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THE EFFECT OF ATMOSPHERIC TURBULENCE
ON THE INTERFERENCE OF ACOUSTIC WAVES
DUE TO THE PRESENCE OF A BOUNDARY

by

GILLES A. DAIGLE

A thesis submitted to the faculty of graduate studies in partial
fulfillment of the requirements for the degree of Master of Science.

Carleton University

Ottawa Ontario

April 1977

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The undersigned recommend to the Faculty of Graduate Studies acceptance of the thesis "THE EFFECT OF ATMOSPHERIC TURBULENCE ON THE INTERFERENCE OF ACOUSTIC WAVES DUE TO THE PRESENCE OF A BOUNDARY", submitted by GILLES A. DAIGLE, B.Sc., in partial fulfilment of the requirements for the degree of Master of Science.

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TABLE OF CONTENTS

Abstract	i
Acknowledgments	ii
Illustrations	iii
Tables	v
Introduction	1
PART I	THEORY

Chapter 1 Historical Perspective

1 - 1 Coherent Wave Theory	2
1 - 2 Incoherent Wave Theory	3
1 - 3 Consequences of Incoherent Wave Theory	4

Chapter 2 Theoretical Treatment

2 - 1 Introduction	6
2 - 2 Statistical Analysis of the Sound Field Above a Boundary by Ingard and Maling	6
2 - 3 Survey of Existing Calculations of Phase and Amplitude Fluctuations in the Absence of a Boundary	
a) Plane Waves	11
b) Spherical Waves	15
2 - 4 Author's Extension of the Theory	
a) Calculation of Phase and Amplitude Fluctuations in a Spherical Wave in the Absence of a Boundary and ∇D .	18
b) Calculation of the Fluctuations in the Index of Refraction .	24
c) Extension of Analysis by Ingard and Maling of the Sound Field Above a Boundary to Include Partial Correlation	29

PART II

EXPERIMENTS

Chapter 3

Measurements and Analysis

3 - 1	Previous Measurements and Analysis	39
3 - 2	Measurements during the Summer, 1976	40
3 - 3	Field Equipment	40
a)	The Sound Source	42
b)	The Sound Receivers	43
c)	The Weather Station	44
3 - 4	Methods for Analysing Field Measurements	44
a)	Sound Pressure Level	45
b)	Wind Velocity and Temperature	
i)	Mean and Standard Deviation	46
ii)	Auto and Cross-Correlation	47
iii)	Spectra	54
3 - 5	Discussion of Errors	56

Chapter 4

Experimental Data

4 - 1	Mean and Standard Deviation of Sound Pressure Levels	57
4 - 2	Wind Velocity and Temperature	61
4 - 3	Discussion of the Data	69

PART III

COMPARISON OF THEORY AND EXPERIMENTS

Chapter 5

Results and Discussion

5 - 1	Comparison of Results with Coherent Theory and the Incoherent Theory of Ingard and Maling	73
5 - 2	Comparison of Results with Incoherent Theory considering Spherical Waves	76

5 - 3 Comparison of Results considering Spherical
Waves with Partial Correlation

79

Conclusion

88

Appendix A

A1

Appendix B

B1

Appendix C

C1

References

89

ABSTRACT

When sound propagates through turbulence caused by wind and temperature variations, it exhibits fluctuations in phase and amplitude. If a boundary is present, certain regions of the interference spectrum are critically dependent on the phase relationship between the direct and reflected waves. Since we find experimentally that the correlation length (~ 3.5 ft) of the measured meteorological fluctuations is comparable to, or greater than the separation between the interfering sound paths, previous theoretical work has been extended to allow for a known amount of partial correlation between the two waves. The theory has been further extended to adapt the calculations of the fluctuations in phase and amplitude of spherical waves as well as to include the explicit calculation of the fluctuating index of refraction from the fluctuating values of temperature and wind velocity.

Measurements (1 to 6 kHz) have been made of the interference spectrum between the direct and reflected waves at several distances in the range 50 to 150 feet from a point source 4 feet above a large asphalt surface. Simultaneously, the fluctuating values of temperature and horizontal wind velocity (atmospheric turbulence) were measured at two related points close to the sound path. There is quantitative agreement between the sound levels calculated from the measured fluctuating meteorological variables and those measured experimentally.

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ILLUSTRATIONS

Figure		Page
1	The Phase and Amplitude Fluctuations of Plane and Spherical Waves	23
2	The Cross-Correlation Coefficient of Phase for Plane and Spherical Waves	38
3	Typical Results found by Ingard and Maling	39
4	Field Equipment	41
5	Typical Transmission Filter	43
6	Instruments used to Analyse the Sound Pressure Levels	45
7	Instruments used to Analyse the Mean and Standard Deviation of the Wind Velocity and Temperature	46
8	Instruments used to obtain the Correlation Functions of the Wind Velocity and Temperature Fluctuations	48
9 a,b,c	Typical Auto-Correlation Function of the Temperature (a) and Wind Velocity (b) Fluctuations and a Typical Cross-Correlation Function (c)	51
10	Instruments used to obtain the Spectra of Turbulence	54
11	Typical Spectrum of Turbulence	55
12 a,b	The Cross-Correlation Coefficient of the Wind Velocity (a) and Temperature (b) Fluctuations	71
13	Fit to the Results for $x=50$ ft, $h_v = h_s = 4$ ft of Table I using Conventional Coherent Theory and the Incoherent Theory of Ingard and Maling	74
14	Standard Deviation of Sound Levels	75

Figure		Page
15	Fit to the Results for $x=50$ ft, $h_r = h_s = 4$ ft of Table I using Incoherent Theory considering Spherical Waves	78
16 a,b,c, d,e,f	Fit to the Results of Tables I (a,b), II (c,d) and III (e,f) using Incoherent Theory considering Spherical Waves with Partial Correlation	82
b1	An Auto-Correlation Function	B2
c1	Experimental Geometry	C1
c2,c3,c4, c5,c6	Photographs showing the equipment in the field	C1

TABLES

Tables I II III	The mean and standard deviation of the measured sound levels for the various geometries considered	58
Tables Ia IIb IIIc	The mean and standard deviation of the wind velocity and temperature and corresponding meteorological parameters	63
Tables IVa Va VIa	Analysis of supplementary meteorological variables	66
Tables Ib IIb. IIIb IVb Vb VIb	The cross-correlation coefficients of the wind velocity and temperature fluctuations vs vertical probe separations	68
Table Vc	The cross-correlation coefficients of the wind velocity and temperature fluctuations vs horizontal probe separations	68
Table VII	Some parameters used to calculate the fluctuations in the index of refraction	61
Table VIII	Parameters used to calculate the predicted sound levels	69

INTRODUCTION

The increasing noise problems facing modern society have led to a re-evaluation of our understanding of sound propagation outdoors. The need is, for example, to predict adequately the sound levels to be expected from installations such as airports and highways. Recent work on sound propagation near the ground has shown the importance of several previously unsuspected or neglected factors such as finite impedance of the ground (leading to such phenomena as ground and surface waves, shadow boundaries, etc...) interference between direct and ground-reflected waves, and atmospheric turbulence.

Lack of coherence due to atmospheric turbulence may be expected to influence many aspects of outdoor sound propagation. One of the conditions more sensitive to atmospheric turbulence is interference phenomena. For this reason a particular case has been chosen for study here, namely the interference between the direct wave and that reflected from a hard surface for short distances of propagation. This, in order to produce sufficient isolation of the physical variables for effective application of theory.

PART I

THEORY

Chapter 1

HISTORICAL PERSPECTIVE

1 - 1 Coherent Wave Theory

In most treatments of acoustic waves propagating through a compressible fluid, simplifying assumptions must be made regarding small amplitudes of density change, condensation, particle displacement, and particle velocity. Furthermore the medium is usually assumed to be homogeneous, isotropic, and perfectly elastic.

With these assumptions it is possible to derive a wave equation satisfied by the pressure disturbances of acoustic waves. In the case of acoustic plane waves propagating in the x-direction, the pressure disturbances satisfy

$$\frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 p}{\partial x^2} = 0 \quad (1)$$

while in the case of acoustic spherical waves the pressure disturbances satisfy

$$\frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} - \nabla^2 p = 0 \quad (2)$$

where c is the velocity of propagation of the wave.

A solution of eq. (2) in the case of diverging spherical waves is represented in complex form by

$$p = \frac{A}{r} e^{i(kr - \omega t)} \quad (3)$$

where A , ω and k are respectively the amplitude, angular frequency and wave number of the acoustic wave; the latter two being related by $\omega = kc$.

If a perfectly rigid boundary of infinite impedance is introduced in the vicinity of the sound source, the sound field at a receiver above the boundary is a combination of the direct and reflected waves.

With no phase change upon reflection, the field at R can be thought of as a combination of two sources; the actual source S and its image I.

Then at R, the resulting sound pressure is given by

$$P = \frac{A_s}{x} e^{i(kx - \omega t)} + \frac{A_r}{r} e^{i(kr - \omega t)} \quad (4)$$

When the path length difference is $\lambda/2$, or an odd multiple of $\lambda/2$, there is destructive interference between the direct and reflected waves producing an interference minimum. Similarly, for path length differences that are multiples of λ , the interference is constructive and produces a pressure maximum. This is illustrated in the sketch.

1 - 2 Incoherent Wave Theory

The assumption of an isotropic and homogeneous medium in the case of

acoustic waves propagating outdoors is in general too stringent a restriction.

The index of refraction of a medium is, by definition

$$n = \frac{c_0}{c} \quad (5)$$

where c_0 is the velocity of propagation of the wave in the medium when it is homogeneous at some reference temperature.

In the case of a gaseous medium, such as the atmosphere, the velocity of propagation c is, in part, a function of the absolute temperature of the gas. Furthermore a sound wave is an alternating disturbance of the propagation medium and c describes the velocity of propagation relative to the medium itself. Sound waves in air are convected by movements of the medium and hence the velocity of propagation relative to some instrument, fixed with respect to the ground, is, in addition, a function of the component of wind velocity in the direction of propagation.

It is known that the wind velocity and temperature undergo random fluctuations in space and time, resulting in atmospheric turbulence. As a consequence c , and hence n , are also random functions of space and time. Therefore a sound wave propagating in such an atmosphere should be expected to contain fluctuations in its phase and amplitude.

1 - 3 Consequences of Incoherent Wave Theory

If the fluctuations in phase and amplitude of a sound wave, given the atmospheric turbulence, are observable, then what is the effect on the interference spectrum produced by the direct and reflected waves near a boundary?

Recent measurements¹ have shown that the fluctuations are significant.

Moreover, theoretical treatment of waves propagating in a turbulent atmosphere in the absence of a boundary has been extensively treated in recent years by several authors^{2,3,4,5}. These, using statistical treatments of the atmosphere, give the magnitude of phase and amplitude fluctuations of plane and

spherical waves as a function of measurable meteorological and acoustical parameters.

Work by Ingard and Maling⁷ has been the only pioneering attempt to explain the effect of atmospheric turbulence on the sound field above a boundary. This work shows that the boundary can produce important effects on the interference spectrum even in the presence of weak turbulence.

Chapter 2

THEORETICAL TREATMENT

2 - 1 Introduction

The work by Ingard and Maling⁷ begins with a theoretical derivation of the sound field above the boundary by taking into account the fluctuations in phase and amplitude produced by atmospheric turbulence. Particular attention was given to the region of destructive interference, since even in the presence of weak turbulence the sound field above a boundary is critically dependent upon the phase relationship between the direct and reflected waves.

Their theoretical treatment is simplified by the assumption of zero correlation between the direct and reflected waves, and by using, in a limited way, a theory² for plane waves to predict the fluctuations in phase and amplitude of spherical waves. Furthermore they did not measure experimentally any meteorological variables and hence used the corresponding meteorological parameters as a constant adjusted for best fit to measured sound levels.

Since it will be shown experimentally that the correlation length of the medium is of the same order as the path separation of the direct and reflected waves it would be expected that partial correlation between the two waves does exist and should be included in the theory. Secondly, the use of a point source and with the distances involved, a theory relating the fluctuations in phase and amplitude of spherical waves should be used.

Finally measurements of the meteorology would be useful to confirm the validity of certain assumptions in the theory above the boundary. Therefore a calculation relating the relevant meteorological parameters should be made in terms of the measured meteorological variables.

2 - 2 Statistical Analysis of the Sound Field Above a Boundary by Ingard and Maling

It would be useful at this point to summarize the theoretical treatment

of Ingard and Maling

As seen in section 1 - 1 for the special case of a homogeneous medium and perfectly reflecting boundary, the sound field at a receiver R, above the boundary, can be treated as arising from the source S and an image source I,

Similarly, in the case of an inhomogeneous medium the sound field at a receiver R arising from a monopole source, both at a height h, above the boundary, can be written

$$p = \frac{A_d}{x} e^{i(k_d x - \omega t)} + \frac{A_r}{r} e^{i(k_r r - \omega t)} \quad (6)$$

where $r = \sqrt{x^2 + 4h^2}$ and where, in the presence of turbulence, the quantities A_d , A_r and k_d , k_r are now respectively the fluctuating amplitudes of direct and reflected waves and the corresponding fluctuating wave numbers.

Ingard and Maling have assumed that both amplitudes and both wave numbers fluctuate about some mean value and have therefore set

$$A_d = 1 + a_d, \quad A_r = 1 + a_r \quad (7a)$$

for a unit amplitude sound field at one foot, and similarly

$$k_d x = Kx + \delta_d, \quad k_r r = Kr + \delta_r \quad (7b)$$

where k is the wave number in the absence of turbulence. It is also assumed that

$$\langle a_d \rangle = \langle a_r \rangle = \langle \delta_d \rangle = \langle \delta_r \rangle = 0$$

Experimentally they consider a $\bar{p}^2$ whose averaging time is long compared with the period of the acoustic wave, but short as compared with a characteristic period of turbulence, and a $\langle \bar{p}^2 \rangle$ whose averaging time extends over many turbulent periods. (Henceforth the overbar, $\bar{\chi}$, will be reserved for an ensemble average while the bracket, $\langle \chi \rangle$, will be used for a time average).

