Comparison Study between Classical ARIMA and Seasonal Adjustment Methods

by

Xu Zhu

**A thesis submitted to the
Faculty of Graduate Studies and Research
in partial fulfilment of the requirements
for the degree of Master of Statistics**

Supervisor: Dr. M. Ould Haye

**Ottawa-Carleton Institute of Mathematics and Statistics
School of Mathematics and Statistics
Carleton University
Ottawa, ON**

Library and
Archives Canada

Published Heritage
Branch

395 Wellington Street
Ottawa ON K1A 0N4
Canada

Bibliothèque et
Archives Canada

Direction du
Patrimoine de l'édition

395, rue Wellington
Ottawa ON K1A 0N4
Canada

Your file Votre référence
ISBN: 978-0-494-52007-9
Our file Notre référence
ISBN: 978-0-494-52007-9

NOTICE:

The author has granted a non-exclusive license allowing Library and Archives Canada to reproduce, publish, archive, preserve, conserve, communicate to the public by telecommunication or on the Internet, loan, distribute and sell theses worldwide, for commercial or non-commercial purposes, in microform, paper, electronic and/or any other formats.

The author retains copyright ownership and moral rights in this thesis. Neither the thesis nor substantial extracts from it may be printed or otherwise reproduced without the author's permission.

AVIS:

L'auteur a accordé une licence non exclusive permettant à la Bibliothèque et Archives Canada de reproduire, publier, archiver, sauvegarder, conserver, transmettre au public par télécommunication ou par l'Internet, prêter, distribuer et vendre des thèses partout dans le monde, à des fins commerciales ou autres, sur support microforme, papier, électronique et/ou autres formats.

L'auteur conserve la propriété du droit d'auteur et des droits moraux qui protègent cette thèse. Ni la thèse ni des extraits substantiels de celle-ci ne doivent être imprimés ou autrement reproduits sans son autorisation.

In compliance with the Canadian Privacy Act some supporting forms may have been removed from this thesis.

Conformément à la loi canadienne sur la protection de la vie privée, quelques formulaires secondaires ont été enlevés de cette thèse.

While these forms may be included in the document page count, their removal does not represent any loss of content from the thesis.

Bien que ces formulaires aient inclus dans la pagination, il n'y aura aucun contenu manquant.

Canada

Abstract:

Autoregressive Moving Average (ARMA) models are widely used to describe time series data. The Autoregressive Integrated Moving Average (ARIMA) models are more likely to be used to describe time series with trend. The ARIMA models which describe series with seasonal effects are called seasonal ARIMA models. However, time series might also be affected by some factors other than seasonal factors or trend, such as leap year effects, trading day effects and so on. The traditional methods (i.e., ARIMA and seasonal ARIMA models) may not satisfy the series with those factors. This thesis will investigate the properties and techniques related to new methods of seasonal adjustment, called X-12-ARIMA methods. We will also compare the fitting potentials of these new methods to the ARIMA ones. Some real-world data will be used in these comparisons.

Acknowledgements

I wish to thank all of those who helped me to complete this thesis, both directly and indirectly, to achieve another peak in my education. I am very grateful to Dr. M. Ould Haye for his encouragement, guidance and comments throughout my undergraduate and graduate studies at Carleton University. During the whole process of this thesis writing, his patient guidance and advice encouraged me and cheered me up. I also owe a great deal to my parents, who patiently supported me with everything I needed throughout my studies from elementary school until now. It is the support and encouragement of all these individuals and others, too many to be mentioned here, that gave me the energy and strength to accomplish my goal of this degree.

Table of Contents

➤ Abstract.....	2
➤ Acknowledgements.....	3
➤ Introduction.....	6-11
➤ Main Parts of the thesis.....	11-91
1. Overview of Traditional Methods	12-23
1.1. Basic concepts and relations of ARMA, ARIMA and seasonal ARIMA models.....	12
1.2. Identification of ARMA, ARIMA and seasonal ARIMA models.	
1.2.1. The computing formulae of the sample ACF, PACF and IACF.....	14
1.2.2. Model identifications, diagnostic checking and model selection.....	18
2. Seasonal ARIMA Models in X-12 Methods.....	24-39
2.1. The three types of decomposition of seasonal time series.....	27
2.1.1. The multiplicative time series decomposition.....	27
2.1.2. The additive decomposition.....	28
2.1.3. The Pseudo-additive decomposition.....	29
2.2. The X-12 methods default calculations.....	29
2.3. Lothian criterion for seasonal filters in X-12 methods.....	33
2.4. The filters for smoothing trend in X-12 methods.....	34
2.4.1. The introduction of symmetric Henderson Filters.....	34
2.4.2. The surrogate Henderson Filters.....	37
3. Properties of X-12 Methods.....	40-80
3.1. Seasonal and trading day effects detection.....	40
3.2. Regressor of trading day effects for the three types decompositions.....	52
3.3. Definitions of outliers and statistics detecting outliers.....	57
3.4. Outlier detection procedure.....	57
3.4.1. Forward adding method.....	58
3.4.2. Backward eliminating method.....	65
3.4.3. Likelihood ratio test.....	66
3.5. Stability diagnostics for seasonal and trend adjustments.....	66
3.5.1. Sliding spans.....	67
3.5.2. Useful computations in sliding span diagnostics.....	69
3.5.3. Revision history.....	73
3.5.4. One application of revision history.....	77
4. Results Comparison between Traditional ARIMA Model and regARIMA Models.....	81-91

4.1. The SGBS series.....	81
4.2. The Hotel series.....	86
4.3. The Trade series.....	89
➤ Conclusion.....	92-93
➤ Bibliography.....	94-96

Introduction:

In time series, Autoregressive Moving Average (ARMA) processes are useful in describing stationary time series. The stationary time series are series with constant mean, constant variance, and a covariance function depending only on the lag (or the lead). If we take differencing on the ARMA processes, we obtain the so-called Autoregressive Integrated Moving Average (ARIMA) model. This model is a large family of time series processes, and it is useful in describing non-stationary time series. Although the differencing technique can transform a non-stationary time series into a stationary one, it is incapable of transforming a non-stationary series caused by non-constant variances into a stationary one. If a series is non-stationary because of non-constant variance, it is likely that this series could use the power transformation functions to transform into a stationary one. The power transformation function method was introduced by Box and Cox, which is also called Box-Cox's transformation function (Wei, 2006). The ARMA and ARIMA models are used under the assumption that the seasonal component of the models is deterministic and uncorrelated to the non-seasonal components, but in practice this assumption does not always hold. Generally, the seasonal component is more likely to be stochastic and correlated to the non-seasonal components. Thus, based on the ARIMA models, researchers have built another model for describing time series with cyclical behaviours, which are correlated to the non-seasonal components. This model is called the seasonal ARIMA (seaARIMA) model. In this thesis, the ARMA models and the latter transformed models (i.e., ARIMA and seaARIMA models) are referred to as the classical or traditional models (methods).

Apart from the above, a time series might also contain other effects excluding the seasonal effects, such as trading day effects, holiday effects, leap year effects, etc.. The trading day effects are caused by the different occurrence of the dates in the same month but in different years. For example, there are four Saturdays in November 2007, but five Saturdays in November 2008. If any products are more likely to sell on weekends, the quantity of sales might be affected by the different occurrence of the trading day. Normally, the more weekends there are in a month, the bigger sales will be achieved in that month. The researchers defined two kinds of holiday effects that are fixed and moving holiday effects. For example, the Easter holiday falls in March 22 to April 25; this kind of holiday is defined as moving holiday effects. Unlike Easter holiday, Christmas is always on the same day of December, it is defined as a fixed holiday effect. Chinese New Year is another example of moving holiday effect, and the adjustment methods of moving holiday were discussed by Graham (2001). The leap year effect results from different length of February. The effects mentioned above are named as calendar effects as well.

The classical methods only identify the seasonal effects and define them as seasonal components in the decomposition of the time series. The other “hidden” effects such as the trading day effects mentioned before are combined with time series errors, which can together be considered as irregular components. The US Census Bureau spent more than fifty years researching and developing an electronic program that is used to do seasonal adjustment and to isolate those hidden effects from irregular components by adding appropriate regressors (i.e. the estimators of those hidden effects) into the classical models. The first program used to achieve seasonal adjustment was introduced in 1954.

Later, the US Census Bureau continued to modify the program and referred to the program as an “X-” series based on its “experimental variants”, (Shiskin, Young and Musgrave, 1967). The X-3 program was the first such variant released to the public in 1960. After that, the U.S. Census continued to modify and improve the program going with X-9, X-10. In October 1965, the X-11 program as a standard one was introduced to the public. Subsequently, Statistics Canada developed the X-11-ARIMA/88 based on the X-11 program. In our current study, we considered the X-12 program/method, developed by the US Census Bureau, as a standard program available to the public.

In this thesis, the ARIMA models from X-12 methods are called regression ARIMA (regARIMA) models. The X-12 method is a new program used to adjust a variety of effects contained in the irregular component of a time series. It is an enhanced version of the X-11 methods (Findley, et al., 1998). The enhancements of the X-12 methods include the following: alternative methods for seasonal, trading day, holiday and leap year effects adjustment that use defined regressors to estimate and adjust effects; more options for seasonal moving average filters and Henderson trend filters, and alternative seasonal decompositions; sliding spans and revision history analysis as new diagnostics to detect the stability of adjusted values; extensive time series modelling and model selection capabilities for linear regression models with ARIMA irregular components, with robust estimation of coefficients. This thesis will introduce more about these enhancements in later discussions.

The purpose of this thesis is to compare the forecasting values obtained by the seaARIMA model and regARIMA model with three applications, so as to examine which

model could give a better forecast for the time series with seasonal effects correlated to non-seasonal components.

The structure of the rest of this thesis is arranged as follows. Section 1 discusses the basic concepts of stationary and non-stationary classical ARIMA methods. These include the techniques of transferring a non-stationary time series to a stationary time series. For example, a non-stationary time series due to non-constant mean could be transferred to a stationary time series by applying differencing techniques. However, for a time series which is non-stationary in the variance and autocovariance, the Box-Cox's (power) transformation can be applied to obtain a stationary time series. We will also discuss the sample autocorrelation function (ACF) and the sample partial autocorrelation function (PACF) used for model identification, and the Portmanteau lack of fit test for diagnostic checking. Finally, we discuss some log-likelihood based criteria, which are used for model selection. These criteria include Akaike's information criterion (AIC), Bayesian information criterion (BIC), Schwartz's SBC (Schwartz's Bayesian criterion) and Parzen's CAT (criterion for autoregressive transfer functions). These criteria are used when there are at least two tentative models which pass all the diagnostic checking. In this case, the model with the smallest information value will be considered as the final model. Also, the X-12 methods use those criteria to check the regARIMA model with and without the regressors which estimate the calendar effects in a time series. If a regARIMA model without regressors has the smallest information values, the users can conclude that the time series does not need to do any adjustments corresponding to those regressors. Otherwise, we have to estimate the effects and add the estimators (i.e.,

regressors) into the model. Among these criteria, the AIC criterion is commonly used to do model selection for both classical and regARIMA models.

Section 2 discusses the decompositions of seasonal time series, which can be listed as multiplicative decomposition, additive decomposition and Pseudo-additive decomposition. The tools used to smooth the time series will be introduced in this section as well. For example, the tools used to smooth seasonal component and trend component such as the symmetric filters and the tools used on the ends of a time series which can not perform symmetric filters such as the asymmetric filters, will be introduced, (Wallis, 1982). The symmetric filters are seasonal moving average filters and Henderson trend filters. The asymmetric filters are Musgrave surrogates Henderson filters (Gray and Thomson, 1997).

Section 3 will discuss some new features and concepts of the X-12 methods. One of the new features is the spectral plots, which are useful in checking seasonal effects and trading-day effects for times series. In this section, we will not only introduce the definition of spectral plots, but also explain the diagnostics which are given by spectral plots. In addition, the log-likelihood base statistics that the X-12 methods use to detect outliers' effects (i.e., Additive outliers, Level shift, and Temporary change) will be discussed in this section. The forms of regressors for trading day and leap year effects will be given in the same section. In this section, we will also see how the X-12 program uses sliding spans and revision history analysis to examine the stability of adjusted values of seasonal adjustment and trend. If a large proportion of adjusted values greater than a given threshold, then the adjusted series is considered as being unstable. The term

“unstable” means that the percentage change of the adjusted values is not within the average of percentage change.

In section 4, we will give three examples: SGBS series, Hotel series and Trade series. The three examples are monthly data with seasonal effects. In each case, we used the classical methods and seasonal adjustment methods to do modelling, respectively. After that, we used both methods to estimate the data of a single year, and compared the forecasted values with the actual values, to see which methods fit the time series data better. Finally, we made conclusions at the end of this thesis.

1. Overview of Traditional Methods

1.1 Basic concepts and relations of ARMA, ARIMA and seasonal ARIMA models

ARMA models are obtained by combining Autoregressive (AR) process and Moving Average (MA) process. The autoregressive model with order p , $AR(p)$, is given by

$$\Phi_p(B)\dot{Z}_t = a_t. \quad (1.1)$$

Here, $\Phi_p(B)$ is the autoregressive polynomial and “B” means backshift, which $\Phi_p(B) = 1 - \phi_1 B^1 - \phi_2 B^2 - \dots - \phi_p B^p$ with $(B\dot{Z}_t = \dot{Z}_{t-1})$ and $\dot{Z}_t = Z_t - \mu$. If the series has zero mean, then $\dot{Z}_t = Z_t$. The moving average model with order q , $MA(q)$, given by

$$\dot{Z}_t = \Theta_q(B)a_t. \quad (1.2)$$

Here, $\Theta_q(B)$ is the moving average polynomial which shifts time series $\dot{Z}_t$ back- q -steps and $\Theta_q(B) = 1 - \theta_1 B^1 - \theta_2 B^2 - \dots - \theta_q B^q$. In AR and MA models, a_t is a sequence of uncorrelated random variables with zero mean and constant variance σ_a^2 . In the entire thesis, a_t will be considered as a white noise and represented as $a_t \sim WN(0, \sigma_a^2)$.

The AR and MA models are basic models in time series. A problem of the two models is that they might contain too many parameters in order to achieve a good approximation. However, if a model contains too many parameters, the efficiency of estimation of the model will be reduced. $ARMA(p, q)$ model, combined $AR(p)$ and $MA(q)$ models, can give a better approximation with less parameter. The expression of $ARMA(p, q)$ is

$$\Phi_p(B)\dot{Z}_t = \Theta_q(B)a_t. \quad (1.3)$$

The ARMA model is an alternative choice when AR and MA models have three or more parameters.

If an $AR(p)$ process has a finite number of nonzero weights π , i.e., $\pi_1 = \phi_1, \pi_2 = \phi_2, \dots, \pi_p = \phi_p$ and $\pi_k = 0$ for any $k > p$, and $\sum_{j=1}^{\infty} |\pi_j| = \sum_{j=1}^p |\phi_j| < \infty$, then the

process is always invertible. Invertible means that a_t can be expressed in terms of past and present value of the time series Z_t . Similarly, if a $MA(q)$ process has a finite number of nonzero weights ψ , i.e., $\psi_1 = -\theta_1$, $\psi_2 = -\theta_2, \dots$, $\psi_q = -\theta_q$ and $\psi_k = 0$ for any $k > q$, and $1 + \theta_1^2 + \dots + \theta_q^2 < \infty$, this finite process is always causal. The term of causal means that a time series Z_t can be represented in terms of past white noise values, $a_t, a_{t-1}, \dots$. For autoregressive process to be causal, the roots of autoregressive polynomial, $\Phi_p(B)$, must lie outside of the unit circle. For moving average process to be invertible, the roots of moving average polynomial, $\Theta_q(B)$, must lie outside of the unit circle as well. Moreover, the ARMA model contains an invertible process AR, and a stationary process MA. Thus, for ARMA process to be causal, the roots of $\Phi_p(B)$ must lie outside unit circle; and for the process to be invertible, the roots of $\Theta_q(B)$ must lie outside of the unit circle. Also, it is necessary to assume the polynomials $\Phi_p(B)$ and $\Theta_q(B)$ do not share common roots.

A time series by which a model described must be a stationary process, because the non-stationary process will affect the model's forecasted accuracy. A time series Z_t can be defined as a stationary time series, if it has (i) a constant mean, i.e., $E(Z_t) = \mu$ for all t , (ii) a constant variance, i.e., $Var(Z_t) = E[(Z_t - \mu)^2] = \sigma^2$, and (iii) the covariance of Z_t and Z_s is only a function of difference $(t - s)$, for $t \neq s$. A non-stationary time series, described by ARMA process, can be transformed into a stationary time series by invoking a proper degree of differencing. After taking differencing, ARMA models lead to a large, useful family in time series area, called the Autoregressive Integrated Moving

Average (ARIMA) models. The ARIMA(p, d, q) model or process can be written as

$$\Phi_p(B)(1-B)^d \dot{Z}_t = \Theta_q(B)a_t, \text{ } d \text{ is any integer and } d \geq 0, \quad (1.4)$$

where, $\Phi_p(B)$, $\Theta_q(B)$, $\dot{Z}_t$ and a_t are defined in previous paragraphs, $(1-B)^d \dot{Z}_t$ means that we take d^{th} differencing on time series $\dot{Z}_t$. For some business and economic time series, which repeat itself after a regular time period, the smallest time period is called the seasonal period. A model or process, used to represent this kind of time series, is called the seasonal ARIMA, and denoted as seaARIMA model. The representations of seaARIMA(p, d, q) $\times$ (P, D, Q) $_s$ is

$$\Phi_p(B)\Phi_P(B^s)(1-B)^d(1-B^s)^D \dot{Z}_t = \Theta_q(B)\Theta_Q(B^s)a_t. \quad (1.5)$$

Here "s" is the seasonal period ($s = 12$ for monthly time series, $s = 4$ for quarterly time series), $\Phi_P(B^s) = 1 - \phi_1 B^s - \phi_2 B^{2s} - \dots - \phi_P B^{Ps}$ is the seasonal AR operator with P^{th} order, $\Theta_Q(B^s) = 1 - \theta_1 B^s - \theta_2 B^{2s} - \dots - \theta_Q B^{Qs}$ is the seasonal MA operator with Q^{th} order, $(1-B^s)^D$ with $D = 0, 1$ or 2 , means taking D^{th} differencing on seasonal parts. And, $\Phi_p(B)$ and $\Theta_q(B)$ are the operators of non-seasonal parts.

