector $y_0 = z$. The optimal policy (see table 2) is that, $s_k = 1$, $k = 0, 1, 2, 3$ (this means that in the first 3 time periods (i.e., 6 years) there should be no harvesting) and, in the 4th time period, we start harvesting. The optimal control s_k

will tend to 0.8966 as k tends to ∞ . On the other hand, the optimal population vector y_k will tend to

$$y_* = [0.3410, 0.2660, 0.2075, 0.1619, 0.1263, 0.0985, 0.3495]^T.$$

Suppose we change $c = 1.5$ and keep α , u and y_0 the same. Then the optimal policy is given by table 3. In this case, in the first 6 time periods (i.e., 12 years) there is no harvesting (i.e., $s_k = 1$, $k = 0, 1, \dots, 6$) and then, in the 7th time period, harvesting starts. The optimal control s_k converges to 0.8966, while the optimal population vector y_k converges to

$$y_* = [0.5116, 0.3991, 0.3113, 0.2428, 0.1894, 0.1478, 0.5242]^T.$$

Compare the above two cases; we find that if the fishing cost is larger, the time we have to wait before we start harvesting is longer, because the catch must be larger to repay the cost.

Suppose $\alpha = 0.8$ and $c = 0.5$ and no change for u and y_0 . Then the optimal policy is given by table 4. We find that harvesting should be started in the first time period, and the first harvesting is not very heavy ($s_0 = 0.8894$). The optimal control s_k will tend to 0.8966 while the optimal population vector y_k tends to

$$y_* = [0.2214, 0.1727, 0.1347, 0.1051, 0.0820, 0.0639, 0.2269]^T.$$

In this case the discounted present value of the stock is 0.2625. In the following we calculate the static profit maximization of the stock. Since no future investment of the stock is considered, we simply maximize the net return of the first year's harvest:

$$\frac{d}{ds_0} [(1 - s_0)uy_0 + c \ln s_0] = 0$$

i.e.,

$$s_0 = c(uy_0)^{-1} = 0.4865$$

We find that the maximum static profit of the stock is 0.1675. The difference in the discounted present value of the stock (corresponding to our policy) versus the static profit maximization is 0.095.

§2.2 Population matrices L_k varying in time.

In the last section the population matrix L is constant.

But, it seems not so realistic to assume that the birth and survival rates do not change with time. However, in the linear model, if these rates are constant over a single time period, then solution and analysis are still possible. Let

$$(2.45) \quad L_k = \begin{bmatrix} f_{0k} & f_{1k} & \cdots & f_{n-2,k} & f_{n-1,k} \\ p_{0k} & 0 & \cdots & 0 & 0 \\ 0 & p_{1k} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & p_{n-2,k} & 0 \end{bmatrix}, \quad k \geq 0$$

be the population matrix of birth and survival rates during the k th time period. Then the population dynamics are given by

$$(2.46) \quad y_{k+1} = s_k L_k y_k, \quad k \geq 0$$

We have the following optimization problem:

$$(2.47) \quad \max_{x, J} = \sum_{k=0}^{\infty} \alpha^k [c_k - s_k] u y_k + c_{n,k} s_k$$

subject to (2.46) and the constraint $0 < s_k \leq 1$, y_0 given.

Theorem 2.10. The sequence of optimal controls $\{s_k\}$ for problem (2.47) is given by

$$(2.48) \quad s_k = \begin{cases} 1 & \text{if } c - \alpha c \geq u y_k - \alpha u L_k y_k \\ (c - \alpha c)(u y_k - \alpha u L_k y_k)^{-1} & \text{if } c - \alpha c \leq u y_k - \alpha u L_k y_k \end{cases}$$

The proof of theorem 2.10 is similar to the proof of theorem 2.3.

By lemma 2.5, if the population matrices L_k are constant in time, i.e., $L_k = L$ for all k and primitive, then $y_k/||y_k||$ converges. In general, if L_k changes with time it is not true that $y_k/||y_k||$ must converge. In the following we shall investigate the conditions under which $y_k/||y_k||$ converges. Golubitsky, Keeler and Rothschild [10] studied such population problems (without harvesting) by means of the projective metric. They present necessary and sufficient conditions for the convergence of the age structure $y_k/||y_k||$ (in a discrete time, one-sex model of population growth). The projective metric provides a way of defining the distance between positive vectors in R^n which has two important features: First, the distance between any two positive vectors depends only on the rays on which the vectors lie; and second, positive matrices act as contractions in this metric [10].

Suppose each population matrix L_k defined by (2.45) has the property that all the survival rates $p_{ik} > 0$, $i = 0, 1, \dots, n-2$,

the last birth rate $f_{n-1,k}$ and birth rates in the middle age groups are non-zero (i.e., two successive age groups, not including the first, have positive birth rates). Then L_k is primitive [10]. By the Perron-Frobenius theorems [13], L_k has a unique positive eigenvalue ξ_k , whose modulus is exceeded by no other eigenvalue of L_k , and a unique positive eigenvector v_k with $L_k v_k = \xi_k v_k$ and $\|v_k\| = 1$.

We assume

$$(2.49) \quad M \leq L_k \leq Q \quad \text{for all } k$$

where M and Q are fixed non-negative matrices and $M^\ell > 0$ for some positive integer ℓ . Under this assumption the survival rates, the middle and last birth rates of each L_k are bounded away from zero. For convenience, we denote

$$(2.50) \quad \tilde{y}_k = y_k / \|y_k\| \quad \text{for all } k.$$

Theorem 2.11. Suppose (2.49) holds. Then $\lim_{k \rightarrow \infty} \tilde{y}_k = v$ for some vector v if and only if $\lim_{k \rightarrow \infty} v_k = v$.

This theorem is proved by Golubitsky, Keeler and Rothschild [10, p. 90].

Suppose

$$(2.51) \quad \lim_{k \rightarrow \infty} L_k = L, \quad \text{where } L^\ell > 0.$$

Let ξ be the simple positive eigenvalue of L which dominates all other eigenvalues of L (in modulus), and v its normalized eigenvector.

Corollary 2.12. Suppose (2.51) holds. Then $\tilde{y}_k = y_k / \|y_k\|$ converges to v [10].

We denote

$$(2.52) \quad y_* = \xi(c - \alpha c)[uv(1 - \alpha\xi)]^{-1} v; \quad \xi \neq \alpha^{-1}.$$

Since this case is quite similar to the previous case (constant population matrix) we shall state the results without proof.

Theorem 2.13 ($1 < \xi < \alpha^{-1}$). Suppose (2.49) and (2.51) hold. Let $\{y_k\}$ be the optimal sequence of population vectors and $\{s_k\}$ its corresponding optimal sequence of controls for problem (2.47).

Then

$$y_k \rightarrow y_* \quad \text{and} \quad s_k \rightarrow \xi$$

as $k \rightarrow \infty$, where y_* is given by (2.52).

Theorem 2.14 ($\xi \leq 1$). Suppose (2.49) and (2.51) hold. Let $\{s_k\}$ be an optimal sequence of controls for problem (2.47). Then $\{s_k\}$ is of the form $\{s_0, s_1, \dots, s_I, 1, 1, \dots\}$ for some integer $I \geq 0$, where

$$s_i = \begin{cases} 1 & \text{if } uy_i - \alpha u L_i y_i \leq c - \alpha c \\ (c - \alpha c)(uy_i - \alpha u L_i y_i)^{-1} & \text{otherwise} \end{cases}$$

$i = 0, 1, \dots, I.$

Suppose $\xi > \alpha^{-1}$. In this case, our linear model is unrealistic as a description of a natural stock, since it requires the population to grow without bound.

§2.3 Cycles

Lefkovich [14] proposed the use of different matrices

$L_0, L_1, \dots, L_m$ to represent a seasonal (or any cyclic) variation in vital rates. We then have a complete cycle of the form,

$$L_m L_{m-1} \cdots L_1 L_0$$

where each L_k , $k = 0, 1, \dots, m$ is defined by (2.45). First we consider the case $m = 1$. Let

$$(2.53) \quad L = L_1 L_0$$

Then, for any initial population vector y_0 , the population dynamics of the stock are given by

$$\begin{aligned}
 y_1 &= s_0 L_0 y_0 \\
 (2.54) \quad y_{2k} &= \left(\prod_{i=0}^{2k-1} s_i \right) L^k y_0, \\
 y_{2k+1} &= \left(\prod_{i=0}^{2k} s_i \right) L_0 L^k y_0
 \end{aligned}$$

for all $k \geq 1$.

In the following we assume that L has a simple positive eigenvalue ξ whose modulus is exceeded by no other eigenvalue of L . Let v and w be the eigencolumn and eigenrow respectively of ξ normalized so that $wv = 1$. Let V be the space spanned by v . By lemma 2.4, we have

$$(2.55) \quad \lim_k (\xi^{-1} L)^k = ww$$

Lemma 2.15. The sequence $y_{2k}/||y_{2k}||$ converges to a multiple of v .

The proof of lemma 2.15 is similar to lemma 2.5.

In the following we consider the optimization problem (2.47) where the population matrix L_k has the property that

$$L_{2k} = L_0 \quad \text{and} \quad L_{2k+1} = L_1, \quad k \geq 0.$$

Obviously theorem 2.10 applies in this case. We shall briefly present our results and omit most of their proofs.

Case (A): $\xi > 1$.

We denote

$$(2.56) \quad x_* = (c - \alpha c)(uv - \alpha u L_0 v)^{-1} L_0 v$$

$$(2.57) \quad s_{1*} = (c - \alpha c)(u x_* - \alpha u L_1 x_*)^{-1}$$

$$(2.58) \quad x_{**} = s_{1*} L_1 x_*$$

$$(2.59) \quad s_{0*} = (c - \alpha c)(u x_{**} - \alpha u L_0 x_{**})^{-1}$$

$$(2.60) \quad z_* = L_1 x_*$$

$$(2.61) \quad s_* = (c - \alpha c)(u z_* - \alpha u L_0 z_*)^{-1}$$

$$(2.62) \quad t_* = (c - \alpha c)(u L_0 x_{**} - \alpha u L_1 L_0 x_{**})^{-1}$$

Case (A-1): $uv - \alpha u L_0 v > 0$.

In this case, we have $x_{**} \in V$. We note that $0 < s_{1*} < 1$ and $0 < uv - \alpha u L_0 v$ imply that $s_{0*} > 0$.

Case (A-1-1): $0 < s_{1*} < 1$ and $s_{0*} < 1$.

In this case, we have $\xi < \alpha^{-2}$.

Theorem 2.16. (A-1-1). Let $\{s_k, y_k\}$ be an optimal sequence for problem

(2.47). Then

$$y_{2k} \rightarrow x_{**},$$

$$y_{2k+1} \rightarrow x_*$$

and

$$s_{2k} \rightarrow s_{0*},$$

$$s_{2k+1} \rightarrow s_{1*}$$

as $k \rightarrow \infty$.

The proof of theorem 2.16 is similar to the proof of theorem 2.8.

Case (A-1-2): $0 < s_{1*} < 1$ and $s_{0*} \geq 1$.

In this case, we also have $\xi < \alpha^{-2}$.

Theorem 2.17 (A-1-2). Let $\{s_k, y_k\}$ be an optimal sequence. Then

$$y_{2k} \rightarrow x_{**}, \quad y_{2k+1} \rightarrow L_0 x_{**}$$

$$s_{2k+1} \rightarrow 1, \quad \text{as } k \rightarrow \infty$$

and $\{s_{2k}\}$ is of the form $\{s_0, s_2, \dots, s_{2K}, 1, 1, \dots\}$ for some non-negative integer K .

The proof of theorem 2.17 is similar to the proof of theorem 2.8.

This case, in fact, just reduces to the ordinary non-periodic case (I) in §2.1, since s_k (for large k) is 1 for alternative years.

Actually, we may assume that the optimal control sequence is of the form $\{1, s_1, 1, s_3, \dots\}$. Suppose we take a "year" to be two years. Then the population dynamics are

$$y, s_1 L_1 L_0 y = s_1 L y, s_3 L^2 y, \dots$$

Case (A-1-3): $s_{1*} \geq 1$ and $s_* < 1$.

Theorem 2.18 (A-1-3). Let $\{s_k, y_k\}$ be an optimal sequence. Then

$$y_{2k} \rightarrow z_*, \quad y_{2k+1} \rightarrow x_*$$

$$s_{2k} \rightarrow s_*, \quad \text{as } k \rightarrow \infty,$$

and

$$\{s_{2k+1}\} = \{s_1, s_3, \dots, s_{2K+1}, 1, 1, \dots\}$$

for some non-negative integer K .

The proof of theorem 2.18 is similar to the proof of theorem 2.8.

This case also reduces to the ordinary non-periodic case (I) in §2.1.

Case (A-1-4): $s_{*1} \geq 1$ and $s_* \geq 1$.

This case does not exist since it is not possible that $s_* \geq 1$.

We note that $s_* > 0$ since $uv - \alpha u L_0 v > 0$.

Case (A-1-5): $s_{*1} < 0$ and $s_* < 1$.

We have the same result as the case (A-1-3).

Case (A-1-6): $s_{*1} < 0$ and $s_* \geq 1$.

This case does not exist since $\xi > 1$.

Case (A-2): $uv - \alpha u L_0 v < 0$.

Case (A-2-1): $u L_0 v - \alpha u L_1 L_0 v > 0$.

We note that $x_{**} \geq 0$ although $uv - \alpha u L_0 v < 0$ (which implies $x_{**} < 0$) in this case.

Theorem 2.19 (A-2-1). Let $\{s_k, y_k\}$ be an optimal sequence. Then

$$y_{2k} \rightarrow x_{**}, \quad y_{2k+1} \rightarrow L_0 x_{**}$$

$$s_{2k+1} \rightarrow t_*, \quad \text{as } k \rightarrow \infty$$

and

$$\{s_{2k}\} = \{s_0, s_2, \dots, s_{2K}, 1, 1, \dots\}$$

for some non-negative integer K .

This case reduces to the ordinary non-periodic case treated in §2.1, since s_k is 1 for alternative years.

Case (A-2-2): $uL_0 - \alpha uL_1L_0v < 0$.

In this case we have $\xi \geq \alpha^{-2}$. Our model is unrealistic as a description of a natural stock since it requires the population to grow without bound.

Case (B): $\xi < 1$.

This case is of limited interest since it implies that species will die out even with no harvesting. Thus such a species would never be observed in nature.

Let W be the space spanned by L_0V . If the initial population y_0 is either in W or in V , then the successive populations y_k oscillate between the two spaces W and V . If the initial population y_0 is neither in W nor in V , then the optimal sequence of populations $\{y_k\}$ has the property that its subsequence $\{y_{2k}\}$ approaches asymptotically to a constant vector

in V while the other subsequence $\{y_{2k+1}\}$ tends to a constant vector in W . We find that some cases of this periodic problem reduce to the ordinary non-periodic cases treated in §2.1, but some do not.

The general case may be treated similarly. Of course, the situation will be much more complicated.

§2.4 The multi-species linear model.

In this section we consider the optimal economic exploitation of a multi-species fishery in which all the species are of economic importance. Suppose there are m different species in the stock. We assume that the fishery can be described by a set of population blocks of different species, and each population block can be divided into different age groups.

Let the lifespan (after recruitment) of the i -species fish be N_i periods, $i = 1, \dots, m$. We denote the population of species i by a column vector $x_i = [x_i^0, x_i^1, \dots, x_i^{N_i-1}]^T$, whose component x_i^j , $j = 0, 1, \dots, N_i-1$ represents the number of individuals of age j .

