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SPLIT EXTENSION RINGS

by

William Gerard Mulvihill

A thesis submitted to the Faculty of
Graduate Studies in partial fulfilment
of the requirements for the degree of
Doctor of Philosophy.

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Ottawa, Ontario

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ABSTRACT

A ring R (associative, with 1) is a split extension ring if R decomposes, additively, as a direct sum of its (Jacobson) radical and a subring of R .

In this paper semiperfect rings are examined with a view to determining which of them are split extension rings. The paper deals almost exclusively with local rings, since it is shown in Chapter four that a semiperfect ring, R , is a split extension ring if and only if each local ring generated by a local idempotent in R is a split extension ring.

The main result in Chapter two shows that a local ring with a central nilpotent radical is a split extension ring if and only if it is commutative and contains a subfield.

In Chapter three, local rings with non central radical elements are considered. In what follows, R is a local ring, $C(R)$ is its center, W is its (Jacobson) radical, $Q = R/W$ is the residue division ring and $\pi : R \longrightarrow Q$ is the canonical projection. Using this notation the following theorems are proved.

THEOREM 3.5 Suppose R is left Artinian, Q is commutative, $W^2 = 0$ and $W \cap C(R) = 0$. Then R is a split extension ring.

THEOREM 3.9 Suppose R is left Artinian, Q is commutative and $W \cap C(R) = 0$. Suppose further that $w_1 w_2 = w_2 w_1$ for all $w_1, w_2 \in W$. Then R is a split extension ring.

THEOREM 3.11. Let R be left Artinian and $C = C(Q)$. Suppose that for each $w \in W^i \setminus W^{i+1}$ there exists $x \in R$ such that $\pi(x) \in C$ and $xw - wx \in W^i \setminus W^{i+1}$. Then R is a split extension ring.

Throughout the paper, examples are presented both to illustrate the theory and to show its limitations.

TABLE OF CONTENTS

<u>CHAPTER</u>		<u>PAGE</u>
1	INTRODUCTION	1
1.1	Definitions and Notation	2
1.2	Split Extension Rings	6
2	LOCAL RINGS WITH CENTRAL NILPOTENT RADICALS	11
3	LOCAL RINGS WITH NON CENTRAL RADICALS	23
4	SEMIPERFECT SPLIT EXTENSION RINGS	47
	BIBLIOGRAPHY	51

CHAPTER ONE

INTRODUCTION

This paper is a study of split extension rings. It is structural in content and addresses the question: "Which rings are split extension rings?" It answers the question for certain classes of left Artinian and local rings and reformulates the problem for semiperfect rings.

The motivation for this study came from a paper of Dlab and Ringel [3], on balanced and exceptional rings. Split extension rings arose in two contexts in that paper. First, the examples of exceptional rings provided were certain split extension rings and the question naturally arose as to whether all exceptional rings are split extension rings. Second, they wished to know whether exceptional rings with isomorphic residue division rings and with isomorphic radicals were isomorphic; a fact which is true for split extension exceptional rings.

In Chapter 2 of this paper, local rings with central nilpotent radicals are investigated. Necessary and sufficient conditions are determined in order for these rings to be split extension rings.

Chapter 3 deals for the most part with local rings having no central radical elements. The discussion here is restricted to left Artinian local rings, in order to obtain

certain finiteness conditions. The most positive results are obtained for rings with commutative residue division rings. The chapter concludes with a discussion of the general case of local rings with square free radicals.

In Chapter 4, necessary and sufficient conditions for semiperfect rings to be split extension rings are formulated in terms of the local components of such rings.

§ 1.1 Definitions and Notation

In this section, most of the definitions required and the notation which will be used throughout the paper are introduced. However, it is expected that the reader have a general knowledge of algebra; especially, linear algebra, field theory and ring theory.

Throughout this paper all rings are associative with unity 1, and subrings have the same unity element.

If R is a ring then $\text{Rad } R$, the (Jacobson) radical of R , is the intersection of all the maximal left ideals of R . The radical of R will always be denoted by W .

Let I be an ideal in R , then $\partial_R I$ denotes the length of (a composition series for) I considered as a left ideal and $\partial_I R$ will denote the length of I considered as a right ideal.

The center of R is denoted by $C(R)$. For a subset S of R and an ideal I of R , the centralizer of S in I , $C_I(S)$ is given by

$$C_I(S) = \{y \in I \mid ys - sy = 0 \text{ for all } s \in S\}.$$

Using this notation we get the following lemma which is used frequently in the paper.

LEMMA 1.1. If $xy - yx \in C_R(x)$, then for every $f(t) \in C(R)[t]$

$$f(x)y - yf(x) = f'(x)(xy - yx),$$

where $f'(x)$ is the formal derivative of $f(t)$ evaluated at x .

Proof. Suppose $xy - yx \in C_R(x)$ then

$$x(xy - yx) + (xy - yx)x = 2x(xy - yx).$$

But,

$$\begin{aligned} x(xy - yx) + (xy - yx)x &= x^2y - xyx + xyx - yx^2 \\ &= x^2y - yx^2 \end{aligned}$$

$$\text{Thus } x^2y - yx^2 = 2x(xy - yx).$$

Using these techniques the proof follows easily by induction.

A ring R is said to be semisimple if R is a sum of minimal left ideals in R . The ring R is left Artinian if every strictly decreasing chain of left ideals in R is finite. It is well known, that if R is left Artinian then

R/W is semisimple and every left ideal in R is finitely generated.

Of foremost interest in this paper are local rings. R is local if it has a unique maximal left ideal. The following characterizations of local rings will be used frequently throughout the paper.

LEMMA 1.2. The following conditions are equivalent

- (i) R is local
- (ii) R/W is a division ring
- (iii) For every x in R , either x or $1-x$ is invertible
- (iv) W consists of all the non-invertible elements in R .

In this next discussion we will extend the common definitions in field theory to local rings.

Suppose $F \subseteq K$ (F is a subfield of K). An element $x \in K$ is said to be algebraic over F , if there exists $f(t) \neq 0 \in F[t]$ such that $f(x) = 0$. An element $x \in K$ is transcendental if it is not algebraic. If x is algebraic then $\text{Irr}(F, x)$ will denote the monic irreducible polynomial satisfied by x .

If x is algebraic over F , then x is separable

over F if its minimal polynomial over F has no multiple roots in any splitting field. An extension K of F is algebraic over F if every element in K is algebraic over F . An algebraic extension K of F is separable if every $x \in K$ is separable over F .

A field F is perfect if every algebraic extension of F is separable. It is well known that

- (i) Finite fields are perfect
- (ii) Fields of characteristic zero are perfect
- (iii) Algebraic extensions of perfect fields are perfect.

Suppose $F \subseteq K$, then the elements $s_1, \dots, s_n$ in K are said to be algebraically independent over F , if for each $f(t_1, \dots, t_n) \in F[t_1, \dots, t_n]$, $f(s_1, \dots, s_n) \neq 0$. It is well known that if K is an extension of F , then there exists a maximal algebraically independent set, called a transcendence basis for K over F . Moreover, any two transcendence basis have the same cardinality (called the transcendence degree of K over F) and K is algebraic over $F(S)$, where S is a transcendence basis.

Now let R be a local ring and suppose R contains a subfield F . Then in fact the subring of R generated

by 1 has a field of quotients called the prime subfield of R .

Suppose F is a subfield of R , and let $\pi : R \longrightarrow Q = R/W$ be the natural projection. Then $x \in R/W$ is algebraic over F if $\pi(x)$ is algebraic over $\pi(F)$. The polynomial $f(t) \in F[t]$ is said to be the minimal polynomial for x , denoted by $\text{Irr}(F, x)$, if

$$\pi(f(t)) = \text{Irr}(\pi(F), \pi(x)).$$

If $f(t) = \text{Irr}(F, x)$, then $f(x) \in W$. Similarly, x is separable or transcendental over F if $\pi(x)$ is, respectively, separable or transcendental over $\pi(F)$.