Therefore from eq. (6)

$$\bar{p}^2 = \frac{A_d^2}{x^2} + \frac{A_r^2}{r^2} + 2 \frac{A_d A_r}{x r} \cos[k(r-x) + \delta_r - \delta_d] \quad (8)$$

Substituting eqs. (7a) and (7b)

$$\begin{aligned} \bar{p}^2 = \frac{1}{x^2} & \left[1 + 2a_d + a_d^2 + \left(\frac{x}{r}\right)^2 (1 + 2a_r + a_r^2) \right. \\ & \left. + 2 \frac{x}{r} (1 + a_r + a_d + a_d a_r) \cos(\varphi + \delta_r - \delta_d) \right] \quad (9) \end{aligned}$$

where $\varphi = k(r-x)$.

At this point it is assumed by the authors that all cross correlation between the amplitude and phase fluctuations of the direct and reflected waves is zero. It is also assumed that

$$\langle a_d^2 \rangle = \langle a_r^2 \rangle = \langle a^2 \rangle \quad (10a)$$

$$\langle \delta_d^2 \rangle = \langle \delta_r^2 \rangle = \sigma_d^2 \quad (10b)$$

Therefore if

$$\delta = \delta_r - \delta_d \quad (11)$$

then

$$\langle \bar{p}^2 \rangle = \frac{1}{x^2} \left[1 + \langle a^2 \rangle + \left(\frac{x}{r} \right)^2 (1 + \langle a^2 \rangle) + a \frac{x}{r} \langle \cos(\varphi + \delta) \rangle \right]$$

(12)

Rearranging terms to show the contribution from the amplitude and phase fluctuations to the constant terms we arrive at

$$\langle \bar{p}^2 \rangle = \frac{2}{x r} \left[\frac{\langle a^2 \rangle}{2} \left(\frac{r}{x} + \frac{x}{r} \right) + \frac{r}{2x} \left(1 - \frac{x}{r} \right)^2 + \langle 1 + \cos(\varphi + \delta) \rangle \right] \quad (13)$$

Ingard and Maling have assumed that both a and δ are normally distributed with standard deviation $(\langle a^2 \rangle)^{1/2}$ and σ , respectively. Then

$$\langle a^2 \rangle = \frac{1}{(\langle a^2 \rangle)^{1/2} \sqrt{2\pi}} \int_{-\infty}^{\infty} a^2 e^{-\frac{a^2}{2\langle a^2 \rangle}} da = \langle a^2 \rangle \quad (14)$$

and

$$\begin{aligned} \langle 1 + \cos(\varphi + \delta) \rangle &= \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{\infty} [1 + \cos(\varphi + \delta)] \exp\left(-\frac{\delta^2}{2\sigma^2}\right) d\delta \\ &= 1 + \cos \varphi \exp\left(-\frac{\sigma^2}{2}\right) \end{aligned} \quad (15)$$

Substituting eq. (15) into eq. (13), we obtain an expression for the time average value of the pressure field

$$\langle \bar{p}^2 \rangle = \frac{2}{x r} \left[\frac{\langle a^2 \rangle}{2} \left(\frac{r}{x} + \frac{x}{r} \right) + \frac{r}{2x} \left(1 - \frac{x}{r} \right)^2 + 1 + \cos \varphi e^{-\frac{\sigma^2}{2}} \right] \quad (18)$$

where, remembering eqs. (10) and (11)

$$\begin{aligned} \langle \delta^2 \rangle &= \langle \delta_d^2 \rangle + 2\langle \delta_r \delta_d \rangle + \langle \delta_r^2 \rangle \approx 2\langle \delta_d^2 \rangle \\ \sigma^2 &= \langle \delta^2 \rangle - \langle \delta \rangle^2 = 2\sigma_d^2 \end{aligned}$$

At this point, Ingard and Maling have gone to the theory of Chernov²,

particularly the version in Morse and Ingard⁸, to estimate the dependence of $\langle a^2 \rangle$ and σ_d^2 on distance, frequency and turbulence level.

Chernov assumes that turbulence^a is homogeneous and isotropic and that the fluctuating index of refraction can be written as

$$n = \frac{c_0}{c} = 1 + \mu \quad (19)$$

where it is assumed that c fluctuates around c_0 , (that is $c = c_0 + \delta c$), and therefore n fluctuates around 1 with the further assumption that $|\mu| \ll 1$. In this case the spatial correlation of the turbulence can be written in the form

$$\langle \mu_1 \mu_2 \rangle = \langle \mu^2 \rangle \exp\left(-\frac{r^2}{L^2}\right) \quad (20)$$

where r is the magnitude of a radius vector and L is a constant which gives a measure of the scale of the turbulence.

Then for what Ingard and Maling assume, from Morse and Ingard⁸, to be deep penetration into the medium^a, $x \gg \sqrt{L\lambda}$, the variance of amplitude and, phase fluctuations in eq. (18) were obtained from

$$\langle \delta_a^2 \rangle = \frac{\sqrt{\pi}}{2} k^2 x \langle \mu^2 \rangle L \quad (21)$$

$$\langle [\ln(1+a)]^2 \rangle = \frac{\sqrt{\pi}}{2} k^2 x \langle \mu^2 \rangle L \quad (22)$$

To determine $\langle a^2 \rangle$ from $\langle [\ln(1+a)]^2 \rangle$ it is necessary to expand the natural logarithm and write the higher-order moments in terms of second-order moments (by assuming that a is normally distributed).

The quantities $\langle \mu^2 \rangle$ and L were not measured experimentally so to compare the theoretical sound pressure levels with the levels actually measured,

^a See section 2 - 3 for a more complete discussion of the statistical treatment of a turbulent medium and for a discussion of the various regions of turbulence described by a wave parameter.

$\langle \mu^2 \rangle L$ was used as a parameter adjusted for best fit.

2 - 3 Survey of Existing Calculations of Phase and Amplitude Fluctuations in the Absence of a Boundary.

a) Plane Waves

Chernov² has done an extensive treatment of phase and amplitude fluctuations of plane waves in the absence of a boundary in the special case of homogeneous isotropic turbulence.

The fluctuations in the refractive index present a random process in space and time, described by the random function of coordinates and time, $\mu(x, y, z, t)$ where we remember that

$$n = 1 + \mu$$

If it is assumed that this random process is stationary in time then it can be characterized by a spatial correlation function

$$N_{12} = \langle \mu(x_1, y_1, z_1, t) \mu(x_2, y_2, z_2, t) \rangle \quad (23)$$

where the bracket designates averaging with respect to time t . For a spatially homogeneous process the correlation function depends only on the coordinate differences $x = x_2 - x_1$, $y = y_2 - y_1$ and $z = z_2 - z_1$. Therefore

$$N_{12} = N_{12}(x, y, z, t) \quad (24)$$

For $x = y = z = 0$ we have by definition

$$N_{12}(0, 0, 0, t) = \langle \mu^2 \rangle \quad (25)$$

The correlation coefficient N is defined as the ratio of the correlation function, N_{12} , to the variance $\langle \mu^2 \rangle$

$$N = \frac{N_{12}}{\langle \mu^2 \rangle} \quad (26)$$

so that

$$N_{12} = \langle \mu^2 \rangle N \quad (27)$$

The only practical case that can be calculated theoretically is the case of homogeneous isotropic turbulence. Therefore Chernov has assumed a Gaussian correlation coefficient for such a case

$$N = \exp\left(-\frac{r^2}{L^2}\right) \quad (28)$$

It is now possible to derive expressions for the variance and covariance of phase and amplitude fluctuations of plane waves in the absence of a boundary. A method due to Rytov² is used, which proceeds in the following manner.

The wave equation satisfied by plane waves can be written from eq. (2)

$$\frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} - \nabla^2 p = 0$$

If the medium is homogeneous and isotropic the solution to (2) is

$$p_0 = A_0 e^{-i(\omega t - S_0(x))} \quad (29)$$

If

$$n = 1 + \mu = \frac{c_0}{c}$$

then the wave equation (2) becomes

$$\frac{(1+\mu)^2}{c_0^2} \frac{\partial^2 p}{\partial t^2} - \nabla^2 p = 0 \quad (30)$$

We look for solution of the form

$$p = A(r) e^{-i(\omega t - S(r))} \quad (31)$$

when $A(r)$ and $S(r)$ are unknown functions. The gist of Rytov's method consists in replacing the wave function p by another function ψ which is connected with p by

$$p = A_0 e^{-i(\omega t - \psi(r))} \quad (32)$$

and therefore

$$\psi(r) = S(r) - i \ln \frac{A(r)}{A_0} \quad (33)$$

Substituting (32) into (30), (29) into (2), and subtracting we obtain a differential equation for

$$\psi' = \psi - \psi_0 \quad (34)$$

which is

$$2(\nabla \psi_0 \cdot \nabla \psi') - \nabla^2 \psi' = 2\mu k^2 + [\mu^2 k^2 - (\nabla \psi')^2] \quad (35)$$

and where $\psi_0 = S(x) = kx$

After a simple approximation and a little rearranging it is possible to find a Green's function solution for (35). The result is

$$\psi' = - \frac{i k^2}{2\pi} \int \frac{1}{r} e^{ik[r - (x - \tau)]} \mu(\tau, \eta, \xi) dv \quad (36)$$

From (33) we identify

$$\delta_p = S - S_0 = \frac{k^2}{2\pi} \int \frac{1}{r} \sin k[r - (x - \tau)] \mu dv \quad (37)$$

$$\ln [1 + a_p(r)] = \frac{k^2}{2\pi} \int \frac{1}{r} \cos k[r - (x - \tau)] \mu dv \quad (38)$$

where we have set $A(r) = A_0 + a_p(r)$ and then $A_0 = 1$ as in eq. (7a). The integration in eqs. (37) and (38) is over the whole volume (τ, η, ξ) and $r^2 = (x - \tau)^2 + y^2 + z^2$.

From eqs. (37) and (38) it is possible to obtain expressions for the mean square amplitude fluctuations, $\langle [\ln(1 + a_p)]^2 \rangle$, and phase fluctuations $\langle \delta_p^2 \rangle$.

If we assume that a receiver is located at point $(x, 0, 0)$, then

$$\delta_p(x, 0, 0) = \int_0^x \int_0^\infty \int_0^\infty \Phi_1(\tau, \eta, \xi) \mu(\tau, \eta, \xi) d\tau d\eta d\xi \quad (39)$$

$$\ln [1 + a_p(x, 0, 0)] = \int_0^x \int_0^\infty \int_0^\infty \Phi_2(\tau, \eta, \xi) \mu(\tau, \eta, \xi) d\tau d\eta d\xi \quad (40)$$

where Φ_1 and Φ_2 represent the integrand in eqs. (37) and (38). Squaring and averaging we obtain

$$\langle \delta_p^2 \rangle = \langle \mu^2 \rangle \int_0^x \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \Phi_1(\tau, \eta, \xi) \Phi_1(\tau', \eta', \xi') N(r) d\tau d\eta d\xi d\tau' d\eta' d\xi' \quad (41)$$

$$\langle [\ln(1 + a_p)]^2 \rangle = \langle \mu^2 \rangle \int_0^x \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \Phi_2(\tau, \eta, \xi) \Phi_2(\tau', \eta', \xi') N(r) d\tau d\eta d\xi d\tau' d\eta' d\xi' \quad (42)$$

where $N(r)$ is the correlation coefficient of the refractive index and

$$r = \sqrt{(\xi_1 - \xi_2)^2 + (\eta_1 - \eta_2)^2 + (\zeta_1 - \zeta_2)^2}$$

In the case where the correlation coefficient has the Gaussian form

$$N(r) = \exp\left(-\frac{r^2}{L^2}\right)$$

the integrals in eqs. (41) and (42) can be evaluated with certain simplifying assumptions. The result is expressions for the amplitude and phase fluctuations of a plane wave

$$\langle \delta_p^2 \rangle = \frac{\sqrt{\pi}}{2} \langle \mu^2 \rangle k^2 x L \left(1 + \frac{1}{D} \arctan D\right) \quad (43)$$

$$\langle [\ln(1+q_p)]^2 \rangle = \frac{\sqrt{\pi}}{2} \langle \mu^2 \rangle k^2 x L \left(1 - \frac{1}{D} \arctan D\right) \quad (44)$$

where

$$D = \frac{4x}{kL^2} \quad (45)$$

is called the wave parameter.

In the case $D \ll 1$ we can expand $\arctan D$ in a Taylor series, $\arctan D \sim D - \frac{D^3}{3} + \dots$

Then eqs. (43) and (44) become

$$\langle \delta_p^2 \rangle = \sqrt{\pi} \langle \mu^2 \rangle k^2 x L \quad (46)$$

$$\langle [\ln(1+q_p)]^2 \rangle = \frac{8\sqrt{\pi}}{3} \langle \mu^2 \rangle \frac{x^3}{L^3} \quad (47)$$

In the case $D \gg 1$

$$\frac{\arctan D}{D} \rightarrow 0$$

and eqs. (43) and (44) reduce to eqs. (21) and (22) used by Ingard and Maling

$$\langle \delta_p^2 \rangle = \frac{\sqrt{\pi}}{2} \langle \mu^2 \rangle k^2 x L \quad (21)$$

$$\langle [\ln(1+q_p)]^2 \rangle = \frac{\sqrt{\pi}}{2} \langle \mu^2 \rangle k^2 x L \quad (22)$$

From eqs. (39) and (40) it is also possible to calculate expressions for cross correlation between amplitude and phase fluctuations at the receiver, $\langle [\ln(1+a_p)] \cdot \delta_p \rangle$, and for cross correlation of amplitude, $\langle [\ln(1+a_{p1})] \cdot [\ln(1+a_{p2})] \rangle$ (or phase $\langle \delta_{p1} \cdot \delta_{p2} \rangle$), fluctuations at different receiving points.