Suppose the population of each species is to grow according to the Leslie growth model [18]. Then, in the absence of harvesting, the population dynamics of species i , $i = 1, \dots, m$, are given by the difference equation

$$(2.63) \quad x_{i,k+1} = L_i x_{ik}$$

where x_{ik} is the population vector of species i in the k th year, and L_i is the corresponding Leslie growth matrix

$$(2.64) \quad L_i = \begin{bmatrix} f_{i0} & f_{i1} & \dots & f_{i,N_i-2} & f_{i,N_i-1} \\ p_{i0} & 0 & \dots & 0 & 0 \\ 0 & p_{i1} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & p_{i,N_i-2} & 0 \end{bmatrix}$$

In order to use matrix formulation to represent population dynamics of the whole stock, we define

$$(2.65) \quad y = [y^0, y^1, \dots, y^n]^T$$

$$= [x_1^0, \dots, x_1^{N_1-1}; x_2^0, \dots, x_2^{N_2-1}; \dots; x_m^0, \dots, x_m^{N_m-1}]^T$$

where $n = N_1 + N_2 + \dots + N_m$. Furthermore, we define a diagonal matrix L by

$$(2.66) \quad L = \begin{bmatrix} L_1 & & \\ & L_2 & 0 \\ 0 & & L_m \end{bmatrix}$$

where each L_i , $i = 1, \dots, m$ is defined by (2.64). Obviously L is a square matrix of order n . Then, in the absence of harvesting, the population dynamics of the whole stock are given by the difference equation

$$(2.67) \quad y_{k+1} = Ly_k$$

We note that this model has no species interactions and therefore is not general.

Suppose non-selective harvesting is applied to the stock. We assume that harvesting has an equal impact on all age classes of all species of the stock. As in §2.1, we have the following optimization problem:

$$(2.68) \quad \max J = \sum_{k=0}^{\infty} \alpha^k [1 - s_k] u y_k + c \sum_{k=0}^{\infty} \alpha^k y_k$$

subject to

$$(2.69) \quad y_{k+1} = s_k Ly_k, \quad y_0 \text{ given}$$

$$(2.70) \quad 0 < s_k \leq 1$$

where L is defined by (2.66).

It is easy to check that all results obtained in the single species linear model (see §2.1) hold for problem (2.68).

In the following we consider the case that species i and j may interact ecologically. This is an important and interesting case in practice. In this case, the growth matrix L is no longer block-diagonal.

For a single species linear model, Lefkovich [14] extended Leslie's formulation by allowing L to be replaced by an arbitrary non-negative square matrix Q . He assumed that the number of individuals (including possibly the progenitor) contributed by one member of group i to group j in one time interval is $Q_{ji} \geq 0$. Then $y_{k+1} = Qy_k$. If Q is irreducible, the same comments (i.e., (a) and (b) in §2.1) apply to Q as to L [9, p. 190].

This can be extended to our multi-species linear model. In the following we assume that L is a non-negative square matrix. Suppose the population dynamics of the stock are given by the difference equation

$$(2.71) \quad y_{k+1} = L y_k, \quad y_0 \text{ given}$$

Consider the problem of maximizing J defined by (2.68) subject to (2.71) and (2.70). It is easy to check that theorem 2.3 holds in this case.

If L is an irreducible matrix, then the growth matrix L has a positive eigenvalue ξ_1 such that (a) and (b) (see §2.1) hold [13, p. 282]. Hence, if L is irreducible, then all the results obtained in §2.1 will hold.

Chapter 3

The one-dimensional model

In this chapter we assume that the reproduction function $f(y)$ is continuously differentiable and satisfies the following conditions; see figure 1

- (a) $f(y)$ increases from $0 = f(0)$ to $m = f(q) = \max\{f(y) : y \geq 0\}$ for $0 \leq y \leq q$, and decreases for $q \leq y \leq f(q) = m$. The possibility $q = f(q) = m$ is admitted.
- (b) $f(y) \geq y$ for $0 \leq y \leq q$, and the graph of $f(y)$ on $0 \leq y \leq q$ is either strictly concave, or else "s-shaped", consisting of a strictly convex and a strictly concave segment.

figure 1.

Reproduction functions of this kind are of common occurrence in various resource management sciences [7]. The dynamics of an unmolested population regulated by such a reproduction curve is easily deduced:

from any population $y_0 \leq q$ the population $y_{k+1} = f(y_k)$ (without harvesting) will increase, and then either converge to, or oscillate about the value $\bar{y}$ determined by $f(\bar{y}) = \bar{y}$. The behaviour of a population from which a harvest is regularly removed, will depend on the sequence $\{s_k\}$ of year harvest controls. Hence, if we know the optimal sequence $\{s_k\}$, then we will know the whole picture of the stock.

The optimization problem we are dealing with is as follows:

$$(3.1) \quad \max J = \sum_{k=0}^{\infty} \alpha^k [(1 - s_k) u y_k + c \ln s_k]$$

subject to

$$(3.2) \quad y_{k+1} = f(s_k y_k) \text{ and } 0 < s_k \leq 1, y_0 \text{ given.}$$

This is actually a special case of Clark's problem (7.56) [8, p. 244].

We shall refer to Clark for arguments and results that substantially parallel his.

Let y_* be an optimal equilibrium population which remains in stock at the end of harvest. By means of formula (7.59) [8, p. 245], we have

$$(3.3) \quad u y_* - c + \frac{\alpha c - \alpha u f(y_*)}{f(y_*)} y_* f'(y_*) = 0$$

where $f'(y_*) = df/dy$ at $y = y_*$. We note that equation (3.3) can be obtained by applying the discrete maximum principle.

In the following we show that equation (3.3) has one and only one zero $y_* \in [0, q]$. We assume that

$$(3.4) \quad c \leq uq$$

i.e., the net stock-value of the largest sustainable stock q is non-negative. We first prove some useful lemmas.

Lemma 3.1. If $s = \{s_j\}$ is optimal, then $s_j y_j \leq q, j = 0, 1, \dots$

Proof: Suppose there is some $j = 0, 1, \dots$ such that $s_j y_j > q$.

Let k be the first integer such that $s_k y_k > q$. Since $f(y)$ is concave, there is an $s_k^* < s_k$ such that

$$(3.5) \quad s_k^* y_k < q \text{ and } f(s_k^* y_k) = f(s_k y_k)$$

Define a new policy $s' = \{s'_i\}$ as follows: $s'_i = s_i, i = 0, 1, \dots, k-1$.

To construct $s'_j, j \geq k$, we denote

$$(3.6) \quad y'_{j+1} = f(s'_j y'_j)$$

$$(3.7) \quad v(t, y) = (1 - t)uy + c \ln t$$

We define s'_k by

$$s'_k = \begin{cases} s_k^* & \text{if } s_k^* \geq c(uy_k)^{-1} \\ c(uy_k)^{-1} & \text{if } s_k^* < c(uy_k)^{-1} \end{cases}$$

By (3.5) we have

$$y_{k+1} \leq y'_{k+1} \quad \text{and} \quad v(s_k, y_k) < v(s'_k, y_k)$$

since the function $v(t, y_k)$ is a strictly decreasing function of t ($(cuy_k)^{-1} \leq t \leq 1$).

Now we construct s'_{k+1} . If $s_{k+1}y_{k+1} \leq q$, we define

$$s'_{k+1} = \begin{cases} s_{k+1} & \text{if } s_{k+1}y'_{k+1} \leq q \\ q/y'_{k+1} & \text{if } s_{k+1}y'_{k+1} > q \end{cases}$$

In this case, it is easy to see that

$$s'_{k+1}y'_{k+1} \geq s_{k+1}y_{k+1} \quad \text{and} \quad s_{k+1} \geq s'_{k+1}$$

and so

$$y_{k+2} \leq y'_{k+2} \quad \text{and} \quad v(s_{k+1}, y_{k+1}) \leq v(s'_{k+1}, y'_{k+1}).$$

On the other hand, if $s_{k+1}y_{k+1} > q$, then there exists an $s^*_{k+1} \leq s_{k+1}$ such that

$$(3.8) \quad s^*_{k+1}y_{k+1} < q \quad \text{and} \quad f(s^*_{k+1}y_{k+1}) = f(s_{k+1}y_{k+1})$$

we take, in this case,

$$s'_{k+1} = \begin{cases} c(uy'_{k+1})^{-1} & \text{if } s^*_{k+1} < c(uy'_{k+1})^{-1} \\ s^*_{k+1} & \text{if } c(uy'_{k+1})^{-1} \leq s^*_{k+1} \leq q/y'_{k+1} \\ q/y'_{k+1} & \text{if } s^*_{k+1} > q/y'_{k+1} \end{cases}$$

Then we have

$$s^*_{k+1}y_{k+1} \leq s'_{k+1}y'_{k+1} \leq q$$

and so, by (3.8),

$$y_{k+2} \leq y'_{k+2}$$

Moreover, if $s^*_{k+1} \geq c(uy'_{k+1})^{-1}$, we have

$$s'_{k+1} \leq s^*_{k+1} < s_{k+1}$$

and so

$$(3.9) \quad v(s_{k+1}, y_{k+1}) < v(s_{k+1}, y'_{k+1}) \leq v(s'_{k+1}, y'_{k+1})$$

It is easy to check that (3.9/) holds, provided $s_{k+1}^* < c(uy'_{k+1})^{-1}$.

Continuing in this way, we can construct a new policy

$s' = \{s'_i\}$ such that

$$v(s_i, y_i) \leq v(s'_{ii}, y'_{ii}), \quad i \neq k$$

and

$$v(s_k, y_k) < v(s'_{kk}, y'_{kk}).$$

We have, therefore,

$$J(s) \equiv \sum_{i=0}^{\infty} \alpha^i v(s_i, y_i) < \sum_{i=0}^{\infty} \alpha^i v(s'_{ii}, y'_{ii}) \equiv J(s')$$

which implies that $s = \{s_i\}$ is not optimal.

For simplicity, we assume that the initial population

$$y_0 \geq cu^{-1}.$$

Lemma 3.2. If $s = \{s_k\}$ is optimal, then $s_k \geq c(uy_k)^{-1}$, $k = 0, 1, \dots$.

Proof. The assertion is equivalent to

$$uy_k \geq \frac{c}{s_k} \quad \text{or} \quad \frac{c}{s_k} - uy_k \leq 0.$$

However,

$$\frac{c}{s_k} - uy_k = \frac{d}{ds} [(1-s)uy_k + c \ln s] \Big|_{s=s_k}$$

If $\frac{c}{s_k} - uy_k > 0$, then by making s_k larger we can increase both

$$(1-s)uy_k + c \ln s \quad (\text{i.e., this year's return})$$

and

$$\sum_{j=k+1}^{\infty} \alpha^j [(1-s_j)uy_j + c \ln s_j]$$

(i.e., the future harvests). Hence s_k cannot be optimal if

$$\frac{c}{s_k} - uy_k > 0.$$

From lemmas 3.1 and 3.2, we see that it is sufficient to consider those controls $s \in [a_k, b_k]$, where

$$a_k = c(uy_k)^{-1} \quad \text{and} \quad b_k = \min\{1, qy_k^{-1}\}.$$

We define

$$(3.10) \quad F_k(s) = uy_k - \frac{c}{s} + \frac{\alpha c - \alpha f(sy_k)}{f(sy_k)} y_k f'(sy_k)$$

for all $s \in [a_k, b_k]$, where $f'(sy_k) = df/d(sy_k)$. Then

$$sy_k f'(sy_k) < f(sy_k) \quad \text{for all } s \in [a_k, b_k]$$

and it is easy to show that

$$F'_k(s) > 0 \quad s \in [a_k, b_k].$$

Thus $F_k(s)$ is an increasing function of $s \in [a_k, b_k]$. Since $F_k(s) = 0$ has at most one zero, it is clear that $sF_k(s) = 0$ has at most one zero over $[a_k, b_k]$. Define

$$(3.11) \quad G(y) = uy - c + \frac{uc - \alpha u f(y)}{f(y)} y f'(y)$$

for all $y \in (0, m]$. Then $G(sy_k) = sF_k(s)$, and so $G(y)$ has at most one zero over $[cu^{-1}, q]$. Since $G(cu^{-1}) < 0$ and $G(q) > 0$, $G(y)$ has one and only one zero $y_* \in [cu^{-1}, q]$. This proves the following theorem.

Theorem 3.3. Equation (3.3) has one and only one zero $y_* \in [cu^{-1}, q]$, provided inequality (3.4) holds and $f(y)$ is concave.

Let y_0 be an initial population. In the absence of fishing, i.e., $s_k = 1$, we have $y_{k+1} = f(y_k)$, $k = 0, 1, \dots$. Since $f(y)$ is concave and increasing over $[0, q]$, there is some integer $k_1 \geq 0$ such that $y_{k_1} \geq y_*$.

Theorem 3.4. Suppose that $c \leq uq$. Let I be the first integer such that $y_I > y_*$ if $s_k = 1$, $k < I$. The optimal sequence of controls is $s_k = 1$, $k = 0, 1, \dots, I-1$, $s_I = y_*/y_I$ and $s_k = y_*/f(y_*)$, $k > I$.

For the proof of theorem 3.4 see Clark's book [8, pp. 244-247].

From theorem 3.4, we see that, for any resource governed by our model and possessing a sufficiently high reproduction potential, there exists an economically optimal equilibrium utilization policy; that is, a policy which, after a finite "adjustment" period, maintains a constant stock level $f(y_*)$, and an optimal sustained economic yield

$$[1 - y_*/f(y_*)]uf(y_*) + c \ln [y_*/f(y_*)].$$

Following Clark's method [8, p. 42], we may interpret equation (3.3) as follows: At the optimal population level y_* , the marginal change in current net return must match the marginal change in the discounted present value of future rent. (Similar discussion for high dimensional cases will be given in detail in chapter 5.)

When the initial population y_0 differs from y_* , according to theorem 3.4, the optimal harvest policy consists simply of reaching the optimal equilibrium population y_* at the earliest possible instant, and thereafter maintaining a constant population.

Suppose the reproduction function $f(y)$ is not concave; its graph is "s-shaped", consisting of two strictly convex segments (see figure 1). In this case lemmas 3.1 and 3.2 still hold. Moreover, $G(y)$ always has a zero $y \in [cu^{-1}, q]$, since $G(cu^{-1}) < 0$, $G(q) > 0$ and $G(y)$ is continuous. However, if $G(y)$ has more than one zero, we proceed as follows: Let $y_* \in [cu^{-1}, q]$ be a solution.

of equation (3.3), i.e., a zero of $G(y)$. Define a sequence of controls s_k with respect to y_* by theorem 3.4. For this sequence of controls s_k , there corresponds a value $J(\{s_k\})$ which is the present value of the stock for the policy $\{s_k\}$. Since there is more than one solution y_* of (3.3), there is more than one policy $\{s_k\}$ that can be determined by theorem 3.4, and so is $J(\{s_k\})$. Compare these values $J(\{s_k\})$ and take the $\{s_k\}$ which gives the maximum value $J(\{s_k\})$ to be our optimal sequence.

We note that there is no such function $f(y)$ such that $G(y) > 0$ for all $y > 0$, if $f(0) = 0$ and $f'(0) > 0$. In fact,

$$\begin{aligned} G(y) &= uy - c + \frac{\alpha c - \alpha u f(y)}{f(y)} y f'(y) \\ &= uy - c + \alpha c \cdot \frac{y}{f(y)} f'(y) - \alpha u y f'(y) \end{aligned}$$

and

$$y/f(y) \rightarrow 1/f'(0), \text{ as } y \rightarrow 0, \text{ and so}$$

$$G(y) \rightarrow (\alpha - 1)c < 0, \text{ as } y \rightarrow 0.$$

Thus, if y is sufficiently small, $G(y) < 0$.

We illustrate the theory by means of the following examples.