Let D be a division ring with center C . Then, clearly, D is a left and right vector space over C . D is finite dimensional over C if there exists $d_1, \dots, d_n \in D$ such that $D = Cd_1 \oplus \dots \oplus Cd_n$ when Cd_i is the subspace of D generated by the element d_i . The element $d \in D$ is algebraic over C if there exists $f(t) \in C[t]$ such that $f(d) = 0$. D is said to be algebraic over C if every $d \in D$ is algebraic over C . If D is finite dimensional over C , then D is algebraic over C .

§ 1.2 Split Extension Rings

Let R be a ring and let $\phi : R \longrightarrow S$ be a ring

homomorphism of R onto S with kernel N . Then R is called an extension of S with kernel N . The extension $\phi: R \longrightarrow S$ is called a split extension if there exists a ring homomorphism $\psi: S \longrightarrow R$ such that $\phi\psi = 1_S$.

(Here, and throughout the paper mappings act on the left.)

Let R be a ring and denote R/W by Q . Then

R is called a split extension ring (or, more concisely, R splits) if the mapping $\pi: R \longrightarrow Q$ is a split extension.

Split extension rings have an alternative formulation which will be more useful in this paper.

LEMMA 1.3. R is a split extension ring if and only if there exists a subring S of R such that, as abelian groups, $R = S \oplus W$.

If $R = S \oplus W$ is a split extension ring then the addition is component wise. Moreover, if (s_1, w_1) and $(s_2, w_2) \in R$, then $(s_1, w_1)(s_2, w_2) = (s_1 s_2, s_1 w_2 + w_1 s_2 + w_1 w_2)$.

Proof. The proof follows easily from a consideration of the exact sequence

$$0 \longrightarrow W \longrightarrow R \begin{array}{c} \xrightarrow{\pi} \\ \xleftarrow{\psi} \end{array} Q \longrightarrow 0$$

The only major result we have found in the literature on split extension rings is the following due to Wedderburn

and Malcev.

THEOREM (Wedderburn, Malcev). Let B be a finite dimensional algebra over a field K with radical N such that the residue class $A = B/N$ is separable.

Then B is a split extension algebra. If S_1 and S_2 are subalgebras such that $B = S_1 \oplus N$ and $B = S_2 \oplus N$, then there exists an element $n \in N$, such that

$$S_1 = (1 - n) S_2 (1 - n)^{-1}$$

Note. The algebra A is separable over K if $A \otimes_K E$ is a semisimple algebra over E for every extension field E of K .

The above result is never referred to in this paper. It has been included here only to indicate the present development of the theory. In fact, the complications avoided by restricting attention to finite dimensional algebras are, for the most part, the problems tackled in this paper.

We conclude this section with two results on split extension rings and an example.

LEMMA 1.4. If $R = S \oplus W$ is a split extension ring and I is an ideal contained in W , then R/I is

a split extension ring.

LEMMA 1.5. Let R be a local ring. Suppose that $W = S \oplus T$ where S and T are ideals in R . Then

- (i) R is a split extension ring if and only if R/S and R/T are split extension rings
- (ii) If R/S is a split extension ring, then there exists a subring R_1 of R such that $R_1 \cap W = S$ and R_1 is a split extension ring if and only if R_1/T is a split extension ring.

Proof. (i) The "only if" part is trivial by Lemma 1.4.

Now suppose R/S and R/T split.

Since R/S splits, we may write

$$R/S = \bar{R}_1 \oplus \bar{U}.$$

Let R_1 be the inverse image of $\bar{R}_1$ under the projection $\alpha : R \rightarrow R/S$. Then $R_1 \oplus T = R$ and $R_1 \cap W = S$.

For suppose $w \in R_1 \cap T$, then $\alpha(w)$ is a nonzero element

of both $\overline{R}_1$ and U , since $S \cap T = 0$, contradicting the fact that $\overline{R}_1 \cap U = 0$.

On the other hand R/T splits. But,

$R/T = (R_1 \oplus T)/T = R_1$. Hence $R_1 = R_2 \oplus S$. Therefore,

$$R = R_2 \oplus S \oplus T = R_2 \oplus (S \oplus T) = R_2 \oplus W.$$

is a split extension ring.

Since (ii) is merely a statement of facts that come out in the proof of (i), the Lemma is proved.

We conclude the introduction with

Example 1.1. Let $F(x)$ be the field in one indeterminate over a field F . Consider the field isomorphism $\psi : F(x) \rightarrow F(x)$ given by

$$\psi(f(x)) = f(x^k)$$

Form the ring R of all ordered pairs (a,b) such that $a,b \in F(x)$ with component wise addition and multiplication as follows

$$(a,b)(c,d) = (ac, ad + b\psi(c))$$

It is easy to check that $W = \{(0,b)\}$ and $W^2 = 0$. Moreover, $\partial_R W = 1$, and $\partial W_R = k$. Clearly, R is a split extension ring since $R = F(x) \oplus F(x)$.

CHAPTER TWO

LOCAL RINGS WITH CENTRAL NILPOTENT RADICALS

The purpose of this chapter is to give necessary and sufficient conditions for local rings with central nilpotent radicals to be split extension rings. The results show that the separability and finiteness conditions imposed on algebras in the Wedderburn Theorem may be removed for these rings. In fact, an example is presented of an algebra which splits when considered as a ring but which does not split as an algebra. The chapter concludes with an example of a local ring with a central nilpotent radical which is not a split extension ring.

Throughout this chapter R is a local ring, W is its radical, $Q = R/W$ is the residue division ring and $\pi : R \longrightarrow Q$ is the canonical projection.

We begin by describing the structure of Q .

LEMMA 2.1. If W is nonzero and central in R then Q is commutative.

Proof. Suppose Q is non-commutative. Then there exist $x, y \in R$ such that $xy - yx$ is invertible in R . Let w be a nonzero element in W ; then, since W is central, $xw - wx = 0$ and, since $yw \in W$, $x(yw) - (yw)x = 0$.

However, $0 = x(yw) - (yw)x = xyw - yxw = (xy - yx)w \neq 0$

This contradiction proves the lemma.

The next two lemmas give some conditions under which subfields of R may be extended.

LEMMA 2.2. Suppose R is commutative, F is a subfield of R and x is algebraic over F . Let $f(t) = \text{Irr}(F, x)$. Then $F[x] \cap W = 0$ if and only if $f(x) = 0$.

Proof. The "only if" part is trivial; since, if $f(x) \neq 0$, then $f(x) \in W$.

Suppose now that $f(x) = 0$ and $F[x] \cap W \neq 0$. Let $g(t) \in F[t]$ be a monic polynomial of minimal degree such that $g(x)$ is nonzero in W . Clearly, $\deg g \geq \deg f$. Otherwise, $\pi(g(x)) = 0$ in Q contradicting the fact that $f(t) = \text{Irr}(F, x)$. Since F is a field, we have

$$g(t) = f(t)h(t) + r(t).$$

where $\deg r < \deg g$. Thus, since $f(x) = 0$, $g(x) = r(x)$ contradicting our choice of g . This proves the lemma.

LEMMA 2.3. Suppose R is commutative, $W^2 = 0$, F is a subfield of R and x is separable over F . Let $f(t) = \text{Irr}(F, x)$. Then there exists a unique $w \in W$ such that $f(x + w) = 0$.

Proof. Since x is separable, $\pi(f'(t)) \neq 0$ in Q ; hence $f'(x)$ is invertible. Suppose $f(x) = u$. Then there exists $w \in W$ such that $f(x+w) = -u$ (take $w = -(f'(x))^{-1} \cdot u$). Suppose further that

$$f(t) = t^n + a_{n-1}t^{n-1} + \dots + a_1t + a_0.$$

$$\text{Then } f(x+w) = (x+w)^n + a_{n-1}(x+w)^{n-1} + \dots + a_1(x+w) + a_0.$$

However, for each i ,

$$a_i(x+w)^i = a_i x^i + i a_i x^{i-1} \cdot w,$$

since $W^2 = 0$. That is,

$$f(x+w) = f(x) + f'(x) \cdot w = u - u = 0.$$

Finally, if $f(x+w) = f(x+v)$, then

$$f(x) + f'(x) \cdot w = f(x) + f'(x) \cdot v. \quad \text{That is,}$$

$$f'(x)(w-v) = 0.$$

Therefore, $w = v$ since $f'(x)$ is invertible. This proves the lemma.

