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Finite Element Formulations for Thin Plate and Shallow Shell Structures

Jeyachandrabose Chinniah
B.Sc. (Madurai University), M.S. (I.I.T., Madras)

*A Thesis submitted to the Faculty of Graduate
Studies and Research in partial fulfillment
of the requirements for the degree
of Doctor of Philosophy in
Mechanical Engineering*

**DEPARTMENT OF MECHANICAL AND AERONAUTICAL ENGINEERING
CARLETON UNIVERSITY
OTTAWA, ONTARIO
CANADA**

October 1985

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The undersigned recommend to the Faculty of Graduate Studies and Research acceptance of the thesis;

*Finite Element Formulations for Thin Plate
and Shallow Shell Structures*

submitted by Jeyachandrabose Chinniah, in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mechanical Engineering.

Thesis Supervisor

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Department of Mechanical and
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Abstract

Finite element displacement type models for analysis of thin plates and shells are studied with the objective of identifying and reformulating some of the best existing elements, and deriving new elements, with improved efficiency of computation and accuracy. Such improved formulations would be particularly attractive for use in finite element analysis packages, including those for personal computers. Thus, in this thesis, the stiffness matrix for all the triangular elements considered is obtained in essentially closed form as a simple matrix triple product, and avoiding costly numerical integration.

The new developments in this thesis are:

- (1) A new eight-node Serendipity plane stress element free from spurious shear effects.
- (2) A method of constructing transition elements for plane stress analysis to analyze structures of varying stress gradients;
- (3) Efficient reformulation of the high precision (HP or T-18), discrete Kirchhoff triangle (DKT), and the hybrid stress (HSM or A-9) plate bending elements;
- (4) Development of new plate bending elements, two triangular (E-1 and E-2), one quadrilateral (IDKQ). These triangular elements are believed to be the best available at present considering simplicity, accuracy and efficiency of computation;
- (5) Development of laminated anisotropic plate/shallow shell elements based on the high precision, DKT and the new E-1 and E-2 triangular plate bending elements.

Numerical examples presented demonstrate the behavior and accuracy of the various formulations and allow comparison with other existing elements.

FORTRAN program listings are appended for all elements presented in the thesis.

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Nomenclature

A, B, C	-	Numerical universal matrices for plane triangular elements.
A', B', C'	-	Numerical universal matrices for transition elements.
C^*	-	diagonal numerical matrix.
D, D_b	-	elasticity matrix.
A, B, D	-	also effective membrane, membrane-bending coupling and bending stiffness matrices for laminate.
$\bar{A}, \bar{B}, \bar{C}$	-	transformed stiffness matrices for laminate.
B	-	also strain-displacement matrix.
$\bar{B}$	-	least squares smoothed strain-displacement matrix.
E	-	Young's modulus. Also transformed elasticity matrix.
E_{11}, E_{22}	-	engineering constants for orthotropic lamina.
F, F_1, F_2, F_3	-	stiffness matrices in local coordinates.
G_1, G_2, H	-	
F	-	also standard integral for triangles.
G_{12}	-	engineering elastic constant for orthotropic lamina.
G, H	-	transformation matrices of shape functions.
H	-	also curvature vector in local coordinates.
$\bar{G}, \bar{H}$	-	shape function vectors.
J	-	Jacobian transformation matrix.
K	-	element stiffness matrix.
K'	-	stiffness matrix for transition element.
L	-	differential operator.

M	-	vector of bending and twisting moments.
N	-	shape function.
$\bar{N}$	-	integral least squares smoothed shape function.
N^*	-	discrete least squares smoothed shape function.
P, P_1, P_2	-	transformation matrices.
Q, Q_2	-	
$Q_{11}, Q_{12}, \dots$ etc.	-	elements of stress-strain matrix for orthotropic lamina in the principal material axes.
$\bar{Q}_{11}, \bar{Q}_{12}, \dots$ etc.	-	elements of stress-strain matrix for orthotropic lamina in the cartesian axes.
Q'	-	transformation matrix for transition element.
R, S	-	transformation matrices.
U	-	strain energy for the element.
U^*	-	complementary strain energy for the element.
V	-	potential of applied loads.
$X_{11}, \dots, X_{33}$	-	elements of stiffness matrix.
X, Y, Z	-	transformation matrices.
<hr/>		
a, b, c	-	intrinsic (geometric) parameters for the element.
f	-	standard integral for the triangle.
h	-	height of shallow shell element above base plane.
m, n	-	powers of polynomial terms.
p, q	-	uniform pressure over the element.
t	-	element thickness.
p, q, r, s, t	-	also intrinsic parameters for triangle or quadrilateral.

u, v	-	tangential displacement functions.
u^t	-	in-plane displacement function for transition element.
$\bar{u}, \bar{v}$	-	least squares smoothed versions of u and v .
w	-	normal displacement.
x, y	-	cartesian coordinates.
z	-	height from the middle surface of the element
Ω	-	consistent load factor.

Greek Symbols

α	-	vector of polynomial terms.
β	-	vector of thickness coefficients.
β_π	-	normal rotation about y -axis.
β_y	-	normal rotation about x -axis.
β_s	-	normal rotation along element side.
β_n	-	normal rotation along the perpendicular to element side.
Δ	-	area of triangle; also area parameter for quadrilateral.
δ	-	nodal displacement vector.
δ_u	-	vector of nodal displacements (x direction).
δ_v	-	vector of nodal displacements (y direction).
δ_w	-	vector of nodal displacements (transverse direction).
δ^*	-	Dirac delta function.
ϵ	-	rational (singular) functions.
$\bar{\epsilon}$	-	least squares smoothed version of ϵ .
ϵ	-	strain vector.

ε^0	-	membrane strain vector.
$\bar{\varepsilon}$	-	least squares smoothed version of ε .
ξ, η, ζ	-	area coordinates of triangle.
κ	-	curvature vector, also functional.
$\kappa_1, \kappa_2, \kappa_3$	-	elements of curvature vector.
$\kappa_x, \kappa_y, \kappa_{xy}$	-	elements of curvature vector.
λ	-	eigenvalue, also intrinsic parameter for triangle.
λ^*	-	shifted eigenvalue.
$\Lambda, \Lambda_1, \Lambda_2$	-	vectors of polynomial terms.
ν	-	Poisson's ratio.
ν_{12}, ν_{21}	-	engineering elastic constants for orthotropic lamina.
Π_p	-	potential energy.
Π_1, Π_2	-	functions.
ρ	-	intrinsic parameter for triangle.
σ	-	stress vector.
θ	-	vector of polynomial terms.
ω	-	solution domain.

¹ Note: Many of the above symbols are used for more than one variable or function; every effort has been made to define clearly in the text the meaning of the symbol.

Chapter 1

Introduction

1.1 The Finite Element Method

During the past twenty five years, the finite element method (FEM) has become the most effective and widely employed numerical technique for solving complex engineering problems. Having been started purely as an extension of matrix structural analysis by civil and aerospace engineers, the FEM is now used as a powerful numerical tool in practically all fields of engineering. The method in general is a process through which a complex domain under consideration is divided into a *finite* number of discrete subdomains or *elements*. The process of subdivision is called finite element discretization or idealization. Then, in each of the elements, the displacement/strain patterns are assumed to be given functions of certain unknown values called degrees of freedom (displacements, slopes, curvatures etc.) at certain selected points on it called the nodes. The nodes are usually, but not always, located on the boundary of the element. A force-displacement relationship is then obtained in terms of an element stiffness matrix for each element by equating the external and internal virtual work or from energy principles. The coefficients of the global stiffness matrix of the complete domain are computed by algebraically adding the relevant coefficients of the element stiffness matrices. Thus a set of simultaneous algebraic equations in terms of the primary unknown nodal degrees of freedom is established which is then solved for these unknowns after introducing the necessary boundary conditions. From these, the nodal quantities corresponding to each element are identified and the strains and stresses are computed.

1.2 Various Formulations of Element Stiffness Matrix

The earliest finite element formulations were for structural applications and were based on the direct approach in which direct consideration was given to conditions of equilibrium and compatibility. But this method could be applied only to simple elements and was not satisfactory for reasonably complex elements, and formulation of symmetric element stiffness matrices became more difficult [1]. Today, variational techniques are almost exclusively used in developing element matrices in the area of structural mechanics. Variational

models usually involve finding the nodal variables that yield a stationary value of a specific integral called the functional. In the displacement approach this integral usually represents the potential energy Π_P . In the force approach, it represents the complementary energy Π_C .

The potential energy Π_P is given by:

$$\Pi_P = U + V \quad (1.2.1)$$

where

U is the strain energy of deformation

V is the potential of applied loads.

It can be shown that, for an element,

$$U = \frac{1}{2} \int_{vol} \{\epsilon\}^T [D] \{\epsilon\} d(vol) \quad (1.2.2)$$

and minimization of the potential energy results in the element stiffness matrix as

$$[K] = \int_{vol} [B]^T [D] [B] d(vol) \quad (1.2.3)$$

where:

$[D]$ is the elasticity matrix.

$[B]$ is the strain-displacement matrix relating the strain vector $\{\epsilon\}$ to the nodal displacement vector $\{\delta\}$ as:

$$\{\epsilon\} = [B] \{\delta\} \quad (1.2.4)$$

In the force method, the complementary energy Π_C is given by

$$\Pi_C = U^* + V \quad (1.2.5)$$

where U^* is the complementary strain energy defined as

$$U^* = \frac{1}{2} \int_{vol} \{\sigma\}^T [D]^{-1} \{\sigma\} d(vol) \quad (1.2.6)$$

with $\{\sigma\}$ is the stress at any point on the element. The flexibility matrix can be obtained from U^* in a similar way as the stiffness matrix from U , which is then normally inverted to obtain the stiffness matrix and the standard stiffness method is thereafter followed [2].

In addition to the displacement and force methods, hybrid and mixed techniques have also been developed and are explained in many standard text books on finite element method.

For the non-structural applications, the weighted residual techniques are increasingly used in the solution of differential equations given by

$$L(\Phi) = Q \quad (1.2.7)$$

and thus the need for the search of mathematically equivalent variational statement can be avoided. If Φ^* is the approximate solution of Φ , then

$$L(\Phi^*) - Q = R \quad (1.2.8)$$

where R is a residual error. Although it is not possible to eliminate the residual term, a weighted integral, over the solution domain, Ω , of the residual can be forced to vanish, i.e.

$$\int_{\Omega} RW \, d\Omega = 0 \quad (1.2.9)$$

where W is the weighting function. When the interpolation function Φ^* and the weighting function W are substituted, a set of algebraic equations is obtained that can be solved for the unknown coefficients in the approximate solution. By selecting $W = \Phi^*$, one obtains the Galerkin's weighted residual method; $W = \frac{\partial R}{\partial \Phi^*}$ gives a least square criterion; while $W = \delta^*$, where δ^* is the Dirac delta function, gives a point collocation procedure.

Since the present work deals with structural analysis of plates and shells, only the variational approach is followed throughout. In spite of the many variational formulations for structural analysis (viz. displacement, force, hybrid and mixed) developed over the years, the displacement method has been and still remains the most popular finite element approach because of its simplicity. Irons and Razzaque [3] have proved that it is possible to develop even hybrid elements as pure displacement-type models by the use of substitute shape functions obtained through least squares procedure. Such new techniques make the displacement method more powerful and attractive than they were originally thought to be. Most users still prefer the displacement approach over other methods. It is therefore

this approach that is paid attention to in this thesis.

1.3 Conditions on Displacement Assumptions (Conforming Elements)

In the finite element displacement method, the critical step is the choice of the element displacement field. The aim is to ensure that the exact solution is progressively approached as more and more elements are used (monotonic convergence to the exact solution in the limit as the element size decreases). The following four conditions are usually postulated to ensure convergence in any displacement approach.

- 1) The element-displacement field should include proper representation of any possible rigid body motion (zero straining modes) of the element.
- 2) The element displacement field should include representation of any possible state of constant strain.

These two conditions are akin to completeness conditions. However, full completeness is achieved only if the displacement field contains all linear terms or all linear and quadratic terms etc. Then the element will have no preferred direction (invariance). Invariance is not mandatory. However it is a very desirable attribute and there is no necessity to violate it.

- 3) Within each element the displacement field should be smooth and continuous.
- 4) At the boundaries between adjacent elements the compatibility condition should be satisfied.

These two conditions are called the compatibility conditions. They ensure that there are no gaps or overlaps or kinks within elements or at element boundaries.

Since the potential energy is a function of displacements and its derivatives, the compatibility condition can be restated as follows. If the displacement variable in a structure enters the potential energy expression with highest derivative of order $m > 0$, then the displacement must be continuous between adjacent compatible elements up to the $(m - 1)$ derivative, in order to evaluate the potential energy uniquely. Elements whose displacement satisfy all compatibility requirements are called conforming elements. As in the case of Rayleigh-Ritz method when the criteria of completeness and compatibility are met, an upperbound solution to the potential energy is obtained and convergence to the true solution takes place with increase in the number of elements. This means with conforming elements, the assumed displacement method yields an approximate structure which is stiffer than the actual structure.

1.4 Nonconforming Elements

Elements whose displacement fields violate the compatibility requirements at the interelement boundaries are called non-conforming elements. The difficulty in meeting the C_1 continuity requirement (Continuity of displacement and slope) for thin plates and shells led to the development of many non-conforming elements. There exist quite a number of successful nonconforming plate and shell elements. It is known that the assumed displacement approach produces a structure which is stiffer than the original one. This approximate structure is softened by allowing separation, overlapping or kinks between elements. The two effects tend to cancel each other and often better results are obtained. Thus the non-conforming elements are no longer based on the minimum potential energy principle and the resultant structure need not always be stiffer than the original structure. Also, convergence may not be monotonic and may be from above or below, and the actual errors involved may well be less than if a conforming element is used. Nevertheless, conformity can be progressively achieved with more mesh refinement if the constant strain condition is satisfied [4,5].

1.5 Plates and Shells

Plates and shells are very commonly used structures by aerospace, civil and mechanical industries. Extensive work has been devoted to finite element analysis of plates and shells in the past and considerable amount of effort is still on in formulating efficient plate and shell elements. References [2] and [6] give a complete survey of plate and shell elements.

Finite elements for plane elasticity problems (plane stress or planar elements) usually have two in-plane degrees of freedom (d.o.f.) per node (u and v in the x and y directions). Plane stress elements with derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$ as additional d.o.f. at the nodes are also sometimes used. For plate bending problems, three out-of-plane displacements (displacement w normal to the $x-y$ plane and slopes $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$) are used as the standard d.o.f. at the nodes. There are many elements which also use second derivatives of w as d.o.f. in addition to w , $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$. The popular high precision triangular plate bending element has six d.o.f. at each node, namely w , $\frac{\partial w}{\partial x}$, $\frac{\partial w}{\partial y}$, $\frac{\partial^2 w}{\partial x^2}$, $\frac{\partial^2 w}{\partial x \partial y}$, $\frac{\partial^2 w}{\partial y^2}$.

All the thin plate bending elements are based on the Kirchhoff's hypothesis which neglects the transverse shear deformation. Thus the mid-surface rotations are given by the first derivatives of the normal displacement, w , and the curvature strains are the second

derivatives of w . More discussions on plate bending elements are given in chapter 3. Superposition of plane and plate bending actions results in elements with a shell type behavior. Such elements are called flat shell elements or facet elements in which there is normally no coupling between in-plane and bending actions. Compatibility is almost always violated in an assembly of facet elements and usually discontinuities of displacement as well as of normal slope are present. In spite of this violation, convergence to correct results always seems to occur, presumably because the discontinuities progressively reduce with mesh refinement. Since a large number of elements are needed to approximate the curved shell geometry, fairly low order elements are adequate for use. Care should be taken to avoid a null stiffness associated with the rotation about the axis normal to a common plane when all the elements meeting at a particular nodal point are coplanar.

Use of shallow shell theory makes elements curved and introduces coupling within elements while avoiding the complexities of differential geometry that prevail in general shell theory. Many practical shell structures are shallow and thus a large number of shallow shell elements have been developed and References [7, 8] contain some representative shallow shell elements. The fact that even in a deep shell each individual element is likely to be small enough to be adequately treated as shallow, has been utilized to analyze deep shells using shallow shell elements. In many test cases good accuracy is obtained for deep shells modeled by elements based on shallow shell theory [9, 10].

Elements specially developed for cylindrical shells need not be shallow but they are mostly restricted to circular cylinders [11-15]. Development of doubly curved elements based on general shell theories involves geometric complexities and thus becomes difficult. Nevertheless, many deep shell elements have been developed [16, 17]. The very sophisticated, doubly curved shell element, 'CURSHL', by Cowper [17] is being used as a standard reference for comparing many other shell elements. All the difficulties associated with curved shell elements based on deep shell theories, can be avoided by degenerating a three dimensional solid element into a doubly curved isoparametric, plate/shell element [18-20]. Both thin as well as moderately thick shells can be analysed by this approach [4,21]. Many plate elements were also developed using Mindlin's shear deformation theory [22]. This approach simplifies the formulation to a great extent since only C_0 continuity is now required. Unfortunately such elements were found to have either some spurious modes or to suffer from a phenomenon called 'shear locking' [23-25], when the thickness becomes very thin. Much research has been devoted to develop elements free from these problems and a few successful elements have been developed [26-31]. However, complete success is yet to

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be achieved in this direction. Another approach is to use the shear deformation theory of Mindlin to start with, but to impose the Kirchhoff conditions at a number of points over the element [32,33]. In this approach (Discrete Kirchhoff Theory) also, C_0 continuity alone is sufficient.

1.6 Objectives of the Present Investigation

This thesis is concerned with displacement type finite element formulations for plane stress, plate bending and shallow shell structures. Thin plate/shell theory is assumed throughout. The broad objective of the study was to examine formulation of element stiffness properties leading to fast, accurate, efficient and overall very economic solutions which would be particularly suitable for implementation in micro-computer or personal computer analyses. For example in the analysis of plate flexure the use of "simple", three-node, nine-degrees-of-freedom elements would be attractive vis-a-vis higher order elements with many more degrees-of-freedom. This problem of plate and shell flexure receives particular emphasis in the thesis.

Before attempting to develop new elements, some of the best existing elements are identified and are completely reformulated so that the algorithm for the element stiffness matrix is fast and efficient. Many new elements have been developed, and to describe these, the thesis is divided into three parts.

Part one deals with plane stress formulations. A new eight noded 'Serendipity' plane stress element is developed by using substitute shape functions derived from least squares techniques. This element does not exhibit any oscillatory nodal shear stresses shown by the original element for certain class of problems. Then a method of constructing mixed order strain triangular elements from degrading a higher order triangle is presented. A sample program listing that generates stiffness matrix for any member of the triangular family (regular or transition element) up to the quadratic strain triangle is appended.

In part two, some of the best existing triangular thin plate bending elements are identified. New efficient formulations are provided for these elements and the stiffness matrices are generated in closed form (no numerical integration) for most of the elements. Both uniform and linearly tapered elements are considered. A laminated anisotropic high precision thin plate bending element is also developed. New efficient plate bending elements (two triangular and one quadrilateral) are then developed and compared with earlier elements. The triangular elements are shown to be the best of their class.

The development of shallow shell elements is covered in part three. A high precision anisotropic thin shallow shell element is developed. Three more anisotropic shallow shell elements based on the DKT (Discrete Kirchhoff Triangle) and the presently developed triangular elements are also presented. In all the cases, the stiffness matrix is provided in closed form.

Finally, conclusions and recommendations regarding the thesis work are presented. Complete computer program listings of all the elements presented in this work are provided in appendices to the thesis.

Chapter 2

Plane Stress Elements

2.1 Introduction

Development of elements for plane elasticity problems is relatively simple. This simplicity is due mainly to the fact that plane elasticity formulations require only the continuity of the displacements (C_0) along element sides, since the highest derivative in the energy expression is of order one. Plane stress elements were originally formulated in terms of the cartesian coordinates x and y , using triangular elements. Subsequently, the formulations were significantly simplified by using area coordinates for triangular elements. However, the introduction of isoparametric elements was the major breakthrough in the formulation of plane stress elements, since curved boundaries could now be treated easily. In isoparametric formulation, the displacements as well as the geometry are interpolated using the same shape functions. The eight node serendipity isoparametric quadrilateral element is now extensively used for solving plane stress problems with curved boundaries. Most of the standard finite element packages have this element in their element library routines. However, this element suffers from spurious shear effects when applied to certain classes of problems, for instance, the analysis of a cantilever beam under uniform loading [21, 34]. Although the computed deflections and axial stresses are in excellent agreement with the theoretical stresses, the shear stresses show a parabolic oscillation within each element, thus providing a poor representation of the actual stresses. Even after nodal averaging, these stresses are much in error. It is therefore desirable to develop an element which is free from such spurious effects. A new eight noded serendipity plane stress element is developed here. Substitute shape functions derived from least squares technique are used to calculate the shear strain which suffers from spurious effects, whereas the original shape functions are used to calculate the normal strains. When the stiffness matrix is evaluated by exact numerical integration, it is found that the spurious shear effects no longer exist. Similar application of least squares shear-alone smoothing to an axisymmetric plate bending element can be found elsewhere [35].

2.2 Substitute Shape Functions by Least Squares Smoothing

A typical displacement function, u , of the eight-node quadrilateral is of the form,

$$u(\xi, \eta) = a_1 + a_2\xi + a_3\eta + a_4\xi^2 + a_5\xi\eta + a_6\eta^2 + a_7\xi^2\eta + a_8\xi\eta^2$$

$$= \sum_{i=1}^8 a_i \xi^{m_i} \eta^{n_i} \quad (2.2.1)$$

with the pair (m_i, n_i) taking values $(0,0)$, $(1,0)$, $(0,1)$, $(2,0)$, $(1,1)$, $(0,2)$, $(2,1)$ and $(1,2)$ for values of $i = 1 \rightarrow 8$. Here ξ, η are the natural coordinates (Figure 1).

In the present formulation, the smoothing function $\bar{u}$, is chosen as a complete quadratic fit one order lower than the actual function, u . Thus,

$$\bar{u}(\xi, \eta) = b_1 + b_2\xi + b_3\eta + b_4\xi^2 + b_5\xi\eta + b_6\eta^2$$

$$= \sum_{i=1}^6 b_i \xi^{m_i} \eta^{n_i} \quad (2.2.2)$$

The unknown coefficients, b_i , are obtained in terms of the known coefficients, a_i , by using the standard least squares smoothing procedure which involves minimization of the functional

$$\kappa = \int \int (u - \bar{u})^2 d\xi d\eta \quad (2.2.3)$$

The problem thus reduces to solving a set of six linear simultaneous equations defined by

$$\frac{\partial \kappa}{\partial b_i} = 0 \quad \text{for } i = 1 \rightarrow 6$$

A detailed description of the above procedure can be found in any standard text on numerical analysis. The smoothed displacement function, $\bar{u}$, results in quadratic shape functions, $\bar{N}_i$, which are presented below along with the original shape functions, N_i .

$$\begin{aligned} N_1 &= \frac{1}{4}(-1 + \xi^2 + \xi\eta + \eta^2 - \xi^2\eta - \xi\eta^2) \\ N_2 &= \frac{1}{2}(1 - \eta - \xi^2 + \xi^2\eta) \\ N_3 &= \frac{1}{4}(-1 + \xi^2 - \xi\eta + \eta^2 - \xi^2\eta + \xi\eta^2) \\ N_4 &= \frac{1}{2}(1 + \xi - \eta^2 - \xi\eta^2) \\ N_5 &= \frac{1}{4}(-1 + \xi^2 + \xi\eta + \eta^2 + \xi^2\eta + \xi\eta^2) \\ N_6 &= \frac{1}{2}(1 + \eta - \xi^2 - \xi^2\eta) \\ N_7 &= \frac{1}{4}(-1 + \xi^2 - \xi\eta + \eta^2 + \xi^2\eta - \xi\eta^2) \\ N_8 &= \frac{1}{2}(1 - \xi - \eta^2 + \xi\eta^2) \end{aligned}$$

Smoothed shape function $\bar{N}_i$

$$\bar{N}_1 = -\frac{1}{12}(\xi + \eta) + \frac{1}{4}(-1 + \xi^2 + \xi\eta + \eta^2)$$

$$\bar{N}_2 = -\frac{1}{3}\eta + \frac{1}{2}(1 - \xi^2)$$

$$\bar{N}_3 = \frac{1}{12}(\xi - \eta) + \frac{1}{4}(-1 + \xi^2 - \xi\eta + \eta^2)$$

$$\bar{N}_4 = \frac{1}{3}\xi + \frac{1}{2}(1 - \eta^2)$$

$$\bar{N}_5 = \frac{1}{12}(\xi + \eta) + \frac{1}{4}(-1 + \xi^2 + \xi\eta + \eta^2)$$

$$\bar{N}_6 = -\frac{1}{3}\eta + \frac{1}{2}(1 - \xi^2)$$

$$\bar{N}_7 = \frac{1}{12}(-\xi + \eta) + \frac{1}{4}(-1 + \xi^2 - \xi\eta + \eta^2)$$

$$\bar{N}_8 = -\frac{1}{3}\xi + \frac{1}{2}(1 - \eta^2)$$

2.3 Stiffness Matrix for the Smoothed Element

The displacements at any point in the element are interpolated from their nodal values as

$$u = \sum_{i=1}^8 N_i u_i \quad (2.3.1)$$

$$v = \sum_{i=1}^8 N_i v_i \quad (2.3.2)$$

$$\bar{u} = \sum_{i=1}^8 \bar{N}_i u_i \quad (2.3.3)$$

$$\bar{v} = \sum_{i=1}^8 \bar{N}_i v_i \quad (2.3.4)$$

The smoothed strain vector $\{\bar{\epsilon}\}$ for the new element is formulated as

$$\begin{aligned} \{\bar{\epsilon}\} &= \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix} \\ &= \sum_{i=1}^8 \begin{Bmatrix} \frac{\partial N_i}{\partial x} u_i \\ \frac{\partial N_i}{\partial y} v_i \\ \frac{\partial N_i}{\partial y} u_i + \frac{\partial N_i}{\partial x} v_i \end{Bmatrix} \\ &= [\bar{B}]\{\delta\} \end{aligned} \quad (2.3.5)$$

where $\{\delta\}$ is the vector of nodal displacements defined as

$$\{\delta\}^T = [u_1, v_1, u_2, v_2, \dots, u_8, v_8] \quad (2.3.6)$$

The smoothed stiffness matrix, $[\bar{K}]$, is computed from the smoothed strain displacement matrix, $[\bar{B}]$, as

$$[\bar{K}] = \int_{vol} [\bar{B}]^T [D] [\bar{B}] d(vol) \quad (2.3.7)$$

For an element of uniform thickness, t ,

$$[\bar{K}] = t \int_{-1}^1 \int_{-1}^1 [\bar{B}]^T [D] [\bar{B}] |J| d\xi d\eta \quad (2.3.8)$$

where J is the Jacobian matrix given by

$$J = \begin{vmatrix} \sum \frac{\partial N_i}{\partial \xi} x_i & \sum \frac{\partial N_i}{\partial \xi} y_i \\ \sum \frac{\partial N_i}{\partial \eta} x_i & \sum \frac{\partial N_i}{\partial \eta} y_i \end{vmatrix} \quad (2.3.9)$$

and $[D]$ is the elasticity matrix. For isotropic plates,

$$[D] = \frac{E}{1-\nu^2} \begin{vmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{vmatrix} \quad (2.3.10)$$

where E is Young's modulus and ν is Poisson's ratio.

Either, full 3×3 , or selective 3×3 direct and 2×2 for shear terms, numerical integration procedure can be used to evaluate $[\bar{K}]$. In both cases, the numerical integration order is exact.

2.4 Eigenanalysis for the Plane Stress Element

The presence of spurious modes in plane isoparametric elements, when reduced or selective integration techniques are used, has been identified by *Picanić* and *Hinton*, [36]. Although exact integration is used for the present element, it is important to check whether the least squares smoothing of shear strain has introduced any spurious modes. Therefore, an eigenanalysis [36] is done on a single unconstrained element as follows.

The standard eigenproblem can be written as,

$$[\bar{K}]\{y\} = \lambda\{y\} \quad (2.4.1)$$

However, the stiffness matrix $[\bar{K}]$ for a single unconstrained element is semidefinite; but is rendered positive definite by adding a shift β to the diagonal terms of $[\bar{K}]$, leading to the eigenproblem

$$([\bar{K}] + \beta[I])\{y\} = (\lambda + \beta)\{y\} \quad (2.4.2)$$

$$[K^*]\{y\} = \lambda^*\{y\} \quad (2.4.3)$$

in which $[K^*]$ is positive definite.

The real eigenvalues λ_i are obtained from λ_i^* as

$$\lambda_i = \lambda_i^* - \beta \quad (2.4.4)$$

Since all the eigenvalues need to be calculated, the Jacobi iteration method is chosen for solving the eigenvalue problem and the program by *Bathe*, [37] is used for this purpose. The complete eigenvalue program listing for the single-element eigenanalysis is appended in Appendix A. Examination of the eigenvalues confirm that there are only three zero eigenvalues corresponding to the three rigid body motions. Thus, there are no spurious modes present in the current formulation.

2.5 Numerical Examples

Two plane stress test problems, flexure of a cantilever and concentrated force on a straight boundary, presented as example problems for local stress smoothing by *Hinton* and *Campbell* [34], are solved here by the present element to demonstrate its effectiveness.

Test 1: Flexure of a Cantilever

A cantilever with slenderness ratio, $h/l = 1/20$, subjected to a distributed loading along its length is analysed using a mesh of four rectangular isoparametric plane stress elements. The computed flexural stresses, σ_x , along the top of the beam for both the unsmoothed and the smoothed cases are indistinguishable from the theoretical stresses. However, the shear stresses in the original element are wildly oscillatory and are much in error even after nodal averaging, whereas the present smoothed element produces a dramatic improvement, giving shear stresses which are in excellent agreement with the average shear stress based upon single beam theory. No nodal averaging was needed for this case and all stresses, normal or shear can be calculated directly at the nodes. These results are plotted in Figure 2.

Test 2: Concentrated Force on a Straight Boundary

The stress distribution in an infinitely large plate subjected to an in-plane concentrated load is given in Reference [38]. For the finite element analysis, a plate of unit thickness subjected to an in-plane concentrated force, $P = 1000$, is chosen (same as chosen by *Hinton*

and Campbell). At a considerable distance from the load, the displacement is assumed to be zero. Because of symmetry, only a half of the infinite plate is analysed. Two types of finite element meshes are considered, (Figure 3);

- (i) Mesh A—A rectangle mesh consisting of nine elements.
- (ii) Mesh B—A mesh of seven elements of general quadrilateral shape.

A fine mesh is used near the loading. The results are plotted in Figures 4 and 5, in which comparisons are given with analytical stresses. Stresses obtained by both the original and present elements are in good agreement with the exact stresses, except in the immediate vicinity of the loading singularity.

The above example problems prove that the present element is free from the spurious effects suffered by the original formulation, and gives good results in all situations. It is therefore this element that is recommended for the finite element packages in place of the original eight-noded serendipity plane stress element.

2.6 Development of Transition Elements

In most engineering structures, the stress gradients vary rapidly only at certain regions, requiring more active nodes to be placed at these zones than at the rest. This could be done by choosing a fine mesh at the stress concentrated zones and a coarse one at the other regions. But in some cases, a low order element may lead to crude approximations even for fine meshes. Practical experience shows that for a given number of total degrees of freedom in a structure, greater accuracy is achieved using fewer higher order elements as against a large number of low order elements. Therefore, higher order elements would be a better choice in these cases. Also, changing the element grid a number of times to achieve convergence is laborious. Instead, a basic grid can be chosen and additional nodes can be placed in selected areas of the initial grid layout. This allows the use of higher as well as lower order elements at different areas of the domain. Such a combination of elements of different orders calls for the development of the mixed-displacement models, also known as matching or transition elements. This idea was originally conceived by Irons [39], and has been successfully used for the combination of linear and constant strain triangles by McNeice and Hunnisett, [40]. In the following sections, a simple method of constructing mixed order elements from a higher order element is presented. The method allows for easy use of elements of different order to model strain variation in the best manner possible, by concentrating additional nodes in elements where there are regions of high strain gradient.

For the purpose of illustration, the triangular family of elements is chosen, but the method is applicable to other shapes and could be extended to three-dimensional and isoparametric elements.

2.7 Transition Elements for the Plane Triangular Family

The first three members of the family of plane triangular elements are shown in Figure 6. However, these are only the regular members of the triangular family since a complete family must include transition elements as well. It is shown here that by choosing a higher order element and simply modifying its shape functions through some transformation matrices, all the other possible lower order elements can be generated. For example, one can choose the ten-noded quadratic strain triangle as the basic higher order element. Following References [41] and [42], the shape function vector, N , for this element can be written in terms of the area coordinates ξ, η (ζ replaced by $1 - \xi - \eta$) as,

$$\{N\}^T = \{\alpha\}^T [Q] \quad (2.7.1)$$

where α is a vector of polynomial terms given by,

$$\{\alpha\}^T = [1, \xi, \eta, \xi^2, \xi\eta, \eta^2, \xi^3, \xi^2\eta, \xi\eta^2, \eta^3] \quad (2.7.2)$$

and Q is a numerical transformation matrix given by

$$Q = \frac{1}{2} \begin{bmatrix} 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & -11 & 0 & 0 & 0 & 0 & 18 & -9 & 0 \\ 0 & 2 & -11 & 0 & 0 & -9 & 18 & 0 & 0 & 0 \\ -9 & 0 & 18 & 0 & 0 & 0 & 0 & -45 & 36 & 0 \\ 0 & 0 & 36 & -9 & -9 & 9 & -45 & -45 & 9 & 54 \\ 0 & -9 & 18 & 0 & 0 & 36 & -45 & 0 & 0 & 0 \\ 9 & 0 & -9 & 0 & 0 & 0 & 0 & 27 & -27 & 0 \\ 0 & 0 & -27 & 27 & 0 & 0 & 27 & 54 & -27 & -54 \\ 0 & 0 & -27 & 0 & 27 & -27 & 54 & 27 & 0 & -54 \\ 0 & 9 & -9 & 0 & 0 & -27 & 27 & 0 & 0 & 0 \end{bmatrix} \quad (2.7.3)$$

A typical displacement, u , at any point in the element can be interpolated from its ten nodal displacements as,

$$u = \{\alpha\}^T [Q] \{\delta_u\} \quad (2.7.4)$$

where

$$\{\delta_u\}^T = [u_1, u_2, \dots, u_{10}] \quad (2.7.5)$$

Suppose it is required to construct a transition element that combines with constant, linear and quadratic strain triangles on its three sides. Such an element will have seven nodes as seen in Figure 7a. The basic ten-noded element is therefore to be transformed into a seven-noded transition element, (Figure 7b).

The variation of a typical displacement function, u , on side AB of the basic element is cubic, uniquely defined by the displacements of the four nodes 1, 4, 5 and 2. But, u varies only linearly on the side AB of the transition element and is given by,

$$u = u'_1 + (u'_2 - u'_1)l \quad (0 \leq l \leq 1) \quad (2.7.6)$$

where l is the normalized length of AB measured from A and u'_1 and u'_2 are the displacements of the transition element at nodes 1' and 2'. The displacement on side AB of the basic elements is therefore constrained to vary linearly as defined by equation (2.7.6), and the following transformation is obtained by substituting the values 0, 1/3, 2/3, 1 for l in equation (2.7.6),

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix} \begin{Bmatrix} u'_1 \\ u'_2 \end{Bmatrix} \quad (2.7.7)$$

Similarly, the variation of u on the side BC of the transition element is quadratic and is given by,

$$u = u'_2 + (4u'_4 - u'_3 - 3u'_2)l + (2u'_2 + 2u'_3 - 4u'_4)l^2 \quad (0 \leq l \leq 1) \quad (2.7.8)$$

and the following transformation is obtained,

$$\begin{Bmatrix} u_3 \\ u_6 \\ u_7 \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 2/9 & -1/9 & 8/9 \\ -1/9 & 2/9 & 8/9 \end{bmatrix} \begin{Bmatrix} u'_2 \\ u'_3 \\ u'_4 \end{Bmatrix} \quad (2.7.9)$$

The other displacements are directly related as,

$$\begin{Bmatrix} u_8 \\ u_9 \\ u_{10} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u'_5 \\ u'_6 \\ u'_7 \end{Bmatrix} \quad (2.7.10)$$

Combining equations (2.7.7), (2.7.9) and (2.7.10), the transformation matrix for the transition element is obtained as,

$$\{\delta_u\} = [R] \{\delta'_u\} \quad (2.7.11)$$

where

$$R = \frac{1}{9} \begin{bmatrix} 9 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 9 & 0 & 0 & 0 & 0 \\ 6 & 3 & 0 & 0 & 0 & 0 & 0 \\ 3 & 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 8 & 0 & 0 & 0 \\ 0 & -1 & 2 & 8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 9 \end{bmatrix} \quad (2.7.12)$$

Substituting for equation (2.7.11) in equation (2.7.4), the shape function vector, N' , for the transition element is obtained as,

$$\{N'\}^T = \{\alpha\}^T [Q'] \quad (2.7.13)$$

and

$$[Q'] = [Q] [R] \quad (2.7.14)$$

or

$$Q' = \frac{1}{2} \begin{bmatrix} 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 2 & 0 & -11 & 0 & 18 & -9 & 0 \\ 0 & -2 & -6 & 8 & 0 & -0 & 0 \\ -9 & 0 & 18 & 0 & -45 & -36 & 0 \\ -9 & -2 & 25 & -32 & -45 & 9 & 54 \\ 0 & 4 & 4 & -8 & 0 & 0 & 0 \\ 9 & 0 & -9 & 0 & 27 & -27 & 0 \\ 18 & 6 & -21 & 24 & 54 & -27 & -54 \\ 9 & 6 & -12 & 24 & 27 & 0 & -54 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (2.7.15)$$

Using this procedure, it is seen that a complete family of plane triangular elements can be obtained as shown in Figure 8.

The transformation matrices for the first ten members of the family with the ten-noded quadratic strain triangle (*element 11*) as the basic element are appended at the end of this chapter for immediate use. Using the above procedure, one can extend the formulation to any order element. Since the shape functions for all the members of the family are now available, the formulation of stiffness matrices can follow the standard numerical integration procedure of any isoparametric element. For the particular case of straight-sided elements, numerical integration procedure can be completely avoided, and the stiffness matrices can be obtained very conveniently in closed form as shown below.

2.8 Stiffness Matrix Formulation for Straight-sided Triangles

Following [41], the strain energy of a plane triangular element can be written as,

$$U = \frac{1}{2} \begin{Bmatrix} \delta_u \\ \delta_v \end{Bmatrix}^T [K] \begin{Bmatrix} \delta_u \\ \delta_v \end{Bmatrix} \quad (2.8.1)$$

where $\{\delta_u\}$ and $\{\delta_v\}$ are the vectors of nodal displacements in the x and y directions and K is the element stiffness matrix given by,

$$[K] = \frac{t}{2\Delta} \begin{vmatrix} X & Y \\ Y^T & Z \end{vmatrix} \quad (2.8.2)$$

and

$$\begin{aligned} X &= g_{11}A + g_{12}(B + B^T) + g_{22}C \\ Y &= g_{13}A + g_{14}B + g_{23}B^T + g_{24}C \\ Z &= g_{33}A + g_{34}(B + B^T) + g_{44}C \end{aligned} \quad (2.8.3)$$

Here A , B and C are numerical universal matrices which are independent of triangle size, shape and the material of the element. The terms t and Δ refer to the thickness and area of the element, respectively. For the ten-noded quadratic strain triangle, they are presented explicitly in Reference [41]. Further, g_{11} , g_{12} , etc. are the elements of the symmetric matrix obtained from the matrix triple product,

$$[G] = [P]^T [D] [P] \quad (2.8.4)$$

where $[D]$ is the elasticity matrix and $[P]$ is the triangle geometry-matrix defined in terms of the nodal coordinates as,

$$P = \frac{1}{2\Delta} \begin{vmatrix} (y_2 - y_3) & (y_3 - y_1) & 0 & 0 \\ 0 & 0 & (x_3 - x_2) & (x_1 - x_3) \\ (x_3 - x_2) & (x_1 - x_3) & (y_2 - y_3) & (y_3 - y_1) \end{vmatrix} \quad (2.8.5)$$

If $[R]$ is the transformation matrix for the transition element under consideration, then the stiffness matrix, $[K']$, for this element can be obtained from that of the basic element, K , as follows,

$$\begin{aligned} [K'] &= \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}^T [K] \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix} \\ &= \frac{t}{2\Delta} \begin{bmatrix} X' & Y' \\ Y'^T & Z' \end{bmatrix} \end{aligned} \quad (2.8.6)$$

where X' , Y' and Z' are obtained from equation (2.8.3) by replacing A , B and C with A' , B' and C' which are defined as follows;

$$\begin{aligned} A' &= R^T A R \\ B' &= R^T B R \\ C' &= R^T C R \end{aligned} \quad (2.8.7)$$

A FORTRAN program listing of a subroutine which generates the stiffness matrix using the above principles for any of the eleven elements in Figure 8 is presented in Appendix B. The program may be readily modified for other order elements by a few simple changes.

2.9 Application to Field Problems

Universal matrices for triangular elements have been published for a variety of field problems from time to time, [43, 44, 45]. In all these cases, only regular members of the triangular family are considered. By pre- and post-multiplying these universal matrices by the appended transformation matrices, the universal matrices of the corresponding transition elements are readily obtained. This broadens the usefulness of the algorithm beyond plane stress problems.

2.10 Discussion and Conclusions—Plane Stress Elements

In this Chapter, a general curved eight-noded isoparametric plane stress element and a number of straight-sided triangular elements have been developed. There is no need to generate still higher order triangular elements, since there would be little, if any, gain in efficiency in using elements with cubic or higher order strain variations. Since no numerical integration is used for these elements, the computational effort required in generating their stiffness matrices is very little. It can be seen that these triangular elements can be used in association with eight-noded isoparametric element also developed in this work, to model any arbitrarily curved planar structural geometry. The interior of the plane body could be modelled with straight-sided triangles with the eight-noded isoparametric element used just where necessary to provide a good match to the structure at its curved boundaries. This mixing of elements is perfectly permissible, so long as the common edges of adjacent elements have the same number and location of nodes and the same number of degrees of freedom at each node.

*Transformation matrices for elements in Figure 8 with QST
as basic element*

Element 1 (CST)

$$\frac{1}{3} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \\ -2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 2 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Element 2

$$\frac{1}{9} \begin{bmatrix} 9 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 6 & 3 & 0 & 0 \\ 3 & 6 & 0 & 0 \\ 0 & 6 & 3 & 0 \\ 0 & 3 & 6 & 0 \\ -1 & 0 & 2 & 8 \\ 2 & 0 & -1 & 8 \\ 1 & 3 & 1 & 4 \end{bmatrix}$$

Element 3

$$\frac{1}{9} \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 & 0 \\ 0 & 0 & 9 & 0 & 0 \\ 6 & 3 & 0 & 0 & 0 \\ 3 & 6 & 0 & 0 & 0 \\ 0 & 2 & -1 & 8 & 0 \\ 0 & -1 & 2 & 8 & 0 \\ -1 & 0 & 2 & 0 & 8 \\ 2 & 0 & -1 & 0 & 8 \\ 1 & 1 & -1 & 4 & 4 \end{bmatrix}$$

Element 4 (LST)

$$\frac{1}{9} \begin{bmatrix} 9 & 0 & 0 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 9 & 0 & 0 & 0 \\ 2 & -1 & 0 & 8 & 0 & 0 \\ -1 & 2 & 0 & 8 & 0 & 0 \\ 0 & 2 & -1 & 0 & 8 & 0 \\ 0 & -1 & 2 & 0 & 8 & 0 \\ -1 & 0 & 2 & 0 & 0 & 8 \\ 2 & 0 & -1 & 0 & 0 & 8 \\ -1 & -1 & -1 & 4 & 4 & 4 \end{bmatrix}$$

Element 5

$$\frac{1}{9} \begin{bmatrix} 9 & 0 & 0 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 9 & 0 & 0 & 0 \\ 6 & 3 & 0 & 0 & 0 & 0 \\ 3 & 6 & 0 & 0 & 0 & 0 \\ 0 & 6 & 3 & 0 & 0 & 0 \\ 0 & 3 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 9 \end{bmatrix}$$

Element 6

$$\frac{1}{9} \begin{bmatrix} 9 & 0 & 0 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 9 & 0 & 0 & 0 \\ 6 & 3 & 0 & 0 & 0 & 0 \\ 3 & 6 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 8 & 0 & 0 \\ 0 & -1 & 2 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 & 0 & 9 \end{bmatrix}$$

Element 7

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Element 8

$$\begin{bmatrix} 9 & 0 & 0 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 9 & 0 & 0 & 0 \\ 2 & -1 & 0 & 8 & 0 & 0 \\ -1 & 2 & 0 & 8 & 0 & 0 \\ 0 & 2 & -1 & 0 & 8 & 0 \\ 0 & -1 & 2 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 9 \end{bmatrix}$$

Element 9

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Element 10

$$\begin{bmatrix} 9 & 0 & 0 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 9 & 0 & 0 & 0 \\ 2 & -1 & 0 & 8 & 0 & 0 \\ -1 & 2 & 0 & 8 & 0 & 0 \\ 0 & 2 & -1 & 0 & 8 & 0 \\ 0 & -1 & 2 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 9 \end{bmatrix}$$

Chapter 3

Plate Bending Elements

3.1 Thin Plate Theory.

The classical thin plate theory is based on the following assumptions (Kirchhoff's hypothesis).

1. Normals to the middle surface before deformation remain straight and normal after deformation. This is equivalent to ignoring the shearing strains in planes perpendicular to the middle surface, that is, $\gamma_{xz} = \gamma_{yz} = 0$ where z is the normal direction.
2. The normals are presumed to have constant length so that the strain perpendicular to the middle surface is ignored, that is, $\epsilon_z = 0$. Therefore the normal deflection, w , at any point in the plate is equal to the normal deflection of the corresponding point on the mid-plane.

Thus, in the thin plate problem, transverse shear deformation is completely neglected and the displacement at any point on the plate is completely defined by the lateral displacement, w , which is a function of the coordinates x, y . The mid-surface rotations are given by the first derivatives of the normal displacement, i.e. by $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$.

The curvatures κ_x , κ_y and κ_{xy} are defined by the rate of change of the rotations, i.e. by the second derivatives of the displacement, given by

$$\begin{aligned}\kappa_x &= -\frac{\partial^2 w}{\partial x^2} \\ \kappa_y &= -\frac{\partial^2 w}{\partial y^2} \\ \kappa_{xy} &= -2\frac{\partial^2 w}{\partial x \partial y}\end{aligned}\tag{3.1.1}$$

The constitutive relations for thin plate flexure are given by,

$$\{M\} = [D_b] \{\kappa\}.\tag{3.1.2}$$

where

$\{M\}^T = [M_x, M_y, M_{xy}]$ is the vector of bending and twisting moments.

$\{\kappa\}^T = [\kappa_y, \kappa_{xy}]$ is the curvature vector and $[D_b]$ is the elasticity matrix.

The governing equilibrium equations are obtained by taking a differential element and considering the equilibrium of forces in the vertical direction and of moments about the x and y axes.

They are given by,

$$\begin{aligned} \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q &= 0 \\ \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x &= 0 \\ \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y &= 0 \end{aligned} \quad (3.1.3)$$

where q is the transverse distributed load and Q_x, Q_y are the shear line loads.

Substituting the moment-curvature relationship into the equilibrium equations, the governing equilibrium equation is obtained as a fourth order differential equation in terms of displacement as follows,

$$\frac{Et^3}{12(1-\nu^2)} \left\{ \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right\} = q \quad (3.1.4)$$

3.2 Compatibility Conditions for Thin Plate Bending Elements

The potential energy for the plate is defined as before as,

$$\Pi_p = U + V$$

where V is the potential of applied loads and U is the strain energy given by the area integral,

$$U = \frac{1}{2} \int_A \{\kappa\}^T [D_b] \{\kappa\} dx dy \quad (3.2.1)$$

The appearance of second derivatives of displacement, w , in the strain energy demands not only continuity of w , but also its first derivatives (C_1 continuity) in developing conforming elements. This requirement of C_1 continuity considerably complicates the element development except in the simplest geometric shapes such as rectangles. For triangles and quadrilaterals, very cumbersome and complicated procedures need to be followed. Some such techniques are use of sub-triangles, midside nodes, higher derivatives as nodal variables or constraints using Lagrange multipliers or penalty functions. A complete discussion

of compatibility for various elements is given by Olson [46]. Rectangular elements have limited applicability because of their shapes and are not discussed in this thesis. The triangular elements are the simplest two-dimensional shapes and are quite convenient for adopting to irregular geometry. They are also convenient for approximating arbitrary shell structures as an assemblage of flat shell elements. For these reasons, more emphasis is placed on triangular elements than on quadrilateral elements in this thesis.

Earlier triangular plate bending elements were formulated in cartesian coordinates. A simple triangular plate bending element has three d.o.f. ($w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}$) per node and a total of nine d.o.f. for the element. Thus, a complete cubic polynomial which has ten terms cannot be chosen unless the term, xy , is omitted or the coefficients of the x^2y and xy^2 terms are equated. Such techniques destroy the completeness (no convergence), or lead to elements which are not geometrically isotropic, or elements which are direction sensitive (singular for certain shapes). Introduction of a tenth displacement at the centroid could not improve convergence. However, Harvey and Kelsey [47] could get converged results for this element by enforcing compatibility at the global level as follows.

Suppose that two adjacent elements A and B have normal slopes $(\frac{\partial w}{\partial n})_A$ and $(\frac{\partial w}{\partial n})_B$ at their midsides respectively. Compatibility of normal slopes can be enforced by equating $(\frac{\partial w}{\partial n})_A$ and $(\frac{\partial w}{\partial n})_B$ at their midpoint. Thus,

$$\left(\frac{\partial w}{\partial n}\right)_A - \left(\frac{\partial w}{\partial n}\right)_B = 0 \quad (3.2.2)$$

In this way, a set of constrained equations are produced which are then imposed on the assembled global equations either by direct substitution or through Lagrange multiplier methods.

With the introduction of area coordinates for triangles, it was possible to develop a nine d.o.f. plate bending element with geometric isotropy and polynomial completeness [48]. The shape functions for this element were provided explicitly. This element was nonconforming, gave reasonably accurate results for many problems, but was later found not to converge for certain mesh patterns. In Reference [48], a conforming twelve d.o.f. element was also presented by providing an additional variable of normal slope of $w, \frac{\partial w}{\partial n}$, at the midside point of each side. The shape functions for this element involved some singular functions of the type,

$$\epsilon_i = \frac{\xi \eta^2 \zeta^2 (1 + \xi)}{(\xi + \eta)(\xi + \zeta)} \quad i = 1, 2, 3 \quad (3.2.3)$$

where ξ, η, ζ are the area coordinates. By constraining the midside normal slope to be the average of the two slopes at the end of that side, a compatible nine d.o.f. plate bending

element was obtained. However, the second derivatives of ϵ_i have singularities at the corners, so that even seven-point numerical integration formula was inadequate. To obtain a converging element, a much higher rule would be needed [3] which is uneconomical. A conforming plate bending element was developed by Clough and Tocher [49] by subdividing the triangle into three parts. A different incomplete (nine-term) cubic polynomial was assumed in each subelement and thus twenty seven displacement coefficients are obtained, out of which eighteen are used in satisfying internal compatibility requirements between adjacent subelements by matching the displacements and slopes at corner points, and normal slopes at the mid-points of interior sides. This element converges monotonically but was found rather over-stiff owing to the restriction of the normal slope to a linear variation along the edge. Addition of midside slope variables allowed quadratic variation of normal slope and resulted in much improved accuracy. This resulted in an element with six nodes, three corner nodes as before and three additional nodes at which only normal slope is specified. Such elements present assembly difficulties and increased bandwidths which can be remedied only with special assembly and solver programs like the Frontal or Skyline solvers [50, 37]. Irons [51] has developed a compatible triangular element with eighteen d.o.f. by considering the values of w , $\frac{\partial w}{\partial n}$ and the cross derivative $\frac{\partial^2 w}{\partial x \partial y}$ at the midside nodes and w , $\frac{\partial w}{\partial x}$, $\frac{\partial w}{\partial y}$ at the corner nodes. This element contains the complete fourth order polynomial terms in addition to three singular functions. By using all the second derivatives as additional nodal variables, a twenty one d.o.f. triangular element (w , $\frac{\partial w}{\partial x}$, $\frac{\partial w}{\partial y}$, $\frac{\partial^2 w}{\partial x^2}$, $\frac{\partial^2 w}{\partial x \partial y}$, $\frac{\partial^2 w}{\partial y^2}$ at the corner nodes and $\frac{\partial w}{\partial n}$ at the midside nodes) based on a complete quintic polynomial was simultaneously but independently developed by Argyris [52], Bell [53], Bosshard [54], Irons [51] and Visser [55] listing the authors alphabetically. Since the existence of midside nodes with their single d.o.f. is an embarrassment, as mentioned earlier, these were constrained by allowing only a cubic variation of the normal slope along each side of the triangle. Such an element with reduced eighteen d.o.f. was again simultaneously but independently developed by Argyris [52], Bell [53], Butlin and Ford [56], and Cowper *et al.* [57]. A family of higher order triangular plate bending elements involving twist-functions, was developed by Caramanlian *et al.* [58], but these elements are seldom used in practice. It is thus seen that conventional formulation of compatible nine d.o.f. triangular plate bending elements is very difficult and one has to resort to special techniques like the sub-elements approach (complicated), inclusion of singular functions (very high order of numerical integration) or alternatively constraining of assembled global matrices using Lagrange multipliers or penalty functions. These difficulties resulted in alternative, unconventional, but simpler approaches. The most promising new methods for

bending elements are development of nonconforming but converging elements or application of Kirchhoff's hypothesis only at discrete points on the element. These methods are discussed in detail later. Attention is paid to the conventional elements first. Inclusion of second derivatives of w appears to be the easiest way to develop C_1 continuous elements.

3.3 The High Precision Plate Bending Element

One of the most useful and popular conventional elements in practice is the eighteen d.o.f. high precision plate bending element [52, 53, 56, 57]. It is conforming and predicts both displacements and stresses reasonably accurately even for moderately coarse grids. The formulation of this element given by Cowper *et al.* [57] is very attractive since the constraint of cubic variation of normal slope along the three sides of the triangle is easily introduced and the stiffness matrix is presented in closed form (numerical integration is avoided). However, the computation time required to form the element stiffness matrix is quite high, since it involves a numerical inversion of a 21×21 matrix and a generation of another 21×21 integral matrix by numerous algebraic operations. Cowper [17], while developing a high precision shell element, presents an explicit inverse of this matrix greatly reducing the computation time. Similar work is also reported by Stavitsky *et al.* [59].

In the following, additional simplification is achieved by providing the integrated matrix explicitly, requiring very few algebraic operations, and reducing the computation time further. Moreover, in this present formulation linearly varying as well as uniform thickness is allowed. It is noted that hitherto, such explicit formulation is available only for simple elements (nine d.o.f.) such as BCIZ1 [60] and DKT [61] plate bending elements. The present formulation is limited to either isotropic or orthotropic plates for which the material constants are independent of the thickness (as in References [60, 61]). Extension to anisotropic cases is presented in the next section.

3.3.1 Explicit Formulation of Stiffness Matrix

The strain energy, U , of the plate bending element is given by the integral over the element (as in equation 3.2.1),

$$U = \frac{1}{2} \int_A \{\kappa\}^T D_b \{\kappa\} dx dy \quad (3.3.1)$$

where D_b is the elasticity and $\{\kappa\}$ is the curvature vector as defined in equation (3.1.1)

In the case of homogeneous orthotropic material, the elasticity matrix is given by

$$[D_b] = \frac{t^3}{12} \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{12} & D_{22} & D_{23} \\ D_{13} & D_{23} & D_{33} \end{bmatrix} = \frac{t^3}{12} [D]$$

For a homogeneous isotropic plate $D_{11} = D_{22} = E/(1 - \nu^2)$, $D_{12} = \nu D_{11}$, $D_{13} = D_{23} = 0$ and $D_{33} = D_{11}(1 - \nu)/2$, where E , ν and t are Young's modulus, Poisson's ratio and thickness respectively.

In terms of area coordinates, ξ and η , with ζ replaced by $1 - \xi - \eta$

$$\{\kappa\} = \frac{1}{(2\Delta)^2} [P] \{H\} \quad (3.3.2)$$

where

$$[P] = \begin{bmatrix} b_1^2 & b_2^2 & 2b_1b_2 \\ c_1^2 & c_2^2 & 2c_1c_2 \\ 2b_1c_1 & 2b_2c_2 & 2(b_1c_2 + b_2c_1) \end{bmatrix}$$

and

$$\{H\}^T = \left[\frac{\partial^2 w}{\partial \xi^2}, \frac{\partial^2 w}{\partial \eta^2}, \frac{\partial^2 w}{\partial \xi \partial \eta} \right]$$

with $b_i = y_j - y_m$ and $c_i = x_m - x_j$, i, j, m taking values of 1, 2, 3 cyclically. Δ is the area of the element.

