1

PM-1 3 1/2" x 4" PHOTOGRAPHIC MICROCOPY TARGET
NBS 1010a ANSI/ISO #2 EQUIVALENT

PRECISIONSM RESOLUTION TARGETS

FILMING AT MICROFILMS BLUE PRINTING SINCE 1957

3000 COUNTY ROAD 4, BATHON, LA 70007, USA
TEL: 504 485 7000 FAX: 504 485 7001 TLA 00000000

National Library
of Canada

Acquisitions and
Bibliographic Services Branch

395 Wellington Street
Ottawa, Ontario
K1A 0N4

Bibliothèque nationale
du Canada

Direction des acquisitions et
des services bibliographiques

395, rue Wellington
Ottawa (Ontario)
K1A 0N4

Your file votre référence

Our file Notre référence

NOTICE

The quality of this microform is heavily dependent upon the quality of the original thesis submitted for microfilming. Every effort has been made to ensure the highest quality of reproduction possible.

If pages are missing, contact the university which granted the degree.

Some pages may have indistinct print especially if the original pages were typed with a poor typewriter ribbon or if the university sent us an inferior photocopy.

Reproduction in full or in part of this microform is governed by the Canadian Copyright Act, R.S.C. 1970, c. C-30, and subsequent amendments.

AVIS

La qualité de cette microforme dépend grandement de la qualité de la thèse soumise au microfilmage. Nous avons tout fait pour assurer une qualité supérieure de reproduction.

S'il manque des pages, veuillez communiquer avec l'université qui a conféré le grade.

La qualité d'impression de certaines pages peut laisser à désirer, surtout si les pages originales ont été dactylographiées à l'aide d'un ruban usé ou si l'université nous a fait parvenir une photocopie de qualité inférieure.

La reproduction, même partielle, de cette microforme est soumise à la Loi canadienne sur le droit d'auteur, SRC 1970, c. C-30, et ses amendements subséquents.

MECHANICS OF RIGID DISC INCLUSIONS IN FLUID SATURATED POROELASTIC MEDIA

by

ZHONGQI YUE

B.Sc.(Hons.), Beijing University, Beijing, China

M.Sc., Beijing University, Beijing, China

A thesis submitted to the
Faculty of Graduate Studies and Research
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

in

Engineering*

**Carleton University
Ottawa, Ontario, Canada
July, 1992.**

©copyright

1992, Zhongqi Yue

***The Doctor of Philosophy in Civil Engineering Program
is a joint program with University of Ottawa,
administrated by the Ottawa-Carleton Institute for Civil Engineering**

National Library
of Canada

Acquisitions and
Bibliographic Services Branch

395 Wellington Street
Ottawa, Ontario
K1A 0N4

Bibliothèque nationale
du Canada

Direction des acquisitions et
des services bibliographiques

395, rue Wellington
Ottawa (Ontario)
K1A 0N4

Your file / Votre référence

Our file / Notre référence

The author has granted an irrevocable non-exclusive licence allowing the National Library of Canada to reproduce, loan, distribute or sell copies of his/her thesis by any means and in any form or format, making this thesis available to interested persons.

L'auteur a accordé une licence irrévocable et non exclusive permettant à la Bibliothèque nationale du Canada de reproduire, prêter, distribuer ou vendre des copies de sa thèse de quelque manière et sous quelque forme que ce soit pour mettre des exemplaires de cette thèse à la disposition des personnes intéressées.

The author retains ownership of the copyright in his/her thesis. Neither the thesis nor substantial extracts from it may be printed or otherwise reproduced without his/her permission.

L'auteur conserve la propriété du droit d'auteur qui protège sa thèse. Ni la thèse ni des extraits substantiels de celle-ci ne doivent être imprimés ou autrement reproduits sans son autorisation.

ISBN 0-315-79846-7

Canada

The undersigned hereby recommend to the
Faculty of Graduate Studies and Research
acceptance of the thesis entitled

**MECHANICS OF RIGID DISC INCLUSIONS
IN FLUID SATURATED POROELASTIC MEDIA**

submitted by

ZHONGQI YUE

in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

Engineering

.....

External Examiner

L.M. Keer

.....

Thesis Supervisor

A.P.S. Selvadurai

.....

Chairman, Department of Civil Engineering

J.L. Humar

Carleton University

September, 1992

Abstract

Mechanics of inclusions are classical mixed boundary value problems in mathematical physics and applied mechanics which have important applications to various branches of engineering and science. However, due to the intrinsic difficulties associated with the mathematical formulations and the ensuing numerical evaluations, the majority of the investigations have concentrated on the study of the static responses of inclusions embedded in linear elastic media. A study of the literature on inclusion problems reveals that only limited results have been obtained with regard to the time-dependent behaviour of inclusions embedded in saturated poroelastic media, visco-elastic media, or thermoelastic media. This thesis presents a rigorous mathematical and analytical investigation of the mechanics of rigid disc inclusions embedded in poroelastic media saturated with compressible pore fluids. Using the classical integral transform technique for the solution of linear partial differential equations, a simple and direct matrix based approach is developed to systematically and rigorously solve the (mixed) boundary value and initial value problems associated with saturated poroelastic layered media. Based on the results, a general Fourier and Laplace transform based approach is further proposed for analytical solution of mixed boundary value and initial value problems in saturated poroelastic layered media. Systems of Fredholm integral equations of the second kind in the Laplace transform domain are systematically formulated for the fundamental problem of a rigid disc inclusion embedded in bonded contact with a poroelastic medium of infinite extent saturated with a compressible fluid. Special techniques are presented for numerical solution in the temporal domain of the systems of Fredholm integral equations of the second kind in the Laplace transform domain. The adopted numerical scheme has high stability and accuracy for the very long time interval associated with the process of soil consolidation. The results of three limiting conditions are rigorously examined in all the above equations, formulations

and solutions. These limiting conditions are the initial response as $t \rightarrow 0^+$, the final response as $t \rightarrow +\infty$ and the limiting response as the drained Poisson's ratio ν approaches the undrained Poisson's ratio ν_u . The time-dependent behaviour of a rigid disc inclusion embedded in bonded contact with a saturated poroelastic medium of infinite extent is analytically and numerically analyzed and illustrated by considering the influence of loading conditions, interface drainage conditions, material parameters, as well as a length parameter. The thesis provides a general framework for the formulation and numerical solutions of the class of rigid disc inclusion problems associated with poroelastic media saturated with compressible pore fluids. The availability of benchmark solutions for these inclusion problems can be an advantage in the verification of computational schemes such as Finite Element Method (FEM) and Boundary Element Method (BEM) and the development of further results for practical problems of engineering interest.

Keywords

3-D consolidation, analytical solutions, compressible pore fluids, mechanics of inclusions, integral equations, integral transforms, numerical integration, numerical inversion, saturated poroelastic layered media.

Dedicated to the memory of my father and to my family

Acknowledgements

The author would like to express his deepest indebtedness and appreciation to his research supervisor, **Dr. A.P.S. Selvadurai**, Professor of Civil Engineering, Carleton University, for his guidance, support, and for his critical review of the manuscript of the thesis.

Thanks are due to **Dr. K.T. Law**, **Dr. G.E. Bauer**, **Dr. J.L. Humar**, **Dr. D.T. Lau**, and **Dr. W. Raman-Nair**, Professors of Civil Engineering, Carleton University, for their encouragement and comments.

Thanks are due to **Dr. J.N. Pandey**, Professor of Mathematics and Statistics, Carleton University, for his appreciation during the thesis proposal examination.

Special thanks are also due to **Dr. M.C. Au** and **Dr. B.M. Singh**, Post-Doctoral Fellows of Civil Engineering, Carleton University, for their advice and discussions.

Warm thanks are due to the author's colleagues, secretarial staffs and technicians in the Department of Civil Engineering, Carleton University, for their helps and friendship over past four years.

The financial supports provided by National Science and Engineering Research Council (Canada) through a research grand awarded to **Dr. A.P.S. Selvadurai** and by a Carleton University Scholarship awarded to the author are also acknowledged.

The author would also like to take this opportunity to thank **Professor Z.Q. Wang**, former Chief Engineer, Comprehensive Institute of Geotechnical Investigation and Surveying, Ministry of Construction China, and **Dr. R. Wang**, Professor of Applied Mechanics, Beijing University, for their continuous encouragement.

Finally, the author is obliged to express his sincere thanks to his mother and family members and friends in China for their long and persevering support.

Table of Contents

Abstract	ii
Key Words	iv
Acknowledgements	vi
Table of Contents	vii
List of Figures	xiv
List of Tables	xx
List of Symbols	xxiii
1 Introduction	1
1.1 General	1
1.2 Literature Review	12
1.2.1 General	12
1.2.2 <i>Rigid disc inclusions in linear elastic media</i>	12
1.2.2.1 General	12
1.2.2.2 <i>Rigid disc inclusion in an infinite space</i>	13
1.2.2.3 <i>Rigid disc inclusion in a halfspace</i>	24
1.2.2.4 <i>Rigid disc inclusion on a halfspace</i>	25
1.2.2.5 <i>Rigid disc inclusion at a bi-material region</i>	33
1.2.3 <i>Rigid disc inclusions in plastic, viscoelastic, and poroelastic media</i>	37
1.2.3.1 General	37
1.2.3.2 <i>Rigid disc inclusion in plastic media</i>	37
1.2.3.3 <i>Rigid disc inclusion in viscoelastic media</i>	39
1.2.3.4 <i>Rigid disc inclusion in poroelastic media</i>	40
1.2.4 Observations and discussions	44
1.3 Objectives and Scope of the Research	45
1.3.1 Objectives of the research	45
1.3.2 Scope of the research	46
2 Integral Transforms	49
2.1 Definitions and Basic Properties	49

2.1.1	<i>Fourier transforms in Cartesian coordinates</i>	49
2.1.2	<i>Fourier transforms in cylindrical coordinates</i>	51
2.1.2	<i>Hankel transforms in cylindrical coordinates</i>	53
2.1.2	<i>Laplace transforms</i>	54
2.2	<i>A General Relation between Hankel and Fourier integral transforms</i>	55
2.3	<i>The Tauberian theorems of Laplace transforms</i>	58
2.4	<i>Algorithms for Numerical Inversions of Laplace Transforms</i>	60
2.4.1	<i>General</i>	60
2.4.2	<i>Algorithms based on the sine-cosine transform for numerical inversions of Laplace transforms</i>	62
2.4.3	<i>A modified algorithm for numerical inversions of Laplace transforms</i>	65
3	Mechanics Of Poroelastic Media	67
3.1	<i>General</i>	67
3.2	<i>Governing Equations in the Physical Domain</i>	68
3.2.1	<i>General</i>	68
3.2.2	<i>Governing equations in general orthogonal curvilinear coordinates</i>	70
3.2.3	<i>Governing equations in Cartesian coordinates</i>	73
3.2.4	<i>Governing equations in cylindrical coordinates</i>	75
3.2.5	<i>Material properties of poroelastic media</i>	76
3.2.6	<i>Governing equations for three limiting responses</i>	77
3.2.6.1	<i>Initial response ($t \rightarrow 0^+$)</i>	79
3.2.6.2	<i>Final response ($t \rightarrow +\infty$)</i>	80
3.2.6.3	<i>Limiting response as $\nu \rightarrow \nu_u$</i>	81
3.3	<i>Governing Equations in the Transform Domain</i>	83
3.3.1	<i>General</i>	83
3.3.2	<i>Governing equations in the transform domains</i>	84
3.3.3	<i>Governing equations in the transform domains for three limiting responses</i>	91
4	Solution Representations In Poroelastic Media	94
4.1	<i>General</i>	94
4.2	<i>Solution Representations in a Poroelastic Layer</i>	96
4.2.1	<i>General</i>	96

4.2.2	<i>Fundamental matrix solution of the governing equations . . .</i>	96
4.2.3	<i>Basic properties of the fundamental matrices of solution . .</i>	100
4.2.4	<i>Solutions representations in a poroelastic Layer</i>	101
4.3	Solution Representation in a Poroelastic Halfspace	103
4.3.1	<i>General</i>	103
4.3.2	<i>Solution representation in the lower halfspace $a^+ \leq z < +\infty$</i>	104
4.3.3	<i>Solution representation in the upper halfspace $-\infty < z \leq b^-$</i>	105
4.3.4	<i>Solution representation in an infinite space</i>	106
4.4	Boundary Algebraic Equations	107
4.4.1	<i>General</i>	107
4.4.2	<i>Independent boundary equations at $z = a^+$</i>	108
4.4.3	<i>Independent boundary equations at $z = b^-$</i>	113
4.5	Boundary Algebraic Equations for the Three Limiting Responses	116
4.5.1	<i>General</i>	116
4.5.2	<i>Boundary equations at $z = a^+$ for the limiting responses . .</i>	116
4.5.3	<i>Boundary equations at $z = b^-$ for the limiting responses . .</i>	118
4.6	Fourier and Laplace Transform Based Approach	119
5	Mechanics Of Inclusions In Poroelastic Media	
	I. Analytical Formulation	122
5.1	General	122
5.2	Integral Equations of a Plane Inclusion in Poroelastic Media . .	123
5.2.1	<i>General mixed boundary and initial value conditions . . .</i>	123
5.2.2	<i>Symmetric solutions for a plane inclusion subjected to direct loads</i>	125
5.2.2.1	<i>Permeable inclusion problem</i>	125
5.2.2.2	<i>Impermeable inclusion problem</i>	130
5.2.3	<i>Systems of 2-D integral equations for plane inclusion problems</i>	135
5.2.3.1	<i>Vertical displacement of a plane inclusion</i>	135
5.2.3.2	<i>Horizontal displacements of a plane inclusion</i>	137
5.2.4	<i>Infinite sets of 1-D integral equations in Fourier series expansions</i>	140
5.2.4.1	<i>Vertical displacement of a disc inclusion</i>	142
5.2.4.2	<i>Horizontal displacements of a disc inclusion</i>	143
5.3	Integral Equations of Rigid Disc Inclusions in Poroelastic Media .	146

5.3.1	General	146
5.3.2	Symmetric solutions for a rigid disc inclusion subjected to direct loads	147
5.3.2.1	Vertical displacement of a rigid disc inclusion	147
5.3.2.2	Horizontal displacement of a rigid disc inclusion	151
5.3.3	Sets of integral equation for fundamental rigid disc inclusion problems	156
5.3.3.1	Vertical displacement of a rigid disc inclusion	156
5.3.3.2	Horizontal displacement of a rigid disc inclusion	158
5.4	Systems of Fredholm Integral Equations of the Second Kind for Rigid Disc Inclusions in Saturated Poroelastic Media	161
5.4.1	General	161
5.4.2	Fredholm integral equations for the axial translation of a rigid disc inclusion	164
5.4.3	Fredholm integral equations for the central rotation of a rigid disc inclusion	167
5.4.4	Fredholm integral equations for the in-plane translation of a rigid disc inclusion	170
5.4.5	Exact solutions for the axial rotation of a rigid disc inclusion	175
5.4.6	Exact solutions of the Fredholm integral equations for the three limiting responses	176
6	Mechanics Of Inclusions In Poroelastic Media	
	II. Numerical Evaluations	179
6.1	General	179
6.2	Axial Translation of a Rigid Disc Inclusion	180
6.2.1	Non-dimensional integral equations	180
6.2.2	Numerical solutions of the integral equations	182
6.3	Central Rotation of a Rigid Disc Inclusion	187
6.3.1	Non-dimensional integral equations	187
6.3.2	Numerical solutions of the integral equations	189
6.4	In-Plane Translation of a Rigid Disc Inclusion	193
6.4.1	Non-dimensional integral equations	193
6.4.2	Numerical solutions of the integral equations	197
6.5	Convergence and Accuracy of the Numerical Solutions	204
7	Mechanics Of Inclusions In Poroelastic Media	
	III. Analytical And Numerical Results	206

7.1	General	206
7.2	Axial Translation of a Rigid Disc Inclusion	208
7.2.1	Axial consolidation displacement of the inclusion	208
7.2.1.1	Influence of interface drainage conditions	208
7.2.1.2	Effects of material and length parameters	209
7.2.2	Axial force relaxation in the inclusion	216
7.2.2.1	Influence of interface drainage conditions	216
7.2.2.2	Effects of material and length parameters	217
7.2.3	Conclusions	218
7.3	Central Rotation of a Rigid Disc Inclusion	223
7.3.1	Central consolidation rotation of the inclusion	223
7.3.1.1	Influence of interface drainage conditions	225
7.3.1.2	Effects of material and length parameters	225
7.3.2	Central moment relaxation in the inclusion	232
7.3.2.1	Influence of interface drainage conditions	232
7.3.2.2	Effects of material and length parameters	233
7.3.3	Conclusions	235
7.4	In-Plane Translation of a Rigid Disc Inclusion	263
7.4.1	In-plane consolidation translation of the inclusion	241
7.4.1.1	Influence of interface drainage conditions	241
7.4.1.2	Effects of material and length parameters	241
7.4.2	In-plane force relaxation in the inclusion	248
7.4.2.1	Influence of interface drainage conditions	248
7.4.2.2	Effects of material and length parameters	249
7.4.3	Conclusions	250
7.5	Further Discussions and Conclusions	256
8	Conclusions And Recommendations	266
8.1	General	266
8.2	Summary and Conclusions	266
8.3	Recommendations for Future Work	272
	References	273
	Appendix A Stability And Accuracy Of The Modified Algorithm For Laplace Transform Inversion	290
A.1	General	290

A.2	Axial Settlement due to a Perfectly Flexible Disc Foundation on a Poroelastic Halfspace	292
A.3	Stability and Accuracy of the Modified Algorithm for Laplace Transform Inversion	295
Appendix B	Bounds For Material Parameters In Saturated Poroelastic Media	304
Appendix C	Integral Equations Of Rigid Disc Inclusions Under Three Limiting Conditions	309
C.1	General	309
C.2	Axial Translation of a Rigid Disc Inclusion	310
C.3	Central Rotation of a Rigid Disc Inclusion	311
C.4	In-plane Translation of a Rigid Disc Inclusion	312
C.5	Axial Rotation of a Rigid Disc Inclusion	313
Appendix D	Numerical Integration For Infinite Integrals With Rapidly Oscillatory Integrands	315
D.1	General	315
D.2	Simpson's Quadrature for Finite Integrals	316
D.3	Proceeding Limit Technique for Infinite Integrals	317
D.4	Approximation Technique for Rapidly Oscillatory Integrands	318
D.5	Evaluations of the Infinite Integrals in the Kernel Functions	320
Appendix E	Convergence And Accuracy Of Numerical Solutions For Fredholm Integral Equations	325
E.1	General	325
E.2	Axial Translation of a Rigid Disc Inclusion	325
E.2.1	<i>General</i>	325
E.2.2	<i>Axial consolidation displacement of the inclusion</i>	226
E.2.3	<i>Axial force relaxation in the inclusion</i>	334
E.2.4	<i>Conclusions</i>	334
E.3	Central Rotation of a Rigid Disc Inclusion	342
E.3.1	<i>General</i>	342
E.3.2	<i>Central consolidation rotation of the inclusion</i>	342
E.3.3	<i>Central moment relaxation in the inclusion</i>	350
E.3.4	<i>Conclusions</i>	350

E.4 In-Plane Translation of a Rigid Disc Inclusion	357
E.4.1 <i>General</i>	357
E.4.2 <i>In-plane consolidation translation of the inclusion</i>	357
E.4.3 <i>In-plane force relaxation in the inclusion</i>	364
E.4.4 <i>Conclusions</i>	364
E.5 Summary Conclusions	371

List of Figures

Figure	Description	Page
1.1.1	A definition of inclusion problems	1
1.1.2	Use of ground anchors for retaining wall supports (after Hanna, 1982)	2
1.1.3	Use of ground anchors for rock slope stabilization (after Hanna, 1982)	3
1.1.4	Constitutive modeling of saturated cohesive soils	6
1.1.5	A generalized displacements of rigid disc inclusions embedded in poroelastic media saturated with compressible fluids	9
1.2.1	A classification of rigid disc inclusion problems in elastic media	14
1.2.2	Geometry and loads of a rigid disc inclusion embedded in bonded contact with an elastic infinite space	16
1.2.3	General system of indirectly loaded rigid disc inclusion problems	16
1.2.4	Geometries and loads of the interaction between an inclusion and a crack in an isotropic elastic medium of infinite extent	21
1.2.5	Geometry and loads of a rigid disc inclusion embedded in an elastic halfspace and oriented parallel to the free surface	26
1.2.6	The four basic displacements of a rigid disc resting on an elastic halfspace	26
1.2.7	Geometry and loads of rigid disc smoothly or roughly resting on an elastic halfspace	28
1.2.8	Rigid disc smoothly resting on an elastic halfspace under indirectly concentrated point loads	28
1.2.9	Disc-shaped anchor zones embedded at bi-material geological interface (after Selvadurai and Au, 1996)	35
1.2.10	Indentation of a debonded interface	35
2.1.1	The coordinate systems in the physical domain and their mirror imaginaries in the transform domain	53
4.1.1	Eight field variables at the horizontal xy (or $r\theta$) plane in poroelastic media saturated with compressible fluids	95
7.2.1	Effects of drainage conditions on the axial translation of a rigid disc embedded in saturated poroelastic media: $\nu_u = 0.5$	211
7.2.2	Effects of drainage conditions on the axial translation of a rigid disc embedded in saturated poroelastic media: $\nu = 0.0$	211

7.2.3	Degrees of consolidation of axial translation of a permeable rigid inclusion against the modified time factor: $\nu_u = 0.5$	212
7.2.4	Degrees of consolidation of axial translation of a permeable rigid inclusion against the modified time factor: $\nu = 0.0$	212
7.2.5	Degrees of consolidation of axial translation of an impermeable rigid inclusion against the modified time factor: $\nu_u = 0.5$	213
7.2.6	Degrees of consolidation of axial translation of an impermeable rigid inclusion against the modified time factor: $\nu = 0.0$	213
7.2.7	Axial translation of a permeable rigid disc inclusion in fluid saturated poroelastic media: $\nu_u = 0.5$	214
7.2.8	Axial translation of a permeable rigid disc inclusion in fluid saturated poroelastic media: $\nu = 0.0$	214
7.2.9	Axial translation of an impermeable rigid disc inclusion in fluid saturated poroelastic media: $\nu_u = 0.5$	215
7.2.10	Axial translation of an impermeable rigid disc inclusion in fluid saturated poroelastic media: $\nu = 0.0$	215
7.2.11	Effects of drainage conditions on the axial force of a rigid disc embedded in saturated poroelastic media: $\nu_u = 0.5$	219
7.2.12	Effects of drainage conditions on the axial force of a rigid disc embedded in saturated poroelastic media: $\nu = 0.0$	219
7.2.13	Degrees of relaxation of axial force of a permeable rigid inclusion against the modified time factor: $\nu_u = 0.5$	220
7.2.14	Degrees of relaxation of axial force of a permeable rigid inclusion against the modified time factor: $\nu = 0.0$	220
7.2.15	Degrees of relaxation of axial force of an impermeable rigid inclusion against the modified time factor: $\nu_u = 0.5$	221
7.2.16	Degrees of relaxation of axial force of an impermeable rigid inclusion against the modified time factor: $\nu = 0.0$	221
7.2.17	Axial force of a permeable rigid disc inclusion in fluid saturated poroelastic media: $\nu_u = 0.5$	222
7.2.18	Axial force of a permeable rigid disc inclusion in fluid saturated poroelastic media: $\nu = 0.0$	222
7.2.19	Axial force of an impermeable rigid disc inclusion in fluid saturated poroelastic media: $\nu_u = 0.5$	223
7.2.20	Axial force of an impermeable rigid disc inclusion in fluid saturated poroelastic media: $\nu = 0.0$	223
7.3.1	Effects of drainage conditions on the central rotation of a rigid disc embedded in saturated poroelastic media: $\nu_u = 0.5$	227
7.3.2	Effects of drainage conditions on the central rotation of a rigid disc embedded in saturated poroelastic media: $\nu = 0.0$	227
7.3.3	Degrees of consolidation of central rotation of a permeable rigid inclusion against the modified time factor: $\nu_u = 0.5$	228
7.3.4	Degrees of consolidation of central rotation of a permeable rigid	

	inclusion against the modified time factor: $\nu = 0.0$	228
7.3.5	Degrees of consolidation of central rotation of an impermeable rigid inclusion against the modified time factor: $\nu_u = 0.5$	229
7.3.6	Degrees of consolidation of central rotation of an impermeable rigid inclusion against the modified time factor: $\nu = 0.0$	229
7.3.7	Central rotation of a permeable rigid disc inclusion in fluid saturated poroelastic media: $\nu_u = 0.5$	230
7.3.8	Central rotation of a permeable rigid disc inclusion in fluid saturated poroelastic media: $\nu = 0.0$	230
7.3.9	Central rotation of an impermeable rigid disc inclusion in fluid saturated poroelastic media: $\nu_u = 0.5$	231
7.3.10	Central rotation of an impermeable rigid disc inclusion in fluid saturated poroelastic media: $\nu = 0.0$	231
7.3.11	Effects of drainage conditions on the central moment of a rigid disc embedded in saturated poroelastic media: $\nu_u = 0.5$	236
7.3.12	Effects of drainage conditions on the central moment of a rigid disc embedded in saturated poroelastic media: $\nu = 0.0$	236
7.3.13	Degrees of relaxation of central moment of a permeable rigid inclusion against the modified time factor: $\nu_u = 0.5$	237
7.3.14	Degrees of relaxation of central moment of a permeable rigid inclusion against the modified time factor: $\nu = 0.0$	237
7.3.15	Degrees of relaxation of central moment of an impermeable rigid inclusion against the modified time factor: $\nu_u = 0.5$	238
7.3.16	Degrees of relaxation of central moment of an impermeable rigid inclusion against the modified time factor: $\nu = 0.0$	238
7.3.17	Central moment of a permeable rigid disc inclusion in fluid saturated poroelastic media: $\nu_u = 0.5$	239
7.3.18	Central moment of a permeable rigid disc inclusion in fluid saturated poroelastic media: $\nu = 0.0$	239
7.3.19	Central moment of an impermeable rigid disc inclusion in fluid saturated poroelastic media: $\nu_u = 0.5$	240
7.3.20	Central moment of an impermeable rigid disc inclusion in fluid saturated poroelastic media: $\nu = 0.0$	240
7.4.1	Effects of drainage conditions on the in-plane translation of a rigid disc embedded in saturated poroelastic media: $\nu_u = 0.5$	243
7.4.2	Effects of drainage conditions on the in-plane translation of a rigid disc embedded in saturated poroelastic media: $\nu = 0.0$	243
7.4.3	Degrees of consolidation of in-plane translation of a permeable rigid inclusion against the modified time factor: $\nu_u = 0.5$	244
7.4.4	Degrees of consolidation of in-plane translation of a permeable rigid inclusion against the modified time factor: $\nu = 0.0$	244
7.4.5	Degrees of consolidation of in-plane translation of an impermeable rigid inclusion against the modified time factor: $\nu_u = 0.5$	245

7.4.6	Degrees of consolidation of in-plane translation of an impermeable rigid inclusion against the modified time factor: $\nu = 0.0$	245
7.4.7	In-plane translation of a permeable rigid disc inclusion in fluid saturated poroelastic media: $\nu_u = 0.5$	246
7.4.8	In-plane translation of a permeable rigid disc inclusion in fluid saturated poroelastic media: $\nu = 0.0$	246
7.4.9	In-plane translation of an impermeable rigid disc inclusion in fluid saturated poroelastic media: $\nu_u = 0.5$	247
7.4.10	In-plane translation of an impermeable rigid disc inclusion in fluid saturated poroelastic media: $\nu = 0.0$	247
7.4.11	Effects of drainage conditions on the in-plane force of a rigid disc embedded in saturated poroelastic media: $\nu_u = 0.5$	251
7.4.12	Effects of drainage conditions on the in-plane force of a rigid disc embedded in saturated poroelastic media: $\nu = 0.0$	251
7.4.13	Degrees of relaxation of in-plane force of a permeable rigid inclusion against the modified time factor: $\nu_u = 0.5$	252
7.4.14	Degrees of relaxation of in-plane force of a permeable rigid inclusion against the modified time factor: $\nu = 0.0$	252
7.4.15	Degrees of relaxation of in-plane force of an impermeable rigid inclusion against the modified time factor: $\nu_u = 0.5$	253
7.4.16	Degrees of relaxation of in-plane force of an impermeable rigid inclusion against the modified time factor: $\nu = 0.0$	253
7.4.17	In-plane force of a permeable rigid disc inclusion in fluid saturated poroelastic media: $\nu_u = 0.5$	254
7.4.18	In-plane force of a permeable rigid disc inclusion in fluid saturated poroelastic media: $\nu = 0.0$	254
7.4.19	In-plane force of an impermeable rigid disc inclusion in fluid saturated poroelastic media: $\nu_u = 0.5$	255
7.4.20	In-plane force of an impermeable rigid disc inclusion in fluid saturated poroelastic media: $\nu = 0.0$	256
7.5.1	Degrees of consolidation displacements for the three basic modes of a permeable or impermeable rigid disc inclusion	260
7.5.2	Degrees of relaxation loads for the three basic modes of a permeable or impermeable rigid disc inclusion	260
7.5.3	Variation of ν_b with the undrained Poisson's ratio ν_u	264
A.1.1	An axisymmetric loading induced by a perfectly flexible circular foundation smoothly resting on a saturated poroelastic halfspace	291
D.5.1	Rapid oscillation of integrand and the approximation technique (real parts of the complex functions)	324
D.5.2	Rapid oscillation of integrand and the approximation technique (imaginary parts of the complex functions)	324

E.2.1	Influence of M on the numerical results of degree of consolidation of axial translation (permeable inclusion)	332
E.2.2	Influence of N on the numerical results of degree of consolidation of axial translation (permeable inclusion)	332
E.2.3	Influence of M on the numerical results of degree of consolidation of axial translation (impermeable inclusion)	333
E.2.4	Influence of N on the numerical results of degree of consolidation of axial translation (impermeable inclusion)	333
E.2.5	Influence of M on the numerical results of degree of relaxation of axial force (permeable inclusion)	340
E.2.6	Influence of N on the numerical results of degree of relaxation of axial force (permeable inclusion)	340
E.2.7	Influence of M on the numerical results of degree of relaxation of axial force (impermeable inclusion)	341
E.2.8	Influence of N on the numerical results of degree of relaxation of axial force (impermeable inclusion)	341
E.3.1	Influence of M on the numerical results of degree of consolidation of central rotation (permeable inclusion)	348
E.3.2	Influence of N on the numerical results of degree of consolidation of central rotation (permeable inclusion)	348
E.3.3	Influence of M on the numerical results of degree of consolidation of central rotation (impermeable inclusion)	349
E.3.4	Influence of N on the numerical results of degree of consolidation of central rotation (impermeable inclusion)	349
E.3.5	Influence of M on the numerical results of degree of relaxation of central moment (permeable inclusion)	355
E.3.6	Influence of N on the numerical results of degree of relaxation of central moment (permeable inclusion)	355
E.3.7	Influence of M on the numerical results of degree of relaxation of central moment (impermeable inclusion)	356
E.3.8	Influence of N on the numerical results of degree of relaxation of central moment (impermeable inclusion)	356
E.4.1	Influence of M on the numerical results of degree of consolidation of in-plane translation (permeable inclusion)	362
E.4.2	Influence of N on the numerical results of degree of consolidation of in-plane translation (permeable inclusion)	362
E.4.3	Influence of M on the numerical results of degree of consolidation of in-plane translation (impermeable inclusion)	363
E.4.4	Influence of N on the numerical results of degree of consolidation of in-plane translation (impermeable inclusion)	363
E.4.5	Influence of M on the numerical results of degree of	

	relaxation of in-plane force (permeable inclusion)	369
E.4.6	Influence of N on the numerical results of degree of relaxation of in-plane force (permeable inclusion)	369
E.4.7	Influence of M on the numerical results of degree of relaxation of in-plane force (impermeable inclusion)	370
E.4.8	Influence of N on the numerical results of degree of relaxation of in-plane force (impermeable inclusion)	370

List of Tables

Table	Description	Page
3.2.1	Relations for material parameters of saturated poroelastic media	78
7.5.1	The time-dependent behaviour of a rigid disc inclusion in a saturated poroelastic medium, case (a): consolidation displacement due to an applied constant load	258
7.5.2	The time-dependent behaviour of a rigid disc inclusion in a saturated poroelastic medium, case (b): force relaxation due to an applied constant displacement	259
7.5.3	A comparison of time for the consolidation process of an inclusion	261
7.5.4	A comparison of time for the relaxation process of an inclusion	261
7.5.5	The time limits for the consolidation process of soil	264
7.5.6	The flexibility or stiffness of a permeable rigid disc inclusion embedded in a saturated poroelastic medium	265
7.5.7	The flexibility or stiffness of an impermeable rigid disc inclusion embedded in a saturated poroelastic medium	265
A.3.1	A comparison between numerical inversion results $\Phi_n(x)$ and their exact values $\Phi_t(x)$: $x = \rho^2 ct/a^2$, $b_y = 0.1$	298
A.3.2	A comparison between numerical inversion results $\Phi_n(x)$ and their exact values $\Phi_t(x)$: $x = \rho^2 ct/a^2$, $b_y = 0.5$	299
A.3.3	A comparison between numerical inversion results $\Phi_n(x)$ and their exact values $\Phi_t(x)$: $x = \rho^2 ct/a^2$, $b_y = 1.0$	300
A.3.4	A comparison between numerical inversion results $U_n(0, ct/a^2)$ and their exact values $U_t(0, ct/a^2)$: $b_y = 0.1$	301
A.3.5	A comparison between numerical inversion results $U_n(0, ct/a^2)$ and their exact values $U_t(0, ct/a^2)$: $b_y = 0.5$	302
A.3.6	A comparison between numerical inversion results $U_n(0, ct/a^2)$ and their exact values $U_t(0, ct/a^2)$: $b_y = 1.0$	303
E.2.1	Influences of M on degree of consolidation $U_z(ct/a^2)$ permeable inclusion	328
E.2.2	Influences of M on axial translation $a\mu D_z(ct/a^2)/P_z$ permeable inclusion	328
E.2.3	Influences of N on degree of consolidation $U_z(ct/a^2)$ permeable inclusion	329
E.2.4	Influences of N on axial translation $a\mu D_z(ct/a^2)/P_z$ permeable inclusion	329
E.2.5	Influences of M on degree of consolidation $U_z(ct/a^2)$ impermeable inclusion	330

E.2.6	Influences of M on axial translation $a\mu D_z(ct/a^2)/P_z$ impermeable inclusion	330
E.2.7	Influences of N on degree of consolidation $U_z(ct/a^2)$ impermeable inclusion	331
E.2.8	Influences of N on axial translation $a\mu D_z(ct/a^2)/P_z$ impermeable inclusion	331
E.2.9	Influences of M on degree of relaxation $V_z(ct/a^2)$ permeable inclusion	336
E.2.10	Influences of M on total axial force $P_z(ct/a^2)/(a\mu D_z)$ permeable inclusion	336
E.2.11	Influences of N on degree of relaxation $V_z(ct/a^2)$ permeable inclusion	337
E.2.12	Influences of N on total axial force $P_z(ct/a^2)/(a\mu D_z)$ permeable inclusion	337
E.2.13	Influences of M on degree of relaxation $V_z(ct/a^2)$ impermeable inclusion	338
E.2.14	Influences of M on total axial force $P_z(ct/a^2)/(a\mu D_z)$ impermeable inclusion	338
E.2.15	Influences of N on degree of relaxation $V_z(ct/a^2)$ impermeable inclusion	339
E.2.16	Influences of N on total axial force $P_z(ct/a^2)/(a\mu D_z)$ impermeable inclusion	339
E.3.1	Influences of M on degree of consolidation $U_y(ct/a^2)$ permeable inclusion	344
E.3.2	Influences of M on central rotation $a^3\mu\Omega_y(ct/a^2)/M_y$ permeable inclusion	344
E.3.3	Influences of N on degree of consolidation $U_y(ct/a^2)$ permeable inclusion	345
E.3.4	Influences of N on central rotation $a^3\mu\Omega_y(ct/a^2)/M_y$ permeable inclusion	345
E.3.5	Influences of M on degree of consolidation $U_y(ct/a^2)$ impermeable inclusion	346
E.3.6	Influences of M on central rotation $a^3\mu\Omega_y(ct/a^2)/M_y$ impermeable inclusion	346
E.3.7	Influences of N on degree of consolidation $U_y(ct/a^2)$ impermeable inclusion	347
E.3.8	Influences of N on central rotation $a^3\mu\Omega_y(ct/a^2)/M_y$ impermeable inclusion	347
E.3.9	Influences of M on degree of relaxation $V_y(ct/a^2)$ permeable inclusion	351
E.3.10	Influences of M on total central moment $M_y(ct/a^2)/(a^3\mu\Omega_y)$ permeable inclusion	351
E.3.11	Influences of N on degree of relaxation $V_y(ct/a^2)$	

	permeable inclusion	352
E.3.12	Influences of N on total central moment $M_y(ct/a^2)/(a^3\mu\Omega_y)$ permeable inclusion	352
E.3.13	Influences of M on degree of relaxation $V_y(ct/a^2)$ impermeable inclusion	353
E.3.14	Influences of M on total central moment $M_y(ct/a^2)/(a^3\mu\Omega_y)$ impermeable inclusion	353
E.3.15	Influences of N on degree of relaxation $V_y(ct/a^2)$ impermeable inclusion	354
E.3.16	Influences of N on total central moment $M_y(ct/a^2)/(a^3\mu\Omega_y)$ impermeable inclusion	354
E.4.1	Influences of M on degree of consolidation $U_x(ct/a^2)$ permeable inclusion	358
E.4.2	Influences of M on in-plane translation $a\mu D_x(ct/a^2)/P_x$ permeable inclusion	358
E.4.3	Influences of N on degree of consolidation $U_x(ct/a^2)$ permeable inclusion	359
E.4.4	Influences of N on in-plane translation $a\mu D_x(ct/a^2)/P_x$ permeable inclusion	359
E.4.5	Influences of M on degree of consolidation $U_x(ct/a^2)$ impermeable inclusion	360
E.4.6	Influences of M on in-plane translation $a\mu D_x(ct/a^2)/P_x$ impermeable inclusion	360
E.4.7	Influences of N on degree of consolidation $U_x(ct/a^2)$ impermeable inclusion	361
E.4.8	Influences of N on in-plane translation $a\mu D_x(ct/a^2)/P_x$ impermeable inclusion	361
E.4.9	Influences of M on degree of relaxation $V_x(ct/a^2)$ permeable inclusion	365
E.4.10	Influences of M on total in-plane force $P_x(ct/a^2)/(a\mu D_x)$ permeable inclusion	365
E.4.11	Influences of N on degree of relaxation $V_x(ct/a^2)$ permeable inclusion	366
E.4.12	Influences of N on total in-plane force $P_x(ct/a^2)/(a\mu D_x)$ permeable inclusion	366
E.4.13	Influences of M on degree of relaxation $V_x(ct/a^2)$ impermeable inclusion	367
E.4.14	Influences of M on total in-plane force $P_x(ct/a^2)/(a\mu D_x)$ impermeable inclusion	367
E.4.15	Influences of N on degree of relaxation $V_x(ct/a^2)$ impermeable inclusion	368
E.4.16	Influences of N on total in-plane force $P_x(ct/a^2)/(a\mu D_x)$ impermeable inclusion	368

List of Symbols

Symbol	Description
MATERIAL PARAMETERS	
a	= radius of circular disc or constant
k	= coefficient of permeability
γ_w	= unit weight of pore fluid
κ	= k/γ_w
μ	= elastic shear modulus of matrix
ν	= drained Poisson's ratio
ν_u	= undrained Poisson's ratio
B	= Skempton's pore pressure coefficient
λ	= drained Lamb's constant
λ_u	= undrained Lamb's constant
K	= drained bulk modulus
K_u	= undrained bulk modulus
α	= Biot's material parameter
β	= Biot's material parameter
c	= generalized consolidation coefficient
c_m	= modified consolidation coefficient
COORDINATES AND OPERATIONS	
C^1	= continuity up to the first order differentiation
C^2	= continuity up to the second order differentiation
$O(x, y, z)$	= Cartesian coordinates in the physical domain
$O(\xi, \eta, z)$	= Cartesian coordinates in the transform domain
$O(r, \theta, z)$	= cylindrical coordinates in the physical domain
$O(\rho, \varphi, z)$	= cylindrical coordinates in the transform domain
O	= centroid location of inclusions or the original
$\mathbf{r}$	= distance vector of location
$\mathbf{r}_A$	= position vector of loads
$\frac{\partial}{\partial n}$	= partial differentiation about n
$\mathbf{n}$	= unit out-normal vector
$\mathbf{I}$	= unit square matrix or tensor
ξ	= coordinate in Fourier transform domain related to x
η	= coordinate in Fourier transform domain related to y
$\mathbf{e}_x$	= unite vector of x -coordinate
∇	= Hamilton operator
$\nabla \cdot \nabla$	= Laplace operator
t	= time

s	= complex independent variable in Laplace transform domain
ω	= imaginary part of complex coordinate s
$\frac{d^n}{dt^n}$	= n order ordinary differentiation about t
D	= domain of surrounding matrix
∂D	= boundary of inclusion and matrix

FIELD VARIABLES

$\mathbf{D}$	= translational vector at the centroid location of a rigid inclusion
D_x	= the component of translational vector in x -coordinate
$\mathbf{F}$	= load vector acting at an inclusion or body force vector acting on pore fluids
$\mathbf{f}$	= body force vector acting on soil skeleton
f_x	= x -component of body force vector acting on soil skeleton
F_x	= x -component of body force vector acting on pore fluids
$\mathbf{g}$	= invariant vector of body force acting on soil skeleton
$\mathbf{G}$	= invariant vector of body force acting on pore fluids
K	= symmetric Fourier kernel function
K^*	= complex conjugate of symmetric Fourier kernel function
$J_m(\rho r)$	= Bessel function of order m
$\mathbf{M}$	= moment vector acting at a rigid inclusion location of Cartesian or cylindrical coordinates
p	= exceed pore water pressure
$\mathbf{P}$	= load vector acting at the centroid of an inclusion
P_x	= component of load vector in x -coordinate
$\mathbf{P}^*$	= load vector acting in surrounding matrix
q	= fluid source
$\mathbf{T}$	= stress tensor
$\mathbf{T}_x$	= stress vector on the horizontal plane
$\mathbf{U}$	= general displacement vector of a rigid inclusion
$U_x(t)$	= degree of consolidation in-plane translation
$U_y(t)$	= degree of consolidation central rotation
$U_z(t)$	= degree of consolidation axial translation
$V_x(t)$	= degree of relaxation in-plane force
$V_y(t)$	= degree of relaxation central moment
$V_z(t)$	= degree of relaxation axial force
$\mathbf{u}$	= displacement vector of surrounding matrix
$u_x(x, y, z)$	= x -component of displacement vector in Cartesian coordinates
$u_x(r, \theta, z)$	= x -component of displacement vector in cylindrical coordinates
$u_r(r, \theta, z)$	= r -component of displacement vector in cylindrical coordinates
$u_\theta(r, \theta, z)$	= θ -component of displacement vector in cylindrical coordinates
$u_z(r, \theta, z)$	= z -component of displacement vector in Cartesian and cylindrical coordinates
$\tilde{u}(\xi, \eta)$	= two-dimensional Fourier transform of function $u(x, y)$ in Cartesian coordinates

$\tilde{u}(\rho, \varphi)$	= two-dimensional Fourier transform of function $u(r, \theta)$ in cylindrical coordinates
$\tilde{u}^m(\rho)$	= Hankel transform of order m
$\hat{u}(s)$	= Laplace transform of function $u(t)$
$\mathbf{v}$	= specific discharge vector of pore fluids
$\mathbf{w}$	= invariant displacement vector in the transform domain
w_1	= first component of $\mathbf{w}$
W	= strain energy density function
Ω	= rotational vector of a rigid inclusion
Ω_x	= component of rotational vector about x -coordinate
Γ	= strain tensor
ζ_v	= volumetric variation of pore fluids
σ_{xx}	= component of normal stress
σ_{xy}	= component of shear stress
ε_{xx}	= normal strain
ε_{xr}	= shear strain
ε_v	= volumetric strain of solid skeleton
τ	= invariant stress vector in the transform domain
τ_1	= first component of the invariant stress vector τ
ϑ	= invariant specific discharge vector vector in the transform domain
ϑ_1	= first component of ϑ

NUMERICAL INTEGRATIONS

n	= integer
m	= integer
T	= maximum time in the numerical inversion of Laplace transforms
S_m	= partial summation of Fourier series
ϵ_a	= relative error in numerical inversion of Laplace transforms
α_t	= constant in numerical inversion of Laplace transforms
α_s	= constant in numerical inversion of Laplace transforms
M	= M in $2M + 1$ the total number of Partial sum of Fourier series in numerical inversion of Laplace transforms
N	= total number of segments in the numerical solutions of integral equations
ϵ_b	= relative or absolute error in Simpson rule
ϵ_r	= absolute error in Proceeding limit technique

Chapter 1

Introduction

1.1 General

Inclusion problems in geomechanics which deal with the mechanics of an inhomogeneity and its surrounding soil media saturated with fluids constitute a special case of the general class of soil-structure interaction problems. In such problems, the shape of the interface region between the inhomogeneity and the saturated soil matrix is known *a priori* and remains constant for all magnitudes and time of the force applied either to the inclusion or to the soil matrix (Fig.1.1.1). The combined response of the soil-inclusion system is governed by (i) the mechanical behaviour of the surrounding saturated soils which are generally represented as media of either an infinite or a semi-infinite extent, (ii) the mechanical response of the inhomogeneities which occupy a subdomain in the saturated soil matrix, (iii) the contact conditions of the inclusion and soil interface, and (iv) other boundary and initial value conditions (such as external loads and displacements). Quite apart from their intrinsic mathematical and theoretical interests, such inclusion problems are of considerable importance to many engineering applications.

In geotechnical engineering, for example, there are many engineering structures which are supported by ground anchors. These structures include guyed towers, bridges, tension roofs, anchored bulkheads, as well as retaining walls (see e.g. Fig.1.1.2 and Fig.1.1.3). As supporting structures, the ground anchors have a significant effect on the stability and serviceability of these structures and facilities. This functional property of ground anchors is governed by their global pullout stiffnesses, *i.e.*, the load-displacement relations. If there is water (or fluid) in the surrounding soil media, these load-displacement relations would change with time due to the dissipation of the pore water pressure and the consolidation of the soil

Fig.1.1.1 A definition of inclusion problems

Fig.1.1.2 Use of Ground anchor for retaining wall support (after Hanna, 1982)

Fig.1.1.3 Use of ground anchors for rock slope stabilization (after Hanna, 1982)

skeleton. Other examples include in situ testing devices such as deep plate load tests and screw plate tests. It is considered that the model of an inclusion embedded in saturated soil media can be applied to analyze the mechanical behaviour of these engineering problems (Selvadurai and Nicholas, 1979, Selvadurai, Bauer, and Nicholas, 1980, Selvadurai and Gopal, 1986).

As indicated previously, the modeling of the mechanical behaviour of the surrounding soil medium is the first essential stage in the theoretical analysis of inclusion problems. The mechanical response of naturally occurring soils can be influenced by a variety of factors (Selvadurai, 1979a). These include (i) the shape, size and mechanical properties of the individual soil particles, (ii) the configuration of the soil fabric, (iii) the intergranular stresses and stress history, and (iv) the presence of soil moisture, the degree of saturation, and the soil permeability. These factors generally contribute to soils which display markedly non-linear, irreversible and time-dependent stress-strain phenomena and to soil masses which exhibit anisotropic and non-homogeneous material properties. Researches in geotechnical engineering have placed great emphasis on the laboratory investigations and mathematical modeling of these fundamental phenomena (see e.g. Desai and Siriwardane, 1984). Due to the complicated nature of these phenomena, it may be very difficult to take into account all possible non-linear and time-dependent material characteristics in modeling and analysing a particular geotechnical engineering problem. In order to obtain meaningful and reliable information for use in engineering practice (such as the global pullout stiffnesses of ground anchors, settlement of foundation, etc.), it is necessary to idealize soils by taking into account their dominant characteristics. The time-dependent coupling between the deformation of the soil skeleton and the flow of pore fluid through the voids is one of such phenomena that is of interest to the scope of this thesis.

Saturated soils (and rocks) consist of an assemblage of particles with different sizes and shapes which form a deformable soil skeleton whose voids are filled with fluid (water or oil) and gas (or air). This multi-phase nature of soils makes the

fundamental properties of soils significantly different from other single phase materials (solid, liquid, or gas). The mechanical behavior of soils is generally a function of the behaviour of the soil fabric and the interaction among the solid, fluid, and gas phases. Under the hypothesis that the pore fluid phase and pore gas phase are mixed in one so-called 'gasified' fluid (or air-water mixture), partially saturated soils can be regarded as a quasi-two phase medium: a solid skeleton containing a compressible fluid (Zaretskii, 1972). When loadings are applied to saturated soils (such as clays) by external agents such as foundations or ground anchors, they cause an increase in pore fluid pressure. Initially the soil skeleton and structure will undergo an immediate deformation with accompanying pore pressures.

With time, the pore pressures will dissipate as pore fluid flows due to the pore pressure gradients within the medium. As the excess pore pressures change, the stresses and strains in the soil skeleton and structure will simultaneously compensate to maintain the system in equilibrium and to satisfy the conditions of mechanical and hydraulic compatibility. This leads to further deformation of soil skeleton and structure. The coupled nature of the problem ensures that the deformation of soil skeleton will have an effect on the excess pore pressures, and vice versa. Consequently, the consolidation process of saturated soils is dominated by the time-dependent coupling between the deformations of the soil skeleton and the flow of pore fluid through the voids within the skeleton.

The initial observation of the process of one-dimensional soil consolidation is due to Terzaghi (1923). Since Terzaghi's initial studies, several formal mathematical theories have been developed to model the process involved in three dimensional consolidation of soils (Fig.1.1.4). Among these, the classical theory of linear, porous, elastic medium saturated with compressible fluids formulated by Biot (1935, 1941, 1955, 1956) is frequently utilized in the analysis of soil-structure interaction problems. This idealized theory is not an exact description of even the gross physical properties of real saturated soil media. However, this theory is rational unification of the classical theory of elastostatics and the seepage flow which can be character-

Fig.1.1.4 Constitutive modeling of saturated cohesive soils

ized by Darcy's law (see e.g. Das, 1983) for porous media. This theory provides a simplified description on dominant characteristics of the interaction between the soil skeleton and the pore fluid of saturated soils. Furthermore, this theory represents a useful first approximation for the mechanics of inclusions embedded in saturated cohesive soils provided the loads imparted to the soil skeleton and/or pore fluid are within the "working load" range of the soil. It is implicitly assumed that under such conditions, the working loads do not induce appreciable plastic deformation of soil skeleton and/or turbulent flow of the pore fluid.

The next essential stage in the theoretical analysis of inclusion problems is the modeling of the mechanical response of inclusions. The behaviour of the inclusion is mainly governed by the deformability characteristics and geometry of the embedded structural elements.

Based on the "relative stiffness" of embedded structural elements and their surrounding soil matrix, the inclusion problems may be classified into at least two distinct classes.

(a) Flexible inclusion problems: In this category, the inclusion is deformable and its stiffness is comparable to that of the soil matrix. The flexible inclusion is usually represented in terms of elastic structural element such as beam, rod, plate, or shell in order to model, for example, the flexural behaviour of buried pipelines, piles, flexible mats and raft foundations, tunnels, etc. Detailed accounts of recent advances of flexible inclusion (structural) problems could be found in the articles by Selvadurai (1979a); Niumpradit and Karasudhi, (1981); and Booker and Small (1984; 1986).

(b) Rigid inclusion problems: In this category, the inclusion is a rigid inhomogeneity within which there is no differential displacement between any two points, i.e., under the action of external or internal loads the inclusion undergoes only rigid body translations and/or rotations. In general, the displacement vector U at any point within the inclusion region can be represented in the form,

$$\mathbf{U} = \mathbf{D} + \mathbf{r} \times \mathbf{\Omega} \quad (1.1.1)$$

where $\mathbf{D} = (D_x, D_y, D_z)$ is the translational vector at the centroid location $O(0, 0, 0)$ of the inclusion and $\mathbf{\Omega} = (\Omega_x, \Omega_y, \Omega_z)$ is the rotational vector of the inclusion about the Cartesian axes x, y, z originated at the centroid O , $\mathbf{r} = (x, y, z)$ is a position vector measured from the centroid O to any field point within the inclusion region (Fig.1.1.5).

The total load vector $\mathbf{F}$ acting on the rigid inclusion can be statically replaced by a force vector $\mathbf{P}$ acting at the centroid O and a moment vector $\mathbf{M}$ acting about the three Cartesian axes x, y, z . The relations between $\mathbf{F}$ and $\mathbf{P}$, $\mathbf{M}$ can be written as

$$\begin{aligned} \mathbf{P} &= \mathbf{F} \\ \mathbf{M} &= \mathbf{r}_a \times \mathbf{F} \end{aligned} \quad (1.1.2)$$

where $\mathbf{r}_a$ is a position vector drawn from centroid O to any point (x_a, y_a, z_a) lying on the line of the total load vector $\mathbf{F}$.

In geotechnical engineering, rigid inclusion model can be considered as a suitable first approximation for the study of the mechanical response of ground anchors, piers, or cylindrical, etc. which possess stiffness much higher than the surrounding soil. The stiffness relations between the displacement vector $\mathbf{U}$ and the total load vector $\mathbf{F}$ of a rigid inclusion and soil system have important applications to engineering practices. Such relationships are time independent for mechanics of rigid inclusions embedded in the elastic or plastic media and are time-dependent for inclusion problems in saturated poroelastic or viscoelastic (due to creep of soil matrix) media. Consequently there are at least two groups of loadings associated with inclusions in saturated poroelastic or viscoelastic media. In one group the time dependent behaviour of total load vector $\mathbf{F}$ can be prescribed whereas in the second the time history of the rigid displacement vector $\mathbf{U}$ can be prescribed.

Several models of symmetrical inclusions, such as spherical, spheroidal, cylin-

Fig.1.1.5 The generalized displacements of rigid disc inclusion embedded in poroelastic media saturated with compressible pore fluids

drical, elliptical disc, annular disc, and circular disc (penny-shaped) inclusions have been utilized quite extensively in studies pertaining to the time independent mechanics of rigid inclusions embedded in linear elastic infinite and/or halfspace media (Collins, 1962; Kanwal and Sharma, 1976; Kassir and Sih, 1968; Luk and Keer, 1980; Selvadurai, 1976, 1978, 1984a, 1984b; Selvadurai and Singh, 1984; Selvadurai and Au, 1986; Zureick, 1988). From these investigations, it is observed that the geometric shapes of rigid inclusions have significant influence not only on the local stress and deformation patterns in the soil matrix surrounding the inclusions but also on the overall pull-out stiffness of the soil-inclusion systems. Also, with inclusions having a three dimensional form, the mathematical operations involved even for linear elasticity problems is usually quite complicated (see, eg. Mura, 1982; 1988). Selvadurai (1976), Kanwal and Sharma (1976), Selvadurai (1987), Selvadurai et al (1987), Selvadurai and Dasupta (1990) have shown that symbolic computational schemes can be used to considerable advantage in the study of 3-D inclusion problems.

Based on these observations, it is recognized that it is of engineering interest and mathematical feasibility to focus attention on the analytical treatment of the embedded rigid disc inclusion problems. The disc or penny-shaped inclusions are a particular limiting case of the general class of three-dimensional rigid inclusions (inhomogeneities) and have the simplest geometry of thin circular disc with finite or zero thickness. Due to its axisymmetric geometry, rigid disc inclusion has only four fundamental displacement modes; these are the axisymmetric and in-plane translations D_z , D_r and the axial and central rotations Ω_z , Ω_y (Fig.1.1.5). Correspondingly, there are only four fundamental loading modes; these are the axial and in-plane loads P_z , P_r and the axial and central moments M_z , M_y . Together with other conditions pertaining to the interfaces, the boundaries and material properties of soil media, these four fundamental displacement modes of rigid disc inclusion constitute a class of mixed boundary value problems which have applications to geomechanics.

The last essential aspect in the theoretical analysis of inclusions embedded in soil media is the specification of conditions at the inclusion-soil interface. There are two kinds of interface conditions for the inclusions embedded in saturated soils. The first represents the bonded contact between the soil skeleton and the inclusion. Such conditions include fully bonded, partially debonded, smoothly bonded contact. The selection of a contact condition for an inclusion problem is dependent on the particular engineering application of interest. Among these conditions, the fully bonded connection is a reasonable approximation and provides a basis for the calibration of other contact conditions (Selvadurai and Yue, 1992).

The second is the drainage conditions at the surface of the inclusion. It is extremely difficult to specify precisely the drainage boundary conditions that may exist in a practical problem related to an embedded foundation element. For this reason, there are two limiting cases which offer the "upper" and "lower" estimates for all the drainage conditions that can occur at the inclusion surfaces. One is the permeable (drained) condition where the excess pore pressures are zero at the inclusion surfaces. The other is the impermeable (undrained) condition where the normal gradient of the excess pore pressure is zero. Mathematically, these two limiting conditions can be expressed as follows,

$$p(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \partial D \quad (1.1.3)$$

or

$$\frac{\partial}{\partial \mathbf{n}} p(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \partial D \quad (1.1.4)$$

where p is the excess pore water pressure in the surrounding saturated soil matrix, ∂D are the surfaces of the inclusion and $\{-\mathbf{n}\}$ is the outward unit normal vector to ∂D .

The focus of this research is to develop analytical solutions to the class of problems related to disc shaped rigid inclusions which are embedded in porous elastic media saturated with a compressible pore fluid. An examination of the related literature indicates that (i) the mechanics of rigid disc inclusion problems have been

appreciated and applied in the geotechnical context for a considerable time, (ii) the majority of the investigations of rigid disc inclusion problems have concentrated on the evaluation of the time-independent response of such inclusions embedded in elastic media of infinite and semi-infinite extent, and (iii) the studies of the mechanics of rigid disc inclusions embedded in saturated soils is limited attention. It is hoped that the results of this research programme will provide a systematic methodology for the analytical formulation and computational study of the mechanics of rigid disc inclusion problems related to poroelastic media and contribute to the understanding of relevant geotechnical engineering problems encountered with ground anchors, rigid foundations and in situ tests which are located in saturated soils and/or rocks.

1.2 Literature Review

1.2.1 General

As mentioned previously, the theory of saturated poroelastic soil model is physically and mathematically based on rational coupling of classical elastostatics together with the seepage flow. It will be shown in Chapter 3 that the governing equations of poroelastic media become uncoupled and reduced to those of classical elastostatics under particular conditions. Therefore, it is instructive to record the results of rigid disc inclusions embedded in linear elastic media of infinite and/or halfspace extents. The publications which deal with the mechanics of rigid disc inclusions located in plastic, viscoelastic, or saturated poroelastic soil media will be reviewed in section 2.3, with emphasis on the influence of time effects of soil consolidation phenomenon.

1.2.2 Rigid disc inclusions in linear elastic media

1.2.2.1 General

In geomechanics studies relating to inclusion problems, it is generally assumed that the surrounding soil medium can be adequately represented by an elastic

medium occupying either an infinite region or a semi-infinite region. An elastic medium of either an infinite or a semi-infinite extent is one of the simplest idealization of the domains related to many problems in engineering applications. These representations are convenient mathematical idealizations for developing analytical solutions for practical problems. For example, the well-known Boussinesq (1885) solution for the static response of an elastic halfspace whose outer surface subjected to a concentrated normal force offers wide applications to loading problems in various branches of engineering including geotechnical engineering. The solutions for rigid disc inclusions problems in current literature can be classified into four categories, namely, (i) rigid disc inclusions in an elastic infinite space; (ii) rigid disc inclusions in an elastic halfspace; (iii) rigid disc inclusions on an elastic halfspace ; (iv) rigid disc inclusions at a bi-material region. In each category, the rigid disc inclusion problem can be further defined on the basis of the interface contact conditions, material anisotropy and external loading (Fig.1.2.1). In the ensuing, the solutions of disc inclusion problems will be described and examined in the light of the following information.

- (1) The mathematical formulations of the appropriate mixed boundary value problems;
- (2) The solution methods and the numerical evaluations;
- (3) The applications of results to geotechnical engineering.

1.2.2.2 Rigid disc inclusion in an infinite space

A basic assumption in the modeling of rigid inclusions embedded in elastic infinite media is that the presence of external boundaries of the surrounding matrix (soil) does not in any way influence the mechanical behaviour of the inclusions. The extent to which this assumption is realized in practice, will depend upon a number of factors, including the geometrical shape of the inclusion, the relative deformability characteristics of the soil matrix and the inclusion and the contact conditions at the inclusion-soil interface. For example, with rigid disc inclusions embedded in elastic media, a key assumption is that the depth of embedment of the inclusions is large

Fig.1.2.1 A classification of rigid disc inclusion problems in elastic media

enough in comparison with the dimensions of the inclusions. The class of problems related to rigid disc inclusions embedded in an elastic infinite space have been widely examined in connection with the estimation of the short-term working load range stiffness of deep ground anchor plates. In such investigations, the influence of complete bonding or partial debonding at the inclusion/soil medium interface, interaction between inclusions and cracks and anisotropy of the soil medium are taken into account. The particular geometry of the inclusion model enables the investigations to be conducted by appeal to mixed boundary value problems associated with a halfspace region. These mixed boundary value problems usually lead to the consideration of systems of Fredholm integral equation of second kind.

The first category of problems relates to the rigid disc inclusion embedded in complete bonded contact with either an isotropic or a transversely isotropic elastic infinite medium. This model also assumes that either the selfweight of the material or the adhesive strength at the inclusion-soil medium interface is sufficient to prevent debonding. According to the location of the external load, this class of problems can be further divided into two sub-cases; namely, (i) the directly loaded rigid disc inclusion problems; (ii) the indirectly loaded rigid disc inclusion problems. It is found that the rotational and translational stiffnesses can be evaluated in exact closed forms for a majority of the complete bonded disc inclusion problems (Selvadurai, 1982a). The regular stress singularity associated with the problems is of order $-\frac{1}{2}$ at the edge of the circular boundary of disc inclusions, $r = a$.

In the sub-case (i), the external loads are applied directly to the rigid inclusion region in four basic fashions which involve axisymmetric or asymmetric loadings located along the axes of geometric or loading symmetry (Fig.1.2.2). It was found that the two translations and two rotations of the rigid disc inclusion induced by these four loads are uncoupled by virtue of the symmetries about the vertical axis z and/or the horizontal plane xy . In other words, there is only one load-translational or moment-rotational relationship in each load fashion, namely, $P_x \sim D_x$; $P_z \sim D_z$; $M_x \sim \Omega_x$; or $M_y \sim \Omega_y$.

Fig.1.2.2 Geometry and loads of a rigid disc inclusion embedded in bonded contact with an elastic infinite space

Fig.1.2.3 General system of indirectly loaded rigid disc inclusion problems

The axisymmetric problem related to a rigid disc inclusion embedded in complete bonded contact with an isotropic elastic infinite medium was first examined by Collins (1962) using complex potential function techniques. The normal stress distribution $\sigma_{zz}(r, \theta, 0^\pm)$ at the interface $z = 0^+$ and $z = 0^-$ is given in cylindrical coordinates $(Or\theta z)$ as

$$\sigma_{zz}(r, \theta, 0^\pm) = \begin{cases} \mp \frac{8\mu(1-\nu)}{\pi(3-4\nu)\sqrt{a^2-r^2}} D_z, & 0 \leq r \leq a \\ 0, & a \leq r < +\infty \end{cases} \quad (1.2.1)$$

where $r = \sqrt{x^2 + y^2}$, μ is the linear elastic shear modulus, ν is Poisson's ratio, and a is the radius of rigid disc. The superscript $(+)$ and $(-)$ refer to the location of the interface normal stresses with respect to the separate halfspace regions $0^+ \leq z < +\infty$ and $-\infty < z \leq 0^-$, respectively.

The relationship between the vertical load P_z and translation D_z is

$$P_z = - \int_0^a \int_0^{2\pi} [\sigma_{zz}(r, \theta, 0^+) - \sigma_{zz}(r, \theta, 0^-)] d\theta r dr = \frac{32\mu(1-\nu)a}{3-4\nu} D_z \quad (1.2.2)$$

This expression of load-translation relationship was also obtained independently as a limiting result of rigid spheroidal inclusions in the works of Kanwal and Sharma (1976) and Selvadurai (1976) who used singularity methods and direct spheroidal harmonic function techniques, respectively.

Keer (1965) investigated the asymmetric problem concerning the in-plane translation D_x (displacement in its own plane) of the bonded rigid disc inclusion in an isotropic elastic infinite medium using complex potential function technique. The displacements and stresses computed on the horizontal plane $z = 0$ are as follows,

$$\begin{aligned} u_x(r, \theta, 0) &= \begin{cases} D_x, & 0 \leq r \leq a \\ \frac{2}{\pi} [\sin^{-1} \frac{a}{r} + \frac{a\sqrt{r^2-a^2}}{(7-8\nu)r^2} \cos 2\theta] D_x, & a < r < +\infty \end{cases} \\ u_y(r, \theta, 0) &= \begin{cases} 0, & 0 \leq r \leq a \\ \frac{2}{\pi} [\frac{a\sqrt{r^2-a^2}}{(7-8\nu)r^2} \sin 2\theta] D_x, & a < r < +\infty \end{cases} \\ u_z(r, \theta, 0) &= 0, & 0 \leq r < +\infty \end{aligned} \quad (1.2.3)$$

$$\begin{aligned} \sigma_{xx}(r, \theta, 0) &= \begin{cases} -\frac{16(1-\nu)\mu}{\pi(7-8\nu)\sqrt{a^2-r^2}} D_x, & 0 \leq r \leq a \\ 0, & a < r < +\infty \end{cases} \\ \sigma_{yy}(r, \theta, 0) &= 0, & 0 \leq r < +\infty \\ \sigma_{zz}(r, \theta, 0) &= \begin{cases} 0, & 0 \leq r \leq a \\ \frac{8(1-2\nu)(1-\nu)\mu}{\pi(7-8\nu)\sqrt{r^2-a^2}} \frac{a}{r} \cos \theta D_x, & a < r < +\infty \end{cases} \end{aligned} \quad (1.2.4)$$

The load-translation relationship for the inclusion takes the form

$$P_z = - \int_0^a \int_0^{2\pi} [\sigma_{zz}(r, \theta, 0^+) - \sigma_{zz}(r, \theta, 0^-)] d\theta r dr = \frac{64\mu(1-\nu)a}{7-8\nu} D_z \quad (1.2.5)$$

The problem of a rigid disc inclusion embedded in complete bonded contact with an isotropic elastic medium of infinite extent and subjected to both a central vertical load P_z and a central moment M_y was considered by Selvadurai (1980a) using the Hankel integral transform technique which reduces the problem into two sets of uncoupled dual integral equations. The load-translation relationship and the contact normal stress at the interfaces due to the vertical load P_z derived are exactly as same as those given Collins (1962) in *eqs.*(1.2.1) and (1.2.2). The additional results due to the central moment M_y are obtained as follows,

$$\begin{aligned} \sigma_{zz}(r, \theta, 0^\pm) &= \begin{cases} \mp \frac{2}{\pi} \left[\frac{8\mu(1-\nu)r}{(3-4\nu)\sqrt{a^2-r^2}} \cos \theta \right] \Omega_y, & 0 \leq r \leq a^- \\ 0, & a^+ \leq r < +\infty \end{cases} \\ M_y &= - \int_0^a \int_0^{2\pi} [\sigma_{zz}(r, \theta, 0^+) - \sigma_{zz}(r, \theta, 0^-)] r^2 \cos \theta d\theta dr \\ &= \frac{64\mu(1-\nu)a^3}{3(3-4\nu)} \Omega_y \end{aligned} \quad (1.2.6)$$

where Ω_y is the resulting rigid rotation of the rigid disc inclusion about the y -axis. These solutions were employed to examine the behaviour of the embedded rigid circular disc foundation under an eccentric load F_z where $P_z = F_z$, $M_y = F_z e$, e = eccentricity of the load.

The axial rotational stiffness (or torsional stiffness) for the embedded rigid disc inclusion in rotationally (or torsionally) symmetric deformation was given by Selvadurai (1981a). The results obtained via Hankel integral transform and dual integral equation techniques give the following:

$$M_z = - \int_0^a \int_0^{2\pi} [\sigma_{\theta z}(r, \theta, 0^+) - \sigma_{\theta z}(r, \theta, 0^-)] r^2 d\theta dr = \frac{32\mu a^3}{3} \Omega_z \quad (1.2.7)$$

Selvadurai (1979d, 1980b) further extended these four rigid disc inclusion problems to take into account the anisotropy of the surrounding soil matrix. The rigid disc inclusion is embedded in complete bonded contact with a transversely isotropic elastic medium of infinite extent whose axis of elastic symmetry is normal to the plane of the disc inclusion. These mixed boundary value problems are also formulated in terms of Hankel integral transforms and reduced to the systems of dual integral equations. The two translational and one rotational stiffnesses as well as the contact stresses for the embedded rigid disc inclusion are derived in exact closed forms, which include the results in equations (1.2.1) ~ (1.2.7) for the rigid disc inclusion embedded in bonded contact with an isotropic elastic matrix.

In the sub-case (ii), the displacements of the rigid disc inclusion are caused by a concentrated point load $\mathbf{P}$ or couple $\mathbf{M}$ which act at location $Q(b, c, d)$ exterior to the inclusion region (Fig.1.2.3). By taking the advantages of symmetries of some specific loads and their locations, several investigators have succeeded in solving some interaction problems between the elastically embedded rigid inhomogeneity and the specific external concentrated forces. Due to the complexity in formulating and solution of the mixed boundary value problems, however, solutions have not been developed for the rigid disc inclusion problems under the action of concentrated point external loads ($\mathbf{P}$ or $\mathbf{M}$) acting at a general location $Q(b, c, d)$.

Using complex potential function techniques, Collins (1962) investigated the axisymmetric interaction problem between a rigid disc inclusion and a concentrated point load P_z located at an exterior point $(0, 0, d)$ along the z -axis. The normal contact stress $\sigma_{zz}(r, \theta, 0^\pm)$ is evaluated in exact closed form and the vertical translational stiffness could be presented in the following form (also see Selvadurai, 1981a),

$$D_z = \frac{P_z}{16\pi(1-\nu)\mu a} \left[(3-4\nu) \tan^{-1} \frac{a}{d} + \frac{ad}{a^2 + d^2} \right] \quad (1.2.8)$$

By using the Hankel integral transform and dual integral equation method, Selvadurai (1979c, 1981a, 1982a) have examined some axisymmetric and asymmetric rigid disc inclusion problems (see Fig.1.2.3) where the disc inclusion is subjected to

concentrated point forces P_y , M_z , and P_z^* located at the exterior points $O_1(0, 0, d)$, $O_2(b, 0, 0)$, respectively. The results for the translations and rotations of the rigid disc inclusion induced by the external load are evaluated in exact closed form as follows.

$$\begin{aligned}
 D_y &= \frac{P_y}{32\pi(1-\nu)\mu a} \left[(7-8\nu) \tan^{-1} \frac{a}{d} - \frac{ad}{a^2+d^2} \right] \\
 \Omega_y &= \frac{3dP_y}{32\pi(1-\nu)\mu a^3} \left[\tan^{-1} \frac{a}{d} - \frac{ad}{a^2+d^2} \right] \\
 \Omega_z &= \frac{3M_z}{16\pi\mu a^3} \left[\tan^{-1} \frac{a}{d} - \frac{ad}{a^2+d^2} \right] \\
 U_z(r, \theta, 0) &= \frac{P_z^*(3-4\nu)}{32(1-\nu)\mu a} \left[\frac{2}{\pi} \sin^{-1} \frac{a}{b} + \frac{3br}{2a^2} R(a, b) \cos \theta \right]
 \end{aligned} \tag{1.2.9a}$$

where

$$R(a, b) = 1 - \frac{2}{\pi} \tan^{-1} \sqrt{\frac{b^2}{a^2} - 1} - \frac{2}{\pi} \sqrt{1 - \frac{a^2}{b^2}} \tag{1.2.9b}$$

It was shown (Selvadurai 1981a; 1982a) that these results in equations (1.2.9) can also be evaluated by appeal to Betti's reciprocal theorem. The auxiliary solutions required for the application of the reciprocal theorem were derived from the analysis of the directly loaded rigid disc inclusion problems.

The second category of problems deals with the interaction between rigid disc inclusions and cracks located at an elastic medium of infinite extent. The physical formulation of these problems are governed by either a partially debonding contact at the inclusion/matrix interface or a crack existing beyond the boundary of rigid disc inclusions (Fig.1.2.4). The presence of the cracked region decreases the elastic stiffness of the embedded inclusion and the ensuing mixed boundary value problems are mathematically complicated. By using the Hankel integral transform, systems of integral equations may be formulated for solving the directly loaded interaction problems between a disc inclusion and an in-plane crack. The induced integral equations may be further solved in an exact fashion, an approximate fashion, or a numerical fashion.

Fig.1.2.4 Geometries and loads of the interaction between an inclusion and a crack in an isotropic elastic medium of infinite extent.

Koer (1975) examined the axisymmetric problem related to a directly loaded rigid disc inclusion in which complete debonding contact occurs at one interface while complete bonded contact occurs at the other interface (Fig.1.2.4.a). By using four auxiliary functions related to the jumps in displacements and stresses across the disc, the solution was achieved through two uncoupled Hilbert problems. The interesting feature of the solution at the interface plane is that there are two stress singularities associated with the problem. They are of order $-\frac{1}{4}$ and $-\frac{3}{4}$. These two stress singularities show oscillations of a type similar to that found in the contact problem related to a rigid punch perfectly bonded to a halfspace. It was noted that the singularities are not oscillatory for an incompressible elastic medium ($\nu = \frac{1}{2}$). The load-translation relationship was evaluated in an exact closed form.

The symmetrical problem of the internal indentation of a penny-shaped crack by a smoothly embedded rigid disc inclusion of small thickness (Fig.1.2.4.b) was investigated by Selvadurai and Singh (1984b). This problem is of interest to the study of thermally induced fracturing and degradation of multiphase composites. The analysis of the problem yielded a system of triple integral equations. These integral equations were further reduced to a single Fredholm integral equation of the second kind which was solved in an approximate fashion. The expression for the stress intensity factor at the boundary of the penny-shaped crack due to the wedging action of the smoothly embedded rigid inclusion was evaluated in the form of a power series which is represented in terms of a parameter which represents the ratio of the radius of the disc to the radius of the crack. Further studies in this class of problems is given by Tan and Selvadurai (1986).

Selvadurai and Singh (1986) extended the above study to include direct axisymmetric loading of the smoothly wedged disc inclusion by a central force P_z (Fig.1.2.4.c). The resulting integral equations for this mixed boundary value problem are solved in an exact fashion from which the closed form results are obtained for axial stiffness of the inclusion and the stress intensity factor at the boundary of the penny-shaped crack. In the limit of material incompressibility (when $\nu = \frac{1}{2}$),

the geometry of the penny-shaped crack has no influence on the axial stiffness.

Selvadurai and Singh (1987) and Selvadurai, Au and Singh (1990) continued the investigation by considering respectively the axisymmetric translation and in-plane translation of a rigid disc inclusion embedded in bonded contact with an isotropic elastic medium of infinite extent where the plane containing the inclusion is weakened by an external circular crack (Fig.1.2.4.d). The integral equations associated with the mixed boundary conditions are reduced to either two coupled Abel-type integral equations or systems of Fredholm integral equations of the second kind which are solved respectively by employing a power series approximation or quadrature schemes. The expressions in power series are derived for the effective stiffness of the inclusion located at the weakened plane and the stress intensity factor at the boundary of the external circular crack. The accuracy of the power series expansion scheme is further verified by a directly numerical scheme of Gaussian quadrature. It is also evident that the extent of cracking at the plane containing the disc inclusion has no effect on the axial stiffness of the inclusion for the incompressible elastic matrix.

Recently, Selvadurai (1989) and Selvadurai and Singh and Au (1989) have examined the influence of a boundary fracture on the elastic stiffness of an inclusion embedded in bonded and partial debonded contact with an isotropic elastic medium of infinite extent (Fig.1.2.4.e and Fig.1.2.4.f), respectively. It was considered that the mathematical formulation based on the Hilbert problem is necessary to accurately evaluate the exact stress distributions or the stress intensity factors at the boundaries of the rigid inclusion. If the results of primary interest focus on the evaluation of global estimates such as load-displacement responses of the inclusion, the Hankel integral transform based scheme yielded accurate results. The mixed boundary value problems were reduced to a system of coupled Fredholm integral equations of the second kind. These integral equations were solved in a numerical fashion to generate the load-displacement relationships for the two embedded disc inclusions which illustrate the manner in which the elastic stiffnesses of the rigid

disc inclusions are influenced by the cracks.

1.2.2.3 Rigid disc inclusion in a halfspace

The rigid disc inclusion problems reviewed in the previous section are based on the assumption that the inclusion is sufficiently far away from the neighboring boundaries and that the redistribution of stresses in the matrix is attributed to either the inclusion geometry or the contact interface condition or the loads applied at the inclusion and matrix system. Mathematically, the boundaries of the solid matrix were assumed to be at an infinite distance from the inclusion. The presence of a free boundary near an inclusion causes the global stiffnesses to decrease with decreasing distance between the free boundary and the inclusion. If the depth of embedment of the rigid inclusion is not sufficiently large in comparison with the diameter of the inclusion, it is more appropriate to use the model of rigid disc inclusion embedded in an elastic halfspace medium as shown in Fig.1.2.5, since, this model accommodates the influences of the external boundary. The solutions developed for this model are of particular importance in designing ground anchors.

The mixed boundary value problems associated with inclusions embedded in halfspace region involve an inordinate amount of computational effort and in the presentation of data the influence of an additional variable namely, the depth of location of the inclusion from the free boundary needs to be considered. In the literature, the examinations for these problems are usually restricted to cases where the rigid disc is embedded in bonded contact with an isotropic halfspace and its plane is parallel to the horizontal free surface.

Utilizing Mindlin's (1936) fundamental solution for a vertical point load acting in the interior of halfspace, Butterfield and Banerjee (1971) presented a direct boundary element method for the evaluation of stress and stiffness resulting from a vertical displacement of a rigid disc, of arbitrary shape, horizontally embedded in smoothly bonded contact with an isotropic halfspace. The results for rigid circular and rectangular discs were presented graphically for a range of disc proportions, burial depths and properties of the halfspace region. It was pointed out that the

stresses and stiffnesses of inclusions remain essentially unchanged if the depth of burial is increased beyond about 4 diameters, and the stiffness at this depth is approximately double the value at the surface.

A similar problem related to rigid circular disc inclusion embedded horizontally in smooth bilateral contact with an isotropic elastic halfspace was examined by Hunter and Gamblen (1974) using the Hankel transform-based integral equation method. The integral equation was solved in an approximate fashion. The vertical displacement field and the stiffness of the rigid disc inclusion were evaluated in the form of power series which involves the ratio of the radius of disc (a) to the depth of embedment (h).

Recently, Selvadurai, Singh, and Au (1991) have examined the asymmetric problem of a rigid disc inclusion which is embedded in perfectly bonded contact with an isotropic elastic halfspace and subjected to an in-plane lateral force P_x . The disc inclusion is embedded at a finite distance h from the free surface of the halfspace and oriented parallel to the free surface. Using the Hankel transforms, the traction free boundary conditions at the free surface and the continuity and boundary conditions at the plane of the inclusion yield a system of three coupled integral equations. These integral equations are solved by a quadrature scheme to generate the in-plane translation D_x and central rotation Ω_y induced by the lateral load P_x . The numerical results for the in-plane stiffness and the rotation which accompanies the in-plane translation indicate that the influence of the traction free boundary becomes insignificant as the depth of embedment (h) to inclusion radius ratio (a) exceeds 4.

1.2.2.4 Rigid disc inclusion on a halfspace

As the depth of embedment $h \rightarrow 0$, the disc inclusion problems reduce to contact (or indentation) problems which have been investigated quite extensively in the literature of classical elasticity and geomechanics. The literature on the contact problems has been well reviewed and/or documented by a number of authors including Ufliand (1965), Selvadurai (1979a), Gladwell (1980), and de Pater and

Fig.1.2.5 Geometry and loads of a rigid disc inclusion embedded in an elastic halfspace and oriented parallel to the free surface

Fig.1.2.6 The four basic displacements of a rigid disc resting on an elastic halfspace

Kalker (1975). The following review is concerned primarily with the study of the rigid disc problems as a topic of interest to applications in geotechnical engineering.

The rigid disc indenting either an isotropic or a transversely isotropic halfspace is an idealized model which has been used to examine the mechanics of rigid, circular, and flat foundations resting on smooth or rough contact with a deep soil or rock medium of semi-infinite extent. Similar to the infinite space problems, there are two translations and two rotations of the rigid disc, *i.e.*, (1) axisymmetric translation $\mathcal{U}_z$ in the z -direction, (2) in-plane translation D_x in the x -direction, (3) central rocking rotation Ω_y about the y -axis, and (4) torsional rotation Ω_z about the z -axis (Fig.1.2.6).

The first problem is that of the rigid disc resting in smooth contact with either an isotropic or a transversely isotropic elastic halfspace. In this case, the torsional rotation problem has no basis. The contact stress at the rigid disc-elastic halfspace interface induced by each basic displacement is uncoupled, *i.e.*, only one non-zero contact stress distribution exists in each basic displacement mode. The single non-zero contact stress has one regular stress singularity of order $-\frac{1}{2}$ at the circular boundary of the smooth rigid disc $r = a$. It was also found that for a majority of this class of frictionless contact rigid disc problems, the rotational and translational stiffnesses can be evaluated in exact closed forms. According to the location of the external load, this class of problems can be further divided into two sub-cases; namely, (i) the directly loaded rigid disc problems; (ii) the indirectly loaded rigid disc problems.

In the sub-case (i), the external loads are applied directly to the rigid disc. These correspond to the four basic displacements of the rigid disc (Fig.1.2.7).

The solution to the axisymmetric translation (in the z -direction) of a rigid disc resting in frictionless contact with an isotropic elastic halfspace was given by Boussinesq (1885) who employed foundations of potential theory, and by Harding and Sneddon (1945) who used Hankel integral transform techniques. A systematic study of the axisymmetric contact problem was also presented by Sneddon (1965)

Fig.1.2.7 Geometry and loads of a rigid disc smoothly or roughly resting on an elastic halfspace

a. A general internal load

b. Two symmetric internal loads

Fig.1.2.8 Rigid disc smoothly resting on an elastic halfspace under indirectly point loads

for a general profile of a smooth indenter. The vertical and radial displacements and normal stress distribution $\sigma_{zz}(r, 0)$ at the surface $z = 0$ were given in cylindrical coordinates ($Or\theta z$) as

$$u_z(r, 0) = \begin{cases} D_z, & 0 \leq r \leq a \\ \frac{2}{\pi} \sin^{-1}\left(\frac{a}{r}\right) D_z, & a < r < +\infty \end{cases} \quad (1.2.10.a)$$

$$u_r(r, 0) = \begin{cases} -\frac{1}{\pi} \frac{1-2\nu}{1-\nu} \frac{a-\sqrt{a^2-r^2}}{r} D_z, & 0 \leq r \leq a \\ -\frac{1}{\pi} \frac{1-2\nu}{1-\nu} \frac{a}{r} D_z, & a < r < +\infty \end{cases}$$

and

$$\sigma_{zz}(r, \theta, 0) = \begin{cases} \frac{-2\mu}{\pi(1-\nu)} \frac{1}{\sqrt{a^2-r^2}} D_z, & 0 \leq r \leq a^- \\ 0, & a^+ \leq r < +\infty \end{cases} \quad (1.2.10.b)$$

The relationship between the vertical load P_z and the vertical rigid disc displacement D_z is

$$P_z = -2\pi \int_0^a \sigma_{zz}(r, 0^+) r dr = \frac{4\mu a}{1-\nu} D_z \quad (1.2.11)$$

The solution to the axisymmetric rigid disc smoothly indenting a transversely isotropic elastic halfspace was obtained by Elliott (1949) and Shield (1951) by considering respectively integral equation methods and results of potential theory.

A partial solution for displacement and stress fields at the surface of an isotropic elastic halfspace due to an in-plane translation D_x of a rigid disc in the x -axis direction was solved by Mindlin (1949) using the well known Boussinesq and Cerruti solutions (Lur'e, 1964):

$$\begin{aligned} u_x(r, \theta, 0) &= D_x \\ u_y(r, \theta, 0) &= 0 \\ u_z(r, \theta, 0) &= \frac{2}{\pi} \frac{1-2\nu}{2-\nu} \frac{r \cos \theta}{a + \sqrt{a^2 - r^2}} D_x \\ \sigma_{xz}(r, \theta, 0) &= -\frac{2}{\pi} \frac{2\mu}{2-\nu} \frac{1}{\sqrt{a^2 - r^2}} D_x \end{aligned} \quad (1.2.12a)$$

where $0 \leq r < a$.

$$\begin{aligned}
u_x(r, \theta, 0) &= \frac{2}{\pi} \sin^{-1}\left(\frac{a}{r}\right) D_x \\
u_y(r, \theta, 0) &= \frac{2}{\pi} \frac{\nu}{2 - \nu} \frac{a}{r^2} \sqrt{r^2 - a^2} D_x \\
u_z(r, \theta, 0) &= \frac{2}{\pi} \frac{1 - 2\nu}{2 - \nu} \frac{a \cos \theta}{r} D_x \\
\sigma_{xz}(r, \theta, 0) &= 0
\end{aligned} \tag{1.2.12b}$$

where $a < r < \infty$.

The relationship between the in-plane load P_x and the translation D_x is given by

$$P_x = -2\pi \int_0^a \sigma_{xz}(r, 0) r dr = \frac{8\mu a}{2 - \nu} D_x \tag{1.2.13}$$

The rotation of rigid disc resting on smooth contact with an isotropic elastic halfspace and subjected to a central moment M_y about the horizontal y -axis was first solved by Abramov (1939) (see Gladwell, 1980), later by Bycroft (1956) and Florence (1961). The Hankel transform based results give the following for the normal stress distribution $\sigma_{zz}(r, \theta, 0)$ at the surface $z = 0$:

$$\sigma_{zz}(r, \theta, 0) = \begin{cases} -\frac{2}{\pi} \frac{2\mu}{1 - \nu} \frac{r \sin \theta}{\sqrt{a^2 - r^2}} \Omega_y, & 0 \leq r \leq a^- \\ 0, & a^+ \leq r < +\infty \end{cases} \tag{1.2.13}$$

The relationship between the central moment M_y and the rotation of rigid disc Ω_y is

$$M_y = - \int_0^a \int_0^{2\pi} \sigma_{zz}(r, \theta, 0) r^2 \cos \theta d\theta dr = \frac{8\mu a^3}{3(1 - \nu)} \Omega_y \tag{1.2.14}$$

This moment-rotation relationship is of interest to the study of the the tilting of towers on circular bases. An assumption made in this solution is that the disc remains in contact with the elastic medium over the entire circle. This may be realized by a suitable superposition of an axial load which would prevent loss of contact at the smooth interface.

In the sub-case (ii), the rigid displacements of the disc at the surface of elastic halfspace are caused by an external concentrated load $\mathbf{P}^*$ located at point $Q(b, c, d)$ either within the halfspace or on the surface of elastic soil medium (Fig.1.2.8.a).

Using Hankel transform based dual integral equations, Selvadurai (1982c, 1983b) examined the problem of the rigid displacement of disc resting in smooth contact with an isotropic elastic halfspace and subjected to a concentrated force P_z^* acting at point $Q(b, 0, 0)$ (Fig.1.2.8.b). The vertical displacement of the rigid disc and the contact normal stress distribution at the rigid disc-elastic halfspace interface were given as follows,

$$\begin{aligned}
 u_z(r, \theta, 0) &= \frac{P_z^*(1-\nu)}{4a\mu} \left\{ \frac{2}{\pi} \sin^{-1}\left(\frac{a}{b}\right) \right. \\
 &\quad \left. + \frac{3br \cos \theta}{2a^2} \left[1 - \frac{2}{\pi} \tan^{-1} \frac{\sqrt{b^2 - a^2}}{a} - \frac{2}{\pi} \frac{a\sqrt{b^2 - a^2}}{b^2} \right] \right\} \\
 \sigma_{zz}(r, \theta, 0) &= \frac{-P_z^*}{\pi^2 a \sqrt{a^2 - r^2}} \left\{ \frac{b^2 - a^2}{b^2 + r^2 - 2br \cos \theta} \right. \\
 &\quad \left. - 2 \sin^{-1}\left(\frac{a}{b}\right) - \frac{3r \cos \theta}{2a^2 b} \left[\pi b^2 - 2b^2 \tan^{-1} \frac{\sqrt{b^2 - a^2}}{a} - 2a\sqrt{b^2 - a^2} \right] \right\}
 \end{aligned} \tag{1.2.15}$$

where $0 \leq r < a, a < b$.

This fundamental solution was further generalized to include other forms of distributed external loading which was achieved by a direct integration within the limits of the loading region. The mutual interaction between a structural foundation and an external load was analysed to assess the settlement of existing foundations due to surcharge loads applied in its vicinity. It was also shown that these deformation due to external load satisfy Betti's reciprocal theorem. Since both a smooth contact and no separation at the interface were assumed in the theoretical treatment of these indentation problem, the practical applicability of the solution is limited to the interaction problems of rigid disc simultaneously subjected to directly and indirectly loads $\mathbf{P}, \mathbf{P}^*$ which together render a compressive normal stress distribution at every location of the contact region $r < a$.

The rigid displacements of the disc at the surface of halfspace due to axial and horizontal concentrated point loads P_z^*, P_z^* located along the z -axis $Q(0, 0, d)$ (Fig.1.2.8.b) were investigated by Selvadurai (1978b, 1979f, 1981b) using the complex potential function formulation and Hankel transform technique. The exact

expression of the vertical disc displacement and the contact normal stress were obtained in a little more complicated form. Here, only the vertical displacement is recorded:

$$u_z(r, 0) = -\frac{P_z^*(1-\nu)}{2\pi a\mu} \left\{ \tan^{-1}\left(\frac{a}{d}\right) + \frac{1}{2(1-\nu)} \frac{ad}{a^2 + d^2} \right\} \quad (1.2.16)$$

where $0 \leq r \leq a$.

These solutions were utilized in analyzing the interaction of a rigid circular foundation and an internal inclined anchor load and in examining the efficiency and the desirable location of anchors.

The second class of problem is that the rigid disc rests in welded, bonded, or adhesive contact with either an isotropic or a transversely isotropic elastic halfspace. In contrast to the first case (smooth contact), the contact stresses beneath the disc induced by the three displacements D_z , D_x , Ω_y are coupled. They also possess an oscillatory stress singularity of the type $\frac{1}{\sqrt{a^2-r^2}} \sin(\epsilon_o \ln \sqrt{a^2-r^2})$ where ϵ_o is a constant which depends on the Poisson's ratio ν . By appeal to a formulation based on the Hilbert transform, it can be shown that the equations for these three bonded contact problems may be reduced to a singular integral equation with Cauchy type kernels.

The solution to the axisymmetric translation D_z along the z -axis of a rigid disc perfectly bonded to an isotropic elastic halfspace was solved by Mossakovskii (1954; 1963), Abramian *et al.* (1966), Keer (1967), and Spence (1968). The general displacements of the rigid disc adhering to a halfspace was examined by Ufliand (1956), and the solution to the in-plane translation D_x in the x -axis was solved through the use of toroidal coordinates and the Mehler-Fok (Hilbert) integral transforms. Later Gladwell (1969, 1980) solved both problems involving the in-plane translation D_x and the central rotation Ω_y using Hankel and Hilbert transforms. The translational and rotational stiffnesses for the three displacements are recorded here in exact closed form:

$$\begin{aligned}
P_z &= \frac{8\mu}{a\beta} D_z \\
16\mu a^2 D_z &= (2 + \beta + 3\omega\alpha) P_z + 3\omega M_y \\
16\mu a^2 \Omega_y &= 3\omega P_z + \frac{3\omega}{\alpha a} M_y
\end{aligned} \tag{1.2.17}$$

where $\alpha = \frac{1}{2\pi} \ln(3 - 4\nu)$, $\beta = \frac{1-2\nu}{\pi\alpha}$, $\omega = \frac{\alpha\beta}{1+\alpha^2}$.

It may be noted that for undrained elastic behaviour of a fully saturated soil medium ($\nu = \nu_u = 0.5$), the elastic solutions for the displacement for both the fully bonded and frictionless interface conditions are identical and the oscillatory part of the contact stresses singularity in the three problems disappears.

By introducing a system of oblate spheroidal coordinates, Reissner and Sagoci (1944) solved the torsional rotation of a rigid disc perfectly bonded to the surface of an isotropic elastic halfspace. This problem is referred to as the Reissner-Sagoci problem. In subsequent treatments, Sneddon (1947, 1966b) and Rostovsev (1955) examined the relevant mixed boundary value problem using Hankel transform based dual integral equation schemes. The only non-zero stress at the surface $z = 0$ and the rotational stiffness were given as

$$\begin{aligned}
\sigma_{\theta z}(r, 0) &= \begin{cases} -\frac{4\mu}{\pi} \frac{r}{\sqrt{a^2 - r^2}} \Omega_z, & 0 \leq r \leq a^- \\ 0, & a^+ \leq r < +\infty \end{cases} \\
M_z &= -\int_0^a \int_0^{2\pi} \sigma_{\theta z}(r, 0) r^2 d\phi dr = \frac{16\mu a^3}{3} \Omega_z
\end{aligned} \tag{1.2.18}$$

1.2.2.5 Rigid disc inclusion at a bi-material region

The literature reviewed in the previous three sub-sections focuses on inclusion problems where the rigid disc is embedded in a homogeneous elastic material of either an infinite or a semi-infinite extent. In this sub-section, we will record some results for the mechanics of rigid disc inclusions which take into account the influences of the nonhomogeneities of the surrounding elastic medium. One of the simplest model of this kind is a bi-material composite made of two bonded homogeneous elastic halfspaces with disc inclusion located at the interface. Such a flat

rigid disc region can be formed at an interface of two geological layers by hydraulic fracturing effects of pressure injected grouting (Fig.1.2.9). The rigid disc represents the hardened cement grout or a resinous material. The elastic stiffness of a rigid disc inclusion at a bi-material regions can provide some insight into the efficiency of the ground anchoring system in stratified soil deposits.

Selvadurai (1984a) examined the problem related to the axial translation of a rigid elliptical disc inclusion (*i.e.* $(x/a)^2 + (y/b)^2 \leq 1$ where a and b are the major and minor semi-axes of the ellipse) which is embedded in bonded contact at the bi-material interface of two transversely isotropic elastic halfspaces. To estimate the load-displacement characteristics of the embedded inclusion, a technique was proposed to develop a set of bounds for the elastic stiffness of the anchor embedded at the bi-material interface. These bounds are developed by imposing displacement and/or traction constraints at the interface. The upper bound imposes an inextensibility constraint at the interface, and the lower bound is obtained by imposing a frictionless bilateral contact at the interface. If the bi-material regions exhibit an isotropic elastic response, the bounds for the axisymmetric translational stiffness P_z/D_z were reduced to the following,

$$\begin{aligned} \frac{\mu_1(1-\nu_2) + \mu_2(1-\nu_1)}{4(1-\nu_1)(1-\nu_2)} &\leq \frac{K(e_o)}{8\pi a} \frac{P_z}{D_z} \\ &\leq \frac{\mu_1(1-\nu_1)(3-4\nu_2) + \mu_2(1-\nu_2)(3-4\nu_1)}{(3-4\nu_1)(3-4\nu_2)} \end{aligned} \quad (1.2.19)$$

where $K(e_o)$ are the complete elliptic integral of the first kind, $e_o = \frac{a^2-b^2}{a^2}$.

The bounds for the asymmetric rotational stiffness M_y/Ω_y of a rigid elliptical disc inclusion embedded in bonded contact at an isotropic bi-material elastic interface were also developed by Selvadurai (1985) in a similar form:

$$\begin{aligned} \frac{\mu_1(1-\nu_2) + \mu_2(1-\nu_1)}{4(1-\nu_1)(1-\nu_2)} &\leq \frac{3[K(e_o) - E(e_o)]}{8\pi a^3} \frac{M_y}{\Omega_y} \\ &\leq \frac{\mu_1(1-\nu_1)(3-4\nu_2) + \mu_2(1-\nu_2)(3-4\nu_1)}{(3-4\nu_1)(3-4\nu_2)} \end{aligned} \quad (1.2.20)$$

Fig.12.9 Disc-shaped anchor zones embedded at bi-material geological interfaces (after Selvadurai and Au, 1986)

Fig.1.2.10 Indentation of a debonded interface (after Selvadurai, 1989)

where $E(e_o)$ are the complete elliptic integral of the second kind.