For example, suppose that two receivers lie in the same x-plane at a distance ℓ from one another; i.e., one receiver at $(x, 0, 0)$ and the other at $(x, 0, \ell)$. Then it is possible to define correlation coefficients; ρ_p in the case of phase, given by

$$\rho_p = \frac{\langle \delta_{p1} \cdot \delta_{p2} \rangle}{\langle \delta_p^2 \rangle} \quad (48)$$

and R_p in the case of amplitude, given by

$$R_p = \frac{\langle [\ln(1+a_{p1})] \cdot [\ln(1+a_{p2})] \rangle}{\langle [\ln(1+a_p)]^2 \rangle} \quad (49)$$

Once $\langle \delta_{p1} \cdot \delta_{p2} \rangle$ has been calculated and using eq. (43) we find in the case $D \gg 1$

$$\rho_p \sim \frac{\exp\left(-\frac{\ell^2}{L^2}\right) + \frac{1}{D} \left[\frac{\pi}{2} - \text{Si}\left(\frac{1}{D} \cdot \frac{\ell^2}{L^2}\right) \right]}{1 + \frac{1}{D} \arctan D} \quad (50)$$

In the case $D \ll 1$

$$\rho_p \sim \exp\left(-\frac{\ell^2}{L^2}\right) \quad (51)$$

Similar expressions are found for R_p .

b) Spherical Waves

A very similar calculation to that of section 2 - 3 can be performed to obtain the variance and covariance of phase and amplitude fluctuations of spherical waves.

Using the same technique, but with a different approach, Tatarski¹ has obtained limited expressions for the variance of amplitude fluctuations of a spherical wave, $\langle [\ln(1+a_s)]^2 \rangle$, in the case where the field of refractive index

fluctuations have a Gaussian correlation function (eq. 20).

In the case $\sqrt{\lambda x} \ll L$ he finds

$$\langle [\ln(1 + a_s)]^2 \rangle = \frac{4\sqrt{\pi}}{15} \langle \mu^2 \rangle \frac{x^3}{L^3} \quad (52)$$

and in the case $\sqrt{\lambda x} \gg L$

$$\langle [\ln(1 + a_s)]^2 \rangle = \frac{\sqrt{\pi}}{2} \langle \mu^2 \rangle k^2 x L \quad (53)$$

At this point it is instructive to show the equivalence of Tatarski's condition, $\frac{\sqrt{\lambda x}}{L}$, and Chernov's wave parameter, D . Remembering that $D = \frac{4x}{kL^2}$, and since $k = \frac{2\pi}{\lambda}$

$$D = \frac{4x}{kL^2} \sim \frac{2\pi x}{kL^2} \sim \frac{\lambda x}{L^2} \quad (54)$$

Therefore the condition $D \ll 1$ is equivalent to $\frac{\lambda x}{L^2} \ll 1$ or $\sqrt{\lambda x} \ll L$ and similarly $D \gg 1$ is equivalent to $\sqrt{\lambda x} \gg L$.

We therefore see from eqs. (52), (53), (47) and (22) that the variance of the amplitude fluctuations is the same for plane and spherical waves in the region $D \gg 1$ but differ by a factor of 10 in the region $D \ll 1$.

Karavainikov³ has done a more extensive calculation of phase and amplitude fluctuations in a spherical wave for the case where the correlation function of the refraction index fluctuations has again a Gaussian form.

The results obtained by Karavainikov for the amplitude fluctuations of a spherical wave in the regions $D \ll 1$ and $D \gg 1$ agree with the calculations done by Tatarski. However his results $\forall D$ do not give correct numerical results, yielding a negative variance. Equivalent but correct expressions are therefore calculated in section 2 - 4 a.

17

Karavainikov also calculates expressions for the covariance of phase and amplitude fluctuations in spherical waves. For example, he finds for the covariance of phase fluctuations for $\forall D$

$$\rho_s \approx \frac{\Phi(l/L)}{l/L} \quad (55)$$

where

$$\Phi = \int_0^{l/L} e^{-\tau^2} d\tau$$

This result will be discussed further in section 2 - 4 c.

2 - 4 Author's Extension of the Theory

a) Calculation of Phase and Amplitude Fluctuations in a Spherical Wave in the Absence of a Boundary and $\forall D$

The calculation is based on Rytov's method as in section 2 - 3 and again in the absence of a boundary. However instead of eq. (31) we substitute

$$p = \frac{A(r)}{r} e^{i[\omega t - S(r)]}$$

into the wave equation. Then by a calculation identical to that of section 2 - 3 for a plane wave we find the equations equivalent to eqs. (41) and (42) to be

$$\langle \delta_s^2 \rangle = \frac{1}{2} (I_1 + I_2) \quad (56)$$

$$\langle [ln(1+\delta_s)]^2 \rangle = \frac{1}{2} (I_1 + I_2) \quad (57)$$

If the correlation coefficient of the refractive index is again a Gaussian given by eq. (28) we have, after some simple integrations

$$I_1 = \langle M^2 \rangle k^2 \int_0^x \int_0^x \frac{1}{1 + \frac{4[(\tau - \tau_1)(x - \tau - \tau_1)]^2}{k^2 L^2 x^2}} \exp\left[-\frac{(\tau - \tau_1)^2}{L^2}\right] d\tau d\tau_1 \quad (58)$$

$$I_2 = \langle M^2 \rangle k^2 \int_0^x \int_0^x \frac{1}{1 + \frac{4[x(\tau + \tau_1) - (\tau_1 + \tau_2)]^2}{k^2 L^2 x^2}} \exp\left[-\frac{(\tau - \tau_1)^2}{L^2}\right] d\tau d\tau_2 \quad (59)$$

We introduce the relative coordinate

$$\gamma = \tau_1 - \tau_2$$

and the coordinate of the center of gravity

$$\gamma = \frac{1}{2}(\tau_1 + \tau_2)$$

Then since $x \gg L$ we can write

$$I_1 = \langle M^2 \rangle k^2 \int_0^x \int_{-\infty}^{\infty} \frac{1}{1 + \frac{4\gamma^2(x - 2\gamma)^2}{k^2 L^2 x^2}} e^{-\frac{\gamma^2}{L^2}} d\gamma d\gamma \quad (60)$$

$$I_2 = \langle M^2 \rangle k^2 \int_0^x \int_{-\infty}^{\infty} \frac{1}{1 + \frac{4(2x\gamma - 2\gamma^2 - \frac{\gamma^2}{2})^2}{k^2 L^2 x^2}} e^{-\frac{\gamma^2}{L^2}} d\gamma d\gamma \quad (61)$$

Since ξ cannot acquire a value greater than L and making use of the condition $KL \gg 1$ it is possible to neglect terms which contain ξ^2 and ξ^4 in the denominators of the integrands of eq. (60) and (61).

In so doing we can immediately integrate eq. (60), first with respect to ξ

$$I_1 = \langle H^2 \rangle k^2 \int_0^x \int_{-\infty}^{\infty} e^{-\frac{\xi^2}{L^2}} d\xi d\gamma = \sqrt{\pi} \langle H^2 \rangle k^2 L \int_0^x d\gamma$$

and then the integration with respect to γ yields

$$I_1 = \sqrt{\pi} \langle H^2 \rangle k^2 L x \quad (62)$$

Similarly eq. (61) can now be written as

$$I_2 = \langle H^2 \rangle k^2 \int_0^x \int_{-\infty}^{\infty} \frac{1}{1 + \frac{4(\alpha x \gamma - \alpha \gamma^2)^2}{k^2 L^4 x^2}} e^{-\frac{\xi^2}{L^2}} d\xi d\gamma \quad (63)$$

Integration with respect to ξ yields

$$I_2 = \sqrt{\pi} \langle H^2 \rangle k^2 L \int_0^x \frac{1}{1 + \frac{4}{k^2 L^4 x^2} (\alpha x \gamma - \alpha \gamma^2)^2} d\gamma \quad (64)$$

For simplification we write

$$\epsilon^2 = \frac{k^2 L^4 x^2}{16} \quad (65)$$

and eq. (64) becomes

$$I_2 = \frac{\sqrt{\pi}}{16} \langle H^2 \rangle k^4 L^5 x^2 \int_0^x \frac{1}{\epsilon^2 + (x\gamma - \gamma^2)^2} d\gamma$$

Concentrating on the integral we write

$$I = \int_0^x \frac{1}{\epsilon^2 + (x\gamma - \gamma^2)^2} d\gamma \quad (66)$$

$$I = \frac{1}{2\epsilon} \int_0^x \left(\frac{1}{x\gamma - \gamma^2 + i\epsilon} - \frac{1}{x\gamma - \gamma^2 - i\epsilon} \right) d\gamma \quad (67)$$

Completing the square in the denominator of the integrand in eq. (67)

$$I = \frac{1}{2\epsilon} \int_0^x \left(\frac{1}{(\gamma - \frac{x}{2})^2 - \gamma_1^2} - \frac{1}{(\gamma - \frac{x}{2})^2 - \gamma_2^2} \right) d\gamma \quad (68)$$

where

$$\gamma_1^2 = \frac{x^2}{4} - i\epsilon \quad (69a)$$

$$\gamma_2^2 = \frac{x^2}{4} + i\epsilon \quad (69b)$$

Separating the integrands of eq. (68) by partial fractions

$$I = \frac{1}{4\epsilon} \int_0^x \left[\frac{1}{\gamma_1} \left(\frac{1}{\gamma - \frac{x}{2} + \gamma_1} - \frac{1}{\gamma - \frac{x}{2} - \gamma_1} \right) - \frac{1}{\gamma_2} \left(\frac{1}{\gamma - \frac{x}{2} + \gamma_2} - \frac{1}{\gamma - \frac{x}{2} - \gamma_2} \right) \right] d\gamma \quad (70)$$

Integrating we obtain

$$I = \frac{1}{4\epsilon} \left(\frac{1}{\gamma_1} \ln \frac{\gamma - \frac{x}{2} + \gamma_1}{\gamma - \frac{x}{2} - \gamma_1} - \frac{1}{\gamma_2} \ln \frac{\gamma - \frac{x}{2} + \gamma_2}{\gamma - \frac{x}{2} - \gamma_2} \right) \Bigg|_0^x \quad (71)$$

To eliminate the imaginary (we set

$$\gamma_i = \alpha + i\beta \quad (72)$$

Solving for α and β from eq. (69a) we find

$$\alpha = -\frac{\epsilon}{2\beta} \quad (73)$$

$$\beta = \sqrt{\frac{1}{2} \sqrt{\frac{x^4}{16} + \epsilon^2} - \frac{x^2}{8}} \quad (74)$$

From eq. (69b) then it is easily seen that

$$\delta_d = -\alpha + i\beta \quad (75)$$

Substituting eqs. (72) and (75) into eq. (71) we obtain

$$I = \frac{1}{4\epsilon(\alpha^2 + \beta^2)} \left[(\alpha - i\beta) \ln \frac{\gamma - \frac{x}{2} + \alpha + i\beta}{\gamma - \frac{x}{2} - \alpha - i\beta} + (\alpha + i\beta) \ln \frac{\gamma - \frac{x}{2} - \alpha + i\beta}{\gamma - \frac{x}{2} + \alpha - i\beta} \right] \Big|_0^x \quad (76)$$

$$= \frac{1}{4\epsilon(\alpha^2 + \beta^2)} \left[\alpha \ln \left(\frac{\gamma - \frac{x}{2} + \alpha + i\beta}{\gamma - \frac{x}{2} - \alpha - i\beta} \cdot \frac{\gamma - \frac{x}{2} - \alpha + i\beta}{\gamma - \frac{x}{2} + \alpha - i\beta} \right) + i\beta \ln \frac{(\gamma - \frac{x}{2} - \alpha)^2 + \beta^2}{(\gamma - \frac{x}{2} + \alpha)^2 + \beta^2} \right] \Big|_0^x \quad (77)$$

Writing

$$\ln \frac{(\gamma - \frac{x}{2} + \alpha) + i\beta}{(\gamma - \frac{x}{2} + \alpha) - i\beta} = 2i \operatorname{arctg} \frac{\beta}{\gamma - \frac{x}{2} + \alpha} \quad (78)$$

$$\ln \frac{(\gamma - \frac{x}{2} - \alpha) + i\beta}{(\gamma - \frac{x}{2} - \alpha) - i\beta} = 2i \operatorname{arctg} \frac{\beta}{\gamma - \frac{x}{2} - \alpha} \quad (79)$$

and evaluating between the limits from 0 to x , eq. (77) becomes

$$I = \frac{1}{4\epsilon(\alpha^2 + \beta^2)} \left[4\alpha \left(\operatorname{arctg} \frac{\beta}{\frac{x}{2} + \alpha} + \operatorname{arctg} \frac{\beta}{\frac{x}{2} - \alpha} \right) + 2\beta \ln \frac{(\frac{x}{2} - \alpha)^2 + \beta^2}{(\frac{x}{2} + \alpha)^2 + \beta^2} \right] \quad (80)$$

We can now substitute for x , β and ϵ from eqs. (73), (74) and (65) respectively into eq. (80). Defining

$$\Delta = \frac{x}{k L^2} = \frac{1}{4} D \quad (81)$$

where D is given by eq. (45), we find, after very cumbersome algebraic manipulations, an expression for I_a

$$I_a = \frac{\sqrt{\pi} \langle \mu^2 \rangle k^2 L^5}{8 x} \left(\frac{(\sqrt{1 + \frac{1}{\Delta^2}} - 1) \sqrt{2\sqrt{1 + \frac{1}{\Delta^2}} - 2}}{(1 + \frac{1}{\Delta^2} - \sqrt{1 + \frac{1}{\Delta^2}}) \frac{1}{\Delta}} \ln \frac{1 + \Delta \sqrt{2\sqrt{1 + \frac{1}{\Delta^2}} - 2}}{1 - \Delta \sqrt{2\sqrt{1 + \frac{1}{\Delta^2}} - 2}} \right. \\ \left. - \frac{2 \sqrt{2\sqrt{1 + \frac{1}{\Delta^2}} - 2}}{(1 + \frac{1}{\Delta^2} - \sqrt{1 + \frac{1}{\Delta^2}}) \left[\operatorname{arctg} \frac{\sqrt{1 + \frac{1}{\Delta^2}} - 1}{\sqrt{2\sqrt{1 + \frac{1}{\Delta^2}} - 2} - \frac{1}{\Delta}} + \operatorname{arctg} \frac{\sqrt{1 + \frac{1}{\Delta^2}} - 1}{\sqrt{2\sqrt{1 + \frac{1}{\Delta^2}} - 2} + \frac{1}{\Delta}} \right]} \right) \quad (82)$$

Substituting the expressions for I_a , eq. (82) and I_1 , eq. (62) into eqs. (56) and (57) we find the expressions for the phase and amplitude fluctuations of a spherical wave which are valid $\forall \Delta$ (and hence $\forall D$).