Substituting from equation (3.3.2) in equation (3.3.1),

$$2U = \frac{1}{(2\Delta)^3} \int_0^1 \int_0^{1-\xi} \frac{t^3}{12} \{H\}^T [E] \{H\} d\xi d\eta \quad (3.3.3)$$

where $[E] = [P]^T [D] [P]$. The displacement, w , anywhere on the element can be written in the form [17, 59],

$$w = \{\Lambda\}^T [S] [R] \{\delta\} \quad (3.3.4)$$

where $\{\delta\}$ is the vector of nodal displacements given by,

$$\{\delta\}^T = [\delta_1^T, \delta_2^T, \delta_3^T] = [\delta_1, \delta_2, \delta_3, \dots, \delta_{18}]$$

$$\{\delta_i\}^T = \left[w_i, \left(\frac{\partial w}{\partial x} \right)_i, \left(\frac{\partial w}{\partial y} \right)_i, \left(\frac{\partial^2 w}{\partial x^2} \right)_i, \left(\frac{\partial^2 w}{\partial x \partial y} \right)_i, \left(\frac{\partial^2 w}{\partial y^2} \right)_i \right]$$

$\{\Lambda\}$ is the vector of polynomial terms given by,

$$\{\Lambda\}^T = [1, \xi, \eta, \xi^2, \xi\eta, \eta^2, \xi^3, \dots, \xi^5, \dots, \eta^5]$$

$[S]$ and $[R]$ are transformation matrices. Noting that every element of $\{\Lambda\}$ is of the form $\xi^m \eta^n$, w can be written as

$$w = \sum_{j=1}^{18} \delta_j \sum_{i=1}^{21} \xi^m \eta^n Q_{ij} \quad (3.3.5)$$

with

$$[Q] = [S] [R]$$

A typical term of the curvature vector $\{H\}$ is in the form

$$\frac{\partial^2 w}{\partial \xi^2} = \sum_{j=1}^{18} \delta_j \sum_{i=1}^{21} m_i (m_i - 1) \xi^{m_i-2} \eta^{n_i} Q_{ij} \quad (3.3.5a)$$

Substituting from an equation of type (3.3.5) into equation (3.3.3) and carrying out integrations of the standard type,

$$f(m, n) = \int_0^1 \int_0^{1-\xi} \xi^m \eta^n d\xi d\eta = \frac{m! n!}{(m+n+2)!} \quad (3.3.6)$$

the expression for the strain energy is obtained as,

$$2U = \frac{1}{12} \frac{1}{(2\Delta)^3} \{\delta\}^T [Q]^T [F] [Q] \{\delta\} \quad (3.3.7)$$

The element of the 21×21 symmetric matrix, $[F]$, is obtained from

$$\begin{aligned} F_{ij} = & E_{11} m_i (m_i - 1) m_j (m_j - 1) f(II - 4, JJ) \\ & + [E_{12} \{m_i (m_i - 1) n_j (n_j - 1) + n_i (n_i - 1) m_j (m_j - 1)\} + E_{33} m_i n_i m_j n_j] \\ & \times f(II - 2, JJ - 2) \\ & + E_{13} \{m_i (m_i - 1) m_j n_j + m_i n_i m_j (m_j - 1)\} f(II - 3, JJ - 1) \\ & + E_{22} n_i (n_i - 1) n_j (n_j - 1) f(II, JJ - 4) \\ & + E_{23} \{n_i (n_i - 1) m_j n_j + m_i n_i n_j (n_j - 1)\} f(II - 1, JJ - 3) \end{aligned} \quad (3.3.8)$$

$i, j = 1 \rightarrow 21$

where

$$II = m_i + m_j$$

$$JJ = n_i + n_j$$

Note that $E_{11} \dots E_{33}$ in the expressions for F_{ij} are the elements of the symmetric matrix, E , in equation (3.3.3). Thus, the stiffness matrix of the plate bending element is given by,

$$[K] = \frac{1}{12} \frac{1}{(2\Delta)^3} [Q]^T [F] [Q]$$

Observing that the elements of the first three rows and columns of the $[F]$ matrix are zeros, the matrix $[K]$ can be expressed compactly as,

$$[K] = \frac{1}{12} \frac{1}{(2\Delta)^3} [Q_1]^T [F_1] [Q_1] \quad (3.3.9)$$

where the 18×18 symmetric matrix, $[F_1]$, is derived by deleting the first three rows and columns of the $[F]$ matrix, and the 18×18 matrix, $[Q_1]$, by deleting the first three rows of the $[Q]$ matrix.

The matrix $[Q_1]$ and $[F_1]$ are presented explicitly using computer coding in Appendix C.

3.3.2 Element with Linearly Varying Thickness

Following [62], the thickness anywhere on the element is interpolated from the nodal thicknesses t_1 , t_2 and t_3 as

$$t = t_3 + t_{13}\xi + t_{23}\eta$$

where

$$t_{13} = (t_1 - t_3) \text{ and } t_{23} = (t_2 - t_3).$$

In equation (3.3.3), t^3 can be expressed as

$$t^3 = \{\alpha\}^T \{\theta\}$$

with

$$\{\alpha\} = [C^*] \{\beta\} \quad (3.3.10)$$

and

$$\{\theta\}^T = [1, \xi, \eta, \xi^2, \xi\eta, \eta^2, \xi^3, \xi^2\eta, \xi\eta^2, \eta^3]$$

The diagonal numerical matrix, $[C^*]$, and the vector, $\{\beta\}$ are given by

$$[C^*] = [1, 3, 3, 3, 6, 3, 1, 3, 3, 1] \quad (3.3.11a)$$

$$\{\beta\}^T = [t_3^3, t_3^2 t_{13}, t_3^2 t_{23}, t_3 t_{13}^2, t_3 t_{13} t_{23}, t_3 t_{23}^2, t_{13}^3, t_{13}^2 t_{23}, t_{13} t_{23}^2, t_{23}^3] \quad (3.3.11b)$$

Substituting from equation (3.3.10) into equation (3.3.3) and carrying out integrations of the standard type, $f(m, n)$, it is found that the final stiffness matrix is in the same form as defined in equation (3.3.9), where the elements of the $[F]$ matrix are given by,

$$\begin{aligned}
 F_{ij} = \sum_{l=1}^{10} \alpha_l \{ & E_{11} m_i (m_i - 1) m_j (m_j - 1) f(II - 4, JJ) \\
 & + [E_{12} \{ m_i (m_i - 1) n_j (n_j - 1) + n_i (n_i - 1) m_j (m_j - 1) \} + E_{33} m_i n_i m_j n_j] \\
 & \times f(II - 2, JJ - 2) \\
 & + E_{13} \{ m_i (m_i - 1) m_j n_j + m_i n_i m_j (m_j - 1) \} f(II - 3, JJ - 1) \\
 & + E_{22} n_i (n_i - 1) n_j (n_j - 1) f(II, JJ - 4) \\
 & + E_{23} \{ n_i (n_i - 1) m_j n_j + m_i n_i n_j (n_j - 1) \} f(II - 1, JJ - 3) \} \\
 & i, j = 1 \rightarrow 21
 \end{aligned} \tag{3.3.12}$$

where

$$II = m_i + m_j + m_l$$

$$JJ = n_i + n_j + n_l$$

3.3.3 Consistent Load Vector

The consistent load vector for the high precision element can be obtained, in the usual manner by equating the work done by the distributed pressure $p(x, y)$, on the displacement over the entire element to that by the equivalent nodal force vector on the corresponding nodal displacements, i.e.

$$\{\mathfrak{S}\}^T \{\delta\} = \int \int p(x, y) w(x, y) dx dy$$

For uniform pressure, p , the consistent load vector is given by,

$$\begin{aligned}
 \{\mathfrak{S}\}^T &= p(2\Delta) \int_0^1 \int_0^{1-\xi} \{\Lambda\}^T [Q] d\xi d\eta \\
 &= \frac{p(2\Delta)}{2520} \{A\}^T [Q]
 \end{aligned} \tag{3.3.13}$$

where

$$\begin{aligned}
 \{A\}^T &= [1260, 420, 420, 210, 105, 210, 126, 42, 42, \\
 & \quad 126, 84, 21, 14, 21, 84, 60, 12, 6, 6, 12, 60]
 \end{aligned}$$

3.3.4 Numerical Results

A numerical example given in [63] is solved here to verify the above formulations for the tapered element. The problem relates to a simply supported, uniformly loaded (intensity

q), square plate of side length a , with its flexural rigidity varying only in the y direction according to the law $D = D_0 e^{\lambda y}$ ($D = D_0$ at $y = 0$ and $8D_0$ at $y = a$). Poisson's ratio is taken to be 0.25, and the computed deflections divided by $qa^4/10^4 D_0$ are presented in Table 1. A (4×8) mesh has been used for one half of the plate, Figure 9. The maximum error in deflection is found to be less than 0.3%.

3.4 A High Precision Laminated Anisotropic Plate Element

Multilayered composites have found wide use in many weight-sensitive structures, where high strength-to-weight and stiffness-to-weight ratios are required. For example, laminated anisotropic plates and shells are now extensively used as aircraft and missile structures. A laminate is a multilayered composite made up of several individual layers (laminae) in each of which the fibres are oriented in a predetermined direction to efficiently meet the required strength and stiffness parameters. The finite element formulation provides a convenient method of solution for such laminated composites having complex geometry and arbitrary loading. The ever increasing use of laminated composites demands development of efficient laminated anisotropic plate and shell finite elements. All the elements presented later could be easily modified into laminated anisotropic elements. In this section, this technique is explained through the development of a high precision triangular laminated anisotropic plate element. In the following chapter (4), formulations for laminated composite shallow shell elements will be presented, and these elements can be used to solve anisotropic plates by prescribing z -coordinates to be zeros or they can be modified to plate elements by neglecting the curvature terms in their formulations.

Many engineering laminates are thin and are analysed by the classical lamination theory, (CLT), by virtue of the Kirchhoff's hypothesis, which neglects the effects due to transverse shears and normal strain in thickness direction. It is, however, noted that these effects are more significant for laminated composite plates than for isotropic plates and a suitable shear deformation theory or elasticity theory is to be used even for moderately thick plates. The present element formulation is based on CLT and, thus, limited to thin plates. The element uses the same d.o.f. as the high precision plate bending element plus the tangential displacements in the x and y directions and their first derivatives for in-plane action. The displacement functions used here are the same as used in the previous high precision elements developed by Cowper *et al.* [8, 9, 10, 17]. Lakshminarayana and Viswanath [64] have modified the high precision cylindrical isotropic shell element [65] into a laminated

anisotropic element. However, their method involves the generation of the 38×38 local stiffness matrix coefficients one by one by numerous algebraic operations including the computation of several integrating functions of the type,

$$F(m, n) = c^{n+1} [a^{m+1} - (-b)^{m+1}] \frac{m! n!}{(m+n+2)!}$$

where m, n are integer numbers and a, b, c are some intrinsic length and height parameters of the element. Further, the stiffness matrix thus formed, is then transformed to the global coordinate system by a double matrix triple product. In the present element formulation, a fast and efficient algorithm is developed similar to that presented for the high precision tapered plate bending element above.

3.4.1 Review of Classical Lamination Theory

In the classical lamination theory [66, 67], the usual Kirchhoff hypothesis is made. That is, the transverse shear strains $\gamma_{xz} = \gamma_{yz} = 0$, and the normal strain in the thickness direction, $\epsilon_z = 0$. Under these assumptions, the strain-displacement relations for a point at a distance z from the middle surface of the laminate are given by,

$$\begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x} - z \frac{\partial^2 w}{\partial x^2} \\ \epsilon_y &= \frac{\partial v}{\partial y} - z \frac{\partial^2 w}{\partial y^2} \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - 2z \frac{\partial^2 w}{\partial x \partial y} \end{aligned} \quad (3.4.1)$$

or

$$\{\epsilon\} = \{\epsilon^0\} + z\{\kappa\}$$

where $\{\epsilon^0\}$ is the mid-plane strain vector and $\{\kappa\}$ is the curvature vector; u, v are tangential mid-plane displacements.

Each layer of the laminate is orthotropic and the stress-strain relations for a typical k^{th} layer with respect to its principal material axes are given by

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}_k = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix}_k \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{Bmatrix}_k \quad (3.4.2)$$

The coefficients of the stiffness matrix, Q , are expressible in terms of the engineering elastic

constants by,

$$\begin{aligned} Q_{11} &= \frac{E_{11}}{(1 - \nu_{12}\nu_{21})} \\ Q_{22} &= \frac{E_{22}}{(1 - \nu_{12}\nu_{21})} \\ Q_{12} &= \frac{\nu_{21}E_{11}}{(1 - \nu_{12}\nu_{21})} \\ Q_{66} &= G_{12} \end{aligned} \quad (3.4.3)$$

Of the five engineering elastic constants, only four are independent and the fifth is evaluated from the relation,

$$\nu_{21}E_{11} = \nu_{12}E_{22} \quad (3.4.4)$$

In terms of global coordinate system (x, y, z) , the elastic stress-strain relations for the orthotropic lamina are obtained as,

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_k = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix}_k \quad (3.4.5)$$

where the transformed reduced stiffnesses, $\bar{Q}_{ij}$, are given by,

$$\begin{aligned} \bar{Q}_{11} &= Q_{11}m^4 + 2(Q_{12} + 2Q_{66})m^2n^2 + Q_{22}n^4 \\ \bar{Q}_{12} &= (Q_{11} + Q_{22} - 4Q_{66})m^2n^2 + Q_{12}(m^4 + n^4) \\ \bar{Q}_{22} &= Q_{11}n^4 + 2(Q_{12} + 2Q_{66})m^2n^2 + Q_{22}m^4 \\ \bar{Q}_{16} &= (Q_{11} - Q_{12} - 2Q_{66})m^3n + (Q_{12} - Q_{22} + 2Q_{66})mn^3 \\ \bar{Q}_{26} &= (Q_{11} - Q_{12} - 2Q_{66})mn^3 + (Q_{12} - Q_{22} + 2Q_{66})m^3n \\ \bar{Q}_{66} &= (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})m^2n^2 + Q_{66}(m^4 + n^4) \end{aligned}$$

In the above

$$m = \cos \theta$$

$$n = \sin \theta$$

and θ is the angle from the global x -axis to the fibre direction of the lamina under consideration (Figure 10). For a laminate composed of n orthotropic laminae with arbitrary orientations and thicknesses, the constitutive relations are given by,

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \epsilon^0 \\ \kappa \end{Bmatrix} \quad (3.4.6)$$

Where, $\{N\}^T$ are the membrane forces, $[N_x, N_y, N_{xy}]$; $\{M\}^T$ are the bending stress resultants, $[M_x, M_y, M_{xy}]$; and $\{\epsilon^0\}$ and $\{\kappa\}$ are as defined earlier. $[A]$, $[D]$, $[B]$ are the effective

membrane, bending and bending-membrane coupling stiffness matrices of the laminate defined as,

$$(A_{ij}, B_{ij}, D_{ij}) = \sum_{k=1}^n \int_{h_k}^{h_{k+1}} (\bar{Q}_{ij})_k (1, z, z^2) dz \quad (3.4.7)$$

Here n is the total number of layers.

The subroutine *ABDCOM* given in Appendix D generates the $[A]$, $[B]$ and $[D]$ stiffness matrices for any arbitrary laminate.

3.4.2 Formulation of Element Stiffness Matrix

Following the procedure given for the tapered high precision element, the displacement functions for the element can be written in terms of the area coordinates ξ, η (ξ replaced by $1 - \xi - \eta$) as,

$$\begin{aligned} u &= \{\Lambda_1\}^T [R] \{\delta_u\} \\ v &= \{\Lambda_1\}^T [R] \{\delta_v\} \\ w &= \{\Lambda_2\}^T [S] \{\delta_w\} \end{aligned} \quad (3.4.8)$$

where the nodal displacement vectors are given by,

$$\begin{aligned} \{\delta_u\}^T &= [u_1, u_{x1}, u_{y1}, u_2, \dots, u_{y3}, u_c] \\ \{\delta_v\}^T &= [v_1, v_{x1}, v_{y1}, v_2, \dots, v_{y3}, v_c] \\ \{\delta_w\}^T &= [w_1, w_{x1}, w_{y1}, w_{xx1}, w_{xy1}, w_{yy1}, w_2, \dots, w_{yy3}] \end{aligned} \quad (3.4.9)$$

$\{\Lambda_1\}$ and $\{\Lambda_2\}$ are vectors of polynomial terms given by,

$$\{\Lambda_1\}^T = [1, \xi, \eta, \xi^2, \xi\eta, \eta^2, \xi^3, \xi^2\eta, \xi\eta^2, \eta^3] \quad (3.4.10)$$

$$\{\Lambda_2\}^T = [1, \xi, \eta, \xi^2, \xi\eta, \eta^2, \xi^3, \dots, \xi^5, \dots, \eta^5] \quad (3.4.11)$$

Here $u_{x1} = (\partial u / \partial x)_1$, $w_{xx1} = (\partial^2 w / \partial x^2)_1$, etc. $[R]$ and $[S]$ are transformation matrices.

Noting that every element of $\{\Lambda_1\}$ and $\{\Lambda_2\}$ is of the form $\xi^{m_i} \eta^{n_i}$, the displacement functions can be written as,

$$\begin{aligned} u &= \sum_{j=1}^{10} \delta_{u_j} \sum_{i=1}^{10} \xi^{m_i} \eta^{n_i} R_{ij} \\ v &= \sum_{j=1}^{10} \delta_{v_j} \sum_{i=1}^{10} \xi^{m_i} \eta^{n_i} R_{ij} \\ w &= \sum_{j=1}^{18} \delta_{w_j} \sum_{i=1}^{21} \xi^{m_i} \eta^{n_i} S_{ij} \end{aligned} \quad (3.4.12)$$

The strain energy of a thin laminated plate element is given by the integral over the element,

$$U = \frac{1}{2} \int \int (\{N\}^T \{\epsilon^0\} + \{M\}^T \{\kappa\}) dx dy \quad (3.4.13)$$

Substituting from equation (3.4.6) into equation (3.4.13),

$$U = \frac{1}{2} \int \int (\{\epsilon^0\}^T [A] \{\epsilon^0\} + \{\epsilon^0\}^T [B] \{\kappa\} + \{\kappa\}^T [B] \{\epsilon^0\} + \{\kappa\}^T [D] \{\kappa\}) dx dy \quad (3.4.14)$$

In terms of area coordinates,

$$\begin{aligned} \{\epsilon^0\} &= \frac{1}{2\Delta} [P_1] \{\alpha\} \\ \{\kappa\} &= \frac{1}{(2\Delta)^2} [P_2] \{\beta\} \end{aligned} \quad (3.4.15)$$

where

$$\begin{aligned} \{\alpha\}^T &= [u_\xi, u_\eta, v_\xi, v_\eta] \\ \{\beta\}^T &= -[w_{\xi\xi}, w_{\eta\eta}, w_{\xi\eta}] \\ [P_1] &= \begin{bmatrix} b_1 & b_2 & 0 & 0 \\ 0 & 0 & c_1 & c_2 \\ c_1 & c_2 & b_1 & b_2 \end{bmatrix} \\ [P_2] &= \begin{bmatrix} b_1^2 & b_2^2 & 2b_1b_2 \\ c_1^2 & c_2^2 & 2c_1c_2 \\ 2b_1c_1 & 2b_2c_2 & 2(b_1c_1 + b_2c_2) \end{bmatrix} \end{aligned}$$

Δ and $b_i, c_i; i = 1 \rightarrow 3$ are as defined previously.

Substituting from equation (3.4.15) into equation (3.4.14),

$$\begin{aligned} 2U &= \int_0^1 \int_0^{1-\xi} (\{\alpha\}^T [\bar{A}] \{\alpha\} + \{\alpha\}^T [\bar{B}] \{\beta\} \\ &\quad + \{\beta\}^T [\bar{B}]^T \{\alpha\} + \{\beta\}^T [\bar{D}] \{\beta\}) d\xi d\eta \end{aligned} \quad (3.4.16)$$

where

$$\begin{aligned} [\bar{A}] &= \frac{1}{2\Delta} [P_1]^T [A] [P_1] \\ [\bar{B}] &= \frac{1}{(2\Delta)^2} [P_1]^T [B] [P_2] \\ [\bar{D}] &= \frac{1}{(2\Delta)^3} [P_2]^T [D] [P_2] \end{aligned}$$

The steps in the derivation are now similar to that of the high precision plate bending element presented earlier and only the final form for the strain energy is given here as,

$$2U = \begin{Bmatrix} \delta_u^T \\ \delta_v^T \\ \delta_w^T \end{Bmatrix} [K] \begin{Bmatrix} \delta_u \\ \delta_v \\ \delta_w \end{Bmatrix} \quad (3.4.17)$$

where the element stiffness matrix, $[K]$, is given by

$$[K] = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12} & X_{22} & X_{23} \\ X_{13} & X_{23} & X_{33} \end{bmatrix}$$

where

$$X_{11} = [R]^T [F_1] [R]$$

$$X_{12} = [R]^T [F_2] [R]$$

$$X_{13} = -[R]^T [G_1] [S]$$

$$X_{22} = [R]^T [F_3] [R]$$

$$X_{23} = -[R]^T [G_2] [S]$$

$$X_{33} = [R]^T [H] [S]$$

The matrices $[F_2]$, $[G_1]$ and $[H]$ are defined as follows;

$$\begin{aligned} F_{2,ij} = & \bar{A}_{13} m_i m_j f(II - 2, JJ) + (\bar{A}_{14} m_i n_j + \bar{A}_{23} n_i m_j) f(II - 1, JJ - 1) \\ & + \bar{A}_{24} n_i n_j f(II, JJ - 2) \end{aligned} \quad (3.4.18)$$

$i, j = 1, 10$

$$\begin{aligned} G_{1,ij} = & \bar{B}_{11} m_i m_j (m_j - 1) f(II - 3, JJ) \\ & + \{ \bar{B}_{12} m_i n_j (n_j - 1) + \bar{B}_{23} n_i m_j n_j \} f(II - 1, JJ - 2) \\ & + \{ \bar{B}_{13} m_i m_j n_j + \bar{B}_{21} n_i m_j (m_j - 1) \} f(II - 2, JJ - 1) \\ & + \bar{B}_{22} n_i n_j (n_j - 1) f(II, JJ - 3) \end{aligned} \quad (3.4.19)$$

$i = 1, 10; j = 1, 21$

$$\begin{aligned} H_{ij} = & \bar{D}_{11} m_i (m_i - 1) m_j (m_j - 1) f(II - 4, JJ) \\ & + \{ \bar{D}_{12} [m_i (m_i - 1) n_j (n_j - 1) + n_i (n_i - 1) m_j (m_j - 1)] \\ & + \bar{D}_{33} m_i n_i m_j n_j \} f(II - 2, JJ - 2) \\ & + \bar{D}_{13} [m_i (m_i - 1) m_j n_j + m_i n_i m_j (m_j - 1)] f(II - 3, JJ - 1) \\ & + \bar{D}_{22} n_i (n_i - 1) n_j (n_j - 1) f(II, JJ - 4) \\ & + \bar{D}_{23} [n_i (n_i - 1) m_j n_j + m_i n_i n_j (n_j - 1)] f(II - 1, JJ - 3) \end{aligned} \quad (3.4.20)$$

$i, j = 1, 21$

where

$$II = m_i + m_j$$

where

$$II = m_i + m_j$$

$$JJ = n_i + n_j$$

and

$$f(M, N) = \frac{M! N!}{(M + N + 2)!}$$

The symmetric matrices $[F_1]$, $[F_3]$ and the matrix $[G_3]$ are obtained from equations (3.4.12) and (3.4.19), respectively, by replacing the elements of $[\bar{A}]$ and $[\bar{B}]$ matrices as shown in the following table.

Elements of the matrices $[F_1]$, $[F_3]$ and $[G_2]$

Originals in F_2	Replacements F_1	for F_3
$\bar{A}_{13}$	$\bar{A}_{11}$	$\bar{A}_{33}$
$\bar{A}_{14}$	$\bar{A}_{12}$	$\bar{A}_{34}$
$\bar{A}_{23}$	$\bar{A}_{12}$	$\bar{A}_{34}$
$\bar{A}_{24}$	$\bar{A}_{22}$	$\bar{A}_{44}$

Originals in G_1	Replacements for G_2
$\bar{B}_{11}$	$\bar{B}_{31}$
$\bar{B}_{12}$	$\bar{B}_{32}$
$\bar{B}_{13}$	$\bar{B}_{33}$
$\bar{B}_{21}$	$\bar{B}_{41}$
$\bar{B}_{22}$	$\bar{B}_{42}$
$\bar{B}_{23}$	$\bar{B}_{43}$

All these matrices are found explicitly and provided in the program listing given in Appendix E. Advantage has been taken of the fact that the first few rows and columns for these matrices contain only zero coefficients. Even though in the derivation of the method, for convenience of illustration, the nodal displacements δ_u , δ_v and δ_w are segregated, by a simple programming logic, the stiffness matrix resulting from the subroutine is made to correspond to the conventional ordering of the d.o.f. at the nodes, viz. $(u_1, u_{z1}, u_{y1}, v_1, v_{z1}, v_{y1}, w_1, w_{z1}, w_{y1}, w_{xz1}, w_{xy1}, w_{yy1}, u_2, \dots, w_{yy3}, u_c, v_c)$. The centroidal displacements, u_c and v_c , can now be easily condensed out and the subroutine readily linked to standard packages.

3.4.3 Numerical Example

In order to verify the above formulations, a sample problem is solved and the results compared with a series (20 terms) solutions [68, 69] considered to be essentially exact. The problem (Figure 11) relates to a two layer (each of thickness 0.01 in.), angle ply ($\pm\theta$) square (side length = 10 in.) laminate, simply supported on all sides and subject to a uniformly distributed transverse load (100 psi). Layer material properties are $E_{11} = 40 \times 10^6$ psi, $E_{22} = 10^6$ psi, $\nu_{12} = 0.25$; $G_{12} = 0.5 \times 10^6$ psi. Displacements obtained with the present element are in excellent agreement with the series solution as shown in the table below.

Results for a two layer simply supported square plate under uniform transverse loading

		Fibre Orientations, $\pm\theta$				
		$\pm 5^\circ$	$\pm 15^\circ$	$\pm 25^\circ$	$\pm 35^\circ$	$\pm 45^\circ$
u at $(a/2, 0)$	Exact	0.265	0.771	1.135	1.465	1.481
	Present	0.263	0.764	1.127	1.459	1.478
v at $(0, a/2)$	Exact	1.871	3.281	2.502	1.610	1.481
	Present	1.870	3.278	2.500	1.609	1.478
w at $(a/2, a/2)$	Exact	592.0	892.7	983.8	945.1	915.2
	Present	591.9	892.8	983.9	945.2	915.3

3.5 Conclusion of Conventional Method

In this section, a review of thin plate bending elements based on the conventional method (method of formulating C_1 continuity elements) was given. It is seen that inclusion of higher order derivatives as additional degrees of freedom is the simplest way of generating C_1 continuity elements. Among the higher order elements, the eighteen d.o.f. high precision element emerges as the most popular thin plate bending element that predicts both displacements and stresses with sufficient accuracy. However, the usual formulation of the stiffness matrix for this element is quite time consuming. Therefore, a fast and efficient formulation has been proposed here. Two basic matrices are presented explicitly and by pre and post multiplying one with the other, the element stiffness matrix is obtained. Both uniform and linearly varying thickness are considered. A brief review of the classical lamination theory is presented and a high precision laminated anisotropic thin plate element has been developed. Its stiffness matrix is obtained in closed form. FORTRAN subroutines are presented to ease implementation of these two elements.

However, many researchers still prefer simple plate bending elements (elements with

nine d.o.f.) which are cost effective and with which the boundary conditions can be imposed with relative ease. In a recent search [70] of plate bending elements with three d.o.f. at the corner nodes only, the DKT (Discrete Kirchhoff Triangle) and HSM (Hybrid Stress Model) elements were found to be the most efficient, cost effective and reliable elements of their class. The HSM element has also been formulated as a pure displacement type model (A-9 element) by *Irons and Razzaque* [3, 71]. Explicit shape functions are also available for this element and therefore this formulation is much preferred by users over the HSM element. Both elements follow a non-conventional approach. For the DKT element, the Kirchhoff's hypothesis is imposed only on selected points on the element. The A-9 is a nonconforming element derived from a conforming element, BC1Z2 [48] by replacing the true derivatives of some singular functions by their least squares smoothed psuedo-derivatives [3]. In the following, new compact and efficient formulations are provided for these two elements.

3.6 The DKT Plate Bending Element

In the Discrete Kirchhoff approach the Mindlin plate theory is followed initially. However, the strain energy due to bending alone is considered (neglecting that due to transverse shear strains) and the Kirchhoff hypothesis imposed at the element nodal points. A more detailed description of this approach is presented later while developing a few new plate bending elements. In Reference [61], *Batoz* presents the stiffness matrix of the DKT element in explicit form, but in a local coordinate system. This means that the stiffness matrix is then to be transformed to a global coordinate system before assembly, through a matrix triple product which is one of the more time consuming procedures in finite element analysis. Further, if a local coordinate system is used to evaluate the element stiffness matrix, then the elasticity matrix for every element must also be transformed to the local coordinates in the case of anisotropic materials. The algorithm presented here obviates these difficulties and produces the stiffness matrix directly in global coordinates. The formulation is thus believed to be much faster than that presented by *Batoz*, since it avoids both stiffness matrix transformation to global, and elasticity matrix transformation to local coordinates.

3.6.1 DKT Element Stiffness Matrix

For the DKT element, the bending curvature vector, $\{\kappa\}$, and the elasticity matrix, D_b , for the most general case of multilayered composite plates and sandwich plates are

given by,

$$\{\kappa\} = \begin{Bmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_{12} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \beta_x}{\partial x} \\ \frac{\partial \beta_y}{\partial y} \\ \frac{\partial \beta_x}{\partial y} + \frac{\partial \beta_y}{\partial x} \end{Bmatrix} \quad (3.6.1)$$

and

$$[D_b] = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{12} & D_{22} & D_{23} \\ D_{13} & D_{23} & D_{33} \end{bmatrix} \quad (3.6.2)$$

Here β_x and β_y are the rotations of the normal to the undeformed middle surface in the $x-z$ and $y-z$ planes respectively, and they are related to the element degrees of freedom [70] as,

$$\begin{aligned} \beta_x &= \langle \bar{G} \rangle \{\delta\} \\ \beta_y &= \langle \bar{H} \rangle \{\delta\} \end{aligned} \quad (3.6.3)$$

where,

$$\{\delta\}^T = \left[w_1, \left(\frac{\partial w}{\partial y} \right)_1, \left(\frac{-\partial w}{\partial x} \right)_1, w_2, \dots, \left(\frac{-\partial w}{\partial x} \right)_3 \right]$$

The shape function vectors $\langle \bar{G} \rangle$ and $\langle \bar{H} \rangle$ presented explicitly in terms of area coordinates ξ, η (ζ replaced by $1 - \xi - \eta$), in Reference [70], can be rewritten as follows;

$$\begin{aligned} \langle \bar{G} \rangle &= [1 \ \xi \ \eta \ \xi^2 \ \xi\eta \ \eta^2] [G] \\ \langle \bar{H} \rangle &= [1 \ \xi \ \eta \ \xi^2 \ \xi\eta \ \eta^2] [H] \end{aligned} \quad (3.6.4)$$

Both $[G]$ and $[H]$ are of order (6×9) .

Simplification results in the terms of the curvature vector as,

$$\begin{aligned} \kappa_1 &= \frac{1}{2\Delta} [1 \ \xi \ \eta] \begin{bmatrix} b_2 \langle G \rangle_2 + b_3 \langle G \rangle_3 \\ 2b_2 \langle G \rangle_4 + b_3 \langle G \rangle_5 \\ b_2 \langle G \rangle_6 + 2b_3 \langle G \rangle_6 \end{bmatrix} \{\delta\} \\ &= \frac{1}{2\Delta} \langle \gamma \rangle [X] \{\delta\} \end{aligned} \quad (3.6.5)$$

Here $b_i = (y_j - y_m)$ and $c_i = (x_m - x_j)$ with i, j, m taking values 1, 2, 3, cyclically. Δ is the area of the triangle with vertices (x_1, y_1) , (x_2, y_2) and (x_3, y_3) . Also, $\langle G \rangle_i$ represents the i^{th} row of the $[G]$ matrix. Similarly,

$$\begin{aligned} \kappa_2 &= \frac{1}{2\Delta} \langle \gamma \rangle [Y] \{\delta\} \\ \kappa_{12} &= \frac{1}{2\Delta} \langle \gamma \rangle [Z] \{\delta\} \end{aligned} \quad (3.6.6)$$

where,

$$[Y] = \begin{bmatrix} c_2 \langle H \rangle_2 + c_3 \langle H \rangle_3 \\ 2c_2 \langle H \rangle_4 + c_3 \langle H \rangle_5 \\ c_2 \langle H \rangle_6 + 2c_3 \langle H \rangle_6 \end{bmatrix} \quad (3.6.7)$$

$$[Z] = \begin{bmatrix} c_2 < G >_2 + c_3 < G >_3 + b_2 < H >_2 + b_3 < H >_3 \\ 2c_2 < G >_4 + c_3 < G >_5 + 2b_2 < H >_4 + b_3 < H >_5 \\ c_2 < G >_5 + 2c_3 < G >_6 + b_2 < H >_5 + 2b_3 < H >_6 \end{bmatrix} \quad (3.6.8)$$

Substituting for the curvature terms into the strain energy equation (3.3.1) and carrying out integrations of the type defined in equation (3.3.6), the strain energy, U , for the DKT element can be written as,

$$2U = \{\delta\}^T [K] \{\delta\}$$

where the global stiffness matrix, $[K]$, is given by,

$$[K] = \frac{1}{2\Delta} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}^T \begin{bmatrix} D_{11}[R] & D_{12}[R] & D_{13}[R] \\ & D_{22}[R] & D_{23}[R] \\ Sym. & & D_{33}[R] \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \quad (3.6.9)$$

with,

$$[R] = \frac{1}{24} \begin{bmatrix} 12 & 4 & 4 \\ 4 & 2 & 1 \\ 4 & 1 & 2 \end{bmatrix} \quad (3.6.10)$$

A FORTRAN subroutine implementing the above algorithm is presented in Appendix F. Since this formulation avoids two matrix triple products needed in an earlier formulation, the algorithm is believed to be faster. Actual time comparison could not be made since FORTRAN coding was not available in Reference [61]. The formulation was verified by many sample problems presented in References [61, 70], but are not presented here to avoid repetition. The introduction of linear taper to this element is very simple, following the procedure outlined earlier for the high precision element.

3.7 The A-9 Plate Bending Element

As mentioned before, the A-9 element is a displacement version of a hybrid element (HSM) originally developed by Allman [72] and reformulated by Batoz *et al.* [70]. The A-9 element is particularly attractive being a displacement model. Also, due to its shape function formulation, any new type of force loading can be introduced easily, and the element can be conveniently adapted to other biharmonic problems. The performance can also be improved by introducing a bubble function, a response without nodal connections [50, 73]. In fact, it is surprising that the HSM element has been chosen in Reference [70] instead of A-9. In the following, new explicit expression of the stiffness matrix is presented for the A-9 element. Both uniform as well as linearly varying thickness are considered. The algorithm is significantly faster (by nearly 30 percent) than the standard three-point numerical integration procedure and should be particularly useful for non-linear analysis.

3.7.1 The A-9 Element Stiffness Matrix

The shape functions N for the A-9 element [3, 48] are essentially a combination of polynomials and three rational functions ϵ_1 , ϵ_2 , ϵ_3 in area coordinates as shown below:

$$\begin{aligned} N_1 &= \xi(1 + \xi - \gamma) - 2\sigma - 6(\lambda_2\epsilon_2 + \rho_3\epsilon_3) \\ N_2 &= c_3\{\xi^2\eta + \epsilon_2 + \epsilon_3(2 + 3\lambda_3)\} - c_2\{\xi^2\zeta + \epsilon_3 + \epsilon_2(2 + 3\rho_2)\} \\ N_3 &= -b_3\{\xi^2\eta + \epsilon_2 + \epsilon_3(2 + 3\lambda_3)\} + b_2\{\xi^2\zeta + \epsilon_3 + \epsilon_2(2 + 3\rho_2)\} \end{aligned} \quad (3.7.1)$$

where

$$\begin{aligned} \sigma &= \epsilon_1 + \epsilon_2 + \epsilon_3 \\ \gamma &= \xi^2 + \eta^2 + \zeta^2 \\ \epsilon_1 &= \frac{\xi\eta^2\zeta^2(1 + \xi)}{(\xi + \eta)(\xi + \zeta)} \\ \rho_1 &= \frac{b_1b_2 + c_1c_2}{b_1^2 + c_1^2} \\ \lambda_1 &= -(1 + \rho_1) \end{aligned}$$

The other shape functions $N_4 \rightarrow N_9$ are obtained by cyclically changing the area coordinates $\xi \rightarrow \eta \rightarrow \zeta$ and the subscripts $1 \rightarrow 2 \rightarrow 3$. The intrinsic parameters b_i and c_i , $i = 1, 2, 3$ of the triangle are as defined in the earlier chapters.

While deriving the stiffness matrix, the true second derivatives of the rational functions are replaced with smoothed pseudo-derivatives which are the least squares polynomial versions of the true derivatives. The smoothed versions, ϵ_i^* , of the rational functions, ϵ_i , are given by

$$\begin{aligned} \epsilon_1^* &= \frac{1}{6}(2\xi^3 - 3\xi^2 + \xi) \\ \epsilon_2^* &= \frac{1}{6}(2\eta^3 - 3\eta^2 + \eta) \\ \epsilon_3^* &= \frac{1}{6}(2\zeta^3 - 3\zeta^2 + \zeta) \end{aligned} \quad (3.7.2)$$

Because of the substitution of the derivatives of smoothed functions, the high order numerical integration scheme needed due to the presence of singularities in the rational function derivatives is no longer required, and a three-point scheme is sufficient for exact integration. It can be noted that the element stiffness matrix can now be obtained explicitly without numerical integration which otherwise is not possible with true derivatives. Using the shape functions N , the normal displacement, w , anywhere on the element is obtained from,

$$w = [N] \{\delta\} \quad (3.7.3)$$

where $\{\delta\}$ is the vector of nodal displacements given by,

$$\{\delta\}^T = \left[w_1, \left(\frac{\partial w}{\partial x} \right)_1, \left(\frac{\partial w}{\partial y} \right)_1, w_2, \dots, \left(\frac{\partial w}{\partial y} \right)_3 \right]$$

The strain energy, U , of the plate bending element can be written in the same form as defined by equation (3.3.3), i.e.

$$2U = \frac{1}{(2\Delta)^3} \int_0^1 \int_0^{1-\xi} \frac{t^3}{12} \{H\}^T [E] \{H\} d\xi d\eta \quad (3.7.4)$$

Replacing the true derivatives of the rational functions with smoothed psuedo-derivatives, i.e.

$$\frac{\partial^2 \epsilon_i}{\partial \xi^2} \text{ instead of } \frac{\partial^2 \epsilon_i}{\partial \xi^2} \text{ etc.,}$$

$\{H\}$ vector can be written as

$$\{H\} = [R] [Q] \{\delta\} \quad (3.7.5)$$

where

$$[R] = \begin{bmatrix} 2 & 0 & 0 & 6\xi & 2\eta & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 2\xi & 6\eta \\ 0 & 1 & 0 & 0 & 2\xi & 2\eta & 0 \end{bmatrix}$$

and $[Q]$ is a (7×9) matrix of constants, dependent upon only the triangle geometry.

An analysis similar to that of the high precision plate bending element leads to the stiffness matrix to a final form as,

$$[K] = \frac{1}{(2\Delta)^3} [Q]^T [F] [Q] \quad (3.7.6)$$

Both the 7×7 $[F]$ matrix and the 7×9 $[Q]$ matrix are presented explicitly in the FORTRAN program listing given in Appendix G. This program generates the stiffness matrix for the linearly tapered or for the uniformly thick element.

3.7.2 Consistent Load Vector for the A-9 Element

The consistent load vector for uniform or linearly varying distributed loads can be found in explicit form if the rational functions are replaced by their smoothed psuedo-functions.

For uniformly distributed loads of intensity q , the consistent load vector, $\{\mathfrak{F}\}$ is given by,

$$\{\mathfrak{F}\}^T = q \frac{(2\Delta)}{60} [p_1, p_2, p_3, \dots, p_9] \quad (3.7.7)$$

where

$$p_1 = 9.0 - \lambda_2 - p_3$$

$$p_2 = c_3(1.5 + 0.5\lambda_3) - c_2(1.5 + 0.5\rho_2)$$

$$p_3 = -b_3(1.5 + 0.5\lambda_3) - b_2(1.5 + 0.5\rho_2)$$

The other six coefficients can be generated by cyclic permutation of the subscripts.

The algorithms presented above for the DKT and A-9 elements are so compact and fast that they can be conveniently used even with small computers. Since the stiffness matrices are obtained in closed form (without numerical integration), the introduction of linear taper to the elements does not increase the computational cost which is usually the case with numerically integrated elements. In general, it is believed that the formulations presented here for the high precision, DKT and A-9 elements are currently superior to all those published earlier.

3.8 Construction of New Efficient Plate Bending Elements

In the previous section, a brief review of development of thin plate bending elements was given. Some of the best plate bending elements were identified, the high precision among the higher order elements, and the DKT and A-9 among simple elements. New efficient formulations were provided for these elements. The current trend is to develop accurate but simple plate bending elements (elements with three d.o.f. at the corner nodes only) rather than to use sophisticated higher order elements. The formulation of DKT and A-9 elements is simple. Even though these elements are reliable, they converge a little slowly, with the A-9 element appearing to be too stiff, and the DKT too flexible, as well as very sensitive to mesh orientations. In fact, the two most accurate simple triangular plate bending elements currently available are not these two, but the following, one developed by *Hansen et al.* [74], and the other by *Fricker* [75]. The former is known as the HBS element and the latter as the composite triangle (CT). However, their formulations are very complicated. The HBS element is based on an entirely new approach. The element is derived directly from the satisfaction of the patch test condition. The CT element is an improved version of the A-9 element with the introduction of a bubble function (a response without nodal connections) to make it more flexible. The idea was originally conceived by *Irons and Khalil* [73], who were able to improve the performance only marginally. Recently using similar ideas, *Fricker* [75] has succeeded in softening the A-9 element sufficiently. Both the HBS and CT elements give highly competitive performance.

The HBS and CT elements, though accurate, have certain drawbacks. The stiffness matrix for the HBS element appears to be quite complex and time consuming. The response to mesh orientation is not fully documented. The formulation of CT element is too 'synthetic' and has no wider applications. It requires more computer operations than the original A-9 element, since a 12×12 stiffness matrix is to be formed first and then condensed later to a 9×9 matrix. In addition, the strain matrix is not independent of the stiffness matrix and thus the calculation of stresses becomes more complex and costly. It is noted here that another recent element, a mixed model developed by *Karamanlidis et al.* [76], as a possible candidate for finite element packages, reproduces the same numerical results for deflections exactly as those obtained with the A-9 element, and thus offers no improvement. A displacement-type element with equal or higher accuracy is preferred by users. The aim of the present chapter is therefore to develop displacement-type plate bending elements with accuracy comparable to those of HBS and CT elements, but with simplicity and economy comparable to those of DKT and A-9 elements. It is shown here that by combining the discrete Kirchhoff theory and the least squares technique, a number of high performance nonconforming elements can be constructed. Three such elements, two triangular and the third a quadrilateral, are presented here. All these elements pass the patch test (condition for assuring convergence) and give equally high or superior performance when compared with the HBS and CT or the DKQ (Discrete Kirchhoff Quadrilateral) element of *Batoz and Tahar* [77]. Other positive features are :

- (1) The algorithm is fast for the triangular elements (thus two 9×9 basic matrices are generated explicitly and by just pre and postmultiplying one with the other, the stiffness matrix is generated).
- (2) Both linearly varying as well as uniform thickness are considered for the triangles.
- (3) The shape functions are presented explicitly for the quadrilateral element.
- (4) FORTRAN codings are provided in Appendices H, I and J to ease implementations of all three elements.

3.8.1 Nonconforming Elements and the Patch Test

Both the HBS and CT are nonconforming nine-degree elements whose performance for degrees of freedom has so far not been bettered by other elements. In fact, nine d.o.f conforming triangles give such poor answers that their use does not appear justified. These nonconforming elements, though violating one of the essential conditions for convergence given earlier, certainly converge to the correct results, giving consistently superior results.

than the equivalent conforming elements. Similar successes experienced with some of the earlier nonconforming elements prompted the search for establishing new convergence criteria for such elements. Today, sufficient condition for convergence of nonconforming elements is established by Iron's patch test [78] and with this, the convergence of most of the non-standard, unconventional elements (i.e. nonconforming elements derived either from substitute shape functions which smooth the approximate original functions or from deliberate inexact numerical integration procedures or from any other principle which violates the required continuity) has now been proved. The patch test in its simplest form is described as follows.

A patch is an assembly of few elements arranged so that at least one node is completely surrounded by elements. Such internal nodes may be positioned so that the elements are of arbitrary shape (Figure 12). Now, displacements and/or forces consistent with a constant strain state are applied to the boundary nodes. Internal nodes are to be neither loaded nor restrained. Finite element analysis is carried out for the patch and the computed displacements, strains and stresses within elements should be consistent with constant strain state. If not, the element fails the patch test and becomes invalid.

For plate bending elements, the constant strain state corresponds to constant curvatures. An arbitrary constant curvature state can now be obtained by choosing the displacement function as,

$$w = a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2 \quad (3.8.1)$$

where $a_1 \rightarrow a_6$ are arbitrary constants. The slopes can be written as,

$$\frac{\partial w}{\partial x} = a_2 + 2a_4x + a_5y \quad (3.8.2)$$

$$\frac{\partial w}{\partial y} = a_3 + a_5x + 2a_6y \quad (3.8.3)$$

At the boundary nodes, the prescribed displacements are given by,

$$w_i = a_1 + a_2x_i + a_3y_i + a_4x_i^2 + a_5x_iy_i + a_6y_i^2 \quad (3.8.4)$$

$$\left(\frac{\partial w}{\partial x}\right)_i = a_2 + 2a_4x_i + a_5y_i \quad (3.8.5)$$

$$\left(\frac{\partial w}{\partial y}\right)_i = a_3 + a_5x_i + 2a_6y_i \quad (3.8.6)$$

where $i = 1 \rightarrow 5$.

A finite element analysis is carried out on this patch with the above prescribed displacements at the boundary nodes. For the element to pass the patch test, the computed displacements at the internal node 6 must be equal to the theoretical values given by,

$$w_6 = a_1 + a_2x_6 + a_3y_6 + a_4x_6^2 + a_5x_6y_6 + a_6y_6^2 \quad (3.8.7)$$

$$\left(\frac{\partial w}{\partial x}\right)_6 = a_2 + 2a_4x_6 + a_5y_6 \quad (3.8.8)$$

$$\left(\frac{\partial w}{\partial y}\right)_6 = a_3 + a_5x_6 + 2a_6y_6 \quad (3.8.9)$$

In addition, the computed curvatures at any point in the patch should be equal to the theoretically constant curvatures, given by

$$\frac{\partial^2 w}{\partial x^2} = 2a_4 \quad (3.8.10)$$

$$\frac{\partial^2 w}{\partial x \partial y} = a_5 \quad (3.8.11)$$

$$\frac{\partial^2 w}{\partial y^2} = 2a_6 \quad (3.8.12)$$

The patch test problems used by *Batoz et al.* [70, 77] for checking the DKT and DKQ (Discrete Kirchhoff Quadrilateral) elements are used for the new elements developed here.

3.8.2 Formulation of New Triangular Elements

The formulation closely resembles that of the DKT [70] element. Starting with Mindlin plate theory where the transverse displacement, w , and the normal rotations, β_x and β_y are independently defined, the strain energy due to bending alone is considered and the Kirchhoff hypothesis is imposed at a discrete number of points. The major difference between the DKT formulation and the present one is in the application of the Kirchhoff hypothesis. For DKT, it is imposed only at the nodes on the element boundary. In the present case, it is applied also at the interior (centroid of the triangle) of the element. This results in a seven noded triangle initially compared with the six noded triangle used by *Batoz et al.* [70], (see Figure 13). The normal rotations are interpolated from their nodal values as,

$$\begin{aligned} \beta_x &= \sum N_i \beta_{x_i} \\ \beta_y &= \sum N_i \beta_{y_i} \end{aligned} \quad i = 1, 2, \dots, 7 \quad (3.8.13)$$

The seven shape functions, which can be found easily by the procedure given in Reference [41], are given by

$$\begin{aligned}
 N_1 &= 1 - 3\xi - 3\eta + 2\xi^2 + 7\xi\eta + 2\eta^2 - 3(\xi^2\eta + \xi\eta^2) \\
 N_2 &= -\xi + 2\xi^2 + 3\xi\eta - 3(\xi^2\eta + \xi\eta^2) \\
 N_3 &= -\eta + 3\xi\eta + 2\eta^2 - 3(\xi^2\eta + \xi\eta^2) \\
 N_4 &= -8\xi\eta + 12(\xi^2\eta + \xi\eta^2) \\
 N_5 &= 4\eta - 16\xi\eta - 4\eta^2 + 12(\xi^2\eta + \xi\eta^2) \\
 N_6 &= 4\xi - 4\xi^2 - 16\xi\eta + 12(\xi^2\eta + \xi\eta^2) \\
 N_7 &= 27(\xi\eta - \xi^2\eta - \xi\eta^2)
 \end{aligned} \tag{3.8.14}$$

The nodal values of β_x and β_y are then directly related to the nine conventional corner node-degrees of freedom vector, $\{\delta\}$, by imposing the following constraints:

(i) The Kirchhoff hypothesis is imposed at;

(a) the corner nodes and the centroid

$$\begin{aligned}
 \beta_x &= -\left(\frac{\partial w}{\partial x}\right) \\
 \beta_y &= -\left(\frac{\partial w}{\partial y}\right)
 \end{aligned} \quad \text{at nodes 1, 2, 3 and 7}$$

(b) the mid-side nodes

$$\beta_s = -\left(\frac{\partial w}{\partial s}\right), \quad j = 4, 5, 6$$

where the suffix, s denotes the component of the rotations along the element side.

(2) The variation of w along the sides is cubic [70]

(3) A linear variation of β_n is imposed along the sides, i.e.

$$\beta_{n_k} = \frac{1}{2}(\beta_{n_i} + \beta_{n_j})$$

where k is the midside node of the side joining the corner nodes i and j

The vector $\{\delta\}$ is defined here as

$$\{\delta\}^T = \left[w_1, \left(\frac{\partial w}{\partial x}\right)_1, \left(\frac{\partial w}{\partial y}\right)_1, w_2, \dots, \left(\frac{\partial w}{\partial y}\right)_3 \right] \tag{3.8.15}$$

Since w is not uniquely defined at the interior of the element, the calculation of $\partial w / \partial x$ and $\partial w / \partial y$ at the centroid of the triangle is carried out by borrowing the shape function

for w , from some existing elements. However, the resulting elements are entirely different from these existing elements. Two elements, E-1 and E-2 are developed here. For E-1, the shape functions of the nonconforming BCIZ1 [48] element are used, whereas for E-2, those of the A-9 element are used.

The elements formed as above were found to be a little stiff. Also, if one chooses to use numerical integration procedure to obtain the element stiffness matrix, then simple 3-point integration schemes for triangles may not be sufficient to assure exact integration, since the shape function N in equation (3.8.14) contain biquadratic terms. To obviate the above difficulties, least squares technique is used to smooth the higher order terms. Again two procedures are used. For element E-2, the shape functions, N_i , are smoothed out using an integral least squares approach as follows.

A typical shape function, N_i is of the form;

$$\begin{aligned} N_i &= a_1 + a_2\xi + a_3\eta + a_4\xi^2 + a_5\xi\eta + a_6\eta^2 + a_7\xi^2\eta + a_8\xi\eta^2 \\ &= \sum_{j=1}^8 a_j \xi^{m_j} \eta^{n_j} \end{aligned} \quad (3.8.16)$$

The smoothing function, $\bar{N}_i$, is chosen as a complete quadratic fit, one order lower than the actual function, N_i . Thus,

$$\bar{N}_i = \sum_{j=1}^6 b_j \xi^{m_j} \eta^{n_j} \quad (3.8.17)$$

The unknown coefficients, b_i are obtained in terms of the known coefficients, a_i by using the standard least squares smoothing procedure by minimizing the functional

$$\Pi_1 = \int_A (N_i - \bar{N}_i)^2 d\xi d\eta \quad (3.8.18)$$

or, in turn, by solving a set of six linear simultaneous equations defined by

$$\frac{\partial \Pi_1}{\partial b_j} = 0 \quad \text{for } j = 1, 2, \dots, 6 \quad (3.8.19)$$

The smoothed shape functions, $\bar{N}_i$, are given as

$$\begin{aligned}
 \bar{N}_1 &= \frac{33}{35} + \frac{1}{7}(-18\xi - 18\eta + 11\xi^2 + 25\xi\eta + 11\eta^2) \\
 \bar{N}_2 &= \frac{-2}{35} + \frac{1}{7}(-4\xi + 3\eta + 11\xi^2 - 3\xi\eta - 3\eta^2) \\
 \bar{N}_3 &= \frac{-2}{35} + \frac{1}{7}(3\xi - 4\eta - 3\xi^2 - 3\xi\eta + 11\eta^2) \\
 \bar{N}_4 &= \frac{8}{35} + \frac{1}{7}(-12\xi - 12\eta + 12\xi^2 + 40\xi\eta + 12\eta^2) \\
 \bar{N}_5 &= \frac{8}{35} + \frac{1}{7}(-12\xi + 16\eta + 12\xi^2 - 16\xi\eta - 16\eta^2) \\
 \bar{N}_6 &= \frac{8}{35} + \frac{1}{7}(16\xi - 12\eta - 16\xi^2 - 16\xi\eta + 12\eta^2) \\
 \bar{N}_7 &= \frac{-18}{35} + \frac{27}{7}(\xi + \eta - \xi^2 - \xi\eta - \eta^2)
 \end{aligned} \tag{3.8.20}$$

For element E-1, the true derivatives, N'_i are substituted by their smoothed derivatives, $N^{*'}_i$ using a discrete least squares fit. The coefficients b of $N^{*'}_i$ are found by minimizing the functional.

$$\Pi_2 = \sum_{p=1}^7 (N'_i - N^{*'}_i)^2 \tag{3.8.21}$$

where p refers to the element node number. The smoothed derivatives for element E-1 are given below as,

$$\begin{aligned}
 N^{*'}_{1,\xi} &= \frac{1}{10}(-27 + 34\xi + 37\eta) \\
 N^{*'}_{2,\xi} &= \frac{1}{10}(-7 + 34\xi - 3\eta) \\
 N^{*'}_{3,\xi} &= \frac{1}{10}(3 - 6\xi - 3\eta) \\
 N^{*'}_{4,\xi} &= \frac{1}{10}(-12 + 24\xi + 52\eta) \\
 N^{*'}_{5,\xi} &= \frac{1}{10}(-12 + 24\xi - 28\eta) \\
 N^{*'}_{6,\xi} &= \frac{1}{10}(28 - 56\xi - 28\eta) \\
 N^{*'}_{7,\xi} &= \frac{1}{10}(27 - 54\xi - 27\eta)
 \end{aligned} \tag{3.8.22}$$

$$\begin{aligned}
N_{1,\eta}' &= \frac{1}{10}(-27 + 37\xi + 34\eta) \\
N_{2,\eta}' &= \frac{1}{10}(3 - 3\xi - 6\eta) \\
N_{3,\eta}' &= \frac{1}{10}(-7 - 3\xi + 34\eta) \\
N_{4,\eta}' &= \frac{1}{10}(-12 + 52\xi + 24\eta) \\
N_{5,\eta}' &= \frac{1}{10}(28 - 28\xi - 56\eta) \\
N_{6,\eta}' &= \frac{1}{10}(-12 - 28\xi + 24\eta) \\
N_{7,\eta}' &= \frac{1}{10}(27 - 27\xi - 54\eta)
\end{aligned} \tag{3.8.23}$$

The generation of the element stiffness matrix is hereafter straight forward and the stiffness matrices for E-1 and E-2 are generated in explicit form following the procedure given earlier in this chapter for the DKT element. Linear taper also has been introduced by the procedures outlined previously for the high precision element. The final form for the stiffness matrices are obtained as

$$[K] = \frac{1}{2\Delta} [Q]^T [E] [Q] \tag{3.8.24}$$

The stiffness matrices in the above explicit form, equation (3.8.24), are provided for elements E-1 and E-2 as FORTRAN codings in Appendices H and I. The $[Q]$ and $[E]$ matrices, both of size 9×9 , can be easily identified since these same variables are also defined in the program.

3.8.3 Numerical Results for New Triangular Elements

The elements are first checked for the patch test, followed by standard square plate examples. Numerical results for rectangular plates are presented and finally an example of a tapered plate is included.

Twisting of a Square Plate:

This problem has become the standard test case to evaluate the ability of a new element to represent constant twisting moment. A square plate with geometry and loading as shown in Figure 14 is analysed for the various meshes considered in References [70, 71]. The present elements like the other successful elements, reproduce the exact thin plate solution ($w = -0.2496$ at C , $w = -0.0624$ at O , $\partial w / \partial x = \partial w / \partial y = -0.0312$ at A and $M_x = 0$, $M_y = 0$, $M_{xy} = 2.5$ everywhere) for all the meshes. Some of the earlier elements, such as the nonconforming BCIZ1 [48], failed in this test.

Patch Test for a Rectangular Plate:

This problem, suggested by *Robinson* [79] was used by *Batoz* [61] to check his DKT formulation. The geometry, boundary conditions and applied loads (Figure 15) give a theoretical constant state of bending moments, $M_x = M_y = M_{xy} = 1$, all over the plate. The correct results are obtained everywhere (at the nodes as well as the centroids of the elements) both with DKT and the present elements. The deflections at the nodes are $w = -12.48$ at node 3 and $w = -1.62$ at node 5.