Example 3.1 (Concave). Consider the curves having the form

$$g(y) = ye^{r(1-y)}$$

where r is a positive parameter. This family of curves, called Ricker curves [21] is used in the management of Pacific salmon (*Oncorhynchus* species) populations. The dynamic behaviour of the stock-recruitment relation for the family of Ricker curves is studied by May [18] and Oster [19], who show that for $0 < r \leq 2$, the point $y = 1$ is a stable equilibrium. We represent the model by the recurrence relation

$$y_{k+1} = y_k e^{r(1-y_k)}$$

Suppose

$$(3.12) \quad r = 1.9, u = 4 \text{ dollars per fish}, \alpha = 0.9$$

$$c = 10^8 \text{ dollars}, y_0 = 0.7 \times 10^8 \text{ fish}$$

Using 10^8 as one unit, we consider the equation corresponding to (3.12),

$$h(x) = 4x - 1 + \frac{0.9 - 3.6 f(x)}{f(x)} x f'(x) = 0$$

where $f(x) = x e^{1.9(1-x)}$. Then we have $x = 0.4254$, and so

$y_* = 0.4254$ units (i.e., 0.4254×10^8 fish). We note that, in this case, $q = 0.5$ units and $cu^{-1} = 0.25$ units, and so $y_* \in [0.25, 0.5]$.

We have, $s_0 = y_*/y_0 = 0.6077$, $s_k = 0.3356$ and $y_k = f(y_*) = 1.2675 \times 10^8$ fish, $k = 1, 2, \dots$.

Suppose $c = 0.5 \times 10^8$ dollars, and r, u, α and y_0 in (3.12) remain the same. Then, as before, we have $y_* = 0.3884 \times 10^8$ fish.

Moreover, in this case, $s_0 = 0.5549$, $s_k = 0.3128$ and $y_k = f(y_*) = 1.2415 \times 10^8$ fish, $k = 1, 2, \dots$.

Suppose $c = 1.8 \times 10^8$ dollars and r, u, α and y_0 in (3.12) remain the same. Then $y_* = 0.4950 \times 10^8$ fish, $s_0 = 0.7071$, $s_k = 0.3831$ and $y_k = f(y_*) = 1.2921 \times 10^8$ fish, $k = 1, 2, \dots$.

By means of theorem 3.3, $y_* \in [cu^{-1}, q]$ exists uniquely in each case.

Chapter 4

A non-linear non-selective model

The linear model is unrealistic as a description of a natural stock provided the unique positive eigenvalue ξ_1 of the Leslie population matrix L is greater than α^{-1} , where α ($0 < \alpha < 1$) denotes the discount rate of money over one time interval (see case (IV) of chapter 2). In this case, the linear model requires the population to grow without bound. This is impossible in the real world. Based on this, we assume that the average birth rate to an individual in every age class becomes less when the population of the stock reaches a certain size.

Let $y = [y^0, y^1, \dots, y^n]^T$ be a population vector of the stock. We denote

$$(4.1) \quad x = \sum_{i=0}^n \beta_i y^i$$

where $\beta_i \geq 0$, $i = 0, 1, \dots, n$ are constants. Suppose the reproduction function $f_0(x)$ of the stock is defined by

$$(4.2) \quad f_0(x) = \begin{cases} \beta x & \text{if } x \leq q \\ \beta q + r(x - q) & \text{if } x \geq q \end{cases}$$

(where x is given by (4.1), and $\beta > 0$, $\beta > r$ and q are constants (see figure 2).

figure 2.

Then the population dynamics are given by the difference equation

$$(4.3) \quad y_{k+1} = [f_0(s_k x_k), 0, \dots, 0]^T + s_k A y_k$$

where $x_k = \sum_{i=0}^n \beta_i y_k^i$ and A is given by (1.11). Consider the optimization problem:

$$(4.4) \quad \max J = \sum_{k=0}^{\infty} \alpha^k [(1 - s_k) u y_k + c] s_k$$

subject to (4.3) and the additional constraint

$$(4.5) \quad 0 < s_k \leq 1, y_0 \text{ given.}$$

For convenience, we define

$$(4.6) \quad L_1 = \begin{bmatrix} \beta\beta_0 & \beta\beta_1 & \dots & \beta\beta_{n-1} & \beta\beta_n \\ p_0 & 0 & \dots & 0 & 0 \\ 0 & p_1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & p_{n-1} & 0 \end{bmatrix}$$

$$(4.7) \quad M = \begin{bmatrix} r\beta_0 & r\beta_1 & \dots & r\beta_{n-1} & r\beta_n \\ p_0 & 0 & \dots & 0 & 0 \\ 0 & p_1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & p_{n-1} & 0 \end{bmatrix}$$

We note that r is a constant which may be positive, negative or zero.

Let η be the largest positive eigenvalue of M . Similar to case (IV) of chapter 2, our model is unrealistic as a description of a natural stock if $\eta \geq \alpha^{-1}$. We assume $\eta < \alpha^{-1}$. Denote

$$(4.8) \quad e_i = (0, \dots, 0, q\beta_i^{-1}, 0, \dots, 0), \quad i = 0, 1, \dots, n$$

provided $\beta_i > 0$, and

$$(4.9) \quad \begin{aligned} Q &= \text{convex hull of } e_i, \quad i = 0, 1, \dots, n \\ \Delta &= \text{convex hull of } Q \text{ and the origin } (0, \dots, 0) \\ \delta &= \text{the interior of } \Delta \\ \chi &= \{y | y \geq 0 \text{ and } y \notin \Delta\}. \end{aligned}$$

Before we go further, let us take a chance to give a brief discussion of the rationale for adopting the type of reproduction function $f_0(x)$ defined by (4.2) (see figure 2).

In the case ($\xi_1 > \alpha^{-1}$) the linear model clearly fails to be realistic (see §2.1 case (IV)). In order that it is (hopefully)

not too serious a distortion of reality while remaining simple

enough for analysis, we reason as follows: The reproduction

function $f_0(x)$ (where $x = \sum_{i=0}^n \beta_i y^i$) is linear if the population y

is in Δ , and again linear (plus a constant) if y is in $\tilde{\Delta}$, but the

slope of $f_0(x)$ in $\tilde{\Delta}$ is less than that in Δ . This assumption is

reasonable since some environmental limitation must force the

birth rate to decline as the population reaches a certain

size. Moreover, if we choose the reproduction function $f_0(x)$

in this way, we can still use a matrix formulation to represent

the population dynamics of the stock.

Suppose the matrix L_1 has a unique positive simple eigenvalue ξ whose modulus is exceeded by no other eigenvalue of L_1 . Let v be the eigenvector corresponding to ξ and $v_1 = v/||v||$. Furthermore, let V be the space spanned by v . We assume

$$(4.10) \quad \xi > \alpha^{-1}.$$

Let

$$(4.11) \quad \{y_\star\} = Q \cap V$$

that is, $y_\star$ is the intersection point of Q and V . From (4.3), (4.6) and (4.7) we have

$$(4.12) \quad y_{k+1} = \begin{cases} s_k L_1 y_k & \text{if } s_k y_k \in \Delta \\ s_k M y_k + b & \text{if } s_k y_k \in \bar{\Delta} \end{cases}$$

where

$$(4.13) \quad b = [b_0, 0, \dots, 0]^T \equiv [(\beta - r)q, 0, \dots, 0]^T.$$

Define the Hamiltonian

$$\begin{aligned}
 (4.14) \quad H_k(y_k, \lambda_{k+1}, s_k, k) \\
 = \alpha^k [(1 - s_k)uy_k + c \ln s_k] + \lambda_{k+1}^T y_{k+1}
 \end{aligned}$$

where y_{k+1} is given by (4.12) and λ_{k+1} is the Lagrange multiplier.

In order to apply the discrete maximum principle to problem (4.4), we consider an N-period case. Then, $\partial H_k / \partial s_k = 0$ implies that

$$(4.15) \quad -\alpha^k uy_k + \frac{\alpha^k c}{s_k} + \lambda_{k+1}^T L_1 y_k = 0, \text{ if } s_k y_k \in \overset{\circ}{\Delta}$$

$$(4.16) \quad -\alpha^k uy_k + \frac{\alpha^k c}{s_k} + \lambda_{k+1}^T M y_k = 0, \text{ if } s_k y_k \in \tilde{\Delta}$$

$\partial H_k / \partial y_k = \lambda_k$ implies that

$$(4.17) \quad \lambda_k^T = \alpha^k (1 - s_k) u + s_k \lambda_{k+1}^T L_1, \text{ if } s_k y_k \in \overset{\circ}{\Delta}$$

$$(4.18) \quad \lambda_k^T = \alpha^k (1 - s_k) u + s_k \lambda_{k+1}^T M, \text{ if } s_k y_k \in \tilde{\Delta}$$

Moreover, $y_{k+1} = \partial H_k / \partial \lambda_{k+1}$ implies that

$$(4.19) \quad y_{k+1} = s_k L_1 y_k \quad \text{if } s_k y_k \in \overset{\circ}{\Delta}$$

$$(4.20) \quad y_{k+1} = s_k M y_k + b \quad \text{if } s_k y_k \in \tilde{\Delta}$$

Furthermore, we have

$$(4.21) \quad \lambda_{N+1} = 0, y_0 \text{ given.}$$

Lemma 4.1. Let $\{s_k\}$ be an optimal control sequence for problem

(4.4). If $s_k y_k \in \overset{0}{\Delta}$, then

$$(4.22) \quad s_k = \begin{cases} (c - \alpha c)(u y_k - \alpha u L_1 y_k)^{-1} & \text{if this is in } (0,1] \\ 1 & \text{if not.} \end{cases}$$

Proof. In order to determine s_k , we consider s_{k+1} . If $s_{k+1} y_{k+1} \in \overset{0}{\Delta}$, from (4.16) and (4.18), we have

$$\lambda_{k+1}^T y_{k+1} = \alpha^{k+1} u y_{k+1} - \alpha^{k+1} c$$

If $s_{k+1} y_{k+1} \in \overset{0}{\Delta}$, from (4.15) and (4.17), we have

$$\lambda_{k+1}^T y_{k+1} = \alpha^{k+1} u y_{k+1} - \alpha^{k+1} c$$

Thus, we have

$$(4.23) \quad \lambda_{k+1}^T y_{k+1} = \alpha^{k+1} u y_{k+1} - \alpha^{k+1} c$$

We note that (4.23) holds only when $s_k \in (0,1)$. Suppose $s_k y_k \in \overset{0}{\Delta}$.

From (4.19),

$$(4.24) \quad y_{k+1} = s_k L_1 y_k$$

By (4.15), we have

$$(4.25) \quad -\alpha^k s_k u y_k + \alpha^k c + \lambda_{k+1}^T (s_k L_1 y_k) = 0.$$

Therefore (4.23), (4.24) and (4.25) imply that

$$s_k = (c - \alpha c)(u y_k - \alpha u L_1 y_k)^{-1}$$

It follows from the concavity of the Hamiltonian as a function of s_k , which implies that if $\partial H_k / \partial s_k = 0$ for some $s \in (0,1]$, then this is the maximum, otherwise the maximum is at $s = 1$.

We define

$$(4.26) \quad g(t, y) = \begin{cases} t L_1 y & \text{if } t y \in \Delta \\ t M y + b & \text{if } t y \notin \Delta \end{cases}$$

for $t \in (0,1]$ and $y \geq 0$. We let

$$(4.27) \quad h_k(t) = \alpha^k [(1-t) u y_k + c] t + \alpha^{k+1} u g(t, y_k) - \alpha^{k+1} c$$

for all $t \in (0,1]$ and fixed y_k .

Let y_0 be an initial population vector. If $y_0 \in \Delta$, then the optimal control $s_0 \in (0,1]$ can be determined by

$$(4.28) \quad s_0 = \begin{cases} 1 & \text{if } c - \alpha c \geq u y_0 - \alpha u L_1 y_0 \\ (c - c)(u y_0 - \alpha u L_1 y_0)^{-1} & \text{if } c - \alpha c < u y_0 - \alpha u L_1 y_0 \end{cases}$$

The proof of this is similar to the proof of theorem 2.2 (chapter 2).

We note that (4.28) includes the case $u y_0 - \alpha u L_1 y_0 \leq 0$. Now we assume, from now on, that

$$(4.29) \quad \tilde{y}_0 \in \tilde{\Delta}.$$

For any $\tilde{y} \in \tilde{\Delta}$, we denote

$$(4.30) \quad \tilde{y} = \mu \tilde{y}_*$$

where $\mu > 1$ is a constant, and

$$(4.31) \quad \{\tilde{y}_*\} = Q \cap \{\tilde{t} \tilde{y}_k \mid t \geq 0\}$$

i.e., $\tilde{y}_*$ is the intersection point of Q and the ray along $\tilde{y}$.

Similarly, we denote

$$(4.32) \quad y_k = \mu_k y_{k*}, \quad \text{if } y_k \in \tilde{\Delta}$$

where $\mu_k > 1$ is a constant and

$$(4.33) \quad \{y_{k*}\} = Q \cap \{t y_k \mid t \geq 0\}$$

If $(c - \alpha c)(uy_0 - \alpha u L_1 y_0)^{-1} y_0 \in \Delta^0$, by lemma 4.1, the optimal control s_0 is given by

$$s_0 = (c - \alpha c)(uy_0 - \alpha u L_1 y_0)^{-1}.$$

If $(c - \alpha c)(uy_0 - \alpha u L_1 y_0)^{-1} y_0 \notin \Delta^0$, then $s_0 y_0 \notin \Delta^0$, i.e., $s_0 \notin (0, \mu_0^{-1})$ where μ_0 is determined by (4.32). This implies that

$$(4.34) \quad s_0 \in [\mu_0^{-1}, 1] \quad \text{if } (c - \alpha c)(uy_0 - \alpha u L_1 y_0)^{-1} y_0 \notin \Delta^0.$$

Lemma 4.2 ($\xi \geq \alpha^{-1}$). Let $\{s_k, y_k\}$ be an optimal sequence for problem (4.4). Then there is some $i = 0, 1, \dots$ such that $s_i y_i \notin \Delta^0$, i.e., $s_i \in [\mu_i^{-1}, 1]$ where μ_i is determined by (4.32).

Proof. By lemma 4.1,

$$(4.35) \quad s_k = (c - \alpha c)(uy_k - \alpha u L_1 y_k)^{-1}$$

provided that $(c - \alpha c)(uy_k - \alpha u L_1 y_k)^{-1} y_k \in \Delta^0$. If $s_k y_k \in \Delta$, then $y_{k+1} = s_k L_1 y_k$ by (4.12). Suppose $s_k y_k \in \Delta$ for all k . Then lemma 2.5 of chapter 2 applies:

$$y_k \|y_k\|^{-1} \rightarrow t v_1, \quad \text{for some } t > 0,$$

and $k \rightarrow \infty$. We note that $t = 1$. We have

$$u(y_k || y_k ||^{-1}) - \alpha u L_1(y_k || y_k ||^{-1}) + u v_1(1 - \alpha \xi) \leq 0$$

as $k \rightarrow \infty$, since $\xi \geq \alpha^{-1}$. This implies that

$$u y_k - \alpha u L_1 y_k \leq 0, \quad \text{for large } k.$$

Hence, s_k cannot be determined by (4.35). This implies that

$s_i y_i \notin \Delta$ for some i ; that is, $s_i y_i \in \lambda U Q$ or $s_i \in [\mu_i^{-1}, 1]$.

Let

$$(4.36) \quad M_1 = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ p_0 & 0 & 0 & \dots & 0 & 0 \\ 0 & p_1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & p_{n-1} & 0 \end{bmatrix}$$

We have found it necessary to restrict our attention to problems satisfying the following assumptions:

$$(4.37) \quad u y - \alpha u L_1 y < c - \alpha c \quad \text{for all } y \in Q$$

$$(4.38) \quad c < u y - \alpha u M_1 y \quad \text{for all } y \in Q$$

and

$$(4.39) \quad c < u y - \alpha u M_1 y \quad \text{for all } y \in Q.$$

The economic significance of the assumptions (4.37), (4.38) and (4.39) will be discussed later.