1.2 Identification of ARMA, ARIMA and seasonal ARIMA models

1.2.1 The Computing Formulae of the Sample ACF, PACF and IACF

The plots of ACF, PACF and IACF are useful tools in model identification. The sample autocorrelation function (ACF), $\hat{\rho}_k$, is

$$\hat{\rho}_k = \frac{\hat{\gamma}(k)}{\hat{\gamma}(0)} = \frac{cov(Z_t, Z_{t+k})}{cov(Z_t, Z_t)} = \frac{cov(Z_t, Z_{t+k})}{var(Z_t)} = \frac{cov(Z_t, Z_{t+k})}{\sqrt{var(Z_t)}\sqrt{var(Z_t)}}. \quad (1.6)$$

Here, $\hat{\rho}_k$ is the estimator of autocorrelation, ρ_k , between Z_t and Z_{t+k} . The properties of ACF are: (i) $\rho(0) = 1$, (ii) $|\rho(k)| \leq 1$, (iii) symmetric, $\rho(k) = \rho(-k)$. $\hat{\gamma}(k)$ is the estimator of autocovariance, $\gamma(k)$, between time series Z_t and Z_{t+k} . $\hat{\gamma}(0)$, is the

estimator of autocovariance between time series Z_t and Z_{t+k} , it is also the estimator of variance of Z_t , $\hat{\gamma}(k) \leq \hat{\gamma}(0)$, $\hat{\gamma}(0) \geq 0$ and $\hat{\gamma}(k) = \hat{\gamma}(-k)$ for all k . W. S. Wei (2006) discussed two biased estimators of autocovariance, $\gamma(k)$, the one is

$$\hat{\gamma}(k) = \frac{1}{n-k} \sum_{t=1}^{n-k} (Z_t - \bar{Z})(Z_{t+k} - \bar{Z}). \quad (1.7)$$

Then,

$$\begin{aligned} \sum_{t=1}^{n-k} (Z_t - \bar{Z})(Z_{t+k} - \bar{Z}) &= \sum_{t=1}^{n-k} [(Z_t - \mu) - (\bar{Z} - \mu)][(Z_{t+k} - \mu) - (\bar{Z} - \mu)] \\ &= \sum_{t=1}^{n-k} (Z_t - \mu)(Z_{t+k} - \mu) - (\bar{Z} - \mu) \sum_{t=1}^{n-k} (Z_t - \mu) \\ &\quad - (\bar{Z} - \mu) \sum_{t=1}^{n-k} (Z_{t+k} - \mu) + (n-k)(\bar{Z} - \mu)^2, \end{aligned} \quad (1.8)$$

in equation (1.8), $\sum_{t=1}^{n-k} (Z_t - \mu)$ and $\sum_{t=1}^{n-k} (Z_{t+k} - \mu)$ approximate $(n-k)(\bar{Z} - \mu)$. Now, the equation (1.8) can be approximated as follows:

$$\sum_{t=1}^{n-k} (Z_t - \bar{Z})(Z_{t+k} - \bar{Z}) \approx \sum_{t=1}^{n-k} (Z_t - \mu)(Z_{t+k} - \mu) - (n-k)(\bar{Z} - \mu)^2. \quad (1.9)$$

Finally, we get

$$E(\hat{\gamma}(k)) \approx \gamma(k) - \text{Var}(\bar{Z}). \quad (1.10)$$

The other biased estimator and the reason why they are biased estimators will be summarized in later discussions.

The partial autocorrelation function:

$$\hat{\phi}_{kk} = \text{corr}[(Z_{k+1} - \hat{Z}_{k+1}|Z_2, \dots, Z_k), (Z_1 - \hat{Z}_1|Z_2, \dots, Z_k)]. \quad (1.11)$$

Here, $\hat{Z}_1$ and $\hat{Z}_{k+1}$ are the best linear predictions of Z_1 and Z_{k+1} in terms of $Z_2, \dots, Z_k$.

$\hat{Z}_1 = \alpha_2 Z_2 + \dots + \alpha_k Z_k$ for some $\alpha_2, \dots, \alpha_k$ and $\hat{Z}_{k+1} = \beta_2 Z_2 + \dots + \beta_k Z_k$ for some $\beta_2, \dots, \beta_k$. $\hat{Z}_1$ is backcast of Z_1 and $\hat{Z}_{k+1}$ is forecast of Z_{k+1} . In other words, $\hat{\phi}_{kk}$ is the autocorrelation between Z_1 and Z_{k+1} after removing the contribution of intermediate values $Z_2, \dots, Z_k$. If Z_t is an AR (p) model, then PACF has the following property:

$$\phi_{kk} = \phi_{pp}, \text{ if } k = p ; \quad \phi_{kk} = 0, \text{ if } k > p.$$

For example, we simulated a time series that has sixty observations by applying AR (p): take $p = 1, \phi_1 = 0.6$. Then, we plot the sample ACF and PACF of the AR (1) process, the outputs are shown in the next page:

ACF

Lag	Covariance	$\hat{\rho}_k$ Correlation	-1	9	8	7	6	5	4	3	2	1	0	1	2	3	4	5	6	7	8	9	1	Std Error
0	1.344123	1.00000																						0
1	0.708436	0.52706																						0.129099
2	0.372181	0.27690																						0.161017
3	0.101524	0.07553																						0.168767
4	0.027719	0.02062																						0.169329
5	-0.226228	-.16831																						0.169371
6	-0.084245	-.06268																						0.172136
7	-0.115625	-.08602																						0.172516
8	-0.126911	-.09442																						0.173229
9	-0.133223	-.09911																						0.174085
10	-0.059137	-.04400																						0.175023
11	-0.089636	-.06669																						0.175207
12	-0.123987	-.09224																						0.175630
13	-0.105308	-.07835																						0.176435
14	-0.056678	-.04217																						0.177014
15	0.058773	0.04373																						0.177182

"," marks two standard errors

PACF

	$\hat{\phi}_{kk}$																								
Lag	Correlation	-1	9	8	7	6	5	4	3	2	1	0	1	2	3	4	5	6	7	8	9	1			
1	0.52706										.														
2	-0.00125										.														
3	-0.09684												**												
4	0.02517												*												
5	-0.23173												****												
6	0.17095																	**							
7	-0.09148												**												
8	-0.07556												**												
9	0.02548													*											
10	-0.04855												*												
11	-0.00350																								
12	-0.09256													**											
13	-0.01029																								
14	0.02073																								
15	0.09118													**											

Obviously, from the output above, the sample ACF decreases and the PACF cuts off after lag 1 for the AR (1) process, meaning that the sample PACF $\hat{\phi}_{kk}$ are negligible for $k > 1$. Similarly, if the ACF cuts off after lag 1 and the sample PACF decreases, then we conclude that the time series is a MA(1) process. The sample Inverse Autocorrelation Function (IACF) is also a tool that can help to model identification. But, in practice, the PACF outperforms the IACF, and therefore, it is not necessary to be used. The sample IACF is only helpful if it is difficult to do model identification based on the sample ACF and the sample PACF. Suppose the time series Z_t following ARMA (p, q) process, then the inverse process of time series Z_t following ARMA (q, p) . The IACF of an AR (p) model will cut off after lag p , which basically looks like PACF. If the IACF decay fast, it indicates that is a MA (q) model. An exact definition of IACF can be the following: if a time series process has spectral density f , then IACF is the ACF of the time series that has spectral density $1/f$ (assuming it exists).

1.2.2 Model Identifications, Diagnostic Checking and Model Selection

Most of the time series are following ARIMA (p, d, q) process, with $p, d, q = 0, 1, \dots$. Taking $d = 0$, the stationary ARIMA $(p, 0, q)$ process is exactly a stationary ARMA (p, q) process, the sample ACF tails off after lag $(q - p)$ and the sample PACF tails off after $(p - q)$. For non-stationary AR (p) , MA (q) and ARMA (p, q) processes, the sample ACF neither cuts off nor decreases exponentially. The non-stationary time series commonly appears in practical situation and affects the accuracy of point estimation. Thus, we need transfer the non-stationary time series into a stationary one before forecasting. The concepts of model transformations and model differencing are commonly used to obtain a stationary time series from non-stationary one. After taking

d th degree differencing, if ACF tails down and PACF cuts off after lag p , the tentative model might be ARIMA $(p, d, 0)$; if ACF cuts off after lag q and PACF tails down, the tentative model might be ARIMA $(0, d, q)$; if the model is seasonal ARIMA $(p, d, 0) \times (0, D, 0)_s$ or seasonal ARIMA $(0, d, q) \times (0, D, 0)_s$, the sample ACF will have periodic stationarity with period $s = 4$ and $s = 12$ for quarterly and monthly time series. The differencing methods could transform non-stationary time series to a stationary one, but the methods do not work well for all non-stationary time series. If a time series is stationary in mean but nonstationary in variance, the Box-Cox power transformation technique is a better choice to obtain a stationary series rather than taking differencing. The Box-Cox power transformation is a variance stabilizing technique and the transformation function is

$$T(Z_t) = \frac{Z_t^\lambda - 1}{\lambda}, \quad 0 \leq \lambda \leq 1. \quad (1.12)$$

Some commonly used values of λ and their corresponding transformations are listed as follows.

Values of λ	Transformation
-1.0	$\frac{1}{Z_t}$
-0.5	$\frac{1}{\sqrt{Z_t}}$
0.0	$\ln Z_t$
0.5	$\sqrt{Z_t}$
1.0	Z_t

To show why we transform time series Z_t to $\ln Z_t$ as $\lambda = 0.0$, let $a^x = e^{x \ln a}$, then

$$(a^x)' = \ln a (e^{x \ln a}) = a^x (\ln a).$$

Hence,

$$\frac{a^x - 1}{x} \rightarrow \ln a, \quad \text{as } x \rightarrow 0,$$

and therefore, we have

$$\frac{Z_t^\lambda - 1}{\lambda} \rightarrow \ln(Z_t), \quad \text{as } \lambda \rightarrow 0.$$

After obtaining the stationary models, we need to check whether a_t 's are white noise or not. If a_t is considered as white noise, it should have zero mean, constant variance σ_a^2 and $\text{cov}(a_t a_s) = 0$, for $t \neq s$. To check if a_t 's follow normal distribution or not, we can construct a Q-Q plot. To check whether the variance of a_t 's is constant, we could check the plot of residuals. Finally, the Portmanteau lack of fit test helps to check whether a_t 's are correlated. The Portmanteau test is a modified Q statistic test, which is discussed by Box and Pierce (1970). The joint null hypothesis is

$$H_0: \rho_1 = \rho_2 = \dots = \rho_k = 0,$$

with the test statistic

$$Q = n(n+2) \sum_{k=1}^K (n-k)^{-1} \hat{\rho}_j^2, \quad 1 \leq j \leq k. \quad (1.13)$$

Here, $\hat{\rho}_j^2$'s are the estimators of square of autocorrelation, ρ_j^2 . Under the null hypothesis, the Q statistic was shown to follow the chi-square distribution with degree $(K - m)$ for large sample size, where $m = p + q$. To check whether a_t 's are white noise, other useful tools are the sample RACF (ACF plot for residuals) and the sample RPACF (PACF plot for residuals). If both the RACF and PACF cut off after lag 0, then we could conclude

that a_t 's are uncorrelated. If the p-values of Q test corresponding to the lags are greater than the significant level, then we can conclude that a_t 's are uncorrelated as well. If a_t 's are not with noise, we have to go back to the RACF and RPACF plots and see which lags have spikes. According to the information from these two plots, we modify the models by taking propitiate differencing.

The model identification methods discussed previously might find out more than one tentative model and those tentative models may possibly pass the model diagnostic checking. Among those models, the tentative model that best fits the time series will be considered as the final model. Now, we need a tool to help us select the “optimal” model, there are several model selection criteria concepts that could help users to select a best fitted model. The standard tool in time series for model selection is the Akaike's Information Criteria (AIC), its computation is available in many time series statistics software packages. Its computing formula is given by:

$$AIC(M) = -2 \ln(L) + 2M \quad (1.14)$$

where L is the conditional maximum likelihood and M is the number of parameters in the model. To show what conditional maximum likelihood estimation is, consider a general stationary ARMA (p, q) process

$$\dot{Z}_t = \phi_1 \dot{Z}_{t-1} + \cdots + \phi_p \dot{Z}_{t-p} + a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q}, \quad (1.15)$$

where $\dot{Z}_t = Z_t - \mu$. We assume that $\{a_t\}$ are i.i.d $N(0, \sigma_a^2)$ white noise. Then the equation (1.15) can be rewrite as

$$a_t = \dot{Z}_t - \phi_1 \dot{Z}_{t-1} - \cdots - \phi_p \dot{Z}_{t-p} + \theta_1 a_{t-1} + \cdots + \theta_q a_{t-q}. \quad (1.16)$$

The data $Z_1, \dots, Z_n$ are known and we define $Z = (Z_1, \dots, Z_n)'$ and assume the initial condition $Z_* = (Z_{1-p}, \dots, Z_{-1}, Z_0)'$ and $a_* = (a_{1-q}, \dots, a_{-1}, a_0)'$. The maximum

likelihood estimation of Φ', Θ', μ and σ_a^2 obtained from $a_1, \dots, a_n$ is calculated through the likelihood function

$$L_*(\Phi', \Theta', \mu, \sigma_a^2) = \frac{1}{(2\pi\sigma_a^2)^{\frac{n}{2}}} \exp \left[-\frac{1}{2\sigma_a^2} (a_1^2 + \dots + a_n^2) \right], \quad (1.17)$$

Here, $\Phi' = (\phi_1, \dots, \phi_p)'$, $\Theta' = (\theta_1, \dots, \theta_q)'$, $\mu = E(Z_t)$, those constants and σ_a^2 are unknown. The conditional log-likelihood function is:

$$Ln(L_*) = -\frac{n}{2} \ln(2\pi\sigma_a^2) - \frac{1}{2\sigma_a^2} S_*(\Phi, \Theta, \mu). \quad (1.18)$$

Where $S_*(\Phi, \Theta, \mu) = \sum_{t=1}^n a_t^2 (\Phi, \Theta, \mu | Z_*, a_*, Z)$. For this equation, we can estimate μ by $\bar{Z}$, we assume Z_* and a_* are given. Then estimates of Φ and Θ will be $\hat{\Phi}$ and $\hat{\Theta}$ which minimize $S_*(\Phi, \Theta)$. Because $\dot{Z}_t$ is stationary, we can replace the unknown $\dot{Z}_t$ by $\bar{\dot{Z}}_t$ with $\bar{\dot{Z}}_t = E(\dot{Z}_t)$ and replace unknown a_t by 0 because of $\{a_t\}$ are i.i.d $N(0, \sigma_a^2)$, (i.e. $E(a_t | \text{available data}) = 0$). Now, only σ_a^2 is unknown, to get estimate for σ_a^2 , we take derivative on equation (1.18) with respect to σ_a^2 , then

$$\frac{\partial Ln(L_*)}{\partial (\sigma_a^2)} = -\frac{n}{2\sigma_a^2} + \frac{S_*(\hat{\Phi}, \hat{\Theta})}{2(\sigma_a^2)^2}. \quad (1.19)$$

Now, let $\frac{\partial Ln(L_*)}{\partial (\sigma_a^2)} = 0$, we obtain the estimator of σ_a^2 , defined as $\hat{\sigma}_a^2$, which is

$$\hat{\sigma}_a^2 = \frac{S_*(\hat{\Phi}, \hat{\Theta})}{n - p - q - 1}.$$

Here, $S_*(\hat{\Phi}, \hat{\Theta})$ follows chi-square distribution asymptotically with the degree of freedom $n - (p + q + 1)$.

Finally, we obtain the maximum conditional log-likelihood function

$$Ln(L) = -\frac{n}{2} \ln(\hat{\sigma}_a^2) - \frac{n}{2} (1 + \ln 2\pi). \quad (1.20)$$

Since the second term of equation (1.20) is constant, the AIC (M) is reduced to:

$$AIC(M) = n \ln(\hat{\sigma}_a^2) + 2M. \quad (1.21)$$

Here M is the number of parameters in the model, which are the optimal order p and q of the model, to obtain a minimum $AIC(M)$ value. The model corresponding to the smallest AIC value will be considered as the best fitting model. Besides the AIC model selection criteria, Akaike's BIC (Bayesian information criterion), SBC (Schwartz's Bayesian criterion) and CAT (criterion for autoregressive transfer functions) are commonly used also. The computing formulae of these criteria are listed as follows:

1. **Akaike's BIC**, defined as $BIC(M)$, the form is

$$BIC(M) = n \ln(\hat{\sigma}_a^2) - (n - M) \ln \left(1 - \frac{M}{n} \right) + M \ln n + M \ln \left[\left(\frac{\hat{\sigma}_z^2}{\hat{\sigma}_a^2} - 1 \right) / M \right]. \quad (1.22)$$

2. **Schwartz's SBC**, defined as $SBC(M)$, the form is

$$SBC(M) = n \ln(\hat{\sigma}_a^2) + M \ln n. \quad (1.23)$$

3. **Parzen's CAT**, defined as $CAT(p)$,

$$CAT(p) = \begin{cases} -\left(1 + \frac{1}{n}\right), & p = 0, \\ \frac{1}{n} \sum_{j=1}^p \frac{1}{\hat{\sigma}_j^2} - \frac{1}{\hat{\sigma}_p^2}, & p = 1, 2, \dots, \end{cases} \quad (1.24)$$

In these criteria, the BIC criteria are less likely to overestimate the order of the autoregression, compared with the AIC criteria. Parzen's CAT criteria are more likely to be used for transformed models. The AIC criteria, as a standard model selection tool, are available in many time series software.

2. Seasonal ARIMA Models in X-12 Methods

In the previous section, we introduced the classical time series models, i.e., $AR(p)$, $MA(q)$, $ARIMA(p, d, q)$ and seasonal $ARIMA(p, d, q) \times (P, D, Q)_s$. We also introduced a series of methods and tools that are relative to the classical time series models. These techniques are useful in model identification, model selection and diagnostic checking. In this section, we will introduce another model which is different from classical models and useful in describing the time series with seasonal and trading day effects, or also some calendar effects (i.e., holiday effect, Ester day effect, etc.). This method is called X-12 method and the model is called X-12 ARIMA or regARIMA (regression ARIMA, is linear regression model with ARIMA time series errors). Figure 2.1 below is the diagrams of main operation procedures of X-12 program.

Figure 2.1 Flow Diagrams for Seasonal Adjustment with X-12 ARIMA.