Clearly, a necessary condition for R to be a split extension ring is that it contain at least one subfield. This is the same as saying that the subring generated by 1 has zero intersection with W . For certain commutative rings this condition is sufficient as the following Theorems demonstrate.

THEOREM 2.4. If R is commutative, $W^2 = 0$ and $\text{char } Q = 0$, then R is a split extension ring.

Proof. Since $\text{char } Q = 0$, R must contain the rational numbers as a subfield. Let

$$F = \{F \mid F \text{ is a subfield of } R\}.$$

By the above remark $F \neq \emptyset$. Moreover, since F can be ordered by set theoretic inclusion, Zorn's Lemma guarantees the existence of a maximal element $F \in F$. We shall show that $R = F \oplus W$.

Suppose $x \in R \setminus F \oplus W$. Then x is invertible in R and hence is either transcendental or algebraic over F . If x is transcendental over F , then $F[x] \cap W = 0$. For, if this is not the case, there exists $g(t) \in F[t]$ such that $g(x) \in W$. This implies, however, that $\pi(g(x)) = 0$, contradicting the transcendentality of x over F . Since $F[x]$ has no zero divisors, it has a field of quotients in R .

If, on the other hand, x is algebraic over F , then it is separable over F , since fields of characteristic zero are perfect. Thus by Lemmas 2.2 and 2.3, there exists $w \in W$ such that $F[x + w] \cap W = 0$. Once again this implies that $F[x + w]$ has a field of quotients in R .

So, in either case above, we get a contradiction

of the maximality of F . Therefore, $R = F \oplus W$, proving the theorem.

THEOREM 2.5. If R is commutative, $W^2 = 0$, $\text{char } Q = p > 0$ and R contains a subfield, then R is a split extension ring.

Note. The condition that R contains a subfield is necessary, since otherwise $Z/_{4Z}$ would satisfy the theorem.

But, it is well known that $Z/_{4Z} \neq Z_2 \oplus Z_2$.

Proof. The difficulty in the proof is that of working with inseparable elements. We guarantee that all p^{th} powers lie in a subfield by the following constructions.

First we shall show that

$$R^p = \{x^p \mid x \in R\}$$

is a subfield of R and Q is algebraic over $\pi(R^p)$.

It is easy to show that all nonzero elements in R^p are invertible. For, if $x \in R$ is invertible, then clearly, $x^p \in R^p$ is also invertible, and if $x \in R$ is not invertible, then $x \in W$ and $x^p = 0$, since $W^2 = 0$ and $p \geq 2$.

Moreover, if x^p and y^p are in R^p , then $(x^p)^{-1}$ and $x^p y^p$ are also in R , since $(x^p)^{-1} = (x^{-1})^p$

and, by the commutativity $x^p y^p = (xy)^p$.

Now suppose that $x^p, y^p \in R^p$ and $x^p + y^p \notin R^p$.

Then, in particular,

$$x^p + y^p \neq (x + y)^p.$$

However, $(x + y)^p = x^p + y^p + pz$ for some $z \in R$. So, by

the inequality above, $pz \neq 0$ and hence $p \cdot 1 \neq 0$. Since

$\text{char } Q = p$, we must have that $p \cdot 1 \in W$. This, however,

contradicts the hypothesis that R contains a subfield. Thus

$x^p + y^p \in R^p$ and R^p is a subfield of R .

The statement that Q is algebraic over $\pi(R^p)$ is trivial, since $x \in Q$ implies that $x^p \in \pi(R^p)$.

Let us denote R^p by F and $\pi(R^p)$ by $\bar{F}$.

Then, since Q is algebraic over $\bar{F}$, there exists a subfield $\bar{F}_1$ of Q such that $\bar{F} \subseteq \bar{F}_1 \subseteq Q$, $\bar{F}_1$ is separable over $\bar{F}$ and Q is purely inseparable over $\bar{F}_1$. We shall show that there is a unique extension of F to a subfield F_1 of R such that $\pi(F_1) = \bar{F}_1$.

Let

$$F_1 = \{x \in R \mid x \in F \text{ or } x \text{ is separable over } F \text{ and } f(x) = 0, \text{ where } f(t) = \text{Irr}(F, x)\}.$$

We must show that $F_1 \cap W = 0$ and F_1 is closed under sums, products and inverses. However, $F_1 \cap W = 0$, since by construction F_1 consists only of invertible elements.

Suppose that $x, y \in F_1$. Then by Lemma 2.3, either $x = y$ or $x - y$ is invertible since only one element in the coset $x + W$ can be in F_1 . Now applying Lemma 2.2 twice we get that $F(x, y) \subseteq F_1$. Hence $x - y$, xy and x^{-1} are all in F_1 .

Now to complete the proof of the theorem we consider the set

$$F = \{G \mid G \subseteq F_1 \text{ and } G \text{ is a subfield of } R\}.$$

$F \neq \emptyset$, since $F_1 \in F$. F can be ordered by set theoretic inclusion. Thus, by Zorn's Lemma, there exists a maximal element $G \in F$.

Suppose $x \in R \setminus G \oplus W$. Then $\deg \text{Irr}(G, x) \leq p$, since $x^p \in F$ and $F \subseteq G$. If $\deg \text{Irr}(G, x) = p$, then $\text{Irr}(G, x) = t^p - x^p = f(t)$, and $f(x) = 0$. Thus, by Lemma 2.2, $G[x] \cap W = 0$. Hence $G[x]$ has a field of quotients in R . On the other hand, if $\deg \text{Irr}(G, x) < p$, then x is separable over G . So, by Lemmas 2.3 and 2.2, there exists $w \in W$ such that $G[x + w] \cap W = 0$. Again, $G[x + w]$ has a field of quotients in R . Thus, in either case, we get a contradiction of the maximality of G . Therefore, $R = G \oplus W$ proving the theorem.

Theorems 2.4 and 2.5 are actually the first step in the induction proof of

THEOREM 2.6. If R is commutative and W is nilpotent, then R is a split extension ring if and only if R contains a subfield.

Proof. The "only if" part is trivial in view of the discussion preceeding Theorem 2.4.

Conversely, suppose R contains a subfield. Then Theorems 2.4 and 2.5 hold in cases where $W^2 = 0$. So let us assume the theorem is true whenever $W^n = 0$.

Suppose now that $W^n \neq 0$ and $W^{n+1} = 0$. By the induction hypothesis R/W^n splits. That is, there exists a subring S of R such that $S \cap W = W^n$,

$R/W^n = S/W^n \oplus W/W^n$ and S/W^n is a field. Clearly, $\text{Rad } S = W^n$ and $(\text{Rad } S)^2 = 0$. Therefore, S splits and we may write

$$S = F \oplus W^n$$

for some subfield F of R . This implies that

$$R = F \oplus W$$

proving the theorem.

THEOREM 2.7. Suppose W is nilpotent and central in R . Then R is a split extension ring if and only if R contains a subfield and R is commutative.

Proof. In view of the previous results, it is

enough to show that if R splits then it must be commutative.

Let $R = F \oplus W$ and suppose $x, y \in R$ and $xy - yx \neq 0$.