We note that (4.39) follows from (4.38) if $r \geq 0$, and (4.38) follows from (4.39) if $r \leq 0$. The general case is considerably more difficult and may not admit an analytic solution. Moreover, we note that $u\tilde{y} - \alpha u_L y$, with $y \in Q$, may be negative if ξ is large. The assumption (4.37) guarantees that there is no optimal control $s_k \in (0,1]$ such that $s_k y_k \in \Delta^0$. This implies that all the Hamiltonians $H_k(y_k, \lambda_{k+1}, t, k)$ are increasing functions of t over $(0, \mu_k^{-1})$, where μ_k is given by (4.32). In fact, assumption (4.37) implies that

$$u\tilde{y} = \alpha u_L \tilde{y} < \mu(c - \alpha c)$$

for all $\tilde{y} = \mu y \in \tilde{\Delta} \cup Q$, where $y \in Q$ and $\mu \geq 1$ is a constant. Hence, under assumption (4.37), if $s_k \in (0,1]$ is optimal then $s_k y_k \in \tilde{\Delta} \cup Q$, i.e., $s_k \in [\mu_k^{-1}, 1]$.

In the following we explain the assumption (4.38). It is easy to see that $h_k(t)$, defined by (4.27), is a continuous concave function of $t \in (0,1]$. Suppose $y_k \in \tilde{\Delta}$. Differentiating $h_k(t)$ with respect to $t \in (0,1] \setminus \{\mu_k^{-1}\}$ (since $h_k(t)$ is not differentiable at $t = \mu_k^{-1}$), we have

$$(4.40) \quad h'_k(t) = -\alpha^k u y_k + \frac{\alpha^k c}{t} + \alpha^{k+1} u g'(t, y_k)$$

i.e.,

$$(4.41) \quad h'_k(t) = \begin{cases} -\alpha^k u y_k + \frac{\alpha^k c}{t} + \alpha^{k+1} u L_1 y_k, & \text{if } t \in (0, \mu_k^{-1}) \\ -\alpha^k u y_k + \frac{\alpha^k c}{t} + \alpha^{k+1} u M y_k, & \text{if } t \in (\mu_k^{-1}, 1] \end{cases}$$

If $(c - \alpha c)(u y_k - \alpha u L_1 y_k)^{-1} y_k \notin \Delta^0$, then $h_k(t)$ cannot take its maximum value in the interval $(0, \mu_k^{-1})$. In fact, if $h'_k(\tilde{t}) = 0$ for some $\tilde{t} \in (0, \mu_k^{-1})$ then

$$\tilde{t} = c(u y_k - \alpha u L_1 y_k)^{-1}$$

which implies that $\tilde{t} \notin (0, \mu_k^{-1})$, i.e., $\tilde{t} y_k \notin \Delta^0$, since $(c - \alpha c)(u y_k - \alpha u L_1 y_k)^{-1} y_k \notin \Delta^0$. Now, since

$$h_k(t) \rightarrow -\infty, \quad \text{as } t \rightarrow 0$$

and $h_k(t)$ is continuous and concave, we see that $h_k(t)$ is an increasing function of $t \in (0, \mu_k^{-1})$. On the other hand, if $t \in (\mu_k^{-1}, 1]$, then

$$h'_k(t) = -\alpha^k u y_k + \frac{\alpha^k c}{t} + \alpha^{k+1} u M y_k$$

and then $h'_k(\tilde{t}) = 0$ implies that

$$(4.42) \quad \tilde{t} = c(u y_k - \alpha u M y_k)^{-1}.$$

Now assumption (4.38) would imply that $\tilde{t} < \mu_k^{-1}$, where $\tilde{t}$ is given

by (4.42). Hence, under the assumption (4.38), $h_k(t)$ is decreasing on $[\mu_k^{-1}, 1]$ provided $y_k \in \tilde{\lambda} \cup Q$.

For convenience, we denote

$$(4.43) \quad H_k(t) = H_k(y_k, \lambda_{k+1}, t, k), \quad k = 0, 1, \dots$$

where $H_k(y_k, \lambda_{k+1}, t, k)$ is defined by (4.14).

We first consider the case $M \geq 0$. Then assumptions (4.38) and $M \geq 0$ imply that

$$(4.44) \quad \alpha u M^2 y \leq u M y \quad \text{for all } y \geq 0.$$

Theorem 4.3 ($\xi > \alpha^{-1}$, $M \geq 0$). Let $\{s_k, y_k\}$ be an optimal sequence for problem (4.4). Then $s_k = \mu_k^{-1}$ for all k , provided (4.37) and (4.38) hold, where μ_k is given by (4.32).

Proof. We assume $y_0 \in \tilde{\lambda}$. By (4.37), we have $s_k y_k \in \tilde{\lambda} \cup Q$, i.e., $s_k \in [\mu_k^{-1}, 1]$ for all k . If $s_k y_k \in \tilde{\lambda}$, then the system of equations (4.16), (4.18), (4.20) and (4.21) apply to the N -period case of the problem. By (4.18) and (4.21), we have

$$\lambda_N^T = \alpha^N (1 - s_N) u$$

Hence

$$\lambda_N^T M y = \alpha^N (1 - s_N) u M y \quad \text{for all } y \geq 0.$$

Since $s_N \in (0, 1]$, we have

$$(4.45) \quad \lambda_N^T M y \leq \alpha^N u M y \quad \text{for all } y \geq 0.$$

By (4.18) again, we have

$$\lambda_{N-1}^T = \alpha^{N-1}(1 - s_{N-1})u + s_{N-1}\lambda_N^T M$$

Thus

$$(4.46) \quad \lambda_{N-1}^T M y = \alpha^{N-1}(1 - s_{N-1})u M y + s_{N-1}\lambda_N^T M^2 y$$

for all $y \geq 0$. Now, by (4.45) and (4.44),

$$\lambda_N^T M^2 y = \lambda_N^T M(My) \leq \alpha^N u M^2 y \leq \alpha^{N-1} u M y$$

Substituting the above inequality into (4.46), we have

$$\lambda_{N-1}^T M y \leq \alpha^{N-1} u M y \quad \text{for all } y \geq 0.$$

Using mathematical induction we can show that

$$(4.47) \quad \lambda_k^T M y \leq \alpha^k u M y \quad \text{for all } y \geq 0,$$

$k = N, N-1, \dots, 1$. In particular, we have

$$(4.48) \quad \lambda_{k+1}^T My_k \leq \alpha^{k+1} u My_k \quad k = N-1, \dots, 1, 0.$$

If $t \in (\mu_k^{-1}, 1)$, we have

$$(4.49) \quad dH_k(t)/dt = -\alpha^k u y_k + \frac{\alpha^k c}{t} + \lambda_{k+1}^T My_k$$

and

$$(4.50) \quad dh_k(t)/dt = -\alpha^k u y_k + \frac{\alpha^k c}{t} + \alpha^{k+1} u My_k$$

where μ_k is given by (4.32). By (4.48), we have

$$(4.51) \quad dH_k(t)/dt \leq dh_k(t)/dt \quad \text{for all } t \in (\mu_k^{-1}, 1).$$

By (4.38), it is easy to see that $c(u y_k - \alpha u My_k)^{-1} < \mu_k^{-1}$.

This implies that $h_k(t)$ is decreasing on $[\mu_k^{-1}, 1]$. Hence (4.51)

implies that $H_k(t)$ is decreasing on $[\mu_k^{-1}, 1]$. On the other hand,

$H_k(t)$ is increasing on $(0, \mu_k^{-1}]$ under the assumption (4.37). Hence

the Hamiltonian $H_k(t)$ takes its maximum at $s_k = \mu_k^{-1}$.

In theorem 4.3 we assume that $M \geq 0$, i.e., $r \geq 0$. Now we consider the other case:

$$(4.52) \quad r < 0:$$

We let

$$(4.53) \quad g_1(t,y) = \begin{cases} tL_1 y & \text{if } ty \in \Delta \\ tM_1 y + b_1 & \text{if } ty \in \lambda \end{cases}$$

for all $t \in (0,1]$ and $y \geq 0$, where b_1 is a constant vector so that $g_1(t,y)$ is a continuous function of t for fixed y . Clearly,

$$(4.54) \quad b_1 = [\beta q, 0, \dots, 0]^T \equiv [b_{10}, 0, \dots, 0]^T.$$

Hence in passing from g to g_1 , we simply replace r by 0. We have

$$(4.55) \quad g(t,y) = g_1(t,y) \quad \text{if } ty \in \Delta$$

$$(4.56) \quad g(t,y) \leq g_1(t,y) \quad \text{if } ty \in \lambda.$$

In fact, by (4.52) and (4.2), we have

$$f(x) = \beta q + r(x - q) < \beta q = b_{10}$$

where $x = \sum_{i=0}^n \beta_i y^i > q$; and hence $g(t,y) \leq g_1(t,y)$ if $ty \in \lambda$. Under assumption (4.39),

$$(4.57) \quad \alpha u M_1^2 y \leq u M_1 y \quad \text{for all } y \geq 0.$$

We consider the following problem:

$$(4.58) \quad \max \hat{J} = \sum_{k=0}^{\infty} \alpha^k [(1 - t_k) u \hat{y}_k + c \ln t_k]$$

subject to

$$(4.59) \quad \hat{y}_{k+1} = g_1(t_k, \hat{y}_k), \quad 0 < t_k \leq 1, \hat{y}_0 \text{ given}$$

Remark: Theorem 4.3 applies to problem (4.58) if (4.37) and (4.39) hold.

Theorem 4.4 ($\xi > \alpha^{-1} > \eta$). Let $\{s_k, y_k\}$ be an optimal sequence for problem (4.4). Then $s_k = \mu_k^{-1}$ for all k , where μ_k is given by (4.32), provided that (4.37), (4.38) and (4.39) hold.

Proof. If $r \geq 0$ (i.e., $M \geq 0$), (4.37) and (4.38) hold, this is theorem 4.3. We assume $r < 0$. For simplicity we assume $y_0 \in \tilde{\lambda}$. Let $s^* = \{s_k^*\}$ be an optimal sequence for problem (4.4). Consider s_0^* . By assumption (4.37), we have $s_0^* y_0 \in Q \cup \tilde{\lambda}$, i.e., $s_0^* \in [\mu_0^{-1}, 1]$, where $\mu_0 > 1$ is given by (4.32).

Suppose $s_0^* > \mu_0^{-1}$, i.e., $s_0^* y_0 \in \tilde{\lambda}$. We apply the control sequence s^* to problem (4.58) whose initial population vector $\hat{y}_0 = y_0$. Then, we have, in problem (4.58)

$$(4.60) \quad \hat{y}_{k+1} = g_1(s_k^*, \hat{y}_k), \quad \hat{y}_0 = y_0$$

$k = 0, 1, \dots$ Now let

$$(4.61) \quad v(t, y) = (1 - t)uy + c \text{ for } t.$$

for all $t > 0$ and $y \geq 0$, and

$$(4.62) \quad y_{k+1}^* = g(s_k^*, y_k^*), \quad y_0^* = y_0$$

$k = 0, 1, \dots$, Then system $\{s_k^*, y_k^*\}$ is for problem (4.4), and system $\{s_k^*, \hat{y}_k\}$ is for problem (4.58). By the assumption $s_0^* > u_0^{-1}$, we have

$$s_0^* \hat{y}_0 = s_0^* y_0^* \in \gamma$$

Hence

$$(4.63) \quad \hat{y}_1^0 = b_{10} > \sum_{i=0}^n r \beta_i s_0^* y_0^{*i} + b_0 = y_1^{*0}$$

where b_{10} and b_0 are given by (4.54) and (4.13) respectively. Moreover, we have

$$(4.64) \quad \hat{y}_1^i = s_0^{*p_{i-1}} \hat{y}_0^{i-1} = s_0^{*p_{i-1}} y_{00}^{*i-1} = y_1^{*i}$$

since $\hat{y}_0 = y_0^* = y_0$, $i = 1, 2, \dots, n$. That is

$$(4.65) \quad s_0^* \hat{A} y_0 = s_0^* A y_0^*$$

where A is the matrix defined by (1.11). Thus, from (4.63) and (4.64), we have

$$(4.66) \quad \hat{y}_1 \geq y^*_1$$

with $\hat{y}_1^0 > y^{*0}_1$. Furthermore, we have

$$(4.67) \quad v(s^*_0, \hat{y}_0) = v(s^*_0, y^{*0}_0).$$

By definition of $v(t, y)$, (4.66) implies that

$$(4.68) \quad v(s^*_1, \hat{y}_1) > v(s^*_1, y^*_1).$$

Using mathematical induction we assume that

$$(4.69) \quad \hat{y}_k \geq y^*_k.$$

By the definition of $f(x)$,

$$(4.70) \quad y^{*0}_{k+1} = f(x_k) \leq \beta q \quad \text{with } x_k = s^*_k \sum_{i=0}^n \beta_i y^{*i}_k$$

since $r < 0$. Hence, if $s^*_k y^*_k \in \Delta$ then $s^*_k \hat{y}_k \in \Delta$ by assumption

(4.69), and then

$$(4.71) \quad \hat{y}^0_{k+1} = \beta q = b_{10}$$

by (4.60) and (4.53). Hence, from (4.70) and (4.71),

$$(4.72) \quad \hat{y}^0_{k+1} \geq y^{*0}_{k+1} \quad \text{if } s^*_k y^*_k \in \Delta.$$

On the other hand, if $s_k^* y_k^* \in \Delta$, then

$$y_{k+1}^{*0} = f(s_k^* \sum_{i=0}^n \beta_i y_k^{*i}) \leq f(s_k^* \sum_{i=0}^n \beta_i \hat{y}_k^i) = \hat{y}_{k+1}^{*0}$$

provided that $s_k^* \hat{y}_k \in \Delta$, and

$$y_{k+1}^{*0} \leq b_0 = b_{10} = \hat{y}_{k+1}^{*0}$$

provided that $s_k^* \hat{y}_k \in \Delta$. Therefore, we have

$$(4.73) \quad \hat{y}_{k+1}^{*0} \geq y_{k+1}^{*0} \quad \text{if } s_k^* y_k^* \in \Delta.$$

Combining (4.72) and (4.73), we have

$$(4.74) \quad \hat{y}_{k+1}^{*0} \geq y_{k+1}^{*0}.$$

Moreover, (4.69) implies

$$(4.75) \quad s_k^* \hat{A} y_k \geq s_k^* A y_k^*.$$

Hence, from (4.74) and (4.75), we have

$$(4.76) \quad \hat{y}_{k+1} \geq y_{k+1}^*.$$

Clearly, we have

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$$(4.77) \quad v(s_k^*, \hat{y}_k) \geq v(s_k^*, y_k^*)$$

and strict inequality holds for $k = 1$. Therefore,

$$\begin{aligned} \hat{J}(s^*) &= \sum_{k=0}^{\infty} \alpha^k v(s_k^*, \hat{y}_k) > \sum_{k=0}^{\infty} \alpha^k v(s_k^*, y_k^*) \\ &= J(s^*) \end{aligned}$$

i.e.,

$$(4.78) \quad \hat{J}(s^*) > J(s^*)$$

where $J(s^*)$ represents the present value of the stock corresponding to s^* . However, by the remark following (4.59), the control sequence $s = \{s_k\} = \{\mu_k^{-1}\}$ (determined by theorem 4.3) is optimal for problem (4.58). Hence, in problem (4.58),

$$(4.79) \quad \hat{J}(s) \geq \hat{J}(s^*).$$

In problem (4.4), the population dynamics corresponding to $\{s_k\} = \{\mu_k^{-1}\}$ are

$$y_{k+1} = s_k L_1 y_k = \mu_k^{-1} L_1 y_k$$

since $\mu_k^{-1} y_k \in Q$ for all k . Hence, corresponding to s , the present values for problems (4.4) and (4.58) coincide, i.e.,

$$(4.80) \quad \hat{J}(s) = J(s).$$

Now, since $\{s^*\}$ with $s_0^* > \mu_0^{-1}$ is optimal for problem (4.4), we have

$$(4.81) \quad J(s^*) \geq J(s).$$

By (4.81), (4.80), (4.79) and (4.78), we have

$$\text{i.e.,} \quad J(s^*) \geq J(s) = \hat{J}(s) \geq \hat{J}(s^*) > J(s^*)$$

i.e.,

$$J(s^*) > J(s^*)$$

a contradiction. This contradiction happens because we assumed that

$s_0^* > \mu_0^{-1}$. Thus, $s_0^* \leq \mu_0^{-1}$. But, we have $s_0^* \geq \mu_0^{-1}$. Hence $s_0^* = \mu_0^{-1}$, i.e., $s_0^* y_0 \in Q$. Repeating the argument, we must have

$$s_k^* = \mu_k^{-1} \quad k = 0, 1, 2, \dots$$

where $y_k = \mu_k y_{k^*}$ with $\mu_k > 1$ and $y_{k^*} \in Q$.