It was observed from equations (1.2.19) and (2.2.20) that if the values of Poisson's ratio ν_1, ν_2 in the two media are the same, then the stiffnesses $P_z/D_z, M_y/\Omega_y$ of the inclusion will depend on the mean value $0.5(\mu_1 + \mu_2)$ not the ratio μ_1/μ_2 of the two shear moduli, however different the two shear moduli may be. If $a = b$, the above results are reduced to two set of bounds for the stiffnesses of a rigid circular disc inclusion at a bi-material interface.

These bounds were compared with the boundary element results which were developed by Selvadurai and Au (1986) to estimate numerically the axial and in-plane translational and central rotational stiffnesses of a rigid elliptical disc anchor which is embedded in perfectly bonded contact at a bi-material interface of two isotropic elastic bodies. The numerical results presented in the paper illustrate the manner in which the various stiffnesses of the rigid anchoring region are influenced by the elasticity mismatch of the surrounding geological media surrounding the anchor.

Recently, Selvadurai, Singh, and Au(1989b) have examined the axial stiffnesses of a rigid disc inclusion which is smoothly located at a debonded region of an deformable medium-rigid boundary interface (Fig.1.2.10). The Hankel transform development of the mixed boundary value problem yields a set of integral equations which can be effectively reduced to a single Fredholm integral equation of the second kind. The numerical results illustrate the manner in which the axial stiffness of the disc is influenced by the Poisson's ratio of the elastic medium and the relative dimensions of the debonded region.

Adopting the bound-technique and Hankel transforms, Selvadurai (1989b) further developed a set of bounds for the axial stiffness of the rigid disc embedded at the cracked region of an deformable medium-rigid boundary interface. The upper and lower bounds were expressed as a power series in terms of the radius ratio of rigid disc and crack region.

1.2.3 Rigid Disc Inclusions in Plastic, Viscoelastic, and Poroelastic Media

1.2.3.1 General

All of the analytical results for the mechanics of a rigid disc inclusion reviewed in section 1.2.2 are restricted to linear elastic behaviour of the geological medium and are useful for predicting time independent behaviour of rigid disc inclusions at or below working loads. However, the mechanical behaviour of soils is not linear elastic or time independent for the entire range of loadings of practical interest. If the failure load of the inclusion is required or the response of soil media under working load is time dependent, the classical theory of linear elastostatics fails to predict the mechanics of inclusion problems. Under these circumstances the phenomena ignored in linear elastic theory, such as plastic deformation or creep or consolidation of the soil matrix become important. The inherent complexities in the behaviour of real soils have led to the development of idealized soil models such as plasticity, viscoelasticity, or poroelasticity. These idealizations are commonly used in the analysis of problems in geomechanics. Some important results related to rigid disc inclusions embedded in plastic, viscoelastic, or poroelastic media will be reviewed in the ensuing section.

1.2.3.2 Rigid disc inclusion in plastic media

The developments of the theories of plasticity and their applications are extensively documented in the engineering literature (Chen and Baladi, 1985; Desai and Siriwardane, 1984; etc). The failure mechanism, ultimate failure loads, and displacements for rigid disc inclusion are of great importance in the design of ground anchors embedded in both frictional (sand) and cohesive (clay) soils. In order to predict these behaviour of ground anchors, it is necessary to represent the soil matrix as a plastic material.

Early investigators (e.g. Balla, 1961, Mariupol'skii, 1965; Matsuo, 1967; Meyerhof and Adams, 1968; Teng, 1962) proposed some approximate evaluations for

the ultimate collapse load (uplift resistance) by assuming a perfectly plastic soil behaviour and using either limit equilibrium techniques or method of characteristics, frequently combined with empirical corrections. Others (e.g. Ladanyi and Johnston, 1974; Vesic, 1971; 1972) suggested alternative applications of cavity expansion theories for predicting the ultimate capacity of anchors. For example, Meyerhof and Adams (1968) derived a method for calculating the ultimate capacity of a shallow anchor based on theoretical considerations and experimental observations. A cylindrical failure surface is assumed and Meyerhof's (1951) bearing capacity factors for deep foundations are employed in this theory to find the ultimate uplift resistance of a deep anchor.

In recent years, the rapid advancement of computational techniques has resulted in the development of finite element methods for linear and non-linear stress analysis in geomechanics. The finite element methods provide a convenient means of numerically analysing the load-displacement behaviour of rigid disc inclusion embedded in plastic media and allows consideration of many factors excluded from such simple analysis.

Davie and Sutherland (1977) performed a finite element analysis to predict the general type of failure of a rigid disc anchor and to estimate the stress fields in the soil mass by using an axisymmetric elasto-plastic analysis which employs a von Mises yield criterion. Rowe and Davis (1982a) have used the finite element method to examine the undrained behaviour of thin rigid strip and disc anchor plates embedded in a saturated clay, by combining the sub-structure approach with an initial stress technique. The cohesive soil medium was assumed to be elasto-plastic material with a Mohr-Coulomb failure criterion and an associative flow rule. Theoretical consideration was given to the effects of anchor embedment, layer depth, overburden pressure and breakaway conditions, as well as anchor roughness, thickness and shape. It was shown that in many cases, the ultimate collapse is preceded by a significant axial displacement of the anchor. It was suggested that the failure load may be defined as the load which would give rise to a displacement four times

that predicted by a linear elastic analysis. The anchor capacity factors for purely frictional or cohesive-friction soils were also presented by Rowe and Davis (1982b).

1.2.3.3 Rigid disc inclusion in viscoelastic media

All the foregoing studies and extensions of the mechanics of rigid disc inclusion problems have explicitly eliminated time effects which are invariably observed in real soil and rock due to the movement of pore-fluid and/or creep. Such time-dependent effects in the inclusion problems are of fundamental importance to the determination of the anchor efficiency under long term loads or in fluid saturated soils, and in the calculation of creep or consolidation deformations of anchorages under sustained load. Here, we shall be concerned with the analysis of a rigid disc inclusion embedded in a soil or rock medium exhibiting creep properties. The time-dependent effects due to the dissipation of pore-water will be reviewed in next sub-section.

Investigations relating to the creep behaviour of soil and rock media have long been the subject of extensive research. Comprehensive accounts of both experimental and theoretical results on the subject are found in many studies (eg. Flugge, 1975; Kravtchenko and Sirieys, 1966; Jaeger and Cook, 1969; Zienkiewicz, Humpheson, and Lewis, 1975). Creep effects in soil and rock media may manifest under natural conditions associated with long term geological processes or as results of interactions with engineering structures such as excavations, mining operations, structural foundations or ground anchors. A variety of phenomenological models and constitutive relations such as viscoelasticity and viscoplasticity have been developed to describe the creep behaviour of soil and rock media under a variety of environmental conditions. The references to such investigations may be found in the literature already cited. The linear theory of viscoelasticity, like its elastic counterpart, provides a useful first approximation-theoretical basis for the analysis of creep phenomena in soil and rock media under working load. It has been effectively employed by many investigators (eg. Booker and Poulos, 1976; Brown and Booker, 1977; de Arriaga, 1968; Selvadurai, 1978c) to provide informative, analytical and

numerical solutions to some problems of engineering interest.

In particular, using the Laplace transform based correspondence principle, Selvadurai (1979g) examined the axisymmetric interaction of a rigid disc resting in smooth contact with a linear viscoelastic halfspace which was, in turn, loaded internally by a distribution of Mindlin type force located along the axis of symmetry. The numerical results illustrated the manner in which the decay of the prestress load in an anchored plate or a rock bolt system can be influenced by the length, depth of location and load distribution in the anchor region and the creep characteristics of the halfspace region. It was assumed that the anchor region in the halfspace was perfectly flexible and was represented by three types of distributions of Mindlin loads.

1.2.3.4 Rigid disc inclusion in poroelastic media

When external loads are applied to a soil-inclusion system the surrounding saturated cohesive soil will experience consolidation. As a result of the consolidation, the rigid disc inclusion will undergo quasi-static translations and/or rotations which are time dependent. This coupling process will continue until all excess pore pressures have dissipated. The process of consolidation in two phase soils (solid skeleton and pore fluid) was first investigated by Terzaghi (1923) for one-dimensional condition. Subsequently Rendulic (1936) generalized Terzaghi's theory to a pseudo-three dimensional situation where the deformation of solid skeleton and the flow of pore fluid is uncoupled. The complete three dimensional theory of soil consolidation was developed by Biot (1941,1955, 1956) to describe the coupled mechanical behaviour of a saturated porous medium.

Investigations have also been made to take into consideration the influence of finite strains or plastic deformations of solid skeleton in consolidation behaviour (eg. Gibson, England, and Hussey, 1967; Smiles and Poulos, 1969 ; Small, Booker and Davis, 1976; Carter, Booker and Small, 1979). Further references on these topics may be found in a very detailed bibliography given by Schiffman (1984).

Extensive research has been carried out on the applications of these linear and

non-linear theories in soil mechanics and geotechnical engineering (see Schiffman, 1984). A recent literature research carried out by Yue (1990) shows that by using integral transform techniques, a great number of analytical studies have been conducted with Biot's theory for poroelastic halfspace media saturated with incompressible pore fluids and subjected to perfectly flexible loads. Some numerical solutions for the consolidation problems where the soil skeleton is either linear or non-linear also have been proposed by various authors using finite element methods or boundary element methods (eg. Sandhu and Wilson, 1969; Carter et al, 1979; Cheng and Liggett, 1984a; Sandhu et al, 1985; Selvadurai and Gopal, 1986; Lewis and Schrefler, 1987; Schrefler and Simoni, 1987; Simoni and Schrefler, 1987. Selvadurai and Karpurapu, 1989).

Due to the complexity of the mathematical formulations, very few attempts have been devoted to the solutions of mixed boundary and initial value problems associated with inclusions. It is noted that in related literature, only two axisymmetric translation problems of rigid disc inclusion embedded in saturated poroelastic media have been examined analytically (Chiarella and Booker, 1974; Gaszynski and Szefer, 1978; Heinrich and Desoyer, 1961; Small and Booker, 1987; Szefer and Gaszynski, 1975; Zaretsky, 1972;). Due to the approximations involved in these studies, it is still desirable to make a comprehensive analysis of these two axisymmetric disc inclusion problems.

The first axisymmetric problem examined by several authors is that a rigid disc resting in smooth contact with an isotropic, linear, poroelastic halfspace saturated with incompressible fluid and subjected to a direct axial loading $P_z H(t)$ (where P_z is the magnitude of the load and $H(t)$ is the Heaviside step function).

Based on the assumptions that (i) the contact stress distribution is uniform or (ii) the contact stress distribution is constant and equal to the correct initial/final value, Heinrich and Desoyer (1961) obtained the approximate time-settlement response of the rigid disc by identifying it with the average settlement over the load area.

Subsequently, Zaretsky (1972) investigated this problem by using a successive approximation method to solve the coupled equations of the theory of consolidation. In the first step of approximation, he utilized the pore pressure distribution obtained from the simplified (without the deformation of soil skeleton) diffusion equation. The approximated solutions were then found through an elastic analysis. This approximation solution scheme avoids the solution of complicated integral equations governing the problem.

Chiarella and Booker (1974) considered the punch indentation problem and formulated a pair of dual integral equations in Laplace transform domain for the case where the surface of the halfspace is fully permeable. By use of the representation given by Sneddon (1966), they reduced the dual integral equations to a Fredholm integral equation of second kind and upon an inversion of the Laplace transforms yielded a double Fredholm-Volterra integral equation. (It is noted that the kernel function in their Fredholm integral equation is unbounded). Applying Galerkin's technique, this double Fredholm-Volterra integral equation was solved in an approximate fashion. The numerical results of the degree of consolidation settlement for the rigid punch and the contact stress distribution were presented for a finite interval of time and compared with the approximated solution developed by Heinrich and Desoyer (1961).

Using Hankel and Laplace transforms, Szefer and Gaszynski (1975) considered the indentation problem for a halfspace region where soil skeleton was assumed to be composed of linear viscoelastic material. In the Laplace transform domain, they formulated the dual integral equations applicable for halfspace region with both permeable and impermeable surfaces. Following the representation of solution for solving the dual integral equations in classical elasticity, they obtained the Fredholm integral equation of the second kind with a continuous and bounded kernel. After assuming the solution in a series form of complete orthogonal functions, they expressed the contact stress and punch displacement in an infinite series form in the Laplace transform domain. The time domain solutions could be obtained through

numerical inversion of the Laplace transforms. A numerical example was given for the case where the purely elastic soil skeleton and for a halfspace region under a linear loading function. It was pointed out that the numerical results for the permeable surface are similar to those for the impermeable surface and the character of singularity of the contact stress beneath the disc in consolidating medium is the same as that for the one-phase elastic solid. Gaszynski and Szefer (1978) also investigated an identical problem for the case of an impermeable rigid disc smoothly indenting halfspace with permeable surface. In this case, the boundary integral equations were reduced to a set of integro-differential equations of Fredholm type with two unknown functions in the Laplace transform domain. It was noted that the mixed boundary permeability conditions do not change the character of singularity of the contact stresses, the differences occur only in the magnitude of the settlements. No numerical results were given in this case.

Agbezuge and Deresiwicz (1974) and Deresiwicz (1977) investigated the axisymmetric indentation of a linear porous elastic halfspace by a smooth rigid *spherical* punch under a constant normal load. Three distinct cases were treated, namely (1) permeable punch on a permeable halfspace; (2) impermeable punch on an impermeable halfspace; (3) impermeable punch on a permeable halfspace. They formulated the dual integral equations for each case with the help of the two displacement functions introduced by McNamee and Gibson (1960). A procedure due to Tranter (1950) was used to reduce the problem of the solution of the system of integral equations to one of obtaining the roots of an infinite set of linear algebraic equations. This set of equations was then solved by approximation. Numerical values in the time domain were obtained for the distribution of the contact pressure, the radius of contact, the largest penetration of the spherical punch and the deflection of the surface outside the region of contact.

The second axisymmetric problem examined by Small and Booker (1987) is that a rigid disc inclusion embedded in an isotropic, linear, poroelastic infinite space saturated with incompressible fluid and subjected to a constant axial loading $P_z H(t)$.

From the considerations of symmetry about the horizontal xy (or $r\theta$) plane, the problem was reduced to that of the axial loading of a rigid disc which is resting on a poroelastic halfspace with special boundary conditions. The boundary equations in the Hankel and Laplace transform domains were solved by approximating the unknown contact normal stress and flow as a series of simple functions with unknown coefficients. The simple functions are assumed to have the same singularity as that of the contact stresses in the theory of elastostatics. After the coefficients were determined by solving a set of linear algebraic equations in the transform domains, the axial translation of the disc was obtained by numerically inverting both the Hankel and Laplace transforms (Talbot, 1979). It was noted in the paper that for very small values of time some errors are introduced into the solutions by numerical problems in the inversion algorithm and the solutions at the initial time $t = 0$ cannot be evaluated. Three different assumptions for the permeability of the soil on the plane $z = 0$ were used; (i) fully permeable; (ii) fully impermeable (a artificial case), or (iii) impermeable over the inclusion $0 \leq r \leq a$ but permeable elsewhere. Computed results showed that for a rigid disc inclusion, the time-settlement behaviour was not extremely sensitive to the value of the Poisson's ratio of the solid skeleton and therefore an accurate knowledge of its value was not important when making predictions of anchor behaviour. Pyrah, Hird and Tanaka (1985) also studied this problem using models and finite element method and also concluded that the Poisson's ratio of the solid skeleton is not an important consideration in the behaviour of ground anchors.

1.2.4 Observations and discussions

Current investigations on the mechanics of rigid disc inclusion problems have been reviewed in the previous sections. The following observations and discussions are presented in order to clarify the objectives of this thesis.

A great majority of studies related to poroelastic media have focused on problems associated with perfectly flexible loadings or rigid punch indentations. Even in these treatments the accuracy of the numerical schemes can be improved.

The formulation and analysis of disc inclusion problems follow a similar format to indentation problems and the reduction to Fredholm integral equations of the second kind in the transform domain is an efficient procedure which allows both a numerical inversion procedure and the application of quadrature schemes.

A majority of mixed boundary value problems in poroelasticity invoke the assumption that the pore fluid is incompressible where the undrained Poisson's ratio $\nu = 0.5$ and the Skempton's pore pressure coefficient $B = 1$. The extension of the assumption to include the values of $\nu < \nu_* \leq 0.5$ and $0 < B \leq 1$ will extend the range of applicability of the classical theory of poroelasticity.

In recent years, purely computational schemes such as finite element and boundary element methods have been developed for the study of the consolidation response of poroelastic media. The reliability of these computational schemes can be greatly enhanced by the development of a group of *benchmark analytical solutions* based on integral transform techniques which accommodate singular behaviour in the stress fields and a range of time scales involved in the Laplace transform inversion schemes. Such verified computational schemes can be applied with confidence to the examination of more complex problems involving practical geotechnical situations.

1.3 Objectives and Scope of the Research

1.3.1 Objectives of the research

Based on the above literature review and discussions, the primary objective of this research has been chosen as the analytical investigation of the time-dependent response of rigid disc inclusions embedded in poroelastic media saturated with compressible fluids. This objective will be achieved by an examination of the following aspects.

1. To develop a systematic Fourier (and Laplace) transform based approach for the solutions of disc inclusions embedded in saturated poroelastic media.

2. To examine the fundamental problems of rigid disc inclusions embedded in perfectly bonded contact with a poroelastic infinite space saturated with compressible fluids, and take into account the two limiting drainage conditions and the two groups of four time-dependent direct loads.

3. To eliminate the numerical problems such as numerical errors in small time responses, and to present a modified numerical inversion algorithm for Laplace transforms which has a high degree of stability and accuracy for the long time process of soil consolidation (seconds to years), and to develop a numerical integration technique for the infinite integrals with rapidly oscillatory integrands found in the kernel functions of Fredholm integral equations of the second kind.

4. To utilize the solutions obtained from the classical elastic analysis, and to investigate the conditions under which the solutions to inclusion problems associated with poroelastic media saturated with compressible fluids can be directly evaluated by using results based on purely elastic analysis.

The main goal of this Ph.D. research program is that the findings will be applied in geotechnical engineering as well as offer a benchmark for the developments of FEM and BEM to solve this class of mixed boundary value problems in poroelastic media.

1.3.2 Scope of the research

In order to succeed in these objectives, and for completeness, it is necessary to review the basic properties of Hankel integral transforms and two-dimensional Fourier integral transforms and examine the relations between the Fourier and Hankel transforms. The three-dimensional partial differential equations governing linear poroelastic media saturated with compressible fluid are formulated by using the two-dimensional Fourier integral transforms and Laplace transforms in both Cartesian and cylindrical coordinates. Next, the solution representations and boundary algebraic equations of the field variables in the transform domain are presented for the governing equations in poroelastic layer and halfspace regions and

in both Cartesian and cylindrical coordinates. A pair of solution representations is suggested by introducing two sets of eight field variables in the transform domain. This pair of solution representations makes the above equations and formulations simple and fundamental. Based on those investigations, a generalized systematic Fourier (and Laplace) transform based approach is proposed for the analysis of the time-dependent mechanics of disc inclusions embedded in poroelastic media saturated with compressible fluids.

The mixed boundary and initial value problems of a general plane inclusion problem are formulated systematically into certain sets of two-dimensional integral equations in the Laplace transform domain. By using the relations between the Fourier and Hankel transforms, those two-dimensional integral equations are then reduced into infinite pairs of one-dimensional integral equations. Also, finite sets of complex integral equations are derived for the rigid disc inclusion problems and then formulated into the systems of Fredholm integral equations of second kind in the Laplace transform domain. These systems of Fredholm integral equations of second kind are numerically solved in the Laplace transform domain. The time domain results are finally evaluated by the numerical inversions of Laplace transforms.

However, there are two unusual numerical problems associated with the solution of Fredholm integral equations of second kind in the Laplace transform domain. One is that the kernel functions in the integral equations are composed of infinite integrals with rapidly oscillatory complex integrands of three coordinate parameters. The second deals with the numerical inversions of Laplace transforms. The first numerical problem is overcome by adopting a particular technique of numerical integration which is based on the proceeding limit technique and approximation expansions.

The second numerical problem of Laplace transform inversion has been investigated with quite extensively. Five numerical inversion algorithms of Laplace transforms have been verified and evaluated in order to obtain an algorithm with high stability and accuracy for the study of mechanics of saturated poroelastic media.

It is found that these algorithms cannot be directly used in the analysis of mechanics in poroelastic media which involve time scales of a very long consolidation process (nondimensional time up to 10^{+3}). A modified algorithm which is based on Crump's (1975) original work is suggested for solving the problem of the numerical inversion procedure. It has been verified that this modified algorithm gives stable and accurate results for the dimensionless time factor in range $10^{-5} \sim 10^{+3}$ which is the main time domain of soil consolidation.

Finally, the governing equations of poroelastic media saturated with compressible fluids have been examined in both the physical and the transform domains under three limiting conditions. Such three limiting conditions are the initial response ($t \rightarrow 0^+$) and the final response ($t \rightarrow +\infty$) as well as the limiting response of $\nu \rightarrow \nu_*$ (where ν is the drained Poisson's ratio of solid skeleton and ν_* is the undrained Poisson's ratio of skeleton-fluid mixture). It is found that the solutions obtained from the classical elastic analysis can be utilized for the problems in poroelastic media saturated with compressible fluids under these three limiting conditions together with some corresponding material parameters. Also, such limiting results are examined and verified for all the integral equations associated with the fundamental inclusion problems formulated in the Laplace transform domain. In turn, some useful procedures and understanding are gained on the mathematical and mechanical properties of the integral equations in the Laplace transform domain. The Tauberian theorems for Laplace transforms are employed in the examinations of the governing equations especially under the conditions applicable to initial and final responses.

Chapter 2

Integral Transforms

2.1 Definitions and Basic Properties

The three integral transforms, (*i.e.* the two-dimensional Fourier transforms, Hankel transforms, and Laplace transforms) are the main tools in the mathematical development of this research. For completeness we shall examine the definitions and characteristics of these three integral transforms. In this section, we simply write down their definitions and some basic properties (Sneddon, 1972).

2.1.1 Fourier transforms in Cartesian coordinates

By introducing a symmetric Fourier *kernel* function K and its complex conjugate K^* in the physical Cartesian coordinates x, y, z and the transformed Cartesian coordinates ξ, η, z ,

$$\begin{aligned} K &= e^{+i(\xi x + \eta y)} \\ K^* &= e^{-i(\xi x + \eta y)} \end{aligned} \tag{2.1.1}$$

we may define the two-dimensional Fourier transform as

$$\tilde{u}(\xi, \eta) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} u(x, y) K^* dx dy \tag{2.1.2.a}$$

and its inverse transform as

$$u(x, y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \tilde{u}(\xi, \eta) K d\xi d\eta \tag{2.1.2.b}$$

where the two-dimensional infinite integrals are interpreted in the sense of the Cauchy principal value. The *kernel* function K and its complex conjugate K^* satisfy the Hamilton equations,

$$\begin{aligned}\nabla \cdot \nabla K + \rho^2 K &= 0 \\ \nabla \cdot \nabla K^* + \rho^2 K^* &= 0\end{aligned}\tag{2.1.3.a}$$

where the invariant vector differential operator ∇ and the invariant Laplace operator $\nabla \cdot \nabla$ in the Cartesian coordinates are

$$\begin{aligned}\nabla &= \mathbf{e}_x \frac{\partial}{\partial x} + \mathbf{e}_y \frac{\partial}{\partial y} + \mathbf{e}_z \frac{\partial}{\partial z} \\ \nabla \cdot \nabla &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\end{aligned}\tag{2.1.3.b}$$

and $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$, are respectively the unit vectors of the x , y , and z coordinates.

Some basic properties of the Fourier transforms are listed below.

1. Linearity

$$\sum_{k=1}^n c_k u_k(x, y) \iff \sum_{k=1}^n c_k \tilde{u}_k(\xi, \eta)\tag{2.1.4}$$

where c_k are arbitrary (complex) constants.

2. Similitude

$$u(ax, ay) \iff \frac{1}{a^2} \tilde{u}\left(\frac{\xi}{a}, \frac{\eta}{a}\right)\tag{2.1.5}$$

where a is a real constant.

3. Shifting

$$u(x \pm x_0, y \pm y_0) \iff e^{\pm i(x_0 \xi + y_0 \eta)} \tilde{u}(\xi, \eta)\tag{2.1.6}$$

4. Products and convolutions

$$\tilde{u}(\xi, \eta) \tilde{v}(\xi, \eta) \iff \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} u(\xi, \eta) v(x - \xi, y - \eta) d\xi d\eta\tag{2.1.7}$$

5. Differentiation

$$\frac{\partial^n \partial^m}{\partial x^n \partial y^m} u(x, y) \iff (i\xi)^n (i\eta)^m \tilde{u}(\xi, \eta)\tag{2.1.8.a}$$

where $n = 0, 1, 2, 3, \dots$; $m = 0, 1, 2, 3, \dots$.

$$\nabla \cdot \nabla u(x, y) \iff -\rho^2 \tilde{u}(\xi, \eta) \quad (2.1.8.b)$$

2.1.2 Fourier transforms in cylindrical coordinates

By introducing a symmetric Fourier *kernel* function K and its complex conjugate K^* in the physical cylindrical coordinates r, θ, z and the transformed cylindrical coordinates ρ, φ, z ,

$$\begin{aligned} K &= e^{+i\rho r \sin(\theta+\varphi)} \\ K^* &= e^{-i\rho r \sin(\theta+\varphi)} \end{aligned} \quad (2.1.9)$$

we may define the two-dimensional Fourier transform as

$$\tilde{u}(\rho, \varphi) = \frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} u(r, \theta) K^* r d\theta dr \quad (2.1.10.a)$$

and its inverse transform as

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} \tilde{u}(\rho, \varphi) K \rho d\varphi d\rho \quad (2.1.10.b)$$

where the two-dimensional infinite integrals are interpreted in the sense of the Cauchy principal value. The *kernel* function K and its complex conjugate K^* satisfy the Hamilton equations,

$$\begin{aligned} \nabla \cdot \nabla K + \rho^2 K &= 0 \\ \nabla \cdot \nabla K^* + \rho^2 K^* &= 0 \end{aligned} \quad (2.1.11.a)$$

where the invariant vector differential operator ∇ and the invariant Laplace operator $\nabla \cdot \nabla$ in the cylindrical coordinates are

$$\begin{aligned} \nabla &= \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{\partial}{r \partial \theta} + \mathbf{e}_z \frac{\partial}{\partial z} \\ \nabla \cdot \nabla &= \frac{\partial}{r \partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{\partial^2}{r^2 \partial \theta^2} + \frac{\partial^2}{\partial z^2} \end{aligned} \quad (2.1.11.b)$$

and $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_z$ are respectively the unit vector of the r , θ , and z coordinates.

Similarly some basic properties of the Fourier transforms are listed below.

1. Linearity

$$\sum_{k=1}^n c_k u_k(r, \theta) \iff \sum_{k=1}^n c_k \tilde{u}_k(\rho, \varphi) \quad (2.1.12)$$

where c_k are arbitrary (complex) constants.

2. Similitude

$$u(ar, \theta) \iff \frac{1}{|a|^2} \tilde{u}\left(\frac{\rho}{a}, \varphi\right) \quad (2.1.13)$$

where a is a real constant.

3. Differentiation

$$\nabla \cdot \nabla u(r, \theta) \iff -\rho^2 \tilde{u}(\rho, \varphi) \quad (2.1.14)$$

The relationships between the Cartesian and cylindrical coordinates in the physical domain are defined by

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \\ z &= z \end{aligned} \quad (2.1.15.a)$$

The relationships between the Cartesian and cylindrical coordinates in the transform domains are defined by

$$\begin{aligned} \xi &= \rho \sin \varphi \\ \eta &= \rho \cos \varphi \\ z &= z \end{aligned} \quad (2.1.15.b)$$

The relationship in equations (2.1.15.b) are suggested by the author. The Fourier transforms physically are a mirror imaginary exchange. The two coordinates in the physical domain are chosen as the right-hand systems. Their mirror imaginaries in the transform domain are then the left-hand systems. In order to make the two domains consistence, we re-set the two coordinates in the transform as right-hand systems. As a result, we suggest the definitions for the two coordinates in the transform domains in equations (2.1.15.b) (see Fig. 2.1.1).

Fig. 2.1.1 The coordinate systems in the physical domain and their mirror imaginaries in the transform domain

2.1.3 Hankel transforms in cylindrical coordinates

By introducing a symmetric and real Fourier *kernel* function $J_m(\rho r)$ in the physical and transform cylindrical coordinates, we may define the Hankel transform of order m as

$$\check{u}^m(\rho) = \int_0^{+\infty} u(r) J_m(\rho r) r dr \quad (2.1.16.a)$$

and its inverse transform

$$u(r) = \int_0^{+\infty} \check{u}^m(\rho) J_m(\rho r) \rho d\rho \quad (2.1.16.b)$$

where the infinite integrals are in the sense of the Cauchy principal value. $J_m(\rho r)$ is the Bessel function of the first kind of order m .

Some basic properties of Hankel transforms are listed below.

1. Linearity

$$\sum_{k=1}^n c_k u_k(r) \Longleftrightarrow \sum_{k=1}^n c_k \check{u}_k^m(\rho) \quad (2.1.17)$$

where c_k are arbitrary (complex) constants.

2. Similitude

$$u(ar) \Longleftrightarrow \frac{1}{|a|^2} \check{u}^m\left(\frac{\rho}{a}\right) \quad (2.1.18)$$

where a is a real constant.

3. Differentiation

$$\int_0^{+\infty} \left[\frac{\partial^2 u(r)}{\partial r^2} + \frac{1}{r} \frac{\partial u(r)}{\partial r} - \frac{m^2}{r^2} u(r) \right] J_m(\rho r) r dr = -\rho^2 \check{u}^m(\rho) \quad (2.1.19)$$

where $f(r) = o(r^{-m-1})$, $r \rightarrow 0^+$.

2.1.4 Laplace transforms

By introducing the Fourier *kernel* function e^{-st} and its reciprocal e^{+st} , we may define the Laplace transform as

$$\hat{u}(s) = \int_0^{+\infty} u(t) e^{-st} dt \quad (2.1.20.a)$$

and its inverse transform as

$$u(t) = \frac{1}{2\pi i} \int_{\omega-i\infty}^{\omega+i\infty} \hat{u}(s) e^{st} ds \quad (2.1.20.b)$$

where s is a complex independent variable in the Laplace transform domain, $Re(s) = \omega > \omega_0$, $u = O(e^{\omega t})$ as $t \rightarrow \infty$, ω_0 is a constant that exceeds the real part of all the singularities of $\hat{u}(s)$, and the two infinite integrals are taken in the sense of the Cauchy principal value.

Some basic properties of the Laplace transforms are listed below.

1. Linearity

$$\sum_{k=1}^n c_k u_k(x, y) \Longleftrightarrow \sum_{k=1}^n c_k \hat{u}_k(\xi, \eta) \quad (2.1.21)$$

where c_k are arbitrary (complex) constants.

2. Similitude

$$u(at) \iff \frac{1}{a} \hat{u}\left(\frac{s}{a}\right) \quad (2.1.22)$$

where a is a complex constant.

3. Shifting

$$u(t)e^{-at} \iff \hat{u}(s+a) \quad (2.1.23)$$

4. Products and convolutions

$$\hat{u}(s) \hat{v}(s) \iff \int_0^t u(\tau) v(t-\tau) d\tau \quad (2.1.24)$$

5. Differentiation

$$\frac{d^n u(t)}{dt^n} \iff s^n \hat{u}(s) - s^{n-1} \frac{du(0)}{dt} - s^{n-2} \frac{d^2 u(0)}{dt^2} \dots - s \frac{d^{n-1} u(0)}{dt^{n-1}} - \frac{d^n u(0)}{dt^n} \quad (2.1.25)$$

where $n = 0, 1, 2, 3, \dots$.

2.2 A General Relation Between Hankel and Fourier Integral Transforms

In order to develop a systematic Fourier transform based approach for disc inclusion problems, it is necessary to examine the relationship between the Hankel and Fourier integral transforms. Such relation provides the mathematical links between the two approaches. Many works have been done on this relation (eg. Erdelyi et al 1954; Sneddon, 1972). Using the coordinate transformations (2.1.15) suggested in the previous section, we shall present a simpler general relation between the two integral transforms in this section.

Based on the integral representations of the Bessel function $J_m(\rho r)$

$$\begin{aligned}
J_m(\rho r) &= \frac{i}{2\pi} \int_0^{2\pi} e^{+i\rho r \sin(\theta+\varphi) - im(\theta+\varphi)} d(\theta + \varphi) \\
&= \frac{1}{2\pi} \int_0^{2\pi} e^{-i\rho r \sin(\theta+\varphi) + im(\theta+\varphi)} d(\theta + \varphi) \\
&= \frac{1}{2\pi} \int_0^{2\pi} \cos[\rho r \sin(\theta + \varphi) - m(\theta + \varphi)] d(\theta + \varphi)
\end{aligned} \tag{2.2.1}$$

we may expand the symmetrical Fourier kernel function K and its complex conjugate K^* in Fourier series,

$$\begin{aligned}
K &= e^{+i(\xi x + \eta y)} = e^{+i\rho r \sin(\theta+\varphi)} = \sum_{m=-\infty}^{m=+\infty} J_m(\rho r) e^{+im(\theta+\varphi)} \\
K^* &= e^{-i(\xi x + \eta y)} = e^{-i\rho r \sin(\theta+\varphi)} = \sum_{m=-\infty}^{m=+\infty} J_m(\rho r) e^{-im(\theta+\varphi)}
\end{aligned} \tag{2.2.2}$$

then the two-dimensional Fourier transform for $u(r, \theta)$ is

$$\begin{aligned}
\tilde{u}(\rho, \varphi) &= \frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} u(r, \theta) e^{-i\rho r \sin(\theta+\varphi)} r d\theta dr \\
&= \sum_{m=-\infty}^{m=+\infty} \int_0^{+\infty} u_m(r) J_m(\rho r) r dr e^{-im\varphi}
\end{aligned} \tag{2.2.3}$$

and its inverse Fourier transform is

$$\begin{aligned}
u(r, \theta) &= \frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} \tilde{u}(\rho, \varphi) e^{+i\rho r \sin(\theta+\varphi)} \rho d\varphi d\rho \\
&= \sum_{m=-\infty}^{m=+\infty} \int_0^{+\infty} \tilde{u}_m(\rho) J_m(\rho r) \rho d\rho e^{+im\theta}
\end{aligned} \tag{2.2.4}$$

where $u_m(r)$ and $\tilde{u}_m(\rho)$ are the m -th coefficients of the Fourier series expansions of

$u(r, \theta)$ and $\tilde{u}(\rho, \varphi)$, respectively,

$$\begin{aligned}
 u(r, \theta) &= \sum_{m=-\infty}^{m=+\infty} u_m(r) e^{+im\theta} \\
 u_m(r) &= \frac{1}{2\pi} \int_0^{2\pi} u(r, \theta) e^{-im\theta} d\theta \\
 \tilde{u}(\rho, \varphi) &= \sum_{m=-\infty}^{m=+\infty} \tilde{u}_m(\rho) e^{-im\varphi} \\
 \tilde{u}_m(\rho) &= \frac{1}{2\pi} \int_0^{2\pi} \tilde{u}(\rho, \varphi) e^{+im\varphi} d\varphi
 \end{aligned} \tag{2.2.5}$$

Consequently, we have the relation

$$\begin{aligned}
 \tilde{u}_m(\rho) &= \int_0^{+\infty} u_m(r) J_m(\rho r) r dr \\
 u_m(r) &= \int_0^{+\infty} \tilde{u}_m(\rho) J_m(\rho r) \rho d\rho
 \end{aligned} \tag{2.2.6}$$

or

$$\begin{aligned}
 \tilde{u}(\rho, \varphi) &= \sum_{m=-\infty}^{m=+\infty} \int_0^{+\infty} u_m(r) J_m(\rho r) r dr e^{-im\varphi} \\
 u(r, \theta) &= \sum_{m=-\infty}^{m=+\infty} \int_0^{+\infty} \tilde{u}_m(\rho) J_m(\rho r) \rho d\rho e^{+im\theta}
 \end{aligned} \tag{2.2.7}$$

In other words, the two-dimensional Fourier transforms can be expressed in terms of the Fourier series expansions whose coefficients consist of the Hankel integrals.

If we follow the conventional coordinate transformations (eg. Sneddon, 1972),

$$\begin{aligned}
 x &= r \cos \theta \\
 y &= r \sin \theta \\
 \xi &= \rho \cos \theta' \\
 \eta &= \rho \sin \theta'
 \end{aligned} \tag{2.2.8}$$

we then have the alternative general relation,

$$\begin{aligned}
 \tilde{u}(\rho, \theta') &= \frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} u(r, \theta) e^{-i\rho r \cos(\theta - \theta')} r d\theta dr \\
 &= \sum_{m=-\infty}^{+\infty} (-i)^m \int_0^{+\infty} u_m(r) J_m(\rho r) r dr e^{+im\theta'} \\
 u(r, \theta) &= \frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} \tilde{u}(\rho, \theta') e^{+i\rho r \cos(\theta - \theta')} \rho d\theta' d\rho \\
 &= \sum_{m=-\infty}^{+\infty} (+i)^m \int_0^{+\infty} \tilde{u}_m(\rho) J_m(\rho r) \rho d\rho e^{+im\theta}
 \end{aligned} \tag{2.2.9}$$

where

$$\begin{aligned}
 u_m(r) &= \frac{1}{2\pi} \int_0^{2\pi} u(r, \theta) e^{-im\theta} d\theta \\
 &= (+i)^m \int_0^{+\infty} \tilde{u}_m(\rho) J_m(\rho r) \rho d\rho \\
 \tilde{u}_m(\rho) &= \frac{1}{2\pi} \int_0^{2\pi} \tilde{u}(\rho, \theta') e^{-im\theta'} d\theta' \\
 &= (-i)^m \int_0^{+\infty} u_m(r) J_m(\rho r) r dr
 \end{aligned} \tag{2.2.10}$$

It is evident from equations (2.2.6) ~ (2.2.10) that the coordinate relationships (2.1.15.b) suggested in this thesis eliminate the coefficients of $(-i)^m$ and $(+i)^m$ in the general relation between Hankel and Fourier integral transforms based on the conventional coordinate transformations (2.2.8).

2.3 Tauberian Theorems of Laplace Transforms

The initial and final responses of the time-dependent mechanics of saturated poroelastic media may be examined by taking the two limiting time $t \rightarrow 0^+$ and $t \rightarrow +\infty$ on the solutions and/or the equations in the time domain, respectively. Since the mathematical formulations are developed in the Laplace transform domain, we have to employ the Tauberian theorems (see e.g. Sneddon, 1972) to analyse the two

limiting responses. Such Tauberian theorems offer a convenient way to perform the evaluations in the Laplace transform domain. The Tauberian theorems recorded below are due to Sneddon (1972),

Theorem 1. *If the function $f(t)$ satisfies the inequality*

$$|f(t)| < Ce^{at} \quad (2.3.1.a)$$

for all $t > 0$, C being a positive constant, then

$$\lim_{s \rightarrow +\infty} s\hat{f}(s) = f(0^+) \quad (2.3.1.b)$$

Theorem 2. *If the function $f(t)$ is bounded for all $t > 0$:*

$$|f(t)| < C \quad (2.3.2.a)$$

and the limit,

$$\lim_{t \rightarrow +\infty} f(t) = f(+\infty) \quad (2.3.2.b)$$

exists, then

$$\lim_{s \rightarrow 0^+} s\hat{f}(s) = f(+\infty) \quad (2.3.2.c)$$

The following two theorems provide a basis for approximating properties of the functions in the transform domain. Such approximating properties are useful in the numerical solutions of the integral equations in the Laplace transform domain.

Theorem 3. *If the function $f(t)$ has the approximation*

$$f(t) \sim At^\gamma, \quad t \rightarrow +\infty, \quad \gamma > -1 \quad (2.3.3.a)$$

then

$$\hat{f}(s) \sim A\Gamma(\gamma + 1)s^{-\gamma-1}, \quad s \rightarrow 0^+ \quad (2.3.3.b)$$

where Γ is the gamma function (Erdelyi et al, 1954).

Theorem 4. *If the function $f(t)$ has the approximation*

$$f(t) \sim At^\gamma, \quad t \rightarrow 0^+, \quad \gamma > -1 \quad (2.3.4.a)$$

then

$$\hat{f}(s) \sim A\Gamma(\gamma + 1)s^{-\gamma-1}, \quad s \rightarrow +\infty \quad (2.3.4.b)$$

where A is a constant.

The following theorem of continuity is important to the examination of the governing equations under some limiting conditions.

Theorem 5. *If the functions $f(s)$ and $g(s)$ have*

$$\lim_{s \rightarrow a} f(s) = A < \infty, \quad \lim_{s \rightarrow a} g(s) = B < \infty \quad (2.3.5.a)$$

then

$$\lim_{s \rightarrow a} [f(s) \pm g(s)] = \lim_{s \rightarrow a} f(s) \pm \lim_{s \rightarrow a} g(s) = A \pm B < \infty \quad (2.3.5.b)$$

$$\lim_{s \rightarrow a} [f(s)g(s)] = \lim_{s \rightarrow a} f(s) \lim_{s \rightarrow a} g(s) = AB < \infty$$

In the investigations of mechanics in poroelastic media, it is assumed that the solutions for the appropriate initial and boundary value problems satisfy the Tauberian theorems.

2.4 Algorithms for the Numerical Inversion of Laplace Transforms

2.4.1 General

It is recognized that the numerical inversion of Laplace transforms plays a key role in the analyses of rigid disc inclusions embedded in saturated poroelastic media. Efforts have been devoted to overcome the numerical problems and to obtain an inversion algorithm with high stability and accuracy for the long time duration

involved in the soil consolidation process. In this section, we shall present a detailed analysis and discussion of the inversion algorithms.

In current literature, there are a great number of algorithms proposed for numerical inversion of Laplace transform. A complete bibliography on numerical procedures for inversion of Laplace transforms and its applications has been compiled by Piessens (1975) and Piessens and Dang (1976), showing over 300 references between the years 1934 and 1976. These algorithms can be broadly classified into the following three groups: (1) interpolation-collocation algorithms; (2) algorithms based on expansion in orthogonal functions; (3) algorithms based on numerical Fourier transforms.

The rationale for many inversion algorithms is that the Laplace transforms offer a powerful tool in the analysis of time-dependent linear systems. On the other hand its numerical inversion is a notoriously ill-conditioned process (Bellman, Kalaba, and Lockett, 1966). Davies and Martin (1979) presented a comprehensive comparison study of 14 algorithms for numerical inversion of Laplace transforms. They concluded that no single algorithm gave optimum results for all purposes and for all occasions and every algorithm will break down with some functions. More recently, Narayanan and Beskos (1982) have presented a systematic discussion on the use of eight inversion algorithms for solving time-dependent linear problems in applied mechanics. It was pointed out in this discussion that the algorithm based on the sine-cosine transform (Durbin, 1974) is the most accurate and most convenient to use, but requires the most computational effort.

In order to adopt a proper and accurate inversion algorithm for application to the mechanics in saturated poroelastic media, we have tested and evaluated five different methods for numerical inversion of Laplace transforms. These five algorithms were developed by Schapery (1962), Durbin (1974), Crump (1976), Talbot (1979), and Wilcox and Gibson (1984). In such examinations we also observe that the algorithms based on the sine-cosine transform (Durbin, 1974; Crump, 1979) is the most accurate and most convenient, even though Schapery and Talbot's algorithms have

been used by several authors to investigate the consolidation behaviour of layered saturated soil media (see e.g., Cheng and Liggett, 1984a; Vardoulakis and Harnpattanapanich, 1986; Booker and Small, 1987; Small and Booker, 1987). However, these algorithms cannot be directly used in the analysis of mechanics in poroelastic media due to the very long (relative to a non-dimensional time) consolidation process. Therefore, further modifications on the algorithms must be made in order to efficiently and accurately invert Laplace transforms encountered in this thesis. In the following sections, we shall briefly introduce these algorithms based on the sine-cosine transform and then develop the modified algorithm which is adopted in the thesis.

2.4.2 Algorithms based on the sine-cosine transform for inverse Laplace transforms

Let $f(t)$ be a real function of t , with $f(t) = 0$ for $t < 0$; the Laplace transform and its inversion formula are defined as follows:

$$\hat{f}(s) = \int_0^{+\infty} f(t)e^{-st}dt \quad (2.4.1)$$

$$f(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \hat{f}(s)e^{st}ds \quad (2.4.2)$$

where $s = a + i\omega$, $a > a_0 > 0$, $f(t) = O(e^{at})$ as $t \rightarrow \infty$, a_0 is a constant that exceeds the real part of all the singularities of $\hat{f}(s)$, and the integrals are in the sense of the Cauchy principal value. In case of singularities of $\hat{f}(s)$ to the right of the origin, a suitable translation of the imaginary axis can always reject those singularities to the left of the origin. In all this discussion it is therefore assumed that equations (2.4.1) and (2.4.2) exist for $\text{Re}(s) \geq a > 0$.

Expanding equation (2.4.1), we have

$$\begin{aligned}\hat{f}(a + i\omega) &= \int_0^{+\infty} f(t)e^{-at} \cos(\omega t) dt - i \int_0^{+\infty} f(t)e^{-at} \sin(\omega t) dt \\ \operatorname{Re}[\hat{f}(a + i\omega)] &= \int_0^{+\infty} f(t)e^{-at} \cos(\omega t) dt \\ \operatorname{Im}[\hat{f}(a + i\omega)] &= - \int_0^{+\infty} f(t)e^{-at} \sin(\omega t) dt\end{aligned}\quad (2.4.3)$$

By expanding equation (2.4.2) with $ds = i d\omega$, and using the parity of $\operatorname{Re}[\hat{f}(a + i\omega)]$ and $\operatorname{Im}[\hat{f}(a + i\omega)]$, we obtain three alternative formulas for the Laplace inverse $f(t)$ corresponding to $\hat{f}(s)$:

$$f(t) = \frac{2}{\pi} e^{at} \int_0^{+\infty} \operatorname{Re}[\hat{f}(a + i\omega)] \cos(\omega t) d\omega \quad (2.4.4)$$

$$f(t) = -\frac{2}{\pi} e^{at} \int_0^{+\infty} \operatorname{Im}[\hat{f}(a + i\omega)] \sin(\omega t) d\omega \quad (2.4.5)$$

$$f(t) = \frac{1}{\pi} e^{at} \int_0^{+\infty} \{ \operatorname{Re}[\hat{f}(a + i\omega)] \cos(\omega t) - \operatorname{Im}[\hat{f}(a + i\omega)] \sin(\omega t) \} d\omega \quad (2.4.6)$$

Based on these three expressions, several algorithms have been proposed for numerical inversion of Laplace transform. These algorithms are essentially formulated using the trapezoidal quadrature in the interval $0 < t < 2T$. The basic approximated formulae can be derived from equations (2.4.4) ~ (2.4.6) as follows,

$$f(t) \approx \frac{2e^{at}}{T} \left\{ \frac{1}{2} \hat{f}(a) + \sum_{n=1}^{+\infty} \operatorname{Re}[\hat{f}(a + i \frac{n\pi}{T})] \cos(\frac{n\pi t}{T}) \right\} \quad (2.4.7)$$

where $0 < t \leq 2T$.

$$f(t) \approx \frac{e^{at}}{T} \left\{ \frac{1}{2} \hat{f}(a) + \sum_{n=1}^{+\infty} \operatorname{Re}[\hat{f}(a + i \frac{n\pi}{T})] \cos(\frac{n\pi t}{T}) - \operatorname{Im}[\hat{f}(a + i \frac{n\pi}{T})] \sin(\frac{n\pi t}{T}) \right\} \quad (2.4.8)$$

where $0 < t \leq 2T$.

The formulae proposed by Dubner and Abate (1968) for numerical inverting $\hat{f}(s)$ is the approximation (2.4.7) which involves only $\operatorname{Re}[\hat{f}(s)]$. Based on the

approximation (2.4.8) which involves both the real and the imaginary parts of $\hat{f}(s)$, Durbin (1974) has developed an efficient improvement to Dubner and Abate's method. This technique employs the Fast Fourier Transform algorithm to speed up the computation and to increase accuracy, *i.e.*

$$f(t_j) \approx \frac{2e^{aj\Delta t}}{T} \left\{ -\frac{1}{2} \text{Re}[\hat{f}(a)] + \text{Re} \left[\sum_{n=0}^{N_1-1} [A(n) + iB(n)] W^{jn} \right] \right\} \quad (2.4.9)$$

where $0 < t \leq T$.

$$\begin{aligned} W &= \cos \frac{2\pi}{N_1} + i \sin \frac{2\pi}{N_1} = e^{i\frac{2\pi}{N_1}}, \quad L = 1, 2, 3, \dots \\ A(n) &= \sum_{l=0}^L \text{Re} \left[\hat{f} \left(a + i(n + lN_1) \frac{2\pi}{T} \right) \right] \\ B(n) &= \sum_{l=0}^L \text{Im} \left[\hat{f} \left(a + i(n + lN_1) \frac{2\pi}{T} \right) \right] \end{aligned} \quad (2.4.10)$$

Thus, $f(t_j)$ is computed for N_1 equidistant points $t_j = j\Delta t = j(T/N_1)$, where $T = t_{max}$ and $j = 0, 1, 2, \dots, N_1 - 1$. It is easily observed that one incremental time, $\Delta t = T/N_1$ is not suitable for the whole range of the soil consolidation process.

Crump (1979) suggested an inversion algorithm by incorporating the epsilon algorithm (MacDonald, 1964) into the approximation (2.4.8) for speeding up its convergence. The epsilon algorithm can be stated as following.

Suppose we wish to approximate the sum of an infinite series $\sum_{n=1}^{+\infty} a_n$ using its $2M + 1$ partial sums

$$S_m = \sum_{n=1}^m a_n, \quad m = 1, 2, 3, \dots, 2M + 1 \quad (2.4.11)$$

By defining

$$\epsilon_{p+1}^{(m)} = \epsilon_{p-1}^{(m+1)} + \frac{1}{\epsilon_p^{(m+1)} - \epsilon_p^{(m)}}, \quad \epsilon_0^{(m)} = 0, \quad \epsilon_1^{(m)} = S_m \quad (2.4.12)$$

we obtain a transform sequence $\epsilon_1^{(1)}, \epsilon_3^{(1)}, \epsilon_5^{(1)} \dots \epsilon_{2M+1}^{(1)}$. This transform sequence is a successive approximation to the sum of the infinite series. and provides better result than the untransform sequence $S_1, S_2, \dots, S_{2M+1}$.

The constant parameter a in equation (2.4.8) is obtained from error analysis as follows,

$$a = \alpha_t - \frac{\ln(\epsilon_a)}{2T} \quad (2.4.13)$$

where ϵ_a is a relative error and α_t is a number larger than $\max\{Re(P); P \text{ is a pole of } \hat{f}(s)\}$. T is chosen so that $2T > t_{max}$. This algorithm can be used to approximate the inverse function on the interval of $0 < t < 2T$. However, it was pointed out in the paper that “the method artificially introduces a discontinuity at $t = 0$ which tends to retard convergence in the neighborhood of the origin”. This is referred to as the “Gibbs phenomenon”. The neighborhood of the origin is a relative distance which depends on t_{max} or T so that this algorithm also cannot be directly used for numerically inverting the Laplace transforms encountered in poroelastic media until further modifications are made.

2.4.3 A modified algorithm for inverse Laplace transforms

Two modifications have been made to adopt Crump's algorithm for the analysis of time-dependent mechanics in poroelastic media. These two modifications allow the inversion algorithm meet the requirements of the very long (non-dimensional time $10^{-5} \sim 10^{+3}$) process of soil consolidation.

The first modification is implemented for reducing the roundoff error induced in the direct calculation of the partial sums of the Fourier series in the Crump's algorithm, i.e.

$$S_m = \sum_{n=1}^m a_n = Re\left[\sum_{n=1}^m \hat{f}\left(a + i\frac{n\pi}{T}\right)e^{in\phi}\right] \quad (2.4.14)$$

where $\phi = \pi t/T$, $m = 1, 2, 3, \dots, 2M + 1$.

Instead of the direct calculation, we incorporate the following algorithm for evaluating more accurately the partial sums (Talbot, 1979).

By defining

$$\begin{aligned} b_{m+2} &= b_{m+1} = 0 \\ b_n &= \hat{f}\left(a + i\frac{n\pi}{T}\right) + 2\cos\phi b_{n+1} - b_{n+2} \\ n &= m, m-1, \dots, 1 \end{aligned} \quad (2.4.15)$$

then the partial sums can be simply expressed as

$$S_m = \operatorname{Re} \sum_{n=1}^m \hat{f}\left(a + i \frac{n\pi}{T}\right) e^{in\phi} = \operatorname{Re}[b_1(\cos \phi + i \sin \phi) - b_2] \quad (2.4.16)$$

where the factors $e^{in\phi}$ satisfy the recurrence relation,

$$e^{i(n+1)\phi} + e^{i(n-1)\phi} = 2 \cos \phi e^{in\phi} \quad (2.4.17)$$

The second modification is suggested for avoiding the Gibbs phenomena. Instead of increasing the total number $2M + 1$ of the partial sum for the convergence, we fix the total number $2M + 1$ and use a time dependent T as $T = \alpha_s t$ where α_s is a constant.

Many functions such as Heaviside step function, exponential function, error function have been employed to verify the modified algorithm for inverting Laplace transforms. In this research, it is observed that by choosing the constant parameters as

$$\begin{aligned} \epsilon_a &= 10^{-6} \sim 10^{-8} \\ M &= 5 \sim 15 \\ \alpha_s &= 0.9 \sim 1.95 \\ \alpha_t &= 0 \end{aligned} \quad (2.4.18)$$

the modified algorithm gives optimal results in the time interval of $10^{-5} \leq t \leq 10^3$. This interval meets the requirements of time dependent mechanics in poroelastic media. A particular example is documented in the Appendix A. In this example, the consolidation settlements due to a perfectly flexible circular load acting at the surface of a poroelastic halfspace are employed and evaluated to demonstrate the high stability and accuracy of the modified inversion algorithm of the Laplace transforms. All the calculations were done in double precision on a Apollo work station. There is no complex double precision arithmetic on this machine, consequently, all complex calculations were done separately by using the real and imaginary parts.

Chapter 3

Mechanics Of Poroelastic Media

3.1 General

In this thesis, Biot's (1941; 1955; 1956) theory of three-dimensional poroelasticity is employed to examine the time dependent mechanics of rigid disc inclusions embedded in linear, isotropic, porous, elastic media saturated with compressible fluids. In this Chapter, we shall discuss this classical Biot's theory and its governing partial differential equations in both the physical and the transform domains. The developments of poroelastic theory are firstly reviewed with emphasis on the physical and mathematical features of Biot's theory. The governing partial differential equations of poroelasticity are then recorded in general orthogonal curvilinear coordinates in tensor form. Such general governing equations are further explicitly expressed in the Cartesian and cylindrical coordinates. Some basic properties are presented for the material parameters of poroelastic media saturated with compressible fluids.

In the second part of this chapter, the governing equations are formulated by using the two-dimensional Fourier transforms and Laplace transforms in both Cartesian and cylindrical coordinates. A set of solution representations is suggested by introducing two sets of eight field variables in the transform domain for all the field variables, in both Cartesian and cylindrical coordinates. The governing partial differential equations in both Cartesian and cylindrical coordinates are then reduced into two systems of first order ordinary differential equations in terms of the independent vertical coordinate z , and referred to the Fourier and Laplace transform domains.

In both the physical and transform domains, attention is focused on the governing equations under limiting physical constraints. There are at least three limiting

physical conditions of engineering significance. They are the initial response $t \rightarrow 0^+$ and the final response $t \rightarrow +\infty$ as well as the limiting response of $\nu \rightarrow \nu_u$. These limiting responses illustrate the mechanical and mathematical relations between the theory of poroelasticity and those of elastostatics and seepage flow governed by Darcy's law.

3.2 Governing Equations in the Physical Domain

3.2.1 General

Terzaghi (1923) first investigated the process of soil consolidation. Through his formulation of the effective stress principle, Terzaghi proposed a fundamental approach to the study of a fully saturated soil and developed the one-dimensional theory of soil consolidation by use of a model of porous media with small deformation. The governing equation for the one-dimensional theory developed by Terzaghi (1923) has the same form as the classical diffusion equation. Terzaghi's one-dimensional theory of soil consolidation has been widely applied to the analysis of practical problems and has become a standard procedure in geomechanical analysis. Many conventional methods for predicting the magnitude and progress of foundation settlements are based on this theory (Lambe and Whitman, 1969; Das, 1983).

A generalization of Terzaghi's theory to include three-dimensional effect was suggested by Rendulic (1936). It is based on the hypothesis that the summation of total normal stresses in soil media is time invariant in the sense of soil consolidation. In other words, the first stress coordinate invariant $\sigma_{11} + \sigma_{22} + \sigma_{33}$ does not change with the dissipation of excess pore pressures. This hypothesis has led to the Terzaghi-Rendulic theory or the pseudo three-dimensional theory of soil consolidation. In this approach, the total stress field in the soil media is treated independently and usually accomplished by assuming that it is governed by the classical theory of elasticity (eg. Zaretsky, 1967).

One of the serious drawbacks of these two theories is the lack of the correct form of coupling between the deformation of soil skeleton and the flow of pore fluids. As a direct result of this deficiency, some special features such as Mandel-Cryer (Mandel, 1953; Cryer, 1963) effect¹ do not appear in these two uncoupled theories. This special feature is a dominant characteristics of three dimensional soil consolidation (Schiffman, Chen, and Jordan, 1969).

The theory of three dimensional linear poroelasticity for a saturated medium was formulated by Biot (1941; 1955; 1956) to model more realistically the mechanical behavior of saturated soils or rocks within the working load range. In this theory, soils or rocks are modeled as deformable, linear, porous, elastic media saturated with compressible fluids. The compressible fluid is a suitable model for the mechanical behaviour of, say, an air-water mixture in a saturated geological medium. The compressibility of the air-water mixture is determined by the dissolved gas in the fluid. A set of partial differential equations was established by Biot (1941; 1955; 1956) to describe the coupled mechanical behavior of saturated porous media and make the linear poroelasticity become a completely self-consistent boundary and initial value problem.

Biot's theory intrinsically takes into account the time dependent interaction of the soil skeleton and pore fluids and/or the coupling between the deformation of solid phase and the flow of pore fluids. The coupled mechanical state is described by mechanical variables (stresses and excess pore pressure) and kinematic variables (displacements and velocities) within each phase. The mechanical and kinematic variables are time-dependent. Four of these variables are independent and represent the coupling characteristics of the governing equations. They could be the three displacement components and the excess pore pressure. Consequently, it is necessary and sufficient to adopt four independent boundary conditions (three for solid phase and one for pore fluid phase) to set up a proper boundary and initial value

¹ In some regions of saturated soils the excess pore pressures would increase beyond their values associated with the start of the consolidation process. With the progress of time, the excess pore pressures will begin to decrease until all excess pore pressures are dissipated.

2

PM-1 3½"x4" PHOTOGRAPHIC MICROCOPY TARGET
NBS 1010a ANSI/ISO #2 EQUIVALENT

PRECISIONSM RESOLUTION TARGETS

PIONEERS IN METHYLENE BLUE TESTING SINCE 1974

1000 COUNTY ROAD 8, DUNEDIN, FL 33511 USA
TEL. 615 485 7857 FAX 615 485 7857 TUL. 015555555

problem for a saturated porous medium. These four independent boundary conditions correctly match the situations encountered with many practical problems associated with saturated soils or rocks.

Under certain special assumptions with regard to the time-dependent variables, the governing equations of Biot's theory reduce to those of Terzaghi's theory and Terzaghi and Rendulic's theory. As mentioned previously, Biot's theory describes the dominant characteristics of many physical phenomena and engineering problems encountered in saturated soils or rocks. In recent years, Biot's theory has been widely applied in geotechnical, hydraulic and petroleum engineering, as well as in geophysics. Many significant results have been obtained particularly in relation to soil consolidation.

3.2.2 Governing equations in general orthogonal curvilinear coordinates

As with Biot's (1941; 1955; 1956) original formulation, we choose the basic mechanical variables to be the total stress tensor $\mathbf{T}$ and the excess pore pressure p (see also, Rice and Cleary, 1976; Cheng and Detournay, 1988). The corresponding kinematic quantities are the strain tensor $\mathbf{\Gamma}$, displacement vector $\mathbf{u}$ in soil skeleton, the volumetric strain in the fluid ζ_v and specific discharge vector $\mathbf{v}$ in the pore fluids.

Based on the assumption of the existence of a strain energy density function W and linear relations between the stresses and strains, the constitutive law for a linear, isotropic, porous, elastic medium saturated with compressible fluids can be expressed, in tensor form, as follows,

$$W = \frac{1}{2}(\mathbf{T}:\mathbf{\Gamma} + p\zeta_v) \quad (3.2.1.a)$$

$$\mathbf{T} = 2\mu\mathbf{\Gamma} + \lambda\epsilon_v\mathbf{I} - \alpha p\mathbf{I} \quad (3.2.1.b)$$

$$p = \beta(\zeta_v - \alpha \epsilon_v) \quad (3.2.1.c)$$

where μ and λ are the two Lamé's constants, α and β are the two material parameters introduced by Biot (see eg. Biot and Willis, 1957), $\mathbf{I}$ is a unit tensor, and ∇ is the vector differential operator

$$\nabla = \mathbf{e}_1 \frac{1}{H_1} \frac{\partial}{\partial x_1} + \mathbf{e}_2 \frac{1}{H_2} \frac{\partial}{\partial x_2} + \mathbf{e}_3 \frac{1}{H_3} \frac{\partial}{\partial x_3} \quad (3.2.1.d)$$

with in Cartesian coordinates, takes the form

$$(x_1 \mathbf{e}_1, x_2 \mathbf{e}_2, x_3 \mathbf{e}_3) = (x \mathbf{e}_x, y \mathbf{e}_y, z \mathbf{e}_z), \quad (H_1, H_2, H_3) = (1, 1, 1) \quad (3.2.1.e)$$

and in cylindrical coordinates, takes the form

$$(x_1 \mathbf{e}_1, x_2 \mathbf{e}_2, x_3 \mathbf{e}_3) = (r \mathbf{e}_r, \theta \mathbf{e}_\theta, z \mathbf{e}_z), \quad (H_1, H_2, H_3) = (1, r, 1) \quad (3.2.1.f)$$

In equations (3.2.1), H_j ($j = 1, 2, 3$) are components of the metric for the coordinates.

Equations (3.2.1.b) show the Biot effective stress tensor, *i.e.*, $\mathbf{T}' = \mathbf{T} + \alpha p \mathbf{I}$, is directly proportional to the strain tensor in the porous solid and has a form similar to the constitutive equations applicable to classical elastic media. Equation (3.2.1.c) indicates that the excess pore pressure depends on both the volumetric strain of the soil skeleton and the volumetric variation of the fluid content.

Equations (3.2.1) are generally referred as the 'compressible constituents' model because they admit fluid and soil skeletal compressibilities in the basic assumptions. The term 'compressible', when applied to the pore fluid, refers to the compressibility of an air-water mixture in a partially saturated soil. When applied to the entire soil medium, it refers to the bulk compressibility of the saturated soil medium where the excess pore pressure is zero under drained condition. The compressibility of pore fluid is reflected in the two Biot material parameters α and β . The pore fluid is incompressible if $\alpha = 1.0$ and $\beta \rightarrow +\infty$.

The additional governing equations needed to complete the poroelasticity theory are:

the quasi-static equilibrium equations,

$$\nabla \cdot \mathbf{T} + \mathbf{f} = \mathbf{0} \quad (3.2.2)$$

the relation of strain tensor and displacement vector,

$$\mathbf{\Gamma} = \frac{1}{2}(\nabla \mathbf{u} + \mathbf{u} \nabla) \quad (3.2.3.a)$$

$$\varepsilon_v = \nabla \cdot \mathbf{u} \quad (3.2.3.b)$$

Darcy law of quasi-static fluid flow,

$$\mathbf{v} = -\kappa(\nabla p - \mathbf{F}) \quad (3.2.4)$$

the continuity equation of quasi-static fluid flow,

$$\frac{\partial \zeta_v}{\partial t} + \nabla \cdot \mathbf{v} = q \quad (3.2.5)$$

and the continuity assumptions for the field variables (see eg., Kupradze, 1979),

$$\mathbf{u}, p \in C^2(D \times [0, +\infty)) \cap C^1(D \cap \partial D \times [0, +\infty)) \quad (3.2.6.a)$$

$$\mathbf{T}, \mathbf{\Gamma}, \mathbf{v} \in C^1(D \times [0, +\infty)) \quad (3.2.6.b)$$

where $\mathbf{f}$ is the body force per unit volume of the bulk material (including both fluid and solid), $\mathbf{F}$ the body force per unit volume of pore fluid, q the rate of injected (or withdrawn) volume due to fluid source (or sink) and $\kappa = k/\gamma_w$ (k is the coefficient of permeability and γ_w is the unit weight of pore fluid). D and ∂D are the domain and boundary of a saturated poroelastic medium, respectively.

Equations (3.2.1) ~ (3.2.6) form a system of linear simultaneous partially differential equations and formulate a completely consistent boundary and initial value

problem for the linear process of a saturated poroelastic medium which can be described by five material parameters, $\mu, \lambda, \alpha, \beta$ and κ . There are seventeen equations in the system with four independent variables (three spatial and one temporal coordinates). Among them there are only four independent equations. These four independent equations are composed of the four independent field variables; the three displacements $\mathbf{u}$ and the pore pressure p . The three elastostatic equations with fluid coupling take the form

$$\mu \nabla \cdot \nabla \mathbf{u} + (\mu + \lambda) \nabla \nabla \cdot \mathbf{u} - \alpha \nabla p + \mathbf{f} = 0 \quad (3.2.7.a)$$

The coupled equation governing pore pressure takes the form $\alpha \beta \frac{\partial \epsilon_v}{\partial t}$,

$$\frac{\partial p}{\partial t} - \kappa \beta \nabla \cdot \nabla p = -\alpha \beta \frac{\partial \epsilon_v}{\partial t} + \beta(q - \kappa \nabla \cdot \mathbf{F}) \quad (3.2.7.b)$$

The coupling characteristics of the governing equations are clearly illustrated in these four independent equations (3.2.7). Consequently, it is necessary and sufficient to use four independent boundary conditions (three for solid phase and one for pore fluid phase) and one initial condition ($\zeta_v|_{t=0} = \zeta_{v0}$) to formulate the boundary and initial value problems for a saturated poroelastic medium. These four independent boundary conditions completely match many of the conditions encountered with the practical problems involving geological media. The initial condition $\zeta_v|_{t=0} = \zeta_{v0}$ usually is assumed to be zero. The simple boundary condition of pore fluid phase is the drainage condition which is usually assumed to be either pervious or impervious.

3.2.3 Governing equations in Cartesian coordinates

The governing equations for a linear, isotropic poroelastic medium saturated with compressible fluid can be explicitly expressed in the Cartesian coordinates ($Oxyz$),

The linear constitutive relations are

$$\begin{aligned}
 \sigma_{xx} &= \lambda \varepsilon_v + 2\mu \varepsilon_{xx} - \alpha p \\
 \sigma_{yy} &= \lambda \varepsilon_v + 2\mu \varepsilon_{yy} - \alpha p \\
 \sigma_{zz} &= \lambda \varepsilon_v + 2\mu \varepsilon_{zz} - \alpha p \\
 p &= \beta \zeta_v - \alpha \beta \varepsilon_v \\
 \sigma_{xz} &= 2\mu \varepsilon_{xz} \\
 \sigma_{yz} &= 2\mu \varepsilon_{yz} \\
 \sigma_{xy} &= 2\mu \varepsilon_{xy}
 \end{aligned} \tag{3.2.8}$$

The quasi-static equilibrium equations are

$$\begin{aligned}
 \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + f_x &= 0 \\
 \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + f_y &= 0 \\
 \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + f_z &= 0
 \end{aligned} \tag{3.2.9}$$

The strain and displacement relationships are

$$\begin{aligned}
 \varepsilon_{xx} &= \frac{\partial u_x}{\partial x} \\
 \varepsilon_{yy} &= \frac{\partial u_y}{\partial y} \\
 \varepsilon_{zz} &= \frac{\partial u_z}{\partial z} \\
 \varepsilon_{xz} &= \frac{1}{2} \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right) \\
 \varepsilon_{yz} &= \frac{1}{2} \left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \right) \\
 \varepsilon_{xy} &= \frac{1}{2} \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) \\
 \varepsilon_v &= \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}
 \end{aligned} \tag{3.2.10}$$

where u_x , u_y , and u_z are the components of the displacement vector $\mathbf{u}$.

Darcy's law for quasi-static fluid flow takes the form

$$\begin{aligned}
 v_x &= -\kappa \left(\frac{\partial p}{\partial x} - F_x \right) \\
 v_y &= -\kappa \left(\frac{\partial p}{\partial y} - F_y \right) \\
 v_z &= -\kappa \left(\frac{\partial p}{\partial z} - F_z \right)
 \end{aligned} \tag{3.2.11}$$

where v_x , v_y , and v_z are the components of the fluid velocity vector $\mathbf{v}$.

The continuity equation for quasi-static fluid flow is given by

$$\frac{\partial \zeta_v}{\partial t} + \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = q \quad (3.2.12)$$

3.2.4 Governing equations in cylindrical coordinates

The equations governing the mechanics of a linear, isotropic poroelastic medium saturated with compressible fluid can also be explicitly expressed in the cylindrical coordinates ($Or\theta z$).

The linear constitutive relationships are

$$\begin{aligned} \sigma_{rr} &= \lambda \epsilon_v + 2\mu \epsilon_{rr} - \alpha p \\ \sigma_{\theta\theta} &= \lambda \epsilon_v + 2\mu \epsilon_{\theta\theta} - \alpha p \\ \sigma_{zz} &= \lambda \epsilon_v + 2\mu \epsilon_{zz} - \alpha p \\ p &= \beta \zeta_v - \alpha \beta \epsilon_v \\ \sigma_{rz} &= 2\mu \epsilon_{rz} \\ \sigma_{\theta z} &= 2\mu \epsilon_{\theta z} \\ \sigma_{r\theta} &= 2\mu \epsilon_{r\theta} \end{aligned} \quad (3.2.13)$$

The quasi-static equilibrium equations are

$$\begin{aligned} \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + f_r &= 0 \\ \frac{\partial \sigma_{\theta r}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2\sigma_{r\theta}}{r} + f_\theta &= 0 \\ \frac{\partial \sigma_{zr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{z\theta}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} + f_z &= 0 \end{aligned} \quad (3.2.14)$$

The strain-displacement relationships are

$$\begin{aligned}
\varepsilon_{rr} &= \frac{\partial u_r}{\partial r} \\
\varepsilon_{\theta\theta} &= \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \\
\varepsilon_{zz} &= \frac{\partial u_z}{\partial z} \\
\varepsilon_{rz} &= \frac{1}{2} \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right) \\
\varepsilon_{\theta z} &= \frac{1}{2} \left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r} \frac{\partial u_z}{\partial \theta} \right) \\
\varepsilon_{r\theta} &= \frac{1}{2} \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) \\
\varepsilon_v &= \varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz}
\end{aligned} \tag{3.2.15}$$

where u_r , u_θ , and u_z are the components of the displacement vector $\mathbf{u}$.

Darcy's law for quasi-static fluid flow takes form

$$\begin{aligned}
v_r &= -\kappa \left(\frac{\partial p}{\partial r} - F_r \right) \\
v_\theta &= -\kappa \left(\frac{1}{r} \frac{\partial p}{\partial \theta} - F_\theta \right) \\
v_z &= -\kappa \left(\frac{\partial p}{\partial z} - F_z \right)
\end{aligned} \tag{3.2.16}$$

where v_r , v_θ , and v_z are the components of the displacement vector $\mathbf{v}$.

The continuity equation for quasi-static fluid flow is given by

$$\frac{\partial \zeta_v}{\partial t} + \frac{\partial v_r}{\partial r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z} = q \tag{3.2.17}$$

3.2.5 Material properties of poroelastic media

From the previous discussion it is evident that five material parameters are necessary and sufficient to characterize the poroelastic behaviour of a soil medium which has an isotropic linear elastic soil skeletal response and compressible ideal pore fluid behaviour.

In the previous section we have introduced the parameters μ , λ , α , β and κ . It is also possible to introduce equivalent combinations of material parameters which refer to parameters such as the drained Poisson's ratio ν , the undrained

Poisson's ratio ν_u , the drained bulk modulus K , the undrained bulk modulus K_u , the undrained Lamé's constant λ_u and the pore pressure parameter B introduced by Skempton (1954). The parameter B is defined as the ratio of the induced pore water pressure to the variation in total isotropic stress, measured under undrained conditions. The Table 3.2.1 gives the various relationships between these sets of material parameters governing the poroelastic response.

In this thesis, the five material parameters μ , λ , α , β and κ are chosen in order to simplify the coefficients occurring in the mathematical formulations. However, due to their greater physical significance, the five material parameters μ , ν , ν_u , B and κ are finally utilized in the presentations of the numerical results associated with the various inclusion problems.

The bounds for the constitutive parameters governing poroelastic behaviour of the fluid saturated medium can be obtained by invoking thermodynamic restrictions (see *e.g.* Bowen, 1976). The bounds for certain parameters can also be determined by using on positive definiteness of the strain energy for saturated poroelastic medium (see Appendix B), *e.g.*,

$$\mu > 0, \quad -1 < \nu < \nu_u \leq 0.5, \quad B \neq 0. \quad (3.2.18.a)$$

When dealing with most geological materials with an elastic response the lower limit of -1 for ν and ν_u can be replaced by zero. Consequently, we have

$$0 \leq \nu < \nu_u \leq 0.5, \quad 0 < B \leq 1 \quad (3.2.18.b)$$

In the specific case of a poroelastic medium which is saturated with an incompressible pore fluid, we have

$$\nu_u = 0.5, \quad B = 1. \quad (3.2.18.c)$$

3.2.6 Governing equations for three limiting responses

It is of mathematical and physical instruction to examine the properties of the structure of the governing equations under certain limiting conditions. There

Table 3.2.1 Relations for material parameters of saturated poroelastic media

	$\nu \quad \nu_u \quad \mu \quad B$	$\lambda \quad \lambda_u \quad \mu \quad B$	$\alpha \quad \beta \quad \mu \quad B$	$K \quad K_u \quad \mu \quad B$
$\nu =$	ν	$\frac{\lambda}{2(\lambda + \mu)}$	$\frac{3\alpha\beta(1 - \alpha\beta) - 2\mu B}{6\alpha\beta + 2\mu B}$	$\frac{3K - 2\mu}{2(3K + \mu)}$
$\nu_u =$	ν_u	$\frac{\lambda_u}{2(\lambda_u + \mu)}$	$\frac{3\alpha\beta - 2\mu B}{6\alpha\beta + 2\mu B}$	$\frac{3K_u - 2\mu}{2(3K_u + \mu)}$
$\lambda =$	$\frac{2\mu\nu}{1 - 2\nu}$	λ	$\alpha\beta(1 - \alpha\beta) - \frac{2}{3}\mu B$	$K - \frac{2}{3}\mu$
$\lambda_u =$	$\frac{2\mu\nu_u}{1 - 2\nu_u}$	λ_u	$\frac{\alpha\beta}{B} - \frac{2}{3}\mu$	$K_u - \frac{2}{3}\mu$
$\alpha =$	$\frac{3(\nu_u - \nu)}{B(1 - 2\nu)(1 + \nu_u)}$	$\frac{3(\lambda_u - \lambda)}{B(3\lambda_u + 2\mu)}$	α	$\frac{1}{B}(1 - \frac{K}{K_u})$
$\beta =$	$\frac{2\mu B^2(1 - 2\nu)(1 + \nu_u)^2}{9(\nu_u - \nu)(1 - 2\nu_u)}$	$\frac{(3\lambda_u + 2\mu)^2 B^2}{9(\lambda_u - \lambda)}$	β	$\frac{B^2 K_u^2}{K_u - K}$
$K =$	$\frac{2\mu(1 + \nu)}{3(1 - 2\nu)}$	$\lambda + \frac{2}{3}\mu$	$\frac{3\alpha\beta(1 - \alpha\beta)}{B}$	K
$K_u =$	$\frac{2\mu(1 + \nu_u)}{3(1 - 2\nu_u)}$	$\lambda_u + \frac{2}{3}\mu$	$\frac{\alpha\beta}{B}$	K_u

are at least three limiting conditions which are of engineering significance. These conditions are the initial response as $t \rightarrow 0^+$ and the final response as $t \rightarrow +\infty$ as well as the limiting response as $\nu \rightarrow \nu_u$.

A special case of initial and final responses was discussed by Davis and Poulos (1963) (see also Booker and Small, 1975) for poroelastic media saturated with an incompressible pore fluid where $B = 1$ and $\nu_u = 0.5$. In their discussion, Davis and

Poulos (1963) concluded that the initial/final solutions can be obtained by performing purely elastostatic analysis and using the undrained/drained elastic constants and initial/final loads, respectively. In the following sections, we shall perform a further examination of the governing equations of poroelastic media saturated with compressible fluids under these three limiting conditions.

3.2.6.1 The initial response ($t \rightarrow 0^+$)

The continuity equation of quasi-static fluid flow (3.2.5) may be re-written in the integral form,

$$\zeta_v + \int_0^t \nabla \cdot \mathbf{v} dt = \int_0^t q dt \quad (3.2.19)$$

where the initial value ζ_{v0} is assumed to be zero.

Taking the limit of $t \rightarrow 0^+$ on the equation (3.2.19) together with the continuity of $\mathbf{v}$ and q , we have

$$\lim_{t \rightarrow 0^+} \zeta_v = 0 \quad (3.2.20)$$

Then the governing equations for the initial response as $t \rightarrow 0^+$ may be obtained as follows,

As $t \rightarrow 0^+$, The linear constitutive relations (3.2.1.b) take the form

$$\mathbf{T}(\mathbf{x}, 0^+) = 2\mu\boldsymbol{\Gamma}(\mathbf{x}, 0^+) + \lambda_{\mathbf{u}}\varepsilon_v(\mathbf{x}, 0^+)\mathbf{I} \quad (3.2.21.a)$$

$$\begin{aligned} p(\mathbf{x}, 0^+) &= -\alpha\beta\varepsilon_v(\mathbf{x}, 0^+) \\ &= -B \frac{\sigma_{11}(\mathbf{x}, 0^+) + \sigma_{22}(\mathbf{x}, 0^+) + \sigma_{33}(\mathbf{x}, 0^+)}{3} \end{aligned} \quad (3.2.21.b)$$

where $\lambda_{\mathbf{u}} = \lambda + \alpha^2\beta$, and $\sigma_{11} + \sigma_{22} + \sigma_{33}$ is the first invariant of total stress tensor $\mathbf{T}$.

Similarly the equilibrium equations (3.2.2) give

$$\nabla \cdot \mathbf{T}(\mathbf{x}, 0^+) + \mathbf{f}(\mathbf{x}, 0^+) = \mathbf{0} \quad (3.2.22)$$

the strain-displacement relationships give

$$\boldsymbol{\Gamma}(\mathbf{x}, 0^+) = \frac{1}{2}[\nabla\mathbf{u}(\mathbf{x}, 0^+) + \mathbf{u}(\mathbf{x}, 0^+)\nabla] \quad (3.2.23)$$

and Darcy's law for fluid flow gives

$$\mathbf{v}(\mathbf{x}, 0^+) = -\kappa[\nabla p(\mathbf{x}, 0^+) - \mathbf{F}(\mathbf{x}, 0^+)] \quad (3.2.24)$$

It is worth noting that equation (3.2.21.b) is exactly the generalized definition of excess pore pressure involving Skempton's pore pressure parameter B for a poroelastic medium saturated with a compressible fluid.

The initial response of stresses and displacements in the soil skeleton is then found by using the classical elastostatics with the undrained elastic constants μ and λ_* (or μ and ν_*). For a homogeneous poroelastic medium, the Navier-Cauchy equations are,

$$\mu \nabla \cdot \nabla \mathbf{u}(\mathbf{x}, 0^+) + (\mu + \lambda_*) \nabla \nabla \cdot \mathbf{u}(\mathbf{x}, 0^+) + \mathbf{f}(\mathbf{x}, 0^+) = \mathbf{0} \quad (3.2.25)$$

Then the initial response of pore pressure $p(\mathbf{x}, 0^+)$ and specific discharge vector $\mathbf{v}(\mathbf{x}, 0^+)$ could be obtained simply through equations (3.2.21.b) and (3.2.24), respectively.

3.2.6.2 The final response ($t \rightarrow +\infty$)

As $t \rightarrow +\infty$, the response of a fluid-saturated porous medium D ($\mu, \lambda, \alpha, \beta, \kappa$) will achieve a steady state, *i.e.*, the volumetric strain of pore fluids ζ_v will not change with time as $t \rightarrow +\infty$,

$$\lim_{t \rightarrow +\infty} \frac{\partial \zeta_v}{\partial t} = 0 \quad (3.2.26)$$

Then the governing equations for the final response may be obtained as follows under the limiting $t \rightarrow +\infty$.

The linear constitutive relations are.

$$\mathbf{T}(\mathbf{x}, +\infty) = 2\mu \boldsymbol{\Gamma}(\mathbf{x}, +\infty) + \lambda \varepsilon_v(\mathbf{x}, +\infty) \mathbf{I} - \alpha p(\mathbf{x}, +\infty) \mathbf{I} \quad (3.2.27.a)$$

and

$$p = \beta \zeta_v(\mathbf{x}, +\infty) - \alpha \beta \varepsilon_v(\mathbf{x}, +\infty) \quad (3.2.27.b)$$

The static equilibrium equations take the form

$$\nabla \cdot \mathbf{T}(\mathbf{x}, +\infty) + \mathbf{f}(\mathbf{x}, +\infty) = \mathbf{0} \quad (3.2.28)$$

The relation of strain tensor and displacement vector are

$$\mathbf{\Gamma}(\mathbf{x}, +\infty) = \frac{1}{2}[\nabla \mathbf{u}(\mathbf{x}, +\infty) + \mathbf{u}(\mathbf{x}, +\infty)\nabla] \quad (3.2.29)$$

Darcy's law of steady state fluid flow takes the form

$$\mathbf{v}(\mathbf{x}, +\infty) = -\kappa[\nabla p(\mathbf{x}, +\infty) - \mathbf{F}(\mathbf{x}, +\infty)] \quad (3.2.30)$$

and the continuity equation of steady state fluid flow is

$$\nabla \cdot \mathbf{v}(\mathbf{x}, +\infty) = q(\mathbf{x}, +\infty) \quad (3.2.31)$$

The steady state solution of pore fluid flow can be obtained from the Poisson equation,

$$\kappa \nabla \cdot \nabla p(\mathbf{x}, +\infty) = -q(\mathbf{x}, +\infty) + \kappa \nabla \cdot \mathbf{F}(\mathbf{x}, +\infty) \quad (3.2.32)$$

The final stresses and displacements in the soil skeleton are then evaluated by using the elasto-static model with the drained elastic constants μ and λ (or μ and ν). The excess pore pressure can be considered as a body force term in the Navier-Cauchy equations of elastostatics,

$$\mu \nabla \cdot \nabla \mathbf{u}(\mathbf{x}, +\infty) + (\mu + \lambda) \nabla \nabla \cdot \mathbf{u}(\mathbf{x}, +\infty) - \alpha \nabla p(\mathbf{x}, +\infty) + \mathbf{f}(\mathbf{x}, +\infty) = \mathbf{0} \quad (3.2.33)$$

By definition, the final excess pore pressure $p(\mathbf{x}, +\infty)$ usually is equal to zero so that the final total stress tensor $\mathbf{T}(\mathbf{x}, +\infty)$ is equal to the final effective stress tensor $\mathbf{T}'(\mathbf{x}, +\infty)$.

3.2.6.3 The limiting response as $\nu \rightarrow \nu_u$

The limiting response as $\nu \rightarrow \nu_u$ indicates the properties of a poroelastic medium if the drained Poisson's ratio ν is equal to the undrained Poisson's ratio ν_u . By taking the limit as $\nu \rightarrow \nu_u$ of the expressions for α and β in terms of ν , ν_u , B , μ (see eg., Table 3.2.1), we have

$$\begin{aligned}
\lim_{\nu \rightarrow \nu_u} \frac{1}{\beta} &= 0 \\
\lim_{\nu \rightarrow \nu_u} \alpha &= 0 \\
\lim_{\nu \rightarrow \nu_u} \alpha\beta &= \frac{2\mu B(1 + \nu)}{3(1 - 2\nu)}
\end{aligned} \tag{3.2.34}$$

Then the governing equations for the limiting response of $\nu \rightarrow \nu_u$ may be obtained as follows,

The linear constitutive relations at $\nu \rightarrow \nu_u$ are,

$$\mathbf{T}(\mathbf{x}, t) = 2\mu\boldsymbol{\Gamma}(\mathbf{x}, t) + \lambda\epsilon_v(\mathbf{x}, t)\mathbf{I} \tag{3.2.35.a}$$

$$\zeta_v(\mathbf{x}, t) = 0. \tag{3.2.35.b}$$

where $0 < t < +\infty$.

In the limit $\nu \rightarrow \nu_u$, the equilibrium equations are unchanged and are given by

$$\nabla \cdot \mathbf{T}(\mathbf{x}, t) + \mathbf{f}(\mathbf{x}, t) = \mathbf{0} \tag{3.2.36}$$

Similarly, the strain tensor and displacement vector are given by

$$\boldsymbol{\Gamma}(\mathbf{x}, t) = \frac{1}{2}[\nabla \mathbf{u}(\mathbf{x}, t) + \mathbf{u}(\mathbf{x}, t)\nabla] \tag{3.2.37.a}$$

$$\epsilon_v = \nabla \cdot \mathbf{u} \tag{3.2.37.b}$$

As $\nu \rightarrow \nu_u$, Darcy's law of fluid flow takes the form

$$\mathbf{v}(\mathbf{x}, t) = -\kappa[\nabla p(\mathbf{x}, t) - \mathbf{F}(\mathbf{x}, t)] \tag{3.2.38}$$

and the continuity equation of quasi-static fluid flow is given by

$$\nabla \cdot \mathbf{v}(\mathbf{x}, t) = q(\mathbf{x}, t) \tag{3.2.39}$$

The equations (3.2.35) ~ (3.2.38) illustrate clearly that as $\nu \rightarrow \nu_u$, the governing equations for a saturated poroelastic medium are now fully decoupled into two parts namely those of elastostatics and seepage flow. In other words, as $\nu \rightarrow \nu_u$,

the limiting response of saturated poroelastic media is determined by either the deformation of soil skeleton or the flow of pore fluid. There is no interaction and/or influence between the two physical phenomena. Consequently, there is no consolidation process at this situation.

Based on the above discussions, we can conclude that the three limiting responses of poroelastic media saturated with compressible fluids can be analyzed by using the corresponding results obtained from the classical elastostatics. However, the initial and final responses of saturated poroelastic media are not fully uncoupled into the elastic deformation of solid skeleton and the seepage flow of pore fluids. At the initial response, seepage flow of pore fluids is influenced by the deformation of soil skeleton while at the final response, the deformation of soil skeleton is uninfluenced by the seepage flow of pore fluids provided $\nabla \lim_{t \rightarrow +\infty} p(\mathbf{x}, t) \rightarrow 0$. This conclusion can be an extension of the observation made by Davis and Poulos (1963) in the consideration of the initial and final responses of poroelastic media saturated with incompressible fluids.

3.3 Governing Equations in the Transform Domain

3.3.1 General

In the section 3.2, the equations governing three-dimensional behaviour of poroelastic media saturated with compressible fluids have been summarized in general orthogonal curvilinear coordinates, Cartesian coordinates as well as cylindrical coordinates. In this section, we shall investigate such partial differential equations by using the two-dimensional Fourier transforms and Laplace transforms. It is well known that integral transforms provide an effective technique for solving analytically the boundary and initial value problems associated with linear partial differential equations (Sneddon, 1972). In general, the integral transforms replace the independent variables in the physical domain by their imaginary parts in the

transform domain and reduce the complicated system of partial differential equations into a system of ordinary differential equations. The ordinary differential equations can be solved easily with by using many conventional techniques (Sneddon, 1972). Finally, the solution in the physical domain can be obtained by using the inversion of the integral transforms.

When dealing with poroelastic media, many investigators have formulated their governing equations in the transform domain by introducing stress or displacement functions. For example, McNamee and Gibson (1960a,b) and Schiffman and Fungaroli (1965) introduced the well known three displacement function technique and developed their governing equations in the transform domain as three sets of fourth-order ordinary differential equations. Recently, Booker and Small (1987) have formulated their governing equations in the transform domain as five linear systems of second-order ordinary differential equations, in terms of the displacements and pore pressure as well as the volumetric strain. In all those studies it was assumed that the pore fluid was incompressible and the behaviour of the fluid saturated poroelastic medium satisfies the constraints $\nu_{\bullet} = 0.5$ and $B = 1$.

In this thesis, with the aid of the Fourier transforms, a set of representations are suggested to simplify and unify the solution procedure. The governing equations in both Cartesian and cylindrical coordinates are then reduced into two sets of first-order ordinary differential equations with independent variable z of the vertical coordinate. Such governing equations in the transform domain are further examined under the three limiting physical conditions ($t \rightarrow 0^+$; $t \rightarrow +\infty$; $\nu \rightarrow \nu_{\bullet}$), where the Tauberian theorems are employed.

3.3.2 Governing equations in the transform domain

First, we define a symmetric matrix kernel function ΠK and its complex conjugate $\Pi^* K^*$ ($\Pi K \cdot \Pi^* K^* = \mathbf{I}$) as (i) in the physical and transform Cartesian

coordinates:

$$K = e^{+i(\xi x + \eta y)}, \quad \Pi = \begin{pmatrix} +\frac{i\xi}{\rho} & +\frac{i\eta}{\rho} & 0 \\ +\frac{i\eta}{\rho} & -\frac{i\xi}{\rho} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (3.3.1.a)$$

$$K^* = e^{-i(\xi x + \eta y)}, \quad \Pi^* = \begin{pmatrix} -\frac{i\xi}{\rho} & -\frac{i\eta}{\rho} & 0 \\ -\frac{i\eta}{\rho} & +\frac{i\xi}{\rho} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and (ii) in the physical and transform cylindrical coordinates:

$$K = e^{+i\rho r \sin(\theta + \varphi)}, \quad \Pi = \begin{pmatrix} +i \sin(\theta + \varphi) & +i \cos(\theta + \varphi) & 0 \\ +i \cos(\theta + \varphi) & -i \sin(\theta + \varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (3.3.1.b)$$

$$K^* = e^{-i\rho r \sin(\theta + \varphi)}, \quad \Pi^* = \begin{pmatrix} -i \sin(\theta + \varphi) & -i \cos(\theta + \varphi) & 0 \\ -i \cos(\theta + \varphi) & +i \sin(\theta + \varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Next, we introduce five invariant vectors $\mathbf{w}$, $\boldsymbol{\tau}$, $\boldsymbol{\vartheta}$, $\mathbf{g}$, $\mathbf{G}$ and a related invariant vector $\mathbf{Y}_p$,

$$\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}, \quad \boldsymbol{\tau} = \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix}, \quad \boldsymbol{\vartheta} = \begin{pmatrix} \vartheta_1 \\ \vartheta_2 \\ \vartheta_3 \end{pmatrix} \quad (3.3.2)$$

$$\mathbf{g} = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} G_1 \\ G_2 \\ G_3 \end{pmatrix}, \quad \mathbf{Y}_p = \begin{pmatrix} w_1 \\ w_2 \\ \tau_3 \\ p_w \end{pmatrix}$$

in both the transform Cartesian coordinates $(\xi, \eta, z, t; -\infty < \xi, \eta < +\infty, a \leq z \leq b, 0 < t < +\infty)$ and the transform cylindrical coordinates $(\rho, \varphi, z, t; 0 \leq \rho < +\infty, 0 \leq \varphi < 2\pi, a \leq z \leq b, 0 < t < +\infty)$.