Then we may write $x = x_1 + w_1$ and $y = y_1 + w_2$ where

$x_1, y_1 \in F$ and $w_1, w_2 \in W$. Therefore,

$$\begin{aligned} xy - yx &= (x_1 + w_1)(y_1 + w_2) - (y_1 + w_2)(x_1 + w_1) \\ &= x_1y_1 + w_1y_1 + x_1w_2 + w_1w_2 - y_1x_1 - w_2x_1 - y_1w_1 - w_2w_1 \\ &= x_1y_1 - y_1x_1, \end{aligned}$$

since W is central. However, Q is commutative and hence

$x_1y_1 - y_1x_1 \in W$. But, $F \cap W = 0$, therefore

$x_1y_1 - y_1x_1 = xy - yx = 0$. This contradiction proves the result.

We now illustrate the previous theorems with

Example 2.1. Let $k = \mathbb{Z}_2(x)$ be the field

in one indeterminate over $\mathbb{Z}/2\mathbb{Z}$. Consider the algebra A

over k generated by $\{1, a, b, c\}$, and having the following multiplication:

	1	a	b	c
1	1	a	b	c
a	a	$x+b$	c	xc
b	b	c	0	0
c	c	xc	0	0

Then A is commutative, local and $(\text{Rad } A)^2 = 0$.

Moreover,

$$A/\text{Rad } A \cong \mathbb{Z}_2(\sqrt{x}),$$

and is not separable over k . If A could split as an algebra, there would exist $\alpha, \beta, \gamma \in k$ with $\alpha \neq 0$, such that

$$k(\alpha a + \beta b + \gamma c) \cap \text{Rad } A = 0.$$

However, $\alpha^2 x \in k$ and

$$\begin{aligned} \alpha^2 x - (\alpha a + \beta b + \gamma c)^2 &= \alpha^2 x - (\alpha a)^2 - \alpha\beta ab - \alpha\gamma ac \\ &= \alpha^2 x - \alpha^2(x+b) - \alpha\beta c - \alpha\gamma xc \\ &= -\alpha^2 b - (\alpha\beta - \alpha\gamma x)c. \end{aligned}$$

Since $\alpha \neq 0$, and b and c are linearly independent over k , $-\alpha^2 b - (\alpha\beta - \alpha\gamma x)c \neq 0$. Thus A cannot split as an algebra.

On the other hand, considering A as a ring, we see that each $a + w$, for $w \in W$, is transcendental over $\mathbb{Z}_2$. Thus $\mathbb{Z}_2[a] \cap W = 0$. Since $\mathbb{Z}_2(a) \cong A/\text{Rad } A$, we get that $A = \mathbb{Z}_2(a) \oplus W$ is a split extension ring.

We shall now show the existence of a ring R containing a subfield and having a central nilpotent radical which is not a split extension ring.

Example 2.2. Let $k(x,y)$ be the field in two indeterminates over a field k . That is, the field of rational functions in two variables over k .

Let

$$A = \left\{ \begin{bmatrix} f(x,y) & \frac{\partial}{\partial y} f(x,y) & 0 \\ 0 & f(x,y) & \frac{\partial}{\partial x} f(x,y) \\ 0 & 0 & f(x,y) \end{bmatrix} \mid f(x,y) \in k(x,y) \right\}$$

and

$$B = \left\{ \begin{bmatrix} 0 & 0 & f(x,y) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid f(x,y) \in k(x,y) \right\}$$

be sets of 3×3 matrices over $k(x,y)$, where $\frac{\partial}{\partial y} f(x,y)$

and $\frac{\partial}{\partial x} f(x,y)$ are the formal partial derivatives of $f(x,y)$

with respect to y and x , respectively. Let R be the

ring generated by A and B using matrix addition and

multiplication. It is easy to show that

$R = \{s + t \mid s \in A \text{ and } t \in B\}$. Moreover, $B = \text{Rad } R$ and

$R/\text{Rad } R$ is a field. Thus R is a local ring with a central

nilpotent radical. Also, since

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix} \in R, \text{ for all } \alpha \in k,$$

R contains a subfield. So in order to show that R is not a split extension ring we need only show that it is not commutative. But

$$\begin{pmatrix} y & 1 & 0 \\ 0 & y & 0 \\ 0 & 0 & y \end{pmatrix} \text{ and } \begin{pmatrix} x & 0 & 0 \\ 0 & x & 1 \\ 0 & 0 & x \end{pmatrix} \text{ are}$$

elements of R ; and,

$$\begin{pmatrix} y & 1 & 0 \\ 0 & y & 0 \\ 0 & 0 & y \end{pmatrix} \begin{pmatrix} x & 0 & 0 \\ 0 & x & 1 \\ 0 & 0 & x \end{pmatrix} - \begin{pmatrix} x & 0 & 0 \\ 0 & x & 1 \\ 0 & 0 & x \end{pmatrix} \begin{pmatrix} y & 1 & 0 \\ 0 & y & 0 \\ 0 & 0 & y \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus R is not a split extension ring.

CHAPTER THREE

LOCAL RINGS WITH NON CENTRAL RADICALS

In this chapter local rings with non central radicals are examined. The theory presented requires, for the most part, finiteness conditions of the type possessed by left Artinian rings. The most positive results are obtained for local left Artinian rings with commutative residue division rings and no central radical elements. Partial results are obtained for other situations and examples are presented to illustrate the difficulties which arise in these cases.

Throughout this chapter R is a local ring, W is its radical, Q is the residue division ring and $\pi : R \longrightarrow Q$ is the canonical projection. Moreover, for each $x \in R$, we define subsets

$$T_x = \{xw - wx \mid w \in W\} \text{ and}$$

$$K_x = \{w \in W \mid xw = wx\}$$

of R , which retain these meanings throughout the chapter.

Suppose $W^2 = 0$. Then $T_{x+w} = T_x$ and $K_{x+w} = K_x$ for all $w \in W$. Also, if $x \in R$ and $\pi(x) \in C(Q)$ (the center of Q), then T_x and K_x are actually ideals of R .

We note that when $W^2 = 0$, W becomes a left and right vector space over Q , since $xw = (x + u)w$ for all $u \in W$. In fact, W has the same structure when considered as a vector space as it does when considered as an ideal.

The following theorem, while important in its own right is crucial in the proofs of more general theorems later on.

THEOREM 3.1. Suppose that $W^2 = 0$ and that there exists an invertible $x \in R$ such that $\pi(x) \in C(Q)$, $T_x = W$ and $K_x = 0$. Then R is a split extension ring.

Note. The theorem does not require that Q be commutative.

Proof. Let us first show the existence of a subring of R which is disjoint from W . Let P be the subring of R generated by 1. Since $K_x = 0$, $W \cap C(R) = 0$. However, $P \subseteq C(R)$ and thus $P \cap W = 0$. Furthermore, since $x \in C_R(P[x])$ and $x \notin C_R(w)$ for any $w \in W$,

$$P[x] \cap W = 0$$

Let

$F = \{S \mid S \text{ is a subring of } R \text{ containing } P[x] \text{ and } S \cap W = 0\}$. Notice that since

$\pi(x) \in C(Q)$, $x \in C_R(S)$, for all $S \in F$. Since

$F \neq \emptyset$ and F can be ordered by set theoretic inclusion,

Zorn's Lemma guarantees the existence of a maximal element

$S \in F$. We shall now show that $R = S \oplus W$.

Suppose $y \in R \setminus S \oplus W$. Then if $xy - yx \neq 0$,
 $xy - yx \in W$, since $\pi(x) \in C(Q)$. However, $W = T_x$,
 so there exists $w \in W$ such that

$$xy - yx = xw - wx$$

That is, $x(y - w) - (y - w)x = 0$. Let $y' = y - w$.

By the maximality of S in F , the subring T of R
 generated by S and y' , contains an element in W .

However, $x \in C_R(T)$ and, by our choice of y' , $x \in C_R(y')$.

Therefore,

$$x \in C_R(T)$$

Again, using the fact that

$$x \notin C_R(w) , \text{ for any } w \in W ,$$

we deduce that $T \cap W = 0$. Thus $T \in F$, contradicting
 our choice of S . This implies that $R = S \oplus W$, is a
 split extension ring, proving the theorem.