From figure 2.1, after inputting data to the X-12 program, the program will exam if the series has any effects, i.e., the effects of seasonal, trading day, calendar, and outliers. When the program detects a time series has any effects, the program will adjust these effects and check the adjusted series again, until the series does not have any effects but seasonal effects. This step is called preadjustments. After the series data finished the preadjustments and passed all diagnostics, the program will do seasonal adjustment for the modified time series (if the seasonal effects are detected). Then the adjusted values will go to the bottom box (in the figure 2.1) and then program will do post-adjustment diagnostics including spectral plot, revision history, etc.. These statistics can be used to indicate the effectiveness of both adjustments and modelling. The details of those statistics will be discussed in section 3.

The general form of $\text{regARIMA}(p, d, q) \times (P, D, Q)_s$, is

$$\Phi_p(B)\Phi_P(B^s)(1-B)^d(1-B^s)^D(y_t - \sum_{i=1}^h \beta_i x_{it}) = \theta_q(B)\theta_Q(B^s)a_t. \quad (2.1)$$

Here, we assume the mean function, μ , of the time series is not zero, but $\mu = \sum_{i=1}^h \beta_i x_{it}$, x_{it} 's are the regressors of any effects (for example, if x_{it} represent trading day effects, $\sum_{i=1}^h \beta_i x_{it} = \beta_1 x_{1t} + \dots + \beta_6 x_{6t}$, we will give more details in section 3.2). Let $Z_t = y_t - \sum_{i=1}^h \beta_i x_{it}$ where Z_t is an error series of regression y_t , (note: $y_t = \sum_{i=1}^h \beta_i x_{it} + Z_t$). The rest of notations in equation (2.1) are the same as the seasonal $\text{ARIMA}(p, d, q) \times (P, D, Q)_s$ introduced in section 1. Later in the thesis, the classical ARIMA model means the seasonal ARIMA model. We assume that $\{a_t\}$ are uncorrelated with zero mean and constant variance σ^2 ; and after applying non-seasonal and/or seasonal differencing on Z_t , the non-stationary series Z_t is transformed to a stationary series. We define $w_t = (1 - B)^d(1 - B^s)^D(y_t - \sum_{i=1}^h \beta_i x_{it})$. Then, the equation (2.1) can be rewritten as

$$\Phi_p(B)\Phi_P(B^s)w_t = \theta_q(B)\theta_Q(B^s)a_t. \quad (2.2)$$

Then, we obtain the expression of w_t from equation (2.2):

$$w_t = \frac{\theta_q(B)\theta_Q(B^s)}{\Phi_p(B)\Phi_P(B^s)}a_t. \quad (2.3)$$

Dividing both sides of equation (2.1) by $\Phi_p(B)\Phi_P(B^s)$, get

$$(1-B)^d(1-B^s)^D \left(y_t - \sum_{i=1}^h \beta_i x_{it} \right) = \frac{\theta_q(B)\theta_Q(B^s)}{\Phi_p(B)\Phi_P(B^s)}a_t. \quad (2.4)$$

Then we rewrite the equation (2.4) in another way, which is

$$(1-B)^d(1-B^s)^D y_t = \sum_{i=1}^h \beta_i (1-B)^d(1-B^s)^D x_{it} + w_t. \quad (2.5)$$

Theoretically, it is true that p, d, q, P, Q and D could take any integer values, but the models containing more parameters than “optimal” ones will not improve the model. When the model contains excessively more parameters, the model becomes unnecessary complicate and results in less effective forecasting. Thus, in practical situation, we try to build an ARIMA model which contains as less parameters as possible. The following five regARIMA models are the default ones in the X-12 methods:

1. Seasonal ARIMA $(0,1,1) \times (0,1,1)_s$ is

$$(1-B)^1(1-B^s)^1 Z_t = (1-\theta_1 B^1)(1-\Theta_1 B^s) a_t.$$

2. Seasonal ARIMA(0,1,2) $\times$ (0,1,1)_s is

$$(1 - B)^1(1 - B^s)^1 Z_t = (1 - \theta_1 B^1 - \theta_2 B^2)(1 - \Theta_1 B^s) a_t.$$

3. Seasonal ARIMA(2,1,0) $\times$ (0,1,1)_s is

$$(1 - \phi_1 B^1 - \phi_2 B^2)(1 - B)^1(1 - B^s)^1 Z_t = (1 - \Theta_1 B^s) a_t.$$

4. Seasonal ARIMA(0,2,2) $\times$ (0,1,1)_s is

$$(1 - B)^2(1 - B^s)^1 Z_t = (1 - \theta_1 B^1 - \theta_2 B^2)(1 - \Theta_1 B^s) a_t.$$

5. Seasonal ARIMA(2,1,2) $\times$ (0,1,1)_s is

$$(1 - \phi_1 B^1 - \phi_2 B^2)(1 - B)^1(1 - B^s)^1 Z_t = (1 - \theta_1 B^1 - \theta_2 B^2)(1 - \Theta_1 B^s) a_t.$$

The five regARIMA models above, they have the same seasonal term (0,1,1)_s with $s = 4$ or 12, but five different non-seasonal terms (0,1,1), (0,1,2), (2,1,0), (0, 2, 2) and (2,1,2). Murdoch et al., (2000) replaced the fifth default model by (1,1,0) $\times$ (0,1,1)_s in Statistics New Zealand. It seems that this model is usually better to fit the time series than (2,1,2) $\times$ (0,1,1)_s model for Australian data.

2.1 The three types of decomposition of seasonal time series

2.1.1 The Multiplicative Time Series Decomposition

The multiplicative seasonal time series is used to describe the data sets with positive values such as sales data, shipments data, etc..

Multiplicative seasonal time series is typically of the form

$$Y_t^m = T_t S_t I_t \quad (2.6)$$

where Y_t^m is a quarterly or monthly multiplicative decomposition time series, T_t is a trend component, S_t is a seasonal component and I_t is an irregular component.

We divide by S_t on the both sides of the equation (2.6) to de-seasonalize and define $A_t^m = \frac{Y_t^m}{S_t}$. Then we obtain the seasonally adjusted time series

$$A_t^m = \frac{Y_t^m}{S_t} = T_t I_t. \quad (2.7)$$

If we divide the both sides of the equation (2.6) by T_t , we obtain adjusted trend series

$$\frac{Y_t^m}{T_t} = S_t I_t. \quad (2.8)$$

2.1.2 The Additive Decomposition

The additive decomposition of time series Y_t^a is the sum of a trend component T_t , a seasonal component S_t and an irregular component I_t . The form of additive decomposition is

$$Y_t^a = T_t + S_t + I_t. \quad (2.9)$$

The seasonally adjusted values of additive decomposition, A_t^a , is the original time series Y_t^a subtract seasonal component S_t , and

$$A_t^a = Y_t^a - S_t. \quad (2.10)$$

To de-trend, we subtract the trend component on the both side of equation (2.9), then we have

$$Y_t^a - T_t = S_t + I_t. \quad (2.11)$$

2.1.3 The Pseudo-additive Decomposition

The pseudo-additive decomposition is used for the time series with non-negative small values, even close to zero values in the same month each year. The pseudo-additive decomposition form is

$$Y_t^p = T_t(S_t + I_t - 1). \quad (2.12)$$

The seasonal adjusted values of pseudo-additive decomposition is

$$A_t^p = Y_t^p - T_t(S_t - 1). \quad (2.13)$$

The de-trend form of the pseudo-additive decomposition is the same as the de-trend form of multiplicative decomposition.

2.2 The X-12 method default calculations

The meanings of notations of three types of decompositions were given in previous paragraphs. The default calculation procedures which were expressed in Findley et al (1998) are summarized in the following:

Stage 1: Initial Estimates

(a) Initial trend estimate with 13-term moving averages (centered 12-term):

$$T_t^{(1)} = \frac{1}{24}Y_{t-6} + \frac{1}{12}Y_{t-5} + \cdots + \frac{1}{12}Y_t + \cdots + \frac{1}{12}Y_{t+5} + \frac{1}{24}Y_{t+6}.$$

In this stage, “(1)” means stage 1. In the following stages, “(2)” means stage 2 and “(3)” means stage 3.

(b) Initial “SI ratio”:

$$(\text{Multiplicative, Pseudo Additive}): SI_t^{(1)} = \frac{Y_t}{T_t^{(1)}}.$$

$$(\text{Additive}): SI_t^{(1)} = Y_t - T_t^{(1)}.$$

(c) Initial preliminary estimated seasonal factor by 3×3 seasonal moving averages. In this stage, the X-12 method default 3×3 seasonal moving average filters are used.

$$\hat{S}_t^{(1)} = \frac{1}{9}SI_{t-24}^{(1)} + \frac{2}{9}SI_{t-12}^{(1)} + \frac{3}{9}SI_t^{(1)} + \frac{2}{9}SI_{t+12}^{(1)} + \frac{1}{9}SI_{t+24}^{(1)}.$$

The estimated seasonal factor $\hat{S}_t^{(1)}$ could be obtained through the following computation.

First, we compute

$$S_t^{3 \times (2n+1)} = \frac{1}{3}(S_{t-12}^{2n+1} + S_t^{2n+1} + S_{t+12}^{2n+1}).$$

where

$$S_t^{2n+1} = \frac{1}{2n+1} \sum_{j=-n}^n SI_{t+12j}.$$

For instance, the 3×3 seasonal moving average filters, here $n = 1$, are:

$$S_t^{3 \times (2 \times 1 + 1)} = \frac{1}{3}(S_{t-12}^{2 \times 1 + 1} + S_t^{2 \times 1 + 1} + S_{t+12}^{2 \times 1 + 1}),$$

which

$$S_{t-12}^{2 \times 1 + 1} = \frac{1}{2 \times 1 + 1} \sum_{j=-1}^1 SI_{t-12+12j} = \frac{1}{3}(SI_{t-12-12} + SI_{t-12} + SI_t).$$

$$S_t^{2 \times 1 + 1} = \frac{1}{2 \times 1 + 1} \sum_{j=-1}^1 SI_{t+12j} = \frac{1}{3}(SI_{t-12} + SI_t + SI_{t+12}).$$

$$S_{t+12}^{2 \times 1 + 1} = \frac{1}{2 \times 1 + 1} \sum_{j=-1}^1 SI_{t+12+12j} = \frac{1}{3}(SI_t + SI_{t+12} + SI_{t+24}).$$

Finally, we yield the 3×3 seasonal moving average filters for $n = 1$ is

$$S_t^{3 \times (2 \times 1 + 1)} = \frac{1}{3}(S_{t-12}^{2 \times 1 + 1} + S_t^{2 \times 1 + 1} + S_{t+12}^{2 \times 1 + 1})$$

$$\begin{aligned}
&= \frac{1}{3} \left[\left(\frac{1}{3} (SI_{t-12-12} + SI_{t-12} + SI_t) \right. \right. \\
&\quad \left. \left. + \frac{1}{3} (SI_{t-12} + SI_t + SI_{t+12}) \right. \right. \\
&\quad \left. \left. + \frac{1}{3} (SI_t + SI_{t+12} + SI_{t+24}) \right) \right] \\
&= \frac{1}{9} SI_{t-24}^{(1)} + \frac{2}{9} SI_{t-12}^{(1)} + \frac{3}{9} SI_t^{(1)} + \frac{2}{9} SI_{t+12}^{(1)} + \frac{1}{9} SI_{t+24}^{(1)}.
\end{aligned}$$

(d) Initial seasonal factor:

$$\begin{aligned}
\text{(Multiplicative, Pseudo Additive): } S_t^{(1)} &= \frac{\hat{S}_t^{(1)}}{\frac{1}{24} \hat{S}_{t-6}^{(1)} + \frac{1}{12} \hat{S}_{t-5}^{(1)} + \dots + \frac{1}{12} \hat{S}_{t+5}^{(1)} + \frac{1}{24} \hat{S}_{t+6}^{(1)}}. \\
\text{(Additive): } S_t^{(1)} &= \hat{S}_t^{(1)} - \left(\frac{1}{24} \hat{S}_{t-6}^{(1)} + \frac{1}{12} \hat{S}_{t-5}^{(1)} + \dots + \frac{1}{12} \hat{S}_{t+5}^{(1)} + \frac{1}{24} \hat{S}_{t+6}^{(1)} \right).
\end{aligned}$$

(e) Initial seasonal adjustment:

$$\text{(Multiplicative): } A_t^{(1)} = \frac{Y_t}{S_t^{(1)}}.$$

$$\text{(Additive): } A_t^{(1)} = Y_t - S_t^{(1)}.$$

$$\text{(Pseudo Additive): } A_t^{(1)} = Y_t - T_t^{(1)}(S_t^{(1)} - 1).$$

Stage 2: Seasonal Factors and Seasonal adjustment

Let the length of Henderson filters be $(2H + 1)$ and the Henderson coefficients be h_j^{2H+1} , $-H \leq j \leq H$. (Note: the Henderson filters are symmetric filters used to smooth trend. We will discuss it in more details in section 2.4).

(a) Intermediate trend: Smoothing initial seasonal adjustment time series convolution with Henderson coefficients,

$$T_t^{(2)} = \sum_{j=-H}^H h_j^{(2H+1)} A_{t+j}^{(1)}.$$

(b) Intermediate “SI ratio”:

$$(\text{Multiplicative, Pseudo Additive}): SI_t^{(2)} = \frac{Y_t}{T_t^{(2)}}.$$

$$(\text{Additive}): SI_t^{(2)} = Y_t - T_t^{(2)}.$$

(c) In this stage, X-12 program selects filters from 3×3 , 3×5 , 3×9 and 3×15 terms length of seasonal moving averages based on Lothian (1984) criteria. The preliminary seasonal factor with 3×5 seasonal moving average:

$$\hat{S}_t^{(2)} = \frac{1}{15} SI_{t-36}^{(2)} + \frac{2}{15} SI_{t-24}^{(2)} + \frac{3}{15} SI_{t-12}^{(2)} + \frac{3}{15} SI_t^{(2)} + \frac{3}{15} SI_{t+12}^{(2)} + \frac{2}{15} SI_{t+24}^{(2)} + \frac{1}{15} SI_{t+36}^{(2)}.$$

Note: Using the same computation strategies which we computed $\hat{S}_t^{(1)}$ in Stage 1 (c) to obtain $\hat{S}_t^{(2)}$ and we substitute $n = 2$ into the formulas instead of $n = 1$ at this time.

(d) Seasonal factor:

$$(\text{Multiplicative, Pseudo Additive}): S_t^{(2)} = \frac{\hat{S}_t^{(2)}}{\frac{1}{24} \hat{S}_{t-6}^{(2)} + \frac{1}{12} \hat{S}_{t-5}^{(2)} + \dots + \frac{1}{12} \hat{S}_{t+5}^{(2)} + \frac{1}{24} \hat{S}_{t+6}^{(2)}}.$$

$$(\text{Additive}): S_t^{(2)} = \hat{S}_t^{(2)} - \left(\frac{1}{24} \hat{S}_{t-6}^{(2)} + \frac{1}{12} \hat{S}_{t-5}^{(2)} + \dots + \frac{1}{12} \hat{S}_{t+5}^{(2)} + \frac{1}{24} \hat{S}_{t+6}^{(2)} \right).$$

(e) Seasonal adjustment:

$$(\text{Multiplicative}): A_t^{(2)} = \frac{Y_t}{S_t^{(2)}}.$$

$$(\text{Additive}): A_t^{(2)} = Y_t - S_t^{(2)}.$$

$$(\text{Pseudo Additive}): A_t^{(2)} = Y_t - T_t^{(2)} (S_t^{(2)} - 1).$$

Stage 3: Final Henderson Trend and Final Irregular

(a) Final trend for pre-adjusted time series:

$$T_t^{(3)} = \sum_{j=-H}^H h_j^{(2H+1)} A_{t+j}^{(2)}.$$

(b) Final irregular:

$$\text{(Multiplicative, Pseudo Additive): } I_t^{(3)} = \frac{A_t^{(2)}}{T_t^{(3)}}.$$

$$\text{(Additive): } I_t^{(3)} = A_t^{(2)} - T_t^{(3)}.$$

The following box shown the representations of three types decompositions for monthly time series Y_t , which are

$\text{(Multiplicative): } Y_t = T_t^{(3)} S_t^{(2)} I_t^{(3)}.$ $\text{(Additive): } Y_t = T_t^{(3)} + S_t^{(2)} + I_t^{(3)}.$ $\text{(Pseudo Additive): } Y_t = T_t^{(2)} (S_t^{(2)} - 1) + T_t^{(3)} I_t^{(3)}.$

2.3 Lothian criterion for seasonal filters in X-12 methods

The Lothian criteria determine the length of seasonal filters based on SI ratios (detrended series values), also called the global moving seasonality ratio (GMSR). The criteria are listed in table 2.3.1 as follows.

Table 2.3.1 The Criteria for Seasonal Moving Average Filters.

The range of SI ratios (SI)	The Length of Seasonal Filters
$2.3 < SI < 4.1$	3×3
$4.1 \leq SI < 5.2$	3×5
$5.2 \leq SI$	3×9

2.4 The filters for smoothing trend in X-12 methods

X-12 methods use Henderson symmetric moving average filters and Surrogate Henderson moving average filters (asymmetric filters) to obtain a smoothing trend. One enhancement of X-12 methods compare with previous methods is that X-12 methods allow users to define any odd length of filters besides selecting the length automatically.

2.4.1 The Introduction of Symmetric Henderson Filters

Henderson filters are symmetric filters, which smooth trend by applying the Henderson coefficients $h_j^{(2H+1)}$ on the (initial) seasonal adjustment component A_t . Findley et al. (1998) showed brief mathematics to illustrate that Henderson filter $h_j^{(2H+1)}$ are symmetric. Gray and Thomson (1996) improved the Kenny and Durbin (1982) derivation on two viewpoints. The two modifications are: it is not necessary to assume symmetric coefficients and the Henderson filters can be used on quadratic trend instead of cubic one. Let

$$A_{t+j} = \alpha + \beta(t+j) + \gamma(t+j)^2 + I_{t+j}, \quad -(H+3) \leq j \leq H. \quad (2.14)$$

Here, we assume irregular components, I_{t+j} , follow normal distribution with zero mean and variance σ^2 , and independent for all different j . Assume Henderson filter

coefficients h_j can produce the unbiased estimated values of A_{t+j} for $-H \leq j \leq H$. In other words, the filter coefficients are constrained by requiring that

$$\sum_{j=-H}^H h_j A_{t+j} = \alpha + \beta t + \gamma t^2, \quad \text{for all } \alpha, \beta, \gamma. \quad (2.15)$$

Define Δ as the differencing operator, $\Delta(A_{t+j}) = A_{t+j} - A_{t+j-1}$ and $\Delta(h_{j+1}) = h_{j+1} - h_j$.