By using the matrix kernel functions and the invariant vectors, we find that the following set of solution representations are true for the field variables in a vertically inhomogeneous poroelastic medium,

$$\begin{aligned}
\mathbf{u}(S, z, t) &= \frac{1}{2\pi} \iint_{\tilde{S}} \frac{1}{\rho} \Pi \mathbf{w} K d\tilde{S} \\
\mathbf{T}_z(S, z, t) &= \frac{1}{2\pi} \iint_{\tilde{S}} \Pi \tau K d\tilde{S} \\
\mathbf{v}(S, z, t) &= \frac{1}{2\pi} \iint_{\tilde{S}} \rho \Pi \vartheta K d\tilde{S} \\
\mathbf{f}(S, z, t) &= \frac{1}{2\pi} \iint_{\tilde{S}} \Pi \mathbf{g} K d\tilde{S} \\
\mathbf{F}(S, z, t) &= \frac{1}{2\pi} \iint_{\tilde{S}} \Pi \mathbf{G} K d\tilde{S} \\
\mathbf{T}_p(S, z, t) &= \frac{1}{2\pi} \iint_{\tilde{S}} \Pi_p \mathbf{Y}_p K d\tilde{S}
\end{aligned} \tag{3.3.3.a}$$

and

$$\begin{aligned}
\mathbf{w}(\tilde{S}, z, t) &= \frac{\rho}{2\pi} \iint_S \Pi^* \mathbf{u} K^* dS \\
\tau(\tilde{S}, z, t) &= \frac{1}{2\pi} \iint_S \Pi^* \mathbf{T}_z K^* dS \\
\vartheta(\tilde{S}, z, t) &= \frac{1}{2\pi\rho} \iint_S \Pi^* \mathbf{v} K^* dS \\
\mathbf{g}(\tilde{S}, z, t) &= \frac{1}{2\pi} \iint_S \Pi^* \mathbf{f} K^* dS \\
\mathbf{G}(\tilde{S}, z, t) &= \frac{1}{2\pi} \iint_S \Pi^* \mathbf{F} K^* dS \\
p_w(\tilde{S}, z, t) &= \frac{1}{2\pi} \iint_S p K^* dS,
\end{aligned} \tag{3.3.3.b}$$

where the integrals are taken in the sense of the Cauchy principal values, and (i) in the Cartesian coordinates:

$$\mathbf{u} = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}, \quad \mathbf{T}_z = \begin{pmatrix} \sigma_{xz} \\ \sigma_{zy} \\ \sigma_{zz} \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$$

$$\mathbf{f} = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix}, \quad \mathbf{T}_p = \begin{pmatrix} \sigma_{xz} \\ \sigma_{zy} \\ \sigma_{yy} \\ p \end{pmatrix}$$

$$\tilde{S} = (\xi, \eta; -\infty < \xi, \eta < \infty), \quad d\tilde{S} = d\xi d\eta$$

$$S = (x, y; -\infty < x, y < \infty), \quad dS = dx dy \quad (3.3.3.c)$$

$$\Pi_p = \begin{pmatrix} \frac{-\mu(3\lambda+2\mu)}{\lambda+2\mu} + \mu \frac{\eta^2 - \xi^2}{\rho^2} & -2\mu \frac{\xi\eta}{\rho^2} & \frac{\lambda}{\lambda+2\mu} & -\alpha \\ -2\mu \frac{\xi\eta}{\rho^2} & -\mu \frac{\eta^2 - \xi^2}{\rho^2} & 0 & 0 \\ \frac{-\mu(3\lambda+2\mu)}{\lambda+2\mu} - \mu \frac{\eta^2 - \xi^2}{\rho^2} & 2\mu \frac{\xi\eta}{\rho^2} & \frac{\lambda}{\lambda+2\mu} & -\alpha \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\det \Pi_p = \frac{-2\mu^2\lambda}{\lambda+2\mu}$$

and (ii) in the cylindrical coordinates:

$$\mathbf{u} = \begin{pmatrix} u_r \\ u_\theta \\ u_z \end{pmatrix}, \quad \mathbf{T}_z = \begin{pmatrix} \sigma_{zr} \\ \sigma_{z\theta} \\ \sigma_{zz} \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} v_r \\ v_\theta \\ v_z \end{pmatrix}$$

$$\mathbf{f} = \begin{pmatrix} f_r \\ f_\theta \\ f_z \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} F_r \\ F_\theta \\ F_z \end{pmatrix}, \quad \mathbf{T}_p = \begin{pmatrix} \sigma_{rrr} \\ \sigma_{r\theta} \\ \sigma_{\theta\theta} \\ p \end{pmatrix}$$

$$\tilde{S} = (\rho, \varphi; 0 \leq \rho < \infty, 0 \leq \varphi < 2\pi), \quad d\tilde{S} = \rho d\varphi d\rho,$$

$$S = (r, \theta; 0 \leq r < \infty, 0 \leq \theta < 2\pi), \quad dS = r d\theta dr, \quad (3.3.3.d)$$

$$\Pi_p = \begin{pmatrix} \frac{-\mu(3\lambda+2\mu)}{\lambda+2\mu} + \mu \cos 2(\theta + \varphi) & -\mu \sin 2(\theta + \varphi) & \frac{\lambda}{\lambda+2\mu} & -\alpha \\ -\mu \sin 2(\theta + \varphi) & -\mu \cos 2(\theta + \varphi) & 0 & 0 \\ \frac{-\mu(3\lambda+2\mu)}{\lambda+2\mu} - \mu \cos 2(\theta + \varphi) & \mu \sin 2(\theta + \varphi) & \frac{\lambda}{\lambda+2\mu} & -\alpha \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\det \Pi_p = \frac{-2\mu^2 \lambda}{\lambda + 2\mu}$$

The 4-D governing partial differential equations of poroelastic media saturated with compressible fluids represented in both Cartesian and cylindrical coordinates can be reduced to the following partially differential equations with only two independent variables z and t in terms of the invariant variables,

$$\frac{\partial}{\partial z} \begin{pmatrix} w_2(z, t) \\ \tau_2(z, t) \end{pmatrix} = \rho \begin{pmatrix} 0 & 1 \\ \mu & 0 \end{pmatrix} \begin{pmatrix} w_2(z, t) \\ \tau_2(z, t) \end{pmatrix} - \begin{pmatrix} 0 \\ g_2(z, t) \end{pmatrix}$$

$$\frac{\partial}{\partial z} \begin{pmatrix} w_1(z, t) \\ w_3(z, t) \\ \tau_3(z, t) \\ \tau_1(z, t) \\ p_w(z, t) \end{pmatrix} = \rho \begin{pmatrix} 0 & -1 & 0 & \frac{1}{\mu} & 0 & 0 \\ \frac{\lambda}{\lambda+2\mu} & 0 & \frac{1}{\lambda+2\mu} & 0 & \frac{\alpha}{\lambda+2\mu} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{4\mu(\lambda+\mu)}{\lambda+2\mu} & 0 & \frac{-\lambda}{\lambda+2\mu} & 0 & \frac{2\mu\alpha}{\lambda+2\mu} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{-1}{\kappa} \end{pmatrix} \begin{pmatrix} w_1(z, t) \\ w_3(z, t) \\ \tau_3(z, t) \\ \tau_1(z, t) \\ p_w(z, t) \\ \vartheta_3(z, t) \end{pmatrix}$$

$$- \begin{pmatrix} 0 \\ 0 \\ g_3(z, t) \\ g_1(z, t) \\ -G_3(z, t) \end{pmatrix}$$

(3.3.4.a)

$$\begin{aligned} \frac{\partial \vartheta_3(z, t)}{\partial z} &= \frac{2\mu\alpha}{\lambda+2\mu} \frac{\partial w_1(z, t)}{\rho \partial t} - \frac{\alpha}{\lambda+2\mu} \frac{\partial \tau_3(z, t)}{\rho \partial t} - \frac{\kappa}{c} \frac{\partial p_w(z, t)}{\rho \partial t} \\ &\quad - \kappa \rho p_w(z, t) + \frac{1}{\rho} \tilde{q}(\tilde{S}, z, t) - \kappa G_1(z, t) \end{aligned}$$

$$\vartheta_1(z, t) = \kappa \rho p_w(z, t) + \kappa G_1(z, t)$$

$$\vartheta_2(z, t) = \kappa G_2(z, t)$$

$$\tilde{q}(\tilde{S}, z, t) = \frac{1}{2\pi} \iint_S q(S, z, t) K^* dS$$

where c is referred to as the generalized consolidation coefficient,

$$\begin{aligned} c &= \frac{\kappa \beta (\lambda + 2\mu)}{\lambda_u + 2\mu} \\ &= \frac{2\mu \kappa B^2 (1 - \nu)(1 + \nu_u)^2}{9(\nu_u - \nu)(1 - \nu_u)} \end{aligned} \quad (3.3.4.b)$$

Further, the application of Laplace transform on equations (3.3.4), we obtain the two sets of first order ordinary differential equations in z , i.e.:

$$\frac{d}{dz} \mathbf{U}_v(z) = \rho \mathbf{C}_v(z) \mathbf{U}_v(z) - \mathbf{G}_v(z) \quad (3.3.5.a)$$

$$\frac{d}{dz} \mathbf{U}_u(z) = \rho \mathbf{C}_u(z) \mathbf{U}_u(z) - \mathbf{G}_u(z)$$

where the variables and the body forces are defined by the invariant variables as

$$\begin{aligned} \mathbf{U}_v(z) &= \begin{pmatrix} \hat{w}_2(z) \\ \hat{\tau}_2(z) \end{pmatrix}, & \mathbf{G}_v(z) &= \begin{pmatrix} 0 \\ \hat{g}_2(z) \end{pmatrix} \\ \mathbf{U}_u(z) &= \begin{pmatrix} \hat{w}_1(z) \\ \hat{w}_3(z) \\ \hat{\tau}_3(z) \\ \hat{\tau}_1(z) \\ \hat{p}_w(z) \\ \hat{\vartheta}_3(z) \end{pmatrix}, & \mathbf{G}_u(z) &= \begin{pmatrix} 0 \\ 0 \\ \hat{g}_3(z) \\ \hat{g}_1(z) \\ -\hat{G}_3(z) \\ \hat{q}_g(z) \end{pmatrix} \end{aligned} \quad (3.3.5.b)$$

and the coefficient matrices are

$$\begin{aligned} \mathbf{C}_v(z) &= \begin{pmatrix} 0 & \frac{1}{\mu} \\ \mu & 0 \end{pmatrix} \\ \mathbf{C}_u(z) &= \begin{pmatrix} 0 & -1 & 0 & \frac{1}{\mu} & 0 & 0 \\ \frac{\lambda}{\lambda+2\mu} & 0 & \frac{1}{\lambda+2\mu} & 0 & \frac{\alpha}{\lambda+2\mu} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{4\mu(\lambda+\mu)}{\lambda+2\mu} & 0 & \frac{-\lambda}{\lambda+2\mu} & 0 & \frac{2\mu\alpha}{\lambda+2\mu} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{-1}{\kappa} \\ \frac{2\mu\alpha c}{\lambda+2\mu}(\gamma^2 - 1) & 0 & \frac{-\alpha c}{\lambda+2\mu}(\gamma^2 - 1) & 0 & -\kappa\gamma^2 & 0 \end{pmatrix} \end{aligned} \quad (3.3.5.c)$$

and

$$\hat{q}_g(z) = \frac{-1}{2\pi\rho} \iint_S [\hat{q}(S, z, s) + \frac{1}{s} \zeta_{v0}(S, z)] K^* dS + \kappa \hat{G}_1(z) \quad (3.3.5.d)$$

$$\gamma = \sqrt{\frac{1}{c} \frac{s}{\rho^2} + 1}$$

Equations (3.3.5) are the governing equations in the transform domain for a vertically inhomogeneous poroelastic medium saturated with compressible fluids. They

are two linear systems of first-order ordinary differential equations in z . Such two sets of equations essentially reflect the anti-symmetric and axisymmetric characteristics of the governing partial differential equations (3.2.1) $\sim$ (3.2.6), respectively.

The two square coefficient matrices $\mathbf{C}_v(z)\{2 \times 2\}$ and $\mathbf{C}_u(z)\{6 \times 6\}$ contain only the variables γ and the five material constants $\mu, \lambda, \alpha, \beta$ and κ . In the two coefficient matrices, there are no spatial gradients of the five material constants. These five material constants are functions of z for a vertically inhomogeneous poroelastic medium. The determinants of the two coefficient matrices $\rho\mathbf{C}_v(z)$ and $\rho\mathbf{C}_u(z)$ are equal to $-\rho^2$ and $\gamma^2\rho^6$, respectively. These two determinants are not equal to zero if $\rho \neq 0$ and $Re(s) > 0$; i.e., the two coefficient matrices are nonsingular only if $\rho \neq 0$ due to $\rho \geq 0$ and $Re(s) > 0$ in their original definitions. The limiting case of $\rho = 0$ can be handled with ease after the solutions are found. This nonsingular property of the two coefficient matrices may imply that there would not be singular poles in the solutions obtained in the transform domain if $\rho > 0$ and $Re(s) > 0$. This property can further provide the mathematical basis in the numerical or analytical inversions of the two-dimensional Fourier transforms and the Laplace transforms. Such nonsingular property may physically correspond with the stability of time-dependent response in saturated poroelastic media.

3.3.3 Governing equations in the transform domain for the three limiting responses

There are at least two ways to examine the governing equations in the transform domain under the three limiting physical conditions. The first procedure follows the above procedure to obtain the results by using the three limiting equations (3.2.19) $\sim$ (3.2.39) in the physical domain. The second procedure may be achieved by imposing directly limits on the governing equations (3.3.5) in the transform domain under the three physical conditions. The second procedure can also be used to check the accuracy of the governing equations in the transform domain. Consequently, we shall focus on the second approach. The governing equations in the transform domain for initial and final responses may be found with the help of

the Tauberian theorems eqs.(2.3.1).

By taking the limit as $s \rightarrow +\infty$ of equations (3.3.5), we have the the governing equations in the Fourier transform domain for the initial response of $t \rightarrow 0^+$,

$$\begin{aligned} \frac{d}{dz} \begin{pmatrix} w_2(z, 0^+) \\ \tau_2(z, 0^+) \end{pmatrix} &= \rho \begin{pmatrix} 0 & \frac{1}{\mu} \\ \mu & 0 \end{pmatrix} \begin{pmatrix} w_2(z, 0^+) \\ \tau_2(z, 0^+) \end{pmatrix} - \begin{pmatrix} 0 \\ g_2(z, 0^+) \end{pmatrix} \\ \frac{d}{dz} \begin{pmatrix} w_1(z, 0^+) \\ w_3(z, 0^+) \\ \tau_3(z, 0^+) \\ \tau_1(z, 0^+) \end{pmatrix} &= \rho \begin{pmatrix} 0 & -1 & 0 & \frac{1}{\mu} \\ \frac{\lambda_u}{\lambda_u + 2\mu} & 0 & \frac{1}{\lambda_u + 2\mu} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{4\mu(\lambda_u + \mu)}{\lambda_u + 2\mu} & 0 & \frac{-\lambda_u}{\lambda_u + 2\mu} & 0 \end{pmatrix} \begin{pmatrix} w_1(z, 0^+) \\ w_3(z, 0^+) \\ \tau_3(z, 0^+) \\ \tau_1(z, 0^+) \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ g_3(z, 0^+) \\ g_1(z, 0^+) \end{pmatrix} \\ p_w(z, 0^+) &= \frac{\alpha\beta}{\lambda_u + 2\mu} [2\mu w_1(z, 0^+) - \tau_3(z, 0^+)] \\ &= -\frac{B}{3} (\tilde{\sigma}_{11}(z, 0^+) + \tilde{\sigma}_{22}(z, 0^+) + \tilde{\sigma}_{33}(z, 0^+)) \end{aligned} \quad (3.3.6)$$

By taking the limit $s \rightarrow 0^+$ of equations (3.3.5), we may have the governing equations in the Fourier transform domain for the final response of $t \rightarrow +\infty^+$,

$$\frac{d}{dz} \begin{pmatrix} p_w(z, +\infty) \\ \vartheta_3(z, +\infty) \end{pmatrix} = \rho \begin{pmatrix} 0 & \frac{-1}{\kappa} \\ -\kappa & 0 \end{pmatrix} \begin{pmatrix} p_w(z, +\infty) \\ \vartheta_3(z, +\infty) \end{pmatrix} - \begin{pmatrix} -G_3(z, +\infty) \\ q_g(z, +\infty) \end{pmatrix} \quad (3.3.7.a)$$

and

$$\begin{aligned} \frac{d}{dz} \begin{pmatrix} w_2(z, +\infty) \\ \tau_2(z, +\infty) \end{pmatrix} &= \rho \begin{pmatrix} 0 & \frac{1}{\mu} \\ \mu & 0 \end{pmatrix} \begin{pmatrix} w_2(z, +\infty) \\ \tau_2(z, +\infty) \end{pmatrix} - \begin{pmatrix} 0 \\ g_2(z, +\infty) \end{pmatrix} \\ \frac{d}{dz} \begin{pmatrix} w_1(z, +\infty) \\ w_3(z, +\infty) \\ \tau_3(z, +\infty) \\ \tau_1(z, +\infty) \end{pmatrix} &= \rho \begin{pmatrix} 0 & -1 & 0 & \frac{1}{\mu} \\ \frac{\lambda}{\lambda + 2\mu} & 0 & \frac{1}{\lambda + 2\mu} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{4\mu(\lambda + \mu)}{\lambda + 2\mu} & 0 & \frac{-\lambda}{\lambda + 2\mu} & 0 \end{pmatrix} \begin{pmatrix} w_1(z, +\infty) \\ w_3(z, +\infty) \\ \tau_3(z, +\infty) \\ \tau_1(z, +\infty) \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 \\ \frac{\alpha}{\lambda + 2\mu} \\ 0 \\ \frac{2\mu\alpha}{\lambda + 2\mu} \end{pmatrix} p_w(z, +\infty) - \begin{pmatrix} 0 \\ 0 \\ g_3(z, +\infty) \\ g_1(z, +\infty) \end{pmatrix} \end{aligned} \quad (3.3.7.b)$$

By taking the limit $\nu \rightarrow \nu_u$ ($c \rightarrow +\infty$ and $\gamma \rightarrow 1$) in equations (3.3.5), we may have the governing equations in the transform domain for the limiting response of $\nu \rightarrow \nu_u$,

$$\frac{d}{dz} \begin{pmatrix} \hat{p}_w(z, s) \\ \hat{v}_3(z, s) \end{pmatrix} = \rho \begin{pmatrix} 0 & -1 \\ -\kappa & 0 \end{pmatrix} \begin{pmatrix} \hat{p}_w(z, s) \\ \hat{v}_3(z, s) \end{pmatrix} - \begin{pmatrix} -\hat{G}_3(z, s) \\ \hat{q}_g(z, s) \end{pmatrix} \quad (3.3.8.a)$$

and

$$\begin{aligned} \frac{d}{dz} \begin{pmatrix} \hat{w}_2(z, s) \\ \hat{r}_2(z, s) \end{pmatrix} &= \rho \begin{pmatrix} 0 & 1 \\ \mu & 0 \end{pmatrix} \begin{pmatrix} \hat{w}_2(z, s) \\ \hat{r}_2(z, s) \end{pmatrix} - \begin{pmatrix} 0 \\ \hat{g}_2(z, s) \end{pmatrix} \\ \frac{d}{dz} \begin{pmatrix} \hat{w}_1(z, s) \\ \hat{w}_3(z, s) \\ \hat{r}_3(z, s) \\ \hat{r}_1(z, s) \end{pmatrix} &= \rho \begin{pmatrix} 0 & -1 & 0 & 1 \\ \frac{\lambda}{\lambda+2\mu} & 0 & \frac{1}{\lambda+2\mu} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{4\mu(\lambda+\mu)}{\lambda+2\mu} & 0 & \frac{-\lambda}{\lambda+2\mu} & 0 \end{pmatrix} \begin{pmatrix} \hat{w}_1(z, s) \\ \hat{w}_3(z, s) \\ \hat{r}_3(z, s) \\ \hat{r}_1(z, s) \end{pmatrix} \\ &\quad - \begin{pmatrix} 0 \\ 0 \\ \hat{g}_3(z, s) \\ \hat{g}_1(z, s) \end{pmatrix} \end{aligned} \quad (3.3.8.b)$$

The independent variable s of Laplace transforms does not appear in equations (3.3.8) so that equations (3.3.8) can be directly re-expressed in the Fourier transform and time domains.

Therefore, the governing equations in the Fourier transform domain for the three limiting responses are same as those of elastostatics and seepage flow by using the undrained or drained elastic constants μ and λ_u or μ and λ .

Chapter 4

Solution Representations In Poroelastic Media

4.1 General

In Chapter 3, the governing partial differential equations of poroelastic media saturated with compressible fluids have been presented generally in matrix form as two sets of first order ordinary differential equations in the transform domain for vertically inhomogeneous media. In this chapter, we shall explicitly discuss the solution representations and boundary algebraic (integral) equations in the transform domain applicable to a homogeneous poroelastic medium of finite layer or halfspace extent.

Many investigators including McNamee and Gibson (1960a,b), Schiffman and Fungaroli (1965), Vardoulakis and Harnpattanapanich (1986), and Booker and Small (1987) have obtained their solution representations in terms of eight unknown variables. These unknowns are arbitrary functions of the three independent variables ξ , η and s and have to be determined by using the conditions applicable in the transform domain. The mathematical properties of such arbitrary functions are not known until their exact expressions are derived in terms of the given conditions.

In this thesis research, we present fundamental solution representations for the equations governing a poroelastic medium of finite layer or halfspace extent saturated with compressible fluids. Instead of eight arbitrary unknowns, the solution representations are expressed in terms of eight boundary field variables and in matrix forms. The eight boundary field variables consist of the three displacements, three tractions, excess pore pressure, and the z -component of specific discharge vector at the plane boundaries of the poroelastic medium (Fig.4.1.1). Since the

Fig.4.1.1 Eight field variables on the horizontal xy (or $r\theta$) plane in poroelastic media saturated with compressible fluids

mathematical properties of such field variables are known prior to the specification of a particular problem, we can investigate generally and quantitatively the solution representations to point out their characteristics such as convergence and singularities. The eight square matrices are equivalent to those in the fundamental singular solution in the transform domain due to the action of an instantaneous concentrated point body forces and a fluid source in a poroelastic medium of infinite extent. Moreover, it is found that there are four independent invariant boundary algebraic equations at the boundary surface of a poroelastic halfspace. Together with the four given boundary conditions, these four boundary algebraic equations yield a completely self-consistent system which governs the behaviour of the eight

boundary field variables. Such boundary algebraic equations are also examined for the three limiting responses. Based on the fundamental solution representations and boundary algebraic equations, we present a systematic Fourier and Laplace transform based approach for disc inclusion problems in poroelastic media saturated with compressible fluids.

4.2 Solution Representations in a Poroelastic Layer

4.2.1 General

In this sections, we shall solve the ordinary differential governing equations (3.3.5) in the transform domain and derive their fundamental matrix solution by using Laplace transforms about the independent variable z of the vertical coordinate. The fundamental matrix solution will be discussed briefly to illustrate its basic properties. Fundamental solution representations will also be presented for the governing equations in a homogeneously poroelastic layer.

4.2.2 The fundamental matrix solution of the governing equations

In a homogeneous poroelastic layer, the governing equations (3.3.5) in the transform domain are two non-homogeneous systems of two and six first-order ordinary differential equations with constant coefficient matrices $\rho C_v \{2 \times 2\}$ and $C_s \{6 \times 6\}$, respectively. By means of elimination, we can obtain two sets of general solutions with two and six integration constants, respectively. Herein, the Laplace transforms are employed again to find the fundamental matrix solution of the governing equations in terms of the eight field variables referred to the horizontal plane $z = 0$.

Taking Laplace transforms on the governing equations (3.3.5) about the independent variable z ,

$$\int_0^\infty \left[\frac{d}{dz} \mathbf{U}_v(z) \right] e^{-\chi z} dz = \int_0^\infty [\rho \mathbf{C}_v \mathbf{V}(z) - \mathbf{G}_v(z)] e^{-\chi z} dz \quad (4.2.1.a)$$

$$\int_0^\infty \left[\frac{d}{dz} \mathbf{U}_u(z) \right] e^{-\chi z} dz = \int_0^\infty [\rho \mathbf{C}_u \mathbf{V}(z) - \mathbf{G}_u(z)] e^{-\chi z} dz$$

we have

$$(\chi \mathbf{I} - \rho \mathbf{C}_v) \mathfrak{Z}\{\mathbf{V}(z)\} = \mathbf{V}(0) - \mathfrak{Z}\{\mathbf{G}_v(z)\} \quad (4.2.1.b)$$

$$(\chi \mathbf{I} - \rho \mathbf{C}_u) \mathfrak{Z}\{\mathbf{U}(z)\} = \mathbf{U}(0) - \mathfrak{Z}\{\mathbf{G}_u(z)\}$$

where $\mathbf{I}$ is a unit matrix, $\mathfrak{Z}$ stands for the Laplace transforms about z , and the characteristic matrices are,

$$(\chi \mathbf{I} - \rho \mathbf{C}_v) = \begin{pmatrix} \chi & \frac{-1}{\mu} \rho \\ -\mu \rho & \chi \end{pmatrix}$$

$$(\chi \mathbf{I} - \rho \mathbf{C}_u) = \begin{pmatrix} \chi & \rho & 0 & \frac{-1}{\mu} \rho & 0 & 0 \\ \frac{-\lambda}{\lambda+2\mu} \rho & \chi & \frac{-1}{\lambda+2\mu} \rho & 0 & \frac{-\alpha}{\lambda+2\mu} \rho & 0 \\ 0 & 0 & \chi & -\rho & 0 & 0 \\ \frac{-4\mu(\lambda+\mu)}{\lambda+2\mu} \rho & 0 & \frac{\lambda}{\lambda+2\mu} \rho & \chi & \frac{-2\mu\alpha}{\lambda+2\mu} \rho & 0 \\ 0 & 0 & 0 & 0 & \chi & \frac{1}{\kappa} \rho \\ \frac{-2\mu\alpha}{\lambda+2\mu} \frac{s}{\rho^2} \rho & 0 & \frac{\alpha}{\lambda+2\mu} \frac{s}{\rho^2} \rho & 0 & \kappa \gamma^2 \rho & \chi \end{pmatrix} \quad (4.2.1.c)$$

The inverse of the characteristic matrices $(\chi \mathbf{I} - \rho \mathbf{C}_v)$ and $(\chi \mathbf{I} - \rho \mathbf{C}_u)$ are exactly derived as follows,

$$(\chi \mathbf{I} - \rho \mathbf{C}_v)^{-1} = \frac{1}{2} \left\{ \frac{1}{\chi - \rho} \mathbf{A}_p + \frac{1}{\chi + \rho} \mathbf{A}_q \right\}$$

$$(\chi \mathbf{I} - \rho \mathbf{C}_u)^{-1} = \frac{1}{2} \left\{ \frac{1}{\chi - \rho} \mathbf{Q}_p + \frac{1}{\chi + \rho} \mathbf{Q}_q + \frac{1}{\chi - \rho \gamma} \mathbf{Q}_s + \frac{1}{\chi + \rho \gamma} \mathbf{Q}_t \right.$$

$$\left. + \frac{\rho}{(\chi - \rho)^2} \mathbf{Q}_u + \frac{\rho}{(\chi + \rho)^2} \mathbf{Q}_v \right\} \quad (4.2.2.a)$$

where the eight square matrices $\mathbf{A}_p$, $\mathbf{A}_q$, $\mathbf{Q}_p$, $\mathbf{Q}_q$, $\mathbf{Q}_s$, $\mathbf{Q}_t$, $\mathbf{Q}_u$, and $\mathbf{Q}_v$ are only functions of γ and the five material constants only,

$$\mathbf{A}_p = \begin{pmatrix} 1 & \frac{1}{\mu} \\ \mu & 1 \end{pmatrix}$$

(4.2.2.b)

$$\mathbf{A}_q = \begin{pmatrix} 1 & -\frac{1}{\mu} \\ -\mu & 1 \end{pmatrix}$$

$$\mathbf{Q}_p = \begin{pmatrix} 1 & \frac{-\mu}{\lambda+2\mu} & 0 & \frac{1}{2\mu} \frac{\lambda+3\mu}{\lambda+2\mu} & 0 & 0 \\ \frac{-\mu}{\lambda+2\mu} & 1 & \frac{1}{2\mu} \frac{\lambda+3\mu}{\lambda+2\mu} & 0 & 0 & 0 \\ 0 & 2\mu \frac{\lambda+\mu}{\lambda+2\mu} & 1 & \frac{\mu}{\lambda+2\mu} & 0 & 0 \\ 2\mu \frac{\lambda+\mu}{\lambda+2\mu} & 0 & \frac{\mu}{\lambda+2\mu} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} \alpha_c & -\frac{3-\gamma^2}{2} \alpha_c & \frac{-1}{2\mu} \alpha_c & \frac{1}{2\mu} \frac{3-\gamma^2}{2} \alpha_c & \frac{-1}{2\mu} \alpha_\gamma & \frac{1}{2\mu\kappa} \alpha_\gamma \\ \frac{1+\gamma^2}{2} \alpha_c & -\alpha_c & \frac{-1}{2\mu} \frac{1+\gamma^2}{2} \alpha_c & \frac{1}{2\mu} \alpha_c & \frac{-1}{2\mu} \alpha_\gamma & \frac{1}{2\mu\kappa} \alpha_\gamma \\ 2\mu \alpha_c & -2\mu \frac{3-\gamma^2}{2} \alpha_c & -\alpha_c & \frac{3-\gamma^2}{2} \alpha_c & -\alpha_\gamma & \frac{1}{\kappa} \alpha_\gamma \\ 2\mu \frac{1+\gamma^2}{2} \alpha_c & -2\mu \alpha_c & -\frac{1+\gamma^2}{2} \alpha_c & \alpha_c & -\alpha_\gamma & \frac{1}{\kappa} \alpha_\gamma \\ 2\mu \alpha_d & -2\mu \alpha_d & -\alpha_d & \alpha_d & 0 & 0 \\ -2\mu \kappa \alpha_d & 2\mu \kappa \alpha_d & \kappa \alpha_d & -\kappa \alpha_d & 0 & 0 \end{pmatrix}$$

(4.2.2.c)

$$\mathbf{Q}_q = \begin{pmatrix} 1 & \frac{\mu}{\lambda+2\mu} & 0 & \frac{-1}{2\mu} \frac{\lambda+3\mu}{\lambda+2\mu} & 0 & 0 \\ \frac{\mu}{\lambda+2\mu} & 1 & \frac{-1}{2\mu} \frac{\lambda+3\mu}{\lambda+2\mu} & 0 & 0 & 0 \\ 0 & -2\mu \frac{\lambda+\mu}{\lambda+2\mu} & 1 & \frac{-\mu}{\lambda+2\mu} & 0 & 0 \\ -2\mu \frac{\lambda+\mu}{\lambda+2\mu} & 0 & \frac{-\mu}{\lambda+2\mu} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} \alpha_c & \frac{3-\gamma^2}{2} \alpha_c & \frac{-1}{2\mu} \alpha_c & \frac{-1}{2\mu} \frac{3-\gamma^2}{2} \alpha_c & \frac{-1}{2\mu} \alpha_\gamma & \frac{-1}{2\mu\kappa} \alpha_\gamma \\ -\frac{1+\gamma^2}{2} \alpha_c & -\alpha_c & \frac{1}{2\mu} \frac{1+\gamma^2}{2} \alpha_c & \frac{1}{2\mu} \alpha_c & \frac{1}{2\mu} \alpha_\gamma & \frac{1}{2\mu\kappa} \alpha_\gamma \\ 2\mu \alpha_c & 2\mu \frac{3-\gamma^2}{2} \alpha_c & -\alpha_c & -\frac{3-\gamma^2}{2} \alpha_c & -\alpha_\gamma & \frac{-1}{\kappa} \alpha_\gamma \\ -2\mu \frac{1+\gamma^2}{2} \alpha_c & -2\mu \alpha_c & \frac{1+\gamma^2}{2} \alpha_c & \alpha_c & \alpha_\gamma & \frac{1}{\kappa} \alpha_\gamma \\ 2\mu \alpha_d & 2\mu \alpha_d & -\alpha_d & -\alpha_d & 0 & 0 \\ 2\mu \kappa \alpha_d & 2\mu \kappa \alpha_d & -\kappa \alpha_d & -\kappa \alpha_d & 0 & 0 \end{pmatrix}$$

$$\mathbf{Q}_s = \begin{pmatrix} -\alpha_c & \frac{1}{\gamma}\alpha_c & \frac{1}{2\mu}\alpha_c & \frac{-1}{2\mu\gamma}\alpha_c & \frac{1}{2\mu}\alpha_\gamma & \frac{-1}{2\mu\kappa\gamma}\alpha_\gamma \\ -\gamma\alpha_c & \alpha_c & \frac{\gamma}{2\mu}\alpha_c & \frac{-1}{2\mu}\alpha_c & \frac{\gamma}{2\mu}\alpha_\gamma & \frac{-1}{2\mu\kappa}\alpha_\gamma \\ -2\mu\alpha_c & \frac{2\mu}{\gamma}\alpha_c & \alpha_c & \frac{-1}{\gamma}\alpha_c & \alpha_\gamma & \frac{-1}{\kappa\gamma}\alpha_\gamma \\ -2\mu\gamma\alpha_c & 2\mu\alpha_c & \gamma\alpha_c & -\alpha_c & \gamma\alpha_\gamma & \frac{-1}{\kappa}\alpha_\gamma \\ -2\mu\alpha_d & \frac{2\mu}{\gamma}\alpha_d & \alpha_d & \frac{-1}{\gamma}\alpha_d & 1 & \frac{-1}{\kappa\gamma} \\ 2\mu\kappa\gamma\alpha_d & -2\mu\kappa\alpha_d & -\kappa\gamma\alpha_d & \kappa\alpha_d & -\kappa\gamma & 1 \end{pmatrix} \quad (4.2.2.d)$$

$$\mathbf{Q}_t = \begin{pmatrix} -\alpha_c & \frac{-1}{\gamma}\alpha_c & \frac{1}{2\mu}\alpha_c & \frac{1}{2\mu\gamma}\alpha_c & \frac{1}{2\mu}\alpha_\gamma & \frac{1}{2\mu\kappa\gamma}\alpha_\gamma \\ \gamma\alpha_c & \alpha_c & \frac{-\gamma}{2\mu}\alpha_c & \frac{-1}{2\mu}\alpha_c & \frac{-\gamma}{2\mu}\alpha_\gamma & \frac{-1}{2\mu\kappa}\alpha_\gamma \\ -2\mu\alpha_c & \frac{-2\mu}{\gamma}\alpha_c & \alpha_c & \frac{1}{\gamma}\alpha_c & \alpha_\gamma & \frac{1}{\kappa\gamma}\alpha_\gamma \\ 2\mu\gamma\alpha_c & 2\mu\alpha_c & -\gamma\alpha_c & -\alpha_c & -\gamma\alpha_\gamma & \frac{-1}{\kappa}\alpha_\gamma \\ -2\mu\alpha_d & \frac{-2\mu}{\gamma}\alpha_d & \alpha_d & \frac{1}{\gamma}\alpha_d & 1 & \frac{1}{\kappa\gamma} \\ -2\mu\kappa\gamma\alpha_d & -2\mu\kappa\alpha_d & \kappa\gamma\alpha_d & \kappa\alpha_d & \kappa\gamma & 1 \end{pmatrix}$$

$$\mathbf{Q}_u = \frac{\lambda_u + \mu}{\lambda_u + 2\mu} \begin{pmatrix} 1 & -1 & \frac{-1}{2\mu} & \frac{1}{2\mu} & 0 & 0 \\ 1 & -1 & \frac{-1}{2\mu} & \frac{1}{2\mu} & 0 & 0 \\ 2\mu & -2\mu & -1 & 1 & 0 & 0 \\ 2\mu & -2\mu & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.2.2.e)$$

$$\mathbf{Q}_v = \frac{\lambda_v + \mu}{\lambda_v + 2\mu} \begin{pmatrix} -1 & -1 & \frac{1}{2\mu} & \frac{1}{2\mu} & 0 & 0 \\ 1 & 1 & \frac{-1}{2\mu} & \frac{-1}{2\mu} & 0 & 0 \\ -2\mu & -2\mu & 1 & 1 & 0 & 0 \\ 2\mu & 2\mu & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\begin{aligned} \alpha_c &= 2\left(\frac{\lambda_u + \mu}{\lambda_u + 2\mu} - \frac{\lambda + \mu}{\lambda + 2\mu}\right) \frac{1}{\gamma^2 - 1} \\ \alpha_d &= \frac{\alpha_c}{(\lambda + 2\mu)\kappa} \\ \alpha_\gamma &= \frac{2\mu\alpha}{\lambda + 2\mu} \frac{1}{\gamma^2 - 1} \\ \alpha_c &= \alpha_d\alpha_\gamma \end{aligned} \quad (4.2.2.f)$$

The fundamental matrix solution of the governing equations (3.3.15) with constant coefficients is obtained in terms of $\mathbf{U}_v(0)$, $\mathbf{U}_u(0)$ by using the inversions of Laplace transforms (2.1.20b) about χ in the following,

$$\begin{aligned} \mathbf{U}_v(z) &= \mathfrak{Z}^{-1}\{(\chi \mathbf{I} - \rho \mathbf{C}_v)^{-1}[\mathbf{U}_v(0) - \mathfrak{Z}\{\mathbf{G}_v(z)\}]\} \\ &= \mathbf{A}(z)\mathbf{U}_v(0) - \mathbf{B}(z, 0) \end{aligned}$$

(4.2.3.a)

$$\begin{aligned} \mathbf{U}_u(z) &= \mathfrak{Z}^{-1}\{(\chi \mathbf{I} - \rho \mathbf{C}_u)^{-1}[\mathbf{U}_u(0) - \mathfrak{Z}\{\mathbf{G}_u(z)\}]\} \\ &= \mathbf{Q}(z)\mathbf{U}_u(0) - \mathbf{D}(z, 0) \end{aligned}$$

where $\mathbf{B}(z, 0)$, $\mathbf{D}(z, 0)$ are the convolution integrals,

$$\begin{aligned} \mathbf{B}(z, 0) &= \int_0^z \mathbf{A}(z - \zeta) \mathbf{G}_v(\zeta) d\zeta \\ \mathbf{D}(z, 0) &= \int_0^z \mathbf{Q}(z - \zeta) \mathbf{G}_u(\zeta) d\zeta \end{aligned}$$

(4.2.3.b)

Using the inverse Laplace transforms below,

$$\begin{aligned} \mathfrak{Z}^{-1}\left\{\frac{1}{\chi \mp \rho}\right\} &= e^{\pm \rho z} \\ \mathfrak{Z}^{-1}\left\{\frac{1}{\chi \mp \rho \gamma}\right\} &= e^{\pm \rho \gamma z} \\ \mathfrak{Z}^{-1}\left\{\frac{\rho}{(\chi \mp \rho)^2}\right\} &= \rho z e^{\pm \rho z} \end{aligned}$$

(4.2.4.a)

we have the fundamental matrices of solution,

$$\begin{aligned} \mathbf{A}(z) &= \mathfrak{Z}^{-1}\{(\chi \mathbf{I} - \rho \mathbf{C}_v)^{-1}\} \\ &= \frac{1}{2}\{e^{+\rho z} \mathbf{A}_p + e^{-\rho z} \mathbf{A}_q\} \end{aligned}$$

$$\begin{aligned} \mathbf{Q}(z) &= \mathfrak{Z}^{-1}\{(\chi \mathbf{I} - \rho \mathbf{C}_u)^{-1}\} \\ &= \frac{1}{2}\{e^{+\rho z} \mathbf{Q}_p + e^{-\rho z} \mathbf{Q}_q + e^{+\rho \gamma z} \mathbf{Q}_s + e^{-\rho \gamma z} \mathbf{Q}_t + \rho z e^{+\rho z} \mathbf{Q}_u + \rho z e^{-\rho z} \mathbf{Q}_v\} \end{aligned}$$

(4.2.4.b)

4.2.3 Basic properties of the fundamental matrices of solution

1. The fundamental matrices of solution are unit matrices if $z = 0$, i.e. :

$$\begin{aligned} \mathbf{A}(0) &= \mathbf{I}\{2 \times 2\} \\ \mathbf{Q}(0) &= \mathbf{I}\{6 \times 6\} \end{aligned}$$

(4.2.5.a)

2. The determinants of the fundamental matrices of solution are unity:

$$\begin{aligned}\det \mathbf{A}(z) &= 1 \\ \det \mathbf{Q}(z) &= 1\end{aligned}\tag{4.2.5.b}$$

3. The inverses of the fundamental matrices of solution are as follows:

$$\begin{aligned}\mathbf{A}(z)\mathbf{A}(-z) &= \mathbf{I}\{2 \times 2\}, & \mathbf{A}^{-1}(z) &= \mathbf{A}(-z) \\ \mathbf{Q}(z)\mathbf{Q}(-z) &= \mathbf{I}\{6 \times 6\}, & \mathbf{Q}^{-1}(z) &= \mathbf{Q}(-z)\end{aligned}\tag{4.2.5.c}$$

4. The transfer characteristics of the fundamental matrices of solution are defined by

$$\begin{aligned}\mathbf{A}(z_1 + z_2) &= \mathbf{A}(z_1)\mathbf{A}(z_2) \\ \mathbf{Q}(z_1 + z_2) &= \mathbf{Q}(z_1)\mathbf{Q}(z_2) \\ \mathbf{B}(z_2 + z_1 + h, h) &= \mathbf{A}(z_1)\mathbf{B}(z_1 + h, h) + \mathbf{B}(z_2 + z_1 + h, z_1 + h) \\ \mathbf{D}(z_2 + z_1 + h, h) &= \mathbf{Q}(z_1)\mathbf{D}(z_1 + h, h) + \mathbf{D}(z_2 + z_1 + h, z_1 + h)\end{aligned}\tag{4.2.5.d}$$

5. No singular poles occur in the fundamental matrices of solution, *i.e.*

$$\begin{aligned}\mathbf{A}(z) &< +\infty \\ \mathbf{Q}(z) &< +\infty\end{aligned}\tag{4.2.5.e}$$

for any values of ρ , s , and z , where $\rho > 0$, $Re(s) > 0$, $-\infty < z < +\infty$.

4.2.4 Solution representations in a poroelastic layer

Using the above results, we have the solution representations in a homogeneously poroelastic layer $a \leq z \leq b$, *i.e.*:

$$\begin{aligned}\mathbf{U}_v(z) &= \mathbf{A}(z - a)\mathbf{U}_v(a) - \mathbf{B}(z, a) \\ \mathbf{U}_u(z) &= \mathbf{Q}(z - a)\mathbf{U}_u(a) - \mathbf{D}(z, a)\end{aligned}\tag{4.2.6.a}$$

or

$$\begin{aligned}\mathbf{U}_v(z) &= \mathbf{A}(z - b)\mathbf{U}_v(b) - \mathbf{B}(z, b) \\ \mathbf{U}_u(z) &= \mathbf{Q}(z - b)\mathbf{U}_u(b) - \mathbf{D}(z, b)\end{aligned}\tag{4.2.6.b}$$

where $a \leq z \leq b$

Taking $z = b$ and $z = a$ in above equations (4.2.6.a) and (4.2.6.b), respectively, we obtain the boundary algebraic equations for a homogeneously poroelastic layer as follows,

$$\begin{aligned} U_v(b) &= A(b-a)U_v(a) - B(b,a) \\ U_u(b) &= Q(b-a)U_u(a) - D(b,a) \end{aligned} \quad (4.2.7.a)$$

or

$$\begin{aligned} U_v(a) &= A(a-b)U_v(b) - B(a,b) \\ U_u(a) &= Q(a-b)U_u(b) - D(a,b) \end{aligned} \quad (4.2.7.b)$$

By introducing the following fundamental solution matrices,

$$\begin{aligned} \Gamma_v(z) &= A_{pq}(z)e^{-\rho|z|}, \\ \Gamma_u(z) &= Q_{pq}(z)e^{-\rho|z|} + Q_{st}(z)e^{-\rho\gamma|z|} + \rho z Q_{uv}(z)e^{-\rho|z|} \end{aligned} \quad (4.2.8.a)$$

where the four coefficient matrices are defined as

$$\begin{aligned} A_{pq}(z) &= \begin{cases} -A_p, & z \leq 0^- \\ +A_q, & z \geq 0^+ \end{cases} \\ Q_{pq}(z) &= \begin{cases} -Q_p, & z \leq 0^- \\ +Q_q, & z \geq 0^+ \end{cases} \\ Q_{st}(z) &= \begin{cases} -Q_s, & z \leq 0^- \\ +Q_t, & z \geq 0^+ \end{cases} \\ Q_{uv}(z) &= \begin{cases} -Q_u, & z \leq 0^- \\ +Q_v, & z \geq 0^+ \end{cases} \end{aligned} \quad (4.2.8.b)$$

we may re-express the algebraic solution representations (4.2.6) as,

$$\begin{aligned} 2U_v(z) &= \Gamma_v(z-a)U_v(a) - \Gamma_v(z-b)U_v(b) - \int_a^b \Gamma_v(z-\zeta)G_v(\zeta)d\zeta \\ 2U_u(z) &= \Gamma_u(z-a)U_u(a) - \Gamma_u(z-b)U_u(b) - \int_a^b \Gamma_u(z-\zeta)G_u(\zeta)d\zeta \end{aligned} \quad (4.2.8.c)$$

where $a \leq z \leq b$,

The boundary algebraic equations (4.2.7) can be expressed as

$$\begin{aligned}
2U_v(a) &= \Gamma_v(0^+)U_v(a) - \Gamma_v(a-b)U_v(a) - \int_a^b \Gamma_v(a-\zeta)G_v(\zeta)d\zeta \\
2U_u(a) &= \Gamma_u(0^+)U_v(a) - \Gamma_u(a-b)U_v(b) - \int_a^b \Gamma_u(a-\zeta)G_u(\zeta)d\zeta
\end{aligned} \tag{4.2.8.d}$$

or

$$\begin{aligned}
2U_v(b) &= \Gamma_v(b-a)U_v(a) - \Gamma_v(0^-)U_v(b) - \int_a^b \Gamma_v(b-\zeta)G_v(\zeta)d\zeta \\
2U_u(b) &= \Gamma_u(b-a)U_v(a) - \Gamma_u(0^-)U_v(b) - \int_a^b \Gamma_u(b-\zeta)G_u(\zeta)d\zeta
\end{aligned} \tag{4.2.8.e}$$

It can be shown that there are only eight independent equations in the boundary algebraic equations (4.2.7) and (4.2.8.d) and (4.2.8.e). These boundary algebraic equations are the 3-D boundary integral equations in the physical domain for a homogeneous poroelastic layer saturated with a compressible fluid.

4.3 Solution Representation in a Poroelastic Halfspace

4.3.1 General

In the last section, we have obtained the fundamental solution representations (4.2.6) or (4.2.8) applicable to a homogeneous poroelastic layer saturated with a compressible fluid. In this section, we shall present the fundamental solution representation applicable to a homogeneous poroelastic halfspace. The solution representation in a poroelastic halfspace is a limiting case of those in a poroelastic layer where one of the two plane boundaries $z = a$ or $z = b$ is allowed to extend to infinity. The four boundary conditions which should be posed at the plane boundary for specific problems become the natural regularity conditions, i.e., the field displacements and excess pore pressure should disappear and the stresses should be bounded as $z \rightarrow \pm\infty$. By using the natural boundary conditions, the fundamental solution representations applicable to a poroelastic layer can be simplified further

for application to a saturated poroelastic medium of halfspace extent, $a^+ \leq z < +\infty$ or $-\infty < z \leq b^-$.

4.3.2 Solution representation in the lower halfspace $a \leq z < +\infty$

As $z \rightarrow +\infty$ in equations (4.2.6.a) or as $b \rightarrow +\infty$ in equations (4.2.8.c), we find the solution representation in the lower poroelastic halfspace $a \leq z < +\infty$ as follows,

$$\begin{aligned} 2U_v(z) &= \Gamma_v(z-a)U_v(a) - \int_a^{+\infty} \Gamma_v(z-\zeta)G_v(\zeta)d\zeta \\ 2U_u(z) &= \Gamma_u(z-a)U_u(a) - \int_a^{+\infty} \Gamma_u(z-\zeta)G_u(\zeta)d\zeta \end{aligned} \quad (4.3.1.a)$$

or more explicitly,

$$\begin{aligned} 2U_v(z) &= e^{-\rho(z-a)}A_q U_v(a) - \int_a^{+\infty} A_{pq}(z-\zeta)e^{-\rho|z-\zeta|}G_v(\zeta)d\zeta \\ 2U_u(z) &= \left[e^{-\rho(z-a)}Q_q + e^{-\rho\gamma(z-a)}Q_t + \rho(z-a)e^{-\rho(z-a)}Q_v \right] U_u(a) \\ &\quad - \int_a^{+\infty} \left[Q_{pq}(z-\zeta)e^{-\rho|z-\zeta|} + Q_{st}(z-\zeta)e^{-\rho\gamma|z-\zeta|} \right. \\ &\quad \left. + \rho(z-\zeta)Q_{uv}(z-\zeta)e^{-\rho|z-\zeta|} \right] G_u(\zeta)d\zeta \end{aligned} \quad (4.3.1.b)$$

The field variables $U_v(a)$, $U_u(a)$ at the top surface $z = a$ of the saturated poroelastic halfspace should satisfy the boundary algebraic equations which can be found either by substituting $z = a^+$ into equations (4.3.1) or by taking the limit $z \rightarrow +\infty$ of the equations (4.2.6.a), i.e :

$$\begin{aligned} A_p U_v(a) &= A_p \int_a^{+\infty} e^{-\rho(\zeta-a)} G_v(\zeta) d\zeta \\ Q_p U_u(a) &= Q_p \int_a^{+\infty} e^{-\rho(\zeta-a)} G_u(\zeta) d\zeta - Q_u \int_a^{+\infty} \rho(\zeta-a) e^{-\rho(\zeta-a)} G_u(\zeta) d\zeta \end{aligned} \quad (4.3.2.a)$$

$$\begin{aligned}
Q_s U_s(a) &= Q_s \int_a^{+\infty} e^{-\rho\gamma(\zeta-a)} G_s(\zeta) d\zeta \\
Q_u U_u(a) &= Q_u \int_a^{+\infty} e^{-\rho(\zeta-a)} G_u(\zeta) d\zeta
\end{aligned}
\tag{4.3.2.b}$$

These boundary algebraic equations are the equivalent for the four natural conditions as $z \rightarrow +\infty$. They are exactly the boundary integral equations in the physical domain at the outer surface of the saturated poroelastic halfspace. Only four of these eighteen equations are independent. In the next section, we shall study in detail the four independent boundary algebraic equations at $z = a^+$.

4.3.3 Solution representation in the upper halfspace $-\infty < z \leq b$

As $z \rightarrow -\infty$ in equations (4.2.6.b) or as $a \rightarrow -\infty$ in equations (4.2.8.d), we similarly find the solution representation in the upper poroelastic halfspace $-\infty < z \leq b$ as follows,

$$\begin{aligned}
2U_v(z) &= -\Gamma_v(z-b)U_v(b) - \int_{-\infty}^b \Gamma_v(z-\zeta)G_v(\zeta)d\zeta \\
2U_u(z) &= -\Gamma_u(z-b)U_u(b) - \int_{-\infty}^b \Gamma_u(z-\zeta)G_u(\zeta)d\zeta
\end{aligned}
\tag{4.3.3.a}$$

or more explicitly,

$$\begin{aligned}
2U_v(z) &= e^{\rho(z-b)} A_p U_v(b) - \int_{-\infty}^b A_{pq}(z-\zeta) e^{-\rho|z-\zeta|} G_v(\zeta) d\zeta \\
2U_u(z) &= \left[e^{\rho(z-b)} Q_p + e^{\rho\gamma(z-b)} Q_s + \rho(z-b) e^{\rho(z-b)} Q_u \right] U_u(b) \\
&\quad - \int_{-\infty}^b \left[Q_{pq}(z-\zeta) e^{-\rho|z-\zeta|} + Q_{st}(z-\zeta) e^{-\rho\gamma|z-\zeta|} \right. \\
&\quad \left. + \rho(z-\zeta) Q_{uv}(z-\zeta) e^{-\rho|z-\zeta|} \right] G_u(\zeta) d\zeta
\end{aligned}
\tag{4.3.3.b}$$

The field variables $U_v(b)$, $U_u(b)$ at the bottom surface $z = b$ of the saturated porous halfspace ($-\infty < z \leq b$) should satisfy the boundary algebraic equations which can

be found either by substituting $z = b^-$ into equations (4.3.3) or by taking limit $z \rightarrow -\infty$ of the equations (4.2.6.b), i.e :

$$\begin{aligned}
 A_q U_v(b) &= -A_q \int_{-\infty}^b e^{\rho(\zeta-b)} G_v(\zeta) d\zeta \\
 Q_q U_u(b) &= -Q_q \int_{-\infty}^b e^{\rho(\zeta-b)} G_u(\zeta) d\zeta + Q_v \int_{-\infty}^b \rho(\zeta-b) e^{\rho(\zeta-b)} G_u(\zeta) d\zeta \\
 Q_t U_u(b) &= -Q_t \int_{-\infty}^b e^{\rho\gamma(\zeta-b)} G_u(\zeta) d\zeta \\
 Q_v U_u(b) &= -Q_v \int_{-\infty}^b e^{\rho(\zeta-b)} G_u(\zeta) d\zeta
 \end{aligned} \tag{4.3.4}$$

These boundary algebraic equations are the equivalent for the four natural conditions as $z \rightarrow -\infty$. They are exactly the boundary integral equations in the physical domain at the outer surface of the saturated poroelastic halfspace. Only four of these eighteen equations are independent. In the next section, we shall study in detail the four independent boundary algebraic equations at $z = b^-$.

4.3.4 Solution representation in an infinite space

The solution representation in a poroelastic infinite space is a particular limiting case of those applicable to either the upper or the lower poroelastic halfspace regions. The four boundary conditions at the out-surface $z = a$ or $z = b$ of the halfspaces are reduced again to the natural regularity conditions. Consequently, the solution representation in a poroelastic infinite space can be easily obtained by taking $a \rightarrow -\infty$ and $b = +\infty$ in equations (4.3.1) and (4.3.3), respectively; i.e :

$$\begin{aligned}
 U_v(z) &= - \int_{-\infty}^{+\infty} \Gamma_v(z - \zeta) G_v(\zeta) d\zeta \\
 U_u(z) &= - \int_{-\infty}^{+\infty} \Gamma_u(z - \zeta) G_u(\zeta) d\zeta
 \end{aligned} \tag{4.3.5.a}$$

or more explicitly,

$$\begin{aligned}
 U_v(z) &= - \int_{-\infty}^{+\infty} A_{pq}(z - \zeta) e^{-\rho|z-\zeta|} G_v(\zeta) d\zeta \\
 U_u(z) &= - \int_{-\infty}^{+\infty} \left\{ Q_{pq}(z - \zeta) e^{-\rho|z-\zeta|} + Q_{st}(z - \zeta) e^{-\rho\gamma|z-\zeta|} \right. \\
 &\quad \left. + \rho(z - \zeta) Q_{uv}(z - \zeta) e^{-\rho|z-\zeta|} \right\} G_u(\zeta) d\zeta
 \end{aligned} \tag{4.3.5.b}$$

where $-\infty < z < +\infty$.

The representation (4.3.5) is exactly the fundamental singular solution in the transform domain obtained for the three-dimensional response of a saturated poroelastic infinite space by the action of instantaneous concentrated point body forces $f\delta(\mathbf{x})\delta(t)$ and fluid source $q\delta(\mathbf{x})\delta(t)$. Using transform inversion, it can be shown that this fundamental singular solution is equal to that given by Cheng and Predeleanu (1987), in the physical domain. The representation (4.3.5) also illustrates that the eight square matrices correspond to those in the fundamental singular solution in the transform domain in a poroelastic infinite space. Furthermore, there is no singular pole in the eight square matrices and in the fundamental matrix solution $\Gamma_v(z)$ and $\Gamma_u(z)$ if $\rho \geq 0$, $Re(s) > 0$ and $-\infty < z < +\infty$. The fundamental matrix solution $\Gamma_v(z)$ and $\Gamma_u(z)$ is convergent if $\rho \rightarrow +\infty$ or $s \rightarrow \infty$ ($Re(s) > 0$) or $z \rightarrow \pm\infty$.

4.4 Boundary Algebraic Equations

4.4.1 General

In the previous section, we have obtained the linear boundary algebraic equations (4.3.3) and (4.3.5) in the transform domain on the outer surfaces of the lower

and upper poroelastic halfspaces. Each set consists of eighteen linear simultaneous algebraic equations. By considering the four natural boundary conditions as $z \rightarrow +\infty$ or $z \rightarrow -\infty$, each set must only have four linear independent simultaneous algebraic equations. In this section we shall discuss the four independent boundary algebraic equations.

4.4.2 Independent boundary equations at $z = a^+$

The four independent boundary algebraic equations at the outer surface $z = a^+$ of the lower poroelastic halfspace $a \leq z < +\infty$ are derived with body forces and fluid source as follows,

$$\begin{pmatrix} 1 & \frac{1}{\mu} \end{pmatrix} \mathbf{U}_v(a) = A_{11}$$

$$\begin{pmatrix} 1 & -1 & \frac{-1}{2\mu} & \frac{1}{2\mu} & 0 & 0 \\ -\alpha_d & \frac{1}{\gamma}\alpha_d & \frac{1}{2\mu}\alpha_d & \frac{-1}{2\mu\gamma}\alpha_d & \frac{1}{2\mu} & \frac{-1}{2\mu\kappa\gamma} \\ \frac{\lambda_u+\mu}{\lambda_u+2\mu} & \frac{\lambda_u+\mu}{\lambda_u+2\mu} & \frac{\lambda_u+3\mu}{2\mu(\lambda_u+2\mu)} & \frac{\lambda_u+3\mu}{2\mu(\lambda_u+2\mu)} & \frac{-1}{\mu}\alpha_\gamma & \frac{1}{\mu\kappa}\alpha_\gamma \end{pmatrix} \mathbf{U}_u(a) = \begin{pmatrix} A_{12} \\ A_{13} \\ A_{14} \end{pmatrix} \quad (4.4.1.a)$$

where the nonhomogeneous terms $A_{11}, A_{12}, A_{13}, A_{14}$ are

$$A_{11} = \begin{pmatrix} 1 & \frac{1}{\mu} \end{pmatrix} \int_a^{+\infty} e^{-\rho(\zeta-a)} \mathbf{G}_v(\zeta) d\zeta$$

$$A_{12} = \begin{pmatrix} 1 & -1 & \frac{-1}{2\mu} & \frac{1}{2\mu} & 0 & 0 \end{pmatrix} \int_a^{+\infty} e^{-\rho(\zeta-a)} \mathbf{G}_u(\zeta) d\zeta$$

$$A_{13} = \begin{pmatrix} -\alpha_d & \frac{1}{\gamma}\alpha_d & \frac{1}{2\mu}\alpha_d & \frac{-1}{2\mu\gamma}\alpha_d & \frac{1}{2\mu} & \frac{-1}{2\mu\kappa\gamma} \end{pmatrix} \int_a^{+\infty} e^{-\rho\gamma(\zeta-a)} \mathbf{G}_u(\zeta) d\zeta$$

$$A_{14} = \begin{pmatrix} \frac{\lambda_u+\mu}{\lambda_u+2\mu} & \frac{\lambda_u+\mu}{\lambda_u+2\mu} & \frac{\lambda_u+3\mu}{2\mu(\lambda_u+2\mu)} & \frac{\lambda_u+3\mu}{2\mu(\lambda_u+2\mu)} & \frac{-1}{\mu}\alpha_\gamma & \frac{1}{\mu\kappa}\alpha_\gamma \end{pmatrix} \int_a^{+\infty} e^{-\rho(\zeta-a)} \mathbf{G}_u(\zeta) d\zeta$$

$$- 2 \frac{\lambda_u + \mu}{\lambda_u + 2\mu} \begin{pmatrix} 1 & -1 & \frac{-1}{2\mu} & \frac{1}{2\mu} & 0 & 0 \end{pmatrix} \int_a^{+\infty} \rho(\zeta - a) e^{-\rho(\zeta-a)} \mathbf{G}_u(\zeta) d\zeta \quad (4.4.1.a)$$

If the body forces $\mathbf{f}(\mathbf{x}, t)$, $\mathbf{F}(\mathbf{x}, t)$ and fluid source $q(\mathbf{x}, t)$ are zero, the four independent boundary algebraic or integral equations at the outer surface $z = a^+$ of the lower halfspace $a \leq z < +\infty$ could be simply written as,

$$\begin{pmatrix} 1 & \frac{1}{\mu} \end{pmatrix} \mathbf{U}_v(a) = 0$$

$$\begin{pmatrix} 1 & -1 & \frac{-1}{2\mu} & \frac{1}{2\mu} & 0 & 0 \\ -\alpha_d & \frac{1}{\gamma}\alpha_d & \frac{1}{2\mu}\alpha_d & \frac{-1}{2\mu\gamma}\alpha_d & \frac{1}{2\mu} & \frac{-1}{2\mu\kappa\gamma} \\ \frac{\lambda_u+\mu}{\lambda_u+2\mu} & \frac{\lambda_u+\mu}{\lambda_u+2\mu} & \frac{\lambda_u+3\mu}{2\mu(\lambda_u+2\mu)} & \frac{\lambda_u+3\mu}{2\mu(\lambda_u+2\mu)} & \frac{-1}{\mu}\alpha_\gamma & \frac{1}{\mu\kappa}\alpha_\gamma \end{pmatrix} \mathbf{U}_u(a) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (4.4.2)$$

It has been pointed out at the very beginning that four boundary conditions must be posed at the outer-surface $z = a$ of the poroelastic halfspace to formulate a specific problem. Among them, three conditions are associated with the solid skeleton and one is associated with the pore fluid phase. At the out surface, there are three displacements and three tractions of the solid skeleton. One of such six boundary variables should be used as the boundary conditions in each directions of the three orthogonal coordinates, where either the two horizontal displacements or the two shear tractions should be used simultaneously. On the other hand, there are one excess pore pressure and the z -component of specific discharge vector of the pore fluid phase at the out surface $z = a$. Only one of these two field variables should be used as the boundary condition of the pore fluid phase. Consequently, we may have the following eight groups of boundary conditions which could be posed at the boundary $z = a$,

$$\begin{aligned}
& \begin{pmatrix} \hat{\tau}_3(a) \\ \hat{\tau}_1(a) \\ \hat{\vartheta}_3(a) \\ \hat{\tau}_2(a) \end{pmatrix}, \quad \begin{pmatrix} \hat{\tau}_3(a) \\ \hat{\tau}_1(a) \\ \hat{p}_w(a) \\ \hat{\tau}_2(a) \end{pmatrix}, \quad \begin{pmatrix} \hat{w}_3(a) \\ \hat{\tau}_1(a) \\ \hat{\vartheta}_3(a) \\ \hat{\tau}_2(a) \end{pmatrix}, \quad \begin{pmatrix} \hat{w}_3(a) \\ \hat{\tau}_1(a) \\ \hat{p}_w(a) \\ \hat{\tau}_2(a) \end{pmatrix} \\
& \begin{pmatrix} \hat{w}_1(a) \\ \hat{w}_3(a) \\ \hat{p}_w(a) \\ \hat{w}_2(a) \end{pmatrix}, \quad \begin{pmatrix} \hat{w}_1(a) \\ \hat{w}_3(a) \\ \hat{\vartheta}_3(a) \\ \hat{w}_2(a) \end{pmatrix}, \quad \begin{pmatrix} \hat{w}_1(a) \\ \hat{\tau}_3(a) \\ \hat{p}_w(a) \\ \hat{w}_2(a) \end{pmatrix}, \quad \begin{pmatrix} \hat{w}_1(a) \\ \hat{\tau}_3(a) \\ \hat{\vartheta}_3(a) \\ \hat{w}_2(a) \end{pmatrix}
\end{aligned} \tag{4.4.3}$$

The four independent algebraic equations (4.4.1) can then be re-organized as the following four sets according to the eight groups of boundary conditions,

$$\begin{aligned}
& \begin{pmatrix} \frac{-1}{2\mu} & \frac{1}{2\mu} & 0 & 0 \\ \frac{1}{2\mu}\alpha_d & \frac{-1}{2\mu\gamma}\alpha_d & \frac{-1}{2\mu\kappa\gamma} & 0 \\ \frac{\lambda_u+3\mu}{2\mu(\lambda_u+2\mu)} & \frac{\lambda_u+3\mu}{2\mu(\lambda_u+2\mu)} & \frac{1}{\mu\kappa}\alpha_\gamma & 0 \\ 0 & 0 & 0 & \frac{1}{\mu} \end{pmatrix} \begin{pmatrix} \hat{\tau}_3(a) \\ \hat{\tau}_1(a) \\ \hat{\vartheta}_3(a) \\ \hat{\tau}_2(a) \end{pmatrix} \\
& + \begin{pmatrix} 1 & -1 & 0 & 0 \\ -\alpha_d & \frac{1}{\gamma}\alpha_d & \frac{1}{2\mu} & 0 \\ \frac{\lambda_u+\mu}{\lambda_u+2\mu} & \frac{\lambda_u+\mu}{\lambda_u+2\mu} & \frac{-1}{\mu}\alpha_\gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{w}_1(a) \\ \hat{w}_3(a) \\ \hat{p}_w(a) \\ \hat{w}_2(a) \end{pmatrix} = \begin{pmatrix} A_{12} \\ A_{13} \\ A_{14} \\ A_{11} \end{pmatrix}
\end{aligned} \tag{4.4.4.a}$$

and

$$\begin{aligned}
& \begin{pmatrix} \frac{-1}{2\mu} & \frac{1}{2\mu} & 0 & 0 \\ \frac{1}{2\mu}\alpha_d & \frac{-1}{2\mu\gamma}\alpha_d & \frac{1}{2\mu} & 0 \\ \frac{\lambda_u+3\mu}{2\mu(\lambda_u+2\mu)} & \frac{\lambda_u+3\mu}{2\mu(\lambda_u+2\mu)} & \frac{-1}{\mu}\alpha_\gamma & 0 \\ 0 & 0 & 0 & \frac{1}{\mu} \end{pmatrix} \begin{pmatrix} \hat{\tau}_3(a) \\ \hat{\tau}_1(a) \\ \hat{p}_w(a) \\ \hat{\tau}_2(a) \end{pmatrix} \\
& + \begin{pmatrix} 1 & -1 & 0 & 0 \\ -\alpha_d & \frac{1}{\gamma}\alpha_d & \frac{-1}{2\mu\kappa\gamma} & 0 \\ \frac{\lambda_u+\mu}{\lambda_u+2\mu} & \frac{\lambda_u+\mu}{\lambda_u+2\mu} & \frac{1}{\mu\kappa}\alpha_\gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{w}_1(a) \\ \hat{w}_3(a) \\ \hat{\vartheta}_3(a) \\ \hat{w}_2(a) \end{pmatrix} = \begin{pmatrix} A_{12} \\ A_{13} \\ A_{14} \\ A_{11} \end{pmatrix}
\end{aligned} \tag{4.4.4.b}$$

and

$$\begin{aligned}
 & \begin{pmatrix} -1 & \frac{1}{2\mu} & 0 & 0 \\ \frac{1}{\gamma}\alpha_d & \frac{-1}{2\mu\gamma}\alpha_d & \frac{-1}{2\mu\kappa\gamma} & 0 \\ \frac{\lambda_u+\mu}{\lambda_u+2\mu} & \frac{\lambda_u+3\mu}{2\mu(\lambda_u+2\mu)} & \frac{1}{\mu\kappa}\alpha_\gamma & 0 \\ 0 & 0 & 0 & \frac{1}{\mu} \end{pmatrix} \begin{pmatrix} \hat{w}_3(a) \\ \hat{\tau}_1(a) \\ \hat{v}_3(a) \\ \hat{\tau}_2(a) \end{pmatrix} \\
 & + \begin{pmatrix} 1 & \frac{-1}{2\mu} & 0 & 0 \\ -\alpha_d & \frac{1}{2\mu}\alpha_d & \frac{1}{2\mu} & 0 \\ \frac{\lambda_u+\mu}{\lambda_u+2\mu} & \frac{\lambda_u+3\mu}{2\mu(\lambda_u+2\mu)} & \frac{-1}{\mu}\alpha_\gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{w}_1(a) \\ \hat{\tau}_3(a) \\ \hat{p}_w(a) \\ \hat{w}_2(a) \end{pmatrix} = \begin{pmatrix} A_{12} \\ A_{13} \\ A_{14} \\ A_{11} \end{pmatrix}
 \end{aligned} \tag{4.4.4.c}$$

and

$$\begin{aligned}
 & \begin{pmatrix} -1 & \frac{1}{2\mu} & 0 & 0 \\ \frac{1}{\gamma}\alpha_d & \frac{-1}{2\mu\gamma}\alpha_d & \frac{1}{2\mu} & 0 \\ \frac{\lambda_u+\mu}{\lambda_u+2\mu} & \frac{\lambda_u+3\mu}{2\mu(\lambda_u+2\mu)} & \frac{-1}{\mu}\alpha_\gamma & 0 \\ 0 & 0 & 0 & \frac{1}{\mu} \end{pmatrix} \begin{pmatrix} \hat{w}_3(a) \\ \hat{\tau}_1(a) \\ \hat{p}_w(a) \\ \hat{\tau}_2(a) \end{pmatrix} \\
 & + \begin{pmatrix} 1 & \frac{-1}{2\mu} & 0 & 0 \\ -\alpha_d & \frac{1}{2\mu}\alpha_d & \frac{-1}{2\mu\kappa\gamma} & 0 \\ \frac{\lambda_u+\mu}{\lambda_u+2\mu} & \frac{\lambda_u+3\mu}{2\mu(\lambda_u+2\mu)} & \frac{1}{\mu\kappa}\alpha_\gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{w}_1(a) \\ \hat{\tau}_3(a) \\ \hat{v}_3(a) \\ \hat{w}_2(a) \end{pmatrix} = \begin{pmatrix} A_{12} \\ A_{13} \\ A_{14} \\ A_{11} \end{pmatrix}
 \end{aligned} \tag{4.4.4.d}$$

The determinants of the eight coefficient square matrices in equations (4.4.4) can be evaluated exactly as follows:

$$\begin{aligned}
 & \det \begin{pmatrix} \frac{-1}{2\mu} & \frac{1}{2\mu} & 0 & 0 \\ \frac{1}{2\mu}\alpha_d & \frac{-1}{2\mu\gamma}\alpha_d & \frac{-1}{2\mu\kappa\gamma} & 0 \\ \frac{\lambda_u+3\mu}{2\mu(\lambda_u+2\mu)} & \frac{\lambda_u+3\mu}{2\mu(\lambda_u+2\mu)} & \frac{1}{\mu\kappa}\alpha_\gamma & 0 \\ 0 & 0 & 0 & \frac{1}{\mu} \end{pmatrix} \\
 & = \frac{-(3-\nu_u)}{8\mu^4\kappa(1-\nu_u)\gamma(\gamma+1)} \left[\gamma+1 + \frac{2(\nu_u-\nu)}{(3-4\nu)(1-\nu)} \right]
 \end{aligned} \tag{4.4.5.a}$$

$$\begin{aligned}
 & \det \begin{pmatrix} 1 & -1 & 0 & 0 \\ -\alpha_d & \frac{1}{\gamma}\alpha_d & \frac{1}{2\mu} & 0 \\ \frac{\lambda_u+\mu}{\lambda_u+2\mu} & \frac{\lambda_u+\mu}{\lambda_u+2\mu} & \frac{-1}{\mu}\alpha_\gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
 & = \frac{-1}{2\mu(1-\nu_u)\gamma(\gamma+1)} \left[\gamma(\gamma+1) - \frac{2(\nu_u-\nu)}{1-\nu} \right]
 \end{aligned}$$

and

$$\det \begin{pmatrix} \frac{-1}{2\mu} & \frac{1}{2\mu} & 0 & 0 \\ \frac{1}{2\mu}\alpha_d & \frac{-1}{2\mu\gamma}\alpha_d & \frac{1}{2\mu} & 0 \\ \frac{\lambda_u+3\mu}{2\mu(\lambda_u+2\mu)} & \frac{\lambda_u+3\mu}{2\mu(\lambda_u+2\mu)} & \frac{-1}{\mu}\alpha_\gamma & 0 \\ 0 & 0 & 0 & \frac{1}{\mu} \end{pmatrix} = \frac{3-\nu_u}{8\mu^4(1-\nu_u)\gamma(\gamma+1)} \left[\gamma(\gamma+1) + \frac{2(\nu_u-\nu)}{(3-4\nu_u)(1-\nu)} \right] \quad (4.4.5.b)$$

$$\det \begin{pmatrix} 1 & -1 & 0 & 0 \\ -\alpha_d & \frac{1}{\gamma}\alpha_d & \frac{-1}{2\mu\kappa\gamma} & 0 \\ \frac{\lambda_u+\mu}{\lambda_u+2\mu} & \frac{\lambda_u+\mu}{\lambda_u+2\mu} & \frac{1}{\mu\kappa}\alpha_\gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \frac{1}{2\mu(1-\nu_u)\gamma(\gamma+1)} \left[\gamma+1 - \frac{2(\nu_u-\nu)}{1-\nu} \right]$$

and

$$\det \begin{pmatrix} -1 & \frac{1}{2\mu} & 0 & 0 \\ \frac{1}{\gamma}\alpha_d & \frac{-1}{2\mu\gamma}\alpha_d & \frac{-1}{2\mu\kappa\gamma} & 0 \\ \frac{\lambda_u+\mu}{\lambda_u+2\mu} & \frac{\lambda_u+3\mu}{2\mu(\lambda_u+2\mu)} & \frac{1}{\mu\kappa}\alpha_\gamma & 0 \\ 0 & 0 & 0 & \frac{1}{\mu} \end{pmatrix} = \frac{-1}{2\mu^3\kappa} \frac{1}{\gamma} \quad (4.4.5.c)$$

$$\det \begin{pmatrix} 1 & \frac{-1}{2\mu} & 0 & 0 \\ -\alpha_d & \frac{1}{2\mu}\alpha_d & \frac{1}{2\mu} & 0 \\ \frac{\lambda_u+\mu}{\lambda_u+2\mu} & \frac{\lambda_u+3\mu}{2\mu(\lambda_u+2\mu)} & \frac{-1}{\mu}\alpha_\gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \frac{-1}{2\mu^2}$$

and

$$\det \begin{pmatrix} -1 & \frac{1}{2\mu} & 0 & 0 \\ \frac{1}{\gamma}\alpha_d & \frac{-1}{2\mu\gamma}\alpha_d & \frac{1}{2\mu} & 0 \\ \frac{\lambda_u+\mu}{\lambda_u+2\mu} & \frac{\lambda_u+3\mu}{2\mu(\lambda_u+2\mu)} & \frac{-1}{\mu}\alpha_\gamma & 0 \\ 0 & 0 & 0 & \frac{1}{\mu} \end{pmatrix} = \frac{1}{2\mu^3} \quad (4.4.5.d)$$

$$\det \begin{pmatrix} 1 & \frac{-1}{2\mu} & 0 & 0 \\ -\alpha_d & \frac{1}{2\mu}\alpha_d & \frac{-1}{2\mu\kappa\gamma} & 0 \\ \frac{\lambda_u+\mu}{\lambda_u+2\mu} & \frac{\lambda_u+3\mu}{2\mu(\lambda_u+2\mu)} & \frac{1}{\mu\kappa}\alpha_\gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \frac{1}{2\mu^2\kappa} \frac{1}{\gamma}$$

It is evident that the determinants of the eight square coefficient matrices in equations (4.4.5) are non-zero under the conditions of $\rho > 0$ and $Re(s) > 0$. This property indicates that the eight square matrices and their inverse matrices are

nonsingular matrices. The the four unknown variables at the surface $z = a$ can be obtained in terms of the given four boundary conditions and the internal body forces and fluid source by using the corresponding independent boundary algebraic equations in (4.4.4). Consequently, under the conditions of $0 < \rho < +\infty$, and $Re(s) > 0$, there is no singular pole in the expressions for the solutions of the four unknown boundary variables in terms of the given conditions. The limiting case of $\rho \rightarrow 0$ can be further examined in the final expressions of solutions.

4.4.3 Independent boundary equations at $z = b^-$

The four independent boundary algebraic or integral equations at the outer surface $z = b^-$ of the upper poroelastic halfspace $-\infty < z \leq b$ are derived with body forces and fluid source as follows,

$$\begin{pmatrix} 1 & \frac{-1}{\mu} \end{pmatrix} \mathbf{U}_v(b) = A_{21}$$

$$\begin{pmatrix} -1 & -1 & \frac{1}{2\mu} & \frac{1}{2\mu} & 0 & 0 \\ -\alpha_d & \frac{-1}{\gamma}\alpha_d & \frac{1}{2\mu}\alpha_d & \frac{1}{2\mu\gamma}\alpha_d & \frac{1}{2\mu} & \frac{1}{2\mu\kappa\gamma} \\ \frac{\lambda_u+\mu}{\lambda_u+2\mu} & -\frac{\lambda_u+\mu}{\lambda_u+2\mu} & \frac{\lambda_u+3\mu}{2\mu(\lambda_u+2\mu)} & -\frac{\lambda_u+3\mu}{2\mu(\lambda_u+2\mu)} & \frac{-1}{\mu}\alpha_\gamma & \frac{-1}{\mu\kappa}\alpha_\gamma \end{pmatrix} \mathbf{U}_u(b) = \begin{pmatrix} A_{22} \\ A_{23} \\ A_{24} \end{pmatrix} \quad (4.4.6.a)$$

where the nonhomogeneous terms $A_{21}, A_{22}, A_{23}, A_{24}$ are

$$\begin{aligned}
A_{21} &= - \left(1 \quad \frac{-1}{\mu} \right) \int_{-\infty}^b e^{\rho(\zeta-b)} \mathbf{G}_v(\zeta) d\zeta \\
A_{22} &= - \left(-1 \quad -1 \quad \frac{1}{2\mu} \quad \frac{1}{2\mu} \quad 0 \quad 0 \right) \int_{-\infty}^b e^{\rho(\zeta-b)} \mathbf{G}_u(\zeta) d\zeta \\
A_{23} &= - \left(-\alpha_d \quad -\frac{1}{\gamma} \alpha_d \quad \frac{1}{2\mu} \alpha_d \quad \frac{1}{2\mu\gamma} \alpha_d \quad \frac{1}{2\mu} \quad \frac{1}{2\mu\kappa\gamma} \right) \int_{-\infty}^b e^{\rho\gamma(\zeta-b)} \mathbf{G}_u(\zeta) d\zeta \\
A_{24} &= \left(\frac{-(\lambda_u+\mu)}{\lambda_u+2\mu} \quad \frac{\lambda_u+\mu}{\lambda_u+2\mu} \quad \frac{-(\lambda_u+3\mu)}{2\mu(\lambda_u+2\mu)} \quad \frac{\lambda_u+3\mu}{2\mu(\lambda_u+2\mu)} \quad \frac{1}{\mu} \alpha_\gamma \quad \frac{1}{\mu\kappa} \alpha_\gamma \right) \int_{-\infty}^b e^{\rho(\zeta-b)} \mathbf{G}_u(\zeta) d\zeta \\
&\quad + 2 \frac{\lambda_u+\mu}{\lambda_u+2\mu} \left(-1 \quad -1 \quad \frac{1}{2\mu} \quad \frac{1}{2\mu} \quad 0 \quad 0 \right) \int_{-\infty}^b \rho(\zeta-b) e^{\rho(\zeta-b)} \mathbf{G}_u(\zeta) d\zeta
\end{aligned} \tag{4.4.6.b}$$

If the body forces $\mathbf{f}(\mathbf{x}, t)$, $\mathbf{F}(\mathbf{x}, t)$ and fluid source $q(\mathbf{x}, t)$ are zero, the four independent boundary algebraic or integral equations at the outer surface $z = b^-$ of the upper halfspace $-\infty < z \leq b$ could be simply written as,

$$\begin{aligned}
&\left(1 \quad \frac{-1}{\mu} \right) \mathbf{U}_v(b) = 0 \\
&\left(\begin{array}{cccccc} -1 & -1 & \frac{1}{2\mu} & \frac{1}{2\mu} & 0 & 0 \\ -\alpha_d & \frac{-1}{\gamma} \alpha_d & \frac{1}{2\mu} \alpha_d & \frac{1}{2\mu\gamma} \alpha_d & \frac{1}{2\mu} & \frac{1}{2\mu\kappa\gamma} \\ \frac{\lambda_u+\mu}{\lambda_u+2\mu} & -\frac{\lambda_u+\mu}{\lambda_u+2\mu} & \frac{\lambda_u+3\mu}{2\mu(\lambda_u+2\mu)} & -\frac{\lambda_u+3\mu}{2\mu(\lambda_u+2\mu)} & \frac{-1}{\mu} \alpha_\gamma & \frac{-1}{\mu\kappa} \alpha_\gamma \end{array} \right) \mathbf{U}_u(b) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\end{aligned} \tag{4.4.7}$$

It is easy to observe that the equations (4.4.6) can be re-expressed as follows,

$$\begin{pmatrix} 1 & \frac{1}{\mu} \end{pmatrix} \begin{pmatrix} +\hat{w}_2(b) \\ -\tau_2(b) \end{pmatrix} = A_{21}$$

$$\begin{pmatrix} 1 & -1 & \frac{-1}{2\mu} & \frac{1}{2\mu} & 0 & 0 \\ -\alpha_d & \frac{1}{\gamma}\alpha_d & \frac{1}{2\mu}\alpha_d & \frac{-1}{2\mu\gamma}\alpha_d & \frac{1}{2\mu} & \frac{-1}{2\mu\kappa\gamma} \\ \frac{\lambda_u+\mu}{\lambda_u+2\mu} & \frac{\lambda_u+\mu}{\lambda_u+2\mu} & \frac{\lambda_u+3\mu}{2\mu(\lambda_u+2\mu)} & \frac{\lambda_u+3\mu}{2\mu(\lambda_u+2\mu)} & \frac{-1}{\mu}\alpha_\gamma & \frac{1}{\mu\kappa}\alpha_\gamma \end{pmatrix} \begin{pmatrix} -\hat{w}_1(b) \\ +\hat{w}_3(b) \\ -\hat{\tau}_3(b) \\ +\hat{\tau}_1(b) \\ -\hat{p}_w(b) \\ +\hat{v}_3(b) \end{pmatrix} = \begin{pmatrix} +A_{22} \\ -A_{23} \\ -A_{24} \end{pmatrix} \quad (4.4.8)$$

where the coefficient matrices are same as those in equations (4.4.1). The same conclusion of no singular poles except at $\rho = 0$ can be made in connection with the solution expressions of the four unknown boundary variables in terms of the four given boundary conditions at the surface $z = b$ and the internal body forces and fluid source.

Based on the above discussion, we can conclude that for any boundary value and initial value problem correctly posed in a poroelastic medium of halfspace extent, the final solution in the transform domain has no singular pole except at $\rho = 0$ and are convergent if $\rho \rightarrow +\infty$ or $s \rightarrow \infty$ ($Re(s) > 0$) or $z \rightarrow \pm\infty$. Furthermore, it can be shown that there is also no singular pole at $\rho = 0$ in the final solution expressed in the inversion transforms. These properties of solution in the transform domain can guarantee that the inverse Fourier and Laplace transforms of the solution are convergent and integrable and the final solution in the spatial and temporal domain is convergent and stable for any correctly posed boundary value and initial value problem. These conclusions can be extended to cover the solution of any appropriate boundary and initial value problem in a homogeneous poroelastic layer or poroelastic layered media saturated with compressible fluids. This extension is relegated to future studies.

4.5 Boundary Algebraic Equations for the Three Limiting Responses

4.5.1 General

The fundamental solution representations and boundary algebraic equations have been presented in the above sections for the mechanics of poroelastic media saturated with compressible fluids. These solution representations and boundary equations in the Fourier and Laplace transform domain can be further examined and simplified for the three limiting physical states of initial and final responses as well as the response of $\nu \rightarrow \nu_u$. Same conclusions as those obtained in Chapter 3 can be made on such representations and equations under these three limiting responses. In particular, the independent boundary algebraic equations will be utilized to formulate the boundary integral equations for the fundamental problems of rigid disc inclusions embedded in a saturated poroelastic infinite space under direct loads. Therefore, we shall write down the limiting results of the boundary algebraic equations (4.4.2) and (4.4.6) at the outer surfaces ($z = a^+$ and $z = b^-$) of the lower and upper poroelastic halfspaces where the effects of internal loads of body forces and fluid source are not considered.

4.5.2 Boundary equations at $z = a^+$ for the limiting responses

Taking the limits as $s \rightarrow +\infty$ of equations (4.4.2), we have the independent boundary algebraic equations at $z = a^+$ for the initial response $t \rightarrow 0^+$; i.e. :

$$w_2(a, 0^+) + \frac{1}{\mu} \tau_2(a, 0^+) = 0$$

$$\begin{pmatrix} 1 & -1 & \frac{-1}{2\mu} & \frac{1}{2\mu} \\ \frac{\lambda_u + \mu}{\lambda_u + 2\mu} & \frac{\lambda_u + \mu}{\lambda_u + 2\mu} & \frac{\lambda_u + 3\mu}{2\mu(\lambda_u + 2\mu)} & \frac{\lambda_u + 3\mu}{2\mu(\lambda_u + 2\mu)} \end{pmatrix} \begin{pmatrix} w_1(a, 0^+) \\ w_3(a, 0^+) \\ \tau_3(a, 0^+) \\ \tau_1(a, 0^+) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (4.5.1.a)$$

and

$$p_w(a, 0^+) - \frac{1}{\kappa} \vartheta_o(a, 0^+) = \frac{\alpha\beta}{\lambda_\infty + 2\mu} [2\mu w_1(a, 0^+) - \tau_3(a, 0^+)] \quad (4.5.1.b)$$

where in particular, it is assumed that

$$\vartheta_o(a, 0^+) = \lim_{s \rightarrow \infty} [s \hat{\vartheta}_3(a, s)/\gamma] < \infty \quad (4.5.1.c)$$

This assumption may indicate that $\vartheta_3(a, t)$ has the singularity of $t^{-\frac{1}{2}}$ as $t \rightarrow 0^+$ (see equations (2.3.1) and (2.3.4)).

Taking the limit as $s \rightarrow 0^+$ of equations (4.4.2), we have the independent boundary algebraic equations at $z = a^+$ for the final response $t \rightarrow +\infty$; i.e.:

$$w_2(a, +\infty) + \frac{1}{\mu} \tau_2(a, +\infty) = 0$$

$$\begin{pmatrix} 1 & -1 & \frac{-1}{2\mu} & \frac{1}{2\mu} \\ \frac{\lambda+\mu}{\lambda+2\mu} & \frac{\lambda+\mu}{\lambda+2\mu} & \frac{\lambda+3\mu}{2\mu(\lambda+2\mu)} & \frac{\lambda+3\mu}{2\mu(\lambda+2\mu)} \end{pmatrix} \begin{pmatrix} w_1(a, +\infty) \\ w_3(a, +\infty) \\ \tau_3(a, +\infty) \\ \tau_1(a, +\infty) \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{-\alpha}{\lambda+2\mu} \end{pmatrix} p_w(a, +\infty) \quad (4.5.2.a)$$

and

$$p_w(a, +\infty) - \frac{1}{\kappa} \vartheta_3(a, +\infty) = 0 \quad (4.5.2.b)$$

Taking the limit as $\nu \rightarrow \nu_\infty$ ($c \rightarrow +\infty$ and $\gamma \rightarrow 1$.) of equations (4.4.2) and inverting the Laplace transforms, we obtain the independent boundary algebraic equations at $z = a^+$ for the limiting response $\nu \rightarrow \nu_\infty$; i.e.:

$$w_2(a, t) + \frac{1}{\mu} \tau_2(a, t) = 0$$

$$\begin{pmatrix} 1 & -1 & \frac{-1}{2\mu} & \frac{1}{2\mu} \\ \frac{\lambda+\mu}{\lambda+2\mu} & \frac{\lambda+\mu}{\lambda+2\mu} & \frac{\lambda+3\mu}{2\mu(\lambda+2\mu)} & \frac{\lambda+3\mu}{2\mu(\lambda+2\mu)} \end{pmatrix} \begin{pmatrix} w_1(a, t) \\ w_3(a, t) \\ \tau_3(a, t) \\ \tau_1(a, t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (4.5.3.a)$$

and

$$p_w(a, t) - \frac{1}{\kappa} \vartheta_3(a, t) = 0 \quad (4.5.3.b)$$

where $0 < t < +\infty$.

4.5.3 Boundary equations at $z = b^-$ for limiting responses

Taking the limit as $s \rightarrow +\infty$ of quations (4.4.6), we have the independent boundary algebraic equations at $z = b^-$ for the initial response $t \rightarrow 0^+$; *i.e.* :

$$w_2(b, 0^+) - \frac{1}{\mu} \tau_2(b, 0^+) = 0$$

$$\begin{pmatrix} -1 & -1 & \frac{1}{2\mu} & \frac{1}{2\mu} \\ \frac{\lambda_u + \mu}{\lambda_u + 2\mu} & -\frac{\lambda_u + \mu}{\lambda_u + 2\mu} & \frac{\lambda_u + 3\mu}{2\mu(\lambda_u + 2\mu)} & -\frac{\lambda_u + 3\mu}{2\mu(\lambda_u + 2\mu)} \end{pmatrix} \begin{pmatrix} w_1(b, 0^+) \\ w_3(b, 0^+) \\ \tau_3(b, 0^+) \\ \tau_1(b, 0^+) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (4.5.4.a)$$

and

$$p_w(b, 0^+) + \frac{1}{\kappa} \vartheta_o(b, 0^+) = \frac{\alpha\beta}{\lambda_u + 2\mu} [2\mu w_1(b, 0^+) - \tau_3(b, 0^+)] \quad (4.5.4.b)$$

where in particular, it is assumed that

$$\vartheta_o(b, 0^+) = \lim_{s \rightarrow \infty} [s \hat{\vartheta}_3(b, s) / \gamma] < \infty \quad (4.5.4.c)$$

This assumption may indicate that $\vartheta_3(b, t)$ has the singularity of $t^{-\frac{1}{2}}$ as $t \rightarrow 0^+$ (see equations (2.3.1) and (2.3.4)).

Taking the limit as $s \rightarrow 0^+$ of quations (4.4.6), we have the independent boundary algebraic equations at $z = b^-$ for the final response $t \rightarrow +\infty$; *i.e.* :

$$w_2(b, +\infty) - \frac{1}{\mu} \tau_2(b, +\infty) = 0$$

$$\begin{pmatrix} -1 & -1 & \frac{1}{2\mu} & \frac{1}{2\mu} \\ \frac{\lambda + \mu}{\lambda + 2\mu} & -\frac{(\lambda + \mu)}{\lambda + 2\mu} & \frac{\lambda + 3\mu}{2\mu(\lambda + 2\mu)} & -\frac{(\lambda + 3\mu)}{2\mu(\lambda + 2\mu)} \end{pmatrix} \begin{pmatrix} w_1(b, +\infty) \\ w_3(b, +\infty) \\ \tau_3(b, +\infty) \\ \tau_1(b, +\infty) \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{-\alpha}{\lambda + 2\mu} \end{pmatrix} p_w(b, +\infty) \quad (4.5.5.a)$$

and

$$p_w(b, +\infty) + \frac{1}{\kappa} \vartheta_3(b, +\infty) = 0 \quad (4.5.5.b)$$

Taking the limit as $\nu \rightarrow \nu_*$ ($c \rightarrow +\infty$ and $\gamma \rightarrow 1$.) of quations (4.4.6) and inverting the Laplace transforms, we obtain the independent boundary algebraic equations at $z = b^-$ for the limiting response $\nu \rightarrow \nu_*$; i.e :

$$w_2(b, t) - \frac{1}{\mu} \tau_2(b, t) = 0$$

$$\begin{pmatrix} -1 & -1 & \frac{1}{2\mu} & \frac{1}{2\mu} \\ \frac{\lambda+\mu}{\lambda+2\mu} & \frac{-(\lambda+\mu)}{\lambda+2\mu} & \frac{\lambda+3\mu}{2\mu(\lambda+2\mu)} & \frac{-(\lambda+3\mu)}{2\mu(\lambda+2\mu)} \end{pmatrix} \begin{pmatrix} w_1(b, t) \\ w_3(b, t) \\ \tau_3(b, t) \\ \tau_1(b, t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (4.5.6.a)$$

and

$$p_w(b, t) + \frac{1}{\kappa} \vartheta_3(b, t) = 0 \quad (4.5.6.b)$$

where $0 < t < +\infty$.

4.6 Fourier and Laplace Transform Based Approach

Up to now, we have presented fundamental solution representations and independent boundary algebraic equations in the transform domain for the equations governing the response of a saturated poroelastic medium of finite layer or half-space extent. These solution representations and boundary equations are of great importance in solving problems in saturated poroelastic media. In particular, they are invariant in the transformed Cartesian and cylindrical coordinates.

In this thesis, the primary objective is the study of the mechanics of rigid disc inclusion embedded in poroelastic media saturated with compressible fluids. Based on the above results, we adopted a systematic Fourier and Laplace transform based approach for analytical solutions of such time-dependent disc inclusion problems. This systematic approach can be briefly summarized as follows.

- (1) Set up the mixed boundary conditions and initial value conditions for a disc inclusion problem under specific loading and drainage conditions in both the physical and the Laplace transform domains;

- (2) Formulate the pairs of two-dimensional Fourier transform based integral equations in the Laplace transform domain by employing the sets of solution representations (3.3.3) and the independent boundary algebraic equations (4.4.1) ~ (4.4.6);
- (3) Reduce the two-dimensional Fourier transform based integral equations into the infinite pairs of one-dimensional Hankel transform based integral equations in the Laplace transform domain, using the general relation between Hankel and Fourier integral transforms (2.2.1) ~ (2.2.6);
- (4) Reduce the infinite pairs of Hankel transform based integral equations into finite pairs of integral equations by using the specific boundary and internal loading conditions in the Laplace transform domain;
- (5) Derive the systems of Fredholm integral equations of the second kind in the Laplace transform domain according to the finite pairs of integral equations given above;
- (6) Solve the systems of complex Fredholm integral equations of the second kind, using a quadrature scheme in the Laplace transform domain;
- (7) Evaluate the time-dependent results of engineering interest such as the degree of consolidation displacement translations and rotations and the degree of relaxation (of forces and moments) and using the modified numerical inversion scheme for Laplace transforms (2.4.1) ~ (2.4.18).
- (8) Analyze influence of material parameters and drainage conditions on the consolidation response of rigid disc inclusions embedded in saturated poroelastic media.

In steps (4) ~ (5), more attention will be focused on the integral equations under the three limiting physical states of the initial and final responses as well as the limiting response of $\nu \rightarrow \nu_*$. It is also instructive to compare the limiting results with those obtained via classical linear elastic analysis.

In this thesis, the proposed Fourier and Laplace transform based approach

provides a successful technique for the analytical investigations of the mechanics of the fundamental disc inclusion problems related to a saturated poroelastic medium. It is shown in such investigations that the proposed systematic approach provides greater convenience and simplifications in the mathematical formulations of the mixed boundary value and initial value problems associated with poroelastic media saturated with compressible fluids.

Chapter 5

Mechanics Of Inclusions In Poroelastic Media

I. Analytical Formulation

5.1 General

In this Chapter, we shall first formulate the systems of two-dimensional (2-D) Fourier transform based integral equations in the Laplace transform domain which will be used in the analysis of the mechanics of a circular disc inclusion embedded in bonded contact with a poroelastic medium of infinite extent and subjected to generalized direct loads. It is found that the integral equations governing any pervious or impervious plane inclusion decouple into two systems associated with the vertical displacement and the horizontal displacements of the inclusion. Such decoupled systems of 2-D Fourier transform based integral equations are further expanded into infinite sets of Hankel transform based integral equations. These infinite sets of integral equations of a general circular disc inclusion are then directly applied to formulate the special integral equations associated with the rigid disc inclusion problems. The systems of Fredholm integral equations of the second kind will finally be derived for the fundamental problems of the rigid disc inclusion. The equations and solutions under the three limiting states of initial and final responses as well as the response as $\nu \rightarrow \nu_u$ will be analyzed and compared with the corresponding results via the classical linear elastic analysis.

5.2 Integral Equations of a Plane Inclusion in Poroelastic Media

5.2.1 General mixed boundary and initial value conditions

We shall now present the boundary conditions appropriate for the problem of a plane inclusion which is embedded in bonded contact with a fluid saturated poroelastic medium of infinite extent. The embedded plane inclusion is subjected to the generalized loadings of force resultants (P_x, P_y, P_z) and moment resultants (M_x, M_y, M_z) . The interface between the saturated poroelastic soil and the inclusion is assumed to be either impermeable or permeable. By considering the symmetry or asymmetry of the embedded plane inclusion problems it can be shown that all of these generalized loading problems can be formulated in relation to mixed boundary value problems associated with the halfspace regions occupying $z \in [0^+, +\infty)$ and $z \in (-\infty, 0^-]$. The notation 0^+ and 0^- are intended to designate the plane surfaces of the halfspace regions corresponding to $z \geq 0$ and $z \leq 0$ respectively. For convenience we introduce two regions $\tilde{A}$ and $\bar{A}$ which correspond to the regions on $z = 0$ occupied by the inclusion and its exterior respectively. It may be noted that $\tilde{A} \cup \bar{A} = A$ where A is the entire plane $z = 0$ containing the plane inclusion and $\tilde{A} \cap \bar{A} = \{0\}$, implying that this is no separation, interpenetration or cracking induced by the loading of the embedded inclusion for any time $t \in (0, +\infty)$. In general terms, the displacements induced at the soil-inclusion interface are denoted by, $U_r(r, \theta, t)$, $U_\theta(r, \theta, t)$ and $U_z(r, \theta, t)$ where the displacements occur in the r , θ and z directions respectively. Considering these generalized displacements we have,

$$\begin{aligned} u_z(r, \theta, 0^+, t) &= u_z(r, \theta, 0^-, t) = U_z(r, \theta, t); & r, \theta \in \tilde{A} \\ u_r(r, \theta, 0^+, t) &= u_r(r, \theta, 0^-, t) = U_r(r, \theta, t); & r, \theta \in \tilde{A} \\ u_\theta(r, \theta, 0^+, t) &= u_\theta(r, \theta, 0^-, t) = U_\theta(r, \theta, t); & r, \theta \in \tilde{A} \end{aligned} \quad (5.2.1.a)$$

In addition to these displacement boundary conditions, pore pressure boundary conditions have to be prescribed at the soil-inclusion interface. These could be

either

$$p(r, \theta, 0^+, t) = p(r, \theta, 0^-, t) = 0; \quad r, \theta \in \tilde{A} \quad (5.2.1.b)$$

or

$$v_z(r, \theta, 0^+, t) = v_z(r, \theta, 0^-, t) = 0; \quad r, \theta \in \tilde{A} \quad (5.2.1.c)$$

which correspond to permeable and impermeable interface respectively.

The definition of the boundary conditions is complete with the specification of continuity conditions associated with the region $\bar{A}$. We have, in general,

$$\begin{aligned} u_z(r, \theta, 0^+, t) &= u_z(r, \theta, 0^-, t); & r, \theta \in \bar{A} \\ u_r(r, \theta, 0^+, t) &= u_r(r, \theta, 0^-, t); & r, \theta \in \bar{A} \\ u_\theta(r, \theta, 0^+, t) &= u_\theta(r, \theta, 0^-, t); & r, \theta \in \bar{A} \\ \sigma_{zz}(r, \theta, 0^+, t) &= \sigma_{zz}(r, \theta, 0^-, t); & r, \theta \in \bar{A} \\ \sigma_{rz}(r, \theta, 0^+, t) &= \sigma_{rz}(r, \theta, 0^-, t); & r, \theta \in \bar{A} \\ \sigma_{\theta z}(r, \theta, 0^+, t) &= \sigma_{\theta z}(r, \theta, 0^-, t); & r, \theta \in \bar{A} \\ p(r, \theta, 0^+, t) &= p(r, \theta, 0^-, t); & r, \theta \in \bar{A} \\ v_z(r, \theta, 0^+, t) &= v_z(r, \theta, 0^-, t); & r, \theta \in \bar{A} \end{aligned} \quad (5.2.1.d)$$

From the solution to the problem of an embedded inclusion, it is possible to evaluate the time dependent force resultants acting on the inclusion, i.e.,

$$\begin{aligned} P_x(t) &= - \iint_{\tilde{A}} [\sigma_{xz}(r, \theta, 0^+, t) - \sigma_{xz}(r, \theta, 0^-, t)] r d\theta dr \\ P_y(t) &= - \iint_{\tilde{A}} [\sigma_{yz}(r, \theta, 0^+, t) - \sigma_{yz}(r, \theta, 0^-, t)] r d\theta dr \\ P_z(t) &= - \iint_{\tilde{A}} [\sigma_{zz}(r, \theta, 0^+, t) - \sigma_{zz}(r, \theta, 0^-, t)] r d\theta dr \end{aligned} \quad (5.2.2.a)$$

Similarly, the moment resultants can be evaluated from the relationships

$$\begin{aligned} M_x(t) &= - \iint_{\tilde{A}} [\sigma_{xz}(r, \theta, 0^+, t) - \sigma_{xz}(r, \theta, 0^-, t)] r^2 \sin \theta d\theta dr \\ M_y(t) &= - \iint_{\tilde{A}} [\sigma_{yz}(r, \theta, 0^+, t) - \sigma_{yz}(r, \theta, 0^-, t)] r^2 \cos \theta d\theta dr \\ M_z(t) &= - \iint_{\tilde{A}} [\sigma_{\theta z}(r, \theta, 0^+, t) - \sigma_{\theta z}(r, \theta, 0^-, t)] r^2 d\theta dr \end{aligned} \quad (5.2.2.b)$$

where the shear stresses in Cartesian coordinates have the following relationships with those in Cylindrical coordinates,

$$\begin{aligned}\sigma_{xz}(r, \theta, 0^\pm, t) &= \cos \theta \sigma_{rz}(r, \theta, 0^\pm, t) - \sin \theta \sigma_{\theta z}(r, \theta, 0^\pm, t) \\ \sigma_{yz}(r, \theta, 0^\pm, t) &= \sin \theta \sigma_{rz}(r, \theta, 0^\pm, t) + \cos \theta \sigma_{\theta z}(r, \theta, 0^\pm, t)\end{aligned}\quad (5.2.2.c)$$

This completes the formal development of the plane inclusion problem.