The subring S in the theorem above is in fact
 a division ring since

$$Q = R/W = S \oplus W/W = S .$$

Theorem 3.1 has a simpler formulation for left Artinian rings.

COROLLARY 3.2. Suppose that R is left Artinian, $W^2 = 0$ and there exists an invertible $x \in R$ such that $\pi(x) \in C(Q)$ and $K_x = 0$, then R is a split extension ring.

Proof. In view of Theorem 3.1 it is enough to show that $T_x = W$. Associate x with its image $\pi(x)$ in Q and consider the mapping.

$$\phi_x : Q^W \longrightarrow Q^W \text{ given by}$$

$$\phi_x(w) = xw - wx$$

The mapping is well defined in view of the remark preceeding Theorem 3.1 concerning the structure of Q^W . ϕ_x is additive and $\ker \phi_x = 0$. Moreover, if $y \in Q$, then $xy = yx$, since x is central in Q . Therefore,

$$\begin{aligned} \phi_x(yw) &= x(yw) - (yw)x = yxw - ywx \\ &= y(xw - wx) = y \phi_x(w) \end{aligned}$$

Thus ϕ_x is a linear transformation. Moreover, since R is left Artinian Q^W is a finite dimensional vector space. Thus, in view of the fact that $\ker \phi_x = 0$, we have

$$\phi_x(W) = W$$

That is, $T_x = W$. This proves the corollary.

In cases where the ideal lattice of R is very restricted we have the following extensions to the Theorem.

COROLLARY 3.3. Let R be left Artinian and suppose that Q is commutative, $\partial_R W \neq \partial W_R$ and $0, W$ and R are the only (two sided) ideals in R . Then R is a split extension ring.

Proof. Since $\partial_R W \neq \partial W_R$, there exists $x \in R$ such that $xw - wx \neq 0$ for some $w \in W$. Thus T_x is a nonzero ideal contained in W . So, by hypothesis, $T_x = W$ and R is a split extension ring.

COROLLARY 3.4. If Q is finitely generated over its center and $\partial W_R = 1$ but $\partial_R W \neq 1$, then R splits.

Proof. It is enough to show the existence of an element $x \in R$, with $\pi(x) \in C(Q)$ which does not commute with some radical element.

Suppose no such element exists and fix $w \in W$. Then, since $\partial W_R = 1$,

$$WQ = W.$$

The mapping $\rho : Q \longrightarrow Q$ given by $x \longrightarrow \rho(x)$ where

$$w\rho(x) = xw$$

is a ring monomorphism. Let $Q' = \rho(Q)$. Since central

elements in Q commute with radical elements, we have

$$C(Q) \subseteq Q'$$

Moreover, since the left and right radical lengths are unequal, Q' is a proper subring of Q . Thus

$$\dim_{C(Q)} Q' \neq \dim_{C(Q)} Q.$$

However, ρ is a $C(Q)$ linear isomorphism of finite dimensional vector spaces. Thus,

$$\dim_{C(Q)} Q' = \dim_{C(Q)} Q.$$

This contradiction proves the result.

We are now in a position to prove

THEOREM 3.5. Suppose R is left Artinian, Q is commutative, $W^2 = 0$ and $W \cap C(R) = 0$. Then R is a split extension ring.

Proof. For each $x \in Q$ the mapping

$$\psi_x : Q^W \longrightarrow Q^W$$

given by $\psi_x(w) = wx$ is a linear transformation. Moreover, since $wxy = wyx$ for all $w \in W$,

$$\psi_x \psi_y = \psi_y \psi_x$$

for all $x, y \in Q$. Also, if x is nonzero in Q then $x^{-1} \in Q$ and

$$\psi_x \psi_x^{-1}(w) = wx^{-1}x = w,$$

for all $w \in W$. Thus

$$\psi_x \psi_x^{-1} = \psi_1$$

Clearly, if $x, y \in Q$, then

$$\psi_x + \psi_y = \psi_{x+y}$$

The above remarks show that $\{\psi_x \mid x \in Q\}$ is a field.

Suppose now that x is nonzero in Q . Then there is a canonical decomposition

$$W = V_x \oplus U_x$$

of QW with respect to ψ_x , such that V_x is the subspace associated with the eigenvalue x of ψ_x and U_x is the subspace associated with all the other eigenvalues of ψ_x .

More precisely, if the minimum polynomial for ψ_x is

$$g(t) = (t - x)^{e_1} g_2(t)^{e_2} \dots g_n(t)^{e_n}$$

where each $g_i(t)$ is irreducible in $Q[t]$, then

$$V_x = \{w \in W \mid (\psi_x - xi)^{e_1} w = 0\}$$

where i is the identity transformation and

$$U_x = \{w \in W \mid g_2(\psi_x)^{e_2} \dots g_n(\psi_x)^{e_n} w = 0\}$$

Let us now show that for each $x \in Q$,

$$W = V_x \oplus U_x$$

is a decomposition of W into two sided ideals in R .

To do this, it is enough to show that V_x and U_x are closed under right multiplication by elements of Q ; for,

by construction they are already left vector spaces. Suppose $w \in V_x$ and $y \in Q$. Then

$$\begin{aligned} (\psi_x - x_i)^{e_1}(wy) &= (\psi_x - x_i)^{e_1} \cdot \psi_y(w) \\ &= \psi_y(\psi_x - x_i)^{e_1}(w) \\ &= \psi_y(0) = 0 \end{aligned}$$

Thus, $wy \in V_x$. Similarly, it is easily shown that U_x is a two sided ideal.

If $V_x = 0$ for some $x \in Q$, then R satisfies the conditions of Corollary 3.2 and hence splits. So, we may suppose that for each $x \in Q$, $V_x \neq 0$.

Choose $x \in Q$, for which $U_x \neq 0$. (The case for which no such x exists will be handled later in the proof.) Then since $W = V_x \oplus U_x$, Lemma 1.5 applies and R splits if and only if R/V_x and R/U_x both split.

However, R/V_x satisfies the conditions of Corollary 3.2

and hence is a split extension ring. This implies, once again by Lemma 1.5, that there exists a subring R_1 of R such that

$$W_1 = \text{Rad } R_1 = V_x \quad \text{and}$$

$$Q_1 = R_1 / W_1 \approx Q$$

Denote R by R_0 . Repeating the above procedure replacing R_i by R_{i+1} for $i = 0, 1, 2, \dots$ we get a strictly decreasing chain

$$R_0 \supset R_1 \supset R_2 \supset \dots$$

of subrings of R , with the property that R_i splits if and only if R_{i+1} splits. It is clear from the way this chain is constructed that the radical chain

$$W_0 \supset W_1 \supset W_2 \supset \dots$$

is a strictly decreasing chain of ideals in R . Since R is left Artinian, the above chain must stop. That is, there exists an integer n for which either

$$W_n = 0, \text{ or}$$

$$V_{n_x} = W_n \text{ for each } x \in Q_n$$

where $V_{n_x} = V_x \cap W_n$ is the subspace associated with the eigenvalue x in the decomposition

$$W_n = V_{n_x} \oplus U_{n_x}$$

In the former case, $R_n \approx Q_n \approx Q$ and hence R_n splits.

We shall now show that the latter case is impossible. For notational convenience, we note that we

may replace R_n by R since $Q_n \cong Q$ under a canonical projection. Thus we may assume that for each invertible $x \in Q$, x is the only eigenvalue for ψ_x . Hence, for each $x \in Q$, there exists an integer $e_x \geq 1$ such that

$$(\psi_x - xi)^{e_x} w = 0$$

for all $w \in W$. Choose $x_1 \in R$ and let

$$S_1 = \{w \in W \mid (\psi_{x_1} - x_1 i) w = 0\}.$$

Clearly, $S_1 \neq 0$. In fact, S_1 is an ideal in W .

For, if $r \in Q$ and $u \in S_1$, then

$$(\psi_{x_1} - x_1 i)(ur) = (\psi_{x_1} - x_1 i)\psi_r u = \psi_r(\psi_{x_1} - x_1 i)u = 0.$$

The other conditions are satisfied in view of the fact that

$(\psi_{x_1} - x_1 i)$ is a linear transformation.