In the case $\Delta \gg 1$, $I_1 \gg I_a$ and therefore eqs. (56) and (57) reduce to

$$\langle \delta_s^2 \rangle = \frac{\sqrt{\pi}}{2} \langle \mu^2 \rangle k^2 x L$$

$$\langle [|\ln(1 + a_s)|]^2 \rangle = \frac{\sqrt{\pi}}{2} \langle \mu^2 \rangle k^2 x L$$

which agrees with results quoted in section 2 - 3a in the case of the amplitude fluctuations. The phase fluctuations are seen to agree with the plane wave case (refer to eqs. (21) and (22)).

In the case $\Delta \ll 1$ we can use the identity

$$\operatorname{arctg} x + \operatorname{arctg} y = \operatorname{arctg} \frac{x + y}{1 + xy}, \quad xy < 1$$

to write

$$\operatorname{arctg} \frac{\sqrt{1 + \frac{1}{\Delta^2}} - 1}{\sqrt{2\sqrt{1 + \frac{1}{\Delta^2}} - 2} - \frac{1}{\Delta}} + \operatorname{arctg} \frac{\sqrt{1 + \frac{1}{\Delta^2}} - 1}{\sqrt{2\sqrt{1 + \frac{1}{\Delta^2}} - 2} + \frac{1}{\Delta}} = \operatorname{arctg} x,$$

where x_1 reduces to

$$x_1 = \frac{(\sqrt{1 + \frac{1}{\Delta^2}} - 1)}{(2\sqrt{1 + \frac{1}{\Delta^2}} - 2) - \frac{1}{\Delta^2}}$$

Then expanding $\text{arctg } x_1$, and $\ln \frac{1+x_1}{1-x_1}$

where $x_1 = 4\sqrt{1 + \frac{1}{\Delta^2}} - 2$, in power series the phase and amplitude fluctuations reduce to

$$\begin{aligned} \langle \delta_s^2 \rangle &= \sqrt{\pi} \langle \mu^2 \rangle k^2 \propto L \\ \langle [\ln(1+q_s)]^2 \rangle &= \frac{4\sqrt{\pi}}{15} \langle \mu^2 \rangle \frac{x^3}{L^3} \end{aligned}$$

Then, the phase fluctuations of a spherical wave also agree with the plane wave case and the amplitude fluctuations again agree with the results quoted in section 2 - 3a.

Figure 1 compares qualitatively the phase and amplitude fluctuations of both plane and spherical waves as a function of D .

b) Calculation of the Fluctuations in the Index of Refraction

It is possible to obtain an expression for $\langle \mu^2 \rangle$ as a function of measurable meteorological parameters.

We remember from eqs. (5) and (19) that the index of refraction is by definition

$$n = 1 + \mu = \frac{c_0}{c}$$

If we write

$$c = c_0 + \delta c \quad (83)$$

Then

$$1 + \mu = \frac{c_0}{c_0 + \delta c} = \frac{1}{1 + \frac{\delta c}{c_0}} \quad (83a)$$

Since $\frac{\delta c}{c_0} \ll 1$ we can expand the right hand side of (83a)

$$1 + \mu \sim 1 - \frac{\delta c}{c_0} \quad (84)$$

Therefore squaring and averaging we obtain

$$\langle \mu^2 \rangle = \frac{\langle \delta c^2 \rangle}{c_0^2} \quad (85)$$

where of course we assume

$$\langle \mu \rangle = \langle \delta c \rangle = 0$$

The velocity of propagation c as discussed in section 1 - 2 is a function of the absolute temperature of the air and of the component of wind velocity in the direction of propagation

$$c = c_0 \sqrt{\frac{T}{T_0}} + \hat{n} \cdot \vec{V} \quad (86)$$

where T and $\vec{V}$ are respectively the absolute temperature in $^{\circ}\text{K}$ and the wind velocity. The unit vector $\hat{n}$ is in the direction of propagation and T_0 is the constant reference temperature at which the velocity of propagation has the constant value of c_0 . Since $c_0 \sqrt{\frac{T}{T_0}} \gg \hat{n} \cdot \vec{V}$ this last c_0 is essentially the same as the one appearing in eqs. (5), (20) and (83).

Let

$$\hat{n} \cdot \vec{V} = |\hat{n}| |\vec{V}| \cos \theta = V \cos \theta = V \quad (87)$$

Then

$$\delta c = \frac{\partial c}{\partial V} \delta V + \frac{\partial c}{\partial T} \delta T \quad (88)$$

Calculating the quantities in eq. (88) we find from eqs. (86) and (87)

$$\frac{\partial c}{\partial V} = 1 \quad (89)$$

$$\frac{\partial c}{\partial T} = \frac{1}{2} c_0 \frac{1}{\sqrt{T_0 T}} = \frac{1}{2} \frac{c_0}{T} \sqrt{\frac{T}{T_0}} \quad (90)$$

and assuming that θ is constant

$$\delta V = \delta v \cos \theta \quad (91)$$

Substituting eqs. (89), (90) and (91) in eq. (88)

$$\delta c = \delta v \cos \theta + \frac{1}{2} c_0 \sqrt{\frac{T}{T_0}} \frac{\delta T}{T} \quad (92)$$

Squaring

$$\delta c^2 = \delta v^2 \cos^2 \theta + \frac{c_0^2}{T} \sqrt{\frac{T}{T_0}} \cos \theta \delta v \delta T + \frac{1}{4} \frac{c_0^2}{T_0} \frac{\delta T^2}{T} \quad (93)$$

Averaging, (again assuming θ constant)

$$\langle \delta c^2 \rangle = \langle \delta v^2 \rangle \cos^2 \theta + \frac{c_0^2 \cos \theta}{\sqrt{T_0}} \left\langle \frac{\sqrt{T}}{T} \delta v \delta T \right\rangle + \frac{1}{4} \frac{c_0^2}{T_0} \left\langle \frac{\delta T^2}{T} \right\rangle \quad (94)$$

We can now define the reference temperature T_0 by writing

$$T = T_0 + \delta T \quad (95)$$

Then

$$\begin{aligned}
 \left\langle \frac{\delta T^2}{T} \right\rangle &= \left\langle \frac{\delta T^2}{T_0 + \delta T} \right\rangle = \frac{1}{T_0} \left\langle \frac{\delta T^2}{1 + \frac{\delta T}{T_0}} \right\rangle \\
 &\sim \frac{1}{T_0} \left\langle \delta T^2 \left(1 - \frac{\delta T}{T_0} \right) \right\rangle \\
 &\sim \frac{1}{T_0} \left\langle \delta T^2 \right\rangle
 \end{aligned} \tag{96a}$$

Similarly

$$\left\langle \frac{\sqrt{T}}{T} \delta v \delta T \right\rangle \sim \frac{1}{\sqrt{T_0}} \left\langle \delta v \delta T \right\rangle \tag{96b}$$

Substituting eqs. (96a) and (96b) into eq. (94) we obtain

$$\langle \delta c^2 \rangle = \langle \delta v^2 \rangle \cos^2 \theta + \frac{c_0}{T_0} \cos \theta \langle \delta v \delta T \rangle + \frac{1}{4} \left(\frac{c_0}{T_0} \right)^2 \langle \delta T^2 \rangle \tag{97}$$

It is possible to arrive at this result by a more direct method than eq. (88) which involves differentials. In eq. (86) we immediately write

$$T = T_0 + \delta T$$

$$V = V_0 + \delta V$$

and it becomes

$$c = c_0 \sqrt{1 + \frac{\delta T}{T_0}} + \cos \theta V_0 \left(1 + \frac{\delta V}{V_0} \right) \tag{98}$$

Since $\frac{\delta T}{T} \ll 1$

$$\sqrt{1 + \frac{\delta T}{T_0}} \sim 1 + \frac{1}{2} \frac{\delta T}{T_0} \tag{99}$$

eq. (98) becomes

$$c = c_0 \left(1 + \frac{1}{2} \frac{\delta T}{T_0} \right) + v_0 \cos \theta \left(1 + \frac{\delta v}{v_0} \right) \quad (100)$$

Separating the fluctuating part in eq. (100) we find

$$\delta c = \frac{1}{2} c_0 \frac{\delta T}{T_0} + \cos \theta \delta v \quad (101)$$

Squaring and averaging we find

$$\langle \delta c^2 \rangle = \langle \delta v^2 \rangle \cos^2 \theta + \frac{c_0^2}{T_0^2} \cos^2 \theta \langle \delta v \delta T \rangle + \frac{1}{4} \left(\frac{c_0}{T_0} \right)^2 \langle \delta T^2 \rangle$$

which agrees exactly with eq. (97).

Since

$$\langle \delta v \rangle = \langle \delta T \rangle = 0$$

we can write

$$\langle \delta v^2 \rangle = \sigma_v^2 \quad \text{and} \quad \langle \delta T^2 \rangle = \sigma_T^2 \quad (102)$$

We also define

$$R = \frac{\langle \delta v \delta T \rangle}{\sqrt{\langle \delta v^2 \rangle \langle \delta T^2 \rangle}} \quad (103a)$$

which is the correlation coefficient of wind and temperature fluctuations.

Therefore

$$\langle \delta v \delta T \rangle = \sigma_v \sigma_T R \quad (103b)$$

Substituting eqs. (102) and (103b) into eq. (97) and from eq. (84) we find

$$\langle H^2 \rangle = \left(\frac{\sigma_v \cos \theta}{c_o} \right)^2 + \frac{\sigma_v \sigma_r R \cos \theta}{T_o c_o} + \left(\frac{1}{2} \frac{\sigma_r}{T_o} \right)^2 \quad (104)$$

which gives the fluctuations in the index of refraction as a function of measurable meteorological variables.

c) Extension of Analysis by Ingard and Maling of the Sound Field Above a Boundary to Include Partial Correlation

We now return to eq. (9) which expresses the sound field above a boundary in the presence of turbulence

$$\bar{p}^* = \frac{1}{x^2} \left[1 + 2a_d + a_d^2 + \left(\frac{x}{r} \right)^2 \left(1 + 2a_r + a_r^2 \right) + 2 \frac{x}{r} (1 + a_r + a_d + a_d a_r) \cos(\varphi + \delta_r - \delta_d) \right]$$

We remember that when calculating $\langle \bar{p}^* \rangle$, Ingard and Maling assumed all correlation between direct and reflected waves to be zero. It will be shown later that a fit with experimental data is improved if at least partial correlation is assumed.

From eqs. (43) and (44), eqs. (56) and (57) and figure 1, we see that the phase fluctuations are more important than the amplitude fluctuations. The measured spectra of turbulence, to be shown in chapter 3, reveal a pre-dominance in low frequency fluctuations and it has been shown¹² that these are the most effective in producing phase fluctuations and least effective in producing amplitude fluctuations. Hence, in calculating $\langle \bar{p}^* \rangle$, we now include cross correlation between the phases of the direct and reflected waves, ie, $\langle \delta_d \delta_r \rangle \neq 0$, while still neglecting other covariance terms.