5.2.2 Symmetric solutions for a plane inclusion subjected to direct loads

From equations (4.4.2) and (4.4.7), we can re-write down the general boundary algebraic equations in the transform domain under application of direct loads at the surfaces $z = 0^+$ and $z = 0^-$ of the lower ($0^+ \leq z < +\infty$) and the upper ($-\infty < z \leq 0^-$) halfspaces, respectively; *i.e.*,

$$\begin{pmatrix} -1 & -1 & \frac{1}{2\mu} & \frac{1}{2\mu} & 0 & 0 \\ -\alpha_d & \frac{-1}{\gamma}\alpha_d & \frac{1}{2\mu}\alpha_d & \frac{1}{2\mu\gamma}\alpha_d & \frac{1}{2\mu} & \frac{1}{2\mu\kappa\gamma} \\ \frac{-(\lambda_u+\mu)}{\lambda_u+2\mu} & \frac{\lambda_u+\mu}{\lambda_u+2\mu} & \frac{-(\lambda_u+3\mu)}{2\mu(\lambda_u+2\mu)} & \frac{\lambda_u+3\mu}{2\mu(\lambda_u+2\mu)} & \frac{-1}{\mu}\alpha_\gamma & \frac{1}{\mu\kappa}\alpha_\gamma \end{pmatrix} \begin{pmatrix} \mp \hat{w}_1(0^\pm) \\ \hat{w}_3(0^\pm) \\ \mp \hat{\tau}_3(0^\pm) \\ \hat{\tau}_1(0^\pm) \\ \mp \hat{p}_w(0^\pm) \\ \hat{\vartheta}_3(0^\pm) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \frac{-1}{\mu} \end{pmatrix} \begin{pmatrix} \hat{w}_2(0^\pm) \\ \mp \hat{\tau}_2(0^\pm) \end{pmatrix} = 0 \quad (5.2.3)$$

In the following, we shall use equations (5.2.3) to formulate the two-dimensional Fourier and Laplace transform based integral equations for the problem of a general plane inclusion subjected to direct loads. Based on these integral equations, we shall present symmetric solutions for the field variables at the plane containing the inclusion in both physical and transform domains.

5.2.2.1 Permeable inclusion problem

Considering the continuity conditions of variables $u_r(0^\pm)$, $u_\theta(0^\pm)$, $u_z(0^\pm)$, and $p(0^\pm)$ at the plane containing the inclusion ($z = 0$, $0 \leq r < +\infty$, $0 \leq \theta < 2\pi$) (see equations (5.2.1)), we have

$$\begin{aligned}
\hat{u}_r(r, \theta, 0^+, s) &= \hat{u}_r(r, \theta, 0^-, s) = \hat{u}_r(r, \theta, 0, s) \\
\hat{u}_\theta(r, \theta, 0^+, s) &= \hat{u}_\theta(r, \theta, 0^-, s) = \hat{u}_\theta(r, \theta, 0, s) \\
\hat{u}_z(r, \theta, 0^+, s) &= \hat{u}_z(r, \theta, 0^-, s) = \hat{u}_z(r, \theta, 0, s) \\
\hat{p}(r, \theta, 0^+, s) &= \hat{p}(r, \theta, 0^-, s) = \hat{p}(r, \theta, 0, s)
\end{aligned} \tag{5.2.4.a}$$

where $0 \leq r < +\infty$, $0 \leq \theta < 2\pi$.

We obtain the corresponding continuity results in the transform domain as follows:

$$\begin{aligned}
\hat{w}_1(0^+) &= \hat{w}_1(0^-) = \hat{w}_1(0) \\
\hat{w}_2(0^+) &= \hat{w}_2(0^-) = \hat{w}_2(0) \\
\hat{w}_3(0^+) &= \hat{w}_3(0^-) = \hat{w}_3(0) \\
\hat{p}_w(0^+) &= \hat{p}_w(0^-) = \hat{p}_w(0)
\end{aligned} \tag{5.2.4.b}$$

The variables $\hat{\tau}_1(0^\pm)$, $\hat{\tau}_2(0^\pm)$, $\hat{\tau}_3(0^\pm)$, and $\hat{\vartheta}_3(0^\pm)$ can then be expressed in terms of $\hat{w}_1(0)$, $\hat{w}_2(0)$, $\hat{w}_3(0)$, and $\hat{p}_w(0)$ by solving the equations (5.2.3); *i.e.*,

$$\begin{pmatrix} \hat{\tau}_3(0^\pm) \\ \hat{\tau}_1(0^\pm) \\ \hat{\vartheta}_3(0^\pm) \\ \hat{\tau}_2(0^\pm) \end{pmatrix} = \begin{pmatrix} b_{11}(\gamma) & \mp b_{12}(\gamma) & b_{13}(\gamma) & 0 \\ \mp b_{21}(\gamma) & b_{22}(\gamma) & \mp b_{23}(\gamma) & 0 \\ \mp b_{31}(\gamma) & b_{32}(\gamma) & \mp b_{33}(\gamma) & 0 \\ 0 & 0 & 0 & \mp \mu \end{pmatrix} \begin{pmatrix} \hat{w}_1(0) \\ \hat{w}_3(0) \\ \hat{p}_w(0) \\ \hat{w}_2(0) \end{pmatrix} \tag{5.2.5.a}$$

where the coefficient matrix is nonsingular and,

$$\begin{pmatrix} b_{11}(\gamma) & b_{12}(\gamma) & b_{13}(\gamma) \\ b_{21}(\gamma) & b_{22}(\gamma) & b_{23}(\gamma) \\ b_{31}(\gamma) & b_{32}(\gamma) & b_{33}(\gamma) \end{pmatrix} = \begin{pmatrix} \frac{1}{2\mu} & \frac{1}{2\mu} & 0 \\ \frac{1}{2\mu}\alpha_d & \frac{1}{2\mu\gamma}\alpha_d & \frac{1}{2\mu\kappa\gamma} \\ \frac{-(\lambda_u+3\mu)}{2\mu(\lambda_u+2\mu)} & \frac{\lambda_u+3\mu}{2\mu(\lambda_u+2\mu)} & \frac{1}{\mu\kappa}\alpha_\gamma \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & 0 \\ \alpha_d & \frac{1}{\gamma}\alpha_d & \frac{-1}{2\mu} \\ \frac{\lambda_u+\mu}{\lambda_u+2\mu} & \frac{-(\lambda_u+\mu)}{\lambda_u+2\mu} & \frac{1}{\mu}\alpha_\gamma \end{pmatrix} \tag{5.2.5.b}$$

Using the sets of solution representations (3.3.3) for the field variables and the boundary conditions in equations (5.2.1.a), (5.2.1.b) and (5.2.1.d), we obtain the

following system of 2-D Fourier transform based integral equations in the Laplace transform domain,

$$\begin{aligned}
 \frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} \frac{i}{\rho} [\cos(\theta + \varphi) \hat{w}_1(0) + \sin(\theta + \varphi) \hat{w}_2(0)] K \rho d\varphi d\rho &= \hat{U}_r(r, \theta, s) \\
 \frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} \frac{i}{\rho} [\sin(\theta + \varphi) \hat{w}_1(0) - \cos(\theta + \varphi) \hat{w}_2(0)] K \rho d\varphi d\rho &= \hat{U}_\theta(r, \theta, s) \\
 \frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} \frac{1}{\rho} \hat{w}_3(0) K \rho d\varphi d\rho &= \hat{U}_z(r, \theta, s) \\
 \frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} \hat{p}_w(0) K \rho d\varphi d\rho &= 0
 \end{aligned} \tag{5.2.6.a}$$

where $r, \theta \in \tilde{A}$, and

$$\begin{aligned}
 \frac{i}{2\pi} \int_0^{+\infty} \int_0^{2\pi} [\cos(\theta + \varphi) [\hat{\tau}_1(0^-) - \hat{\tau}_1(0^+)] + \sin(\theta + \varphi) [\hat{\tau}_2(0^-) - \hat{\tau}_2(0^+)]] K \rho d\varphi d\rho &= 0 \\
 \frac{i}{2\pi} \int_0^{+\infty} \int_0^{2\pi} [\sin(\theta + \varphi) [\hat{\tau}_1(0^-) - \hat{\tau}_1(0^+)] - \cos(\theta + \varphi) [\hat{\tau}_2(0^-) - \hat{\tau}_2(0^+)]] K \rho d\varphi d\rho &= 0 \\
 \frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} [\hat{\tau}_3(0^-) - \hat{\tau}_3(0^+)] K \rho d\varphi d\rho &= 0 \\
 \frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} \rho [\hat{\vartheta}_3(0^-) - \hat{\vartheta}_3(0^+)] K \rho d\varphi d\rho &= 0
 \end{aligned} \tag{5.2.6.b}$$

where $r, \theta \in \bar{A}$, and the symmetric Fourier kernel function $K = e^{i\rho r \sin(\theta + \varphi)}$.

Substituting the expressions (5.2.5) for $\hat{\tau}_1(0^\pm)$, $\hat{\tau}_2(0^\pm)$, $\hat{\tau}_3(0^\pm)$, and $\hat{\vartheta}_3(0^\pm, s)$, into the equations (5.2.6), we decouple the system of integral equations (5.2.5) as the following two systems associated with either the vertical displacement or the horizontal displacements of plane inclusion in terms of $\hat{w}_3(0)$ and $(\hat{w}_1(0), \hat{w}_2(0), \text{ and } \hat{p}_w(0))$, respectively. The first system is

$$\begin{aligned}
\frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} \frac{1}{\rho} \hat{w}_3(0) K \rho d\varphi d\rho &= \hat{U}_z(r, \theta, s); \quad r, \theta \in \tilde{A} \\
\frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} b_{12} \hat{w}_3(0) K \rho d\varphi d\rho &= 0; \quad r, \theta \in \bar{A}
\end{aligned} \tag{5.2.7}$$

the second system is

$$\begin{aligned}
\frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} \frac{i}{\rho} [\cos(\theta + \varphi) \hat{w}_1(0) + \sin(\theta + \varphi) \hat{w}_2(0)] K \rho d\varphi d\rho &= \hat{U}_r(r, \theta, s); \quad r, \theta \in \tilde{A} \\
\frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} \frac{i}{\rho} [\sin(\theta + \varphi) \hat{w}_1(0) - \cos(\theta + \varphi) \hat{w}_2(0)] K \rho d\varphi d\rho &= \hat{U}_\theta(r, \theta, s); \quad r, \theta \in \tilde{A} \\
\frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} \hat{p}_w(0) K \rho d\varphi d\rho &= 0; \quad r, \theta \in \tilde{A} \\
\frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} i [\cos(\theta + \varphi) [b_{21} \hat{w}_1(0) + b_{23} \hat{p}_w(0)] + \sin(\theta + \varphi) \mu \hat{w}_2(0)] K \rho d\varphi d\rho &= 0; \quad r, \theta \in \bar{A} \\
\frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} i [\sin(\theta + \varphi) [b_{21} \hat{w}_1(0) + b_{23} \hat{p}_w(0)] - \cos(\theta + \varphi) \mu \hat{w}_2(0)] K \rho d\varphi d\rho &= 0; \quad r, \theta \in \bar{A} \\
\frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} \rho [b_{31} \hat{w}_1(0) + b_{33} \hat{p}_w(0)] K \rho d\varphi d\rho &= 0; \quad r, \theta \in \bar{A}
\end{aligned} \tag{5.2.8}$$

The following symmetric solutions can be found by using the two decoupled systems of integral equations (5.2.7) and (5.2.8) and the existence of a solution in a poroelastic medium.

1. If $U_z(r, \theta, t) = 0$, from the equations (5.2.7) we have the symmetric solutions: (i) in the transform domain,

$$\dot{w}_3(0) = 0$$

$$\hat{\tau}_3(0^-) = \hat{\tau}_3(0^+) = \hat{\tau}_3(0) = b_{11}(\gamma)\hat{w}_1(0) + b_{13}(\gamma)\hat{p}_w(0)$$

$$\hat{\tau}_1(0^-) = -\hat{\tau}_1(0^+) = -\hat{\tau}_1(0) = b_{21}(\gamma)\hat{w}_1(0) + b_{23}(\gamma)\hat{p}_w(0) \quad (5.2.9.a)$$

$$\hat{\vartheta}_3(0^-) = -\hat{\vartheta}_3(0^+) = -\hat{\vartheta}_3(0) = b_{31}(\gamma)\hat{w}_1(0) + b_{33}(\gamma)\hat{p}_w(0)$$

$$\hat{\tau}_2(0^-) = -\hat{\tau}_2(0^+) = -\hat{\tau}_2(0) = \mu\hat{w}_2(0)$$

and (ii) in the physical domain,

$$\begin{aligned} u_z(r, \theta, 0, t) &= 0 & r, \theta &\in \tilde{A} \cup \overline{A} \\ \sigma_{zz}(r, \theta, 0^-, t) &= \sigma_{zz}(r, \theta, 0^+, t) & r, \theta &\in \tilde{A} \cup \overline{A} \\ \sigma_{rz}(r, \theta, 0^-, t) &= -\sigma_{rz}(r, \theta, 0^+, t) & r, \theta &\in \tilde{A} \\ \sigma_{\theta z}(r, \theta, 0^-, t) &= -\sigma_{\theta z}(r, \theta, 0^+, t) & r, \theta &\in \tilde{A} \\ v_z(r, \theta, 0^-, t) &= -v_z(r, \theta, 0^+, t) & r, \theta &\in \tilde{A} \\ \sigma_{rz}(r, \theta, 0^-, t) &= \pm \sigma_{rz}(r, \theta, 0^+, t) = 0 & r, \theta &\in \overline{A} \\ \sigma_{\theta z}(r, \theta, 0^-, t) &= \pm \sigma_{\theta z}(r, \theta, 0^+, t) = 0 & r, \theta &\in \overline{A} \\ v_z(r, \theta, 0^-, t) &= \pm v_z(r, \theta, 0^+, t) = 0 & r, \theta &\in \overline{A} \end{aligned} \quad (5.2.9.b)$$

2. If $U_r(r, \theta, t) = U_\theta(r, \theta, t) = 0$, from the equations (5.2.8) we have the symmetric solutions: (i) in the transform domain,

$$\hat{w}_1(0) = 0$$

$$\hat{w}_2(0) = 0$$

$$\hat{p}_w(0) = 0$$

$$\hat{\tau}_3(0^-) = -\hat{\tau}_3(0^+) = -\hat{\tau}_3(0) = b_{12}(\gamma)\hat{w}_3(0) \quad (5.2.10.a)$$

$$\hat{\tau}_1(0^-) = \hat{\tau}_1(0^+) = \hat{\tau}_1(0) = b_{22}(\gamma)\hat{w}_3(0)$$

$$\hat{\vartheta}_3(0^-) = \hat{\vartheta}_3(0^+) = \hat{\vartheta}_3(0) = b_{32}(\gamma)\hat{w}_3(0)$$

$$\hat{\tau}_2(0^-) = 0$$

and (ii) in the physical domain,

$$\begin{aligned}
u_r(r, \theta, 0, t) &= 0 & r, \theta &\in \tilde{A} \cup \bar{A} \\
u_\theta(r, \theta, 0, t) &= 0 & r, \theta &\in \tilde{A} \cup \bar{A} \\
p(r, \theta, 0, t) &= 0 & r, \theta &\in \tilde{A} \cup \bar{A} \\
\sigma_{rz}(r, \theta, 0^-, t) &= \sigma_{rz}(r, \theta, 0^+, t) & r, \theta &\in \tilde{A} \cup \bar{A} \\
\sigma_{\theta z}(r, \theta, 0^-, t) &= \sigma_{\theta z}(r, \theta, 0^+, t) & r, \theta &\in \tilde{A} \cup \bar{A} \\
v_z(r, \theta, 0^-, t) &= v_z(r, \theta, 0^+, t) & r, \theta &\in \tilde{A} \cup \bar{A} \\
\sigma_{zz}(r, \theta, 0^-, t) &= -\sigma_{zz}(r, \theta, 0^+, t) & r, \theta &\in \tilde{A} \\
\sigma_{zz}(r, \theta, 0^-, t) &= \pm \sigma_{zz}(r, \theta, 0^+, t) = 0 & r, \theta &\in \bar{A}
\end{aligned} \tag{5.2.10.b}$$

5.2.2.2 Impermeable inclusion problem

Similarly, considering the continuity conditions of variables $u_r(0^\pm)$, $u_\theta(0^\pm)$, $u_z(0^\pm)$, and $\hat{v}_z(0^\pm)$ at the plane containing the inclusion ($z = 0$, $0 \leq r < +\infty$, $0 \leq \theta < 2\pi$) (see equations (5.2.1)), we have

$$\begin{aligned}
\hat{u}_r(r, \theta, 0^+, s) &= \hat{u}_r(r, \theta, 0^-, s) = \hat{u}_r(r, \theta, 0, s) \\
\hat{u}_\theta(r, \theta, 0^+, s) &= \hat{u}_\theta(r, \theta, 0^-, s) = \hat{u}_\theta(r, \theta, 0, s) \\
\hat{u}_z(r, \theta, 0^+, s) &= \hat{u}_z(r, \theta, 0^-, s) = \hat{u}_z(r, \theta, 0, s) \\
\hat{v}_z(r, \theta, 0^+, s) &= \hat{v}_z(r, \theta, 0^-, s) = \hat{v}_z(r, \theta, 0, s)
\end{aligned} \tag{5.2.11.a}$$

where $0 \leq r < +\infty$, $0 \leq \theta < 2\pi$.

We obtain the corresponding continuity results in the transform domain as follows,

$$\begin{aligned}
\hat{w}_1(0^+, s) &= \hat{w}_1(0^-, s) = \hat{w}_1(0) \\
\hat{w}_2(0^+, s) &= \hat{w}_2(0^-, s) = \hat{w}_2(0) \\
\hat{w}_3(0^+, s) &= \hat{w}_3(0^-, s) = \hat{w}_3(0) \\
\hat{\vartheta}_3(0^+, s) &= \hat{\vartheta}_3(0^-, s) = \hat{\vartheta}_3(0)
\end{aligned} \tag{5.2.11.b}$$

We can express the variables $\hat{r}_1(0^\pm)$, $\hat{r}_2(0^\pm)$, $\hat{r}_3(0^\pm)$, and $\hat{p}_w(0^\pm)$ in terms of $\hat{w}_1(0)$, $\hat{w}_2(0)$, $\hat{w}_3(0)$, and $\hat{\vartheta}_3(0)$ by solving the equations (5.2.3); i.e.,

$$\begin{pmatrix} \hat{r}_3(0^\pm) \\ \hat{r}_1(0^\pm) \\ \hat{p}_w(0^\pm) \\ \hat{r}_2(0^\pm) \end{pmatrix} = \begin{pmatrix} c_{11}(\gamma) & \mp c_{12}(\gamma) & \mp c_{13}(\gamma) & 0 \\ \mp c_{21}(\gamma) & c_{22}(\gamma) & c_{23}(\gamma) & 0 \\ c_{31}(\gamma) & \mp c_{32}(\gamma) & \mp c_{33}(\gamma) & 0 \\ 0 & 0 & 0 & \mp \mu \end{pmatrix} \begin{pmatrix} \hat{w}_1(0) \\ \hat{w}_3(0) \\ \hat{v}_3(0) \\ \hat{w}_2(0) \end{pmatrix} \quad (5.2.12.a)$$

where the coefficient matrix is nonsingular and,

$$\begin{pmatrix} c_{11}(\gamma) & c_{12}(\gamma) & c_{13}(\gamma) \\ c_{21}(\gamma) & c_{22}(\gamma) & c_{23}(\gamma) \\ c_{31}(\gamma) & c_{32}(\gamma) & c_{33}(\gamma) \end{pmatrix} = \begin{pmatrix} \frac{1}{2\mu} & \frac{1}{2\mu} & 0 \\ \frac{1}{2\mu}\alpha_d & \frac{1}{2\mu\gamma}\alpha_d & \frac{1}{2\mu} \\ \frac{-(\lambda_u+3\mu)}{2\mu(\lambda_u+2\mu)} & \frac{\lambda_u+3\mu}{2\mu(\lambda_u+2\mu)} & \frac{-1}{\mu}\alpha_\gamma \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & 0 \\ \alpha_d & \frac{1}{\gamma}\alpha_d & \frac{-1}{2\mu\kappa\gamma} \\ \frac{\lambda_u+\mu}{\lambda_u+2\mu} & \frac{-(\lambda_u+\mu)}{\lambda_u+2\mu} & \frac{-1}{\mu\kappa}\alpha_\gamma \end{pmatrix} \quad (5.2.12.b)$$

Using the sets of solution representations (3.3.3) for the field variables and the boundary conditions in equations (5.2.1.a), (5.2.1.c) and (5.2.1.d), we obtain the following system of two-dimensional boundary integral equations in the Laplace transform domain,

$$\begin{aligned} \frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} \frac{i}{\rho} [\cos(\theta + \varphi) \hat{w}_1(0) + \sin(\theta + \varphi) \hat{w}_2(0)] K \rho d\varphi d\rho &= \hat{U}_r(r, \theta, s) \\ \frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} \frac{i}{\rho} [\sin(\theta + \varphi) \hat{w}_1(0) - \cos(\theta + \varphi) \hat{w}_2(0)] K \rho d\varphi d\rho &= \hat{U}_\theta(r, \theta, s) \\ \frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} \frac{1}{\rho} \hat{w}_3(0) K \rho d\varphi d\rho &= \hat{U}_z(r, \theta, s) \\ \frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} \rho \hat{v}_3(0) K \rho d\varphi d\rho &= 0 \end{aligned} \quad (5.2.13.a)$$

where $r, \theta \in \tilde{A}$ and,

$$\begin{aligned}
\frac{i}{2\pi} \int_0^{+\infty} \int_0^{2\pi} [\cos(\theta + \varphi)[\hat{\tau}_1(0^-) - \hat{\tau}_1(0^+)] + \sin(\theta + \varphi)[\hat{\tau}_2(0^-) - \hat{\tau}_2(0^+)] K \rho d\varphi d\rho &= 0 \\
\frac{i}{2\pi} \int_0^{+\infty} \int_0^{2\pi} [\sin(\theta + \varphi)[\hat{\tau}_1(0^-) - \hat{\tau}_1(0^+)] - \cos(\theta + \varphi)[\hat{\tau}_2(0^-) - \hat{\tau}_2(0^+)] K \rho d\varphi d\rho &= 0 \\
\frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} [\hat{\tau}_3(0^-) - \hat{\tau}_3(0^+)] K \rho d\varphi d\rho &= 0 \\
\frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} [\hat{p}_w(0^-) - \hat{p}_w(0^+)] K \rho d\varphi d\rho &= 0
\end{aligned}
\tag{5.2.13.b}$$

where $r, \theta \in \bar{A}$ and the symmetric Fourier kernel function $K = e^{i\rho r \sin(\theta + \varphi)}$.

Substituting the expressions (5.2.12) of $\hat{\tau}_1(0^\pm)$, $\hat{\tau}_2(0^\pm)$, $\hat{\tau}_3(0^\pm)$, and $\hat{p}_w(0^\pm, s)$ into the equations (5.2.13), we decouple the boundary integral equations as the following two systems associated with either the vertical displacement or the horizontal displacements of plane inclusion in terms of $(\hat{w}_3(0), \hat{\vartheta}_3(0))$, and $(\hat{w}_1(0), \hat{w}_2(0))$, respectively. The first system is

$$\begin{aligned}
\frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} \frac{1}{\rho} \hat{w}_3(0) K \rho d\varphi d\rho &= \hat{U}_z(r, \theta, s); \quad r, \theta \in \tilde{A} \\
\frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} \rho \hat{\vartheta}_3(0) K \rho d\varphi d\rho &= 0; \quad r, \theta \in \tilde{A} \\
\frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} [c_{12} \hat{w}_3(0) + c_{13} \hat{\vartheta}_3(0)] K \rho d\varphi d\rho &= 0; \quad r, \theta \in \bar{A} \\
\frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} \rho [c_{32} \hat{w}_1(0) + c_{33} \hat{\vartheta}_3(0)] K \rho d\varphi d\rho &= 0; \quad r, \theta \in \bar{A}
\end{aligned}
\tag{5.2.14}$$

and the second system is

$$\begin{aligned}
\frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} \frac{i}{\rho} [\cos(\theta + \varphi) \hat{w}_1(0) + \sin(\theta + \varphi) \hat{w}_2(0)] K \rho d\varphi d\rho &= \hat{U}_r(r, \theta, s); r, \theta \in \tilde{A} \\
\frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} \frac{i}{\rho} [\sin(\theta + \varphi) \hat{w}_1(0) - \cos(\theta + \varphi) \hat{w}_2(0)] K \rho d\varphi d\rho &= \hat{U}_\theta(r, \theta, s); r, \theta \in \tilde{A} \\
\frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} i \{ \cos(\theta + \varphi) c_{21} \hat{w}_1(0) + \sin(\theta + \varphi) \mu \hat{w}_2(0) \} K \rho d\varphi d\rho &= 0; \quad r, \theta \in \bar{A} \\
\frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} i \{ \sin(\theta + \varphi) c_{21} \hat{w}_1(0) - \cos(\theta + \varphi) \mu \hat{w}_2(0) \} K \rho d\varphi d\rho &= 0; \quad r, \theta \in \bar{A}
\end{aligned} \tag{5.2.15}$$

The following symmetric solutions can be found by using the two decoupled systems of integral equations (5.2.14) and (5.2.15) and the existence of a solution in a poroelastic medium.

1. If $U_z(r, \theta, t) = 0$, from equations (5.2.14) we have the symmetric solutions: (i) in the transform domain,

$$\begin{aligned}
\hat{w}_3(0) &= 0 \\
\hat{\vartheta}_3(0) &= 0 \\
\hat{\tau}_3(0^-) &= \hat{\tau}_3(0^+) = \hat{\tau}_3(0) = c_{11}(\rho, s) \hat{w}_1(0) \\
\hat{\tau}_1(0^-) &= -\hat{\tau}_1(0^+) = -\hat{\tau}_1(0) = c_{21}(\rho, s) \hat{w}_1(0) \\
\hat{p}_w(0^-) &= \hat{p}_w(0^+) = \hat{p}_w(0) = c_{31}(\rho, s) \hat{w}_1(0) \\
\hat{\tau}_2(0^-) &= -\hat{\tau}_2(0^+) = -\hat{\tau}_2(0) = \mu \hat{w}_2(0)
\end{aligned} \tag{5.2.16.a}$$

and (ii) in the physical domain,

$$\begin{aligned}
u_z(r, \theta, 0, t) &= 0 & r, \theta &\in \tilde{A} \cup \bar{A} \\
v_z(r, \theta, 0, t) &= 0 & r, \theta &\in \tilde{A} \cup \bar{A} \\
p(r, \theta, 0^-, t) &= p(r, \theta, 0^+, t) & r, \theta &\in \tilde{A} \cup \bar{A} \\
\sigma_{zz}(r, \theta, 0^-, t) &= \sigma_{zz}(r, \theta, 0^+, t) & r, \theta &\in \tilde{A} \cup \bar{A} \\
\sigma_{rz}(r, \theta, 0^-, t) &= -\sigma_{rz}(r, \theta, 0^+, t) & r, \theta &\in \tilde{A} \\
\sigma_{\theta z}(r, \theta, 0^-, t) &= -\sigma_{\theta z}(r, \theta, 0^+, t) & r, \theta &\in \tilde{A} \\
\sigma_{rz}(r, \theta, 0^-, t) &= \pm \sigma_{rz}(r, \theta, 0^+, t) = 0 & r, \theta &\in \bar{A} \\
\sigma_{\theta z}(r, \theta, 0^-, t) &= \pm \sigma_{\theta z}(r, \theta, 0^+, t) = 0 & r, \theta &\in \bar{A}
\end{aligned} \tag{5.2.16.b}$$

2. If $U_r(r, \theta, t) = U_\theta(r, \theta, t) = 0$, from equations (5.2.15) we have the symmetric solutions; (i) in the transform domain,

$$\begin{aligned}
\hat{w}_1(0) &= 0 \\
\hat{w}_2(0) &= 0 \\
\hat{\tau}_3(0^-) &= -\hat{\tau}_3(0^+) = -\hat{\tau}_3(0) = c_{12}(\rho, s)\hat{w}_3(0) + c_{13}(\rho, s)\hat{\vartheta}_3(0) \\
\hat{\tau}_1(0^-) &= \hat{\tau}_1(0^+) = \hat{\tau}_1(0) = c_{22}(\rho, s)\hat{w}_3(0) + c_{23}(\rho, s)\hat{\vartheta}_3(0) \\
\hat{p}_w(0^-) &= -\hat{p}_w(0^+) = -\hat{p}_w(0) = c_{32}(\rho, s)\hat{w}_3(0) + c_{33}(\rho, s)\hat{\vartheta}_3(0) \\
\hat{\tau}_2(0^-) &= 0
\end{aligned} \tag{5.2.17.a}$$

and (ii) in the physical domain,

$$\begin{aligned}
u_r(r, \theta, 0, t) &= 0 & r, \theta &\in \tilde{A} \cup \bar{A} \\
u_\theta(r, \theta, 0, t) &= 0 & r, \theta &\in \tilde{A} \cup \bar{A} \\
\sigma_{rz}(r, \theta, 0^-, t) &= \sigma_{rz}(r, \theta, 0^+, t) & r, \theta &\in \tilde{A} \cup \bar{A} \\
\sigma_{\theta z}(r, \theta, 0^-, t) &= \sigma_{\theta z}(r, \theta, 0^+, t) & r, \theta &\in \tilde{A} \cup \bar{A} \\
\sigma_{zz}(r, \theta, 0^-, t) &= -\sigma_{zz}(r, \theta, 0^+, t) & r, \theta &\in \tilde{A} \\
p(r, \theta, 0^-, t) &= p(r, \theta, 0^+, t) & r, \theta &\in \tilde{A} \\
\sigma_{zz}(r, \theta, 0^-, t) &= \pm \sigma_{zz}(r, \theta, 0^+, t) = 0 & r, \theta &\in \bar{A} \\
p(r, \theta, 0^-, t) &= \pm p(r, \theta, 0^+, t) = 0 & r, \theta &\in \bar{A}
\end{aligned} \tag{5.2.17.b}$$

5.2.3 Systems of 2-D integral equations for plane inclusion problems

The systems of 2-D Fourier transform based integral equations for the plane inclusion problems associated with the mixed boundary conditions (5.2.1) decouple into two systems of integral equations associated with the vertical and horizontal displacements of the inclusion, respectively. Using the symmetric solutions obtained above, we can re-express the systems of integral equations for the plane inclusion problems according to the displacements of the inclusion.

5.2.3.1 Vertical displacement of a plane inclusion

The system of 2-D Fourier transform based integral equations (5.2.7) for the vertical displacement of a *permeable* plane inclusion embedded in a saturated poroelastic medium can be re-written as,

$$\begin{aligned} \frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} \frac{1}{\rho} a_1(\gamma) \hat{\tau}_3(0) K \rho d\varphi d\rho &= \hat{U}_z(r, \theta, s), & r, \theta \in \tilde{A} \\ \frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} \hat{\tau}_3(0) K \rho d\varphi d\rho &= 0, & r, \theta \in \bar{A} \end{aligned} \quad (5.2.18.a)$$

And the system of 2-D Fourier transform based integral equations (5.2.14) for the vertical displacement of an *impermeable* plane inclusion embedded in a saturated poroelastic medium can be re-written as,

$$\begin{aligned}
\frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} \frac{1}{\rho} [a_1(\gamma) \hat{r}_3(0) + a_2(\gamma) \hat{p}_w(0)] K \rho d\varphi d\rho &= \hat{U}_z(r, \theta, s), \quad r, \theta \in \tilde{A} \\
\frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} \rho [a_3(\gamma) \hat{r}_3(0) + a_4(\gamma) \hat{p}_w(0)] K \rho d\varphi d\rho &= 0, \quad r, \theta \in \tilde{A} \\
\frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} \hat{r}_3(0) K \rho d\varphi d\rho &= 0, \quad r, \theta \in \bar{A} \\
\frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} \hat{p}_w(0) K \rho d\varphi d\rho &= 0, \quad r, \theta \in \bar{A}
\end{aligned} \tag{5.2.18.b}$$

where the kernel functions $a_j(\gamma)$ ($j = 1 \sim 4$) in the above equations are explicitly expressed as,

$$\begin{aligned}
a_1(\gamma) &= \frac{1}{\alpha_a} [1 + \alpha_2 k_1(\gamma)] \\
a_2(\gamma) &= \frac{\alpha_3}{B\alpha_a} [1 + k_1(\gamma)] \\
a_3(\gamma) &= \kappa B \alpha_4 k_2(\gamma) \\
a_4(\gamma) &= \kappa [1 + k_2(\gamma)] \\
k_1(\gamma) &= \frac{1 - \gamma}{1 + \gamma} \\
k_2(\gamma) &= \gamma - 1 \\
\alpha_2 &= \frac{\nu_u - \nu}{(1 - \nu_u)(3 - 4\nu)} \\
\alpha_3 &= \frac{3(\nu_u - \nu)}{(1 + \nu_u)(3 - 4\nu)} \\
\alpha_4 &= \frac{1 + \nu_u}{3(1 - \nu_u)} \\
\alpha_a &= -\frac{4\mu(1 - \nu)}{3 - 4\nu} \\
\beta_a &= -\frac{4\mu(1 - \nu_u)}{3 - 4\nu_u}
\end{aligned} \tag{5.2.18.c}$$

There is no singular pole in the kernel functions $a_j(\gamma)$ if $\rho > 0$ and $Re(s) > 0$.

The non-zero force and moments acting on the permeable or impermeable plane inclusion associated with the vertical displacement can be expressed as the following in the Laplace transform domain,

$$\begin{aligned}
\hat{P}_z(s) &= -2 \iint_{\tilde{A}} \hat{\sigma}_{zz}(r, \theta, 0, s) r d\theta dr \\
\hat{M}_x(s) &= -2 \iint_{\tilde{A}} \hat{\sigma}_{zz}(r, \theta, 0, s) r^2 \sin \theta d\theta dr \\
\hat{M}_y(s) &= -2 \iint_{\tilde{A}} \hat{\sigma}_{zz}(r, \theta, 0, s) r^2 \cos \theta d\theta dr
\end{aligned} \tag{5.2.19.a}$$

where the normal stress is

$$\hat{\sigma}_{zz}(r, \theta, 0, s) = \frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} \hat{\tau}_3(\rho, \varphi, 0, s) K \rho d\varphi d\rho \tag{5.2.19.b}$$

5.2.3.2 Horizontal displacements of a plane inclusion

The system of 2-D Fourier transform based integral equations (5.2.8) for the horizontal displacements of a *permeable* plane inclusion embedded in a saturated poroelastic medium can be re-written as,

$$\begin{aligned}
\frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} \frac{i}{\rho} \left[\sin(\varphi + \theta) [b_1(\gamma) \hat{\tau}_1(0) + b_2(\gamma) \hat{\vartheta}_3(0)] - \cos(\varphi + \theta) \frac{1}{\mu} \hat{\tau}_2(0) \right] K \rho d\varphi d\rho \\
= \hat{U}_r(r, \theta, s), \quad r, \theta \in \tilde{A} \\
\frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} \frac{i}{\rho} \left[\cos(\varphi + \theta) [b_1(\gamma) \hat{\tau}_1(0) + b_2(\gamma) \hat{\vartheta}_3(0)] + \sin(\varphi + \theta) \frac{1}{\mu} \hat{\tau}_2(0) \right] K \rho d\varphi d\rho \\
= \hat{U}_\theta(r, \theta, s), \quad r, \theta \in \tilde{A} \\
\frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} \left[b_3(\gamma) \hat{\tau}_1(0) + b_4(\gamma) \hat{\vartheta}_3(0) \right] K \rho d\varphi d\rho = 0, \quad r, \theta \in \tilde{A} \\
\frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} i \left[\sin(\varphi + \theta) \hat{\tau}_1(0) + \cos(\varphi + \theta) \hat{\tau}_2(0) \right] K \rho d\varphi d\rho = 0, \quad r, \theta \in \bar{A} \\
\frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} i \left[\cos(\varphi + \theta) \hat{\tau}_1(0) - \sin(\varphi + \theta) \hat{\tau}_2(0) \right] K \rho d\varphi d\rho = 0, \quad r, \theta \in \bar{A} \\
\frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} \rho \hat{\vartheta}_3(0) K \rho d\varphi d\rho = 0, \quad r, \theta \in \bar{A}
\end{aligned} \tag{5.2.20.a}$$

The system of 2-D Fourier transform based integral equations (5.2.15) for the

horizontal displacements of an *impermeable* plane inclusion embedded in a saturated poroelastic medium can be re-written as,

$$\begin{aligned}
 & \frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} \frac{i}{\rho} \left[\sin(\varphi + \theta) b_1(\gamma) \hat{r}_1(0) - \cos(\varphi + \theta) \frac{1}{\mu} \hat{r}_2(0) \right] K \rho d\varphi d\rho \\
 & \quad = \hat{U}_r(r, \theta, s), \quad r, \theta \in \tilde{A} \\
 & \frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} \frac{i}{\rho} \left[\cos(\varphi + \theta) b_1(\gamma) \hat{r}_1(0) + \sin(\varphi + \theta) \frac{1}{\mu} \hat{r}_2(0) \right] K \rho d\varphi d\rho \\
 & \quad = \hat{U}_\theta(r, \theta, s), \quad r, \theta \in \tilde{A} \\
 & \frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} i \left[\sin(\varphi + \theta) \hat{r}_1(0) + \cos(\varphi + \theta) \hat{r}_2(0) \right] K \rho d\varphi d\rho \\
 & \quad = 0, \quad r, \theta \in \overline{A} \\
 & \frac{1}{2\pi} \int_0^{+\infty} \int_0^{2\pi} i \left[\cos(\varphi + \theta) \hat{r}_1(0) - \sin(\varphi + \theta) \hat{r}_2(0) \right] K \rho d\varphi d\rho \\
 & \quad = 0, \quad r, \theta \in \overline{A}
 \end{aligned} \tag{5.2.20.b}$$

where the kernel functions $b_j(\gamma)$ ($j = 1 \sim 6$) in the above equations are explicitly expressed as,

$$\begin{aligned}
b_1(\gamma) &= \frac{2}{\alpha_b} [1 + \alpha_6 k_3(\gamma)] + \frac{1}{\mu} \\
b_2(\gamma) &= \frac{\alpha_5}{\kappa B \alpha_b} [1 + k_3(\gamma)] \\
b_3(\gamma) &= 2B \alpha_8 k_4(\gamma) \\
b_4(\gamma) &= \frac{1}{\kappa} [1 + k_4(\gamma)] \\
b_5(\gamma) &= \frac{1}{\alpha_b} [1 + \alpha_6 k_3(\gamma)] \\
b_6(\gamma) &= \frac{\alpha_1}{\alpha_b} [1 + \beta_7 k_3(\gamma)] \\
k_3(\gamma) &= \frac{2}{(1 + \gamma)\gamma} - 1 \\
k_4(\gamma) &= \frac{1 - \gamma}{\gamma} \\
\alpha_1 &= \frac{-1}{7 - 8\nu}, \quad \beta_1 = \frac{-1}{7 - 8\nu_{\mathbf{u}}} \\
\alpha_5 &= \frac{6(\nu_{\mathbf{u}} - \nu)}{(1 + \nu_{\mathbf{u}})(7 - 8\nu)} \\
\alpha_6 &= \frac{\nu_{\mathbf{u}} - \nu}{(1 - \nu_{\mathbf{u}})(7 - 8\nu)} \\
\alpha_7 &= \frac{\nu - \nu_{\mathbf{u}}}{1 - \nu_{\mathbf{u}}} \\
\alpha_8 &= \frac{1 + \nu_{\mathbf{u}}}{6(1 - \nu_{\mathbf{u}})} \\
\alpha_b &= -\frac{8\mu(1 - \nu)}{7 - 8\nu}, \quad \beta_b = -\frac{8\mu(1 - \nu_{\mathbf{u}})}{7 - 8\nu_{\mathbf{u}}}
\end{aligned} \tag{5.2.20.c}$$

Again, there is no singular pole in the kernel functions $b_j(\gamma)$ if $\rho > 0$ and $Re(s) > 0$.

The non-zero forces and moment acting on the permeable or impermeable plane inclusion associated with the horizontal displacements can be expressed as the following in the Laplace transform domain,

$$\begin{aligned}
\hat{P}_x(s) &= -2 \iint_{\tilde{A}} [\cos \theta \hat{\sigma}_{rz}(r, \theta, 0, s) - \sin \theta \hat{\sigma}_{rz}(r, \theta, 0, s)] r d\theta dr \\
\hat{P}_y(s) &= -2 \iint_{\tilde{A}} [\sin \theta \hat{\sigma}_{rz}(r, \theta, 0, s) + \cos \theta \hat{\sigma}_{rz}(r, \theta, 0, s)] r d\theta dr \\
\hat{M}_z(s) &= -2 \iint_{\tilde{A}} \hat{\sigma}_{\theta z}(r, \theta, 0, s) r^2 d\theta dr
\end{aligned} \tag{5.2.21.a}$$

where the shear stresses are,

$$\begin{aligned}\hat{\sigma}_{rz}(r, \theta, 0, s) &= \frac{i}{2\pi} \int_0^{+\infty} \int_0^{2\pi} [\sin(\theta + \varphi) \hat{\tau}_1(\rho, \varphi, 0, s) + \cos(\theta + \varphi) \hat{\tau}_2(\rho, \varphi, 0, s)] K \rho d\varphi d\rho \\ \hat{\sigma}_{\theta z}(r, \theta, 0, s) &= \frac{i}{2\pi} \int_0^{+\infty} \int_0^{2\pi} [\cos(\theta + \varphi) \hat{\tau}_1(\rho, \varphi, 0, s) - \sin(\theta + \varphi) \hat{\tau}_2(\rho, \varphi, 0, s)] K \rho d\varphi d\rho\end{aligned}\quad (5.2.21.b)$$

5.2.4 Infinite sets of 1-D integral equations in Fourier series expansions

The systems of 2-D Fourier transform based integral equations (5.2.18) and (5.2.20) are valid for an arbitrary geometric shape of a plane inclusion subjected to generalized direct loads. In particular, if the shape of the plane inclusion is a circular disc of radius a , then the 2-D integral equations can be further simplified into infinite sets of Hankel transform based integral equations by using the general relation between the Fourier and Hankel transforms. We assume that the generalized displacement vector for the disc inclusion, in the Laplace transform domain has the following Fourier series expansion,

$$\begin{aligned}\hat{U}_z(r, \theta, s) &= \sum_{m=-\infty}^{m=+\infty} \hat{U}_{zm}(r, s) e^{im\theta} \\ \hat{U}_r(r, \theta, s) &= \sum_{m=-\infty}^{m=+\infty} \hat{U}_{rm}(r, s) e^{im\theta} \\ \hat{U}_\theta(r, \theta, s) &= \sum_{m=-\infty}^{m=+\infty} \hat{U}_{\theta m}(r, s) e^{im\theta}\end{aligned}\quad (5.2.22.a)$$

where

$$\begin{aligned}\hat{U}_{zm}(r, s) &= \frac{1}{2\pi} \int_0^{2\pi} \hat{U}_z(r, \theta, s) e^{-im\theta} d\theta \\ \hat{U}_{rm}(r, s) &= \frac{1}{2\pi} \int_0^{2\pi} \hat{U}_r(r, \theta, s) e^{-im\theta} d\theta \\ \hat{U}_{\theta m}(r, s) &= \frac{1}{2\pi} \int_0^{2\pi} \hat{U}_\theta(r, \theta, s) e^{-im\theta} d\theta\end{aligned}\quad (5.2.22.b)$$

As a result, the unknown variables in the 2-D Fourier transform based integral equations have the following solutions of Fourier series expansions,

$$\begin{aligned}
 \hat{\tau}_1(\rho, \varphi, 0, s) &= \sum_{m=-\infty}^{m=+\infty} \hat{\tau}_{1m}(\rho, s) e^{-im\varphi} \\
 \hat{\tau}_2(\rho, \varphi, 0, s) &= \sum_{m=-\infty}^{m=+\infty} \hat{\tau}_{2m}(\rho, s) e^{-im\varphi} \\
 \hat{\tau}_3(\rho, \varphi, 0, s) &= \sum_{m=-\infty}^{m=+\infty} \hat{\tau}_{3m}(\rho, s) e^{-im\varphi} \\
 \hat{p}_w(\rho, \varphi, 0, s) &= \sum_{m=-\infty}^{m=+\infty} \hat{p}_{wm}(\rho, s) e^{-im\varphi} \\
 \hat{\vartheta}_3(\rho, \varphi, 0, s) &= \sum_{m=-\infty}^{m=+\infty} \hat{\vartheta}_{3m}(\rho, s) e^{-im\varphi}
 \end{aligned} \tag{5.2.23.a}$$

where

$$\begin{aligned}
 \hat{\tau}_{1m}(\rho, s) &= \frac{1}{2\pi} \int_0^{2\pi} \hat{\tau}_1(\rho, \varphi, 0, s) e^{im\varphi} d\varphi \\
 \hat{\tau}_{2m}(\rho, s) &= \frac{1}{2\pi} \int_0^{2\pi} \hat{\tau}_2(\rho, \varphi, 0, s) e^{im\varphi} d\varphi \\
 \hat{\tau}_{3m}(\rho, s) &= \frac{1}{2\pi} \int_0^{2\pi} \hat{\tau}_3(\rho, \varphi, 0, s) e^{im\varphi} d\varphi \\
 \hat{p}_{wm}(\rho, s) &= \frac{1}{2\pi} \int_0^{2\pi} \hat{p}_w(\rho, \varphi, 0, s) e^{im\varphi} d\varphi \\
 \hat{\vartheta}_{3m}(\rho, s) &= \frac{1}{2\pi} \int_0^{2\pi} \hat{\vartheta}_3(\rho, \varphi, 0, s) e^{im\varphi} d\varphi
 \end{aligned} \tag{5.2.23.b}$$

Substituting these Fourier series expansions of the variables into the systems of 2-D Fourier transform based integral equations (5.2.18) and (5.2.20), we reduce them into infinite sets of Hankel transform based integral equations. To aid this, we utilize the integral representations (2.2.1) for the Bessel functions $J_m(\rho r)$,

$$\begin{aligned}
\frac{1}{2\pi} \int_0^{2\pi} e^{\pm[i\rho r \sin(\varphi+\theta) - im(\varphi+\theta)]} d\varphi &= J_m(\rho r) \\
\frac{1}{2\pi} \int_0^{2\pi} \sin(\varphi + \theta) e^{i\rho r \sin(\varphi+\theta) - im(\varphi+\theta)} d\varphi &= \frac{1}{2i} [J_{m-1}(\rho r) - J_{m+1}(\rho r)] \\
\frac{1}{2\pi} \int_0^{2\pi} \cos(\varphi + \theta) e^{i\rho r \sin(\varphi+\theta) - im(\varphi+\theta)} d\varphi &= \frac{1}{2} [J_{m-1}(\rho r) + J_{m+1}(\rho r)]
\end{aligned} \tag{5.2.24}$$

where the integer $m = 0, \pm 1, \pm 2, \dots, \pm \infty$.

5.2.4.1 Vertical displacement of a disc inclusion

The infinite sets of Hankel transform based integral equations for the vertical displacement of a disc inclusion may be presented as follows:

(i) for the *permeable* disc inclusion problem,

$$\begin{aligned}
\int_0^{+\infty} a_1(\gamma) \hat{\tau}_{3m}(\rho, s) J_m(\rho r) d\rho &= \hat{U}_{zm}(\rho, s), & 0 \leq r \leq a \\
\int_0^{+\infty} \hat{\tau}_{3m}(\rho, s) J_m(\rho r) \rho d\rho &= 0, & a < r < +\infty
\end{aligned} \tag{5.2.25.a}$$

and (ii) for the *impermeable* disc inclusion problem,

$$\begin{aligned}
\int_0^{+\infty} [a_1(\gamma) \hat{\tau}_{3m}(\rho, s) + a_2(\gamma) \hat{p}_{wm}(\rho, s)] J_m(\rho r) d\rho &= \hat{U}_{zm}(r, s), & 0 \leq r \leq a \\
\int_0^{+\infty} [a_3(\gamma) \hat{\tau}_{3m}(\rho, s) + a_4(\gamma) \hat{p}_{wm}(\rho, s)] J_m(\rho r) \rho^2 d\rho &= 0, & 0 \leq r < a \\
\int_0^{+\infty} \hat{\tau}_{3m}(\rho, s) J_m(\rho r) \rho d\rho &= 0, & a < r < +\infty \\
\int_0^{+\infty} \hat{p}_{wm}(\rho, s) J_m(\rho r) \rho d\rho &= 0, & a < r < +\infty
\end{aligned} \tag{5.2.25.b}$$

where the integer $m = 0, \pm 1, \pm 2, \dots, \pm \infty$.

The associated non-zero total force and moments acting on the disc inclusion are,

$$\begin{aligned}
 \hat{P}_z(s) &= -4\pi a \int_0^{+\infty} \hat{\tau}_{30}(\rho, s) J_1(\rho a) d\rho \\
 \hat{M}_z(s) &= -2\pi i a^2 \int_0^{+\infty} [\hat{\tau}_{31}(\rho, s) + \hat{\tau}_{3(-1)}(\rho, s)] J_2(\rho a) d\rho \\
 \hat{M}_y(s) &= -2\pi a^2 \int_0^{+\infty} [\hat{\tau}_{31}(\rho, s) - \hat{\tau}_{3(-1)}(\rho, s)] J_2(\rho a) d\rho
 \end{aligned} \tag{5.2.26.a}$$

and the contact normal stress is

$$\hat{\sigma}_{zz}(r, \theta, 0, s) = \sum_{m=-\infty}^{m=+\infty} \int_0^{+\infty} \hat{\tau}_{3m}(\rho, s) J_m(\rho r) \rho d\rho e^{im\theta} \tag{5.2.26.b}$$

5.2.4.2 Horizontal displacements of a disc inclusion

The infinite sets of Hankel transform based integral equations for the horizontal displacements of a disc inclusion may be presented as follows:

(i) for the *permeable* disc inclusion problem,

$$\begin{aligned}
& \int_0^{+\infty} \left[b_1(\gamma) \hat{\tau}_{1m}(\rho, s) + b_2(\gamma) \hat{\vartheta}_{3m}(\rho, s) - i \frac{1}{\mu} \hat{\tau}_{2m}(\rho, s) \right] J_{m-1}(\rho r) d\rho \\
& \quad = \hat{U}_{rm}(r, s) - i \hat{U}_{\theta m}(r, s), \quad 0 \leq r \leq a \\
& \int_0^{+\infty} \left[b_1(\gamma) \hat{\tau}_{1m}(\rho, s) + b_2(\gamma) \hat{\vartheta}_{3m}(\rho, s) + i \frac{1}{\mu} \hat{\tau}_{2m}(\rho, s) \right] J_{m+1}(\rho r) d\rho \\
& \quad = -\hat{U}_{rm}(r, s) - i \hat{U}_{\theta m}(r, s), \quad 0 \leq r \leq a \\
& \int_0^{+\infty} \left[b_3(\gamma) \hat{\tau}_{1m}(\rho, s) + b_4(\gamma) \hat{\vartheta}_{3m}(\rho, s) \right] J_m(\rho r) \rho d\rho \\
& \quad = 0, \quad 0 \leq r < a \\
& \int_0^{+\infty} \left[\hat{\tau}_{1m}(\rho, s) + i \hat{\tau}_{2m}(\rho, s) \right] J_{m-1}(\rho r) \rho d\rho = 0, \quad a < r < +\infty \\
& \int_0^{+\infty} \left[\hat{\tau}_{1m}(\rho, s) - i \hat{\tau}_{2m}(\rho, s) \right] J_{m+1}(\rho r) \rho d\rho = 0, \quad a < r < +\infty \\
& \int_0^{+\infty} \hat{\vartheta}_{3m}(\rho, s) J_m(\rho r) \rho^2 d\rho = 0, \quad a < r < +\infty
\end{aligned} \tag{5.2.27.a}$$

and (ii) for the *impermeable* plane inclusion problem,

$$\begin{aligned}
& \int_0^{+\infty} \left[b_1(\gamma) \hat{\tau}_{1m}(\rho, s) - i \frac{1}{\mu} \hat{\tau}_{2m}(\rho, s) \right] J_{m-1}(\rho r) d\rho \\
& \quad = \hat{U}_{rm}(r, s) - i \hat{U}_{\theta m}(r, s), \quad 0 \leq r \leq a \\
& \int_0^{+\infty} \left[b_1(\gamma) \hat{\tau}_{1m}(\rho, s) + i \frac{1}{\mu} \hat{\tau}_{2m}(\rho, s) \right] J_{m+1}(\rho r) d\rho \\
& \quad = -\hat{U}_{\theta m}(r, s) - i \hat{U}_{rm}(r, s), \quad 0 \leq r \leq a \\
& \int_0^{+\infty} \left[\hat{\tau}_{1m}(\rho, s) + i \hat{\tau}_{2m}(\rho, s) \right] J_{m-1}(\rho r) \rho d\rho \\
& \quad = 0, \quad a < r < +\infty \\
& \int_0^{+\infty} \left[\hat{\tau}_{1m}(\rho, s) - i \hat{\tau}_{2m}(\rho, s) \right] J_{m+1}(\rho r) \rho d\rho \\
& \quad = 0, \quad a < r < +\infty
\end{aligned} \tag{5.2.27.b}$$

where the integer $m = 0, \pm 1, \pm 2, \dots, \pm \infty$.

The associated non-zero total force and moments acting on the disc inclusion are,

$$\begin{aligned}\hat{P}_z(s) &= -2\pi a \int_0^{+\infty} [\hat{\tau}_{11}(\rho, s) + i\hat{\tau}_{21}(\rho, s) - \hat{\tau}_{1(-1)}(\rho, s) + i\hat{\tau}_{2(-1)}(\rho, s)] J_1(\rho a) d\rho \\ \hat{P}_y(s) &= -2\pi a \int_0^{+\infty} [i\hat{\tau}_{11}(\rho, s) - \hat{\tau}_{21}(\rho, s) + i\hat{\tau}_{1(-1)}(\rho, s) + \hat{\tau}_{2(-1)}(\rho, s)] J_1(\rho a) d\rho \\ \hat{M}_z(s) &= -4\pi a^2 \int_0^{+\infty} \hat{\tau}_{20}(\rho, s) J_2(\rho a) d\rho\end{aligned}\tag{5.2.28.a}$$

and the contact shear stresses are

$$\begin{aligned}\hat{\sigma}_{rz}(r, \theta, 0, s) &= \frac{1}{2} \sum_{m=-\infty}^{m=+\infty} \int_0^{+\infty} \left[[J_{m-1}(\rho r) - J_{m+1}(\rho r)] \hat{\tau}_{1m}(\rho, s) \right. \\ &\quad \left. + i[J_{m-1}(\rho r) + J_{m+1}(\rho r)] \hat{\tau}_{2m}(\rho, s) \right] \rho d\rho e^{im\theta}\end{aligned}\tag{5.2.28.b}$$

$$\begin{aligned}\hat{\sigma}_{\theta z}(r, \theta, 0, s) &= \frac{1}{2} \sum_{m=-\infty}^{m=+\infty} \int_0^{+\infty} \left[i[J_{m-1}(\rho r) + J_{m+1}(\rho r)] \hat{\tau}_{1m}(\rho, s) \right. \\ &\quad \left. - [J_{m-1}(\rho r) - J_{m+1}(\rho r)] \hat{\tau}_{2m}(\rho, s) \right] \rho d\rho e^{im\theta}\end{aligned}$$

5.3 Integral Equations of Rigid Disc Inclusions in Poroelastic Media

5.3.1 General

In the section 5.2, the systems of 2-D Fourier transform based integral equations for a plane inclusion embedded in bonded contact with a saturated poroelastic medium subjected to general direct loads have been formulated and decoupled into two systems of integral equations associated with the vertical and horizontal displacements of inclusion, respectively. For a plane inclusion of circular disc shape, such two systems of 2-D Fourier transform based integral equations are further reduced into infinite sets of Hankel transform based integral equations in Fourier series form. Such infinite sets of 1-D integral equations are correct for the general displacement vector of a rigid or flexible disc inclusion. In particular, the general displacement vector of a rigid disc inclusion can be represented as the following in the Laplace transform domain;

$$\begin{aligned}\hat{U}_z(r, \theta, s) &= \frac{r}{2} \hat{\Omega}_y(s) e^{-i\theta} + \hat{D}_z(s) + \frac{r}{2} \hat{\Omega}_y(s) e^{i\theta} = \hat{D}_z(s) + \hat{\Omega}_y(s) r \cos \theta \\ \hat{U}_r(r, \theta, s) &= \frac{1}{2} \hat{D}_x(s) e^{-i\theta} + \frac{1}{2} \hat{D}_x(s) e^{i\theta} = \hat{D}_x(s) \cos \theta \\ \hat{U}_\theta(r, \theta, s) &= \frac{1}{2i} \hat{D}_x(s) e^{-i\theta} + r \hat{\Omega}_z(s) - \frac{1}{2i} \hat{D}_x(s) e^{i\theta} = -\hat{D}_x(s) \sin \theta + r \hat{\Omega}_z(s)\end{aligned}\tag{5.3.1}$$

where $0 \leq r \leq a$, $0 \leq \theta < 2\pi$, a is the radius of rigid disc, $\hat{D}_x(s)$ and $\hat{D}_z(s)$ are the horizontal and vertical translations at the centre of rigid disc inclusion, and $\hat{\Omega}_y(s)$ and $\hat{\Omega}_z(s)$ are the central and axial rotations about y and z axes of rigid disc inclusion, respectively.

As stated in Chapter 1, these four basic displacement modes ($\hat{D}_x(s)$, $\hat{D}_z(s)$, $\hat{\Omega}_y(s)$, and $\hat{\Omega}_z(s)$) together with the two limiting drainage conditions at the inclusion

surfaces and the two groups of direct loading conditions constitute the sixteen fundamental problems for the time-dependent mechanics of inclusions in poroelastic elastic media saturated with compressible fluids. In this section, we shall present the special sets of integral equations for such fundamental rigid disc inclusion problems by using particular displacement vectors.

5.3.2 Symmetric solutions for a rigid disc inclusion subjected to direct loads

Substituting the rigid displacements (5.3.1) into the infinite sets of Hankel transform based integral equations (5.2.25) ~ (5.2.28), we can obtain the special sets of integral equations for the fundamental problems of a rigid disc inclusion embedded in bonded contact with a poroelastic infinite space and subjected to direct loads. Due to the particular displacements of rigid disc inclusion, we shall first obtain symmetric solutions of the unknown variables.

5.3.2.1 Vertical displacement of a rigid disc inclusion

For the vertical displacement of a *permeable rigid disc* inclusion, the infinite sets of Hankel transform based integral equations (5.2.25.a) can be written as the following separated forms.

For $m = -1$, we have

$$\begin{aligned} \int_0^{+\infty} \frac{1}{\rho} a_1(\gamma) \hat{\tau}_{3(-1)}(\rho, s) J_1(\rho r) \rho d\rho &= -\frac{r}{2} \hat{\Omega}_\mathbf{r}(s), & 0 \leq r \leq a \\ \int_0^{+\infty} \hat{\tau}_{3(-1)}(\rho, s) J_1(\rho r) \rho d\rho &= 0, & a < r < +\infty \end{aligned} \quad (5.3.2.a)$$

for $m = 0$, we have

$$\begin{aligned}
\int_0^{+\infty} a_1(\gamma) \hat{\tau}_{30}(\rho, s) J_0(\rho r) d\rho &= \hat{D}_z(s), & 0 \leq r \leq a \\
\int_0^{+\infty} \hat{\tau}_{30}(\rho, s) J_0(\rho r) \rho d\rho &= 0, & a < r < +\infty
\end{aligned} \tag{5.3.2.b}$$

for $m = 1$, we have

$$\begin{aligned}
\int_0^{+\infty} \frac{1}{\rho} a_1(\gamma) \hat{\tau}_{31}(\rho, s) J_1(\rho r) \rho d\rho &= \frac{r}{2} \hat{\Omega}_y(s), & 0 \leq r \leq a \\
\int_0^{+\infty} \hat{\tau}_{31}(\rho, s) J_1(\rho r) \rho d\rho &= 0, & a < r < +\infty
\end{aligned} \tag{5.3.2.c}$$

and for $m = \pm 2, \pm 3, \dots, \pm \infty$, we have

$$\begin{aligned}
\int_0^{+\infty} a_1(\gamma) \hat{\tau}_{3m}(\rho, s) J_m(\rho r) d\rho &= 0, & 0 \leq r \leq a \\
\int_0^{+\infty} \hat{\tau}_{3m}(\rho, s) J_m(\rho r) \rho d\rho &= 0, & a < r < +\infty
\end{aligned} \tag{5.3.2.d}$$

Considering the sets of integral equations (5.3.2), we obtain the following partial solutions for the unknown variable $\hat{\tau}_3(\rho, \varphi, 0, s)$,

$$\begin{aligned}
\hat{\tau}_{3m}(\rho, s) &= 0, & m &= \pm 2, \pm 3, \dots, \pm \infty \\
\hat{\tau}_{3(-1)}(\rho, s) &= -\hat{\tau}_{31}(\rho, s) \\
\hat{\tau}_3(\rho, \varphi, 0, s) &= \hat{\tau}_{30}(\rho, s) - 2i \sin \varphi \hat{\tau}_{31}(\rho, s)
\end{aligned} \tag{5.3.3}$$

The variables $\hat{\tau}_{30}(\rho, s)$ and $\hat{\tau}_{31}(\rho, s)$ satisfy the dual integral equations (5.3.2.b) and (5.3.2.c), respectively.

For the vertical displacements of an *impermeable rigid disc* inclusion, the infinite sets of Hankel transform based integral equations (5.2.25.b) can be written as the

following forms: for $m = -1$, we have

$$\begin{aligned}
 \int_0^{+\infty} \frac{1}{\rho} \{a_1(\gamma) \hat{\tau}_{3(-1)}(\rho, s) + a_2(\gamma) \hat{p}_{w(-1)}(\rho, s)\} J_1(\rho r) \rho d\rho &= -\frac{r}{2} \hat{\Omega}_y(s), & 0 \leq r \leq a \\
 \int_0^{+\infty} \rho \{a_3(\gamma) \hat{\tau}_{3(-1)}(\rho, s) + a_4(\gamma) \hat{p}_{w(-1)}(\rho, s)\} J_1(\rho r) \rho d\rho &= 0, & 0 \leq r < a \\
 \int_0^{+\infty} \hat{\tau}_{3(-1)}(\rho, s) J_1(\rho r) \rho d\rho &= 0, & a < r < \infty \\
 \int_0^{+\infty} \hat{p}_{w(-1)}(\rho, s) J_1(\rho r) \rho d\rho &= 0, & a < r < \infty
 \end{aligned}
 \tag{5.3.4.a}$$

for $m = 0$, we have

$$\begin{aligned}
 \int_0^{+\infty} \{a_1(\gamma) \hat{\tau}_{30}(\rho, s) + a_2(\gamma) \hat{p}_{w0}(\rho, s)\} J_0(\rho r) d\rho &= \hat{D}_z(s), & 0 \leq r \leq a \\
 \int_0^{+\infty} \{a_3(\gamma) \hat{\tau}_{30}(\rho, s) + a_4(\gamma) \hat{p}_{w0}(\rho, s)\} J_0(\rho r) \rho^2 d\rho &= 0, & 0 \leq r < a \\
 \int_0^{+\infty} \hat{\tau}_{30}(\rho, s) J_0(\rho r) \rho d\rho &= 0, & a < r < \infty \\
 \int_0^{+\infty} \hat{p}_{w0}(\rho, s) J_0(\rho r) \rho d\rho &= 0, & a < r < \infty
 \end{aligned}
 \tag{5.3.4.b}$$

for $m = 1$, we have

$$\begin{aligned}
\int_0^{+\infty} \frac{1}{\rho} \{a_1(\gamma) \hat{\tau}_{31}(\rho, s) + a_2(\gamma) \hat{p}_{w1}(\rho, s)\} J_1(\rho r) \rho d\rho &= \frac{r}{2} \hat{\Omega}_y(s), & 0 \leq r \leq a \\
\int_0^{+\infty} \rho \{a_3(\gamma) \hat{\tau}_{31}(\rho, s) + a_4(\gamma) \hat{p}_{w1}(\rho, s)\} J_1(\rho r) \rho d\rho &= 0, & 0 \leq r < a \\
\int_0^{+\infty} \hat{\tau}_{31}(\rho, s) J_1(\rho r) \rho d\rho &= 0, & a < r < \infty \\
\int_0^{+\infty} \hat{p}_{w1}(\rho, s) J_1(\rho r) \rho d\rho &= 0, & a < r < \infty
\end{aligned} \tag{5.3.4.c}$$

and for $m = \pm 2, \pm 3, \dots, \pm \infty$, we have

$$\begin{aligned}
\int_0^{+\infty} \{a_1(\gamma) \hat{\tau}_{3m}(\rho, s) + a_2(\gamma) \hat{p}_{wm}(\rho, s)\} J_m(\rho r) d\rho &= 0, & 0 \leq r \leq a \\
\int_0^{+\infty} \{a_3(\gamma) \hat{\tau}_{3m}(\rho, s) + a_4(\gamma) \hat{p}_{wm}(\rho, s)\} J_m(\rho r) \rho^2 d\rho &= 0, & 0 \leq r < a \\
\int_0^{+\infty} \hat{\tau}_{3m}(\rho, s) J_m(\rho r) \rho d\rho &= 0, & a < r < \infty \\
\int_0^{+\infty} \hat{p}_{wm}(\rho, s) J_m(\rho r) \rho d\rho &= 0, & a < r < \infty
\end{aligned} \tag{5.3.4.d}$$

Considering the sets of integral equations (5.3.4), we obtain the following partial solutions for the two unknown variables $\hat{\tau}_3(\rho, \varphi, 0, s)$, $\hat{p}_w(\rho, \varphi, 0, s)$,

$$\begin{aligned}
\hat{\tau}_{3(-1)}(\rho, s) &= -\hat{\tau}_{31}(\rho, s) \\
\hat{p}_{w(-1)}(\rho, s) &= -\hat{p}_{w1}(\rho, s) \\
\hat{\tau}_{3m}(\rho, s) &= 0, & m = \pm 2, \pm 3, \dots, \pm \infty \\
\hat{p}_{wm}(\rho, s) &= 0, & m = \pm 2, \pm 3, \dots, \pm \infty \\
\hat{\tau}_3(\rho, \varphi, 0, s) &= \hat{\tau}_{30}(\rho, s) - 2i \sin \varphi \hat{\tau}_{31}(\rho, s) \\
\hat{p}_w(\rho, \varphi, 0, s) &= \hat{p}_{w0}(\rho, s) - 2i \sin \varphi \hat{p}_{w1}(\rho, s)
\end{aligned} \tag{5.3.5}$$

where the unknown variables $\hat{\tau}_{30}(\rho, s)$, $\hat{p}_{w0}(\rho, s)$ and $\hat{\tau}_{31}(\rho, s)$, $\hat{p}_{w1}(\rho, s)$ satisfy the sets of integral equations (5.3.4.b) and (5.3.4.c), respectively.

5.3.2.2 Horizontal displacement of a rigid disc inclusion

For the horizontal displacement of a *permeable rigid disc* inclusion, the infinite sets of Hankel transform based integral equations (5.2.27.a) can be written as the following forms: for $m = -1$, we have

$$\begin{aligned}
 & \int_0^{+\infty} \left\{ b_1(\gamma) \hat{\tau}_{1(-1)}(\rho, s) + \frac{i}{\mu} \hat{\tau}_{2(-1)}(\rho, s) + b_2(\gamma) \hat{\vartheta}_{3(-1)}(\rho, s) \right\} J_0(\rho r) d\rho \\
 & \qquad \qquad \qquad = -\hat{D}_x(s), \quad 0 \leq r \leq a \\
 & \int_0^{+\infty} \left\{ b_1(\gamma) \hat{\tau}_{1(-1)}(\rho, s) - \frac{i}{\mu} \hat{\tau}_{2(-1)}(\rho, s) + b_2(\gamma) \hat{\vartheta}_{3(-1)}(\rho, s) \right\} J_2(\rho r) d\rho \\
 & \qquad \qquad \qquad = 0, \quad 0 \leq r \leq a \\
 & \int_0^{+\infty} \left\{ b_3(\gamma) \hat{\tau}_{1(-1)}(\rho, s) + b_4(\gamma) \hat{\vartheta}_{3(-1)}(\rho, s) \right\} J_1(\rho r) \rho d\rho \\
 & \qquad \qquad \qquad = 0, \quad 0 \leq r < a \quad (5.3.6.a) \\
 & \int_0^{+\infty} \left\{ \hat{\tau}_{1(-1)}(\rho, s) - i \hat{\tau}_{2(-1)}(\rho, s) \right\} J_0(\rho r) \rho d\rho \\
 & \qquad \qquad \qquad = 0, \quad a < r < \infty \\
 & \int_0^{+\infty} \left\{ \hat{\tau}_{1(-1)}(\rho, s) + i \hat{\tau}_{2(-1)}(\rho, s) \right\} J_2(\rho r) \rho d\rho \\
 & \qquad \qquad \qquad = 0, \quad a < r < \infty \\
 & \int_0^{+\infty} \hat{\vartheta}_{3(-1)}(\rho, s) J_1(\rho r) \rho^2 d\rho = 0, \quad a < r < \infty
 \end{aligned}$$

for $m = 0$, we have

$$\begin{aligned}
& \int_0^{+\infty} \{b_1(\gamma)\hat{\tau}_{10}(\rho, s) + b_2(\gamma)\hat{\vartheta}_{30}(\rho, s)\} J_1(\rho r) d\rho = 0, & 0 \leq r \leq a \\
& \int_0^{+\infty} \{b_3(\gamma)\hat{\tau}_{10}(\rho, s) + b_4(\gamma)\hat{\vartheta}_{30}(\rho, s)\} J_0(\rho r) \rho d\rho = 0, & 0 \leq r < a \\
& \int_0^{+\infty} \hat{\tau}_{10}(\rho, s) J_1(\rho r) \rho d\rho = 0, & a < r < \infty \\
& \int_0^{+\infty} \hat{\vartheta}_{30}(\rho, s) J_0(\rho r) \rho^2 d\rho = 0, & a < r < \infty
\end{aligned} \tag{5.3.6.b}$$

and

$$\begin{aligned}
& \int_0^{+\infty} \hat{\tau}_{20}(\rho, s) J_1(\rho r) d\rho = -\mu r \hat{\Omega}_r(s), & 0 \leq r \leq a \\
& \int_0^{+\infty} \hat{\tau}_{20}(\rho, s) J_1(\rho r) \rho d\rho = 0, & a < r < \infty
\end{aligned} \tag{5.3.6.c}$$

for $m = 1$, we have

$$\begin{aligned}
& \int_0^{+\infty} \left\{ b_1(\gamma)\hat{\tau}_{11}(\rho, s) - \frac{i}{\mu}\hat{\tau}_{21}(\rho, s) + b_2(\gamma)\hat{\vartheta}_{31}(\rho, s) \right\} J_0(\rho r) d\rho = \hat{D}_r(s), & 0 \leq r \leq a \\
& \int_0^{+\infty} \left\{ b_1(\gamma)\hat{\tau}_{11}(\rho, s) + \frac{i}{\mu}\hat{\tau}_{21}(\rho, s) + b_2(\gamma)\hat{\vartheta}_{31}(\rho, s) \right\} J_2(\rho r) d\rho = 0, & 0 \leq r \leq a \\
& \int_0^{+\infty} \{b_3(\gamma)\hat{\tau}_{11}(\rho, s) + b_4(\gamma)\hat{\vartheta}_{31}(\rho, s)\} J_1(\rho r) \rho d\rho = 0, & 0 \leq r < a \\
& \int_0^{+\infty} \{ \hat{\tau}_{11}(\rho, s) + i\hat{\tau}_{21}(\rho, s) \} J_0(\rho r) \rho d\rho = 0, & a < r < \infty \\
& \int_0^{+\infty} \{ \hat{\tau}_{11}(\rho, s) - i\hat{\tau}_{21}(\rho, s) \} J_2(\rho r) \rho d\rho = 0, & a < r < \infty \\
& \int_0^{+\infty} \hat{\vartheta}_{31}(\rho, s) J_1(\rho r) \rho^2 d\rho = 0, & a < r < \infty
\end{aligned} \tag{5.3.6.d}$$

and for $m = \pm 2, \pm 3, \dots, \pm \infty$, we have

$$\begin{aligned}
 \int_0^{+\infty} \left\{ b_1(\gamma) \hat{\tau}_{1m}(\rho, s) - \frac{i}{\mu} \hat{\tau}_{2m}(\rho, s) + b_2(\gamma) \hat{\vartheta}_{3m}(\rho, s) \right\} J_{m-1}(\rho r) d\rho &= 0, \quad 0 \leq r \leq a \\
 \int_0^{+\infty} \left\{ b_1(\gamma) \hat{\tau}_{1m}(\rho, s) + \frac{i}{\mu} \hat{\tau}_{2m}(\rho, s) + b_2(\gamma) \hat{\vartheta}_{3m}(\rho, s) \right\} J_{m+1}(\rho r) d\rho &= 0, \quad 0 \leq r \leq a \\
 \int_0^{+\infty} \left\{ b_3(\gamma) \hat{\tau}_{1m}(\rho, s) + b_4(\gamma) \hat{\vartheta}_{3m}(\rho, s) \right\} J_m(\rho r) \rho d\rho &= 0, \quad 0 \leq r < a \\
 \int_0^{+\infty} \left\{ \hat{\tau}_{1m}(\rho, s) + i \hat{\tau}_{2m}(\rho, s) \right\} J_{m-1}(\rho r) \rho d\rho &= 0, \quad a < r < \infty \\
 \int_0^{+\infty} \left\{ \hat{\tau}_{1m}(\rho, s) - i \hat{\tau}_{2m}(\rho, s) \right\} J_{m+1}(\rho r) \rho d\rho &= 0, \quad a < r < \infty \\
 \int_0^{+\infty} \hat{\vartheta}_{3m}(\rho, s) J_m(\rho r) \rho^2 d\rho &= 0, \quad a < r < \infty
 \end{aligned} \tag{5.3.6.e}$$

Considering the sets of integral equations (5.3.6), we obtain the following partial solutions for the three unknown variables $\hat{\tau}_1(\rho, \varphi, 0, s)$, $\hat{\tau}_2(\rho, \varphi, 0, s)$, $\hat{\vartheta}_3(\rho, \varphi, 0, s)$,

$$\begin{aligned}
 \hat{\tau}_{1(-1)}(\rho, s) &= -\hat{\tau}_{11}(\rho, s) \\
 \hat{\tau}_{2(-1)}(\rho, s) &= \hat{\tau}_{21}(\rho, s) \\
 \hat{\vartheta}_{3(-1)}(\rho, s) &= -\hat{\vartheta}_{31}(\rho, s) \\
 \hat{\tau}_{10}(\rho, s) &= 0 \\
 \hat{\vartheta}_{30}(\rho, s) &= 0 \\
 \hat{\tau}_{1m}(\rho, s) &= 0, \quad m = \pm 2, \pm 3, \dots, \pm \infty \\
 \hat{\tau}_{2m}(\rho, s) &= 0, \quad m = \pm 2, \pm 3, \dots, \pm \infty \\
 \hat{\vartheta}_{3m}(\rho, s) &= 0, \quad m = \pm 2, \pm 3, \dots, \pm \infty \\
 \hat{\tau}_1(\rho, \varphi, 0, s) &= -2i \sin \varphi \hat{\tau}_{11}(\rho, s) \\
 \hat{\tau}_2(\rho, \varphi, 0, s) &= \hat{\tau}_{20}(\rho, s) + 2 \cos \varphi \hat{\tau}_{21}(\rho, s) \\
 \hat{\vartheta}_3(\rho, \varphi, 0, s) &= -2i \sin \varphi \hat{\vartheta}_{31}(\rho, s)
 \end{aligned} \tag{5.3.7}$$

where the unknown variables $\hat{\tau}_{20}(\rho, s)$ and $\hat{\tau}_{11}(\rho, s)$, $\hat{\tau}_{21}(\rho, s)$, $\hat{\nu}_{31}(\rho, s)$ satisfy the sets of integral equations (5.3.6.c) and (5.3.6.d), respectively.

For the horizontal displacement of an *impermeable rigid disc* inclusion, the infinite sets of Hankel transform based integral equations (5.2.27.b) can be written as the following forms: for $m = -1$, we have

$$\begin{aligned}
 & \int_0^{+\infty} \left\{ b_1(\gamma) \hat{\tau}_{1(-1)}(\rho, s) + \frac{i}{\mu} \hat{\tau}_{2(-1)}(\rho, s) \right\} J_0(\rho r) d\rho = -\hat{D}_r(s), \quad 0 \leq r \leq a \\
 & \int_0^{+\infty} \left\{ b_1(\gamma) \hat{\tau}_{1(-1)}(\rho, s) - \frac{i}{\mu} \hat{\tau}_{2(-1)}(\rho, s) \right\} J_2(\rho r) d\rho = 0, \quad 0 \leq r \leq a \\
 & \int_0^{+\infty} \left\{ \hat{\tau}_{1(-1)}(\rho, s) - i \hat{\tau}_{2(-1)}(\rho, s) \right\} J_0(\rho r) \rho d\rho = 0, \quad a < r < \infty \\
 & \int_0^{+\infty} \left\{ \hat{\tau}_{1(-1)}(\rho, s) + i \hat{\tau}_{2(-1)}(\rho, s) \right\} J_2(\rho r) \rho d\rho = 0, \quad a < r < \infty
 \end{aligned} \tag{5.3.8.a}$$

for $m = 0$, we have

$$\begin{aligned}
 & \int_0^{+\infty} b_1(\gamma) \hat{\tau}_{10}(\rho, s) J_1(\rho r) d\rho = 0, \quad 0 \leq r \leq a \\
 & \int_0^{+\infty} \hat{\tau}_{10}(\rho, s) J_1(\rho r) \rho d\rho = 0, \quad a < r < \infty
 \end{aligned} \tag{5.3.8.b}$$

and

$$\begin{aligned}
 & \int_0^{+\infty} \hat{\tau}_{20}(\rho, s) J_1(\rho r) d\rho = -\mu r \hat{\Omega}_z(s), \quad 0 \leq r \leq a \\
 & \int_0^{+\infty} \hat{\tau}_{20}(\rho, s) J_1(\rho r) \rho d\rho = 0, \quad a < r < \infty
 \end{aligned} \tag{5.3.8.c}$$

for $m = 1$, we have

$$\begin{aligned}
& \int_0^{+\infty} \left\{ b_1(\gamma) \hat{\tau}_{11}(\rho, s) - \frac{i}{\mu} \hat{\tau}_{21}(\rho, s) \right\} J_0(\rho r) d\rho = \hat{D}_r(s), \quad 0 \leq r \leq a \\
& \int_0^{+\infty} \left\{ b_1(\gamma) \hat{\tau}_{11}(\rho, s) + \frac{i}{\mu} \hat{\tau}_{21}(\rho, s) \right\} J_2(\rho r) d\rho = 0, \quad 0 \leq r \leq a \\
& \int_0^{+\infty} \left\{ \hat{\tau}_{11}(\rho, s) + i \hat{\tau}_{21}(\rho, s) \right\} J_0(\rho r) \rho d\rho = 0, \quad a < r < \infty \\
& \int_0^{+\infty} \left\{ \hat{\tau}_{11}(\rho, s) - i \hat{\tau}_{21}(\rho, s) \right\} J_2(\rho r) \rho d\rho = 0, \quad a < r < \infty
\end{aligned} \tag{5.3.8.d}$$

for $m = \pm 2, \pm 3, \dots, \pm \infty$, we have

$$\begin{aligned}
& \int_0^{+\infty} \left\{ b_1(\gamma) \hat{\tau}_{1m}(\rho, s) - \frac{i}{\mu} \hat{\tau}_{2m}(\rho, s) \right\} J_{m-1}(\rho r) d\rho = 0, \quad 0 \leq r \leq a \\
& \int_0^{+\infty} \left\{ b_1(\gamma) \hat{\tau}_{1m}(\rho, s) + \frac{i}{\mu} \hat{\tau}_{2m}(\rho, s) \right\} J_{m+1}(\rho r) d\rho = 0, \quad 0 \leq r \leq a \\
& \int_0^{+\infty} \left\{ \hat{\tau}_{1m}(\rho, s) + i \hat{\tau}_{2m}(\rho, s) \right\} J_{m-1}(\rho r) \rho d\rho = 0, \quad a < r < \infty \\
& \int_0^{+\infty} \left\{ \hat{\tau}_{1m}(\rho, s) - i \hat{\tau}_{2m}(\rho, s) \right\} J_{m+1}(\rho r) \rho d\rho = 0, \quad a < r < \infty
\end{aligned} \tag{5.3.8.e}$$

Considering the sets of integral equations (5.3.8), we obtain the following partial solutions for the two unknown variables $\hat{\tau}_1(\rho, \varphi, 0, s)$ and $\hat{\tau}_2(\rho, \varphi, 0, s)$,

$$\begin{aligned}
& \hat{\tau}_{1(-1)}(\rho, s) = -\hat{\tau}_{11}(\rho, s) \\
& \hat{\tau}_{2(-1)}(\rho, s) = \hat{\tau}_{21}(\rho, s) \\
& \hat{\tau}_{10}(\rho, s) = 0 \\
& \hat{\tau}_{1m}(\rho, s) = 0, \quad m = \pm 2, \pm 3, \dots, \pm \infty \\
& \hat{\tau}_{2m}(\rho, s) = 0, \quad m = \pm 2, \pm 3, \dots, \pm \infty \\
& \hat{\tau}_1(\rho, \varphi, 0, s) = -2i \sin \varphi \hat{\tau}_{11}(\rho, s) \\
& \hat{\tau}_2(\rho, \varphi, 0, s) = \hat{\tau}_{20}(\rho, s) + 2 \cos \varphi \hat{\tau}_{21}(\rho, s)
\end{aligned} \tag{5.3.9}$$

where the unknown variables $\hat{r}_{20}(\rho, s)$ and $\hat{r}_{11}(\rho, s)$, $\hat{r}_{21}(\rho, s)$, satisfy the sets of integral equations (5.3.8.c) and (5.3.8.d), respectively.

5.3.3 Sets of integral equations for fundamental rigid disc inclusion problems

Using the above symmetric solutions, we can develop the special sets of Hankel transform based integral equations for the fundamental problems of a rigid disc inclusion subjected to direct loads and simplify the resulting expressions for the total forces and moments acting on the disc inclusion.

5.3.3.1 Vertical displacement of a rigid disc inclusion

The special sets of Hankel transform based integral equations in the Laplace transform domain for the vertical displacement of a *permeable rigid disc* inclusion embedded in bonded contact with a saturated poroelastic medium are

$$\begin{aligned} \int_0^{+\infty} a_1(\gamma) \hat{r}_{30}(\rho, s) J_0(\rho r) d\rho &= \hat{D}_z(s), & 0 \leq r \leq a \\ \int_0^{+\infty} \hat{r}_{30}(\rho, s) J_0(\rho r) \rho d\rho &= 0, & a < r < +\infty \end{aligned} \quad (5.3.10.a)$$

and

$$\begin{aligned} \int_0^{+\infty} \frac{1}{\rho} a_1(\gamma) \hat{r}_{31}(\rho, s) J_1(\rho r) \rho d\rho &= \frac{r}{2} \hat{\Omega}_y(s), & 0 \leq r \leq a \\ \int_0^{+\infty} \hat{r}_{31}(\rho, s) J_1(\rho r) \rho d\rho &= 0, & a < r < +\infty \end{aligned} \quad (5.3.10.b)$$

The corresponding sets of Hankel transform based integral equations in the Laplace transform domain for the vertical displacement of an *impermeable rigid disc* inclusion embedded in bonded contact with a saturated poroelastic medium are

$$\begin{aligned}
\int_0^{+\infty} \{a_1(\gamma)\hat{\tau}_{30}(\rho, s) + a_2(\gamma)\hat{p}_{w0}(\rho, s)\} J_0(\rho r) d\rho &= \hat{D}_z(s), & 0 \leq r \leq a \\
\int_0^{+\infty} \{a_3(\gamma)\hat{\tau}_{30}(\rho, s) + a_4(\gamma)\hat{p}_{w0}(\rho, s)\} J_0(\rho r) \rho^2 d\rho &= 0, & 0 \leq r < a \\
\int_0^{+\infty} \hat{\tau}_{30}(\rho, s) J_0(\rho r) \rho d\rho &= 0, & a < r < \infty \\
\int_0^{+\infty} \hat{p}_{w0}(\rho, s) J_0(\rho r) \rho d\rho &= 0, & a < r < \infty
\end{aligned} \tag{5.3.11.a}$$

and

$$\begin{aligned}
\int_0^{+\infty} \frac{1}{\rho} \{a_1(\gamma)\hat{\tau}_{31}(\rho, s) + a_2(\gamma)\hat{p}_{w1}(\rho, s)\} J_1(\rho r) \rho d\rho &= \frac{r}{2} \hat{\Omega}_y(s), & 0 \leq r \leq a \\
\int_0^{+\infty} \rho \{a_3(\gamma)\hat{\tau}_{31}(\rho, s) + a_4(\gamma)\hat{p}_{w1}(\rho, s)\} J_1(\rho r) \rho d\rho &= 0, & 0 \leq r < a \\
\int_0^{+\infty} \hat{\tau}_{31}(\rho, s) J_1(\rho r) \rho d\rho &= 0, & a < r < \infty \\
\int_0^{+\infty} \hat{p}_{w1}(\rho, s) J_1(\rho r) \rho d\rho &= 0, & a < r < \infty
\end{aligned} \tag{5.3.11.b}$$

For both *permeable and impermeable* rigid disc inclusion problems, we can express the associated total force and moments acting on the inclusion in the following forms:

$$\begin{aligned}
\hat{P}_z(s) &= -4\pi a \int_0^{+\infty} \hat{\tau}_{30}(\rho, s) J_1(\rho a) d\rho \\
\hat{M}_x(s) &= 0 \\
\hat{M}_y(s) &= -4\pi a^2 \int_0^{+\infty} \hat{\tau}_{31}(\rho, s) J_2(\rho a) d\rho
\end{aligned} \tag{5.3.12.a}$$

and the contact normal stress is

$$\hat{\sigma}_{zz}(r, \theta, 0, s) = \int_0^{+\infty} \hat{\tau}_{30}(\rho, s) J_0(\rho r) \rho d\rho + 2 \cos \theta \int_0^{+\infty} \hat{\tau}_{31}(\rho, s) J_1(\rho r) \rho d\rho \tag{5.3.12.b}$$

The equations (5.3.10) ~ (5.3.12) indicate that the fundamental problems for the vertical displacement of a rigid disc inclusion are fully decoupled into an axial translational problem and a central rotational problem of the rigid disc inclusion. In other words, there is only one fundamental loading mode P_z (or M_y) corresponding to the basic displacement mode D_z (or Ω_y).

5.3.3.2 Horizontal displacement of a rigid disc inclusion

Introducing the following variables $\hat{N}(\rho, s)$ and $\hat{M}(\rho, s)$ such that

$$\begin{aligned}\hat{\tau}_{11}(\rho, s) &= \frac{1}{2}[\hat{N}(\rho, s) + \hat{M}(\rho, s)] \\ i\hat{\tau}_{21}(\rho, s) &= \frac{1}{2}[\hat{N}(\rho, s) - \hat{M}(\rho, s)] \\ \hat{N}(\rho, s) &= \hat{\tau}_{11}(\rho, s) + i\hat{\tau}_{21}(\rho, s) \\ \hat{M}(\rho, s) &= \hat{\tau}_{11}(\rho, s) - i\hat{\tau}_{21}(\rho, s)\end{aligned}\tag{5.3.13}$$

we can further simplify the special sets of integral equations associated with the horizontal displacement of a rigid disc inclusion.

The special sets of Hankel transform based integral equations in the Laplace transform domain for the horizontal displacement of a *permeable rigid disc* inclusion embedded in bonded contact with a saturated poroelastic medium are

$$\begin{aligned}
& \int_0^{+\infty} \left[b_6(\gamma) \hat{N}(\rho, s) + b_6(\gamma) \hat{M}(\rho, s) + b_2(\gamma) \hat{v}_{31}(\rho, s) \right] J_0(\rho r) d\rho \\
& \qquad \qquad \qquad = \hat{D}_z(s), \quad 0 \leq r \leq a \\
& \int_0^{+\infty} \left[b_6(\gamma) \hat{N}(\rho, s) + b_5(\gamma) \hat{M}(\rho, s) + b_2(\gamma) \hat{v}_{31}(\rho, s) \right] J_2(\rho r) d\rho \\
& \qquad \qquad \qquad = 0, \quad 0 \leq r \leq a \\
& \int_0^{+\infty} \left[\frac{1}{2} b_3(\gamma) \hat{N}(\rho, s) + \frac{1}{2} b_3(\gamma) \hat{M}(\rho, s) + b_4(\gamma) \hat{v}_{31}(\rho, s) \right] J_1(\rho r) \rho d\rho \\
& \qquad \qquad \qquad = 0, \quad 0 \leq r < a \\
& \int_0^{+\infty} \hat{N}(\rho, s) J_0(\rho r) \rho d\rho = 0, \quad a < r < \infty \\
& \int_0^{+\infty} \hat{M}(\rho, s) J_2(\rho r) \rho d\rho = 0, \quad a < r < \infty \\
& \int_0^{+\infty} \hat{v}_{31}(\rho, s) J_1(\rho r) \rho^2 d\rho = 0, \quad a < r < \infty
\end{aligned} \tag{5.3.14.a}$$

and

$$\begin{aligned}
& \int_0^{+\infty} \frac{1}{\rho} \hat{\tau}_{20}(\rho, s) J_1(\rho r) \rho d\rho = -\mu r \hat{\Omega}_z(s), \quad 0 \leq r \leq a \\
& \int_0^{+\infty} \hat{\tau}_{20}(\rho, s) J_1(\rho r) \rho d\rho = 0, \quad a < r < \infty
\end{aligned} \tag{5.3.14.b}$$

The corresponding sets of Hankel transform based integral equations in the Laplace transform domain for the horizontal displacement of an *impermeable rigid disc* inclusion embedded in bonded contact with a saturated poroelastic medium are

$$\begin{aligned}
\int_0^{+\infty} [b_5(\gamma)\hat{N}(\rho, s) + b_6(\gamma)\hat{M}(\rho, s)]J_0(\rho r)d\rho &= \hat{D}_r(s), & 0 \leq r \leq a \\
\int_0^{+\infty} [b_6(\gamma)\hat{N}(\rho, s) + b_5(\gamma)\hat{M}(\rho, s)]J_2(\rho r)d\rho &= 0, & 0 \leq r \leq a \\
\int_0^{+\infty} \hat{N}(\rho, s)J_0(\rho r)\rho d\rho &= 0, & a < r < \infty \\
\int_0^{+\infty} \hat{M}(\rho, s)J_2(\rho r)\rho d\rho &= 0, & a < r < \infty
\end{aligned} \tag{5.3.15.a}$$

and

$$\begin{aligned}
\int_0^{+\infty} \frac{1}{\rho} \hat{\tau}_{20}(\rho, s)J_1(\rho r)\rho d\rho &= -\mu r \hat{\Omega}_z(s), & 0 \leq r \leq a \\
\int_0^{+\infty} \hat{\tau}_{20}(\rho, s)J_1(\rho r)\rho d\rho &= 0, & a < r < \infty
\end{aligned} \tag{5.3.15.b}$$

For both *permeable and impermeable* rigid disc inclusion problems, the associated forces and moment acting on the inclusion can be expressed as the follow forms:

$$\begin{aligned}
\hat{P}_x(s) &= -4\pi a \int_0^{+\infty} \hat{N}(\rho, s)J_1(\rho a)d\rho \\
\hat{P}_y(s) &= 0 \\
\hat{M}_z(s) &= -4\pi a^2 \int_0^{+\infty} \hat{\tau}_{20}(\rho, s)J_2(\rho a)d\rho
\end{aligned} \tag{5.3.16.a}$$

and the associated shear stresses at the interface plane of rigid disc inclusion are,

$$\begin{aligned}
\hat{\sigma}_{rz}(r, \theta, 0, s) &= \cos \theta \left[\int_0^{+\infty} \hat{N}(\rho, s)J_0(\rho r)\rho d\rho - \int_0^{+\infty} \hat{M}(\rho, s)J_2(\rho r)\rho d\rho \right] \\
\hat{\sigma}_{\theta z}(r, \theta, 0, s) &= -\sin \theta \left[\int_0^{+\infty} \hat{N}(\rho, s)J_0(\rho r)\rho d\rho + \int_0^{+\infty} \hat{M}(\rho, s)J_2(\rho r)\rho d\rho \right] \\
&\quad + \int_0^{+\infty} \hat{\tau}_{20}(\rho, s)J_1(\rho r)\rho d\rho
\end{aligned} \tag{5.3.16.b}$$

Equations (5.3.14) ~ (5.3.16) indicate that the fundamental problems for the horizontal displacement of a rigid disc inclusion are also fully decoupled into an in-plane translational problem and an axial rotational problem of the rigid disc inclusion. In other words, there is only one fundamental loading mode P_z (or M_z) responsible for the basic displacement mode D_z (or Ω_z). Furthermore, the set of integral equations for the axial rotational problem is same for both permeable and impermeable rigid disc inclusions.

Prior to reducing the special sets of Hankel transform based integral equations into the systems of Fredholm integral equations of the second kind, we make an examination on the special sets of the integral equations under the three limiting conditions which pertain to the initial ($t \rightarrow 0^+$), the final ($t \rightarrow +\infty$) responses as well as the limiting response as $\nu \rightarrow \nu_u$. It is shown that the limiting integral equations are exactly same as those obtained from the linear elastic analysis. We document the derivation and results in the Appendix C.

5.4 Systems of Fredholm Integral Equations of the Second Kind for Rigid Disc Inclusions in Saturated Poroelastic Media

5.4.1 General

The special sets of Hankel transform based integral equations in the Laplace transform domain have been systematically formulated for the mechanics of a rigid disc inclusion embedded in bonded contact with a poroelastic medium of infinite space extent saturated with a compressible fluid. From such integral equations, it is found that the time-dependent behaviour of the fundamental inclusion problems decouple into the symmetric problems associated with the four basic displacement and loading modes for both permeable and impermeable inclusions. However, the sets of Hankel transform based integral equations are singular integral equations with unknown singularities¹ (see eg. Atkinson, 1979). Such singularities require

¹ As stated in Literature Review, Small and Booker (1987) numerically evaluated this

detailed consideration in the numerical evaluation of the sets of integral equations. Further simplifications and reductions have to be made to eliminate and separate the unknown singularities from the integral equations. It is found that all the finite sets of Hankel transform based integral equations can be reduced into the systems of Fredholm integral equations of the second kind in the Laplace transform domain. Such systems of complex Fredholm integral equations of the second kind are standard and regular integral equations and can be numerical evaluated with many numerical techniques if their closed-form solution is not possible (see eg., Sneddon and Lowengrub, 1969; Kanwal, 1971; Atkinson, 1976; Baker, 1977; Selvadurai, 1989a; b).

In this section, we shall present the systems of Fredholm integral equations of the second kind for the fundamental problems of a rigid disc inclusion in a poroelastic medium saturated with compressible fluid. Prior to deriving the systems of complex Fredholm integral equations of the second kind, it is of importance to record the formulae below (see eg. Watson, 1962; Tranter, 1968; Sneddon, 1972; Gradshteyn and Ryzhik, 1980).