Now, we adjust Henderson filter coefficients to get the minimum variance of time series.

After taking the first differencing on A_{t+j} , we have

$$\Delta(A_{t+j}) = \beta + 2\gamma(t+j-1) + \Delta(I_{t+j}), \quad (2.16)$$

the second differencing function of A_{t+j} is

$$\Delta^2(A_{t+j}) = \Delta(\Delta A_{t+j}) = 2\gamma + \Delta^2(I_{t+j}), \quad (2.17)$$

the third differencing function of A_{t+j} is

$$\Delta^3(A_{t+j}) = \Delta(\Delta(\Delta A_{t+j})) = \Delta^3(I_{t+j}) = I_{t+j} - 3I_{t+j-1} + 3I_{t+j-2} - I_{t+j-3}. \quad (2.18)$$

By a similar idea, we also could obtain the third differencing of h_j , which is

$$\Delta^3 h_j = h_j - 3h_{j-1} + 3h_{j-2} - h_{j-3}. \quad (2.19)$$

Under the conditions $\sum_{j=-H}^H h_j = 1$ for $-H \leq j \leq H$ and $h_j = 0$ for $j = \pm(H+1), \pm(H+2), \pm(H+3)$, we obtain

$$\Delta^3 \left(\sum_{j=-H}^H h_j A_{t+j} \right) = \Delta^3 \left(\sum_{j=-H}^{H+3} h_j I_{t+j} \right), \quad (2.20)$$

after taking variance on the both sides of equation (2.20), we get

$$E \left[\Delta^3 \left(\sum_{j=-H}^H h_j A_{t+j} \right) \right]^2 = \sigma^2 \sum_{j=-H}^{H+3} (\Delta^3 h_j)^2. \quad (2.21)$$

Let

$$W = \sigma^2 \sum_{j=-H}^{H+3} (\Delta^3 h_j)^2$$

Where σ^2 is the variance of I_{t+j} and $h_j=0$ with $j = \pm(H+1), \pm(H+2), \pm(H+3)$.

Gray and Thomson (1996) used Lagrange multipliers to minimize W subject to h_j under the constraints are

$$\sum_{j=-H}^H h_j = 1, \quad \sum_{j=-H}^H j h_j = 0, \quad \sum_{j=-H}^H j^2 h_j = 0, \quad (2.22)$$

and got

$$\Delta^6 h_{j+3} = \lambda_1 + \lambda_2 j + \lambda_3 j^2, \quad (-H \leq j \leq H), \quad (2.23)$$

where the λ_1, λ_2 and λ_3 are denoted as the Lagrange multipliers and the solutions for equation (2.23). Kenny and Durbin (1982) solved equation (2.23) and got

$$h_j = [(H+1)^2 - j^2][(H+2)^2 - j^2][(H+3)^2 - j^2](b_1 + b_2 j + b_3 j^2),$$

and the b_1, b_2 and b_3 are determined from the equation (2.22). These are the smoothing filters used by X-11 and X-12 methods and they do not need to be symmetric. Findley et al. (1998) summarized Gray and Thomson's approach and got the Henderson coefficients

$$h_j = [(H+1)^2 - j^2][(H+2)^2 - j^2][(H+3)^2 - j^2](a + b j^2),$$

where a and b can be computed by

$$a \sum_{j=-H}^H q_j(H) + b \sum_{j=-H}^H q_j(H) j^2 = 1$$

and

$$a \sum_{j=-H}^H q_j(H) j^2 + b \sum_{j=-H}^H q_j(H) j^4 = 0,$$

here, $q_j(H) = [(H+1)^2 - j^2][(H+2)^2 - j^2][((H+3)^2 - j^2)]$. The Henderson coefficients from Findley et al. (1998) conclude $h_j^{(2H+1)} = h_{-j}^{(2H+1)}$, thus the Henderson coefficients are symmetric. For the cubic trend the above Henderson filters are not held at the ends of the series. The Surrogate Henderson filter solves this problem and this asymmetric filter will be introduced in following paragraphs.

2.4.2 The Surrogate Henderson Filters

The asymmetric (surrogate) Henderson filters criteria were given by Musgrave (1964 a, b) in the paper of Doherty (2001). Musgrave's assumed that:

1. A set of filter weights $w_1, w_2, \dots, w_n$ such that $\sum_{r=1}^n w_r = 1$ is given.
2. Suppose a time series X_r and let $X_r = \alpha + \beta r + I_r$, where I_r are independent identically distribution errors with mean 0 and variance σ^2 , here α and β are constants.

For a set of finite weights $v_1, \dots, v_q$, with $q < n$, and $\sum_{r=1}^q v_r = 1$, the minimum mean square errors (MSE) between the initial and the final time series estimate $E(\sum_{r=1}^q v_r X_r - \sum_{r=1}^n w_r X_r)^2$ is achieved when

$$v_i = w_i + \frac{1}{q} \sum_{r=q+1}^n w_r + \frac{(i - (\frac{q+1}{2})) \frac{4}{\pi} R^2}{1 + \frac{q(q-1)(q+1) \frac{4}{\pi} R^2}{12}} \times \sum_{r=q+1}^n (r - (\frac{q+1}{2})) w_r.$$

Here $R^2 = (\frac{\beta}{2\sigma/\sqrt{\pi}})^2$. Let $\bar{I} = E(|\hat{I}_{t+1} - \hat{I}_t|)$, $\bar{I}$ is the average of the absolute month to month change in the estimated irregular component. Let $|x| = |\hat{I}_{t+1} - \hat{I}_t|$ and x follows normal distribution with mean zero and variance $2\sigma^2$. Then

$$\begin{aligned} E(|\hat{I}_{t+1} - \hat{I}_t|) &= E(|x|) = \frac{1}{\sqrt{2\pi}\sqrt{2\sigma^2}} \int_{-\infty}^{\infty} |x| e^{-\frac{x^2}{4\sigma^2}} dx \\ &= \frac{2}{\sqrt{2\pi}\sqrt{2\sigma^2}} \int_0^{\infty} x e^{-\frac{x^2}{4\sigma^2}} dx, \end{aligned}$$

let $(x/2\sigma) = y$, so that $x = 2\sigma y$ and $dx = 2\sigma dy$. Then,

$$\begin{aligned} E(|x|) &= \frac{2}{\sqrt{2\pi}\sqrt{2\sigma^2}} \int_0^{\infty} 2\sigma y e^{-y^2} 2\sigma dy \\ &= \frac{2}{\sqrt{2\pi}\sqrt{2\sigma^2}} 4\sigma^2 \int_0^{\infty} y e^{-y^2} dy \\ &= \frac{4\sigma}{\sqrt{\pi}} \left[-\frac{1}{2} e^{-y^2} \right]_0^{\infty} = \frac{1}{2} \frac{4\sigma}{\sqrt{\pi}} \\ &= \frac{2\sigma}{\sqrt{\pi}} \end{aligned}$$

Finally, we get $E(|\hat{I}_{t+1} - \hat{I}_t|) = 2\sigma/\sqrt{\pi}$. Let $\bar{C} = E(|\hat{T}_{t+1} - \hat{T}_t|)$, with $\hat{T}_t = \gamma + \beta t$ and $\bar{C}$ is the average of the absolute month to month change in the estimated trend. Then, R^2 can be simplified to $R^2 = (\bar{C}/\bar{I})^2$. The inverse of R , R^{-1} , is called the noise-to-ratio. The noise-to-ratio helps to determine the length of surrogate Henderson filters used to smooth trend. Generally, the length of filters can take any odd length $2H + 1$, the X-12 method default value are 9-term, 13-term and 23-term. The Shiskin's (1967) criterion for monthly asymmetric moving average is that: if $R^{-1} \leq 0.99$, 9-term filters be used; if $1 \leq R^{-1} \leq 3.49$, suggest 13-term filters; if $R^{-1} \geq 3.5$, 23-term moving average. The Musgrave's criterion for monthly asymmetric moving average suggested that: for $R^{-1} = 1$ use 9-term

filters; for $R^{-1} = 3.5$ use 13-term; for $R^{-1} = 4.5$ use 23-term. For quarterly series, $R^{-1} = 0.001$ and 4.5, the 5-term and 7-term quarterly filters are used, respectively.

3. Properties of X-12 Methods

This section discusses some advantages of regARIMA models compared with classical models. In X-12 methods, users can define regressors and add them to regARIMA models. The X-12 program can automatically detect seasonal and calendar effects through spectral plots, then add corresponding regressors to regARIMA models in order to adjust those effects.

3.1 Seasonal and trading day effects detection

The spectral plots are helpful in so-called frequency domain analysis. they are a useful tool for detecting seasonal and trading day effects if the effects are approximately periodic. Seasonal effects mean that some time series data repeat periodically, especially, economic and business data. Trading day effects are caused by the frequency of dates in each month. For instance, people are more likely to go grocery shopping on weekends. If a month has five Saturdays or Sundays while the same month in other years has four Saturdays or Sundays, sales of grocery will be greater in the month with more Saturdays or Sundays. X-12 methods calculate spectrum estimators and print spectral plots with a warning message about “visually significant” seasonal peaks. The message alerts users of probable seasonal effects, and seasonal effects are predictable (stable). Generally speaking, one or more peak of seasonal frequency, indicate seasonal effects exist. Similarly, one or more peak of trading day frequency, indicate that trading day effects exist. We will discuss the spectral plots later in the thesis.

Two types of spectrum estimators can be used, called periodogram spectrum estimators and autoregressive spectrum estimators. The X-12 program draws spectral plots using periodogram and autoregressive spectrum estimators in decibel units. The use of decibel

units makes the peaks sharp (if time series have seasonal and/or trading day effects and they are detected).

The periodogram analysis is helpful in detecting hidden periodicities or non-random periodic component. To better understand the periodogram spectrum estimators, the definition of spectral density will be give firstly. Let Z_t is a real value stationary time series with absolutely summable autocovariance function $\gamma(k)$, which means $\sum_{h=-\infty}^{\infty} |\gamma(k)| < \infty$. Then, the spectral density of Z_t from Brockwell and Davis (2002, pp. 112) is

$$f(\lambda) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma(k) e^{-ik\lambda}, \quad -\infty < \lambda < \infty, \quad (3.1)$$

where $e^{-ik\lambda} = \cos k\lambda - i \sin k\lambda$, and the spectral density function absolutely converges because of $|e^{ik\lambda}|^2 = \sin^2(k\lambda) + \cos^2(k\lambda) = 1$. Since $\gamma(k) = \gamma(-k)$, $\sin \lambda(k) = \sin \lambda(-k)$ and $\cos \lambda(k) = \cos \lambda(-k)$, then equation (3.1) can be rewritten as

$$f(\lambda) = \frac{1}{2\pi} \left[\gamma(0) + 2 \sum_{k=1}^{\infty} \gamma(k) \cos k\lambda \right]. \quad (3.2)$$

Here, $f(\lambda) = f(-\lambda)$ and $f(\lambda) \geq 0$ for all $\lambda \in (-\pi, \pi]$. A function $f(\lambda)$ is the spectral density function of a stationary time series Z_t with autocovariance function $\gamma(k)$ if

$$\gamma(k) = \int_{-\pi}^{\pi} e^{ik\lambda} f(\lambda) d\lambda, \quad \text{for all integers } k, \quad (3.3)$$

Specially, the variance of the time series is

$$\gamma(0) = \int_{-\pi}^{\pi} f(\lambda) d\lambda.$$

To estimate $f(\lambda)$, we replace autocovariance $\gamma(k)$ by the sample autocovariance $\hat{\gamma}(k)$ for $k = 0, 1, \dots, (n-1)$. The function of sample autocovariance $\hat{\gamma}(k)$ is

$$\hat{\gamma}(k) = \frac{1}{n} \sum_{t=1}^{n-k} (Z_t - \bar{Z})(Z_{t+k} - \bar{Z}),$$

and

$$\begin{aligned} \sum_{t=1}^{n-k} (Z_t - \bar{Z})(Z_{t+k} - \bar{Z}) &= \sum_{t=1}^{n-k} [(Z_t - \mu) - (\bar{Z} - \mu)][(Z_{t+k} - \mu) - (\bar{Z} - \mu)] \\ &= \sum_{t=1}^{n-k} (Z_t - \mu)(Z_{t+k} - \mu) - (\bar{Z} - \mu) \sum_{t=1}^{n-k} (Z_t - \mu) \\ &\quad - (\bar{Z} - \mu) \sum_{t=1}^{n-k} (Z_{t+k} - \mu) + (n-k)(\bar{Z} - \mu)^2. \end{aligned}$$

Since $\sum_{t=1}^{n-k} (Z_t - \mu)$ and $\sum_{t=1}^{n-k} (Z_{t+k} - \mu)$ are approximately the same as $(n-k)(\bar{Z} - \mu)$, then yield

$$\sum_{t=1}^{n-k} (Z_t - \bar{Z})(Z_{t+k} - \bar{Z}) \approx \sum_{t=1}^{n-k} (Z_t - \mu)(Z_{t+k} - \mu) - (n-k)(\bar{Z} - \mu)^2.$$

Then, we have

$$\begin{aligned} E(\hat{\gamma}(k)) &\approx \frac{1}{n} E \left[\sum_{t=1}^{n-k} (Z_t - \mu)(Z_{t+k} - \mu) - (n-k)(\bar{Z} - \mu)^2 \right] \\ &\approx \frac{1}{n} E \left[\sum_{t=1}^n (Z_t - \mu)(Z_{t+k} - \mu) - \sum_{t=1}^k (Z_t - \mu)(Z_{t+k} - \mu) \right. \\ &\quad \left. - (n-k)(\bar{Z} - \mu)^2 \right] \end{aligned}$$

$$\begin{aligned}
&\approx \frac{1}{n} \sum_{t=1}^n E[(Z_t - \mu)(Z_{t+k} - \mu)] - \frac{1}{n} \sum_{t=1}^k E[(Z_t - \mu)(Z_{t+k} - \mu)] \\
&\quad - \frac{1}{n} (n-k) E(\bar{Z} - \mu)^2 \\
&\approx \gamma(k) - \frac{k}{n} \gamma(k) - \left(\frac{n-k}{n} \right) \text{var}(\bar{Z}),
\end{aligned}$$

as $n \rightarrow \infty$, $\lim_{n \rightarrow \infty} \frac{k}{n} \gamma(k) = \lim_{n \rightarrow \infty} \left(\frac{n-k}{n} \right) \text{var}(\bar{Z}) = 0$, the sample autocovariance $\hat{\gamma}(k)$ is an approximate unbiased estimator of autocovariance $\gamma(k)$. Using the estimator of autocovariance $\hat{\gamma}(k)$, the estimated spectral density function $\hat{f}(\lambda)$ can be obtained, which is

$$\hat{f}(\lambda) = \frac{1}{2\pi} \sum_{k=-(n-1)}^{n-1} \hat{\gamma}(k) e^{-ik\lambda} \quad (3.4 a)$$

$$= \frac{1}{2\pi} \left[\hat{\gamma}(0) + 2 \sum_{k=1}^{n-1} \hat{\gamma}(k) \cos k\lambda \right]. \quad (3.4 b)$$

The general form of the periodogram function for $\{Z_1, \dots, Z_n\}$ is

$$I_n(\lambda) = \frac{1}{n} \left| \sum_{t=1}^n Z_t e^{-it\lambda} \right|^2.$$

Define λ_k as Fourier frequency with $\lambda_k = 2\pi k/n$, $k = 1, \dots, [n/2]$. Then, the periodogram function with Fourier frequency λ_k can be written as

$$I(\lambda_k) = \frac{1}{n} \left[\sum_{t=1}^n Z_t e^{i\lambda_k t} \right] \left[\sum_{t=1}^n Z_t e^{-i\lambda_k t} \right] \quad (3.5 a)$$

$$= \frac{1}{n} \left[\sum_{t=1}^n (Z_t - \bar{Z}) e^{i\lambda_k t} \right] \left[\sum_{t=1}^n (Z_t - \bar{Z}) e^{-i\lambda_k t} \right]. \quad (3.5 \text{ b})$$

From following computations, equation (3.5 b) can be achieved:

$$\begin{aligned} & \left[\sum_{t=1}^n (Z_t - \bar{Z}) e^{i\lambda_k t} \right] \left[\sum_{t=1}^n (Z_t - \bar{Z}) e^{-i\lambda_k t} \right] \\ &= \left(\sum_{t=1}^n Z_t e^{i\lambda_k t} - \bar{Z} \sum_{t=1}^n e^{i\lambda_k t} \right) \left(\sum_{t=1}^n Z_t e^{-i\lambda_k t} - \bar{Z} \sum_{t=1}^n e^{-i\lambda_k t} \right) \\ &= \left(\sum_{t=1}^n Z_t e^{i\lambda_k t} \right) \left(\sum_{t=1}^n Z_t e^{-i\lambda_k t} \right) - \left(\bar{Z} \sum_{t=1}^n e^{i\lambda_k t} \right) \left(\sum_{t=1}^n Z_t e^{-i\lambda_k t} \right) \\ &\quad - \left(\sum_{t=1}^n Z_t e^{i\lambda_k t} \right) \left(\bar{Z} \sum_{t=1}^n e^{-i\lambda_k t} \right) + \left(\bar{Z} \sum_{t=1}^n e^{i\lambda_k t} \right) \left(\bar{Z} \sum_{t=1}^n e^{-i\lambda_k t} \right). \quad (3.6) \end{aligned}$$

Since

$$\sum_{t=1}^n e^{i\lambda_k t} = \sum_{t=1}^n \cos \lambda_k t + i \sum_{t=1}^n \sin \lambda_k t,$$

at $\lambda_k = 2\pi k/n$, we have

$$\begin{aligned} \sum_{t=1}^n \cos \frac{2\pi k}{n} t &= \begin{cases} n, & k = 0 \\ 0, & k \neq 0 \end{cases} \\ \sum_{t=1}^n \sin \frac{2\pi k}{n} t &= 0, k = 0, 1, \dots, \left[\frac{n}{2} \right]. \end{aligned}$$

Here, to obtain equation (3.5 b), we use the facts that

$$\sum_{t=1}^n e^{i\lambda_k t} = \sum_{t=1}^n e^{-i\lambda_k t} = 0,$$

and then

$$\left[\sum_{t=1}^n (Z_t - \bar{Z}) e^{i\lambda_k t} \right] \left[\sum_{t=1}^n (Z_t - \bar{Z}) e^{-i\lambda_k t} \right] = \left(\sum_{t=1}^n Z_t e^{i\lambda_k t} \right) \left(\sum_{t=1}^n Z_t e^{-i\lambda_k t} \right).$$

Back to equation (3.5 a, b), let's $j = t - s$, then we rewrite equation (3.5 a, b) as

$$I(\lambda_k) = \frac{1}{n} \sum_{j=-(n-1)}^{n-1} \sum_{s=1}^{n-j} ((Z_s - \bar{Z}) (Z_{s+j} - \bar{Z}) e^{-i\lambda_k(j+s)}). \quad (3.7)$$

Since,

$$\hat{\gamma}(j) = \frac{1}{n} \sum_{s=1}^{n-j} ((Z_s - \bar{Z}) (Z_{s+j} - \bar{Z})).$$

The periodogram then can be written as

$$\hat{I}(\lambda_k) = \sum_{j=-(n-1)}^{n-1} \hat{\gamma}(j) e^{-i\lambda_k j} \quad (3.8 a)$$

$$= \hat{\gamma}(0) + 2 \sum_{k=1}^{\infty} \hat{\gamma}(k) \cos \lambda_k j. \quad (3.8 b)$$

Hence, from equation (3.4 b), we have

$$\hat{I}(\lambda_k) = 2\pi \hat{f}(\lambda_k). \quad (3.9)$$

Then,

$$\hat{f}(\lambda_k) = \frac{1}{2\pi} I(\lambda_k). \quad (3.10)$$

Finally, the discrete spectral estimator of the spectral density $f(\lambda)$ can be expressed as

$$\hat{f}(\lambda) = \frac{1}{2\pi} \sum_{|j| \leq m_n} W_n(j) I_n(\lambda_k),$$

where λ_k is the closet Fourier frequency to λ and the bandwidths m_n converge to infinite,

and m_n/n converge to 0 as $n \rightarrow \infty$. The weight functions $W_n(j)$ are symmetric for

$W_n(j) \geq 0$ for all j , $\sum_{|j| \leq m_n} W_n(j) = 1$ and $\sum_{|j| \leq m_n} W_n^2(j) \rightarrow 0$ as $n \rightarrow \infty$. The purpose of making conditions on the weight functions is to ensure that $\hat{f}(\lambda)$ is consistent and asymptotically unbiased estimator for $f(\lambda)$:

$$\lim_{n \rightarrow \infty} \hat{f}(\lambda) = f(\lambda)$$

in probability,

$$\lim_{n \rightarrow \infty} E\hat{f}(\lambda) = f(\lambda),$$

Brockwell and Davis (1991).