We therefore rewrite eq. (12) in the following way

$$\langle \bar{p}^2 \rangle = \frac{1}{x^2} \left[1 + \langle a^2 \rangle + \left(\frac{x}{r} \right)^2 (1 + \langle a^2 \rangle) + 2 \frac{x}{r} \langle \cos(\varphi + \delta_r - \delta_d) \rangle \right] \quad (105)$$

We again assume a to be normally distributed with standard deviation $(\langle a^2 \rangle)^{1/2}$. But this time we assume δ_r and δ_d to be normally distributed with standard deviation σ_r and σ_d respectively and we define the cross-correlation coefficient between the phases of the direct and reflected waves

$$\rho = \frac{\langle \delta_r \delta_d \rangle}{\sigma_r \sigma_d} \quad (106)$$

Therefore

$$\langle \cos(\varphi + \delta_r - \delta_d) \rangle = \frac{1}{2\pi \sigma_r \sigma_d \sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cos(\varphi + \delta_r - \delta_d) \exp \left\{ -\frac{1}{2(1-\rho^2)} \left[\frac{\delta_r^2}{\sigma_r^2} - \frac{2\rho \delta_r \delta_d}{\sigma_r \sigma_d} + \frac{\delta_d^2}{\sigma_d^2} \right] \right\} d\delta_r d\delta_d \quad (107)$$

We write

$$\begin{aligned} \cos(\varphi + \delta_r - \delta_d) &= \cos \varphi (\cos \delta_r \cos \delta_d + \sin \delta_r \sin \delta_d) \\ &\quad - \sin \varphi (\sin \delta_r \cos \delta_d - \cos \delta_r \sin \delta_d) \end{aligned}$$

and substitute into eq. (107) to evaluate the integrals. Eq. (107) then becomes

$$\begin{aligned} \langle \cos(\varphi + \delta_r - \delta_d) \rangle &= \frac{1}{2\pi \sigma_r \sigma_d \sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\cos \varphi (\cos \delta_r \cos \delta_d + \sin \delta_r \sin \delta_d) \right. \\ &\quad \left. - \sin \varphi (\sin \delta_r \cos \delta_d - \cos \delta_r \sin \delta_d) \right] \exp \left\{ -\frac{1}{2(1-\rho^2)} \left[\frac{\delta_r^2}{\sigma_r^2} - \frac{2\rho \delta_r \delta_d}{\sigma_r \sigma_d} + \frac{\delta_d^2}{\sigma_d^2} \right] \right\} d\delta_r d\delta_d \end{aligned}$$

We proceed by evaluating the integrals for each of the four terms in (108) by writing

$$\langle \cos(\varphi + \delta_r - \delta_d) \rangle = \frac{1}{2\pi\sigma_r\sigma_d\sqrt{1-\rho^2}} \left[\cos\varphi(I_1 + I_2) - \sin\varphi(I_3 - I_4) \right] \quad (109)$$

where I_1 , I_2 , I_3 and I_4 are defined and evaluated as follows. First

$$I_1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cos\delta_r \cos\delta_d e^{-\frac{1}{2} \frac{1}{1-\rho^2} \left(\frac{\delta_r^2}{\sigma_r^2} - \frac{2\rho\delta_r\delta_d}{\sigma_r\sigma_d} + \frac{\delta_d^2}{\sigma_d^2} \right)} d\delta_r d\delta_d$$

The first integral in I_1 is evaluated in the following way

$$\begin{aligned} \int_{-\infty}^{\infty} \cos\delta_r e^{-\frac{1}{2} \frac{1}{1-\rho^2} \left(\frac{\delta_r^2}{\sigma_r^2} - \frac{2\rho\delta_r\delta_d}{\sigma_r\sigma_d} \right)} d\delta_r \\ = \int_{-\infty}^{\infty} \cos\delta_r e^{-\frac{1}{2} \frac{1}{1-\rho^2} \left(\frac{\delta_r^2}{\sigma_r^2} - \frac{2\rho\delta_r\delta_d}{\sigma_r\sigma_d} \right)} d\delta_r \\ + \int_{-\infty}^{\infty} \cos\delta_r e^{-\frac{1}{2} \frac{1}{1-\rho^2} \left(\frac{\delta_r^2}{\sigma_r^2} - \frac{2\rho\delta_r\delta_d}{\sigma_r\sigma_d} \right)} d\delta_r \end{aligned}$$

In the first integral of the right hand side of this expression we set

$$\delta_r = -\Gamma$$

so that the limits of integration can be changed from $-\infty$ to ∞ . After a few manipulations we obtain

$$\begin{aligned} \int_{-\infty}^{\infty} \cos\delta_r e^{-\frac{1}{2} \frac{1}{1-\rho^2} \left(\frac{\delta_r^2}{\sigma_r^2} - \frac{2\rho\delta_r\delta_d}{\sigma_r\sigma_d} \right)} d\delta_r &= \int_{-\infty}^{\infty} \cos\delta_r e^{-\frac{1}{2} \frac{1}{1-\rho^2} \frac{\delta_r^2}{\sigma_r^2}} \left(e^{\frac{\rho\delta_r\delta_d}{\sigma_r\sigma_d}} + e^{-\frac{\rho\delta_r\delta_d}{\sigma_r\sigma_d}} \right) d\delta_r \\ &= 2 \int_{-\infty}^{\infty} \cos\delta_r \cosh\left(\frac{\rho\delta_r\delta_d}{\sigma_r\sigma_d}\right) e^{-\frac{1}{2} \frac{1}{1-\rho^2} \frac{\delta_r^2}{\sigma_r^2}} d\delta_r \quad (110) \end{aligned}$$

The integral on the right hand side of eq. (110) is now in a standard form which can be found in tables⁹.

$$2 \int_0^{\infty} \cos \delta_r \cosh \left(\frac{1}{1-\rho^2} \frac{\rho \delta_d}{\sigma_r \sigma_d} \delta_r \right) \exp \left(-\frac{1}{2} \frac{1}{1-\rho^2} \frac{\delta_r^2}{\sigma_r^2} \right) d\delta_r$$

$$= \sigma_r \sqrt{2\pi(1-\rho^2)} \exp \left(\frac{1}{2} \frac{\rho^2 \delta_d^2}{1-\rho^2 \sigma_d^2} \right) \exp \left(-\frac{\sigma_d^2 (1-\rho^2)}{2} \right) \cos \left(\frac{\rho \delta_d \sigma_r}{\sigma_d} \right) \quad (111)$$

Substituting eq. (111) into the expression for I_1 ,

$$I_1 = \sigma_r \sqrt{2\pi(1-\rho^2)} e^{-\frac{\sigma_d^2 (1-\rho^2)}{2}} \int_0^{\infty} \cos \delta_d \cos \left(\frac{\rho \sigma_r}{\sigma_d} \delta_d \right) e^{\frac{1}{2} \frac{\rho^2 \delta_d^2}{1-\rho^2 \sigma_d^2}} e^{-\frac{1}{2} \frac{1}{1-\rho^2} \frac{\delta_d^2}{\sigma_d^2}} d\delta_d$$

$$= \sigma_r \sqrt{2\pi(1-\rho^2)} e^{-\frac{\sigma_d^2 (1-\rho^2)}{2}} \int_0^{\infty} \cos \delta_d \cos \left(\frac{\rho \sigma_r}{\sigma_d} \delta_d \right) \exp \left(-\frac{1}{2} \frac{\delta_d^2}{\sigma_d^2} \right) d\delta_d \quad (112)$$

The integral in eq. (112) is again a standard form found in tables⁹. Since the integrand is even

$$2 \int_0^{\infty} \cos \delta_d \cos \left(\frac{\rho \sigma_r}{\sigma_d} \delta_d \right) \exp \left(-\frac{1}{2} \frac{\delta_d^2}{\sigma_d^2} \right) d\delta_d$$

$$= \frac{\sigma_d}{2} \sqrt{2\pi} \left[e^{-\frac{\sigma_d^2}{2} \left(\frac{\rho \sigma_r}{\sigma_d} - 1 \right)^2} + e^{-\frac{\sigma_d^2}{2} \left(\frac{\rho \sigma_r}{\sigma_d} + 1 \right)^2} \right]$$

Therefore

$$I_1 = \frac{\sigma_r \sigma_d}{2} \sqrt{2\pi(1-\rho^2)} e^{-\frac{\sigma_d^2 (1-\rho^2)}{2}} \left[e^{-\frac{\sigma_d^2}{2} \left(\frac{\rho \sigma_r}{\sigma_d} - 1 \right)^2} + e^{-\frac{\sigma_d^2}{2} \left(\frac{\rho \sigma_r}{\sigma_d} + 1 \right)^2} \right] \quad (113)$$

In a similar way we calculate I_2 , where

$$I_2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sin \delta_r \sin \delta_d e^{-\frac{1}{2} \frac{1}{1-\rho^2} \left(\frac{\delta_r^2}{\sigma_r^2} - \frac{2\rho \delta_r \delta_d}{\sigma_r \sigma_d} + \frac{\delta_d^2}{\sigma_d^2} \right)} d\delta_r d\delta_d$$

The first integral is written

$$\begin{aligned}
 & \int_{-\infty}^{\infty} \sin \delta_r e^{-\frac{1}{2} \frac{1}{(1-p^2)} \left(\frac{\delta_r^2}{\sigma_r^2} - \frac{2 p \delta_r \delta_d}{\sigma_r \sigma_d} \right)} d \delta_r \\
 &= 2 \int_0^{\infty} \sin \delta_r \sinh \left(\frac{1}{1-p^2} \frac{p \delta_d}{\sigma_r \sigma_d} \delta_r \right) \exp \left(-\frac{1}{2} \frac{1}{1-p^2} \frac{\delta_r^2}{\sigma_r^2} \right) d \delta_r \\
 &= \sigma_r \sqrt{2\pi(1-p^2)} e^{-\frac{\delta_d^2}{2} (1-p^2)} \sin \left(\frac{p \sigma_r}{\sigma_d} \delta_d \right) e^{\frac{1}{2} \frac{p^2}{1-p^2} \frac{\delta_d^2}{\sigma_d^2}} \quad (114)
 \end{aligned}$$

Substituting eq. (114) into I_1

$$I_2 = \sigma_r \sqrt{2\pi(1-p^2)} e^{-\frac{\delta_d^2}{2} (1-p^2)} \int_{-\infty}^{\infty} \sin \delta_d \sin \left(\frac{p \sigma_r}{\sigma_d} \delta_d \right) e^{-\frac{1}{2} \frac{\delta_d^2}{\sigma_d^2}} d \delta_d \quad (115)$$

Evaluating the integral in eq. (115) we find

$$I_2 = \frac{\sigma_r \sigma_d}{2} 2\pi \sqrt{1-p^2} e^{-\frac{\delta_d^2}{2} (1-p^2)} \left[e^{-\frac{\delta_d^2}{2} \left(\frac{p \sigma_r}{\sigma_d} - 1 \right)^2} - e^{-\frac{\delta_d^2}{2} \left(\frac{p \sigma_r}{\sigma_d} + 1 \right)^2} \right] \quad (116)$$

Now we quickly find I_3

$$I_3 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sin \delta_r \cos \delta_d e^{-\frac{1}{2} \frac{1}{1-p^2} \left(\frac{\delta_r^2}{\sigma_r^2} - \frac{2 p \delta_r \delta_d}{\sigma_r \sigma_d} + \frac{\delta_d^2}{\sigma_d^2} \right)} d \delta_r d \delta_d$$

In a now familiar way we perform the integration with respect to δ_r and find

$$I_3 = \sigma_r \sqrt{2\pi(1-p^2)} e^{-\frac{\delta_d^2}{2} (1-p^2)} \int_{-\infty}^{\infty} \cos \delta_d \sin \left(\frac{p \sigma_r}{\sigma_d} \delta_d \right) e^{-\frac{1}{2} \frac{\delta_d^2}{\sigma_d^2}} d \delta_d \quad (117)$$

We immediately see that the integrand in eq. (117) is odd, therefore

$$I_3 = 0 \quad (118)$$

Similarly for I_4 , where

$$I_4 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cos \delta_r \sin \delta_d e^{-\frac{1}{2}(1-P^2) \left(\frac{\delta_r^2}{\sigma_r^2} - \frac{2P\delta_r\delta_d}{\sigma_r\sigma_d} + \frac{\delta_d^2}{\sigma_d^2} \right)} d\delta_r d\delta_d$$

We find after first integration

$$I_4 = \sigma_r \sqrt{2\pi(1-P^2)} e^{-\frac{\sigma_d^2}{2}(1-P^2)} \int_{-\infty}^{\infty} \sin \delta_d \cos \left(\frac{P\sigma_r}{\sigma_d} \delta_d \right) e^{-\frac{1}{2} \frac{\delta_d^2}{\sigma_d^2}} d\delta_d \quad (119)$$

where again the integrand in eq. (119) is odd and

$$I_4 = 0 \quad (120)$$

Finally by substituting the results for I_1 , I_2 , I_3 and I_4 into eq. (109) we find

$$\begin{aligned} \langle \cos(\varphi + \delta_r - \delta_d) \rangle &= \frac{1}{2\pi\sigma_r\sigma_d\sqrt{1-P^2}} \left[\cos \varphi \frac{\sigma_r\sigma_d}{2} 2\pi\sqrt{1-P^2} e^{-\frac{\sigma_d^2}{2}(1-P^2)} \left(e^{-\frac{\sigma_d^2}{2} \left(\frac{P\sigma_r}{\sigma_d} - 1 \right)^2} \right. \right. \\ &\quad \left. \left. + e^{-\frac{\sigma_d^2}{2} \left(\frac{P\sigma_r}{\sigma_d} + 1 \right)^2} + e^{-\frac{\sigma_d^2}{2} \left(\frac{P\sigma_r}{\sigma_d} - 1 \right)^2} - e^{-\frac{\sigma_d^2}{2} \left(\frac{P\sigma_r}{\sigma_d} + 1 \right)^2} \right) \right] \\ &= \cos \varphi e^{-\frac{\sigma_d^2}{2}(1-P^2)} e^{-\frac{\sigma_d^2}{2} \left(\frac{P\sigma_r}{\sigma_d} - 1 \right)^2} \end{aligned} \quad (121)$$

If we assume $\sigma_r = \sigma_d$ eq. (121) reduces to the simple form

$$\langle \cos(\varphi + \delta_r - \delta_d) \rangle = \cos \varphi e^{-\sigma_d^2(1-P)} \quad (122)$$

Substituting eq. (122) into eq. (105) we find

$$\langle \bar{p}^2 \rangle = \frac{1}{x^2} \left[1 + \langle a^2 \rangle + \left(\frac{x}{r} \right)^2 (1 + \langle a^2 \rangle) + \frac{2x}{r} \cos \varphi e^{-\sigma_a^2(1-\rho)} \right] \quad (123)$$

Finally eq. (123) can be written as

$$\langle \bar{p}^2 \rangle = \frac{2}{xr} \left[\frac{\langle a^2 \rangle}{2} \left(\frac{r}{x} + \frac{x}{r} \right) + \frac{r}{2x} \left(1 - \frac{x}{r} \right)^2 + 1 + \cos \varphi e^{-\sigma_a^2(1-\rho)} \right] \quad (124)$$

where we note that eq. (124) differs from eq. (18), derived by Løgard and Maling, only by the presence of the term $(1-\rho)$ in the exponent. The addition of the partial correlation effectively reduces, as expected, the effect of the phase fluctuations, particularly in the region of destructive interference. Finally eq. (124) reduces, as it should, to eq. (18) for $\rho = 0$.