The first formula is

$$J_q(\rho_1 r_1) = \frac{2^{1-g} \rho_1^g}{\Gamma(g) r_1^g} \int_0^{r_1} \frac{x_1^{q-g+1}}{(r_1^2 - x_1^2)^{1-g}} J_{q-g}(\rho_1 x_1) dx_1 \quad (5.4.1.a)$$

where $Re(g) > 0$, $Re(q - g) > -1$.

The second is

$$\int_0^\infty \rho_1^{q-g+1} J_g(\rho_1 r_1) J_q(\rho_1 x_1) d\rho_1 = \frac{2^{q+1-g}}{\Gamma(g-q)} \frac{x_1^q r_1^{-g}}{(r_1^2 - x_1^2)^{q+1-g}} H(r_1 - x_1) \quad (5.4.1.b)$$

where $Re(g) > Re(q) > -1$ with $r_1 \neq x_1$ or $Re(g) > Re(q + 1) > 0$ with $r_1 = x_1$, and $H(r_1 - x_1)$ is the Heaviside step function.

type of singular integral equations for the axisymmetric translation of a rigid disc inclusion by assuming the unknown variables have the singularity as same as those in the classical elastostatics.

The third is the pair of integral equations of Abel type,

$$\begin{aligned} g(y_1) &= \sqrt{\frac{2}{\pi}} \int_0^{y_1} \frac{f(x_1)}{\sqrt{y_1^2 - x_1^2}} dx_1 \\ f(x_1) &= \sqrt{\frac{2}{\pi}} \frac{d}{dx_1} \int_0^{x_1} \frac{y_1 g(y_1)}{\sqrt{x_1^2 - y_1^2}} dy_1 \end{aligned} \quad (5.4.1.c)$$

We also note the particular formulae

$$\begin{aligned} J_1(\rho_1 r_1) &= \sqrt{\frac{2}{\pi}} \frac{\sqrt{\rho_1}}{r_1} \int_0^{r_1} \frac{x_1^{\frac{3}{2}}}{\sqrt{r_1^2 - x_1^2}} J_{\frac{1}{2}}(\rho_1 x_1) dx_1 \\ J_2(\rho_1 r_1) &= \sqrt{\frac{2}{\pi}} \frac{\sqrt{\rho_1}}{r_1^2} \int_0^{r_1} \frac{x_1^{\frac{5}{2}}}{\sqrt{r_1^2 - x_1^2}} J_{\frac{3}{2}}(\rho_1 x_1) dx_1 \\ \int_0^\infty \sqrt{\rho_1} J_1(\rho_1 r_1) J_{\frac{1}{2}}(\rho_1 x_1) d\rho_1 &= \sqrt{\frac{2}{\pi}} \frac{\sqrt{x_1}}{r_1 \sqrt{r_1^2 - x_1^2}} H(r_1 - x_1) \\ \int_0^\infty \sqrt{\rho_1} J_2(\rho_1 r_1) J_{\frac{3}{2}}(\rho_1 x_1) d\rho_1 &= \sqrt{\frac{2}{\pi}} \frac{x_1^{\frac{3}{2}}}{r_1^2 \sqrt{r_1^2 - x_1^2}} H(r_1 - x_1) \end{aligned} \quad (5.4.1.d)$$

and integrals,

$$\begin{aligned} \int_0^{x_1} \frac{r_1}{\sqrt{x_1^2 - r_1^2}} dr_1 &= x_1 \\ \frac{\partial}{\partial x_1} \int_0^{x_1} \frac{r_1}{\sqrt{x_1^2 - r_1^2}} dr_1 &= 1 \\ \frac{\partial}{\partial x_1} \int_0^{x_1} \frac{r_1^3}{\sqrt{x_1^2 - r_1^2}} dr_1 &= 2x_1^2 \\ \frac{\partial}{\partial x_1} \int_0^{x_1} \frac{r_1^3 J_2(\rho_1 r_1)}{\sqrt{x_1^2 - r_1^2}} dr_1 &= \sqrt{\frac{\pi}{2}} \sqrt{\rho_1} x_1^{\frac{3}{2}} J_{\frac{3}{2}}(\rho_1 x_1) \\ \frac{\partial}{\partial x_1} \int_0^{x_1} \frac{r_1^2 J_1(\rho_1 r_1)}{\sqrt{x_1^2 - r_1^2}} dr_1 &= x_1 \sin(\rho_1 x_1) \\ \int_0^{x_1} \frac{r_1 J_0(\rho_1 r_1)}{\sqrt{x_1^2 - r_1^2}} dr_1 &= \frac{\sin(\rho_1 x_1)}{\rho_1} \\ \frac{\partial}{\partial x_1} \int_0^{x_1} \frac{r_1 J_0(\rho_1 r_1)}{\sqrt{x_1^2 - r_1^2}} dr_1 &= \cos(\rho_1 x_1) \\ \int_0^{x_1} \frac{r_1^2 J_1(\rho_1 r_1)}{\sqrt{x_1^2 - r_1^2}} dr_1 &= \sqrt{\frac{\pi}{2}} x_1 \sqrt{\frac{x_1}{\rho_1}} J_{\frac{3}{2}}(\rho_1 x_1) \end{aligned} \quad (5.4.1.e)$$

5.4.2 Fredholm integral equations for the axial translation of a rigid disc inclusion

Using the functions $k_1(\gamma)$ and $k_2(\gamma)$, we can re-write the sets of integral equations (5.3.2.a) and (5.3.3.a) for the axial translation of a *permeable* or an *impermeable* rigid disc inclusion embedded in bonded contact with a poroelastic infinite medium: (i) for the *permeable* inclusion problem, we have

$$\begin{aligned} \int_0^{+\infty} [1 + \alpha_2 k_1(\gamma)] \hat{\tau}_{30}(\rho, s) J_0(\rho r) d\rho &= \alpha_a \hat{D}_z(s), & 0 \leq r \leq a \\ \int_0^{+\infty} \hat{\tau}_{30}(\rho, s) J_0(\rho r) \rho d\rho &= 0, & a < r < \infty \end{aligned} \quad (5.4.2.a)$$

and (ii) for the *impermeable* inclusion problem, we have

$$\begin{aligned} \int_0^{+\infty} \left\{ [1 + \alpha_2 k_1(\gamma)] \hat{\tau}_{30}(\rho, s) + \frac{\alpha_3}{B} [1 + k_1(\gamma)] \hat{p}_{w0}(\rho, s) \right\} J_0(\rho r) d\rho \\ = \alpha_a \hat{D}_z(s), & 0 \leq r \leq a \\ \int_0^{+\infty} \left\{ \alpha_4 k_2(\gamma) \hat{\tau}_{30}(\rho, s) + \frac{1}{B} [1 + k_2(\gamma)] \hat{p}_{w0}(\rho, s) \right\} J_0(\rho r) \rho^2 d\rho \\ = 0, & 0 \leq r < a \\ \int_0^{+\infty} \hat{\tau}_{30}(\rho, s) J_0(\rho r) \rho d\rho &= 0, & a < r < \infty \\ \int_0^{+\infty} \hat{p}_{w0}(\rho, s) J_0(\rho r) \rho d\rho &= 0, & a < r < \infty \end{aligned} \quad (5.4.2.b)$$

The last two equations in (5.4.2.b) are satisfied automatically by the following representations of the unknown functions $\hat{\tau}_{30}(\rho, s)$ and $\hat{p}_{w0}(\rho, s)$ in terms of two auxiliary functions $\hat{\phi}_0(r, s)$ and $\hat{\psi}_0(r, s)$

$$\begin{aligned}
\hat{\tau}_{30}(\rho, s) &= \int_0^a \hat{\phi}_0(x, s) \cos(\rho x) dx \\
&= \frac{1}{\rho} \left\{ \hat{\phi}_0(a, s) \sin(\rho a) - \int_0^a \frac{\partial \hat{\phi}_0(x, s)}{\partial x} \sin(\rho x) dx \right\} \\
\hat{p}_{w0}(\rho, s) &= \frac{B}{\rho} \int_0^a \hat{\psi}_0(x, s) \sin(\rho x) dx \\
&= \frac{B}{\rho^2} \left\{ -\hat{\psi}_0(a, s) \cos(\rho a) + \int_0^a \frac{\partial \hat{\psi}_0(x, s)}{\partial x} \cos(\rho x) dx \right\}
\end{aligned} \tag{5.4.3}$$

where $\hat{\psi}_0(0, s) = 0$.

We also note that

$$\begin{aligned}
\int_0^{+\infty} \hat{\tau}_{30}(\rho, s) J_0(\rho r) d\rho &= \int_0^r \frac{\hat{\phi}_0(x, s)}{\sqrt{r^2 - x^2}} dx \\
\int_0^{+\infty} \hat{p}_{w0}(\rho, s) J_0(\rho r) \rho^2 d\rho &= B \int_0^r \frac{\partial \hat{\psi}_0(x, s)}{\partial x} \frac{dx}{\sqrt{r^2 - x^2}}
\end{aligned} \tag{5.4.4}$$

The sets of integral equations (5.4.2.b) can then be reduced to the integral equations of the Abel type (5.4.1.c),

$$\begin{aligned}
\int_0^r \frac{\hat{\phi}_0(x, s)}{\sqrt{r^2 - x^2}} dx + \int_0^{+\infty} \left\{ \alpha_2 k_1(\gamma) \hat{\tau}_{30}(\rho, s) \right. \\
\left. + \frac{\alpha_3}{B} [1 + k_1(\gamma)] \hat{p}_{w0}(\rho, s) \right\} J_0(\rho r) d\rho &= \alpha_a \hat{D}_z(s) \\
\int_0^r \frac{\partial \hat{\psi}_0(x, s)}{\partial x} \frac{dx}{\sqrt{r^2 - x^2}} + \int_0^{+\infty} \left\{ \alpha_4 k_2(\gamma) \hat{\tau}_{30}(\rho, s) \right. \\
\left. + \frac{1}{B} k_2(\gamma) \hat{p}_{w0}(\rho, s) \right\} J_0(\rho r) \rho^2 d\rho &= 0
\end{aligned} \tag{5.4.5}$$

The solution of the Abel type integral equations (5.4.5) can be written as

$$\begin{pmatrix} \hat{\phi}_0(r, s) \\ \hat{\psi}_0(r, s) \end{pmatrix} + \int_0^a \begin{pmatrix} K_{01}(r, y, s) & K_{02}(r, y, s) \\ K_{03}(r, y, s) & K_{04}(r, y, s) \end{pmatrix} \begin{pmatrix} \hat{\phi}_0(y, s) \\ \hat{\psi}_0(y, s) \end{pmatrix} dy = \begin{pmatrix} \frac{2}{\pi} \alpha_a \hat{D}_z(s) \\ 0 \end{pmatrix} \tag{5.4.6.a}$$

where $0 \leq r < a$, and the symmetric kernel functions are

$$\begin{aligned}
 K_{01}(r, y, s) &= \alpha_2 \frac{2}{\pi} \int_0^{+\infty} k_1(\gamma) \cos(\rho r) \cos(\rho y) d\rho \\
 K_{02}(r, y, s) &= \alpha_3 \frac{2}{\pi} \int_0^{+\infty} [1 + k_1(\gamma)] \cos(\rho r) \sin(\rho y) \frac{1}{\rho} d\rho \\
 K_{03}(r, y, s) &= \alpha_4 \frac{2}{\pi} \int_0^{+\infty} k_2(\gamma) \sin(\rho r) \cos(\rho y) \rho d\rho \\
 K_{04}(r, y, s) &= \frac{2}{\pi} \int_0^{+\infty} k_2(\gamma) \sin(\rho r) \sin(\rho y) d\rho
 \end{aligned} \tag{5.4.6.b}$$

Equations (5.4.6.a) are a system of Fredholm integral equations of the second kind in the Laplace transform domain for the axial translation of an *impermeable rigid disc* inclusion in a poroelastic medium.

The Fredholm integral equation of the second kind in the Laplace transform domain for the axial translation of a *permeable rigid disc* inclusion in a poroelastic medium can be similarly found by using the auxiliary functions $\hat{\phi}_0(r, s)$ in equations (5.4.3); i.e.,

$$\hat{\phi}_0(r, s) + \int_0^a K_{01}(r, y, s) \hat{\phi}_0(y, s) dy = \frac{2}{\pi} \alpha_a \hat{D}_z(s) \tag{5.4.6.c}$$

For both *permeable and impermeable rigid disc* inclusion problems, the associated total force and moments acting on the inclusion in the Laplace transform domain have the following relationships with the auxiliary function $\hat{\phi}_0(x, s)$,

$$\begin{aligned}
 \hat{P}_z(s) &= -4\pi \int_0^a \hat{\phi}_0(x, s) dx \\
 \hat{M}_x(s) &= 0 \\
 \hat{M}_y(s) &= 0
 \end{aligned} \tag{5.4.7.a}$$

3

PM-1 3½"x4" PHOTOGRAPHIC MICROCOPY TARGET
NBS 1010a ANSI/ISO #2 EQUIVALENT

PRECISIONSM RESOLUTION TARGETS

PIONEERS IN METHYLBLE BLUE TESTING SINCE 1974

METRO

1000 COUNTY ROAD 5, SUITE 100, WILSON, NJ 07094
TEL: 912 685 7000 FAX: 912 685 7001 TOLL FREE: 800 368 6868

and the related contact normal stress in the Laplace transform domain is

$$\begin{aligned}\hat{\sigma}_{zz}(r, \theta, 0, s) &= \int_0^{\infty} \hat{\tau}_{30}(\rho, s) J_0(\rho r) \rho d\rho \\ &= \frac{\hat{\phi}_0(a, s)}{\sqrt{a^2 - r^2}} - \int_r^a \frac{\partial \hat{\phi}_0(x, s)}{\partial x} \frac{1}{\sqrt{x^2 - r^2}} dx\end{aligned}\quad (5.4.7.b)$$

where $0 \leq r \leq a$.

5.4.3 Fredholm integral equations for the central rotation of a rigid disc inclusion

Using the functions $k_1(\gamma)$ and $k_2(\gamma)$, we can re-write the sets of integral equations (5.3.2.b) and (5.3.3.b) for the central rotation of a *permeable* or an *impermeable* rigid disc inclusion embedded in bonded contact with a poroelastic medium: (i) for the *permeable* inclusion problem, we have

$$\begin{aligned}\int_0^{+\infty} [1 + \alpha_2 k_1(\gamma)] \hat{\tau}_{31}(\rho, s) J_1(\rho r) d\rho &= \alpha_a \frac{r}{2} \hat{\Omega}_y(s), \quad 0 \leq r \leq a \\ \int_0^{+\infty} \hat{\tau}_{31}(\rho, s) J_1(\rho r) \rho d\rho &= 0, \quad a < r < \infty\end{aligned}\quad (5.4.8.a)$$

and (ii) for the *impermeable* inclusion problem, we have

$$\begin{aligned}\int_0^{+\infty} \left\{ [1 + \alpha_2 k_1(\gamma)] \hat{\tau}_{31}(\rho, s) + \frac{\alpha_3}{B} [1 + k_1(\gamma)] \hat{p}_{w1}(\rho, s) \right\} J_1(\rho r) d\rho \\ = \alpha_a \frac{r}{2} \hat{\Omega}_y(s), \quad 0 \leq r \leq a \\ \int_0^{+\infty} \left\{ \alpha_4 k_2(\gamma) \hat{\tau}_{31}(\rho, s) + \frac{1}{B} [1 + k_2(\gamma)] \hat{p}_{w1}(\rho, s) \right\} J_1(\rho r) \rho^2 d\rho \\ = 0, \quad 0 \leq r < a \\ \int_0^{+\infty} \hat{\tau}_{31}(\rho, s) J_1(\rho r) \rho d\rho &= 0, \quad a < r < \infty \\ \int_0^{+\infty} \hat{p}_{w1}(\rho, s) J_1(\rho r) \rho d\rho &= 0, \quad a < r < \infty\end{aligned}\quad (5.4.8.b)$$

The last two equations in (5.4.8.b) are satisfied automatically by the following representations of the unknown functions $\hat{r}_{31}(\rho, s)$ and $\hat{p}_{w1}(\rho, s)$ in terms of two auxiliary functions $\hat{\phi}_1(r, s)$ and $\hat{\psi}_1(r, s)$,

$$\begin{aligned}\hat{r}_{31}(\rho, s) &= \int_0^a \hat{\phi}_1(x, s) \sin(\rho x) dx \\ \hat{p}_{w1}(\rho, s) &= \sqrt{\frac{\pi}{2}} \frac{B}{\sqrt{\rho}} \int_0^a \hat{\psi}_1(x, s) J_{\frac{3}{2}}(\rho x) dx \\ &= \frac{B}{\rho^2} \left\{ -\frac{\hat{\psi}_1(a, s)}{\sqrt{a}} \sin(\rho a) + \int_0^a \frac{\partial[\sqrt{x}\hat{\psi}_1(x, s)]}{\partial x} \frac{\sin(\rho x)}{x} dx \right\}\end{aligned}\quad (5.4.9)$$

where $\lim_{x \rightarrow 0} \sqrt{x}\hat{\psi}_1(x, s) = 0$.

We also note that

$$\begin{aligned}\int_0^{+\infty} \hat{r}_{31}(\rho, s) J_1(\rho r) d\rho &= \frac{1}{r} \int_0^r \frac{x \hat{\phi}_1(x, s)}{\sqrt{r^2 - x^2}} dx \\ \int_0^{+\infty} \hat{p}_{w1}(\rho, s) J_1(\rho r) \rho^2 d\rho &= \frac{B}{r} \int_0^r \frac{\partial[\sqrt{x}\hat{\psi}_1(x, s)]}{\partial x} \frac{dx}{\sqrt{r^2 - x^2}}\end{aligned}\quad (5.4.10)$$

The sets of integral equations (5.4.8.b) can then be reduced to the integral equations of the Abel type,

$$\begin{aligned}\int_0^r \frac{x \hat{\phi}_1(x, s)}{\sqrt{r^2 - x^2}} dx + \int_0^{+\infty} \left\{ \alpha_2 k_1(\gamma) \hat{r}_{31}(\rho, s) \right. \\ \left. + \frac{\alpha_3}{B} [1 + k_1(\gamma)] \hat{p}_{w1}(\rho, s) \right\} r J_1(\rho r) d\rho = \alpha_a \frac{r^2}{2} \hat{\Omega}_y(s) \\ \int_0^r \frac{\partial[\sqrt{x}\hat{\psi}_1(x, s)]}{\partial x} \frac{dx}{\sqrt{r^2 - x^2}} + \int_0^{+\infty} \left\{ \alpha_4 k_2(\gamma) \hat{r}_{31}(\rho, s) \right. \\ \left. + \frac{1}{B} k_2(\gamma) \hat{p}_{w1}(\rho, s) \right\} r J_1(\rho r) \rho^2 d\rho = 0\end{aligned}\quad (5.4.11)$$

The solution of the Abel type integral equations (5.4.11) can be written as

$$\begin{pmatrix} \hat{\phi}_1(r, s) \\ \hat{\psi}_1(r, s) \end{pmatrix} + \int_0^a \begin{pmatrix} K_{11}(r, y, s) & K_{12}(r, y, s) \\ K_{13}(r, y, s) & K_{14}(r, y, s) \end{pmatrix} \begin{pmatrix} \hat{\phi}_1(y, s) \\ \hat{\psi}_1(y, s) \end{pmatrix} dy = \begin{pmatrix} \frac{2}{\pi} \alpha_a \hat{\Omega}_y(s) r \\ 0 \end{pmatrix} \quad (5.4.12.a)$$

where $0 \leq r < a$, and the symmetric kernel functions are

$$\begin{aligned} K_{11}(r, y, s) &= \alpha_2 \frac{2}{\pi} \int_0^{+\infty} k_1(\gamma) \sin(\rho r) \sin(\rho y) d\rho \\ K_{12}(r, y, s) &= \alpha_3 \sqrt{\frac{2}{\pi}} \int_0^{+\infty} \frac{1}{\sqrt{\rho}} [1 + k_1(\gamma)] \sin(\rho r) J_{\frac{3}{2}}(\rho y) d\rho \\ K_{13}(r, y, s) &= \alpha_4 \sqrt{\frac{2}{\pi}} r \int_0^{+\infty} k_2(\gamma) J_{\frac{3}{2}}(\rho r) \sin(\rho y) \rho^{\frac{3}{2}} d\rho \\ K_{14}(r, y, s) &= r \int_0^{+\infty} k_2(\gamma) J_{\frac{3}{2}}(\rho r) J_{\frac{3}{2}}(\rho y) \rho d\rho \end{aligned} \quad (5.4.12.b)$$

The equations (5.4.12.a) are the system of Fredholm integral equations of the second kind in the Laplace transform domain for the central rotation of an *impermeable rigid disc* inclusion in a poroelastic medium.

The Fredholm integral equation of the second kind in the Laplace transform domain for the central rotation of a *permeable rigid disc* inclusion in a poroelastic medium can be similarly found by using the auxiliary functions $\hat{\phi}_1(r, s)$ in equations (5.4.9); i.e.,

$$\hat{\phi}_1(r, s) + \int_0^a K_{11}(r, y, s) \hat{\phi}_1(y, s) dy = \frac{2}{\pi} \alpha_a \hat{\Omega}_y(s) r \quad (5.4.12.c)$$

where $0 \leq r < a$.

For the both *permeable and impermeable rigid disc* inclusion problems, the associated total force and moments in the Laplace transform domain have the following

relationships in terms of the auxiliary function $\hat{\phi}_1(x, s)$,

$$\begin{aligned}\hat{P}_z(s) &= 0 \\ \hat{M}_x(s) &= 0 \\ \hat{M}_y(s) &= -8\pi \int_0^a x \hat{\phi}_1(x, s) dx\end{aligned}\tag{5.4.13.a}$$

and the related contact normal stress in the Laplace transform domain is,

$$\begin{aligned}\hat{\sigma}_{zz}(r, \theta, 0, s) &= 2 \cos \theta \int_0^{+\infty} \hat{\tau}_{31}(\rho, s) J_1(\rho r) \rho d\rho \\ &= -2 \cos \theta \frac{d}{dr} \int_r^a \frac{\hat{\phi}_1(x, s)}{\sqrt{x^2 - r^2}} dx\end{aligned}\tag{5.4.13.b}$$

where $0 \leq r \leq a$.

5.4.4 Fredholm integral equations for the in-plane translation of a rigid disc inclusion

Using the functions $k_3(\gamma)$ and $k_4(\gamma)$, we can re-write the sets of integral equations (5.3.6.a) and (5.3.7.a) for the in-plane translation of a *permeable* or an *impermeable* rigid disc inclusion embedded in bonded contact with a poroelastic medium;

(i) for the *permeable* inclusion problem, we have

$$\begin{aligned}
& \int_0^{+\infty} \left[[1 + \alpha_6 k_3(\gamma)] \hat{N}(\rho, s) + \alpha_1 [1 + \alpha_7 k_3(\gamma)] \hat{M}(\rho, s) \right. \\
& \quad \left. + \frac{\alpha_5}{\kappa B} [1 + k_3(\gamma)] \hat{\vartheta}_{31}(\rho, s) \right] J_0(\rho r) d\rho = \alpha_b \hat{D}_x(s), \quad 0 \leq r \leq a \\
& \int_0^{+\infty} \left[\alpha_1 [1 + \alpha_7 k_3(\gamma)] \hat{N}(\rho, s) + [1 + \alpha_6 k_3(\gamma)] \hat{M}(\rho, s) \right. \\
& \quad \left. + \frac{\alpha_5}{\kappa B} [1 + k_3(\gamma)] \hat{\vartheta}_{31}(\rho, s) \right] J_2(\rho r) d\rho = 0, \quad 0 \leq r \leq a \\
& \int_0^{+\infty} \left[\alpha_8 k_4(\gamma) \hat{N}(\rho, s) + \alpha_8 k_4(\gamma) \hat{M}(\rho, s) \right. \\
& \quad \left. + \frac{1}{\kappa B} [1 + k_4(\gamma)] \hat{\vartheta}_{31}(\rho, s) \right] J_1(\rho r) \rho d\rho = 0, \quad 0 \leq r < a \\
& \int_0^{+\infty} \hat{N}(\rho, s) J_0(\rho r) \rho d\rho = 0, \quad a < r < \infty \\
& \int_0^{+\infty} \hat{M}(\rho, s) J_2(\rho r) \rho d\rho = 0, \quad a < r < \infty \\
& \int_0^{+\infty} \rho \hat{\vartheta}_{31}(\rho, s) J_1(\rho r) \rho d\rho = 0, \quad a < r < \infty
\end{aligned} \tag{5.4.14.a}$$

and (ii) for the *impermeable* inclusion problem, we have

$$\begin{aligned}
& \int_0^{+\infty} \left[[1 + \alpha_6 k_3(\gamma)] \hat{N}(\rho, s) + \alpha_1 [1 + \alpha_7 k_3(\gamma)] \hat{M}(\rho, s) \right] J_0(\rho r) \rho d\rho \\
& \quad = \alpha_b \hat{D}_x(s), \quad 0 \leq r \leq a \\
& \int_0^{+\infty} \left[\alpha_1 [1 + \alpha_7 k_3(\gamma)] \hat{N}(\rho, s) + [1 + \alpha_6 k_3(\gamma)] \hat{M}(\rho, s) \right] J_2(\rho r) d\rho \\
& \quad = 0, \quad 0 \leq r \leq a \\
& \int_0^{+\infty} \hat{N}(\rho, s) J_0(\rho r) \rho d\rho = 0, \quad a < r < \infty \\
& \int_0^{+\infty} \hat{M}(\rho, s) J_2(\rho r) \rho d\rho = 0, \quad a < r < \infty
\end{aligned} \tag{5.4.14.b}$$

The last three equations in (5.4.14.a) are satisfied automatically by the following

representations of the unknown functions $\hat{N}(\rho, s)$, $\hat{M}(\rho, s)$, and $\hat{\vartheta}_{31}(\rho, s)$ in terms of three auxiliary functions $\hat{\phi}(x, s)$, $\hat{\psi}(x, s)$ and $\hat{\varphi}(x, s)$, respectively,

$$\begin{aligned}\hat{N}(\rho, s) &= \int_0^a \hat{\phi}(x, s) \cos(\rho x) dx \\ &= \frac{1}{\rho} \left\{ \hat{\phi}(a, s) \sin(\rho a) - \int_0^a \frac{\partial \hat{\phi}(x, s)}{\partial x} \sin(\rho x) dx \right\}\end{aligned}\quad (5.4.15)$$

$$\hat{M}(\rho, s) = \sqrt{\frac{\pi}{2}} \sqrt{\rho} \int_0^a \hat{\psi}(x, s) J_{\frac{3}{2}}(\rho x) dx$$

$$\hat{\vartheta}_{31}(\rho, s) = \frac{\kappa B}{\rho} \sqrt{\frac{a^2 s}{c}} \int_0^a \hat{\varphi}(x, s) \sin(\rho x) dx$$

We also note the following relations:

$$\begin{aligned}\int_0^{+\infty} \hat{N}(\rho, s) J_0(\rho r) d\rho &= \int_0^r \frac{\hat{\phi}(x, s)}{\sqrt{r^2 - x^2}} dx \\ \int_0^{+\infty} \hat{M}(\rho, s) J_2(\rho r) d\rho &= \int_0^r \frac{x^{\frac{3}{2}} \hat{\psi}(x, s)}{r^2 \sqrt{r^2 - x^2}} dx \\ \int_0^{+\infty} \hat{\vartheta}_{31}(\rho, s) J_1(\rho r) \rho d\rho &= \kappa B \sqrt{\frac{a^2 s}{c}} \int_0^r \frac{x \hat{\varphi}(x, s)}{r \sqrt{r^2 - x^2}} dx\end{aligned}\quad (5.4.16)$$

The sets of integral equations (5.4.14.a) can then be reduced to the following sets of integral equations of the Abel type;

$$\begin{aligned}\int_0^r \frac{\hat{\phi}(x, s)}{\sqrt{r^2 - x^2}} dx + \int_0^{+\infty} \left\{ \alpha_6 k_3 \hat{N}(\rho, s) + \alpha_1 [1 + \alpha_7 k_3] \hat{M}(\rho, s) \right. \\ \left. + \frac{\alpha_5}{\kappa B} [1 + k_3(\gamma)] \hat{\vartheta}_{31}(\rho, s) \right\} J_0(\rho r) d\rho = \alpha_6 \hat{D}_x(s) \\ \int_0^r \frac{x^{\frac{3}{2}} \hat{\psi}(x, s)}{r^2 \sqrt{r^2 - x^2}} dx + \int_0^{+\infty} \left\{ \alpha_1 [1 + \alpha_7 k_3(\gamma)] \hat{N}(\rho, s) + \alpha_6 k_3(\gamma) \hat{M}(\rho, s) \right. \\ \left. + \frac{\alpha_5}{\kappa B} [1 + k_3(\gamma)] \hat{\vartheta}_{31}(\rho, s) \right\} J_2(\rho r) d\rho = 0 \\ \int_0^r \frac{x \hat{\varphi}(x, s)}{r \sqrt{r^2 - x^2}} dx + \int_0^{+\infty} \left\{ \alpha_8 k_4(\gamma) \sqrt{\frac{c}{a^2 s}} \hat{N}(\rho, s) + \alpha_8 k_4(\gamma) \sqrt{\frac{c}{a^2 s}} \hat{M}(\rho, s) \right. \\ \left. + \frac{1}{\kappa B} k_4(\gamma) \hat{\vartheta}_{31}(\rho, s) \right\} J_1(\rho r) \rho d\rho = 0\end{aligned}\quad (5.4.17)$$

The solution of the Abel type integral equations (5.4.17) can be written as

$$\begin{pmatrix} \hat{\phi}(r, s) \\ \hat{\psi}(r, s) \\ \hat{\varphi}(r, s) \end{pmatrix} + \int_0^a \begin{pmatrix} K_1(r, y, s) & K_2(r, y, s) & K_3(r, y, s) \\ K_4(r, y, s) & K_5(r, y, s) & K_6(r, y, s) \\ K_7(r, y, s) & K_8(r, y, s) & K_9(r, y, s) \end{pmatrix} \begin{pmatrix} \hat{\phi}(y, s) \\ \hat{\psi}(y, s) \\ \hat{\varphi}(y, s) \end{pmatrix} dy = \begin{pmatrix} \frac{2}{\pi} \alpha_b \hat{D}_z(s) \\ 0 \\ 0 \end{pmatrix} \quad (5.4.18.a)$$

where $0 \leq r < a$, and the symmetric kernel functions are

$$\begin{aligned} K_1(r, y, s) &= \alpha_6 \frac{2}{\pi} \int_0^{+\infty} \hat{k}_3(\gamma) \cos(\rho r) \cos(\rho y) d\rho \\ K_2(r, y, s) &= \alpha_1 \sqrt{\frac{2}{\pi}} \int_0^{+\infty} \sqrt{\rho} [1 + \alpha_7 \hat{k}_3(\gamma)] \cos(\rho r) J_{\frac{3}{2}}(\rho y) d\rho \\ K_3(r, y, s) &= \alpha_5 \frac{2}{\pi} \sqrt{\frac{a^2 s}{c}} \int_0^{+\infty} \frac{1}{\rho} [1 + \hat{k}_3(\gamma)] \cos(\rho r) \sin(\rho y) d\rho \\ K_4(r, y, s) &= \alpha_1 r \sqrt{\frac{2}{\pi}} \int_0^{+\infty} \sqrt{\rho} [1 + \alpha_7 \hat{k}_3(\gamma)] J_{\frac{3}{2}}(\rho r) \cos(\rho y) d\rho \\ K_5(r, y, s) &= \alpha_6 r \int_0^{+\infty} \rho \hat{k}_3(\gamma) J_{\frac{3}{2}}(\rho r) J_{\frac{3}{2}}(\rho y) d\rho \\ K_6(r, y, s) &= \alpha_5 r \sqrt{\frac{2}{\pi}} \sqrt{\frac{a^2 s}{c}} \int_0^{+\infty} \frac{1}{\sqrt{\rho}} [1 + \hat{k}_3(\gamma)] J_{\frac{3}{2}}(\rho r) \sin(\rho y) d\rho \\ K_7(r, y, s) &= \alpha_8 \frac{2}{\pi} \sqrt{\frac{c}{a^2 s}} \int_0^{+\infty} \hat{k}_4(\gamma) \sin(\rho r) \cos(\rho y) \rho d\rho \\ K_8(r, y, s) &= \alpha_8 \sqrt{\frac{2}{\pi}} \sqrt{\frac{c}{a^2 s}} \int_0^{+\infty} \hat{k}_4(\gamma) \sin(\rho r) J_{\frac{3}{2}}(\rho y) \rho^{\frac{3}{2}} d\rho \\ K_9(r, y, s) &= \frac{2}{\pi} \int_0^{+\infty} \hat{k}_4(\gamma) \sin(\rho r) \sin(\rho y) d\rho \end{aligned} \quad (5.4.18.b)$$

The equations (5.4.18.a) are the system of Fredholm integral equations of the second kind in the Laplace transform domain for the in-plane translation of a *permeable rigid disc* inclusion in a poroelastic medium.

The system of Fredholm integral equations of the second kind in the Laplace transform domain for the in-plane translation of an *impermeable rigid disc* inclusion in a poroelastic medium can be found similarly by using the auxiliary functions $\hat{\phi}(r, s)$ and $\hat{\psi}(r, s)$ in equations (5.4.15). We have,

$$\begin{pmatrix} \hat{\phi}(r, s) \\ \hat{\psi}(r, s) \end{pmatrix} + \int_0^a \begin{pmatrix} K_1(r, y, s) & K_2(r, y, s) \\ K_4(r, y, s) & K_5(r, y, s) \end{pmatrix} \begin{pmatrix} \hat{\phi}(y, s) \\ \hat{\psi}(y, s) \end{pmatrix} dy = \begin{pmatrix} \frac{2}{\pi} \alpha_b \hat{D}_x(s) \\ 0 \end{pmatrix} \quad (5.4.18.c)$$

where $0 \leq r < a$.

For both *permeable and impermeable rigid disc* inclusion problems, the related total forces and moment in the Laplace transform domain have the following relations with the auxiliary function $\hat{\phi}(x, s)$,

$$\begin{aligned} \hat{P}_x(s) &= -4\pi \int_0^a \hat{\phi}(x, s) dx \\ \hat{P}_y(s) &= 0 \\ \hat{M}_z(s) &= 0 \end{aligned} \quad (5.4.19.a)$$

and the related contact shear stresses in the Laplace transform domain are,

$$\begin{aligned} \hat{\sigma}_{xz}(r, \theta, 0, s) &= -\cos 2\theta \int_0^{+\infty} \hat{M}(\rho, s) J_2(\rho r) \rho d\rho + \int_0^{+\infty} \hat{N}(\rho, s) J_0(\rho r) \rho d\rho \\ &= \cos 2\theta \, r \frac{d}{dr} \int_r^a \frac{\hat{\psi}(x, s)}{x^{\frac{3}{2}} \sqrt{x^2 - r^2}} dx + \frac{\hat{\phi}(a, s)}{\sqrt{a^2 - r^2}} - \int_r^a \frac{\partial \hat{\phi}(x, s)}{\partial x} \frac{1}{\sqrt{x^2 - r^2}} dx \\ \hat{\sigma}_{yz}(r, \theta, 0, s) &= -\sin 2\theta \int_0^{+\infty} \hat{M}(\rho, s) J_2(\rho r) \rho d\rho \\ &= \sin 2\theta \, r \frac{d}{dr} \int_r^a \frac{\hat{\psi}(x, s)}{x^{\frac{3}{2}} \sqrt{x^2 - r^2}} dx \end{aligned} \quad (5.4.19.b)$$

where $0 \leq x < a$.

It may be noted that the shear stress $\sigma_{yz}(r, \theta, 0, t)$ is not equal to zero at the plane of the inclusion. This is not identical to the observation made by Keer (1965) and Selvadurai (1980b) in connection with the elastostatic problem where $\sigma_{yz}(r, \theta, 0)$ is

equal to zero. In both poroelastic and pure elastic cases, the total horizontal force P_y in the direction of y -axis is zero.

5.4.5 Exact solution for the axial rotation of a rigid disc inclusion

The pairs of integral equations (5.3.6.b) and (5.3.7.b) respectively for the axial rotation of a *permeable* or an *impermeable* rigid disc inclusion embedded in bonded contact with a poroelastic medium of infinite extent can be written into the following in the temporal domain,

$$\begin{aligned} \int_0^{+\infty} \frac{1}{\rho} \tau_{20}(\rho, t) J_1(\rho r) \rho d\rho &= -\mu r \Omega_z(t), & 0 \leq r \leq a \\ \int_0^{+\infty} \tau_{20}(\rho, t) J_1(\rho r) \rho d\rho &= 0, & a < r < \infty \end{aligned} \quad (5.4.20)$$

The last equation in (5.4.20) is satisfied automatically by the following representation of solution in terms of an auxiliary functions $\phi(x, t)$,

$$\tau_{20}(\rho, t) = \int_0^a \sin(\rho x) \phi(x, t) dx \quad (5.4.21)$$

The dual integral equations (5.4.20) can then be reduced to an integral equations of the Abel type,

$$\int_0^r \frac{x \phi(x, t)}{\sqrt{r^2 - x^2}} dx = -\frac{2\mu}{\pi} r^2 \Omega_z(t), \quad 0 \leq r \leq a \quad (5.4.22)$$

The solution of the Abel type integral equations (5.4.22) is

$$\begin{aligned} \phi(x, t) &= -\frac{2\mu}{\pi} \Omega_z(t) \frac{d}{x dx} \int_0^x \frac{r^3}{\sqrt{x^2 - r^2}} dr \\ &= -\frac{4\mu}{\pi} \Omega_z(t) x \end{aligned} \quad (5.4.23)$$

The related forces and moment acting on the rigid disc inclusion are

$$\begin{aligned}
P_z(t) &= 0 \\
P_y(t) &= 0 \\
M_z(t) &= -8\pi \int_0^a \hat{\phi}(x, s) x dx \\
&= \frac{32\mu a^3}{3} \Omega_z(t)
\end{aligned} \tag{5.4.24.a}$$

The shear stress at the inclusion and poroelastic medium interface is

$$\begin{aligned}
\sigma_{\theta z}(r, t) &= \frac{4}{\pi} \mu \Omega_z(t) \frac{d}{dr} \int_r^a \frac{x}{\sqrt{x^2 - r^2}} dx \\
&= -\frac{4}{\pi} \frac{\mu r}{\sqrt{a^2 - r^2}} \Omega_z(t)
\end{aligned} \tag{5.4.24.b}$$

These solutions are identical to those given in the classical elasticity (see Sneddon 1966b; Gladwell, 1980). From these exact solutions, we may conclude that the mechanical behaviour of the two phase model of poroelastic media saturated with compressible fluids is same as that of one phase model of elastic media under the load of axial rotation. There is no consolidation effects associated with the torsional loading of a rigid disc inclusion embedded in bonded contact with a poroelastic medium saturated with a compressible pore fluid.

5.4.6 Exact solutions of the Fredholm integral equations for the three limiting responses

The solutions of the Fredholm integral equations of the second kind can be derived in exact closed-form for the three limiting physical conditions of initial and final responses as well as the limiting case of $\nu \rightarrow \nu_u$.

1. The closed-form solutions for the Fredholm integral equations (5.4.6) under the three limiting responses for the axial translation of a *permeable* or an *impermeable* rigid disc inclusion are

$$\begin{aligned}
\phi_0(x, 0^+) &= \frac{2}{\pi} \beta_a D_z(0^+), & t \rightarrow 0^+ \\
\phi_0(x, +\infty) &= \frac{2}{\pi} \alpha_a D_z(+\infty), & t \rightarrow +\infty \\
\phi_0(x, t) &= \frac{2}{\pi} \alpha_a D_z(t), & \nu \rightarrow \nu_u
\end{aligned} \tag{5.4.25.a}$$

and the axial translational stiffnesses of the rigid disc inclusions are

$$\begin{aligned}
P_z(0^+) &= \frac{32\mu(1-\nu_u)a}{3-4\nu_u} D_z(0^+), & t \rightarrow 0^+ \\
P_z(+\infty) &= \frac{32\mu(1-\nu)a}{3-4\nu} D_z(+\infty), & t \rightarrow +\infty \\
P_z(t) &= \frac{32\mu(1-\nu)a}{3-4\nu} D_z(t), & \nu \rightarrow \nu_u
\end{aligned} \tag{5.4.25.b}$$

2. The closed-form solutions for the Fredholm integral equations (5.4.12) under the three limiting responses of central rotation of a *permeable* or an *impermeable* rigid disc inclusion are

$$\begin{aligned}
\phi_1(x, 0^+) &= \frac{2}{\pi} \beta_a \Omega_y(0^+) x, & t \rightarrow 0^+ \\
\phi_1(x, +\infty) &= \frac{2}{\pi} \alpha_a \Omega_y(+\infty) x, & t \rightarrow +\infty \\
\phi_1(x, t) &= \frac{2}{\pi} \alpha_a \Omega_y(t) x, & \nu \rightarrow \nu_u
\end{aligned} \tag{5.4.26.a}$$

and the central rotational stiffnesses of the rigid disc inclusions are

$$\begin{aligned}
M_y(0^+) &= \frac{64\mu a^3(1-\nu_u)}{3(3-4\nu_u)} \Omega_y(0^+), & t \rightarrow 0^+ \\
M_y(+\infty) &= \frac{64\mu a^3(1-\nu)}{3(3-4\nu)} \Omega_y(+\infty), & t \rightarrow +\infty \\
M_y(t) &= \frac{64\mu a^3(1-\nu)}{3(3-4\nu)} \Omega_y(t), & \nu \rightarrow \nu_u
\end{aligned} \tag{5.4.26.b}$$

3. The system of Fredholm integral equations of the second kind (5.4.18) for the in-plane translation of a *permeable* or an *impermeable* rigid disc inclusion can be reduced to the following forms: for $t \rightarrow 0^+$ and $s \rightarrow +\infty$, we have

$$\begin{aligned}
\phi(x, 0^+) - \frac{\beta_1}{\sqrt{x}} \psi(x, 0^+) + \beta_1 \int_x^{\infty} \frac{\psi(y, 0^+)}{y^{\frac{3}{2}}} dy &= \frac{2}{\pi} \beta_b D_z(0^+) \\
\psi(x, 0^+) - \beta_1 \sqrt{x} \phi(x, 0^+) + \frac{\beta_1}{\sqrt{x}} \int_0^x \phi(y, 0^+) dy &= 0
\end{aligned} \tag{5.4.27.a}$$

for $t \rightarrow +\infty$ and $s \rightarrow 0^+$, we have

$$\begin{aligned}\phi(x, +\infty) - \frac{\alpha_1}{\sqrt{x}}\psi(x, +\infty) + \alpha_1 \int_x^a \frac{\psi(y, +\infty)}{y^{\frac{3}{2}}} dy &= \frac{2}{\pi} \alpha_b D_x(+\infty) \\ \psi(x, +\infty) - \alpha_1 \sqrt{x} \phi(x, +\infty) + \frac{\alpha_1}{\sqrt{x}} \int_0^x \phi(y, +\infty) dy &= 0\end{aligned}\quad (5.4.27.b)$$

and for $0^+ \leq t < +\infty$ and $\nu \rightarrow \nu_u$, we have

$$\begin{aligned}\phi(x, t) - \frac{\alpha_1}{\sqrt{x}}\psi(x, t) + \alpha_1 \int_x^a \frac{\psi(y, t)}{y^{\frac{3}{2}}} dy &= \frac{2}{\pi} \alpha_b D_x(t) \\ \psi(x, t) - \alpha_1 \sqrt{x} \phi(x, t) + \frac{\alpha_1}{\sqrt{x}} \int_0^x \phi(y, t) dy &= 0\end{aligned}\quad (5.4.27.c)$$

where $0 \leq x \leq a$.

The closed-form solutions of the integral equations (5.4.27) are

$$\begin{aligned}\phi(x, 0^+) &= \frac{2}{\pi} \beta_b D_x(0^+), & \psi(x, 0^+) &= 0, & t &\rightarrow 0^+ \\ \phi(x, +\infty) &= \frac{2}{\pi} \alpha_b D_x(+\infty), & \psi(x, +\infty) &= 0, & t &\rightarrow +\infty \\ \phi(x, t) &= \frac{2}{\pi} \alpha_b D_x(t), & \psi(x, t) &= 0, & \nu &\rightarrow \nu_u\end{aligned}\quad (5.4.28.a)$$

and the in-plane translational stiffnesses of the rigid disc inclusions are

$$\begin{aligned}P_x(0^+) &= \frac{64\mu(1-\nu_u)a}{7-8\nu_u} D_x(0^+), & t &\rightarrow 0^+ \\ P_x(+\infty) &= \frac{64\mu(1-\nu)a}{7-8\nu} D_x(+\infty), & t &\rightarrow +\infty \\ P_x(t) &= \frac{64\mu(1-\nu)a}{7-8\nu} D_x(t), & \nu &\rightarrow \nu_u\end{aligned}\quad (5.4.28.b)$$

It may be noted that the above limiting results are exactly same as those obtained for the classical elasticity solutions of the inclusion taken with the related elastic constants (see eg., Keer (1965); Seivadurai (1980a; b)).

Chapter 6

Mechanics Of Inclusions In Poroelastic Media

II. Numerical Evaluations

6.1 General

In Chapter 5, we have formulated the systems of integral equations for a permeable or an impermeable plane inclusion subjected to general direct loads. These systems of integral equations are reduced to systems of Fredholm integral equations of the second kind in the Laplace transform domain for the particular cases of a rigid disc inclusion which is subjected to axial translational, rotational and in-plane translational modes. It is found that the integral equations for the axial rotation of a rigid disc inclusion can be solved in exact closed-form. By considering the structure of the kernel functions in the integral equations for axial translation, central rotation and in-plane translation of a rigid disc inclusion, it becomes evident that the systems of complex Fredholm integral equations of the second kind are not amenable to exact solution. Consequently, it is necessary to adopt numerical techniques to generate the relevant results in time domain.

In this Chapter, we shall present numerical solutions for the integral equations governing the three modes of deformation of a rigid disc inclusion by adopting a numerical scheme based on the numerical inversion of Laplace transforms. First, we formulate the systems of complex Fredholm integral equations of the second kind in the non-dimensional forms. We then reduce the complex integral equations into the systems of real Fredholm integral equations of the second kind by separating the complex unknown variables and kernel functions into their real and imaginary parts. The real integral equations are converted into the systems of linear algebraic simultaneous equations. These systems of linear algebraic simultaneous equations

will be solved numerically in the Laplace transform domain. The time-dependent results of translational and rotational stiffnesses and flexibilities of the rigid disc inclusion will be evaluated with the aid of the modified inversion algorithm of Laplace transforms.

In the related derivations, the following integrals are used,

$$\begin{aligned}
 \int_0^{+\infty} \sin(\rho x) \sin(\rho y) d\rho &= \frac{\pi}{2} \delta(x - y) \\
 \int_0^{+\infty} \cos(\rho x) \cos(\rho y) d\rho &= \frac{\pi}{2} \delta(x - y) \\
 \int_0^{+\infty} \frac{1}{\rho} \sin(\rho x) \cos(\rho y) d\rho &= \begin{cases} \frac{\pi}{2} & 0 \leq y < x \\ \frac{\pi}{4} & x = y \\ 0 & x < y < +\infty \end{cases} \quad (6.1.1) \\
 \int_0^{+\infty} \frac{1}{\rho^2} \sin(\rho x) \sin(\rho y) d\rho &= \begin{cases} \frac{\pi}{2} y & 0 \leq y \leq x \\ \frac{\pi}{2} x & x < y < \infty \end{cases}
 \end{aligned}$$

where the first two integrals are in the sense of generalized function $\delta(x)$, i.e., the Dirac delta function.

6.2 Axial Translation of a Rigid Disc Inclusion

6.2.1 Non-dimensional integral equations

Introducing the non-dimensional variables $\hat{\phi}(x, s_1)$, $\hat{\psi}(x, s_1)$, $\hat{X}_z(s_1)$ and $\hat{Y}_z(s_1)$ defined by,

$$\begin{aligned}
 x &= \frac{r}{a} \\
 s_1 &= \frac{a^2 s}{c} \\
 \hat{\phi}_0(r, s) &= -\frac{P_z}{4\pi a} \hat{\phi}(x, s_1) \\
 \hat{\psi}_0(r, s) &= -\frac{P_z}{4\pi a^2} \sqrt{s_1} \hat{\psi}(x, s_1) \\
 \frac{2}{\pi} \alpha_a \hat{D}_z(s) &= -\frac{P_z}{4\pi a} \hat{X}_z(s_1) \\
 \hat{P}_z(s) &= P_z \hat{Y}_z(s_1)
 \end{aligned} \quad (6.2.1)$$

we can re-write the two systems of Fredholm integral equations of the second kind (5.4.6) for the axial translation of a rigid disc inclusion in the following non-dimensional form: (i) for a permeable inclusion, we have

$$\begin{aligned} \hat{\phi}(x, s_1) + \int_0^1 \hat{K}_1(x, y, s_1) \hat{\phi}(y, s_1) dy &= \hat{X}_z(s_1) \\ \int_0^1 \hat{\phi}(x, s_1) dx &= \hat{Y}_z(s_1) \end{aligned} \quad (6.2.2.a)$$

and (ii) for an impermeable inclusion, we have

$$\begin{aligned} \begin{pmatrix} \hat{\phi}(x, s_1) \\ \hat{\psi}(x, s_1) \end{pmatrix} + \int_0^1 \begin{pmatrix} \hat{K}_1(x, y, s_1) & \hat{K}_2(x, y, s_1) \\ \hat{K}_3(x, y, s_1) & \hat{K}_4(x, y, s_1) \end{pmatrix} \begin{pmatrix} \hat{\phi}(y, s_1) \\ \hat{\psi}(y, s_1) \end{pmatrix} dy &= \begin{pmatrix} \hat{X}_z(s_1) \\ 0 \end{pmatrix} \\ \int_0^1 \hat{\phi}(y, s_1) dy &= \hat{Y}_z(s_1) \end{aligned} \quad (6.2.2.b)$$

where $0 \leq x \leq 1$, and the non-dimensional kernel functions are

$$\begin{aligned} \hat{K}_1(x, y, s_1) &= -\alpha_2 I_1(x, y, s_1) \\ \hat{K}_2(x, y, s_1) &= -\alpha_3 \sqrt{s_1} [I_2(x, y, s_1) - H(y - x)] \\ \hat{K}_3(x, y, s_1) &= -\frac{\alpha_4}{2} \sqrt{s_1} [I_3(x, y, s_1) - H(x - y)] \\ \hat{K}_4(x, y, s_1) &= I_4(x, y, s_1) \end{aligned} \quad (6.2.2.c)$$

where $H(x)$ is the Heaviside step function: i.e.,

$$H(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & x = 0 \\ 1 & x > 0 \end{cases} \quad (6.2.2.d)$$

and

$$\begin{aligned}
 I_1(x, y, s_1) &= s_1 \frac{2}{\pi} \int_0^{+\infty} \frac{\cos(\rho x) \cos(\rho y)}{(\rho + \sqrt{\rho^2 + s_1})^2} d\rho \\
 I_2(x, y, s_1) &= s_1 \frac{2}{\pi} \int_0^{+\infty} \frac{\cos(\rho x) \sin(\rho y)}{\rho(\rho + \sqrt{\rho^2 + s_1})^2} d\rho \\
 I_3(x, y, s_1) &= s_1 \frac{2}{\pi} \int_0^{+\infty} \frac{\sin(\rho x) \cos(\rho y)}{\rho(\rho + \sqrt{\rho^2 + s_1})^2} d\rho \\
 I_4(x, y, s_1) &= s_1 \frac{2}{\pi} \int_0^{+\infty} \frac{\sin(\rho x) \sin(\rho y)}{\rho(\rho + \sqrt{\rho^2 + s_1})} d\rho
 \end{aligned}
 \tag{6.2.2.e}$$

The infinite integrals in equations (6.2.2.e) with the independent variables x , y , and s_1 are uniformly convergent if and only if $Re(s_1) > 0$. Furthermore, the material parameters do not occur in these infinite integrals. However, it is not possible to find the exact integrations of such infinite integrals.

6.2.2 Numerical solutions of the integral equations

It can be shown that these two systems of complex Fredholm integral equations of the second kind (6.2.2) are regular and standard integral equations with non-singular kernel functions. Considering the structure of the kernel function in equations (6.2.2.d) it becomes evident that the complex Fredholm integral equation of the second kind has no known exact solution. Consequently it is necessary to adopt a numerical scheme to generate the relevant results. In this thesis, we suggest a numerical scheme for the time-dependent solutions of the complex integral equations in the Laplace transform domain. Analogous procedures for the numerical treatment of system of Fredholm integral equations of the second kind associated with the inclusion problems related to purely elastic media were given by Selvadurai and Singh and Au (1990) and Selvadurai and Au and Singh (1991).

The complex Fredholm integral equations of the second kind (6.2.2) can be

reduced into the systems of real Fredholm integral equation of the second kind by separating the complex variables and kernel function into their real and imaginary parts, i.e:

$$\begin{aligned}
 s_1 &= s_r + is_i \\
 \hat{\phi}(x, s_1) &= \hat{\phi}_1(x) + i\hat{\phi}_2(x) \\
 \hat{\psi}(x, s_1) &= \hat{\psi}_1(x) + i\hat{\psi}_2(x) \\
 \hat{K}_1(x, y, s_1) &= \hat{K}_{11}(x, y) + i\hat{K}_{12}(x, y) \\
 \hat{K}_2(x, y, s_1) &= \hat{K}_{21}(x, y) + i\hat{K}_{22}(x, y) \\
 \hat{K}_3(x, y, s_1) &= \hat{K}_{31}(x, y) + i\hat{K}_{32}(x, y) \\
 \hat{K}_4(x, y, s_1) &= \hat{K}_{41}(x, y) + i\hat{K}_{42}(x, y) \\
 \hat{X}_z(s_1) &= \hat{X}_{z1} + i\hat{X}_{z2} \\
 \hat{Y}_z(s_1) &= \hat{Y}_{z1} + i\hat{Y}_{z2}
 \end{aligned} \tag{6.2.3}$$

Substituting these formulae into the equations (6.2.2), we have the two systems of simultaneous real Fredholm integral equations of the second kind as follows:

(i) for an inclusion with permeable surfaces, we have

$$\begin{aligned}
 \int_0^1 \begin{pmatrix} \hat{K}_{11}(x, y) & -\hat{K}_{12}(x, y) \\ \hat{K}_{12}(x, y) & \hat{K}_{11}(x, y) \end{pmatrix} \begin{pmatrix} \hat{\phi}_1(y) \\ \hat{\phi}_2(y) \end{pmatrix} dy + \begin{pmatrix} \hat{\phi}_1(x) \\ \hat{\phi}_2(x) \end{pmatrix} &= \begin{pmatrix} \hat{X}_{z1} \\ \hat{X}_{z2} \end{pmatrix} \\
 \int_0^1 \begin{pmatrix} \hat{\phi}_1(y) \\ \hat{\phi}_2(y) \end{pmatrix} dy &= \begin{pmatrix} \hat{Y}_{z1} \\ \hat{Y}_{z2} \end{pmatrix}
 \end{aligned} \tag{6.2.4.a}$$

and (ii) for an inclusion with impermeable surfaces, we have

$$\begin{aligned}
 & \int_0^1 \begin{pmatrix} \hat{K}_{11}(x, y) & -\hat{K}_{12}(x, y) & \hat{K}_{21}(x, y) & -\hat{K}_{22}(x, y) \\ \hat{K}_{12}(x, y) & \hat{K}_{11}(x, y) & \hat{K}_{22}(x, y) & \hat{K}_{21}(x, y) \\ \hat{K}_{31}(x, y) & -\hat{K}_{32}(x, y) & \hat{K}_{41}(x, y) & -\hat{K}_{42}(x, y) \\ \hat{K}_{32}(x, y) & \hat{K}_{31}(x, y) & \hat{K}_{42}(x, y) & \hat{K}_{41}(x, y) \end{pmatrix} \begin{pmatrix} \hat{\phi}_1(y) \\ \hat{\phi}_2(y) \\ \hat{\psi}_1(y) \\ \hat{\psi}_2(y) \end{pmatrix} dy \\
 & + \begin{pmatrix} \hat{\phi}_1(x) \\ \hat{\phi}_2(x) \\ \hat{\psi}_1(x) \\ \hat{\psi}_2(x) \end{pmatrix} = \begin{pmatrix} \hat{X}_{x1} \\ \hat{X}_{x2} \\ 0 \\ 0 \end{pmatrix} \tag{6.2.4.b} \\
 & \int_0^1 \begin{pmatrix} \hat{\phi}_1(y) \\ \hat{\phi}_2(y) \end{pmatrix} dy = \begin{pmatrix} \hat{Y}_{z1} \\ \hat{Y}_{z2} \end{pmatrix}
 \end{aligned}$$

where $0 \leq x < 1$.

For the numerical solution of the integral equations, the interval $[0, 1]$ is divided into N segments with ends defined by $r_k = (k - 1)/N$; $k = 1, 2, 3 \dots (N + 1)$, and the collocation points are $x_k = (r_k + r_{k+1})/2$, $k = 1, 2, 3 \dots N$. Consequently we can convert the above integral equations into two systems of linear algebraic simultaneous equations. (i) For the inclusion with permeable surfaces, we obtain the linear algebraic equations:

$$\begin{aligned}
 & \sum_{l=1}^N \frac{1}{N} \begin{pmatrix} \hat{K}_{11}(x_k, x_l) & -\hat{K}_{12}(x_k, x_l) \\ \hat{K}_{12}(x_k, x_l) & \hat{K}_{11}(x_k, x_l) \end{pmatrix} \begin{pmatrix} \hat{\phi}_1(x_l) \\ \hat{\phi}_2(x_l) \end{pmatrix} + \begin{pmatrix} \hat{\phi}_1(x_k) \\ \hat{\phi}_2(x_k) \end{pmatrix} = \begin{pmatrix} \hat{X}_{x1} \\ \hat{X}_{x2} \end{pmatrix} \\
 & \sum_{l=1}^N \frac{1}{N} \begin{pmatrix} \hat{\phi}_1(x_l) \\ \hat{\phi}_2(x_l) \end{pmatrix} = \begin{pmatrix} \hat{Y}_{z1} \\ \hat{Y}_{z2} \end{pmatrix} \tag{6.2.5.a}
 \end{aligned}$$

(i) For the inclusion with impermeable surfaces, we have

$$\sum_{l=1}^N \frac{1}{N} \begin{pmatrix} \hat{K}_{11}(x_k, x_l) & -\hat{K}_{12}(x_k, x_l) & \hat{K}_{21}(x_k, x_l) & -\hat{K}_{22}(x_k, x_l) \\ \hat{K}_{12}(x_k, x_l) & \hat{K}_{11}(x_k, x_l) & \hat{K}_{22}(x_k, x_l) & \hat{K}_{21}(x_k, x_l) \\ \hat{K}_{31}(x_k, x_l) & -\hat{K}_{32}(x_k, x_l) & \hat{K}_{41}(x_k, x_l) & -\hat{K}_{42}(x_k, x_l) \\ \hat{K}_{32}(x_k, x_l) & \hat{K}_{31}(x_k, x_l) & \hat{K}_{42}(x_k, x_l) & \hat{K}_{41}(x_k, x_l) \end{pmatrix} \begin{pmatrix} \hat{\phi}_1(x_l) \\ \hat{\phi}_2(x_l) \\ \hat{\psi}_1(x_l) \\ \hat{\psi}_2(x_l) \end{pmatrix} + \begin{pmatrix} \hat{\phi}_1(x_k) \\ \hat{\phi}_2(x_k) \\ \hat{\psi}_1(x_k) \\ \hat{\psi}_2(x_k) \end{pmatrix} = \begin{pmatrix} \hat{X}_{z1} \\ \hat{X}_{z2} \\ 0 \\ 0 \end{pmatrix}$$

$$\sum_{l=1}^N \frac{1}{N} \begin{pmatrix} \hat{\phi}_1(x_l) \\ \hat{\phi}_2(x_l) \end{pmatrix} = \begin{pmatrix} \hat{Y}_{z1} \\ \hat{Y}_{z2} \end{pmatrix} \quad (6.2.5.b)$$

where $k = 1, 2, 3, \dots, N$.

These linear algebraic simultaneous equations can be further uniformly written in matrix form as

$$\sum_{j=1}^L [\mathbf{A}_{lj}] \{\hat{\mathbf{X}}_j\} = \{\mathbf{B}_l\} \quad (6.2.6)$$

where $l = 1, 2, 3, \dots, L$, $L = 2N + 2$ for a permeable inclusion and $L = 4N + 2$ for an impermeable inclusion.

The matrix equation (6.2.6) can be solved numerically to generate the unknown variables $\hat{\phi}(x_l, s_1)$, $\hat{\psi}(x_l, s_1)$, $\hat{X}_z(s_1)$ (or $\hat{Y}_z(s_1)$) in the Laplace transform domain. The time-dependent results of the variables $\phi(x_l, t)$, $\psi(x_l, t)$, $X_z(t)$ (or $Y_z(t)$) can be evaluated by using the modified algorithm for Laplace transform inversion.

For the axial translation of the inclusion, there are at least two types of loading conditions which are of interest to geotechnical engineering. The first type of loading is one where the time-dependent variation of total axial force $P_z(t)(= P_z Y_z(t))$ applied on the rigid disc inclusion is known *a priori*. In the second category the

time-dependent behaviour of axial translation $D_z(t)(= D_z X_z(t))$ of the rigid disc inclusion is prescribed.

The most basic condition in the first category is that of a time-independent constant axial load, i.e., $P_z(t) = P_z H(t)$. For this condition, we can write down the time-dependent response of the axial translation $D_z(t)$ as follows,

$$\frac{a\mu D_z(t)}{P_z} = \frac{3-4\nu}{32(1-\nu)} X_z(t) \quad (6.2.7.a)$$

and the degree of axial consolidation associated with the displacement of the rigid disc can be expressed in the conventional form

$$\begin{aligned} U_z(t) &= \frac{D_z(t) - D_z(0^+)}{D_z(\infty) - D_z(0^+)} \\ &= \frac{X_z(t) - \alpha_9}{1 - \alpha_9} \end{aligned} \quad (6.2.7.b)$$

where $0 < t < +\infty$, $D_z(t) - D_z(0^+)$ is the consolidation displacement at any time t and, $D_z(\infty) - D_z(0^+)$ is the total consolidation displacement, and the constant α_9 is given by,

$$\alpha_9 = \frac{(3-4\nu_u)(1-\nu)}{(3-4\nu)(1-\nu_u)} \quad (6.2.7.c)$$

The most basic form in the second category loading is that of the constant axial displacement, i.e., $D_z(t) = D_z H(t)$. For this condition, we can write down the time-dependent response of the axial total force $P_z(t)$ as follows,

$$\frac{P_z(t)}{a\mu D_z} = \frac{32(1-\nu)}{3-4\nu} Y_z(t) \quad (6.2.8.a)$$

and the degree of relaxation of axial force can be expressed in the form

$$\begin{aligned} V_z(t) &= \frac{P_z(t) - P_z(0^+)}{P_z(\infty) - P_z(0^+)} \\ &= \frac{\alpha_9 Y_z(t) - 1}{\alpha_9 - 1} \end{aligned} \quad (6.2.8.b)$$

where $0 < t < +\infty$, $P_z(t) - P_z(0^+)$ is the relaxation force at any time t , and $P_z(\infty) - P_z(0^+)$ is the total relaxation force.

For the two conditions, the time-dependent normal stress at the interface can be evaluated by using, respectively, the following two equations;

$$\begin{aligned}\sigma_{zz}(x, \theta, 0, t) &= -\frac{P_z}{4\pi a^2} \left[\frac{\phi(1, t)}{\sqrt{1-x^2}} - \int_x^1 \frac{\partial \phi(y, t)}{\partial y} \frac{1}{\sqrt{y^2-x^2}} dy \right] \\ \sigma_{zz}(x, \theta, 0, t) &= -\frac{8\mu(1-\nu)D_z}{\pi a(3-4\nu)} \left[\frac{\phi(1, t)}{\sqrt{1-x^2}} - \int_x^1 \frac{\partial \phi(y, t)}{\partial y} \frac{1}{\sqrt{y^2-x^2}} dy \right]\end{aligned}\quad (6.2.9)$$

where $0 \leq x < 1$, $0 \leq \theta < 2\pi$.

6.3 Central Rotation of a Rigid Disc Inclusion

6.3.1 Non-dimensional integral equations

Introducing the non-dimensional variables $\hat{\phi}(x, s_1)$, $\hat{\psi}(x, s_1)$, $\hat{X}_z(s_1)$ and $\hat{Y}_z(s_1)$ defined by

$$\begin{aligned}x &= \frac{r}{a} \\ s_1 &= \frac{a^2 s}{c} \\ \hat{\phi}_1(r, s) &= -\frac{M_y}{8\pi a^2} \hat{\phi}(x, s_1) \\ \hat{\psi}_1(r, s) &= -\frac{M_y}{8\pi a^{\frac{3}{2}}} \sqrt{x s_1} \hat{\psi}(x, s_1) \\ \frac{2}{\pi} \alpha_a \hat{\Omega}_y(s) &= -\frac{M_y}{8\pi a^3} \hat{X}_y(x, s_1) \\ \hat{M}_y(s) &= M_y \hat{Y}_y(s_1)\end{aligned}\quad (6.3.1)$$

we can re-write the two systems of Fredholm integral equations of the second kind (5.4.12) for the central rotation of a rigid disc inclusion in the following non-dimensional form: (i) for a permeable inclusion, we have

$$\begin{aligned}\hat{\phi}(x, s_1) + \int_0^1 \hat{K}_1(x, y, s_1) \hat{\phi}(y, s_1) dy &= x \hat{X}_y(s_1) \\ \int_0^1 y \hat{\phi}(y, s_1) dy &= \hat{Y}_y(s_1)\end{aligned}\quad (6.3.2.a)$$

and (ii) for an impermeable inclusion, we have

$$\begin{pmatrix} \hat{\phi}(x, s_1) \\ \hat{\psi}(x, s_1) \end{pmatrix} + \int_0^1 \begin{pmatrix} \hat{K}_1(x, y, s_1) & \hat{K}_2(x, y, s_1) \\ \hat{K}_3(x, y, s_1) & \hat{K}_4(x, y, s_1) \end{pmatrix} \begin{pmatrix} \hat{\phi}(y, s_1) \\ \hat{\psi}(y, s_1) \end{pmatrix} dy = \begin{pmatrix} xX_y(s_1) \\ 0 \end{pmatrix}$$

$$\int_0^1 y \hat{\phi}(y, s_1) dy = \hat{Y}_y(s_1)$$
(6.3.2.b)

where $0 \leq x \leq 1$, and the non-dimensional kernel functions are

$$\begin{aligned} \hat{K}_1(x, y, s_1) &= -\alpha_2 I_1(x, y, s_1) \\ \hat{K}_2(x, y, s_1) &= -\alpha_3 \sqrt{s_1} [I_2(x, y, s_1) - \frac{x}{y} H(y - x)] \\ \hat{K}_3(x, y, s_1) &= -\frac{\alpha_4}{2} \sqrt{s_1} [I_3(x, y, s_1) - \frac{y}{x} H(x - y)] \\ \hat{K}_4(x, y, s_1) &= I_4(x, y, s_1) \end{aligned}$$
(6.3.2.c)

where

$$\begin{aligned} I_1(x, y, s_1) &= s_1 \frac{2}{\pi} \int_0^{+\infty} \frac{\sin(\rho x) \sin(\rho y)}{(\rho + \sqrt{\rho^2 + s_1})^2} d\rho \\ I_2(x, y, s_1) &= s_1 \frac{2}{\pi} \int_0^{+\infty} \frac{\sin(\rho x) [\frac{\sin(\rho y)}{\rho y} - \cos(\rho y)]}{\rho(\rho + \sqrt{\rho^2 + s_1})^2} d\rho \\ I_3(x, y, s_1) &= s_1 \frac{2}{\pi} \int_0^{+\infty} \frac{[\frac{\sin(\rho x)}{\rho x} - \cos(\rho x)] \sin(\rho y)}{\rho(\rho + \sqrt{\rho^2 + s_1})^2} d\rho \\ I_4(x, y, s_1) &= s_1 \frac{2}{\pi} \int_0^{+\infty} \frac{[\frac{\sin(\rho x)}{\rho x} - \cos(\rho x)] [\frac{\sin(\rho y)}{\rho y} - \cos(\rho y)]}{\rho(\rho + \sqrt{\rho^2 + s_1})} d\rho \end{aligned}$$
(6.3.2.d)

The infinite integrals in equations (6.3.2.d) with the independent variables x , y , and s_1 are uniformly convergent if and only if $\text{Re}(s_1) > 0$. Furthermore, the material parameters do not occur in these infinite integrals. However, it is not possible to find the exact integrations of such infinite integrals.

6.3.2 Numerical solutions of the integral equations

It can be shown that these two systems of Fredholm integral equations of the second kind (6.3.2) are regular and standard integral equations with non-singular kernel functions. Again, by considering the structure of the kernel function in equations (6.3.2.c) it becomes evident that the system of complex Fredholm integral equation of the second kind has no known exact solution. Consequently it is necessary to adopt a numerical scheme to generate the relevant results. We basically follow the procedure outlined in section 6.2.

The systems of complex Fredholm integral equations of the second kind (6.3.2) can be reduced into the systems of real Fredholm integral equations of the second kind by separating the complex variables and kernel function into their real and imaginary parts,

$$\begin{aligned}
 s_1 &= s_r + is_i \\
 \hat{\phi}(x, s_1) &= \hat{\phi}_1(x) + i\hat{\phi}_2(x) \\
 \hat{\psi}(x, s_1) &= \hat{\psi}_1(x) + i\hat{\psi}_2(x) \\
 \hat{K}_1(x, y, s_1) &= \hat{K}_{11}(x, y) + i\hat{K}_{12}(x, y) \\
 \hat{K}_2(x, y, s_1) &= \hat{K}_{21}(x, y) + i\hat{K}_{22}(x, y) \\
 \hat{K}_3(x, y, s_1) &= \hat{K}_{31}(x, y) + i\hat{K}_{32}(x, y) \\
 \hat{K}_4(x, y, s_1) &= \hat{K}_{41}(x, y) + i\hat{K}_{42}(x, y) \\
 \hat{X}_y(s_1) &= \hat{X}_{y1} + i\hat{X}_{y2} \\
 \hat{Y}_y(s_1) &= \hat{Y}_{y1} + i\hat{Y}_{y2}
 \end{aligned} \tag{6.3.3}$$

Substituting these formulae into the equations (6.3.2) we have the two systems of simultaneous real Fredholm integral equations of the second kind as follows: (i) for

an inclusion with permeable surfaces, we have

$$\int_0^1 \begin{pmatrix} \hat{K}_{11}(x, y) & -\hat{K}_{12}(x, y) \\ \hat{K}_{12}(x, y) & \hat{K}_{11}(x, y) \end{pmatrix} \begin{pmatrix} \hat{\phi}_1(y) \\ \hat{\phi}_2(y) \end{pmatrix} dy + \begin{pmatrix} \hat{\phi}_1(x) \\ \hat{\phi}_2(x) \end{pmatrix} = \begin{pmatrix} x\hat{X}_{y1} \\ x\hat{X}_{y2} \end{pmatrix} \quad (6.3.4.a)$$

$$\int_0^1 \begin{pmatrix} y\hat{\phi}_1(y) \\ y\hat{\phi}_2(y) \end{pmatrix} dy = \begin{pmatrix} \hat{Y}_{y1} \\ \hat{Y}_{y2} \end{pmatrix}$$

and (ii) for an inclusion with impermeable surfaces, we have

$$\int_0^1 \begin{pmatrix} \hat{K}_{11}(x, y) & -\hat{K}_{12}(x, y) & \hat{K}_{21}(x, y) & -\hat{K}_{22}(x, y) \\ \hat{K}_{12}(x, y) & \hat{K}_{11}(x, y) & \hat{K}_{22}(x, y) & \hat{K}_{21}(x, y) \\ \hat{K}_{31}(x, y) & -\hat{K}_{32}(x, y) & \hat{K}_{41}(x, y) & -\hat{K}_{42}(x, y) \\ \hat{K}_{32}(x, y) & \hat{K}_{31}(x, y) & \hat{K}_{42}(x, y) & \hat{K}_{41}(x, y) \end{pmatrix} \begin{pmatrix} \hat{\phi}_1(y) \\ \hat{\phi}_2(y) \\ \hat{\psi}_1(y) \\ \hat{\psi}_2(y) \end{pmatrix} dy$$

$$+ \begin{pmatrix} \hat{\phi}_1(x) \\ \hat{\phi}_2(x) \\ \hat{\psi}_1(x) \\ \hat{\psi}_2(x) \end{pmatrix} = \begin{pmatrix} x\hat{X}_{y1} \\ x\hat{X}_{y2} \\ 0 \\ 0 \end{pmatrix} \quad (6.3.4.b)$$

$$\int_0^1 \begin{pmatrix} y\hat{\phi}_1(y) \\ y\hat{\phi}_2(y) \end{pmatrix} dy = \begin{pmatrix} \hat{Y}_{y1} \\ \hat{Y}_{y2} \end{pmatrix}$$

where $0 \leq x < 1$.

For the numerical solution of the problem, the interval $[0, 1]$ is divided into N segments with ends defined by $r_k = (k - 1)/N$; $k = 1, 2, 3 \dots (N + 1)$, and the collocation points are $x_k = (r_k + r_{k+1})/2$, $k = 1, 2, 3 \dots N$. Consequently we can convert the above integral equations into two systems of linear algebraic simultaneous equations. (i) For an inclusion with permeable surfaces, we obtain

the following linear algebraic equations,

$$\sum_{l=1}^N \frac{1}{N} \begin{pmatrix} K_{11}(x_k, x_l) & -K_{12}(x_k, x_l) \\ K_{12}(x_k, x_l) & K_{11}(x_k, x_l) \end{pmatrix} \begin{pmatrix} \hat{\phi}_1(x_l) \\ \hat{\phi}_2(x_l) \end{pmatrix} + \begin{pmatrix} \hat{\phi}_1(x_k) \\ \hat{\phi}_2(x_k) \end{pmatrix} = \begin{pmatrix} x_k \hat{X}_{y1} \\ x_k \hat{X}_{y2} \end{pmatrix}$$

$$\sum_{l=1}^N \frac{1}{N} \begin{pmatrix} x_l \hat{\phi}_1(x_l) \\ x_l \hat{\phi}_2(x_l) \end{pmatrix} = \begin{pmatrix} \hat{Y}_{y1} \\ \hat{Y}_{y2} \end{pmatrix} \quad (6.3.5.a)$$

and (ii) for an inclusion with impermeable surfaces, we have

$$\sum_{l=1}^N \frac{1}{N} \begin{pmatrix} \hat{K}_{11}(x_k, x_l) & -\hat{K}_{12}(x_k, x_l) & \hat{K}_{21}(x_k, x_l) & -\hat{K}_{22}(x_k, x_l) \\ \hat{K}_{12}(x_k, x_l) & \hat{K}_{11}(x_k, x_l) & \hat{K}_{22}(x_k, x_l) & \hat{K}_{21}(x_k, x_l) \\ \hat{K}_{31}(x_k, x_l) & -\hat{K}_{32}(x_k, x_l) & \hat{K}_{41}(x_k, x_l) & -\hat{K}_{42}(x_k, x_l) \\ \hat{K}_{32}(x_k, x_l) & \hat{K}_{31}(x_k, x_l) & \hat{K}_{42}(x_k, x_l) & \hat{K}_{41}(x_k, x_l) \end{pmatrix} \begin{pmatrix} \hat{\phi}_1(x_l) \\ \hat{\phi}_2(x_l) \\ \hat{\psi}_1(x_l) \\ \hat{\psi}_2(x_l) \end{pmatrix}$$

$$+ \begin{pmatrix} \hat{\phi}_1(x_k) \\ \hat{\phi}_2(x_k) \\ \hat{\psi}_1(x_k) \\ \hat{\psi}_2(x_k) \end{pmatrix} = \begin{pmatrix} x_k \hat{X}_{y1} \\ x_k \hat{X}_{y2} \\ 0 \\ 0 \end{pmatrix}$$

$$\sum_{l=1}^N \frac{1}{N} \begin{pmatrix} x_l \hat{\phi}_1(x_l) \\ x_l \hat{\phi}_2(x_l) \end{pmatrix} = \begin{pmatrix} \hat{Y}_{y1} \\ \hat{Y}_{y2} \end{pmatrix} \quad (6.3.5.b)$$

where $k = 1, 2, 3, \dots, N$.

These linear algebraic simultaneous equations can be further uniformly written in matrix form as

$$\sum_{j=1}^L [A_{lj}] \{\hat{X}_j\} = \{B_l\} \quad (6.3.6)$$

where $l = 1, 2, 3, \dots, L$, $L = 2N + 2$ for a permeable inclusion problem and $L = 4N + 2$ for impermeable inclusion problem.

The matrix equation (6.3.6) can be solved numerically to generate the unknown variables $\hat{\phi}(x_l, s_1)$, $\hat{\psi}(x_l, s_1)$, $\hat{X}_y(s_1)$ (or $\hat{Y}_y(s_1)$) in the Laplace transform domain. The time-dependent results of the variables $\phi(x_l, t)$, $\psi(x_l, t)$, $X_y(t)$ (or $Y_y(t)$) can be evaluated by using the modified algorithm for Laplace transform inversion.

For the central rotation of the inclusion, there are at least two types of loading conditions which are of engineering interest. The first loading condition is one where the time-dependent behaviour of total central moment $M_y(t)(= M_y Y_y(t))$ applied on the rigid disc inclusions is known *a priori*. The second is that where the time-dependent history of central rotation $\Omega_y(t)(= \Omega_y X_y(t))$ of the rigid disc inclusion is prescribed. The most basic condition in the first category is a constant central moment, i.e., $M_y(t) = M_y H(t)$. For this condition, we can write down the time-dependent response of the central rotation $\Omega_y(t)$ as follows,

$$\frac{a^3 \mu \Omega_y(t)}{M_y} = \frac{3 - 4\nu}{64(1 - \nu)} X_y(t) \quad (6.3.7.a)$$

and the degree of axial consolidation of the central rotation can be expressed in the conventional form

$$\begin{aligned} U_y(t) &= \frac{\Omega_y(t) - \Omega_y(0^+)}{\Omega_y(\infty) - \Omega_y(0^+)} \\ &= \frac{X_y(t) - 3\alpha_9}{3 - 3\alpha_9} \end{aligned} \quad (6.3.7.b)$$

where $0 < t < +\infty$, $\Omega_y(t) - \Omega_y(0^+)$ is the consolidation rotation at any time t , and $\Omega_y(\infty) - \Omega_y(0^+)$ is the total consolidation rotation.

The most basic condition in the second category of loading is a constant central rotation, i.e., $\Omega_y(t) = \Omega_y H(t)$. For this condition, we can write down the time-dependent response of the total central moment $M_y(t)$ as follows,

$$\frac{M_y(t)}{a^3 \mu \Omega_y} = \frac{64(1 - \nu)}{3 - 4\nu} Y_y(t) \quad (6.3.8.a)$$

and the degree of relaxation of central moment can be expressed in the form

$$\begin{aligned} V_y(t) &= \frac{M_y(t) - M_y(0^+)}{M_y(\infty) - M_y(0^+)} \\ &= \frac{3\alpha_9 Y_y(t) - 1}{\alpha_9 - 1} \end{aligned} \quad (6.3.8.b)$$

where $0 < t < +\infty$, $M_y(t) - M_y(0^+)$ is the relaxation moment at any time t , and $M_y(\infty) - M_y(0^+)$ is the total relaxation moment.

For the two loading conditions, the time-dependent normal stress at the interface can be evaluated by using, respectively, the following two equations;

$$\sigma_{zz}(x, \theta, 0, t) = -\frac{M_y}{4\pi a^3} \frac{d}{dx} \int_x^1 \frac{\phi(y, t)}{\sqrt{y^2 - x^2}} dy \cos \theta \quad (6.3.9)$$

$$\sigma_{zz}(x, \theta, 0, t) = -\frac{16\mu(1-\nu)\Omega_y}{\pi(3-4\nu)} \frac{d}{dx} \int_x^1 \frac{\phi(y, t)}{\sqrt{y^2 - x^2}} dy \cos \theta$$

where $0 \leq x < 1$, $0 \leq \theta < 2\pi$.

6.4 In-Plane Translation of a Rigid Disc Inclusion

6.4.1 Non-dimensional integral equations

Introducing the non-dimensional variables $\hat{\phi}(x, s_1)$, $\hat{\varphi}(x, s_1)$, $\hat{\psi}(x, s_1)$, $\hat{X}_x(s_1)$, and $\hat{Y}_x(s_1)$ defined by

$$\begin{aligned} x &= \frac{r}{a} \\ s_1 &= \frac{a^2 s}{c} \\ \hat{\phi}(r, s) &= -\frac{P_z}{4\pi a} \hat{\phi}(x, s_1) \\ \hat{\psi}(r, s) &= -\frac{P_z}{4\pi\sqrt{a}} \sqrt{x} \hat{\psi}(x, s_1) \\ \hat{\varphi}(r, s) &= -\frac{P_z}{4\pi} \hat{\varphi}(x, s_1) \\ \frac{2}{\pi} \alpha_b \hat{D}_x(s) &= -\frac{P_z}{4\pi a} \hat{X}_x(x, s_1) \\ \hat{P}_x(s) &= P_z \hat{Y}_x(s_1) \end{aligned} \quad (6.4.1)$$

we can re-write the two systems of Fredholm integral equations of the second kind (5.4.18) for the in-plane translation of a rigid disc inclusion in the following non-dimensional form: (i) for a permeable inclusion, we have

$$\begin{aligned}
 & \begin{pmatrix} 1 & -\alpha_1 & 0 \\ -\alpha_1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\phi}(x, s_1) \\ \hat{\psi}(x, s_1) \\ \hat{\varphi}(x, s_1) \end{pmatrix} \\
 & + \int_0^1 \begin{pmatrix} K_1(x, y, s_1) & K_2(x, y, s_1) & K_3(x, y, s_1) \\ K_4(x, y, s_1) & K_5(x, y, s_1) & K_6(x, y, s_1) \\ K_7(x, y, s_1) & K_8(x, y, s_1) & K_9(x, y, s_1) \end{pmatrix} \begin{pmatrix} \hat{\phi}(y, s_1) \\ \hat{\psi}(y, s_1) \\ \hat{\varphi}(y, s_1) \end{pmatrix} dy = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \hat{X}_x(s_1) \\
 & \int_0^1 \hat{\phi}(y, s_1) dy = \hat{Y}_x(s_1)
 \end{aligned} \tag{6.4.2.a}$$

and (ii) for an impermeable inclusion, we have

$$\begin{aligned}
 & \begin{pmatrix} 1 & -\alpha_1 \\ -\alpha_1 & 1 \end{pmatrix} \begin{pmatrix} \hat{\phi}(x, s_1) \\ \hat{\psi}(x, s_1) \end{pmatrix} \\
 & + \int_0^1 \begin{pmatrix} K_1(x, y, s_1) & K_2(x, y, s_1) \\ K_4(x, y, s_1) & K_5(x, y, s_1) \end{pmatrix} \begin{pmatrix} \hat{\phi}(y, s_1) \\ \hat{\psi}(y, s_1) \end{pmatrix} dy = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \hat{X}_x(s_1) \\
 & \int_0^1 \hat{\phi}(y, s_1) dy = \hat{Y}_x(s_1)
 \end{aligned} \tag{6.4.2.b}$$

where $0 \leq x < 1$, and the non-dimensional kernel functions are

$$K_1(x, y, s_1) = -\alpha_6 I_1(x, y, s_1)$$

$$K_2(x, y, s_1) = \alpha_6 [I_1(x, y, s_1) - I_2(x, y, s_1)] + \frac{\alpha_1}{y} H(y - x)$$

$$K_3(x, y, s_1) = 2\alpha_5 [I_5(x, y, s_1) + \frac{\sqrt{s_1}}{2} H(y - x)]$$

$$K_4(x, y, s_1) = \alpha_6 [I_1(x, y, s_1) - I_3(x, y, s_1)] + \frac{\alpha_1}{x} H(x - y)$$

$$K_5(x, y, s_1) = -\alpha_6 [I_1(x, y, s_1) - I_2(x, y, s_1) - I_3(x, y, s_1) + I_4(x, y, s_1)] \quad (6.4.2.c)$$

$$K_6(x, y, s_1) = 2\alpha_5 \left[\frac{1}{x\sqrt{s_1}} I_7(x, y, s_1) - I_5(x, y, s_1) - \frac{\sqrt{s_1}}{2} H(y - x) \right]$$

$$K_7(x, y, s_1) = -\alpha_8 \left[I_6(x, y, s_1) + \frac{\sqrt{s_1}}{2} H(x - y) \right]$$

$$K_8(x, y, s_1) = -\alpha_8 \left[\frac{1}{y\sqrt{s_1}} I_7(x, y, s_1) - I_6(x, y, s_1) - \frac{\sqrt{s_1}}{2} H(x - y) \right]$$

$$K_9(x, y, s_1) = I_7(x, y, s_1)$$

where

$$I_1(x, y, s_1) = \frac{2}{\pi} s_1 \int_0^{+\infty} [f_3(\rho, s_1) + f_4(\rho, s_1)] \cos(\rho x) \cos(\rho y) d\rho$$

$$I_2(x, y, s_1) = \frac{2}{\pi} s_1 \int_0^{+\infty} \frac{1}{\rho y} [f_3(\rho, s_1) + f_4(\rho, s_1)] \cos(\rho x) \sin(\rho y) d\rho$$

$$I_3(x, y, s_1) = \frac{2}{\pi} s_1 \int_0^{+\infty} \frac{1}{\rho x} [f_3(\rho, s_1) + f_4(\rho, s_1)] \sin(\rho x) \cos(\rho y) d\rho$$

$$I_4(x, y, s_1) = \frac{2}{\pi} s_1 \int_0^{+\infty} \frac{1}{\rho^2 xy} [f_3(\rho, s_1) + f_4(\rho, s_1)] \sin(\rho x) \sin(\rho y) d\rho$$

$$I_5(x, y, s_1) = \frac{2}{\pi} \sqrt{s_1} \int_0^{+\infty} f_5(\rho, s_1) \cos(\rho x) \sin(\rho y) d\rho$$

(6.4.2.d)

$$I_6(x, y, s_1) = \frac{2}{\pi} \sqrt{s_1} \int_0^{+\infty} f_5(\rho, s_1) \sin(\rho x) \cos(\rho y) d\rho$$

$$I_7(x, y, s_1) = \frac{2}{\pi} s_1 \int_0^{+\infty} f_3(\rho, s_1) \sin(\rho x) \sin(\rho y) d\rho$$

$$f_3(\rho, s_1) = \frac{1}{\rho^2 + s_1 + \rho \sqrt{\rho^2 + s_1}}$$

$$f_4(\rho, s_1) = \frac{\rho}{2\rho(\rho^2 + s_1) + (2\rho^2 + s_1)\sqrt{\rho^2 + s_1}}$$

$$f_5(\rho, s_1) = \frac{\rho^2 - s_1 - \rho \sqrt{\rho^2 + s_1}}{\rho^2 + s_1 + \rho \sqrt{\rho^2 + s_1}}$$

The infinite integrals in equations (6.4.2.d) with the independent variables x , y , and s_1 are uniformly convergent if and only if $\text{Re}(s_1) > 0$. Furthermore, the material

parameters do not occur in these infinite integrals. However, it is not possible to find the exact integrations of such infinite integrals.

6.4.2 Numerical solutions of the integral equations

It can be shown that these two systems of Fredholm integral equations of the second kind (6.4.2) are regular and standard integral equations with non-singular kernel functions. Again, by considering the structure of the kernel function in equations (6.3.2.c) it becomes evident that the system of complex Fredholm integral equation of the second kind has no known exact solution. Consequently it is necessary to adopt a numerical scheme to generate the relevant results. We basically follow the procedure outlined in section 6.2.

The systems of complex Fredholm integral equations of the second kind (6.4.2) can be reduced into the systems of real Fredholm integral equation of the second kind by separating the complex variables and kernel function into their real and imaginary parts; *i.e.* :

$$\begin{aligned}
 \hat{K}_1(x, y, s_1) &= \hat{K}_{11}(x, y) + i\hat{K}_{12}(x, y) \\
 \hat{K}_2(x, y, s_1) &= \hat{K}_{21}(x, y) + i\hat{K}_{22}(x, y) \\
 \hat{K}_3(x, y, s_1) &= \hat{K}_{31}(x, y) + i\hat{K}_{32}(x, y) \\
 \hat{K}_4(x, y, s_1) &= \hat{K}_{41}(x, y) + i\hat{K}_{42}(x, y) \\
 \hat{K}_5(x, y, s_1) &= \hat{K}_{51}(x, y) + i\hat{K}_{52}(x, y) \\
 \hat{K}_6(x, y, s_1) &= \hat{K}_{61}(x, y) + i\hat{K}_{62}(x, y) \\
 \hat{K}_7(x, y, s_1) &= \hat{K}_{71}(x, y) + i\hat{K}_{72}(x, y) \\
 \hat{K}_8(x, y, s_1) &= \hat{K}_{81}(x, y) + i\hat{K}_{82}(x, y) \\
 \hat{K}_9(x, y, s_1) &= \hat{K}_{91}(x, y) + i\hat{K}_{92}(x, y)
 \end{aligned}
 \tag{6.4.3.a}$$

and

$$\begin{aligned}
 s_1 &= s_r + is_i \\
 \hat{\phi}(x, s_1) &= \hat{\phi}_1(x) + i\hat{\phi}_2(x) \\
 \hat{\psi}(x, s_1) &= \hat{\psi}_1(x) + i\hat{\psi}_2(x) \\
 \hat{\varphi}(x, s_1) &= \hat{\varphi}_1(x) + i\hat{\varphi}_2(x) \\
 \hat{X}_x(s_1) &= \hat{X}_{x1} + i\hat{X}_{x2} \\
 \hat{Y}_x(s_1) &= \hat{Y}_{x1} + i\hat{Y}_{x2}
 \end{aligned} \tag{6.4.3.b}$$

Substituting these formulae into the equations (6.4.2), we have the two systems of real Fredholm integral equations of the second kind as follows;

(i) for an inclusion with permeable surfaces, we have

$$\begin{aligned}
 &\int_0^1 \begin{pmatrix} K_{11}(x, y) & -K_{12}(x, y) & K_{21}(x, y) & -K_{22}(x, y) & K_{31}(x, y) & -K_{32}(x, y) \\ K_{12}(x, y) & K_{11}(x, y) & K_{22}(x, y) & K_{21}(x, y) & K_{32}(x, y) & K_{31}(x, y) \\ K_{41}(x, y) & -K_{42}(x, y) & K_{51}(x, y) & -K_{52}(x, y) & K_{61}(x, y) & -K_{62}(x, y) \\ K_{42}(x, y) & K_{41}(x, y) & K_{52}(x, y) & K_{51}(x, y) & K_{62}(x, y) & K_{61}(x, y) \\ K_{71}(x, y) & -K_{72}(x, y) & K_{81}(x, y) & -K_{82}(x, y) & K_{91}(x, y) & -K_{92}(x, y) \\ K_{72}(x, y) & K_{71}(x, y) & K_{82}(x, y) & K_{81}(x, y) & K_{92}(x, y) & K_{91}(x, y) \end{pmatrix} \begin{pmatrix} \hat{\phi}_1(y) \\ \hat{\phi}_2(y) \\ \hat{\psi}_1(y) \\ \hat{\psi}_2(y) \\ \hat{\varphi}_1(y) \\ \hat{\varphi}_2(y) \end{pmatrix} dy \\
 &+ \begin{pmatrix} 1 & 0 & -\alpha_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\alpha_1 & 0 & 0 \\ -\alpha_1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -\alpha_1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\phi}_1(x) \\ \hat{\phi}_2(x) \\ \hat{\psi}_1(x) \\ \hat{\psi}_2(x) \\ \hat{\varphi}_1(x) \\ \hat{\varphi}_2(x) \end{pmatrix} = \begin{pmatrix} \hat{X}_{x1} \\ \hat{X}_{x2} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
 &\int_0^1 \begin{pmatrix} \hat{\phi}_1(y) \\ \hat{\phi}_2(y) \end{pmatrix} dy = \begin{pmatrix} \hat{Y}_{x1} \\ \hat{Y}_{x2} \end{pmatrix}
 \end{aligned} \tag{6.4.4.a}$$

and (ii) for an inclusion with impermeable surfaces, we have

$$\begin{aligned}
 & \int_0^1 \begin{pmatrix} K_{11}(x, y) & -K_{12}(x, y) & K_{21}(x, y) & -K_{22}(x, y) \\ K_{12}(x, y) & K_{11}(x, y) & K_{22}(x, y) & K_{21}(x, y) \\ K_{41}(x, y) & -K_{42}(x, y) & K_{51}(x, y) & -K_{52}(x, y) \\ K_{42}(x, y) & K_{41}(x, y) & K_{52}(x, y) & K_{51}(x, y) \end{pmatrix} \begin{pmatrix} \hat{\phi}_1(y) \\ \hat{\phi}_2(y) \\ \hat{\psi}_1(y) \\ \hat{\psi}_2(y) \end{pmatrix} dy \\
 & + \begin{pmatrix} 1 & 0 & -\alpha_1 & 0 \\ 0 & 1 & 0 & -\alpha_1 \\ -\alpha_1 & 0 & 1 & 0 \\ 0 & -\alpha_1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\phi}_1(x) \\ \hat{\phi}_2(x) \\ \hat{\psi}_1(x) \\ \hat{\psi}_2(x) \end{pmatrix} = \begin{pmatrix} \hat{X}_{x1} \\ \hat{X}_{x2} \\ 0 \\ 0 \end{pmatrix} \quad (6.4.4.b) \\
 & \int_0^1 \begin{pmatrix} \hat{\phi}_1(y) \\ \hat{\phi}_2(y) \end{pmatrix} dy = \begin{pmatrix} \hat{Y}_{x1} \\ \hat{Y}_{x2} \end{pmatrix}
 \end{aligned}$$

where $0 \leq x < 1$.

For the numerical solution of the problem, the interval $[0, 1]$ is divided into N segments with ends defined by $r_k = (k - 1)/N$; $k = 1, 2, 3 \dots (N + 1)$, and the collocation points are $x_k = (r_k + r_{k+1})/2$, $k = 1, 2, 3 \dots N$. Consequently we can convert the above two systems of integral equations into two systems of linear algebraic simultaneous equations. (i) For the inclusion with permeable surfaces, we

obtain the linear algebraic equations:

$$\sum_{l=1}^N \frac{1}{N} \begin{pmatrix} K_{11}(x_k, x_l) & -K_{12}(x_k, x_l) & K_{21}(x_k, x_l) & -K_{22}(x_k, x_l) & K_{31}(x_k, x_l) & -K_{32}(x_k, x_l) \\ K_{12}(x_k, x_l) & K_{11}(x_k, x_l) & K_{22}(x_k, x_l) & K_{21}(x_k, x_l) & K_{32}(x_k, x_l) & K_{31}(x_k, x_l) \\ K_{41}(x_k, x_l) & -K_{42}(x_k, x_l) & K_{51}(x_k, x_l) & -K_{52}(x_k, x_l) & K_{61}(x_k, x_l) & -K_{62}(x_k, x_l) \\ K_{42}(x_k, x_l) & K_{41}(x_k, x_l) & K_{52}(x_k, x_l) & K_{51}(x_k, x_l) & K_{62}(x_k, x_l) & K_{61}(x_k, x_l) \\ K_{71}(x_k, x_l) & -K_{72}(x_k, x_l) & K_{81}(x_k, x_l) & -K_{82}(x_k, x_l) & K_{91}(x_k, x_l) & -K_{92}(x_k, x_l) \\ K_{72}(x_k, x_l) & K_{71}(x_k, x_l) & K_{82}(x_k, x_l) & K_{81}(x_k, x_l) & K_{92}(x_k, x_l) & K_{91}(x_k, x_l) \end{pmatrix}$$

$$\begin{pmatrix} \hat{\phi}_1(x_l) \\ \hat{\phi}_2(x_l) \\ \hat{\psi}_1(x_l) \\ \hat{\psi}_2(x_l) \\ \hat{\varphi}_1(x_l) \\ \hat{\varphi}_2(x_l) \end{pmatrix} + \begin{pmatrix} 1 & 0 & -\alpha_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\alpha_1 & 0 & 0 \\ -\alpha_1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -\alpha_1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\phi}_1(x_k) \\ \hat{\phi}_2(x_k) \\ \hat{\psi}_1(x_k) \\ \hat{\psi}_2(x_k) \\ \hat{\varphi}_1(x_k) \\ \hat{\varphi}_2(x_k) \end{pmatrix} = \begin{pmatrix} \hat{X}_{x1} \\ \hat{X}_{x2} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\sum_{l=1}^N \frac{1}{N} \begin{pmatrix} \hat{\phi}_1(x_l) \\ \hat{\phi}_2(x_l) \end{pmatrix} dy = \begin{pmatrix} \hat{Y}_{x1} \\ \hat{Y}_{x2} \end{pmatrix}$$

(6.4.5.a)

and (ii) for the inclusion with impermeable surfaces, we have

$$\sum_{l=1}^N \frac{1}{N} \begin{pmatrix} K_{11}(x_k, x_l) & -K_{12}(x_k, x_l) & K_{21}(x_k, x_l) & -K_{22}(x_k, x_l) \\ K_{12}(x_k, x_l) & K_{11}(x_k, x_l) & K_{22}(x_k, x_l) & K_{21}(x_k, x_l) \\ K_{41}(x_k, x_l) & -K_{42}(x_k, x_l) & K_{51}(x_k, x_l) & -K_{52}(x_k, x_l) \\ K_{42}(x_k, x_l) & K_{41}(x_k, x_l) & K_{52}(x_k, x_l) & K_{51}(x_k, x_l) \end{pmatrix} \begin{pmatrix} \hat{\phi}_1(x_l) \\ \hat{\phi}_2(x_l) \\ \hat{\psi}_1(x_l) \\ \hat{\psi}_2(x_l) \end{pmatrix} + \begin{pmatrix} 1 & 0 & -\alpha_1 & 0 \\ 0 & 1 & 0 & -\alpha_1 \\ -\alpha_1 & 0 & 1 & 0 \\ 0 & -\alpha_1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\phi}_1(x_k) \\ \hat{\phi}_2(x_k) \\ \hat{\psi}_1(x_k) \\ \hat{\psi}_2(x_k) \end{pmatrix} = \begin{pmatrix} \hat{X}_{x1} \\ \hat{X}_{x2} \\ 0 \\ 0 \end{pmatrix}$$

$$\sum_{l=1}^N \frac{1}{N} \begin{pmatrix} \hat{\phi}_1(x_l) \\ \hat{\phi}_2(x_l) \end{pmatrix} dy = \begin{pmatrix} \hat{Y}_{x1} \\ \hat{Y}_{x2} \end{pmatrix} \quad (6.4.5.b)$$

where $k = 1, 2, 3, \dots, N$.