Square Plates:

Since HBS and CT have been identified as the two most accurate 3-noded nine d.o.f. thin plate bending elements, the accuracy of the present elements, E-1 and E-2, are mostly compared with those two. Two mesh orientations, A and B, shown in Figure 16 are used to represent one quarter of the plate with different numbers of elements. The results for the HBS element are available for mesh orientation A only. Moreover, no numerical values are available for HBS, and only graphical data is presented in References [74, 75].

Simply Supported Square Plate with Central Concentrated Load:

Figure 17 shows the percentage error in the central deflection for the HBS, CT, E-1 and E-2 elements for mesh orientation A. The CT element gives the best result for this case. The E-1 and E-2 elements give performance similar to that of HBS.

Clamped Square Plate with Central Concentrated Load:

The percentage error in central deflection for this case is shown in Figure 18. Elements E-1 and E-2 give the best performance for this case, followed by the HBS and the CT element.

Other Results for Square and Rectangular Plates:

Numerical results for deflections and moments obtained with the present elements are summarised in Tables 2— 15 extensively along with those obtained with the CT element for a number of square and rectangular plate problems. In the case of uniform loads, lumped nodal forces were used for all elements. It can be seen from these results that E-1 and E-2 elements give equally high performance with the CT element. The CT element, however, shows a little sensitivity for mesh orientation, *B* for the particular case of clamped plates.

Simply Supported Tapered Plate:

Finally, to verify the accuracy of the new elements for tapered plates, the same numerical example solved using tapered eighteen d.o.f. high precision element is analysed and results are compared in Table 16. The deflections obtained with the E-1 and E-2 elements are found to be better than those obtained with the high precision element for this example. A consistent load vector for uniform loading is used in this example. This load vector is obtained using the shape functions of the A-9 element and neglecting the singular rational functions. The load vector is,

$$\{S\}^T = -q \frac{(2\Delta)}{60} \left[10, (c_3 - c_2), (b_2 - b_3), 10, (c_1 - c_3), (b_3 - b_1), 10, (c_2 - c_1), (b_1 - b_2) \right] \quad (3.8.25)$$

and gives excellent results for other problems involving uniform loads. In the above, b_i and c_i , $i = 1, 2, 3$ are as defined earlier, for triangles.

3.8.4 A New Quadrilateral Plate Bending Element

In Reference [77], Batoz and Tahar have presented the formulation for a thin quadrilateral plate bending element which is based on a generalization of the DKT element. This element, called DKQ (Discrete Kirchhoff Quadrilateral), has three d.o.f. at the corner nodes only. The performance of DKQ is demonstrated through many standard tests and compared with other twelve d.o.f. elements. Even though the performance of this element is generally good, the convergence rate for displacements is not quite satisfactory. It was shown in the previous section that by combining the discrete Kirchhoff theory and the least squares technique, efficient triangular plate bending elements could be developed. This idea is extended here to attempt development of an improved discrete Kirchhoff quadrilateral (IDKQ) element. The results for typical displacements and stresses of IDKQ are compared with those of DKQ.

3.8.5 Formulation of the IDKQ Element

The formulation of the IDKQ element follows exactly those of the E-1 and E-2 elements. That is, it closely resembles the DKT and DKQ formulations, however, with a difference in the application of Kirchhoff hypothesis. For the DKT and DKQ, the Kirchhoff hypothesis is imposed only at the nodes on the element boundary. For IDKQ, it is applied also at the interior of the element. Thus the nine-noded Lagrange element is used here initially instead of the eight-noded Serendipity element used by Batoz and Tahar [77], (Figure 19).

Mindlin plate theory is assumed initially where the transverse displacement, w , and the normal rotations, β_x and β_y are independently defined. However, the strain energy of the element due to bending alone is considered. Thus,

$$U = \frac{1}{2} \int_A \{\kappa\}^T [D_b] \{\kappa\} dx dy \quad (3.8.26)$$

where the curvature vector, $\{\kappa\}$ and the bending elasticity matrix, $[D_b]$ are as defined earlier for the E-1 and E-2 elements.

The normal rotations are interpolated from their nodal values as:

$$\begin{aligned} \beta_x &= \sum_{i=1}^9 \bar{N}_i \beta_{x_i} \\ \beta_y &= \sum_{i=1}^9 \bar{N}_i \beta_{y_i} \end{aligned} \quad (3.8.27)$$

Here the shape functions, $\bar{N}_i$, are the substitute shape functions derived as the least squares polynomial versions of the original shape functions, N_i , of the Lagrange element. While the original shape functions, N_i , can be found in any standard text on the finite element

method, the substitute shape functions are presented below explicitly.

$$\begin{aligned}
 \bar{N}_1 &= -\frac{1}{36} + \frac{1}{12}\xi^2 + \frac{1}{4}\xi\eta + \frac{1}{12}\eta^2 - \frac{1}{4}\xi^2\eta - \frac{1}{4}\xi\eta^2 \\
 \bar{N}_2 &= -\frac{1}{36} + \frac{1}{12}\xi^2 - \frac{1}{4}\xi\eta + \frac{1}{12}\eta^2 - \frac{1}{4}\xi^2\eta + \frac{1}{4}\xi\eta^2 \\
 \bar{N}_3 &= -\frac{1}{36} + \frac{1}{12}\xi^2 + \frac{1}{4}\xi\eta + \frac{1}{12}\eta^2 + \frac{1}{4}\xi^2\eta + \frac{1}{4}\xi\eta^2 \\
 \bar{N}_4 &= -\frac{1}{36} + \frac{1}{12}\xi^2 - \frac{1}{4}\xi\eta + \frac{1}{12}\eta^2 + \frac{1}{4}\xi^2\eta - \frac{1}{4}\xi\eta^2 \\
 \bar{N}_5 &= \frac{1}{18} - \frac{1}{2}\eta - \frac{1}{6}\xi^2 + \frac{1}{3}\eta^2 + \frac{1}{2}\xi^2\eta \\
 \bar{N}_6 &= \frac{1}{18} + \frac{1}{2}\xi + \frac{1}{3}\xi^2 - \frac{1}{6}\eta^2 - \frac{1}{2}\xi\eta^2 \\
 \bar{N}_7 &= \frac{1}{18} + \frac{1}{2}\eta - \frac{1}{6}\xi^2 + \frac{1}{3}\eta^2 - \frac{1}{2}\xi^2\eta \\
 \bar{N}_8 &= \frac{1}{18} - \frac{1}{2}\xi + \frac{1}{3}\xi^2 - \frac{1}{6}\eta^2 + \frac{1}{2}\xi\eta^2 \\
 \bar{N}_9 &= \frac{8}{9} - \frac{2}{3}\xi^2 - \frac{2}{3}\eta^2
 \end{aligned} \tag{3.8.28}$$

The reason for using $\bar{N}_i$ instead of N_i is that, while the usage of N_i in the present approach improves the convergence rate for displacements, it also slows down the convergence rate for the stresses, whereas, using $\bar{N}_i$ improves convergence of both.

The nodal values of β_x and β_y of the straight sided quadrilateral are then directly related to the twelve conventional corner-node-degrees of freedom vector, $\{\delta\}$, by imposing the following constraints;

(1) The Kirchhoff hypothesis is imposed at

(a) the corner nodes and at the interior node

$$\begin{aligned}
 \beta_x &= -\left(\frac{\partial w}{\partial x}\right) \\
 \beta_y &= -\left(\frac{\partial w}{\partial y}\right)
 \end{aligned} \quad \text{at nodes } 1, 2, 3, 4, 9 \tag{3.8.29}$$

(b) the mid-side nodes

$$\beta_{s_j} = -\left(\frac{\partial w}{\partial s}\right), \quad j = 5, 6, 7, 8 \tag{3.8.30}$$

where the suffix, s , denotes the component of the rotations along the element side.

(2) The variation of w along the sides is cubic (details are given in reference 70).

(3) A linear variation of β_n is imposed along the sides, i.e.

$$\beta_{n_k} = \frac{1}{2}(\beta_{n_1} + \beta_{n_2}) \tag{3.8.31}$$

where k is the mid-side node of side joining the corner nodes i and j .

The vector $\{\delta\}$, is defined here as

$$\{\delta\}^T = \left[w_1, \left(\frac{\partial w}{\partial x} \right)_1, \left(\frac{\partial w}{\partial y} \right)_1, w_2, \dots, \left(\frac{\partial w}{\partial y} \right)_4 \right] \quad (3.8.32)$$

Since w is not uniquely defined at the interior of the element, the calculation of $\partial w / \partial x$ and $\partial w / \partial y$ at the interior node (node 9) is done assuming a bi-cubic expansion for w as follows.

$$w = a_1 + a_2\xi + a_3\eta + a_4\xi^2 + a_5\xi\eta + a_6\eta^2 + a_7\xi^3 + a_8\xi^2\eta + a_9\xi\eta^2 + a_{10}\eta^3 + a_{11}\xi^3\eta + a_{12}\xi\eta^3 \quad (3.8.33)$$

The coefficients $a_1 \rightarrow a_{12}$ can be uniquely expressed in terms of the nodal displacement vector and $\partial w / \partial x$, $\partial w / \partial y$ can be obtained at $\xi = 0$, $\eta = 0$ without any difficulty. The same expression for w is used for calculating the consistent load vector.

Using the above constraints and simplifying, the explicit expression of the rotations, β_x and β_y of a general four corner-noded quadrilateral in terms of the conventional nodal degrees of freedom vector, $\{\delta\}$, is obtained as;

$$\begin{aligned} \beta_x &= \sum_{i=1}^{12} G_i \delta_i = \{G\}^T \{\delta\} \\ \beta_y &= \sum_{i=1}^{12} H_i \delta_i = \{H\}^T \{\delta\} \end{aligned} \quad (3.8.34)$$

where the new shape functions, G and H are presented in terms of $\bar{N}$ as follows;

$$\begin{aligned} G_1 &= \frac{p_1}{4} \bar{N}_5 - \frac{p_4}{4} \bar{N}_8 + \frac{3}{2\Delta} y_{42} \bar{N}_9 \\ G_2 &= -\bar{N}_1 + \frac{(s_1 - 2)}{4} \bar{N}_5 + \frac{(s_4 - 2)}{4} \bar{N}_8 + \frac{y_{42}}{4\Delta} (x_{21} + x_{41}) \bar{N}_9 \\ G_3 &= -\frac{q_1}{4} \bar{N}_5 - \frac{q_4}{4} \bar{N}_8 + \frac{y_{42}}{4\Delta} (y_{21} + y_{41}) \bar{N}_9 \\ G_4 &= -\frac{p_1}{4} \bar{N}_5 + \frac{p_2}{4} \bar{N}_6 + \frac{3}{2\Delta} y_{13} \bar{N}_9 \\ G_5 &= -\bar{N}_2 + \frac{(s_1 - 2)}{4} \bar{N}_5 + \frac{(s_2 - 2)}{4} \bar{N}_6 + \frac{y_{13}}{4\Delta} (x_{12} + x_{32}) \bar{N}_9 \\ G_6 &= -\frac{q_1}{4} \bar{N}_5 - \frac{q_2}{4} \bar{N}_6 + \frac{y_{13}}{4\Delta} (y_{12} + y_{32}) \bar{N}_9 \\ G_7 &= -\frac{p_2}{4} \bar{N}_6 + \frac{p_3}{4} \bar{N}_7 + \frac{3}{2\Delta} y_{24} \bar{N}_9 \\ G_8 &= -\bar{N}_3 + \frac{(s_2 - 2)}{4} \bar{N}_6 + \frac{(s_3 - 2)}{4} \bar{N}_7 + \frac{y_{24}}{4\Delta} (x_{23} + x_{43}) \bar{N}_9 \end{aligned} \quad (3.8.35)$$

$$\begin{aligned}
G_9 &= -\frac{q_2}{4}\bar{N}_6 - \frac{q_3}{4}\bar{N}_7 + \frac{y_{24}}{4\Delta}(y_{23} + y_{43})\bar{N}_9 \\
G_{10} &= -\frac{p_3}{4}\bar{N}_7 + \frac{p_4}{4}\bar{N}_8 + \frac{3}{2\Delta}y_{31}\bar{N}_9 \\
G_{11} &= -\bar{N}_4 + \frac{(s_3 - 2)}{4}\bar{N}_7 + \frac{(s_4 - 2)}{4}\bar{N}_8 + \frac{y_{31}}{4\Delta}(x_{14} + x_{34})\bar{N}_9 \\
G_{12} &= -\frac{q_3}{4}\bar{N}_7 - \frac{q_4}{4}\bar{N}_8 + \frac{y_{31}}{4\Delta}(y_{14} + y_{34})\bar{N}_9
\end{aligned}$$

$$\begin{aligned}
H_1 &= -\frac{t_1}{4}\bar{N}_5 + \frac{t_4}{4}\bar{N}_8 + \frac{3}{2\Delta}x_{24}\bar{N}_9 \\
H_2 &= -\frac{q_1}{4}\bar{N}_5 - \frac{q_4}{4}\bar{N}_8 + \frac{x_{24}}{4\Delta}(x_{21} + x_{41})\bar{N}_9 \\
H_3 &= -\bar{N}_1 + \frac{(r_1 - 2)}{4}\bar{N}_5 + \frac{(r_4 - 2)}{4}\bar{N}_8 + \frac{x_{24}}{4\Delta}(y_{21} + y_{41})\bar{N}_9 \\
H_4 &= \frac{t_1}{4}\bar{N}_5 - \frac{t_2}{4}\bar{N}_6 + \frac{3}{2\Delta}x_{31}\bar{N}_9 \\
H_5 &= -\frac{q_1}{4}\bar{N}_5 - \frac{q_2}{4}\bar{N}_6 + \frac{x_{31}}{4\Delta}(x_{12} + x_{32})\bar{N}_9 \\
H_6 &= -\bar{N}_2 + \frac{(r_1 - 2)}{4}\bar{N}_5 + \frac{(r_2 - 2)}{4}\bar{N}_6 + \frac{x_{31}}{4\Delta}(y_{12} + y_{32})\bar{N}_9 \\
H_7 &= \frac{t_2}{4}\bar{N}_6 - \frac{t_3}{4}\bar{N}_7 + \frac{3}{2\Delta}x_{42}\bar{N}_9 \\
H_8 &= -\frac{q_2}{4}\bar{N}_6 - \frac{q_3}{4}\bar{N}_7 + \frac{x_{42}}{4\Delta}(x_{23} + x_{43})\bar{N}_9 \\
H_9 &= -\bar{N}_3 + \frac{(r_2 - 2)}{4}\bar{N}_6 + \frac{(r_3 - 2)}{4}\bar{N}_7 + \frac{x_{42}}{4\Delta}(y_{23} + y_{43})\bar{N}_9 \\
H_{10} &= \frac{t_3}{4}\bar{N}_7 - \frac{t_4}{4}\bar{N}_8 + \frac{3}{2\Delta}x_{13}\bar{N}_9 \\
H_{11} &= -\frac{q_3}{4}\bar{N}_7 - \frac{q_4}{4}\bar{N}_8 + \frac{x_{13}}{4\Delta}(x_{14} + x_{34})\bar{N}_9 \\
H_{12} &= -\bar{N}_4 + \frac{(r_3 - 2)}{4}\bar{N}_7 + \frac{(r_4 - 2)}{4}\bar{N}_8 + \frac{x_{13}}{4\Delta}(y_{14} + y_{34})\bar{N}_9
\end{aligned} \tag{3.8.36}$$

where

$$x_{ij} = x_i - x_j \quad \text{and} \quad y_{ij} = y_i - y_j$$

$$\Delta = (x_1y_{24} + x_2y_{31} + x_3y_{42} + x_4y_{13})$$

Also, for $i = 1 \rightarrow 4$

$$\begin{aligned} p_i &= \frac{6c_i}{(b_i^2 + c_i^2)^{3/2}} \\ \bar{q}_i &= \frac{3b_i c_i}{(b_i^2 + c_i^2)^{3/2}} \\ r_i &= \frac{3b_i^2}{(b_i^2 + c_i^2)^{3/2}} \\ s_i &= \frac{3c_i^2}{(b_i^2 + c_i^2)^{3/2}} \\ t_i &= \frac{6b_i}{(b_i^2 + c_i^2)^{3/2}} \end{aligned} \quad (3.8.37)$$

Here $b_1 = y_{12}$; $b_2 = y_{23}$; $b_3 = y_{34}$; $b_4 = y_{41}$; and $c_1 = x_{21}$; $c_2 = x_{32}$; $c_3 = x_{43}$; $c_4 = x_{14}$

The curvature vector, $\{\kappa\}$, can now be written as

$$\{\kappa\} = [B] \{\delta\} \quad (3.8.38)$$

where

$$[B] = [\{B_1\}, \{B_2\}, \dots, \{B_{12}\}]$$

with

$$\{B_i\} = \begin{Bmatrix} \frac{\partial G_i}{\partial x} \\ \frac{\partial H_i}{\partial y} \\ \frac{\partial G_i}{\partial y} + \frac{\partial H_i}{\partial x} \end{Bmatrix} \quad (3.8.39)$$

The Jacobian matrix, $[J]$, of the transformation between the cartesian coordinates (x, y) and the natural coordinates (ξ, η) is given by

$$[J] = \frac{1}{4} \begin{vmatrix} x_{21} + x_{34} + \eta(x_{12} + x_{34}) & y_{21} + y_{34} + \eta(y_{12} + y_{34}) \\ x_{32} + x_{41} + \xi(x_{12} + x_{34}) & y_{32} + y_{41} + \xi(y_{12} + y_{34}) \end{vmatrix} \quad (3.8.40)$$

The cartesian derivatives in the $[B]$ matrix are transformed into derivatives in natural coordinates and the stiffness matrix is obtained using the standard procedure as

$$\begin{aligned} [K] &= \int_A [B]^T [D_b] [B] dx dy \\ &= \int_{-1}^{+1} \int_{-1}^{+1} [B]^T [D_b] [B] |J| d\xi d\eta \end{aligned} \quad (3.8.41)$$

A standard 2×2 Gauss integration scheme is found to be sufficient for the integration, even though a 3×3 integration is theoretically necessary. No spurious mode is present in the element. The 2×2 scheme should not be confused with the standard reduced integration scheme, deliberately introduced in some Mindlin thick/thin plate formulations such as in

References [25, 26]. A FORTRAN program listing is provided in Appendix J implementing the above algorithm.

The bending moments, $\{M\}$, at any point x, y in the element are obtained as

$$\{M\} = \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = [D_b] [B(x, y)] \{\delta\} \quad (3.8.42)$$

3.8.6 Numerical Results for the Quadrilateral Element

Patch Test for Rectangular Plate

This is the same patch test to which the elements E-1 and E-2 were subjected. The arbitrary mesh used in the present case is shown in Figure 20. The correct results ($M_x = M_y = M_{xy} = 1$ at all the nodes and $w = -12.48$ at node 3) are obtained with IDKQ. Thus, the element passes the patch test.

Square Plates

As stated in Reference [77], the convergence of displacements for the DKQ element is not quite satisfactory. For the particular case of simply supported plates with uniform lumped loads, the deflections are excellent (Figure 21), but this appears to be only a peculiarity of this case. For although the use of lumped loads gives excellent results for displacements, poor results are obtained for the corner reactions (Figure 22). The use of consistent loads gives sufficiently accurate results, both for displacements as well as stress resultants, Figures 23 and 24. Therefore, only consistent loads are used throughout the following while comparing the performance of DKQ and IDKQ elements. Figures 25→32 show the comparison of the performance of the two elements for other standard problems. It is clear that the IDKQ element gives superior performance in almost all cases, and equal performance in the two exceptions, Figures 28 and 32. Performance of IDKQ is also compared with those of other 12 d.o.f. rectangular or quadrilateral elements in Figures 33→38 by reproducing the plots given in Reference [77], where the references for these elements can also be found.

Rectangular Plates

Rectangular plates with aspect ratios 1, 2 and 3 are analysed. Figure 39 shows the convergence of central deflection for clamped rectangular plates subjected to a central point load. Only for this case, there is noticeable difference between the performances of DKQ and IDKQ. The IDKQ element gives much superior results for this case. For all other rectangular plate problems, both the elements give essentially identical performance.

Single Element Tests

Batoz and Tahar [77] have shown that the performance of DKQ is excellent in the single element test suggested by Robinson [79]. The present element is also subjected to the same tests and the deflections obtained with both the elements for aspect ratios 1 and 12 are plotted for both the elements. The geometry and loading are also shown in Figures 40 and 41. All the tests carried out here clearly indicate the superiority of the IDKQ element in predicting both displacements as well as stresses.

3.9 Discussion and Conclusion—Plate Bending Elements

In this chapter, a review of thin plate bending elements based on the displacement approach was given. The difficulties associated with developing simple conforming plate bending elements are explained. A successful higher order plate bending element, the high precision element, and two nine d.o.f. plate bending elements, one based on the discrete Kirchhoff theory (DKT) and the other, a nonconforming but converging element, derived from substitute shape functions, have been identified as some of the best plate bending elements currently available. These elements have been completely reformulated so that the algorithms are more efficient and faster than they originally were. Two new nine d.o.f. triangular plate bending elements and a new twelve d.o.f. quadrilateral element have been developed. These triangular elements are shown to be the best of their class, taking into account the overall consideration of accuracy, simplicity and economy. The quadrilateral element is shown to give improved results over the DKQ element developed earlier on the same discrete Kirchhoff theory. Due to the importance of composites in modern structures, the method of development of a high precision laminated anisotropic thin plate element has been presented. This technique is developed for anisotropic versions of the other triangular elements in Chapter 4. For all the triangular elements presented here, the stiffness matrices are generated in closed form (no numerical integration). FORTRAN codings of the algorithms developed here, are appended for immediate use.

Chapter 4

Thin Laminated Anisotropic Shallow Shell Elements

In this chapter, a number of laminated anisotropic thin triangular shallow shell elements are developed. For all these elements, the membrane displacements, u and v , are represented by complete cubic polynomials. For the bending action, the triangular elements presented in the previous chapter are considered. The high precision isotropic shallow shell element developed by Cowper *et al.* [8] is first modified into a laminated anisotropic element, however, using a fast algorithm similar to the one presented for the plate bending element. Following this, laminated anisotropic shallow shell elements based on DKT, E-1 and E-2 elements are developed. The use of three noded triangular elements enables the determination of their stiffness matrices in closed form, thus completely avoiding numerical integration.

These elements can be readily used for isotropic cases by considering a single layer and constant elastic properties in all directions. A review of the classical lamination theory has already been presented in Chapter 3. A program listing for generating the effective membrane, bending and bending-membrane coupling stiffness matrices has also been presented in Appendix D. They are not repeated, and only the shallow shell equations necessary for the development of element stiffness matrices are reviewed here.

4.1 Shallow Shell Equations

As in the case of laminated plates, the strain energy, U , of a thin laminated shallow shell element is given by the integral over the element;

$$U = \frac{1}{2} \int \int \left(\{N\}^T \{\epsilon^0\} + \{M\}^T \{\kappa\} \right) dx dy \quad (4.1.1)$$

The constitutive relations for an arbitrary laminate are given by;

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \epsilon^0 \\ \kappa \end{Bmatrix} \quad (4.1.2)$$

where the membrane forces, $\{N\}$, bending stress resultants, $\{M\}$, and the elastic stiffness matrices $[A]$, $[B]$, $[D]$ for the arbitrary laminate are all as defined for the laminated plate element in Chapter 3.

Following the shallow shell theory of *Novozhilov* [80], the midplane membrane strain vector, $\{\epsilon^0\}$, and the curvature vector, $\{\kappa\}$, are defined in the cartesian base plane coordinate (x, y) as;

$$\{\epsilon^0\} = \begin{Bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} - w \frac{\partial^2 h}{\partial x^2} \\ \frac{\partial v}{\partial y} - w \frac{\partial^2 h}{\partial y^2} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - 2w \frac{\partial^2 h}{\partial x \partial y} \end{Bmatrix} \quad (4.1.3)$$

$$\{\kappa\} = - \begin{Bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ 2 \frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} \quad (4.1.4)$$

where u and v are the tangential displacements measured parallel to the x and y directions, respectively, w is the normal displacement and $h(x, y)$ is the height above the base plane defining the shell shape (Figure 42). The height function, h , is assumed to be of quadratic form as;

$$h(x, y) = c_1 + c_2x + c_3y + c_4x^2 + c_5xy + c_6y^2 \quad (4.1.5)$$

This implies that the shell element has constant curvatures and this is consistent with the approximation of shallow shell theory. In Reference [8], *Cowper et al.* show that, for a smooth shallow shell where $\partial h/\partial x$ and $\partial h/\partial y$ are continuous everywhere, the continuity of displacement and edgewise rotation is assured if the individual displacements u , v and w and the derivatives $\partial w/\partial x$ and $\partial w/\partial y$ are continuous. That is, they are exactly similar to those for the flat plate elements. Thus, the conforming high precision and DKT elements can readily be used to simulate the bending action in shallow shells. The nonconforming elements E-1 and E-2, though violating the continuity requirements, can also be used, since they have been proved to give converging as well as highly superior performance in plate bending situations. First, the formulation of the high precision anisotropic shallow shell element is given in detail. The formulation for the other elements is essentially same with a few changes; thus only the equations that are to be modified are presented to avoid repetition of the steps. However, complete FORTRAN codings are appended for all these elements to ease implementation.

4.2 A High Precision Laminated Anisotropic Shallow Shell Element

The formulation for this element is based on the isotropic shallow shell element developed by *Cowper et al.* [8]. The displacement functions and the nodal degrees of freedom

are the same as those for the laminated plate element. Thus, the displacement functions in area coordinates (ξ, η) of the base triangle (Node-3 is taken as the origin of the ξ, η axes) are given by;

$$u = \sum_{j=1}^{10} \delta_{u_j} \sum_{i=1}^{10} \xi^{m_i} \eta^{n_i} R_{ij} \quad (4.2.1)$$

$$v = \sum_{j=1}^{10} \delta_{v_j} \sum_{i=1}^{10} \xi^{m_i} \eta^{n_i} R_{ij} \quad (4.2.2)$$

$$w = \sum_{j=1}^{12} \delta_{w_j} \sum_{i=1}^{21} \xi^{m_i} \eta^{n_i} S_{ij} \quad (4.2.3)$$

The transformation matrices, $[R]$ and $[S]$, and the nodal displacement vectors, $\{\delta_u\}$, $\{\delta_v\}$ and $\{\delta_w\}$ are all as defined for the high precision laminated anisotropic plate element in Section 3.4.2.

It may be noted from equations (4.1.3) and (4.1.4) that the shell curvatures are the only shape quantities required in calculating the strain energy. Therefore, they may be specified for an element directly or they may be calculated from equation (4.1.5). One convenient way to do this is to specify the height of the shell element at the three vertices 1, 2, 3 and at the mid-edge points 4, 5, 6 of the base triangle of Figure 42; this is the approach followed in the present work.

The height, h , can be interpolated from the nodal heights as;

$$h = \sum_{i=1}^6 N_i h_i \quad (4.2.4)$$

where N_i are the standard shape functions for the six-noded base triangle. Thus,

$$\left\{ \begin{array}{c} \frac{\partial^2 h}{\partial \xi^2} \\ \frac{\partial^2 h}{\partial \eta^2} \\ \frac{\partial^2 h}{\partial \xi \partial \eta} \end{array} \right\} = \left\{ \begin{array}{c} 4(h_1 + h_3) - 8h_6 \\ 4(h_2 + h_3) - 8h_5 \\ 4(h_3 + h_4 - h_5 - h_6) \end{array} \right\} \quad (4.2.5)$$

The shell element curvatures are then obtained as;

$$\left\{ \begin{array}{c} \frac{\partial^2 h}{\partial x^2} \\ \frac{\partial^2 h}{\partial y^2} \\ 2 \frac{\partial^2 h}{\partial x \partial y} \end{array} \right\} = \frac{1}{(2\Delta)^2} \left| \begin{array}{ccc} b_1^2 & b_2^2 & 2b_1b_2 \\ c_1^2 & c_2^2 & 2c_1c_2 \\ 2b_1c_1 & 2b_2c_2 & 2(b_1c_2 + b_2c_1) \end{array} \right| \left\{ \begin{array}{c} \frac{\partial^2 h}{\partial \xi^2} \\ \frac{\partial^2 h}{\partial \eta^2} \\ \frac{\partial^2 h}{\partial \xi \partial \eta} \end{array} \right\}$$

or

$$\{\rho\} = \frac{1}{(2\Delta)^2} [Q] \{H\} \quad (4.2.6)$$

with $b_i = y_j - y_m$ and $c_i = x_m - x_j$, i, j, m taking values 1, 2, 3 cyclically. Δ is the area of the base triangle.

Again in terms of area coordinates;

$$\{\epsilon^0\} = \frac{1}{(2\Delta)} [P] \{\alpha\} \quad (4.2.7)$$

$$\{\kappa\} = \frac{1}{(2\Delta)^2} [Q] \{\beta\} \quad (4.2.8)$$

where

$$\{\alpha\}^T = \left[\frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}, \frac{\partial v}{\partial \xi}, \frac{\partial v}{\partial \eta}, w \right]$$

$$\{\beta\}^T = - \left[\frac{\partial^2 w}{\partial \xi^2}, \frac{\partial^2 w}{\partial \eta^2}, \frac{\partial^2 w}{\partial \xi \partial \eta} \right]$$

and

$$[P] = \begin{bmatrix} b_1 & b_2 & 0 & 0 & -2\Delta \frac{\partial^2 w}{\partial x^2} \\ 0 & 0 & c_1 & c_2 & -2\Delta \frac{\partial^2 w}{\partial y^2} \\ c_1 & c_2 & b_1 & b_2 & -4\Delta \frac{\partial^2 w}{\partial x \partial y} \end{bmatrix}$$

Substituting from equations (4.1.2), (4.2.7) and (4.2.8) into equation (4.1.1),

$$2U = \int_0^1 \int_0^{1-\xi} (\{\alpha\}^T [\bar{A}] \{\alpha\} + \{\alpha\}^T [\bar{B}] \{\beta\} + \{\beta\}^T [\bar{B}]^T \{\alpha\} + \{\beta\}^T [\bar{D}] \{\beta\}) d\xi d\eta \quad (4.2.9)$$

where

$$[\bar{A}] = \frac{1}{(2\Delta)} [P]^T [A] [P]$$

$$[\bar{B}] = \frac{1}{(2\Delta)^2} [P]^T [B] [Q] \quad (4.2.10)$$

$$[\bar{D}] = \frac{1}{(2\Delta)^3} [Q]^T [D] [Q]$$

By differentiating the displacement functions u, v, w defined in equations (4.2.1) to (4.2.3), the elements of the $\{\alpha\}$ and $\{\beta\}$ vectors are obtained and substituted into the strain energy equation (4.2.9). Carrying out the standard integration of the type

$$f(m, n) = \int_0^1 \int_0^{1-\xi} \xi^m \eta^n d\xi d\eta = \frac{m! n!}{(m+n+2)!}$$

the strain energy is obtained in the form;

$$2U = \begin{Bmatrix} \delta_u^T \\ \delta_v^T \\ \delta_w^T \end{Bmatrix} [K] \begin{Bmatrix} \delta_u \\ \delta_v \\ \delta_w \end{Bmatrix} \quad (4.2.11)$$

where the element stiffness matrix, $[K]$, is given by;

$$[K] = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12} & X_{22} & X_{23} \\ X_{13} & X_{23} & X_{33} \end{bmatrix} \quad (4.2.12)$$

where

$$X_{11} = [R]^T [F_1] [R]$$

$$X_{12} = [R]^T [F_2] [R]$$

$$X_{13} = -[R]^T [G_1] [S]$$

$$X_{22} = [R]^T [F_3] [R]$$

$$X_{23} = -[R]^T [G_2] [S]$$

$$X_{33} = [S]^T [H] [S]$$

The matrices $[F_2]$, $[G_1]$ and $[H]$ are defined as follows;

$$F_{2,ij} = \bar{A}_{13} m_i m_j f(II-2, JJ) + \{\bar{A}_{14} m_i n_j + \bar{A}_{23} n_i m_j\} f(II-1, JJ-1) \\ + \bar{A}_{24} n_i n_j f(II, JJ-2) \quad (4.2.13) \\ i, j = 1, 10$$

$$G_{1,ij} = \bar{B}_{11} m_i m_j (m_j - 1) f(II-3, JJ) \\ + \{\bar{B}_{12} m_i n_j (n_j - 1) + \bar{B}_{23} n_i m_j n_j\} f(II-1, JJ-2) \\ + \{\bar{B}_{13} m_i m_j n_j + \bar{B}_{21} n_i m_j (m_j - 1)\} f(II-2, JJ-1) \\ + \bar{B}_{22} n_i n_j (n_j - 1) f(II, JJ-3) \quad (4.2.14) \\ + \bar{A}_{15} m_i f(II-1, JJ) + \bar{A}_{25} n_i f(II, JJ-1) \\ i = 1, 10, j = 1, 21$$

$$\begin{aligned}
H_{ij} = & \bar{D}_{11}m_i(m_i - 1)m_j(m_j - 1)f(II - 4, JJ) \\
& + \{\bar{D}_{12}[m_i(m_i - 1)n_j(n_j - 1) + n_i(n_i - 1)m_j(m_j - 1)] \\
& + \bar{D}_{33}m_in_im_jn_j\}f(II - 2, JJ - 2) \\
& + \bar{D}_{13}[m_i(m_i - 1)m_jn_j + m_in_im_j(m_j - 1)]f(II - 3, JJ - 1) \\
& + \bar{D}_{22}n_i(n_i - 1)n_j(n_j - 1)f(II, JJ - 4) \\
& + \bar{D}_{23}[n_i(n_i - 1)m_jn_j + m_in_in_j(n_j - 1)]f(II - 1, JJ - 3) \\
& + \bar{A}_{55}f(II, JJ) \\
& + \bar{B}_{51}\{m_i(m_i - 1) + m_j(m_j - 1)\}f(II - 2, JJ) \\
& + \bar{B}_{52}\{n_i(n_i - 1) + n_j(n_j - 1)\}f(II, JJ - 2) \\
& + \bar{B}_{53}\{m_in_i + m_jn_j\}f(II - 1, JJ - 1) \\
& i, j = 1, 21
\end{aligned} \tag{4.2.15}$$

where

$$II = m_i + m_j$$

$$JJ = n_i + n_j$$

The symmetric matrices $[F_1]$, $[F_3]$ and the matrix $[G_2]$ are obtained from equations (4.2.13) and (4.2.14), respectively by replacing the elements of $[\bar{A}]$ and $[\bar{B}]$ matrices as shown in the table below.

Elements of the Matrices $[F_1]$, $[F_3]$ and $[G_2]$

Originals in F_2	Replacements F_1	for F_3
$\bar{A}_{13}$	$\bar{A}_{11}$	$\bar{A}_{33}$
$\bar{A}_{14}$	$\bar{A}_{12}$	$\bar{A}_{34}$
$\bar{A}_{23}$	$\bar{A}_{12}$	$\bar{A}_{34}$
$\bar{A}_{24}$	$\bar{A}_{22}$	$\bar{A}_{44}$

Originals in G_1 Replacements for G_2

$\bar{B}_{11}$	$\bar{B}_{31}$
$\bar{B}_{12}$	$\bar{B}_{32}$
$\bar{B}_{13}$	$\bar{B}_{33}$
$\bar{B}_{21}$	$\bar{B}_{41}$
$\bar{B}_{22}$	$\bar{B}_{42}$
$\bar{B}_{23}$	$\bar{B}_{43}$
$\bar{A}_{15}$	$\bar{A}_{35}$
$\bar{A}_{25}$	$\bar{A}_{45}$

All these matrices are found explicitly and provided in the program listing given in Appendix K. Advantage has been taken from the fact that the first few rows and columns for these matrices contain only zero coefficients. Eventhough in the derivation of the method for convenience of illustration the nodal displacements δu , δv and δw are segregated, by a simple programming logic the stiffness matrix resulting from the subroutine is made to correspond to the conventional ordering of degrees of freedom, viz $u_1, (\frac{\partial u}{\partial x})_1, (\frac{\partial u}{\partial y})_1, v_1, (\frac{\partial v}{\partial x})_1, (\frac{\partial v}{\partial y})_1, w_1, (\frac{\partial w}{\partial x})_1, (\frac{\partial w}{\partial y})_1, (\frac{\partial^2 w}{\partial x^2})_1, (\frac{\partial^2 w}{\partial x \partial y})_1, (\frac{\partial^2 w}{\partial y^2})_1, u_2, \dots, (\frac{\partial^2 w}{\partial y^2})_3, u_c, v_c$. This way the centroidal displacements, u_c and v_c , can be easily condensed out and the subroutine readily linked to standard packages.

4.3 DKT, E-1 and E-2 Shallow Shell Elements

For these elements, the tangential displacements u and v are the same polynomial functions as for the high precision element. However, the curvature vector, $\{\kappa\}$, is defined similar to their plate bending versions as;

$$\{\kappa\} = \left\{ \begin{array}{c} \frac{\partial \beta_x}{\partial x} \\ \frac{\partial \beta_x}{\partial y} \\ \frac{\partial \beta_y}{\partial x} + \frac{\partial \beta_y}{\partial y} \end{array} \right\} \quad (4.3.1)$$

and the same discrete Kirchoff conditions are imposed. The normal displacement, w , is not uniquely defined for these elements and therefore is borrowed from an existing element. For all the three elements, the shape functions of the nonconforming BCIZ1 [48] are used. The normal displacement, w , can therefore be written in terms of area coordinates (Node-1 is taken as the origin of the (ξ, η) coordinates); as

$$w = \sum_{i=1}^9 \delta_{w_i} \sum_{j=1}^{10} \xi^{m_j} \eta^{n_j} S_{ij} \quad (4.3.2)$$

The strain energy for the element is given by

$$2U = \int \int (\{\epsilon^0\}^T [A] \{\epsilon\} + \{\epsilon^0\}^T [B] \{\kappa\} + \{\kappa\}^T [B] \{\epsilon\} + \{\kappa\}^T [D] \{\kappa\}) dx dy \quad (4.3.3)$$

The membrane strains, $\{\epsilon\}$, can be written as;

$$\{\epsilon\} = \frac{1}{(2\Delta)} [P_1] \{\alpha\} \quad (4.3.4)$$

where

$$[P_1] = \frac{1}{2\Delta} \begin{bmatrix} b_2 & b_3 & 0 & 0 & -2\Delta \frac{\partial^2 h}{\partial x^2} \\ 0 & 0 & c_2 & c_3 & -2\Delta \frac{\partial^2 h}{\partial y^2} \\ c_2 & c_3 & b_2 & b_3 & -4\Delta \frac{\partial^2 h}{\partial x \partial y} \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{Bmatrix}$$

A typical element of the curvature vector is given by;

$$\kappa_1 = \frac{1}{(2\Delta)} [1 \ \xi \ \eta] [X] \{\delta_w\} \quad (4.3.5)$$

(same as given by equation 3.6.5).

The formulation of the stiffness matrices for these elements is now essentially the same as for the high precision element and the stiffness matrices are obtained explicitly in identical forms. FORTRAN codings for these elements are provided in Appendix L for immediate use.

4.4 Numerical Results

A number of plate and shallow shell problems are analysed with these elements and results compared with exact or existing solutions.

Orthotropic Laminated Plates

The first problem relates to a symmetrically laminated graphite-epoxy plate made up of nine layers with alternate fibre orientations of 0 and 90 degrees and with the 0-degree layers as outside layers. The geometry and material properties are shown in Figure 43. The total thickness of the 0-degree and 90-degree layers is the same. Boundary conditions for both simply supported and clamped plates are considered. In Reference [81] Noor and Mathers provide an analytical solution for the simply supported case, and a converged solution for the clamped case using a shear flexible quadrilateral element with an 8×8 grid in the quarter plate. Tables 17 and 18 show the results obtained with the present elements along with that of Noor and Mathers [81]. Good agreement is seen between the results.

Anisotropic Laminated Plates

To study the effect of anisotropy on the performance of the present elements, a simply supported graphite-epoxy plate with the same material property as above, but with fibre orientations of (45, -45, 45, -45, 45, -45, 45, -45, 45) degrees is analysed. The results are presented in Table 19 along with that of *Noor and Mathers* [81]. It can be seen from Table 19 that there is no need to satisfy the stress boundary conditions. While the central deflection and bending moment are quite accurate even for coarse meshes with kinematic boundary conditions alone, non-zero moments and forces are also obtained at the boundary where they should be zero. However, these quantities rapidly converge to zero as a finite modelling is systematically refined [8].

Isotropic Spherical Cap

A shallow spherical cap (Figure 44) loaded by uniform pressure, p_0 , and freely supported on a boundary of square platform. This problem is analysed in Reference [8] where an analytical solution is also presented. The parameter $Rt/L^2 = 0.02$ is used here. Results are presented in Tables 20a, 20b and 20c. This example is the test case for comparing the accuracies of all the elements presented here.

Orthotropic Shallow Shell

Finally, a nine layered symmetrically laminated graphite-epoxy shallow shell with material properties, layer thickness and fibre orientations the same as those for the orthotropic plate is analysed for simply supported boundary conditions. The geometry is shown in Figure 45. Table 21 shows the results obtained with the present elements along with analytical solution of *Noor and Mathers* [81]. Good agreement is observed for this case also.

Conclusion-Shallow Shell Elements

In this chapter, four thin triangular laminated anisotropic shallow shell finite elements are presented. All these elements use a complete cubic polynomial for in plane action. The bending behavior is based on the plate bending elements developed in Chapter 3. Numerical examples are presented to demonstrate the accuracy of the present elements.

The high precision isotropic shallow shell element by *Cowper et al.* [8] is one of the most accurate finite elements developed for shallow shells and has become the standard element for comparing the performance of other elements. The high precision element presented here is the laminated anisotropic version of this element. The other elements are based on the DKT, E-1 and E-2 elements. As expected, the high precision element is the most accurate of the four. The other three elements, though less accurate than the high precision

element, predict displacements and stresses with accuracy sufficient for practical purposes with reasonable mesh sizes. Also, they are less complex, relatively faster, and applying the boundary conditions is easier. For the high precision element, as stated in Reference [8], there is no need to satisfy stress boundary conditions. The zero moments and forces at the boundary of some problems can be obtained even with displacement boundary conditions as the finite element modelling is progressively refined. The results at the interior vary only very marginally between the two sets of boundary conditions as seen from Table 19. The three lower order elements presented here give almost equal performance and no element is particularly superior among them under all circumstances.

Chapter 5

CONCLUSION

This thesis is concerned with finite element formulations for plates and shallow shell structures. Even though various approaches have been put forward and applied with success by researchers, the displacement-type formulation predominates over the other methods in practice. Therefore, this thesis is exclusively concerned with displacement-type analysis.

In the opening chapter, a brief introduction of the finite element method is given. Plane stress analysis is described in Chapter 2. A new eight-noded isoparametric plane stress element has been developed. This element uses the standard serendipity shape functions for the normal strains and substitute shape functions derived from least squares technique for the shear strain. The use of substitute shape functions smooths out the spurious terms in the shear strain, which existed in the earlier formulation, for certain classes of problems. Numerical results for the well known cantilever problem indicate that the violent oscillations of shear stresses found in the standard element no longer exist with the present one, and average shear stress based upon simple beam theory is now obtained at the nodes without nodal averaging. In the latter part of Chapter 2, a convenient method of constructing transition finite elements is presented. The method is illustrated by constructing all possible members of the plane triangular family up to the quadratic strain triangle. A compact program listing is appended which generates the stiffness matrix for any of the eleven members. Since straight sided triangles are considered, the stiffness matrix is obtained in closed form. By combining these triangular elements and the shear-strain-smoothed isoparametric element, practical plane stress problems can be analysed efficiently, particularly where regions of high stress gradients exist, and structures with curved boundaries can also be analysed conveniently.

Chapter 3 is devoted to plate bending problems based on the classical thin plate theory. The perceived need for, and difficulty in constructing, C_1 continuous elements are described. Fully compatible plate models based on the Kirchhoff hypothesis can only be generated by either using a sub-region approach where different incomplete nine term cubic polynomials are assumed for each sub-region in such a way that the normal slope for the exterior boundary varies linearly, or, by introducing singular functions to ensure normal slope continuity, or, by using higher order derivatives at corners. Among higher order elements, only the

eighteen d.o.f. constrained quintic (high precision) element is commonly used in practice, although still higher order elements (sixth or above order polynomial displacement field) can be generated. However, the earlier formulation of this element involved considerable algebraic operations and was therefore quite time consuming. In this thesis, an efficient and faster formulation is presented for this element after including linear taper. Cowper [82] has also developed a faster algorithm for this element (perhaps faster than the present one) which was not published. However, his formulation is not as general in that it is limited to the isotropic case and to uniform thickness. A high precision laminated plate bending element is also developed. In the next section, some of the most successful simple (nine d.o.f elements) plate bending elements are identified. Two triangular elements, one based on the discrete Kirchhoff theory (DKT), and the other based on least squares-smoothed derivatives, are found to be simple to formulate and yet give reasonably good accuracy. Recent publications have proved the popularity of these elements. In this thesis, complete reformulation is presented for these two elements so that the algorithms are very fast and easy to use even in minicomputers. The two most accurate nine d.o.f. plate bending elements are the HBS and CT elements. Both elements appear to be complicated and time consuming. It is therefore desirable to develop elements with simpler formulations and yet giving accuracies comparable to those of the HBS and CT elements. It is shown in this thesis that a number of high performance simple plate bending elements can be generated by combining the discrete Kirchhoff theory and least squares technique. Two such successful nonconforming elements E-1 and E-2 are developed. They are shown to pass the patch test and give excellent results for many standard test problems. The idea is later extended to develop an improved discrete Kirchhoff quadrilateral (IDKQ) element. Numerical results indicate that good accuracy is also obtained with this element. Unlike the CT element approach which has no further use, the approach given in the present thesis can be further explored to form many new simple plate bending elements only two of which are presented here.

In Chapter 4, a number of triangular laminated anisotropic shallow shell elements are presented. These elements are based on the plate bending elements considered in Chapter 3. The high precision element is found to be the most accurate among the four elements present. However, the other elements demonstrate very good engineering accuracy for reasonable mesh sizes.

Further Scope

The present work is limited to static analysis of plates and shells. An obvious extension is to solve eigenvalue problems (Vibration and buckling). Vibration analysis requires the development of a mass matrix to represent the effect of dynamic loading. The mass matrix is used in conjunction with the conventional stiffness matrix to obtain a set of algebraic equations which govern the structural behaviour. Their solution yields the values of the square of the natural frequencies (eigenvalues) and corresponding sets of model vectors (eigenvectors) defining the deformed mode shapes. A consistent mass matrix for an element can be obtained from the kinetic energy in the same way as the stiffness matrix from the potential energy. Mass matrices for the high precision (HP or T-18) and the A-9 elements are readily available. For the other elements, the displacement function, w , is not uniquely defined and therefore consistent mass matrices are not available. For these elements, either lumped mass matrices or the consistent mass matrices of other similar order elements like BCIZ1 etc. can be used.

The other eigenvalue problem is the lateral buckling of plates subjected to in-plane compressive force. This is a linear stability problem. To solve such problems, the geometric stiffness matrix which accounts for the effect of in-plane loading on the conventional bending stiffness is required, and the same arguments regarding the mass matrix hold good for this case also. The eigenvalues in this case represent the critical values of loading magnitude at which buckling occurs. Usually, the lowest eigenvalue is of practical interest. The accuracy of the new elements developed here should be assessed with the other existing elements for the above eigenvalue problems.

Nonlinear analysis using these new elements is also an important area. Recently, the use of simpler facet elements, rather than sophisticated curvilinear elements, for linear and nonlinear analysis of shell structures is becoming popular again. By combining the constant strain triangle or other effective membrane element with E-1 and E-2 plate bending elements, simple but efficient facet shell elements could be developed, and their effectiveness studied.

Finally, the least squares technique which has been successfully used here to smooth spurious shear strains for the plane stress element and to produce least positive plate bending stiffness matrices in the displacement formulation that is known to usually overestimate the actual stiffness of the structure, should be investigated to eliminate the "shear-locking" phenomenon, commonly found in Mindlin type plate bending elements.

REFERENCES

Notation for References

PERIODICALS

AeroJ	Aeronautical Journal of the Royal Aeronautical Society.
AIAAJ	Journal, American Institute of Aeronautics and Astronautics.
CMAME	Computer Methods in Applied Mechanics and Engineering.
CompSt	Computer and Structures, An International Journal.
ECJ	Engineering Computation Journal.
IJESci	International Journal of Engineering Sciences
IJMSci	International Journal of Mechanical Sciences.
IJNME	International Journal for Numerical Methods in Engineering.
IJSoS	International Journal of Solids and Structures.
JApplM	Journal of Applied Mechanics, Transactions of ASME.
JCM	Journal of Composite Materials
JEMDiv	Journal of Engineering Mechanics Division, Proceedings of ASCE.
JSA	Journal of Strain Analysis

CONFERENCES AND SYMPOSIA

AFEMCE Symp. *Application of Finite Element Methods in Civil Engineering*, Proceedings of the Symposium held at Nashville, Tennessee, W.H. Rowan Jr. and R.M. Hackett editors; ASCE, 1969.

MAFELAP3 Conf. *International Conference on the Mathematics of Finite Elements and Applications*, Volume III, J.R. Whiteman (Editor), Academic Press, 1978.

VME Conf. *International Conference of Variational Methods in Engineering*, held at the University of Southampton, 1972.

WPAFB1 Conf. *Proceedings of the (first) Conference in Structural Mechanics*, Wright-Patterson Air Force Base, Ohio, Oct. 1965 (AFFDL-TR-66-80, Nov. 1966; AD-646-300, N.T.I.S.).

WPAFB2 Conf. *Proceedings of the second Conference on Matrix Methods in Structural Mechanics*, Wright-Patterson Air Force Base, Ohio, Oct. 1968 (AFFDL-TR-68-150, Dec. 1969; AD-703-685; N.T.I.S.).

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IUTAM Symp. IUTAM Symposium on High Speed Computing of Elastic Structures, Univ. of Liège, 1971.