The variance of the phase fluctuations, σ_a^2 , is calculated from eq. (56) for spherical waves.

The variance of the amplitude fluctuations, $\langle a^2 \rangle$ is obtained from eq. (57). However it is necessary to determine $\langle a^2 \rangle$ from $\langle [\ln(1+a)]^2 \rangle$. This is achieved by expanding the natural logarithm and writing the higher-order moments in terms of second-order moments

$$\ln(1+a) \sim a - \frac{1}{2} a^2 + \frac{1}{3} a^3 - \dots$$

$$[\ln(1+a)]^2 \sim a^2 - a^3 + \frac{11}{12} a^4 - \dots$$

Then averaging

$$\langle [\ln(1+a)]^2 \rangle \sim \langle a^2 \rangle + \frac{11}{12} \langle a^4 \rangle \quad (125)$$

Assuming a to be normally distributed with standard deviation $(\langle a^2 \rangle)^{\frac{1}{2}}$ we find after a simple integration

$$\langle a^4 \rangle = \frac{1}{\sqrt{2\pi\langle a^2 \rangle}} \int_{-\infty}^{\infty} a^4 e^{-\frac{a^2}{2\langle a^2 \rangle}} da = 3\langle a^2 \rangle^2 \quad (126)$$

Substituting eq. (126) into eq. (125)

$$\langle [\ln(1+a)]^2 \rangle \sim \langle a^2 \rangle \left(1 + \frac{33}{12} \langle a^2 \rangle \right)$$

Therefore

$$\langle a^2 \rangle \sim \frac{\langle [\ln(1+a)]^2 \rangle}{\left(1 + \frac{33}{12} \langle a^2 \rangle \right)} \quad (127)$$

If

$$\frac{33}{12} \langle a^2 \rangle < 1 \quad \text{or} \quad \langle a^2 \rangle < 0.4 \quad (128)$$

then

$$\frac{1}{1 + \frac{33}{12} \langle a^2 \rangle} \sim 1 - \frac{33}{12} \langle a^2 \rangle$$

and eq. (127) becomes

$$\langle a^2 \rangle \sim \langle [\ln(1+a)]^2 \rangle \left(1 - \frac{33}{12} \langle a^2 \rangle \right) \quad (129)$$

From which

$$\langle a^2 \rangle \sim \frac{\langle [\ln(1+a)]^2 \rangle}{1 + \frac{33}{12} \langle [\ln(1+a)]^2 \rangle} \quad (130)$$

when, from eq. (128)

$$\langle [\ln(1+a)]^2 \rangle < 0.7$$

The cross-correlation coefficient ρ is obtained from eq. (55) for spherical waves. If we compare ρ_p for plane waves with ρ_s for spherical waves as a function of ℓ/L we find as expected that the phases of spherical waves are more strongly correlated than those of plane waves (See figure 2). However the result for ρ_s holds $\forall D$. Since D is a function of x we would expect ρ_s to tend toward ρ_p for large x . This is not, however, evident in Karavainikov's result and care will be taken in using eq. (55).

PART II

EXPERIMENTS

Chapter 3

EXPERIMENTS

3 - 1 Previous Measurements and Analysis

Ingard and Maling⁷ obtained field data to compare with their theoretical analysis. A pure tone of sound at frequencies 500, 1000 or 2000 Hz was transmitted between a source and a receiver, both 4 ft. above a plane boundary. The received signal was recorded on magnetic tape for a length of time at a fixed source/receiver separation, x , which was discretely varied from 20 ft. to 150 ft.

The measurements were later analysed with the aid of a level recorder from which the mean and variance of the sound levels were evaluated.

This analysis was then compared with eq. (18) where, as previously mentioned, the quantities σ^2 and $\langle a^2 \rangle$ were obtained from eqs. (21) and (22) in which $\langle u^2 \rangle L$, not having been measured, was used as a parameter adjusted for best fit. A sample of the results obtained by Ingard and Maling is shown in figure 3.

Experimental data and the calculated rms sound-pressure level for a small source over a plane boundary ($f = 1000 \text{ cps}$).

Figure 3 Typical results found by Ingard and Maling. The dashed curve is conventional coherent theory while the solid curve is a fit from their theory. They also show similar results at 500 and 2000 Hz.

3 - 2 Measurements during the Summer 1976

Improved and more complete measurements, similar to those made by Ingard and Maling, were carried out in the field during the summer of 1976, on a disused airport runway (Rockcliffe Airfield, Ottawa).

The validity of eq. (4) requires a plane rigid boundary such that its reflection coefficient approaches unity. Previous measurements of the propagation of sound over asphalt¹¹ indicated it behaves as a perfect reflector, its acoustic impedance in fact being too high to measure with standard techniques. Because of this and the abundance of large and relatively flat surfaces covered with asphalt, such a surface was chosen over which to do the measurements. A point sound source was placed at a height of 4 ft. above this boundary while two receivers were placed variously at heights of 4 ft. and 2 ft. and at horizontal separations of 50 ft., 100 ft. and 150 ft. For convenience the frequency was the parameter varied over a series of discrete values between 1 and 6 kHz in order to outline the shape of the interference spectrum and then fixed, for a length of time, in order to localise the interference minima and maxima.

Simultaneously the wind velocity and temperature as well as their fluctuations were monitored to obtain L and in order to calculate $\langle \mu^2 \rangle$.

High quality magnetic tape recordings were made of the acoustical signal and of the meteorological variables for later analysis in the laboratory.

The sketch of figure c1 in Appendix C illustrates the general configuration of the experiment.

3 - 3 Field Equipment

The equipment used for the field measurements can be blocked into 3 groups, the sound source, the sound receivers and the weather station (See figure 4).

The whole system is mobile and battery powered. All the equipment shown in figure 4, except the oscillator and single channel chart recorder, is DC powered by either a car battery or appropriate dry cells. The oscillator and recorder require AC voltage and are therefore powered from a Topaz static inverter connected to a 12v heavy duty truck battery. The photographs of figure c2, c3, and c4 illustrate some of the equipment.

Figure 4 The field equipment showing the sound source (a), the sound receivers (b) and the weather station (c).

a) The Sound Source

The sound generating system consists of a Brüel and Kjaer, (B & K), beat frequency oscillator type 1014 whose frequency was continuously monitored by a Racal digital frequency meter (SA-520). On occasion a simpler battery powered Wavetek oscillator, model 30, was used. The signal was applied through a Bogen (BT-35A) power amplifier to a University horn driver feeding into one end of an ebonite pipe 1 in. internal diameter and 30 in. long. The other end, which was the point source, is fitted with felt and wire mesh to damp the resonances of the pipe. It was found in an anechoic chamber to obey classic theory for radiation from a pipe and be sufficiently omnidirectional for present purposes (output within 1 dB through a solid angle of 90 degrees) up to a frequency of 5 kHz.

A General Radio piezoelectric microphone-preamplifier unit (type 1560-P40), with its 4 in. diameter foam wind shield, was placed 1 foot from the end of the pipe. The electrical signal was fed through a variable gain amplifier (B & K sound level meter type 2203) to drive a compressor circuit in the oscillator. This compressor circuit is a feedback device designed to operate at constant voltage at the compressor input, hence at constant sound pressure one foot from the source (to within 1 dB from 1 kHz to 5 kHz). The flatness of the source can be checked by recording the sound pressure level of the compressor microphone on a B & K level recorder type 2305 as a function of frequency. The level recorder is mechanically coupled to the oscillator to provide a calibrated frequency sweep on the horizontal scale of the chart. The output of the sound level meter is fed into the B & K level recorder where the vertical scale of the chart is graduated in decibels.

Finally, the level of the compressor microphone can be used as a reference with which to normalise the levels at distances x , to a unit amplitude sound field at one foot. A General Radio calibrator (type 1562-A) is used for calibration and when coupled to the microphone produces a sound level of 114 dB in an enclosed cavity of known volume at either 125, 250, 500, 1000 or 2000 Hz. The sound level meter can then be adjusted to produce the correct reading (and correct corresponding output voltage) using the A-weighted network. The A-weighting is used to suppress the low frequency response and hence unwanted low frequency background sound.

Figure c5 in Appendix C illustrates most of the sound source equipment.

b) The Sound Receivers

The receiving system consisted of two similar sets of microphone, power supply and amplifier as in section 3 - 3a, plus a magnetic tape recorder.

These microphones and sound level meters are also calibrated in the same way. The transmission filter of the medium and boundary for each microphone position is obtained by means of a frequency sweep which is recorded on the B & K level recorder. A typical filter is illustrated in figure 5.

With the frequency of the minima and maxima established for each microphone position, fixed frequencies are set and maintained for periods of about 3 minutes.

The signal received by both microphones was recorded on two channels of a magnetic tape recorder. Two different tape recorders were used in the course of the measurements. The first is a 4 channel B & K "FM" tape recorder, type 7003 with frequency range of 0 Hz to 10 kHz at 15 ips. Agfa Professional PEM 368 magnetic tape, or the equivalent Scotch brand, was used with this recorder. The alternative tape recorder was a Nagra IV-SJ on which the acoustic data could be recorded on the two AM channels which have a frequency response which is flat to within 1 dB from 30 Hz to 10 kHz at 3 $\frac{1}{2}$ ips. Scotch AV 177 magnetic tape was used for these recordings.

Figure 5 A typical transmission filter. This is a record of the interference spectrum on a given site for a particular source and receiver position.

c) The Weather Station

The weather station consists of two Wallac thermoanemometers type GGA 23F which are capable of measuring either air temperature or wind velocity, but not both simultaneously. The sensitive element is the variable resistance component of a Wheatstone bridge circuit operating as a resistance thermometer or as a hot-wire anemometer.

They are mounted above the ground near the propagation path on a structure, designed to minimize perturbations in the air flow, at heights comparable with those of the acoustic sound source and receivers (See figure 6 in Appendix C). Their horizontal or vertical separation can be discretely varied and they are free to rotate through $\sim 180^\circ$ about a vertical axis. To follow the sometimes varying wind direction, they are also fitted with vanes.

The signal of each thermoanemometer was passed through a dual channel, fixed gain, amplifier whose outputs were connected to the two channels of a Brush 222 pen recorder. The response time of this measuring system was approximately 0.03 sec, and thus sufficiently fast to record essentially the full spectrum of the turbulence.

A tap was found in the Brush recorder circuit where the signal from each thermoanemometer has been amplified to a level and dynamic range compatible with the B & K tape recorder. This signal was then recorded continuously for about 9 minutes, simultaneously with acoustic data, at 15 ips. on the remaining two channels. Occasionally, in order to save tape, the meteorological data was recorded at 1.5 ips. (frequency range of 0 Hz to 1 kHz) while the acoustic data was recorded on the Nagra. Synchronizing pips were recorded on the two tape recorders at the beginning of each run when the latter procedure was used.

3 - 4 Methods for Analysing Field Measurements

The data recorded in the field, both acoustical and especially the meteorological, requires extensive statistical analysis in order to be compatible

with the theory.

a) Sound Pressure Level

The three minute recordings of the fluctuating sound levels need to be averaged in time to yield their mean (and standard deviation). The diagram of figure 6 illustrates the equipment used.

The recorded signal is first played through an amplifier-filter system to eliminate the various background noises encountered in the field. This system consists of a B & K measuring amplifier, type 2608, electrically coupled to a B & K heterodyne slave filter, type 2020, coupled in turn to a B & K beat frequency oscillator, type 1024. The bandwidth of the filter was set at 100 Hz to accommodate the fluctuating signal and its center frequency was tuned to that of the signal on the tape with the oscillator. The frequencies were monitored with the Racal frequency meter.

Figure 6 Instruments used to analyse the sound pressure levels in order to obtain their mean and standard deviation.

The output of the amplifier was then passed through a fixed attenuator to a General Radio sound level meter, type 1933, in-order to obtain a DC level proportional, in decibels, to the level of the AC at the input. The attenuator simply matches the dynamic ranges of both instruments.

The fluctuating DC output of the sound level meter is then fed to a Saicor correlation and probability analyser, model Sai-43A. In the probability distribution mode the analyser, which contains 400 bins, yields the mean and standard deviation of the input signal to a resolution of 0.1 dB.

The recorded sound levels are averaged for ~3 minutes. The output of the analyser, which is the cumulative distribution function, can be displayed on an oscilloscope and can also be read digitally, bin by bin, with a digital voltmeter. The bins are calibrated in dB with the 114 dB calibration signal recorded in the field.

b) Wind Velocity and Temperature

1) Mean and Standard Deviation : The wind velocity and temperature recordings are also analysed to yield their mean value and standard deviation in a similar way as described in section 3 - 4a. The differences being that filtering is not necessary and since the recordings are already a fluctuating DC level, there is no need to rectify the signal.