Theorem 4.5 ($\xi > \alpha^{-1} > \eta$). Suppose (4.37), (4.38) and (4.39) hold.

If $\{s_k, y_k\}$ is optimal for problem (4.4), then

$$s_k \rightarrow \xi^{-1} \text{ and } y_k \rightarrow \xi y_*$$

as $k \rightarrow \infty$, where y_* is given by (4.11).

Proof. By theorem 4.4, we have $s_k = \mu_k^{-1}$ for all k . Hence, $s_k y_k \in Q \subset \Delta$ for all k . Now, $\mu_k^{-1} y_k$ and $y_k \|y_k\|^{-1}$ have the same direction, and $y_* = \|y_*\| v_1$. Similar to lemma 4.2, we have

$$y_k \|y_k\|^{-1} \rightarrow v_1 \text{ as } k \rightarrow \infty.$$

Since all $\mu_k^{-1} y_k \in Q$ and $y_* \in Q$, they are bounded. We must have

$$\mu_k^{-1} y_k \rightarrow y_* \text{ as } k \rightarrow \infty.$$

Hence,

$$y_{k+1} = s_k L_1 y_k = \mu_k^{-1} L_1 y_k \rightarrow L_1 y_* = \xi y_*$$

and thus,

$$\mu_k^{-1} \rightarrow \xi^{-1} \text{ as } k \rightarrow \infty.$$

But $s_k = \mu_k^{-1}$ for all k , we have

$$s_k \rightarrow \xi^{-1} \text{ as } k \rightarrow \infty.$$

In the following we shall give some more illustration for the assumptions (4.37), (4.38) and (4.39), and the main results of theorems 4.3 and 4.4. Assumption (4.37) implies that

$$(uL_1y - c)/(uy - c) > \frac{1}{\alpha} \quad \text{for } y \in \Delta$$

provided $uy - c > 0$. Thus, whenever the population y is in Δ , the net stock-value of y is growing more rapidly than the growth of money. Hence one should harvest no stock, whenever y is in Δ .

Suppose the population y is in the exterior $\bar{\Delta}$ of Δ . Consider the case $r \geq 0$. Assumption (4.38) is equivalent to

$$u\tilde{y} - c > \alpha M\tilde{y} \quad \tilde{y} \in Q$$

i.e., the net stock-value of population $\tilde{y} \in Q$ is greater than the discounted value of $M\tilde{y}$. By theorem 4.3, the optimal harvest would reduce the population $y \in \bar{\Delta}$ to the boundary Q of Δ . Optimal fishing would not reduce the population y into the interior Δ of Δ , for then fishing would cease as soon as the reduced population reaches Q , and the reduced population would grow according to the Leslie population matrix L_1 . Similar illustration applies to the case $r < 0$ (theorem 4.4).

Following the harvest policy of reducing the population in the exterior $\bar{\Delta}$ of Δ to the boundary Q of Δ , the optimal sequence $\{y_k\}$ of populations approaches asymptotically to ξy_* , while the optimal control sequence $\{s_k\}$ approaches asymptotically to $\frac{1}{\xi}$.

Chapter 5

Selective harvesting models

In this chapter we consider the optimal economic exploitation of a multi-species ecosystem in which some or all of the interacting species are of economic importance. We shall consider that harvesting has different impacts on the population blocks of the ecosystem, i.e., harvesting is selective. We assume that the ecosystem can be described by a set of population blocks, which may be the populations of different species or sets of species on the one hand, or may be age classes within a single species on the other. We then assume that the state of the ecosystem can be adequately described by specifying the size (in terms of numbers or biomass) of each of these blocks. For example, a system of cattle and feed may be characterized by the number of cows, the number of calves, the mass of grass.

Our model can deal with the following harvesting problems: whales, seals, deer [22], forestry [5, 8], etc. We shall give some examples later.

§5.1 The selective harvesting model.

In any given time period (or "year") the population is described by the number of individuals in each population group:

Suppose there are $n > 1$ population groups. Then the ecosystem is described by a (column) vector $y = [y^0, y^1, \dots, y^{n-1}]^T$ where the component y^i , $i = 0, \dots, n-1$, represents the size of the i th population group. We denote by y_k the population-vector in the k th year. We assume that the dynamics of the community is represented by the system of difference equations (in the absence of harvesting).

$$y_{k+1}^i = f_i(y_k^0, \dots, y_k^{n-1}) \equiv f_i(y_k) \quad i = 0, \dots, n-1$$

We assume that each f_i is continuously differentiable with respect to its arguments. In vector form, the $(k+1)$ th population vector is

$$y_{k+1} = [f_0, f_1, \dots, f_{n-1}]^T$$

where $f_i \equiv f_i(y_k)$, $i = 0, 1, \dots, n-1$.

Now we consider harvesting: If we assume that a unit effort devoted to the i th population group results in a total catch of $E_i y^i$, where E_i is a constant between 0 and 1, then an effort $t(i)$ produces a catch of $[1 - (1 - E_i)^{t(i)}] y_0^i$ from an initial i -population group y_0^i of y_0 , and reduces the i -group y_0^i to $(1 - E_i)^{t(i)} y_0^i$, $i = 0, \dots, n-1$. We assume that the cost of an effort $t(i)$ is proportional to $t(i)$, with a constant of proportionality $c_0^i > 0$.

Let u_i be the price of an individual of group i . Then we have a row vector $u = [u_0, u_1, \dots, u_{n-1}]$. If $t_k(i)$ is the effort devoted

to the i th population group for year k , let

$$s_k^i = (1 - E_i)^{t_k(i)}$$

Then, the effort $t_k(i)$ produces a catch of $(1 - s_k^i)y_k^i$ and reduces the i th population to $s_k^i y_k^i$. It is easy to see

$$t_k(i) = \frac{1}{\ln(1 - E_i)} \ln s_k^i$$

Let

$$c_i = -\frac{1}{\ln(1 - E_i)} c_0^i$$

Then c_i is the cost of raising the proportional catch rate of the i th population by one unit. The fishing cost, corresponding to the effort $t_k(i)$ for population i in year k , is

$$c_0^i t_k(i) = -c_i \ln s_k^i$$

Therefore, the net revenue on a year's harvest for the i -population group, starting with an i -population y_k^i , is

$$(1 - s_k^i)u_i y_k^i + c_i \ln s_k^i$$

We note here that harvesting is selective, in contrast to previous chapters in which harvesting is unselective.

For convenience, we denote s_k as a row vector of controls

$$(5.1) \quad s_k = [s_k^0, s_k^1, \dots, s_k^{n-1}]$$

and

$$(5.2) \quad y_{k+1} = [f_0(s_k, y_k), \dots, f_{n-1}(s_k, y_k)]^T$$

where

$$(5.3) \quad f_i(s_k, y_k) = f_i(s_k^0 y_k^0, \dots, s_k^{n-1} y_k^{n-1}),$$

$$i = 0, \dots, n-1.$$

We denote the total net revenue on a year's harvest, starting with population y_k , by

$$(5.4) \quad v_k(s_k, y_k) = \sum_{i=0}^{n-1} (1 - s_k^i) u_i y_k^i + c_i \ln s_k^i.$$

Then the present value of the resource stock equals

$$(5.5) \quad J = \sum_{k=0}^{\infty} \alpha^k v_k(s_k, y_k)$$

where y_0 is the initial population vector and the subsequent population vectors are determined by (5.2). This, then, is our optimization problem: maximize J subject to (5.2), and the additional constraints

$$(5.6) \quad 0 < s_k^i \leq 1, \quad i = 0, \dots, n-1$$

which are required by the definition of s_k^i .

Before solving this optimal problem, we introduce some examples from natural (as opposed to domestic) ecosystems.

(1). This example is given in Doubleday's paper [9].

Bosch [5] considers the harvesting of California redwoods. He classifies the trees by age into three groups $[0, 200)$, $[200, 800)$, $[800, \infty)$, and uses a time interval of 50 years. He considers harvesting half of the redwoods in the first two classes and none in the third. In matrix notation,

$$y_{k+1} = M s_k y_k$$

where

$$\hat{s}_k = \begin{bmatrix} s_k^0 & 0 & 0 \\ 0 & s_k^1 & 0 \\ 0 & 0 & s_k^2 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 12 & 26 & 6 \\ 0.30 & 0.92 & 0 \\ 0 & 0.18 & 0.67 \end{bmatrix}$$

(2). Harvesting the blue whales (example 2.2, chapter 2): Doubleday considers that harvesting is selective [9]. In matrix notation

$$y_{k+1} = \hat{s}_k M y_k$$

where M is given in example 2.2 and $\hat{s}_k$ is a diagonal matrix whose diagonal entries are s_k^i , $i = 0, 1, \dots, 6$. This example will be solved later.

(3). Rorres and Fair [22] consider an example of selective harvesting. They consider a (fictitious) population of female animals divided into 5 age groups whose Leslie growth matrix is

$$M = \begin{bmatrix} 0 & 0 & 0.93 & 0.65 & 0 \\ 0.98 & 0 & 0 & 0 & 0 \\ 0 & 0.95 & 0 & 0 & 0 \\ 0 & 0 & 0.65 & 0 & 0 \\ 0 & 0 & 0 & 0.61 & 0 \end{bmatrix}$$

Then

$$y_{k+1} = \hat{s}_k M y_k$$

where the diagonal harvesting matrix $\hat{s}_k = (s_k^0, s_k^1, \dots, s_k^4)$.

§5.2 The solution.

Consider the finite N-horizon problem:

$$(5.7) \quad \max J_N = \sum_{k=0}^N \alpha^k v_k(s_k, y_k)$$

subject to (5.2) and (5.6), with y_0 given.

Using the discrete maximum principle, we define the Hamiltonian

$$(5.8) \quad H_k(y_k, s_k, \lambda_{k+1}, k) = \alpha^k v_k(s_k, y_k) + \lambda_{k+1}^T y_{k+1}$$

where λ_{k+1} is a vector of Lagrange multipliers and y_{k+1} is given by (5.2). Let $\{s_k\}$ be a sequence of optimal control vectors. Then s_k maximizes the Hamiltonian H_k subject to the conditions

$$y_{k+1} = \partial H_k / \partial \lambda_{k+1}, \lambda_k = \partial H_k / \partial y_k \text{ and } \lambda_{N+1} = 0.$$

If each component $s_k^i \in (0,1)$, $i = 0, 1, \dots, n-1$, then we have the following necessary conditions on y_k , s_k and λ_{k+1} :

$$(5.9) \quad y_{k+1} = [f_0(s_k, y_k), \dots, f_{n-1}(s_k, y_k)]^T$$

$$(5.10) \quad \alpha^k Q_k + R_k \lambda_{k+1} = 0, \text{ if } s_k^i \in (0,1)$$

$$(5.11) \quad \lambda_k = \alpha^k M_k + x_k \lambda_{k+1}, \quad \lambda_{N+1} = 0$$

where

$$(5.12) \quad Q_k = \begin{bmatrix} -u_0 y_k^0 + \frac{c_0}{s_k^0} \\ \dots \\ -u_{n-1} y_k^{n-1} + \frac{c_{n-1}}{s_k^{n-1}} \end{bmatrix}, \quad R_k = \begin{bmatrix} -y_k^{0f'}{}_{0,0} & \dots & y_k^{0f'}{}_{n-1,0} \\ \dots & \dots & \dots \\ y_k^{n-1f'}{}_{0,n-1} & \dots & y_k^{n-1f'}{}_{n-1,n-1} \end{bmatrix}$$

$$(5.13) \quad M_k = \begin{bmatrix} (1 - s_k^0) u_0 \\ \dots \\ (1 - s_k^{n-1}) u_{n-1} \end{bmatrix}, \quad X_k = \begin{bmatrix} s_k^{0f'}{}_{0,0} & \dots & s_k^{0f'}{}_{n-1,0} \\ \dots & \dots & \dots \\ s_k^{n-1f'}{}_{0,n-1} & \dots & s_k^{n-1f'}{}_{n-1,n-1} \end{bmatrix}$$

Here we have used the notation

$$(5.14) \quad f'_{ij} = \frac{\partial f_i}{\partial (s_k^j y_k^j)}, \quad i, j = 0, 1, \dots, n-1.$$

From (5.10) and (5.11) respectively,

$$(5.15) \quad \alpha^k s_k Q_k + s_k R_k \lambda_{k+1} = 0$$

$$(5.16) \quad \lambda_k^T y_k = \alpha^k y_k^T M_k + y_k^T X_k \lambda_{k+1}$$

Since $s_k^R \lambda_{k+1} = y_k^T \lambda_{k+1}$, we have

$$\begin{aligned}
 (5.17) \quad \lambda_k^T y_k &= \alpha^k y_k^T M_k - \alpha^k s_k Q_k \\
 &= \alpha^k u y_k - \alpha^k \sum_{i=0}^{n-1} c_i
 \end{aligned}$$

and similarly

$$(5.18) \quad \lambda_{k+1}^T y_{k+1} = \alpha^{k+1} u y_{k+1} - \alpha^{k+1} c$$

where

$$(5.19) \quad c = \sum_{i=0}^{n-1} c_i$$

This implies that if $\{y_k, s_k\}$ is an optimal sequence, then it is necessary that there exists a sequence $\{\lambda_k\}$ for which each λ_{k+1} satisfies (5.18). Hence, if $(y_k, s_k, \lambda_{k+1})$ is optimal for year k , then the maximum value of the Hamiltonian is given by

$$(5.20) \quad H_k(y_k, s_k, k) = \alpha^k v_k(s_k, y_k) + \alpha^{k+1} u y_{k+1} - \alpha^{k+1} c$$

where y_{k+1} is given by (5.9).

For the optimal multiplier vector λ_{k+1} , $k = 0, 1, \dots, N-1$, we may take

$$(5.21) \quad \lambda_{k+1} = \alpha^{k+1} [z_{0k}, z_{1k}, \dots, z_{n-1,k}]^T$$

where

$$(5.22) \quad z_{ik} = [1 - (u_i y_{k+1}^i)] u_i, \quad i = 0, \dots, n-1$$

and $\lambda_{N+1} = 0$. We note that $y_{k+1}^i = f_i(s_k, y_k)$ is a function of s_k and y_k . It is easy to check that our chosen λ_{k+1} satisfies (5.18). Substituting (5.21) into (5.10) and multiplying by s_k^i , we have a system of n simultaneous equations in s_k^i for each $k = 0, 1, \dots, N-1$:

$$(5.23) \quad u_i s_k^i y_k^i - c_i - \alpha s_k^i y_k^i \sum_{j=0}^{n-1} z_{jk} f'_{ji} = 0$$

$i = 0, 1, \dots, n-1$, where f_{ji} and z_{jk} are defined respectively by (5.14) and (5.22).

Lemma 5.1. For each fixed $k = 0, 1, \dots, N-1$, the equation

$\lambda_k = \alpha^k M_k + \lambda_k \lambda_{k+1}$ holds if and only if the system of equations (5.23) holds, where λ_{k+1} is defined by (5.21).