The periodogram estimator of spectrum (in decibel unit) is

$$\begin{aligned} I(\lambda) &= 10 \log_{10} \left\{ \frac{n}{2} \left[\frac{2}{n} \sum_{t=1}^n Z_t (\cos \lambda t - i \sin \lambda t) \right] \left[\frac{2}{n} \sum_{t=1}^n Z_t (\cos \lambda t + i \sin \lambda t) \right] \right\} \\ &= 10 \log_{10} \left\{ \frac{2}{n} \left(\sum_{t=1}^n Z_t e^{-i\lambda t} \right) \left(\sum_{t=1}^n Z_t e^{i\lambda t} \right) \right\}, \quad 0 \leq \lambda \leq 0.5. \end{aligned} \quad (3.11)$$

Equation (3.11) can be used on all types of stationary series with continuous spectral density, and it is not necessary to assume a specific parametric model for non-parametric series data. If an autoregressive process can be confirmed for a time series, then an alternative spectrum estimator can be used, called an autoregressive spectrum estimator, to estimate the density function of a spectrum of a series. Suppose the series Z_t is confirmed as an $AR(p)$ model, then the estimated spectral density function, Priestley (1981, pp 601), is

$$\hat{f}_Z(\lambda) = \frac{\hat{\sigma}_Z}{2\pi} \frac{1}{|1 - \hat{\phi}_j e^{-ij\lambda}|^2}, \quad 1 \leq j \leq p. \quad (3.12a)$$

The autoregressive estimator of spectrum (in decibel units) is

$$\hat{s}(\lambda) = 10 \log_{10} \left\{ \frac{\sigma_m^2}{2\pi |1 - \sum_{j=1}^m \hat{\phi}_j e^{i2\pi j\lambda}|^2} \right\}, 0 \leq \lambda \leq 0.5. \quad (3.12b)$$

Here, coefficients $\hat{\phi}_j$ are the estimated values of ϕ_j , $1 \leq j \leq m$, and σ_m^2 is the sample variance computed from regression residuals, (Findley et al., 1998). The X-12 program uses $m = 30$, this value can produce sharp peaks and the program defaults autoregressive spectrum estimator $\hat{s}(\lambda)$ to draw spectral plots.

The plots are given at radian frequencies $\lambda_t = k/120$, $0 \leq k \leq 60$. The three trading day frequencies for monthly time series are located at 0.304, 0.348, and 0.432. At trading day frequencies, the spectral plot prints as columns of “ T ”. The average length of a month is $365.25/12=30.4375$ days, and the daily component repeats every seven days, a month has $30.4375/7 = 4.348$ cycles, with 0.348 cycles residual. In addition, the trading day frequency 0.304 comes from $(1 - 2 \times 0.348 = 0.304)$, resulting from empirical experiences by W. P. Cleveland. The frequency 0.432 is weaker and it is also unreliable. The detection methods of trading day effects are the same as the methods of seasonal effects. If peaks occur at these trading day frequencies, it means that the series has trading day effects. For quarterly time series, the trading day frequency is 0.04464 cycles/quarter, (assuming the quarter has equal length, $365.25/4 = 91.3125$ and $91.3125/7 = 13.044$). The monthly seasonal frequencies are $\frac{1}{12}, \frac{2}{12}, \dots, \frac{6}{12}$ cycles/month (i.e., Fourier frequency $\lambda_t = k/s$, $k = \pm 1, \dots, \pm(s/2)$). The spectral plots print as columns of “ S ” at the seasonal frequencies. For other non-seasonal frequencies and non-trading day frequencies, the plots print as columns of asterisks (“*”).

Figure 1 shows spectral plots for irregular components of the construction series of single- and multi-unit housing in the Midwest region of the United States, from January 1964 to December 1982. Seasonal adjustments have been made to this series, but not trading-day adjustments. Figure 3.1 (a) shows a peak at trading day frequency 0.348, but it does not appear in Figure 3.1 (b). This case means the trading day effect is not significant in the later of this time series.

Figure 3.1: The autoregressive spectrum estimator plot of the irregulars of North Central Housing Starts for (a) the full series and (b) the last eight years.

If a peak occurs at any seasonal frequency, it indicates that the series has a seasonal effect. It is possible that the series has both seasonal and trading-day effects. Figure 3.2 (a) shows the spectral plots of an irregular component of dry wine monthly sales data in Australia from January 1987 to December 1994. A peak occurring at seasonal frequency $\frac{5}{6}$ means that the dry wine series has seasonal effects. Figure 3.2 (b) is a spectral plot of irregular components of the dry wine series after first differencing, log-transformation, and seasonal adjustment. In figure 3.2 (b), the peak of column *S* never appears at seasonal frequency $\frac{5}{6}$, which means that the seasonal adjustment of the series is adequate. But an “insignificant” peak occurs at trading day frequency 0.432, and this frequency is weak and unreliable, because the peak might be caused by white noise. The X-12 methods use Akaike’s information criteria, mentioned in section 1, to test the regARIMA model with and without trading day regressor. The result from AIC criterion suggests that the regARIMA model does not have to contain the trading day regressor (trading day regressor will be shown later). Even if trading day regressors are added to the regARIMA model, a significant peak still exists in the spectral plots of the modified irregular components at trading-day frequency. This situation means that the adjustments of trading-day effect are unsatisfactory.

Figure 3.2 (a): The spectrum plots using autoregressive spectrum estimator $\hat{s}(\lambda)$ for the unmodified irregular component of dry wine monthly sale in Australia (1987/01 to 1994/12).

Figure 3.2 (b): The spectrum plots using autoregressive spectrum estimator $\hat{s}(\lambda)$ for the modified irregular component of dry wine monthly sale in Australia (1987/01 to 1994/12).

The X-12 methods use the last (most recent) 96 observations in spectrum estimation, not only because seasonal and trading day patterns change over time, but also because modelling adjustment and modelling adequacy checking are most sensitive for the most

recent data. Nevertheless, the X-12 methods also allow users to define the start date and the end date in format *year.month*, and then the program will draw the spectral plots using series data within the defined time interval. Additionally, the program might draw the spectral plots with a warning message “Visually significant seasonal peaks”; this message means that the seasonal effects are predictable. Because of difficulties in using statistical significance tests for cyclical components in autocorrelated series, mentioned by Priestley (1981, chap. 8), the message of “Visually significant” is defined by the range $\hat{s}^{max} - \hat{s}^{min}$ of the autoregressive estimated values $\hat{s}(\lambda)$. This approximate concept is obtained in an empirical way. “Visually significant” means that the value $\hat{s}(\lambda_t)$ at seasonal or trading-day frequency λ_t (except seasonal frequency $\lambda_{60} = 0.5$, because the peaks at frequency $\lambda_{60} = 0.5$ are considered as “noise” rather than identified seasonal components in practice) must be higher than the median of the values of $\hat{s}(\lambda)$ and must be higher than both $\hat{s}(\lambda_{t-1})$ and $\hat{s}(\lambda_{t+1})$ by at least six “asterisks”.

3.2 Regressors of trading day effects for the three types of decompositions

When the X-12 methods detect a time series with any calendar effects, the methods will add the corresponding regressors to the regARIMA models. The regressors of each effect are discussed in the following paragraphs. The definitions of any notations used in the following equations, see table 3.2.1 after the equation (3.21) below.

Denoting the cumulative effect of a month is $\sum_{j=1}^7 \alpha_j D_{jt}$, where α_j means the effect of j -th day of the week, so that $j = 1$ indicates Monday, $j = 2$ Tuesday, and so on. Also, D_{jt} denotes the number of occurrences of day j in month t . Let $\bar{\alpha} = \sum_{j=1}^7 \alpha_j / 7$ and the length of month t is $N_t = \sum_{j=1}^7 D_{jt}$. The cumulative effect

$$\begin{aligned}
\sum_{j=1}^7 \alpha_j D_{jt} &= \bar{\alpha} N_t + \sum_{j=1}^7 (\alpha_j - \bar{\alpha}) D_{jt} \\
&= \bar{\alpha} N_t + \sum_{j=1}^6 (\alpha_j - \bar{\alpha}) (D_{jt} - D_{j7}), \tag{3.13}
\end{aligned}$$

with $\sum_{j=1}^7 (\alpha_j - \bar{\alpha}) = 0$.

The regressors are used in the regression models for irregulars without seasonality and trends. To de-seasonalize and de-trend, we divide by $\bar{\alpha} N_t^*$ on equation (3.13) and defining $\beta_j = (\alpha_j / \bar{\alpha}) - 1$, then we obtain the trading-day regressor for multiplicative decomposition:

$$\frac{N_t}{N_t^*} + \sum_{j=1}^6 \beta_j \left(\frac{D_{jt} - D_{7t}}{N_t^*} \right) = \frac{\sum_{j=1}^7 (1 + \beta_j) D_{jt}}{N_t^*}. \tag{3.14}$$

The coefficients $(1 + \beta_j)$ are the weights of daily, $\beta_7 = -\sum_{j=1}^6 \beta_j$, N_t is the length of month t ($N_t = \sum_{j=1}^7 D_{jt}$), N_t^* satisfies the following:

$$N_t^* = \begin{cases} N_t, & \text{if } t \text{ is not February} \\ 28.25, & \text{if } t \text{ is February} \end{cases}. \tag{3.15}$$

In fact, the $\frac{N_t}{N_t^*}$ term is leap year (LY_t) variable and can be negligible and $N_{t+48} = N_t$.

The leap year effect caused by the length of February, the length of February in the every fourth year is 29 days. We treat leap year (LY_t) effects as trading day effects also. The coefficients β_j can be estimated by the ordinary least squares (OLS) estimation fitting of the regression model

$$N_t^* \hat{I}_t - N_t = \sum_{j=1}^6 \beta_j (D_{jt} - D_{7t}) + e_t. \tag{3.16}$$

Here, e_t 's are the errors after removing the trading day effects. Subtracting $\bar{\alpha}N_t^*$ in equation (3.13), the trading-day effect regressor for additive decomposition is obtained, which is

$$\hat{I}_t = \beta_0(N_t - N_t^*) + \sum_{j=1}^6 \beta_j(D_{jt} - D_{7t}) + e_t. \quad (3.17)$$

Here, $\beta_0 = \bar{\alpha}$ and $\beta_j = \alpha_j - \bar{\alpha}$, $1 \leq j \leq 6$.

Since the log transformation for ARIMA, seaARIMA and regARIMA is commonly used in time series, thus we give the trading-day effect form after taking log on the multiplicative decomposition, called log-additive decomposition. Taking log on the left of equation (3.14), the trading-day effect formula for log-additive decomposition is

$$\log \left\{ 1 + \frac{N_t - N_t^*}{N_t^*} + \sum_{j=1}^6 \beta_j \left(\frac{D_{jt} - D_{7t}}{N_t^*} \right) \right\} \approx \frac{N_t - N_t^*}{N_t^*} + \sum_{j=1}^6 \beta_j \left(\frac{D_{jt} - D_{7t}}{N_t^*} \right). \quad (3.18)$$

Here, using the property $\log(1 + x) \approx x$, for small values of x . The length of February is 29 in every four year, so $N_{t+48} = N_t$. Let the average length of month t be $\bar{N}_t = (1/48) \sum_{k=1}^{48} N_{t+k} = 30.4375$. If we use $e^x \approx 1 + x$ instead of $\log(1 + x) \approx x$ and set $\beta'_j = \beta_j / \bar{N}_t$, the another expression of log-additive decomposition is

$$\exp \left(\frac{N_t - N_t^*}{N_t^*} \right) \exp \left\{ \sum_{j=1}^6 \beta_j \left(\frac{D_{jt} - D_{7t}}{N_t^*} \right) \right\} \approx \frac{N_t}{N_t^*} \exp \left\{ \sum_{j=1}^6 \beta'_j (D_{jt} - D_{7t}) \frac{\bar{N}_t}{N_t^*} \right\}. \quad (3.19)$$

The trading-day regressor of pseudo-additive decomposition is

$$1 + \frac{N_t - N_t^*}{\bar{N}_t} + \sum_{j=1}^6 \beta_j \left(\frac{D_{jt} - D_{7t}}{\bar{N}_t} \right). \quad (3.20)$$

Simplifying equation (3.19) by treating $\bar{N}_t/N_t^* = 1.0$, the default regARIMA(p, d, q) $\times$ (P, D, Q)_s model with only trading-day regressor for the multiplicative decomposition of a monthly series Y_t with log transformation is then

$$(1 - B)^d (1 - B^{12})^D \left\{ \log \left(\frac{N_t^*}{N_t} Y_t \right) - \sum_{j=1}^6 \beta_j (D_{it} - D_{7t}) \right\} = \omega_t. \quad (3.21)$$

Here, $\omega_t = \frac{\theta_q(B)\theta_Q(B^s)}{\Phi_p(B)\Phi_P(B^s)} a_t$.

Table 3.2.1 Predefined Regression Variables in X-12-ARIMA

Regression Effect	Variable Definition(s)
Additive outlier at t_0	$AO_t^{(t_0)} = \begin{cases} 1, & \text{for } t = t_0 \\ 0, & \text{for } t \neq t_0 \end{cases}$
Level shift at t_0	$LS_t^{(t_0)} = \begin{cases} -1, & \text{for } t < t_0 \\ 0, & \text{for } t \geq t_0 \end{cases}$
Temporary change at t_0	$TC_t^{(t_0)} = \begin{cases} 0, & \text{for } t < t_0 \\ \alpha^{t-t_0}, & \text{for } t \geq t_0 \end{cases}$
where α is the rate of decay back to the previous level ($0 < \alpha < 1$).	
Innovation outliers, t_0 to t_1	$IO_t^{(t_0, t_1)} = \begin{cases} -1, & \text{for } t \leq t_0 \\ \frac{(t-t_0)}{(t_1-t_0)} - 1, & \text{for } t_0 < t < t_1 \\ 0, & \text{for } t \geq t_1 \end{cases}$
*Fixed seasonal	$M_{i,t} = \begin{cases} 1, & \text{in month } i \\ -1, & \text{in December, } 1 \leq i \leq 11 \\ 0, & \text{otherwise} \end{cases}$
(For example, the Christmas in December and New Year in January, these holidays which happen in the same month in every year are treated as fixed season effects.)	
Trading day	$T_{1,t} = (\text{no. of Mondays}) - (\text{no. of Sundays})$
	□
	$T_{6,t} = (\text{no. of Saturdays}) - (\text{no. of Sundays})$
*Length of month	$N_t - \bar{N}_t$, where N_t = length of month t [in days] and
(monthly flow)	$\bar{N}_t = 30.4375$ [Average length of month]
Leap year	$N_t - N_t^*$, where $N_t^* = (N_t + N_{t-12} + N_{t-24} + N_{t-36})/4$
(monthly flow)	(Note: This variable is 0 except in February.)

*The variables shown are for monthly series. Corresponding variables are available for quarterly series.

3.3 Definitions of outliers and statistics detecting outliers

X-12 methods provide four types of regressive variables to deal with temporary or permanent abrupt changes in the level of time series: additive outliers (AOs), level shifts (LSs), temporary changes (TCs), and innovation outliers (IOs). The additive outliers affect only one time series observation. If there is an additive outlier at date t_0 , for month t , at $t = t_0$, $AO_t^{(t_0)} = 1$, otherwise, $AO_t^{(t_0)} = 0$. The level shift outliers shift all series observations from certain time point t_0 onward by a constant amount, for $t < t_0$, $LS_t^{(t_0)} = -1$, otherwise, $LS_t^{(t_0)}$ is zero. This is also equivalent to $LS_t^{(t_0)} = 0$ and $LS_t^{(t_0)} = 1$ for $t < t_0$. The temporary change outliers mean an abrupt change in the level of the series and return to its previous level quickly. Assume a temporary change at date t_0 in month t , for $t < t_0$, define $TC_t^{(t_0)} = 0$, otherwise, the temporary change at date t_0 will be defined as $TC_t^{(t_0)} = \alpha^{t-t_0}$, where α is the rate of decay back to the previous level, in which $0 < \alpha < 1$. The formula for computing decay rate of temporary change is $\alpha = 0.7 ** (12/period)$, with period $s = 12$ for monthly time series and $s = 4$ for quarterly time series. Compared with level shift outliers, innovation outliers ($IO_t^{(t_0, t_1)}$) also change the level of the series also. Unlike level shifts, innovation outliers are a linear increase or decrease in the level of the series over a certain time interval. If a time series has an innovation outlier effect from date t_0 to date t_1 in month t , for $t \leq t_0$, $IO_t^{(t_0, t_1)} = -1$; if $t_0 < t < t_1$, $IO_t^{(t_0, t_1)} = (t - t_0)/(t_1 - t_0) - 1$, otherwise, $IO_t^{(t_0, t_1)} = 0$.