The simplified diagram is illustrated in figure 7.

Figure 7 Instruments used to analyse the mean and standard deviation of the wind velocity and temperature.

u) Auto and Cross-Correlation Functions : The fluctuations in the wind velocity and temperature must be further analysed to obtain their covariance function, from which the scale of the wind velocity fluctuations, L_v , and temperature fluctuations, L_T , can be found. The relationship between L_v and L_T with L , the scale of the fluctuations in the index of refraction, is shown in Appendix A.

The covariance functions are found with the Saicor analyser in the correlation mode and by essentially two means: by obtaining the auto-correlation function in time of the fluctuating signal of one probe, or the cross-correlation function in space and time of the fluctuating signals of two probes separated by a vertical distance ξ .

I- In the auto-correlation mode, the analyser obtains the auto-correlation functions

$$\langle \delta T(r, t_1) \delta T(r, t_2) \rangle = \int \delta T(r, t) \delta T(r, t + \tau) dt \quad (133 a)$$

in the case of the temperature fluctuations, and

$$\langle \delta V(r, t_1) \delta V(r, t_2) \rangle = \int \delta V(r, t) \delta V(r, t + \tau) dt \quad (133 b)$$

in the case of the wind velocity fluctuations and where $t_2 = t_1 + \tau$.

Eq. (133a) for example is computed by taking the product $\delta T(r, t) \delta T(r, t + \tau)$ for 400 values of τ , at preselected intervals $d\tau$, and then summing the products. As the signal from tape is fed to the input of the analyser, the correlation function is continually updated for a selected number of summations.

The block diagram is shown in figure 8.

Figure 8 Instruments used to obtain the correlation functions of the wind velocity and temperature fluctuations.

Since only the fluctuating part of recordings is of interest, a capacitor is introduced between the tape recorder output and analyser input to provide adequate AC coupling.

The plot obtained on the X-Y recorder (Hewlett Packard, 7035-B) is assumed to approximate a Gaussian of the form

$$\exp \left(- \frac{r^2}{t_T^2} \right)$$

in the case of the temperature fluctuations, and

$$\exp \left(- \frac{r^2}{t_v^2} \right)$$

in the case of wind velocity fluctuations where, respectively, t_T and t_v are the characteristic times of the fluctuations. These are related to the corresponding scale of turbulence by using the "Frozen Turbulence" hypothesis which essentially assumes that the fluctuations are swept past the probes by the wind velocity.

Then, using the average wind velocity

$$L_T = \langle V \rangle t_T \quad (134a)$$

$$L_V = \langle V \rangle t_V \quad (134b)$$

Figures 9a and 9b illustrate typical auto-correlation functions in the case of temperature and wind velocity respectively (See also Appendix B).

II- In the cross-correlation mode, the recorded signals of the fluctuating temperature (or wind velocity) from two probes separated by a vertical distance ξ are each fed to an input of the analyser (the same AC coupling is used, see figure 8). The two inputs are cross correlated in an operation similar to that previously discussed for eqs. (133a, b). The equivalent equation is, in the case of temperature

$$\langle \delta T(\tau_1, t) \delta T(\tau_2, t) \rangle = \int \delta T(\tau_1, t) \delta T(\tau_2, t + \tau) dt \quad (135)$$

where $\xi = \xi_1 - \xi_2$

Eq. (135) for a given ξ is normalized to the corresponding auto correlation function at $\tau = 0$ to yield the spatial cross-correlation coefficient

$$N_T = \frac{\langle \delta T(\tau_1) \delta T(\tau_2) \rangle}{\sqrt{\langle \delta T(\tau_1)^2 \rangle} \sqrt{\langle \delta T(\tau_2)^2 \rangle}} \bigg|_{\tau=0} = \exp\left(-\frac{\tau^2}{L_T^2}\right) \quad (136a)$$

Similarly, in the case of the wind velocity fluctuations

$$N_V = \frac{\langle \delta V(\tau_1) \delta V(\tau_2) \rangle}{\sqrt{\langle \delta V(\tau_1)^2 \rangle} \sqrt{\langle \delta V(\tau_2)^2 \rangle}} \bigg|_{\tau=0} = \exp\left(-\frac{\tau^2}{L_V^2}\right) \quad (136b)$$

Figure 9c illustrates a typical cross-correlation function. (Note that scales of figures 9a, b and 9c are not the same and that these are adjusted for the normalization of eq. (136). See also Appendix B.)

Finally it is noted here that the correlation coefficient R , between the temperature and wind velocity fluctuations is obtained in this way (see section 2 - 4b, eq. (103a)).

$$R = \frac{\langle \delta T(\tau) \delta V(\tau) \rangle}{\sqrt{\langle \delta T(\tau)^2 \rangle} \sqrt{\langle \delta V(\tau)^2 \rangle}} \bigg|_{\tau=0}$$

Figure 9a Typical auto-correlation function of the temperature fluctuations as a function of delay time τ .

Figure 9b Typical auto-correlation function of the wind velocity fluctuations as a function of delay time τ .

Figure 9c Typical cross-correlation function. In this case, the top probe is delayed with respect to the bottom probe.

iii) Spectra : In order to have more confidence in the correlation functions obtained from the wind velocity and temperature fluctuations, the spectra of the fluctuations were obtained. These were obtained (see figure 10) with a Federal Scientific Ubiquitous Spectrum Analyser UA-6B which essentially transforms a signal in the time domain to the frequency domain. The signal is continuously updated and then averaged by a Federal Scientific Spectrum Averager Option 129 B. The spectra are plotted with Hewlett-Packard X-Y recorder.

Figure 11 illustrates a typical wind velocity and temperature spectrum drawn on the same graph.

Figure 10 Instruments used to obtain the spectra of wind velocity and temperature fluctuations.

3 - 5 Discussion of Errors

To avoid confusion with fluctuating parameters a discussion of errors has, up till now, been intentionally omitted. Before presenting the data, however it would be desirable to discuss these.

Besides the frequency response of the equipment, of importance is their stability, repeatability of measurement and in the case of analysing equipment, confidence in the results.

As mentioned previously the acoustical equipment is calibrated in the field and similarly for the meteorological equipment. Prescribed calibration checks are performed on the filters and analysers used in the analysis - for this a simple oscillator and oscilloscope can be useful. Then, for example, the signal from the compressor microphone recorded on tape can be analysed to verify that the analysis reproduces the correct frequency. Furthermore it is found that it fluctuates by no more than 0.5 dB, verifying at once the stability of the oscillator and sound source as well as the noise in the receiver, tape recorder, amplifiers and filter and in effect the analyser itself. Similar confidence can be gained with the meteorological analysis (and see section 3 - 3a).

Since the following chapters continue to discuss fluctuating quantities, errors will again be omitted to avoid future confusion. However this discussion should be kept in mind.

Chapter 4

EXPERIMENTAL DATA

4 - 1 Mean and Standard Deviation of Sound Pressure Levels

Acoustic data was obtained, as described in sections 3 - 2 and 3 - 3, on three separate days. These were then analysed to obtain the mean, $\langle \bar{p}^2 \rangle$, and standard deviation, $\sigma_{\bar{p}}$, of the sound levels as described in section 3 - 4a (note again, averaging time ~ 3 minutes).

The results are presented in tables I, II and III (October 1, 4 and 5, respectively) where the symbols h_s and h_r denote respectively the source height and receiver height.

The frequencies are listed in the order in which the measurements were made.

TABLE I

Receiver 1			Receiver 2	
$x = 50$ ft.			$x = 50$ ft.	
$h_s = h_r = 4$ ft.			$h_s = 4$ ft. $h_r = 2$ ft.	
Frequency	$\langle \bar{p} \rangle$	$\sigma_{\bar{p}}$	$\langle \bar{p} \rangle$	$\sigma_{\bar{p}}$
Hz	dB		dB	
	re level at 1 ft.		re level at 1 ft.	
3000	-35.4	2.6	-32.5	1.6
2906	-36.8	3.6	-29.6	1.3
2800	-40.6	3.5	-34.4	1.8
1807	-29.0	0.4	-43.1	3.5
2243	-29.3	0.9	-45.0	4.3
2485	-36.4	2.8	-38.5	3.0
2705	-43.3	4.5	-34.2	1.7
2617	-42.6	4.7	-33.7	1.5
2100	-29.2	0.6	-48.4	4.2
3501	-28.9	0.9	-29.8	1.1
1514	-29.3	0.7	-37.6	1.4
3008	-36.3	3.7	-33.0	1.4
2811	-45.7	6.2	-34.4	2.0
2896	-38.9	4.3	-30.0	1.3
4499	-36.3	4.2	-31.4	1.2
4996	-34.2	3.5	-32.0	1.7
3994	-30.1	0.8	-27.2	1.0

TABLE II

Receiver 1

 $x = 50$ ft. $h_s = h_r = 4$ ft.

Receiver 2

 $x = 100$ ft. $h_s = h_r = 4$ ft.

Frequency	$\langle \bar{p} \rangle$	σ_{dB}	$\langle \bar{p} \rangle$	σ_{dB}
Hz	dB		dB	
	re level at 1 ft.		re level at 1 ft.	
2600	-39.1	2.9	-31.3	.07
1800	-28.8	0.4	-37.4	2.0
2400	-41.4	3.7	-32.1	0.7
1500	-27.9	0.3	-41.0	3.0
3000	-36.7	0.7	-34.6	1.4
3500	-31.0	0.4	-33.7	0.7
2600	-36.2	1.9	-31.4	0.6
1500	-	-	-47.7	3.6
2000	-30.9	0.6	-38.1	0.9
2300	-40.3	2.8	-	-

TABLE III

Receiver 1

x = 50 ft.

 $h_s = h_r = 4$ ft.

Receiver 2

x = 150 ft.

 $h_s = h_r = 4$ ft.

Frequency Hz	$\langle \bar{p} \rangle$ dB re level at 1 ft.	σ_{dB}	$\langle \bar{p} \rangle$ dB re level at 1 ft.	σ_{dB}
3000	-38.6	2.5	-38.3	2.3
2800	-38.8	3.3	-39.0	1.6
2600	-36.2	2.5	-40.0	1.7
2000	-27.2	0.6	-38.7	1.7
1800	-28.0	0.8	-38.2	1.3
3500	-34.1	2.5	-39.3	1.7
4000	-30.8	0.8	-38.7	2.8
1200	-36.7	1.5	-38.6	0.7
1500	-30.2	0.7	-36.8	1.0
2400	-31.7	0.9	-39.3	1.6
3250	-37.3	2.6	-35.0	2.3
3750	-31.3	2.1	-39.4	2.9
4250	-33.2	-	-37.7	2.5
5000	-37.9	2.4	-37.0	2.3
2800	-43.1	5.2	-40.3	1.6
2200	-31.0	0.6	-41.2	1.5
4500	-45.0	5.0	-51.1	4.5
4250	-34.2	3.3	-44.7	5.0
4750	-41.2	3.2	-45.4	3.5
2600	-40.8	2.3	-42.4	1.8
1800	-28.1	0.3	-40.0	0.9

4 - 2 Wind Velocity and Temperature

These variables were analysed as described in section 3 - 4b. The data obtained simultaneously with acoustic measurements is listed in tables Ia, IIa and IIIa, where $\langle V \rangle$, σ_V and $\langle T \rangle$, σ_T are respectively the mean and standard deviation of wind velocity and temperature and h_p is the height of the probe above the ground. As mentioned in chapter 3, sound levels were recorded for three minutes while the meteorological measurements were recorded for nine minutes. It was therefore possible to obtain recordings of 3 different frequencies in the time of one uninterrupted meteorological recording. The frequencies of tables I, II, and III are therefore listed again to maintain the chronological information of the measurements.

The two probes both measured alternatively wind velocity and temperature in order to obtain the cross-correlation functions. Two values of $\langle u^2 \rangle$ were calculated from eq.(104), one for each probe height, using the alternating measurements. The coefficient R from eq.(103a) was obtained once for each day by measuring wind velocity with one probe and temperature with the other. Table VII lists the relevant values of R as well as $\bar{\theta}$, the average angle between the wind direction and propagation path. Finally T_p was taken as $\langle T \rangle$ for each height and the corresponding velocity of propagation, c_p , was calculated.

TABLE VII

	R	$\bar{\theta}$
Table Ia	-0.13	13°
Table IIa	-0.13	00°
Table IIIa	-0.00	135°

For reasons discussed in Appendix B, I_v and I_T were obtained from the corresponding auto-correlation function by finding the τ value at which the peak value dropped to $e^{-1/4}$. In order to complete the information on I_v and I_T Tables IVa, Va and VIa (from June 20, Aug 4 and Aug 12 respectively) list further meteorological analysis.

Tables Ib, IIb, IIIb, IVb, Vb, and VIb list the cross-correlation coefficients of the wind velocity and temperature fluctuations as defined in eq. (136a) and (136b). They are listed as a function of vertical probe separation, z .

Finally Table Vc lists a few cross-correlation coefficients as a function of horizontal probe separations, r , with both probes 4 ft. above the ground.

As a final remark an effort was made to cover a variety of meteorological conditions from bright sunny days (Aug 4, Oct 1) and a hazy warm day with high winds (Aug 12), to an overcast day (Oct 5), and a calm evening (Oct 4). A quick scan of these tables will then illustrate the typical variation of weather encountered during an Ottawa summer.