Proof. For each $k = 0, 1, \dots, N-1$,

$$\lambda_k = \alpha^k M_k + \lambda_k \lambda_{k+1}$$

holds, if and only if

$$\alpha^k [z_{0,k-1}, z_{1,k-1}, \dots, z_{n-1,k-1}]^T$$

$$\approx \alpha^k M_k + \alpha^{k+1} \chi_k [z_{0k}, \dots, z_{n-1,k}]^T$$

if and only if the system of equations with s_k^i as unknowns and y_k^i fixed

$$u_i s_k^i y_k^i - c_i - \alpha s_k^i y_k^i \sum_{j=0}^{n-1} z_{jk} f'_{ji} = 0$$

$i = 0, 1, \dots, n-1$ holds, which is exactly (5.23).

If $s_k = [s_k^0, \dots, s_k^{n-1}]$, $k = 0, 1, \dots, N-1$, is a solution of the system (5.23) with y_k given, and $s_N^i = c_i (u_i y_N^i)^{-1}$, $i = 0, 1, \dots, n-1$, then $(y_k, s_k, \lambda_{k+1})$ satisfies all the necessary conditions (5.9), (5.10) and (5.11) of the discrete maximum principle.

Suppose we consider $s_k^i y_k^i = x_i$, $i = 0, \dots, n-1$ as unknowns. Corresponding to system (5.23), we have a system of equations which are independent of k :

$$(5.24) \quad u_i x_i - c_i - \alpha x_i \sum_{j=0}^{n-1} [1 - c_j (u_j f_j)^{-1}] u_j f'_{ji} = 0$$

$i = 0, 1, \dots, n-1$, where

$$(5.25) \quad f_j = f_j(x_0, \dots, x_{n-1}) \text{ and } f'_{ji} = \partial f_j / \partial x_i.$$

We may apply numerical methods (e.g., Picard-iteration method) to solve this system of equations. We note that (5.24) is independent of k , i.e., we solve (5.24) for $x_0, x_1, \dots, x_{n-1}$ once only; from then

on, s_0^i is determined from y_0^i by $s_0^i = x_i/y_0^i$; then y_1 is determined by

$$\begin{aligned} y_1^i &= f_i(s_0^0 y_0^0, \dots, s_0^{n-1} y_0^{n-1}) \\ &= f_i(x_0, \dots, x_{n-1}) \end{aligned}$$

provided all $0 < s_0^i = x_i/y_0^i < 1$, $i = 0, 1, \dots, n-1$, and indeed

$$y_{k+1} = [f_0(x_0, \dots, x_{n-1}), \dots, f_{n-1}(x_0, \dots, x_{n-1})]^T$$

for all $k = 0, 1, \dots$. Hence, $y_1 = y_2 = y_3 = \dots$ and the population stabilizes after 1 harvest, provided $0 < x_i < y_0^i$, $i = 0, 1, \dots, n-1$. This is an extension of the results obtained in chapter 3 (see theorem 3.4).

There are n equations and n unknowns $x_0, x_1, \dots, x_{n-1}$ in system (5.24). When $n = 1$, system (5.24) is exactly the equation (3.3) which has one and only one zero provided that $f(y)$ is concave. Thus (5.24) (or (5.23)) is an extension of (3.3). In order to do some investigation of the existence theory for solutions of (5.24), we consider the following example.

Example 5.1. We choose the biological data given in example 2.2.

Then, system (5.24) in this case, has the following form

$$(5.26) \quad \mu_0 x_0 - c_0 - \alpha[0.87u_1 x_0 - c_1] = 0$$

$$(5.27) \quad u_1 x_1 - c_1 - \alpha[0.87u_2 x_1 - c_2] = 0$$

$$(5.28) \quad u_2 x_2 - c_2 - \alpha[0.19u_0 x_2 + 0.87u_3 x_2 - c_3 \\ - 0.19c_0 x_2 g(x_2, x_3, x_4, x_5, x_6)] = 0$$

$$(5.29) \quad u_3 x_3 - c_3 - \alpha[0.44u_0 x_3 + 0.87u_4 x_3 - c_4 \\ - 0.44c_0 x_3 g(x_2, \dots, x_6)] = 0$$

$$(5.30) \quad u_4 x_4 - c_4 - \alpha[0.5u_0 x_4 + 0.87u_5 x_4 - c_5 \\ - 0.5c_0 x_4 g(x_2, \dots, x_6)] = 0$$

$$(5.31) \quad u_5 x_5 - c_5 - \alpha[0.5u_0 x_5 + 0.87u_6 x_5 - \frac{c_6 x_5}{x_5 + x_6} \\ - 0.5c_0 x_5 g(x_2, \dots, x_6)] = 0$$

$$(5.32) \quad u_6 x_6 - c_6 - \alpha[0.45u_0 x_6 + 0.87u_6 x_6 - \frac{c_6 x_6}{x_5 + x_6} \\ - 0.45c_0 x_6 g(x_2, \dots, x_6)] = 0$$

where

$$(5.33) \quad g(x_2, \dots, x_6) = 0.19x_2 + 0.44x_3 + 0.5x_4 + 0.5x_5 + 0.45x_6$$

Suppose

$$(5.34) \quad \alpha = 0.9, c_i = 1, i = 0, 1, \dots, 6$$

$$u = [2.95, 2.96, 2.97, 2.98, 2.99, 3, 3.01].$$

Then, from (5.26) and (5.27), respectively, we have

$$(5.35) \quad x_0 = 0.1584 \text{ and } x_1 = 0.1576$$

To calculate x_i , $i = 2, 3, \dots, 6$, we use

$$(5.36) \quad x^0 = (0.14, 0.11, 0.08, 0.07, 0.2)$$

as an initial approximation. From equations (5.28)-(5.32), we have

$$(5.37) \quad x_2 = -18.79, x_3 = -0.1617, x_4 = -0.1269,$$

$$x_5 = -0.7240, x_6 = -0.8234$$

(see table 5). We note that $x_i < 0$, $i = 2, \dots, 6$. In theorem 5.2 we shall discuss how to choose the optimal control $s_k^i \in \{0,1\}$ if $x_i \leq 0$.

Suppose

$$(5.38) \quad \alpha = 0.9, c_i = 1, u_i = 3, i = 0, 1, \dots, 6.$$

Then, similarly we have

$$(5.39) \quad x_0 = 0.1536, x_1 = 0.1536, x_2 = -1.786,$$

$$x_3 = -0.1014, x_4 = -0.0826, x_5 = -0.4526,$$

$$x_6 = -0.5108$$

where x_0 and x_1 are obtained directly from (5.26) and (5.27) respectively, and x_i , $i = 2, \dots, 6$ from (5.28) to (5.32) (see table 6).

In the following we assume that each reproduction function $f_i(y)$, $i = 0, 1, \dots, n-1$ is a concave function of y . It follows from straightforward weak*-compactness arguments that an optimal sequence $\{s_k\}$ exists.

Theorem 5.2. If system (5.24) has a solution $x = [x_0, \dots, x_{n-1}]$, the controls $s_k = [s_k^0, \dots, s_k^{n-1}]$, $k = 0, 1, \dots, N-1$, defined by

$$(5.40) \quad s_k^i = \begin{cases} x_i / y_k^i & \text{if } y_k^i > x_i > 0 \\ 1 & \text{otherwise} \end{cases}$$

$$(5.41) \quad s_N^i = c_i (u_i y_N^i)^{-1}, i = 0, 1, \dots, n-1$$

are admissible controls for maximizing J .

Proof. Suppose $x_i > 0$, $i = 0, 1, \dots, n-1$. If $y_k^i > x_i$ for all i , then we take $s_k^i = x_i/y_k^i$, so that $s_k^i y_k^i = x_i$, $i = 0, 1, \dots, n-1$ satisfy (5.23). If $y_k^i \leq x_i$ for some i , we take $s_k^i = 1$ which means there is no harvest from the i th population block in year k . The reason is as follows: The Hamiltonian H_k , which can be represented by (5.20), is a concave function of s_k^i . The Hamiltonian will take its maximum value at some point $s_k^i = x_i/y_k^i \geq 1$. Hence H_k is increasing over $s_k^i \in (0, 1]$, and so the best choice for s_k^i to be optimal (for maximizing H_k) is 1 if $x_i \geq y_k^i$.

Suppose $x_i \leq 0$ for some $i = 0, \dots, n-1$. Since the admissible control $s_k^i \in (0, 1]$, $x_i/y_k^i \leq 0$ is not an admissible control. Thus there is no $s_k^i \in (0, 1)$ such that $[x_0, \dots, x_{i-1}, s_k^i y_k^i, x_{i+1}, \dots, x_{n-1}]$ satisfies (5.24). Since H_k (5.20) is a concave continuous function of $s_k^i \in (0, 1]$, we see that H_k is an increasing function of $s_k^i \in (0, 1]$. Hence the best choice for s_k^i to be optimal (for maximizing H_k) is 1 if $x_i \leq 0$.

At last, when $k = N$, we have $\lambda_{N+1} = 0$ by (5.11), and so $s_N^i = c_i(u_i y_k^i)^{-1}$ by (5.10), $i = 0, \dots, n-1$.

If system (5.24) has more than one solution, we proceed as follows: Let x be a solution of (5.24). Define a control $s_k = [s_k^0, \dots, s_k^{n-1}]$, $k = 0, 1, \dots, N-1$, with respect to x by (5.40), and s_N by (5.41). For this sequence of controls $\{s_k\}$, there corresponds a value $J(\{s_k\})$ which is the present value of the stock for the policy $\{s_k\}$. Since there is more than one solution x of (5.24), there is more than one policy $\{s_k\}$ that can be determined through (5.40) and (5.41), and so is $J(\{s_k\})$. Compare these values $J(\{s_k\})$ and take the $\{s_k\}$ which gives the maximum value $J(\{s_k\})$ to be

our optimal sequence. Note that this is easy since s_k and y_k are usually constant for $k \geq$ some positive integer, hence computing $J(\{s_k\})$ is just a matter of summing a geometric series.

Corollary 5.3. Suppose system (5.24) has solutions. There is a solution x of (5.24) from which we can determine the optimal controls s_k , $k = 0, \dots, N$ by (5.40) and (5.41).

As in chapter 2, the N -horizon case can approximate the ∞ -horizon case as $N \rightarrow \infty$. Hence we have the following theorem.

Theorem 5.4. Suppose system (5.24) has solutions. Then there is a solution $x = [x_0, x_1, \dots, x_{n-1}]$ of (5.24) from which the optimal sequence s_k is defined by (5.40), $k = 0, 1, \dots$.

In the following, $x = [x_0, \dots, x_{n-1}]$ will denote the solution of (5.24), from which the optimal sequence $\{s_k\}$ is defined. We call such x the optimal solution of (5.24).

In order to give some economic interpretation of our main results, we extend Clark's discussion [8, p. 42] to our high dimensional case. We may rewrite equation (5.18) as

$$\lambda_{k+1}^T y_{k+1} = \alpha^{k+1} \sum_{i=0}^{n-1} (u_i y_{k+1}^i - c_i)$$

Then equation (5.18) means that the value $\lambda_{k+1}^T y_{k+1}$ of the capital

asset y_{k+1} in year $k+1$ is equal to the sum of the net stock-values of y_{k+1}^i discounted by α^{k+1} . Let us consider the system of equations (5.24). Suppose $x = [x_0, x_1, \dots, x_{n-1}]$, with $x_i > 0$ for all $i = 0, 1, \dots, n-1$, is the optimal solution of the system of equations (5.24). Moreover, we suppose that the population is in equilibrium at the population level x . This implies a constant harvest control $s_i = x_i/f_i$, $i = 0, 1, \dots, n-1$ where $f_i = f_i(x_0, x_1, \dots, x_{n-1})$ (see (5.25)). We note that it takes one year for the i -population x_i to reach the level f_i , $i = 0, 1, \dots, n-1$. Then we have a constant net revenue on a year's harvest, starting with population vector $[f_0, \dots, f_{n-1}]^T$,

$$R(x) = \sum_{i=0}^{n-1} [(1 - x_i/f_i)u_i f_i + c_i \ln(x_i/f_i)]$$

For simplicity, we denote

$$R_i' = \frac{\partial R(x)}{\partial x_i} \quad \text{and} \quad f_{ji}' = \frac{\partial f_j}{\partial x_i}$$

(see (5.25)). Then

$$R_i' = \sum_{j=0}^{n-1} \left(1 - \frac{c_j}{u_j f_j}\right) u_j f_{ji}' - \left(u_i - \frac{c_i}{x_i}\right)$$

Hence system (5.24) can be written as

$$(5.42) \quad u_i - \frac{c_i}{x_i} = \frac{\alpha}{1-\alpha} R_i' \quad i = 0, 1, \dots, n-1.$$

These equations have a "marginal" interpretation. Consider a "marginal" change in x_i , for example a unit decrease to $x_i - 1$. There are two effects: it will increase the immediate revenue from the i -population group and it will alter the future rent of the i -population group. The increase in current net revenue from the i -population group is

$$\left(1 - \frac{x_i - 1}{x_i}\right) u_i x_i + c_i \ln \frac{x_i - 1}{x_i} \approx u_i - c_i/x_i$$

On the other hand, the change in the future rent of the i -population group is

$$R(\dots, x_i, \dots) - R(\dots, x_i - 1, \dots) \approx R_i'.$$

The discounted present value of this change in rent (of group i) is given by

$$\alpha \sum_{k=0}^{\infty} \alpha^k R_i' = \frac{\alpha}{1 - \alpha} R_i'$$

Equation (5.42) therefore asserts that, at the optimal population level x_i of the population group i , the marginal change in current net revenue (from the i -population) must match the marginal change in the discounted present value of the future rent of the i -population. However, W.G. Doubleday pointed out that equation (5.24) can be re-written to show that the unit dividend from a small incremental

- harvest after the stock has been reduced to x_i at time t equals the discounted dividend achieved by foregoing the small increment in harvest at time t but taking the resulting additional animals at time $t+1$.

Now let us turn back to our problem. Define

$$D_i = \{y = [y^0, \dots, y^{n-1}]^T : 0 \leq y^i \leq x_i\}$$

$i = 0, 1, \dots, n-1$. Suppose that each concave reproduction function

f_i , $i = 0, \dots, n-1$ is increasing in the sense that, for any $y_k \in D_i$,

$$(5.43) \quad f_i(s_k, y_k) \geq y_k^i$$

where s_k is optimal, defined by (5.40) with respect to x . If

$y_k \in D_i$, then $y_k^i \leq x_i$ and so $s_k^i = 1$ by (5.40). Then, in the

$(k+1)$ th year, the population of block i is given by

$$y_{k+1}^i = f_i(\dots, y_k^i, \dots) > y_k^i.$$

Theorem 5.5. Suppose each reproduction function f_i , $i = 0, 1, \dots, n-1$ is increasing in the sense of (5.43). Then there is an integer $I \geq 0$ such that the optimal controls s_k , $k = 0, 1, \dots, I-1$ are defined by (5.40), and

$$s_k^i = x_i / f_i(x) = \text{constant}, \quad k \geq I+1,$$

$i = 0, 1, \dots, n-1$, where $x > 0$ is the optimal solution of (5.24).

Proof. If $y_0 \in D_i$ for some $i = 0, 1, \dots, n-1$, then $y_0^i \leq x_i$ and so $s_0^i = 1$ by (5.40). By (5.43), we have

$$y_1^i = f_i(s_0, y_0) > y_0^i$$

Hence there is some integer $j \geq 0$ such that $y_j \notin D_i$, i.e., $y_j^i > x_i$ for all $i = 0, 1, \dots, n-1$. Let I be the first such integer j such that $y_I^i > x_i$ for all i . Then $s_I^i = x_i / y_I^i < 1$, $i = 0, \dots, n-1$. Moreover, for each i ,

$$y_{I+1}^i = f_i(s_I, y_I) = f_i(x) > x_i$$

since $s_I^i y_I^i = x_i$. Hence

$$s_{I+1}^i = x_i / f_i(x) \quad i = 0, 1, \dots, n-1.$$

Mathematical induction shows that $s_k^i = x_i / f_i(x)$ for each $i = 0, \dots, n-1$, and $k \geq I + 1$.