Since $W \cap C(R) = 0$, there exists $x_2 \in R$, such

that

$$(\psi_{x_2} - x_2 i)u \neq 0$$

for some $u \in S_1$. Form

$$S_2 = \{w \in S_1 \mid (\psi_{x_2} - x_2 i)w = 0\}.$$

S_2 is a proper subideal of S_1 . Moreover, $S_2 \neq 0$. For, if $w \in S_1$, there is a smallest integer e such that

$$(\psi_{x_2} - x_2 i)^e w = 0$$

If $e = 1$ we are finished; if not, then

$$(\psi_{x_2} - x_2 i)^{e-1} w \neq 0$$

and

$$(\psi_{x_2} - x_2 i) [(\psi_{x_2} - x_2 i)^{e-1} w] = 0$$

Repeating this procedure, we get a strictly decreasing sequence

$$S_1 \supset S_2 \supset S_3 \supset \dots$$

of ideals in R . Since R is left Artinian, this sequence stops. That is, for some k , S_k has the property that

$$S_k \neq 0 \text{ and}$$

$$(\psi_x - xi)w = 0$$

for all $w \in S_k$ and all $x \in Q$. That is, $S_k \subseteq C(R)$.

This contradiction proves the theorem.

The construction used in the theorem prove immediately

COROLLARY 3.6. If R satisfies the conditions of the theorem, then every element in W is a sum of commutators of elements in Q with elements in W . (That is, a sum of elements of the form $xw - wx$, for $w \in W$ and $x \in Q$.)

Another application of the theorem is

THEOREM 3.7. If R is left Artinian, Q is commutative and $W^2 = 0$, then R is a subring of a full matrix ring over Q .

Proof. Let $\{w_1, \dots, w_n\}$ be a basis for QW . Then for each $x \in Q$, there is a matrix M_x for the linear transformation ψ_x relative to this basis. Then it is easy to check that $R_1 \cong R$, where R_1 is the ring generated by:

$$\left\{ \left[\begin{array}{c|c} x & 0 \\ \hline 0 & M_x \end{array} \right] \mid x \in F \right\} \quad \text{and}$$

$$\left\{ \left[\begin{array}{c|c} 0 & \alpha_1, \dots, \alpha_n \\ \hline 0 & 0 \end{array} \right] \mid \alpha_1, \dots, \alpha_n \in F \right\}.$$

The following technical lemma will be required in the proof of the next theorem.

LEMMA 3.8. Suppose R is left Artinian and $W^n = 0$. For each $x \in R$ define a map

$$\phi_x : W \longrightarrow W \quad \text{by}$$

$$\phi_x(w) = xw - wx.$$

Suppose further that V is a subset of W with the property that for each $v \in V$, there exists an integer m_v such that for all $x \in R$,

$$\phi_x^{m_v}(v) \in W^2.$$

Then for each set of $n-1$ elements in V , there exists an integer k such that

$$\phi_x^k(v_1 \cdot v_2 \cdots v_{n-1}) = 0, \text{ for all}$$

$$x \in R.$$

Proof. Let $v_1, \dots, v_{n-1} \in V$. Then for any

$$x \in R$$

$$\begin{aligned} \phi_x(v_1 \cdots v_{n-1}) &= xv_1 \cdots v_{n-1} - v_1 \cdots v_{n-1}x \\ &= xv_1 \cdots v_{n-1} - v_1(v_2 \cdots v_{n-1}x \\ &\quad + xv_2 \cdots v_{n-1} - xv_2 \cdots v_{n-1}) \\ &= xv_1 \cdots v_{n-1} - v_1xv_2 \cdots v_{n-1} \\ &\quad + v_1(xv_2 \cdots v_{n-1} - v_2 \cdots v_{n-1}x) \\ &= \phi_x(v_1) \cdot v_2 \cdots v_{n-1} + v_1(\phi_x(v_2 \cdots v_{n-1})). \end{aligned}$$

Expanding the second term $n-2$ times we get:

$$\begin{aligned} \phi_x(v_1 \cdots v_{n-1}) &= \phi_x(v_1) \cdot v_2 \cdots v_{n-1} + v_1 \phi_x(v_2) \cdot v_3 \cdots v_{n-1} + \cdots \\ &\quad + v_1 \cdots v_{n-1} \phi_x(v_{n-1}). \end{aligned}$$

If we denote v_i by $\phi_x^0(v_i)$, then we may write

$$\begin{aligned} \phi_x(v_1 \cdots v_{n-1}) &= \sum_{\alpha_1 + \dots + \alpha_{n-1} = 1} \phi_x^{\alpha_1}(v_1) \cdot \phi_x^{\alpha_2}(v_2) \cdots \phi_x^{\alpha_{n-1}}(v_{n-1}). \end{aligned}$$

Inductively, it can be shown that

$$\phi_x^k(v_1 \dots v_{n-1}) = \sum_{\alpha_1 + \dots + \alpha_{n-1} = k} \phi_x^{\alpha_1}(v_1) \dots \phi_x^{\alpha_{n-1}}(v_{n-1})$$

Let $m = \max \{m_{v_1}, \dots, m_{v_n}\}$. Then we claim that

$$\phi_x^{m(n-1)}(v_1 \dots v_{n-1}) = 0$$

For,

$$\phi_x^{m(n-1)}(v_1 \dots v_{n-1}) = \sum_{\alpha_1 + \dots + \alpha_{n-1} = m(n-1)} \prod_{i=1}^{n-1} \phi_x^{\alpha_i}(v_i)$$

and clearly, in each term of the summation at least one of the α_i 's is greater than or equal to m . That is, for each term

$$\phi_x^{\alpha_1}(v_1) \dots \phi_x^{\alpha_n}(v_n)$$

at least one of the elements is an element from W^2 .

Since $V \subseteq W$, each term is contained in $V^{n-2} W^2$. But,

$$V^{n-2} W^2 \subseteq W^{n-2} W^2 \subseteq W^n = 0$$

This proves the result.

We are now ready to extend Theorem 3.5 to the case where W is nilpotent.

THEOREM 3.9. Suppose R is left Artinian, Q is commutative and $W \cap C(R) = 0$. Suppose further that $w_1 w_2 = w_2 w_1$ for all $w_1, w_2 \in W$. Then R is a split extension ring.

Proof. The theorem will be proved using induction on the degree of nilpotency of W . If $W^2 = 0$, the result follows from Theorem 3.5. Let us assume the result is true whenever $W^n = 0$.

Now let R be a ring satisfying the conditions of the theorem such that $W^{n+1} = 0$ and $W^n \neq 0$. Let

$$\bar{R} = R/W^2, \quad \bar{W} = \text{Rad}(\bar{R}), \quad \text{and} \quad \bar{Q} = \bar{R}/\bar{W}.$$

There are two cases to be considered.

Case I. $\bar{W} \cap C(\bar{R}) = 0$.

In this case $\bar{R}$ splits by Theorem 3.5. That is, there exists a subring S of $\bar{R}$ such that $S + \bar{W} = \bar{R}$, $S \cap \bar{W} = 0$ and $S/W^2 = \bar{Q}$. However

$$(\text{Rad } S)^n \subseteq (W^2)^n = 0.$$

Moreover, it is easy to check that $(\text{Rad } S) \cap C(S) = 0$.

Therefore, by the induction hypothesis, there exists a subfield F of S such that $F \oplus W^2 = S$. From this we conclude that

$$R = S + W = (F \oplus W^2) + W = F \oplus W,$$

since $F \cap W \subseteq F \cap W^2 = 0$. Thus R is a split extension ring.