These linear algebraic simultaneous equations can be further uniformly written in matrix form as

$$\sum_{j=1}^L [\mathbf{A}_{lj}] \{\hat{\mathbf{X}}_j\} = \{\mathbf{B}_l\} \quad (6.4.6)$$

where $l = 1, 2, 3, \dots, L$, $L = 6N + 2$ for a permeable inclusion and $L = 4N + 2$ for an impermeable inclusion.

The matrix equation (6.4.6) can be solved numerically to generate the unknown variables $\hat{\phi}(x_l, s_1)$, $\hat{\psi}(x_l, s_1)$, $\hat{\phi}(x_l, s_1)$, $\hat{X}_x(s_1)$ (or $\hat{Y}_x(s_1)$) in the Laplace transform domain. The time-dependent results of the variables $\phi(x_l, t)$, $\psi(x_l, t)$, $\phi(x_l, t)$, $X_x(t)$ (or $Y_x(t)$) can be evaluated by using the modified algorithm for Laplace transform inversion.

For the in-plane translation of the inclusion, there are at least two types of loading conditions which are of engineering interests. The first type of loading conditions is one where the time-dependent behaviour of total in-plane force $P_x(t)$ (=

$P_z Y_z(t)$ applied on the rigid disc inclusions is known *a priori*. The second is that where the time-dependent history of in-plane translation $D_z(t)(= D_z X_z(t))$ of the rigid disc inclusion is prescribed. The most basic condition in the first category is that of a constant in-plane load, i.e., $P_z(t) = P_z H(t)$. For this condition, we can write down the time-dependent response of the in-plane translation $D_z(t)$ as follows,

$$\frac{a\mu D_z(t)}{P_z} = \frac{7-8\nu}{64(1-\nu)} X_z(t) \quad (6.4.7.a)$$

and the degree of in-plane consolidation displacement of the rigid disc can be expressed in the conventional form

$$\begin{aligned} U_z(t) &= \frac{D_z(t) - D_z(0^+)}{D_z(\infty) - D_z(0^+)} \\ &= \frac{X_z(t) - \alpha_{10}}{1 - \alpha_{10}} \end{aligned} \quad (6.4.7.b)$$

where $0 < t < +\infty$, $D_z(t) - D_z(0^+)$ is the consolidation translation at any time t , $D_z(\infty) - D_z(0^+)$ is the total consolidation translation, and the constant α_{10} is given by,

$$\alpha_{10} = \frac{(7-8\nu_u)(1-\nu)}{(7-8\nu)(1-\nu_u)} \quad (6.4.7.c)$$

The most basic condition in the second category is that of a constant in-plane displacement, i.e., $D_z(t) = D_z H(t)$. For this condition, we can write down the time-dependent response of the in-plane total force $P_z(t)$ as follows,

$$\frac{P_z(t)}{a\mu D_z} = \frac{64(1-\nu)}{7-8\nu} Y_z(t) \quad (6.4.8.a)$$

and the degree of relaxation of in-plane force can be expressed in the form

$$\begin{aligned}
 V_z(t) &= \frac{P_z(t) - P_z(0^+)}{P_z(\infty) - P_z(0^+)} \\
 &= \frac{\alpha_{10} Y_z(t) - 1}{\alpha_{10} - 1}
 \end{aligned}
 \tag{6.4.8.b}$$

where $0 < t < +\infty$, $P_z(t) - P_z(0^+)$ is the relaxation force at any time t , and $P_z(\infty) - P_z(0^+)$ is the total relaxation force.

For the two conditions, the time-dependent shear stresses¹ at the interface can be evaluated by using, respectively, the following two equations,

$$\begin{aligned}
 \sigma_{xz}(x, \theta, 0, t) &= \frac{-P_z}{4\pi a^2} \left[\frac{\phi(1, t)}{\sqrt{1-x^2}} - \int_x^1 \frac{\partial \phi(y, t)}{\partial y} \frac{1}{\sqrt{y^2-x^2}} dy \right. \\
 &\quad \left. + x \frac{d}{dx} \int_x^1 \frac{\psi(y, t)}{y \sqrt{y^2-x^2}} dy \cos 2\theta \right]
 \end{aligned}
 \tag{6.4.9.a}$$

$$\sigma_{yz}(x, \theta, 0, t) = \frac{-P_z}{4\pi a^2} \left[x \frac{d}{dx} \int_x^1 \frac{\psi(y, t)}{y \sqrt{y^2-x^2}} dy \right] \sin 2\theta$$

and

$$\begin{aligned}
 \sigma_{xz}(x, \theta, 0, t) &= \frac{-16\mu(1-\nu)D_z}{\pi a(7-8\nu)} \left[\frac{\phi(1, t)}{\sqrt{1-x^2}} - \int_x^1 \frac{\partial \phi(y, t)}{\partial y} \frac{1}{\sqrt{y^2-x^2}} dy \right. \\
 &\quad \left. + x \frac{d}{dx} \int_x^1 \frac{\psi(y, t)}{y \sqrt{y^2-x^2}} dy \cos 2\theta \right]
 \end{aligned}
 \tag{6.4.9.b}$$

$$\sigma_{yz}(x, \theta, 0, t) = \frac{-16\mu(1-\nu)D_z}{\pi a(7-8\nu)} \left[x \frac{d}{dx} \int_x^1 \frac{\psi(y, t)}{y \sqrt{y^2-x^2}} dy \right] \sin 2\theta$$

where $0 \leq x < 1$, $0 \leq \theta < 2\pi$.

¹ It is noted that the shear stress $\sigma_{yz}(x, \theta, 0, t)$ may not equal to zero while this shear stress is identical to zero for the corresponding inclusion problem in a purely elastic medium.

6.5 Convergence and Accuracy of the Numerical Solutions

The numerical scheme discussed in above three sections consists of three computational steps which are essentially based on three-dimensional repeated numerical integrations.

The first computational step involves the numerical integration of the infinite integrals (6.2.2.e), (6.3.2.d) and (6.4.2.d), respectively, in the kernel functions (6.2.2.c), (6.3.2.c) and (6.4.2.c) for the systems of Fredholm integral equations of the second kind. As stated previously, these infinite integrals are uniformly convergent. However, there are two unusual problems in the numerical integration. One is the infinite limit and the other is associated with the integrands. As the values of the independent variables s and ρ become larger, the integrands will become rapidly oscillatory functions. This property of the integrands will slow down the convergence of the numerical integration and cause the numerical integration unstable. The large values of the complex variable s are associated with small values of time t in the numerical inversion of Laplace transforms. A particular technique adopted in this thesis overcomes these two numerical problems associated with the infinite limit and the rapidly oscillatory integrands. This particular technique consists of a Simpson quadrature based proceeding limit technique and an approximation technique, the details of which are documented in the Appendix D.

The second repeated numerical integration involves the systems of Fredholm integral equations whose properties have been well investigated by many researchers (e.g., Baker, 1978; Delves and Mohamed, 1985; Selvadurai and Singh and Au, (1990); and Selvadurai and Au and Singh, 1991). With an increase in the segment number N we can obtain more accurate and convergent results for the integral equations.

The last computational step is the numerical inversion of Laplace transforms. A modified algorithm is proposed and verified for the numerical inversion of Laplace

transforms (see Section 2.4 and Appendix A).

Due to the accumulations of the errors in each repeated numerical integration, however, it is necessary to evaluate and test the convergence and accuracy of the time-dependent results for the systems of Fredholm integral equations of the second kind in the Laplace transform domain. These calculations are documented in the Appendix E. It is concluded from this verification that the numerical scheme and techniques adopted in this thesis provide highly stable and accurate solutions in the time domain for the systems of Fredholm integral equations of the second kind in the Laplace transform domain and particularly overcome the numerical problems at the very initial responses ($10^{-4} \leq ct/a^2 \leq 10^{-2}$) of the poroelastic medium encountered by other investigators.

Chapter 7

Mechanics Of Inclusions In Poroelastic Media

III. Analytical And Numerical Results

7.1 General

In this Chapter, we shall present analytical and numerical results for the time-dependent behaviour of a rigid disc inclusion embedded in a poroelastic medium of infinite extent saturated with a compressible pore fluid.

The time-dependent behaviour of an embedded rigid disc inclusion is represented by the relationship between the fundamental loading mode $P_z(t)$ ($M_y(t)$, $P_z(t)$, or $M_z(t)$) and the associated fundamental displacement mode $D_z(t)$ ($\Omega_y(t)$, $D_z(t)$, or $\Omega_z(t)$). The time-dependent relationship between the axial moment $M_z(t)$ and the axial rotation $\Omega_z(t)$ of the embedded inclusion is simply expressed in an exact closed-form, *i.e.*,

$$M_z(t) = \frac{32\mu a^3}{3} \Omega_z(t) \quad (7.1.1)$$

In order to obtain the behaviour of the axial translation and central rotation and in-plane translation for the embedded inclusion, we assume that the embedded rigid disc inclusion is subjected to the most basic loading, *i.e.*, $P_z H(t)$ ($M_y H(t)$, $P_z H(t)$, $D_z H(t)$, $\Omega_y H(t)$, or $D_z H(t)$). We then analyze the time-dependent response of the associated fundamental mode, *i.e.*, $D_z(t)$ ($\Omega_y(t)$, $D_z(t)$, $P_z(t)$ $M_y(t)$, or $P_z(t)$), by considering separately the influence of the interface drainage conditions and the six parameters, μ , ν , ν_* , B , κ , and a . The drainage conditions at the inclusion-poroelastic medium interface are assumed to be either complete permeable or complete impermeable. These two drainage conditions are the limiting cases which can offer the “upper” and “lower” estimates for all the drainage conditions

that can occur at the inclusion-soil interface. The parameters μ , ν , ν_u , B , and κ are the five basic material parameters describing the consolidation response of a poroelastic medium saturated with a compressible pore fluid. The other parameter a is the radius of the rigid disc inclusion which represents the length parameter for the fundamental inclusion problems. Finally, we discuss and particularly comment on the mechanics of fundamental inclusion problems by considering together the four time-dependent displacement or force modes associated with an embedded rigid disc inclusion.

For clarification, we define in this thesis that (i) consolidation (or relaxation) rate is the transient rate of the consolidation displacement (or relaxation force) associated with an embedded inclusion and (ii) consolidation (or relaxation) time is the time used by the embedded inclusion for a certain degree of consolidation displacement (or relaxation force).

Two non-dimensional time factors are utilized in the presentation of the numerical results. One is the time factor ct/a^2 and the other is a modified time factor $c_m t/a^2$; i.e.,

$$\begin{aligned}\frac{ct}{a^2} &= \frac{(1-\nu)(1+\nu_u)^2}{9(\nu_u-\nu)(1-\nu_u)} \frac{2\mu B^2 \kappa}{a^2} t \\ \frac{c_m t}{a^2} &= \frac{2\mu B^2 \kappa}{a^2} t\end{aligned}\tag{7.1.2}$$

where c is the generalized coefficient of consolidation and c_m is a modified coefficient of consolidation. They have the following relationship,

$$c = \nu_b c_m\tag{7.1.3}$$

where,

$$\nu_b = \frac{(1-\nu)(1+\nu_u)^2}{9(\nu_u-\nu)(1-\nu_u)}\tag{7.1.4}$$

The modified time factor $c_m t/a^2$ eliminates the dependency on Poisson's ratios ν and ν_u in the time factor ct/a^2 .

Furthermore, according to the differentiations:

$$\frac{dG(x)}{dx} = \ln(10) t \frac{dG(x)}{dt}\tag{7.1.5}$$

where $x = \log(ct/a^2)$ or $\log(c_m t/a^2)$, we can observe that the consolidation (or relaxation) rate at a certain time is indicated by the gradient of the plots for the consolidation displacement (or relaxation force) or the associated degree of consolidation (or relaxation) against the logarithm of one of the two time factors.

7.2 Axial Translation of a Rigid Disc Inclusion

7.2.1 Axial consolidation displacement of the inclusion

We examine the time-dependent axial displacement $D_z(t)$ and the associated degree of consolidation $U_z(t)$ for either a permeable or an impermeable embedded rigid disc inclusion due to the action of a constant total axial force $P_z H(t)$.

7.2.1.1 Influence of interface drainage conditions

The influence of interface drainage conditions on the axial consolidation displacement of a rigid disc inclusion embedded in the saturated poroelastic medium can be observed in Fig.7.2.1 and Fig.7.2.2 in which the degree of consolidation is plotted against the logarithm of the non-dimensional time factor ct/a^2 . The drainage conditions at the inclusion-poroelastic medium interface have a significant effect on the consolidation rate of axial translation. At the initial stages, the consolidation rate of axial translation of a *permeable* inclusion is much faster than that of an *impermeable* inclusion. At a certain time ($ct/a^2 = 0.07$ for $\nu = 0.0$ and $\nu_u = 0.5$), the rate of consolidation for axial translation of a *permeable* inclusion turns out to be slower than that of an *impermeable* inclusion. When the time factor ct/a^2 is larger than 10, the consolidation behaviour of the permeable and impermeable inclusions are virtually identical and the drainage conditions almost have no observable effect on the consolidation associated with the axial translation.

The difference $\Delta U(ct/a^2)$ between the degrees of axial consolidation displacement for the *permeable* and *impermeable* inclusions monotonically increases at the initial stages and reaches its maximum value of 17.32% (= 43.90% - 26.58%) at the turning point where $ct/a^2 = 0.07$ and then monotonically decreases to zero as

$ct/a^2 \rightarrow +\infty$. At $ct/a^2 = 10$, the degrees of consolidation in axial translation are equal to 92.48% and 92.35% for the *permeable* and *impermeable* inclusions, respectively.

7.2.1.2 Effects of material and length parameters

The role of the four parameters μ , B , κ , and a on the axial translation can be determined analytically from the non-dimensional axial displacement, i.e.,

$$\frac{a\mu}{P_z} D_z(ct/a^2) = \frac{3-4\nu}{32(1-\nu)} X_z(ct/a^2) \quad (7.2.1)$$

where $X_z(x)$ is a non-dimensional function evaluated from the integral equations. This function $X_z(x)$ includes only the material parameters of the drained and undrained Poisson's ratios, i.e., ν and ν_u . The equations (7.2.1) imply that the time-dependent flexibility $D_z(\nu_b t)/P_z$ is inversely proportional to $a\mu$ and the consolidation time is also inversely proportional to $2\mu B^2 \kappa/a^2 (= c_m)$.

The functions of the other two parameters ν , ν_u , i.e., the drained and undrained Poisson's ratios will be discussed in the following. From Fig.7.2.1 and Fig.7.2.2, it may be observed that the degree of consolidation translation $U_z(ct/a^2)$ is relatively insensitive to the values of the drained and undrained Poisson's ratios. For the following range of values of $(\nu, \nu_u) = (0.0, 0.5), (0.3, 0.5), (0.49, 0.5), (0.0, 0.01)$, and $(0.0, 0.3)$, the differences are not easily discernible in the plots given in Fig.7.2.1 and Fig.7.2.2. Similar results were also observed by Pyrah, Hird and Tanaka (1985) and Small and Booker (1987) in the analysis of rigid disc underreamed anchor embedded in a poroelastic infinite space saturated with an incompressible fluid. Based on this phenomenon, the above authors concluded that "an accurate knowledge of Poisson's ratio ν is not important when making predictions of anchor behaviour in saturated soil media". However, this conclusion is inaccurate since the time factor ct/a^2 is a function of Poisson's ratios ν , ν_u (see, eg. equations (7.1.2)).

The degree of consolidation associated with the axial translation of the rigid disc inclusion is then re-plotted with the logarithm of the modified time factor $c_m t/a^2$ (see, Fig.7.2.3 ~ Fig.7.2.6). These figures show that the degree of consolidation

associated with the axial translation of the rigid disc inclusion is extremely sensitive to the values of Poisson's ratios ν, ν_u . As ν approaches ν_u , the consolidation rate is much faster and the consolidation time is much shorter.

The variations in the non-dimensional axial displacement $a\mu D_z(ct/a^2)/P_z$ are given in Fig.7.2.7 ~ Fig.7.2.10 for a series pairs of Poisson's ratios. We observe that the non-dimensional axial displacement $a\mu D_z(ct/a^2)/P_z$ is also extremely sensitive to the values of Poisson's ratios ν, ν_u .

Based on these results, the influence of Poisson's ratios on the time-dependent behaviour of the axial translation of the rigid disc inclusion can be summarized as follows:

- i. The undrained Poisson's ratio ν_u governs the initial translation of the rigid disc inclusion. The initial translation decreases with increase in ν_u .
- ii. The drained Poisson's ratio ν governs the final translation of the rigid disc inclusion. The final translation decreases with increase in ν .
- iii. The consolidation translation of the inclusion can be expressed in the form:

$$\frac{a\mu}{P_z}[D_z(ct/a^2) - D_z(0^+)] = \frac{\nu_u - \nu}{32(1 - \nu)(1 - \nu_u)} U_z(ct/a^2) \quad (7.2.2)$$

From the equation (7.2.2), we observe that the difference between the undrained and drained Poisson's ratios, $(\nu_u - \nu)$, governs the magnitude and rate of the non-dimensional axial displacement induced by consolidation. The smaller the difference $(\nu_u - \nu)$, the smaller the consolidation displacement. The consolidation rate and time associated with the axial translation of the rigid disc inclusion is not very sensitive to the Poisson's ratios occurring in the integral equations (6.2.2). The value of ν_b in equations (7.1.3) controls the consolidation rate and time by using the modified time factor $c_m t/a^2$. The higher the value of ν_b , the faster the consolidation rate and the shorter the consolidation time.

- iv. The maximum percentage of total consolidation induced translation over initial translation is 50% at $\nu = 0.0$ and $\nu_u = 0.5$ provided $0 \leq \nu < \nu_u \leq 0.5$.

Fig.7.2.1 Effects of drainage conditons on the axial translation of a rigid disc embedded in saturated poroelastic media

Fig.7.2.2 Effects of drainage conditons on the axial translation of a rigid disc embedded in saturated poroelastic media

Fig.7.2.3 Degrees of consolidation of axial translation of a permeable rigid disc inclusion against the modified time factor

Fig.7.2.4 Degrees of consolidation of axial translation of a permeable rigid disc inclusion against the modified time factor

Fig.7.2.5 Degrees of consolidation of axial translation of an impermeable rigid disc against the modified time factor

Fig.7.2.6 Degrees of consolidation of axial translation of an impermeable rigid disc against the modified time factor

Fig.7.2.7 Axial translation of a permeable rigid disc inclusion in poroelastic media saturated with compressible pore fluids

Fig.7.2.8 Axial translation of a permeable rigid disc inclusion in poroelastic media saturated with compressible pore fluids

Fig.7.2.9 Axial translation of an impermeable rigid disc in poroelastic media saturated with compressible pore fluids

Fig.7.2.10 Axial translation of an impermeable rigid disc embedded in poroelastic media saturated with compressible pore fluids

- v. At the limiting condition as $\nu \rightarrow \nu_u$, consolidation time is zero and the complete axial translation of the rigid disc inclusion occurs at the application of the axial load. The coupled two phase model of saturated poroelastic theory decouples into one solid phase model of elastostatics and one fluid phase model of seepage flow.

7.2.2 Axial force relaxation in the inclusion

We examine the time-dependent axial force $P_z(t)$ and the associated degree of relaxation $V_z(t)$ for either a permeable or an impermeable embedded rigid disc inclusion due to the action of a constant axial translation $D_z H(t)$.

7.2.2.1 Influence of interface drainage conditions

The influence of interface drainage conditions on the relaxation of total axial force induced by a rigid disc inclusion embedded in a saturated poroelastic medium is shown in Fig.7.2.11 and Fig.7.2.12 where the degree of relaxation is plotted against the logarithm of the time factor ct/a^2 . The drainage conditions have a significant effect on the relaxation rate of axial force. At the initial stages, the relaxation rate of induced axial force by a *permeable* rigid disc inclusion is much faster than that of an *impermeable* rigid disc inclusion. At a certain time (e.g., $ct/a^2 = 0.05$ for $\nu = 0$ and $\nu_u = 0.5$), the relaxation rate of axial force for a *permeable* rigid disc inclusion turns out to be slower than that for an *impermeable* rigid disc inclusion. If ct/a^2 is larger than a certain value (e.g., $ct/a^2 = 7$ for $\nu = 0$ and $\nu_u = 0.5$), the relaxation behaviour of the *permeable* and *impermeable* inclusions becomes almost the same and the drainage conditions have a negligible effect on the relaxation of axial force.

The difference $\Delta V(ct/a^2)$ between the degrees of relaxation of the force applicable to the *permeable* and *impermeable* inclusions monotonically increases at the initial stages and reaches its maximum value ($19.63\% = 50.53\% - 30.9\%$ for $\nu = 0$ and $\nu_u = 0.5$) at the turning point ($ct/a^2 = 0.05$) and then monotonically decreases to zero as $ct/a^2 \rightarrow +\infty$. At $ct/a^2 = 7$ for $\nu = 0$ and $\nu_u = 0.5$, the degrees of

relaxation of the axial force are equal to 93.98% and 93.89% for the *permeable* and *impermeable* disc inclusions, respectively.

7.2.2.2 Effects of material and length parameters

The effect of the four parameters μ , B , κ , and a on the axial force can be determined from the non-dimensional axial force:

$$\frac{P_z(ct/a^2)}{a\mu D_z} = \frac{32(1-\nu)}{3-4\nu} Y_z(ct/a^2) \quad (7.2.3)$$

where $Y_z(x)$ is a non-dimensional function evaluated from the integral equations. This function $Y_z(x)$ includes only the material parameters of the drained and undrained Poisson's ratios, i.e., ν and ν_u . The equations (7.2.3) imply that the time-dependent stiffness $P_z(\nu_b t)/D_z$ is directly proportional to $a\mu$ and the consolidation time is inversely proportional to $2\mu B^2 \kappa/a^2$.

The influence of the drained and undrained Poisson's ratios ν , ν_u can be observed in the Fig.7.2.11 ~ 7.2.20. The Fig.7.2.11 and Fig.7.2.12 illustrate that relaxation rate decreases and relaxation time increases with the decrease in the difference $(\nu_u - \nu)$, where the degree of relaxation force $V_z(ct/a^2)$ is plotted against the logarithm of the time factor ct/a^2 . However, when the degree of relaxation force $V_z(c_m t/a^2)$ is re-plotted against the modified time factor $c_m t/a^2$ (Fig.7.2.13 and Fig.7.2.15), it is evident that the relaxation rate increases and relaxation time decreases as ν approaches ν_u . These results indicate that the Poisson's ratios in the integral equations (6.2.2) and in the time factor ct/a^2 (i.e., ν_b) produce counteracting effects on the relaxation rate and time of the axial force. Such counteracting effects cause the irregular shift of curves in Fig.7.2.12 as we re-plot them into Fig.7.2.14 and Fig.7.2.16 using the modified time factor $c_m t/a^2$.

The behaviour of the non-dimensional axial force $P_z(ct/a^2)/(a\mu D_z)$ can be clearly observed from the plots in Fig.7.2.17 ~ Fig.7.2.20 for a series pairs of Poisson's ratios. The non-dimensional axial force $P_z(ct/a^2)/(a\mu D_z)$ is also extremely sensitive to the values of Poisson's ratios ν , ν_u . Based on these results, the ef-

fects of Poisson's ratios on the time-dependent behaviour of the axial force can be summarized as follows.

- i. The undrained Poisson's ratio ν_u governs the initial force acting on the rigid disc inclusion. The larger the undrained Poisson's ratio ν_u , the larger the initial force.
- ii. The drained Poisson's ratio ν governs the final force acting on the rigid disc inclusion. The larger the drained Poisson's ratio ν , the larger the final force.
- iii. The non-dimensional relaxation force can be expressed in terms of the degree of consolidation:

$$\frac{1}{a\mu D_s}[P_z(ct/a^2) - P_z(0^+)] = -\frac{32(\nu_u - \nu)}{(3 - 4\nu)(3 - 4\nu_u)}V_z(ct/a^2) \quad (7.2.4)$$

The difference between the undrained and drained Poisson's ratios, $(\nu_u - \nu)$, governs the magnitude and time of relaxation force. The smaller the difference $\nu_u - \nu$, the smaller the relaxation force. The Poisson's ratios in the integral equations (7.2.2) and in the time factor ct/a^2 produce counteracting effects on the relaxation time of axial force. As a combined result, the relaxation time of the induced axial force decreases to zero as ν approaches ν_u in the sense of the modified time factor.

- iv. The maximum percentage of total relaxation force over initial force is 33.33% at $\nu = 0.0$ and $\nu_u = 0.5$ with the constraint of $0 \leq \nu < \nu_u \leq 0.5$.
- v. Under the limiting condition as $\nu \rightarrow \nu_u$, relaxation time is zero and the rigid disc inclusion mobilizes its total axial force once the constant axial displacement is applied. The coupled two phase model of saturated poroelastic theory decouples into one solid phase model of elastostatics and one fluid phase model of seepage flow.

7.2.3 Conclusions

The time-dependent behaviour of a embedded rigid disc inclusion due to applied axial load or applied axial displacement has been well examined in the proceeding

Fig.7.2.11 Effects of drainage conditons on the axial force of a rigid disc embedded in saturated poroelastic media

Fig.7.2.12 Effects of drainage conditons on the axial force of a rigid disc embedded in saturated poroelastic media

Fig.7.2.13 Degrees of relaxation of axial force of a permeable rigid disc inclusion against the modified time factor

Fig.7.2.14 Degrees of relaxation of axial force of a permeable rigid disc inclusion against the modified time factor

Fig.7.2.15 Degrees of relaxation of axial force of an impermeable rigid disc inclusion against the modified time factor

Fig.7.2.16 Degrees of relaxation of axial force of an impermeable rigid disc inclusion against the modified time factor

Fig.7.2.17 Axial force of a permeable rigid disc inclusion in poroelastic media saturated with compressible pore fluids

Fig.7.2.18 Axial force of a permeable rigid disc inclusion in poroelastic media saturated with compressible pore fluids

Fig.7.2.19 Axial force of an impermeable rigid disc inclusion in poroelastic media saturated with compressible pore fluids

Fig.7.2.20 Axial force of an impermeable rigid disc inclusion in poroelastic media saturated with compressible pore fluids

sub-sections by considering the influence of drainage conditions and material parameters as well as length parameter. Such time-dependent behaviour of a rigid disc inclusion can be correctly interpreted due to the analytical approach adopted in this thesis.

It is of theoretical and practical interest to observe that the Poisson's ratios play a very important role in the time-dependent response of rigid disc inclusions embedded in saturated poroelastic media and that the dominant interval of axial consolidation (or relaxation) process associated with a rigid disc inclusion is $10^{-4} \leq ct/a^2 \leq 10^{+3}$. With such limits on the time factor, we can summarize the degree of consolidation (or relaxation) associated with axial translation (or axial force) of the inclusion due to an applied constant load (or an applied constant displacement), as follows:

$$\begin{aligned}
 U_z(10^{-4}) &= \begin{cases} 4.41\% & \text{for permeable inclusion} \\ 1.02\% & \text{for impermeable inclusion} \end{cases} \\
 U_z(10^{+3}) &= 99.24\% \quad \text{for both permeable and impermeable inclusions} \\
 V_z(10^{-4}) &= \begin{cases} 5.94\% & \text{for permeable inclusion} \\ 1.51\% & \text{for impermeable inclusion} \end{cases} \\
 V_z(10^{+3}) &= 99.49\% \quad \text{for both permeable and impermeable inclusions}
 \end{aligned} \tag{7.2.5}$$

These results are the exact values when $\nu = 0.0$, $\nu_u = 0.5$ and are applicable to any pair of Poisson's ratios in $0 \leq \nu < \nu_u \leq 0.5$.

7.3 Central Rotation of a Rigid Disc Inclusion

7.3.1 Central consolidation rotation of the inclusion

We examine the time-dependent central rotation $\Omega_y(t)$ and the associated degree of consolidation $U_y(t)$ for either a permeable or an impermeable embedded rigid disc inclusion due to the action of a constant total central moment $M_y H(t)$.

7.3.1.1 Influence of interface drainage conditions

The influence of interface drainage conditions on the consolidation induced central rotation of a rigid disc inclusion embedded in a saturated poroelastic medium is illustrated in Fig.7.3.1 and Fig.7.3.2. The drainage conditions at the inclusion-poroelastic medium interface have a significant effect on the consolidation rate of the central rotation. At the initial stages, the consolidation rate of the central rotation of a *permeable* rigid disc inclusion is much faster than that of an *impermeable* rigid disc inclusion. At a certain time (e.g., $ct/a^2 = 0.012$ for $\nu = 0$ and $\nu_u = 0.5$), the rate of consolidation for the central rotation of a *permeable* rigid disc inclusion becomes slower than that for an *impermeable* rigid disc inclusion. As ct/a^2 becomes larger (e.g. $ct/a^2 = 2$ for $\nu = 0$ and $\nu_u = 0.5$), the consolidation behaviour of both *permeable* and *impermeable* inclusions is nearly identical, which implies that the drainage conditions have a negligible effect on the consolidation induced central rotation.

The difference $\Delta U(ct/a^2)$ between the degrees of central consolidation rotation associated with the two limiting drainage conditions monotonically increases at the initial stages and reaches its maximum value (21.15% = 51.70% – 30.55% for $\nu = 0$ and $\nu_u = 0.5$) at a certain time (eg., $ct/a^2 = 0.012$) and then monotonically decreases to zero as $ct/a^2 \rightarrow +\infty$. At $ct/a^2 = 2$ for $\nu = 0$ and $\nu_u = 0.5$, the degrees of consolidation induced rotation are respectively equal to 99.49% and 99.46% for the *permeable* and *impermeable* disc inclusions.

7.3.1.2 Effects of material and length parameters

The role of the four parameters μ , B , κ , and a on the central rotation can be determined from the non-dimensional central rotation; i.e.,

$$\frac{a^3 \mu}{M_y} \Omega_y(ct/a^2) = \frac{3 - 4\nu}{64(1 - \nu)} X_y(ct/a^2) \quad (7.3.1)$$

where $X_y(x)$ is a non-dimensional function evaluated from the integral equations. This function $X_y(x)$ includes only Poisson's ratios, i.e., ν and ν_u . The expres-

sion (7.3.1) indicates that the magnitude and consolidation time of the flexibility $\Omega_y(t)/M_y$ is inversely proportional to $a^3\mu$ and $2\mu B^2\kappa/a^2$, respectively.

The influences of the drained and undrained Poisson's ratios ν , ν_u can be observed in the Fig.7.3.1 ~ Fig.7.3.10. The Fig.7.3.1 and Fig.7.3.2 illustrate that the degree of consolidation induced rotation $U_y(ct/a^2)$ is not sensitive to the values of the Poisson's ratios. For pairs of $(\nu, \nu_u) = (0.0, 0.5)$, $(0.3, 0.5)$, $(0.49, 0.5)$, $(0.0, 0.01)$, and $(0.0, 0.3)$, the differences among the associated $U_y(ct/a^2)$ are not readily discernible on the plots shown in Fig.7.3.1 and Fig.7.3.2.

Using the modified time factor $c_m t/a^2$ which is independent of Poisson's ratios, the degree of consolidation induced rotation can be re-plotted (Fig.7.3.3 ~ Fig.7.3.6). These figures show that the degree of consolidation induced rotation is extremely sensitive to the values of Poisson's ratios ν, ν_u . As the difference of $(\nu_u - \nu)$ become smaller, the consolidation rate becomes much faster and the consolidation time becomes much shorter.

The behaviour of the non-dimensional central rotation $a^3\mu\Omega_y(ct/a^2)/M_y$ can be clearly observed from the plots in Fig.7.3.7 ~ Fig.7.3.10 for a series of combinations of Poisson's ratios. The non-dimensional central rotation $a^3\mu\Omega_y(ct/a^2)/M_y$ is also extremely sensitive to the values of Poisson's ratios ν, ν_u . On the basis of these results, the influences of Poisson's ratios on the time-dependent behaviour of central rotation can be summarized as follows.

- i. The undrained Poisson's ratio ν_u governs the initial rotation of rigid disc inclusion. The larger the value of the undrained Poisson's ratio ν_u , the smaller the initial rotation.
- ii. The drained Poisson's ratio ν governs the final rotation of rigid disc inclusion. The larger the value of drained Poisson's ratio ν , the smaller the final rotation.
- iii. The non-dimensional consolidation rotation can be expressed in terms of the degree of consolidation as

Fig.7.3.1 Effects of drainage conditons on the central rotation of a rigid disc embedded in saturated poroelastic media

Fig.7.3.2 Effects of drainage conditons on the central rotation of a rigid disc embedded in saturated poroelastic media

Fig.7.3.3 Degrees of consolidation of central rotation of a permeable rigid disc inclusion against the modified time factor

Fig.7.3.4 Degrees of consolidation of central rotation of a permeable rigid disc inclusion against the modified time factor

Fig.7.3.5 Degrees of consolidation of central rotation of an impermeable rigid disc against the modified time factor

Fig.7.3.6 Degrees of consolidation of central rotation of an impermeable rigid disc against the modified time factor

Fig.7.3.7 Central rotation of a permeable rigid disc inclusion in poroelastic media saturated with compressible pore fluids

Fig.7.3.8 Central rotation of a permeable rigid disc inclusion in poroelastic media saturated with compressible pore fluids

Fig.7.3.9 Central rotation of an impermeable rigid disc in poroelastic media saturated with compressible pore fluids

Fig.7.3.10 Central rotation of an impermeable rigid disc embedded in poroelastic media saturated with compressible pore fluids

$$\frac{a^3 \mu}{M_y} [\Omega_y(ct/a^2) - \Omega_y(0^+)] = \frac{3(\nu_u - \nu)}{64(1 - \nu)(1 - \nu_u)} U_y(ct/a^2) \quad (7.3.2)$$

The difference between the undrained and drained Poisson's ratios, $(\nu_u - \nu)$, governs the magnitude and time of consolidation induced rotation. The smaller the difference $(\nu_u - \nu)$, the smaller the consolidation rotation. The consolidation rate and time of central rotation is not very sensitive to the Poisson's ratios in the integral equations (6.3.2). The value of ν_u in equations (7.2.14.a) almost determines the consolidation rate and time in the sense of the modified time factor. The larger the value of ν_u , the faster the consolidation rate and the shorter the consolidation time.

- iv. The maximum percentage of total consolidation rotation over the initial rotation is 50% at $\nu = 0.0$ and $\nu_u = 0.5$ with the associated limits of $0 \leq \nu < \nu_u \leq 0.5$.
- v. At the limiting condition $\nu \rightarrow \nu_u$, consolidation time is zero and final rotation of the rigid disc inclusion occurs with the application of the central moment. The coupled two phase model of saturated poroelastic theory decouples into one solid phase model of elastostatics and one fluid phase model of seepage flow.

7.3.2 Central moment relaxation in the inclusion

We examine the time-dependent central moment $M_y(t)$ and the associated degree of relaxation $V_y(t)$ for either a permeable or an impermeable embedded rigid disc inclusion due to the action of a constant central rotation $\Omega_y H(t)$.

7.3.2.1 Influence of interface drainage conditions

The influence of interface drainage conditions on the relaxation of central moment in a rigid disc inclusion embedded in a saturated poroelastic medium are shown in Fig.7.3.11 and Fig.7.3.12 where $V_y(ct/a^2)$ is plotted against $\log(ct/a^2)$. The interface drainage conditions have a significant effect on the relaxation rate of the induced central moment. At the initial stages, the relaxation rate of central moment of a *permeable* rigid disc inclusion is much faster than that of an *impermeable* rigid disc inclusion. At a certain time (e.g., $ct/a^2 = 0.007$ for $\nu = 0$ and

$\nu_u = 0.5$), the relaxation rate for the central moment of a *permeable* rigid disc inclusion is slower than that for an *impermeable* rigid disc inclusion. If ct/a^2 is larger than a certain value (e.g., $ct/a^2 = 1.2$ for $\nu = 0$ and $\nu_u = 0.5$), the relaxation behaviour of the *permeable* and *impermeable* inclusions becomes almost the same and the drainage conditions almost have no effect on the relaxation of central moment.

The difference $\Delta V(ct/a^2)$ between the degrees of relaxation moment associated with the two limiting drainage conditions monotonically increases at the initial stages and reaches its maximum value ($22.64\% = 55.07\% - 32.43\%$ for $\nu = 0$ and $\nu_u = 0.5$) at the certain time (e.g., $ct/a^2 = 0.007$) and then monotonically decreases to zero as $ct/a^2 \rightarrow +\infty$. At $ct/a^2 = 1.2$ for $\nu = 0$ and $\nu_u = 0.5$, the degrees of relaxation induced central moment are equal to 99.34% and 99.30% for the *permeable* and *impermeable* disc inclusions, respectively.

7.3.2.2 Effects of material and length parameters

The role of the four parameters μ , B , κ , and a on the central moment can be determined from the non-dimensional central moment, i.e.,

$$\frac{M_y(ct/a^2)}{a^3 \mu \Omega_y} = \frac{64(1-\nu)}{3-4\nu} Y_y(ct/a^2) \quad (7.3.3)$$

where $Y_y(x)$ is a non-dimensional function evaluated from the integral equations. This function includes only the Poisson's ratios ν and ν_u . This result shows that the magnitude and relaxation time of the stiffness $M_y(t)/\Omega_y$ are respectively directly and inversely proportional to $a^3 \mu$ and $2\mu B^2 \kappa/a^2$.

The influences of the drained and undrained Poisson's ratios ν , ν_u can be observed in the Fig.7.3.11~Fig.7.3.20.

The Fig.7.3.11 and Fig.7.3.12 illustrate that relaxation time increases with the decrease in the difference $(\nu_u - \nu)$, where the degree of relaxation moment $V_y(ct/a^2)$ is plotted against the logarithm of the time factor ct/a^2 .

However, when the degree of relaxation moment $V_y(c_m t/a^2)$ is re-plotted against the logarithm of the modified time factor $c_m t/a^2$, Fig.7.3.13 and Fig.7.3.15 illustrate that relaxation time decreases as ν approaches ν_u . These results indicate that the

Poisson's ratios in the integral equations (6.3.2) and in the time factor ct/a^2 (i.e., ν_b) produce counteracting effects on the relaxation rate of the central moment. Such counteracting effects cause the irregular shift of curves in Fig.7.3.12 as we re-plot them into Fig.7.3.14 and Fig.7.3.16 using the modified time factor $c_m t/a^2$.

The behaviour of the non-dimensional central moment $M_y(ct/a^2)/(a^3\mu\Omega_y)$ can be clearly observed from the plots in Fig.7.3.17 ~ Fig.7.3.20 for certain pairs of Poisson's ratios. The non-dimensional central moment $M_y(ct/a^2)/(a^3\mu\Omega_y)$ is also extremely sensitive to the values of Poisson's ratios ν , ν_u . Based on these results, the influences of Poisson's ratios on the time-dependent behaviour of central moment can be summarized as follows.

- i. The undrained Poisson's ratio ν_u governs the initial moment of the rigid disc inclusion. The larger the undrained Poisson's ratio ν_u , the larger the initial moment.
- ii. The drained Poisson's ratio ν governs the final moment of the rigid disc inclusion. The larger the drained Poisson's ratio ν , the larger the final moment.
- iii. The non-dimensional relaxation moment can be expressed in terms of the degree of consolidation as

$$\frac{1}{a^3\mu\Omega_y}[M_y(ct/a^2) - M_y(0^+)] = -\frac{64(\nu_u - \nu)}{3(3 - 4\nu)(3 - 4\nu_u)}V_y(ct/a^2) \quad (7.3.4)$$

The difference between the undrained and drained Poisson's ratios, $(\nu_u - \nu)$, governs the magnitude and time of relaxation moment. The smaller the difference $\nu_u - \nu$, the smaller the relaxation moment. The Poisson's ratios in the integral equations (7.3.2) and in the time factor ct/a^2 (i.e., ν_b) produce counteracting effects on the relaxation time of central moment. As combined results, the relaxation time of induced central moment decreases as ν approaches ν_u in the sense of the modified time factor.

- iv. The maximum percentage of total relaxation moment over initial moment is 33.33% at $\nu = 0.0$ and $\nu_u = 0.5$ with the constraint of $0 \leq \nu < \nu_u \leq 0.5$.

- v. Under the limiting condition $\nu_u = \nu$, relaxation time is zero and the rigid disc inclusion mobilizes its total central moment once the central rotation is applied. The coupled two phase model of saturated poroelastic theory decouples into one solid phase model of elastostatics and one fluid phase model of seepage flow.

7.3.3 Conclusions

The time-dependent behaviour of a rigid disc inclusion due to applied central moment or applied central rotation has been well examined in the proceeding subsections by considering the influence of interface drainage conditions and material parameters as well as length parameter. Such time-dependent behaviour of an embedded inclusion can be clearly interpreted due to the analytical approach adopted in this thesis.

It is of theoretical and practical interest to observe that (i) the Poisson's ratios plays a very important role in the time-dependent response of rigid disc inclusions embedded in saturated poroelastic media and (ii) that the dominant interval of central rotation process associated with the embedded inclusion is $10^{-4} \leq ct/a^2 \leq 10$. With such limits on the time factor ct/a^2 , we can summarize the degree of consolidation (or relaxation) associated with the central rotation (or central moment) of the inclusion due to an applied constant moment (or an applied constant rotation), as

$$\begin{aligned}
 U_y(10^{-4}) &= \begin{cases} 8.65\% & \text{for permeable inclusion} \\ 3.00\% & \text{for impermeable inclusion} \end{cases} \\
 U_y(10) &= 99.96\% \quad \text{for both permeable and impermeable inclusions} \\
 \\
 V_y(10^{-4}) &= \begin{cases} 13.01\% & \text{for permeable inclusion} \\ 4.69\% & \text{for impermeable inclusion} \end{cases} \\
 V_y(10) &= 99.97\% \quad \text{for both permeable and impermeable inclusions}
 \end{aligned}
 \tag{7.3.5}$$

Fig.7.3.11 Effects of drainage conditons on the central moment of a rigid disc embedded in saturated poroelastic media

Fig.7.3.12 Effects of drainage conditons on the central moment of a rigid disc embedded in saturated poroelastic media

Fig.7.3.13 Degrees of relaxation of central moment of a permeable rigid disc inclusion against the modified time factor

Fig.7.3.14 Degrees of relaxation of central moment of a permeable rigid disc inclusion against the modified time factor

Fig.7.3.15 Degrees of relaxation of central moment of an impermeable rigid disc inclusion against the modified time factor

Fig.7.3.16 Degrees of relaxation of central moment of an impermeable rigid disc inclusion against the modified time factor

Fig.7.3.17 Central moment of a permeable rigid disc inclusion in poroelastic media saturated with compressible pore fluids

Fig.7.3.18 Central moment of a permeable rigid disc inclusion in poroelastic media saturated with compressible pore fluids

Fig.7.3.19 Central moment of an impermeable rigid disc inclusion in poroelastic media saturated with compressible pore fluids

Fig.7.3.20 Central moment of an impermeable rigid disc inclusion in poroelastic media saturated with compressible pore fluids

These results are the exact values when $\nu = 0.0$, $\nu_u = 0.5$ and are applicable to any pair of Poisson's ratios in $0 \leq \nu < \nu_u \leq 0.5$.

7.4 In-Plane Translation of a Rigid Disc Inclusion

7.4.1 In-plane consolidation translation of the inclusion

We examine the time-dependent in-plane translation $D_x(t)$ and the associated degree of consolidation $U_x(t)$ for either a permeable or an impermeable rigid disc inclusion due to the action of a constant total in-plane force $P_x H(t)$.

7.4.1.1 Influence of interface drainage conditions

The influence of interface drainage conditions on the in-plane consolidation translation of a rigid disc inclusion embedded in a saturated poroelastic medium can be observed in Fig.7.4.1 and Fig.7.4.2, where the degree of consolidation translation is plotted against the logarithm of the time factor ct/a^2 . The drainage conditions at the inclusion-poroelastic medium interface have a limit influence on the time-dependent behaviour of in-plane translation of the embedded inclusion. The limited influence appears in the interval of $10^{-3} \leq ct/a^2 \leq 10$. The maximum value of the difference $\Delta U(ct/a^2)$ between the degrees of in-plane consolidation translation for the *permeable* and *impermeable* inclusions is equal to 4.78% (= 41.81% – 37.03%) at the time factor $ct/a^2 = 0.3$.

7.4.1.2 Effects of material and length parameters

The role of the four parameters μ , B , κ , and a on the in-plane translation can be determined from the non-dimensional in-plane translation, *i.e.*,

$$\frac{a\mu D_x(ct/a^2)}{P_x} = \frac{7 - 8\nu}{64(1 - \nu)} X_x(ct/a^2) \quad (7.4.1)$$

where $X_x(x)$ is a non-dimensional function evaluated from the integral equations. This function $X_x(x)$ includes only the Poisson's ratios ν and ν_u . The formula (7.4.1)

indicates that the magnitude of the flexibility $D_x(t)/P_x$ is inversely proportional to $a\mu$ and the associated consolidation time is also inversely proportional to $2\mu B^2\kappa/a^2$.

The influences of the drained and undrained Poisson's ratios ν , ν_u can be observed in the Fig.7.4.1~10. The Fig.7.4.1 and Fig.7.4.2 show that the degree of consolidation $U_x(ct/a^2)$ is not sensitive to the values of the Poisson's ratios. For values of $(\nu, \nu_u) = (0.0, 0.5)$, $(0.3, 0.5)$, $(0.49, 0.5)$, $(0.0, 0.01)$, and $(0.0, 0.3)$, the differences are not discernible on the plots in Fig.7.4.1 and Fig.7.4.2.

Using the modified time factor c_mt/a^2 , we re-plot the degree of consolidation translation in Fig.7.4.3 ~ Fig.7.4.6. The figures illustrate that $U_x(c_mt/a^2)$ is extremely sensitive to the values of Poisson's ratios ν , ν_u . As the difference of $(\nu_u - \nu)$ is smaller, the consolidation rate is much faster and the consolidation time is much shorter.

The variations in the non-dimensional in-plane translation $a\mu D_x(ct/a^2)/P_x$ are given in Fig.7.4.7 ~ Fig.7.4.10 for certain pairs of Poisson's ratios. We observe that $a\mu D_x(ct/a^2)/P_x$ is also extremely sensitive to the values of Poisson's ratios ν and ν_u .

Consequently, the effects of Poisson's ratios on the time-dependent behaviour of in-plane translation can be summarized as follows.

- i. The undrained Poisson's ratio ν_u governs the initial in-plane translation of the embedded inclusion. The larger the undrained Poisson's ratio ν_u , the smaller the initial in-plane translation.
- ii. The drained Poisson's ratio ν governs the final in-plane translation of the embedded inclusion. The larger the drained Poisson's ratio ν , the smaller the final in-plane translation.
- iii. The non-dimensional consolidation translation can be expressed in terms of the degree of consolidation as

$$\frac{a\mu}{P_x}[D_x(ct/a^2) - D_x(0^+)] = \frac{(\nu_u - \nu)}{64(1 - \nu)(1 - \nu_u)}U_x(ct/a^2) \quad (7.4.2)$$

Fig.7.4.1 Effects of drainage conditons on the in-plane translation of a rigid disc embedded in saturated poroelastic media

Fig.7.4.2 Effects of drainage conditons on the in-plane translation of a rigid disc embedded in saturated poroelastic media

Fig.7.4.3 Degrees of consolidation of in-plane translation of a permeable rigid disc inclusion against the modified time factor

Fig.7.4.4 Degrees of consolidation of in-plane translation of a permeable rigid disc inclusion against the modified time factor

Fig.7.4.5 Degrees of consolidation of in-plane translation of an impermeable rigid disc against the modified time factor

Fig.7.4.6 Degrees of consolidation of in-plane translation of an impermeable rigid disc against the modified time factor

Fig.7.4.7 In-plane translation of a permeable rigid disc inclusion in poroelastic media saturated with compressible pore fluids

Fig.7.4.8 In-plane translation of a permeable rigid disc inclusion in poroelastic media saturated with compressible pore fluids

Fig.7.4.9 Axial translation of an impermeable rigid disc in poroelastic media saturated with compressible pore fluids

Fig.7.4.10 In-plane translation of an impermeable rigid disc embedded in poroelastic media saturated with compressible pore fluids

The difference between the undrained and drained Poisson's ratios, $(\nu_u - \nu)$, governs the magnitude and time of the non-dimensional consolidation translation. The smaller the difference $\nu_u - \nu$, the smaller the magnitude of the in-plane consolidation translation. The consolidation time of the in-plane translation is not very sensitive to the values of the Poisson's ratios in the integral equations (6.4.2). The value of ν_b in equation (7.1.4) controls the consolidation time in the sense of the modified time factor. The larger the value of ν_b , the shorter the consolidation time $c_m t / a^2$.

- iv. The maximum percentage of total in-plane consolidation translation over initial in-plane translation is 17.67% at $\nu = 0.0$ and $\nu_u = 0.5$ with the constraint of $0 \leq \nu < \nu_u \leq 0.5$.
- v. As $\nu \rightarrow \nu_u$, the consolidation time is zero and the complete in-plane translation of the embedded inclusion occurs with the application of the in-plane force. The coupled two phase model of saturated poroelastic theory decouples into one solid phase model of elastostatics and one fluid phase model of seepage flow.

7.4.2 In-plane force relaxation in the inclusion

We examine the time-dependent in-plane force $P_x(t)$ and the associated degree of relaxation $V_x(t)$ for either a permeable or an impermeable rigid disc inclusion due to the action of a constant in-plane translation $D_x H(t)$.

7.4.2.1 Influence of interface drainage conditions

In Fig.7.4.11 and Fig.7.4.12, we observe that the drainage conditions at the inclusion-poroelastic medium interface have a limited influence on the degree of the in-plane relaxation force in the interval of $10^{-3} \leq ct/a^2 \leq 10$. The maximum value of the difference $\Delta V(ct/a^2)$ between the degrees of in-plane relaxation force for the *permeable* and *impermeable* inclusions is equal to 4.98% ($= 47.14\% - 41.16\%$) at the time factor $ct/a^2 = 0.3$ for $\nu = 0.0$ and $\nu_u = 0.5$.

7.4.2.2 Effects of material and length parameters

The role of the four parameters μ , B , κ , and a on the in-plane force can be determined from the non-dimensional in-plane force, *i.e.*,

$$\frac{P_x(t)}{a\mu D_x} = \frac{64(1-\nu)}{7-8\nu} Y_x(t) \quad (7.4.3)$$

where $Y_x(x)$ is a non-dimensional function evaluated from the integral equations. This function $Y_x(x)$ includes only the Poisson's ratios ν and ν_u . The formula (7.4.3) indicates that the magnitude of the stiffness $P_x(t)/D_x$ is directly proportional to $a\mu$ and the associated consolidation time is inversely proportional to $2\mu B^2\kappa/a^2$.

The influence of the drained and undrained Poisson's ratios ν , ν_u can be observed in Fig.7.4.11 ~ 7.4.20. The Fig.7.4.11 and Fig.7.4.12 illustrate that relaxation time has a little increase with the decrease in the difference $(\nu_u - \nu)$, where $V_x(ct/a^2)$ is plotted against the logarithm of the time factor ct/a^2 . However, when $V_x(c_mt/a^2)$ is re-plotted against $\log(c_mt/a^2)$ in Fig.7.4.13 and Fig.7.4.15, we observe that the relaxation time decreases as ν approaches ν_u . These results indicate that the Poisson's ratios in the integral equations (7.4.2) and in the time factor ct/a^2 (*i.e.*, ν_b) produce counteracting effects on the relaxation time of in-plane force relaxation. Such counteracting effects cause the irregular shift of curves in Fig.7.4.12 as we re-plot them into Fig.7.4.14 and Fig.7.4.16 using $\log(c_mt/a^2)$.

The Fig.7.4.17 ~ Fig.7.4.20 clearly illustrate that the behaviour of the non-dimensional in-plane force $P_x(ct/a^2)/(a\mu\Omega_x)$ with certain pairs of Poisson's ratios. These figures indicate that $P_x(ct/a^2)/(a\mu D_x)$ is also extremely sensitive to the values of Poisson's ratios ν , ν_u .

Based on the above analysis, we can summarize the effects of Poisson's ratios on the time-dependent behaviour of in-plane force relaxation as follows.

- i. The undrained Poisson's ratio ν_u governs the initial in-plane force induced by the embedded inclusion. The larger the undrained Poisson's ratio ν_u , the larger the initial in-plane force.

- ii. The drained Poisson's ratio ν governs the final in-plane force induced by the inclusion. The larger the drained Poisson's ratio ν , the larger the final in-plane force.
- iii. The non-dimensional relaxation force can be expressed in terms of the degree of consolidation as

$$\frac{1}{a\mu D_z}[P_z(ct/a^2) - P_z(0^+)] = -\frac{64(\nu_u - \nu)}{(7 - 8\nu)(7 - 8\nu_u)}V_z(ct/a^2) \quad (7.4.4)$$

The difference between the undrained and drained Poisson's ratios, $(\nu_u - \nu)$, governs the magnitude and time of non-dimensional relaxation force. The smaller the difference $\nu_u - \nu$, the smaller the relaxation force. The Poisson's ratios in the integral equations (7.4.2) and in the time factor ct/a^2 (i.e., ν_b) produce counteracting effects on the relaxation time of the in-plane force relaxation. As combined results, the relaxation time of the in-plane force decreases as ν approaches ν_u in the sense of the modified time factor.

- iv. The maximum percentage of total in-plane relaxation force over initial in-plane force is 14.29% at $\nu = 0.0$ and $\nu_u = 0.5$ with the constraint of $0 \leq \nu < \nu_u \leq 0.5$.
- v. As $\nu \rightarrow \nu_u$, relaxation time is zero and the rigid disc inclusion mobilizes its total in-plane force when the in-plane translation is applied. The coupled two phase model of saturated poroelastic theory decouples into one solid phase model of elastostatics and one fluid phase model of seepage flow.

7.4.3 Conclusions

The time-dependent behaviour of an embedded rigid disc inclusion due to applied in-plane load or applied in-plane displacement has been well examined in the proceeding sub-sections by considering the influence of interface drainage conditions and material parameters as well as length parameter. Such time-dependent behaviour of an embedded inclusion can be neatly interpreted due to the analytical approach adopted in this thesis.

It is of theoretical and practical interest to indicate that (i) the Poisson's ratios

Fig.7.4.11 Effects of drainage conditons on the in-plane force of a rigid disc embedded in saturated poroelastic media

Fig.7.4.12 Effects of drainage conditons on the in-plane force of a rigid disc embedded in saturated poroelastic media

Fig.7.4.13 Degrees of relaxation of in-plane force of a permeable rigid disc inclusion against the modified time factor

Fig.7.4.14 Degrees of relaxation of in-plane force of a permeable rigid disc inclusion against the modified time factor

Fig.7.4.15 Degrees of relaxation of in-plane force of an impermeable rigid disc inclusion against the modified time factor

Fig.7.4.16 Degrees of relaxation of in-plane force of an impermeable rigid disc inclusion against the modified time factor

Fig.7.4.17 In-plane force of a permeable rigid disc inclusion in poroelastic media saturated with compressible pore fluids

Fig.7.4.18 In-plane force of a permeable rigid disc inclusion in poroelastic media saturated with compressible pore fluids

Fig.7.4.19 In-plane force of an impermeable rigid disc inclusion in poroelastic media saturated with compressible pore fluids

Fig.7.4.20 In-plane force of an impermeable rigid disc inclusion in poroelastic media saturated with compressible pore fluids

play a very important role in the time-dependent response of inclusions embedded in saturated poroelastic media and (ii) the interface drainage conditions have a limited effect on the time-dependent behaviour of in-plane translation for an embedded rigid disc inclusion.

At the limits $10^{-4} \leq ct/a^2 \leq 10^{+3}$, we have the degrees of consolidation translation and relaxation force as follows,

$$\begin{aligned}
 U_x(10^{-4}) &= \begin{cases} 0.83\% & \text{for permeable inclusion} \\ 0.63\% & \text{for impermeable inclusion} \end{cases} \\
 U_x(10^3) &= 98.49\% \quad \text{for both permeable and impermeable inclusions} \\
 \\
 V_x(10^{-4}) &= \begin{cases} 0.91\% & \text{for permeable inclusion} \\ 0.71\% & \text{for impermeable inclusion} \end{cases} \\
 V_x(10^3) &= 98.70\% \quad \text{for both permeable and impermeable inclusions}
 \end{aligned} \tag{7.4.5}$$

These results are the exact values when $\nu = 0.0$, $\nu_u = 0.5$ and are applicable to any pair of Poisson's ratios in $0 \leq \nu < \nu_u \leq 0.5$.

7.5 Further Discussions and Conclusions

In the above sections, we have successfully examined the time-dependent behaviour of a rigid disc inclusion which is embedded in a poroelastic medium saturated with a compressible pore fluid by taking into account the influence of the interface drainage conditions and the five basic material parameters as well as the length parameter. The interesting results associated with the time-dependent behaviour of the embedded inclusion are further summarized in Table 7.5.1 and Table 7.5.2 by considering together all the four basic displacement and loading modes.

In particular, the degrees of consolidation or relaxation for the three basic loading modes of the embedded inclusion are re-plotted in Fig.7.5.1 and Fig.7.5.2 to illustrate more effectively the time difference in the consolidation process associ-

ated with the four fundamental modes. The numerical values of the time factor ct/a^2 for certain degrees of consolidation or relaxation are also presented in Table 7.5.3 and Table 7.5.4. It is apparent from these Figures and Tables that the consolidation (or relaxation) process during the axial translation mode is much slower than that associated with the central rotation mode, and is much faster than that for the in-plane translation mode. For example, at the 50% consolidation (or relaxation) level, the time of axial translation mode is about ten times longer and five times shorter than those respectively for the central rotation mode and the in-plane translation mode.

As stated in Chapter 3, the presence of a compressible pore fluid is represented by the two material constants namely, Skempton's pore pressure parameter B and the undrained Poisson's ratio ν_u where $0 < B < 1$ and $\nu < \nu_u < 0.5$. The role of the compressible pore fluid can be examined by observing the effects of B and ν_u on the time-dependent response of the embedded inclusion.

When $B < 1$, Skempton's pore pressure parameter simply reduce the consolidation time of the soil-inclusion system. The undrained Poisson's ratio ν_u has an influence on the consolidation time and initial displacements (see, Table 7.5.1 and Table 7.5.2). The variation of ν_b is plotted in Fig.7.5.3 against the undrained Poisson's ratio ν_u for five values of drained Poisson's ratio ν . As $\nu_u \rightarrow \nu$, the consolidation rate approaches infinity since $\nu_b \sim 1/(\nu_u - \nu) \rightarrow +\infty$. On the other hand, the consolidation displacements or relaxation forces approach zero as $\nu_u \rightarrow \nu$ since their magnitudes are directly proportional to the difference of $\nu_u - \nu$. This result is consistent with the intuitive reasoning that a lower magnitude of consolidation displacement or relaxation force require less time to complete.

In order to consider the effects of the undrained Poisson's ratio ν_u on the consolidation rate together with the consolidation magnitude, we introduce an average time as the time $T_{99.5\%}$ divided by the total consolidation displacement or relaxation force, where $T_{99.5\%}$ is the time for achieving 99.5% of the total consolidation displacement or relaxation force. We then can obtain the mathematical expressions

Table 7.5.1 The time-dependent behaviour of a rigid disc inclusion in a saturated poroelastic medium
 case (A): consolidation displacement due to an applied constant load

Displacement Modes		Vertical Displacement modes		Horizontal Displacement modes	
Conditions and Phenomena		axial translation	central rotation	in-plane translation	axial rotation
solution		$a\mu D_z(t)/P_z$	$a^3\mu\Omega_y(t)/M_y$	$a\mu D_x(t)/P_z$	$a^3\mu\Omega_z(t)/M_z$
time for consolidation process		numerical results from integral equations, $N > 1/\sqrt{ct/a^2}$			
drainage	consolidation rate	longer	shorter	much longer	shortest 0
	magnitude	significant	more significant	limited	no effect
	maximum $\Delta U(ct/a^2)$	no effect on the initial and final values of displacement			
The four parameters		17.3%	21.2%	4.8%	no effect
μ, κ, B, a		consolidation time is inversely proportional to $2\mu\kappa B^2/a^2$			
Poisson's ratios ν, ν_u	initial displacement	$\frac{3-4\nu_u}{32(1-\nu_u)}$	$\frac{3(3-4\nu_u)}{64(1-\nu_u)}$	$\frac{7-8\nu_u}{64(1-\nu_u)}$	$\frac{3}{64}$
	final displacement	$\frac{3-4\nu}{32(1-\nu)}$	$\frac{3(3-4\nu)}{64(1-\nu)}$	$\frac{7-8\nu}{64(1-\nu)}$	$\frac{3}{64}$
	consolidation displacement	$\frac{(\nu_u-\nu)U_z(ct/a^2)}{32(1-\nu)(1-\nu_u)}$	$\frac{3(\nu_u-\nu)U_y(ct/a^2)}{64(1-\nu)(1-\nu_u)}$	$\frac{(\nu_u-\nu)U_x(ct/a^2)}{64(1-\nu)(1-\nu_u)}$	0
consolidation time		very limited effect by using the time factor ct/a^2			
		significant effect due to $\nu_0 = \frac{(1-\nu)(1+\nu_u)^2}{9(\nu_u-\nu)(1-\nu_u)}$ in $c = \nu_0 c_m$			
		by using the modified time factor $c_m t/a^2$			
		no effect			

Table 7.5.2 The time-dependent behaviour of a rigid disc inclusion in a saturated poroelastic medium
case (B): force relaxation due to an applied constant displacement

Displacement Modes		Vertical Displacement modes		Horizontal Displacement modes	
Conditions and Phenomena		axial force	central moment	in-plane force	axial moment
solution		$P_z(t)/(a\mu D_z)$	$M_y(t)/(a^3\mu\Omega_y)$	$P_z(t)/(a\mu D_z)$	$M_z(t)/(a^3\mu\Omega_z)$
time for relaxation process		numerical results from integral equations, $N > 1/\sqrt{ct/a^2}$			
drainage	relaxation rate	longer	shorter	much longer	shortest 0
	magnitude	significant	more significant	limited	no effect
	maximum $\Delta V(ct/a^2)$	20.0%	22.6%	5.0%	no effect
the four parameters		relaxation time is inversely proportional to			
μ, κ, B, a		$2\mu\kappa B^2/a^2$			
Poisson's ratios ν, ν_u	initial force	$\frac{32(1-\nu_u)}{(3-4\nu_u)}$	$\frac{64(1-\nu_u)}{3(3-4\nu_u)}$	$\frac{64(1-\nu_u)}{7-8\nu_u}$	$\frac{64}{3}$
	final force	$\frac{32(1-\nu)}{3-4\nu}$	$\frac{64(1-\nu)}{3(3-4\nu)}$	$\frac{64(1-\nu)}{3-4\nu}$	$\frac{64}{3}$
	relaxation force	$\frac{32(\nu-\nu_u)}{(3-4\nu)} \frac{V_z(ct/a^2)}{(3-4\nu_u)}$	$\frac{64(\nu-\nu_u)}{3(3-4\nu)} \frac{V_y(ct/a^2)}{(3-4\nu_u)}$	$\frac{64(\nu-\nu_u)}{(7-8\nu)(7-8\nu_u)} \frac{V_z(ct/a^2)}{(7-8\nu_u)}$	0
	degree by ct/a^2	significant effect		limited effect	no effect
	degree by $c_m t/a^2$	significant due to the counteracting effects of ν, ν_u in ν_j and in the integral equations

Fig.7.5.1 Degrees of consolidation displacement for the three basic modes of a permeable or impermeable rigid disc inclusion

Fig.7.5.2 Degrees of relaxation loads for the three basic modes of a permeable or impermeable rigid disc inclusion

Table 7.5.3 A comparison of time for the consolidation process of an inclusion(1)= permeable inclusion; (2)= impermeable inclusion, and $\nu = 0.0$, $\nu_u = 0.5$

Degrees	Time Factor ct/a^2 for Degree of Consolidation					
	$U_z(ct/a^2)$		$U_y(ct/a^2)$		$U_x(ct/a^2)$	
	(1)	(2)	(1)	(2)	(1)	(2)
20%	.0063	.041	.00073	.0049	.061	.08
50%	.11	.27	.011	.034	.49	.62
70%	.52	.79	.040	.09	1.95	2.1
90%	5.5	6.8	.19	.27	28.5	28.7

Table 7.5.4 A comparison of time for the relaxation process of an inclusion(1)= permeable inclusion; (2)= impermeable inclusion, and $\nu = 0.0$, $\nu_u = 0.5$

Degrees	Time Factor ct/a^2 for Degree of Relaxation					
	$V_z(ct/a^2)$		$V_y(ct/a^2)$		$V_x(ct/a^2)$	
	(1)	(2)	(1)	(2)	(1)	(2)
20%	.0028	.02	.00027	.0022	.045	.064
50%	.049	.13	.0049	.0195	.37	.49
70%	.23	.42	.020	.053	.149	.16
90%	3.3	3.5	.105	.18	17.4	17.5

of the average time for the three fundamental modes of deformation of the rigid disc inclusion problems under the two basic groups of loading conditions; i.e.,

$$\begin{aligned}\frac{T_{99.5\%}}{a\mu[D_z(+\infty) - D_z(0^+)]/P_z} &= \frac{(1 - \nu_u)^2}{(1 + \nu_u)^2} \frac{144a^2}{\mu B^2 \kappa} 10^{+3} \\ \frac{T_{99.5\%}}{a^3\mu[\Omega_y(+\infty) - \Omega_y(0^+)]/M_y} &= \frac{(1 - \nu_u)^2}{(1 + \nu_u)^2} \frac{96a^2}{\mu B^2 \kappa} 10 \\ \frac{T_{99.5\%}}{a\mu[D_z(+\infty) - D_z(0^+)]/P_z} &= \frac{(1 - \nu_u)^2}{(1 + \nu_u)^2} \frac{288a^2}{\mu B^2 \kappa} 10^{+4}\end{aligned}\quad (7.5.1.a)$$

and

$$\begin{aligned}\frac{T_{99.5\%}}{[P_z(0^+) - P_z(+\infty)]/(a^2\mu D_z)} &= \frac{(3 - 4\nu)(3 - 4\nu_u)(1 - \nu_u)}{(1 - \nu)(1 + \nu_u)^2} \frac{9a^2}{64\mu B^2 \kappa} 10^{+3} \\ \frac{T_{99.5\%}}{[M_y(0^+) - M_y(+\infty)]/(a^3\mu \Omega_y)} &= \frac{(3 - 4\nu)(3 - 4\nu_u)(1 - \nu_u)}{(1 - \nu)(1 + \nu_u)^2} \frac{27a^2}{128\mu B^2 \kappa} 10 \\ \frac{T_{99.5\%}}{[P_z(0^+) - P_z(+\infty)]/(a\mu D_z)} &= \frac{(7 - 8\nu)(7 - 8\nu_u)(1 - \nu_u)}{(1 - \nu)(1 + \nu_u)^2} \frac{9a^2}{128\mu B^2 \kappa} 10^{+4}\end{aligned}\quad (7.5.1.b)$$

It can be observed from equations (7.5.1) that the average time for unit magnitude (of either consolidation displacement or relaxation force) increases with decrease in the values of the undrained Poisson's ratio ν_u and the Skempton's pore water pressure B . Such average time for the consolidation process of soil/inclusion systems reaches the minimum value for the limiting case of incompressible pore fluids ($B = 1$, $\nu_u = 0.5$). In particular, if $B \rightarrow 0$ or $\kappa \rightarrow 0$, the consolidation time will increase to infinity, which physically means there is no consolidation process for dry soils ($B = 0$) or impervious soils ($\kappa = 0$). The above results also illustrate that the average time for unit consolidation (or relaxation) increases with the increase of the undrained compressibility in saturated soils or rocks. Consequently, the dissolved gas phase in the pore fluid reduces the potential of soil consolidation deformation and contributes to an 'apparent' retardation in the consolidation (or relaxation) process.

Finally, as an example, we consider the problem of a ground flat anchor embedded a deep stiff clay layer. The ground anchor can be modeled as a rigid circular disc of radius 2 metres. The values for the material parameters given below are

based on the work of Lee, White and Ingles (1983),

$$\begin{aligned}
 a &= 2 \times 10^3 \text{ mm} \\
 \mu &= 2.5 \frac{\text{N}}{\text{mm}^2} \\
 k &= 2.91 \times 10^{-5} \frac{\text{mm}}{\text{sec.}} \\
 \kappa &= \frac{k}{\gamma_w} = 2.98 \frac{\text{mm}^4}{\text{N sec.}} \\
 B &= 1.0 \\
 c_m &= 2\mu\kappa B = 14.9 \frac{\text{mm}^2}{\text{sec.}} \\
 c_m/a^2 &= 3.725 \times 10^{-6} \frac{1}{\text{sec.}}
 \end{aligned} \tag{7.5.2}$$

The time for the limits of $ct/a^2 = 10^{-4}$, 10, and 10^{+3} is presented in Table 7.5.5, where eleven pairs of Poisson's ratios are used. This time indicates that the consolidation process of an embedded anchor takes weeks and/or years to complete. The values of the global flexibilities or stiffnesses associated with the ground flat anchor are also presented in Table 7.5.6 and Table 7.5.7.

YORK COUNTY NEWS & BROADCASTING, INC. 1980
TEL. 612-425-7500 FAX: 612-425-7500 712 P. O. BOX 1000

Table 6.5.5 The time limits for the consolidation process of soil

ν	ν_u	ν_b	$a^2 c^{-1} 10^{-4}$ (minutes)	$a^2 c^{-1} 10$ (weeks)	$a^2 c^{-1} 10^{+3}$ (years)
.00	.50	1.000	.447	4.439	8.513
.10	.50	1.125	.398	3.946	7.567
.20	.50	1.333	.336	3.329	6.385
.30	.50	1.750	.256	2.536	4.864
.40	.50	3.000	.149	1.480	2.838
.49	.50	25.500	.018	0.174	0.334
.00	.01	11.449	.039	0.388	0.744
.00	.10	1.494	.300	2.971	5.699
.00	.20	1.000	.447	4.439	8.513
.00	.30	0.894	.500	4.964	9.520
.00	.40	0.907	.493	4.892	9.381

Fig.7.5.3 Variation of ν_b with the undrained Poisson's ratio ν_u

Table 7.5.6 The flexibility or stiffness of a permeable rigid disc inclusion embedded in a saturated poroelastic medium: $\nu = 0.0$, $\nu_u = 0.5$

$t_1 = ct/a^2$	Flexibilities of Displacements				Stiffnesses of Loads			
	Axial		Horizontal		Axial		Horizontal	
	$\frac{D_x(t_1)}{10^{-4}P_s}$ ($\frac{mm}{N}$)	$\frac{\Omega_x(t_1)}{10^{-11}M_s}$ ($\frac{1}{mm N}$)	$\frac{D_x(t_1)}{10^{-4}P_s}$ ($\frac{mm}{N}$)	$\frac{\Omega_x(t_1)}{10^{-11}M_s}$ ($\frac{1}{mm N}$)	$\frac{P_x(t_1)}{10^4 D_s}$ ($\frac{N}{mm}$)	$\frac{M_x(t_1)}{10^{13} \Omega_s}$ ($mm N$)	$\frac{P_x(t_1)}{10^4 D_s}$ ($\frac{N}{mm}$)	$\frac{M_x(t_1)}{10^{13} \Omega_s}$ ($mm N$)
0	.1250	.4688	.1875	.4688	.8000	.2133	.5333	.2133
10^{-4}	.1273	.4890	.1878	.4688	.7840	.2040	.5326	.2133
10^{-3}	.1307	.5219	.1882	.4688	.7645	.1913	.5312	.2133
10^{-2}	.1395	.5836	.1899	.4688	.7160	.1706	.5265	.2133
10^{-1}	.1554	.6643	.1954	.4688	.6410	.1499	.5116	.2133
10^0	.1734	.7000	.2066	.4688	.5749	.1428	.4837	.2133
10^1	.1928	.7030	.2142	.4688	.5468	.1422	.4668	.2133
10^2	.1860	.7030	.2173	.4688	.5376	.1422	.4603	.2133
10^3	.1870	.7030	.2183	.4688	.5347	.1422	.4581	.2133
$+\infty$	.1875	.7031	.2188	.4688	.5333	.1422	.4571	.2133

Table 7.5.7 The flexibility or stiffness of an impermeable rigid disc inclusion embedded in a saturated poroelastic medium: $\nu = 0.0$, $\nu_u = 0.5$

$t_1 = ct/a^2$	Flexibilities of Displacements				Stiffnesses of Loads			
	Axial		Horizontal		Axial		Horizontal	
	$\frac{D_x(t_1)}{10^{-4}P_s}$ ($\frac{mm}{N}$)	$\frac{\Omega_x(t_1)}{10^{-11}M_s}$ ($\frac{1}{mm N}$)	$\frac{D_x(t_1)}{10^{-4}P_s}$ ($\frac{mm}{N}$)	$\frac{\Omega_x(t_1)}{10^{-11}M_s}$ ($\frac{1}{mm N}$)	$\frac{P_x(t_1)}{10^4 D_s}$ ($\frac{N}{mm}$)	$\frac{M_x(t_1)}{10^{13} \Omega_s}$ ($mm N$)	$\frac{P_x(t_1)}{10^4 D_s}$ ($\frac{N}{mm}$)	$\frac{M_x(t_1)}{10^{13} \Omega_s}$ ($mm N$)
0	.1250	.4688	.1875	.4688	.8000	.2133	.5333	.2133
10^{-4}	.1256	.4763	.1878	.4688	.7959	.2101	.5326	.2133
10^{-3}	.1269	.4904	.1881	.4688	.7877	.2038	.5316	.2133
10^{-2}	.1312	.5345	.1896	.4688	.7618	.1865	.5275	.2133
10^{-1}	.1448	.6389	.1943	.4688	.6873	.1548	.5145	.2133
10^0	.1712	.6996	.2055	.4688	.5802	.1429	.4861	.2133
10^1	.1827	.7030	.2141	.4688	.5469	.1423	.4670	.2133
10^2	.1860	.7030	.2173	.4688	.5376	.1423	.4603	.2133
10^3	.1870	.7030	.2183	.4688	.5347	.1423	.4581	.2133
$+\infty$	.1875	.7031	.2188	.4688	.5333	.1422	.4571	.2133

Chapter 8

Conclusions And Recommendations

8.1 General

In the proceeding Chapters, we have presented a rigorous mathematical and analytical investigation of the mechanics of rigid disc inclusions embedded in poroelastic media saturated with compressible pore fluids. In this final Chapter, we shall summarize the main achievements of this research and present recommendations for future work.

8.2 Summary and Conclusions

The inclusion problems discussed here are classical mixed boundary value (and initial value) problems in mathematical physics and applied mechanics which have important applications to various branches of engineering and science. However, due to the intrinsic difficulties associated with the mathematical formulations and the ensuing numerical evaluations, the majority of the investigations have concentrated on the study of the static responses of inclusions embedded in linear elastic media. A study of the literature on inclusion problems reveals that only limited results have been obtained with regard to the time-dependent behaviour of inclusions embedded in saturated poroelastic media, visco-elastic media, or thermoclastic media.

In this thesis, we have developed a valuable extension to the scope of inclusion problems, namely, the time-dependent response of a rigid disc inclusion embedded in a poroelastic medium saturated with a compressible pore fluid. In order to achieve the objectives of the study, we have completed the following phases of the research programme.

1. Using the classical integral transform technique for the solution of linear partial differential equations, we have developed a simple and direct matrix based approach to systematically and rigorously solve the (mixed) boundary value and initial value problems associated with saturated poroelastic layered media. The main contributions in the mathematical development of the adopted approach can be summarized in the following.

- The coordinate systems in the Fourier transform domain and the expressions for the 2-D Fourier transforms in the cylindrical coordinates are re-defined. A simpler general relation is obtained for 2-D Fourier transforms and 1-D Hankel transforms;
- A set of symmetric Fourier matrix kernel functions are introduced in both physical and transformed Cartesian and cylindrical coordinates. A set of invariant variable vectors are introduced in the transformed Cartesian and cylindrical coordinates. A set of matrix solution representations are developed for all the field variables in the physical domain and the introduced invariant variables in the transform domain in both the Cartesian and the cylindrical coordinates;
- Two invariant sets of two and six first-order ordinate differential equations in the transform domain are derived for the seventeen four-dimensional partial differential equations with body forces and fluid sources in both Cartesian and cylindrical coordinates. These seventeen partial differential equations govern the behaviour of vertically inhomogeneous poroelastic media saturated with compressible pore fluids;
- Invariant algebraic fundamental solution representations in matrix form in the transform domain are obtained for the partial differential equations governing the behaviour of a saturated homogeneous poroelastic medium of finite layer extent or halfspace extent or infinite extent;
- Invariant boundary algebraic equations in matrix form in the transform domain are obtained for the field variables at the outer surface(s) of a saturated poroelastic medium of finite layer extent or halfspace extent in both Cartesian and

cylindrical coordinates;

- The eight square matrices in the algebraic fundamental solution representations are equivalent to those in the fundamental singular solutions in the transform domain due to the action of instantaneous concentrated point body forces and fluid source in a saturated homogeneous poroelastic medium of infinite extent;
- For any correctly posed boundary and initial value condition in a saturated poroelastic medium of halfspace extent, the solution in the transform domain satisfies the mathematical requirements for the inverse Fourier and Laplace transforms.

2. Based on the above results, we have proposed a general and systematic Fourier and Laplace transform based approach for analytical solution of mixed boundary value and initial value problems in saturated poroelastic layered media and have succeeded in systematically formulating the integral equations for the fundamental problem of a rigid disc inclusion embedded in bonded contact with a poroelastic medium of infinite extent saturated with a compressible fluid. The main contributions in the mathematical formulation of the integral equations can be summarized in following:

- Systems of 2-D Fourier transform based integral equations in the Laplace transform domain are formulated for either a permeable or an impermeable plane inclusion in a saturated poroelastic medium of infinite extent subjected to general direct loads. Such plane inclusion problems fully decouple into the individual modes associated with the vertical and the horizontal displacements of the plane inclusion;
- By using the general relation between Fourier and Hankel transforms, the systems of 2-D Fourier transform based integral equations are reduced to infinite sets of Hankel transform based integral equations in the Laplace transform domain for a circular disc inclusion embedded in a saturated poroelastic medium of infinite extent;
- Finite sets of Hankel transformed based integral equations in the Laplace trans-

form domain are systematically obtained for the fundamental problem of a rigid disc inclusion embedded in a saturated poroelastic medium of infinite extent subjected to direct loads. Such fundamental inclusion problem fully decouple into the four basic problems: (i) axisymmetric displacement (or axial translation); (ii) asymmetric rotation about the plane of the inclusion (or central rotation); (iii) in-plane translation; (iv) rotation about the axis of axisymmetry (or axial rotation);

- Systems of Fredholm integral equations of the second kind in the Laplace transform domain are derived for the four basic modes of a rigid disc inclusion embedded in a saturated poroelastic medium of infinite extent. The kernel functions in the standard Fredholm integral equations are uniformly absolutely convergent and regular with the three independent variables x, y, s where $Re(s) > 0$;
- The exact solution is obtained in closed-form for the axial rotation of a permeable or an impermeable rigid disc inclusion embedded in a saturated poroelastic medium;
- The regular type $(a^2 - r^2)^{-\frac{1}{2}}$ of singularity is derived analytically for all the non-zero contact normal or shear stresses or excess pore pressure at the soil-inclusion interface.

3. We have presented techniques for numerical solution of the systems of Fredholm integral equations of the second kind in the Laplace transform domain. These techniques are used to generate the results of engineering interesting in the temporal domain. The adopted numerical scheme has high stability and accuracy for the very long time interval associated with the process of soil consolidation. The main contributions in the numerical evaluations of the integral equations can be summarized in the following.

- A modified algorithm is developed for Laplace transform inversion. This modified algorithm displays high stability and accuracy for the long time interval associated with the process of soil consolidation;
- A Simpson's quadrature based proceeding limit technique is adopted and an

approximation technique is developed for fast and accurate evaluation of the infinite integrals, with rapidly oscillatory integrands, encountered in the kernel functions of the Fredholm integral equations;

- The numerical scheme is verified. This process is achieved by the selection of optimum values for the integral parameters in all the three-dimensions of the repeated numerical integration technique;
- A range of numerical solutions are developed, in which the influence of singular field on the numerical scheme is documented.

4. In all the above equations, formulations and solutions, we have rigorously examined the results of the three limiting conditions — the initial response as $t \rightarrow 0^+$; the final response as $t \rightarrow +\infty$; and the limiting response as $\nu \rightarrow \nu_u$. The main results associated with these limits are summarized in the following.

- The initial ($t \rightarrow 0^+$) responses of stresses and displacements in the soil skeleton can be evaluated by using the results in classical elastostatics with the undrained values of the elastic constants μ, ν_u . The initial excess pore pressure can be obtained from the definition of Skempton's pore pressure parameter B ;
- The final ($t \rightarrow +\infty$) responses of stresses and displacements in the soil skeleton can be evaluated by using the results in classical elastostatics with the drained values of the elastic constants μ, ν . The final excess pore pressure decays to zero;
- In the limiting case of $\nu \rightarrow \nu_u$, the two phase model for a saturated poroelastic medium fully decouples into two parts: an elastostatic response for the solid phase and a steady state seepage flow response for the fluid phase. Under this limiting condition, there is no interaction between the soil skeleton phase and the pore fluid phase.

5. We have successfully analyzed and illustrated the time-dependent behaviour of a rigid disc inclusion embedded in bonded contact with a saturated poroelastic medium of infinite extent by considering the influence of loading conditions, interface drainage conditions, material parameters, as well as length parameter.

The points of interest, related to the time-dependent behaviour of the embedded inclusion, are summarized in the following.

- The consolidation processes associated with the rotation of the embedded inclusion are much shorter than those associated with the translation of the embedded inclusion. In particular, there is no consolidation process induced by the axial rotation of the disc inclusion. The consolidation process for the axial translation of the inclusion occurs more rapidly than that associated with the in-plane translation of the inclusion;
- The drainage conditions at the inclusion-poroelastic medium interface have a significant influence on the rate of consolidation particularly for the central rotation and the axial translation of the embedded inclusion. Such permeability conditions have only a very limited effect on the consolidation rate for the in-plane translation of the disc inclusion;
- The time and magnitudes of consolidation induced displacements or relaxation induced forces of the embedded inclusion are directly proportional to the difference between the undrained and drained Poisson's ratios ($\nu_u - \nu$);
- When the time factor ct/a^2 is used, the values of ν and ν_u do not introduce an easily discernible effect on the degrees of consolidation induced displacements of the embedded inclusion subjected to the various loading modes. The values of Poisson's ratios have a limited effect on the degree of relaxation induced forces or moment in the inclusion which is subjected to the various displacement modes;
- The influence of ν and ν_u can be best illustrated by the adoption of a modified time factor $c_m t/a^2$ where $c_m = 2\mu\kappa B^2/a^2$;
- the dissolved gas phase in pore fluids increases its undrained compressibility and reduces the potential of soil consolidation deformation and contributes to an 'apparent' retardation in the consolidation (or relaxation) process.

8.3 Recommendations for Future Work

In the proceeding sections, we have summarized the main achievements of this research. The research has provided a general framework for the formulation and numerical solutions of the class of rigid disc inclusion problems associated with poroelastic media saturated with compressible pore fluids. The availability of benchmark solutions for these inclusion problems can be used to advantage in the verification of computational schemes and the development of further results for practical problems of geotechnical interest. In the ensuing, we recommend some specific extensions.

- Examination of the mechanics of rigid disc inclusions embedded in saturated poroelastic media induced by varying direct loads, indirect loads, debonded contacts, boundary effects, or inhomogeneous (layered) media;
- Examination of the time-dependent behaviour of the contact normal or shear stresses and excess pore pressures at the soil/inclusion interfaces or in the surrounding soil matrix;
- Development of solutions for the mechanics of flexible plane (or disc) inclusions embedded in saturated poroelastic media by considering the elastic flexural deformation of inclusions;
- Examination of the time-dependent characteristics of hydraulic fractures in saturated poroelastic media;
- Utilization of the modified numerical inversion of Laplace transforms in the development of effective Boundary Element Methods or Finite Element Methods for the studies of more complex problems in saturated poroelastic media;
- Application of the mathematical approaches and numerical techniques presented in this thesis in the investigations of the mechanics of inclusions or fracture mechanics in viscoelastic media, thermoelastic media or other processes which can be characterized by linear partial differential equations of parabolic type.

References

- [1] Abramian, B.L., Arutiunian, N.Kn., and Babloin, A.A., 1966, On symmetric pressure of a circular stamp on an elastic half-space in the presence of adhesion, *Prikl. Mat. Mekh., Journal of Applied Mathematics and Mechanics*, 30, 143-147.
- [2] Agbezuge, L.K., and Deresiewicz, H., 1974, On the indentation of a consolidation half-space, *Israel Journal of Technology*, 12, 322-338.
- [3] Anwar, M.N., and Sherief, H.H., 1988, Boundary integral equation formulation of generalized thermoelasticity in a Laplace-transform domain, *Appl. Math. Modeling*, 12, 161-166.
- [4] Atkinson, C., 1979, Stress singularities and fracture mechanics, *Appl. Mech. Rev.*, 32, 123-135.
- [5] Atkinson, K., 1976, *A Survey of Numerical Methods for the Solution of Fredholm Integral Equations Of the Second-Kind*, Soc. Industrial and Appl. Math., Philadelphia, PA.
- [6] Baker, C.T.H., 1977, *The Numerical Treatment of Integral Equations*, Clarendon Press, Oxford.
- [7] Balla, A., 1961, The resistance to breaking out of mushroom foundations for pylons. *Proc. 5th Int. Conf. Soil mech. and Found. Eng.*, Paris, 1, 569-576.
- [8] Bellman, R., Kalaba, R.E., and Lockett, J.A., 1966, *Numerical Inversion of the Laplace Transform: Applications to Biology, Economics, Engineering, and Physics*, American Elsevier, New York.
- [9] Biot, M.A., 1941, General theory of three-dimensional consolidation, *J. Appl. Phys.*, 12, 155-164.
- [10] Biot, M.A., 1955, Theory of elasticity and consolidation for a porous anisotropic

- solid, *J. Appl. Phys.*, 26, 182-185.
- [11] Biot, M.A., 1956, General solution of the equations of elasticity and consolidation for a porous material, *J. Appl. Mech.*, ASME, 23, 91-95.
 - [12] Biot, M.A., and Willis, D.G., 1957, The elastic coefficients of the theory of consolidation. *J. Appl. Mech.*, ASME, 44, 594-601.
 - [13] Booker, J.R., and Poulos, H.G., 1976, Analysis of creep settlement of pile foundations. *Journal of Geotechnical Engineering*, ASCE, 102, 1-14.
 - [14] Booker, J.R., and Small, J.C., 1975, An investigation of the stability of numerical solutions of Biot's equations of consolidation, *Int. J. Solids Struct.*, 11, 907-917.
 - [15] Booker, J.R., and Small, J.C., 1984, The time-behaviour of a circular raft of finite flexibility on a deep clay layer, *Int. j. numer. anal. methods in geomech.*, 8, 343-357.
 - [16] Booker, J.R., and Small, J.C., 1986, The behaviour of an impermeable flexible raft on deep clay layer of consolidating soil, *Int. j. numer. anal. methods in geomech.*, 10, 311-327.
 - [17] Booker, J.R., and Small, J.C., 1987, A method of computing the consolidation behaviour of layered soils using direct numerical inversion of Laplace transforms, *Int. j. numer. anal. methods in geomech.*, 11, 363-380.
 - [18] Boussinesq, J., 1885, *Application des potentiels a l'etude de l'equilibre et du mouvement des solides elastique*, Gaythier-Villars, Paris.
 - [19] Bowen, R.M. 1976, Theory of mixtures, in *Continuum Physics*, 3, (Ed. A.C. Eringen), Academic Press, New York.
 - [20] Brown, P.T., and Booker, J.R., 1979, Numerical analysis of rafts on visco-elastic media using eigenvector expansions, *Int. j. numer. anal. methods in geomech.*, 3, 363-
 - [21] Butterfield, R, and Banerjee, P.K., 1971, A rigid disc embedded in an elastic halfspace, *Geotechnical Engineering*, Southeast Asian Society of Soil Engineer-

ing, 2, 35-49.

- [22] Bycroft, G.N., 1956, Forced vibrations of a rigid circular plate on a semi-infinite elastic space and on an elastic stratum, *Phil. Trans. Roy. Soc.*, 248, 327-368.
- [23] Carter, J.P., and Booker, J.R., and Small, J.C., 1979, The analysis of finite elasto-plastic consolidation *Int. j. numer. anal. methods in geomech.*, 3, 107-129.
- [24] Chen, W.F., and Baladi, G.Y., 1985, *Soil Plasticity: Theory and Implementation*, Elsevier Science Publishers B. V., New York.
- [25] Cheng, A.H.D., and Liggett, J.A., 1984a, Boundary integral equation method for linear porous-elasticity with applications to soil consolidation. *Int. J. numer. methods eng.*, 20, 255-278.
- [26] Cheng, A.H.D., and Liggett, J.A., 1984b, Boundary integral equation method for linear porous elasticity with applications to fracture propagation, *Int. J. numer. methods eng.*, 20, 279-296.
- [27] Cheng, A.H.D., and Predeleanu, M., 1987, Transient boundary element formulation for linear porous-elasticity with applications to soil consolidation. *Appl. Math. Model.*, 11, 285.
- [28] Cheng, A.H.D., and Detournay, E., 1988, A direct boundary element method for plane stress poroelasticity, *Int. j. numer. anal. methods in geomech.*, 12, 551-572.
- [29] Chiarella, C., and Booker, J.R., 1975, The time-settlement behavior of a rigid die resting a on a deep clay layer, *Q. J. Mech. Appl. Math.*, 28, 317-328.
- [30] Collins, W.D., 1962, Some axially symmetric stress distributions in elastic solids containing penny-shaped cracks I., Cracks in an infinite solid and a thick plate, *Proc. R. Soc. A*, 203, 359-386.
- [31] Crump, K.S., 1976, Numerical inversion of Laplace transforms using a Fourier series approximation, *J. Assoc. Comp. Machin.*, 10, 311-327.
- [32] Cryer, C.W., 1963, A comparison of the three-dimensional consolidation theo-

- ries of Biot and Terzaghi, *Q. J. Mech. Appl. Math.*, 16, 401-412.
- [33] Das, B.M., 1983, *Advanced Soil Mechanics*, Hemisphere Publishing Corporation, Washington.
 - [34] Davie, J.R., and Sutherland, H.B., 1977, Uplift resistance of cohesive soils, *Journal of Geotechnical Engineering*, ASCE, 103, 935-952.
 - [35] Davies, B, and Martin, B., 1979, Numerical inversion of the Laplace transform: a survey and comparison of methods, *Journal of Computational Physics*, 33, 1-32.
 - [36] Davis, E.H., and Poulos, H.G., 1963, *4-th Aust-N.Z. Conf. Soil Mech. Found. Eng.*, 233.
 - [37] Delves, L.M., and Mohamed, J.L., 1985, *Computational Methods for Integral Equations*, Cambridge University Press, Cambridge.
 - [38] de Arriaga, F.J., 1968, Stress analysis for gravity dams and foundations including rheological behaviour, *Proc. Int. Symp. Rock Mech.* Madrid, 329-334.
 - [39] de Pater, A.D., and Kalker, J.J., 1975, *The Mechanics of the Contact between Deformable Bodies*, Delft University Press, Belgium.
 - [40] Deresiewicz, H., 1977, On the indentation of a consolidating half space, II. effect of Poisson's ratio, *Israel Journal of Technology*, 15, 89-97.
 - [41] Desai, C.S., and Siriwardane, H.J., 1984, *Constitutive Laws For Engineering Materials (with Emphasis on Geological Materials)*, Prentice-Hall, Inc., Englewood Cliff.
 - [42] Dubner, R., and Abate, J., 1968, Numerical inversion of Laplace transforms by relating them to the finite Fourier Cosine transform, *J. Assoc. Comp. Machin.*, 15, 115-123.
 - [43] Durbin, F., 1974, Numerical inversion of Laplace transforms: an efficient improvement to Dubner and Abate's method, *The Computer Journal*, 17, 371-376
 - [44] Elliott, H.A., 1949, Axial symmetric stress distributions in aeolotropic hexagonal crystals; The problem of the plane and related problems, *Proc. Camb. Phil.*

Soc., 45, 621-630.

- [45] Erdelyi, A., et al, 1954, *Tables of Integral Transforms*, Vol.I and II, McGraw-Hill Book Co. Inc., New York.
- [46] Eshelby, J.D., 1961, Elastic inclusions and inhomogeneities, In *Progress in Solid Mechanics* (Edited by I. N. Sneddon and R. Hill), North-Holland Amsterdam, 2, 89-142,
- [47] Florence, A.L., 1961, Two contact problems for an elastic layer, *Quart. J. Mech. Appl. Math.*, 14, 453-459
- [48] Flugge, W., 1975, *Viscoelasticity*, Waltham, Mass., Blaisdell.
- [49] Gaszynski, J., and Szefer, G., 1978, Axisymmetric problem of the punch for the consolidating semi-space with mixed boundary permeability conditions, *Archivum Mechaniki Stosowanej*, 30, 17-26.
- [50] Ghaboussi, J., and Wilson, E.L., 1973, Flow of compressible fluid in porous elastic solids, *Int. J. numer. methods eng.*, 5, 419-442.
- [51] Gibson, R.E., England G.L., and Hussey, M.J.L., 1967, The theory of one dimensional consolidation of clays, *Geotechnique*, 17, 261-273.
- [52] Gibson, R.E., Schiffman, R.L., and Pu, S.L., 1970, Plane strain and axially symmetric consolidation of a clay layer on a smooth impervious base, *Q. J. Mech. Appl. Math.*, 23, 505-520.
- [53] Gladwell, G.M.L., 1969, A contact problem for a circular cylindrical punch in adhesive contact with an elastic halfspaces: the case of rocking and translation parallel to the planes, *Int. J. Engng. Sci.*, 7, 295-307.
- [54] Gladwell, G.M.L., 1980, *Contact Problems in the Theory of Elasticity*, Sijthoff and Noordhoff, The Netherlands.
- [55] Gradshteyn, I.S., and Ryzhik, I.M., 1980, *Table of Integrals, Series, and Products*, Academic Press, Inc., New York.
- [56] Green, A.E., 1949, On Boussinesq's problem and penny-shaped cracks, *Proc. Cam. Phil. Soc.*, 45, 251-257.