We note that theorem 5.5 is an extension of theorem 3.4.

§5.3 Application to age-structured model.

We apply our method to the age-structured harvesting model based on the Leslie matrix model (Lewis [17], Leslie [15]) of population growth. This is a single species model. Following the notations of chapters 1 and 4, the population dynamics are given by the difference equation

$$(5.44) \quad y_{k+1} = [f_0(s_k, y_k), 0, \dots, 0]^T + A[s_k^0 y_k^0, \dots, s_k^{n-1} y_k^{n-1}]^T$$

where A is given by (1.11), and

$$(5.45) \quad f_0(s_k, y_k) = \sum_{i=0}^{n-1} g_i(s_k^i y_k^i) = y_{k+1}^0$$

where each reproduction function $g_i(y^i)$, $i = 0, \dots, n-1$ is assumed.

to be continuously differentiable. From system (5.23), we have a system of equations

$$\begin{aligned}
 & u_0 s_k^0 y_k^0 - c_0 - \alpha s_k^0 y_k^0 g'_{0k} \left(1 - \frac{c_0}{u_0 y_{k+1}^0}\right) u_0 \\
 & \quad - \alpha p_0 s_k^0 y_k^0 \left(1 - \frac{c_1}{u_1 y_{k+1}^1}\right) u_1 = 0 \\
 (5.46) \quad & u_1 s_k^1 y_k^1 - c_1 - \alpha s_k^1 y_k^1 g'_{1k} \left(1 - \frac{c_0}{u_0 y_{k+1}^0}\right) u_0 \\
 & \quad - \alpha p_1 s_k^1 y_k^1 \left(1 - \frac{c_2}{u_2 y_{k+1}^2}\right) u_2 = 0 \\
 & \quad \dots \dots \dots \\
 & u_{n-1} s_k^{n-1} y_k^{n-1} - c_{n-1} - \alpha s_k^{n-1} y_k^{n-1} g'_{n-1,k} \left(1 - \frac{c_0}{u_0 y_{k+1}^0}\right) u_0 = 0
 \end{aligned}$$

where $g'_{ik} = dg_i/d(s_k^i y_k^i)$, $i = 0, 1, \dots, n-1$, and y_{k+1}^0 is given by (5.45) with

$$y_{k+1}^i = p_{i-1} s_k^{i-1} y_k^{i-1} \quad i = 1, 2, \dots, n-1.$$

As before, if we consider $x_i = s_k^i y_k^i$, $i = 0, 1, \dots, n-1$ as unknowns, we have the following system of equations.

$$\begin{aligned}
 & u_0 x_0 - c_0 - \alpha \left(1 - \frac{c_0}{u_0 g(x)}\right) u_0 x_0 g'_0(x_0) \\
 & \quad - \alpha \left(1 - \frac{c_1}{u_1 p_0 x_0}\right) p_0 u_1 x_0 = 0 \\
 (5.47) \quad & u_1 x_1 - c_1 - \alpha \left(1 - \frac{c_0}{u_0 g(x)}\right) u_0 x_1 g'_1(x_1) \\
 & \quad - \alpha \left(1 - \frac{c_2}{u_2 p_1 x_1}\right) p_1 u_2 x_1 = 0 \\
 & \quad \vdots \\
 & u_{n-1} x_{n-1} - c_{n-1} - \alpha \left(1 - \frac{c_0}{u_0 g(x)}\right) u_0 x_{n-1} g'_{n-1}(x_{n-1}) = 0
 \end{aligned}$$

where

$$(5.48) \quad g(x) = \sum_{i=0}^{n-1} g_i(x_i)$$

We note that (5.47) is a system of n nonlinear equations with n unknowns $x_0, \dots, x_{n-1}$. We may apply numerical methods to solve it. When $n = 1$, there is only one equations with one unknown in (5.47) i.e.,

$$u_0 x_0 - c_0 - \alpha \left(1 - \frac{c_0}{u_0 g(x)}\right) u_0 x_0 g'_0(x_0) = 0$$

This is exactly (3.3) if we set $x_0 = y$, $c_0 = c$, $u = u_0$ and $g_0(y) = f(y)$. In chapter 3, we show that (3.3) or $G(y) = 0$ (see (3.19)) has one and only one solution provided $f(y)$ is concave. Hence, when $n = 1$ and $g_0(x_0)$ is concave, (5.47) has a unique solution.

By theorem 5.4 we have the following theorem.

Theorem 5.6. Suppose the system (5.47) has solutions. Then there is a solution $x = [x_0, \dots, x_{n-1}]$ from which the optimal sequence of controls $s_k = [s_k^0, s_k^1, \dots, s_k^{n-1}]$ is defined by (5.40).

Doubleday [9] studied the problem of harvesting the blue whales. He considered that harvesting is selective. We shall study the same problem given by Doubleday.

Example 5.2. Suppose that the biological data are the same as in example 2.2. The system (5.24) (or (5.36)), in this case, has the form given by (5.26), ..., (5.32) (see example 5.1).

Suppose that the economic data α, c_i and $u_i, i = 0, 1, \dots, 6$ are given by (5.34). We have a solution $x = (x_0, x_1, \dots, x_6)$, for the system of equations (5.26), ..., (5.32), given by (5.35) and (5.37), i.e.,

$$(5.49) \quad x = (0.1584, 0.1576, -18.79, -0.1617, -0.1269, -0.7240, -0.8234)$$

Let

$$(5.50) \quad y_0 = [0.221, 0.175, 0.138, 0.11, 0.087, 0.069, 0.201]^T$$

be the initial population vector. Since $x_i < 10, i = 2, 3, 4, 5, 6$, by

corollary 5.3, we take the optimal controls (corresponding to x given by (5.49))

$$s_k^i = 1, \quad i = 2, \dots, 6 \text{ and } k = 0, 1, \dots$$

We note that $s = 1$ means no harvesting. To determine s_k^0 and s_k^1 , we use (5.40). We have

$$s_0^0 = x_0/y_0^0 = 0.7167$$

and

$$s_0^1 = x_1/y_0^1 = 0.9006$$

However, we find that the net return, from the population of year 0, is

$$(1 - s_0^0)u_0 y_0^0 + c_0 \ln s_0^0 = -0.1484$$

which is negative. This implies that $s_0^0 = 0.7167$ is not optimal. Why? This happens because the fishing cost coefficient $c_0 = 1$ is so large compared with the price $u_0 = 2.95$. We note that

$$u_0 x_0 - c_0 = 0.4672 - 1 = -0.5327$$

i.e., the net stock-value of x_0 is negative. Thus, there must be some relation between u , c and x .

Let $s_k = (s_k^0, \dots, s_k^{n-1})$ be optimal. Then it is necessary that

$$(5.51) \quad s_k^i \geq c_i (u_i y_k^i)^{-1} \quad \text{for all } i \text{ and } k.$$

(see lemma 3.2). In fact, (5.51) is equivalent to

$$c_i/s_k^i - u_i y_k^i \leq 0.$$

However,

$$c_i/s_k^i - u_i y_k^i = \frac{\partial}{\partial s^i} v_k(s, y_k) \Big|_{s^i = s_k^i}$$

where $v_k(s, y_k)$ is defined by (5.4). If

$$(5.52) \quad c_i/s_k^i - u_i y_k^i > 0 \text{ for some } i = 0, 1, \dots, n-1$$

then by making s_k^i larger we can increase both

$$v_k(s_k, y_k) = \sum_{j=0}^{n-1} (1 - s_k^j) u_j y_k^j + c_j \ln s_k^j$$

(i.e., this year's return) and

$$\sum_{j=k+1}^{\infty} \alpha^j v_j(s_j, y_j)$$

(i.e., the future harvests) where $v_j(s_j, y_j)$ is given by (5.4), provided that all the reproduction functions $f_i(s_k, y_k)$ (defined by (5.3)) have the property that

$$(5.53) \quad f_i(\dots, z', \dots) \geq f_i(\dots, z, \dots) \quad \text{if } z' \geq z.$$

Hence, s_k^i cannot be optimal if (5.52) holds. This proves that (5.51) holds.

Therefore, if s_k^i is optimal, by (5.51) we have

$$(5.54) \quad u_i x_i - c_i \geq 0 \quad i = 0, 1, \dots, n-1$$

where $x_i = s_k^i y_k^i$ for all k . (5.54) tells us that the net stock-value of x_i , $i = 0, 1, \dots, n-1$ must be non-negative, if $x_i > 0$ is optimal.

Example 5.3. Suppose that the biological data are the same as in example 2.2. The system (5.24) (or (5.36)), in this case, has the form given by (5.26), ..., (5.32) (see example 5.1).

Suppose that the economic data are

$$(5.55) \quad \begin{aligned} \alpha &= 0.9 \\ c_0 &= c_1 = 0.2, c_i = 0.4, i = 2, \dots, 6 \\ u_0 &= 2.5, u_1 = 3, u_2 = 3.5 \\ u_3 &= 3.7, u_4 = 3.8, u_5 = 3.9, u_6 = 4 \end{aligned}$$

We have a solution $x = (x_0, x_1, \dots, x_6)$ for the system of equations (5.26), ..., (5.32):

$$x_0 = 0.1325 \quad (\text{from (5.26)})$$

$$x_1 = 0.077 \quad (\text{from (5.27)})$$

and

$$(5.56) \quad \begin{aligned} x_2 &= 0.1448, x_3 = -0.1013, x_4 = -0.0761 \\ x_5 &= -0.4950, x_6 = -0.6886 \end{aligned}$$

which are obtained by solving equations (5.28) - (5.32) with an initial approximation

$$x^0 = (0.14, 0.11, -0.08, 0.07, 0.2)$$

(see table 7). Using many different initial approximations for x_i , $i = 2, \dots, 6$, we have the same solution (5.56). This suggests that the solution x may be unique in this case. In this sense we may call this solution x optimal. We have

$$u_0 x_0 - c_0 = 0.1313$$

$$u_1 x_1 - c_1 = 0.031$$

$$u_2 x_2 - c_2 = 0.1068$$

Let

$$y_0 = [0.221, 0.175, 0.138, 0.11, 0.087, 0.069, 0.201]^T$$

be the initial population vector. Since $x_i < 0$, $i = 3, 4, 5, 6$ by corollary 5.3, we take the optimal controls

$$s_k^i = 1 \quad i = 3, 4, 5, 6 \text{ and } k = 0, 1, \dots$$

We note that $s = 1$ means no harvesting. To determine s_k^0 and s_k^1 , we use the formula

$$s_k^i = x_i / y_k^i \quad i = 0, 1; k = 0, 1, 2, \dots$$

and their values are given in table 8. Since

$$x_2 / y_k^2 > 1 \quad \text{for all } k$$

(see table 8), we take

$$s_k^2 = 1 \quad \text{for all } k.$$

Moreover, the corresponding population dynamics y_k^i , $i = 0, 1, \dots, 6$ and $k = 0, 1, \dots$ are given in table 8. Corresponding to our optimal control sequence $\{s_k\}$, the first five harvest net returns are

$$v_0(s_0, y_0) = 0.2487$$

$$v_1(s_1, y_1) = 0.1892$$

$$v_2(s_2, y_2) = 0.2128$$

$$v_3(s_3, y_3) = 0.2074$$

$$v_4(s_4, y_4) = 0.1967$$

The discounted sum of the above five net returns is 0.87159.

In order to calculate the undiscounted profit maximization of the stock, we simply maximize the net return in the first year's

harvest. Then

$$\frac{\partial v_0}{\partial s^i}(s, y_0) = 0, \quad i = 0, 1, \dots, 6$$

implies that

$$s^i = c_i (u_i y_0^i)^{-1} \quad i = 0, 1, \dots, 6.$$

We have

$$s^0 = 0.3619, \quad s^1 = 0.3809, \quad s^2 = 0.8282$$

$$s^3 = 0.9828, \quad s^4 = 1, \quad s^5 = 1, \quad s^6 = 0.495$$

and the undiscounted profit maximization is 0.4135.

In this chapter we consider that many different species or subpopulations are involved in a single stock. If selective harvesting is feasible we may be able to make optimal harvest policies to deal with multispecies of the stock independently. However, if selective harvesting is not feasible, then we have to consider non-selective harvesting.

s_k	y_k^0	y_k^1	y_k^2
1.00000000	21.00000000	7.00000000	3.50000000
.00518852	105.00000000	7.00001431	3.50000000
1.00000000	.54479533	.18159854	.01815985
.39673352	1.85230350	.18159878	.09079922
1.00000000	1.08069324	.24495739	.03602316
.36487901	2.63689327	.36023182	.12247866
.85291457	1.71924686	.32071620	.08572050
.35581291	3.13453770	.48879111	.13677174
.59723192	2.14924431	.37177038	.08695906
.45112008	2.62151432	.42786652	.11101651
.52848148	2.33815384	.39420654	.09650958
.48490258	2.48701954	.41189104	.10416547
.50832736	2.40366364	.40198815	.09988347
.49555022	2.44823360	.40798343	.10217877
.50239688	2.42403223	.40420834	.10091434
.49872261	2.43695068	.40594274	.10158670
.50068086	2.43003750	.40512151	.10122639
.49963993	2.43371391	.40555853	.10141826
.50018889	2.43176937	.40532756	.10131657
.49989957	2.43279076	.40544868	.10137016
.50005215	2.43225002	.40538436	.10134178
.49997187	2.43253326	.40541810	.10135663
.50001478	2.43238554	.40540022	.10134882
.49999255	2.43246460	.40541005	.10135305
.50000209	2.43242645	.40540540	.10135094
.49999887	2.43243980	.40540689	.10135174
.50000048	2.43243504	.40540653	.10135144
.49999887	2.43243980	.40540701	.10135168
.50000125	2.43243504	.40540653	.10135150
.49999887	2.43244362	.40540761	.10135186

STOP

TABLE 1

s_k	y_k^0	y_k^1	y_k^2	y_k^3	y_k^4	y_k^5	y_k^6
1.00000	.22100	.17500	.13800	.11000	.08700	.06900	.20100
1.00000	.24307	.19227	.15225	.12006	.09570	.07569	.23490
1.00000	.27315	.21147	.16727	.13246	.10445	.08326	.27021
1.00000	.30551	.23764	.18398	.14553	.11524	.09087	.30752
.90795	.34043	.26580	.20675	.16006	.12661	.10026	.34660
.88663	.34422	.26891	.20996	.16332	.12644	.10001	.35298
.89578	.34030	.26552	.20743	.16195	.12598	.09753	.34942
.89868	.34010	.26521	.20693	.16166	.12622	.09818	.34832
.89776	.34095	.26590	.20735	.16178	.12639	.09868	.34910
.89636	.34134	.26630	.20768	.16195	.12636	.09872	.34974
.89623	.34119	.26619	.20767	.16196	.12630	.09854	.34972
.89656	.34103	.26603	.20755	.16192	.12628	.09848	.34952
.89681	.34100	.26600	.20751	.16189	.12630	.09850	.34944
.89672	.34106	.26606	.20754	.16190	.12631	.09854	.34950
.89663	.34109	.26608	.20756	.16191	.12631	.09854	.34953
.89663	.34107	.26607	.20756	.16191	.12630	.09853	.34953
.89666	.34106	.26606	.20755	.16191	.12630	.09852	.34952
.89667	.34106	.26606	.20755	.16191	.12630	.09853	.34951
.89666	.34107	.26606	.20755	.16191	.12631	.09853	.34952
.89666	.34107	.26606	.20755	.16191	.12631	.09853	.34952