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Table 1

Deflections

$$\left(\frac{q a^4}{10^4 D_0} \right)$$

Plate with four edges simply supported, carrying a uniform load

y	x		
	1/4a	1/2a	Solution
3/4a	5.92	8.20	Ref. 63
	5.90	8.18	HP
1/2a	10.14	14.01	Ref. 63
	10.11	13.97	HP
1/4a	9.31	12.75	Ref. 63
	9.30	12.72	HP

HP — High Precision

Central deflection for simply supported square plate
under concentrated load

Element	Element E-1		Element E-2		Element CT	
Mesh	Mesh A	Mesh B	Mesh A	Mesh B	Mesh A	Mesh B
1 × 1	11602 (0.0 %)	11787 (+1.6 %)	11743 (+1.2 %)	12078 (+4.1 %)	11473 (-1.1 %)	10841 (-6.6 %)
2 × 2	11405 (-1.7 %)	12010 (+3.5 %)	11449 (-1.3 %)	12120 (+4.4 %)	11551 (-0.4 %)	11629 (+0.2 %)
4 × 4	11549 (-0.4 %)	11733 (+1.1 %)	11562 (-0.3 %)	11768 (+1.4 %)	11613 (+0.1 %)	11608 (+0.1 %)
6 × 6	11578 (-0.2 %)	11666 (+0.6 %)	11585 (-0.1 %)	11684 (+0.7 %)	11615 (+0.1 %)	11604 (+0.0 %)
Analytical Solution	11601					
Multiplier	$(\frac{PL^2}{D}) \cdot 10^{-6}$					

Table 3

Central deflection for clamped square plate
under concentrated load

Element	Element E-1		Element E-2		Element CT	
Mesh	Mesh A	Mesh B	Mesh A	Mesh B	Mesh A	Mesh B
1 × 1	45086 (-19.7 %)	39747 (-29.2 %)	46899 (-16.4 %)	42653 (-24.0 %)	42941 (-23.5 %)	30286 (-46.0 %)
2 × 2	55493 (-1.1 %)	55586 (-1.0 %)	55957 (-0.3 %)	56642 (+0.9 %)	57249 (+2.0 %)	52086 (-7.2 %)
4 × 4	56046 (-0.1 %)	56488 (+0.7 %)	56198 (+0.1 %)	56838 (+1.3 %)	56779 (+1.2 %)	55365 (-1.3 %)
6 × 6	56093 (0.0 %)	56357 (+0.4 %)	56166 (+0.1 %)	56532 (+0.7 %)	56507 (+0.7 %)	55789 (-0.6 %)
Analytical Solution	56120					
Multiplier	$(\frac{PL^2}{D}) \cdot 10^{-7}$					

Table 4

Central deflection for simply supported square plate
under uniform loading

Element	E-1		E-2		CT	
Mesh	Mesh A	Mesh B	Mesh A	Mesh B	Mesh A	Mesh B
1 × 1	38674 (-4.8 %)	19645 (-51.6 %)	39143 (-3.6 %)	20130 (-50.4 %)	38244 (-5.6 %)	18069 (-55.5 %)
2 × 2	39919 (-1.7 %)	35620 (-12.3 %)	40019 (-1.5 %)	35773 (-11.9 %)	39930 (-1.7 %)	35118 (-13.6 %)
4 × 4	40471 (-0.4 %)	39417 (-3.0 %)	40498 (-0.3 %)	39458 (-2.9 %)	40439 (-0.5 %)	39280 (-3.3 %)
6 × 6	40561 (-0.2 %)	40092 (-1.3 %)	40573 (-0.1 %)	40111 (-1.3 %)	40540 (-0.2 %)	40028 (-1.5 %)
Analytical Solution	40624					
Multiplier	$(\frac{qL^4}{D}) \cdot 10^{-7}$					

Table 5

*Central deflection for clamped square plate
under uniform loading*

Element	Element E-1		Element E-2		Element CT	
Mesh	Mesh A	Mesh B	Mesh A	Mesh B	Mesh A	Mesh B
1, x 1	15029 (+18.8 %)	6624 (-47.6 %)	15633 (+23.6 %)	7108 (-43.8 %)	14314 (+13.1 %)	5048 (-60.1 %)
2 x 2	14641 (+15.7 %)	11094 (-12.3 %)	14770 (+16.7 %)	11235 (-11.2 %)	14750 (+16.6 %)	10732 (-15.2 %)
4 x 4	13214 (+4.4 %)	12299 (-2.8 %)	13252 (+4.7 %)	12338 (-2.5 %)	13221 (+4.5 %)	12232 (-3.3 %)
6 x 6	12906 (+2.0 %)	12495 (-1.2 %)	12924 (+2.1 %)	12513 (-1.1 %)	12912 (+2.0 %)	12468 (-1.5 %)
Analytical Solution	12653					
Multiplier	$(\frac{qL^4}{D}) \cdot 10^{-7}$					

Table 6

*Twisting moment at corner for simply supported square
plate under concentrated load*

Element	Element E-1		Element E-2		Element CT	
Mesh	Mesh A	Mesh B	Mesh A	Mesh B	Mesh A	Mesh B
1 × 1	81596 (+33.9 %)	43750 (-28.2 %)	81239 (+33.3 %)	43750 (-28.2 %)	90372 (+48.3 %)	43750 (-28.2 %)
2 × 2	63705 (+4.5 %)	58521 (-4.0 %)	63550 (+4.3 %)	58476 (-4.1 %)	66188 (+8.6 %)	58439 (-4.1 %)
4 × 4	61548 (+1.0 %)	61015 (+0.1 %)	61561 (+1.0 %)	60981 (+0.0 %)	61811 (+1.4 %)	61064 (+0.2 %)
6 × 6	61202 (+0.4 %)	61059 (+0.2 %)	61213 (+0.4 %)	61042 (+0.1 %)	61291 (+0.6 %)	61090 (+0.2 %)
Analytical Solution	60953					
Multiplier	$P \cdot 10^{-6}$					

Table 7

*Midside bending moment for clamped square plate
under concentrated load*

Element	Element E-1		Element E-2		Element CT	
Mesh	Mesh A	Mesh B	Mesh A	Mesh B	Mesh A	Mesh B
1 × 1	12931 (+2.8 %)	6201 (-50.7 %)	13085 (+4.0 %)	6398 (-49.1 %)	9905 (-21.2 %)	3818 (-69.6 %)
2 × 2	13278 (+5.6 %)	9980 (-20.6 %)	13307 (+5.8 %)	10043 (-20.1 %)	12246 (-2.6 %)	8601 (-31.6 %)
4 × 4	12891 (+2.5 %)	11548 (-8.2 %)	12909 (+2.6 %)	11573 (-8.0 %)	12225 (-2.8 %)	10493 (-16.6 %)
6 × 6	12804 (+1.8 %)	11843 (-5.8 %)	12818 (+1.9 %)	11891 (-5.5 %)	12337 (-1.9 %)	11094 (-11.8 %)
Analytical Solution	12577					
Multiplier	$P \cdot 10^{-5}$					

Table 8

Central bending moment for simply supported square
plate under uniform load

Element	Element E-1		Element E-2		Element CT	
Mesh	Mesh A	Mesh B	Mesh A	Mesh B	Mesh A	Mesh B
1 × 1	62432 (+30.4 %)	40411 (-15.6 %)	62849 (+31.2 %)	41266 (-13.8 %)	51433 (+7.4 %)	30515 (-36.3 %)
2 × 2	51301 (+7.1 %)	46871 (-2.1 %)	51338 (+7.2 %)	47106 (-1.6 %)	49988 (+4.4 %)	43958 (-8.2 %)
4 × 4	48731 (+1.8 %)	47797 (-0.2 %)	48751 (+1.8 %)	47860 (-0.1 %)	48347 (+1.0 %)	47005 (-1.8 %)
6 × 6	48244 (+0.7 %)	47871 (0.0 %)	48253 (+0.8 %)	47900 (0.0 %)	48090 (+0.4 %)	47493 (-0.8 %)
Analytical Solution	47886					
Multiplier	$(qL^2) \cdot 10^{-6}$					

Table 9

*Twisting moment at corner for simply supported square
plate under uniform load*

Element	Element E-1		Element E-2		Element CT	
Mesh	Mesh A	Mesh B	Mesh A	Mesh B	Mesh A	Mesh B
1 × 1	27199 (-16.3 %)	7292 (-77.6 %)	27080 (-16.6 %)	7292 (-77.6 %)	30124 (-7.3 %)	7292 (-77.6 %)
2 × 2	30597 (-5.8 %)	22583 (-30.5 %)	30549 (-6.0 %)	22559 (-30.5 %)	32080 (-1.2 %)	22766 (-29.9 %)
4 × 4	31961 (-1.6 %)	29150 (-10.3 %)	31958 (-1.6 %)	29131 (-10.3 %)	32524 (+0.1 %)	29254 (-9.9 %)
6 × 6	32234 (-0.8 %)	30778 (-5.2 %)	32235 (-0.8 %)	30765 (-5.3 %)	32541 (+0.2 %)	30848 (-5.0 %)
Analytical Solution	32482					
Multiplier	$(qL^2) \cdot 10^{-6}$					

Table 10

*Central bending moment for clamped square plate
under uniform loading*

Element	Element E-1		Element E-2		Element CT	
Mesh	Mesh A	Mesh B	Mesh A	Mesh B	Mesh A	Mesh B
1 × 1	38684 (+68.9 %)	26869 (+17.3 %)	39630 (+73.0 %)	27724 (+21.0 %)	25803 (+12.7 %)	16974 (-25.9 %)
2 × 2	30870 (+34.8 %)	23360 (+2.0 %)	30837 (+34.6 %)	23569 (+2.9 %)	29510 (+28.8 %)	20527 (-10.4 %)
4 × 4	25082 (+9.5 %)	23122 (+0.9 %)	25107 (+9.6 %)	23179 (+1.2 %)	24671 (+7.7 %)	22389 (-2.3 %)
6 × 6	23910 (+4.4 %)	22960 (+0.2 %)	23920 (+4.4 %)	22990 (+0.4 %)	23751 (+3.7 %)	22605 (-1.3 %)
Analytical Solution	22905					
Multiplier	$(qL^2) \cdot 10^{-6}$					

Table 11

Midside bending moment for clamped square plate
under uniform load

Element	Element E-1		Element E-2		Element CT	
Mesh	Mesh A	Mesh B	Mesh A	Mesh B	Mesh A	Mesh B
1 × 1	43103 (-16.0 %)	10334 (-79.9 %)	43616 (-15.0 %)	10663 (-79.2 %)	33017 (-35.7 %)	6363 (-87.6 %)
2 × 2	52808 (+2.9 %)	36600 (-28.7 %)	53045 (+3.3 %)	37056 (-27.8 %)	45621 (-11.1 %)	28249 (-45.0 %)
4 × 4	53017 (+3.3 %)	44376 (-13.6 %)	53163 (+3.6 %)	44771 (-12.9 %)	48850 (-4.8 %)	38626 (-24.8 %)
6 × 6	52512 (+2.3 %)	46720 (-9.0 %)	52606 (+2.5 %)	47021 (-8.4 %)	49682 (-3.2 %)	42547 (-17.1 %)
Analytical Solution	51334					
Multiplier	$(qL^2) \cdot 10^{-6}$					

Table 12

Central deflection for simply supported rectangular
plate ($a/b = 2$) under concentrated load

Element	Element E-1		Element E-2		Element CT	
Mesh	Mesh A	Mesh B	Mesh A	Mesh B	Mesh A	Mesh B
1 × 1	13402 (-18.9 %)	14933 (-9.6 %)	13873 (-16.0 %)	15324 (-7.3 %)	11075 (-33.0 %)	13749 (-16.8 %)
2 × 2	15890 ⁴ (-3.8 %)	16749 (+1.4 %)	16009 (-3.1 %)	16946 (+2.6 %)	15731 (-4.8 %)	16077 (-2.7 %)
4 × 4	16335 (-1.1 %)	16647 (+0.7 %)	16369 (-0.9 %)	16714 (+1.1 %)	16395 (-0.8 %)	16410 (-0.7 %)
6 × 6	16442 ² (-0.5 %)	16592 (+0.4 %)	16459 (-0.4 %)	16625 (+0.6 %)	16482 (-0.3 %)	16473 (-0.3 %)
Analytical Solution	16524					
Multiplier	$(\frac{Pb^2}{D}) \cdot 10^{-6}$					

Table 13

Central deflection for clamped rectangular
plate ($a/b = 2$) under concentrated load

Element	Element E-1		Element E-2		Element CT	
Mesh	Mesh A	Mesh B	Mesh A	Mesh B	Mesh A	Mesh B
1 × 1	49805 (-31.0 %)	46071 (-36.1 %)	52800 (-26.8 %)	49603 (-31.2 %)	44416 (-38.4 %)	36554 (-49.3 %)
2 × 2	66676 (-7.6 %)	65711 (-8.9 %)	67856 (-6.0 %)	67633 (-6.3 %)	65590 (-9.1 %)	57504 (-20.3 %)
4 × 4	71237 (-1.3 %)	71624 (-0.7 %)	71565 (-0.8 %)	72318 (+0.2 %)	72135 (+0.0 %)	68810 (-4.6 %)
6 × 6	71831 (-0.4 %)	72190 (+0.1 %)	72000 (-0.2 %)	72540 (+0.5 %)	72380 (+0.3 %)	70870 (-1.8 %)
Analytical Solution	72150					
Multiplier	$(\frac{Pb^2}{D}) \cdot 10^{-7}$					

Table 14

Twisting moment at the corner of a simply supported
rectangular plate ($a/b = 2$) under concentrated load

Element	Element E-1		Element E-2		Element CT	
Mesh	Mesh A	Mesh B	Mesh A	Mesh B	Mesh A	Mesh B
1 × 1	15632 (+52.1 %)	21968 (-26.8 %)	16471 (+54.9 %)	22006 (-26.6 %)	42040 (+40.1 %)	24084 (-19.7 %)
2 × 2	29810 (-0.6 %)	26417 (-11.9 %)	29616 (-1.3 %)	27044 (-9.9 %)	30148 (+0.5 %)	23323 (-22.3 %)
4 × 4	30587 (+2.0 %)	28632 (-4.6 %)	30540 (+1.8 %)	28794 (-4.0 %)	30960 (+3.2 %)	27427 (-8.6 %)
6 × 6	30277 (+0.9 %)	29391 (-2.0 %)	30250 (+0.8 %)	29465 (-1.8 %)	30500 (+1.7 %)	28792 (-4.0 %)
Analytical Solution	30000					
Multiplier	$P \cdot 10^{-6}$					

Table 15

*Bending moment at the longer midside for clamped
rectangular plate ($a/b = 2$) under concentrated load*

Element	Element E-1		Element E-2		Element CT	
Mesh	Mesh A	Mesh B	Mesh A	Mesh B	Mesh A	Mesh B
1 × 1	15037 (-10.2 %)	7187 (-57.1 %)	15385 (-8.0 %)	7441 (-55.5 %)	10569 (-36.9 %)	4539 (-72.9 %)
2 × 2	19043 (+13.8 %)	11444 (-31.6 %)	19191 (+14.6 %)	11672 (-30.3 %)	16910 (+1.0 %)	8929 (-46.7 %)
4 × 4	17860 (+6.7 %)	14310 (-14.5 %)	17950 (+7.2 %)	14428 (-13.8 %)	17074 (+2.0 %)	12851 (-23.2 %)
6 × 6	17590 (+5.1 %)	15319 (-8.5 %)	17660 (+5.5 %)	15410 (-7.9 %)	16800 (+0.4 %)	14245 (-14.9 %)
Analytical Solution	16740					
Multiplier	$P \cdot 10^{-5}$					

Table 16

Deflections

$$\left(\frac{qa^4}{10^4 D_0}\right)$$

Plate with four edges simply supported, carrying a uniform load

y	x		
	1/4a	1/2a	Solution
3/4a	5.92	8.20	Ref. 63
	5.90	8.18	HP
	5.91	8.20	E-1
	5.91	8.21	E-2
1/2a	10.14	14.01	Ref. 63
	10.11	13.97	HP
	10.13	14.01	E-1
	10.14	14.02	E-2
1/4a	9.31	12.75	Ref. 63
	9.30	12.72	HP
	9.31	12.74	E-1
	9.33	12.75	E-2

HP — High Precision

Table 17

9 layered orthotropic simply supported plate under uniform pressure:
Central deflection and central bending moment

Grid Size in Quarter Plate	$\bar{W}$				$\bar{M}_z$			
	HP	DKT	E-1	E-2	HP	DKT	E-1	E-2
2×2	4.472	4.610	4.547	4.556	8.873	9.609	9.499	9.515
3×3	-	4.534	4.504	4.509	-	9.229	9.160	9.170
4×4	-	4.508	4.490	4.493	-	9.068	9.027	9.033
Exact [81]	4.486				8.879			

$$\bar{W} = \left(\frac{W E_T t^3}{P a^4} \right) \times 10^3 .$$

$$\bar{M}_z = \left(\frac{M_z}{P a^2} \right) \times 10^2$$

HP — High Precision

Table 18

9 layered orthotropic clamped plate under uniform pressure:
Central deflection and bending moment at centre of clamped edge

Grid Size in Quarter Plate	$\bar{W}$				$\bar{M}_x$			
	HP	DKT	E-1	E-2	HP	DKT	E-1	E-2
2 × 2	9.488	10.500	10.010	10.080	-6.643	-7.435	-7.253	-7.295
3 × 3	9.492	9.976	9.730	9.764	-6.637	-7.353	-7.255	-7.281
4 × 4	-	9.772	9.626	9.646	-	-7.237	-7.169	-7.188
F.E. solutions [81]	9.634				-6.607			

$$\bar{W} = \left(\frac{W E_T t^3}{P a^4} \right) \times 10^4$$

$$\bar{M}_x = \left(\frac{M_x}{P a^2} \right) \times 10^2$$

HP — High Precision

Table 19

9 layered anisotropic simply supported plate:
Central deflection and central bending moment

Grid Size for Full Plate	$\bar{W}$				$\bar{M}_z$			
	HP	DKT	E-1	E-2	HP	DKT	E-1	E-2
4 × 4	2.448 (2.432)	2.569	2.499	2.509	3.660 (3.556)	4.108	4.103	4.106
6 × 6	2.448 (2.440)	2.484	2.462	2.465	3.638 (3.637)	3.812	3.794	3.797
8 × 8	-	2.469	2.456	2.458	-	3.722	3.715	3.716
F.E. solution [81]	2.403				3.604			

$$\bar{W} = \left(\frac{W E_T t^3}{P a^4} \right) \times 10^3$$

$$\bar{M}_z = \left(\frac{M_z}{P a^2} \right) \times 10^2$$

HP — High Precision

Note: Values in the parenthesis are for stress boundary conditions.

i.e. $\frac{\partial^2 w}{\partial y^2} = 0$ on edge $y = 0, a$ etc.

Table 20a

Deflections for isotropic shallow spherical cap

Grid Size in Quarter Shell	$\frac{EtW_C}{PR^2}$				$\frac{10EtV_B}{PRL}$			
	HP	DKT	E-1	E-2	HP	DKT	E-1	E-2
2 × 2	-1.011	-0.970	-0.974	-0.973	-3.670	-3.653	-3.653	-3.645
3 × 3	-1.008	-0.997	-0.994	-0.994	-3.673	-3.668	-3.665	-3.667
4 × 4	-1.008	-1.003	-1.002	-1.002	-3.674	-3.673	-3.669	-3.670
Exact [8]	-1.0098				-3.6794			

HP — High Precision

Table 20b

*Membrane stress resultants for isotropic shallow
spherical cap*

Grid Size in Quarter Shell	$\frac{10N_{xx}C}{PR}$				$\frac{N_{xy}A}{PR}$			
	HP	DKT	E-1	E-2	HP	DKT	E-1	E-2
2 × 2	-5.207	-4.814	-4.845	-4.844	-1.298	-1.228	-1.224	-1.225
3 × 3	-5.096	-5.051	-5.045	-5.055	-1.228	-1.220	-1.213	-1.214
4 × 4	-5.065	-5.044	-5.036	-5.036	-1.196	-1.200	-1.193	-1.194
Exact [8]	-5.049				-1.159			

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Table 20c

Bending stress resultants for isotropic shallow spherical cap

Grid Size in Quarter Shell	$\frac{10^3 M_{xx} C}{PRt}$				$\frac{10 M_{xy} A}{PRt}$			
	HP	DKT	E-1	E-2	HP	DKT	E-1	E-2
2 × 2	-1.076	-17.85	-18.38	-18.29	-1.236	-0.623	-0.652	-0.647
3 × 3	-8.715	-11.41	-13.18	-12.86	-1.160	-0.848	-0.881	-0.876
4 × 4	-8.529	-9.873	-10.46	-10.36	-1.118	-0.955	-0.981	-0.974
Exact [8]	-8.487				-1.059			

HP — High Precision

Table 21

Simply supported 9 layered orthotropic shallow shell:
 Central deflection and central membrane and bending stress resultants

Grid Size in Quarter Shell	$\bar{W}$			$\bar{N}_z$			$\bar{M}_z$		
	HP	DKT	E-1	E-2	HP	DKT	E-1	E-2	E-2
2 x 2	2.705	2.749	2.727	2.730	2.192	2.336	2.354	2.106	5.515
3 x 3	2.707	2.726	2.716	2.717	2.742	2.681	2.706	2.646	5.360
4 x 4	2.707	2.718	2.712	2.713	2.887	2.812	2.826	2.805	5.299
Exact [81]	2.713			3.005			5.239		

$$\bar{W} = \left(\frac{W E_T t^3}{P a^4} \right) \times 10^3$$

$$\bar{M}_z = \left(\frac{M_z}{P a^2} \right) \times 10^2$$

$$\bar{N}_z = \frac{N_z}{P a}$$

HP — High Precision

Figure 1 A two-dimensional eight noded parabolic isoparametric element.

Figure 2 Bending of a slender beam - variation of shear stress, τ along the neutral axis.

(a) Loading

(b) Geometry and boundary conditions of section analyzed

(c) Regular mesh
(MESH A)

100
0
Scale

(d) Irregular mesh
(MESH B)

Figure 3 Analysis of infinitely long plate subjected to concentrated line load

Figure 4: Infinite plate analysis - vertical stress σ_x along x axis (mesh A)

Figure 5 Infinite plate analysis - vertical stress σ_x along x axis (mesh

Figure 6 Regular members of the plane triangular family

Figure 7 Combination of different elements.

Figure 8 Complete family of plane triangular elements.

Figure 9 Finite element mesh for the tapered square plate

Figure 10 A composite laminate with a typical lamina.

$$h_1 = -0.01; \quad h_2 = 0.0; \quad h_3 = 0.01$$

Figure 11 2-layer simply supported square plate, and the grid used for the complete plate.

Figure 12 An arbitrary patch of elements with a fully connected node 6

Figure 13 Geometry and co-ordinate system; w positive upwards; ϵ co-ordinates in parentheses.

$E = 10,000 \text{ psi}; \quad t = 1 \text{ in}; \quad \nu = 0.3$

Boundary condition: $w = 0$ at A, B, D.

Loading at C = - 5 lbs.

Figure 14, Twisting of a square plate - various meshes

$$E = 1000; \quad t = 1; \quad \nu = 0.3$$

Boundary condition: $w = 0$ at nodes 1, 2, 4

Loading: $(0, 10, 20)$ at node 1
 $(0, -10, 20)$ at node 2
 $(-2, -10, -20)$ at node 3
 $(0, 10, -20)$ at node 4

Figure 15 Patch test problem for rectangular plate

MESH A

MESH B

 $N = 2$

Figure 16 Typical finite element mesh for one-quarter of the plate

Figure 17 Percentage error in central deflection for a simply supported square plate under concentrated load; mesh A

Figure 18 Percentage error in central deflection for a clamped square plate under concentrated load; mesh A

Figure 19 Geometry of the IDKQ element

$E = 1000$; $t = 1$; $\nu = 0.3$

Boundary condition: $w = 0$ at nodes 1, 2, 4

Loading: $(0, 10, 20)$ at node 1
 $(0, -10, 20)$ at node 2
 $(-2, -10, -20)$ at node 3
 $(0, 10, -20)$ at node 4

Figure 20 Patch test problem for rectangular plate with IDKQ element

Figure 21 Simply supported square plate. Lumped load; central deflection.

Figure 22 Simply supported square plate. Lumped load; corner reaction.

Figure 23 Simply supported square plate. Consistent load; central deflection.

Figure 24 Simply supported square plate. Consistent load; corner reaction.

Figure 25 Simply supported square plate. Point load; central deflection.

Figure 26 Clamped square plate. Point load; central deflection.

Figure 27 Clamped square plate. Uniform load; central deflection.

Figure 28 Simply supported square plate. Point load; corner reaction.

Figure 29 Simply supported square plate. Uniform load; central bending moment.

Figure 30 Clamped square plate. Point load; midside bending moment.

Figure 31 Clamped square plate, Uniform load; central bending moment.

Figure 32 Clamped square plate. Uniform load; midside bending moment.

Figure 33 Simply supported square plate. Point load; central deflection.

Figure 34 Clamped square plate. Point load; central deflection.

Figure 35 Simply supported square plate. Uniform load; central deflection.

Figure 36 Clumped square plate. Uniform load; central deflection.

Figure 37 Clamped square plate. Point load; midside bending moment.

Figure 38 Simply supported square plate. Point load; corner reaction.

Figure 39 Clamped rectangular plate. Point load; central deflection: Influence of aspect ratio b/a .

Figure 40 Performances of various elements in single element test - A.

Figure 41 Performances of various elements in single element test - B.

Figure 42. Shell element geometry

Figure 43 Characteristics of laminated graphite-epoxy plates;

$$\frac{E_L}{E_T} = 40; \quad \frac{G_{LT}}{E_T} = 0.6; \quad \nu = 0.25; \quad \frac{h}{a} = 0.01$$

$$Rt/L^2 = 0.02; \quad \nu = 0.3$$

Figure 44 Spherical cap geometry.

Figure 45 Characteristics of laminated graphite-epoxy shallow spherical shell;

$$\frac{E_L}{E_T} = 40; \quad \frac{G_{LT}}{E_T} = 0.6; \quad \nu_{LT} = 0.25; \quad \frac{R}{a} = 10;$$

$$\frac{b}{a} = 0.01252; \quad \frac{h}{a} = 0.01$$

APPENDIX-A

```
C*** .Eigenanalysis Program for a Single Unconstrained Smoothed
C*** eight-noded Serendipity Plane Stress Element.
C***
C*** INPUT
C***
C*** IFPR = Flag for printing during iteration
C***      = 0 No printing
C***      = 1 Print
C*** NGAUS1 = Integration order for normal strains
C***      = 3
C*** NGAUS2 = Integration order for shear Strain
C***      = 3 or 2
C*** BETA = Shift to the diagonal terms of the element stiffness
C***      matrix
C*** NEL = Number of elements = 1 (single element)
C*** NNODE = Total number of nodes = 8 (single element)
C*** NDFPN = Number of degrees of freedom per node = 2 (u & v)
C*** NNPE = Number of nodes per element = 8
C*** YMOD, PNU, THIK = Young's modulus, Poisson's ratio, Element
C***                    thickness.
C*** XCO, YCO = x and y coordinates of the nodes
C*** NODE = Nodal connectivity for the elements (Here only one element)
C***
C*** OUTPUT
C*** Printing of eigenvalues during Jacobi iteration, if opted
C*** Printing of shifted eigenvalues
C*** Printing of original eigenvalues
C*** Printing of eigenvectors
```

C***

```

DIMENSION A(1000),B(1000),X(27,27),EIGV(27),Z(27),T(27,27)
COMMON/TWO/NDFPN,NDFPE,NNPE
COMMON/PP/NEL,NNODE,NBNODE
COMMON/BLK1/NODE(100,8),ID(2,120),XCO(120),
1      YCO(120),YMOD,PNU,THIK
COMMON/BLK2/S(136)
READ(5,*)IFPR,NGAUS1,NGAUS2
WRITE(6,43)NGAUS1,NGAUS2
43 FORMAT(2X,'INTEGRATION ORDER=',I3,'X',I2//)
READ(5,*)BETA
READ(5,*)NEL,NNODE,NDFPN,NNPE
WRITE(6,10)NEL,NNODE,NDFPN,NNPE
NDFPE=NDFPN*NNPE
NWA=NDFPE*(NDFPE+1)/2
N=NDFPE
READ(5,*)YMOD,PNU,THIK
WRITE(6,6)YMOD,PNU,THIK
DO 45 I=1,NNODE
READ(5,*)I,XCO(I),YCO(I)
45 WRITE(6,8)I,XCO(I),YCO(I)
8      FORMAT(I10,2F15.4)
6      FORMAT(2X,5E12.5)
10     FORMAT(20I4)
DO 141 IEL=1,NEL
READ(5,*)(NODE(IEL,J),J=1,NNPE)
WRITE(6,10)(IEL,(NODE(IEL,J),J=1,NNPE))
141 CONTINUE
NDFPE=NNPE*NDFPN
CALL ELSTIF(1,NGAUS1,NGAUS2)
N1=N+1
II=1

```

```
DO 19 I=1,N
S(II)=S(II)+BETA
19 II=II+N1-I
DO 50 I=1,N
DO 50 J=1,N
T(I,J)=0.
50 T(I,I)=1.
K=1
DO 60 I=1,N
DO 60 J=I,N
B(K)=T(I,J)
60 K=K+1
DO 71 I=1,NWA
71 A(I)=S(I)
CALL JACOBI(A,B,X,EIGV,Z,N,NWA,RTOL,NSMAX,IFPR)
WRITE(6,1234)
1234 FORMAT(///)
WRITE(6,1233)
1233 FORMAT(15X,' THE SHIFTED EIGENVALUES ARE: '/15X,27('-')//)
WRITE(6,13)(EIGV(I),I=1,N)
DO 90 I=1,N
90 EIGV(I)=EIGV(I)-BETA
WRITE(6,1234)
WRITE(6,1235)
1235 FORMAT(15X,' THE ORIGINAL EIGENVALUES ARE: '/15X,29('-')//)
WRITE(6,13)(EIGV(I),I=1,N)
WRITE(6,1234)
WRITE(6,1237)
1237 FORMAT(10X,' THE EIGENVECTORS ARE: '/10X,20('-')//)
DO 95 I=1,N
WRITE(6,1227)
1227 FORMAT(//)
```

```
95 WRITE(6,13)(X(I,J),J=1,N)
```

```
13 FORMAT(5E13.6/)
```

```
STOP
```

```
END
```

```
SUBROUTINE SHAPE2(T,S)
```

```
C
```

```
C**** CALCULATES SHAPE FUNCTIONS AND THEIR
```

```
C DERIVATIVES FOR 2D ELEMENTS
```

```
C
```

```
COMMON/SP/ELCOD(2,8),SHAP2(8),DERIV2(2,8),
```

```
1 CARTD2(2,8),DJACB2,XJACM2(2,2),XJACI2(2,2)
```

```
S2=S*2.0
```

```
T2=T*2.0
```

```
SS=S*S
```

```
TT=T*T
```

```
ST=S*T
```

```
SST=S*S*T
```

```
STT=S*T*T
```

```
ST2=S*T*2.0
```

```
C
```

```
C**** SHAPE FUNCTIONS
```

```
C
```

```
SHAP2(1)=(-1.0+ST+SS+TT-SST-STT)/4.0
```

```
SHAP2(2)=(1.0-T-SS+SST)/2.0
```

```
SHAP2(3)=(-1.0-ST+SS+TT-SST+STT)/4.0
```

```
SHAP2(4)=(1.0+S-TT-STT)/2.0
```

```
SHAP2(5)=(-1.0+ST+SS+TT+SST+STT)/4.0
```

```
SHAP2(6)=(1.0+T-SS-SST)/2.0
```

```
SHAP2(7)=(-1.0-ST+SS+TT+SST-STT)/4.0
```

SHAP2(8)=(1.0-S-TT+STT)/2.0

C

C**** SHAPE FUNCTIONS DERIVATIVES

C

DERIV2(1,1)=(T+S2-ST2-TT)/4.0

DERIV2(1,2)=-S+ST

DERIV2(1,3)=(-T+S2-ST2+TT)/4.0

DERIV2(1,4)=(1.0-TT)/2.0

DERIV2(1,5)=(T+S2+ST2+TT)/4.0

DERIV2(1,6)=-S-ST

DERIV2(1,7)=(-T+S2+ST2-TT)/4.0

DERIV2(1,8)=(-1.0+TT)/2.0

DERIV2(2,1)=(S+T2-SS-ST2)/4.0

DERIV2(2,2)=(-1.0+SS)/2.0

DERIV2(2,3)=(-S+T2-SS+ST2)/4.0

DERIV2(2,4)=-T-ST

DERIV2(2,5)=(S+T2+SS+ST2)/4.0

DERIV2(2,6)=(1.0-SS)/2.0

DERIV2(2,7)=(-S+T2+SS-ST2)/4.0

DERIV2(2,8)=-T+ST

RETURN

END

SUBROUTINE ELSTIF(IEL,NGAUS1,NGAUS2)

COMMON/SP/ELCOD(2,8),SHAP2(8),DERIV2(2,8),

1 CARTD2(2,8),DJACB2,XJACM2(2,2),XJACI2(2,2)

COMMON/TWO/NDFPN,NDFPE,NNPE

COMMON/S3/SHAP3(8),DERIV3(2,8),CARTD3(2,8)

COMMON/TP/NEL,NNODE,NBNODE

COMMON/BLK1/NODE(100,8),ID(2,120),XCO(120),

```

1      YCO(120),YMOD,PNU,THIK
      COMMON/BLK2/S(136)
      DIMENSION POSGP(4),WEIGP(4)
      DIMENSION DM(3,3),BM(3,16),DB(3,16),ES(16,16)
      DO 15 I=1,136
15  S(I)=0.
      DO 10 I=1,8
      L=NODE(IEL,I)
      ELCOD(1,I)=XCO(L)
10  ELCOD(2,I)=YCO(L)
      DO 30 I=1,3
      DO 30 J=1,3
30  DM(I,J)=0.
      CON=(YMOD*THIK)/(1.-PNU*PNU)
      DM(1,1)=CON
      DM(2,2)=CON
      DM(1,2)=CON*PNU
      DM(2,1)=CON*PNU
      DM(3,3)=CON*(1.-PNU)/2.
      CALL GAUSSQ(NGAUS1,POSGP,WEIGP)
      DO 50 IG=1,NGAUS1
      DO 50 JG=1,NGAUS1
      EXISP=POSGP(IG)
      ETASP=POSGP(JG)
      CALL SHAPE2(ETASP,EXISP)
      CALL JACOB2(IEL)
      DAREA=DJACB2*WEIGP(IG)*WEIGP(JG)
      DO 40 I=1,3
      DO 40 J=1,16
40  BM(I,J)=0.
      DO 46 I=1,8
      II=2*(I-1)+1

```

```
BM(1,II)=CARTD2(1,I)
BM(2,II+1)=CARTD2(2,I)
46 CONTINUE
DO 75 I=1,3
DO 75 J=1,16
DB(I,J)=0.
DO 75 K=1,3
75 DB(I,J)=DB(I,J)+DM(I,K)*BM(K,J)
DO 80 I=1,16
DO 80 J=1,16
ES(I,J)=0.
DO 80 K=1,3
80 ES(I,J)=ES(I,J)+BM(K,I)*DB(K,J)*DAREA
K=1
DO 90 I=1,16
DO 90 J=1,16
S(K)=S(K)+ES(I,J)
90 K=K+1
50 CONTINUE
CALL GAUSSQ(NGAUS2,POSGP,WEIGP)
DO 51 IG=1,NGAUS2
DO 51 JG=1,NGAUS2
EXISP=POSGP(IG)
ETASP=POSGP(JG)
CALL SHAPE2(ETASP,EXISP)
CALL SHAPE3(ETASP,EXISP)
CALL JACOB2(IEL)
CALL JACOB3(IEL)
DAREA=DJACB2*WEIGP(IG)*WEIGP(JG)
DO 41 I=1,3
DO 41 J=1,16
41 BM(I,J)=0.
```

```
DO 47 I=1,8
  II=2*(I-1)+1
  BM(3,II)=CARTD3(2,I)
  BM(3,II+1)=CARTD3(1,I)
47 CONTINUE
  DO 76 I=1,3
    DO 76 J=1,16
      DB(I,J)=0.
      DO 76 K=1,3
66 DB(I,J)=DB(I,J)+DM(I,K)*BM(K,J)
      DO 81 I=1,16
        DO 81 J=1,16
          ES(I,J)=0.
          DO 81 K=1,3
81 ES(I,J)=ES(I,J)+BM(K,I)*DB(K,J)*DAREA
            K=1
            DO 91 I=1,16
              DO 91 J=1,16
                S(K)=S(K)+ES(I,J)
91 K=K+1
51 CONTINUE
  RETURN
END
```

```
SUBROUTINE GAUSSQ(NGAUS,POSGP,WEIGP)
DIMENSION POSGP(4),WEIGP(4)
IF(NGAUS.GT.2) GO TO 40
POSGP(1)=-.5/7350269189626
WEIGP(1)=1.
GO TO 20
```



```

10  POSGP(1)=-0.774596669241483
    POSGP(2)=0.
    WEIGP(1)=0.5555555555555556
    WEIGP(2)=.8888888888888889
20  KGAUS=NGAUS/2
    DO 30 IGASH=1,KGAUS
    JGASH=NGAUS+1-IGASH
    POSGP(JGASH)=-POSGP(IGASH)
    WEIGP(JGASH)=WEIGP(IGASH)
30  CONTINUE
    RETURN
    END

```

SUBROUTINE SHAPE3(T,S)

C***

C*** Subroutine for smoothed shape functions and derivatives.

C***

COMMON/S3/SHAP3(8),DERIV3(2,8),CARTD3(2,8)

B=1./12.

E=1./3.

SHAP3(1)=-B*(1.+S+T)

SHAP3(2)=(1.-T)*E

SHAP3(3)=B*(-1.+S-T)

SHAP3(4)=E*(1.+S)

SHAP3(5)=B*(-1.+S+T)

SHAP3(6)=E*(1.+T)

SHAP3(7)=B*(-1.-S+T)

SHAP3(8)=E*(1.-S)

DERIV3(1,1)=-1./12+.5*S+.25*T

```

DERIV3(1,2)=-S
DERIV3(1,3)=1./12.+5*S-.25*T
DERIV3(1,4)=1./3.
DERIV3(1,5)=1./12.+5*S+.25*T
DERIV3(1,6)=-S
DERIV3(1,7)=-1./12.+5*S-.25*T
DERIV3(1,8)=-1./3.
DERIV3(2,1)=-1./12.+25*S+.5*T
DERIV3(2,2)=-1./3.
DERIV3(2,3)=-1./12.-.25*S+.5*T
DERIV3(2,4)=-T
DERIV3(2,5)=1./12+.25*S+.5*T
DERIV3(2,6)=1./3.
DERIV3(2,7)=1./12.-.25*S+.5*T
DERIV3(2,8)=-T
RETURN
END

```

```

SUBROUTINE JACOB2(IEL)
COMMON/SP/ELCOD(2,8),SHAP2(8),DERIV2(2,8),
1      CARTD2(2,8),DJACB2,XJACM2(2,2),XJACI2(2,2)
DO 20 IDIME=1,2
DO 20 JDIME=1,2
XJACM2(IDIME,JDIME)=0.0
DO 20 INODE=1,8
XJACM2(IDIME,JDIME)=XJACM2(IDIME,JDIME)+
, DERIV2(IDIME,INODE)*ELCOD(JDIME,INODE)
20 CONTINUE
DJACB2=XJACM2(1,1)*XJACM2(2,2)-XJACM2(1,2)*
, XJACM2(2,1)

```

```

      IF(DJACB2.GT.0.0) GO TO 30
      WRITE (6,900) IEL
      STOP
30  XJACI2(1,1)=XJACM2(2,2)/DJACB2
      XJACI2(2,2)=XJACM2(1,1)/DJACB2
      XJACI2(1,2)=-XJACM2(1,2)/DJACB2
      XJACI2(2,1)=-XJACM2(2,1)/DJACB2
      DO 40 IDIME=1,2
      DO 40 INODE=1,8
      CARTD2(IDIME,INODE)=0.0
      DO 40 JDIME=1,2
      CARTD2(IDIME,INODE)=CARTD2(IDIME,INODE)+
      XJACI2(IDIME,JDIME)*DERIV2(JDIME,INODE)
40  CONTINUE
900  FORMAT(/,24HPROGRAM HALTED IN JACOB2,
      /,11X,22H ZERO OR NEGATIVE AREA,/,
      10X,16H ELEMENT NUMBER,,I5)
      RETURN
      END

```

```

SUBROUTINE JACOB3(IEE)
  COMMON/SP/ELCOU(2,8),SHAP2(8),DERIV2(2,8),
1  CARTD2(2,8),DJACB2,XJACM2(2,2),XJACI2(2,2)
  COMMON/S3/SHAP3(8),DERIV3(2,8),CARTD3(2,8)
  DO 40 IDIME=1,2
  DO 40 INODE=1,8
  CARTD3(IDIME,INODE)=0.0
  DO 40 JDIME=1,2
  CARTD3(IDIME,INODE)=CARTD3(IDIME,INODE)+
  XJACI2(IDIME,JDIME)*DERIV3(JDIME,INODE)

```

40 CONTINUE

RETURN

END

SUBROUTINE JACOBI(A,B,X,EIGV,Z,N,NWA,RTOL,NSMAX,IFPR)

C***

C*** Program to Solve Generalized Eigenproblem using the Generalized

C*** Jacobi Iteration

C***

DIMENSION A(1000),B(1000),X(27,27),EIGV(27),Z(27)

NSMAX=16

RTOL=1.E-06

N1=N+1

II=1

DO 10 I=1,N

? IF(A(II).GT.0.AND.B(II).GT.0.)GO TO 4

WRITE(6,2020)II,A(II),B(II)

STOP

4 Z(I)=A(II)/B(II)

EIGV(I)=Z(I)

10 II=II+N1-I

DO 20 I=1,N

DO 20 J=1,N

20 X(I,J)=0.

30 X(I,I)=1.

IF(N.EQ.1)RETURN

NSWEEP=0

NR=N-1

40 NSWEEP=NSWEEP+1

```
IF(IFPR.EQ.1)WRITE(6,2000)NSWEEP
EPS=(.01**NSWEEP)**2
DO 210 J=1,NR
JP1=J+1
JM1=J-1
LJK=JM1*N-JM1*J/2
JJ=LJK+J
DO 210 K=JP1,N
KP1=K+1
KM1=K-1
JK=LJK+K
KK=KM1*N-KM1*K/2+K
EPTOLA=(A(JK)*A(JK))/(A(JJ)*A(KK))
EPTOLB=(B(JK)*B(JK))/(B(JJ)*B(KK))
IF((EPTOLA.LT.EPS).AND.(EPTOLB.LT.EPS))GO TO 210
AKK=A(KK)*B(JK)-B(KK)*A(JK)
AJJ=A(JJ)*B(JK)-B(JJ)*A(JK)
AB=A(JJ)*B(KK)-A(KK)*B(JJ)
CHECK=(AB*AB+4.*AKK*AJJ)/4.
IF(CHECK)50,60,60
50 WRITE(6, 2020)
STOP
60 SQCH=SQRT(CHECK)
D1=AB/2.+SQCH
D2=AB/2.-SQCH
DEN=D1
IF(ABS(D2).GT.ABS(D1))DEN=D2
IF(DEN)80,70,80
70 CA=0.
CG=-A(JK)/A(KK)
GO TO 90
80 CA=AKK/DEN
```

```
CG=-AJJ/DEN
90 IF(N-2)100,190,100
100 JM11=JM1-1
    IF(JM11)130,110,110
110 DO 120 I=1,JM1
    IM1=I-1
    IJ=IM1*N-IM1*I/2+J
    IK=IM1*N-IM1*I/2+K
    AJ=A(IJ)
    BJ=B(IJ)
    AK=A(IK)
    BK=B(IK)
    A(IJ)=AJ+CG*AK
    B(IJ)=BJ+CG*BK
    A(IK)=AK+CA*AJ
120 B(IK)=BK+CA*BJ
130 IF(KP1-N)140,140,160
140 LJ1=JM1*N-JM1*J/2
    LK1=KM1*N-KM1*K/2
    DO 150 I=KP1,N
    JI=LJ1+I
    KI=LK1+I
    AJ=A(JI)
    BJ=B(JI)
    AK=A(KI)
    BK=B(KI)
    A(JI)=AJ+CG*AK
    B(JI)=BJ+CG*BK
    A(KI)=AK+CA*AJ
150 B(KI)=BK+CA*BJ
160 IF(JP1-KM1)170,170,190
170 LJ1=JM1*N-JM1*J/2
```

```

DO 180 I=JP1,KM1
  JI=LJI+I
  IM1=I-1
  IK=IM1*N-IM1*I/2+K
  AJ=A(JI)
  BJ=B(JI)
  AK=A(IK)
  BK=B(IK)
  A(JI)=AJ+CG*AK
  B(JI)=BJ+CG*BK
  A(IK)=AK+CA*AJ
180 B(IK)=BK+CA*BJ
190 AK=A(KK)
  BK=B(KK)
  A(KK)=AK+2.*CA*A(JK)+CA*CA*A(JJ)
  B(KK)=BK+2.*CA*B(JK)+CA*CA*B(JJ)
  A(JJ)=A(JJ)+2.*CG*A(JK)+CG*CG*AK
  B(JJ)=B(JJ)+2.*CG*B(JK)+CG*CG*BK
  A(JK)=0.
  B(JK)=0.
DO 200 I=1,N
  XJ=X(I,J)
  XK=X(I,K)
  X(I,J)=XJ+CG*XK
200 X(I,K)=XK+CA*XJ
210 CONTINUE
  II=1
DO 220 I=1,N
  IF(A(II).GT.0..AND.B(II).GT.0.)GO TO 215
  WRITE(6, 2020)II,A(II),B(II)
  STOP
215 EIGV(I)=A(II)/B(II)

```

```

220 II=II+N1-I
    IF(IFPR.EQ.O) GO TO 230
    WRITE(6, 2030)
    WRITE(6, 2010)(EIGV(I),I=1,N)
230 DO 240 I=1,N
    TOL=RTOL*Z(I)
    DIF=ABS(EIGV(I)-Z(I))
    IF(DIF.GT.TOL) GO TO 280
240 CONTINUE
    EPS=RTOL**2
    DO 250 J=1,NR
    JM1=J-1
    JP1=J+1
    LJK=JM1*N-JM1*J/2
    JJ=LJK+J
    DO 250 K=JP1,N
    KM1=K-1
    JK=LJK+K
    KK=KM1*N-KM1*K/2+K
    EPSA=(A(JK)*A(JK))/(A(JJ)*A(KK))
    EPSB=(B(JK)*B(JK))/(B(JJ)*B(KK))
    IF((EPSA.LT.EPS).AND.(EPSB.LT.EPS)) GO TO 250
    GO TO 280
250 CONTINUE
255 II=1
    DO 275 I=1,N
    BB=SQRT(B(II))
    DO 270 K=1,N
270 X(K,I)=X(K,I)/BB
275 II=II+N1-I
    RETURN
280 DO 290 I=1,N

```



```
290 Z(I)=EIGV(I)
    IF(NSWEEP.LT.NSMAX) GO TO 40
    GO TO 255
2000 FORMAT(//1X,23HSWEEP NUMBER IN JACOBI#,I4)
2010 FORMAT(/1X,5E13.8)
2020 FORMAT(//1X,19HERROR SOLUTION STOP//
    11X,30HMATRICES NOT POSITIVE DEFINITE//
    11X,3HII=,I5,6HA(II)=,E20.12,3X,6HB(II)=,E20.12)
2030 FORMAT(1X,34HCURRENT EIGEN VALUES IN JACOBI ARE)
    END
```

APPENDIX-B

SUBROUTINE ELSTIF(LCOD)

C***

C*** Element Stiffness Matrix Subroutine for the Regular or Transition

C*** Member of the Plane Triangular Family.

C***

C**** LCOD element number(Fig.8)example LCOD=4 for LST,LCOD=1 for CST

C**** X(3),Y(3) X and Y coordinates of the vertices of the triagle

C**** THK thickness of the element

C**** D11,D12,D13,...,D33 coefficients of the elasticity matrix

C**** S array containing stiffness coefficients stored row wise(con

C ventional ordering of d.o.f. viz. ($u_1, v_1, u_2, v_2, \dots$)

C**** NDFPN number of degrees of freedom per node =2

C**** T matrix is the transformation matrix

COMMON/ONE/D11,D12,D13,D22,D23,D33

COMMON/TWO/NDFPN,NDFPE,X(3),Y(3),S(210),THK

COMMON/TRS/T(10,9)

DIMENSION IA(10,10),IB(10,10),KOD(10),A(10,10),B(10,10),C(10,10)

1,NNPE(11)

DATA KOD/2,1,3,5,4,2,8,7,6,10/

DATA NNPE/3,4,5,6,6,7,8,8,9,10/

DATA IA/68,0,-14,6,6,-6,-6,54,-108,11*0,-14,0,68,-6,-6,6,6,-108,

154,0,6,0,-6,270,-54,54,54,0,0,-324,6,0,-6,-54,270,-270,54,3*0,

2-6,0,6,54,-270,270,-54,3*0,-6,0,6,54,54,-54,270,0,0,-324,54,0,

3-108,4*0,270,-216,0,-108,0,54,4*0,-216,270,4*0,-324,0,0,-324,

40,0,648/

DATA IB/10*0,14,0,-14,-48,114,-114,48,3*0,-14,0,68,-6,-6,6,6,-108,

154,0,114,0,-6,135,-27,3*27,-135,-162,-48,0,-6,2*135,-135,3*27,

2-162,0,0,54,27,-135,135,-189,2*27,162,0,0,-108,27,27,-27,2*135

3,-27,-162,48,0,6,27,27,-27,2*135,-189,-162,-114,0,6,-135,27,3*
 4-27,135,162,3*0,2*-162,162,2*-162,162,324/

CC=160.0

DO 100 I=1,10

II=KOD(I)

DO 100 J=1,10

JJ=KOD(J)

A(I,J)=IA(I,J)

B(I,J)=IB(I,J)

100 C(I,J)=IA(II,JJ)

NNPEL=NNPE(LCOD)

IF(LCOD.EQ.11)GO TO 300

CALL TRANS(LCOD)

CALL MULT(A,NNPEL)

CALL MULT(B,NNPEL)

CALL MULT(C,NNPEL)

300 CONTINUE

B1=Y(2)-Y(3)

B2=Y(3)-Y(1)

C1=X(3)-X(2)

C2=X(1)-X(3)

DET=ABS(B1*C2-B2*C1)

G11=D11*B1*B1+2.*D13*B1*C1+D33*C1*C1

G12=D11*B1*B2+D13*(B1*C2+B2*C1)+D33*C1*C2

G13=D13*B1*B1+D23*C1*C1+(D12+D33)*B1*C1

G14=D12*B1*C2+D13*B1*B2+D23*C1*C2+D33*B2*C1

G22=D11*B2*B2+2.*D13*B2*C2+D33*C2*C2

G23=D12*B2*C1+D13*B1*B2+D23*C1*C2+D33*B1*C2

G24=D13*B2*B2+(D12+D33)*B2*C2+D23*C2*C2

G33=D22*C1*C1+2.*D23*B1*C1+D33*B1*B1

G34=D22*C1*C2+D23*(B1*C2+B2*C1)+D33*B1*B2

G44=D22*C2*C2+2.*D23*B2*C2+D33*B2*B2

```

CON=THK/(DET*CC)
NDFPE=NDFPN*NNPEL
NDF=NDFPE*(NDFPE+1)/2
IIN=2*(NDFPE+1)
DO 20 I=1,NNPEL
II=NDFPN*(I-1)+1
IIX=(IIN-II)*(II-1)/2+1
DO 20 J=I,NNPEL
JJ=NDFPN*(J-I)
N=IIX+JJ
M=N+NDFPE-II+1
S(N)=G11*A(I,J)+G12*(B(I,J)+B(J,I))+G22*C(I,J)
S(M)=G33*A(I,J)+G34*(B(I,J)+B(J,I))+G44*C(I,J)
S(N+1)=G13*A(I,J)+G14*B(I,J)+G23*B(J,I)+G24*C(I,J)
IF(I.EQ.J)GO TO 20
K=II*(IIN-II-1)/2+JJ
S(K)=G13*A(J,I)+G14*B(J,I)+G23*B(I,J)+G24*C(J,I)
20 CONTINUE
DO 40 I=1,NDF
40 S(I)=S(I)*CON
RETURN
END

```

SUBROUTINE TRANS(LCOD)

COMMON/TRS/T(10,9)

DIMENSION NNOS(3,4),NA(3,8),NB(4,6),KOD(10),NODS(3)

DATA NNOS/1,2,3,2,3,1,6*0/

DATA NA/6,2,0,3,-1,0,3,8,9,6,-1,0,0,2,0,0,8,6*0,9/

DATA NB/3,1,1,-1,3,3,1,-1,3,1,-1,-1,0,3*4,0,0,4,4,3*0,4/

DATA KOD/222,223,233,333,224,234,243,334,244,344/

```

NODS(1)=KOD(LCOD)/100
NODS(2)=(KOD(LCOD)-NODS(1)*100)/10
NODS(3)=(KOD(LCOD)-NODS(1)*100-NODS(2)*10)
KL=3
DO 100 ISIDE=1,3
NP=NODS(ISIDE)-2
IF(NP.EQ.0)GO TO 100
DO 200 J=1,NP
KL=KL+1
200 NNOS(ISIDE,J+2)=KL
100 CONTINUE
IF(LCOD.GT.4)KL=KL+1
DO 300 I=1,10
DO 300 J=1,KL
300 T(I,J)=0.
DO 350 ISIDE=1,3
T(ISIDE,ISIDE)=1.
NROW=2*ISIDE+2
NNS=NODS(ISIDE)
DO 400 I=1,NNS
II=NNOS(ISIDE,I)
T(NROW,II)=NA(NNS-1,I)/9.
400 T(NROW+1,II)=NA(NNS-1,I+NNS)/9.
350 CONTINUE
IF(LCOD.LE.4) GO TO 475
T(10,KL)=1.
GO TO 500
475 DO 450 I=1,KL
450 T(10,I)=NB(LCOD,I)/9.
500 RETURN
END

```

```
SUBROUTINE MULT(A,N)
COMMON/TRS/T(10,9)
DIMENSION A(10,10),B(10,10)
DO 100 I=1,10
DO 100 J=1,N
B(I,J)=0
DO 100 K=1,10
100 B(I,J)=B(I,J)+A(I,K)*T(K,J)
DO 200 I=1,N
DO 200 J=1,N
A(I,J)=0
DO 200 K=1,10
200 A(I,J)=A(I,J)+T(K,I)*B(K,J)
RETURN
END
```

APPENDIX-C

SUBROUTINE ELSTIF(X1,X2,X3,Y1,Y2,Y3,T1,T2,T3)

C***

C*** Element Stiffness Matrix Subroutine for the Tapered or Uniformly
C*** Thick High Precision Plate Bending Element.

C***

C*** INPUT

C***

C*** X1,X2,X3,Y1,Y2,Y3,T1,T2,T3 are the x,y coordinates and thicknesses
C*** at the triangle vertices. For uniform thickness, T1=T2=T3

C*** D11,D12,D13,D22,D23,D33 are the elements of the upper half of the
C*** elasticity matrix excluding the thickness term. For example:

C*** $D11=Y/(12*(1-PNU*PNU))$; $D12=PNU*D11$; $D13=0.0$ etc. for the

C*** isotropic case. Here, Y- Young's modulus , PNU- Poisson's ratio.

C***

C*** OUTPUT

C*** ESTIF is the (18 X 18) element stiffness matrix

C***

COMMON/THRE/Q1(18,18)

COMMON/ONE/E11,E12,E13,E22,E23,E33

COMMON/FOUR/F1(18,18),G(28)

COMMON/BLK2/ESTIF(18,18),D11,D12,D13,D22,D23,D33

DIMENSION P(18,18),IT(28,10),T(10)

DATA IT/831600,2*554400,3*138600,55440,2*166320,2*55440,27720,

19240,27720,55440,79200,15840,2*7920,15840,79200,118800,59400,

223760,5940,23760,59400,118800,2*831600,415800,249480,166320,

32*83160,332640,83160,55440,118800,47520,11880,2*23760,178200,

429700,11880,8910,11880,29700,277200,118800,39600,7920,23760,

52*39600,831600,415800,831600,83160,166320,249480,55440,83160,

6332340,83160,2*23760,11880,47520,118800,29700,11880,8910,11880,
729700,178200,2*39600,23760,7920,39600,118800,277200,415800,
8498960,2*166320,83160,27720,47520,237600,2*23760,89100,29700,
95940,8910,5940,138600,19800,6600,2*3960,6600,221760,83160,
123760,3960,9504,11880,7920,415800,2*332640,83160,110880,
283160,47520,2*95040,47520,29700,23760,8910,23760,29700,39600,
313200,2*7920,13200,39600,55440,47520,23760,6336,23760,47520,
455440,415800,166320,498960,27720,83160,166320,2*23760,237600,
547520,5940,8910,5940,29700,89100,6600,2*3960,6600,19800,138600,
67920,11880,9504,3960,23760,83160,221760,83160,110880,27720,
739600,15840,3960,9900,59400,2970,3960,23100,6600,1100,1320,660,
836960,4620,1320,660,528,660,60480,20160,5040,720,2*1440,720,
92*83160,55440,2*23760,2*11880,29700,11880,8910,9900,6600,1980,
13960,3300,13860,3960,1980,1584,1980,3960,20160,15120,6480,1440,
24320,6480,5040,83160,55440,83160,11880,2*23760,8910,11880,29700,
311880,3300,3960,1980,6600,9900,3960,1980,1584,1980,3960,13860,
45040,6480,4320,1440,6480,15120,20160,83160,27720,110880,3960,
515840,39600,3960,2970,59400,9900,660,1320,1100,6600,23100,
6660,528,660,1320,4620,36960,720,2*1440,720,5040,20160,60480/

B1=Y2-Y3

B2=Y3-Y1

B3=Y1-Y2

C1=X3-X2

C2=X1-X3

C3=X2-X1

DET=ABS(B1*C2-B2*C1)

T13=T1-T3

T23=T2-T3

T(1)=T3**3

T(2)=T3*T3*T13

T(3)=T3*T3*T23

T(4)=T3*T13*T13

3

MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS
STANDARD REFERENCE MATERIAL 1010a
(ANSI and ISO TEST CHART No. 2)

T(5)=T3*T13*T23

T(6)=T3*T23*T23

T(7)=T13**3

T(8)=T13*T13*T23

T(9)=T13*T23*T23

T(10)=T23**3

JJ=10

IF(T1.EQ.T2.AND.T1.EQ.T3)JJ=1

DO 810 MX=1,28

SUM=0.