- [57] Hanna, T.H., 1982, *Foundations In Tension: Ground Anchors*, Co-published by Trans Tech Publications and McGraw-Hill Book Company,
- [58] Harding, J.W. and Sneddon, I.N., 1945, The elastic stress produced by the indentation of the plane surface of a semi-infinite elastic solid by a rigid punch, *Proc. Camb. Phil. Soc.*, 41, 16-26.
- [59] Hunter, S.C., and Gamblen, D., 1974, The theory of a rigid circular disc ground anchor buried in an elastic soil either with adhesion or without adhesion, *J. Mech. Phy. Solids*, 22, 371-399.
- [60] Heinrich, G., and Desoyer, K., 1961, Theorie dreidimensionalen sezungsvorgange in tongschichten, *Ing. Arch.*, 30, 225-253.
- [61] Jaeger, J.C., and Cook, N.G.W., 1969, *Fundamentals of Rock Mechanics*, Methuen, London, 1963.
- [62] Kanwal, R.P., 1971, *Linear Integral equations: Theory and Technique*, Academic Press, Inc., New York.
- [63] Kanwal, R.P., and Sharma, D.L., 1976, Singularity methods for elastostatics. *J. Elasticity*, 6, 405-418.
- [64] Kassir, M.K., and Sih, G.C., 1968, Some three-dimensional inclusions problems in elasticity theory, *Int. J. Solids Struct.*, 4, 225-241.
- [65] Kassir, M.K., and Sih, G. C., 1975, *Mechanics of Fracture: Three-Dimensional Crack Problems*, Vol. 2, Noordhoff International Publishing, Leyden.
- [66] Keer, L.M., 1965, A note on the solution of two asymmetric boundary value problems, *Int. J. Solids Struct.*, 1, 257-264.
- [67] Keer, L.M., 1967, Mixed boundary-value problems for an elastic halfspace, *Proc. Camb. Phil. Soc.*, 63, 1379-1386.
- [68] Keer, L.M., 1975, Mixed boundary value problems for a penny-shaped cut, *J. Elasticity*, 5, 89-98.
- [69] Kravtchenko, J., and Sirieys, P.M.,(Eds) 1966, *Rheology and Soil Mechanics; Proc. I.U.T.A. M. Conf.*, Grenoble, Springer, Berlin.

- [70] Kupradze, V.D., 1979, *Three-Dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity*, North-Holland, New York.
- [71] Ladanyi, B., and Johnston, G.H., 1974, Behaviour of circular footing and plate anchors embedded in permafrost, *Canadian Geotechnical Journal*, 11, 531-553.
- [72] Lambe, T.W., and Whitman, R.V., 1969, *Soil Mechanics*, John Wiley & Sons, Inc., New York.
- [73] Lee, I.K, White, W., and Ingles, O.G., 1983, *Geotechnical Engineering*, Pitman, Boston.
- [74] Lewis, R.W., and Schrefler, B.A., 1987, *The Finite Element Methods in the Deformation and Consolidation of Porous Media*, John Wiley and Sons, New York.
- [75] Lur'e, A.I., 1964, *Three-Dimensional Problems of the Theory of Elasticity*, John Wiley, New York.
- [76] MacDonald, J.R., 1964, Accelerated convergence, divergence, iteration, extrapolation, and curve fitting, *J. Appl. Phys.* 10, 3034-3041.
- [77] Mandel, J., 1953, Consolidation des sols (Etude mathematique), *Geotechnique*, 3, 287-299.
- [78] Mariupol'skii, L.G., 1965, The bearing capacity of anchor foundations, *Osnovaniya Fundamenty i Mekhanika Gruntov*, 3, 14-18 (available in English translation from Consultants Bureau, New York, N. Y., 26-32).
- [79] Matsuo, M., 1967, Study on the uplift resistance of footing (1), *Soil and Foundation*, Tokyo, Japan, 7, 1-37.
- [80] Meyerhof, G.G., 1951, The ultimate bearing capacity of foundations, *Geotechnique*, 2, 301-332.
- [81] Meyerhof, G.G., and Adams, J.I., 1968, The ultimate uplift capacity of foundations, *Canadian Geotechnical Journal*, 5, 225-244.
- [82] McNamee, J., and Gibson, R.E., 1960a, Displacement functions and linear transforms applied to diffusion through porous elastic media, *Q. J. Mech. Appl.*

- Math.*, 13, 98-111.
- [83] McNamee, J., and Gibson, R.E., 1960b, Plane strain and axially symmetric problems of consolidation of a semi-infinite clay stratum, *Q. J. Mech. Appl. Math.*, 13, 210-227.
 - [84] Mindlin, R.D., 1936, Force at a point in the interior of a semi-infinite solid, *J. Appl. Mech.*, ASME, 7, 195-202.
 - [85] Mindlin, R.D., 1949, Compliance of elastic bodies in contact, *J. Appl. Mech.*, ASME, 16, 259-268.
 - [86] Mossakovskii, V.I., 1954, The fundamental mixed problem of the theory of elasticity for a half-space with a circular line separating the boundary conditions, *Prikl. Mat. Mekh.*, *Journal of Applied Mathematics and Mechanics*, 18, 187-196.
 - [87] Mura, T., 1982, *Micromechanics of Defects in Solids*, Martinus Nijhoff, The Hague.
 - [88] Mura, T., 1988, Inclusion problems, *Appl. Mech. Rev.*, 41, 15-20.
 - [89] Narayanan, G.V., and Beskos, D.E., 1982, Numerical operational methods for time-dependent linear problems, *Int. J. numer. methods eng.*, 18, 1829-1954.
 - [90] Niumpradit, B., and Karasudhi, P., 1981, Load transfer from an elastic pile to a saturated porous elastic soil, *Int. j. numer. anal. methods in geomech.*, 5, 115-138.
 - [91] Piessens, R., 1975, A bibliography on numerical inversion of the Laplace transform and applications, *J. Comp. Appl. Math.*, 1, 115-128.
 - [92] Piessens, R., and Dang, N.D.P., 1976, A bibliography on numerical inversion of the Laplace transform and applications: a supplement, *J. Comp. Appl. Math.*, 2, 225-228
 - [93] Pyrah, I.C., Hird, C.C., and Tanaka, Y., 1985, Consolidation behaviour of a single under-reamed anchor, *Proc. Fifth Int. Conf. on Num. Meth. in Geom.*, Nogoya, Vol.1, 629-636.

- [94] Reissner, E., and Sagoci, H.F., 1944, Forced torsional oscillations of an elastic half-space; I. *J. of Appl. Phys.*, 15, 652-654.
- [95] Rice, J.R., and Cleary, M.P., 1976, Some basic stress diffusion solutions for fluid-saturated elastic porous media with compressible constituents, *Rev. Geophys. Space Phys.*, 14, No.2, 227-241.
- [96] Rice, J.R., and Simons, D. A., 1976, The stabilization of spreading shear faults by coupled deformation-diffusion effects in fluid-saturated porous material, *J. Geophys. Res.*, 81(29), 5322-5334.
- [97] Rostovtsev, N.A., 1955, On the problem of torsion of an elastic halfspace, *Prikl. Mat. Mekh.*, 19, 55-60 (in Russian).
- [98] Rowe, R.K., and Davis, E.H., 1982a, The behaviour of anchor plates in clay, *Geotechnique*, 32, 9-23.
- [99] Rowe, R.K., and Davis, E.H., 1982b, The behaviour of anchor plates in sand, *Geotechnique*, 32, 25-41.
- [100] Sandhu, R.S., Lee, S.C., and The, H., 1985, Special finite elements for analysis of soil consolidation, *Int. j. numer. anal. methods in geomech.*, 9: 125-147.
- [101] Sandhu, R.S., and Wilson, E.L., 1969, Finite element analysis of seepage in elastic media, *J. Engng. Mech. Div., A.S.C.E.*, 95, 641-
- [102] Schapery, R.A., 1962, Approximate methods of transform inversion for viscoelastic stress analysis, *Proc. 4th U.S. Natl. Congr. Appl. Mech.*, Berkeley, California, 2, 1075-1085.
- [103] Schrefler, B.A., and Simoni, L., 1987, Non-isothermal consolidation of unbounded porous media using mapped infinite elements, *Comm. in Appl. Num. Methods*, 3, 445-452.
- [104] Schiffman, R.L., 1984. A bibliography of consolidation, in *Fundamentals of Transport Phenomena in Porous Media*, edited by Jacob Bear and M. Yavuz Corapcioglu, 617-669
- [105] Schiffman, R.L., Chen, A.T-F, Jordan, J.C., 1969, An analysis of consolidation

- theories, *J. Soil Mech. Found. Divi., ASCE*, 95(SM1), 285-312.
- [106] Schiffman, R.L., and Fungaroli, A.A., 1965, Consolidation due to tangential loads, *Proc. 6th. Int. Conf. Soil Found. Eng.*, Montreal, Canada, 1, 188-192.
 - [107] Selvadurai, A.P.S., 1976, The load-deflexion characteristics of a deep rigid anchor in an elastic medium, *Geotechnique*, 26, 603-612.
 - [108] Selvadurai, A.P.S., 1978a, The response of a deep rigid anchor due to undrained elastic deformation of the surrounding soil medium, *Int. j. numer. anal. methods in geomech.*, 2, 189-197.
 - [109] Selvadurai, A.P.S., 1978b, Interaction between a rigid circular punch on an elastic halfspace and a Mindlin force, *Mech. Res. Commun.*, 5, 57-64.
 - [110] Selvadurai, A.P.S., 1978c, The time-dependent response of a deep rigid anchor in a viscoelastic medium, *Int. J. Rock Mech. Min. Sci. & Geomech. Abstr.*, 15, 11-19.
 - [111] Selvadurai, A.P.S., 1979a, *Elastic Analysis of Soil-Foundation Interaction*, Developments in Geotechnical Engineering, Vol.17, Elsevier Sci. Publ. Co. Amsterdam.
 - [112] Selvadurai, A.P.S., 1979b, An energy estimate of the flexural behaviour of a circular foundation embedded in an isotropic elastic medium, *Int. j. numer. anal. methods in geomech.*, 3, 285-292.
 - [113] Selvadurai, A.P.S., 1979c, The elastic displacements of a rigid disc inclusion embedded in an isotropic elastic medium due to the action of an external force, *Mech. Res. Commun.*, 6, 379-385.
 - [114] Selvadurai, A.P.S., 1979d, On the displacement of a penny-shaped rigid inclusion embedded in a transversely isotropic elastic medium, *Solid Mechanics Archives*, 4, 163-172.
 - [115] Selvadurai, A.P.S., 1979e, The cable method of in situ testing: influence of the anchor region, *Proc. 3rd. Int. Conf. Num. Methods in Geomech.*, Aachen, 3, 1237-1243.

- [116] Selvadurai, A.P.S., 1979f, The displacement of a rigid circular foundation anchored to an isotropic elastic halfspace, *Geotechnique*, 29, 195-202.
- [117] Selvadurai, A.P.S., 1979g, Some results concerning the viscoelastic relaxation of prestress in a surface rock anchor, *Int. J. Rock Mech. Min. Sci. & Geomech. Abstr.*, 16, 309-317.
- [118] Selvadurai, A.P.S., 1980a, The eccentric loading of a rigid circular foundation embedded in an isotropic elastic medium, *Int. j. numer. anal. methods in geomech.*, 4, 121-129.
- [119] Selvadurai, A.P.S., 1980b, Asymmetric displacement of a rigid disc inclusion embedded in a transversely isotropic, elastic medium of infinite extent, *Int. J. Engng. Sci.*, 18, 979-986.
- [120] Selvadurai, A.P.S., 1981a, Betti's reciprocal relationships for displacements of an elastic infinite space bounded internally by a rigid inclusion, *Journal Structural Mechanics*, 9, 199-210.
- [121] Selvadurai, A.P.S., 1981b, A theoretical estimate of the tilt of a surface foundation due to inclined anchor load applied at the interior of the soil medium, *Int. j. numer. anal. methods in geomech.*, 5, 369-378
- [122] Selvadurai, A.P.S., 1981c, The displacements of a flexible inhomogeneity embedded in a transversely isotropic elastic medium, *Fibre Science and Technology*, 14, 251-259.
- [123] Selvadurai, A.P.S., 1982a, On the interaction between an elastically embedded rigid inhomogeneity and a laterally placed concentrated force, *J. Appl. Math. Phys. (ZAMP)* 33, 241-250.
- [124] Selvadurai, A.P.S., 1982b, Axial displacement of a rigid elliptical disc inclusion embedded in a transversely isotropic elastic solid, *Mech. Res. Comm.*, 9, 39-45.
- [125] Selvadurai, A.P.S., 1982c, The additional settlement of a rigid circular foundation on an isotropic elastic halfspace due to multiple distributed external loads, *Geotechnique*, 32, 1-7.

- [126] Selvadurai, A.P.S., 1983a, Asymmetric rotation of a penny-shaped rigid inclusion embedded at a bi-material elastic interface, *Advances at Aerospace Structures, Materials and Dynamics*, AD-06, Winter Annual Meeting of ASME, Boston, 201-207.
- [127] Selvadurai, A.P.S., 1983b, Fundamental results concerning the settlement of a rigid foundation on an elastic medium due to an adjacent surface load, *Int. j. numer. anal. methods in geomech.*, 7, 209-223.
- [128] Selvadurai, A.P.S., 1984a, Elastostatic bounds for the stiffness of an elliptical disc inclusion embedded at a transversely isotropic bi-material interface, *Journal of Applied Mathematics and Physics*(ZAMP), 35, 13-23.
- [129] Selvadurai, A.P.S., 1984b, The rotation of a rigid elliptical disc inclusion embedded in a transversely isotropic elastic solid, *Mech. Res. Commun.*, 11, 41-48.
- [130] Selvadurai, A.P.S., 1985, Rotational stiffness of a rigid elliptical disc inclusion embedded at bi-material elastic interface, *Solid Mech. Arch.*, 10, 3-16.
- [131] Selvadurai, A.P.S., 1987, Computer algebra in applied mathematics: applications to problems in non-linear elasticity. *Studies in Applied Mechanics*, A.P.S. Selvadurai, ed. 16, Elsevier Scientific Publishing Co., Amsterdam, the Netherlands, 69-88.
- [132] Selvadurai, A.P.S., 1989a, The influence of a boundary fracture on the elastic stiffness of a deeply embedded anchor plate, *Int. j. numer. anal. methods in geomech.*, 13, 159-170.
- [133] Selvadurai, A.P.S., 1989b, Elastic stiffness of flat anchor region at cracked geological interface, *Journal of Engineering Mechanics*, ASCE, 115, 18-32.
- [134] Selvadurai, A.P.S., 1993, Mechanics of a rigid circular anchor plate embedded in an elastic halfspace, *Int. j. numer. anal. methods in geomech.*, (in press)
- [135] Selvadurai, A.P.S., and Au, M.C., 1986, Generalized displacements of a rigid elliptical anchor embedded at a bi-material geological interface, *Int. j. numer.*

- anal. methods in geomech.*, 10, 633-652.
- [136] Selvadurai, A.P.S., Au, M.C., and Singh, B.M., 1990, Asymmetric loading of an externally cracked elastic solid by an in-plane penny-shaped inclusion, *Theoretical and Applied Fracture Mechanics*, 14, 253-266.
 - [137] Selvadurai, A.P.S., Bauer, G.E., and Nicholas, T.J., 1980, Screw plate testing of a soft clay, *Canadian Geotechnical Journal*, 17, 465-472.
 - [138] Selvadurai, A.P.S., and Dasupta, G., 1990, Harmonic response of smoothly embedded rigid sphere, *Journal of Engineering Mechanics*, 116, 1945-1957.
 - [139] Selvadurai, A.P.S., and Gopal, K.P., 1986, Consolidation analysis of the screw plate test, *Proc. 39th Canadian Geotech. Conference*, Ottawa, Canada, 167-178.
 - [140] Selvadurai, A.P.S., and Karpurapu, R., 1989, Composite infinite element for modeling unbounded saturated-soil media, *Journal of Geotechnical Engineering*, ASCE, 115, No. 11, 1633-11646.
 - [141] Selvadurai, A.P.S., and Nicholas, T.J., 1979, A theoretical assessment of the screw plate test, *Proc. 3rd Int. Conf. Num. Anal. Meth. Geomech.*, Aachen, 2, 1245-1252.
 - [142] Selvadurai, A.P.S., and Singh, B.M., 1984a, Some annular disc inclusion problems in elasticity, *Int. J. Solids structures*, 20, 129-139.
 - [143] Selvadurai, A.P.S., and Singh, B.M., 1984b, On the expansion of a penny-shaped crack by a rigid circular disc inclusion, *Int. J. Fracture Mechanics*, 25, 69-77.
 - [144] Selvadurai, A.P.S., and Singh, B.M., 1986, The axial displacement of a disc inclusion embedded in a penny-shaped crack, *J. App. Math. Phys.(ZAMP)*, 37, 64-77.
 - [145] Selvadurai, A.P.S., and Singh, B.M., 1987, Axisymmetrical problems for an externally cracks elastic solid II. Effect of a penny-shaped inclusion, *Int. J. Engng. Sci.*, 25, 1477-1490.
 - [146] Selvadurai, A.P.S., Singh, B.M., and Au, M.C., 1989a, Axial loading of a rigid

- disc inclusion with a debonded region, *Int. J. Solids Structures*, 25, 783-795.
- [147] Selvadurai, A.P.S., Singh, B.M., and Au, M.C., 1989b, Boussinesq's problem for a debonded boundary, *J. Appl. Math. and Phys. (ZAMP)*, 42, 1-14.
- [148] Selvadurai, A.P.S., Singh, B.M., and Au, M.C., 1991, The in-plane loading of a rigid disc inclusion embedded in an elastic halfspace, *J. Appl. Mech.*, ASME, 1-8
- [149] Selvadurai, A.P.S., Spencer, A.J.M., and Rudgyard, M.A., 1988, Second-order elasticity with axial symmetry, II. Spherical cavity and spherical rigid inclusion problems, *Int. J. Engrg. Sci.*, 25, 1477-1490.
- [150] Selvadurai, A.P.S., and Yue, Z.Q., 1992, Consolidation behaviour of embedded rigid foundations, *Proceedings of the 46th Canadian Conference of Geotechnical Engineering*, Toronto, Canada.
- [151] Shield, R. T., 1951, Notes on problems in hexagonal anisotropic materials, *Cambridge Philosophical Society, Proceedings*, 47, 401-409.
- [152] Sih, G.C., and Chen, E.P. 1981, *Mechanics of Fracture, Vol. 6, Cracks in Composite Materials*, Martinus Nijhoff, The Hague.
- [153] Simoni, L., and Schrefler, B.A. 1987, Mapped infinite elements in soil consolidation, *Int. J. numer. methods eng.*, 24: 513-527.
- [154] Skempton, A.W., 1954, The pore pressure coefficients *A* and *B*, *Geotechnique*, 4, 143-147.
- [155] Small, J.C., and Booker, J.R., 1987, The time-deflection behaviour of a rigid under-reamed anchor in a deep clay layer, *Int. j. numer. anal. methods in geomech.*, 11, 269-281.
- [156] Small, J.C., Booker, J.R., and Davis, E.H., 1976, Elasto-plastic consolidation of soil, *Int. J. Solids Struct.*, 12, 431-448.
- [157] Smiles, D.E., and Poulos, H.G., 1969, The one dimensional consolidation of columns of soil of finite length, *Australia J. Soil Res.*, 7, 285-291.
- [158] Sneddon, I.N., 1947, A note on a boundary value problem of Reissner and

- Sagoci, *J. App. Phys.*, 18, 130-132.
- [159] Sneddon, I.N., 1965, The relation between load and penetration in the axisymmetric Boussinesq problem for a punch of arbitrary profile. *Int. J. Engng Sci.* 3, 47-57.
 - [160] Sneddon, I.N., 1966a, *Mixed Boundary Value Problems in Potential Theory*, Amsterdam: North Holland.
 - [161] Sneddon, I.N., 1966b, The Reissner-Sagoci problem, *Proceedings of Glasgow Mathematical Association*, 7, 136-144.
 - [162] Sneddon, I.N., 1972, *The use of integral transforms*, McGraw-Hill Company.
 - [163] Sneddon, I.N., and Lowengub, M., 1969, *Crack Problems in the Theory of Elasticity*, Wiley, New York,
 - [164] Spence, D.A., 1968, Self-similar solutions to adhesive contact problems with incremental loading, *Royal Society of London, Proceedings, Series A*, 30, 55-80.
 - [165] Szefer, G., and Gaszynski, J., 1975, Axisymmetric punch problem under condition of consolidation, *Archiwum Mechaniki Stosowanej*, 27, 497-515.
 - [166] Tabaddor, F., 1978, Plates on saturated porous elastic foundations, *Int. j. numer. anal. methods in geomech.*, 2, 177-187.
 - [167] Talbot, A., 1979, The accurate numerical inversion of Laplace transforms, *Journal of the Institute of Mathematics and its Applications*, 23, 97-120.
 - [168] Tan, C.L., and Selvadurai, A.P.S., 1986, Internally indented penny-shaped cracks: A comparison of analytical and boundary integral equation estimates, *Theoretical and Applied Fracture Mechanics*, 5, 181-188.
 - [169] Teng, W.C., 1962, Foundation design, in *Civil Engineering and Engineering Mechanics Series*, N. M. Newmark and W. J. Hall, Eds., Prentice-Hall, Inc., Englewood Cliffs, N. J. p. 466.
 - [170] Terzaghi, K., 1923, Die Berechnung der Durchlassigkeitszaiffer des Tones aus dem Verlaug der Hydrohynamischen Spannungsercheinungen, Akademie der Wissenschaften in Wein, *Sitzungsberichte Math Naturwiss. Klasse*, Part IIa,

132(3/4), 125-138.

- [171] Tranter, C.J., 1968, *Bessel Functions with Some Physical Applications*, London, English U. P.
- [172] Ufliand, Ia, S., 1956, The contact problem of the theory of elasticity for a die, circular in its plane, in the presence of adhesion, *Prikl. Mat. Mekh.*, 20, 578-587 (in Russian).
- [173] Vardoulakis, I., and Harnpattanapanich, T., 1986, Numerical Laplace-Fourier transform inversion technique for layered-soil consolidation problems: I fundamental solutions and validation, *Int. j. numer. anal. methods in geomech.*, 10, 347-365.
- [174] Vesic, A.S., 1971, Breakout resistance of objects embedded in ocean bottom, *Journal of the Soil Mechanics and Foundations Division, ASCE*, 97, 1181-1206.
- [175] Vesic, A.S., 1972, Expansion of cavities in infinite soil mass, *Journal of the Soil Mechanics and Foundations Division, ASCE*, 98, 265-290.
- [176] Watson, G.N., 1962, *A Treatise on the Theory of Bessel Functions*, Cambridge, Eng. U. P.
- [177] Wilcox, D.J., and Gibson, I. S., 1984, Numerical Laplace transformation and inversion in the analysis of physical systems, *Int. J. numer. methods eng.*, 20, 1507-1519.
- [178] Yue, Z.Q., 1988, Solution for the thermoelastic problem in vertically inhomogeneous media, *Acta Mechanica Sinica*, 4, 182-189.
- [179] Yue, Z.Q., 1990, A review of three-dimensional analytical solutions in saturated poroelastic media under perfectly flexible loads, *A directed study under A. P. S. Selvadurai*, Dept. of Civil Engineering, Carleton University, Ottawa, Canada.
- [180] Yue, Z.Q., and Selvadurai, A.P.S., 1992, The role of Poisson's ratios on the consolidation response of soils, *Proceedings of the 46th Canadian Conference of Geotechnical Engineering*, Toronto, Canada.
- [181] Yue, Z.Q., and Wang, R., 1988, Analytic solution for transversely isotropic elas-

tic N-Layered systems, *Acta Scientiarum Naturalium, Universitatis Pekinensis*, 24, 202-211 (in Chinese).

- [182] Zaretskii, Yu. K., 1972, *Theory of Soil Consolidation*, (Translated from the Russian), U.S. Department of Commerce national Technical Information Service, Springfield, Virginia, TT70-50131.
- [183] Zienkiewicz, O.C., Humpheson, C., and Lewis, R.W., 1975, Associated and non-associated visco-plasticity and plasticity in soil mechanics, *Geotechnique*, 25, 671-689.
- [184] Zureick, A.K., 1988, Transversely isotropic medium with a rigid spheroidal inclusion under an axial pull, *J. Appl. Mech.*, 55, 495-497.

Appendix A

Stability And Accuracy Of The Modified Algorithm For Laplace Transform Inversion

A.1 General

As stated in Chapter 2, the numerical inversion of Laplace transforms is a notoriously ill-conditioned process. A modified algorithm for inverse Laplace transforms is suggested in this thesis to overcome the numerical problem encountered in the analysis of mechanics in saturated poroelastic media. In this Appendix, we shall present a representative example to show that the modified algorithm for Laplace transform inversion is of high stability and accuracy for the problems encountered in saturated poroelastic media. This representative example is the well-known consolidation problem of a perfectly flexible disc foundation acting at the permeable surface of a poroelastic medium of halfspace extent saturated with a compressible fluid¹(see Fig.A.1.1). The axial settlement at the surface of the poroelastic halfspace will be evaluated by using the modified inversion algorithm of Laplace transforms and such numerical inversion results will then be compared with their exact values. In this problems, the initial condition is assumed to be zero, i.e., $\zeta_{v0} = \zeta_v|_{t \rightarrow 0+} = 0$. The boundary conditions at the surface $z = 0$ of the poroelastic halfspace $0 \leq z < +\infty$ are given in the physical domain as follows,

¹ By using an analytical inversion of Laplace transform, McNamee and Gibson (1960a,b) have given an analytical solution for this problem under the conditions of incompressible pore fluids ($B = 1$, $\nu_u = 0.5$).

Fig.A.1.1 An axisymmetric load induced by a perfectly flexible foundation resting at the permeable surface of a poroelastic halfspace

$$\begin{aligned}
 \sigma_{zz}(r, 0, t) &= \begin{cases} f(r, t) & 0 \leq r < a \\ 0 & a \leq r < +\infty \end{cases} \\
 \sigma_{rz}(r, 0, t) &= 0 \quad 0 \leq r < +\infty \\
 p(r, 0, t) &= 0 \quad 0 \leq r < +\infty
 \end{aligned} \tag{A.1.1}$$

where $0 < t < +\infty$, and $f(r, t)$ is an arbitrary axisymmetric load distribution induced by a perfectly flexible circular foundation. The corresponding boundary conditions in the transform domain are:

$$\begin{aligned}
 \hat{\tau}_3(\rho, 0, s) &= \int_0^a \hat{f}(r, s) J_0(\rho r) r dr \\
 \hat{\tau}_1(\rho, 0, s) &= 0 \\
 \hat{p}_w(\rho, 0, s) &= 0
 \end{aligned} \tag{A.1.2}$$

where $0 \leq \rho < +\infty$, and $Re(s) > \omega_0 > 0$.

A.2 Axial Settlement due to a Perfectly Flexible Foundation on a Poroelastic Halfspace

From equations (4.4.2), we have the boundary algebraic equations at the outer surface $z = 0$ of the poroelastic halfspace $0 \leq z < +\infty$ as follows,

$$\begin{pmatrix} 1 & -1 & 0 \\ -\alpha_d & \frac{1}{\gamma}\alpha_d & \frac{-1}{2\mu\kappa\gamma} \\ \frac{\lambda_u+\mu}{\lambda_u+2\mu} & \frac{\lambda_u+\mu}{\lambda_u+2\mu} & \frac{1}{\mu\kappa}\alpha_\gamma \end{pmatrix} \begin{pmatrix} \hat{w}_1(\rho, \varphi, 0, s) \\ \hat{w}_3(\rho, \varphi, 0, s) \\ \hat{v}_3(\rho, \varphi, 0, s) \end{pmatrix} \quad (A.2.1)$$

$$= \begin{pmatrix} \frac{1}{2\mu} & \frac{-1}{2\mu} & 0 \\ \frac{-1}{2\mu}\alpha_d & \frac{1}{2\mu\gamma}\alpha_d & \frac{-1}{2\mu} \\ \frac{-(\lambda_u+3\mu)}{2\mu(\lambda_u+2\mu)} & \frac{-(\lambda_u+3\mu)}{2\mu(\lambda_u+2\mu)} & \frac{1}{\mu}\alpha_\gamma \end{pmatrix} \begin{pmatrix} \hat{\tau}_3(\rho, \varphi, 0, s) \\ \hat{\tau}_1(\rho, \varphi, 0, s) \\ \hat{p}_w(\rho, \varphi, 0, s) \end{pmatrix}$$

Solving the algebraic equations (A.2.1), we can express $\hat{w}_1(\rho, \varphi, 0, s)$, $\hat{w}_3(\rho, \varphi, 0, s)$, and $\hat{v}_3(\rho, \varphi, 0, s)$ in terms of $\hat{\tau}_3(\rho, \varphi, 0, s)$, $\hat{\tau}_1(\rho, \varphi, 0, s)$, and $\hat{p}_w(\rho, \varphi, 0, s)$. Substituting the boundary conditions (A.1.2) into the expressions, we obtain the expression of the transformed axial displacement $\hat{w}_3(\rho, \varphi, 0, s)$ ($= \hat{w}_3(\rho, 0, s)$) in terms of the transformed normal traction $\hat{\tau}_3(\rho, \varphi, 0, s)$ ($= \hat{\tau}_3(\rho, 0, s)$): i.e.,

$$\hat{w}_3(\rho, 0, s) = \frac{-(1-\nu)}{\mu} \left\{ 1 - \frac{b_y}{2} \frac{\gamma-1}{\gamma+1-b_y} \right\} \hat{\tau}_3(\rho, 0, s) \quad (A.2.2)$$

where $b_y = \frac{2(\nu_u-\nu)}{1-\nu}$.

By using the expression for $\hat{w}_3(\rho, 0, s)$ and the inverse integral transforms, the axial settlement in the spatial and temporal domain can be expressed in terms of two-dimensional repeated integrals with different integral sequences; i.e.,

$$u_z(r, 0, t) = -\frac{1-\nu}{\mu} \frac{1}{2\pi i} \int_0^{+\infty} \int_{\omega-i\infty}^{\omega+i\infty} \left\{ 1 - \frac{b_y}{2} \frac{\gamma-1}{\gamma+1-b_y} \right\} \hat{\tau}_3(\rho, 0, s) e^{st} J_0(\rho r) ds d\rho \quad (A.2.3.a)$$

$$u_z(r, 0, t) = -\frac{1-\nu}{\mu} \frac{1}{2\pi i} \int_{\omega-i\infty}^{\omega+i\infty} \int_0^{+\infty} \left\{ 1 - \frac{b_y}{2} \frac{\gamma-1}{\gamma+1-b_y} \right\} \hat{\tau}_3(\rho, 0, s) e^{st} J_0(\rho r) d\rho ds \quad (A.2.3.b)$$

where $0^+ \leq t < +\infty$, $0 \leq r < +\infty$.

The initial and final settlements can be found by taking the limits of $s \rightarrow +\infty$ and $s \rightarrow 0^+$ in equation (A.2.2); i.e.,

$$\begin{aligned} u_z^I(r) &= u_z(r, 0, 0^+) = -\frac{1-\nu_u}{\mu} \int_0^{+\infty} \tau_3(\rho, 0, 0^+) J_0(\rho r) d\rho \\ u_z^F(r) &= u_z(r, 0, +\infty) = -\frac{1-\nu}{\mu} \int_0^{+\infty} \tau_3(\rho, 0, +\infty) J_0(\rho r) d\rho \end{aligned} \quad (\text{A.2.4})$$

The axial settlement can then be expressed in the forms:

$$u_z(r, 0, t) = u_z^F(r) H(t) + \frac{\nu_u - \nu}{\mu} \frac{1}{2\pi i} \int_0^{+\infty} \int_{\omega-i\infty}^{\omega+i\infty} \frac{\gamma-1}{\gamma+1-b_y} \hat{\tau}_3(\rho, 0, s) e^{st} J_0(\rho r) ds d\rho \quad (\text{A.2.5.a})$$

$$u_z(r, 0, t) = u_z^F(r) H(t) + \frac{\nu_u - \nu}{\mu} \frac{1}{2\pi i} \int_{\omega-i\infty}^{\omega+i\infty} \int_0^{+\infty} \frac{\gamma-1}{\gamma+1-b_y} \hat{\tau}_3(\rho, 0, s) e^{st} J_0(\rho r) d\rho ds \quad (\text{A.2.5.b})$$

where $H(t)$ is the Heaviside step function and has the Laplace transform $\hat{H}(s) = \frac{1}{s}$.

Up to now, we have obtained the integral expressions of axial settlement at the permeable surface of a poroelastic medium subjected to the action of an axisymmetric normal traction. In the following, we shall make a detailed analysis on the axial settlement due to a constant uniform circular load by using both analytical and numerical methods. In other words, we take the axisymmetric normal load in the following form,

$$f(r, t) = -\frac{P_z}{\pi a^2} H(t) \quad (\text{A.2.6})$$

where P_z is the total load.

The normal traction in the transform domain is

$$\hat{\tau}_3(\rho, 0, s) = \int_0^a \hat{f}(r, s) J_0(\rho r) r dr = -\frac{P_z}{\pi a} \frac{1}{s} \frac{J_1(a\rho)}{\rho} \quad (\text{A.2.7})$$

Substituting equation (A.2.7) into equations (A.2.5), we have the axial settlement:

$$u_z(r, 0, t) = u_z^f(r) H(t) - \frac{\nu_u - \nu}{\mu} \frac{P_z}{\pi a} \frac{1}{2\pi i} \int_0^{+\infty} \int_{\omega-i\infty}^{\omega+i\infty} \frac{\gamma-1}{\gamma+1-b_y} \frac{e^{st}}{s} ds \frac{1}{\rho} J_1(\rho a) J_0(\rho r) d\rho \quad (\text{A.2.8.a})$$

$$u_z(r, 0, t) = u_z^f(r)H(t) - \frac{\nu_u - \nu}{\mu} \frac{P_z}{\pi a} \frac{1}{2\pi i} \int_{\omega-i\infty}^{\omega+i\infty} \int_0^{+\infty} \frac{\gamma-1}{\gamma+1-b_y} \frac{1}{\rho} J_1(\rho a) J_0(\rho r) d\rho \frac{e^{st}}{s} ds \quad (\text{A.2.8.b})$$

where $0^+ \leq t < +\infty$, $0 \leq r < +\infty$, and the initial and final axial settlements are

$$\begin{aligned} u_z^i(r) &= \frac{1-\nu_u}{\mu} \frac{P_z}{\pi a} E_k(r/a) \\ u_z^f(r) &= \frac{1-\nu}{\mu} \frac{P_z}{\pi a} E_k(r/a) \\ E_k(r/a) &= \frac{2}{\pi} \begin{cases} E(\frac{r}{a}), & 0 \leq r \leq a \\ \frac{r}{a} [E(\frac{a}{r}) - (1 - \frac{a^2}{r^2}) K(\frac{a}{r})], & a < r < \infty \end{cases} \end{aligned} \quad (\text{A.2.8.c})$$

where $K(x)$, $E(x)$ are the complete elliptic integrals of the first and the second kinds, respectively.

The inverse Laplace transform in equation (A.2.8.a) can be analytically integrated; i.e.,

$$\begin{aligned} \Psi^f(\rho^2 ct) &= \frac{1}{2\pi i} \int_{\omega-i\infty}^{\omega+i\infty} \frac{b_y}{2-b_y} \frac{\gamma-1}{\gamma+1-b_y} \frac{1}{s} e^{st} ds \\ &= \frac{1-\nu}{2(1-\nu_u)} [\text{Erfc}(\rho\sqrt{ct}) - (1-b_y)e^{-b_y(2-b_y)\rho^2 ct} \text{Erfc}[(1-b_y)\rho\sqrt{ct}]] \end{aligned} \quad (\text{A.2.9.a})$$

where $\text{Erf}(x)$ and $\text{Erfc}(x)$ are the error function and the complementary error function, respectively; i.e.,

$$\begin{aligned} \text{Erfc}(x) &= \frac{2}{\sqrt{\pi}} \int_x^{+\infty} e^{-u^2} du \\ \text{Erf}(x) &= \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du \\ \text{Erf}(x) + \text{Erfc}(x) &= 1 \end{aligned} \quad (\text{A.2.9.b})$$

The function $\Psi^f(x)$ has the initial and final limits; i.e.,

$$\begin{aligned} \lim_{x \rightarrow 0} \Psi^f(x) &= \frac{\nu_u - \nu}{1 - \nu_u} \\ \lim_{x \rightarrow \infty} \Psi^f(x) &= 0 \end{aligned} \quad (\text{A.2.10})$$

The axial settlement can then be exactly expressed in the following Hankel transform integral:

$$u_z(r, 0, t) = u_z^f(r)H(t) + \frac{1 - \nu_u}{\mu} \frac{P_z}{\pi a} \int_0^{+\infty} \Psi^f(\rho^2 ct) \frac{1}{\rho} J_1(\rho a) J_0(\rho r) d\rho \quad (A.2.11)$$

which is consistently convergent provided $0 < t < +\infty$ and can be evaluated with high accuracy (up to 10^{-6}) by using the Simpson's rule based proceeding limit technique.

Considering equations (A.2.4), (A.2.8.c), and (A.2.10), we observe that the limits in both the transform and the physical domain give the same initial and final settlements. These settlements are identical to those obtained from the classical theory of elastostatics by using the undrained and drained values of Poisson's ratios, respectively.

A.3 Stability and Accuracy of the Modified Algorithm for Laplace Transform Inversion

In order to examine the stability and accuracy of the modified inversion algorithm of Laplace transforms, we design two test schemes using the axial settlement expressed in equations (A.2.8) and (A.2.10).

Test scheme 1

In this test, we evaluate the integrand $\Phi_n(x)$ in the (A.2.8.a) using the modified inversion algorithm of Laplace transforms; i.e.,

$$\Phi_n(x) = \frac{1}{2\pi i} \int_{\omega-i\infty}^{\omega+i\infty} \frac{\sqrt{s+1}-1}{\sqrt{s+1}+1-b_y} \frac{1}{s} e^{sx} ds \quad (A.3.1.a)$$

We then compare the numerical inversion results with their exact values; i.e.,

$$\Phi_t(x) = \frac{1}{b_y} \left[\text{Erfc}(\sqrt{x}) - (1 - b_y) e^{-b_y(2-b_y)x} \text{Erfc}[(1 - b_y)\sqrt{x}] \right] \quad (A.3.1.b)$$

Test scheme 2

In this test, we evaluate the degree of consolidation settlement using the modified inversion algorithm of Laplace transforms; i.e.,

$$U_n(r_1, T) = 1 - \frac{1}{E_k(r_1)} \frac{1}{2\pi i} \int_{\omega-i\infty}^{\omega+i\infty} \Psi_n(r_1, s) \frac{e^{sT}}{s} ds \quad (A.3.2.a)$$

We then compare these numerical results with their exactly values which are obtained using the following Hankel transform integral,

$$U_t(r_1, T) = 1 - \frac{1}{E_k(r_1)} \int_0^{+\infty} \Phi_t(\rho^2 T) \frac{1}{\rho} J_1(\rho) J_0(\rho r_1) d\rho \quad (A.3.2.b)$$

where the dimensionless time factor $T = ct/a^2$ and radial coordinate $r_1 = r/a$, and

$$\Psi_n(r_1, s) = \int_0^{+\infty} \frac{\sqrt{s + \rho^2} - \rho}{\sqrt{s + \rho^2} + (1 - b_y)\rho} \frac{1}{\rho} J_1(\rho) J_0(\rho r_1) d\rho \quad (A.3.2.c)$$

In designing the second test scheme, we also intend to show that the small errors in the numerical integration of the infinite integral (A.3.2.c) do not induce any unacceptable and unstable results in the numerical inversion of the Laplace transform (A.3.2.a). In other words, the modified inversion algorithm of Laplace transforms is of high accuracy and stability even through there are small errors in the inverse function Ψ_n . The infinite integrals (A.3.2.b) and (A.3.2.c) are evaluated by using the Simpson's rule based proceeding limit technique (see Appendix E).

In all the evaluations, we set the following values for the integral parameters in the two numerical integrations,

$$\begin{aligned} \epsilon_a &= 10^{-8} \\ \epsilon_b &= 10^{-6} \\ \epsilon_c &= 10^{-6} \\ M &= 10 \\ \alpha_s &= 1.95 \\ \alpha_t &= 0 \end{aligned} \quad (A.3.3)$$

where ϵ_a is a relative error and α_t and α_s are constants in the modified inversion algorithm of Laplace transforms; ϵ_b is an absolute and/or relative error in the Simpson quadrature and ϵ_c is an absolute error in the proceeding limit technique (see Appendix E).

The results for the first test are given in Tables A.3.1 ~ A.3.3 for certain values of Poisson's ratios ν, ν_* . The absolute and relative errors between the numerical inversion results $\Phi_n(x)$ and their exact values $\Phi_t(x)$ are less than 10^{-6} and 10^{-5} for any value of the non-dimensional time factor ct/a^2 in the interval of $10^{-5} \leq x \leq 10^{+2}$, respectively.

In the second test, the results are presented in Tables A.3.4 ~ A.3.6 for the degree of consolidation settlement at the center of the circular uniform load ($r_1 = 0$). It is shown that the absolute and relative errors between the numerical inversion results $U_n(0, ct/a^2)$ and their exact values $U_t(0, ct/a^2)$ are less than 10^{-3} for the time factor interval $10^{-4} \leq ct/a^2 \leq 10^{+3}$. Consequently the truncating errors induced in the numerical integration of the Hankel transform integral (A.3.2.c) do not introduce any unacceptable and/or unstable result in the numerical inversion of Laplace transforms.

Based on the above analysis, we can conclude that the modified algorithm for Laplace transform inversion provides highly stable and accurate results and can be used in the analysis of consolidation problems in saturated poroelastic media.

Table A.3.1 A comparison between numerical inversion results $\Phi_n(x)$ and their exact values $\Phi_i(x)$: $x = \rho^2 ct/a^2$, $b_g = 0.1$

x	$\Phi_1(x)$	$\Psi_1(x)$	$ \Phi_1(x) - \Psi_1(x) $
0.00001	0.9932364925E+00	0.9932374143E+00	0.9218157456E-06
0.00003	0.9883075715E+00	0.9883084913E+00	0.9197870581E-06
0.00005	0.9849246179E+00	0.9849255363E+00	0.9183891836E-06
0.00007	0.9821811973E+00	0.9821821146E+00	0.9172534148E-06
0.00009	0.9798135892E+00	0.9798145055E+00	0.9162704458E-06
0.00010	0.9787304353E+00	0.9787313511E+00	0.9158202834E-06
0.00030	0.9633759830E+00	0.9633768923E+00	0.9093877726E-06
0.00050	0.9529095834E+00	0.9529104883E+00	0.9049516099E-06
0.00070	0.9444650120E+00	0.9444659133E+00	0.9013396596E-06
0.00090	0.9372083972E+00	0.9372092954E+00	0.8982133273E-06
0.00100	0.9338982164E+00	0.9338991132E+00	0.8967784378E-06
0.00300	0.8876275512E+00	0.8876284277E+00	0.8764857042E-06
0.00500	0.8567905458E+00	0.8567914079E+00	0.8620948436E-06
0.00700	0.8323288789E+00	0.8323297294E+00	0.8505075069E-06
0.00900	0.8116083604E+00	0.8116092010E+00	0.8405377914E-06
0.01000	0.8022488605E+00	0.8022496965E+00	0.8359504453E-06
0.03000	0.6775294130E+00	0.6775301835E+00	0.7705692165E-06
0.05000	0.6008038360E+00	0.6008045601E+00	0.7241441599E-06
0.07000	0.5436077716E+00	0.5436084594E+00	0.6877491511E-06
0.09000	0.4977038109E+00	0.4977044671E+00	0.6561787153E-06
0.10000	0.4777341199E+00	0.4777347618E+00	0.6418831046E-06
0.30000	0.2565307327E+00	0.2565311781E+00	0.4453898239E-06
0.50000	0.1602774209E+00	0.1602777434E+00	0.3224838217E-06
0.70000	0.1065100253E+00	0.1065102658E+00	0.2404615193E-06
0.90000	0.7335449984E-01	0.7335466295E-01	0.1631043873E-06
1.00000	0.6145444978E-01	0.6145460113E-01	0.1513512569E-06
3.00000	0.3160889013E-02	0.3160783821E-02	0.1051920597E-06
5.00000	0.2468245538E-03	0.2467502942E-03	0.7425957622E-07
7.00000	0.2254348145E-04	0.2252557851E-04	0.1790294067E-07
8.00000	0.7048148441E-05	0.7043919605E-05	0.4228836191E-08
10.00000	0.7157445497E-06	0.7213829658E-06	0.5638416040E-08
30.00000	0.1785765813E-10	-0.1492256356E-15	0.1785780736E-10
50.00000	0.6413450501E-13	0.0000000000E+00	0.6413450501E-13
70.00000	-0.3615403094E-14	0.0000000000E+00	0.3615403094E-14
90.00000	0.7296910604E-15	0.0000000000E+00	0.7296910604E-15
100.00000	-0.3203521729E-15	0.0000000000E+00	0.3203521729E-15

Table A.3.2 A comparison between numerical inversion results

 $\Phi_n(x)$ and their exact values $\Phi_1(x)$: $x = \rho^2 ct/a^2$, $b_y = 0.5$

x	$\Phi_1(x)$	$\Psi_1(x)$	$ \Phi_1(x) - \Psi_1(x) $
0.00001	0.9946542141E+00	0.9946551365E+00	0.9223988821E-06
0.00003	0.9907510445E+00	0.9907519653E+00	0.9207981678E-06
0.00005	0.9880684109E+00	0.9880693306E+00	0.9196951417E-06
0.00007	0.9858906989E+00	0.9858916177E+00	0.9187989891E-06
0.00009	0.9840097035E+00	0.9840106215E+00	0.9180250119E-06
0.00010	0.9831486741E+00	0.9831495918E+00	0.9176681739E-06
0.00030	0.9709093771E+00	0.9709102897E+00	0.9125966055E-06
0.00050	0.9625301865E+00	0.9625310956E+00	0.9091001392E-06
0.00070	0.9557480202E+00	0.9557489264E+00	0.9062562730E-06
0.00090	0.9499044218E+00	0.9499053256E+00	0.9037932140E-06
0.00100	0.9472340124E+00	0.9472349151E+00	0.9026642412E-06
0.00300	0.9095871146E+00	0.9095880012E+00	0.8865172056E-06
0.00500	0.8841593259E+00	0.8841599791E+00	0.6532081356E-06
0.00700	0.8637900115E+00	0.8637908778E+00	0.8662258493E-06
0.00900	0.8463958810E+00	0.8463967393E+00	0.8583286391E-06
0.01000	0.8384960279E+00	0.8384968826E+00	0.8547058367E-06
0.03000	0.7305478543E+00	0.7305486566E+00	0.8023174651E-06
0.05000	0.6614730301E+00	0.6614737956E+00	0.7655082301E-06
0.07000	0.6085210680E+00	0.6085218035E+00	0.7355078235E-06
0.09000	0.5650485427E+00	0.5650492520E+00	0.7092472813E-06
0.10000	0.5458494039E+00	0.5458501011E+00	0.6971818397E-06
0.30000	0.3193637164E+00	0.3193642394E+00	0.5230144543E-06
0.50000	0.2105115252E+00	0.2105119303E+00	0.4051816287E-06
0.70000	0.1456582316E+00	0.1456585475E+00	0.3159026772E-06
0.90000	0.1036576810E+00	0.1036579220E+00	0.2406591229E-06
1.00000	0.8809838513E-01	0.8809859617E-01	0.2110428941E-06
3.00000	0.5353289767E-02	0.5353166269E-02	0.1234988927E-06
5.00000	0.4535129188E-03	0.4533962320E-03	0.1166867917E-06
7.00000	0.4362196184E-04	0.4358720682E-04	0.3475502210E-07
9.00000	0.4492767681E-05	0.4494200823E-05	0.1433142514E-08
10.00000	0.1461765541E-05	0.1469227049E-05	0.7461508430E-08
30.00000	0.3237270830E-10	0.6839977223E-15	0.3237202430E-10
50.00000	0.1444467948E-12	-0.2967161951E-22	0.1444467948E-12
70.00000	-0.4896811785E-14	-0.5219941882E-31	0.4896811785E-14
90.00000	0.4289159648E-14	-0.9542562037E-40	0.4289159648E-14
100.00000	-0.4004402161E-15	-0.4118235158E-44	0.4004402161E-15

Table A.3.3 A comparison between numerical inversion results $\Phi_n(x)$ and their exact values $\Phi_1(x)$: $x = \rho^2 ct/a^2$, $b_g = 1.0$

x	$\Phi_1(x)$	$\Psi_1(x)$	$ \Phi_1(x) - \Psi_1(x) $
0.00001	0.9964308405E+00	0.9964317637E+00	0.9231268480E-06
0.00003	0.9938187525E+00	0.9938196746E+00	0.9220604283E-06
0.00005	0.9920203660E+00	0.9920212874E+00	0.9213267058E-06
0.00007	0.9905586021E+00	0.9905595228E+00	0.9207292170E-06
0.00009	0.9892946562E+00	0.9892955764E+00	0.9202129305E-06
0.00010	0.9887156645E+00	0.9887165844E+00	0.9199770930E-06
0.00030	0.9804569372E+00	0.9804578538E+00	0.9166008074E-06
0.00050	0.9747719651E+00	0.9747728794E+00	0.9142769081E-06
0.00070	0.9701519455E+00	0.9701528579E+00	0.9123864430E-06
0.00090	0.9661578669E+00	0.9661587777E+00	0.9107504474E-06
0.00100	0.9643284983E+00	0.9643294083E+00	0.9099994082E-06
0.00300	0.9382569767E+00	0.9382578760E+00	0.8992909050E-06
0.00500	0.9203434336E+00	0.9203443255E+00	0.8918894420E-06
0.00700	0.9058119609E+00	0.9058128467E+00	0.8858526085E-06
0.00900	0.8932719495E+00	0.8932728302E+00	0.8806416394E-06
0.01000	0.8875362056E+00	0.8875370840E+00	0.8783884652E-06
0.03000	0.8064950970E+00	0.8064959405E+00	0.8434764078E-06
0.05000	0.7518288151E+00	0.7518296340E+00	0.8188923563E-06
0.07000	0.7082802138E+00	0.7082810123E+00	0.7985650251E-06
0.09000	0.6713724600E+00	0.6713732406E+00	0.7805811666E-06
0.10000	0.6547200740E+00	0.6547208461E+00	0.7720819941E-06
0.30000	0.4385773821E+00	0.4385780262E+00	0.6440486047E-06
0.50000	0.3173099613E+00	0.3173105077E+00	0.5464249622E-06
0.70000	0.2367231071E+00	0.2367235710E+00	0.4639467166E-06
0.90000	0.1797121047E+00	0.1797124946E+00	0.3898997629E-06
1.00000	0.1572988507E+00	0.1572992080E+00	0.3572657881E-06
3.00000	0.1430597853E-01	0.1430587839E-01	0.1001405901E-06
5.00000	0.1565627547E-02	0.1565402261E-02	0.2252853363E-06
7.00000	0.1829248644E-03	0.1828106327E-03	0.1142317568E-06
9.00000	0.2211233970E-04	0.2209049696E-04	0.2184273686E-07
10.00000	0.7743185474E-05	0.7744216447E-05	0.1030973259E-08
30.00000	0.1110597447E-09	0.9436895709E-14	0.1110503078E-09
50.00000	0.8466587577E-12	-0.2141342490E-37	0.8466587577E-12
70.00000	-0.1226491176E-13	-0.4413635830E-46	0.1226491176E-13
90.00000	-0.1601760864E-14	-0.9097181479E-55	0.1601760864E-14
100.00000	0.3166360877E-12	-0.4130114002E-59	0.3166360877E-12

Table A.3.4 A comparison between numerical inverse results $U_n(0, ct/a^2)$ and their exact values $U_i(0, ct/a^2)$ for the degree of consolidation: $b_y = 0.1$

ct/a^2	$U_n(ct/a^2)$	$U_i(ct/a^2)$	$ U_n(ct/a^2) - U_i(ct/a^2) $
0.0001	0.2125233507E-01	0.2126812463E-01	0.1578956E-04
0.0003	0.3548825353E-01	0.3662035149E-01	0.1132098E-02
0.0005	0.4702701946E-01	0.4708422312E-01	0.5720366E-04
0.0007	0.5555534301E-01	0.5552543731E-01	0.2990570E-04
0.0009	0.6282400186E-01	0.6277813519E-01	0.4586667E-04
0.0010	0.6609796922E-01	0.6608616059E-01	0.1180863E-04
0.0030	0.1123100305E+00	0.1122928687E+00	0.1716180E-04
0.0050	0.1434088604E+00	0.1430367115E+00	0.3721489E-03
0.0070	0.1670504379E+00	0.1673792888E+00	0.3288509E-03
0.0090	0.1882252118E+00	0.1879607904E+00	0.2644214E-03
0.0100	0.1974134015E+00	0.1972428664E+00	0.1705351E-03
0.0300	0.3195855079E+00	0.3195863742E+00	0.8663000E-06
0.0500	0.3925036646E+00	0.3926408225E+00	0.1371579E-03
0.0700	0.4451345511E+00	0.4451972743E+00	0.6272320E-04
0.0900	0.4858735994E+00	0.4858305014E+00	0.4309800E-04
0.1000	0.5029481353E+00	0.5030223078E+00	0.7417250E-04
0.3000	0.6731625285E+00	0.6731825051E+00	0.1997660E-04
0.5000	0.7389715091E+00	0.7389823087E+00	0.1079960E-04
0.7000	0.7763677544E+00	0.7763614060E+00	0.6348400E-05
1.0000	0.8109128485E+00	0.8109165760E+00	0.3727500E-05
3.0000	0.8889891184E+00	0.8889898239E+00	0.7055000E-06
5.0000	0.9137204543E+00	0.9137180767E+00	0.2377600E-05
7.0000	0.9269684396E+00	0.9269713772E+00	0.2937600E-05
10.0000	0.9388326894E+00	0.9388323898E+00	0.2996000E-06
30.0000	0.9646257165E+00	0.9646240270E+00	0.1689500E-05
50.0000	0.9725878276E+00	0.9725884301E+00	0.6025000E-06
70.0000	0.9768299267E+00	0.9768295611E+00	0.3656000E-06
90.0000	0.9795650339E+00	0.9795639052E+00	0.1128700E-05
100.0000	0.9806106601E+00	0.9806120553E+00	0.1395200E-05
300.0000	0.9888023550E+00	0.9888044395E+00	0.2084500E-05
500.0000	0.9913250469E+00	0.9913276443E+00	0.2597400E-05
700.0000	0.9926675491E+00	0.9926703975E+00	0.2848400E-05
900.0000	0.9935377562E+00	0.9935358265E+00	0.1929700E-05
1000.0000	0.9938689907E+00	0.9938675185E+00	0.1472200E-05

Table A.3.5 A comparison between numerical inverse results $U_n(0, ct/a^2)$ and their exact values $U_t(0, ct/a^2)$ for the degree of consolidation: $b_y = 0.5$

ct/a^2	$U_n(ct/a^2)$	$U_t(ct/a^2)$	$ U_n(ct/a^2) - U_t(ct/a^2) $
0.0001	0.1683261044E-01	0.1685054407E-01	0.1793363E-04
0.0003	0.2894649762E-01	0.2909053619E-01	0.1440386E-03
0.0005	0.3754894130E-01	0.3747153297E-01	0.7740833E-04
0.0007	0.4420264297E-01	0.4425556560E-01	0.5292263E-04
0.0009	0.5013527341E-01	0.5010101460E-01	0.3425881E-04
0.0010	0.5270784473E-01	0.5277277159E-01	0.6492686E-04
0.0030	0.9045925101E-01	0.9044809188E-01	0.1115913E-04
0.0050	0.1159621754E+00	0.1159137753E+00	0.4840010E-04
0.0070	0.1366029595E+00	0.1363181185E+00	0.2848410E-03
0.0090	0.1534260343E+00	0.1537504800E+00	0.3244457E-03
0.0100	0.1614937793E+00	0.1616695872E+00	0.1758079E-03
0.0300	0.2698020194E+00	0.2697923279E+00	0.9691500E-05
0.0500	0.3384037769E+00	0.3382742634E+00	0.1295135E-03
0.0700	0.3894463993E+00	0.3895701458E+00	0.1237465E-03
0.0900	0.4304786754E+00	0.4303752550E+00	0.1034204E-03
0.1000	0.4480195075E+00	0.4479264246E+00	0.9308290E-04
0.3000	0.6293507913E+00	0.6293518976E+00	0.1106300E-05
0.5000	0.7023840104E+00	0.7024017034E+00	0.1769300E-04
0.7000	0.7444143865E+00	0.7444018116E+00	0.1257490E-04
1.0000	0.7834808794E+00	0.7834896692E+00	0.8789800E-05
3.0000	0.8725017138E+00	0.8725024685E+00	0.7547000E-06
5.0000	0.9008415148E+00	0.9008418871E+00	0.3723000E-06
7.0000	0.9160532471E+00	0.9160504327E+00	0.2814400E-05
10.0000	0.9296702752E+00	0.9296710120E+00	0.7368000E-06
30.0000	0.9593147213E+00	0.9593128210E+00	0.1900300E-05
50.0000	0.9684700094E+00	0.9684709537E+00	0.9443000E-06
70.0000	0.9733478422E+00	0.9733484350E+00	0.5928000E-06
90.0000	0.9764940379E+00	0.9764932497E+00	0.7882000E-06
100.0000	0.9777014514E+00	0.9776987830E+00	0.2668400E-05
300.0000	0.9871169687E+00	0.9871217521E+00	0.4783400E-05
500.0000	0.9900271874E+00	0.9900241524E+00	0.3035000E-05
700.0000	0.9915661831E+00	0.9915687323E+00	0.2549200E-05
900.0000	0.9925618881E+00	0.9925642602E+00	0.2372100E-05
1000.0000	0.9929469084E+00	0.9929458204E+00	0.1088000E-05

Table A.3.6 A comparison between numerical inverse results $U_n(0, ct/a^2)$ and their exact values $U_i(0, ct/a^2)$ for the degree of consolidation: $b_y = 1.0$

ct/a^2	$U_n(ct/a^2)$	$U_i(ct/a^2)$	$ U_n(ct/a^2) - U_i(ct/a^2) $
0.0001	0.1139178164E-01	0.1128449515E-01	0.1072865E-03
0.0003	0.1950381450E-01	0.1954349356E-01	0.3967906E-04
0.0005	0.2510690914E-01	0.2523089672E-01	0.1239876E-03
0.0007	0.2993677567E-01	0.2985395664E-01	0.8281903E-04
0.0009	0.3376103811E-01	0.3385116282E-01	0.9012471E-04
0.0010	0.3562721902E-01	0.3568229880E-01	0.5507978E-04
0.0030	0.6178244659E-01	0.6180388117E-01	0.2143458E-04
0.0050	0.7979352730E-01	0.7978838804E-01	0.5139260E-05
0.0070	0.9445955609E-01	0.9440765375E-01	0.5190234E-04
0.0090	0.1074438315E+00	0.1070474751E+00	0.3963564E-03
0.0100	0.1132088439E+00	0.1128378898E+00	0.3709541E-03
0.0300	0.1954626688E+00	0.1954377109E+00	0.2495790E-04
0.0500	0.2521700625E+00	0.2521786457E+00	0.8583200E-05
0.0700	0.2978117090E+00	0.2976736687E+00	0.1380403E-03
0.0900	0.3357427840E+00	0.3358883189E+00	0.1455349E-03
0.1000	0.3527519473E+00	0.3528822018E+00	0.1302545E-03
0.3000	0.5461322147E+00	0.5461457532E+00	0.1353850E-04
0.5000	0.6312512641E+00	0.6312536135E+00	0.2349400E-05
0.7000	0.6815367126E+00	0.6815547814E+00	0.1806880E-04
1.0000	0.7290868867E+00	0.7290967077E+00	0.9821000E-05
3.0000	0.8393619867E+00	0.8393574065E+00	0.4580200E-05
5.0000	0.8748846728E+00	0.8748843311E+00	0.3417000E-06
7.0000	0.8940078778E+00	0.8940083471E+00	0.4693000E-06
10.0000	0.9111677386E+00	0.9111636488E+00	0.4089800E-05
30.0000	0.9485684613E+00	0.9485682488E+00	0.2125000E-06
50.0000	0.9601409336E+00	0.9601389923E+00	0.1941300E-05
70.0000	0.9663018462E+00	0.9663032925E+00	0.1446300E-05
90.0000	0.9702781245E+00	0.9702783770E+00	0.2525000E-06
100.0000	0.9718027680E+00	0.9718022925E+00	0.4755000E-06
300.0000	0.9837159328E+00	0.9837155189E+00	0.4139000E-06
500.0000	0.9873826650E+00	0.9873854051E+00	0.2740100E-05
700.0000	0.9893418160E+00	0.9893384810E+00	0.3335000E-05
900.0000	0.9905945221E+00	0.9905973154E+00	0.2793300E-05
1000.0000	0.9910780625E+00	0.9910797978E+00	0.1735300E-05

Appendix B

Bounds For Material Parameters In Saturated Poroelastic Media

It is of engineering interest to figure out the bounds for the material parameters in a poroelastic medium saturated with a compressible fluid. In this Appendix, we shall investigate the bounds for the material parameters using the definition of a positive strain energy in a saturated poroelastic medium.

In the theory of poroelasticity, a strain energy density function W exists in a poroelastic medium saturated with a compressible fluid. This strain energy density function W can be expressed in terms of the stress and strain tensors and the pore pressure and the volumetric strain of pore fluid, *i.e.*,

$$W = \frac{1}{2}(\mathbf{T}:\mathbf{\Gamma} + p\zeta_v) \quad (B.1)$$

which can be written in the following matrix form,

$$W = \frac{1}{2} \begin{pmatrix} \sigma_{11} & \sigma_{22} & \sigma_{33} & p & \sigma_{12} & \sigma_{13} & \sigma_{23} \end{pmatrix} \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \zeta_v \\ 2\epsilon_{12} \\ 2\epsilon_{13} \\ 2\epsilon_{23} \end{pmatrix} \quad (B.2)$$

Using the assumption of linear relations between the stresses and strains,

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ p \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{pmatrix} = \begin{pmatrix} \lambda_u + 2\mu & \lambda_u & \lambda_u & -\alpha\beta & 0 & 0 & 0 \\ \lambda_u & \lambda_u + 2\mu & \lambda_u & -\alpha\beta & 0 & 0 & 0 \\ \lambda_u & \lambda_u & \lambda_u + 2\mu & -\alpha\beta & 0 & 0 & 0 \\ -\alpha\beta & -\alpha\beta & -\alpha\beta & \beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mu \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \zeta_v \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{pmatrix} \quad (B.3)$$

we can re-express the strain energy density function W in the following quadratic form with a symmetric square matrix,

$$W = \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \zeta_v \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{pmatrix}^T \begin{pmatrix} \lambda_u + 2\mu & \lambda_u & \lambda_u & -\alpha\beta & 0 & 0 & 0 \\ \lambda_u & \lambda_u + 2\mu & \lambda_u & -\alpha\beta & 0 & 0 & 0 \\ \lambda_u & \lambda_u & \lambda_u + 2\mu & -\alpha\beta & 0 & 0 & 0 \\ -\alpha\beta & -\alpha\beta & -\alpha\beta & \beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mu \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \zeta_v \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{pmatrix} \quad (B.4)$$

where the superscript T stands for the matrix transpose.

According to thermodynamic law, the strain energy density function W must be positive if the strain fields are not zero. To satisfy this physical requirement, the quadratic form W must be positive definite. The quadratic form W will be positive if all the principal minors in the top left-hand corner of the symmetric square matrix are positive so that we have the following inequalities,

$$\lambda_u + 2\mu > 0$$

$$\det \begin{pmatrix} \lambda_u + 2\mu & \lambda_u \\ \lambda_u & \lambda_u + 2\mu \end{pmatrix} = 4\mu(\lambda_u + \mu) > 0$$

$$\det \begin{pmatrix} \lambda_u + 2\mu & \lambda_u & \lambda_u \\ \lambda_u & \lambda_u + 2\mu & \lambda_u \\ \lambda_u & \lambda_u & \lambda_u + 2\mu \\ -\alpha\beta & -\alpha\beta & -\alpha\beta \end{pmatrix} = 4\mu^2(3\lambda_u + 2\mu) > 0$$

$$\det \begin{pmatrix} \lambda_u + 2\mu & \lambda_u & \lambda_u & -\alpha\beta \\ \lambda_u & \lambda_u + 2\mu & \lambda_u & -\alpha\beta \\ \lambda_u & \lambda_u & \lambda_u + 2\mu & -\alpha\beta \\ -\alpha\beta & -\alpha\beta & -\alpha\beta & \beta \end{pmatrix} = 4\mu^2\beta(3\lambda_u + 2\mu) > 0$$

$$\det \begin{pmatrix} \lambda_u + 2\mu & \lambda_u & \lambda_u & -\alpha\beta & 0 \\ \lambda_u & \lambda_u + 2\mu & \lambda_u & -\alpha\beta & 0 \\ \lambda_u & \lambda_u & \lambda_u + 2\mu & -\alpha\beta & 0 \\ -\alpha\beta & -\alpha\beta & -\alpha\beta & \beta & 0 \\ 0 & 0 & 0 & 0 & \mu \end{pmatrix} = 4\mu^3\beta(3\lambda_u + 2\mu) > 0$$

$$\det \begin{pmatrix} \lambda_u + 2\mu & \lambda_u & \lambda_u & -\alpha\beta & 0 & 0 \\ \lambda_u & \lambda_u + 2\mu & \lambda_u & -\alpha\beta & 0 & 0 \\ \lambda_u & \lambda_u & \lambda_u + 2\mu & -\alpha\beta & 0 & 0 \\ -\alpha\beta & -\alpha\beta & -\alpha\beta & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{pmatrix} = 4\mu^4\beta(3\lambda_u + 2\mu) > 0$$

$$\det \begin{pmatrix} \lambda_u + 2\mu & \lambda_u & \lambda_u & -\alpha\beta & 0 & 0 & 0 \\ \lambda_u & \lambda_u + 2\mu & \lambda_u & -\alpha\beta & 0 & 0 & 0 \\ \lambda_u & \lambda_u & \lambda_u + 2\mu & -\alpha\beta & 0 & 0 & 0 \\ -\alpha\beta & -\alpha\beta & -\alpha\beta & \beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mu \end{pmatrix} = 4\mu^5\beta(3\lambda_u + 2\mu) > 0$$

(B.5)

Consequently, the following bounds are found for the material parameters in a saturated poroelastic medium,

$$\begin{aligned}
 \mu &> 0 \\
 3\lambda + 2\mu &> 0 \\
 3\lambda_u + 2\mu &> 0 \\
 \beta &> 0
 \end{aligned}
 \tag{B.6}$$

The bounds for the shear modulus μ , drained Poisson's ratio ν , and undrained Poisson's ratio ν_u , and the Skempton's pore pressure parameter B are,

$$\begin{aligned}
 \mu &> 0 \\
 -1 < \nu &< 0.5 \\
 -1 < \nu_u &< 0.5 \\
 \nu &< \nu_u \\
 B &\neq 0.
 \end{aligned}
 \tag{B.7}$$

Appendix C

Integral Equations Of Rigid Disc Inclusions Under Three Limiting Responses

C.1 General

It is instructive to examine the finite sets of Hankel transform based integral equations for the fundamental problems of an embedded rigid disc inclusion under the three limiting states which pertain to the initial ($t \rightarrow 0^+$), the final ($t \rightarrow +\infty$) responses as well as the limiting response as $\nu \rightarrow \nu_u$.

The kernel functions $a_j(\gamma)$ ($j = 1, 2, 3, 4$) and $b_j(\gamma)$ ($j = 2, 3, 4, 6, 7$) in the equations (5.2.18.c) and (5.2.20.c) have the following limiting results: as $s \rightarrow +\infty$,

$$\begin{aligned}
 \lim_{s \rightarrow +\infty} a_1(\gamma) &= -\frac{3 - 4\nu_u}{4\mu(1 - \nu_u)} \\
 \lim_{s \rightarrow +\infty} a_2(\gamma) &= 0 \\
 \lim_{s \rightarrow +\infty} \frac{1}{\sqrt{s}} a_3(\gamma) &= \frac{\kappa B(1 + \nu_u)}{3(1 - \nu_u)} \frac{1}{\sqrt{c\rho}} \\
 \lim_{s \rightarrow +\infty} \frac{1}{\sqrt{s}} a_4(\gamma) &= \kappa \frac{1}{\sqrt{c\rho}} \\
 \lim_{s \rightarrow +\infty} b_2(\gamma) &= 0 \\
 \lim_{s \rightarrow +\infty} b_3(\gamma) &= -\frac{B(1 + \nu_u)}{3(1 - \nu_u)} \\
 \lim_{s \rightarrow +\infty} b_4(\gamma) &= 0 \\
 \lim_{s \rightarrow +\infty} b_6(\gamma) &= -\frac{7 - 8\nu_u}{8\mu(1 - \nu_u)} \\
 \lim_{s \rightarrow +\infty} b_7(\gamma) &= \frac{1}{8\mu(1 - \nu_u)}
 \end{aligned} \tag{C.1.1}$$

as $s \rightarrow 0^+$,

$$\begin{aligned}
\lim_{s \rightarrow 0^+} a_1(\gamma) &= -\frac{3-4\nu}{4\mu(1-\nu)} \\
\lim_{s \rightarrow 0^+} a_2(\gamma) &= -\frac{3-4\nu}{4\mu(1-\nu)} \frac{3(\nu_u - \nu)}{B(1+\nu_u)(3-4\nu)} \\
\lim_{s \rightarrow 0^+} a_3(\gamma) &= 0 \\
\lim_{s \rightarrow 0^+} a_4(\gamma) &= \kappa \\
\lim_{s \rightarrow 0^+} b_2(\gamma) &= -\frac{7-8\nu}{8\mu(1-\nu)} \frac{6(\nu_u - \nu)}{\kappa B(1+\nu_u)(7-8\nu)} \\
\lim_{s \rightarrow 0^+} b_3(\gamma) &= 0 \\
\lim_{s \rightarrow 0^+} b_4(\gamma) &= \frac{1}{\kappa} \\
\lim_{s \rightarrow 0^+} b_6(\gamma) &= -\frac{7-8\nu}{8\mu(1-\nu)} \\
\lim_{s \rightarrow 0^+} b_7(\gamma) &= \frac{1}{8\mu(1-\nu)}
\end{aligned} \tag{C.1.2}$$

and as $\nu \rightarrow \nu_u$,

$$\begin{aligned}
\lim_{\nu \rightarrow \nu_u} a_1(\gamma) &= -\frac{3-4\nu}{4\mu(1-\nu)} \\
\lim_{\nu \rightarrow \nu_u} a_2(\gamma) &= 0 \\
\lim_{\nu \rightarrow \nu_u} a_3(\gamma) &= 0 \\
\lim_{\nu \rightarrow \nu_u} a_4(\gamma) &= \kappa \\
\lim_{\nu \rightarrow \nu_u} b_2(\gamma) &= 0 \\
\lim_{\nu \rightarrow \nu_u} b_3(\gamma) &= 0 \\
\lim_{\nu \rightarrow \nu_u} b_4(\gamma) &= \frac{1}{\kappa} \\
\lim_{\nu \rightarrow \nu_u} b_6(\gamma) &= -\frac{7-8\nu}{8\mu(1-\nu)} \\
\lim_{\nu \rightarrow \nu_u} b_7(\gamma) &= \frac{1}{8\mu(1-\nu)}
\end{aligned} \tag{C.1.3}$$

The special sets of integral equations under the three limiting conditions can then be summarized as follows.

C.2 Axial Translation of a Rigid Disc Inclusion

The pairs of integral equations for the three limiting responses of axial transla-

tion of either a permeable or an impermeable rigid disc inclusion are the following: for $t \rightarrow 0^+$,

$$\begin{aligned} \int_0^{+\infty} \frac{1}{\rho} \tau_{30}(\rho, 0^+) J_0(\rho r) \rho d\rho &= \beta_a D_z(0^+), & 0 \leq r \leq a \\ \int_0^{+\infty} \tau_{30}(\rho, 0^+) J_0(\rho r) \rho d\rho &= 0, & a < r < +\infty \end{aligned} \quad (C.2.1)$$

for $t \rightarrow +\infty$,

$$\begin{aligned} \int_0^{+\infty} \frac{1}{\rho} \tau_{30}(\rho, +\infty) J_0(\rho r) \rho d\rho &= \alpha_a D_z(+\infty), & 0 \leq r \leq a \\ \int_0^{+\infty} \tau_{30}(\rho, +\infty) J_0(\rho r) \rho d\rho &= 0, & a < r < +\infty \end{aligned} \quad (C.2.2)$$

and for $\nu \rightarrow \nu_u$,

$$\begin{aligned} \int_0^{+\infty} \frac{1}{\rho} \tau_{30}(\rho, t) J_0(\rho r) \rho d\rho &= \alpha_a D_z(t), & 0 \leq r \leq a \\ \int_0^{+\infty} \tau_{30}(\rho, t) J_0(\rho r) \rho d\rho &= 0, & a < r < +\infty \end{aligned} \quad (C.2.3)$$

C.3 Central Rotation of a Rigid Disc Inclusion

The pairs of integral equations for the three limiting responses of central rotation of either a permeable or an impermeable rigid disc inclusion are the following: for $t \rightarrow 0^+$,

$$\begin{aligned} \int_0^{+\infty} \frac{1}{\rho} \tau_{31}(\rho, 0^+) J_1(\rho r) \rho d\rho &= \beta_a \frac{r}{2} \Omega_y(0^+), & 0 \leq r \leq a \\ \int_0^{+\infty} \tau_{31}(\rho, 0^+) J_1(\rho r) \rho d\rho &= 0, & a < r < +\infty \end{aligned} \quad (C.3.1)$$

for $t \rightarrow +\infty$,

$$\begin{aligned} \int_0^{+\infty} \frac{1}{\rho} \tau_{31}(\rho, +\infty) J_1(\rho r) \rho d\rho &= \alpha_a \frac{r}{2} \Omega_y(+\infty), & 0 \leq r \leq a \\ \int_0^{+\infty} \tau_{31}(\rho, +\infty) J_1(\rho r) \rho d\rho &= 0, & a < r < +\infty \end{aligned} \quad (C.3.2)$$

and for $\nu \rightarrow \nu_u$,

$$\begin{aligned} \int_0^{+\infty} \frac{1}{\rho} \tau_{31}(\rho, t) J_1(\rho r) \rho d\rho &= \alpha_a \frac{r}{2} \Omega_y(t), & 0 \leq r \leq a \\ \int_0^{+\infty} \tau_{31}(\rho, t) J_1(\rho r) \rho d\rho &= 0, & a < r < +\infty \end{aligned} \quad (C.3.3)$$

C.4 In-Plane Translation of a Rigid Disc Inclusion

The sets of integral equations for the three limiting responses of in-plane translation of *either a permeable or an impermeable rigid disc inclusion* are the following: for $t \rightarrow 0^+$,

$$\begin{aligned} \int_0^{+\infty} \frac{1}{\rho} \{N(\rho, 0^+) + \beta_1 M(\rho, 0^+)\} J_0(\rho r) \rho d\rho &= \beta_b D_x(0^+), & 0 \leq r \leq a \\ \int_0^{+\infty} \frac{1}{\rho} \{\beta_1 N(\rho, 0^+) + M(\rho, 0^+)\} J_2(\rho r) \rho d\rho &= 0, & 0 \leq r \leq a \\ \int_0^{+\infty} N(\rho, 0^+) J_0(\rho r) \rho d\rho &= 0, & a < r < \infty \\ \int_0^{+\infty} M(\rho, 0^+) J_2(\rho r) \rho d\rho &= 0, & a < r < \infty \end{aligned} \quad (C.4.1)$$

for $t \rightarrow +\infty$,

$$\begin{aligned}
 \int_0^{+\infty} \frac{1}{\rho} \{N(\rho, +\infty) + \alpha_1 M(\rho, +\infty)\} J_0(\rho r) \rho d\rho &= \alpha_b D_z(+\infty), & 0 \leq r \leq a \\
 \int_0^{+\infty} \frac{1}{\rho} \{\alpha_1 N(\rho, +\infty) + M(\rho, +\infty)\} J_2(\rho r) \rho d\rho &= 0, & 0 \leq r \leq a \\
 \int_0^{+\infty} N(\rho, +\infty) J_0(\rho r) \rho d\rho &= 0, & a < r < \infty \\
 \int_0^{+\infty} M(\rho, +\infty) J_2(\rho r) \rho d\rho &= 0, & a < r < \infty
 \end{aligned} \tag{C.4.2}$$

and for $\nu \rightarrow \nu_u$,

$$\begin{aligned}
 \int_0^{+\infty} \frac{1}{\rho} \{N(\rho, t) + \alpha_1 M(\rho, t)\} J_0(\rho r) \rho d\rho &= \alpha_b D_z(t), & 0 \leq r \leq a \\
 \int_0^{+\infty} \frac{1}{\rho} \{\alpha_1 N(\rho, t) + M(\rho, t)\} J_2(\rho r) \rho d\rho &= 0, & 0 \leq r \leq a \\
 \int_0^{+\infty} N(\rho, t) J_0(\rho r) \rho d\rho &= 0, & a < r < \infty \\
 \int_0^{+\infty} M(\rho, t) J_2(\rho r) \rho d\rho &= 0, & a < r < \infty
 \end{aligned} \tag{C.4.3}$$

C.5 Axial Rotation of a Rigid Disc Inclusion

4. The pair of integral equations for the three limiting responses of axial rotation of either a permeable or an impermeable rigid disc inclusion is the following: for $0^+ \leq t < +\infty$,

$$\begin{aligned}
 \int_0^{+\infty} \frac{1}{\rho} \tau_{20}(\rho, t) J_1(\rho r) \rho d\rho &= -\mu r \Omega_z(t), & 0 \leq r \leq a \\
 \int_0^{+\infty} \tau_{20}(\rho, t) J_1(\rho r) \rho d\rho &= 0, & a < r < \infty
 \end{aligned} \tag{C.5.1}$$

The six constants in the above equations are

$$\begin{aligned}
 \alpha_a &= -\frac{4\mu(1-\nu)}{3-4\nu} \\
 \alpha_b &= -\frac{8\mu(1-\nu)}{7-8\nu} \\
 \alpha_1 &= \frac{-1}{7-8\nu} \\
 \beta_a &= -\frac{4\mu(1-\nu_u)}{3-4\nu_u} \\
 \beta_b &= -\frac{8\mu(1-\nu_u)}{7-8\nu_u} \\
 \beta_1 &= \frac{-1}{7-8\nu_u}
 \end{aligned}
 \tag{C.5.2}$$

It may be noted that the above limiting integral equations are exactly same as those obtained for the classical elasticity solutions of the inclusion taken with the related elastic constants (see eg., Keer (1965); Selvadurai (1980a; b); Sneddon (1966b)).

Appendix D

Numerical Integration For Infinite Integrals With Rapidly Oscillatory Integrands

D.1 General

In the analytical analysis of mechanics in saturated poroelastic media, we often encounter the following two kinds of infinite integrals associated with the Hankel and Fourier sine (or cosine) transforms,

$$\Phi(s, r) = \int_0^{+\infty} P(\rho, s) J_m(\rho r) d\rho \quad (D.1.1)$$

and

$$\begin{aligned} K(s, r) &= \int_0^{+\infty} F(\rho, s) \sin(\rho r) d\rho \\ K(s, r) &= \int_0^{+\infty} F(\rho, s) \cos(\rho r) d\rho \end{aligned} \quad (D.1.2)$$

where $m = 0, 1, 2, \dots$, and $s = s_r + is_i$ ($s_r > 0$) is the independent complex variable of Laplace transforms.

Due to complexity of the integrands, it is not possible to find exact solutions for these infinite integrals with two independent parameters s, r . Consequently it is necessary to adopt numerical techniques to generate the relevant results. There are two unusual problems encountered in the numerical evaluation of the infinite integrals. One is the infinite limit and the other is associated with the integrands. As the values of s and ρ are larger, the integrands will become rapidly oscillatory

functions. This property of the integrands will slow down the convergence of the numerical integration and cause the numerical integration unstable.

In this Appendix, we shall introduce a particular technique for overcoming the two problems in the numerical integration. This particular technique is consisted of a Simpson's quadrature based proceeding limit technique and an approximation technique which handle the infinite limitation and the rapidly oscillatory integrands, respectively. We first briefly write down the well known Simpson's quadrature and then introduce the particular technique.

D.2 Simpson's Quadrature for Finite Integrals

According to the theory of numerical integration, the following finite integral can be evaluated

$$S = \int_a^b f(x)dx \quad (D.2.1)$$

by using the Simpson's quadrature; *i.e.*,

$$S \sim \frac{h}{3} \left[f(a) + f(b) + 2 \sum_{j=1}^{n-1} f(x_{2j}) + 4 \sum_{j=1}^n f(x_{2j-1}) \right] \quad (D.2.2)$$

where $h = \frac{b-a}{2n}$, $x_j = a + jh$, $j = 1, 2, \dots, 2n-1$.

Based on the Simpson's rule, an adaptively iterative integration is further adopted to calculate automatically the summation (D.2.1) until a proposed accuracy. This adaptively iterative integration can be expressed in the following. At the first iteration, the following value is calculated,

$$T_1 = \frac{b-a}{2} [f(a) + f(b)] \quad (D.2.3)$$

At the k -th iteration, the sampling points are automatically doubled; *i.e.*,

$$h = \frac{b-a}{2^k}, \quad k = 1, 2, 3, \dots \quad (D.2.4)$$

The following summation is then calculated,

$$T_{2^k} = \frac{1}{2} T_{2^{k-1}} + h \sum_{j=1}^{2^{k-1}} f(a + (2j-1)h) \quad (D.2.5)$$

The approximate evaluation for the finite integral S (D.2.1) at the k -th iterative is,

$$S_{2k} = (4T_{2k} - T_{2k-1})/3 \quad (D.2.6)$$

This iteration will keep going until a proposed accuracy is satisfied; i.e.,

$$|S_{2k} - S_{2k-1}| \leq \epsilon_b(1 + |S_{2k}|) \quad (D.2.7)$$

which ϵ_b is an absolute and/or relative error.

An advantage of this algorithm is that the $(k-1)$ -th iterative result is used and only half of the total sampling points is evaluated at the k -th iteration .

D.3 Proceeding Limit Technique for Infinite Integrals

This technique is based on the basic definition of infinite integrals, i.e.,

$$\int_0^{+\infty} F(\rho) d\rho = \lim_{A \rightarrow \infty} \int_0^A F(\rho) d\rho \quad (D.3.1)$$

This definition suggests a primitive mode of procedure.

If $0 < A_0 < A_1 \dots$ be a sequence of numbers that approaches to infinite, we can re-write the infinite integral (D.3.1) below,

$$\int_0^{+\infty} F(\rho) d\rho = \int_0^{A_0} F(\rho) d\rho + \int_{A_0}^{A_1} F(\rho) d\rho + \dots \quad (D.3.2)$$

Each of the finite integrals on the right-hand side is proper and can be calculated by using the Simpson's quadrature based adaptively iterative integration. The evaluation of these proceeding finite integrals is automatically terminated provided the following condition is satisfied,

$$\left| \int_{A_n}^{A_{n+1}} F(\rho) d\rho \right| \leq \epsilon_c \quad (D.3.3)$$

where ϵ_c is an absolute error.

Using this proceeding limit technique, we should make sure the integrand satisfies the convergence condition of infinite integrals; i.e.,

$$\lim_{\rho \rightarrow \infty} \frac{F(\rho)}{\rho^q} < +\infty \quad (D.3.4)$$

where $q > 1$.

In this thesis, we use $A_j = bA_{j-1}$, $b = 1.5 \sim 3$, $A_0 = 1 \sim 2$, for the Hankel transform integral (D.1.1). For the infinite integral of Fourier sine or cosine transforms (D.1.2), the limits for the proceeding finite integrals are chosen as the zeros of the oscillatory part of the integrands, i.e., either $\sin \rho r$ or $\cos \rho r$.

D.4 Approximation Technique for Rapidly Oscillatory Integrands

Adopting the proceeding limit technique, we can overcome the first numerical integration problem of infinite interval. However, the integrands in the proper integrals of finite interval, i.e.,

$$\begin{aligned} F(\rho, s) \sin(\rho r) \\ F(\rho, s) \cos(\rho r) \end{aligned} \quad (D.4.1)$$

can become rapidly oscillatory functions which have numerous local maxima and minima over the interval of integration.

If we change the integral variable ρ with $\rho_1 \sqrt{|s|}$, we can observe this numerical problem from the expressions below,

$$\begin{aligned} \int_a^b F(\rho_1, s_u) \sin(\rho_1 x) d\rho_1 \\ \int_a^b F(\rho_1, s_u) \cos(\rho_1 x) d\rho_1 \end{aligned} \quad (D.4.2)$$

where $x = r\sqrt{|s|}$, $s_u = s/|s|$, $|s| = \sqrt{s_r^2 + s_i^2}$, $|s_u| = 1$.

As $x \rightarrow +\infty$ due to $|s| \rightarrow +\infty$ (time $t \rightarrow +0$), the graphs of $F(\rho_1, s_u) \sin(\rho_1 x)$ or $F(\rho_1, s_u) \cos(\rho_1 x)$ will consist of plus areas and minus areas of nearly equal size and the resulting cancellation of area is attended by a loss of significance. Furthermore, as x is very large, the integrands $F(\rho_1, s_u) \sin(\rho_1 x)$ and $F(\rho_1, s_u) \cos(\rho_1 x)$ become polynomials of very high degree (see e.g. Fig.D.5.1 and Fig.D.5.2). This indicates that further special techniques should be developed to evaluate the infinite integrals of Fourier sine and cosine transforms (D.1.2) as $x = r\sqrt{|s|}$ is very large (or t is very small). In this thesis, the following approximation technique is adopted to evaluate stably and accurately the infinite integrals of Fourier sine and cosine transforms (D.1.2).

At first, we derive an approximation function $f(\rho_1, s_u)$ for the function $F(\rho_1, s_u)$ as $\rho_1 \rightarrow +\infty$, i.e.,

$$\lim_{\rho_1 \rightarrow +\infty} [F(\rho_1, s_u) - f(\rho_1, s_u)] = o[\lim_{\rho_1 \rightarrow +\infty} F(\rho_1, s_u)] \quad (D.4.3)$$

The infinite integrals (D.1.2) of Fourier sine or cosine transforms can be re-expressed in the forms:

$$\begin{aligned} I_s(x, s_u) &= \int_0^{+\infty} [F(\rho_1, s_u) - f(\rho_1, s_u)] \sin(\rho_1 x) d\rho_1 + I_s^o(x, s_u) \\ I_c(x, s_u) &= \int_0^{+\infty} [F(\rho_1, s_u) - f(\rho_1, s_u)] \cos(\rho_1 x) d\rho_1 + I_c^o(x, s_u) \end{aligned} \quad (D.4.4)$$

where $I_s^o(x, s_u)$ and $I_c^o(x, s_u)$ are the following infinite integrals, respectively,

$$\begin{aligned} I_s^o(x, s_u) &= \int_0^{+\infty} f(\rho_1, s_u) \sin(\rho_1 x) d\rho_1 \\ I_c^o(x, s_u) &= \int_0^{+\infty} f(\rho_1, s_u) \cos(\rho_1 x) d\rho_1 \end{aligned} \quad (D.4.5)$$

We can obtain the exact closed-form integrations for the two introduced infinite integrals (D.4.5) by properly choosing the approximation function $f(\rho_1, s_u)$. The

adjusted integrand function $[F(\rho_1, s_*) - f(\rho_1, s_*)]$ has a much higher convergent speed as $\rho_1 \rightarrow +\infty$. As a result, the adjusted infinite integrals in equations (D.4.4) can be evaluated with much higher accuracy and faster convergence by using the Simpson's quadrature based proceeding limit technique. Thus, we can overcome the two numerical problems arisen in the numerical integration for the infinite integrals with rapidly oscillatory integrands.

D.5 Evaluation of the Infinite Integrals in the Kernel Functions

In this section, we discuss how to evaluate the infinite integrals of Fourier sine and cosine transforms in the kernel functions of the systems of Fredholm integral equations of the second. Generally, the infinite integrals in the kernel functions can be written as follows with three independent variables x, y , and s ,

$$\begin{aligned} I_{1j}(x, y, s) &= \frac{2}{\pi} \int_0^{+\infty} f_j(\rho, s) \cos(\rho x) \cos(\rho y) d\rho \\ I_{2j}(x, y, s) &= \frac{2}{\pi} \int_0^{+\infty} \frac{1}{\rho} f_j(\rho, s) \sin(\rho x) \cos(\rho y) d\rho \\ I_{3j}(x, y, s) &= \frac{2}{\pi} \int_0^{+\infty} \frac{1}{\rho} f_j(\rho, s) \cos(\rho x) \sin(\rho y) d\rho \\ I_{4j}(x, y, s) &= \frac{2}{\pi} \int_0^{+\infty} f_j(\rho, s) \sin(\rho x) \sin(\rho y) d\rho \end{aligned} \tag{D.5.1}$$

where $0 \leq x, y < 1$, $s = s_r + is_i$, $s_r > 0$, $-\infty < s_i < +\infty$, $i = \sqrt{-1}$, and the functions $f_j(\rho, s)$ ($j = 1, 2, 3, 4, 5$)¹ are

¹ it is worth to note herein: $j \neq 2$ for $I_{2j}(x, y, s)$ and $I_{3j}(x, y, s)$.

$$\begin{aligned}
f_1(\rho, s) &= \frac{s}{(\rho + \sqrt{\rho^2 + s})^2} \\
f_2(\rho, s) &= \frac{s}{\rho(\rho + \sqrt{\rho^2 + s})} \\
f_3(\rho, s) &= \frac{s}{\rho^2 + s + \rho\sqrt{\rho^2 + s}} \\
f_4(\rho, s) &= \frac{\rho s}{2\rho(\rho^2 + s) + (2\rho^2 + s)\sqrt{\rho^2 + s}} \\
f_5(\rho, s) &= \frac{\rho^2 - s - \rho\sqrt{\rho^2 + s}}{\rho^2 + s + \rho\sqrt{\rho^2 + s}}
\end{aligned} \tag{D.5.2}$$

The infinite integrals (D.5.1) with independent variables x, y , and s are uniformly convergent provided $\text{Re}(s) > 0$. Furthermore, the integrands $f_j(\rho, s)$ ($j = 1, 2, 3, 4, 5$) have the following approximations as $\frac{|s|}{\rho^2} \rightarrow 0$,

$$\begin{aligned}
f_1(\rho, s) &\sim a_1(\rho, s) = \frac{1}{4} \frac{s}{\rho^2 + \frac{s}{2}} \\
f_2(\rho, s) &\sim a_2(\rho, s) = \frac{1}{2} \frac{s}{\rho^2 + \frac{s}{4}} \\
f_3(\rho, s) &\sim a_3(\rho, s) = \frac{1}{2} \frac{s}{\rho^2 + \frac{3s}{4}} \\
f_4(\rho, s) &\sim a_4(\rho, s) = \frac{1}{4} \frac{s}{\rho^2 + s} \\
f_5(\rho, s) &\sim a_5(\rho, s) = \frac{3}{4} \frac{-s}{\rho^2 + \frac{5s}{6}}
\end{aligned} \tag{D.5.3}$$

As a result, we re-write the infinite integrals (D.5.1) into the following forms:

$$\begin{aligned}
I_{1j}(x, y, s) &= \frac{2}{\pi} \int_0^{+\infty} [f_j(\rho, s) - a_j(\rho, s)] \cos(\rho x) \cos(\rho y) d\rho + I_{1j}^o(x, y, s) \\
I_{2j}(x, y, s) &= \frac{2}{\pi} \int_0^{+\infty} \frac{1}{\rho} [f_j(\rho, s) - a_j(\rho, s)] \sin(\rho x) \cos(\rho y) d\rho + I_{2j}^o(x, y, s) \\
I_{3j}(x, y, s) &= \frac{2}{\pi} \int_0^{+\infty} \frac{1}{\rho} [f_j(\rho, s) - a_j(\rho, s)] \cos(\rho x) \sin(\rho y) d\rho + I_{3j}^o(x, y, s) \\
I_{4j}(x, y, s) &= \frac{2}{\pi} \int_0^{+\infty} [f_j(\rho, s) - a_j(\rho, s)] \sin(\rho x) \sin(\rho y) d\rho + I_{4j}^o(x, y, s)
\end{aligned} \tag{D.5.4}$$

where $0 \leq x, y < 1$, $Re(s) > 0$, and,

$$\begin{aligned}
I_{1j}^o(x, y, s) &= \frac{2}{\pi} \int_0^{+\infty} a_j(\rho, s) \cos(\rho x) \cos(\rho y) d\rho \\
I_{2j}^o(x, y, s) &= \frac{2}{\pi} \int_0^{+\infty} \frac{1}{\rho} a_j(\rho, s) \sin(\rho x) \cos(\rho y) d\rho \\
I_{3j}^o(x, y, s) &= \frac{2}{\pi} \int_0^{+\infty} \frac{1}{\rho} a_j(\rho, s) \cos(\rho x) \sin(\rho y) d\rho \\
I_{4j}^o(x, y, s) &= \frac{2}{\pi} \int_0^{+\infty} a_j(\rho, s) \sin(\rho x) \sin(\rho y) d\rho
\end{aligned} \tag{D.5.5}$$

Such infinite integrals (D.5.5) are integrated in exact closed-form with the help of the formulae below (see Erdelyi, et al, 1954),

$$\begin{aligned}
\cos(\rho x) \cos(\rho y) &= \frac{1}{2} [\cos(\rho x - \rho y) + \cos(\rho x + \rho y)] \\
\sin(\rho x) \sin(\rho y) &= \frac{1}{2} [\cos(\rho x - \rho y) - \cos(\rho x + \rho y)] \\
\sin(\rho x) \cos(\rho y) &= \frac{1}{2} [\sin(\rho x + \rho y) + \sin(\rho x - \rho y)] \\
\cos(\rho x) \sin(\rho y) &= \frac{1}{2} [\sin(\rho x + \rho y) - \sin(\rho x - \rho y)]
\end{aligned} \tag{D.5.6}$$

$$\begin{aligned}
\int_0^{+\infty} \frac{s}{\rho^2 + s} \cos(\rho c) d\rho &= \frac{\pi}{2} \sqrt{s} e^{-c\sqrt{s}} \\
\int_0^{+\infty} \frac{s}{\rho(\rho^2 + s)} \sin(\rho c) d\rho &= \frac{\pi}{2} (1 - e^{-c\sqrt{s}})
\end{aligned}$$

where $c \geq 0$, $Re(s) > 0$.

The infinite integrals in equations (D.5.4) are evaluated by using the proceeding limit technique discussed above. According to this discussion, we evaluate successfully the infinite integrals (D.5.1) in the kernel functions with high accuracy and fast convergence.

An example is given in Fig.D.4.1 and Fig.D.4.2 which illustrates the oscillation and the approximations of the integrands in the infinite integrals.

Fig.D.5.1 Rapid oscillation of integrand and the approximation technique (real parts of the complex functions)

Fig.D.5.2 Rapid oscillation of integrand and the approximation technique (imaginary parts of the complex functions)

Appendix E

Convergence And Accuracy Of Numerical Solutions For Fredholm Integral Equations

E.1 General

The numerical scheme adopted in this thesis for the solutions of the systems of Fredholm integral equations of the second kind in the Laplace transform domain is essentially based on three-dimensional repeated numerical integrations. As discussed in Section 6.5, due to the accumulations of the errors in each repeated numerical integration, it is necessary to verify the convergence and accuracy of the numerical solutions in time domain for the integral equations. In this Appendix, we shall present the results associated with the verifications. In all these tests, the most basic condition in each of the two types of loadings, *i.e.*, either the constant load or the constant displacement prescribed at the inclusion, is used.

E.2 Axial Translation of a Rigid Disc Inclusion

E.2.1 General

In the evaluation, the following values are first selected for the integral parameters in the numerical integration and the modified algorithm for Laplace transform

inversion

$$\begin{aligned}
 \epsilon_a &= 10^{-8} \\
 \alpha_t &= 0 \\
 \alpha_s &= 1.95 \\
 \epsilon_b &= 10^{-5} \\
 \epsilon_c &= 10^{-5}
 \end{aligned}
 \tag{E.2.1}$$

where ϵ_a is a relative error and α_t and α_s are constants in the modified inversion algorithm of Laplace transforms; ϵ_b is an absolute and/or relative error in Simpson's quadrature and ϵ_c is an absolute error in the proceeding limit technique.

The time factor ct/a^2 is assigned the limits:

$$10^{-4} \leq ct/a^2 \leq 10^{+3} \tag{E.2.2}$$

The convergence and accuracy of the solutions to the problem of the time-dependent axial translation of a rigid disc inclusion are then evaluated by considering the influences of the numbers of M and N , where $2M + 1$ is the total number of terms of the Fourier series expansion in the numerical inversion of Laplace transforms and N is the total number of the segments in the numerical solution of the integral equations in the Laplace transform domain.