3.4 Outlier detection procedure

X-12 methods use either forward adding methods or backward eliminating methods to detect outliers. The general idea of forward adding is to assume a regARIMA model

contains no outliers' regressors. If the maximum, absolute calculated values $\lambda_{I,t}$ and $\lambda_{A,t}$ for IO and AO outliers ($\lambda_{I,t}$ and $\lambda_{A,t}$ are obtained from likelihood ratio test, the test will be discussed later), respectively, are greater than or equal to the X-12 method default critical value C (values of C can be found in table 3.4.1 below), we put the proper outliers' regressor in the regARIMA model. The idea of the backward deletion procedure is to assume the regARIMA model contains outliers' regressor. If the minimum, absolute calculated values $\lambda_{I,t}$ and $\lambda_{A,t}$ are smaller than or equal to critical value C , then we remove the regressor from the model. An important equation for forward and backward outliers' detection procedures is obtained from equation (2.1), which is

$$a_t(\beta, \psi) = \frac{\Phi_p(B)\Phi_p(B^s)}{\theta_q(B)\theta_q(B^s)} (1-B)^d(1-B^s)^D \left(y_t - \sum_{i=1}^r \beta_i x_{it} \right). \quad (3.22)$$

Here, β is the vector of the regression coefficients and ψ is the coefficients vector of ARMA model. The two procedures will be summarized in sections 3.4.2 and 3.4.3, respectively.

3.4.1 Likelihood Ratio Test.

X-12 methods use the likelihood ratio test (LRT), also called t-statistics, to detect the outliers. In Chang et al. (1988), they defined the model for an IO as

$$Z_t = x_t + \frac{\theta(B)}{\phi(B)\alpha(B)} \omega I_t^{(T)}. \quad (3.23)$$

Here $\alpha(B) = (1-B)^d(1-B^s)^D$, $I_t^{(T)}$ is an indicator function, $I_t^{(T)} = 1$ at $t = T$, otherwise, $I_t^{(T)} = 0$. The model for AO is

$$Z_t = x_t + \omega I_t^{(T)}. \quad (3.24)$$

Where $x_t = \frac{\theta(B)}{\phi(B)\alpha(B)}a_t$, $\theta(B)$ and $\phi(B)$ are the operators of ARIMA models, and $\alpha(B) = (1-B)^d(1-B^s)^D$ is the operator of seasonal and non-seasonal differencing, and a_t 's are white noise. Then, we substitute x_t back into equation (3.23) and (3.24), respectively, get

$$(IO) \quad Z_t = \frac{\theta(B)}{\phi(B)\alpha(B)} \{a_t + \omega I_t^{(T)}\}. \quad (3.25)$$

$$(AO) \quad Z_t = \frac{\theta(B)}{\phi(B)\alpha(B)} a_t + \omega I_t^{(T)}. \quad (3.26)$$

Let $\pi(B) = (\frac{\theta(B)}{\phi(B)\alpha(B)})^{-1} = 1 - \pi_1 B - \pi_2 B^2 - \dots$, and multiply by $\pi(B)$ on both sides of equations (3.25) and (3.26), then obtain:

$$(IO) \quad \pi(B)Z_t = a_t + \omega I_t^{(T)}, \quad (3.27)$$

$$(AO) \quad \pi(B)Z_t = \omega I_t^{(T)} \pi(B) + a_t, \quad (3.28)$$

Now, let $e_t = \pi(B)Z_t$, for $t = 1, \dots, n$, and $\hat{\omega}_I$ and $\hat{\omega}_A$ be the estimator of impact ω for IO and AO outliers. When outliers occur at $t = T$, then

$$(IO) \quad \hat{\omega}_I = e_{n-T}, \quad (3.29)$$

$$(AO) \quad \hat{\omega}_A = \pi(F)e_T/\rho_A^2 = (1 - \pi_1 F - \pi_2 F^2 - \dots - \pi_T F^T)e_{n-T}/\rho_A^2, \quad (3.30)$$

where $\rho_A^2 = 1 + \pi_1^2 + \pi_2^2 + \dots + \pi_T^2$. Equation (3.30) also can be written as

$$\hat{\omega}_A = \pi(F)\pi(B)Z_t/\rho_A^2. \quad (3.31)$$

Thus, the variances of $\hat{\omega}_I$ and $\hat{\omega}_A$ are

$$(IO) \quad \text{var}(\hat{\omega}_I) = \text{var}(e_T) = \hat{\sigma}_a^2, \quad (3.32)$$

$$(AO) \quad \text{var}(\hat{\omega}_A) = \text{var}(\pi(F)e_T/\rho_A^2) = \hat{\sigma}_a^2/\rho_A^2, \quad (3.33)$$

where, $\hat{\sigma}_a$ is the robust estimator of σ_a .

To test if there is IO effects, the likelihood ratio test hypothesis is

$$H_o: \omega = 0 \quad \text{vs.} \quad H_A: \omega \neq 0$$

the general likelihood function for ω :

$$L(\omega) = \left(\frac{1}{2\pi\sigma_a^2}\right)^{\frac{1}{2}} \times \exp\left(-\frac{(e_t - \omega)^2}{2\sigma_a^2}\right), \quad (3.34)$$

where, e_t follows normal distribution with constant variance σ_a^2 and mean zero.

Under $H_0: \omega = 0$, the likelihood ratio function

$$L(\omega = 0) = \left(\frac{1}{2\pi\sigma_a^2}\right)^{\frac{1}{2}} \times \exp\left(-\frac{(e_t)^2}{2\sigma_a^2}\right), \quad (3.35)$$

and under $H_A: \omega \neq 0$, the likelihood ration function

$$L(\omega \neq 0) = \left(\frac{1}{2\pi\sigma_a^2}\right)^{\frac{1}{2}} \times \exp\left(-\frac{(e_t - \omega)^2}{2\sigma_a^2}\right). \quad (3.36)$$

Let $\hat{\omega}$ be the least square estimate of the impact ω . Then, the likelihood function for $\hat{\omega}$ is

$$L(\hat{\omega}) = \left(\frac{1}{2\pi\sigma_a^2}\right)^{\frac{1}{2}} \times \exp\left(-\frac{(e_t - \hat{\omega})^2}{2\sigma_a^2}\right). \quad (3.37)$$

Because of $\hat{\omega}_I = e_t$ for IO case, thus equation (3.37) becomes

$$L(\hat{\omega}) = \left(\frac{1}{2\pi\sigma_a^2}\right)^{\frac{1}{2}} \times \exp\left(-\frac{(e_t - \hat{e}_t)^2}{2\sigma_a^2}\right). \quad (3.38)$$

Finally, we get the likelihood ratio Λ , which is equal to

$$\Lambda = \frac{L(\omega=0)}{L(\omega \neq 0)} = \frac{L(\omega=0)}{L(\hat{\omega})} = \frac{\left(\frac{1}{2\pi\sigma_a^2}\right)^{\frac{1}{2}} \times \exp\left(-\frac{(e_t)^2}{2\sigma_a^2}\right)}{\left(\frac{1}{2\pi\sigma_a^2}\right)^{\frac{1}{2}}} = \exp\left(-\frac{(e_t)^2}{2\sigma_a^2}\right). \quad (3.39)$$

Then, taking the log function of the likelihood ratio Λ , we get

$$\log \Lambda = -\frac{1}{2} \frac{(e_t)^2}{\sigma_a^2}, \quad (3.40)$$

and,

$$-2\log \Lambda = \frac{(e_t)^2}{\sigma_a^2} \sim \chi_{\alpha}^2(1). \quad (3.41)$$

When $-2\log \Lambda \geq \chi_{\alpha}^2(1)$ at significant level α , we reject the null hypothesis and conclude that there is IO impact in the series. The calculations of other three impacts are quite complicate and they are not shown in this thesis.

After diagnosing the types of impact in a time series, the next step will detect outliers. To detect outliers, one simple method is to compare the studentized residuals $\lambda_{A,t}$, $\lambda_{I,t}$, $\lambda_{L,t}$ and $\lambda_{T,t}$ with t-statistics,

$$\lambda_{A,t} = \frac{\hat{\omega}_A}{\hat{\sigma}_a / \rho_A^2},$$

and

$$\lambda_{I,t} = \frac{\hat{\omega}_I}{\hat{\sigma}_a}.$$

Also, we use likelihood ratio test to detect LS and TC impacts, Chen and Liu (1991) denoted the models of LS and TC through empirical examples, which are

$$\text{LS:} \quad Z_t = X_t + \frac{1}{(1-B)} \omega_L I_t^{(T)}. \quad (3.42)$$

$$\text{TC:} \quad Z_t = X_t + \frac{1}{(1-\delta B)} \omega_T I_t^{(T)}, \quad 0 < \delta < 1. \quad (3.43)$$

In TC model, δ is the pace of the dynamic dampening impact. The paper recommends $\delta = 0.7$ that in order to separate the nature of TC from an AO or LS, because AO and LS are the boundary case of a TC when $\delta = 0$ and $\delta = 1$, respectively.

The test statistics for LS and TC are

$$\lambda_{L,t} = \frac{\hat{\omega}_L}{\hat{\sigma}_a / \rho_L^2}. \quad (3.44)$$

$$\lambda_{T,t} = \frac{\hat{\omega}_T}{\hat{\sigma}_a / \rho_T^2}. \quad (3.45)$$

Here,

$$\hat{\omega}_L = [1 + (1 - \pi_1 F) + \cdots + (1 - \pi_1 F^T - \cdots - \pi_T F^T)] e_{n-T} / \rho_L^2.$$

$$\hat{\omega}_T = [1 + (\delta - \pi_1 F) + \cdots + (\delta^T - \delta^{T-1} \pi_1 F^T - \cdots - \pi_T F^T)] e_{n-T} / \rho_T^2.$$

And

$$\rho_L^2 = 1 + (1 - \pi_1)^2 + \cdots + (1 - \pi_1 - \cdots - \pi_T)^2.$$

$$\rho_T^2 = 1 + (\delta - \pi_1)^2 + \cdots + (\delta^T - \delta^{T-1} \pi_1 - \cdots - \pi_T)^2.$$

From above, we obtain the likelihood ratio test statistics for additive outlier(s) given by

$$\lambda_{A,t} = \frac{\hat{\omega}_A}{\hat{\sigma}_a / \rho_A^2}.$$

When the locations of outliers are unknown, we need to use the outlier detection procedure either step 4 in the forward adding methods or step 3 in the backward

eliminating methods, and set $\lambda = \max_{1 \leq t \leq n}(\lambda_{A,t})$. However, the distribution of λ for finite sample size data is complicated due to the correlation of $\lambda_{A,t}$. In the present study, the critical values C are computed by the following asymptotic formula, from Berman (1964).

Let ρ_k be the correlation between $\lambda_{A,t}$ and $\lambda_{A,t-1}$ i.e. $\rho_k = \text{corr}(\lambda_{A,t}, \lambda_{A,t-1})$. Berman (1964) showed that if either $\sum_{k=1}^{\infty} \rho_k^2 < \infty$ or $\rho_k \log k \rightarrow 0$ as $k \rightarrow \infty$ then

$$\lim_{n \rightarrow \infty} [P\{a_n(\lambda - b_n) \leq x\}] = \exp\{-\exp(-x)\},$$

and we have

$$\begin{aligned} a_n(\lambda - b_n) &\leq x \\ \Rightarrow \lambda &\leq \frac{x}{a_n} + b_n, \end{aligned}$$

which gives the critical value C with

$$a_n = (2 \log n)^{1/2}$$

and

$$b_n = (2 \log n)^{1/2} - \frac{1}{2} (2 \log n)^{-1/2} \{\log(\log n) + \log(4\pi)\}.$$

The critical values are obtained by Ljung (1993) based on Berman's formula above and listed in table 3.4.1 as follows.

Table3.4.1: The Default Critical Values based on the Number of Observations for Outliers' Detection. (Source: REFERENCE MANUAL, 2007).

No. of Observations Tested	Outlier Critical Value	No. of Observations Tested	Outlier Critical Value
1	1.96	48	3.63
2	2.24	72	3.73
3	2.44	96	3.80
4	2.62	120	3.85
5	2.74	144	3.89
6	2.84	168	3.92
7	2.92	192	3.95
8	2.99	216	3.97
9	3.04	240	3.99
10	3.09	264	4.01
11	3.13	288	4.03
12	3.16	312	4.04
24	3.42	336	4.05
36	3.55	360	4.07

3.4.2 Forward Adding Method

1. Estimating the coefficients β and ψ from equation (3.22). Compute robust standard deviation for error $\sigma_a^R = 1.483 \times \text{median}|a_t(\beta, \psi) - \text{med}[a_t(\beta, \psi)]|$, Hampel (1986, pp.105) for the current β and ψ . Note: to compute the coefficient 1.483, let $\sigma_a^R = K \times MAD$, where MAD is median absolute deviation, $MAD = \text{median}|a_t(\beta, \psi) - \text{med}[a_t(\beta, \psi)]|$. For a symmetric random variable X , the MAD satisfies:

$$\begin{aligned}\frac{1}{2} &= P(|X - \mu| \leq MAD), \\ \Rightarrow \frac{1}{4} &= P(0 < X - \mu < MAD), \\ \Rightarrow \frac{1}{4} &= P\left(0 < \frac{X - \mu}{\sigma} < \frac{MAD}{\sigma}\right).\end{aligned}$$

If X is a normal random variable, then the ratio term MAD/σ satisfies

$$\Phi\left(\frac{MAD}{\sigma}\right) = \frac{3}{4},$$

where, Φ is the standard normal cumulative distribution function. Then

$$\Phi^{-1}\left(\frac{3}{4}\right) = \frac{MAD}{\sigma} = 0.6745,$$

the σ under the form $\sigma_a^R = K \times MAD$, we got

$$\sigma_a^R = \frac{1}{\Phi^{-1}\left(\frac{3}{4}\right)} MAD,$$

which $K = \frac{1}{\Phi^{-1}\left(\frac{3}{4}\right)} \approx 1.483$.

2. Using the current ψ and σ_a^R to compute $\lambda_{I,t}$ and $\lambda_{A,t}$ for IO and AO regressor not currently in the regARIMA model.

3. Using $\max|\lambda_{I,t}|$ and $\max|\lambda_{A,t}|$ to determine IO and AO outliers, respectively.
4. If $\max|\lambda_{I,t}| > C$, we add IO outlier regressor in the regARIMA model;
if $\max|\lambda_{A,t}| > C$, we add AO outlier regressor in the regARIMA model. (Note: C is a critical value, for finite sample, the critical values of C are shown in Table 3.4.1 above.)
5. And then re-estimate β and ψ , redo the steps 2 to 5. Otherwise stop.

3.4.3 Backward Eliminating Method

For backward method, we assume that the regARIMA model including all outlier regressors, and then

1. Estimate β, ψ and σ_a by maximum likelihood estimation (MLE).
2. Using estimated β, ψ and σ_a to compute $\lambda_{I,t}$ and $\lambda_{A,t}$ for all IO and AO regressors.
Determine which of these regressor has $\min|\lambda_{I,t}|$ and $\min|\lambda_{A,t}|$.
3. Use the minimum $\lambda_{I,t}$ and $\lambda_{A,t}$ calculated from step 2 to compare with critical value C . If $\min|\lambda_{I,t}| < C$ or $\min|\lambda_{A,t}| < C$, we eliminate the regressor corresponding with the minimum values from the regARIMA model and then go back to the first step. Otherwise, we stop.

3.5 Stability diagnostics for seasonal and trend adjustments

From computations of the decompositions discussed in section 2.2, after applying seasonal moving average filters to a series, we got seasonally adjusted values and final trend adjusted values. Since unstable adjusted values will highly vary the magnitude of seasonal and trend in the adjusted time series, it is necessary to check the stability of the adjusted values. Besides Quality control statistics (Q-statistics), the X-12 methods allow

for two new informative diagnostics to detect the stability of adjusted series: sliding spans and revision history. A discussion of Q-statistics is provided by Lothian and Morry (1978), so we will not discuss it further in this thesis. Sliding spans and revision history diagnostics are detailed in the following.

3.5.1 Sliding Spans

Sliding spans analysis uses the most current data to investigate stability of adjusted values. As new observations become available and are added to the series, the adjusted values will be re-computed (hence, changed) for any given time point. Sliding spans analysis calculates the percentage change of seasonal adjustment for up to four overlapping sub-spans of the series; each month is common to at least two of the sub-spans (see figure 3.3 below). The uses of sliding spans are to check whether adjusted values are stable and to confirm the choice of length of the seasonal moving average filters (Findley et al., 1998). We will detail these two uses a bit later.

Figure 3.3 The Diagram of Sliding Spans for a Series Using 3×5 Seasonal Moving Filters.

Sliding spans analysis is used for the seasonally adjusted values of series that are all positive. Generally, a series data is distributed to four overlapping spans (if enough data are available), and the length of the first span could be determined by the length of filters used in seasonally adjusted procedures. The table 3.5.1 listed in the following shows

empirical criteria of the length of the first span according to the length of seasonal moving average filters.

Table3.5.1: The Criteria of Choosing the Length for the First Span.

Length of seasonal moving average filters	Length of years for the first span(in years)
3×3	6
3×5	7 or 8
3×9	11

From table 3.5.1, we know that if we use 3×3 seasonal average moving filters to adjust the series, the first span contains six years of data, from year $j + 1$ to year $j + 6$. After we obtain the first span, the second span will start with the second data in the first span and the first available data excluded from the first span, which means the second span includes $j + 2$ to $j + 7$. Similarly, the third span includes $j + 3$ to $j + 8$, and the fourth span includes $j + 4$ to $j + 9$; year $j + 9$ should be the most recent year. The structures of sliding spans of 8 years' length in the first span are shown in figure 3.3 on the previous page. The last span contains the most recent data in the time series.