DO 820 MY=1,JJ

820 SUM=SUM+IT(MX,MY)*T(MY)

810 G(MX)=SUM/1663200.

BC12=B1*C2

BC21=B2*C1

BB11=B1*B1

BB22=B2*B2

CC11=C1*C1

CC22=C2*C2

BC11=B1*C1

BC112=2.*BC11

BB122=2.*B1*B2

CC122=2.*C1*C2

BC222=2.*B2*C2

BCX122=2.*(BC12+BC21)

DET3=(DET**3)

FF1=D11*BB22+D12*CC22+D13*BC222

FF2=D12*BB22+D22*CC22+D23*BC222

FF3=D13*BB22+D23*CC22+D33*BC222

FF4=D11*BB122+D12*CC122+D13*BCX122

FF5=D12*BB122+D22*CC122+D23*BCX122

FF6=D13*BB122+D23*CC122+D33*BCX122

```

E11=(BB11*(D11*BB11+D12*CC11+D13*BC112)+CC11*(D12*BB11+D22*CC11+
1D23*BC112)+BC112*(D13*BB11+D23*CC11+D33*BC112))/DET3
E12=(BB11*FF1+CC11*FF2+BC112*FF3)/DET3
E13=(BB11*FF4+CC11*FF5+BC112*FF6)/DET3
E22=(BB22*FF1+CC22*FF2+BC222*FF3)/DET3
E23=(BB22*FF4+CC22*FF5+BC222*FF6)/DET3
E33=(BB122*FF4+CC122*FF5+BCX122*FF6)/DET3
CALL QMAT(B1,B2,B3,C1,C2,C3)
CALL FMAT
DO 600 I=1,18
DO 600 J=1,18
P(I,J)=0.
DO 600 K=1,18
600 P(I,J)=P(I,J)+F1(I,K)*Q1(K,J)
DO 700 I=1,18
DO 700 J=1,18
ESTIF(I,J)=0.
DO 710 K=1,18
710 ESTIF(I,J)=ESTIF(I,J)+Q1(K,I)*P(K,J)
700 ESTIF(J,I)=ESTIF(I,J)
RETURN
END

```

SUBROUTINE FMAT

COMMON/ONE/E11,E12,E13,E22,E23,E33

COMMON/FOUR/F1(18,18),G(28)

F1(1,1)=4.*G(1)*E11

F1(1,2)=2.*G(1)*E13

F1(1,3)=4.*G(1)*E12

F1(1,4)=6.*G(2)*E11

$$F1(1,5)=2. \cdot (G(3) \cdot E11 + G(2) \cdot E13)$$

$$F1(1,6)=2. \cdot (G(2) \cdot E12 + G(3) \cdot E13)$$

$$F1(1,7)=6. \cdot G(3) \cdot E12$$

$$F1(1,8)=24. \cdot G(4) \cdot E11$$

$$F1(1,9)=6. \cdot (G(5) \cdot E11 + G(4) \cdot E13)$$

$$F1(1,10)=4. \cdot (G(6) \cdot E11 + G(4) \cdot E12 + G(5) \cdot E13)$$

$$F1(1,11)=6. \cdot (G(5) \cdot E12 + G(6) \cdot E13)$$

$$F1(1,12)=24. \cdot G(6) \cdot E12$$

$$F1(1,13)=20. \cdot G(8) \cdot E11$$

$$F1(1,14)=12. \cdot G(7) \cdot E11 + 4. \cdot G(8) \cdot E13$$

$$F1(1,15)=6. \cdot G(10) \cdot E11 + 2. \cdot G(8) \cdot E12 + 6. \cdot G(7) \cdot E13$$

$$F1(1,16)=2. \cdot G(9) \cdot E11 + 6. \cdot G(7) \cdot E12 + 8. \cdot G(10) \cdot E13$$

$$F1(1,17)=12. \cdot G(10) \cdot E12 + 4. \cdot G(9) \cdot E13$$

$$F1(1,18)=20. \cdot G(9) \cdot E12$$

$$F1(2,2)=G(1) \cdot E33$$

$$F1(2,3)=2. \cdot G(1) \cdot E23$$

$$F1(2,4)=3. \cdot G(2) \cdot E13$$

$$F1(2,5)=G(3) \cdot E13 + G(2) \cdot E33$$

$$F1(2,6)=G(2) \cdot E23 + G(3) \cdot E33$$

$$F1(2,7)=3. \cdot G(3) \cdot E23$$

$$F1(2,8)=12. \cdot G(4) \cdot E13$$

$$F1(2,9)=3. \cdot (G(5) \cdot E13 + G(4) \cdot E33)$$

$$F1(2,10)=2. \cdot (G(6) \cdot E13 + G(4) \cdot E23 + G(5) \cdot E33)$$

$$F1(2,11)=3. \cdot (G(5) \cdot E23 + G(6) \cdot E33)$$

$$F1(2,12)=12. \cdot G(6) \cdot E23$$

$$F1(2,13)=10. \cdot G(8) \cdot E13$$

$$F1(2,14)=6. \cdot G(7) \cdot E13 + 2. \cdot G(8) \cdot E33$$

$$F1(2,15)=3. \cdot G(10) \cdot E13 + G(8) \cdot E23 + 3. \cdot G(7) \cdot E33$$

$$F1(2,16)=G(9) \cdot E13 + 3. \cdot (G(7) \cdot E23 + G(10) \cdot E33)$$

$$F1(2,17)=6. \cdot G(10) \cdot E23 + 2. \cdot G(9) \cdot E33$$

$$F1(2,18)=10. \cdot G(9) \cdot E23$$

$$F1(3,3)=4. \cdot G(1) \cdot E22$$

$$F1(3,4) = 6 * G(2) * E12$$

$$F1(3,5) = 2 * (G(3) * E12 + G(2) * E23)$$

$$F1(3,6) = 2 * (G(2) * E22 + G(3) * E23)$$

$$F1(3,7) = 6 * G(3) * E22$$

$$F1(3,8) = 24 * G(4) * E12$$

$$F1(3,9) = 6 * (G(5) * E12 + G(4) * E23)$$

$$F1(3,10) = 4 * (G(6) * E12 + G(4) * E22 + G(5) * E23)$$

$$F1(3,11) = 6 * (G(5) * E22 + G(6) * E23)$$

$$F1(3,12) = 24 * G(6) * E22$$

$$F1(3,13) = 20 * G(8) * E12$$

$$F1(3,14) = 12 * G(7) * E12 + 4 * G(8) * E23$$

$$F1(3,15) = 6 * G(10) * E12 + 2 * G(8) * E22 + 6 * G(7) * E23$$

$$F1(3,16) = 2 * G(9) * E12 + 6 * G(7) * E22 + 6 * G(10) * E23$$

$$F1(3,17) = 12 * G(10) * E22 + 4 * G(9) * E23$$

$$F1(3,18) = 20 * G(9) * E22$$

$$F1(4,4) = 36 * G(4) * E11$$

$$F1(4,5) = 6 * G(5) * E11 + 12 * G(4) * E13$$

$$F1(4,6) = 12 * G(4) * E12 + 6 * G(5) * E13$$

$$F1(4,7) = 18 * G(5) * E12$$

$$F1(4,8) = 36 * G(8) * E11$$

$$F1(4,9) = 18 * G(7) * E11 + 9 * G(8) * E13$$

$$F1(4,10) = 6 * G(10) * E11 + 6 * G(8) * E12 + 12 * G(7) * E13$$

$$F1(4,11) = 18 * G(7) * E12 + 9 * G(10) * E13$$

$$F1(4,12) = 36 * G(10) * E12$$

$$F1(4,13) = 120 * G(11) * E11$$

$$F1(4,14) = 36 * G(12) * E11 + 24 * G(11) * E13$$

$$F1(4,15) = 36 * G(13) * E11 + 12 * G(11) * E12 + 18 * G(12) * E13$$

$$F1(4,16) = 6 * G(14) * E11 + 18 * G(12) * E12 + 36 * G(13) * E13$$

$$F1(4,17) = 72 * G(13) * E12 + 12 * G(14) * E13$$

$$F1(4,18) = 60 * G(14) * E12$$

$$F1(5,5) = 4 * (G(6) * E11 + G(5) * E13 + G(4) * E33)$$

$$F1(5,6) = 2 * G(5) * (E12 + E33) + 4 * (G(6) * E13 + G(4) * E23)$$

$$F1(5,7)=12.*G(6)*E12+6.*G(5)*E23$$

$$F1(5,8)=12.*(G(7)*E11+G(8)*E13)$$

$$F1(5,9)=6.*G(10)*E11+9.*G(7)*E13+3.*G(8)*E33$$

$$F1(5,10)=2.*G(9)*E11+2.*G(7)*(E12+2.*E33)+6.*G(10)*E13+2.*G(8)*E23$$

$$F1(5,11)=G(10)*(6.*E12+3.*E33)+3.*G(9)*E13+6.*G(7)*E23$$

$$F1(5,12)=12.*(G(9)*E12+G(10)*E23)$$

$$F1(5,13)=20.*G(12)*E11+40.*G(11)*E13$$

$$F1(5,14)=24.*G(13)*E11+16.*G(12)*E13+8.*G(11)*E33$$

$$F1(5,15)=6.*G(14)*E11+G(12)*(2.*E12+6.*E33)+24.*G(13)*E13$$

$$1 \quad +4.*G(11)*E23$$

$$F1(5,16)=4.*G(15)*E11+12.*G(13)*(E12+E33)+8.*G(14)*E13$$

$$1 \quad +6.*G(12)*E23$$

$$F1(5,17)=G(14)*(12.*E12+4.*E33)+8.*G(15)*E13+24.*G(13)*E23$$

$$F1(5,18)=40.*G(15)*E12+20.*G(14)*E23$$

$$F1(6,6)=4.*(G(4)*E22+G(5)*E23+G(6)*E33)$$

$$F1(6,7)=6.*G(5)*E22+12.*G(6)*E23$$

$$F1(6,8)=12.*(G(8)*E12+G(7)*E13)$$

$$F1(6,9)=G(7)*(6.*E12+3.*E33)+6.*G(10)*E13+3.*G(8)*E23$$

$$F1(6,10)=G(10)*(2.*E12+4.*E33)+2.*G(9)*E13+2.*G(8)*E22+6.*G(7)*E23$$

$$F1(6,11)=6.*G(7)*E22+9.*G(10)*E23+3.*G(9)*E33$$

$$F1(6,12)=12.*(G(10)*E22+G(9)*E23)$$

$$F1(6,13)=40.*G(11)*E12+20.*G(12)*E13$$

$$F1(6,14)=G(12)*(12.*E12+4.*E33)+24.*G(13)*E13+8.*G(11)*E23$$

$$F1(6,15)=12.*G(13)*(E12+E33)+6.*G(14)*E13+4.*G(11)*E22$$

$$1 \quad +8.*G(12)*E23$$

$$F1(6,16)=G(14)*(2.*E12+6.*E33)+4.*G(15)*E13+6.*G(12)*E22$$

$$1 \quad +24.*G(13)*E23$$

$$F1(6,17)=24.*G(13)*E22+16.*G(14)*E23+8.*G(15)*E33$$

$$F1(6,18)=20.*G(14)*E22+40.*G(15)*E23$$

$$F1(7,7)=36.*G(6)*E22$$

$$F1(7,8)=36.*G(7)*E12$$

$$F1(7,9)=18.*G(10)*E12+9.*G(7)*E23$$

$$F1(7,10)=6.*G(9)*E12+G(7)*E22+12.*G(10)*E23$$

$$F1(7,11)=18.*G(10)*E22+9.*G(9)*E23$$

$$F1(7,12)=36.*G(9)*E22$$

$$F1(7,13)=60.*G(12)*E12$$

$$F1(7,14)=72.*G(13)*E12+12.*G(12)*E23$$

$$F1(7,15)=18.*G(14)*E12+6.*G(12)*E22+36.*G(13)*E23$$

$$F1(7,16)=12.*G(15)*E12+36.*G(13)*E22+18.*G(14)*E23$$

$$F1(7,17)=36.*G(14)*E22+24.*G(15)*E23$$

$$F1(7,18)=120.*G(15)*E22$$

$$F1(8,8)=144.*G(11)*E11$$

$$F1(8,9)=36.*G(12)*E11+G(11)*E13$$

$$F1(8,10)=24.*G(13)*E11+G(11)*E12+G(12)*E13$$

$$F1(8,11)=36.*G(12)*E12+G(13)*E13$$

$$F1(8,12)=144.*G(13)*E12$$

$$F1(8,13)=120.*G(16)*E11$$

$$F1(8,14)=72.*G(17)*E11+24.*G(16)*E13$$

$$F1(8,15)=36.*G(18)*E11+G(17)*E13+12.*G(16)*E12$$

$$F1(8,16)=12.*G(19)*E11+36.*G(17)*E12+G(18)*E13$$

$$F1(8,17)=72.*G(18)*E12+24.*G(19)*E13$$

$$F1(8,18)=120.*G(19)*E12$$

$$F1(9,9)=36.*G(13)*E11+18.*G(12)*E13+9.*G(11)*E33$$

$$F1(9,10)=6.*G(14)*E11+G(11)*E23+G(12)*(E12+E33)+30.*G(13)*E13$$

$$F1(9,11)=G(13)*(36.*E12+9.*E33)+9.*G(14)*E13+G(12)*E23$$

$$F1(9,12)=36.*G(14)*E12+G(13)*E23$$

$$F1(9,13)=60.*G(17)*E11+30.*G(16)*E13$$

$$F1(9,14)=36.*G(18)*E11+30.*G(17)*E13+6.*G(16)*E33$$

$$F1(9,15)=18.*G(19)*E11+27.*G(18)*E13+G(17)*(6.*E12+9.*E33)$$

$$1 \quad +3.*G(16)*E23$$

$$F1(9,16)=6.*G(20)*E11+G(18)*(18.*E12+9.*E33)+21.*G(19)*E13$$

$$1 \quad +9.*G(17)*E23$$

$$F1(9,17)=G(19)*(36.*E12+6.*E33)+12.*G(20)*E13+18.*G(18)*E23$$

$$F1(9,18)=60.*G(20)*E12+30.*G(19)*E23$$

$F1(10,10)=4.*G(15)*E11+G(13)*(8.*E12+16.*E33)+8.*G(14)*E13$
 $1+4.*G(11)*E22+8.*G(12)*E23$
 $F1(10,11)=6.*(G(14)*(E12+E33)+G(15)*E13+G(12)*E22)+30.*G(13)*E23$
 $F1(10,12)=24.*(G(15)*E12+G(13)*E22+G(14)*E23)$
 $F1(10,13)=20.*(G(18)*E11+G(16)*E12)+40.*G(17)*E13$
 $F1(10,14)=12.*G(19)*E11+G(17)*(12.*E12+8.*E33)+28.*G(18)*$
 $1E13+4.*G(16)*E23$
 $F1(10,15)=6.*G(20)*E11+G(18)*(8.*E12+12.*E33)+18.*G(19)*$
 $1E13+2.*G(16)*E22+10.*G(17)*E23$
 $F1(10,16)=2.*G(21)*E11+G(19)*(8.*E12+12.*E33)+10.*G(20)*E13$
 $1+6.*G(17)*E22+18.*G(18)*E23$
 $F1(10,17)=G(20)*(12.*E12+8.*E33)+4.*G(21)*E13+12.*G(18)*E22$
 $1+28.*G(19)*E23$
 $F1(10,18)=20.*(G(21)*E12+G(19)*E22)+40.*G(20)*E23$
 $F1(11,11)=36.*G(13)*E22+18.*G(14)*E23+9.*G(15)*E33$
 $F1(11,12)=36.*(G(14)*E22+G(15)*E23)$
 $F1(11,13)=60.*G(17)*E12+30.*G(18)*E13$
 $F1(11,14)=G(18)*(36.*E12+6.*E33)+18.*G(19)*E13+12.*G(17)*E23$
 $F1(11,15)=G(19)*(18.*E12+9.*E33)+9.*G(20)*E13+6.*G(17)*E22+$
 $121.*G(18)*E23$
 $F1(11,16)=G(20)*(6.*E12+9.*E33)+3.*G(21)*E13+18.*G(18)*E22+$
 $127.*G(19)*E23$
 $F1(11,17)=36.*G(19)*E22+30.*G(20)*E23+6.*G(21)*E33$
 $F1(11,18)=60.*G(20)*E22+30.*G(21)*E23$
 $F1(12,12)=144.*G(15)*E22$
 $F1(12,13)=120.*G(18)*E12$
 $F1(12,14)=72.*G(19)*E12+24.*G(18)*E23$
 $F1(12,15)=36.*(G(20)*E12+G(19)*E23)+12.*G(18)*E22$
 $F1(12,16)=12.*G(21)*E12+36.*(G(19)*E22+G(20)*E23)$
 $F1(12,17)=72.*G(20)*E22+24.*G(21)*E23$
 $F1(12,18)=120.*G(21)*E22$
 $F1(13,13)=100.*G(22)*E11$

$$F1(13,14)=20.*(G(23)*E11+G(22)*E13)$$

$$F1(13,15)=10.*(G(24)*E11+G(22)*E12+G(23)*E13)$$

$$F1(13,16)=10.*(G(25)*E11+G(23)*E12+G(24)*E13)$$

$$F1(13,17)=20.*(G(24)*E12+G(25)*E13)$$

$$F1(13,18)=100.*G(25)*E12$$

$$F1(14,14)=12.*G(24)*E11+8.*G(23)*E13+4.*G(22)*E33$$

$$F1(14,15)=18.*G(25)*E11+2.*G(23)*(E12+E33)+8.*G(24)*E13$$

$$1 \quad +2.*G(22)*E23$$

$$F1(14,16)=2.*G(26)*E11+G(24)*(6.*E12+2.*E33)+20.*G(25)*E13$$

$$1 \quad +2.*G(23)*E23$$

$$F1(14,17)=G(25)*(36.*E12+4.*E33)+4.*(G(26)*E13+G(24)*E23)$$

$$F1(14,18)=20.*(G(26)*E12+G(25)*E23)$$

$$F1(15,15)=3.*G(26)*E11+G(24)*(2.*E12+3.*E33)+18.*G(25)*E13$$

$$1 \quad +G(22)*E22+2.*G(23)*E23$$

$$F1(15,16)=G(27)*E11+G(25)*(10.*E12+9.*E33)+4.*G(26)*E13$$

$$1 \quad +G(23)*E22+4.*G(24)*E23$$

$$F1(15,17)=G(26)*(6.*E12+2.*E33)+2.*(G(27)*E13+G(24)*E22)$$

$$1 \quad +20.*G(25)*E23$$

$$F1(15,18)=10.*(G(27)*E12+G(25)*E22+G(26)*E23)$$

$$F1(16,16)=G(28)*E11+G(26)*(2.*E12+3.*E33)+2.*G(27)*E13$$

$$1 \quad +3.*G(24)*E22+18.*G(25)*E23$$

$$F1(16,17)=2.*(G(27)*(E12+E33)+G(28)*E13)+18.*G(25)*E22$$

$$1 \quad +8.*G(26)*E23$$

$$F1(16,18)=10.*(G(28)*E12+G(26)*E22+G(27)*E23)$$

$$F1(17,17)=12.*G(26)*E22+8.*G(27)*E23+4.*G(28)*E33$$

$$F1(17,18)=20.*(G(27)*E22+G(28)*E23)$$

$$F1(18,18)=100.*G(28)*E22$$

$$DO \ 550 \ I=1,18$$

$$DO \ 550 \ J=I,18$$

$$550 \ F1(J,I)=F1(I,J)$$

$$RETURN$$

$$END$$

```

SUBROUTINE QMAT(B1,B2,B3,C1,C2,C3)
COMMON/THRE/Q1(18,18)
DIMENSION BC(2,3),AB(3,4)
D1=-(B1*B3+C1*C3)/(B3*B3+C3*C3)
D2=(B1*B2+C1*C2)/(B1*B1+C1*C1)
D3=(B1*B2+C1*C2)/(B2*B2+C2*C2)
DO 100 I=1,18
DO 100 J=1,18
100 Q1(I,J)=0.
Q1(4,1)=10.
Q1(5,1)=-30.*D3
Q1(8,1)=-15.
Q1(9,1)=60.*D3
Q1(10,1)=30.*(D1+2.*D3)
Q1(13,1)=6.
Q1(14,1)=-30.*D3
Q1(15,1)=-30.*(D1+2.*D3)
Q1(16,1)=-30.*(D3+D1)
Q1(6,7)=-30.*D2
Q1(7,7)=10.
Q1(10,7)=-30.*(1.-D1+2.*D2)
Q1(11,7)=60.*D2
Q1(12,7)=-15.
Q1(15,7)=30.*(D1-D2-1.)
Q1(16,7)=30.*(D1-2.*D2-1.)
Q1(17,7)=-30.*D2
Q1(18,7)=6.
Q1(4,13)=-10.
Q1(5,13)=30.*D3

```

Q1(6,13)=30.*D2
Q1(7,13)=-10.
Q1(8,13)=15.
Q1(9,13)=-60.*D3
Q1(10,13)=-30.*(1.+2.*(D2+D3))
Q1(11,13)=-60.*D2
Q1(12,13)=15.
Q1(13,13)=-6.
Q1(14,13)=30.*D3
Q1(15,13)=30.*(1.+D2)+60.*D3
Q1(16,13)=30.*(1.+D3)+60.*D2
Q1(17,13)=30.*D2
Q1(18,13)=-6.
BC(1,1)=C1
BC(1,2)=C2
BC(1,3)=C3
BC(2,1)=-B1
BC(2,2)=-B2
BC(2,3)=-B3
AB(1,1)=C1*C1
AB(1,2)=C2*C2
AB(1,3)=C3*C3
AB(1,4)=C1*C2
AB(3,1)=B1*B1
AB(3,2)=B2*B2
AB(3,3)=B3*B3
AB(3,4)=B1*B2
AB(2,1)=-2.*B1*C1
AB(2,2)=-2.*B2*C2
AB(2,3)=-2.*B3*C3
AB(2,4)=- (B1*C2+B2*C1)
DO 79 I=1,2

II=I-1
XX=BC(I,1)
YY=BQ(I,2)
ZZ=BC(I,3)
Q1(4,2+II)=-4.*YY
Q1(5,2+II)=15.*YY*D3-3.*XX
Q1(8,2+II)=7.*YY
Q1(9,2+II)=2.*XX-30.*YY*D3
Q1(10,2+II)=9.*XX+15.*ZZ*D1-30.*YY*D3
Q1(13,2+II)=-3.*YY
Q1(14,2+II)=15.*YY*D3
Q1(15,2+II)=30.*YY*D3-15.*ZZ*D1-9.*XX
Q1(16,2+II)=15.*(YY*D3-ZZ*D1)-6.*XX
Q1(6,8+II)=3.*YY-15.*XX*D2
Q1(7,8+II)=4.*XX
Q1(10,8+II)=15.*ZZ*D1+30.*XX*D2+6.*YY+15.*XX
Q1(11,8+II)=30.*XX*D2-2.*YY
Q1(12,8+II)=-7.*XX
Q1(15,8+II)=-15.*(ZZ*D1+XX*D2+XX)-9.*YY
Q1(16,8+II)=-15.*(ZZ*D1+XX)-30.*XX*D2-6.*YY
Q1(17,8+II)=-15.*XX*D2
Q1(18,8+II)=3.*XX
Q1(4,14+II)=-6.*YY
Q1(5,14+II)=15.*YY*D3+3.*XX
Q1(6,14+II)=-3.*YY-15.*XX*D2
Q1(7,14+II)=6.*XX
Q1(8,14+II)=8.*YY
Q1(9,14+II)=-30.*YY*D3-2.*XX
Q1(10,14+II)=6.*(XX-YY)+30.*(XX*D2-YY*D3)
Q1(11,14+II)=2.*YY+30.*XX*D2
Q1(12,14+II)=-8.*XX
Q1(13,14+II)=-3.*YY

Q1(14,14+II)=15.*YY*D3

Q1(15,14+II)=9.*YY-6.*XX+30.*YY*D3-15.*XX*D2

Q1(16,14+II)=6.*YY-9.*XX+15.*YY*D3-30.*XX*D2

Q1(17,14+II)=-15.*XX*D2

Q1(18,14+II)=3.*XX

79 CONTINUE

DO 56 I=1,3

II=I-1

XX=AB(I,1)

YY=AB(I,2)

ZZ=AB(I,3)

XY=AB(I,4)

Q1(4,4+II)=.5*YY

Q1(5,4+II)=XY-2.5*YY*D3

Q1(8,4+II)=-YY

Q1(9,4+II)=5.*YY*D3-XY

Q1(10,4+II)=2.5*ZZ*D1+5.*YY*D3-3.*XY-XX

Q1(13,4+II)=.5*YY

Q1(14,4+II)=-2.5*YY*D3

Q1(15,4+II)=3.*XY+1.5*XX-2.5*ZZ*D1-5.*YY*D3

Q1(16,4+II)=2.*XY+XX-2.5*D1*ZZ-2.5*D3*YY

Q1(6,10+II)=XY-2.5*XX*D2

Q1(7,10+II)=.5*XX

Q1(10,10+II)=2.5*(XX-ZZ*D1)+1.5*YY+2.*XY+5.*XX*D2

Q1(11,10+II)=5.*XX*D2-XY

Q1(12,10+II)=-XX

Q1(15,10+II)=2.5*ZZ*D1-2.5*XX*(D2+1.)-1.5*YY-3.*XY

Q1(16,10+II)=2.5*(ZZ*D1-XX)-5.*XX*D2-YY-2.*XY

Q1(17,10+II)=-2.5*XX*D2

Q1(18,10+II)=.5*XX

Q1(1,16+II)=.5*YY

Q1(2,16+II)=-XY

```

Q1(3,16+II)=.5*XX
Q1(4,16+II)=-1.5*YY
Q1(5,16+II)=2.5*YY*D3+2.*XY
Q1(6,16+II)=2.*XY+2.5*XX*D2
Q1(7,16+II)=-1.5*XX
Q1(8,16+II)=1.5*YY
Q1(9,16+II)=-XY-5.*YY*D3
Q1(10,16+II)=-YY*(1.5+5.*D3)-2.*XY-XX*(1.5+5.*D2)
Q1(11,16+II)=-XY-5.*XX*D2
Q1(12,16+II)=1.5*XX
Q1(13,16+II)=-.5*YY
Q1(14,16+II)=2.5*YY*D3
Q1(15,16+II)=YY*(1.5+5.*D3)+XX*(1.0+2.5*D2)
Q1(16,16+II)=YY*(1.+2.5*D3)+XX*(1.5+5.*D2)
Q1(17,16+II)=2.5*XX*D2
Q1(18,16+II)=-.5*XX

```

```

56  CONTINUE
      RETURN
      END

```

```

SUBROUTINE FORCE(X1,X2,X3,Y1,Y2,Y3)

```

```

C***

```

```

C*** Element consistent load (uniform pressure) vector subroutine for
C*** the high precision plate bending element

```

```

C***

```

```

C*** INPUT

```

```

C***

```

```

C*** X1,X2,X3,Y1,Y2,Y3 are the x,y coordinates of the triangle vertices.

```

```

C*** PRESS is the uniform pressure acting over the element

```

C***

C*** OUTPUT

C***

C*** ELOAD is the (18 x 1) consistent load vector.

C***

COMMON/THRE/Q1(18,18)

COMMON/FIVE/ELOAD(18),PRESS

COMMON/BLK2/ESTIF(18,18),D11,D12,D13,D22,D23,D33

DIMENSION QQ(21,18),IP(21)

DATA IP/1260,2*420,210,105,210,126,42,42,126,84,21,14,21,84,60,

1 42,6,6,12,60/

B1=Y2-Y3

B2=Y3-Y1

B3=Y1-Y2

C1=X3-X2

C2=X1-X3

C3=X2-X1

DET=ABS(B1*C2-B2*C1)

CALL QMAT(B1,B2,B3,C1,C2,C3)

DO 100 I=1,3

DO 100 J=1,18

100 QQ(I,J)=0.

DO 708 I=1,18

DO 708 J=1,18

708 QQ(I+3,J)=Q1(I,J)

QQ(1,13)=1.

QQ(2,14)=C2

QQ(3,14)=C1

QQ(2,15)=-B2

QQ(3,15)=B1

DO 605 J=1,18

ELOAD(J)=0.

DO 606 K=1,21

606 ELOAD(J)=ELOAD(J)+I²(K)*QQ(K,J)

605 ELOAD(J)=ELOAD(J)*PRESS*DET/2520.

607 CONTINUE

RETURN

END

APPENDIX-D

SUBROUTINE ABDGOM(NLAY)

C***

C*** Subroutine for Generating the Membrane, Bending and
C*** Membrane - Bending Coupling Stiffness Matrices.

C***

C*** INPUT

C***

C*** NLAY is number of layers

C*** EL(I),ET(I),PNULT(I),GLT(I),THETA(I) (I=1,NLAY) are the material
C*** properties and fibre orientation angle of each layer.

C*** HT(I) (I=1,NLAY+1) are the heights of the (NLAY+1) surfaces
C*** measured from the reference mid-surface.

C***

C*** OUTPUT

C***

C*** AAM,ADM,ABM are the membrane,bending and membrane-bending coupling
C*** stiffness matrices of the laminate.

C***

COMMON/LAY/HT(11),EL(10),ET(10),PNULT(10),GLT(10),THETA(10).

1 QM(10,3,3)

COMMON/ABD/AAM(3,3),ABM(3,3),ADM(3,3)

DO 730 I=1,3

DO 730 J=1,3

AAM(I,J)=0.

ABM(I,J)=0.

730 ADM(I,J)=0.

DO 740 K=1,NLAY

K1=K+1

```

PNUTL=ET(K)*PNULT(K)/EL(K)
TT=THETA(K)*0.0174533
CM=COS(TT)
CN=SIN(TT)
IF (ABS(CM).LT.1.E-04) CM=0.
IF (ABS(CN).LT.1.E-04) CN=0.
CM2=CM*CM
CN2=CN*CN
CM4=CM2*CM2
CN4=CN2*CN2
CM3CN=CM2*CM*CN
CN3CM=CN2*CN*CM
DE=1.-PNULT(K)*PNUTL
CP11=EL(K)/DE
CP22=ET(K)/DE
CP12=PNUTL*CP11
CP66=GLT(K)
QM(K,1,1)=CP11*CM4+2.*(CP12+2.*CP66)*CM2*CN2+CP22*CN4
QM(K,1,2)=(CP11+CP22-4.*CP66)*CM2*CN2+CP12*(CM4+CN4)
QM(K,1,3)=(CP11-CP12-2.*CP66)*CM3CN+(CP12-CP22+2.*CP66)*CN3CM
QM(K,2,2)=CP11*CN4+2.*(CP12+2.*CP66)*CM2*CN2+CP22*CM4
QM(K,2,3)=(CP11-CP12-2.*CP66)*CN3CM+(CP12-CP22+2.*CP66)*CM3CN
QM(K,3,3)=(CP11+CP22-2.*(CP12+CP66))*CM2*CN2+CP66*(CM4+CN4)
DO 745 LK=1,3
DO 745 LJ=LK,3
745 QM(K,LJ,LK)=QM(K,LK,LJ)
DO 750 IA=1,3
DO 750 IB=IA,3
AAM(IA,IB)=AAM(IA,IB)+QM(K,IA,IB)*(HT(K1)-HT(K))
ABM(IA,IB)=ABM(IA,IB)+0.5*QM(K,IA,IB)*(HT(K1)+HT(K1)-HT(K)+HT(K))
750 ADM(IA,IB)=ADM(IA,IB)+QM(K,IA,IB)*(HT(K1)**3-HT(K)**3)/3.
740 CONTINUE

```

```
DO 760 IX=1,3  
DO 760 JX=IX,3  
AAM(JX,IX)=AAM(IX,JX)  
ABM(JX,IX)=ABM(IX,JX)  
ADM(JX,IX)=ADM(IX,JX)  
760 CONTINUE  
RETURN  
END
```

APPENDIX-E

SUBROUTINE ELSTIF(X1,X2,X3,Y1,Y2,Y3)

C*** Element Stiffness Matrix Subroutine for Laminated Anisotropic High

C*** Precision Plate Element

C***

C*** INPUT

C***

C*** X1,X2,X3,Y1,Y2,Y3 are the x and y coordinates of the vertices of
C*** the triangle.

C*** AAM,ADM,ABM are the membrane,bending and membrane-bending coupling
C*** stiffness matrices of the laminate.

C***

C*** OUTPUT

C***

C*** ESM is the (38 X 38) element stiffness matrix

C***

COMMON/ST/ESM(38,38)

COMMON/CODE/KOD(38)

COMMON/ABD/AAM(3,3),ABM(3,3),ADM(3,3)

DIMENSION P(3,4),Q(3,3),AM(4,4),BM(4,3),DM(3,3),R(9,10).

1 S(18,18)

DATA KOD/1,2,3,13,14,15,25,26,27,37,4,5,6,16,17,18,28,29,30,38.

1 7,8,9,10,11,12,19,20,21,22,23,24,31,32,33,34,35,36/

B1=Y2-Y3

B2=Y3-Y1

B3=Y1-Y2

C1=X3-X2

C2=X1-X3

C3=X2-X1

```
DET=ABS(B1*C2-B2*C1)
P(1,1)=B1
P(1,2)=B2
P(1,3)=0.
P(1,4)=0.
P(2,1)=0.
P(2,2)=0.
P(2,3)=C1
P(2,4)=C2
P(3,1)=C1
P(3,2)=C2
P(3,3)=B1
P(3,4)=B2
Q(1,1)=B1*B1
Q(1,2)=B2*B2
Q(1,3)=2.*B1*B2
Q(2,1)=C1*C1
Q(2,2)=C2*C2
Q(2,3)=2.*C1*C2
Q(3,1)=2.*B1*C1
Q(3,2)=2.*B2*C2
Q(3,3)=2.*(B1*C2+B2*C1)
CALL TRMULT(P, AAM, P, AM, 4, 3, 3, 4, 1)
CALL TRMULT(Q, ADM, Q, DM, 3, 3, 3, 3, 1)
CALL TRMULT(P, ABM, Q, BM, 4, 3, 3, 3, 0)
DO 25 I=1,4
DO 25 J=1,4
25 AM(I,J)=AM(I,J)/DET
DO 30 I=1,3
DO 30 J=1,3
30 DM(I,J)=DM(I,J)/DET**3
DO 35 I=1,4
```

DO 35 J=1,3

35 BM(I,J)=BM(I,J)/DET**2

CALL RMAT(B1,B2,B3,C1,C2,C3,R)

CALL SMAT(B1,B2,B3,C1,C2,C3,S)

CALL UV(R,AM)

CALL UVW(BM,R,S)

CALL WW(S,DM(1,1),DM(1,2),DM(1,3),DM(2,2),DM(2,3),DM(3,3))

RETURN

END

SUBROUTINE RMAT(B1,B2,B3,C1,C2,C3,R)

DIMENSION R(9,10)

DO 25 I=1,9

DO 25 J=1,10

25 R(I,J)=0.

R(1,8)=C2

R(1,9)=-B2

R(2,8)=-C1

R(2,9)=B1

R(3,1)=3.

R(3,2)=-C2

R(3,3)=B2

R(3,7)=-3.

R(3,8)=-2.*C2

R(3,9)=2.*B2

R(4,1)=-7.

R(4,2)=C1+2.*C2

R(4,3)=-B1-2.*B2

R(4,4)=-7.

R(4,5)=-C2+2.*C1

$$R(4,6) = B2 + 2 * B1$$

$$R(4,7) = -13$$

$$R(4,8) = 3 * (C1 - C2)$$

$$R(4,9) = 3 * (B2 - B1)$$

$$R(4,10) = 27$$

$$R(5,4) = 3$$

$$R(5,5) = C1$$

$$R(5,6) = -B1$$

$$R(5,7) = -3$$

$$R(5,8) = 2 * C1$$

$$R(5,9) = -2 * B1$$

$$R(6,1) = -2$$

$$R(6,2) = C2$$

$$R(6,3) = -B2$$

$$R(6,7) = 2$$

$$R(6,8) = C2$$

$$R(6,9) = -B2$$

$$R(7,1) = 7$$

$$R(7,2) = -2 * (C1 + C2)$$

$$R(7,3) = 2 * (B1 + B2)$$

$$R(7,4) = 7$$

$$R(7,5) = C2 + 2 * C1$$

$$R(7,6) = -B2 - 2 * B1$$

$$R(7,7) = 13$$

$$R(7,8) = 3 * C2 - 2 * C1$$

$$R(7,9) = -3 * B2 + 2 * B1$$

$$R(7,10) = -27$$

$$R(8,1) = 7$$

$$R(8,2) = -C1 - 2 * C2$$

$$R(8,3) = B1 + 2 * B2$$

$$R(8,4) = 7$$

$$R(8,5) = 2 * (C1 + C2)$$

```

R(8,6)=-2.*(B1+B2)
R(8,7)=13.
R(8,8)=2.*C2-3.*C1
R(8,9)=-2.*B2+3.*B1
R(8,10)=-27.
R(9,4)=-2.
R(9,5)=-51
R(9,6)=B1
R(9,7)=2.
R(9,8)=-C1
R(9,9)=B1
*RETURN
END

```

```

SUBROUTINE SMAT(B1,B2,B3,C1,C2,C3,S)
DIMENSION BC(2,3),AB(3,4),S(18,18)
D1=-(B1*B3+C1*C3)/(B3*B3+C3*C3)
D2=(B1*B2+C1*C2)/(B1*B1+C1*C1)
D3=(B1*B2+C1*C2)/(B2*B2+C2*C2)
DO 20 I=1,18
DO 20 J=1,18
20 S(I,J)=0.
S(4,1)=10.
S(5,1)=-30.*D3
S(8,1)=-15.
S(9,1)=60.*D3
S(10,1)=30.*(D1+2.*D3)
S(13,1)=6.
S(14,1)=-30.*D3
S(15,1)=-30.*(D1+2.*D3)

```


$$S(16,1) = -30. * (D3 + D1)$$

$$S(6,7) = -30. * D2$$

$$S(7,7) = 10.$$

$$S(10,7) = 30. * (1. - D1 + 2. * D2)$$

$$S(11,7) = 60. * D2$$

$$S(12,7) = -15.$$

$$S(15,7) = 30. * (D1 - D2 - 1.)$$

$$S(16,7) = 30. * (D1 - 2. * D2 - 1.)$$

$$S(17,7) = -30. * D2$$

$$S(18,7) = 6.$$

$$S(4,13) = -10.$$

$$S(5,13) = 30. * D3$$

$$S(6,13) = 30. * D2$$

$$S(7,13) = -10.$$

$$S(8,13) = 15.$$

$$S(9,13) = -60. * D3$$

$$S(10,13) = -30. * (1. + 2. * (D2 + D3))$$

$$S(11,13) = -60. * D2$$

$$S(12,13) = 15.$$

$$S(13,13) = -6.$$

$$S(14,13) = 30. * D3$$

$$S(15,13) = 30. * (1. + D2) + 60. * D3$$

$$S(16,13) = 30. * (1. + D3) + 60. * D2$$

$$S(17,13) = 30. * D2$$

$$S(18,13) = -6.$$

$$BC(1,1) = C1$$

$$BC(1,2) = C2$$

$$BC(1,3) = C3$$

$$BC(2,1) = -B1$$

$$BC(2,2) = -B2$$

$$BC(2,3) = -B3$$

$$AB(1,1) = C1 * C1$$

AB(1,2)=C2*C2
AB(1,3)=C3*C3
AB(1,4)=C1*C2
AB(3,1)=B1*B1
AB(3,2)=B2*B2
AB(3,3)=B3*B3
AB(3,4)=B1*B2
AB(2,1)=-2.*B1*C1
AB(2,2)=-2.*B2*C2
AB(2,3)=-2.*B3*C3
AB(2,4)=- (B1*C2+B2*C1)
DO 79 I=1,2
II=I-1
IX=BC(I,1)
YY=BC(I,2)
ZZ=BC(I,3)
S(4,2+II)=-4.*YY
S(5,2+II)=15.*YY*D3-3.*XX
S(8,2+II)=7.*YY
S(9,2+II)=2.*XX-30.*YY*D3
S(10,2+II)=9.*XX+15.*ZZ*D1-30.*YY*D3
S(13,2+II)=-3.*YY
S(14,2+II)=15.*YY*D3
S(15,2+II)=30.*YY*D3-15.*ZZ*D1-9.*XX
S(16,2+II)=15.*(YY*D3-ZZ*D1)-6.*XX
S(6,8+II)=3.*YY-15.*XX*D2
S(7,8+II)=4.*XX
S(10,8+II)=15.*ZZ*D1+30.*XX*D2+6.*YY+15.*XX
S(11,8+II)=30.*XX*D2-2.*YY
S(12,8+II)=-7.*XX
S(15,8+II)=-15.*(ZZ*D1+XX*D2+XX)-9.*YY
S(16,8+II)=-15.*(ZZ*D1+XX)-30.*XX*D2-6.*YY

$S(17, 8+II) = -15 * XX * D2$
 $S(18, 8+II) = 3 * XX$
 $S(4, 14+II) = -6 * YY$
 $S(5, 14+II) = 15 * YY * D3 + 3 * XX$
 $S(6, 14+II) = -3 * YY - 15 * XX * D2$
 $S(7, 14+II) = 6 * XX$
 $S(8, 14+II) = 8 * YY$
 $S(9, 14+II) = -30 * YY * D3 - 2 * XX$
 $S(10, 14+II) = 6 * (XX - YY) + 30 * (XX * D2 - YY * D3)$
 $S(11, 14+II) = 2 * YY + 30 * XX * D2$
 $S(12, 14+II) = -8 * XX$
 $S(13, 14+II) = -3 * YY$
 $S(14, 14+II) = 15 * YY * D3$
 $S(15, 14+II) = 9 * YY - 6 * XX + 30 * YY * D3 - 15 * XX * D2$
 $S(16, 14+II) = 6 * YY - 9 * XX + 15 * YY * D3 - 30 * XX * D2$
 $S(17, 14+II) = -15 * XX * D2$
 $S(18, 14+II) = 3 * XX$

79 CONTINUE

DO 56 I=1,3

II=I-1

XX=AB(I,1)

YY=AB(I,2)

ZZ=AB(I,3)

XY=AB(I,4)

$S(4, 4+II) = .5 * YY$

$S(5, 4+II) = XY - 2.5 * YY * D3$

$S(8, 4+II) = -YY$

$S(9, 4+II) = 5 * YY * D3 - XY$

$S(10, 4+II) = 2.5 * ZZ * D1 + 5 * YY * D3 - 3 * XY - XX$

$S(13, 4+II) = .5 * YY$

$S(14, 4+II) = -2.5 * YY * D3$

$S(15, 4+II) = 3 * XY + 1.5 * XX - 2.5 * ZZ * D1 - 5 * YY * D3$

$S(16,4+II) = 2.*XY + XX - 2.5*D1*ZZ - 2.5*D3*YY$
 $S(6,10+II) = XY - 2.5*XX*D2$
 $S(7,10+II) = .5*XX$
 $S(10,10+II) = 2.5*(XX - ZZ*D1) + 1.5*YY + 2.*XY + 5.*XX*D2$
 $S(11,10+II) = 5.*XX*D2 - XY$
 $S(12,10+II) = -XX$
 $S(15,10+II) = 2.5*ZZ*D1 - 2.5*XX*(D2+1.) - 1.5*YY - 3.*XY$
 $S(16,10+II) = 2.5*(ZZ*D1 - XX) - 5.*XX*D2 - YY - 2.*XY$
 $S(17,10+II) = -2.5*XX*D2$
 $S(18,10+II) = .5*XX$
 $S(1,16+II) = .5*YY$
 $S(2,16+II) = -XY$
 $S(3,16+II) = .5*XX$
 $S(4,16+II) = -1.5*YY$
 $S(5,16+II) = 2.5*YY*D3 + 2.*XY$
 $S(6,16+II) = 2.*XY + 2.5*XX*D2$
 $S(7,16+II) = -1.5*XX$
 $S(8,16+II) = 1.5*YY$
 $S(9,16+II) = -XY - 5.*YY*D3$
 $S(10,16+II) = -YY*(1.5+5.*D3) - 2.*XY - XX*(1.5+5.*D2)$
 $S(11,16+II) = -XY - 5.*XX*D2$
 $S(12,16+II) = 1.5*XX$
 $S(13,16+II) = -.5*YY$
 $S(14,16+II) = 2.5*YY*D3$
 $S(15,16+II) = YY*(1.5+5.*D3) + XX*(1.0+2.5*D2)$
 $S(16,16+II) = YY*(1.+2.5*D3) + XX*(1.5+5.*D2)$
 $S(17,16+II) = 2.5*XX*D2$
 $S(18,16+II) = -.5*XX$

56 CONTINUE

RETURN

END

```

SUBROUTINE UV(R,AM)
COMMON/ST/ESM(38,38)
COMMON/COE/KOD(38)
DIMENSION TEMP(10,10),F(9,9),R(9,10),AM(4,4)
CALL FMAT(AM(1,1),AM(1,2),AM(1,2),AM(2,2),F)
CALL TRMULT(R,F,R,TEMP,10,9,9,10,1)
DO 100 I=1,10
DO 100 J=1,10
ESM(KOD(I),KOD(J))=TEMP(I,J)
100 ESM(KOD(J),KOD(I))=ESM(KOD(I),KOD(J))
CALL FMAT(AM(1,3),AM(1,4),AM(2,3),AM(2,4),F)
CALL TRMULT(R,F,R,TEMP,10,9,9,10,0)
DO 150 I=1,10
DO 150 J=1,10
ESM(KOD(I),KOD(J+10))=TEMP(I,J)
150 ESM(KOD(J+10),KOD(I))=ESM(KOD(I),KOD(J+10))
CALL FMAT(AM(3,3),AM(3,4),AM(3,4),AM(4,4),F)
CALL TRMULT(R,F,R,TEMP,10,9,9,10,1)
DO 200 I=1,10
DO 200 J=1,10
ESM(KOD(I+10),KOD(J+10))=TEMP(I,J)
200 ESM(KOD(J+10),KOD(I+10))=ESM(KOD(I+10),KOD(J+10))
RETURN
END

```

```

SUBROUTINE UVW(BM,R;S)
COMMON/ST/ESM(38,38)
COMMON/COE/KOD(38)

```

```

DIMENSION G(9,18),TEMP(10,18),BM(4,3),R(9,10),S(18,18)
CALL GMAT(BM(1,1),BM(1,2),BM(1,3),BM(2,1),BM(2,2),BM(2,3),G)
CALL TRMULT(R,G,S,TEMP,10,9,18,18,0)
DO 10 I=1,10
DO 10 J=1,18
    ESM(KOD(I),KOD(J+20))=-TEMP(I,J)
10 ESM(KOD(J+20),KOD(I))=ESM(KOD(I),KOD(J+20))
CALL GMAT(BM(3,1),BM(3,2),BM(3,3),BM(4,1),BM(4,2),BM(4,3),G)
CALL TRMULT(R,G,S,TEMP,10,9,18,18,0)
DO 20 I=1,10
DO 20 J=1,18
    ESM(KOD(I+10),KOD(J+20))=-TEMP(I,J)
20 ESM(KOD(J+20),KOD(I+10))=ESM(KOD(I+10),KOD(J+20))
RETURN
END

```

```

SUBROUTINE WW(S,E11,E12,E13,E22,E23,E33)
COMMON/ST/ESM(38,38)
COMMON/CODE/KOD(38)
DIMENSION TEMP(18,18),H(18,18),S(18,18)
H(1,1)=2.*E11
H(1,2)=E13
H(1,3)=2.*E12
H(1,4)=2.*E11
H(1,5)=2.*(E11+E13)/3.
H(1,6)=2.*(E12+E13)/3.
H(1,7)=2.*E12
H(1,8)=2.*E11
H(1,9)=.5*(E11+E13)
H(1,10)=(E11+E12+E13)/3.

```

$$H(1,11) = .5 * (E12 + E13)$$

$$H(1,12) = 2. * E12$$

$$H(1,13) = 2. * E11$$

$$H(1,14) = .4 * (E11 + E13)$$

$$H(1,15) = 0.2 * (E11 + E12 + E13)$$

$$H(1,16) = 0.2 * (E11 + E12 + E13)$$

$$H(1,17) = 0.4 * (E12 + E13)$$

$$H(1,18) = 2. * E12$$

$$H(2,2) = .5 * E33$$

$$H(2,3) = E23$$

$$H(2,4) = E13$$

$$H(2,5) = (E13 + E33) / 3.$$

$$H(2,6) = (E23 + E33) / 3.$$

$$H(2,7) = E23$$

$$H(2,8) = E13$$

$$H(2,9) = .25 * (E13 + E33)$$

$$H(2,10) = (E13 + E23 + E33) / 6.$$

$$H(2,11) = .25 * (E23 + E33)$$

$$H(2,12) = E23$$

$$H(2,13) = E13$$

$$H(2,14) = 0.2 * (E13 + E33)$$

$$H(2,15) = 0.1 * (E13 + E23 + E33)$$

$$H(2,16) = 0.1 * (E13 + E23 + E33)$$

$$H(2,17) = 0.2 * (E23 + E33)$$

$$H(2,18) = E23$$

$$H(3,3) = 2. * E22$$

$$H(3,4) = 2. * E12$$

$$H(3,5) = 2. * (E12 + E23) / 3.$$

$$H(3,6) = 2. * (E22 + E23) / 3.$$

$$H(3,7) = 2. * E22$$

$$H(3,8) = 2. * E12$$

$$H(3,9) = .5 * (E12 + E23)$$

$$H(3,10) = (E12 + E22 + E23) / 3$$

$$H(3,11) = .5 * (E22 + E23)$$

$$H(3,12) = 2 * E22$$

$$H(3,13) = 2 * E12$$

$$H(3,14) = 0.4 * (E12 + E23)$$

$$H(3,15) = 0.2 * (E12 + E22 + E23)$$

$$H(3,16) = 0.2 * (E12 + E22 + E23)$$

$$H(3,17) = 0.4 * (E22 + E23)$$

$$H(3,18) = 2 * E22$$

$$H(4,4) = 3 * E11$$

$$H(4,5) = .5 * E11 + E13$$

$$H(4,6) = E12 + .5 * E13$$

$$H(4,7) = 1.5 * E12$$

$$H(4,8) = 3.6 * E11$$

$$H(4,9) = 0.6 * E11 + 0.9 * E13$$

$$H(4,10) = 0.2 * E11 + 0.6 * E12 + 0.4 * E13$$

$$H(4,11) = 0.6 * E12 + 0.3 * E13$$

$$H(4,12) = 1.2 * E12$$

$$H(4,13) = 4 * E11$$

$$H(4,14) = 0.6 * E11 + 0.8 * E13$$

$$H(4,15) = 0.2 * E11 + 0.4 * E12 + 0.3 * E13$$

$$H(4,16) = 0.1 * E11 + 0.3 * E12 + 0.2 * E13$$

$$H(4,17) = 0.4 * E12 + 0.2 * E13$$

$$H(4,18) = E12$$

$$H(5,5) = (E11 + E12 + E33) / 3$$

$$H(5,6) = (E12 + E33) / 6 + (E13 + E23) / 3$$

$$H(5,7) = E12 + .5 * E23$$

$$H(5,8) = 0.4 * E11 + 1.2 * E13$$

$$H(5,9) = 0.2 * E11 + 0.3 * (E13 + E33)$$

$$H(5,10) = (.2 * E12 + .4 * E33) / 3 + 0.2 * (E11 + E13 + E23)$$

$$H(5,11) = 0.2 * (E12 + E23) + 0.1 * E33 + 0.3 * E13$$

$$H(5,12) = 1.2 * E12 + 0.4 * E23$$

$$H(5,13) = (E11 + 4 * E13) / 3.$$

$$H(5,14) = (.4 * E11 + .8 * (E13 + E33)) / 3.$$

$$H(5,15) = (.3 * (E11 + E33) + .1 * E12 + 0.4 * (E13 + E23)) / 3.$$

$$H(5,16) = (0.2 * (E12 + E33) + 0.4 * (E11 + E13) + 0.3 * E23) / 3.$$

$$H(5,17) = (.6 * E12 + .2 * E33 + 0.8 * E13 + 0.4 * E23) / 3.$$

$$H(5,18) = (4 * E12 + E23) / 3.$$

$$H(6,6) = (E22 + E23 + E33) / 3.$$

$$H(6,7) = .5 * E22 + E23$$

$$H(6,8) = 1.2 * E12 + 0.4 * E13$$

$$H(6,9) = 0.2 * (E12 + E13) + 0.1 * E33 + 0.3 * E23$$

$$H(6,10) = (.2 * E12 + .4 * E33) / 3 + 0.2 * (E13 + E22 + E23)$$

$$H(6,11) = 0.2 * E22 + 0.3 * (E23 + E33)$$

$$H(6,12) = 0.4 * E22 + 1.2 * E23$$

$$H(6,13) = (4 * E12 + E13) / 3.$$

$$H(6,14) = 0.2 * E12 + (.2 * E33 + 0.4 * E13 + 0.8 * E23) / 3.$$

$$H(6,15) = (.2 * (E12 + E33) + 0.4 * (E22 + E23)) / 3 + .1 * E13$$

$$H(6,16) = (.1 * E12 + 0.4 * (E13 + E23)) / 3 + 0.1 * (E22 + E33)$$

$$H(6,17) = (.4 * E22 + 0.8 * (E23 + E33)) / 3.$$

$$H(6,18) = (E22 + 4 * E23) / 3.$$

$$H(7,7) = 3 * E22$$

$$H(7,8) = 1.2 * E12$$

$$H(7,9) = 0.6 * E12 + 0.3 * E23$$

$$H(7,10) = 0.6 * E12 + 0.2 * E22 + 0.4 * E23$$

$$H(7,11) = 0.6 * E22 + 0.9 * E23$$

$$H(7,12) = 3.6 * E22$$

$$H(7,13) = E12$$

$$H(7,14) = 0.4 * E12 + 0.2 * E23$$

$$H(7,15) = 0.3 * E12 + 0.1 * E22 + 0.2 * E23$$

$$H(7,16) = 0.4 * E12 + 0.2 * E22 + 0.3 * E23$$

$$H(7,17) = 0.6 * E22 + 0.8 * E23$$

$$H(7,18) = 4 * E22$$

$$H(8,8) = 4.8 * E11$$

$$H(8,9)=0.6*E11+1.2*E13$$

$$H(8,10)=0.4*E11/3.+0.8*E12+0.4*E13$$

$$H(8,11)=0.6*E12+0.2*E13$$

$$H(8,12)=0.8*E12$$

$$H(8,13)=40.*E11/7.$$

$$H(8,14)=24.*E11/35.+8.*E13/7.$$

$$H(8,15)=(6.*E11+12.*E13)/35.+4.*E12/7.$$

$$H(8,16)=(2.*E11+12.*E12+6.*E13)/35.$$

$$H(8,17)=(12.*E12+4.*E13)/35.$$

$$H(8,18)=20.*E12/35.$$

$$H(9,9)=0.2*E11+0.3*(E13+E33)$$

$$H(9,10)=0.1*(E11+E12+E33)+0.2*E23+E13/6.$$

$$H(9,11)=0.2*E12+.05*E33+0.15*(E13+E23)$$

$$H(9,12)=0.6*E12+0.2*E23$$

$$H(9,13)=20.*E11/35.+10.*E13/7.$$

$$H(9,14)=(6.*E11+10.*E13)/35.+2.*E33/7.$$

$$H(9,15)=(3.*E11+4.5*E13+2.*E12+3.*E33)/35.+E23/7.$$

$$H(9,16)=(2.*E11+3.*E12+1.5*E33+3.5*E13+3.*E23)/35.$$

$$H(9,17)=(6.*E12+E33+4.*E13+3.*E23)/35.$$

$$H(9,18)=(20.*E12+5.*E23)/35.$$

$$H(10,10)=(2.*E12+4.*E33)/45.+0.4*(E11+E13+E22+E23)/3.$$

$$H(10,11)=0.1*(E12+E33+E22)+0.2*E13+E23/6.$$

$$H(10,12)=0.8*E12+0.4*E22/3.+0.4*E23$$

$$H(10,13)=(10.*E11+40.*E13)/105.+20.*E12/21.$$

$$H(10,14)=(6.*E11+12.*E12+8.*E33+14.*E13)/105.+4.*E23/21.$$

$$H(10,15)=(6.*E11+4.*E12+6.*E33+9.*E13+10.*E23)/105.+2.*E22/21.$$

$$H(10,16)=2.*E11/21.+(4.*E12+6.*E33+10.*E13+6.*E22+9.*E23)/105.$$

$$H(10,17)=(12.*E12+8.*E33+6.*E22+14.*E23)/105.+4.*E13/21.$$

$$H(10,18)=20.*E12/21.+(10.*E22+40.*E23)/105.$$

$$H(11,11)=0.2*E22+0.3*(E23+E33)$$

$$H(11,12)=0.6*E22+1.2*E23$$

$$H(11,13)=(60.*E12+15.*E13)/105.$$

$$H(11,14) = (18 * E12 + 3 * E33 + 9 * E13 + 12 * E23) / 105.$$

$$H(11,15) = (9 * E12 + 4.5 * E33 + 9 * E13 + 6 * E22 + 10.5 * E23) / 105.$$

$$H(11,16) = (6 * E12 + 9 * (E33 + E22) + 13.5 * E23) / 105 + E13 / 7.$$

$$H(11,17) = (6 * E22 + 10 * E23) / 35 + 2 * E33 / 7.$$

$$H(11,18) = 20 * E22 / 35 + 10 * E23 / 7.$$

$$H(12,12) = 4.8 * E22$$

$$H(12,13) = 20 * E12 / 35.$$

$$H(12,14) = (12 * E12 + 4 * E23) / 35.$$

$$H(12,15) = (12 * E12 + 6 * E23 + 2 * E22) / 35.$$

$$H(12,16) = 4 * E12 / 7 + (6 * E22 + 12 * E23) / 35.$$

$$H(12,17) = 24 * E22 / 35 + 8 * E23 / 7.$$

$$H(12,18) = 40 * E22 / 7.$$

$$H(13,13) = 60 * E11 / 7.$$

$$H(13,14) = (5 * E11 + 10 * E13) / 7.$$

$$H(13,15) = 5 * E11 / 35 + (30 * E12 + 15 * E13) / 42.$$

$$H(13,16) = (E11 + 10 * E12) / 28 + 5 * E13 / 35.$$

$$H(13,17) = (40 * E12 + 10 * E13) / 140.$$

$$H(13,18) = 10 * E12 / 28.$$

$$H(14,14) = 2 * (E13 + E33) / 7 + 6 * E11 / 35.$$

$$H(14,15) = (.5 * (E12 + E33) + E23) / 7 + (16 * E13 + 9 * E11) / 140.$$

$$H(14,16) = (4 * (E11 + E33) + 12 * E12 + 10 * E13) / 140 + .5 * E23 / 7.$$

$$H(14,17) = (4.5 * E12 + .5 * E33 + 2 * (E13 + E23)) / 35.$$

$$H(14,18) = (40 * E12 + 10 * E23) / 140.$$

$$H(15,15) = (6 * E11 + 4 * E12 + 6 * E33 + 9 * E13) / 140 + (E22 + E23) / 14.$$

$$H(15,16) = (E11 + E22) / 28 + (10 * E12 + 9 * E33) / 280 + 2 * (E13 + E23) / 35.$$

$$H(15,17) = (12 * E12 + 4 * (E33 + E22) + 10 * E23) / 140 + E13 / 14.$$

$$H(15,18) = (10 * E12 + E22 + 4 * E23) / 28.$$

$$H(16,16) = (E11 + E13) / 14 + (4 * E12 + 6 * (E22 + E33) + 9 * E23) / 140.$$

$$H(16,17) = (.5 * (E12 + E33) + E13) / 7 + (9 * E22 + 16 * E23) / 140.$$

$$H(16,18) = (20 * E12 + 10 * E23) / 28 + 5 * E22 / 35.$$

$$H(17,17) = 6 * E22 / 35 + 2 * (E23 + E33) / 7.$$

$$H(17,18) = (6 * E22 + 10 * E23) / 7.$$

```
H(18,18)=50.*E22/7.  
DO 550 I=1,18  
DO 550 J=I,18  
550 H(J,I)=H(I,J)  
CALL TRMULT(S,H,S,TEMP,18,18,18,18,1)  
DO 600 I=1,18  
DO 600 J=1,18  
ESM(KOD(I+20),KOD(J+20))=TEMP(I,J)  
600 ESM(KOD(J+20),KOD(I+20))=ESM(KOD(I+20),KOD(J+20))  
RETURN  
END
```

```
SUBROUTINE FMAT(X13,X14,X23,X24,C)
```

```
DIMENSION C(9,9)
```

```
C(1,1)=180.*X13
```

```
C(1,2)=180.*X14
```

```
C(1,3)=120.*X13
```

```
C(1,4)=60.*(X13+X14)
```

```
C(1,5)=120.*X14
```

```
C(1,6)=90.*X13
```

```
C(1,7)=30.*(X13+X14)
```

```
C(1,8)=30.*(X13+X14)
```

```
C(1,9)=90.*X14
```

```
C(2,1)=180.*X23
```

```
C(2,2)=180.*X24
```

```
C(2,3)=120.*X23
```

```
C(2,4)=60.*(X23+X24)
```

```
C(2,5)=120.*X24
```

```
C(2,6)=90.*X23
```

```
C(2,7)=30.*(X23+X24)
```

$$C(2,8)=30. \cdot (X23+X24)$$

$$C(2,9)=90. \cdot X24$$

$$C(3,1)=120. \cdot X13$$

$$C(3,2)=120. \cdot X14$$

$$C(3,3)=120. \cdot X13$$

$$C(3,4)=30. \cdot X13+60. \cdot X14$$

$$C(3,5)=60. \cdot X14$$

$$C(3,6)=108. \cdot X13$$

$$C(3,7)=24. \cdot X13+36. \cdot X14$$

$$C(3,8)=12. \cdot X13+24. \cdot X14$$

$$C(3,9)=36. \cdot X14$$

$$C(4,1)=60. \cdot (X13+X23)$$

$$C(4,2)=60. \cdot (X14+X24)$$

$$C(4,3)=30. \cdot X13+60. \cdot X23$$

$$C(4,4)=30. \cdot (X13+X24)+15. \cdot (X14+X23)$$

$$C(4,5)=60. \cdot X14+30. \cdot X24$$

$$C(4,6)=18. \cdot X13+54. \cdot X23$$

$$C(4,7)=12. \cdot (X13+X23)+6. \cdot X14+18. \cdot X24$$

$$C(4,8)=18. \cdot X13+12. \cdot (X14+X24)+6. \cdot X23$$

$$C(4,9)=54. \cdot X14+18. \cdot X24$$

$$C(5,1)=120. \cdot X23$$

$$C(5,2)=120. \cdot X24$$

$$C(5,3)=60. \cdot X23$$

$$C(5,4)=60. \cdot X23+30. \cdot X24$$

$$C(5,5)=120. \cdot X24$$

$$C(5,6)=36. \cdot X23$$

$$C(5,7)=24. \cdot X23+12. \cdot X24$$

$$C(5,8)=36. \cdot X23+24. \cdot X24$$

$$C(5,9)=108. \cdot X24$$

$$C(6,1)=90. \cdot X13$$

$$C(6,2)=90. \cdot X14$$

$$C(6,3)=108. \cdot X13$$

$$C(6,4)=18.*X13+54.*X14$$

$$C(6,5)=36.*X14$$

$$C(6,6)=108.*X13$$

$$C(6,7)=18.*X13+36.*X14$$

$$C(6,8)=6.*X13+18.*X14$$

$$C(6,9)=18.*X14$$

$$C(7,1)=30.*(X13+X23)$$

$$C(7,2)=30.*(X14+X24)$$

$$C(7,3)=24.*X13+36.*X23$$

$$C(7,4)=12.*(X13+X14)+6.*X23+18.*X24$$

$$C(7,5)=24.*X14+12.*X24$$

$$C(7,6)=18.*X13+36.*X23$$

$$C(7,7)=8.*X13+6.*(X14+X23)+12.*X24$$

$$C(7,8)=6.*(X13+X24)+8.*X14+2.*X23$$

$$C(7,9)=18.*X14+6.*X24$$

$$C(8,1)=30.*(X13+X23)$$

$$C(8,2)=30.*(X14+X24)$$

$$C(8,3)=12.*X13+24.*X23$$

$$C(8,4)=18.*X13+6.*X14+12.*(X23+X24)$$

$$C(8,5)=36.*X14+24.*X24$$

$$C(8,6)=6.*X13+18.*X23$$

$$C(8,7)=6.*(X13+X24)+2.*X14+8.*X23$$

$$C(8,8)=12.*X13+6.*(X14+X23)+8.*X24$$

$$C(8,9)=36.*X14+18.*X24$$

$$C(9,1)=90.*X23$$

$$C(9,2)=90.*X24$$

$$C(9,3)=36.*X23$$

$$C(9,4)=54.*X23+18.*X24$$

$$C(9,5)=108.*X24$$

$$C(9,6)=18.*X23$$

$$C(9,7)=18.*X23+6.*X24$$

$$C(9,8)=36.*X23+18.*X24$$

```
C(9,9)=108.*X24  
DO 50 I=1,9  
DO 50 J=1,9  
50 C(I,J)=C(I,J)/360.  
RETURN  
END
```

```
SUBROUTINE GMAT(Y11,Y12,Y13,Y21,Y22,Y23,G)
```

```
DIMENSION G(9,18)
```

```
G(1,1)=1260.*Y11
```

```
G(1,2)=630.*Y13
```

```
G(1,3)=1260.*Y12
```

```
G(1,4)=G(1,1)
```

```
G(1,5)=420.*(Y11+Y13)
```

```
G(1,6)=420.*(Y12+Y13)
```

```
G(1,7)=1260.*Y12
```

```
G(1,8)=1260.*Y11
```

```
G(1,9)=315.*(Y11+Y13)
```

```
G(1,10)=210.*(Y11+Y12+Y13)
```

```
G(1,11)=315.*(Y12+Y13)
```

```
G(1,12)=1260.*Y12
```

```
G(1,13)=1260.*Y11
```

```
G(1,14)=252.*(Y11+Y13)
```

```
G(1,15)=126.*(Y11+Y12+Y13)
```

```
G(1,16)=G(1,15)
```

```
G(1,17)=252.*(Y12+Y13)
```

```
G(1,18)=1260.*Y12
```

```
G(2,1)=1260.*Y21
```

```
G(2,2)=630.*Y23
```

```
G(2,3)=1260.*Y22
```