Case II. $\bar{W} \cap C(\bar{R}) \neq 0$

Using the techniques and notations of Theorem 3.5

we may write

$$\bar{W} = \bar{V} \oplus \bar{U},$$

when $\bar{V} = \sum_{x \in \bar{Q}} \bar{V}_x$ and $\bar{U} = \sum_{x \in \bar{Q}} \bar{U}_x$. Clearly, $\bar{R}/\bar{V}$ splits,

$$\text{since } \text{Rad}(\bar{R}/\bar{V}) \cap C(\bar{R}/\bar{V}) = \bar{U} \cap C(\bar{R}) = 0.$$

Let U and V be the inverse images under the canonical projection of $\bar{U}$ and $\bar{V}$ respectively. Then by Lemma 1.5 and the work above, there exists a subring S of R such that $S \cap W = V + W^2$ and $S/\text{Rad } S \cong Q$, with the property that R is a split extension ring if and only if S is a split extension ring. In view of the induction hypothesis, the theorem will be proved if we can show that

$$\text{Rad } S \cap C(S) = 0 \text{ and}$$

$$(\text{Rad } S)^n = 0.$$

Suppose that $w \in \text{Rad } S$. Then since $W \subseteq C_R(w)$, there exists an invertible $x \in R$ such that $xw - wx \neq 0$. Moreover, since $S/\text{Rad } S \cong Q$, there exists $u \in W$, such that $x - u \in S$. Therefore,

$$(x - u)w - w(x - u) = xw - wx - uw + wu = xw - wx \neq 0.$$

We now show that $(\text{Rad } S)^n = 0$. Since $\text{Rad } S = V$ and $(W^2)^n = 0$, it is clearly enough to show that the product of any n elements in $V \setminus W^2$ is zero. Let $v_1, \dots, v_n \in V \setminus W^2$, and let $k = \dim_{\bar{Q}} \bar{V}$. Then by the

construction of V , we have

$$(\psi_x - xi)^k(v) \in W^2 \text{ for}$$

all $v \in V \setminus W^2$ and all $x \in R$. Since $(\psi_x - xi)^k = -\phi_x^k$

as defined in Lemma 3.8, there exists an integer k such that

$$\phi_x^k(v_1 \dots v_n) = 0 \text{ for all}$$

$x \in R$ and all products of n elements from $V \setminus W^2$.

Suppose that $v_1 \dots v_n \neq 0$. Then $v_1 \dots v_n \in W^n$.

Moreover, since W^n is a vector space over Q , $(W^n)^2 = 0$ and $W^n \cap C(R) = 0$, we know from the proof of Theorem 3.5, that

$$W^n = \sum_{x \in Q} U_x,$$

where, as usual, U_x is the subspace associated with the eigenvalue other than x of the mapping $\psi_x : W^n \rightarrow W^n$ given by $\psi_x(w) = xw - wx$. Hence for some $x \in Q$

$$(\psi_x - xi)^k(v_1 \dots v_n) \neq 0 \text{ for any integer } k.$$

That is $\phi_x^k(v_1 \dots v_n) \neq 0$ for any k . Since this contradicts the statement above, we must have $v_1 \dots v_n = 0$.

Thus $(\text{Rad } S)^n = 0$ and the theorem is proved.

THEOREM 3.10. Suppose R is left Artinian, and Q is commutative. Suppose further that for each

$w \in W^i \setminus W^{i+1}$, there exists $x \in R$ such that $xw - wx \in W^i \setminus W^{i+1}$. Then R is a split extension ring.

Proof. Again, the proof is by induction on the degree of nilpotency of W . As in the proof of the previous theorem, we let $\bar{R} = R/W^2$, $\bar{W} = \text{Rad } \bar{R}$ and $\bar{Q} = \bar{R}/\bar{W}$. Clearly, from the hypotheses of the theorem $\bar{W} \cap C(\bar{R}) = 0$. Thus $\bar{R}$ splits, and there exists a subring S of R such that $S \cap W = W^2$, $R = S + W$ and $S/W^2 \cong \bar{Q}$. Moreover, R splits if and only if S splits. Once again, it is easy to show that $(\text{Rad } S) \cap C(S) = 0$. Hence, by our induction hypothesis, S is a split extension ring and the theorem is proved.

The previous results have implications when Q is not commutative.

Theorem 3.11. Let R be left Artinian and $C = C(Q)$. Suppose that for each $w \in W^i \setminus W^{i+1}$ there exists $x \in R$ such that $\pi(x) \in C$ and $xw - wx \in W^i \setminus W^{i+1}$. Then R is a split extension ring.

Proof. The proof follows from the fact that Theorems 3.5 and 3.10 both hold in this situation. The

proofs follow exactly since we can restrict our attention to elements whose projections are central in Q .

The following example shows that for the general case where Q is commutative and $W \cap C(R) \neq 0$, R need not be commutative.

Example 3.1. Let $k(x,y)$ be the field two indeterminates over a field k . Let $\theta : k(x,y) \rightarrow k(x,y)$ be the mapping defined by

$$\theta(f(x,y)) = f(x^2, y).$$

Clearly, θ is a monomorphism of fields, hence, $k(x,y) = \theta(k(x,y))$.

Consider the following sets of 4×4 matrices over $k(x,y)$.

$$A = \left\{ \begin{pmatrix} f(x,y) & \frac{\partial f(x,y)}{\partial y} & 0 & 0 \\ 0 & f(x,y) & \frac{\partial f(x,y)}{\partial x} & 0 \\ 0 & 0 & f(x,y) & 0 \\ 0 & 0 & 0 & \theta f(x,y) \end{pmatrix} \mid f(x,y) \in k(x,y) \right\}.$$

and

$$B = \left\{ \begin{bmatrix} 0 & 0 & f(x,y) & g(x,y) \\ 0 & 0 & 0 & h(x,y) \\ 0 & 0 & 0 & l(x,y) \\ 0 & 0 & 0 & 0 \end{bmatrix} \mid f(x,y), g(x,y), h(x,y), l(x,y) \in k(x,y) \right\}$$

Let R be the ring generated under matrix addition and multiplication by A and B . As was the case in Example 2.2 we can show that $R = \{a + b \mid a \in A \text{ and } b \in B\}$.

Moreover, $W = B$ and

$$W^2 = \left\{ \begin{bmatrix} 0 & 0 & 0 & f(x,y) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mid f(x,y) \in k(x,y) \right\}$$

As well, it is easily seen, that $W \cap C(R) = 0$.

$$\text{Let } \underline{x} = \begin{bmatrix} x & 0 & 0 & 0 \\ 0 & x & 1 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x^2 \end{bmatrix} \text{ and } \underline{y} = \begin{bmatrix} y & 1 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & y \end{bmatrix}$$

Then $\underline{x}$ and $\underline{y} \in R$ and for any $\underline{w}_1, \underline{w}_2 \in W$.

We get that

$$\underline{z} = (\underline{x} + \underline{w}_1)(\underline{y} + \underline{w}_2) - (\underline{y} + \underline{w}_2)(\underline{x} + \underline{w}_1)$$

has a -1 in the $(1,3)^{th}$ component.

To conclude this chapter we present a short discussion of the problems which are encountered in the general case where R is left Artinian, Q is commutative and $W^2 = 0$.

We have already seen that

$$W = V \oplus U,$$

where V has the property, that for each $x \in Q$,

$$(\psi_x - xi)^n v = 0, \text{ for all } v \in V, \text{ where } n = \dim_Q V.$$

We have already considered the case where

$V \subseteq C(R)$. So it remains now to consider the general case for V . The following example shows that rings with such a radical will not always split.

Example 3.2. Let $k(x,y,z)$ be the field in three indeterminates over a field k . Consider the following sets of 5×5 matrices over $k(x,y,z)$.