3.5.2 Useful Computations in Sliding Span Diagnostics

Two important computations are made in sliding spans analysis, $A\%$ and $MM\%$. " $A\%$ ", denotes the proportion of the percent change of seasonal adjusted values greater than 0.03. " $MM\%$ ", denotes the proportion of the month-to-month percent change of adjusted values greater than 0.03. If $A\% \leq 15\%$, which means the most of seasonal adjusted values (around 85%) are stable. The meaning of the term "stable" is that seasonal movements are not highly variable. Because most seasonally adjusted values are stable,

we can perform the seasonal adjustment on the series. If $15\% < A\% \leq 25\%$, it indicates that the adjusted values are less likely to be stable and we should carefully consider performing the seasonal adjustment on the series; if $A\% > 25\%$ and/or $MM\% > 40\%$, the seasonal adjusted values are unstable and we should not perform seasonal adjustment on the series. Similarly, we define $YY\%$ as the proportion of the percent change in year-to-year seasonal adjusted values greater than 0.03. The seasonal adjustment of year-to-year percentage change is $100 \times [A_t(k) - A_{t-12}(k)/A_{t-12}(k)]$.

To calculate the percentage change of the seasonally adjusted values, let A_t denote a seasonal adjusted value (if applicable, taking into consideration trading-day) obtained from the entire series. Let $A_t(k)$ denote seasonal (or trading-day) adjusted values in span k at month t in different years. The seasonal adjusted value A_t is unstable (or unreliable), if

$$\frac{\max_k A_t(k) - \min_k A_t(k)}{\min_k A_t(k)} > 0.03. \quad (3.46)$$

The seasonally adjustment month-to-month in span k at month t ,

$$MM_t(k) = \frac{A_t(k) - A_{t-1}(k)}{A_{t-1}(k)}, \quad (3.47)$$

is considered as unreliable if,

$$\max_k MM_t(k) - \min_k MM_t(k) > 0.03. \quad (3.48)$$

The range of index k in (3.46) over all spans contains month t ; in (3.47) the range of index k must contain both month t and month $t - 1$. The threshold value 0.03 is obtained from empirical experience (economic data) by Findley, Monsell, Shulman and Pugh (1990).

In the previous section, we mentioned that Quality control statistics (Q-statistics) can be used to examine the adequacy of adjusted values as well. It is possible that the diagnostics of sliding spans analysis conflict with the diagnostics of the Q-statistics. Findley, Monsell, Shulman and Pugh (1990) gave a good example (shipments of building papers series), illustrating the conflict between the two tests. Figure 3.4 below is a line plot of shipments of building papers (SBP) series (initial data). The plot shows sudden changes near the end of the series. For this series, Q-statistics suggest that the seasonal adjustment methods can be used in the SBP series, while sliding spans diagnostics gave an opposite suggestion. As we mentioned, sliding spans diagnostics examine the most current observations, and the current observations are usually informative. Thus, it would be better for users to accept sliding spans diagnostics rather than those of Q-statistics.

Figure 3.4 The Line Plot of Shipments of Building Papers Series.

Besides diagnosing the stability of seasonal adjustment, sliding spans analysis can be used to select the length of average seasonal moving factors as well. Table 2.3.1 shows the criteria by which X-12 methods use the *SI* ratio (global moving seasonality ratio, GMSR) to determine the length of seasonal moving average filters. Table 3.5.2 below, from Findley et al. (1990) shows the diagnostics of three series resulting from sliding spans analysis and *SI* ratio, as well as the moving average filters applied to the three series.

Table 3.5.2. The Criterion of Selecting the Length of Seasonal Filters Based on The Sliding-Spans Diagnostics. (Source: Journal of the American Statistical Association, 1990).

Series (GMSR)	Seasonal filter	A (%)	MM (%)
EDMISC (6.47)	3×5	35.3	42.6
	3×9	15.1	29.8
UOOG (3.75)	3×3	41.7	32.5
	3×5	37.0	28.1
	3×9	10.4	9.1
TOTIMP (5.1)	3×3	8.5	28.6
	3×5	11.0	41.9

The GMSR of EDMISC (total consumption of noncategorized edible products) series is 6.47. According to the criteria from table 2.3.1 in section 2, this GMSR value indicates that the series should use 3×9 seasonal moving filters and the result of sliding-spans analysis is consistent with the *SI* ratio criterion. For UOOG (unfilled orders of ophthalmic goods, watches, and watch cases) series, GMSR is 3.75, which suggests 3×5 seasonal filters for the series. Sliding-spans analysis indicates that only 3×9 moving average filters are acceptable, because value *A*% is 10.4% less than 15% and *MM*% is 9.1% less than 40% under this length of moving average filters. Since the seasonal factors with

3×9 filters are more reliable than the seasonal factors with 3×5 filters, it would be better to use 3×9 seasonal filters with this series to adjust seasonal effects. The GMSR of TOTIMP (total U.S. imports, including freight and insurance) series is 5.1, which suggests 3×5 filters, but $MM\%$ is 41.9% and thus greater than 40%, which means that the 3×5 filters are not acceptable. Sliding-spans analysis suggests that 3×3 filters are acceptable for the series because value $A\%$ is less than 15% and $MM\%$ is less than 40%.

3.5.3 Revision History

When new series data become available and join the series, the initial adjusted values for each given time point will be recalculated, the procedure is called revision. The set of revision values is called revision history. To explain the process of calculating the revision, we denote Y_t as an unadjusted series, $1 \leq t \leq M$, and let $A_{t|v}$ be the seasonally adjusted values of time t , computed from series Y_t , $1 \leq t \leq v$, where v is any integer between time t and M (assume only v observations are available and $Y_{v+1}, \dots, Y_M$ are withheld at this time). We also denote $A_{t|t}$ the concurrent seasonally adjusted value obtained through time t and denote $A_{t|M}$ the most current seasonally adjusted value obtained through time M . Generally, $A_{t|t}$ is the initial (earliest) adjusted value and $A_{t|M}$ is the final seasonally adjusted value.

The percentage of revision between initial and final seasonal adjustment for multiplicative decomposition time series can be expressed as:

$$R_{t|M}^A = \frac{A_{t|M} - A_{t|t}}{A_{t|t}} \times 100. \quad (3.49)$$

For given time points M_0 and M_1 where $0 < M_0 < M_1 < M$, the sequence of $R_{t|M}^A$, with time point t between M_0 and M_1 , is called the revision history of the seasonal adjustment from time M_0 to time M_1 . The percentage of month-to-month change in the seasonally adjusted series at time t calculated from the series $Y_{v+1}, \dots, Y_M$:

$$C_{t|v}^A = \frac{A_{t|v} - A_{t-1|v}}{A_{t-1|v}} \times 100. \quad (3.50)$$

The revision history of month-to-month revisions:

$$R_{M-t|t}^A = C_{t|M} - C_{t|t}, \quad M_0 \leq t \leq M_1. \quad (3.51)$$

The program can calculate the basic revisions which are the difference between the initial adjustment for any given time point (the adjusted value of a given time point is concurrent adjustment in a series) and final adjustment (the last adjusted value when series undergoing the data set reach the effective length; more details about effective length will be found in a later discussion). Similarly, the revision could be obtained from month-to-month change, trend estimates and trend change.

To obtain more accurate information from revision history analysis, X-12 methods allow the user to choose the number of forecasting year(s) in order to extend the series to reach an effective length. Effective length is determined by the length of moving average filters. For a series using $3 \times m$ seasonal moving average filters for seasonal adjustments, the number of forecasting years to achieve the effective length is $1 + m/2$ years, with $m = 3, 5$ or 9 for seasonal moving average filters and $m = (n + 3)/2$ for Henderson trend filters, here $n = 9, 13$, or 23 . In this thesis the decimal value will be rounded down

the value of $m/2$. The effective length of seasonal moving average filters and Henderson trend filters is $12(2 + m)$, (Bell and Monsell, 1992).

The program also calculates the adjusted values of final Henderson trend, $T_{t|v}$, the month-to-month change in the trend series $C_{t|v}^T$ and revision $R_{t|M}^T$ and revision history $R_{M-t|t}^T$. The interval of trend revision is slightly wider than the revision interval of seasonal adjustment, because the effective length for trend filters needs one or two years longer than seasonal moving average filters (see the effective length computing formula in the previous paragraph).

Near the end of the series, the number of observations is insufficient for performing the symmetric filters for seasonal adjustments or trend adjustments, (the symmetric filters are considered the best). In this situation, besides applying asymmetric filters, the X-12 methods also provide a forecasting option to extend the series, resulting in use of symmetric filters. The alternative meaning of effective length is that the length of the series is long enough to perform symmetric filters at the end of series data.

Without the forecasting option, we have to wait three years to achieve effective length for series using 3×5 moving filters and wait five years to achieve effective length for use of 3×9 moving filters, (Bobbitt and Otto, 1990). Symmetric moving average filters are considered the best adjustment for the series, because they help reduce the total revisions. The forecasting extension series have smaller revisions than the series without forecasting extension, proved by Pierce (1980). For the observed series y_t which is generated by a stochastic model, $y_t = \psi(B)a_t$, $y_t \leq t - v$, a_t is white noise with mean 0 and variance σ_a^2 , and v is an index which means the number of available data. The estimation of seasonal component through v observations, $S_t^{(v)}$, can be expressed as:

$$S_t^{(v)} = \lambda^{(v)}(B)y_t = \sum_{k=v}^{\infty} \lambda_k^{(v)} y_{t-k} \quad (3.52)$$

$$= \eta^{(v)}(B)a_t = \sum_{k=v}^{\infty} \eta_k^{(v)} a_{t-k}, \quad (3.53)$$

where

$$\eta^v(z) = \psi(z)\lambda^{(v)}(z). \quad (3.54)$$

At $v = 0$, we obtained the initial estimate of seasonal component. For $v < 0$, $S_t^{(v)}$ are revised when future data become available (or the series need forecasting extension), and $v > 0$, $S_t^{(v)}$ are computed based on observations through $t - v$. The revision of the seasonal component as of time $v - n$, $r_t^{(v,n)}$, which $n < v$, is based on the difference between seasonal component estimates $S_t^{(n)}$ and $S_t^{(v)}$:

$$\begin{aligned} r_t^{(v,n)} &= S_t^{(n)} - S_t^{(v)} = [\eta^{(n)}(B) - \eta^{(v)}(B)]a_t \\ &= \sum_{j=n}^{\infty} \eta_j^{(n)} a_{t-j} - \sum_{j=v}^{\infty} \eta_j^{(v)} a_{t-j} \\ &= \sum_{j=n}^{v-1} \eta_j^{(n)} a_{t-j} + \sum_{j=v}^{\infty} [\eta_j^{(n)} - \eta_j^{(v)}] a_{t-j}. \end{aligned} \quad (3.55)$$

Assume the revision $r_t^{(v,n)}$ are stationary (i.e., $E(r_t^{(v,n)}) = 0$, $var(r_t^{(v,n)}) = \sigma_r^2$ and $cov(r_t^{(v,n)}, r_s^{(v,n)}) = \gamma(t - s)$ which is only a function of the difference $(t - s)$). Then the mean square error (MSE) of total revision is,

$$E(r_t^{(v)})^2 = \sigma_a^2 \left(\sum_{j=-\infty}^{v-1} \eta_j^2 \right) + \sigma_a^2 \sum_{j=v}^{\infty} (\eta_j - \eta_j^{(v)})^2, \quad (3.56)$$

it is minimized when $\eta_j = \eta_j^{(v)}$, for $j \geq v$.

3.5.4. One Application of Revision History

Give a series, monthly sales data of dry wine in Australia from January 1980 to November 1994 with 3×5 term moving average filters. According to the criteria introduced in the previous paragraph, $m = 5$ in this example, the effective length of dry wine series is obtained through a 36-month forecasting extension (which is $1 + m/2 = 3.5$ years, rounded down is 3 years). In table 3.5.3 below, the absolute average values of $R_{t|M}^A$ of dry wine series, denoted as $\text{Avg}|R_{t|M}^A|$, converge to 1.96 when a 48-month forecasting extension is done. This is the smallest average value, compared with 2.48 (series without forecasting extension), 2.04 (12-month forecasting extension), and 1.97 (36-month forecasting extension). The differences between the 36-month forecasting extension and 48-month forecasting extension, in the absolute average values of $R_{t|M}^A$, are not large. Therefore, we used a three-year forecasting extension method on the series. Table 3.5.3 also lists the absolute average values of trend revision $R_{t|M}^T$, denoted as $\text{Avg}|R_{t|M}^T|$. The absolute average trend revision is 2.79 without forecasting extension and the $\text{Avg}|R_{t|M}^T|$ of the other three forecasting extension lengths is 2.24. This example illustrates that the method of forecasting extension reduces revisions of seasonal and trend adjustments in magnitude in practical situation.

The absolute seasonal adjustment revision (X-12 output table R1) and trend revision (X-12 output table R4) values are considered extreme revisions if those values are greater than 5.0% (this value is usually takes more than twice the $\text{Avg}|R_{t|M}^A|$). The table includes the number of extreme seasonal adjustment and Henderson trend revisions, denoted as

No. $|R_{t|M}^A|$ and No. $|R_{t|M}^T|$, respectively. In this example the number of extreme revisions is approximately decreasing as the number of forecasting extensions increases.

Table 3.5.3 Average Absolute Percent Revision of the Seasonal Adjustment and Concurrent Henderson Trend and Number of Extreme revision for dry wine sales data in Australia from 1980.01 to 1994.11, obtained from Different Number of Forecasting.

No. forecasts	Avg $ R_{t M}^A $	Avg $ R_{t M}^T $	No. $ R_{t M}^A $ > 5.0%	No. $ R_{t M}^T $ > 5.0%
0	2.48	2.79	13	13
12	2.04	2.24	8	2
36	1.97	2.24	6	3
48	1.96	2.24	6	3

Revision history analysis helps users determine whether the final Henderson trend estimates are stable enough to serve as an alternative to the seasonal adjustment estimates. Figure 3.5 is (a) the percent revision of the concurrent Henderson trend for the dry wine series with 36-month forecasting extension and (b) the percent revision of seasonal adjustment with 36-month forecasting extension. In figure 3.5, the vertical lines connecting initial values and final values show the size of each revision, in addition, the vertical lines crossing the horizontal axis mean that a change in sign occurs. Excessive sign changing indicates that the adjusted values are not very stable, in other words, it is an unfavourable situation. The magnitudes of revisions are indicated on the vertical axis in (a) and (b) in different scales.

Figure 3.5 Initial (concurrent) and Final (most recent) Estimates of Percent change in (a) Henderson Trend and in (b) Seasonal Adjustment. The connecting vertical lines show the size of revision.

In figure 3.5 (a) and (b), the number of trend revisions that change sign is 36 and the number of seasonal adjustment revisions changing sign is 16. For the dry wine series, the number of signs changing in trend revisions is almost twice as large as in seasonal adjustment revisions, indicating that seasonally adjusted values are more stable (the earlier adjusted values are not changed or are slightly changed when new data become available) than the final Henderson trend estimates. For forecasting leads 0, 12, 36, 48, the average absolute revision history of month-to-month seasonal adjustment and trend revisions, denoted as $\text{Avg}|R_{M-t|t}^A|$ and $\text{Avg}|R_{M-t|t}^T|$, are 3.43, 2.8, 2.81, 2.82 and 0.82, 0.6, 0.61, 0.61, respectively. From these average values we know that the concept of forecasting extension reduces the revision in magnitude comparable to the series without forecasting extension.

4. Results Comparison between Traditional ARIMA Models and regARIMA Models

In this section, we performed traditional methods and seasonal adjustment methods on three seasonal series to see which methods would give better forecasting models to users.

4.1. The SGBS series

The seasonal adjustment methods of monthly retail data of Sports Goods and Bicycle Shops (SGBS) with multiplicative decomposition is regARIMA $(0,1,1) \times (0,1,1)_{12}$. X-12 methods detected that the series have leap year effect $\frac{N_t^*}{N_t}$, trading-day effects $\sum_{j=1}^6 \beta_j (D_{it} - D_{7t})$, and temporary change effects $TC_{1993.Feb}$. The regARIMA $(0,1,1) \times (0,1,1)_{12}$ is

$$(1 - B)(1 - B^{12}) \left[\log \left(\frac{N_t^*}{N_t} Y_t \right) - \sum_{j=1}^6 \beta_j (D_{it} - D_{7t}) - k TC_{1993.Feb} \right] = w_t, \quad (4.1)$$

here, $w_t = (1 - \theta_1 B)(1 - \Theta_1 B^{12})a_t$, with $\theta_1 = 0.2364$ and $\Theta_1 = 0.1941$. The estimates of parameter β_j and k are calculated by ordinary least square (OLS), listed in Table 4.1.1 below.

Table 4.1.1 The Estimated Values of Parameter β_j and k for SGBS Series (From 1987/01 to 1998/12).

Variable	Parameter Estimate	Standard Error	t-value	**p-value
Trading Day (β_j)				
Mon	0.0013	0.00331	0.40	0.345
Tue	-0.0036	0.00336	-1.06	0.146
Wed	0.0067	0.00341	1.96	0.026
Thu	-0.0033	0.00331	-0.98	0.165
Fri	0.0086	0.00335	2.57	0.006
Sat	0.0057	0.00334	1.70	0.046
*Sun (derived)	-0.0154	0.00334	-4.63	<0.0001
TC1993. Feb (k)	-0.1129	0.02175	-5.19	<0.0001

* $\beta_7 = -\sum_{j=1}^6 \beta_j$.

** If the p-value of a regressor is less than 0.05 ($\alpha = 5\%$), then the model should contain that regressor.

To examine the capability of forecasting of the regARIMA model, we remove the last twelve observations (observation in year 1998) and use equation (4.1) to forecast the removed observations. The new estimated values of parameters θ_1 and Θ_1 are 0.2412 and 0.1828, respectively. The new estimated values of parameters β_j and k are listed in table 4.1.2 as follows.

Table 4.1.2 The Estimated Values of Parameter β_j and k for SGBS Series (From 1987/01 to 1997/12).

Variable	Parameter Estimate	Standard Error	t-value	**p-value
Trading Day (β_j)				
Mon	0.0021	0.00357	0.58	0.282
Tue	-0.0046	0.00365	-1.26	0.105
Wed	0.0062	0.00369	1.68	0.048
Thu	-0.0014	0.00353	-0.40	0.345
Fri	0.0073	0.00354	2.07	0.020
Sat	0.0066	0.00360	1.83	0.035
*Sun (derived)	-0.0162	0.00354	-4.58	<0.0001
TC1993. Feb (k)	-0.1136	0.02210	-5.14	<0.0001

* $\beta_7 = -\sum_{j=1}^6 \beta_j$.