$$G(2,4)=1260.*Y21$$

$$G(2,5)=420.*(Y21+Y23)$$

$$G(2,6)=420.*(Y22+Y23)$$

$$G(2,7)=1260.*Y22$$

$$G(2,8)=1260.*Y21$$

$$G(2,9)=315.*(Y21+Y23)$$

$$G(2,10)=210.*(Y21+Y22+Y23)$$

$$G(2,11)=315.*(Y22+Y23)$$

$$G(2,12)=1260.*Y22$$

$$G(2,13)=1260.*Y21$$

$$G(2,14)=252.*(Y21+Y23)$$

$$G(2,15)=126.*(Y21+Y22+Y23)$$

$$G(2,16)=126.*(Y21+Y22+Y23)$$

$$G(2,17)=252.*(Y22+Y23)$$

$$G(2,18)=1260.*Y22$$

$$G(3,1)=840.*Y11$$

$$G(3,2)=420.*Y13$$

$$G(3,3)=840.*Y12$$

$$G(3,4)=1260.*Y11$$

$$G(3,5)=210.*Y11+420.*Y13$$

$$G(3,6)=420.*Y12+210.*Y13$$

$$G(3,7)=630.*Y12$$

$$G(3,8)=1512.*Y11$$

$$G(3,9)=252.*Y11+378.*Y13$$

$$G(3,10)=84.*Y11+252.*Y12+168.*Y13$$

$$G(3,11)=252.*Y12+126.*Y13$$

$$G(3,12)=504.*Y12$$

$$G(3,13)=1680.*Y11$$

$$G(3,14)=252.*Y11+336.*Y13$$

$$G(3,15)=84.*Y11+168.*Y12+126.*Y13$$

$$G(3,16)=42.*Y11+126.*Y12+84.*Y13$$

$$G(3,17)=168.*Y12+84.*Y13$$

$$G(3,18)=420.*Y12$$

$$G(4,1)=420.*(Y11+Y21)$$

$$G(4,2)=210.*(Y13+Y23)$$

$$G(4,3)=420.*(Y12+Y22)$$

$$G(4,4)=315.*Y11+630.*Y21$$

$$G(4,5)=210.*(Y11+Y23)+105.*(Y13+Y21)$$

$$G(4,6)=105.*(Y12+Y23)+210.*(Y13+Y22)$$

$$G(4,7)=630.*Y12+315.*Y22$$

$$G(4,8)=252.*Y11+756.*Y21$$

$$G(4,9)=126.*(Y11+Y21)+63.*Y13+189.*Y23$$

$$G(4,10)=126.*(Y11+Y22)+42.*(Y12+Y21)+84.*(Y13+Y23)$$

$$G(4,11)=126.*(Y12+Y22)+189.*Y13+63.*Y23$$

$$G(4,12)=756.*Y12+252.*Y22$$

$$G(4,13)=210.*Y11+840.*Y21$$

$$G(4,14)=84.*Y11+42.*Y13+126.*Y21+168.*Y23$$

$$G(4,15)=63.*Y11+21.*Y12+42.*(Y13+Y21)+84.*Y22+63.*Y23$$

$$G(4,16)=84.*Y11+42.*(Y12+Y23)+63.*(Y13+Y22)+21.*Y21$$

$$G(4,17)=126.*Y12+168.*Y13+84.*Y22+42.*Y23$$

$$G(4,18)=840.*Y12+210.*Y22$$

$$G(5,1)=840.*Y21$$

$$G(5,2)=420.*Y23$$

$$G(5,3)=840.*Y22$$

$$G(5,4)=630.*Y21$$

$$G(5,5)=420.*Y21+210.*Y23$$

$$G(5,6)=210.*Y22+420.*Y23$$

$$G(5,7)=1260.*Y22$$

$$G(5,8)=504.*Y21$$

$$G(5,9)=252.*Y21+126.*Y23$$

$$G(5,10)=252.*Y21+84.*Y22+168.*Y23$$

$$G(5,11)=252.*Y22+378.*Y23$$

$$G(5,12)=1512.*Y22$$

$$G(5,13)=420.*Y21$$

$$G(5,14)=168.*Y21+84.*Y23$$

$$G(5,15)=126.*Y21+42.*Y22+84.*Y23$$

$$G(5,16)=168.*Y21+84.*Y22+126.*Y23$$

$$G(5,17)=252.*Y22+336.*Y23$$

$$G(5,18)=1580.*Y22$$

$$G(6,1)=630.*Y11$$

$$G(6,2)=315.*Y13$$

$$G(6,3)=830.*Y12$$

$$G(6,4)=1134.*Y11$$

$$G(6,5)=126.*Y11+378.*Y13$$

$$G(6,6)=378.*Y12+126.*Y13$$

$$G(6,7)=378.*Y12$$

$$G(6,8)=1512.*Y11$$

$$G(6,9)=189.*Y11+378.*Y13$$

$$G(6,10)=42.*Y11+252.*Y12+126.*Y13$$

$$G(6,11)=189.*Y12+63.*Y13$$

$$G(6,12)=252.*Y12$$

$$G(6,13)=1800.*Y11$$

$$G(6,14)=216.*Y11+360.*Y13$$

$$G(6,15)=54.*Y11+180.*Y12+108.*Y13$$

$$G(6,16)=18.*Y11+108.*Y12+54.*Y13$$

$$G(6,17)=108.*Y12+36.*Y13$$

$$G(6,18)=180.*Y12$$

$$G(7,1)=210.*(Y11+Y21)$$

$$G(7,2)=105.*(Y13+Y23)$$

$$G(7,3)=210.*(Y12+Y22)$$

$$G(7,4)=252.*Y11+378.*Y21$$

$$G(7,5)=84.*(Y11+Y13)+42.*Y21+126.*Y23$$

$$G(7,6)=84.*(Y12+Y13)+126.*Y22+42.*Y23$$

$$G(7,7)=252.*Y12+126.*Y22$$

$$G(7,8)=252.*Y11+504.*Y21$$

$$G(7,9)=84.*Y11+63.*(Y13+Y21)+126.*Y23$$

$$G(7,10)=42 \cdot (Y_{11}+Y_{12}+Y_{23})+56 \cdot Y_{13}+14 \cdot Y_{21}+84 \cdot Y_{22}$$

$$G(7,11)=84 \cdot Y_{13}+63 \cdot (Y_{13}+Y_{22})+21 \cdot Y_{23}$$

$$G(7,12)=252 \cdot Y_{12}+84 \cdot Y_{22}$$

$$G(7,13)=240 \cdot Y_{11}+600 \cdot Y_{21}$$

$$G(7,14)=72 \cdot (Y_{11}+Y_{21})+48 \cdot Y_{13}+120 \cdot Y_{23}$$

$$G(7,15)=36 \cdot (Y_{11}+Y_{13}+Y_{23})+24 \cdot Y_{12}+18 \cdot Y_{21}+60 \cdot Y_{22}$$

$$G(7,16)=24 \cdot Y_{11}+36 \cdot (Y_{12}+Y_{13}+Y_{22})+6 \cdot Y_{21}+18 \cdot Y_{23}$$

$$G(7,17)=72 \cdot Y_{12}+48 \cdot Y_{13}+36 \cdot Y_{22}+12 \cdot Y_{23}$$

$$G(7,18)=240 \cdot Y_{12}+60 \cdot Y_{22}$$

$$G(8,1)=210 \cdot (Y_{11}+Y_{21})$$

$$G(8,2)=105 \cdot (Y_{11}+Y_{23})$$

$$G(8,3)=210 \cdot (Y_{12}+Y_{22})$$

$$G(8,4)=126 \cdot Y_{11}+252 \cdot Y_{21}$$

$$G(8,5)=126 \cdot Y_{11}+42 \cdot Y_{13}+84 \cdot (Y_{21}+Y_{23})$$

$$G(8,6)=42 \cdot Y_{12}+126 \cdot Y_{13}+84 \cdot (Y_{22}+Y_{23})$$

$$G(8,7)=378 \cdot Y_{12}+252 \cdot Y_{22}$$

$$G(8,8)=84 \cdot Y_{11}+252 \cdot Y_{21}$$

$$G(8,9)=63 \cdot (Y_{11}+Y_{23})+21 \cdot Y_{13}+84 \cdot Y_{21}$$

$$G(8,10)=84 \cdot Y_{11}+14 \cdot Y_{12}+42 \cdot (Y_{13}+Y_{21}+Y_{22})+56 \cdot Y_{23}$$

$$G(8,11)=63 \cdot (Y_{12}+Y_{23})+126 \cdot Y_{13}+84 \cdot Y_{22}$$

$$G(8,12)=504 \cdot Y_{12}+252 \cdot Y_{22}$$

$$G(8,13)=60 \cdot Y_{11}+240 \cdot Y_{21}$$

$$G(8,14)=36 \cdot Y_{11}+12 \cdot Y_{13}+72 \cdot Y_{21}+48 \cdot Y_{23}$$

$$G(8,15)=36 \cdot (Y_{11}+Y_{21}+Y_{23})+6 \cdot Y_{12}+18 \cdot Y_{13}+24 \cdot Y_{22}$$

$$G(8,16)=60 \cdot Y_{11}+18 \cdot Y_{12}+36 \cdot (Y_{13}+Y_{22}+Y_{23})+24 \cdot Y_{21}$$

$$G(8,17)=72 \cdot (Y_{12}+Y_{22})+120 \cdot Y_{13}+48 \cdot Y_{23}$$

$$G(8,18)=600 \cdot Y_{12}+240 \cdot Y_{22}$$

$$G(9,1)=630 \cdot Y_{21}$$

$$G(9,2)=315 \cdot Y_{23}$$

$$G(9,3)=630 \cdot Y_{22}$$

$$G(9,4)=378 \cdot Y_{21}$$

$$G(9,5)=378 \cdot Y_{21}+126 \cdot Y_{23}$$

```

      G(9,6)=126.*Y22+378.*Y23
      G(9,7)=1134.*Y22
      G(9,8)=252.*Y21
      G(9,9)=189.*Y21+63.*Y23
      G(9,10)=252.*Y21+42.*Y22+126.*Y23
      G(9,11)=189.*Y22+378.*Y23
      G(9,12)=1512.*Y22
      G(9,13)=180.*Y21
      G(9,14)=108.*Y21+36.*Y23
      G(9,15)=108.*Y21+18.*Y22+54.*Y23
      G(9,16)=180.*Y21+54.*Y22+108.*Y23
      G(9,17)=216.*Y22+360.*Y23
      G(9,18)=1800.*Y22
      DO 100 I=1,9
      DO 100 J=1,18
100  G(I,J)=G(I,J)/1260.
      RETURN
      END

```

```

      SUBROUTINE TRMULT(A,B,C,D,K,M,N,L,ISYM)
      DIMENSION A(M,K),B(M,N),C(N,L),D(K,L),E(18,18)
      DO 10 I=1,M
      DO 10 J=1,L
      E(I,J)=0.
      DO 10 KK=1,N
10  E(I,J)=E(I,J)+B(I,KK)*C(KK,J)
      IF(ISYM.NE.0) GO TO 75
      DO 20 I=1,K
      DO 20 J=1,L
      D(I,J)=0.

```

```
DO 20 KK=1,M
20 D(I,J)=D(I,J)+A(KK,I)*E(KK,J)
RETURN
75 DO 30 I=1,K
DO 30 J=I,L
D(I,J)=0.
DO 35 KK=1,M
35 D(I,J)=D(I,J)+A(KK,I)*E(KK,J)
30 D(J,I)=D(I,J)
RETURN
END
```

- APPENDIX-F

SUBROUTINE ELSTIF(X1,X2,X3,Y1,Y2,Y3)

C***

C*** Element Stiffness Matrix Subroutine for the DKT Plate Bending

C*** Element.

C***

C*** INPUT

C*** x1,x2,x3,y1,y2,y3 are the x and y coordinates of the triangle.

C*** D is the elasticity matrix. For example, for isotropic case

C*** $D(1,1) = E \cdot T^3 / (12 \cdot (1 - \nu \cdot \nu))$. $D(1,2) = D(1,1) \cdot \nu$ etc.

C***

C*** OUTPUT

C*** STIF(9,9) is the stiffness matrix

C***

C*** GG(10,9) contains both G and H matrices without their first rows

C***

COMMON/PQ/D(3,3),STIF(9,9)

DIMENSION DD(9,9),SS(9,9),QQ(9,9),PP(3,3),PT(2,3),RS(2,3),Q(3)

DIMENSION GG(10,9),KOD(2,9),B(3),C(3),ALS(3)

DATA KOD/1,1,2,3,3,2,4,4,5,6,6,6,7,7,8,9,9,8/

DATA PP/12,3*4,2,1,4,1,2,1/

B(1)=Y2-Y3

B(2)=Y3-Y1

B(3)=Y1-Y2

C(1)=X3-X2

C(2)=X1-X3

C(3)=X2-X1

DET=(B(1)*C(2)-B(2)*C(1))*24.

DO 2F I=1,3

```

DO 25 J=1,3
DO 25 K1=1,3
II=(J-1)*3+K1
DO 25 K2=1,3
JJ=(J-1)*3+K2
25 DD(II,JJ)=D(I,J)*PP(K1,K2)/DET
DO 30 I=1,3
ALS(I)=B(I)*B(I)+C(I)*C(I)
PT(1,I)=6.*C(I)/ALS(I)
PT(2,I)=6.*B(I)/ALS(I)
RS(1,I)=3.*C(I)*C(I)/ALS(I)
RS(2,I)=3.*B(I)*B(I)/ALS(I)
30 Q(I)=3.*B(I)*C(I)/ALS(I)
DO 720 I=1,10
DO 720 J=1,9
720 GG(I,J)=0.
DO 730 I=1,2
II=(I-1)*5
GG(II+1,KOD(I,1))=PT(I,3)
GG(II+2,KOD(I,1))=-PT(I,2)
GG(II+3,KOD(I,1))=-PT(I,3)
GG(II+4,KOD(I,1))=PT(I,2)-PT(I,3)
GG(II+5,KOD(I,1))=PT(I,2)
GG(II+1,KOD(I,2))=-Q(3)
GG(II+2,KOD(I,2))=-Q(2)
GG(II+3,KOD(I,2))=Q(3)
GG(II+4,KOD(I,2))=Q(2)+Q(3)
GG(II+5,KOD(I,2))=Q(2)
GG(II+1,KOD(I,3))=-1.-RS(I,3)
GG(II+2,KOD(I,3))=-1.-RS(I,2)
GG(II+3,KOD(I,3))=RS(I,3)
GG(II+4,KOD(I,3))=RS(I,2)+RS(I,3)

```

```

GG(II+5,KOD(I,3))=RS(I,2)
GG(II+1,KOD(I,4))=-PT(I,3)
GG(II+3,KOD(I,4))=PT(I,3)
GG(II+4,KOD(I,4))=PT(I,1)+PT(I,3)
GG(II+1,KOD(I,5))=-Q(3)
GG(II+3,KOD(I,5))=Q(3)
GG(II+4,KOD(I,5))=Q(3)-Q(1)
GG(II+1,KOD(I,6))=1.-RS(I,3)
GG(II+3,KOD(I,6))=RS(I,3)
GG(II+4,KOD(I,6))=RS(I,3)-RS(I,1)
GG(II+2,KOD(I,7))=PT(I,2)
GG(II+4,KOD(I,7))=-PT(I,1)-PT(I,2)
GG(II+5,KOD(I,7))=-PT(I,2)
GG(II+2,KOD(I,8))=-Q(2)
GG(II+4,KOD(I,8))=Q(2)-Q(1)
GG(II+5,KOD(I,8))=Q(2)
GG(II+2,KOD(I,9))=1.-RS(I,2)
GG(II+4,KOD(I,9))=RS(I,2)-RS(I,1)
GG(II+5,KOD(I,9))=RS(I,2)

```

730 CONTINUE

```

DO 150 I=1,9
QQ(1,I)=B(2)*GG(1,I)+B(3)*GG(2,I)
QQ(2,I)=2.*B(2)*GG(3,I)+B(3)*GG(4,I)
QQ(3,I)=B(2)*GG(4,I)+2.*B(3)*GG(5,I)
QQ(4,I)=-C(2)*GG(6,I)-C(3)*GG(7,I)
QQ(5,I)=-2.*C(2)*GG(8,I)-C(3)*GG(9,I)
QQ(6,I)=-C(2)*GG(9,I)-2.*C(3)*GG(10,I)
QQ(7,I)=C(2)*GG(1,I)+C(3)*GG(2,I)-B(2)*GG(6,I)-B(3)*GG(7,I)
QQ(8,I)=2.*C(2)*GG(3,I)+C(3)*GG(4,I)-2.*B(2)*GG(8,I)-B(3)*GG(9,I)
QQ(9,I)=C(2)*GG(4,I)+2.*C(3)*GG(5,I)-B(2)*GG(9,I)-2.*B(3)*GG(10,I)

```

150 CONTINUE

DO 55 I=1,9


```
DO 55 J=1,9
  SS(I,J)=0.
  DO 55 K=1,9
    55 SS(I,J)=SS(I,J)+DD(I,K)*QQ(K,J)
  DO 60 I=1,9
    DO 60 J=1,9
      STIF(I,J)=0.
      DO 60 K=1,9
        60 STIF(I,J)=STIF(I,J)+QQ(K,I)*SS(K,J)
      RETURN
    END
```

APPENDIX-G

SUBROUTINE ELSTIF(X1,X2,X3,Y1,Y2,Y3,T1,T2,T3)

C***

C*** Element Stiffness Matrix Subroutine for the A-9 Plate Bending

C*** Element

C*** INPUT

C*** X1,X2,X3,Y1,Y2,Y3,T1,T2,T3 are the x,y coordinates and thicknesses

C*** at the triangle vertices; For uniform thickness, T1=T2=T3

C*** DMOD is the elasticity matrix excluding the thickness term. For

C*** example: DMOD(1,1)=Y/(12*(1.-PNU*PNU)) Y- Young's Modulus and

C*** PNU-Poisson's ratio

C*** OUTPUT

C*** EK is the (9 X 9) element stiffness matrix

COMMON/ST/EK(9,9),DMOD(3,3)

DIMENSION IT(6,10),F(7,7),BLS(3),AMDA(3),BETA(10),BC(2,3)

DIMENSION XX(6),FQ(7,9),P(3,3),E(6),DP(3,3),RHO(3),Q(7,9)

DATA IT/210,2*140,70,35,70,2*210,105,126,2*42,210,105,210,42,42,

1 126,105,126,42,84,21,14,105,2*84,42,28,42,105,42,126,14,21,84,

2 21,28,7,20,4,2,21,21,14,12,6,6,21,14,21,6,6,12,21,7,28,2,4,20/

C

C*** Generate the P matrix

C

BC(2,1)=Y3-Y2

BC(2,2)=Y1-Y3

BC(2,3)=Y2-Y1

BC(1,1)=X3-X2

BC(1,2)=X1-X3

BC(1,3)=X2-X1

P(1,1)=BC(2,1)*BC(2,1)

```

P(1,2)=BC(2,2)*BC(2,2)
P(1,3)=2.*BC(2,1)*BC(2,2)
P(2,1)=BC(1,1)*BC(1,1)
P(2,2)=BC(1,2)*BC(1,2)
P(2,3)=2.*BC(1,1)*BC(1,2)
P(3,1)=-2.*BC(2,1)*BC(1,1)
P(3,2)=-2.*BC(2,2)*BC(1,2)
P(3,3)=-2.*(BC(2,1)*BC(1,2)+BC(2,2)*BC(1,1))
DET3=210.*(BC(2,2)*BC(1,1)-BC(2,1)*BC(1,2))*3

```

C

C*** Generate upper half of E matrix. Include some constants

C

```

DO 100 I=1,3
  DO 100 J=1,3
    DP(I,J)=0.
    DO 100 K=1,3
100  DP(I,J)=DP(I,J)+DMOD(I,K)*P(K,J)
  L=1
  DO 120 I=1,3
    DO 120 J=I,3
      E(L)=0.
      DO 110 K=1,3
110  E(L)=E(L)+P(K,I)*DP(K,J)/DET3
120  L=L+1

```

C

C*** Generate the Lamda's and Rho's appearing in shape functions

C

```

DO 130 I=1,3
  BLS(I)=BC(2,I)*BC(2,I)+BC(1,I)*BC(1,I)
  J=I+1
  K=I+2
  IF (J.GT.3)J=J-3

```

```

      IF (K.GT.3)K=K-3
      AMDA(I)=(BC(2,I)*BC(2,K)+BC(1,I)*BC(1,K))/BLS(I)
130  RHO(I)=(BC(2,I)*BC(2,J)+BC(1,I)*BC(1,J))/BLS(I)
C
C*** Generate the Q matrix .
C
      Q(1,1)=3.*(1.-RHO(3))
      Q(2,1)=-6.*RHO(3)
      Q(3,1)=3.*(AMDA(2)-RHO(3))
      Q(4,1)=2.*(RHO(3)-1.)
      Q(5,1)=6.*RHO(3)
      Q(6,1)=6.*RHO(3)
      Q(7,1)=2.*(RHO(3)-AMDA(2))
      Q(1,4)=3.*(RHO(1)-AMDA(3))
      Q(2,4)=-6.*AMDA(3)
      Q(3,4)=3.*(1.-AMDA(3))
      Q(4,4)=2.*(AMDA(3)-RHO(1))
      Q(5,4)=6.*AMDA(3)
      Q(6,4)=6.*AMDA(3)
      Q(7,4)=2.*(AMDA(3)-1.)
      DO 140 I=1,7
140  Q(I,7)=-Q(I,1)-Q(I,4)
      DO 150 I=1,2
      Q(1,I+1)=BC(I,3)*(1.+1.5*AMDA(3))-1.5*BC(I,2)
      Q(2,I+1)=-BC(I,2)+BC(I,3)*(2.+3.*AMDA(3))
      Q(3,I+1)=-.5*BC(I,1)+1.5*(BC(I,3)*AMDA(3)+BC(I,2)*RHO(2))
      Q(4,I+1)=4./3.*BC(I,2)-BC(I,3)*(2./3.+AMDA(3))
      Q(5,I+1)=2.*BC(I,2)-BC(I,3)*(1.+3.*AMDA(3))
      Q(6,I+1)=BC(I,2)-BC(I,3)*(2.+3.*AMDA(3))
      Q(7,I+1)=BC(I,1)/3.-BC(I,3)*AMDA(3)-BC(I,2)*RHO(2)
      Q(1,I+4)=.5*BC(I,2)-1.5*(BC(I,1)*AMDA(1)+BC(I,3)*RHO(3))
      Q(2,I+4)=BC(I,1)-BC(I,3)*(2.+3.*RHO(3))

```

```

Q(3,I+4)=1.5*BC(I,1)-BC(I,3)*(1.+1.5*RHO(3))
Q(4,I+4)=-BC(I,2)/3.+BC(I,1)*AMDA(1)+BC(I,3)*RHO(3)
Q(5,I+4)=-BC(I,1)+BC(I,3)*(2.+3.*RHO(3))
Q(6,I+4)=-2.*BC(I,1)+BC(I,3)*(1.+3.*RHO(3))-
Q(7,I+4)=-4./3.*BC(I,1)+BC(I,3)*(2./3.+RHO(3))
Q(1,I+7)=-2.5*BC(I,2)+BC(I,1)*(1.+1.5*RHO(1))
Q(2,I+7)=2.*(BC(I,1)-BC(I,2))
Q(3,I+7)=2.5*BC(I,1)-BC(I,2)*(1.+1.5*AMDA(2))
Q(4,I+7)=4./3.*BC(I,2)-BC(I,1)*(2./3.+RHO(1))
Q(5,I+7)=2.*BC(I,2)-BC(I,1)
Q(6,I+7)=BC(I,2)-2.*BC(I,1)
Q(7,I+7)=-4./3.*BC(I,1)+BC(I,2)*(2./3.+AMDA(2))

```

150 CONTINUE

C

C*** Generate the Beta vector.

C

T13=T1-T3

T23=T2-T3

BETA(1)=T3*T3*T3

BETA(2)=T3*T3*T13

BETA(3)=T3*T3*T23

BETA(4)=T3*T13*T13

BETA(5)=T3*T13*T23

BETA(6)=T3*T23*T23

BETA(7)=T13*T13*T13

BETA(8)=T13*T13*T23

BETA(9)=T13*T23*T23

BETA(10)=T23*T23*T23

C

C*** Generate F matrix for uniform thickness(T1=T2=T3) or linear taper

C

JJ=10

```

IF (T1.EQ.T2.AND.T1.EQ.T3)JJ=1
DO 160 I=1,6
  XX(I)=0.
  DO 160 J=1,JJ
160  XX(I)=XX(I)+IT(I,J)*BETA(J)
    F(1,1)=2.*E(1)*XX(1)
    F(1,2)=E(3)*XX(1)
    F(1,3)=2.*E(2)*XX(1)
    F(1,4)=3.*E(1)*XX(2)
    F(1,5)=E(1)*XX(3)+E(3)*XX(2)
    F(1,6)=E(2)*XX(2)+E(3)*XX(3)
    F(1,7)=3.*E(2)*XX(3)
    F(2,2)=.5*E(6)*XX(1)
    F(2,3)=E(5)*XX(1)
    F(2,4)=1.5*E(3)*XX(2)
    F(2,5)=.5*(E(3)*XX(3)+E(6)*XX(2))
    F(2,6)=.5*(E(5)*XX(2)+E(6)*XX(3))
    F(2,7)=1.5*E(5)*XX(3)
    F(3,3)=2.*E(4)*XX(1)
    F(3,4)=3.*E(2)*XX(2)
    F(3,5)=E(2)*XX(3)+E(5)*XX(2)
    F(3,6)=E(4)*XX(2)+E(5)*XX(3)
    F(3,7)=3.*E(4)*XX(3)
    F(4,4)=9.*E(1)*XX(4)
    F(4,5)=3.*(E(1)*XX(5)+E(3)*XX(4))
    F(4,6)=3.*(E(2)*XX(4)+E(3)*XX(5))
    F(4,7)=9.*E(2)*XX(5)
    F(5,5)=E(1)*XX(6)+E(6)*XX(4)+2.*E(3)*XX(5)
    F(5,6)=E(3)*XX(6)+E(5)*XX(4)+(E(2)+E(6))*XX(5)
    F(5,7)=3.*(E(2)*XX(6)+E(5)*XX(5))
    F(6,6)=E(4)*XX(4)+E(6)*XX(6)+2.*E(5)*XX(5)
    F(6,7)=3.*(E(4)*XX(5)+E(5)*XX(6))

```

```
F(7,7)=9.*E(4)*XX(6)
DO 170 I=1,7
    DO 170 J=1,7
170  F(J,I)=F(I,J)
C
C*** Generate the element stiffness matrix EK
C
DO 180 I=1,7
    DO 180 J=1,9
        FQ(I,J)=0.
        DO 180 K=1,7
180  FQ(I,J)=FQ(I,J)+F(I,K)*Q(K,J)
DO 200 I=1,9
    DO 200 J=1,9
        EK(I,J)=0.
        DO 190 K=1,7
190  EK(I,J)=EK(I,J)+Q(K,I)*FQ(K,J)
200  EK(J,I)=EK(I,J)
RETURN
END
```

APPENDIX-H

SUBROUTINE ELSTIF(X1,X2,X3,Y1,Y2,Y3,H1,H2,H3)

C*** Element Stiffness Matrix Subroutine for Element E-1

C*** INPUT

C*** X1,X2,X3,Y1,Y2,Y3,H1,H2,H3 are the x,y coordinates and thicknesses

C*** at the triangle vertices; For uniform thickness, H1=H2=H3

C*** D is the elasticity matrix excluding the thickness term. For

C*** example: $D(1,1) = Y / (12 * (1 - PNU * PNU))$ Y- Young's modulus and

C*** PNU- Poisson's ratio

C*** OUTPUT

C*** ESTIF is the (9 X 9) element stiffness matrix

COMMON/PQ/D(3,3),ESTIF(9,9)

DIMENSION S(9,9),Q(9,9),IT(6,10),GAMA(10),XX(3,3)

DIMENSION E(9,9),G(10,9)

DATA IT/420.,140.,140.,70.,35.,70.,420.,210.,105.,126.,42.,42.,

1 420.,105.,210.,42.,42.,126.,210.,126.,42.,84.,21.,14.,210.

2 ,84.,84.,42.,28.,42.,210.,42.,126.,14.,21.,84.,42.,28.,7.,

3 20.,4.,2.,42.,21.,14.,12.,6.,6.,42.,14.,21.,6.,6.,12.,42.,

4 7.,28.,2.,4.,20./

B1=Y2-Y3

B2=Y3-Y1

B3=Y1-Y2

C1=X3-X2

C2=X1-X3

C3=X2-X1

DET=B2*C3-B3*C2

C

C*** Generate the Gamma vector.

C


```
H21=H2-H1
H31=H3-H1
GAMA(1)=H1**3
GAMA(2)=H1*H1*H21
GAMA(3)=H1*H1*H31
GAMA(4)=H1*H21*H21
GAMA(5)=H1*H21*H31
GAMA(6)=H1*H31*H31
GAMA(7)=H21**3
GAMA(8)=H21*H21*H31
GAMA(9)=H21*H31*H31
GAMA(10)=H31**3
C
C*** Generate the E matrix.
C
JJ=10
IF (H1.EQ.H2.AND.H1.EQ.H3)JJ=1
K=1
DO 110 I=1,3
  DO 110 J=1,3
    IX(I,J)=0
    DO 100 L=1,JJ
100   XX(I,J)=XX(I,J)+IT(K,L)*GAMA(L)
    K=K+1
110   XX(J,I)=XX(I,J)
DO 120 I=1,3
  DO 120 J=1,3
    DO 120 K1=1,3
      II=(I-1)*3+K1
      DO 120 K2=1,3
        JJ=(J-1)*3+K2
        E(II,JJ)=D(I,J)*XX(K1,K2)/DET/840.
```

120 E(JJ,II)=E(II,JJ)

C

C*** Generate the Q matrix.

C

$$P1=6.*C1/(B1**2+C1**2)$$

$$P2=6.*C2/(B2**2+C2**2)$$

$$P3=6.*C3/(B3**2+C3**2)$$

$$Q1=3.*B1*C1/(B1**2+C1**2)$$

$$Q2=3.*B2*C2/(B2**2+C2**2)$$

$$Q3=3.*B3*C3/(B3**2+C3**2)$$

$$R1=3.*B1**2/(B1**2+C1**2)$$

$$R2=3.*B2**2/(B2**2+C2**2)$$

$$R3=3.*B3**2/(B3**2+C3**2)$$

$$S1=3.*C1**2/(B1**2+C1**2)$$

$$S2=3.*C2**2/(B2**2+C2**2)$$

$$S3=3.*C3**2/(B3**2+C3**2)$$

$$T1=6.*B1/(B1**2+C1**2)$$

$$T2=6.*B2/(B2**2+C2**2)$$

$$T3=6.*B3/(B3**2+C3**2)$$

$$A1=27.*(B1*(C2-C3)/6.+C1*(B2-B3)/18.)/DET$$

$$A2=27.*(B2*C3/3.-B1*C2/6.+C2*(B2-B3)/18.)/DET$$

$$A3=27.*(B1*C3/6.-B3*C2/3.+C3*(B2-B3)/18.)/DET$$

$$A4=27.*(C1*(B3-B2)/6.+B1*(C3-C2)/18.)/DET$$

$$A5=27.*(B2*C1/6.-B3*C2/3.+B2*(C3-C2)/18.)/DET$$

$$A6=27.*(B2*C3/3.-B3*C1/6.+B3*(C3-C2)/18.)/DET$$

$$G(1,1)=3.*P2+7.*P3-36.*B1/DET$$

$$G(1,2)=19.-3.*S2+7.*S3+A1$$

$$G(1,3)=3.*Q2-7.*Q3+6.*B1*(B3-B2)/DET$$

$$G(1,4)=-3.*P1-7.*P3-36.*B2/DET$$

$$G(1,5)=-1-3.*S1+7.*S3+A2$$

$$G(1,6)=3.*Q1-7.*Q3+6.*(B3*B3-B1*B1)/DET$$

$$G(1,7)=3.*(P1-P2)-36.*B3/DET$$

$$G(1,8)=9.-3.*(S1+S2)+A3$$

$$G(1,9)=3.*(Q1+Q2)+6.*(B1*B1-B2*B2)/DET$$

$$G(2,1)=-7.*P2-3.*P3-36.*B1/DET$$

$$G(2,2)=19.+7.*S2-3.*S3+A1$$

$$G(2,3)=3.*Q3-7.*Q2+6.*B1*(B3-B2)/DET$$

$$G(2,4)=3.*(P3-P1)-36.*B2/DET$$

$$G(2,5)=9.-3.*(S1+S3)+A2$$

$$G(2,6)=3.*(Q1+Q3)+6.*(B3*B3-B1*B1)/DET$$

$$G(2,7)=3.*P1+7.*P2-36.*B3/DET$$

$$G(2,8)=-1.-3.*S1+7.*S2+A3$$

$$G(2,9)=3.*Q1-7.*Q2+6.*(B1*B1-B2*B2)/DET$$

$$G(4,1)=7.*(P2-P3)+36.*B1/DET$$

$$G(4,2)=-9.-7.*(S2+S3)-A1$$

$$G(4,3)=7.*(Q2+Q3)-6.*B1*(B3-B2)/DET$$

$$G(4,4)=13.*P1+7.*P3+36.*B2/DET$$

$$G(4,5)=13.*S1-9.-7.*S3-A2$$

$$G(4,6)=7.*Q3-13.*Q1-6.*(B3*B3-B1*B1)/DET$$

$$G(4,7)=-13.*P1-7.*P2+36.*B3/DET$$

$$G(4,8)=13.*S1-9.-7.*S2-A3$$

$$G(4,9)=7.*Q2-13.*Q1-6.*(B1*B1-B2*B2)/DET$$

$$G(6,1)=-3.*T2-7.*T3-36.*C1/DET$$

$$G(6,2)=3.*Q2-7.*Q3+6.*C1*(C2-C3)/DET$$

$$G(6,3)=19.-3.*R2+7.*R3+A4$$

$$G(6,4)=3.*T1+7.*T3-36.*C2/DET$$

$$G(6,5)=3.*Q1-7.*Q3+6.*(C1*C1-C3*C3)/DET$$

$$G(6,6)=7.*R3-1.-3.*R1+A5$$

$$G(6,7)=3.*T2-3.*T1-36.*C3/DET$$

$$G(6,8)=3.*(Q1+Q2)+6.*(C2*C2-C1*C1)/DET$$

$$G(6,9)=9.-3.*(R1+R2)+A6$$

$$G(7,1)=7.*T2+3.*T3-36.*C1/DET$$

$$G(7,2)=3.*Q3-7.*Q2+6.*C1*(C2-C3)/DET$$

$$G(7,3)=19.+7.*R2-3.*R3+A4$$

$G(7,4)=3.*T1-3.*T3-36.*C2/DET$
 $G(7,5)=3.*(Q1+Q3)+6.*(C1*C1-C3*C3)/DET$
 $G(7,6)=9.-3.*(R1+R3)+A5$
 $G(7,7)=-3.*T1-7.*T2-36.*C3/DET$
 $G(7,8)=3.*Q1-7.*Q2+6.*(C2*C2-C1*C1)/DET$
 $G(7,9)=7.*R2-1.-3.*R1+A6$
 $G(8,1)=7.*(T3-T2)+36.*C1/DET$
 $G(9,2)=7.*(Q2+Q3)-6.*C1*(C2-C3)/DET$
 $G(9,3)=-9.-7.*(R2+R3)-A4$
 $G(9,4)=-13.*T1-7.*T3+36.*C2/DET$
 $G(9,5)=-13.*Q1+7.*Q3-6.*(C1*C1-C3*C3)/DET$
 $G(9,6)=13.*R1-9.-7.*R3-A5$
 $G(9,7)=13.*T1+7.*T2+36.*C3/DET$
 $G(9,8)=-13.*Q1+7.*Q2-6.*(C2*C2-C1*C1)/DET$
 $G(9,9)=13.*R1-9.-7.*R2-A6$

DO 130 I=1,9

$G(3,I)=-G(1,I)$

$G(5,I)=-G(2,I)$

$G(8,I)=-G(6,I)$

130 $G(10,I)=-G(7,I)$

$G(3,2)=G(3,2)+10$

$G(3,5)=G(3,5)-10$

$G(5,2)=G(5,2)+10$

$G(5,8)=G(5,8)-10$

$G(8,3)=G(8,3)+10$

$G(8,6)=G(8,6)-10$

$G(10,3)=G(10,3)+10$

$G(10,9)=G(10,9)-10$

DO 140 I=1,9

$Q(1,I)=(B2*G(1,I)+B3*G(2,I))/10.$

$Q(2,I)=(2.*B2*G(3,I)+B3*G(4,I))/10.$

$Q(3,I)=(B2*G(4,I)+2.*B3*G(5,I))/10.$

```

Q(4,I)=(C2*G(6,I)+C3*G(7,I))/10.
Q(5,I)=(2.*C2*G(8,I)+C3*G(9,I))/10.
Q(6,I)=(C2*G(9,I)+2.*C3*G(10,I))/10.
Q(7,I)=(C2*G(1,I)+C3*G(2,I)+B2*G(6,I)+B3*G(7,I))/10.
Q(8,I)=(2.*C2*G(3,I)+C3*G(4,I)+2.*B2*G(8,I)+B3*G(9,I))/10.
Q(9,I)=(C2*G(4,I)+2.*C3*G(5,I)+B2*G(9,I)+2.*B3*G(10,I))/10.

```

```

140  CONTINUE

```

```

C

```

```

C*** Generate the stiffness matrix  ESTIF

```

```

C

```

```

    DO 150 I=1,9

```

```

        DO 150 J=1,9

```

```

            S(I,J)=0.

```

```

            DO 150 K=1,9

```

```

150      S(I,J)=S(I,J)+E(I,K)*Q(K,J)

```

```

    DO 170 I=1,9

```

```

        DO 170 J=1,9

```

```

            ESTIF(I,J)=0.

```

```

            DO 160 K=1,9

```

```

160      ESTIF(I,J)=ESTIF(I,J)+S(K,I)*Q(K,J)

```

```

170      ESTIF(J,I)=ESTIF(I,J)

```

```

    RETURN

```

```

END

```

APPENDIX-I

SUBROUTINE ELSTIF(X1,X2,X3,Y1,Y2,Y3,H1,H2,H3)

C*** Element Stiffness Matrix Subroutine for Element E-2

C*** INPUT

C*** X1,X2,X3,Y1,Y2,Y3,H1,H2,H3 are the x,y coordinates and thicknesses

C*** at the triangle vertices; For uniform thickness, H1=H2=H3

C*** D is the elasticity matrix excluding the thickness term. For

C*** example: $D(1,1) = Y / (12 * (1 - \text{PNU} * \text{PNU}))$ Y- Young's modulus and

C*** PNU- Poisson's ratio

C*** OUTPUT

C*** ESTIF is the (9 X 9) element stiffness matrix

COMMON/PQ/D(3,3),ESTIF(9,9)

DIMENSION S(9,9),Q(9,9),IT(6,10),GAMA(10),XX(3,3)

DIMENSION E(9,9),G(10,9),AA(2,9),CC(2,2)

DATA IT/420.,140.,140.,70.,35.,70.,420.,210.,105.,126.,42.,42.,

1 420.,105.,210.,42.,42.,126.,210.,126.,42.,84.,21.,14.,210.

2 ,84.,84.,42.,28.,42.,210.,42.,126.,14.,21.,84.,42.,28.,7.,

3 20.,4.,2.,42.,21.,14.,12.,6.,6.,42.,14.,21.,6.,6.,12.,42.,

4 7.,28.,2.,4.,20./

B1=Y2-Y3

B2=Y3-Y1

B3=Y1-Y2

C1=X3-X2

C2=X1-X3

C3=X2-X1

DET=B2*C3-B3*C2

CC(1,1)=-B2/DET/4.

CC(1,2)=-B3/DET/4.

CC(2,1)=-C2/DET/4.

CC(2,2)=-C3/DET/4.

C

C*** Generate the Gamma vector.

C

H21=H2-H1

H31=H3-H1

GAMA(1)=H1**3

GAMA(2)=H1*H1*H21

GAMA(3)=H1*H1*H31

GAMA(4)=H1*H21*H21

GAMA(5)=H1*H21*H31

GAMA(6)=H1*H31*H31

GAMA(7)=H21**3

GAMA(8)=H21*H21*H31

GAMA(9)=H21*H31*H31

GAMA(10)=H31**3

C

C*** Generate the E matrix.

C

JJ=10

IF (H1.EQ.H2.AND.H1.EQ.H3)JJ=1

K=1

DO 210 I=1,3

DO 210 J=1,3

XX(I,J)=0.

DO 200 L=1,JJ

200 XX(I,J)=XX(I,J)+IT(K,L)*GAMA(L)

K=K+1

210 XX(J,I)=XX(I,J)

DO 220 I=1,3

DO 220 J=1,3

DO 220 K1=1,3

$$II=(I-1)*3+K1$$

$$DO\ 220\ K2=1,3$$

$$JJ=(J-1)*3+K2$$

$$E(II,JJ)=D(I,J)*XX(K1,K2)/DET/840.$$

$$220\ E(JJ,II)=E(II,JJ)$$

C

C*** Generate the Q matrix.

C

$$P1=6.*C1/(B1**2+C1**2)$$

$$P2=6.*C2/(B2**2+C2**2)$$

$$P3=6.*C3/(B3**2+C3**2)$$

$$Q1=3.*B1*C1/(B1**2+C1**2)$$

$$Q2=3.*B2*C2/(B2**2+C2**2)$$

$$Q3=3.*B3*C3/(B3**2+C3**2)$$

$$R1=3.*B1**2/(B1**2+C1**2)$$

$$R2=3.*B2**2/(B2**2+C2**2)$$

$$R3=3.*B3**2/(B3**2+C3**2)$$

$$S1=3.*C1**2/(B1**2+C1**2)$$

$$S2=3.*C2**2/(B2**2+C2**2)$$

$$S3=3.*C3**2/(B3**2+C3**2)$$

$$T1=6.*B1/(B1**2+C1**2)$$

$$T2=6.*B2/(B2**2+C2**2)$$

$$T3=6.*B3/(B3**2+C3**2)$$

$$R01=(B1*B2+C1*C2)/(B1*B1+C1*C1)$$

$$R02=(B2*B3+C2*C3)/(B2*B2+C2*C2)$$

$$R03=(B3*B1+C3*C1)/(B3*B3+C3*C3)$$

$$DO\ 230\ I=1,2$$

$$E1=CC(I,1)$$

$$E2=CC(I,2)$$

$$AA(I,1)=E1*(30.*R02-114.)-E2*(30.*R03+144.)$$

$$AA(I,4)=E1*(30.*R01+144.)+E2*30.*(R01+R03+1.)$$

$$AA(I,7)=-E1*30.*(R01+R02+1.)+E2*(114.-30.*R01)$$

$$AA(I,2)=E1*(C2*(14.-15.*R02)-7.*C3)+E2*(7.*C2-C3*(29.+15.*R03))$$

$$AA(I,3)=-E1*(B2*(14.-15.*R02)-7.*B3)-E2*(7.*B2-B3*(29.+15.*R03))$$

1)

$$AA(I,5)=E1*(C1*(29.+15.*R01)-7.*C3)+E2*(C1*(22.+15.*R01)-C3*$$

1 (15.*R03-7.))

$$AA(I,6)=-E1*(B1*(29.+15.*R01)-7.*B3)-E2*(B1*(22.+15.*R01)-B3*$$

1 (15.*R03-7.))

$$AA(I,8)=E1*(C1*(15.*R01-7.)-C2*(15.*R02+22.))+E2*(7.*C2+C1*$$

1 (15.*R01-14.))

$$230 \quad AA(I,9)=-E1*(B1*(15.*R01-7.)-B2*(15.*R02+22.))-E2*(7.*B2+B1* \\ 1(15.*R01-14.))$$

$$G(1,1)=3.*P2+4.*P3+AA(1,1)$$

$$G(1,2)=16.-3.*S2+4.*S3+AA(1,2)$$

$$G(1,3)=3.*Q2-4.*Q3+AA(1,3)$$

$$G(1,4)=-3.*P1-4.*P3+AA(1,4)$$

$$G(1,5)=2.-3.*S1+4.*S3+AA(1,5)$$

$$G(1,6)=3.*Q1-4.*Q3+AA(1,6)$$

$$G(1,7)=3.*(P1-P2)+AA(1,7)$$

$$G(1,8)=9.-3.*(S1+S2)+AA(1,8)$$

$$G(1,9)=3.*(Q1+Q2)+AA(1,9)$$

$$G(2,1)=-4.*P2-3.*P3+AA(1,1)$$

$$G(2,2)=16.+4.*S2-3.*S3+AA(1,2)$$

$$G(2,3)=3.*Q3-4.*Q2+AA(1,3)$$

$$G(2,4)=3.*(P3-P1)+AA(1,4)$$

$$G(2,5)=9.-3.*(S1+S3)+AA(1,5)$$

$$G(2,6)=3.*(Q1+Q3)+AA(1,6)$$

$$G(2,7)=3.*P1+4.*P2+AA(1,7)$$

$$G(2,8)=2.-3.*S1+4.*S2+AA(1,8)$$

$$G(2,9)=3.*Q1-4.*Q2+AA(1,9)$$

$$G(4,1)=4.*(P2-P3)-AA(1,1)$$

$$G(4,2)=-9.-4.*(S2+S3)-AA(1,2)$$

$$G(4,3)=4.*(Q2+Q3)-AA(1,3)$$

$G(4,4)=10.*P1+4.*P3-AA(1,4)$
 $G(4,5)=10.*S1-9.-4.*S3-AA(1,5)$
 $G(4,6)=4.*Q3-10.*Q1-AA(1,6)$
 $G(4,7)=-10.*P1-4.*P2-AA(1,7)$
 $G(4,8)=10.*S1-9.-4.*S2-AA(1,8)$
 $G(4,9)=4.*Q2-10.*Q1-AA(1,9)$
 $G(6,1)=-3.*T2-4.*T3+AA(2,1)$
 $G(6,2)=3.*Q2-4.*Q3+AA(2,2)$
 $G(6,3)=16.-3.*R2+4.*R3+AA(2,3)$
 $G(6,4)=3.*T1+4.*T3+AA(2,4)$
 $G(6,5)=3.*Q1-4.*Q3+AA(2,5)$
 $G(6,6)=2.+4.*R3-3.*R1+AA(2,6)$
 $G(6,7)=3.*(T2-T1)+AA(2,7)$
 $G(6,8)=3.*(Q1+Q2)+AA(2,8)$
 $G(6,9)=9.-3.*(R1+R2)+AA(2,9)$
 $G(7,1)=4.*T2+3.*T3+AA(2,1)$
 $G(7,2)=3.*Q3-4.*Q2+AA(2,2)$
 $G(7,3)=16.+4.*R2-3.*R3+AA(2,3)$
 $G(7,4)=3.*(T1-T3)+AA(2,4)$
 $G(7,5)=3.*(Q1+Q3)+AA(2,5)$
 $G(7,6)=9.-3.*(R1+R3)+AA(2,6)$
 $G(7,7)=-3.*T1-4.*T2+AA(2,7)$
 $G(7,8)=3.*Q1-4.*Q2+AA(2,8)$
 $G(7,9)=2.+4.*R2-3.*R1+AA(2,9)$
 $G(9,1)=4.*(T3-T2)-AA(2,1)$
 $G(9,2)=4.*(Q2+Q3)-AA(2,2)$
 $G(9,3)=7.-4.*(R2+R3)-AA(2,3)$
 $G(9,4)=-10.*T1-4.*T3-AA(2,4)$
 $G(9,5)=-10.*Q1+4.*Q3-AA(2,5)$
 $G(9,6)=10.*R1-9.-4.*R3-AA(2,6)$
 $G(9,7)=10.*T1+4.*T2-AA(2,7)$
 $G(9,8)=-10.*Q1+4.*Q2-AA(2,8)$

```

      G(9,9)=10.*R1-9.-4.*R2-AA(2,9)
      DO 240 I=1,9
        G(3,I)=-G(1,I)
        G(5,I)=-G(2,I)
        G(8,I)=-G(6,I)
240    G(10,I)=-G(7,I)
        G(3,2)=G(3,2)+7.
        G(3,5)=G(3,5)-7.
        G(5,2)=G(5,2)+7.
        G(5,8)=G(5,8)-7.
        G(8,3)=G(8,3)+7.
        G(8,6)=G(8,6)-7.
        G(10,3)=G(10,3)+7.
        G(10,9)=G(10,9)-7.
      DO 250 I=1,9
        Q(1,I)=(B2*G(1,I)+B3*G(2,I))/7.
        Q(2,I)=(2.*B2*G(3,I)+B3*G(4,I))/7.
        Q(3,I)=(B2*G(4,I)+2.*B3*G(5,I))/7.
        Q(4,I)=(C2*G(6,I)+C3*G(7,I))/7.
        Q(5,I)=(2.*C2*G(8,I)+C3*G(9,I))/7.
        Q(6,I)=(C2*G(9,I)+2.*C3*G(10,I))/7.
        Q(7,I)=(C2*G(1,I)+C3*G(2,I)+B2*G(6,I)+B3*G(7,I))/7.
        Q(8,I)=(2.*C2*G(3,I)+C3*G(4,I)+2.*B2*G(8,I)+B3*G(9,I))/7.
        Q(9,I)=(C2*G(4,I)+2.*C3*G(5,I)+B2*G(9,I)+2.*B3*G(10,I))/7.
250    CONTINUE
      C
      C*** Generate the stiffness matrix ESTIF
      C
      DO 280 I=1,9
        DO 260 J=1,9
          S(I,J)=0.
          DO 260 K=1,9

```

```
260  S(I,J)=S(I,J)+E(I,K)*Q(K,J)
      DO 280 I=1,9
          DO 280 J=1,9
              ESTIF(I,J)=0.
              DO 270 K=1,9
270          ESTIF(I,J)=ESTIF(I,J)+S(K,I)*Q(K,J)
280  ESTIF(J,I)=ESTIF(I,J)
      RETURN
      END
```

APPENDIX-J

SUBROUTINE ELSTIF (X1,X2,X3,X4,Y1,Y2,Y3,Y4,NGAUS)

C***

C*** Element Stiffness Matrix Subroutine for the IDKQ Plate Bending.

C*** Element.

C***

C*** INPUT

C***

C*** X1,X2,X3,X4,Y1,Y2,Y3,Y4 are the x and y coordinates of the four

C*** nodes of the quadrilateral.

C*** NGAUS = Gauss integration order = 3 or 2

C***

C*** OUTPUT

C***

C*** ESTIF is the (12 X 12) element stiffness matrix.

C***

COMMON/TWO/NDFPN,NDFPE,NNPE

COMMON/PQ/DFLEX(3,3),ESTIF(12,12),THIK,YMOD,PNU

COMMON/SQ/CD(2,9),DERIV(2,9),XJAC(2,2),XJACI(2,2),DJACB1

COMMON/BMX/BM(3,12),P1,P2,P3,P4,Q1,Q2,Q3,Q4,R1,R2,R3,R4,S1,S2,S3,

1 S4,T1,T2,T3,T4

DIMENSION B(4),C(4),STF(12,12),DFB(3,12)

DIMENSION POSGP(4),WEIGP(4)

DO 15 I = 1,12

DO 1 J = 1,12

15 ESTIF(I,J) = 0.

B(1) = Y1 - Y2

B(2) = Y2 - Y3

B(3) = Y3 - Y4

```

B(4) = Y4 - Y1
C(1) = X2 - X1
C(2) = X3 - X2
C(3) = X4 - X3
C(4) = X1 - X4
P1 = 6.*C(1)/(B(1)**2 + C(1)**2)
P2 = 6.*C(2)/(B(2)**2 + C(2)**2)
P3 = 6.*C(3)/(B(3)**2 + C(3)**2)
P4 = 6.*C(4)/(B(4)**2 + C(4)**2)
Q1 = 3.*B(1)*C(1)/(B(1)**2 + C(1)**2)
Q2 = 3.*B(2)*C(2)/(B(2)**2 + C(2)**2)
Q3 = 3.*B(3)*C(3)/(B(3)**2 + C(3)**2)
Q4 = 3.*B(4)*C(4)/(B(4)**2 + C(4)**2)
R1 = 3.*B(1)**2/(B(1)**2 + C(1)**2)
R2 = 3.*B(2)**2/(B(2)**2 + C(2)**2)
R3 = 3.*B(3)**2/(B(3)**2 + C(3)**2)
R4 = 3.*B(4)**2/(B(4)**2 + C(4)**2)
S1 = 3.*C(1)**2/(B(1)**2 + C(1)**2)
S2 = 3.*C(2)**2/(B(2)**2 + C(2)**2)
S3 = 3.*C(3)**2/(B(3)**2 + C(3)**2)
S4 = 3.*C(4)**2/(B(4)**2 + C(4)**2)
T1 = 6.*B(1)/(B(1)**2 + C(1)**2)
T2 = 6.*B(2)/(B(2)**2 + C(2)**2)
T3 = 6.*B(3)/(B(3)**2 + C(3)**2)
T4 = 6.*B(4)/(B(4)**2 + C(4)**2)
DO 30 I =1,3
DO 30 J =1,3
30 DFLEX(I,J) = 0.
CON = (YMOD*THIK**3)/(12.*(1. - PNU*PNU))
DFLEX(1,1) = CON
DFLEX(2,2) = CON
DFLEX(1,2) = CON*PNU

```

```

DFLEX(2,1) = CON*PNU
DFLEX(3,3) = CON*(1. - PNU)/2.
CALL GAUSSQ (NGAUS, POSGP, WEIGP)
DO 1000 IG = 1,NGAUS
DO 1000 JG = 1,NGAUS
EXI = POSGP(IG)
ETA = POSGP(JG)
CALL BMAT(EXI,ETA,X1,X2,X3,X4,Y1,Y2,Y3,Y4)
DAREA = DJACB1*WEIGP(IG)*WEIGP(JG)
DO 75 I = 1,3
DO 75 J = 1,12
DFB(I,J) = 0.
DO 75 K = 1,3
75 DFB(I,J) = DFB(I,J) + DFLEX(I,K)*BM(K,J)
DO 80 I = 1,12
DO 80 J = 1,12
STF(I,J) = 0.
DO 80 K = 1,3
80 STF(I,J) = STF(I,J) + BM(K,I)*DFB(K,J)*DAREA
DO 85 I = 1,12
DO 85 J = 1,12
85 ESTIF(I,J) = ESTIF(I,J) + STF(I,J)
1000 CONTINUE
RETURN
END

```

SUBROUTINE GAUSSQ(NGAUS,POSGP,WEIGP)

DIMENSION POSGP(4),WEIGP(4)

IF(NGAUS.GT.2) GO TO 10

POSGP(1) = -.577350269189626

```

WEIGP(1)=1.
GO TO 20
10 POSGP(1)=-0.774598669241483
   POSGP(2)=0.
   WEIGP(1)=0.5555555555555556
   WEIGP(2)=.8888888888888889
20 KGAUS=NGAUS/2
   DO 30 IGASH=1,KGAUS
   JGASH=NGAUS+1-IGASH
   POSGP(JGASH)=-POSGP(IGASH)
   WEIGP(JGASH)=WEIGP(IGASH)
30 CONTINUE
   RETURN
   END

```

SUBROUTINE SHAPE(T,S)

```
COMMON/SQ/CD(2,9),DERIV(2,9),XJAC(2,2),XJACI(2,2),DJACB1
```

```
DERIV(1,1)=S/6.+T/4.-S*T/2.-T*T/4.
```

```
DERIV(1,5)=-S/3.+S*T
```

```
DERIV(1,2)=S/6.-T/4.-S*T/2.+T*T/4.
```

```
DERIV(1,6)=.5+2.*S/3.-T*T/2.
```

```
DERIV(1,3)=S/6.+T/4.+S*T/2.+T*T/4.
```

```
DERIV(1,7)=-S/3.-S*T
```

```
DERIV(1,4)=S/6.-T/4.+S*T/2.-T*T/4.
```

```
DERIV(1,8)=-.5+2.*S/3.+T*T/2.
```

```
DERIV(1,9)=-4.*S/3.
```

```
DERIV(2,1)=S/4.+T/6.-S*S/4.-S*T/2.
```

```
DERIV(2,5)=-.5+2.*T/3.+S*S/2.
```

```
DERIV(2,2)=-S/4.+T/6.-S*S/4.+S*T/2.
```

```
DERIV(2,6)=-T/3.-S*T
```


DERIV(2,3)=S/4.+T/6.+S*S/4.+S*T/2.