TABLE 2

s_k	y_k^0	y_k^1	y_k^2	y_k^3	y_k^4	y_k^5	y_k^6
1.00000	.22100	.17500	.13800	.11000	.08700	.06900	.20100
1.00000	.24307	.19227	.15225	.12006	.09570	.07569	.23490
1.00000	.27315	.21147	.16727	.13246	.10445	.08326	.27021
1.00000	.30551	.23764	.18398	.14553	.11524	.09087	.30752
1.00000	.34043	.26580	.20675	.16006	.12661	.10026	.34660
1.00000	.37911	.29617	.23124	.17987	.13925	.11015	.38877
1.00000	.42273	.32983	.25767	.20118	.15649	.12115	.43406
.97209	.47162	.36777	.28695	.22417	.17503	.13615	.48303
.89776	.51142	.39886	.31103	.24268	.18959	.14802	.52365
.89635	.51201	.39945	.31153	.24293	.18954	.14808	.52461
.89623	.51178	.39928	.31150	.24294	.18944	.14781	.52458
.89656	.51154	.39905	.31133	.24288	.18942	.14771	.52428
.89681	.51150	.39900	.31126	.24284	.18945	.14775	.52416
.89672	.51159	.39908	.31131	.24285	.18947	.14781	.52424
.89664	.51163	.39912	.31135	.24287	.18946	.14781	.52430
.89662	.51162	.39911	.31134	.24287	.18946	.14780	.52430
.89665	.51159	.39909	.31133	.24287	.18946	.14779	.52427
.89667	.51159	.39909	.31133	.24286	.18946	.14779	.52427
.89667	.51160	.39909	.31133	.24287	.18946	.14780	.52427
.89664	.51161	.39910	.31133	.24287	.18946	.14780	.52428

TABLE 3

s_k	y_k^0	y_k^1	y_k^2	y_k^3	y_k^4	y_k^5	y_k^6
.88946	.22100	.17500	.13800	.11000	.08700	.06900	.20100
.89125	.21620	.17102	.13542	.10879	.08512	.06732	.20894
.90054	.22171	.17165	.13577	.10751	.08478	.06758	.21933
.89334	.22332	.17371	.13443	.10637	.08423	.06842	.22478
.89341	.22229	.17353	.13500	.10452	.08267	.06547	.22636
.89649	.22118	.17279	.13491	.10474	.08124	.06426	.22681
.89752	.22110	.17251	.13477	.10522	.08185	.06336	.22702
.89711	.22132	.17264	.13470	.10535	.08216	.06391	.22675
.89847	.22156	.17279	.13474	.10513	.08213	.06413	.22685
.89845	.22148	.17279	.13473	.10506	.08199	.06405	.22693
.89866	.22141	.17274	.13475	.10510	.08196	.06395	.22694
.89673	.22141	.17272	.13475	.10513	.08177	.06395	.22692
.89664	.22143	.17273	.13475	.10513	.08201	.06396	.22691
.89664	.22144	.17274	.13475	.10512	.08201	.06398	.22691
.89664	.22144	.17274	.13475	.10512	.08200	.06397	.22692
.89666	.22143	.17274	.13475	.10512	.08200	.06397	.22692
.89667	.22143	.17273	.13475	.10512	.08200	.06397	.22692
.89665	.22143	.17274	.13475	.10512	.08200	.06397	.22692
.89666	.22143	.17274	.13475	.10512	.08200	.06397	.22692
.89666	.22143	.17274	.13475	.10512	.08200	.06397	.22692

TABLE 4

	x_2	x_3	x_4	x_5	x_6
1	.1334E+00	.7946E-01	.7497E-01	.1748E+00	.3476E+00
2	.1914E+00	.1390E+00	.2721E+00	.5130E+00	.1141E+01
3	.5095E+00	-.4917E+00	-.3728E+00	.3471E+00	.1689E+01
4	.2982E+00	-.5602E-01	.1151E+01	-.2780E+02	-.1627E+01
5	.4768E+01	-.1289E+00	-.1542E+00	-.7773E+00	-.1016E+02
6	-.3623E+01	-.1673E+00	-.1305E+00	-.3158E+00	-.2188E-02
7	.1768E+02	-.2506E+00	-.1874E+00	-.2597E+00	-.2610E+00
8	-.4571E+02	-.1745E+00	-.1351E+00	-.4372E+00	-.5700E+00
9	-.2139E+02	-.1629E+00	-.1277E+00	-.6102E+00	-.7413E+00
10	-.1942E+02	-.1618E+00	-.1270E+00	-.6385E+00	-.8004E+00
11	-.1897E+02	-.1617E+00	-.1269E+00	-.7139E+00	-.8172E+00
12	-.1884E+02	-.1617E+00	-.1269E+00	-.7212E+00	-.8218E+00
13	-.1881E+02	-.1617E+00	-.1269E+00	-.7232E+00	-.8230E+00
14	-.1880E+02	-.1617E+00	-.1269E+00	-.7238E+00	-.8233E+00
15	-.1879E+02	-.1617E+00	-.1269E+00	-.7239E+00	-.8234E+00
16	-.1879E+02	-.1617E+00	-.1269E+00	-.7239E+00	-.8234E+00
17	-.1879E+02	-.1617E+00	-.1269E+00	-.7240E+00	-.8234E+00
18	-.1879E+02	-.1617E+00	-.1269E+00	-.7240E+00	-.8234E+00
19	-.1879E+02	-.1617E+00	-.1269E+00	-.7240E+00	-.8234E+00

STOP

TABLE 5

	x_2	x_3	x_4	x_5	x_6
WITH>					
1	.1420E+00	.1393E+00	.1611E+00	.2678E+00	.6713E+00
2	.2406E+00	.2559E+01	-.2200E+00	.2107E+01	-.7429E+01
3	-.1528E+02	-.5261E-01	-.1274E+00	-.8061E-01	-.5265E+00
4	-.4620E+01	-.1173E+00	-.9439E-01	-.3420E+00	-.5287E+00
5	-.2816E+01	-.1089E+00	-.8823E-01	-.4359E+00	-.5357E+00
6	-.2038E+01	-.1042E+00	-.8470E-01	-.4547E+00	-.5237E+00
7	-.1719E+01	-.1011E+00	-.8243E-01	-.4517E+00	-.5104E+00
8	-.1817E+01	-.1017E+00	-.8290E-01	-.4533E+00	-.5120E+00
9	-.1766E+01	-.1012E+00	-.8256E-01	-.4523E+00	-.5103E+00
10	-.1799E+01	-.1015E+00	-.8276E-01	-.4529E+00	-.5113E+00
11	-.1778E+01	-.1013E+00	-.8263E-01	-.4525E+00	-.5106E+00
12	-.1791E+01	-.1014E+00	-.8271E-01	-.4527E+00	-.5110E+00
13	-.1783E+01	-.1014E+00	-.8266E-01	-.4526E+00	-.5107E+00
14	-.1788E+01	-.1014E+00	-.8269E-01	-.4527E+00	-.5109E+00
15	-.1785E+01	-.1014E+00	-.8267E-01	-.4526E+00	-.5108E+00
16	-.1787E+01	-.1014E+00	-.8268E-01	-.4527E+00	-.5109E+00
17	-.1785E+01	-.1014E+00	-.8267E-01	-.4526E+00	-.5108E+00
18	-.1786E+01	-.1014E+00	-.8268E-01	-.4526E+00	-.5108E+00
19	-.1786E+01	-.1014E+00	-.8268E-01	-.4526E+00	-.5108E+00
20	-.1786E+01	-.1014E+00	-.8268E-01	-.4526E+00	-.5108E+00
20	-.1786E+01	-.1014E+00	-.8268E-01	-.4526E+00	-.5108E+00
STOP					

TABLE 6

k	x ₂	x ₃	x ₄	x ₅	x ₆
1	.8238E-01	.4373E+01	-.1186E+00	.5569E+00	.1561E+01
2	.1163E+00	.1602E+00	-.1383E+00	-.2164E+01	.9143E+00
3	.1435E+00	-.9404E-01	-.7901E-01	.1272E+02	-.3664E+01
4	.1179E+00	-.1611E+00	-.1114E+00	.1717E+01	-.1566E+02
5	.1225E+00	-.1439E+00	-.1018E+00	-.9724E+00	.4974E+00
6	.1714E+00	-.7480E-01	-.5695E-01	-.2502E+01	-.3012E+00
7	.1285E+00	-.1278E+00	-.9256E-01	-.2385E+00	-.4034E+00
8	.1664E+00	-.8041E-01	-.6131E-01	-.3712E+00	-.5559E+00
9	.1541E+00	-.9273E-01	-.7075E-01	-.4480E+00	-.6439E+00
10	.1475E+00	-.9859E-01	-.7440E-01	-.4807E+00	-.6761E+00
11	.1456E+00	-.1006E+00	-.7582E-01	-.4912E+00	-.6853E+00
12	.1450E+00	-.1011E+00	-.7597E-01	-.4940E+00	-.6877E+00
13	.1449E+00	-.1013E+00	-.7607E-01	-.4947E+00	-.6883E+00
14	.1449E+00	-.1013E+00	-.7609E-01	-.4949E+00	-.6885E+00
15	.1448E+00	-.1013E+00	-.7609E-01	-.4950E+00	-.6885E+00
16	.1448E+00	-.1013E+00	-.7610E-01	-.4950E+00	-.6885E+00
17	.1448E+00	-.1013E+00	-.7610E-01	-.4950E+00	-.6886E+00
18	.1448E+00	-.1013E+00	-.7610E-01	-.4950E+00	-.6886E+00
19	.1448E+00	-.1013E+00	-.7610E-01	-.4950E+00	-.6886E+00
20	.1448E+00	-.1013E+00	-.7610E-01	-.4950E+00	-.6886E+00
21	.1448E+00	-.1013E+00	-.7610E-01	-.4950E+00	-.6886E+00
22	.1448E+00	-.1013E+00	-.7610E-01	-.4950E+00	-.6886E+00
23	.1448E+00	-.1013E+00	-.7610E-01	-.4950E+00	-.6886E+00
24	.1448E+00	-.1013E+00	-.7610E-01	-.4950E+00	-.6886E+00
25	.1448E+00	-.1013E+00	-.7610E-01	-.4950E+00	-.6886E+00
26	.1448E+00	-.1013E+00	-.7610E-01	-.4950E+00	-.6886E+00
27	.1448E+00	-.1013E+00	-.7610E-01	-.4950E+00	-.6886E+00
28	.1448E+00	-.1013E+00	-.7610E-01	-.4950E+00	-.6886E+00
29	.1448E+00	-.1013E+00	-.7610E-01	-.4950E+00	-.6886E+00
30	.1448E+00	-.1013E+00	-.7610E-01	-.4950E+00	-.6886E+00
31	.1448E+00	-.1013E+00	-.7610E-01	-.4950E+00	-.6886E+00
32	.1448E+00	-.1013E+00	-.7610E-01	-.4950E+00	-.6886E+00
33	.1448E+00	-.1013E+00	-.7610E-01	-.4950E+00	-.6886E+00
34	.1448E+00	-.1013E+00	-.7610E-01	-.4950E+00	-.6886E+00
35	.1448E+00	-.1013E+00	-.7610E-01	-.4950E+00	-.6886E+00

STOP

TABLE 7

s_k^0	s_k^1	y_k^0	y_k^1	y_k^2	y_k^3	y_k^4	y_k^5	y_k^6
.5995	.4400	.2210	.1750	.1380	.1100	.0870	.0690	.2010
.5451	.6680	.2431	.1153	.0670	.1201	.0957	.0757	.2349
.5157	.6680	.2570	.1153	.0670	.0583	.1045	.0833	.2702
.5220	.6680	.2538	.1153	.0670	.0583	.0507	.0909	.3075
.5353	.6680	.2475	.1153	.0670	.0583	.0507	.0441	.3466
.5481	.6680	.2418	.1153	.0670	.0583	.0507	.0441	.3399
.5550	.6680	.2387	.1153	.0670	.0583	.0507	.0441	.3341
.5611	.6680	.2361	.1153	.0670	.0583	.0507	.0441	.3291
.5666	.6680	.2339	.1153	.0670	.0583	.0507	.0441	.3247
.5714	.6680	.2319	.1153	.0670	.0583	.0507	.0441	.3208
.5757	.6680	.2302	.1153	.0670	.0583	.0507	.0441	.3175
.5795	.6680	.2287	.1153	.0670	.0583	.0507	.0441	.3146
.5828	.6680	.2274	.1153	.0670	.0583	.0507	.0441	.3121
.5857	.6680	.2262	.1153	.0670	.0583	.0507	.0441	.3099
.5883	.6680	.2252	.1153	.0670	.0583	.0507	.0441	.3080
.5905	.6680	.2244	.1153	.0670	.0583	.0507	.0441	.3063
.5925	.6680	.2236	.1153	.0670	.0583	.0507	.0441	.3049
.5942	.6680	.2230	.1153	.0670	.0583	.0507	.0441	.3036
.5957	.6680	.2224	.1153	.0670	.0583	.0507	.0441	.3025
.5971	.6680	.2219	.1153	.0670	.0583	.0507	.0441	.3016
STOP								

TABLE 8

BIBLIOGRAPHY

1. Bellman, R.E.; Dynamic Programming, Princeton University Press, 1957.
2. Beverton, R.J.H. and S.J. Holt; The Dynamics of Exploited Fish Populations, London, Her Majesty's Stationery Office, 1957.
3. Birkhoff, G.; Extensions of Jentzsch's Theorem, Trans.-Amer. Math. Soc. 85, 219-227, 1957.
4. Bodewig, E.; Matric Calculus, Amsterdam, North Holland Pub. Co. New York, Interscience Publishers, 1959.
5. Bosch, C.A.; Redwoods: a population model. Science 1972, 345-9, 1971.
6. Clark, C.W., G. Edwards and M. Friedlaender; Beverton-Holt Model of a Commercial Fishery: Optimal Dynamics, Journal of the Fisheries Research Board of Canada 30, 1973.
7. Clark, C.W.; Economically optimal policies for the Utilization of Biologically Renewable Resources, Math. Biosciences, 12, 1971.
8. Clark, C.W.; Mathematical Bioeconomics, A Wiley-Interscience Publication, New York, 1976.
9. Doubleday, W.G.; Harvesting in Matrix Population Models, Biometrics 31, March 1975.
10. Golubitsky, M., E.B. Keeler and M. Rothschild; Convergence of the Age Structure: Application of the Projective Metric, Theoretical Population Biology 7, 1975.
11. Hughes, E.; An Optimization Problem Arising in Resource Management, to appear.
12. Hirshleifer, J.; Investment, Interest, and Capital, Prentice-Hall, Inc., 1970.
13. Lancaster, P.; Theory of Matrices, Academic, New York, 1969.
14. Lefkovich, L.P.; A Theoretical Evaluation of Population Growth after Removing Individuals from some age groups, Bull Ent. Res. 57, 437-445.

15. Leslie, P.H.; On the use of matrices in certain population mathematics. *Biometrika*, 33 (1945), 184-313.
16. Leslie, P.H.; Some further notes on the use of matrices in population mathematics. *Biometrika*, 35 (1948), 213-45.
17. Lewis, E.G.; On the generation and growth of a population. *Sankhya*, 6 (1942), 93-6.
18. May, R.M.; Biological Populations with Nonoverlapping Generations: Stable Points, Stable Cycles, and Chaos, *Science* 186 (1974), 645-647.
19. Oster, G.; Stochastic Behavior of Deterministic Models, in S.A. Levin (Ed.), *Ecosystem Analysis and Prediction*, SIAM-SIMS Conference Proceedings (Alta, Utah, 1975), pp. 24-37.
20. Reed, W.J.; Optimal Age-Specific Harvesting in a Non-linear Population Model, to appear.
21. Ricker, W.E.; Stock and Recruitment, *Journal of the Fisheries Research Board of Canada*, 11 (1954), 559-623.
22. Rorres, C. and Fair, W.; Optimal Harvesting Policy for an Age-Specific Population, *Math. Biosciences* 24 (1975), 31-47.
23. Rudin, W.; *Functional Analysis*, McGraw-Hill Book Company, 1973.
24. Sage, A.P. and Chelsea, C. White, III; *Optimal systems control*, Prentice-Hall, Inc., 1977.
25. Sivert, W. and W.R. Smith; Optimal Exploitation of a Multi-species Community, *Math. Biosciences* 33 (1977), 121-134.

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