E.2.2 Axial consolidation displacement of the inclusion

For this problem, two sets of the numbers M and N are selected. The first set is $M = 4, 6, 8, 10$ where $N = 50$ and the second set is $N = 10, 20, 30, 40, 50, 75, 100$ where $M = 10$. The drained and undrained Poisson's ratios are assumed as $\nu = 0.0$ and $\nu = 0.5$, respectively.

The numerical results for the non-dimensional axial translation $a\mu D_z(ct/a^2)/P_z$ and the associated degree of consolidation $U_z(ct/a^2)$ are presented in Tables (E.2.1) ~ (E.2.8) for the two sets of the numbers M and N under the conditions applicable to either a permeable or an impermeable rigid disc inclusion. The patterns of influence of M and/or N on the axial translation $D_z(t)$ are also plotted using the

degree of consolidation and the logarithm time factor ct/a^2 in the Fig. E.2.1 ~ E.2.4.

By calculating the absolute and relative differences between the numerical results for $M = 4$ (6, or 8) and $M = 10$ in the first set and $N = 10$ (20, 30, 40, 50, or 75) and $N = 100$ in the second set, it is observed that the numerical results are progressively convergent with high stability for the extensive non-dimensional time interval $10^{-4} \leq ct/a^2 \leq 10^{+3}$. The absolute and relative differences in the first set indicate that each M has a very uniform influence on the numerical results at every time factor ct/a^2 in the entire range of $10^{-4} \leq ct/a^2 \leq 10^{+3}$ (see e.g., Fig.E.2.1 and Fig.E.2.3). The absolute and relative differences between the non-dimensional axial translations of $M = 8$ and $M = 10$ are less than 10^{-5} and 10^{-4} , respectively. In other words, the $M = 10$ gives very accurate results in the numerical inversion of Laplace transforms for the integral equations. The absolute and relative differences in the second set also indicate that the number N depends on the value of the time factor ct/a^2 for a convergent and accurate evaluation of the integral equations at a particular time factor. In general, the following relationship between the number N and the time factor ct/a^2 is valid for a stable, convergent and accurate solution of the integral equations;

$$N > \frac{1}{\sqrt{ct/a^2}} \quad (E.2.3)$$

This inequality gives a lower constraint on the choice of the value of N for solving the integral equations particularly at small values of the time factor ($0 < ct/a^2 \leq 10^{-2}$). This property can also be observed in Fig.E.2.2 and Fig.E.2.4.

Table.E.2.1 Influence of M on degree of consolidation $U_r(ct/a^2)$
(permeable case, $N = 50$, $\nu = 0.0$, $\nu_u = 0.5$)

ct/a^2	$M = 4$	$M = 6$	$M = 8$	$M = 10$
0.0001	0.0671	0.0639	0.0674	0.0640
0.0003	0.0251	0.0238	0.0238	0.0265
0.0005	0.0642	0.0701	0.0702	0.0702
0.0007	0.0721	0.0802	0.0798	0.0771
0.0010	0.0905	0.0958	0.0968	0.0963
0.0030	0.1382	0.1447	0.1452	0.1452
0.0050	0.1724	0.1786	0.1801	0.1797
0.0100	0.2243	0.2310	0.2318	0.2318
0.0500	0.3895	0.3959	0.3968	0.3968
0.1000	0.4783	0.4848	0.4857	0.4858
0.5000	0.6904	0.6974	0.6984	0.6984
1.0000	0.7664	0.7739	0.7749	0.7750
5.0000	0.8846	0.8931	0.8942	0.8943
10.0000	0.9146	0.9235	0.9247	0.9248
50.0000	0.9553	0.9649	0.9661	0.9662
100.0000	0.9650	0.9748	0.9760	0.9761
500.0000	0.9779	0.9880	0.9892	0.9893
1000.0000	0.9809	0.9911	0.9924	0.9924

Table.E.2.2 Influence of M on axial translation $\frac{a\mu D_s(ct/a^2)}{p_s}$
(permeable case, $N = 50$, $\nu = 0.0$, $\nu_u = 0.5$)

ct/a^2	$M = 4$	$M = 6$	$M = 8$	$M = 10$
0 exact	0.06250	0.06250	0.06250	0.06250
0.0001	0.06460	0.06450	0.06461	0.06450
0.0003	0.06328	0.06324	0.06324	0.06333
0.0005	0.06450	0.06469	0.06469	0.06469
0.0007	0.06475	0.06501	0.06499	0.06491
0.0010	0.06533	0.06549	0.06552	0.06551
0.0030	0.06682	0.06702	0.06704	0.06704
0.0050	0.06789	0.06808	0.06813	0.06812
0.0100	0.06951	0.06972	0.06974	0.06974
0.0500	0.07467	0.07487	0.07490	0.07490
0.1000	0.07745	0.07765	0.07768	0.07768
0.5000	0.08407	0.08429	0.08432	0.08433
1.0000	0.08645	0.08668	0.08672	0.08672
5.0000	0.09014	0.09041	0.09045	0.09045
10.0000	0.09108	0.09136	0.09140	0.09140
50.0000	0.09235	0.09265	0.09269	0.09269
100.0000	0.09266	0.09296	0.09300	0.09300
500.0000	0.09306	0.09337	0.09341	0.09342
1000.0000	0.09315	0.09347	0.09351	0.09351
$+\infty$ exact	0.09375	0.09375	0.09375	0.09375

Table.E.2.5 Influence of M on degree of consolidation $U_z(ct/a^2)$
(impermeable case, $N = 50$, $\nu = 0.0$, $\nu_u = 0.5$)

ct/a^2	$M = 4$	$M = 6$	$M = 8$	$M = 10$
0.0001	0.00994187	0.00751484	0.00927092	0.00709829
0.0003	0.01255685	0.01635534	0.02016604	0.01907279
0.0005	0.01626346	0.02127122	0.02199336	0.02199061
0.0007	0.01978895	0.02547887	0.02632620	0.02636714
0.0010	0.02433449	0.03029367	0.03113335	0.03110250
0.0030	0.04666728	0.05314163	0.05398350	0.05402156
0.0050	0.06247784	0.06903856	0.06988542	0.06989580
0.0100	0.09170064	0.09827355	0.09912160	0.09916259
0.0500	0.21711724	0.22340272	0.22425151	0.22429977
0.1000	0.31036958	0.31640857	0.31725982	0.31730962
0.5000	0.61567616	0.62149793	0.62241036	0.62146460
1.0000	0.73211712	0.73856932	0.73955435	0.73961240
5.0000	0.88139117	0.88956410	0.89068848	0.89075309
10.0000	0.91363370	0.92228729	0.92344844	0.92351472
50.0000	0.95536435	0.96478146	0.96599746	0.96606606
100.0000	0.96505207	0.97472638	0.97595870	0.97602797
500.0000	0.97789162	0.98796004	0.98921615	0.98928636
1000.0000	0.98092437	0.99109197	0.99235380	0.99242431

Table.E.2.6 Influence of M on axial translation $\frac{a\mu D_z(ct/a^2)}{P_s}$
(impermeable case, $N = 50$, $\nu = 0.0$, $\nu_u = 0.5$)

ct/a^2	$M = 4$	$M = 6$	$M = 8$	$M = 10$
0 exact	0.06250000	0.06250000	0.06250000	0.06250000
0.0001	0.06281068	0.06273484	0.06278972	0.06272182
0.0003	0.06289240	0.06301111	0.06313019	0.06309603
0.0005	0.06300823	0.06316473	0.06318729	0.06318721
0.0007	0.06311841	0.06329622	0.06332269	0.06332397
0.0010	0.06326045	0.06344668	0.06347292	0.06347195
0.0030	0.06395835	0.06416068	0.06418698	0.06418817
0.0050	0.06445243	0.06465746	0.06468392	0.06468424
0.0100	0.06536565	0.06557105	0.06559755	0.06559883
0.0500	0.06928491	0.06948134	0.06950786	0.06950937
0.1000	0.07219905	0.07238777	0.07241437	0.07241593
0.5000	0.08173988	0.08192181	0.08195032	0.08195202
1.0000	0.08537866	0.08558029	0.08561107	0.08561289
5.0000	0.09004347	0.09029888	0.09033402	0.09033603
10.0000	0.09105105	0.09132148	0.09135776	0.09135984
100.0000	0.09265788	0.09296020	0.09299871	0.09300088
500.0000	0.09305911	0.09337375	0.09341300	0.09341520
1000.0000	0.09315389	0.09347162	0.09351106	0.09351326
$+\infty$ exact	0.09375000	0.09375000	0.09375000	0.09375000

Table.E.2.7 Influence of N on degree of consolidation $U_z(ct/a^2)$
(impermeable case, $M = 10$, $\nu = 0.0$, $\nu_u = 0.5$)

ct/a^2	$N = 10$	$N = 25$	$N = 50$	$N = 75$	$N = 100$
0.0001	-.41301459	-.00416730	.00709829	.00815863	.01018118
0.0003	-.07165990	.02206961	.01907279	.01703114	.01667711
0.0005	.01646680	.02007961	.02199061	.02215912	.02186816
0.0007	.02466598	.02914578	.02636714	.02597389	.02603760
0.0010	.03406332	.03112294	.03110250	.03111641	.03101233
0.0030	.05880824	.05428573	.05402156	.05395665	.05389859
0.0050	.07219699	.07025394	.06989580	.06983017	.06979159
0.0100	.10141848	.09936868	.09916259	.09908881	.09908856
0.0500	.22512774	.22440369	.22429977	.22427899	.22427304
0.1000	.31773800	.31734377	.31730962	.31730202	.31729686
0.5000	.62242949	.62246180	.62246460	.62246513	.62246561
1.0000	.73960590	.73961139	.73961240	.73961288	.73961288
5.0000	.89075726	.89075357	.89075309	.89075303	.89075297
10.0000	.92351615	.92351496	.92351472	.92351472	.92351472
50.0000	.96606636	.96606606	.96606606	.96606600	.96606606
100.0000	.97602797	.97602791	.97602797	.97602803	.97602803
500.0000	.98928642	.98928636	.98928636	.98928636	.98928642
1000.0000	.99242425	.99242431	.99242431	.99242431	.99242431

Table.E.2.8 Influence of N on axial translation $\frac{a\mu D_z(ct/a^2)}{P_i}$
(impermeable case, $M = 10$, $\nu = 0.0$, $\nu_u = 0.5$)

ct/a^2	$N = 10$	$N = 25$	$N = 50$	$N = 75$	$N = 100$
0 exact	.06250000	.06250000	.06250000	.06250000	.06250000
0.0001	.04959329	.06236977	.06272182	.06275496	.06281816
0.0003	.06026063	.06318968	.06309603	.06303222	.06302116
0.0005	.06301459	.06312749	.06318721	.06319247	.06318338
0.0007	.06327081	.06341081	.06332397	.06331168	.06331368
0.0010	.06356448	.06347259	.06347195	.06347239	.06346914
0.0030	.06433776	.06419643	.06418817	.06418615	.06418433
0.0050	.06475616	.06469544	.06468424	.06468219	.06468099
0.0100	.06566933	.06560527	.06559883	.06559653	.06559652
0.0500	.06953524	.06951262	.06950937	.06950872	.06950853
0.1000	.07242931	.07241699	.07241593	.07241569	.07241553
0.5000	.08195092	.08195193	.08195202	.08195204	.08195205
1.0000	.08561268	.08561286	.08561289	.08561290	.08561290
5.0000	.09033616	.09033605	.09033603	.09033603	.09033603
10.0000	.09135988	.09135984	.09135984	.09135984	.09135983
50.0000	.09268957	.09268956	.09268956	.09268956	.09268957
100.0000	.09300088	.09300087	.09300088	.09300088	.09300088
500.0000	.09341520	.09341520	.09341520	.09341520	.09341520
1000.0000	.09351326	.09351326	.09351326	.09351326	.09351326
+∞ exact	.09375000	.09375000	.09375000	.09375000	.09375000

Fig. E.2.1 Influence of M on the numerical results of degree of consolidation of axial translation

Fig. E.2.2 Influence of N on the numerical results of degree of consolidation of axial translation

Fig. E.2.3 Influence of M on the numerical results of degree of consolidation of axial translation

Fig. E.2.4 Influence of N on the numerical results of degree of consolidation of axial translation

E.2.3 Axial force relaxation in the inclusion

For this problem, two sets of the numbers M and N are also selected. The first set is $M = 4, 6, 8, 10$ where $N = 50$ and the second set is $N = 10, 25, 50, 75, 100$ where $M = 10$. The drained and undrained Poisson's ratios are assumed as $\nu = 0.49$ and $\nu = 0.5$, respectively.

The numerical results for the non-dimensional axial force $P_z(ct/a^2)/(a\mu D_z)$ and the associated degree of relaxation $V_z(ct/a^2)$ are presented in Tables (E.2.9) ~ (E.2.16) for the two sets of the numbers M and N , under the conditions applicable to either a permeable or an impermeable rigid disc inclusion. The patterns of influence of M and/or N on the axial force $P_z(ct/a^2)$ are also plotted using the values of the degree of relaxation and the logarithm of the time factor ct/a^2 (see Fig. E.2.5 ~ E.2.8).

By calculating the absolute and relative differences between the numerical results of $M = 4$ (6, or 8) and $M = 10$ in the first set and $N = 10$ (25, 50, or 75) and $N = 100$ in the second set, it is also observed that the numerical results are progressively convergent with high stability over the long time interval of $10^{-4} \leq ct/a^2 \leq 10^{+3}$. Similar conclusions can be drawn from these test results about the influences of M and N on the convergence and accuracy of the time-dependent axial force. In particular, the inequality (E.2.3) is also applicable in choosing the value of N for accurately solving the integral equations.

E.2.4 Conclusions

The numerical scheme adopted in this thesis can provide highly stable, convergent and accurate results for the axial translation of a permeable or an impermeable rigid disc inclusion under the action of a constant axial load or a constant axial displacement loading by properly selecting the integral parameters and the numbers M and N . The following sets of M , N and ct/a^2 together with the integral parameters in equations (E.2.1) are suggested for examining the consolidation response associated with the axial translation of a rigid disc inclusion:

$$\begin{array}{lll}
M = 10, & N = 110, & 10^{-4} \leq ct/a^2 \leq 10^{-3} \\
M = 10, & N = 90, & 10^{-3} \leq ct/a^2 \leq 10^{-2} \\
M = 10, & N = 70, & 10^{-2} \leq ct/a^2 \leq 10^{-1} \\
M = 10, & N = 50, & 10^{-1} \leq ct/a^2 \leq 10^{+3}
\end{array}
\tag{E.2.4}$$

Table.E.2.9 Influence of M on degree of relaxation $V_z(ct/a^2)$
(permeable case, $N = 50$, $\nu = 0.49$, $\nu_u = 0.5$)

ct/a^2	$M = 4$	$M = 6$	$M = 8$	$M = 10$
0.0001	0.27797270	0.09039562	0.06998667	0.07390682
0.0002	0.25550914	0.06899184	0.04405291	0.04287352
0.0005	0.27944991	0.09825508	0.07627641	0.07414502
0.0007	0.28789228	0.10878385	0.08816500	0.08335666
0.0010	0.30702531	0.12495156	0.10188928	0.10140426
0.0020	0.33662248	0.15563887	0.13499330	0.13204700
0.0050	0.39120924	0.20996428	0.18790074	0.18711707
0.0100	0.44364098	0.26296359	0.24086572	0.23969305
0.0500	0.60808182	0.42712575	0.40507033	0.40384349
0.1000	0.69591200	0.51503754	0.49299145	0.49177328
0.5000	0.90639526	0.72603154	0.70407391	0.70285207
1.0000	0.98176271	0.80185050	0.77994025	0.77872413
5.0000	1.09815037	0.91935527	0.89753950	0.89632738
10.0000	1.12765694	0.94924611	0.92745984	0.92624909
50.0000	1.16759491	0.98982590	0.96808612	0.96687728
100.0000	1.17708743	0.99951226	0.97778547	0.97657716
500.0000	1.18972969	1.01246047	0.99075323	0.98954570
1000.0000	1.19271696	1.01553047	0.99382836	0.99262106

Table.E.2.10 Influence of M on total axial force $\frac{P_z(ct/a^2)}{a\mu D_z}$
(permeable case, $N = 50$, $\nu = 0.49$, $\nu_u = 0.5$)

ct/a^2	$M = 4$	$M = 6$	$M = 8$	$M = 10$
0 exact	16.000000	16.000000	16.000000	16.000000
0.0001	15.914470	15.972186	15.978466	15.977259
0.0002	15.921382	15.978772	15.986445	15.986808
0.0005	15.914015	15.969768	15.976530	15.977186
0.0007	15.911418	15.966528	15.972872	15.974352
0.0010	15.905531	15.961553	15.968649	15.968799
0.0020	15.896424	15.952111	15.958464	15.959370
0.0050	15.879628	15.935396	15.942184	15.942426
0.0100	15.863495	15.919088	15.925887	15.926248
0.0500	15.812898	15.868577	15.875363	15.875740
0.1000	15.785873	15.841527	15.848310	15.848685
0.5000	15.721109	15.776606	15.783362	15.783738
1.0000	15.697919	15.753277	15.760018	15.760393
5.0000	15.662108	15.717121	15.723834	15.724207
10.0000	15.653029	15.707924	15.714628	15.715000
50.0000	15.640740	15.695438	15.702127	15.702499
100.0000	15.637819	15.692458	15.699143	15.699515
500.0000	15.633929	15.688474	15.695153	15.695524
1000.0000	15.633010	15.687529	15.694207	15.694578
$+\infty$ exact	15.692308	15.692308	15.692308	15.692308

Table.E.2.11 Influence of N on degree of relaxation $V_r(ct/a^2)$
(permeable case, $M = 10$, $\nu = 0.49$, $\nu_u = 0.5$)

ct/a^2	$N = 10$	$N = 25$	$N = 50$	$N = 75$	$N = 100$
0.0001	-.41876829	.01519996	.07390682	.01983780	.02824979
0.0002	-.07126430	.05349918	.04287352	.04842557	.05742112
0.0005	-.02948675	.05509995	.07414502	.07579813	.06709909
0.0007	.10965710	.08598271	.08335666	.09092676	.08062824
0.0010	.07197101	.09817968	.10140426	.09496211	.09725851
0.0020	.12668988	.12436849	.13204700	.13031334	.12785733
0.0050	.19009833	.18912219	.18711707	.18668094	.18680523
0.0100	.23218410	.24063197	.23969305	.23957653	.23930122
0.0500	.40316167	.40397123	.40384349	.40402830	.40396264
0.1000	.49179628	.49160308	.49177328	.49175614	.49175239
0.5000	.70281839	.70282030	.70285207	.70284498	.70285022
1.0000	.77870667	.77872586	.77872413	.77872086	.77872318
5.0000	.89632612	.89632744	.89632738	.89632732	.89632732
10.0000	.92624635	.92624897	.92624909	.92624915	.92624915
50.0000	.96687734	.96687722	.96687728	.96687728	.96687734
100.0000	.97657734	.97657704	.97657716	.97657716	.97657722
500.0000	.98954576	.98954570	.98954570	.98954570	.98954576
1000.0000	.99262100	.99262106	.99262106	.99262106	.99262106

Table.E.2.12 Influence of N on total axial force $\frac{P_z(ct/a^2)}{a\mu D_s}$
(permeable case, $M = 10$, $\nu = 0.49$, $\nu_u = 0.5$)

ct/a^2	$N = 10$	$N = 25$	$N = 50$	$N = 75$	$N = 100$
0 exact	16.000000	16.000000	16.000000	16.000000	16.000000
0.0001	16.128852	15.995323	15.977259	15.993896	15.991308
0.0002	16.021927	15.983539	15.986808	15.985100	15.982332
0.0005	16.009073	15.983046	15.977186	15.976677	15.979354
0.0007	15.966259	15.973544	15.974352	15.972023	15.975191
0.0010	15.977855	15.969791	15.968799	15.970781	15.970074
0.0020	15.961018	15.961733	15.959370	15.959904	15.960659
0.0050	15.941508	15.941809	15.942426	15.942560	15.942521
0.0100	15.928559	15.925959	15.926248	15.926284	15.926369
0.0500	15.875950	15.875701	15.875740	15.875684	15.875704
0.1000	15.848678	15.848738	15.848685	15.848690	15.848692
0.5000	15.783748	15.783748	15.783738	15.783740	15.783738
1.0000	15.760398	15.760392	15.760393	15.760394	15.760393
5.0000	15.724207	15.724207	15.724207	15.724207	15.724207
10.0000	15.715001	15.715000	15.715000	15.715000	15.715000
50.0000	15.702499	15.702499	15.702499	15.702499	15.702499
100.0000	15.699515	15.699515	15.699515	15.699515	15.699515
500.0000	15.695524	15.695524	15.695524	15.695524	15.695524
1000.0000	15.694578	15.694578	15.694578	15.694578	15.694578
+∞ exact	15.692308	15.692308	15.692308	15.692308	15.692308

Table.E.2.13 Influence of M on degree of relaxation $V_z(ct/a^2)$
(impermeable case, $N = 50$, $\nu = 0.49$, $\nu_u = 0.5$)

ct/a^2	$M = 4$	$M = 6$	$M = 8$	$M = 10$
0.0001	0.21935	0.03373	0.01099	0.00989
0.0002	0.22387	0.04322	0.02197	0.01997
0.0005	0.23125	0.04911	0.02687	0.02562
0.0007	0.23548	0.05390	0.03180	0.03058
0.0010	0.24084	0.05947	0.03735	0.03647
0.0020	0.25515	0.07414	0.05201	0.05076
0.0050	0.28391	0.10307	0.08095	0.07975
0.0100	0.31596	0.13514	0.11302	0.11179
0.0500	0.44621	0.26510	0.24299	0.24176
0.1000	0.53719	0.35589	0.33378	0.33255
0.5000	0.83123	0.64965	0.62760	0.62638
1.0000	0.94721	0.76627	0.74429	0.74307
5.0000	1.09500	0.91591	0.89408	0.89287
10.0000	1.12660	0.94804	0.92624	0.92503
50.0000	1.16760	0.98972	0.96798	0.96677
100.0000	1.17710	0.99948	0.97775	0.97654
500.0000	1.18980	1.01250	0.99075	0.98954
1000.0000	1.19270	1.01550	0.99383	0.99262

Table.E.2.14 Influence of M on total axial force $\frac{P_z(ct/a^2)}{a\mu D_z}$
(impermeable case, $N = 50$, $\nu = 0.49$, $\nu_u = 0.5$)

ct/a^2	$M = 4$	$M = 6$	$M = 8$	$M = 10$
0 exact	16.000	16.000	16.000	16.000
0.0001	15.933	15.990	15.997	15.997
0.0002	15.931	15.987	15.993	15.994
0.0005	15.929	15.985	15.992	15.992
0.0007	15.928	15.983	15.990	15.991
0.0010	15.926	15.982	15.989	15.989
0.0020	15.921	15.977	15.984	15.984
0.0050	15.913	15.968	15.975	15.975
0.0100	15.903	15.958	15.965	15.966
0.0500	15.863	15.918	15.925	15.926
0.1000	15.835	15.890	15.897	15.898
0.5000	15.744	15.800	15.807	15.807
1.0000	15.709	15.764	15.771	15.771
5.0000	15.663	15.718	15.725	15.725
10.0000	15.653	15.708	15.715	15.715
50.0000	15.641	15.695	15.702	15.703
100.0000	15.638	15.692	15.699	15.700
500.0000	15.634	15.688	15.695	15.696
1000.0000	15.633	15.688	15.694	15.695
$+\infty$ exact	15.692	15.692	15.692	15.692

Table.E.2.15 Influence of N on degree of relaxation $V_z(ct/a^2)$
(impermeable case, $M = 10$, $\nu = 0.49$, $\nu_w = 0.5$)

ct/a^2	$N = 15$	$N = 25$	$N = 50$	$N = 75$
0.00010	-0.14010	-0.00196	0.00989	0.00990
0.00020	-0.00869	0.02206	0.01997	0.01608
0.00050	0.03202	0.02342	0.02562	0.02575
0.00070	0.03487	0.03333	0.03058	0.03011
0.00100	0.04012	0.03717	0.03647	0.03608
0.00200	0.05057	0.05081	0.05076	0.05072
0.00500	0.08056	0.08009	0.07975	0.07962
0.01000	0.11301	0.11196	0.11179	0.11171
0.05000	0.24205	0.24184	0.24176	0.24175
0.10000	0.33270	0.33259	0.33255	0.33255
0.50000	0.62637	0.62638	0.62638	0.62638
1.00000	0.74308	0.74307	0.74307	0.74307
5.00000	0.89287	0.89287	0.89287	0.89287
10.00000	0.92503	0.92503	0.92503	0.92503
50.00000	0.96677	0.96677	0.96677	0.96677
100.00000	0.97654	0.97654	0.97654	0.97654
200.00000	0.98343	0.98343	0.98343	0.98343
500.00000	0.98954	0.98954	0.98954	0.98954
1000.00000	0.99262	0.99262	0.99262	0.99262

Table.E.2.16 Influence of N on total axial force $\frac{P_z(ct/a^2)}{a\mu D_z}$
(impermeable case, $M = 10$, $\nu = 0.49$, $\nu_w = 0.5$)

ct/a^2	$N = 15$	$N = 25$	$N = 50$	$N = 75$
0 exact	16.000	16.000	16.000	16.00000
0.0001	16.043	16.001	15.997	15.99696
0.0002	16.003	15.993	15.994	15.99505
0.0005	15.990	15.993	15.992	15.99208
0.0007	15.989	15.990	15.991	15.99074
0.0010	15.988	15.989	15.989	15.98890
0.0020	15.984	15.984	15.984	15.98439
0.0050	15.975	15.975	15.975	15.97550
0.0100	15.965	15.966	15.966	15.96563
0.0500	15.926	15.926	15.926	15.92562
0.1000	15.898	15.898	15.898	15.89768
0.5000	15.807	15.807	15.807	15.80727
1.0000	15.771	15.771	15.771	15.77136
5.0000	15.725	15.725	15.725	15.72527
10.0000	15.715	15.715	15.715	15.71538
50.0000	15.703	15.703	15.703	15.70253
100.0000	15.700	15.700	15.700	15.69953
500.0000	15.696	15.696	15.696	15.69553
1000.0000	15.695	15.695	15.695	15.69458
$+\infty$ exact	15.692	15.692	15.692	15.69231

Fig. E.2.5 Influence of M on the numerical results of degree of relaxation of axial force

Fig. E.2.6 Influence of N on the numerical results of degree of relaxation of axial force

Fig. E.2.8 Influence of M on the numerical results of degree of relaxation of axial force

Fig. 6.2.18 Influence of N on the numerical results of degree of relaxation of axial force

E.3 Central Rotation of a Rigid Disc Inclusion

E.3.1 General

In the evaluation, the integral parameters in the numerical integration techniques are first assigned the following values:

$$\begin{aligned}\epsilon_a &= 10^{-8} \\ \alpha_s &= 1.95 \\ \alpha_t &= 0 \\ \epsilon_b &= 10^{-6} \\ \epsilon_c &= 10^{-6}\end{aligned}\tag{E.3.1}$$

The time factor ct/a^2 is in the interval:

$$10^{-4} \leq ct/a^2 \leq 10\tag{E.3.2}$$

Two sets of the numbers M and N are selected. The first set is $M = 4, 6, 8, 10$ where $N = 50$ and the second set is $N = 10, 20, 30, 40, 50, 75, 100$ where $M = 10$. The drained and undrained Poisson's ratios are assumed as $\nu = 0.49$ and $\nu = 0.5$, respectively. The convergence and accuracy of the solutions to the problem of the time-dependent central rotation of a rigid disc inclusion are then evaluated by considering the influences of the numbers of M and N .

E.3.2 Central consolidation rotation of the inclusion

The numerical results for the central rotation $a^3\mu\Omega_y(ct/a^2)/M_y$ and the associated degree of consolidation $U_y(ct/a^2)$ are presented in Tables (E.3.1) ~ (E.3.8) for the two sets of the numbers M and N under the conditions applicable to either a permeable or an impermeable rigid disc inclusion. The patterns of influence of M and/or N on the central rotation $\Omega_y(ct/a^2)$ are also plotted using the degree of consolidation rotation and the logarithm time factor ct/a^2 in the Fig. E.3.1 ~ E.3.4.

By calculating the absolute and relative differences between the numerical results of $M = 4$ (6, or 8) and $M = 10$ in the first set and $N = 10$ (25, 50, or 75) and

$N = 100$ in the second set, it is observed that the numerical results are progressively convergent with highly stability for the long non-dimensional time interval $10^{-4} \leq ct/a^2 \leq 10$. Such absolute and relative differences indicate that both M and N have very uniform influences on the numerical results for every time factor ct/a^2 in the entire range of $10^{-4} \leq ct/a^2 \leq 10$ (see e.g., Fig.E.3.1 ~ Fig.E.3.4). The absolute and relative differences between the non-dimensional central rotations of $M = 8$ and $M = 10$ are less than 10^{-5} and 10^{-4} , respectively. In other words, the $M = 10$ gives very accurate results in the numerical inversion of Laplace transforms for the integral equations. In particular, the inequality (E.2.3) is also valid for selecting the number N at small values of the time factor.

Table.E.3.1 Influence of M on degree of consolidation $U_y(ct/a^2)$
(permeable case, $N = 50$, $\nu = 0.49$, $\nu_u = 0.5$)

ct/a^2	$M = 4$	$M = 6$	$M = 8$	$M = 10$
0.0001	-0.09045985	0.07497025	0.08981656	0.10033031
0.0002	-0.03570146	0.14264588	0.21391405	0.17673945
0.0005	-0.02312807	0.15353103	0.17728294	0.17535461
0.0007	0.00912837	0.18319562	0.20090592	0.20322713
0.0010	0.02196936	0.19640817	0.22002514	0.22235815
0.0020	0.08906953	0.26596233	0.28899640	0.28967041
0.0050	0.20741647	0.38387132	0.40414995	0.40604660
0.0070	0.25175264	0.42715177	0.44789895	0.44938043
0.0100	0.30068636	0.47738320	0.49630176	0.50034964
0.0200	0.40298498	0.57933432	0.60079020	0.60245460
0.0500	0.54272169	0.71949339	0.74133134	0.74252754
0.0700	0.59120733	0.76816422	0.79002589	0.79124457
0.1000	0.63848060	0.81568193	0.83757472	0.83881193
0.3000	0.74361694	0.92188156	0.94386309	0.94508868
0.5000	0.76994884	0.94872206	0.97074163	0.97197175
0.7000	0.78107989	0.96014470	0.98218322	0.98341060
1.0000	0.78895980	0.96828419	0.99033821	0.99156588
5.0000	0.80046535	0.98044288	1.00253606	1.00376594
7.0000	0.80092221	0.98097092	1.00306809	1.00429797
10.0000	0.80120981	0.98132455	1.00342512	1.00465512

Table.E.3.2 Influence of M on central rotation $\frac{a^3 \mu \Omega_y(ct/a^2)}{M_y}$
(permeable case, $N = 50$, $\nu = 0.49$, $\nu_u = 0.5$)

ct/a^2	$M = 4$	$M = 6$	$M = 8$	$M = 10$
0 exact	0.09375000	0.09375000	0.09375000	0.09375000
0.0001	0.09358371	0.09388781	0.09391510	0.09393443
0.0002	0.09368437	0.09401222	0.09414322	0.09407489
0.0005	0.09370748	0.09403223	0.09407589	0.09407234
0.0007	0.09376678	0.09408676	0.09411931	0.09412358
0.0010	0.09379039	0.09411104	0.09415446	0.09415875
0.0020	0.09391373	0.09423890	0.09428124	0.09428248
0.0050	0.09413128	0.09445565	0.09449292	0.09449641
0.0070	0.09421278	0.09453521	0.09457334	0.09457607
0.0100	0.09430273	0.09462754	0.09466783	0.09466976
0.0200	0.09449078	0.09481495	0.09485439	0.09485745
0.0500	0.09474765	0.09507260	0.09511274	0.09511494
0.0700	0.09483678	0.09516207	0.09520225	0.09520449
0.1000	0.09492368	0.09524942	0.09528966	0.09529193
0.3000	0.09511694	0.09544464	0.09548504	0.09548730
0.5000	0.09516535	0.09549397	0.09553445	0.09553671
0.7000	0.09518581	0.09551497	0.09555548	0.09555774
1.0000	0.09520029	0.09552993	0.09557048	0.09557273
5.0000	0.09522144	0.09555229	0.09559290	0.09559516
7.0000	0.09522228	0.09555326	0.09559388	0.09559614
10.0000	0.09522281	0.09555391	0.09559453	0.09559679
+∞ exact	0.09558824	0.09558824	0.09558824	0.09558824

Table.E.3.3 Influence of N on degree of consolidation $U_v(ct/a^2)$
(permeable case, $M = 10$, $\nu = 0.49$, $\nu_u = 0.5$)

ct/a^2	$N = 10$	$N = 25$	$N = 50$	$N = 75$	$N = 100$
0.0001	-0.20808993	0.10446570	0.10033031	0.09436812	0.08955778
0.0002	0.07308602	0.15769438	0.17673945	0.12461358	0.12006629
0.0005	0.27029994	0.19796100	0.17535461	0.18306316	0.18647191
0.0007	0.31535745	0.22381227	0.20322713	0.20606937	0.20467979
0.0010	0.34517816	0.25009084	0.22235815	0.23791277	0.23426814
0.0020	0.42071569	0.31757456	0.28967041	0.30048680	0.29843843
0.0050	0.52666450	0.42057189	0.40604660	0.94219631	0.40277088
0.0070	0.56933522	0.46578309	0.44938043	0.95891148	0.44630221
0.0100	0.62115633	0.51435375	0.50034964	0.40324783	0.49589649
0.0200	0.72660476	0.61807877	0.60245460	0.49703145	0.59886044
0.0500	0.86619711	0.75806707	0.74252754	0.52351928	0.73879880
0.0700	0.91552663	0.80681413	0.79124457	0.59960210	0.78736651
0.1000	0.96330553	0.85431838	0.83881193	0.73981100	0.83486670
0.3000	1.07021475	0.94219631	0.97197175	0.85804445	0.94118851
0.5000	1.09704053	0.98756564	0.99156588	0.96907818	0.96806806
0.7000	1.10851741	0.99902928	0.99454743	0.98053175	0.97952026
1.0000	1.11669135	1.00718451	0.99993205	0.98868775	0.98767751
5.0000	1.12889707	1.01937985	1.00376594	1.00088286	0.99987078
7.0000	1.12943602	1.01991892	1.00429797	1.00142193	1.00040984
10.0000	1.12978125	1.02026391	1.00465512	1.00176656	1.00075543

Table.E.3.4 Influence of N on central rotation $\frac{a^3 \mu \Omega_v(ct/a^2)}{M_v}$
(permeable case, $M = 10$, $\nu = 0.49$, $\nu_u = 0.5$)

ct/a^2	$N = 10$	$N = 25$	$N = 50$	$N = 75$	$N = 100$
0 exact	0.09375000	0.09375000	0.09375000	0.09375000	0.09375000
0.0001	0.09336748	0.09394203	0.09393443	0.09392347	0.09391463
0.0002	0.09388435	0.09403988	0.09407489	0.09397907	0.09397071
0.0005	0.09424688	0.09411390	0.09407234	0.09408651	0.09409278
0.0007	0.09432970	0.09416142	0.09412358	0.09412880	0.09412625
0.0010	0.09438452	0.09420973	0.09415875	0.09418734	0.09418064
0.0020	0.09452337	0.09433378	0.09428248	0.09430237	0.09429860
0.0050	0.09471813	0.09452311	0.09449641	0.09548198	0.09449039
0.0070	0.09479657	0.09460622	0.09457607	0.09551271	0.09457041
0.0100	0.09489183	0.09469550	0.09466976	0.09449126	0.09466157
0.0200	0.09508567	0.09488617	0.09485745	0.09466366	0.09485085
0.0500	0.09534227	0.09514351	0.09511494	0.09471235	0.09510809
0.0700	0.09543295	0.09523311	0.09520449	0.09485221	0.09519737
0.1000	0.09552078	0.09532044	0.09529193	0.09510995	0.09528468
0.3000	0.09571731	0.09548198	0.09553671	0.09532729	0.09548013
0.5000	0.09576662	0.09556538	0.09557273	0.09553139	0.09552954
0.7000	0.09578772	0.09558645	0.09557821	0.09555245	0.09555059
1.0000	0.09580274	0.09560144	0.09558811	0.09556744	0.09556558
5.0000	0.09582518	0.09562386	0.09559516	0.09558986	0.09558800
7.0000	0.09582617	0.09562485	0.09559614	0.09559085	0.09558899
10.0000	0.09582680	0.09562549	0.09559679	0.09559148	0.09558962
+∞ exact	0.09558824	0.09558824	0.09558824	0.09558824	0.09558824

Table.E.3.5 Influence of M on degree of consolidation $U_v(ct/a^2)$
(impermeable case, $N = 50$, $\nu = 0.49$, $\nu_u = 0.5$)

ct/a^2	$M = 4$	$M = 6$	$M = 8$	$M = 10$
0.0001	-0.15458369	0.01633202	0.03630931	0.03739511
0.0002	-0.14135770	0.03761703	0.06000084	0.06028640
0.0005	-0.11968500	0.05562294	0.07736651	0.07844969
0.0007	-0.10624760	0.06944875	0.09116795	0.09208559
0.0010	-0.09074729	0.08518309	0.10687350	0.10793230
0.0020	-0.05084482	0.12535340	0.14703690	0.14820260
0.0050	0.02588020	0.20212990	0.22385360	0.22507280
0.0070	0.06235048	0.23856451	0.26026231	0.26146641
0.0100	0.10671960	0.28287169	0.30457351	0.30577609
0.0200	0.21127529	0.38726550	0.40897331	0.41018730
0.0500	0.38536540	0.56118828	0.58292472	0.58414042
0.0700	0.45639589	0.63221270	0.65396369	0.65517730
0.1000	0.53197849	0.70786601	0.72964150	0.73085582
0.3000	0.71798909	0.89505363	0.91696268	0.91818577
0.5000	0.76134008	0.93936259	0.96133888	0.96256429
0.7000	0.77742118	0.95593941	0.97794789	0.97917467
1.0000	0.78767151	0.96656990	0.98860258	0.98983037
5.0000	0.80062300	0.98042738	1.00251198	1.00374198
7.0000	0.80110472	0.98097801	1.00306702	1.00429595
10.0000	0.80137891	0.98132747	1.00342000	1.00465000

Table.E.3.6 Influence of M on central rotation $\frac{a^3 \mu \Omega_v(ct/a^2)}{M_v}$
(impermeable case, $N = 50$, $\nu = 0.49$, $\nu_u = 0.5$)

ct/a^2	$M = 4$	$M = 6$	$M = 8$	$M = 10$
0 exact	0.09375000	0.09375000	0.09375000	0.09375000
0.0001	0.09346584	0.09378002	0.09381675	0.09381874
0.0002	0.09349015	0.09381915	0.09386030	0.09386082
0.0005	0.09352999	0.09385225	0.09389222	0.09389421
0.0007	0.09355469	0.09387766	0.09391759	0.09391927
0.0010	0.09358319	0.09390659	0.09394646	0.09394840
0.0020	0.09365654	0.09398043	0.09402029	0.09402243
0.0050	0.09379757	0.09412156	0.09416150	0.09416374
0.0070	0.09386461	0.09418854	0.09422842	0.09423064
0.0100	0.09394618	0.09426998	0.09430988	0.09431209
0.0200	0.09413837	0.09446189	0.09450179	0.09450402
0.0500	0.09445839	0.09478160	0.09482155	0.09482379
0.0700	0.09458896	0.09491216	0.09495214	0.09495437
0.1000	0.09472790	0.09505122	0.09509125	0.09509348
0.3000	0.09506983	0.09539532	0.09543559	0.09543784
0.5000	0.09514952	0.09547677	0.09551717	0.09551942
0.7000	0.09517908	0.09550724	0.09554770	0.09554995
1.0000	0.09519793	0.09552678	0.09556728	0.09556954
5.0000	0.09522173	0.09555226	0.09559285	0.09559511
7.0000	0.09522262	0.09555327	0.09559387	0.09559613
10.0000	0.09522312	0.09555391	0.09559452	0.09559678
+∞ exact	0.09558824	0.09558824	0.09558824	0.09558824

Table.E.3.7 Influence of N on degree of consolidation $U_v(ct/a^2)$
(impermeable case, $M = 10$, $\nu = 0.49$, $\nu_u = 0.5$)

ct/a^2	$N = 10$	$N = 25$	$N = 50$	$N = 75$	$N = 100$
0.0001	-0.27148971	0.05443712	0.03739511	0.03540557	0.03350455
0.0002	0.03737263	0.07971426	0.06028640	0.04760762	0.05115623
0.0005	0.19242971	0.09088833	0.07844969	0.07605218	0.07430242
0.0007	0.22029170	0.11255750	0.09208559	0.08866063	0.08802939
0.0010	0.23568790	0.12225160	0.10793230	0.10516000	0.10359440
0.0020	0.27797100	0.16378860	0.14820260	0.14536440	0.14442720
0.0050	0.34627080	0.24035969	0.22507280	0.22219820	0.22115961
0.0070	0.38280061	0.27673629	0.26146641	0.25866249	0.25767651
0.0100	0.42578241	0.32062510	0.30577609	0.30299440	0.30201611
0.0200	0.53021139	0.42517951	0.41018730	0.40739560	0.40642831
0.0500	0.70516539	0.59921718	0.58414042	0.58133483	0.58036000
0.0700	0.77689350	0.67037052	0.65517730	0.65236330	0.65137857
0.1000	0.85338902	0.74614829	0.73085582	0.72803098	0.72703779
0.3000	1.04279006	0.93372732	0.91818577	0.91530818	0.91430068
0.5000	1.08753300	0.97815341	0.96256429	0.95967919	0.95866930
0.7000	1.10423696	0.99477541	0.97917467	0.97628742	0.97527653
1.0000	1.11493599	1.00543499	0.98983037	0.98694158	0.98593038
5.0000	1.12886703	1.01934898	1.00374198	1.00085199	0.99984092
7.0000	1.12942195	1.01990294	1.00429595	1.00140703	1.00039601
10.0000	1.12977600	1.02025795	1.00465000	1.00176096	1.00074899

Table.E.3.8 Influence of N on central rotation $\frac{a^3 \mu \Omega_v(ct/a^2)}{M_v}$
(permeable case, $M = 10$, $\nu = 0.49$, $\nu_u = 0.5$)

ct/a^2	$N = 10$	$N = 25$	$N = 50$	$N = 75$	$N = 100$
0 exact	0.09375000	0.09375000	0.09375000	0.09375000	0.09375000
0.0001	0.09325094	0.09385007	0.09381874	0.09381508	0.09381159
0.0002	0.09381870	0.09389653	0.09386082	0.09383751	0.09384404
0.0005	0.09410373	0.09391707	0.09389421	0.09388980	0.09388659
0.0007	0.09415495	0.09395691	0.09391927	0.09391298	0.09391182
0.0010	0.09418325	0.09397473	0.09394840	0.09394331	0.09394043
0.0020	0.09426098	0.09405108	0.09402243	0.09401721	0.09401549
0.0050	0.09438653	0.09419184	0.09416374	0.09415845	0.09415654
0.0070	0.09445368	0.09425871	0.09423064	0.09422548	0.09422367
0.0100	0.09453269	0.09433938	0.09431209	0.09430697	0.09430518
0.0200	0.09472465	0.09453158	0.09450402	0.09449889	0.09449711
0.0500	0.09504626	0.09485150	0.09482379	0.09481863	0.09481684
0.0700	0.09517811	0.09498230	0.09495437	0.09494920	0.09494739
0.1000	0.09531873	0.09512160	0.09509348	0.09508829	0.09508647
0.3000	0.09566689	0.09546641	0.09543784	0.09543255	0.09543070
0.5000	0.09574914	0.09554808	0.09551942	0.09551412	0.09551226
0.7000	0.09577985	0.09557863	0.09554995	0.09554465	0.09554279
1.0000	0.09579951	0.09559823	0.09556954	0.09556423	0.09556237
5.0000	0.09582512	0.09562380	0.09559511	0.09558980	0.09558794
7.0000	0.09582614	0.09562482	0.09559613	0.09559082	0.09558896
10.0000	0.09582679	0.09562547	0.09559678	0.09559147	0.09558961
$+\infty$ exact	0.09558824	0.09558824	0.09558824	0.09558824	0.09558824

Fig. E.3.1 Influence of M on the numerical results of degree of consolidation of central rotation

Fig. E.3.2 Influence of N on the numerical results of degree of consolidation of central rotation

Fig. E.3.3 Influence of M on the numerical results of degree of consolidation of central rotation

Fig. E.3.4 Influence of N on the numerical results of degree of consolidation of central rotation

E.3.3 Central moment relaxation in the inclusion

The numerical results for the central moment $M_y(ct/a^2)/(a^3\mu\Omega_y)$ and the associated degree of relaxation $V_y(ct/a^2)$ are presented in Tables (E.3.9) ~ (E.3.16) for the two sets of the numbers M and N under the conditions applicable to either a permeable or an impermeable rigid disc inclusion. The patterns of influence of M and/or N on the central moment $M_y(ct/a^2)$ are also plotted using the values of the degree of relaxation and the logarithm of the time factor ct/a^2 (see Fig. E.3.5 ~ E.3.8).

By calculating the absolute and relative differences between the numerical results of $M = 4$ (6, or 8) and $M = 10$ in the first set and $N = 10$ (25, 50, or 75) and $N = 100$ in the second set, it is also observed that the numerical results are progressively convergent with high stability over the long non-dimensional time interval of $10^{-4} \leq ct/a^2 \leq 10$. Similar conclusions can be drawn from these test results about the influences of M and N on the convergence and accuracy of the time-dependent central moment. In particular, the inequality (E.2.3) is also valid in choosing the value of N for accurately solving the integral equations in small time factor ct/a^2 .

E.3.4 Conclusions

The numerical scheme adopted in this thesis can provide highly stable, convergent and accurate results for the central rotation of a permeable or an impermeable rigid disc inclusion under the action of a constant central moment or a constant central rotation loading by properly selecting the numbers M and N . The following values of M and N together with the integral parameters in equations (E.3.1) are suggested for examining the consolidation response associated with the central rotation of a rigid disc inclusion.

$$\begin{aligned}
 M = 10, \quad N = 110, \quad 10^{-4} \leq ct/a^2 \leq 10^{-3} \\
 M = 10, \quad N = 90, \quad 10^{-3} \leq ct/a^2 \leq 10^{-2} \\
 M = 10, \quad N = 70, \quad 10^{-2} \leq ct/a^2 \leq 10
 \end{aligned}
 \tag{E.3.3}$$

Table.EE.3.9 Influence of M on degree of relaxation $V_y(ct/a^2)$
(permeable case, $N = 50$, $\nu = 0.49$, $\nu_u = 0.5$)

ct/a^2	$M = 4$	$M = 6$	$M = 8$	$M = 10$
0.0001	0.31684199	0.12786870	0.10167240	0.10030282
0.0002	0.31999561	0.13737749	0.13338177	0.11514467
0.0005	0.39067250	0.20771588	0.18768108	0.18534701
0.0007	0.41547385	0.23484327	0.21247728	0.20939448
0.0010	0.44633102	0.26465750	0.24410795	0.24214910
0.0020	0.51247191	0.33116603	0.30924803	0.30843422
0.0050	0.61563480	0.43444958	0.41240463	0.41015959
0.0070	0.66028172	0.47925290	0.45711732	0.45594910
0.0100	0.70987803	0.52891529	0.50685561	0.50557733
0.0200	0.81218982	0.63124758	0.60919011	0.60798264
0.0500	0.95087981	0.77028304	0.74831855	0.74710125
0.0700	0.99873012	0.81832522	0.79638362	0.79517072
0.1000	1.04533815	0.86518264	0.84327066	0.84205556
0.3000	1.14859807	0.96949542	0.94767582	0.94646424
0.5000	1.17432225	0.99572581	0.97394258	0.97273242
0.7000	1.18519795	1.00688815	0.98512405	0.98391438
1.0000	1.19289160	1.01483643	0.99308848	0.99187982
5.0000	1.20412707	1.02671206	1.00500214	1.00379467
7.0000	1.20457995	1.02723467	1.00552857	1.00432134
10.0000	1.20485115	1.02757061	1.00586796	1.00466084

Table.E.3.10 Influence of M on total central moment $\frac{M_y(ct/a^2)}{a^3 \mu \Omega_y}$
(permeable case, $N = 50$, $\nu = 0.49$, $\nu_u = 0.5$)

ct/a^2	$M = 4$	$M = 6$	$M = 8$	$M = 10$
0 exact	10.666667	10.666667	10.666667	10.666667
0.0001	10.601673	10.640437	10.645811	10.646092
0.0002	10.601027	10.638487	10.639306	10.643047
0.0005	10.586529	10.624058	10.628168	10.628647
0.0007	10.581441	10.618494	10.623082	10.623714
0.0010	10.575112	10.612378	10.616593	10.616995
0.0020	10.561544	10.598735	10.603231	10.603398
0.0050	10.540383	10.577549	10.582071	10.582531
0.0070	10.531224	10.568358	10.572899	10.573139
0.0100	10.521051	10.558171	10.562696	10.562958
0.0200	10.500064	10.537180	10.541705	10.541952
0.0500	10.471614	10.508660	10.513165	10.513415
0.0700	10.461799	10.498805	10.503306	10.503555
0.1000	10.452238	10.489193	10.493688	10.493937
0.3000	10.431057	10.467796	10.472272	10.472520
0.5000	10.425780	10.462415	10.466884	10.467132
0.7000	10.423549	10.460126	10.464590	10.464838
1.0000	10.421971	10.458495	10.462956	10.463204
5.0000	10.419666	10.456059	10.460512	10.460760
7.0000	10.419573	10.455952	10.460404	10.460652
10.0000	10.419518	10.455883	10.460335	10.460582
$+\infty$ exact	10.461538	10.461538	10.461538	10.461538

Table.E.3.11 Influence of N on degree of relaxation $V_y(ct/a^2)$
(permeable case, $M = 10$, $\nu = 0.49$, $\nu_s = 0.5$)

ct/a^2	$N = 10$	$N = 25$	$N = 50$	$N = 75$	$N = 100$
0.0001	-0.18242411	0.10199367	0.10030282	0.09970012	0.08850697
0.0002	0.08651491	0.15599424	0.11514467	0.12531096	0.12191349
0.0005	0.27662715	0.19912113	0.18534701	0.18574011	0.18927573
0.0007	0.32128096	0.22739853	0.20939448	0.20922896	0.20780978
0.0010	0.35034123	0.25368619	0.24214910	0.24136189	0.23824508
0.0020	0.42585111	0.32280499	0.30843422	0.30335090	0.30291775
0.0050	0.53202462	0.42587432	0.41015959	0.40870211	0.40791932
0.0070	0.57448208	0.47128344	0.45594910	0.45303506	0.45166993
0.0100	0.62643981	0.51931119	0.50557733	0.50249922	0.50133795
0.0200	0.73124760	0.62339222	0.60798264	0.60496432	0.60423070
0.0500	0.86924356	0.76239640	0.74710125	0.74431211	0.74330938
0.0700	0.91780126	0.81058598	0.79517072	0.79235226	0.79135770
0.1000	0.96471447	0.85742509	0.84205556	0.83922607	0.83822745
0.3000	1.06915176	0.96177429	0.94646424	0.94362122	0.94263059
0.5000	1.09523904	0.98804337	0.97273242	0.96989626	0.96890438
0.7000	1.10639000	0.99922907	0.98391438	0.98108011	0.98008728
1.0000	1.11433125	1.00718558	0.99187982	0.98904312	0.98805195
5.0000	1.12620115	1.01909566	1.00379467	1.00096154	0.99996901
7.0000	1.12672627	1.01962328	1.00432134	1.00148952	1.00049586
10.0000	1.12706292	1.01996100	1.00466084	1.00182712	1.00083542

Table.E.3.12 Influence of N on total central moment $\frac{M_y(ct/a^2)}{a^3 \mu \Omega_y}$
(permeable case, $M = 10$, $\nu = 0.49$, $\nu_s = 0.5$)

ct/a^2	$N = 10$	$N = 25$	$N = 50$	$N = 75$	$N = 100$
0 exact	10.666667	10.666667	10.666667	10.666667	
0.0001	10.704087	10.645745	10.646092	10.646215	10.648511
0.0002	10.648920	10.634668	10.643047	10.640962	10.641659
0.0005	10.609923	10.625821	10.628647	10.628566	10.627841
0.0007	10.600763	10.620021	10.623714	10.623748	10.624039
0.0010	10.594802	10.614628	10.616995	10.617157	10.617796
0.0020	10.579313	10.600450	10.603398	10.604441	10.604530
0.0050	10.557533	10.579308	10.582531	10.582830	10.582991
0.0070	10.548824	10.569993	10.573139	10.573736	10.574016
0.0100	10.538166	10.560141	10.562958	10.563590	10.563828
0.0200	10.516667	10.538791	10.541952	10.542571	10.542722
0.0500	10.488360	10.510278	10.513415	10.513987	10.514193
0.0700	10.478400	10.500393	10.503555	10.504133	10.504337
0.1000	10.468777	10.490785	10.493937	10.494518	10.494723
0.3000	10.447353	10.469380	10.472520	10.473103	10.473307
0.5000	10.442002	10.463991	10.467132	10.467714	10.467917
0.7000	10.439715	10.461697	10.464838	10.465419	10.465623
1.0000	10.438086	10.460064	10.463204	10.463786	10.463989
5.0000	10.435651	10.457621	10.460760	10.461341	10.461545
7.0000	10.435543	10.457513	10.460652	10.461233	10.461437
10.0000	10.435474	10.457444	10.460582	10.461164	10.461367
+∞ exact	10.461538	10.461538	10.461538	10.461538	10.461538

Table.E.3.13 Influence of M on degree of relaxation $V_y(ct/a^2)$
(impermeable case, $N = 50$, $\nu = 0.49$, $\nu_\infty = 0.5$)

ct/a^2	$M = 4$	$M = 6$	$M = 8$	$M = 10$
0.0001	0.26449579	0.08155928	0.06083797	0.05826460
0.0002	0.26353478	0.08516266	0.06413471	0.06212919
0.0005	0.28554258	0.10341568	0.08135560	0.08012690
0.0007	0.29920325	0.11747430	0.09541944	0.09358741
0.0010	0.31495336	0.13346466	0.11135277	0.11000964
0.0020	0.35546643	0.17425083	0.15213229	0.15087201
0.0050	0.43322578	0.25206375	0.22998480	0.22880696
0.0070	0.47011948	0.28892207	0.26681682	0.26560116
0.0100	0.51494247	0.33368322	0.31158200	0.31034881
0.0200	0.62029445	0.43887466	0.41677961	0.41555685
0.0500	0.79479200	0.61321235	0.59114683	0.58992547
0.0700	0.86562186	0.68404180	0.66199112	0.66076761
0.1000	0.94071984	0.75921839	0.73719281	0.73597020
0.3000	1.12391293	0.94361979	0.92172903	0.92051494
0.5000	1.16610909	0.98677450	0.96495020	0.96373844
0.7000	1.18171513	1.00286865	0.98107594	0.97986555
1.0000	1.19166529	1.01319134	0.99142230	0.99021292
5.0000	1.20427620	1.02669168	1.00497341	1.00376558
7.0000	1.20474792	1.02723038	1.00551617	1.00430858
10.0000	1.20501626	1.02757275	1.00586236	1.00465488

Table.E.3.14 Influence of M on total central moment $\frac{M_y(ct/a^2)}{a^3\mu\Omega_y}$
(impermeable case, $N = 50$, $\nu = 0.49$, $\nu_\infty = 0.5$)

ct/a^2	$M = 4$	$M = 6$	$M = 8$	$M = 10$
0 exact	10.666667	10.666667	10.666667	10.666667
0.0001	10.615331	10.653602	10.658441	10.658729
0.0002	10.612608	10.649197	10.653511	10.653922
0.0005	10.608094	10.645453	10.649978	10.650230
0.0007	10.605292	10.642569	10.647093	10.647469
0.0010	10.602061	10.639289	10.643825	10.644101
0.0020	10.593750	10.630923	10.635460	10.635719
0.0050	10.577800	10.614961	10.619490	10.619732
0.0070	10.570232	10.607401	10.611935	10.612184
0.0100	10.561037	10.598219	10.602752	10.603005
0.0200	10.539427	10.576641	10.581173	10.581424
0.0500	10.503632	10.540880	10.545406	10.545656
0.0700	10.489103	10.526350	10.530874	10.531125
0.1000	10.473698	10.510930	10.515448	10.515698
0.3000	10.436120	10.473104	10.477594	10.477843
0.5000	10.427465	10.464251	10.468728	10.468977
0.7000	10.424264	10.460950	10.465420	10.465669
1.0000	10.422223	10.458833	10.463298	10.463546
5.0000	10.419636	10.456063	10.460518	10.460766
7.0000	10.419539	10.455953	10.460407	10.460655
10.0000	10.419484	10.455883	10.460336	10.460584
$+\infty$ exact	10.461538	10.461538	10.461538	10.461538

Table.E.3.15 Influence of N on degree of relaxation $V_p(ct/a^2)$
(permeable case, $M = 10$, $\nu = 0.49$, $\nu_u = 0.5$)

ct/a^2	$N = 10$	$N = 25$	$N = 50$	$N = 75$	$N = 100$
0.0001	-0.27699384	0.057777981	0.03869614	0.03535700	0.03415911
0.0002	0.05431500	0.08153184	0.06212919	0.06594820	0.05188536
0.0005	0.19758457	0.09233879	0.08012690	0.07761099	0.07372644
0.0007	0.22285932	0.11447835	0.09358741	0.09038691	0.08931589
0.0010	0.23958910	0.12383932	0.11000964	0.10758409	0.10554756
0.0020	0.28181535	0.16645427	0.15087201	0.14798900	0.14704490
0.0050	0.35374877	0.24424033	0.22880696	0.22586119	0.22481085
0.0070	0.38690317	0.28100783	0.26560116	0.26275256	0.26175958
0.0100	0.43127358	0.32531130	0.31034881	0.30754277	0.30655634
0.0200	0.53575522	0.43060344	0.41555685	0.41275388	0.41178292
0.0500	0.71029067	0.60495079	0.58992547	0.58712828	0.58615643
0.0700	0.78148401	0.67586720	0.66076761	0.65797007	0.65699095
0.1000	0.85713851	0.75112444	0.73597020	0.73317003	0.73218536
0.3000	1.04282701	0.93580258	0.92051494	0.91768342	0.91669184
0.5000	1.08619034	0.97904575	0.96373844	0.96090454	0.95991242
0.7000	1.10232842	0.99517405	0.97986555	0.97703129	0.97603893
1.0000	1.11266625	1.00551927	0.99021292	0.98737836	0.98638612
5.0000	1.12617207	1.01906586	1.00376558	1.00093186	0.99994028
7.0000	1.12671292	1.01960802	1.00430858	1.00147486	1.00048316
10.0000	1.12705743	1.01995540	1.00465488	1.00182152	1.00082970

Table.E.3.16 Influence of N on total central moment $\frac{M_y(ct/a^2)}{a^3 \mu \Omega_y}$
(permeable case, $M = 10$, $\nu = 0.49$, $\nu_u = 0.5$)

ct/a^2	$N = 10$	$N = 25$	$N = 50$	$N = 75$	$N = 100$
0 exact	10.666667	10.666667	10.666667	10.666667	
0.0001	10.723486	10.654814	10.658729	10.659414	10.659660
0.0002	10.655525	10.649942	10.653922	10.653139	10.656024
0.0005	10.626136	10.647725	10.650230	10.650746	10.651543
0.0007	10.620952	10.643184	10.647469	10.648126	10.648345
0.0010	10.617520	10.641264	10.644101	10.644598	10.645016
0.0020	10.608858	10.632522	10.635719	10.636310	10.636504
0.0050	10.594103	10.616566	10.619732	10.620336	10.620552
0.0070	10.587302	10.609024	10.612184	10.612769	10.612972
0.0100	10.578200	10.599936	10.603005	10.603581	10.603783
0.0200	10.556768	10.578338	10.581424	10.581999	10.582198
0.0500	10.520966	10.542574	10.545656	10.546230	10.546429
0.0700	10.506362	10.528027	10.531125	10.531698	10.531899
0.1000	10.490843	10.512590	10.515698	10.516273	10.516475
0.3000	10.452753	10.474707	10.477843	10.478424	10.478627
0.5000	10.443858	10.465837	10.468977	10.469558	10.469762
0.7000	10.440548	10.462528	10.465669	10.466250	10.466454
1.0000	10.438427	10.460406	10.463546	10.464128	10.464331
5.0000	10.435657	10.457628	10.460766	10.461347	10.461551
7.0000	10.435546	10.457516	10.460655	10.461236	10.461439
10.0000	10.435475	10.457445	10.460584	10.461165	10.461368
+∞ exact	10.461538	10.461538	10.461538	10.461538	10.461538

Fig. E.3.5 Influence of M on the numerical results of degree of relaxation of central moment

Fig. E.3.6 Influence of N on the numerical results of degree of relaxation of central moment

Fig. E.3.7 Influence of M on the numerical results of degree of relaxation of central moment

Fig. E.3.8 Influence of N on the numerical results of degree of relaxation of central moment

E.4 In-Plane Translation of a Rigid Disc Inclusion

E.4.1 general

In the evaluation, the values of the integral parameters in equations (E.2.1) and the limits for time factor ct/a^2 in equation (E.2.2) are first selected. The two sets of the numbers M and N and the drained and undrained Poisson's ratios are selected as same as those in the evaluation for the central rotation of a rigid disc inclusion (Section E.3.3).

E.4.2 In-plane consolidation displacement of the inclusion

The numerical results for the in-plane translation $a\mu D_x(ct/a^2)/P_r$ and the associated degree of consolidation $U_x(ct/a^2)$ are presented in Tables (E.4.1) ~ (E.4.8) for the two sets of the numbers M and N under the conditions applicable to either a permeable or an impermeable rigid disc inclusion. The influence patterns of M and/or N on the in-plane translation $D_x(ct/a^2)$ are also plotted using the degree of consolidation translation and the logarithm time factor ct/a^2 in the Fig. E.4.1 ~ E.4.4.

By calculating the absolute and relative differences between the numerical results of $M = 4$ (6, or 8) and $M = 10$ in the first set and $N = 10$ (25, 50, or 75) and $N = 100$ in the second set, it is observed that the numerical results are progressively convergent with high stability for the long non-dimensional time interval $10^{-4} \leq ct/a^2 \leq 10^{+3}$. The absolute and relative differences indicate that each M has a very uniform influence on the numerical results at every time factor ct/a^2 in the entire range of $10^{-4} \leq ct/a^2 \leq 10^{+3}$ (see e.g., Fig.E.4.1 ~ Fig.E.4.4). The absolute and relative differences between the non-dimensional in-plane translations of $M = 8$ and $M = 10$ are less than 10^{-5} and 10^{-4} , respectively. In other words, the $M = 10$ gives very accurate results in the numerical inversion of Laplace transforms for the integral equations. In particular, the inequality (E.2.3) is also valid for a stable, convergent and accurate solution of the integral equations.

Table.E.4.1 Influence of M on degree of consolidation $U_x(ct/a^2)$
(permeable case, $N = 50$, $\nu = 0.49$, $\nu_u = 0.5$)

ct/a^2	$M = 4$	$M = 6$	$M = 8$	$M = 10$
0.0001	-0.57632756	-0.05469878	0.01044452	0.01413967
0.0002	-0.58323944	-0.05662397	0.00853121	0.01191589
0.0005	-0.57866228	-0.05142980	0.01348846	0.01707276
0.0007	-0.57484460	-0.04670501	0.01824821	0.02172275
0.0010	-0.57153958	-0.04297885	0.02206009	0.02571705
0.0020	-0.56172693	-0.03217565	0.03284745	0.0346146
0.0050	-0.54088211	-0.01068280	0.05436517	0.05800326
0.0070	-0.53036046	-0.00003839	0.06501801	0.06864013
0.0100	-0.51698762	0.01339053	0.07844071	0.08206703
0.0500	-0.41651607	0.11381007	0.17887545	0.18249618
0.0700	-0.38392493	0.14632378	0.21139482	0.21501614
0.1000	-0.34363088	0.18652897	0.25159752	0.25521952
0.5000	-0.09627428	0.43358323	0.49869159	0.50231713
1.0000	0.01096146	0.54124228	0.60641193	0.61004055
5.0000	0.19701979	0.72853476	0.79381567	0.79745036
10.0000	0.25179496	0.78381217	0.84913707	0.85277379
50.0000	0.33052662	0.86349422	0.92889440	0.93253464
100.0000	0.34976187	0.88304675	0.94846976	0.95211101
500.0000	0.37555414	0.90935302	0.97481084	0.97845352
1000.0000	0.38166234	0.91560787	0.98107505	0.98471814

Table.E.4.2 Influence of M on in-plane translation $\frac{a\mu D_x(ct/a^2)}{p_x}$
(permeable case, $N = 50$, $\nu = 0.49$, $\nu_u = 0.5$)

ct/a^2	$M = 4$	$M = 6$	$M = 8$	$M = 10$
0 exact	0.09375000	0.09375000	0.09375000	0.09375000
0.0001	0.09339686	0.09371648	0.09375640	0.09375866
0.0002	0.09339262	0.09371530	0.09375523	0.09375730
0.0005	0.09339543	0.09371849	0.09375827	0.09376046
0.0007	0.09339777	0.09372138	0.09376118	0.09376331
0.0010	0.09339979	0.09372367	0.09376352	0.09376576
0.0020	0.09340581	0.09373029	0.09377013	0.09377234
0.0050	0.09341858	0.09374345	0.09378331	0.09378554
0.0070	0.09342502	0.09374998	0.09378984	0.09379206
0.0100	0.09343322	0.09375821	0.09379806	0.09380029
0.0500	0.09349478	0.09381974	0.09385961	0.09386182
0.0700	0.09351475	0.09383966	0.09387953	0.09388175
0.1000	0.09353944	0.09386430	0.09390417	0.09390639
0.5000	0.09369101	0.09401568	0.09405557	0.09405779
1.0000	0.09375672	0.09408165	0.09412158	0.09412380
5.0000	0.09387072	0.09419641	0.09423641	0.09423863
10.0000	0.09390429	0.09423028	0.09427031	0.09427253
50.0000	0.09395253	0.09427910	0.09431918	0.09432141
100.0000	0.09396432	0.09429108	0.09433117	0.09433340
200.0000	0.09397269	0.09429961	0.09433971	0.09434194
1000.0000	0.09398386	0.09431103	0.09435115	0.09435338
$+\infty$ exact	0.09436275	0.09436275	0.09436275	0.09436275

Table.E.4.3 Influence of N on degree of consolidation $U_r(ct/a^2)$
(permeable case, $M = 10$, $\nu = 0.49$, $\nu_u = 0.5$)

ct/a^2	$N = 10$	$N = 25$	$N = 50$	$N = 75$	$N = 100$
0.0001	-1.35955107	-0.08343199	0.01413967	0.00666742	0.00851347
0.0002	-0.67789322	0.00423928	0.01191589	0.01052726	0.01119331
0.0005	-0.14970990	0.01230675	0.01707276	0.01776414	0.01798662
0.0007	-0.05639257	0.02150864	0.02172275	0.02136104	0.02196473
0.0010	-0.00251295	0.02355306	0.02571705	0.02563675	0.02582696
0.0020	0.02905257	0.03554962	0.03646146	0.03661959	0.03663778
0.0050	0.05195477	0.05754820	0.05800326	0.05808425	0.05806480
0.0070	0.06487686	0.06831945	0.06864013	0.06875579	0.06877026
0.0100	0.07851221	0.08172328	0.08206703	0.08214922	0.08217429
0.0500	0.18113822	0.18233465	0.18249618	0.18252940	0.18254139
0.0700	0.21388859	0.21488152	0.21501614	0.21504284	0.21505427
0.1000	0.25424477	0.25509563	0.25521952	0.25524172	0.25525135
0.5000	0.50201875	0.50228119	0.50231713	0.50232458	0.50232667
1.0000	0.60989606	0.61002314	0.61004055	0.61004436	0.61004537
5.0000	0.79742664	0.79744738	0.79745036	0.79745096	0.79745108
10.0000	0.85276401	0.85277253	0.85277379	0.85277414	0.85277420
50.0000	0.93253368	0.93253446	0.93253464	0.93253469	0.93253469
100.0000	0.95211065	0.95211095	0.95211101	0.95211095	0.95211101
500.0000	0.97845346	0.97845352	0.97845352	0.97845352	0.97845352
1000.0000	0.98471814	0.98471814	0.98471814	0.98471814	0.98471814

Table.E.4.4 Influence of N on in-plane translation $\frac{a\mu D_r(ct/a^2)}{p_r}$
(permeable case, $M = 10$, $\nu = 0.49$, $\nu_u = 0.5$)

ct/a^2	$N = 10$	$N = 25$	$N = 50$	$N = 75$	$N = 100$
0 exact	0.09375000	0.09375000	0.09375000	0.09375000	0.09375000
0.0001	0.09291694	0.09369888	0.09375866	0.09375409	0.09375522
0.0002	0.09333462	0.09375260	0.09375730	0.09375645	0.09375686
0.0005	0.09365827	0.09375754	0.09376046	0.09376089	0.09376102
0.0007	0.09371545	0.09376318	0.09376331	0.09376309	0.09376346
0.0010	0.09374846	0.09376443	0.09376576	0.09376571	0.09376583
0.0020	0.09376780	0.09377178	0.09377234	0.09377244	0.09377245
0.0050	0.09378184	0.09378526	0.09378554	0.09378559	0.09378558
0.0070	0.09378975	0.09379186	0.09379206	0.09379213	0.09379214
0.0100	0.09379811	0.09380008	0.09380029	0.09380034	0.09380035
0.0500	0.09386099	0.09386172	0.09386182	0.09386184	0.09386185
0.0700	0.09388106	0.09388167	0.09388175	0.09388177	0.09388177
0.1000	0.09390579	0.09390631	0.09390639	0.09390640	0.09390640
0.5000	0.09405761	0.09405777	0.09405779	0.09405780	0.09405780
1.0000	0.09412371	0.09412379	0.09412380	0.09412380	0.09412380
5.0000	0.09423862	0.09423863	0.09423863	0.09423863	0.09423863
10.0000	0.09427253	0.09427253	0.09427253	0.09427253	0.09427253
50.0000	0.09432141	0.09432141	0.09432141	0.09432141	0.09432141
100.0000	0.09433340	0.09433340	0.09433340	0.09433340	0.09433340
500.0000	0.09434954	0.09434954	0.09434954	0.09434954	0.09434954
1000.0000	0.09435338	0.09435338	0.09435338	0.09435338	0.09435338
+∞ exact	0.09436275	0.09436275	0.09436275	0.09436275	0.09436275

Table.E.4.5 Influence of M on degree of consolidation $U_z(ct/a^2)$
(impermeable case, $N = 50$, $\nu = 0.49$, $\nu_u = 0.5$)

ct/a^2	$M = 4$	$M = 6$	$M = 8$	$M = 10$
0.0001	-0.59266835	-0.07387308	-0.01024481	-0.00998385
0.0002	-0.58712208	-0.06100935	0.00350674	0.00939085
0.0005	-0.57974708	-0.04964869	0.01663191	0.02425475
0.0007	-0.57676929	-0.04901411	0.01661856	0.02048537
0.0010	-0.57419264	-0.04543022	0.01973475	0.02351834
0.0020	-0.56716102	-0.03797553	0.02779530	0.03241578
0.0050	-0.54914421	-0.01908145	0.04592097	0.04966964
0.0070	-0.53906149	-0.00864032	0.05640233	0.05997799
0.0100	-0.52793288	0.00239257	0.06729802	0.07088275
0.0500	-0.44031802	0.09009169	0.15517680	0.15879774
0.0700	-0.41178885	0.11853117	0.18359505	0.18721673
0.1000	-0.37615672	0.15408774	0.21914902	0.22276923
0.5000	-0.14199428	0.38781351	0.45290357	0.45652834
1.0000	-0.02332352	0.50665736	0.57179028	0.57541770
5.0000	0.19102715	0.72231078	0.78757805	0.79121214
10.0000	0.24956174	0.78144902	0.84676611	0.85040265
50.0000	0.33035529	0.86327380	0.92867142	0.93231153
100.0000	0.34973013	0.88296980	0.94839048	0.95203161
500.0000	0.37556839	0.90934688	0.97480369	0.97844636
1000.0000	0.38167256	0.91560590	0.98107255	0.98471558

Table.E.4.6 Influence of M on in-plane translation $\frac{\sigma \mu D_z(ct/a^2)}{p_z}$
(impermeable case, $N = 50$, $\nu = 0.49$, $\nu_u = 0.5$)

ct/a^2	$M = 4$	$M = 6$	$M = 8$	$M = 10$
0 exact	0.09375000	0.09375000	0.09375000	0.09375000
0.0001	0.09338685	0.09370474	0.09374372	0.09374388
0.0002	0.09339024	0.09371262	0.09375215	0.09375575
0.0005	0.09339476	0.09371958	0.09376019	0.09376486
0.0007	0.09339659	0.09371997	0.09376018	0.09376255
0.0010	0.09339817	0.09372216	0.09376209	0.09376441
0.0020	0.09340248	0.09372673	0.09376703	0.09376986
0.0050	0.09341352	0.09373831	0.09377814	0.09378044
0.0070	0.09341969	0.09374471	0.09378456	0.09378675
0.0100	0.09342651	0.09375147	0.09379124	0.09379343
0.0500	0.09348020	0.09380520	0.09384508	0.09384730
0.0700	0.09349768	0.09382263	0.09386250	0.09386472
0.1000	0.09351951	0.09384442	0.09388428	0.09388650
0.5000	0.09366299	0.09398763	0.09402751	0.09402974
1.0000	0.09373571	0.09406045	0.09410036	0.09410258
5.0000	0.09386705	0.09419259	0.09423259	0.09423481
10.0000	0.09390292	0.09422883	0.09426885	0.09427108
50.0000	0.09395242	0.09427897	0.09431904	0.09432127
100.0000	0.09396430	0.09429104	0.09433112	0.09433335
500.0000	0.09398013	0.09430720	0.09434731	0.09434954
1000.0000	0.09398387	0.09431103	0.09435115	0.09435338
$+\infty$ exact	0.09436275	0.09436275	0.09436275	0.09436275

Table.E.4.7 Influence of N on degree of consolidation $U_r(ct/a^2)$
(impermeable case, $M = 10$, $\nu = 0.49$, $\nu_u = 0.5$)

ct/a^2	$N = 10$	$N = 25$	$N = 50$	$N = 75$	$N = 100$
0.0001	-1.32022369	-0.10520060	-0.00998385	0.00307348	-0.01449024
0.0002	-0.71806914	0.00746165	0.00939085	0.01460677	0.01642363
0.0005	-0.16735294	0.00893709	0.02425475	0.01338490	0.01461596
0.0007	-0.08040696	0.02092799	0.02048537	0.01920454	0.01744417
0.0010	0.01102408	0.02198030	0.02351834	0.02051149	0.02443713
0.0020	0.03558056	0.02840145	0.03241578	0.03031438	0.03196874
0.0050	0.04717933	0.05146394	0.04966964	0.05051522	0.05035975
0.0070	0.05716381	0.05824209	0.05997799	0.05982552	0.05978260
0.0100	0.07004559	0.07180034	0.07088275	0.07139525	0.07114945
0.0500	0.15808190	0.15861981	0.15879774	0.15875469	0.15878937
0.0700	0.18642277	0.18718532	0.18721673	0.18728387	0.18728034
0.1000	0.22207841	0.22266299	0.22276923	0.22278614	0.22278978
0.5000	0.45630944	0.45649910	0.45652834	0.45653155	0.45653337
1.0000	0.57531303	0.57540458	0.57541770	0.57542044	0.57542109
5.0000	0.79119968	0.79121053	0.79121214	0.79121244	0.79121244
10.0000	0.85039824	0.85040206	0.85040265	0.85040283	0.85040289
50.0000	0.93231112	0.93231142	0.93231153	0.93232125	0.93231153
100.0000	0.95203149	0.95203161	0.95203161	0.95203161	0.95203161
500.0000	0.97844636	0.97844636	0.97844636	0.97844636	0.97844636
1000.0000	0.98471558	0.98471558	0.98471558	0.98471558	0.98471558

Table.E.4.8 Influence of N on in-plane translation $\frac{u\mu D_r(ct/a^2)}{p_r}$
(impermeable case, $M = 10$, $\nu = 0.49$, $\nu_u = 0.5$)

ct/a^2	$N = 10$	$N = 25$	$N = 50$	$N = 75$	$N = 100$
0 exact	0.09375000	0.09375000	0.09375000	0.09375000	0.09375000
0.0001	0.09394104	0.09368554	0.09374388	0.09375188	0.09374112
0.0002	0.09331001	0.09375457	0.09375575	0.09375895	0.09376006
0.0005	0.09364746	0.09375548	0.09376486	0.09375820	0.09375896
0.0007	0.09370073	0.09376282	0.09376255	0.09376177	0.09376069
0.0010	0.09375676	0.09376347	0.09376441	0.09376257	0.09376497
0.0020	0.09377180	0.09376740	0.09376986	0.09376858	0.09376959
0.0050	0.09377891	0.09378153	0.09378044	0.09378095	0.09378086
0.0070	0.09378503	0.09378569	0.09378675	0.09378666	0.09378663
0.0100	0.09379292	0.09379400	0.09379343	0.09379375	0.09379360
0.0500	0.09384686	0.09384719	0.09384730	0.09384728	0.09384730
0.0700	0.09386423	0.09386470	0.09386472	0.09386476	0.09386476
0.1000	0.09388608	0.09388644	0.09388650	0.09388651	0.09388651
0.5000	0.09402960	0.09402972	0.09402974	0.09402974	0.09402974
1.0000	0.09410252	0.09410258	0.09410258	0.09410259	0.09410259
5.0000	0.09423480	0.09423481	0.09423481	0.09423481	0.09423481
10.0000	0.09427108	0.09427108	0.09427108	0.09427108	0.09427108
50.0000	0.09432127	0.09432127	0.09432127	0.09432128	0.09432127
100.0000	0.09433335	0.09433335	0.09433335	0.09433335	0.09433335
500.0000	0.09434954	0.09434954	0.09434954	0.09434954	0.09434954
1000.0000	0.09435338	0.09435338	0.09435338	0.09435338	0.09435338
$+\infty$ exact	0.09436275	0.09436275	0.09436275	0.09436275	0.09436275

5 1 of/de 5

PM-1 3 1/2" x 4" PHOTOGRAPHIC MICROCOPY TARGET
NBS 1010a ANSI/ISO #2 EQUIVALENT

PRECISIONSM RESOLUTION TARGETS

PIONEERS IN METHYLENE BLUE TESTING SINCE 1974

1000 COUNTY ROAD 8, BETHLEHEM, PA 18015 USA
TEL: 610 485 1200 FAX: 610 485 1201 TOLL: 800 551 2896

Fig. E.4.1 Influence of M on the numerical results of degree of consolidation of in-plane translation

Fig. E.4.2 Influence of N on the numerical results of degree of consolidation of in-plane translation

Fig. E.4.3 Influence of M on the numerical results of degree of consolidation of in-plane translation

Fig. E.4.4 Influence of N on the numerical results of degree of consolidation of in-plane translation

E.4.3 In-plane relaxation force in the inclusion

The numerical results for the in-plane force $P_x(ct/a^2)/(a\mu D_x)$ and the associated degree of relaxation $V_x(ct/a^2)$ are presented in Tables (E.4.9) ~ (E.4.16) for the two sets of the numbers M and N under the conditions applicable to either a permeable or an impermeable rigid disc inclusion. The patterns of influence of M and/or N on the in-plane force $P_x(ct/a^2)$ are also plotted using the degree of relaxation and the logarithm time factor ct/a^2 (see Fig. E.4.5 ~ E.4.8).

By calculating the absolute and relative differences between the numerical results of $M = 4$ (6, or 8) and $M = 10$ in the first set and $N = 10$ (25, 50, or 75) and $N = 100$ in the second set, it is observed that the numerical results are progressively convergent with high stability in the long non-dimensional time interval of $10^{-4} \leq ct/a^2 \leq 10^{+3}$. Similar conclusions can be drawn from these test results about the influences of M and N on the convergence and accuracy of the time-dependent in-plane force. In particular, the inequality (E.2.1) is also applicable in choosing the value of N for accurately solving the integral equations.

E.4.4 Conclusions

The numerical scheme adopted in this thesis can provide very stable, convergent and accurate results for the in-plane translation of a permeable or an impermeable rigid disc inclusion under the action of a constant in-plane load or a constant in-plane displacement loading by properly selecting the numbers M and N . The values of M and N in equations (E.2.4) and the integral parameters in equations (E.2.1) are suggested for examining the consolidation response associated with the in-plane translation of a rigid disc inclusion.

Table.E.4.9 Influence of M on degree of relaxation $V_z(ct/a^2)$
(permeable case, $N = 50$, $\nu = 0.49$, $\nu_u = 0.5$)

ct/a^2	$M = 4$	$M = 6$	$M = 8$	$M = 10$
0.0001	0.62736911	0.08367895	0.01828515	0.01487504
0.0002	0.62024230	0.08158918	0.01624308	0.01262975
0.0005	0.62485409	0.08679479	0.02115225	0.01747330
0.0007	0.62869561	0.09155089	0.02594617	0.02217555
0.0010	0.63201874	0.09529945	0.02977790	0.02621868
0.0020	0.64189035	0.10616835	0.04063124	0.03698401
0.0050	0.66285980	0.12779005	0.06227794	0.05865158
0.0070	0.67344296	0.13849676	0.07299317	0.06935103
0.0100	0.68689245	0.15200266	0.08649281	0.08285472
0.0500	0.78788137	0.25293961	0.18744506	0.18380135
0.0700	0.82061750	0.28560308	0.22010919	0.21646607
0.1000	0.86107475	0.32596627	0.26047492	0.25683245
0.5000	1.10897350	0.57356501	0.50811386	0.50447494
1.0000	1.21612036	0.68114746	0.61575097	0.61211514
5.0000	1.40156031	0.86782026	0.80254215	0.79891235
10.0000	1.45605671	0.92281908	0.85758489	0.85395718
50.0000	1.53432310	1.00203335	0.93687427	0.93325001
100.0000	1.55343628	1.02146256	0.95632607	0.95270282
500.0000	1.57906234	1.04759991	0.98249793	0.97887617
1000.0000	1.58513093	1.05381417	0.98872161	0.98510021

Table.E.4.10 Influence of M on total in-plane force $\frac{P_x(ct/a^2)}{a\mu D_x}$
(permeable case, $N = 50$, $\nu = 0.49$, $\nu_u = 0.5$)

ct/a^2	$M = 4$	$M = 6$	$M = 8$	$M = 10$
0 exact	10.666667	10.666667	10.666667	10.666667
0.0001	10.623213	10.660871	10.665400	10.665636
0.0002	10.623706	10.661015	10.665542	10.665792
0.0005	10.623387	10.660655	10.665202	10.665456
0.0007	10.623121	10.660325	10.664870	10.665131
0.0010	10.622890	10.660066	10.664604	10.664851
0.0020	10.622207	10.659313	10.663852	10.664105
0.0050	10.620754	10.657815	10.662353	10.662604
0.0070	10.620021	10.657074	10.661611	10.661863
0.0100	10.619090	10.656138	10.660676	10.660928
0.0500	10.612095	10.649147	10.653683	10.653936
0.0700	10.609827	10.646885	10.651421	10.651673
0.1000	10.607025	10.644089	10.648625	10.648877
0.5000	10.589855	10.626939	10.631473	10.631725
1.0000	10.582433	10.619488	10.624017	10.624269
5.0000	10.569589	10.606558	10.611079	10.611331
10.0000	10.565814	10.602748	10.607267	10.607518
20.0000	10.562984	10.599888	10.604404	10.604655
100.0000	10.559069	10.595916	10.600428	10.600679
500.0000	10.557294	10.594106	10.598615	10.598866
1000.0000	10.556874	10.593675	10.598184	10.598435
$+\infty$ exact	10.597403	10.597403	10.597403	10.597403

Table.E.4.11 Influence of N on degree of relaxation $V_x(ct/a^2)$
(permeable case, $M = 10$, $\nu = 0.49$, $\nu_u = 0.5$)

ct/a^2	$N = 10$	$N = 25$	$N = 50$	$N = 75$	$N = 100$
0.0001	-1.33995318	-0.07881742	0.01487504	-0.00330205	0.00909177
0.0002	-0.66632432	0.00557586	0.01262975	0.01093594	0.01160106
0.0005	-0.14557424	0.01273520	0.01747330	0.01816720	0.01839771
0.0007	-0.05444947	0.02201878	0.02217555	0.02178605	0.02248672
0.0010	-0.00113596	0.02432007	0.02621868	0.02608727	0.02628461
0.0020	0.02966828	0.03661661	0.03698401	0.03716695	0.03715595
0.0050	0.05262008	0.05820490	0.05865158	0.05873212	0.05871229
0.0070	0.06562000	0.06903762	0.06935103	0.06946661	0.06948056
0.0100	0.07932376	0.08251330	0.08285472	0.08293656	0.08296159
0.0500	0.18246222	0.18364112	0.18380135	0.18383434	0.18384624
0.0700	0.21534860	0.21633279	0.21646607	0.21649253	0.21650392
0.1000	0.25586620	0.25670958	0.25683245	0.25685447	0.25686404
0.5000	0.50418067	0.50443953	0.50447494	0.50448233	0.50448436
1.0000	0.61197305	0.61209810	0.61211514	0.61211896	0.61211991
5.0000	0.79888904	0.79890943	0.79891235	0.79891294	0.79891306
10.0000	0.85394764	0.85395592	0.85395718	0.85395753	0.85395759
50.0000	0.93324906	0.93324989	0.93325001	0.93325007	0.93325007
100.0000	0.95270252	0.95270276	0.95270282	0.95270282	0.95270288
500.0000	0.97887611	0.97887617	0.97887617	0.97887617	0.97887617
1000.0000	0.98510021	0.98510021	0.98510021	0.98510021	0.98510021

Table.E.4.12 Influence of N on total in-plane force $\frac{P_x(ct/a^2)}{a\mu D_x}$
(permeable case, $M = 10$, $\nu = 0.49$, $\nu_u = 0.5$)

ct/a^2	$N = 10$	$N = 25$	$N = 50$	$N = 75$	$N = 100$
0 exact	10.666667	10.666667	10.666667	10.666667	10.666667
0.0001	10.759477	10.672126	10.665636	10.666895	10.666037
0.0002	10.712819	10.666280	10.665792	10.665909	10.665863
0.0005	10.676750	10.665785	10.665456	10.665408	10.665392
0.0007	10.670438	10.665142	10.665131	10.665158	10.665109
0.0010	10.666745	10.664982	10.664851	10.664860	10.664846
0.0020	10.664612	10.664130	10.664105	10.664092	10.664093
0.0050	10.663022	10.662635	10.662604	10.662599	10.662600
0.0070	10.662122	10.661885	10.661863	10.661855	10.661854
0.0100	10.661172	10.660951	10.660928	10.660922	10.660920
0.0500	10.654029	10.653947	10.653936	10.653934	10.653933
0.0700	10.651751	10.651683	10.651673	10.651672	10.651671
0.1000	10.648944	10.648886	10.648877	10.648876	10.648875
0.5000	10.631745	10.631727	10.631725	10.631724	10.631724
1.0000	10.624279	10.624270	10.624269	10.624269	10.624269
5.0000	10.611332	10.611331	10.611331	10.611331	10.611331
10.0000	10.607519	10.607518	10.607518	10.607518	10.607518
50.0000	10.602026	10.602026	10.602026	10.602026	10.602026
100.0000	10.600679	10.600679	10.600679	10.600679	10.600679
500.0000	10.598866	10.598866	10.598866	10.598866	10.598866
1000.0000	10.598435	10.598435	10.598435	10.598435	10.598435
+∞ exact	10.597403	10.597403	10.597403	10.597403	10.597403

Table.E.4.13 Influence of M on degree of relaxation $V_z(ct/a^2)$
(impermeable case, $N = 50$, $\nu = 0.49$, $\nu_u = 0.5$)

ct/a^2	$M = 4$	$M = 6$	$M = 8$	$M = 10$
0.0001	0.61043584	0.06383584	-0.00197199	-0.01146461
0.0002	0.61638516	0.07720684	0.01168556	0.00885192
0.0005	0.62376207	0.08858740	0.02431060	0.02079503
0.0007	0.62675828	0.08921921	0.02429616	0.01996507
0.0010	0.62934828	0.09282951	0.02744383	0.02394691
0.0020	0.63642156	0.10033157	0.03554508	0.03330597
0.0050	0.65454787	0.11934065	0.05378295	0.05031647
0.0070	0.66469079	0.12984414	0.06432746	0.06064418
0.0100	0.67588413	0.14094126	0.07528759	0.07163020
0.0500	0.76396620	0.22910838	0.16363370	0.15999095
0.0700	0.79263008	0.25768223	0.19218618	0.18854523
0.1000	0.82841849	0.29339480	0.22789615	0.22425190
0.5000	1.06322849	0.52776879	0.46229917	0.45865953
1.0000	1.18191171	0.64662910	0.58120275	0.57756567
5.0000	1.39562249	0.86165136	0.79635960	0.79272932
10.0000	1.45384824	0.92048120	0.85523927	0.85161132
50.0000	1.53415418	1.00181580	0.93665403	0.93302965
100.0000	1.55340528	1.02138662	0.95624787	0.95262450
500.0000	1.57907653	1.04759383	0.98249090	0.97886914
1000.0000	1.58514106	1.05381227	0.98871911	0.98509771

Table.E.4.14 Influence of M on total in-plane force $\frac{P_z(ct/a^2)}{a\mu D_z}$
(impermeable case, $N = 50$, $\nu = 0.49$, $\nu_u = 0.5$)

ct/a^2	$M = 4$	$M = 6$	$M = 8$	$M = 10$
0 exact	10.666667	10.666667	10.666667	10.666667
0.0001	10.624385	10.662245	10.666803	10.667461
0.0002	10.623973	10.661319	10.665857	10.666054
0.0005	10.623462	10.660531	10.664983	10.665226
0.0007	10.623255	10.660487	10.664984	10.665284
0.0010	10.623075	10.660237	10.664766	10.665008
0.0020	10.622586	10.659717	10.664205	10.664360
0.0050	10.621330	10.658401	10.662941	10.663182
0.0070	10.620627	10.657673	10.662211	10.662466
0.0100	10.619852	10.656905	10.661452	10.661705
0.0500	10.613751	10.650798	10.655333	10.655585
0.0700	10.611766	10.648819	10.653355	10.653607
0.1000	10.609287	10.646345	10.650882	10.651134
0.5000	10.593023	10.630111	10.634646	10.634898
1.0000	10.584803	10.621879	10.626410	10.626662
5.0000	10.570000	10.606985	10.611508	10.611759
10.0000	10.565967	10.602910	10.607429	10.607681
50.0000	10.560405	10.597277	10.601790	10.602041
100.0000	10.559071	10.595921	10.600433	10.600684
500.0000	10.557293	10.594106	10.598615	10.598866
1000.0000	10.556873	10.593675	10.598184	10.598435
+∞ exact	10.597403	10.597403	10.597403	10.597403

Table.E.4.15 Influence of N on degree of relaxation $V_z(ct/a^2)$
(impermeable case, $M = 10$, $\nu = 0.49$, $\nu_u = 0.5$)

ct/a^2	$N = 10$	$N = 25$	$N = 50$	$N = 75$	$N = 100$
0.0001	-1.31942463	-0.12540595	-0.01146461	0.00253583	-0.01541573
0.0002	-0.71778005	0.00936909	0.00885192	0.01418288	0.01322601
0.0005	-0.16673744	0.00914629	0.02079503	0.01290852	0.01408171
0.0007	-0.04801994	0.01986569	0.01996507	0.01841499	0.01700559
0.0010	0.01138519	0.02174670	0.02394691	0.02335936	0.02003167
0.0020	0.03554878	0.02889007	0.03330597	0.03083809	0.03249540
0.0050	0.04856896	0.05117515	0.05031647	0.05112208	0.05104182
0.0070	0.05777974	0.05885784	0.06064418	0.06052436	0.06043953
0.0100	0.07095301	0.07365803	0.07163020	0.07214002	0.07187607
0.0500	0.15929723	0.15981425	0.15999095	0.15994650	0.15998127
0.0700	0.18775877	0.18851334	0.18854523	0.18861137	0.18860696
0.1000	0.22357458	0.22414680	0.22425190	0.22426873	0.22427221
0.5000	0.45844403	0.45863071	0.45865953	0.45866263	0.45866442
1.0000	0.57746321	0.57755286	0.57756567	0.57756841	0.57756907
5.0000	0.79271722	0.79272777	0.79272932	0.79272962	0.79272962
10.0000	0.85160708	0.85161078	0.85161132	0.85161150	0.85161155
50.0000	0.93302929	0.93302959	0.93302965	0.93303943	0.93302971
100.0000	0.95262438	0.95262450	0.95262450	0.95262450	0.95262450
500.0000	0.97886908	0.97886914	0.97886914	0.97886914	0.97886914
1000.0000	0.98509777	0.98509771	0.98509771	0.98509771	0.98509771

Table.E.4.16 Influence of N on total in-plane force $\frac{P_z(ct/a^2)}{a\mu D_z}$
(impermeable case, $M = 10$, $\nu = 0.49$, $\nu_u = 0.5$)

ct/a^2	$N = 10$	$N = 25$	$N = 50$	$N = 75$	$N = 100$
0 exact	10.666667	10.666667	10.666667	10.666667	10.666667
0.0001	10.758055	10.675353	10.667461	10.666491	10.667734
0.0002	10.716383	10.666018	10.666054	10.665684	10.665751
0.0005	10.678216	10.666033	10.665226	10.665773	10.665691
0.0007	10.669993	10.665291	10.665284	10.665391	10.665489
0.0010	10.665878	10.665160	10.665008	10.665049	10.665279
0.0020	10.664204	10.664666	10.664360	10.664531	10.664416
0.0050	10.663303	10.663122	10.663182	10.663126	10.663131
0.0070	10.662665	10.662590	10.662466	10.662475	10.662480
0.0100	10.661752	10.661565	10.661705	10.661670	10.661688
0.0500	10.655633	10.655597	10.655585	10.655588	10.655586
0.0700	10.653662	10.653609	10.653607	10.653603	10.653603
0.1000	10.651181	10.651141	10.651134	10.651133	10.651133
0.5000	10.634913	10.634900	10.634898	10.634898	10.634898
1.0000	10.626669	10.626663	10.626662	10.626662	10.626662
5.0000	10.611760	10.611759	10.611759	10.611759	10.611759
10.0000	10.607681	10.607681	10.607681	10.607681	10.607681
50.0000	10.602041	10.602041	10.602041	10.602041	10.602041
100.0000	10.600684	10.600684	10.600684	10.600684	10.600684
500.0000	10.598866	10.598866	10.598866	10.598866	10.598866
1000.0000	10.598435	10.598435	10.598435	10.598435	10.598435
+∞ exact	10.597403	10.597403	10.597403	10.597403	10.597403

Fig. E.4.5 Influence of M on the numerical results of degree of relaxation of in-plane force

Fig. E.4.6 Influence of N on the numerical results of degree of relaxation of in-plane force

Fig. E.4.7 Influence of M on the numerical results of degree of relaxation of in-plane force

Fig. E.4.8 Influence of N on the numerical results of degree of relaxation of in-plane force

E.5 Summary Conclusions

Based on the above results and discussions, it can be concluded that the numerical scheme and techniques adopted in this thesis provide highly stable and accurate solutions in time domain for the systems of Fredholm integral equations of the second kind in the Laplace transform domain.

In this numerical scheme, the most time consuming step is the evaluation of the infinite integrals in the kernel functions. The total computations involved in the calculation of the infinite integrals are equal to $(2M + 1)N(N + 1)$, $4(2M + 1)N(N + 1)$ or $7(2M + 1)N(N + 1)$ for one time factor according to the systems of the integral equations. If $M = 10$ and $N = 100$, for instance, the total computations are then equal to 212,100, 848,400, or 1,484,700. With the help of the approximation technique, such large numbers of numerical integration of the infinite integrals with rapidly oscillatory integrands require approximately 0.5, 2 or 3.5 hours CPU time to complete in the Sun SPARC work station 1+, respectively. For larger values of the time factor ct/a^2 , the CPU time becomes less due to infrequent oscillation of the integrands. Furthermore, the absence of material parameters in the infinite integrals greatly reduces the computation time and effort associated with the numerical integration.

END

2|5 0|5 193

FIN