Let

$$A = \left\{ \begin{bmatrix} f(x,y,z) & \frac{\partial f(x,y,z)}{\partial y} & 0 & 0 & 0 \\ 0 & f(x,y,z) & \frac{\partial f(x,y,z)}{\partial z} & 0 & 0 \\ 0 & 0 & f(x,y,z) & 0 & 0 \\ 0 & 0 & \frac{\partial f(x,y,z)}{\partial z} & f(x,y,z) & \frac{\partial f(x,y,z)}{\partial x} \\ 0 & 0 & 0 & 0 & f(x,y,z) \end{bmatrix} \middle| \begin{matrix} f(x,y,z) \\ \in k(x,y,z) \end{matrix} \right\}$$

and

$$B = \left\{ \begin{bmatrix} 0 & 0 & f(x,y,z) & g(x,y,z) & h(x,y,z) \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \middle| \begin{matrix} f(x,y,z) \\ g(x,y,z) \\ h(x,y,z) \in k(x,y,z) \end{matrix} \right\}$$

Let R be the ring generated by A and B under matrix addition and multiplication. Then, as in the previous examples of this sort, we can show that

$$R = \{a + b \mid a \in A \text{ and } b \in B\}.$$

Moreover, $W = B$, $W^2 = 0$ and Q is commutative. Also, it is easy to show that for each $s \in Q$, $(\psi_s - sI)^2 w = 0$ for all $w \in W$.

Let

$$\underline{z} = \begin{pmatrix} z & 0 & 0 & 0 & 0 \\ 0 & z & 1 & 0 & 0 \\ 0 & 0 & z & 0 & 0 \\ 0 & 0 & 1 & z & 0 \\ 0 & 0 & 0 & 0 & z \end{pmatrix}, \quad \underline{y} = \begin{pmatrix} y & 1 & 0 & 0 & 0 \\ 0 & y & 0 & 0 & 0 \\ 0 & 0 & y & 0 & 0 \\ 0 & 0 & 0 & y & 0 \\ 0 & 0 & 0 & 0 & y \end{pmatrix}$$

$$\text{and } \underline{x} = \begin{pmatrix} x & 0 & 0 & 0 & 0 \\ 0 & x & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 \\ 0 & 0 & 0 & x & 1 \\ 0 & 0 & 0 & 0 & x \end{pmatrix}$$

Then $\underline{x}$, $\underline{y}$ and $\underline{z}$ are in R . We shall show that there do not exist $\underline{w}_1$, $\underline{w}_2$ and $\underline{w}_3$ in W such that

- (a). $(\underline{x} + \underline{w}_1)(\underline{y} + \underline{w}_2) - (\underline{y} + \underline{w}_2)(\underline{x} + \underline{w}_1) = 0$,
- (b). $(\underline{x} + \underline{w}_1)(\underline{z} + \underline{w}_3) - (\underline{z} + \underline{w}_3)(\underline{x} + \underline{w}_1) = 0$ and
- (c). $(\underline{y} + \underline{w}_2)(\underline{z} + \underline{w}_3) - (\underline{z} + \underline{w}_3)(\underline{y} + \underline{w}_2) = 0$

are all zero. This will then imply that R is not a split extension ring.

Since $\underline{y}$ commutes with radical elements $\underline{w}_2$ must have a -1 in the $(1,4)^{\text{th}}$ component if (c) is to be zero. But if (a) is to be zero, then $\underline{w}_2$ must have a zero in the $(1,4)^{\text{th}}$ component. Since these two conditions cannot be satisfied, R is not a split extension ring.

The fact that R does not split in the above Example is surprising because R has many strong properties; namely:

$R / \sum_{s \in Q} T_s$ is commutative and

$st - ts \in T_s + T_t$ for all $s, t \in R$.

These facts suggest that any splitting theorems for rings of this type will have to be very restrictive.

CHAPTER FOUR

SEMIPERFECT SPLIT EXTENSION RINGS

The purpose of this chapter is to provide necessary and sufficient conditions for a semiperfect ring to be a split extension ring. These conditions are formulated in terms of the local components of semiperfect rings. Most of the background material required in this chapter is presented without proof. A more complete discussion of semiperfect rings can be found, for example, in Lambek [11].

A ring R is semiperfect if R/W is semisimple and idempotents can be lifted modulo W .

Here, we say that idempotents can be lifted modulo W if for every element $x \in R$ such that $x^2 - x \in W$, there exists an element $e^2 = e \in R$ such that $e - x \in W$. It is well known that every left Artinian ring is semiperfect. The following results are presented without proofs.

LEMMA 4.1. If R is a semiperfect ring then

$$1 = e_1 + \dots + e_n$$

where the e_i 's are an orthogonal set of primitive idempotents.

LEMMA 4.2. Every primitive idempotent e in a semiperfect ring R is local. That is, eRe is a local ring.

LEMMA 4.3. Let R be a semiperfect ring and let e_1 and e_2 be two local idempotents in R , then $e_1Re_2 \neq e_2Re_1$ if and only if $e_1Re_2 \subseteq e_1We_2$. Moreover, $e_1Re_1 = e_2Re_2$ if and only if e_1Re_1 and e_2Re_2 are isomorphic as rings.

An idempotent e in R is called basic if $e = e_1 + \dots + e_n$ where $\{e_i\}$ is an orthogonal set of primitive idempotents such that $e_iRe_j = e_jRe_i$ if and only if $i = j$ and if e' is any other primitive idempotent in R then there exists e_i such that $e'Re' = e_iRe_i$. The ring eRe is called the basic ring associated with R . A ring R is basic if it is its own basic ring.

We are now ready to prove our first main result.

THEOREM 4.4. Let R be a basic semiperfect ring and let $1 = e_1 + \dots + e_n$ be a decomposition of 1 into orthogonal primitive idempotents. Then R is a split extension ring if and only if each of the local rings e_iRe_i is a split extension ring.

Proof. The "only if" part is trivial since if

$R = S \oplus W$ is a split extension ring, $e_i Re_i = e_i Se_i \oplus e_i We_i$ is also a split extension ring.

Now suppose that each $e_i Re_i$ is a split extension ring. Then

$$\begin{aligned} R/W &\cong \sum_{i=1}^n \sum_{j=1}^n e_i Re_j / e_i We_j \\ &\cong \sum_{i=1}^n e_i Re_i / e_i We_i \end{aligned}$$

since $e_i Re_j / e_i We_j = 0$ for $i \neq j$, by Lemma 4.3. Now

let $e_i Re_i = e_i F_i e_i \oplus e_i We_i$ for $i = 1, 2, \dots, n$. Then

we claim that $R = F \oplus W$ where $F = \sum_{i=1}^n e_i F_i e_i$. For,

suppose $x \in F \cap W$. Then since F is a direct sum, we may

suppose $x = e_i x e_i$. Now $e_i x e_i \in W$ implies that

$e_i x e_i \in e_i We_i$, contradicting the fact that $e_i Re_i$ splits.

THEOREM 4.5. Let R be a semiperfect ring whose basic ring R^0 is local. Then R is a split extension ring if and only if R^0 is a split extension ring.

Proof. Let $1 = e_1 + \dots + e_n$, for orthogonal primitive idempotents in R . Since R^0 is local, $Re_i \cong Re_j$ for all i, j . Thus R is isomorphic to a

full matrix ring over R^0 . Now suppose that R^0 splits.

Then

$$R = (F \oplus W)_n, \quad (n \times n \text{ matrices over } F \oplus W).$$

We claim that $R = (F)_n \oplus (W)_n$. For, suppose

$x \in (F)_n \cap (W)_n$. Then some component of x lies in W ,

contradicting the fact that $F \cap W = 0$. The theorem is

proved in view of the fact that the "only if" part is trivial.

The previous two theorems imply.

THEOREM 4.6. Let R be a semiperfect ring and suppose $1 = e_1 + \dots + e_n$ for a set $\{e_i\}$ of orthogonal primitive idempotents in R . Then R is a split extension ring if and only if each local ring $e_i R e_i$ is a split extension ring.

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NOTE

The author has been made aware after this paper was written, that the results in Chapter 2 overlap with work done by A. Geddes. The reference is:

Geddes, A., "A Short Proof of the Existence of Coefficient Fields for Complete, Equicharacteristic Local Rings", Journal London Math. Soc., 29 (1954), pp. 334-341.

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28 • 01 • 75

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