DERIV(2,7)=.5+2.*T/3.-S*S/2.

DERIV(2,4)=-S/4.+T/6.+S*S/4.-S*T/2.

DERIV(2,8)=-T/3.+S*T

DERIV(2,9)=-4.*T/3.

RETURN

END

SUBROUTINE JACOB1(X1,X2,X3,X4,Y1,Y2,Y3,Y4,EXI,ETA)

COMMON/SQ/CD(2,9),DERIV(2,9),XJAC(2,2),XJACI(2,2),DJACB1

C

XJAC(1,1)=0.25*(X2+X3-X1-X4+ETA*(X1+X3-X2-X4)).

XJAC(1,2)=0.25*(Y2+Y3-Y1-Y4+ETA*(Y1+Y3-Y2-Y4))

XJAC(2,1)=0.25*(X3+X4-X1-X2+EXI*(X1+X3-X2-X4))

XJAC(2,2)=0.25*(Y3+Y4-Y1-Y2+EXI*(Y1+Y3-Y2-Y4))

C

C**** CALCULATE DETERMINANT AND INVERSE OF

C JACOBIAN MATRIX

C

DJACB1=XJAC(1,1)*XJAC(2,2)-XJAC(1,2)*

XJAC(2,1)

XJACI(1,1)=XJAC(2,2)/DJACB1

XJACI(2,2)=XJAC(1,1)/DJACB1

XJACI(1,2)=-XJAC(1,2)/DJACB1

XJACI(2,1)=-XJAC(2,1)/DJACB1

C

C**** CALCULATE CARTESIAN DERIVATIVES

C

DO 40 IDIME=1,2

DO 40 INODE=1,9

```

CD(IDIME, INODE)=0.0
DO 40 JDIME=1,2
  CD(IDIME, INODE)=CD(IDIME, INODE)+
    XJACI(IDIME, JDIME)*DERIV(JDIME, INODE)
40 CONTINUE
RETURN
END

```

```

SUBROUTINE BMAT (XI, ETA, X1, X2, X3, X4, Y1, Y2, Y3, Y4)
COMMON/SQ/CD(2,9), DERIV(2,9), XJAC(2,2), XJACI(2,2), DJACB1
COMMON/BMX/BM(3,12), P1, P2, P3, P4, Q1, Q2, Q3, Q4, R1, R2, R3, R4, S1, S2, S3,
1      S4, T1, T2, T3, T4
CALL SHAPE(ETA, XI)
CALL JACOB1(X1, X2, X3, X4, Y1, Y2, Y3, Y4, XI, ETA)
DELTA=X1*(Y2-Y4)+X2*(Y3-Y1)+X3*(Y4-Y2)+X4*(Y1-Y3)
BM(1,1) = P1/4.*CD(1,5) - P4/4.*CD(1,8) +
1 3./(2.*DELTA)*(Y4-Y2)*CD(1,9)
BM(1,2) = -CD(1,1) + (S1-2.)/4.*CD(1,5) + (S4-2.)/4.*
1 CD(1,8) + (Y4-Y2)*(X2+X4-2.*X1)/(4.*DELTA)*CD(1,9)
BM(1,3) = -Q1/4.*CD(1,5) - Q4/4.*CD(1,8) + (Y4-Y2)*
1 (Y2+Y4-2.*Y1)/(4.*DELTA)*CD(1,9)
BM(1,4) = -P1/4.*CD(1,5) + P2/4.*CD(1,6) + 3./(2.*DELTA)
1 *(Y1-Y3)*CD(1,9)
BM(1,5) = -CD(1,2) + (S1-2.)/4.*CD(1,5) + (S2-2.)/4.*
1 CD(1,6) + (Y1-Y3)*(X1+X3-2.*X2)/(4.*DELTA)*CD(1,9)
BM(1,6) = -Q1/4.*CD(1,5) - Q2/4.*CD(1,6) + (Y1-Y3)*
1 (Y1+Y3-2.*Y2)/(4.*DELTA)*CD(1,9)
BM(1,7) = -P2/4.*CD(1,6) + P3/4.*CD(1,7) + 3./(2.*DELTA)
1 *(Y2-Y4)*CD(1,9)
BM(1,8) = -CD(1,3) + (S2-2.)/4.*CD(1,6) + (S3-2.)/4.*

```

$$1 \quad CD(1,7) + (Y2-Y4)*(X2+X4-2.*X3)/(4.*DELTA)*CD(1,9)$$

$$BM(1,9) = -Q2/4.*CD(1,6) - Q3/4.*CD(1,7) + (Y2-Y4)*$$

$$1 \quad (Y2+Y4-2.*Y3)/(4.*DELTA)*CD(1,9)$$

$$BM(1,10) = -P3/4.*CD(1,7) + P4/4.*CD(1,8) + 3./(2.*DELTA)$$

$$1 \quad *(Y3-Y1)*CD(1,9)$$

$$BM(1,11) = -CD(1,4) + (S3-2.)/4.*CD(1,7) + (S4-2.)/4.*$$

$$1 \quad CD(1,8) + (Y3-Y1)*(X1+X3-2.*X4)/(4.*DELTA)*CD(1,9)$$

$$BM(1,12) = -Q3/4.*CD(1,7) - Q4/4.*CD(1,8) + (Y3-Y1)*$$

$$1 \quad (Y1+Y3-2.*Y4)/(4.*DELTA)*CD(1,9)$$

$$BM(2,1) = -T1/4.*CD(2,5) + T4/4.*CD(2,8) + 3./(2.*DELTA)$$

$$1 \quad *(X2-X4)*CD(2,9)$$

$$BM(2,2) = -Q1/4.*CD(2,5) - Q4/4.*CD(2,8) + (X2-X4)*$$

$$1 \quad (X2+X4-2.*X1)/(4.*DELTA)*CD(2,9)$$

$$BM(2,3) = -CD(2,1) + (R1-2.)/4.*CD(2,5) + (R4-2.)/4.*$$

$$1 \quad CD(2,8) + (X2-X4)*(Y2+Y4-2.*Y1)/(4.*DELTA)*CD(2,9)$$

$$BM(2,4) = T1/4.*CD(2,5) - T2/4.*CD(2,6) + 3./(2.*DELTA)$$

$$1 \quad *(X3-X1)*CD(2,9)$$

$$BM(2,5) = -Q1/4.*CD(2,5) - Q2/4.*CD(2,6) + (X3-X1)*$$

$$1 \quad (X1+X3-2.*X2)/(4.*DELTA)*CD(2,9)$$

$$BM(2,6) = -CD(2,2) + (R1-2.)/4.*CD(2,5) + (R2-2.)/4.*$$

$$1 \quad CD(2,6) + (X3-X1)*(Y1+Y3-2.*Y2)/(4.*DELTA)*CD(2,9)$$

$$BM(2,7) = T2/4.*CD(2,6) - T3/4.*CD(2,7) + 3./(2.*DELTA)$$

$$1 \quad *(X4-X2)*CD(2,9)$$

$$BM(2,8) = -Q2/4.*CD(2,6) - Q3/4.*CD(2,7) + (X4-X2)*$$

$$1 \quad (X2+X4-2.*X3)/(4.*DELTA)*CD(2,9)$$

$$BM(2,9) = -CD(2,3) + (R2-2.)/4.*CD(2,6) + (R3-2.)/4.*$$

$$1 \quad *CD(2,7) + (X4-X2)*(Y2+Y4-2.*Y3)/(4.*DELTA)*CD(2,9)$$

$$BM(2,10) = T3/4.*CD(2,7) - T4/4.*CD(2,8) + 3./(2.*DELTA)$$

$$1 \quad *(X1-X3)*CD(2,9)$$

$$BM(2,11) = -Q3/4.*CD(2,7) - Q4/4.*CD(2,8) + (X1-X3)*$$

$$1 \quad (X1+X3-2.*X4)/(4.*DELTA)*CD(2,9)$$

$$BM(2,12) = -CD(2,4) + (R3-2.)/4.*CD(2,7) + (R4-2.)/4.*$$

$$1 \quad *CD(2,8) + (X1-X3)*(Y1+Y3-2.*Y4)/(4.*DELTA)*CD(2,9)$$

$$BM(3,1) = P1/4.*CD(2,5) - P4/4.*CD(2,8) +$$

$$1 \quad 3./(2.*DELTA)*(Y4-Y2)*CD(2,9)$$

$$2 \quad -T1/4.*CD(1,8) + T4/4.*CD(1,8) + 3./(2.*DELTA)$$

$$3 \quad *(X2-X4)*CD(1,9)$$

$$BM(3,2) = -CD(2,1) + (S1-2.)/4.*CD(2,5) + (S4-2.)/4.*$$

$$1 \quad CD(2,8) + (Y4-Y2)*(X2+X4-2.*X1)/(4.*DELTA)*CD(2,9)$$

$$2 \quad -Q1/4.*CD(1,5) - Q4/4.*CD(1,8) + (X2-X4)*$$

$$3 \quad (X2+X4-2.*X1)/(4.*DELTA)*CD(1,9)$$

$$BM(3,3) = -Q1/4.*CD(2,5) - Q4/4.*CD(2,8) + (Y4-Y2)*$$

$$1 \quad (Y2+Y4-2.*Y1)/(4.*DELTA)*CD(2,9)$$

$$2 \quad -CD(1,1) + (R1-2.)/4.*CD(1,5) + (R4-2.)/4.*$$

$$3 \quad QD(1,8) + (X2-X4)*(Y2+Y4-2.*Y1)/(4.*DELTA)*CD(1,9)$$

$$BM(3,4) = -P1/4.*CD(2,5) + P2/4.*CD(2,6) + 3./(2.*DELTA)$$

$$1 \quad *(Y1-Y3)*CD(2,9) +$$

$$2 \quad T1/4.*CD(1,5) - T2/4.*CD(1,6) + 3./(2.*DELTA)$$

$$3 \quad *(X3-X1)*CD(1,9)$$

$$BM(3,5) = -CD(2,2) + (S1-2.)/4.*CD(2,5) + (S2-2.)/4.*$$

$$1 \quad CD(2,6) + (Y1-Y3)*(X1+X3-2.*X2)/(4.*DELTA)*CD(2,9) -$$

$$2 \quad Q1/4.*CD(1,5) - Q2/4.*CD(1,6) + (X3-X1)*$$

$$3 \quad (X1+X3-2.*X2)/(4.*DELTA)*CD(1,9)$$

$$BM(3,6) = -Q1/4.*CD(2,5) - Q2/4.*CD(2,6) + (Y1-Y3)*$$

$$1 \quad (Y1+Y3-2.*Y2)/(4.*DELTA)*CD(2,9)$$

$$2 \quad -CD(1,2) + (R1-2.)/4.*CD(1,5) + (R2-2.)/4.*$$

$$3 \quad CD(1,6) + (X3-X1)*(Y1+Y3-2.*Y2)/(4.*DELTA)*CD(1,9)$$

$$BM(3,7) = -P2/4.*CD(2,6) + P3/4.*CD(2,7) + 3./(2.*DELTA)$$

$$1 \quad *(Y2-Y4)*CD(2,9) +$$

$$2 \quad T2/4.*CD(1,6) - T3/4.*CD(1,7) + 3./(2.*DELTA)$$

$$3 \quad *(X4-X2)*CD(1,9)$$

$$BM(3,8) = -CD(2,3) + (S2-2.)/4.*CD(2,6) + (S3-2.)/4.*$$

$$1 \quad CD(2,7) + (Y2-Y4)*(X2+X4-2.*X3)/(4.*DELTA)*CD(2,9)$$

$$2 \quad -Q2/4.*CD(1,6) - Q3/4.*CD(1,7) + (X4-X2)*$$

3 $(X2+X4-2.*X3)/(4.*DELTA)*CD(1,9)$

BM(3,9) = $-Q2/4.*CD(2,6) - Q3/4.*CD(2,7) + (Y2-Y4)*$

1 $(Y2+Y4-2.*Y3)/(4.*DELTA)*CD(2,9)$

2 $-CD(1,3) + (R2-2.)/4.*CD(1,6) + (R3-2.)/4.$

3 $*CD(1,7) + (X4-X2)*(Y2+Y4-2.*Y3)/(4.*DELTA)*CD(1,9)$

BM(3,10) = $-P3/4.*CD(2,7) + P4/4.*CD(2,8) + 3./(2.*DELTA)$

1 $*(Y3-Y1)*CD(2,9) +$

2 $T3/4.*CD(1,7) - T4/4.*CD(1,8) + 3./(2.*DELTA)$

3 $*(X1-X3)*CD(1,9)$

BM(3,11) = $-CD(2,4) + (S3-2.)/4.*CD(2,7) + (S4-2.)/4.*$

1 $CD(2,8) + (Y3-Y1)*(X1+X3-2.*X4)/(4.*DELTA)*CD(2,9)$

2 $-Q3/4.*CD(1,7) - Q4/4.*CD(1,8) + (X1-X3)*$

3 $(X1+X3-2.*X4)/(4.*DELTA)*CD(1,9)$

BM(3,12) = $-Q3/4.*CD(2,7) - Q4/4.*CD(2,8) + (Y3-Y1)*$

1 $(Y1+Y3-2.*Y4)/(4.*DELTA)*CD(2,9)$

2 $-CD(1,4) + (R3-2.)/4.*CD(1,7) + (R4-2.)/4.$

3 $*CD(1,8) + (X1-X3)*(Y1+Y3-2.*Y4)/(4.*DELTA)*CD(1,9)$

RETURN

END

APPENDIX-K

SUBROUTINE ELSTIP(X1,X2,X3,Y1,Y2,Y3,Z,S,CUR)

```

C***
C*** Element Stiffness Matrix Subroutine for Laminated Anisotropic High
C*** Precision Shallow Shell Element
C***
C*** INPUT
C***
C*** X1,X2,X3,Y1,Y2,Y3 are the x and y coordinates of the vertices of
C*** the triangle.
C*** AAM,ADM,ABM are the membrane,bending and membrane-bending coupling
C*** stiffness matrices of the laminate.
C***
C*** OUTPUT
C***
C*** ESM is the (38 X 38) element stiffness matrix
C***
COMMON/ST/ESM(38,38)
COMMON/CODE/KOD(38)
COMMON/ABD/AAM(3,3),ABM(3,3),ADM(3,3)
DIMENSION P(3,6),Q(3,3),AM(5,3),BM(5,3),DM(3,3),R(9,10),
1 S(21,18),Z(6),ZV(3),CUR(3)
DATA KOD/1,2,3,13,14,15,25,26,27,37,4,5,6,16,17,18,28,29,30,38,
1 7,8,9,10,11,12,19,20,21,22,23,24,31,32,33,34,35,36/
B1=Y2-Y3
B2=Y3-Y1
B3=Y1-Y2
C1=X3-X2
C2=X1-X3

```

C3=X2-X1

DET=ABS(B1*C2-B2*C1)

Q(1,1)=B1*B1

Q(1,2)=B2*B2

Q(1,3)=2.*B1*B2

Q(2,1)=C1*C1

Q(2,2)=C2*C2

Q(2,3)=2.*C1*C2

Q(3,1)=2.*B1*C1

Q(3,2)=2.*B2*C2

Q(3,3)=2.*(B1*C2+B2*C1)

ZV(1)=4.*(Z(1)+Z(3))-8.*Z(6)

ZV(2)=4.*(Z(2)+Z(3))-8.*Z(5)

ZV(3)=4.*(Z(3)+Z(4)-Z(5)-Z(6))

DO 15 I=1,3

CUR(I)=0.

DO 15 J=1,3

15 CUR(I)=CUR(I)+Q(I,J)*ZV(J)/DET**2

P(1,1)=B1

P(1,2)=B2

P(1,3)=0.

P(1,4)=0.

P(1,5)=CUR(1)*DET

P(2,1)=0.

P(2,2)=0.

P(2,3)=C1

P(2,4)=C2

P(2,5)=CUR(2)*DET

P(3,1)=C1

P(3,2)=C2

P(3,3)=B1

P(3,4)=B2

```

P(3,5)=CUR(3)*DET
CALL TRMULT(P, AAM, P, AM, 5, 3, 3, 5, 1)
CALL TRMULT(Q, ADM, Q, DM, 3, 3, 3, 3, 1)
CALL TRMULT(P, ABM, Q, BM, 5, 3, 3, 3, 0)
DO 25 I=1,5
DO 25 J=1,5
25 AM(I,J)=AM(I,J)/DET
DO 30 I=1,3
DO 30 J=1,3
30 DM(I,J)=DM(I,J)/DET**3
DO 35 I=1,5
DO 35 J=1,3
35 BM(I,J)=BM(I,J)/DET**2
CALL RMAT(B1,B2,B3,C1,C2,C3,R)
CALL SMAT(B1,B2,B3,C1,C2,C3,S)
CALL UV(R,AM)
CALL UVW(AM,BM,R,S)
CALL WW(S,DM(1,1),DM(1,2),DM(1,3),DM(2,2),DM(2,3),DM(3,3),AM(5,5),
1      BM(5,1),BM(5,2),BM(5,3))
RETURN
END
SUBROUTINE RMAT(B1,B2,B3,C1,C2,C3,R)

```

```

DIMENSION R(9,10)
DO 25 I=1,9
DO 25 J=1,10
25 R(I,J)=0.
R(1,8)=C2
R(1,9)=-B2
R(2,8)=-C1

```


$$R(2,9)=B1$$

$$R(3,1)=3.$$

$$R(3,2)=-C2$$

$$R(3,3)=B2$$

$$R(3,7)=-3.$$

$$R(3,8)=-2.*C2$$

$$R(3,9)=2.*B2$$

$$R(4,1)=-7.$$

$$R(4,2)=C1+2.*C2$$

$$R(4,3)=-B1-2.*B2$$

$$R(4,4)=-7.$$

$$R(4,5)=-C2-2.*C1$$

$$R(4,6)=B2+2.*B1$$

$$R(4,7)=-13..$$

$$R(4,8)=3.*(C1-C2)$$

$$R(4,9)=3.*(B2-B1)$$

$$R(4,10)=27.$$

$$R(5,4)=3.$$

$$R(5,5)=C1$$

$$R(5,6)=-B1$$

$$R(5,7)=-3.$$

$$R(5,8)=2.*C1$$

$$R(5,9)=-2.*B1$$

$$R(6,1)=-2.$$

$$R(6,2)=C2$$

$$R(6,3)=-B2$$

$$R(6,7)=2.$$

$$R(6,8)=C2$$

$$R(6,9)=-B2$$

$$R(7,1)=7.$$

$$R(7,2)=-2.*(C1+C2)$$

$$R(7,3)=2.*(B1+B2)$$

R(7,4)=7.

R(7,5)=C2+2.*C1

R(7,6)=-B2-2.*B1

R(7,7)=13.

R(7,8)=3.*C2-2.*C1

R(7,9)=-3.*B2+2.*B1

R(7,10)=-27.

R(8,1)=7.

R(8,2)=-C1-2.*C2

R(8,3)=B1+2.*B2

R(8,4)=7.

R(8,5)=2.*(C1+C2)

R(8,6)=-2.*(B1+B2)

R(8,7)=13.

R(8,8)=2.*C2-3.*C1

R(8,9)=-2.*B2+3.*B1

R(8,10)=-27.

R(9,4)=-2.

R(9,5)=-C1

R(9,6)=B1

R(9,7)=2.

R(9,8)=-C1

R(9,9)=B1

RETURN

END

SUBROUTINE SMAT(B1,B2,B3,C1,C2,C3,S)

DIMENSION BC(2,3),AB(3,4),S(21,18)

D1=-(B1*B3+C1*C3)/(B3*B3+C3*C3)

D2=(B1*B2+C1*C2)/(B1*B1+C1*C1)

```
D3=(B1*B2+C1*C2)/(B2*B2+C2*C2)
DO 20 I=1,21
DO 20 J=1,18
20 S(I,J)=0.
S(1,13)=1.
S(2,14)=C2
S(2,15)=-B2
S(3,14)=-C1
S(3,15)=B1
S(7,1)=10.
S(8,1)=-30.*D3
S(11,1)=-15.
S(12,1)=60.*D3
S(13,1)=30.*(D1+2.*D3)
S(16,1)=6.
S(17,1)=-30.*D3
S(18,1)=-30.*(D1+2.*D3)
S(19,1)=-30.*(D3+D1)
S(9,7)=-30.*D2
S(10,7)=10.
S(13,7)=30.*(1.-D1+2.*D2)
S(14,7)=60.*D2
S(15,7)=-15.
S(18,7)=30.*(D1-D2-1.)
S(19,7)=30.*(D1-2.*D2-1.)
S(20,7)=-30.*D2
S(21,7)=6.
S(7,13)=-10.
S(8,13)=30.*D3
S(9,13)=30.*D2
S(10,13)=-10.
S(11,13)=5.
```

$$S(12,13)=-60.*D3$$

$$S(13,13)=-30.*(1.+2.*(D2+D3))$$

$$S(14,13)=-60.*D2$$

$$S(15,13)=15.$$

$$S(16,13)=-6.$$

$$S(17,13)=30.*D3$$

$$S(18,13)=30.*(1.+D2)+60.*D3$$

$$S(19,13)=30.*(1.+D3)+60.*D2$$

$$S(20,13)=30.*D2$$

$$S(21,13)=-6.$$

$$BC(1,1)=C1$$

$$BC(1,2)=C2$$

$$BC(1,3)=C3$$

$$BC(2,1)=-B1$$

$$BC(2,2)=-B2$$

$$BC(2,3)=-B3$$

$$AB(1,1)=C1+C1$$

$$AB(1,2)=C2+C2$$

$$AB(1,3)=C3+C3$$

$$AB(1,4)=C1+C2$$

$$AB(3,1)=B1*B1$$

$$AB(3,2)=B2*B2$$

$$AB(3,3)=B3*B3$$

$$AB(3,4)=B1*B2$$

$$AB(2,1)=-2.*B1*C1$$

$$AB(2,2)=-2.*B2*C2$$

$$AB(2,3)=-2.*B3*C3$$

$$AB(2,4)=- (B1*C2+B2*C1)$$

$$DO 79 I=1,2$$

$$II=I-1$$

$$XX=BC(I,1)$$

$$YY=BC(I,2)$$

ZZ=BC(I,3)

S(7,2+II)=-4.*YY

S(8,2+II)=15.*YY*D3-3.*XX

S(11,2+II)=7.*YY

S(12,2+II)=2.*XX-30.*YY*D3

S(13,2+II)=9.*XX+15.*ZZ*D1-30.*YY*D3

S(16,2+II)=-3.*YY

S(17,2+II)=15.*YY*D3

S(18,2+II)=30.*YY*D3-15.*ZZ*D1-9.*XX

S(19,2+II)=15.*(YY*D3-ZZ*D1)-6.*XX

S(9,8+II)=3.*YY-15.*XX*D2

S(10,8+II)=4.*XX

S(13,8+II)=15.*ZZ*D1+30.*XX*D2+6.*YY+15.*XX

S(14,8+II)=30.*XX*D2-2.*YY

S(15,8+II)=-7.*XX

S(18,8+II)=-15.*(ZZ*D1+XX*D2+XX)-9.*YY

S(19,8+II)=-15.*(ZZ*D1+XX)-30.*XX*D2-6.*YY

S(20,8+II)=-15.*XX*D2

S(21,8+II)=3.*XX

S(7,14+II)=-8.*YY

S(8,14+II)=15.*YY*D3+3.*XX

S(9,14+II)=-3.*YY-15.*XX*D2

S(10,14+II)=6.*XX

S(11,14+II)=8.*YY

S(12,14+II)=-30.*YY*D3-2.*XX

S(13,14+II)=6.*(XX-YY)+30.*(XX*D2-YY*D3)

S(14,14+II)=2.*YY+30.*XX*D2

S(15,14+II)=-8.*XX

S(16,14+II)=-3.*YY

S(17,14+II)=15.*YY*D3

S(18,14+II)=9.*YY-6.*XX+30.*YY*D3-15.*XX*D2

S(19,14+II)=6.*YY-9.*XX+15.*YY*D3-30.*XX*D2

S(20,14+II)=-15.*XX*D2

S(21,14+II)=3.*XX

79 CONTINUE

DO 56 I=1,3

II=I-1

XX=AB(I,1)

YY=AB(I,2)

ZZ=AB(I,3)

XY=AB(I,4)

S(7,4+II)=.5*YY

S(8,4+II)=XY-2.5*YY*D3

S(11,4+II)=-YY

S(12,4+II)=5.*YY*D3-XY

S(13,4+II)=2.5*ZZ*D1+5.*YY*D3-3.*XY-XX

S(16,4+II)=.5*YY

S(17,4+II)=-2.5*YY*D3

S(18,4+II)=3.*XY+1.5*XX-2.5*ZZ*D1-5.*YY*D3

S(19,4+II)=2.*XY+XX-2.5*D1*ZZ-2.5*D3*YY

S(9,10+II)=XY-2.5*XX*D2

S(10,10+II)=.5*XX

S(13,10+II)=2.5*(XX-ZZ*D1)+1.5*YY+2.*XY+5.*XX*D2

S(14,10+II)=5.*XX*D2-XY

S(15,10+II)=-XX

S(18,10+II)=2.5*ZZ*D1-2.5*XX*(D2+1.)-1.5*YY-3.*XY

S(19,10+II)=2.5*(ZZ*D1-XX)-5.*XX*D2-YY-2.*XY

S(20,10+II)=-2.5*XX*D2

S(21,10+II)=.5*XX

S(4,16+II)=.5*YY

S(5,16+II)=-XY

S(6,16+II)=.5*XX

S(7,16+II)=-1.5*YY

S(8,16+II)=2.5*YY*D3+2.*XY

```

S(9,16+II)=2.*XY+2.5*XX*D2
S(10,16+II)=-1.5*XX
S(11,16+II)=1.5*YY
S(12,16+II)=-XY-5.*YY*D3
S(13,16+II)=-YY*(1.5+5.*D3)-2.*XY-XX*(1.5+5.*D2)
S(14,16+II)=-XY-5.*XX*D2
S(15,16+II)=1.5*XX
S(16,16+II)=-.5*YY
S(17,16+II)=2.5*YY*D3
S(18,16+II)=YY*(1.5+5.*D3)+XX*(1.0+2.5*D2)
S(19,16+II)=YY*(1.+2.5*D3)+XX*(1.5+5.*D2)
S(20,16+II)=2.5*XX*D2
S(21,16+II)=-.5*XX

```

56

CONTINUE

RETURN

END

```

SUBROUTINE UV(R,AM)
COMMON/ST/ESM(38,38)
COMMON/CODE/KOD(38)
DIMENSION TEMP(10,10),F(9,9),R(9,10),AM(5,5)
CALL FMAT(AM(1,1),AM(1,2),AM(1,2),AM(2,2),F)
CALL TRMULT(R,F,R,TEMP,10,9,9,10,1)
DO 100 I=1,10
DO 100 J=1,10
100 ESM(KOD(I),KOD(J))=TEMP(I,J)
CALL FMAT(AM(1,3),AM(1,4),AM(2,3),AM(2,4),F)
CALL TRMULT(R,F,R,TEMP,10,9,9,10,0)
DO 150 I=1,10
DO 150 J=1,10

```

```

    ESM(KOD(I),KOD(J+10))=TEMP(I,J)
150 ESM(KOD(J+10),KOD(I))=ESM(KOD(I),KOD(J+10))
    CALL FMAT(AM(3,3),AM(3,4),AM(3,4),AM(4,4),F)
    CALL TRMULT(R,F,R,TEMP,10,9,9,10,1)
    DO 200 I=1,10
    DO 200 J=1,10
200 ESM(KOD(I+10),KOD(J+10))=TEMP(I,J)
    RETURN
    END

```

```

SUBROUTINE UVW(AM,BM,R,S)
COMMON/ST/ESM(38,38)
COMMON/CODE/KOD(38)
    DIMENSION G(9,21),TEMP(10,18),AM(5,5),BM(5,3),R(9,10),S(21,18)
    CALL GMAT(BM(1,1),BM(1,2),BM(1,3),BM(2,1),BM(2,2),BM(2,3),AM(1,5),
1      AM(2,5),G)
    CALL TRMULT(R,G,S,TEMP,10,9,21,18,0)
    DO 10 I=1,10
    DO 10 J=1,18
    ESM(KOD(I),KOD(J+20))=-TEMP(I,J)
10 ESM(KOD(J+20),KOD(I))=ESM(KOD(I),KOD(J+20))
    CALL GMAT(BM(3,1),BM(3,2),BM(3,3),BM(4,1),BM(4,2),BM(4,3),AM(3,5),
1      AM(4,5),G)
    CALL TRMULT(R,G,S,TEMP,10,9,21,18,0)
    DO 20 I=1,10
    DO 20 J=1,18
    ESM(KOD(I+10),KOD(J+20))=-TEMP(I,J)
20 ESM(KOD(J+20),KOD(I+10))=ESM(KOD(I+10),KOD(J+20))
    RETURN
    END

```


SUBROUTINE WW(S,A,B,C,D,E,F,G,P,Q,R)

COMMON/ST/ESM(38,38)

COMMON/COE/KOD(38)

DIMENSION TEMP(18,18),H(21,21),S(21,18)

H(1,1)=0.5*G

H(1,2)=G/6.

H(1,3)=G/6.

H(1,4)=G/12.+P

H(1,5)=G/24.+0.5*R

H(1,6)=G/12.+Q

H(1,7)=G/20.+P

H(1,8)=G/60.+(P+R)/3.

H(1,9)=G/60.+(Q+R)/3.

H(1,10)=G/20.+Q

H(1,11)=G/30.+P

H(1,12)=G/120.+0.25*(P+R)

H(1,13)=G/180.+(P+Q+R)/6.

H(1,14)=G/120.+0.25*(Q+R)

H(1,15)=G/30.+Q

H(1,16)=G/42.+P

H(1,17)=G/210.+0.2*(P+R)

H(1,18)=G/420.+0.1*(P+Q+R)

H(1,19)=G/420.+0.1*(P+Q+R)

H(1,20)=G/210.+0.2*(Q+R)

H(1,21)=G/42.+Q

H(2,2)=G/12.

H(2,3)=G/24.

H(2,4)=G/20.+P/3.

H(2,5)=G/60.+R/6.

$$H(2,6)=G/60.+Q/3.$$

$$H(2,7)=G/30.+0.5*P$$

$$H(2,8)=G/120.+P/12.+R/6.$$

$$H(2,9)=G/180.+Q/6.+R/12.$$

$$H(2,10)=G/120.+0.25*Q$$

$$H(2,11)=G/42.+0.6*P$$

$$H(2,12)=G/210.+0.1*P+0.15*R$$

$$H(2,13)=G/420.+P/30.+0.1*Q+R/15.$$

$$H(2,14)=G/420.+0.1*Q+0.05*R$$

$$H(2,15)=G/210.+0.2*Q$$

$$H(2,16)=G/56.+2.*P/3.$$

$$H(2,17)=G/336.+0.1*P+R/7.5$$

$$H(2,18)=G/840.+P/30.+Q/15.+0.05*R$$

$$H(2,19)=G/1120.+P/60.+0.05*Q+R/30.$$

$$H(2,20)=G/840.+Q/15.+R/30$$

$$H(2,21)=G/336.+Q/6.$$

$$H(3,3)=G/12.$$

$$H(3,4)=G/60.+P/3.$$

$$H(3,5)=G/60.+R/6.$$

$$H(3,6)=G/20.+Q/3.$$

$$H(3,7)=G/120.+0.25*P$$

$$H(3,8)=G/180.+P/6.+R/12.$$

$$H(3,9)=G/120.+Q/12.+R/6.$$

$$H(3,10)=G/30.+0.5*Q$$

$$H(3,11)=G/210.+0.2*P$$

$$H(3,12)=G/420.+0.1*P+0.05*R$$

$$H(3,13)=G/420.+Q.1*P+Q/30.+R/15.$$

$$H(3,14)=G/210.+0.1*Q+0.15*R$$

$$H(3,15)=G/42.+0.6*Q$$

$$H(3,16)=G/336.+P/6.$$

$$H(3,17)=G/840.+P/15.+R/30.$$

$$H(3,18)=G/1120.+0.05*P+Q/60.+R/30.$$

$$H(3,19)=G/840.+P/15.+Q/30.+0.05*R$$

$$H(3,20)=G/336.+Q/1*Q+R/7.5$$

$$H(3,21)=G/56.+Q/1.5$$

$$H(4,4)=2.*A+G/30.+P/3.$$

$$H(4,5)=C+G/120.+(P+R)/12.$$

$$H(4,6)=2.*B+G/180.+(P+Q)/6.$$

$$H(4,7)=2.*A+G/42.+0.4*P$$

$$H(4,8)=2.*(A+C)/3.+G/210.+P/15.+R/10.$$

$$H(4,9)=2.*(B+C)/3.+G/420.+(P+R)/30.+0.1*Q$$

$$H(4,10)=2.*B+G/420.+0.1*(P+Q)$$

$$H(4,11)=2.*A+G/56.+7.*P/15.$$

$$H(4,12)=.5*(A+C)+G/336.+P/15.+0.1*R$$

$$H(4,13)=(A+B+C)/3.+G/840.+P/45.+Q/15.+R/30.$$

$$H(4,14)=.5*(B+C)+G/1120.+(P+R)/60.+Q/0.05*Q$$

$$H(4,15)=2.*B+G/840.+(P+Q)/15.$$

$$H(4,16)=2.*A+G/72.+11.*P/21.$$

$$H(4,17)=.4*(A+C)+G/504.+P/15.+R/10.5.$$

$$H(4,18)=0.2*(A+B+C)+G/1512.+P/52.5+Q/21.+R/35.$$

$$H(4,19)=0.2*(A+B+C)+G/2520.+P/105.+Q/35.+R/70.$$

$$H(4,20)=0.4*(B+C)+G/2520.+(P+R)/105.+Q/35.$$

$$H(4,21)=2.*B+G/1512.+(P+Q)/21.$$

$$H(5,5)=.5*F+G/180.+R/12.$$

$$H(5,6)=E+G/120.+(Q+R)/12.$$

$$H(5,7)=C+G/210.+0.1*P+0.05*R$$

$$H(5,8)=(C+F)/3.+G/420.+P/30.+0.05*R$$

$$H(5,9)=(E+F)/3.+G/420.+Q/30.+0.05*R$$

$$H(5,10)=E+G/210.+0.1*Q+0.05*R$$

$$H(5,11)=C+G/836.+0.1*P+R/30.$$

$$H(5,12)=.25*(C+F)+G/840.+(P+R)/30.$$

$$H(5,13)=(C+E+F)/6.+G/1120.+(P+Q)/60.+R/36.$$

$$H(5,14)=.25*(E+F)+G/840.+(Q+R)/30.$$

$$H(5,15)=E+G/336.+0.1*Q+R/30.$$

$$H(5,16)=C+G/504.+P/10.5+R/42..$$

$$H(5,17)=0.2*(C+F)+G/1512.+P/35.+R/42.$$

$$H(5,18)=0.1*(C+E+F)+G/2520.+P/70.+Q/105.+R/60..$$

$$H(5,19)=0.1*(C+E+F)+G/2520.+P/105.+Q/70.+R/60.$$

$$H(5,20)=0.2*(E+F)+G/1512.+Q/35.+R/42.$$

$$H(5,21)=E+G/504.+Q/10.5+R/42.$$

$$H(6,6)=2.*D+G/30.+Q/3.$$

$$H(6,7)=2.*B+G/420.+0.1*(P+Q)$$

$$H(6,8)=2.*(B+E)/3.+G/420.+0.1*P+(Q+R)/30.$$

$$H(6,9)=2.*(D+E)/3.+G/210.+Q/15.+0.1*R$$

$$H(6,10)=2.*D+G/42.+0.4*Q$$

$$H(6,11)=2.*B+G/840.+(P+Q)/15.$$

$$H(6,12)=.5*(B+E)+G/1120.+0.05*P+(Q+R)/60.$$

$$H(6,13)=(B+D+E)/3.+G/840.+P/15.+Q/45.+R/30.$$

$$H(6,14)=.5*(D+E)+G/336.+Q/15.+0.1*R$$

$$H(6,15)=2.*D+G/56.+7.*Q/15.$$

$$H(6,16)=2.*B+G/1512.+(P+Q)/21.$$

$$H(6,17)=0.4*(B+E)+G/2520.+P/35.+(Q+R)/105.$$

$$H(6,18)=0.2*(B+D+E)+G/2520.+P/35.+Q/105.+R/70.$$

$$H(6,19)=0.2*(B+D+E)+G/1512.+P/21.+Q/52.5+R/35.$$

$$H(6,20)=0.4*(D+E)+G/504.+Q/15.+R/10.5$$

$$H(6,21)=2.*D+G/72.+11.*Q/21.$$

$$H(7,7)=3.*A+G/56.+P/2.5$$

$$H(7,8)=.5*A+C+G/336.+(P+R)/15..$$

$$H(7,9)=B+.5*C+G/840.+P/30.+Q/15.+R/60.$$

$$H(7,10)=1.5*B+G/1120.+(P+Q)/20.$$

$$H(7,11)=3.6*A+G/72.+3.*P/7.$$

$$H(7,12)=0.6*A+0.9*C+G/504.+P/17.5+R/14.$$

$$H(7,13)=0.2*A+0.6*B+0.4*C+G/1512.+(P+R)/52.5+Q/21.$$

$$H(7,14)=0.6*B+0.3*C+G/2520.+P/70.+Q/35.+R/140.$$

$$H(7,15)=1.2*B+G/2520.+(P+Q)/35..$$

$$H(7,16)=4.*A+G/90.+13.*P/28.$$

$$H(7,17) = 0.6A + 0.8C + G/720 + P/56 + R/14.$$

$$H(7,18) = 0.2A + 0.4B + 0.3C + G/2520 + P/70 + Q/28 + R/56.$$

$$H(7,19) = 0.1A + 0.3B + 0.2C + G/5040 + (P+R)/140 + Q/56.$$

$$H(7,20) = 0.4B + 0.2C + G/6300 + P/140 + Q/70 + R/280.$$

$$H(7,21) = B + G/5040 + (P+Q)/56.$$

$$H(8,8) = (A+C+F)/3 + G/840 + P/45 + R/30.$$

$$H(8,9) = (B+F)/6 + (C+E)/3 + G/1120 + (P+Q)/60 + R/45.$$

$$H(8,10) = B + .5E + G/840 + P/15 + Q/30 + R/60.$$

$$H(8,11) = 0.4A + 1.2C + G/504 + P/15 + R/21.$$

$$H(8,12) = 0.2A + 0.3(C+F) + G/1512 + P/52.5 + R/42.$$

$$H(8,13) = (.2B + .4F)/3 + 0.2(A+C+E) + G/2520 + (P+Q)/105 + R/70.$$

$$H(8,14) = 0.2(B+E) + 0.1F + 0.3C + G/2520 + P/105 + Q/70 + R/84.$$

$$H(8,15) = 1.2B + 0.4E + G/1512 + P/21 + Q/35 + R/105.$$

$$H(8,16) = (A + 4C)/3 + G/720 + 11P/168 + R/28.$$

$$H(8,17) = (.4A + .8(C+F))/3 + G/2520 + P/60 + R/56.$$

$$H(8,18) = (.3(A+F) + .1B + 0.4(C+E))/3 + G/5040 + P/140 + Q/168 + R/105.$$

$$H(8,19) = (0.2(B+F) + 0.4(A+C) + 0.3E)/3 + G/6300 + P/210 + (Q+R)/140.$$

$$H(8,20) = (.6B + .2F + 0.8C + 0.4E)/3 + G/5040 + P/168 + 3Q/280 + R/140.$$

$$H(8,21) = (4B + E)/3 + G/2520 + P/28 + Q/42 + R/168.$$

$$H(9,9) = (D+E+F)/3 + G/840 + Q/45 + R/30.$$

$$H(9,10) = .5D + E + G/336 + (Q+R)/15.$$

$$H(9,11) = 1.2B + 0.4C + G/1512 + P/35 + Q/21 + R/105.$$

$$H(9,12) = 0.2(B+C) + 0.1F + 0.3E + G/2520 + P/70 + Q/105 + R/84.$$

$$H(9,13) = (.2B + .4F)/3 + 0.2(C+D+E) + G/2520 + (P+Q)/105 + R/70.$$

$$H(9,14) = 0.2D + 0.3(E+F) + G/1512 + Q/52.5 + R/42.$$

$$H(9,15) = 0.4D + 1.2E + G/504 + Q/15 + R/21.$$

$$H(9,16) = (4B + C)/3 + G/2520 + P/42 + Q/28 + R/168.$$

$$H(9,17) = 0.2B + (.2F + .4C + .8E)/3 + G/5040 + 3P/280 + Q/168 + R/140.$$

$$H(9,18) = (.2(B+F) + 0.4(D+E))/3 + .1C + G/6300 + (P+R)/140 + Q/210.$$

$$H(9,19) = (.1B + 0.4(C+E))/3 + 0.1(D+F) + G/5040 + P/168 + Q/140 + R/105.$$

$$H(9,20) = (.4D + 0.8(E+F))/3 + G/2520 + Q/60 + R/56.$$

$$H(9,21) = (D + 4E)/3 + G/720 + 11Q/168 + R/28.$$

$$H(10,10)=3.*D+G/56.+0.4*Q$$

$$H(10,11)=1.2*B+G/2520.+(P+Q)/35.$$

$$H(10,12)=0.6*B+0.3*E+G/2520.+P/35.+Q/70.+R/140.$$

$$H(10,13)=0.6*B+0.2*D+0.4*E+G/1512.+P/21.+(Q+R)/52.5$$

$$H(10,14)=0.6*D+0.9*E+G/504.+Q/17.5+R/14.$$

$$H(10,15)=3.6*D+G/72.+3.*Q/7.$$

$$H(10,16)=B+G/5040.+(P+Q)/56.$$

$$H(10,17)=0.4*B+0.2*E+G/6300.+P/70.+Q/140.+R/280.$$

$$H(10,18)=0.3*B+0.1*D+0.2*E+G/5040.+P/56.+(Q+R)/140.$$

$$H(10,19)=0.4*B+0.2*D+0.3*E+G/2520.+P/28.+Q/70.+R/56.$$

$$H(10,20)=0.6*D+0.8*E+G/720.+3.*Q/56.+R/14.$$

$$H(10,21)=4.*D+G/90.+13.*Q/28.$$

$$H(11,11)=4.8*A+G/90.+3.*P/7.$$

$$H(11,12)=0.6*A+1.2*C+G/720.+3.*(P+R)/56.$$

$$H(11,13)=0.4*A/3.+0.8*B+0.4*C+G/2520.+P/60.+Q/28.+R/84.$$

$$H(11,14)=0.6*B+0.2*C+G/5040.+3.*P/280.+Q/56.+R/280.$$

$$H(11,15)=0.8*B+G/6300.+(P+Q)/70.$$

$$H(11,16)=40.*A/7.+G/110.+P/2.25$$

$$H(11,17)=24.*A/35.+8.*C/7.+G/990.+P/21.+R/18.$$

$$H(11,18)=(6.*A+12.*C)/35.+4.*B/7.+G/3960.+(P+R)/84.+Q/36.$$

$$H(11,19)=(2.*A+12.*B+6.*C)/35.+G/9240.+P/180.+Q/84.+R/252.$$

$$H(11,20)=(12.*B+4.*C)/35.+G/13860.+P/210.+Q/126.+R/630.$$

$$H(11,21)=20.*B/35.+G/13860.+(P+Q)/126.$$

$$H(12,12)=0.2*A+0.3*(C+F)+G/2520.+P/70.+R/56.$$

$$H(12,13)=0.1*(A+B+F)+0.2*E+C/6.+G/5040.+P/140.+Q/168.+R/120.$$

$$H(12,14)=0.2*B+.05*F+0.15*(C+E)+G/6300.+(P+Q)/140.+3.*R/560.$$

$$H(12,15)=0.6*B+0.2*E+G/5040.+P/56.+3.*Q/280.+R/280.$$

$$H(12,16)=20.*A/35.+10.*C/7.+G/990.+13.*P/252.+R/24.$$

$$H(12,17)=(6.*A+10.*C)/35.+2.*F/7.+G/3960.+P/84.+R/72.$$

$$H(12,18)=(3.*A+4.5*C+2.*B+3.*F+5.*E)/35.+G/9240.+P/210.+Q/252.$$

$$1 \quad +R/168.$$

$$H(12,19)=(2.*A+3.*B+1.5*F+3.5*C+3.*E)/35.+G/13860.+P/315.+Q/252.$$

$$1 \quad +R/280.$$

$$H(12,20) = (6.*B+F+4.*C+3.*E)/35.+G/13860.+P/252.+Q/210.+R/360.$$

$$H(12,21) = (20.*B+5.*E)/35.+G/9240.+P/84.+Q/128.+R/504.$$

$$H(13,13) = (2.*B+4.*F)/45.+4*(A+C+D+E)/3.+G/6300.+(P+Q)/210.+R/140.$$

$$H(13,14) = 0.1*(B+F+D)+0.2*C+E/6.+G/5040.+P/168.+Q/140.+R/120.$$

$$H(13,15) = 0.8*B+0.4*D/3.+0.4*E+G/2520.+P/28.+Q/60.+R/84.$$

$$H(13,16) = (10.*A+40.*C)/105.+20.*B/21.+G/3960.+11.*P/756.+Q/36.$$

$$1 \quad +R/126.$$

$$H(13,17) = (6.*A+12.*B+8.*F+14.*C)/105.+4.*E/21.+G/9240.+P/180.$$

$$1 \quad +Q/252.+R/189.$$

$$H(13,18) = (6.*A+4.*B+6.*F+9.*C+10.*E)/105.+2.*D/21.+G/13860.+P/315.$$

$$1 \quad +Q/378.+R/252.$$

$$H(13,19) = 2.*A/21.+(4.*B+6.*F+10.*C+6.*D+9.*E)/105.+G/13860..$$

$$1 \quad +P/378.+Q/315.+R/252.$$

$$H(13,20) = (12.*B+8.*F+6.*D+14.*E)/105.+4.*C/21.+G/9240.+P/252.$$

$$1 \quad +Q/180.+R/189.$$

$$H(13,21) = 20.*B/21.+(10.*D+40.*E)/105.+G/3960.+P/36.+11.*Q/756.$$

$$1 \quad +R/126.$$

$$H(14,14) = 0.2*D+0.3*(E+F)+G/2520.+Q/70.+R/56.$$

$$H(14,15) = 0.6*D+1.2*E+G/720.+3.*(Q+R)/56.$$

$$H(14,16) = (60.*B+15.*C)/105.+G/9240.+P/126.+Q/84.+R/504.$$

$$H(14,17) = (18.*B+3.*F+9.*C+12.*E)/105.+G/13860.+P/210.+Q/252.$$

$$1 \quad +R/360.$$

$$H(14,18) = (9.*B+4.5*F+9.*C+6.*D+10.5*E)/105.+G/13860.+P/252.$$

$$1 \quad +Q/315.+R/280.$$

$$H(14,19) = (6.*B+9.*F+D)+13.5*E)/105.+C/7.+G/9240.+P/252.+Q/210.$$

$$1 \quad +R/168.$$

$$H(14,20) = (6.*D+10.*E)/35.+2.*F/7.+G/3960.+Q/84.+R/72.$$

$$H(14,21) = 20.*D/35.+10.*E/7.+G/990.+13.*Q/252.+R/24.$$

$$H(15,15) = 4.8*D+G/90.+3.*Q/7.$$

$$H(15,16) = 20.*B/35.+G/13860.+(P+Q)/126.$$

$$H(15,17) = (12.*B+4.*E)/35.+G/13860.+P/126.+Q/210.+R/630.$$

$$H(15,18) = (12 \cdot B + 6 \cdot E + 2 \cdot D) / 35 + G / 9240 + P / 84 + Q / 180 + R / 252$$

$$H(15,19) = 4 \cdot B / 7 + (6 \cdot D + 12 \cdot E) / 35 + G / 3960 + P / 36 + (Q + R) / 84$$

$$H(15,20) = 24 \cdot D / 35 + 8 \cdot E / 7 + G / 990 + Q / 21 + R / 18$$

$$H(15,21) = 40 \cdot D / 7 + G / 110 + Q / 2.25$$

$$H(16,16) = 50 \cdot A / 7 + G / 132 + P / 2.25$$

$$H(16,17) = (5 \cdot A + 10 \cdot C) / 7 + G / 1320 + (P + R) / 22.5$$

$$H(16,18) = 5 \cdot A / 35 + (30 \cdot B + 15 \cdot C) / 42 + G / 5940 + 13 \cdot P / 1260 + Q / 45$$

$$1 \quad + R / 120$$

$$H(16,19) = (A + 10 \cdot B) / 28 + 5 \cdot C / 35 + G / 15840 + 11 \cdot P / 2520 + Q / 120$$

$$1 \quad + R / 420$$

$$H(16,20) = (40 \cdot B + 10 \cdot C) / 140 + G / 27720 + P / 315 + Q / 210 + R / 1260$$

$$H(16,21) = 10 \cdot B / 28 + G / 33264 + (P + Q) / 252$$

$$H(17,17) = 2 \cdot (C + F) / 7 + 6 \cdot A / 35 + G / 5940 + P / 105 + R / 90$$

$$H(17,18) = (.5 \cdot (B + F) + E) / 7 + (16 \cdot C + 9 \cdot A) / 140 + G / 15840 + P / 280$$

$$1 \quad + Q / 360 + R / 252$$

$$H(17,19) = (4 \cdot (A + F) + 12 \cdot B + 10 \cdot C) / 140 + .5 \cdot E / 7 + G / 27720 + P / 450$$

$$1 \quad + Q / 420 + R / 504$$

$$H(17,20) = (4.5 \cdot B + .5 \cdot F + 2 \cdot (C + E)) / 35 + G / 33264 + (P + Q) / 420 + R / 787.5$$

$$H(17,21) = (40 \cdot B + 10 \cdot E) / 140 + G / 27720 + P / 210 + Q / 315 + R / 1260$$

$$H(18,18) = (6 \cdot A + 4 \cdot B + 6 \cdot F + 9 \cdot C) / 140 + (D + E) / 14 + G / 27720 + P / 525$$

$$1 \quad + Q / 630 + R / 420$$

$$H(18,19) = (A + D) / 28 + (10 \cdot B + 9 \cdot F) / 280 + 2 \cdot (C + E) / 35 + G / 33264$$

$$1 \quad + (P + Q) / 630 + R / 525$$

$$H(18,20) = (12 \cdot B + 4 \cdot (F + D) + 10 \cdot E) / 140 + C / 14 + G / 27720 + P / 420$$

$$1 \quad + Q / 450 + R / 504$$

$$H(18,21) = (10 \cdot B + D + 4 \cdot E) / 28 + G / 15840 + P / 120 + 11 \cdot Q / 2520 + R / 420$$

$$H(19,19) = (A + C) / 14 + (4 \cdot B + 6 \cdot (D + F) + 9 \cdot E) / 140 + G / 27720 + P / 630$$

$$1 \quad + Q / 525 + R / 420$$

$$H(19,20) = (.5 \cdot (B + F) + C) / 7 + (9 \cdot D + 16 \cdot E) / 140 + G / 15840 + P / 360$$

$$1 \quad + Q / 280 + R / 252$$

$$H(19,21) = (20 \cdot B + 10 \cdot E) / 28 + 5 \cdot D / 35 + G / 5940 + P / 45 + 13 \cdot Q / 1260$$

$$1 \quad + R / 120$$


```

H(20,20)=6.*D/35.+2.*(E+F)/7.+G/5940.+Q/105.+R/90.
H(20,21)=(6.*D+10.*E)/7.+G/1320.+(Q+R)/22.5
H(21,21)=50.*D/7.+G/132.+Q/2.25
DO 550 I=1,21
DO 550 J=I,21
550 H(J,I)=H(I,J)
CALL TRMILT(S,H,S,TEMP,18,21,21,18,1)
DO 600 I=1,18
DO 600 J=1,18
600 FSM(KOD(I+20),KOD(J+20))=TEMP(I,J)
RETURN
END

```

```

SUBROUTINE FMAT(X13,X14,X23,X24,C)

```

```

  DIMENSION C(6,9)

```

```

  C(1,1)=180.*X13

```

```

  C(1,2)=180.*X14

```

```

  C(1,3)=120.*X13

```

```

  C(1,4)=60.*(X13+X14)

```

```

  C(1,5)=120.*X14

```

```

  C(1,6)=90.*X13

```

```

  C(1,7)=30.*(X13+X14)

```

```

  C(1,8)=30.*(X13+X14)

```

```

  C(1,9)=90.*X14

```

```

  C(2,1)=180.*X23

```

```

  C(2,2)=180.*X24

```

```

  C(2,3)=120.*X23

```

```

  C(2,4)=60.*(X23+X24)

```

```

  C(2,5)=120.*X24

```

```

  C(2,6)=90.*X23

```

$$C(2,7)=30. \cdot (X23+X24)$$

$$C(2,8)=30. \cdot (X23+X24)$$

$$C(2,9)=90. \cdot X24$$

$$C(3,1)=120. \cdot X13$$

$$C(3,2)=120. \cdot X14$$

$$C(3,3)=120. \cdot X13$$

$$C(3,4)=30. \cdot X13+60. \cdot X14$$

$$C(3,5)=60. \cdot X14$$

$$C(3,6)=108. \cdot X13$$

$$C(3,7)=24. \cdot X13+36. \cdot X14$$

$$C(3,8)=12. \cdot X13+24. \cdot X14$$

$$C(3,9)=36. \cdot X14$$

$$C(4,1)=60. \cdot (X13+X23)$$

$$C(4,2)=60. \cdot (X14+X24)$$

$$C(4,3)=30. \cdot X13+60. \cdot X23$$

$$C(4,4)=30. \cdot (X13+X24)+15. \cdot (X14+X23)$$

$$C(4,5)=60. \cdot X14+30. \cdot X24$$

$$C(4,6)=18. \cdot X13+54. \cdot X23$$

$$C(4,7)=12. \cdot (X13+X23)+6. \cdot X14+18. \cdot X24$$

$$C(4,8)=18. \cdot X13+12. \cdot (X14+X24)+6. \cdot X23$$

$$C(4,9)=54. \cdot X14+18. \cdot X24$$

$$C(5,1)=120. \cdot X23$$

$$C(5,2)=120. \cdot X24$$

$$C(5,3)=60. \cdot X23$$

$$C(5,4)=60. \cdot X23+30. \cdot X24$$

$$C(5,5)=120. \cdot X24$$

$$C(5,6)=36. \cdot X23$$

$$C(5,7)=24. \cdot X23+12. \cdot X24$$

$$C(5,8)=36. \cdot X23+24. \cdot X24$$

$$C(5,9)=108. \cdot X24$$

$$C(6,1)=90. \cdot X13$$

$$C(6,2)=90. \cdot X14$$

4 OF/DE 4

MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS
STANDARD REFERENCE MATERIAL 1010a
(ANSI and ISO TEST CHART No. 2)