** If the p-value of a regressor is less than 0.05 ($\alpha = 5\%$), then the model should contain that regressor.

The forecasted values of the year 1998 through the regARIMA model, based on data from January 1987 to December 1997, are listed in table 4.1.3 below.

In traditional methods, the model of SGBS series is seasonal ARIMA(0,1,1) $\times$ (0,1,1)₁₂ after applying log-transformation on original SGBS series, Y_t . The seaARIMA(0,1,1) $\times$ (0,1,1)₁₂ is

$$(1 - B)(1 - B^{12}) \log Y_t = (1 - \theta_1 B^1)(1 - \Theta_1 B^{12}) a_t, \quad (4.2)$$

here, $\theta_1 = 0.45603$ and $\Theta_1 = 0.37832$. After removing the last year observations from the series data, the new estimated values of parameters θ_1 and Θ_1 are 0.47282 and 0.39678, respectively. The forecasted values of year 1998 obtained through the seasonal ARIMA model with the new estimated parameters are show on table 4.1.3 below.

Table 4.1.3 The Forecasted Values of 1998 Obtained from traditional methods (seaARIMA) and X-12 Methods (regARIMA) and the Accurate Values of year 1998.

	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
seaARIMA	1480	1480	1808	1901	1978	2143	2039	2186	1790	1686	1845	3165
regARIMA	1480	1480	1772	1920	1939	2186	2059	2165	1790	1686	1826	3165
Actual	1458	1485	1848	1919	2043	2231	2081	2143	1779	1678	1764	2979

From table 4.1.3 we could see that the forecasted values calculated by the regARIMA models and seaARIMA models are not dramatically different, and some of them are the same. It is not easy to conclude which model fits the SGBS series better. The 95% prediction intervals (PIs) of forecasting values, using formula: *predicted values* $\pm$ $1.96 \times \text{standard deviation}$, are listed in table 4.1.4 below.

Table 4.1.4 The 95% Prediction Intervals (PIs) of Two Methods for None Transformed Models.

	Seasonal ARIMA			regARIMA		
Forecast 1998	Low CIs (L)	High CIs (H)	Difference (H-I)	Low CIs (L)	High CIs (H)	Difference (H-I)
January	1371	1599	228	1393	1573	180
February	1357	1615	258	1374	1595	221
March	1642	1990	348	1623	1936	313
April	1713	2110	397	1737	2122	385
May	1769	2213	444	1738	2164	426
June	1902	2414	512	1944	2459	515
July	1798	2312	514	1813	2339	526
August	1915	2496	581	1895	2473	578
September	1559	2056	497	1554	2061	507
October	1459	1947	488	1452	1957	505
November	1588	2143	555	1564	2132	568
December	2710	3697	987	2690	3724	1034

From table 4.1.4, all actual values fall in the 95% prediction intervals. The regARIMA models provide the narrower prediction intervals at 95% significance level for short range forecasting. From June to December, the seaARIMA model provides narrower confidence interval (excluding August) than the regARIMA model. Thus, one possible suggestion is that the users could use regARIMA model to do short range forecasting and use seaARIMA model to do long range forecasting.

4.2. The Hotel series

The hotel series is the average number of occupied rooms per day of four hotels in a Midwestern city in USA for 14 year period. Here, we use 13-year observations to investigate the fitting models by traditional methods and seasonal adjustment methods, respectively. Then, we use the tentative models obtained from these two methods to forecast the fourteenth year observations and compare the forecasted values with the real values.

For seasonal adjustment methods, the model used to fit the hotel series is regARIMA $(3,0,1) \times (0,1,1)_{12}$ with regression constant 0.0333. Also, we should take log on the hotel series Y_t , and the regARIMA $(3,0,1) \times (0,1,1)_{12}$ model for hotel series can be written as

$$\Phi_p(B)(1 - B^{12})[\log(Y_t) - 0.0333] = \theta_q(B)\theta_Q(B^s)a_t. \quad (4.3)$$

Here, $\Phi_p(B) = 1 - 0.0895B - 0.2239B^2 + 0.2891B^3$, $\theta_q(B) = 1 - 0.2128B$ and $\theta_Q(B^s) = 1 - 0.4846B^{12}$. The fourteenth year observations after taking exponential (for anti-log), forecasted by regARIMA model (equation (4.3) above), are listed in table 4.1.5 below.

The traditional method found the seaARIMA model for the hotel series is seaARIMA $(3,0,0) \times (0,1,0)_{12}$ and be written as

$$\Phi_p(B)(Y_t^{0.25} - 0.0430) = \theta_Q(B^s)a_t. \quad (4.4)$$

Here, $\Phi_p(B) = 1 - 0.2429B + 0.2992B^3 + 0.1574B^5$ and $\theta_Q(B^s) = 1 - 0.3852B^{12}$.

The forecasted observations through equation (4.4) after transferring $Y_t^{0.25}$ to Y_t are listed in table 4.1.5 also.

Table 4.1.5 The Forecasted Values the Fourteenth Year Observations Obtained from traditional methods (seaARIMA) and X-12 Methods (regARIMA) and the Accurate Values.

	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
seaARIMA	775	759	770	857	829	963	1110	1150	870	886	751	848
regARIMA	773	750	765	854	829	963	1119	1164	880	889	750	854
Actual	811	732	745	844	833	935	1110	1124	868	860	762	877

From table 4.1.5, the forecasted observations obtained from the two methods are not dramatically different. The 95% prediction intervals of forecasted values are also calculated and shown in the table 4.1.6 below. All actual values fall in the prediction intervals except January. The prediction intervals of the forecasted observations obtained from seasonal ARIMA model are slightly narrower than the forecasted values obtained by regARIMA model. Thus, the seaARIMA model seems better fit for the hotel series data.

Table 4.1.6 The 95% Prediction Intervals (PIs) of Two Methods for None Transformed Models.

	seaARIMA			regARIMA		
Forecast 14 th Obs.	Low CIs (L)	High CIs (H)	Difference (H-I)	Low CIs (L)	High CIs (H)	Difference (H-I)
January	747	804	57	745	802	57
February	731	788	57	721	780	59
March	741	800	59	736	796	60
April	825	891	66	820	890	70
May	798	862	64	795	864	69
June	927	1001	74	924	1003	79
July	1070	1152	82	1074	1166	92
August	1108	1192	84	1117	1213	96
September	836	904	68	845	917	72
October	852	922	70	853	926	73
November	721	782	61	720	781	62
December	815	882	67	820	890	70

4.3. The Trade series

The Trade series is a monthly business trade in 13 year period, from January 1961 to December 1973. The X-12 method diagnoses that the trade series has trading day and leap year effects and a temporary change (TC) outlier ($TC_{1970.Sep}$). The regARIMA model of the trade series with additive decomposition, regARIMA $(0,1,1) \times (0,1,1)_{12}$, is written as

$$(1 - B)(1 - B^{12})[Y_t - \beta_0(N_t^* - N_t) - \sum_{j=1}^6 \beta_j (D_{it} - D_{7t}) - kTC_{1970.Sep}] = w_t, \quad (4.5)$$

here, $w_t = (1 - \theta_1 B^1)(1 - \Theta_1 B^{12})a_t$, the non-seasonal MA coefficient $\theta_1 = -0.2168$, and the seasonal coefficient $\Theta_1 = 0.2275$. The parameters β_0 , β_j and k are the coefficients of leap year effect, trading day effects and temporary change outlier, respectively. The estimated values of those parameters are listed in table 4.1.7 as follows.

Table 4.1.7 The Estimated Values of Parameter β_0 , β_j and k for Trade Series (From 1961/01 to 1973/12).

Variable	Parameter Estimate	Standard Error	t-value	**p-value
Trading Day (β_j)				
Mon	0.1392	0.10083	1.38	0.085
Tue	-0.0063	0.09779	-0.06	0.476
Wed	-0.4066	0.09763	-4.16	<0.0001
Thu	0.1566	0.09652	1.62	0.054
Fri	0.2755	0.09631	2.86	0.003
Sat	0.0546	0.09419	0.58	0.282
*Sun (derived)	-0.2131	0.09821	-2.17	0.016
Leap Year (β_0)	0.0569	0.33128	0.17	0.433
TC1970. Sep (k)	3.9463	0.81581	4.84	<0.0001

* $\beta_7 = -\sum_{j=1}^6 \beta_j$.

** If the p-value of a regressor is less than 0.05 ($\alpha = 5\%$), then the model should contain that regressor.

The trade series can be described by the seasonal ARIMA $(0,1,1) \times (0,1,1)_{12}$ without any transformations, trading day and leap year effects regressor. The seasonal ARIMA $(0,1,1) \times (0,1,1)_{12}$ is written as

$$(1 - B)(1 - B^{12})Y_t = (1 - \theta_1 B^1)(1 - \Theta_1 B^{12})a_t. \quad (4.6)$$

Here, the non-seasonal MA coefficient $\theta_1 = 0.1076$ and seasonal MA coefficient $\Theta_1 = -0.3238$. The one year ahead forecasted values that are obtained by equation (4.6) listed in table 4.1.8 and the 95% prediction confidence intervals are listed in table 4.1.9 below.

In table 4.1.8 and table 4.1.9 below, we conclude that the regARIMA models are better than seasonal ARIMA models for the trade series, because the forecasted values obtained by regARIMA models are slightly closer to the actual values compared to seaARIMA models. Moreover, the 95% prediction intervals of trade series obtained from regARIMA models are little narrower than the intervals obtained from seaARIMA models. The actual values of January, February and March did not fall in the 95% prediction intervals for both models and the actual value of April only falls in the prediction intervals of regARIMA methods.

Table 4.1.8 The Forecasted Values of the Thirteenth Year Observations for Trade series (1961/01-1973/12), Obtained from traditional methods (seaARIMA) and X-12 Methods (regARIMA) and the Accurate Values.

	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
seaARIMA	365.1	362.1	365.5	372.2	379.3	383.3	383.2	386.7	387.4	390.8	396.1	400.9
regARIMA	365.2	362.3	365.6	372.7	379.9	383.4	383.4	387.9	388.1	391.5	396.9	401.5
Actual	360.6	354.4	357.4	367.0	375.7	381.0	381.2	383.0	384.3	387.0	391.7	396.0

Table 4.1.9 The 95% Prediction Intervals (PIs) of Two Methods for Trade Series.

	seaARIMA			regARIMA		
Forecast 14 th Obs.	Low CIs (L)	High CIs (H)	Difference (H-I)	Low CIs (L)	High CIs (H)	Difference (H-I)
January	362.5	367.7	5.2	362.9	367.5	4.6
February	358.2	366	7.8	358.8	365.8	7
March	360.6	370.4	9.8	361.2	370	8.8
April	366.6	377.8	11.2	367.5	377.9	10.4
May	373	385.6	12.6	374	385.8	11.8
June	376.3	390.3	14	377	389.8	12.8
July	375.6	390.8	15.2	376.4	390.4	14
August	378.6	394.8	16.2	380.4	395.4	15
September	378.8	396	17.2	380.1	396.1	16
October	381.7	399.9	18.2	383.1	399.9	16.8
November	386.6	405.6	19	388.1	405.7	17.6
December	391	410.8	19.8	392.3	410.7	18.4

Conclusion

This thesis introduced the forecasting capabilities of both traditional methods (seaARIMA model) and X-12 methods (regARIMA model). We use the two models to forecast the SGBS series, Hotel series, and Trade series, and then compare the forecasted values with the actual values. In the three applications, SGBS and Trade series not only have seasonal effects, but also have trading day and leap year effects. For these two series, roughly speaking, the regARIMA models are better for describing the two series, because regARIMA models give slightly narrower prediction intervals for the SGBS series (in short run forecast) and the Trade series at 5% significance levels. The Hotel series has only seasonal effects, so the seaARIMA model gives a better description, since the 95% prediction intervals obtained from the seaARIMA model are slightly narrower than those obtained from the regARIMA model; however, the seaARIMA models forecast better for the SGBS series in long-run forecasts. Based on the applications in this thesis, when the series have not only seasonal effects, but other effects as well, such as trading day, leap year, etc., the regARIMA models seem a better choice. If the situation allows it, we suggest use of the series in both seaARIMA and regARIMA models to see which model gives a better description.

The X-12 program and its use manual can be downloaded from the *ftp. census. gov*. A partial of basic program is listed below for readers who are interested in.

```
series {title= 'SGBSF.spc' file= 'SGBSF.dat' format= 'free' period=12 start=1987.01
      savelog=peaks}
transform {function=log}
regression {variables=( td TC1993.Feb) aictest=(td easter) savelog=aictest}
outlier{ }
```

```

arima{model= (0 1 1)(0 1 1)}
forecast{maxlead=12 probability=0.95 print=() save=(variances) }
estimate{ }
check{ }
slidingspans{ }
history{ }

```

The above program can be explained quickly as follows. The **series** {.} is used to create a .spc file, **transform** {.} allows users to do model transformations (such as log-transformation, first differencing, etc.), **regression** {.} used to estimate the parameters for regression part in regARIMA models, **outlier** {.} used to check if a series contains any of the four types of outliers, **arima** {.} indicates the types of model for a series, **forecast** {.} and **estimate** {.} allows users to do forecasting extension, and estimates the forecasted values, respectively. The **check** {.} used to do model checking. The **slidingspans** {.} and **history** {.} used to do sliding spans and revision history diagnostics, respectively. The users could modify and write programs based on their requirements, for more information about program, please read the manual of X-12 methods.

Bibliography

- Bell, W. R., and Monsell, B. C. (1992), "X-11's symmetric linear filters and their transfer functions," Research Report 92/15, Bureau of the Census, Washington, DC: Statistical Research Division.
- Berman, S. M. (1964), "Limit theorems for the maximum term in stationary sequences," *Ann. Math. Statist.*, 35, 502-516.
- Bobbitt, L. and Otto, M. (1985), "Effects of forecasts on the revisions of concurrent seasonally adjusted data using the X-11 seasonal adjustment procedure," *U.S. Bureau of the Census*.
- Box, G. E. P., and Pierce, D. A. (1970), "Distribution of residual autocorrelations in autoregressive integrated moving average time series models," *J. Amer. Statist. Assoc.*, 65, 1509-1526.
- Brockwell, P. J., and Davis, R. A., (1991), "Time Series: Theory and Methods," *second Edition*.
- Brockwell, P. J., and Davis, R. A., (2002), "Introduction to Time Series and Forecasting," Springer-Verlag New York, inc., NY, USA, pp.112.
- Chang, Ih, Tiao, G. C., and Chen, C. (1988), "Estimation of time series parameters in the presence of outliers," *Technometrics*, 30, pp. 193-204.
- Chen, C., and Liu, L. M. (1991), "Forecasting time series with outliers," *Journal of Forecasting*, 12, 13-35.
- Chu, Y., Tiao G. C., and Bell W. R. (2007), "A mean squared error criterion for comparing X-12-ARIMA and model-based Seasonal adjustment filters," *The U.S. Census Bureau*.
- Dagum, E. B. (1983), "Spectral properties of the concurrent and forecasting seasonal linear filters of the X-11-ARIMA method," *The Canadian Journal of Statistics*, 11, 73-90.
- Doherty, M. (2001), "The surrogate Henderson filters in X-11," *Australian Bureau of Statistics*, 43(4). 2001. 901-999.
- Findley, D. F., Monsell, B. C., Bell, W.R., Otto, M.C., and Chen, B.C. (1998), "New capabilities and methods of the X-12-ARIMA seasonal-adjustment program," *Journal of Business and Economic Statistics*, 16, pp. 127-152.
- Findley, D. F., Monsell, B. C., Shulman, H. B., and Pugh, M. G. (1990), "Sliding spans diagnostics for seasonal and related adjustments," *Journal of the American Statistical Association*, 85, 345-355.

- Graham, P. (2001), "Seasonal adjustment within statistics New Zealand," *The Analytical Support Division of Statistics New Zealand*, pp. 9-12.
- Gray, A., Thomson, P. (1996), "Design of moving-average trend filters using fidelity, smoothness and minimum revision criteria," *Bureau of the Census Statistical Research Division*, No. RR96/01.
- Hampel, F. R., Ronchetti, E. M., Rousseeuw, P. J., and Stahel, W. A. (1986), "Robust Statistics: The Approach Based on Influence Functions," John Wiley and Sons, Inc. pp.105.
- Kenny, P. B., and Durbin, J. (1982), "Local trend estimation and seasonal adjustment of Economic and Social Time Series," *Journal of the Royal Statistical Society, Ser. A*, 145, 1-41.
- Ljung, G. M. (1993), "On outlier detection in time series," *Journal of Royal Statistical Society B* 55, 559-567.
- Lothian, J., and Morry, M. (1978), "A set of quality control statistics for the X-11-ARIMA seasonal adjustment method," *Statistics Canada*.
- Lothian, J. (1984), "The identification and treatment of moving seasonality in X-11 ARIMA," *Proceedings of the 1984 Business and Economics Statistics Section, American Statistical Association*, pp. 167-171.
- Murdoch, K., Wilcox, V., Kuzmich, G., and Krsinich, F. (2000), "The effect of ARIMA forecasting on revisions to seasonally adjusted time series," *Statistics New Zealand, Analytical Support Division, Wellington, New Zealand*.
- Musgrave, J. C. (1964 a), "A set of end weights to end all end weights," *US Department of Commerce*.
- Musgrave, J. C. (1964 b), "Alternative sets of weights for proposed X-11 seasonal factor curve moving averages," *US Department of Commerce*.
- Pierce, D. A. (1980), "Data revision with moving average seasonal adjustment procedures," *Journal of Econometrics*, 14, pp. 95-114.
- Priestley, M. B. (1981), "Spectral Analysis and Time Series," London: Academic Press. V.1, pp.601.
- Shiskin, J., Young, A. H., and Musgrave, J. C., (1967), "The X-11 variant of the census method II seasonal adjustment program," Technical paper 15, *Bureau of the Census, U.S. Department of Commerce, Washington, DC*.

U.S. Census Bureau (2007), *X-12-ARIMA Reference Manual, Final Version 0.3*, Washington, DC: U.S. Census Bureau.

Wallis, K. F. (1982), "Seasonal Adjustment and Revision of Current Data: Linear Filters for the X-11 Method," *Journal of the Royal Statistical Society*, Vol. 145, No. 1 (1982), pp. 74-85.

Wei, W. W. S., Lynch, D., Oliver, S., and Hampton, R., (2006), "Time Series Analysis: Univariate and Multivariate Methods," New York, NY: Tobin Greg.