$C(6,3)=108.*X13$
 $C(6,4)=18.*X13+54.*X14$
 $C(6,5)=36.*X14$
 $C(6,6)=108.*X13$
 $C(6,7)=18.*X13+36.*X14$
 $C(6,8)=6.*X13+18.*X14$
 $C(6,9)=18.*X14$
 $C(7,1)=30.*(X13+X23)$
 $C(7,2)=30.*(X14+X24)$
 $C(7,3)=24.*X13+36.*X23$
 $C(7,4)=12.*(X13+X14)+6.*X23+18.*X24$
 $C(7,5)=24.*X14+12.*X24$
 $C(7,6)=18.*X13+36.*X23$
 $C(7,7)=8.*X13+6.*(X14+X23)+12.*X24$
 $C(7,8)=6.*(X13+X24)+8.*X14+2.*X23$
 $C(7,9)=18.*X14+6.*X24$
 $C(8,1)=30.*(X13+X23)$
 $C(8,2)=30.*(X14+X24)$
 $C(8,3)=12.*X13+24.*X23$
 $C(8,4)=18.*X13+6.*X14+12.*(X23+X24)$
 $C(8,5)=36.*X14+24.*X24$
 $C(8,6)=6.*X13+18.*X23$
 $C(8,7)=6.*(X13+X24)+2.*X14+8.*X23$
 $C(8,8)=12.*X13+6.*(X14+X23)+8.*X24$
 $C(8,9)=36.*X14+18.*X24$
 $C(9,1)=90.*X23$
 $C(9,2)=90.*X24$
 $C(9,3)=36.*X23$
 $C(9,4)=54.*X23+18.*X24$
 $C(9,5)=108.*X24$
 $C(9,6)=18.*X23$
 $C(9,7)=18.*X23+6.*X24$

```
C(9,8)=36.*X23+18.*X24
C(9,9)=108.*X24
DO 50 I=1,9
DO 50 J=1,9
50 C(I,J)=C(I,J)/360.
RETURN
END
```

```
SUBROUTINE GMAT(A,B,C,D,E,F,P,Q,G)
DIMENSION G(9,21)
G(1,1)=630.*P
G(1,2)=210.*P
G(1,3)=210.*P
G(1,4)=1260.*A+105.*P
G(1,5)=630.*C+52.5*P
G(1,6)=1260.*B+105.*P
G(1,7)=1260.*A+63.*P
G(1,8)=420.*(A+C)+21.*P
G(1,9)=420.*(B+C)+21.*P
G(1,10)=1260.*B+63.*P
G(1,11)=1260.*A+42.*P
G(1,12)=315.*(A+C)+10.5*P
G(1,13)=210.*(A+B+C)+7.*P
G(1,14)=315.*(B+C)+10.5*P
G(1,15)=1260.*B+42.*P
G(1,16)=1260.*A+30.*P
G(1,17)=252.*(A+C)+6.*P
G(1,18)=126.*(A+B+C)+3.*P
G(1,19)=126.*(A+B+C)+3.*P
G(1,20)=252.*(B+C)+6.*P
```

$$G(1,21)=1260.*B+30.*P$$

$$G(2,1)=630.*Q$$

$$G(2,2)=210.*Q$$

$$G(2,3)=210.*Q$$

$$G(2,4)=1260.*D+105.*Q$$

$$G(2,5)=630.*F+52.5*Q$$

$$G(2,6)=1260.*E+105.*Q$$

$$G(2,7)=1260.*D+63.*Q$$

$$G(2,8)=420.*(D+F)+21.*Q$$

$$G(2,9)=420.*(E+F)+21.*Q$$

$$G(2,10)=1260.*E+63.*Q$$

$$G(2,11)=1260.*D+42.*Q$$

$$G(2,12)=315.*(D+F)+10.5*Q$$

$$G(2,13)=210.*(D+E+F)+7.*Q$$

$$G(2,14)=315.*(E+F)+10.5*Q$$

$$G(2,15)=1260.*E+42.*Q$$

$$G(2,16)=1260.*D+30.*Q$$

$$G(2,17)=252.*(D+F)+6.*Q$$

$$G(2,18)=126.*(D+E+F)+3.*Q$$

$$G(2,19)=126.*(D+E+F)+3.*Q$$

$$G(2,20)=252.*(E+F)+6.*Q$$

$$G(2,21)=1260.*E+30.*Q$$

$$G(3,1)=420.*P$$

$$G(3,2)=210.*P$$

$$G(3,3)=105.*P$$

$$G(3,4)=840.*A+126.*P$$

$$G(3,5)=420.*C+42.*P$$

$$G(3,6)=840.*B+42.*P$$

$$G(3,7)=1260.*A+84.*P$$

$$G(3,8)=210.*A+420.*C+21.*P$$

$$G(3,9)=420.*B+210.*C+14.*P$$

$$G(3,10)=630.*B+21.*P$$

$$G(3,11)=1512.*A+60.*P$$

$$G(3,12)=252.*A+378.*C+12.*P$$

$$G(3,13)=84.*A+252.*B+168.*C+6.*P$$

$$G(3,14)=252.*B+126.*C+6.*P$$

$$G(3,15)=504.*B+12.*P$$

$$G(3,16)=1680.*A+45.*P$$

$$G(3,17)=252.*A+336.*C+7.5*P$$

$$G(3,18)=84.*A+168.*B+126.*C+3.*P$$

$$G(3,19)=42.*A+126.*B+84.*C+2.25*P$$

$$G(3,20)=168.*B+84.*C+3.*P$$

$$G(3,21)=420.*B+7.5*P$$

$$G(4,1)=210.*(P+Q)$$

$$G(4,2)=52.5*P+105.*Q$$

$$G(4,3)=105.*P+52.5*Q$$

$$G(4,4)=420.*(A+D)+21.*P+63.*Q$$

$$G(4,5)=210.*(C+F)+21.*(P+Q)$$

$$G(4,6)=420.*(B+E)+63.*P+21.*Q$$

$$G(4,7)=315.*A+630.*D+10.5*P+42.*Q$$

$$G(4,8)=210.*(A+F)+105.*(C+D)+7.*P+10.5*Q$$

$$G(4,9)=105.*(B+F)+210.*(C+E)+10.5*P+7.*Q$$

$$G(4,10)=630.*B+315.*E+42.*P+10.5*Q$$

$$G(4,11)=252.*A+756.*D+6.*P+30.*Q$$

$$G(4,12)=126.*(A+D)+63.*C+189.*F+3.*P+6.*Q$$

$$G(4,13)=126.*(A+E)+42.*(B+D)+84.*(C+F)+3.*(P+Q)$$

$$G(4,14)=126.*(B+E)+189.*C+63.*F+6.*P+3.*Q$$

$$G(4,15)=756.*B+252.*E+30.*P+6.*Q$$

$$G(4,16)=210.*A+840.*D+3.75*P+22.5*Q$$

$$G(4,17)=84.*A+42.*C+126.*D+168.*F+1.5*P+3.75*Q$$

$$G(4,18)=63.*A+21.*B+42.*(C+D)+84.*E+63.*F+1.125*P+1.5*Q$$

$$G(4,19)=84.*A+42.*(B+F)+63.*(C+E)+21.*D+1.5*P+1.125*Q$$

$$G(4,20)=126.*B+168.*C+84.*E+42.*F+3.75*P+1.5*Q$$

$$G(4,21)=840.*B+210.*E+22.5*P+3.75*Q$$

$$G(5,1)=420.*Q$$

$$G(5,2)=105.*Q$$

$$G(5,3)=210.*Q$$

$$G(5,4)=840.*D+42.*Q$$

$$G(5,5)=420.*F+42.*Q$$

$$G(5,6)=840.*E+126.*Q$$

$$G(5,7)=630.*D+21.*Q$$

$$G(5,8)=42Q.*D+210.*F+14.*Q$$

$$G(5,9)=210.*E+420.*F+21.*Q$$

$$G(5,10)=1260.*E+84.*Q$$

$$G(5,11)=504.*D+12.*Q$$

$$G(5,12)=252.*D+126.*F+6.*Q$$

$$G(5,13)=252.*D+84.*E+168.*F+6.*Q$$

$$G(5,14)=252.*E+378.*F+12.*Q$$

$$G(5,15)=1512.*E+60.*Q$$

$$G(5,16)=420.*D+7.5*Q$$

$$G(5,17)=168.*D+84.*F+3.*Q$$

$$G(5,18)=126.*D+42.*E+84.*F+2.25*Q$$

$$G(5,19)=168.*D+84.*E+126.*F+3.*Q$$

$$G(5,20)=252.*E+336.*F+7.5*Q$$

$$G(5,21)=1680.*E+45.*Q$$

$$G(6,1)=315.*P$$

$$G(6,2)=189.*P$$

$$G(6,3)=63.*P$$

$$G(6,4)=630.*A+126.*P$$

$$G(6,5)=315.*C+31.5*P$$

$$G(6,6)=630.*B+21.*P$$

$$G(6,7)=1134.*A+90.*P$$

$$G(6,8)=126.*A+378.*C+18.*P$$

$$G(6,9)=378.*B+126.*C+9.*P$$

$$G(6,10)=378.*B+9.*P$$

$$G(6,11)=1512.*A+67.5*P$$

$$G(6,12)=189.*A+378.*C+11.25*P$$

$$G(6,13)=42.*A+252.*B+126.*C+4.5*P$$

$$G(6,14)=189.*B+63.*C+3.375*P$$

$$G(6,15)=252.*B+4.5*P$$

$$G(6,16)=1800.*A+52.5*P$$

$$G(6,17)=216.*A+360.*C+7.5*P$$

$$G(6,18)=54.*A+180.*B+108.*C+2.5*P$$

$$G(6,19)=18.*A+108.*B+54.*C+1.5*P$$

$$G(6,20)=108.*B+36.*C+1.5*P$$

$$G(6,21)=180.*B+2.5*P$$

$$G(7,1)=105.*(P+Q)$$

$$G(7,2)=42.*P+63.*Q$$

$$G(7,3)=42.*P+21.*Q$$

$$G(7,4)=210.*(A+D)+21.*P+42.*Q$$

$$G(7,5)=105.*(C+F)+14.*P+10.5*Q$$

$$G(7,6)=210.*(B+E)+21.*P+7.*Q$$

$$G(7,7)=252.*A+378.*D+12.*P+30.*Q$$

$$G(7,8)=84.*(A+C)+42.*D+126.*F+6.*(P+Q)$$

$$G(7,9)=84.*(B+C)+126.*E+42.*F+6.*P+3.*Q$$

$$G(7,10)=252.*B+126.*E+12.*P+3.*Q$$

$$G(7,11)=252.*A+504.*D+7.5*P+22.5*Q$$

$$G(7,12)=84.*A+63.*(C+D)+126.*F+3.*P+3.75*Q$$

$$G(7,13)=42.*(A+B+F)+56.*C+14.*D+84.*E+2.25*P+1.5*Q$$

$$G(7,14)=84.*B+63.*(C+E)+21.*F+3.*P+1.125*Q$$

$$G(7,15)=252.*B+84.*E+7.5*P+1.5*Q$$

$$G(7,16)=240.*A+600.*D+5.*P+17.5*Q$$

$$G(7,17)=72.*(A+D)+48.*C+120.*F+5.*P/3.+2.5*Q$$

$$G(7,18)=36.*(A+C+F)+24.*B+18.*D+60.*E+P+Q/1.2$$

$$G(7,19)=24.*A+36.*(B+C+E)+6.*D+18.*F+P+.5*Q$$

$$G(7,20)=72.*B+48.*C+36.*E+12.*F+5.*P/3.+5*Q$$

$$G(7,21)=240.*B+60.*E+5.*P+Q/1.2$$

$$G(8,1)=105.*(P+Q)$$

$G(8,2)=21.*P+42.*Q$
 $G(8,3)=63.*P+42.*Q$
 $G(8,4)=210.*(A+D)+7.*P+21.*Q$
 $G(8,5)=105.*(C+F)+10.5*P+14.*Q$
 $G(8,6)=210.*(B+E)+42.*P+21.*Q$
 $G(8,7)=126.*A+252.*D+3.*P+12.*Q$
 $G(8,8)=126.*A+42.*C+84.*(D+F)+3.*P+6.*Q$
 $G(8,9)=42.*B+126.*C+84.*(E+F)+6.*(P+Q)$
 $G(8,10)=378.*B+252.*E+30.*P+12.*Q$
 $G(8,11)=84.*A+252.*D+1.5*P+7.5*Q$
 $G(8,12)=63.*(A+F)+21.*C+84.*D+1.125*P+3.*Q$
 $G(8,13)=84.*A+14.*B+42.*(C+D+E)+56.*F+1.5*P+2.25*Q$
 $G(8,14)=63.*(B+F)+126.*C+84.*E+3.75*P+3.*Q$
 $G(8,15)=504.*B+252.*E+22.5*P+7.5*Q$
 $G(8,16)=60.*A+240.*D+P/1.2+5.*Q$
 $G(8,17)=36.*A+12.*C+72.*D+48.*F+.5*P+5.*Q/3.$
 $G(8,18)=36.*(A+D+F)+6.*B+18.*C+24.*E+.5*P+Q$
 $G(8,19)=60.*A+18.*B+36.*(C+E+F)+24.*D+P/1.2+Q$
 $G(8,20)=72.*(B+E)+120.*C+48.*F+2.5*P+5.*Q/3.$
 $G(8,21)=600.*B+240.*E+17.5*P+5.*Q$
 $G(9,1)=315.*Q$
 $G(9,2)=63.*Q$
 $G(9,3)=189.*Q$
 $G(9,4)=630.*D+21.*Q$
 $G(9,5)=315.*F+31.5*Q$
 $G(9,6)=630.*E+126.*Q$
 $G(9,7)=378.*D+9.*Q$
 $G(9,8)=378.*D+126.*F+9.*Q$
 $G(9,9)=126.*E+378.*F+18.*Q$
 $G(9,10)=1134.*E+90.*Q$
 $G(9,11)=252.*D+4.5*Q$
 $G(9,12)=189.*D+63.*F+3.375*Q$

```

G(9,13)=252.*D+42.*E+126.*F+4.5*Q
G(9,14)=189.*E+378.*F+11.25*Q
G(9,15)=1512.*E+67.5*Q
G(9,16)=180.*D+2.5*Q
G(9,17)=108.*D+36.*F+1.5*Q
G(9,18)=108.*D+18.*E+54.*F+1.5*Q
G(9,19)=180.*D+54.*E+108.*F+2.5*Q
G(9,20)=216.*E+360.*F+7.5*Q
G(9,21)=1800.*E+52.5*Q
DO 100 I=1,9
DO 100 J=1,21
100 G(I,J)=G(I,J)/1260.
RETURN
END

```

```

SUBROUTINE TRMULT(A,B,C,D,K,M,N,L,ISYM)
DIMENSION A(M,K),B(M,N),C(N,L),D(K,L),E(38,38)
DO 10 I=1,M
DO 10 J=1,L
E(I,J)=0.
DO 10 KK=1,N
10 E(I,J)=E(I,J)+B(I,KK)*C(KK,J)
IF(ISYM.NE.0)GO TO 75
DO 20 I=1,K
DO 20 J=1,L
D(I,J)=0.
DO 20 KK=1,M
20 D(I,J)=D(I,J)+A(KK,I)*E(KK,J)
RETURN
75 DO 30 I=1,K

```

```
DO 30 J=I,L
D(I,J)=0.
DO 35 KK=1,M
35 D(I,J)=D(I,J)+A(KK,I)*E(KK,J)
30 D(J,I)=D(I,J) .
RETURN
END
```

APPENDIX - L

SUBROUTINE ELSTIF(IELTYP,X1,X2,X3,Y1,Y2,Y3,H,S,CUR)

C*** Element Stiffness Matrix Subroutine for Laminated Anisotropic

C*** Shallow Shell Elements

C*** Generates stiffness matrices for DKT, E-1 and E-2 shallow shell

C*** elements.

C***

C*** INPUT

C***

C*** IELTYP is the element type.

C*** = 0 DKT shallow shell element

C*** = 1 E-1 shallow shell element

C*** = 2 E-2 shallow shell element

C*** X1,X2,X3,Y1,Y2,Y3 are the x and y coordinates of the vertices of
C*** the triangle.

C*** H(1),H(2),.....H(6) are the nodal heights. (See Figure 42 and
C*** equation 4.2.4.

C*** AAM,ADM,ABM are the membrane,bending and membrane-bending coupling
C*** stiffness matrices of the laminate.

C***

C*** OUTPUT

C***

C*** ESM is the (29 X 29) element stiffness matrix

C***

COMMON/ST/ESM(29,29)

COMMON/XAB/B1,B2,B3,C1,C2,C3

COMMON/CODE/KOD(29)

COMMON/ABD/AAM(3,3),ABM(3,3),ADM(3,3)

DIMENSION P(3,5),Q(3,3),AM(5,5),BM(5,3),DM(3,3),H(6),ZV(3),R(9,10)

```
DIMENSION S(10,9),T(9,9),CUR(3)
DATA KOD/1,2,3,10,11,12,19,20,21,28,4,5,6,13,14,15,22,23,24,29,
1      7,8,9,16,17,18,25,26,27/
DO 10 I=1,29
DO 10 J=1,29
10 ESM(I,J)=0.
   B1=Y2-Y3
   B2=Y3-Y1
   B3=Y1-Y2
   C1=X3-X2
   C2=X1-X3
   C3=X2-X1
   DET=ABS(B1*C2-B2*C1)
   Q(1,1)=B2*B2
   Q(1,2)=B3*B3
   Q(1,3)=2.*B2*B3
   Q(2,1)=C2*C2
   Q(2,2)=C3*C3
   Q(2,3)=2.*C2*C3
   Q(3,1)=2.*B2*C2
   Q(3,2)=2.*B3*C3
   Q(3,3)=2.*(B2*C3+B3*C2)
   ZV(1)=4.*(H(1)+H(2))-8.*H(4)
   ZV(2)=4.*(H(1)+H(3))-8.*H(6)
   ZV(3)=4.*(H(1)+H(5)-H(4)-H(6))
DO 15 I=1,3
CUR(I)=0.
DO 15 J=1,3
15 CUR(I)=CUR(I)+Q(I,J)*ZV(J)/DET**2
   P(1,1)=B2
   P(1,2)=B3
   P(1,3)=0.
```

```
P(1,4)=0.
P(1,5)=CUR(1)*DET
P(2,1)=0.
P(2,2)=0.
P(2,3)=C2
P(2,4)=C3
P(2,5)=CUR(2)*DET
P(3,1)=C2
P(3,2)=C3
P(3,3)=B2
P(3,4)=B3
P(3,5)=CUR(3)*DET
DO 20 I=1,5
DO 20 J=1,3
BM(I,J)=0.
DO 20 K=1,3
20 BM(I,J)=BM(I,J)+P(K,I)*ABM(K,J)
CALL TRMULT(P,AAM,P,AM,5,3,3,5,1)
CALL TRMULT(Q,ADM,Q,DM,3,3,3,3,1)
DO 25 I=1,5
DO 25 J=1,5
25 AM(I,J)=AM(I,J)/DET
DO 30 I=1,3
DO 30 J=1,3
30 DM(I,J)=DM(I,J)/DET
DO 35 I=1,5
DO 35 J=1,3
35 BM(I,J)=BM(I,J)/DET
CALL RMAT(B1,B2,B3,C1,C2,C3,R)
CALL SMAT(S)
IF(IELTYP.EQ.0)CALL TMDKT(T)
IF(IELTYP.EQ.1)CALL TME1(T)
```

```
IF(IELTYP.EQ.2)CALL TME2(T)
CALL UV(R,AM)
CALL UVKAPA(BM,R,T)
CALL UVW(AM,R,S)
CALL WW(AM(5,5),S)
CALL KAPA(DET,T)
CALL WKAPA(BM(5,1),BM(5,2),BM(5,3),T,S)
RETURN
END
```

```
SUBROUTINE RMAT(B1,B2,B3,C1,C2,C3,R)
DIMENSION R(9,10)
DO 25 I=1,9
DO 25 J=1,10
25 R(I,J)=0.
R(1,2)=C3
R(1,3)=-B3
R(2,2)=-C2
R(2,3)=B2
R(3,1)=-3.
R(3,2)=-2.*C3
R(3,3)=2.*B3
R(3,4)=3.
R(3,5)=-C3
R(3,6)=B3
R(4,1)=-13.
R(4,2)=3.*(C2-C3)
R(4,3)=3.*(B3-B2)
R(4,4)=-7.
R(4,5)=2.*C3+C2
```


$$R(4,6) = -2 * B3 - B2$$

$$R(4,7) = -7$$

$$R(4,8) = -C3 - 2 * C2$$

$$R(4,9) = B3 + 2 * B2$$

$$R(4,10) = 27$$

$$R(5,1) = -3$$

$$R(5,2) = 2 * C2$$

$$R(5,3) = -2 * B2$$

$$R(5,7) = 3$$

$$R(5,8) = C2$$

$$R(5,9) = -B2$$

$$R(6,1) = 2$$

$$R(6,2) = C3$$

$$R(6,3) = -B3$$

$$R(6,4) = -2$$

$$R(6,5) = C3$$

$$R(6,6) = -B3$$

$$R(7,1) = 13$$

$$R(7,2) = 3 * C3 - 2 * C2$$

$$R(7,3) = 2 * B2 - 3 * B3$$

$$R(7,4) = 7$$

$$R(7,5) = -2 * (C2 + C3)$$

$$R(7,6) = 2 * (B2 + B3)$$

$$R(7,7) = 7$$

$$R(7,8) = C3 + 2 * C2$$

$$R(7,9) = -(B3 + 2 * B2)$$

$$R(7,10) = -27$$

$$R(8,1) = 13$$

$$R(8,2) = 2 * C3 - 3 * C2$$

$$R(8,3) = 3 * B2 - 2 * B3$$

$$R(8,4) = 7$$

$$R(8,5) = C2 - 2 * C3$$

R(8,6)=B2+2.*B3

R(8,7)=7.

R(8,8)=2.*(C2+C3)

R(8,9)=-2.*(B2+B3)

R(8,10)=-27.

R(9,1)=2.

R(9,2)=-C2

R(9,3)=B2

R(9,7)=-2.

R(9,8)=-C2

R(9,9)=B2

RETURN

END

SUBROUTINE SMAT(S)

COMMON/XAB/B(3),C(3)

DIMENSION S(10,9),PP(2,3)

DO 125 I=1,10

DO 125 J=1,9

125 S(I,J)=0.

DO 130 I=1,3

PP(1,I)=-C(I)

130 PP(2,I)=B(I)

S(1,1)=1.

S(4,1)=-3.

S(5,1)=-4.

S(6,1)=-3.

S(7,1)=2.

S(8,1)=4.

S(9,1)=4.

S(10,1)=2.

S(4,4)=3.

S(5,4)=2.

S(7,4)=-2.

S(8,4)=-2.

S(9,4)=-2.

S(5,7)=2.

S(6,7)=3.

S(8,7)=-2.

S(9,7)=-2.

S(10,7)=-2.

DO 135 I=1,2

II=I-1

S(2,2+II)=-PP(I,3)

S(3,2+II)=PP(I,2)

S(4,2+II)=2.*PP(I,3)

S(5,2+II)=1.5*(PP(I,3)-PP(I,2))

S(6,2+II)=-2.*PP(I,2)

S(7,2+II)=-PP(I,3)

S(8,2+II)=0.5*PP(I,2)-1.5*PP(I,3)

S(9,2+II)=1.5*PP(I,2)-0.5*PP(I,3)

S(10,2+II)=PP(I,2)

S(4,5+II)=PP(I,3)

S(5,5+II)=0.5*(PP(I,3)-PP(I,1))

S(7,5+II)=-PP(I,3)

S(8,5+II)=-0.5*PP(I,1)-1.5*PP(I,3)

S(9,5+II)=0.5*(PP(I,1)-PP(I,3))

S(5,8+II)=0.5*(PP(I,1)-PP(I,2))

S(6,8+II)=-PP(I,2)

S(8,8+II)=0.5*(PP(I,2)-PP(I,1))

S(9,8+II)=1.5*PP(I,2)+0.5*PP(I,1)

S(10,8+II)=PP(I,2)

135 CONTINUE

RETURN

END

SUBROUTINE UV(R,AM)

COMMON/ST/ESM(29,29)

COMMON/CODE/KOD(29)

DIMENSION TEMP(10,10),F(9,9),AM(5,5),R(9,10)

CALL FMAT(AM(1,1),AM(1,2),AM(1,2),AM(2,2),F)

CALL TRMULT(R,F,R,TEMP,10,9,9,10,1)

DO 100 I=1,10

DO 100 J=1,10

100 ESM(KOD(I),KOD(J))=TEMP(I,J)

CALL FMAT(AM(1,3),AM(1,4),AM(2,3),AM(2,4),F)

CALL TRMULT(R,F,R,TEMP,10,9,9,10,0)

DO 150 I=1,10

DO 150 J=1,10

ESM(KOD(I),KOD(J+10))=TEMP(I,J)

150 ESM(KOD(J+10),KOD(I))=ESM(KOD(I),KOD(J+10))

CALL FMAT(AM(3,3),AM(3,4),AM(3,4),AM(4,4),F)

CALL TRMULT(R,F,R,TEMP,10,9,9,10,1)

DO 200 I=1,10

DO 200 J=1,10

200 ESM(KOD(I+10),KOD(J+10))=TEMP(I,J)

RETURN

END

SUBROUTINE UVW(AM,R,S)

```

COMMON/ST/ESM(29,29)
COMMON/COE/KOD(29)
DIMENSION AM(5,5),AK(9,10),TEMP(10,9),R(9,10),S(10,9)
CALL UVWMAT(AM(1,5),AM(2,5),AK)
CALL TRMULT(R,AK,S,TEMP,10,9,10,9,0)
DO 10 I=1,10
DO 10 J=1,9
ESM(KOD(I),KOD(J+20))=ESM(KOD(I),KOD(J+20))-TEMP(I,J)
10 ESM(KOD(J+20),KOD(I))=ESM(KOD(I),KOD(J+20))
CALL UVWMAT(AM(3,5),AM(4,5),AK)
CALL TRMULT(R,AK,S,TEMP,10,9,10,9,0)
DO 20 I=1,10
DO 20 J=1,9
ESM(KOD(I+10),KOD(J+20))=ESM(KOD(I+10),KOD(J+20))-TEMP(I,J)
20 ESM(KOD(J+20),KOD(I+10))=ESM(KOD(I+10),KOD(J+20))
RETURN
END

```

SUBROUTINE WW(A,S)

COMMON/ST/ESM(29,29)

COMMON/COE/KOD(29)

DIMENSION IT(10,10),TEMP(9,9),B(10,10),S(10,9)

DATA IT/5040,2*1680,840,420,840,504,2*168,504,1680,840,420,504,2*

```

1      168,336,84,56,84,1680,420,840,2*168,504,84,56,84,336,840,
2      504,168,336,84,56,240,48,24,24,420,2*168,84,56,84,48,24,24
3      ,48,840,168,504,56,84,336,24,24,48,240,504,336,84,240,48,
4      24,180,30,12,9,168,84,56,48,24,24,30,12,9,12,168,56,84,24,
5      24,48,12,9,12,30,504,84,336,24,48,240,9,12,30,180/

```

DO 100 I=1,10

DO 100 J=1,10

100 B(I,J)=A*IT(I,J)/10080.

CALL TRMULT(S,B,S,TEMP,9,10,10,9,1)

DO 200 I=1,9

DO 200 J=1,9

200 ESM(KOD(I+20),KOD(J+20))=ESM(KOD(I+20),KOD(J+20))+TEMP(I,J)

RETURN

END

SUBROUTINE TRMULT(A,B,C,D,K,M,N,L,ISYM)

DIMENSION A(M,K),B(M,N),C(N,L),D(K,L),E(29,29)

DO 10 I=1,M

DO 10 J=1,L

E(I,J)=0.

DO 10 KK=1,N

10 E(I,J)=E(I,J)+B(I,KK)*C(KK,J)

IF(ISYM.NE.0) GO TO 75

DO 20 I=1,K

DO 20 J=1,L

D(I,J)=0.

DO 20 KK=1,M

20 D(I,J)=D(I,J)+A(KK,I)*E(KK,J)

RETURN

75 DO 30 I=1,K

DO 30 J=1,L

D(I,J)=0.

DO 30 KK=1,M

D(I,J)=D(I,J)+A(KK,I)*E(KK,J)

30 D(J,I)=D(I,J)

RETURN

END

```

SUBROUTINE UVK(A1,A2,A3,B1,B2,B3,C)
COMMON/COE/KOD(29)
DIMENSION IT(9,3),JT(9,3),C(9,9)
DATA IT/60,0,40,20,0,30,10,10,0,20,0,20,5,0,18,4,2,0,20,0,10,
1      10,0,6,4,6,0/
DATA JT/0,60,0,20,40,0,10,10,30,0,20,0,10,10,0,6,4,6,0,20,0,5,
1      20,0,2,4,18/
DO 100 I=1,9
DO 100 J=1,3
C(I,J)=(A1*IT(I,J)+B1*JT(I,J))/120.
C(I,J+3)=(A2*IT(I,J)+B2*JT(I,J))/120.
C(I,J+6)=(A3*IT(I,J)+B3*JT(I,J))/120.
100 CONTINUE
RETURN
END

```

```

SUBROUTINE TMDKT(T)
COMMON/XAB/B(3),C(3)
DIMENSION T(9,9),ALS(3),PT(2,3),RS(2,3),GG(10,9),Q(3)
DIMENSION KOD(2,9)
DATA KOD/1,1,2,3,3,2,4,4,5,6,6,5,7,7,8,9,9,8/
DO 30 I=1,3
ALS(I)=B(I)*B(I)+C(I)*C(I)
PT(1,I)=6.*C(I)/ALS(I)

```

```

PT(2,I)=6.*B(I)/ALS(I)
RS(1,I)=3.*C(I)*C(I)/ALS(I)
RS(2,I)=3.*B(I)*B(I)/ALS(I)
30 Q(I)=3.*B(I)*C(I)/ALS(I)
DO 720 I=1,10
DO 720 J=1,9
720 GG(I,J)=0.
DO 730 I=1,2
AJ=3.-2.*I
II=(I-1)*6
GG(II+1,KOD(I,1))=PT(I,3)
GG(II+2,KOD(I,1))=-PT(I,2)
GG(II+3,KOD(I,1))=-PT(I,3)
GG(II+4,KOD(I,1))=PT(I,2)-PT(I,3)
GG(II+5,KOD(I,1))=PT(I,2)
GG(II+1,KOD(I,3))=-Q(3)*AJ
GG(II+2,KOD(I,3))=-Q(2)*AJ
GG(II+3,KOD(I,3))=Q(3)*AJ
GG(II+4,KOD(I,3))=(Q(2)+Q(3))*AJ
GG(II+5,KOD(I,3))=Q(2)*AJ
GG(II+1,KOD(I,2))=(1.+RS(I,3))*AJ
GG(II+2,KOD(I,2))=(RS(I,2)+1.)*AJ
GG(II+3,KOD(I,2))=-RS(I,3)*AJ
GG(II+4,KOD(I,2))=-(RS(I,2)+RS(I,3))*AJ
GG(II+5,KOD(I,2))=-RS(I,2)*AJ
GG(II+1,KOD(I,4))=-PT(I,3)
GG(II+3,KOD(I,4))=PT(I,3)
GG(II+4,KOD(I,4))=PT(I,1)+PT(I,3)
GG(II+1,KOD(I,6))=-Q(3)*AJ
GG(II+3,KOD(I,6))=Q(3)*AJ
GG(II+4,KOD(I,6))=(Q(3)-Q(1))*AJ
GG(II+1,KOD(I,5))=(RS(I,3)+1.)*AJ

```



```

GG(II+3,KOD(I,5))=-RS(I,3)*AJ
GG(II+4,KOD(I,5))=(RS(I,1)-RS(I,3))*AJ
GG(II+2,KOD(I,7))=PT(I,2)
GG(II+4,KOD(I,7))=-PT(I,1)-PT(I,2)
GG(II+5,KOD(I,7))=-PT(I,2)
GG(II+2,KOD(I,9))=-Q(2)*AJ
GG(II+4,KOD(I,9))=(Q(2)-Q(1))*AJ
GG(II+5,KOD(I,9))=Q(2)*AJ
GG(II+2,KOD(I,8))=(RS(I,2)-1.)*AJ
GG(II+4,KOD(I,8))=(RS(I,1)-RS(I,2))*AJ
GG(II+5,KOD(I,8))=-RS(I,2)*AJ

```

730 CONTINUE

DO 150 I=1,9

```

T(1,I)=B(2)*GG(1,I)+B(3)*GG(2,I)
T(2,I)=2.*B(2)*GG(3,I)+B(3)*GG(4,I)
T(3,I)=B(2)*GG(4,I)+2.*B(3)*GG(5,I)
T(4,I)=-C(2)*GG(6,I)-C(3)*GG(7,I)
T(5,I)=-2.*C(2)*GG(8,I)-C(3)*GG(9,I)
T(6,I)=-C(2)*GG(9,I)-2.*C(3)*GG(10,I)
T(7,I)=C(2)*GG(1,I)+C(3)*GG(2,I)-B(2)*GG(6,I)-B(3)*GG(7,I)
T(8,I)=2.*C(2)*GG(3,I)+C(3)*GG(4,I)-2.*B(2)*GG(8,I)-B(3)*GG(9,I)
T(9,I)=C(2)*GG(4,I)+2.*C(3)*GG(5,I)-B(2)*GG(9,I)-2.*B(3)*GG(10,I)

```

150 CONTINUE

RETURN

END

SUBROUTINE TME1(T)

COMMON/XAB/B1,B2,B3,C1,C2,C3

DIMENSION T(9,9),G(10,9)

DET=B2*C3-B3*C2

$$P1=6.*C1/(B1**2+C1**2)$$

$$P2=6.*C2/(B2**2+C2**2)$$

$$P3=6.*C3/(B3**2+C3**2)$$

$$Q1=3.*B1*C1/(B1**2+C1**2)$$

$$Q2=3.*B2*C2/(B2**2+C2**2)$$

$$Q3=3.*B3*C3/(B3**2+C3**2)$$

$$R1=3.*B1**2/(B1**2+C1**2)$$

$$R2=3.*B2**2/(B2**2+C2**2)$$

$$R3=3.*B3**2/(B3**2+C3**2)$$

$$S1=3.*C1**2/(B1**2+C1**2)$$

$$S2=3.*C2**2/(B2**2+C2**2)$$

$$S3=3.*C3**2/(B3**2+C3**2)$$

$$T1=6.*B1/(B1**2+C1**2)$$

$$T2=6.*B2/(B2**2+C2**2)$$

$$T3=6.*B3/(B3**2+C3**2)$$

$$A1=27.*(B1*(C2-C3)/6.+C1*(B2-B3)/18.)/DET$$

$$A2=27.*(B2*C3/3.-B1*C2/6.+C2*(B2-B3)/18.)/DET$$

$$A3=27.*(B1*C3/6.-B3*C2/3.+C3*(B2-B3)/18.)/DET$$

$$A4=27.*(C1*(B3-B2)/6.+B1*(C3-C2)/18.)/DET$$

$$A5=27.*(B2*C1/6.-B3*C2/3.+B2*(C3-C2)/18.)/DET$$

$$A6=27.*(B2*C3/3.-B3*C1/6.+B3*(C3-C2)/18.)/DET$$

$$G(1,1)=3.*P2+7.*P3-36.*B1/DET$$

$$G(1,2)=19.-3.*S2+7.*S3+A1$$

$$G(1,3)=3.*Q2-7.*Q3+6.*B1*(B3-B2)/DET$$

$$G(1,4)=-3.*P1-7.*P3-36.*B2/DET$$

$$G(1,5)=-1.-3.*S1+7.*S3+A2$$

$$G(1,6)=3.*Q1-7.*Q3+6.*(B3*B3-B1*B1)/DET$$

$$G(1,7)=3.*(P1-P2)-36.*B3/DET$$

$$G(1,8)=9.-3.*(S1+S2)+A3$$

$$G(1,9)=3.*(Q1+Q2)+6.*(B1*B1-B2*B2)/DET$$

$$G(2,1)=-7.*P2-3.*P3-36.*B1/DET$$

$$G(2,2)=19.+7.*S2-3.*S3+A1$$

$$G(2,3)=3.*Q3-7.*Q2+6.*B1*(B3-B2)/DET$$

$$G(2,4)=3.*(P3-P1)-36.*B2/DET$$

$$G(2,5)=9.-3.*(S1+S3)+A2$$

$$G(2,6)=3.*(Q1+Q3)+6.*(B3*B3-B1*B1)/DET$$

$$G(2,7)=3.*P1+7.*P2-36.*B3/DET$$

$$G(2,8)=-1.-3.*S1+7.*S2+A3$$

$$G(2,9)=3.*Q1-7.*Q2+6.*(B1*B1-B2*B2)/DET$$

$$G(4,1)=7.*(P2-P3)+36.*B1/DET$$

$$G(4,2)=-9.-7.*(S2+S3)-A1$$

$$G(4,3)=7.*(Q2+Q3)-6.*B1*(B3-B2)/DET$$

$$G(4,4)=13.*P1+7.*P3+36.*B2/DET$$

$$G(4,5)=13.*S1-9.-7.*S3-A2$$

$$G(4,6)=7.*Q3-13.*Q1-6.*(B3*B3-B1*B1)/DET$$

$$G(4,7)=-13.*P1-7.*P2+36.*B3/DET$$

$$G(4,8)=13.*S1-9.-7.*S2-A3$$

$$G(4,9)=7.*Q2-13.*Q1-6.*(B1*B1-B2*B2)/DET$$

$$G(6,1)=-3.*T2-7.*T3-36.*C1/DET$$

$$G(6,2)=3.*Q2-7.*Q3+6.*C1*(C2-C3)/DET$$

$$G(6,3)=19.-3.*R2+7.*R3+A4$$

$$G(6,4)=3.*T1+7.*T3-36.*C2/DET$$

$$G(6,5)=3.*Q1-7.*Q3+6.*(C1*C1-C3*C3)/DET$$

$$G(6,6)=7.*R3-1.-3.*R1+A5$$

$$G(6,7)=3.*T2-3.*T1-36.*C3/DET$$

$$G(6,8)=3.*(Q1+Q2)+6.*(C2*C2-C1*C1)/DET$$

$$G(6,9)=9.-3.*(R1+R2)+A6$$

$$G(7,1)=7.*T2+3.*T3-36.*C1/DET$$

$$G(7,2)=3.*Q3-7.*Q2+6.*C1*(C2-C3)/DET$$

$$G(7,3)=19.+7.*R2-3.*R3+A4$$

$$G(7,4)=3.*T1-3.*T3-36.*C2/DET$$

$$G(7,5)=3.*(Q1+Q3)+6.*(C1*C1-C3*C3)/DET$$

$$G(7,6)=9.-3.*(R1+R3)+A5$$

$$G(7,7)=-3.*T1-7.*T2-36.*C3/DET$$

```

G(7,8)=3.*Q1-7.*Q2+6.*(C2*C2-C1*C1)/DET
G(7,9)=7.*R2-1.-3.*R1+A6
G(9,1)=7.*(T3-T2)+36.*C1/DET
G(9,2)=7.*(Q2+Q3)-6.*C1*(C2-C3)/DET
G(9,3)=-9.-7.*(R2+R3)-A4
G(9,4)=-13.*T1-7.*T3+36.*C2/DET
G(9,5)=-13.*Q1+7.*Q3-6.*(C1*C1-C3*C3)/DET
G(9,6)=13.*R1-9.-7.*R3-A5
G(9,7)=13.*T1+7.*T2+36.*C3/DET
G(9,8)=-13.*Q1+7.*Q2-6.*(C2*C2-C1*C1)/DET
G(9,9)=13.*R1-9.-7.*R2-A6

```

```
DO 130 I=1,9
```

```
    G(3,I)=-G(1,I)
```

```
    G(5,I)=-G(2,I)
```

```
    G(8,I)=-G(6,I)
```

```
130  G(10,I)=-G(7,I)
```

```
    G(3,2)=G(3,2)+10
```

```
    G(3,5)=G(3,5)-10
```

```
    G(5,2)=G(5,2)+10
```

```
    G(5,8)=G(5,8)-10
```

```
    G(8,3)=G(8,3)+10
```

```
    G(8,6)=G(8,6)-10
```

```
    G(10,3)=G(10,3)+10
```

```
    G(10,9)=G(10,9)-10
```

```
DO 140 I=1,9
```

```
    T(1,I)=(B2*G(1,I)+B3*G(2,I))/10.
```

```
    T(2,I)=(2.*B2*G(3,I)+B3*G(4,I))/10.
```

```
    T(3,I)=(B2*G(4,I)+2.*B3*G(5,I))/10.
```

```
    T(4,I)=(C2*G(6,I)+C3*G(7,I))/10.
```

```
    T(5,I)=(2.*C2*G(8,I)+C3*G(9,I))/10.
```

```
    T(6,I)=(C2*G(9,I)+2.*C3*G(10,I))/10.
```

```
    T(7,I)=(C2*G(1,I)+C3*G(2,I)+B2*G(6,I)+B3*G(7,I))/10.
```

$$T(8,I) = (2.*C2*G(3,I) + C3*G(4,I) + 2.*B2*G(8,I) + B3*G(9,I))/10.$$

$$T(9,I) = (C2*G(4,I) + 2.*C3*G(5,I) + B2*G(9,I) + 2.*B3*G(10,I))/10.$$

140 CONTINUE

RETURN

END

SUBROUTINE TME2(T)

COMMON/XAB/B(3),C(3)

DIMENSION T(9,9),G(10,9),AA(2,9),CC(2,2)

DET=B(2)*C(3)-B(3)*C(2)

CC(1,1)=-B(2)/DET/4.

CC(1,2)=-B(3)/DET/4.

CC(2,1)=-C(2)/DET/4.

CC(2,2)=-C(3)/DET/4.

P1=6.*C(1)/(B(1)**2+C(1)**2)

P2=6.*C(2)/(B(2)**2+C(2)**2)

P3=6.*C(3)/(B(3)**2+C(3)**2)

Q1=3.*B(1)*C(1)/(B(1)**2+C(1)**2)

Q2=3.*B(2)*C(2)/(B(2)**2+C(2)**2)

Q3=3.*B(3)*C(3)/(B(3)**2+C(3)**2)

R1=3.*B(1)**2/(B(1)**2+C(1)**2)

R2=3.*B(2)**2/(B(2)**2+C(2)**2)

R3=3.*B(3)**2/(B(3)**2+C(3)**2)

S1=3.*C(1)**2/(B(1)**2+C(1)**2)

S2=3.*C(2)**2/(B(2)**2+C(2)**2)

S3=3.*C(3)**2/(B(3)**2+C(3)**2)

T1=6.*B(1)/(B(1)**2+C(1)**2)

T2=6.*B(2)/(B(2)**2+C(2)**2)

T3=6.*B(3)/(B(3)**2+C(3)**2)

R01=(B(1)*B(2)+C(1)*C(2))/(B(1)*B(1)+C(1)*C(1))

```

RO2=(B(2)*B(3)+C(2)*C(3))/(B(2)*B(2)+C(2)*C(2))
RO3=(B(3)*B(1)+C(3)*C(1))/(B(3)*B(3)+C(3)*C(3))
DO 230 I=1,2
  E1=CC(I,1)
  E2=CC(I,2)
  AA(I,1)=E1*(30.*RO2-114.)-E2*(30.*RO3+144.)
  AA(I,4)=E1*(30.*RO1+144.)+E2*30.*(RO1+RO3+1.)
  AA(I,7)=-E1*30.*(RO1+RO2+1.)+E2*(114.-30.*RO1)
  AA(I,2)=E1*(C(2)*(14.-15.*RO2)-7.*C(3))+E2*(7.*C(2)-C(3)*(29.
1      +15.*RO3))
  AA(I,3)=-E1*(B(2)*(14.-15.*RO2)-7.*B(3))-E2*(7.*B(2)-B(3)*(29.
1      +15.*RO3))
  AA(I,5)=E1*(C(1)*(29.+15.*RO1)-7.*C(3))+E2*(C(1)*(22.+15.*RO1)
1      -C(3)*(15.*RO3-7.))
  AA(I,6)=-E1*(B(1)*(29.+15.*RO1)-7.*B(3))-E2*(B(1)*(22.+15.*RO1)
1      -B(3)*(15.*RO3-7.))
  AA(I,8)=E1*(C(1)*(15.*RO1-7.)-C(2)*(15.*RO2+22.))+E2*(7.*C(2)
1      +C(1)*(15.*RO1-14.))
230  AA(I,9)=-E1*(B(1)*(15.*RO1-7.)-B(2)*(15.*RO2+22.))-E2*(7.*B(2)+
1      B(1)*(15.*RO1-14.))
  G(1,1)=3.*P2+4.*P3+AA(1,1)
  G(1,2)=16.-3.*S2+4.*S3+AA(1,2)
  G(1,3)=3.*Q2-4.*Q3+AA(1,3)
  G(1,4)=-3.*P1-4.*P3+AA(1,4)
  G(1,5)=2.-3.*S1+4.*S3+AA(1,5)
  G(1,6)=3.*Q1-4.*Q3+AA(1,6)
  G(1,7)=3.*(P1-P2)+AA(1,7)
  G(1,8)=9.-3.*(S1+S2)+AA(1,8)
  G(1,9)=3.*(Q1+Q2)+AA(1,9)
  G(2,1)=-4.*P2-3.*P3+AA(1,1)
  G(2,2)=16.+4.*S2-3.*S3+AA(1,2)
  G(2,3)=3.*Q3-4.*Q2+AA(1,3)

```

$G(2,4)=3.*(P3-P1)+AA(1,4)$
 $G(2,5)=9.-3.*(S1+S3)+AA(1,5)$
 $G(2,6)=3.*(Q1+Q3)+AA(1,6)$
 $G(2,7)=3.*P1+4.*P2+AA(1,7)$
 $G(2,8)=2.-3.*S1+4.*S2+AA(1,8)$
 $G(2,9)=3.*Q1-4.*Q2+AA(1,9)$
 $G(4,1)=4.*(P2-P3)-AA(1,1)$
 $G(4,2)=-9.-4.*(S2+S3)-AA(1,2)$
 $G(4,3)=4.*(Q2+Q3)-AA(1,3)$
 $G(4,4)=10.*P1+4.*P3-AA(1,4)$
 $G(4,5)=10.*S1-9.-4.*S3-AA(1,5)$
 $G(4,6)=4.*Q3-10.*Q1-AA(1,6)$
 $G(4,7)=-10.*P1-4.*P2-AA(1,7)$
 $G(4,8)=10.*S1-9.-4.*S2-AA(1,8)$
 $G(4,9)=4.*Q2-10.*Q1-AA(1,9)$
 $G(6,1)=-3.*T2-4.*T3+AA(2,1)$
 $G(6,2)=3.*Q2-4.*Q3+AA(2,2)$
 $G(6,3)=16.-3.*R2+4.*R3+AA(2,3)$
 $G(6,4)=3.*T1+4.*T3+AA(2,4)$
 $G(6,5)=3.*Q1-4.*Q3+AA(2,5)$
 $G(6,6)=2.+4.*R3-3.*R1+AA(2,6)$
 $G(6,7)=3.*(T2-T1)+AA(2,7)$
 $G(6,8)=3.*(Q1+Q2)+AA(2,8)$
 $G(6,9)=9.-3.*(R1+R2)+AA(2,9)$
 $G(7,1)=4.*T2+3.*T3+AA(2,1)$
 $G(7,2)=3.*Q3-4.*Q2+AA(2,2)$
 $G(7,3)=16.+4.*R2-3.*R3+AA(2,3)$
 $G(7,4)=3.*(T1-T3)+AA(2,4)$
 $G(7,5)=3.*(Q1+Q3)+AA(2,5)$
 $G(7,6)=9.-3.*(R1+R3)+AA(2,6)$
 $G(7,7)=-3.*T1-4.*T2+AA(2,7)$
 $G(7,8)=3.*Q1-4.*Q2+AA(2,8)$

$$G(7,9)=2.+4.*R2-3.*R1+AA(2,9)$$

$$G(9,1)=4.*(T3-T2)-AA(2,1)$$

$$G(9,2)=4.*(Q2+Q3)-AA(2,2)$$

$$G(9,3)=-9.-4.*(R2+R3)-AA(2,3)$$

$$G(9,4)=-10.*T1-4.*T3-AA(2,4)$$

$$G(9,5)=-10.*Q1+4.*Q3-AA(2,5)$$

$$G(9,6)=10.*R1-9.-4.*R3-AA(2,6)$$

$$G(9,7)=10.*T1+4.*T2-AA(2,7)$$

$$G(9,8)=-10.*Q1+4.*Q2-AA(2,8)$$

$$G(9,9)=10.*R1-9.-4.*R2-AA(2,9)$$

DO 240 I=1,9

$$G(3,I)=-G(1,I)$$

$$G(5,I)=-G(2,I)$$

$$G(8,I)=-G(6,I)$$

240 G(10,I)=-G(7,I)

$$G(3,2)=G(3,2)+7.$$

$$G(3,5)=G(3,5)-7.$$

$$G(5,2)=G(5,2)+7.$$

$$G(5,8)=G(5,8)-7.$$

$$G(8,3)=G(8,3)+7.$$

$$G(8,6)=G(8,6)-7.$$

$$G(10,3)=G(10,3)+7.$$

$$G(10,9)=G(10,9)-7.$$

DO 250 I=1,9

$$T(1,I)=(B(2)*G(1,I)+B(3)*G(2,I))/7.$$

$$T(2,I)=(2.*B(2)*G(3,I)+B(3)*G(4,I))/7.$$

$$T(3,I)=(B(2)*G(4,I)+2.*B(3)*G(5,I))/7.$$

$$T(4,I)=(C(2)*G(6,I)+C(3)*G(7,I))/7.$$

$$T(5,I)=(2.*C(2)*G(8,I)+C(3)*G(9,I))/7.$$

$$T(6,I)=(C(2)*G(9,I)+2.*C(3)*G(10,I))/7.$$

$$T(7,I)=(C(2)*G(1,I)+C(3)*G(2,I)+B(2)*G(6,I)+B(3)*G(7,I))/7.$$

$$T(8,I)=(2.*C(2)*G(3,I)+C(3)*G(4,I)+2.*B(2)*G(8,I)+B(3)*G(9,I))$$

1)/7.

$T(9,I) = (C(2)*G(4,I) + 2.*C(3)*G(5,I) + B(2)*G(9,I) + 2.*B(3)*G(10,I)$

1)/7.

250 CONTINUE

RETURN

END

SUBROUTINE UVKAPA(BM,R,T)

COMMON/ST/ESM(29,29)

COMMON/CODE/KOD(29)

DIMENSION BM(5,3),R(9,10),T(9,9),TEMP(10,9),BK(9,9)

CALL UVK(BM(1,1),BM(1,2),BM(1,3),BM(2,1),BM(2,2),BM(2,3),BK)

CALL TRMULT(R,BK,T,TEMP,10,9,9,9,0)

DO 100 I=1,10

DO 100 J=1,9

ESM(KOD(I),KOD(J+20))=ESM(KOD(I),KOD(J+20))+TEMP(I,J)

100 ESM(KOD(J+20),KOD(I))=ESM(KOD(I),KOD(J+20))

CALL UVK(BM(3,1),BM(3,2),BM(3,3),BM(4,1),BM(4,2),BM(4,3),BK)

CALL TRMULT(R,BK,T,TEMP,10,9,9,9,0)

DO 200 I=1,10

DO 200 J=1,9

ESM(KOD(I+10),KOD(J+20))=ESM(KOD(I+10),KOD(J+20))+TEMP(I,J)

200 ESM(KOD(J+20),KOD(I+10))=ESM(KOD(I+10),KOD(J+20))

RETURN

END

```

SUBROUTINE WKAPA(B51,B52,B53,T,S)
COMMON/ST/ESM(29,29)
COMMON/CODE/KOD(29)
DIMENSION IT(10,3),BK(10,9),TEMP(9,9),T(9,9),S(10,9)
DATA IT/180,60,60,30,15,30,18,6,6,18,60,30,15,18,6,6,12,3,2,3,60,
1    15,30,6,6,18,3,2,3,12/
DO 100 I=1,10
DO 100 J=1,3
J3=J+3
J6=J+6
BK(I,J)=IT(I,J)*B51/360.
BK(I,J3)=IT(I,J)*B52/360.
100 BK(I,J6)=IT(I,J)*B53/360.
CALL TRMULT(S,BK,T,TEMP,9,10,9,9,0)
DO 200 I=1,9
DO 200 J=1,9
200 ESM(KOD(I+20),KOD(J+20))=ESM(KOD(I+20),KOD(J+20))-TEMP(I,J)-
1    TEMP(J,I)
RETURN
END

```

```

SUBROUTINE FMAT(X13,X14,X23,X24,C)
DIMENSION C(9,9)
C(1,1)=180.*X13
C(1,2)=180.*X14
C(1,3)=120.*X13
C(1,4)=60.*(X13+X14)
C(1,5)=120.*X14
C(1,6)=90.*X13
C(1,7)=30.*(X13+X14)

```

$$C(1,8)=30. \cdot (X13+X14)$$

$$C(1,9)=90. \cdot X14$$

$$C(2,1)=180. \cdot X23$$

$$C(2,2)=180. \cdot X24$$

$$C(2,3)=120. \cdot X23$$

$$C(2,4)=60. \cdot (X23+X24)$$

$$C(2,5)=120. \cdot X24$$

$$C(2,6)=90. \cdot X23$$

$$C(2,7)=30. \cdot (X23+X24)$$

$$C(2,8)=30. \cdot (X23+X24)$$

$$C(2,9)=90. \cdot X24$$

$$C(3,1)=120. \cdot X13$$

$$C(3,2)=120. \cdot X14$$

$$C(3,3)=120. \cdot X13$$

$$C(3,4)=30. \cdot X13+60. \cdot X14$$

$$C(3,5)=60. \cdot X14$$

$$C(3,6)=108. \cdot X13$$

$$C(3,7)=24. \cdot X13+36. \cdot X14$$

$$C(3,8)=12. \cdot X13+24. \cdot X14$$

$$C(3,9)=36. \cdot X14$$

$$C(4,1)=60. \cdot (X13+X23)$$

$$C(4,2)=60. \cdot (X14+X24)$$

$$C(4,3)=30. \cdot X13+60. \cdot X23$$

$$C(4,4)=30. \cdot (X13+X24)+15. \cdot (X14+X23)$$

$$C(4,5)=60. \cdot X14+30. \cdot X24$$

$$C(4,6)=18. \cdot X13+54. \cdot X23$$

$$C(4,7)=12. \cdot (X13+X23)+6. \cdot X14+18. \cdot X24$$

$$C(4,8)=18. \cdot X13+12. \cdot (X14+X24)+6. \cdot X23$$

$$C(4,9)=54. \cdot X14+18. \cdot X24$$

$$C(5,1)=120. \cdot X23$$

$$C(5,2)=120. \cdot X24$$

$$C(5,3)=60. \cdot X23$$

$$C(5,4)=60.*X_{23}+30.*X_{24}$$

$$C(5,5)=120.*X_{24}$$

$$C(5,6)=36.*X_{23}$$

$$C(5,7)=24.*X_{23}+12.*X_{24}$$

$$C(5,8)=36.*X_{23}+24.*X_{24}$$

$$C(5,9)=108.*X_{24}$$

$$C(6,1)=90.*X_{13}$$

$$C(6,2)=90.*X_{14}$$

$$C(6,3)=108.*X_{13}$$

$$C(6,4)=18.*X_{13}+54.*X_{14}$$

$$C(6,5)=36.*X_{14}$$

$$C(6,6)=108.*X_{13}$$

$$C(6,7)=18.*X_{13}+36.*X_{14}$$

$$C(6,8)=6.*X_{13}+18.*X_{14}$$

$$C(6,9)=18.*X_{14}$$

$$C(7,1)=30.*(X_{13}+X_{23})$$

$$-C(7,2)=30.*(X_{14}+X_{24})$$

$$C(7,3)=24.*X_{13}+36.*X_{23}$$

$$C(7,4)=12.*(X_{13}+X_{14})+6.*X_{23}+18.*X_{24}$$

$$C(7,5)=24.*X_{14}+12.*X_{24}$$

$$C(7,6)=18.*X_{13}+36.*X_{23}$$

$$C(7,7)=8.*X_{13}+6.*(X_{14}+X_{23})+12.*X_{24}$$

$$C(7,8)=6.*(X_{13}+X_{24})+8.*X_{14}+2.*X_{23}$$

$$C(7,9)=18.*X_{14}+6.*X_{24}$$

$$C(8,1)=30.*(X_{13}+X_{23})$$

$$C(8,2)=30.*(X_{14}+X_{24})$$

$$C(8,3)=12.*X_{13}+24.*X_{23}$$

$$C(8,4)=18.*X_{13}+6.*X_{14}+12.*(X_{23}+X_{24})$$

$$C(8,5)=36.*X_{14}+24.*X_{24}$$

$$C(8,6)=6.*X_{13}+18.*X_{23}$$

$$C(8,7)=6.*(X_{13}+X_{24})+2.*X_{14}+8.*X_{23}$$

$$C(8,8)=12.*X_{13}+6.*(X_{14}+X_{23})+8.*X_{24}$$

```

C(8,9)=36.*X14+18.*X24
C(9,1)=90.*X23
C(9,2)=90.*X24
C(9,3)=36.*X23
C(9,4)=54.*X23+18.*X24
C(9,5)=108.*X24
C(9,6)=18.*X23
C(9,7)=18.*X23+6.*X24
C(9,8)=36.*X23+18.*X24
C(9,9)=108.*X24
DO 50 I=1,9
DO 50 J=1,9
50 C(I,J)=C(I,J)/360.
RETURN
END

```

```

SUBROUTINE UVWMAT(A,B,C)
COMMON/CODE/KOD(29)
DIMENSION IT(10,9),IR(9),IC(10),C(9,10)
DATA IT/1260,2*420,210,105,210,126,42,42,126,10*0,840,420,210,252,
1      84,84,168,42,28,42,420,105,210,42,42,126,21,14,21,84,10*0,
2      630,378,126,252,63,42,180,36,18,18,210,84,84,42,28,42,24,
3      12,12,24,210,42,126,14,21,84,8,8,12,60,10*0/
DATA IR/2,1,5,4,3,9,8,7,6/IC/1,3,2,6,5,4,10,9,8,7/
DO 100 I=1,9
DO 100 J=1,10
100 C(I,J)=(A*IT(J,I)+B*IT(IC(J),IR(I)))/2520.
RETURN
END

```

```
SUBROUTINE KAPA(DET,T)
COMMON/ST/ESM(29,29)
COMMON/ABD/AAM(3,3),ABM(3,3),ADM(3,3)
COMMON/CODE/KOD(29)
DIMENSION PP(3,3),TEMP(9,9),DD(9,9),T(9,9)
DATA PP/12.,3*4.,2.,1.,4.,1.,27/
DO 25 I=1,3
DO 25 J=1,3
DO 25 K1=1,3
II=(I-1)*3+K1
DO 25 K2=1,3
JJ=(J-1)*3+K2
25 DD(II,JJ)=ADM(I,J)*PP(K1,K2)/DET/24.
CALL TRMULT(T,DD,T,TEMP,9,9,9,9,1)
DO 100 I=1,9
DO 100 J=1,9
100 ESM(KOD(I+20),KOD(J+20))=ESM(KOD(I+20),KOD(J+20))+TEMP(I,J)
RETURN
END
```

END

FIN

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