TABLE Ia

Frequency Hz	$\langle V \rangle$ ft/sec	σ_V	$\langle T \rangle$ °F	σ_T	h_p ft	$\langle M^2 \rangle$ $\times 10^{-6}$	L_v ft	L_T ft
3000 2906 2800	7.4	1.7	56.8	1.5	4	8.0	2.7	
1807 2243 2485			61.4 59.4	2.2 2.3	2 4	16.4		- -
2705	4.7	1.9			2	15.6	1.5	
2617	4.1	1.6			4		1.5	
2100 3501 1514	8.2 8.6	2.1 1.8			.5 4	8.1	1.6 2.4	
3008 2811 2896			67.4 61.4	2.2 1.5	.5 4	15.7		3.3 4.1
4499 4996 3994	8.7	1.9			2 4	6.7		4.4 1.7
	6.1 5.5	1.6 1.5			3 4		2.0 2.0	
			64.1 63.1 65.0 62.9	1.6 1.6 1.6 1.2	3 4 1 4			2.6 2.3 2.7 2.1
	6.7 5.7 6.5 5.2	1.5 1.6 1.2 1.2			1 4 4 6 4 6		1.6 1.6 1.8 1.2	
			64.6 60.4	0.9 1.0	4 6			4.2 2.9

TABLE IIa

Frequency Hz	$\langle V \rangle$ ft/sec	σ_v	$\langle T \rangle$ °F	σ_T	h_p ft	$\langle M^2 \rangle$ $\times 10^{-6}$	L_v ft	L_T ft
2600	3.3	0.8			2		1.3	
1800								
2400	3.7	1.0			4	-	1.7	
1500			59.8	1.9	2	10.8		-
3000								
3500			57.4	-	4			-
2600								
1500	4.7	0.4	57.9	0.4	4	0.5	1.7	-
2000								
2300	-	-	-	-				

TABLE IIIa

Frequency Hz	$\langle V \rangle$ ft/sec	σ_v	$\langle T \rangle$ °F	σ_T	h_p ft	$\langle M^2 \rangle$ $\times 10^{-6}$	L_v ft	L_T ft
3000								
2800	13.8	2.9	70.6	0.8	4	6.5	4.1	5.8
2600								
2000			71.6	1.1	2			3.5
1800								
3500			67.5	0.8	4	5.2		2.3
4000	9.7	2.1			2	5.5	3.7	
1200								
1500	11.6	2.9			4		4.4	
2400	9.1	2.1			.5		3.5	
3250								
3750	14.2	3.9			4	7.7	4.3	
4250			70.9	0.8	.5	3.6		-
5000								
2800			67.5	0.8	4			-
2200	7.9	1.6			1		3.0	
4500								
4250	9.6	2.0			4	3.1	2.8	
4750			69.1	0.6	1			-
2600								
1800			66.6	0.7	4	2.0		-

TABLE IVa

$\langle V \rangle$	σ_v	$\langle T \rangle$	σ_T	h_p	L_v	L_T
ft/sec		$^{\circ}\text{F}$		ft	ft	
9.0	2.3			3.3	1.8	
8.4	2.6			4.9	1.7	
		71.0	1.3	3.3		2.9
		74.3	1.3	4.9		2.7
8.4	2.5			3.3	1.7	
8.8	2.3			"	1.6	
		70.3	1.2	"		2.7
		72.3	0.8	"		2.8
8.7	2.0			"	1.6	
7.9	2.2			6.6	1.6	
		70.1	1.2	3.3		3.1
		73.4	1.0	6.6		2.4
8.4	1.9			3.3	1.7	
7.2	2.1			8.9	1.4	
		71.4	1.6	3.3		4.7
		75.7	1.5	8.9		3.6
3.9	1.7			3.3	1.2	
3.8	2.1			"	1.4	
		75.3	2.0	"		-
		77.8	2.9	"		-

TABLE Va

$\langle V \rangle$	σ_v	$\langle T \rangle$	σ_T	h_p	L_v	L_T
ft/sec		$^{\circ}\text{F}$		ft	ft	
		82.5	2.5	0.5		5.8
		77.4	1.5	4.0		5.1
		82.5	2.1	2.0		5.1
		78.9	1.7	4.0		4.2
		82.3	2.4	3.0		4.2
		79.3	2.3	2.5		4.2
		81.8	2.0	4.0		5.1
		81.0	2.3	"		5.1
		82.5	1.7	"		4.5
		80.2	2.0	"		4.5
16.5	3.2			"	4.3	
16.7	4.2			"	3.7	
		82.8	1.6	"		3.6
		80.3	1.7	"		3.7
13.8	3.2			"	4.4	
11.6	3.2			"	3.2	
		83.2	1.8	"		3.9
		80.7	1.9	"		2.8

TABLE VIa

$\langle V \rangle$	σ_v	$\langle T \rangle$	σ_T	h_p	L_v	L_T
ft/sec		$^{\circ}\text{F}$		ft	ft	
		78.0	1.7	1.5		3.5
		75.9	1.4	4.0		3.7
10.9	2.5			1.5	2.8	
13.1	3.6			4.0	4.7	
		80.8	1.6	0.5		3.1
		76.9	1.1	4.0		4.6
10.4	2.3			0.5	1.7	
15.2	3.4			4.0	3.3	
		79.9	1.1	3.0		4.4
		78.5	1.2	4.0		5.7
14.7	2.3			3.0	3.2	
18.9	3.7			4.0	5.3	
		79.4	1.1	3.75		8.8
		78.0	1.2	4.0		9.9
15.2	2.1			3.75	3.0	
19.0	3.5			4.0	3.8	

	N_v	N_T	r ft
TABLE Ib	.63	.81	1.0
	.38	-	2.0
	.33	.54	"
	.28	.42	3.0
	.20	.42	3.5

TABLE IIb	.55	-	2.0
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TABLE IIIb	.56	.55	2.0
	.37	-	3.0
	.28	-	3.5

TABLE IVb	.79	1.05	0.0
	.40	.62	1.6
	.23	-	"
	.23	.34	3.3
	.16	.19	5.6

TABLE Vb		.87	0.5
		.64	2.0
		.29	3.5

TABLE VIb	.95	.95	.25
	.81	.91	1.0
	.48	.52	2.5
	.30	.28	3.5

TABLE Vc

$h_s = 4 \text{ ft.}$		
N_v	N_T	r ft
	.87	1.5
	.77	3.0
.48	.52	5.0
.44	.44	8.0

4 - 3 Discussion of the Data

It can be seen from the tables in the preceding section, with a few exceptions, that $\langle \mu^2 \rangle$ does not vary significantly throughout any particular day. Keeping in mind also that these measurements are not of the temperature and component of wind velocity along the propagation path, but are measured at a single location, the values of $\langle \mu^2 \rangle$ can be averaged for each day.

Similar arguments can be made for L_V and L_T . Therefore Table VIII lists, for each day, the average values, $\langle \mu^2 \rangle$, $\bar{L}_V$ and $\bar{L}_T$ of all the values of $\langle \mu^2 \rangle$, L_V and L_T respectively. Certain extreme but briefly occurring values of $\langle \mu^2 \rangle$ are quoted in brackets. In addition the average velocity of propagation $\bar{c}$, calculated from each $\langle T \rangle$, is also included.

TABLE VIII

	$\langle \mu^2 \rangle$ $\times 10^{-6}$	$\bar{L}_V$ ft	$\bar{L}_T$ ft	$\bar{c}$ ft/sec
Table I a	7.7 (16)	1.8	3.7	1120.2
Table II a	10.8 (0.5)	1.6	-	1116.6
Table III a	4.8	3.9	-	1127.5
Table IV a		1.6	2.8	
Table V a		3.9	4.5	
Table VI a		3.5	4.2	

From Table VIII it can be seen that $\bar{L}_V < \bar{L}_T$. This is confirmed by plotting the cross-correlation coefficients of Table VII as a function of r . This is done in Figures 12 and 13 from which it is found that $\bar{L}_V \sim 2.8$ ft. and $\bar{L}_T \sim 3.1$ ft. In fact this difference is expected from measured spectra of fluctuations (see figure 11) which shows that the wind velocity fluctuations contain higher frequency components than the temperature fluctuations. The data of Table VI is not plotted because of the scarcity of measurements.

To fit the acoustic data, however a value of L must be found from the $\bar{L}_V$ and $\bar{L}_T$ values. In calculating $\langle \mu^2 \rangle$ it is found that the term due to the temperature fluctuations is the most important, and therefore from Appendix A, L is more heavily weighted towards L_T than L_V . Furthermore from the discussion of section 2 - 4 c, L_T should be more representative of the fluctuations producing phase fluctuations than L_V .

This argument leads to the problems of the scarcity of values for $\bar{L}_T$ in Table VIII. Because of this scarcity an overall average of the $\bar{L}_T$ values will be used to fit the acoustic data which is

$$L \sim 3.5 \text{ ft.}$$

PART III

COMPARISON of THEORY

and

EXPERIMENTS

5 - 1 Comparison of Results with Coherent Theory and the Incoherent Theory of Ingard and Maling.

Shown in figure 13 is a comparison of some of the results found in Table I with conventional coherent acoustic theory and also the incoherent theory of Ingard and Maling.

It is obvious that both theories fail to predict the sound levels in the interference minima. The results are similar for the other data of tables I, II, and III.

For illustrative purposes the standard deviation σ_{δ} of the sound levels are plotted in figure 14. This further justifies abandoning coherent theory especially in the region of the interference minima (see section 2 - 2). These results are typical¹ and Ingard and Maling do show that the predicted σ_{δ} is larger in the interference minimum than in the interference maximum. No effort is made here to improve the prediction of σ_{δ} .

Figure 13 A fit to some of the results of Table I using the conventional coherent theory and the incoherent theory of Ingard and Maling.

Figure 14 Standard deviations, σ_{dB} , of sound levels found in Tables I, II and III.

5 - 2 Comparison of Results with Incoherent Theory considering Spherical Waves.

The use of a point source is inconsistent with a theory that considers the phase and amplitude fluctuations of plane waves. As seen in section 2 - 3 and 2 - 4 a and figure 1, plane wave theory overestimates the amplitude fluctuations of spherical waves in the region $D \sim 1$ and $D \ll 1$. However the phase fluctuations are relatively unchanged and since the phase fluctuations are most important, especially in the presence of a boundary (see section 2-2), a better fit in figure 13 should not be expected with spherical wave theory.

Furthermore Ingard and Maling have used eqs. (21) and (22) to estimate the phase and amplitude fluctuations under the assumption that they are valid for deep penetration into the medium⁸ (see section 2 - 2)

$$x \gg \sqrt{\lambda L} \quad (131)$$

It has been shown in section 2 - 3 a that eqs. (21) and 20) are valid for $D \gg 1$. Since D varies from values of .2 to 10 in Ingard and Maling's measurements (see figure 3) the two conditions are inconsistent.

A close look at Morse and Ingard⁸ reveals that the condition of eq. (131) comes by finding a critical distance, x_c , at which the r.m.s. amplitude fluctuations change from a 3/2 power dependence in depth of penetration x , (eq. (47)), to a 1/2 power law, (eq. (44)). This critical distance is written as

$$x_c \approx \frac{4L\lambda}{\sqrt{2\pi}} \approx \sqrt{L\lambda} \quad (132)$$

Equating eq. (44) and (47) in order to verify eq. (132) the critical distance is found to be

$$x'_c \approx \frac{L^2}{\lambda} \quad (132 a)$$

This is consistent with Tatarski's condition $\sqrt{L\lambda}/L \sim 1$ and Chernov's wave

parameter (see eq. (54)) for justifying the use of the various equations of section 2 - 3 a for plane waves, but inconsistent with eq.(132).

Since the minimum at 2720 Hz of figure 13 has a value for the wave parameter of $D = 1$, the use of eq. (21) and (22) underestimates the phases fluctuations while it overestimates the amplitude fluctuations (and even more so for spherical waves).

Figure 15 is a fit to the data of Table I using eqs. (56) and (57) to estimate the phase and amplitude fluctuations of the spherical waves produced by the point source. Interestingly enough the fit isn't any better than with the theory of Ingard and Maling.

Figure 15 A fit to the results for $x = 50$ ft, $h_s = h_r = 4$ ft, of Table I using incoherent theory considering spherical waves.

5 - 3 Comparison of Results considering Spherical Waves with Partial Correlation.

The discussion of section 5 - 2 shows the need to include partial correlation between the direct and reflected waves. This deduction is further supported by finding that L is of the order of 3.5 ft, i.e., of the order of the path separation of the two waves.

The sound levels in Tables I, II and III are compared with those predicted by eq. (124). The phase and amplitude fluctuations are estimated with eqs. (56) and (57) using $L = 3.5$ ft and for $\langle \mu^2 \rangle$, the corresponding values listed in Table VIII. The cross-correlation coefficient ρ is estimated from eq. (55). This equation gives the correlation between the phase fluctuations of a spherical wave arriving at two receivers separated by a distance ℓ during propagation well away from a boundary. In the present case ρ is the correlation between the phase fluctuations of the direct and reflected waves arriving at one receiver. Hence ℓ must be chosen empirically. It is reasonable to assume that ℓ has a value somewhere between the mean path separation and the maximum path separation.

In the case $h_r = h_x = 4$ ft, ℓ should have a value between 2 and 4 ft while in the case $h_s = 4$ ft, $h_r = 2$ ft, it should lie between 1.5 and 3 ft.

The results are presented in figures 16a, 16b, 16c, 16d, 16e, and 16f. All points labeled anomalous are so for any of several specific reasons that include experimental artifacts or other propagation effects such as refraction, which enter from time to time into the measurements. Adjustments are made for the site to site variation in the position of the interference minima caused by the ground not being perfectly planar (This variation being greatest at the longer propagation distance $x = 150$ ft).

Figure 16a: A reasonable fit is obtained for $\ell = 2$ ft. the points at which $\langle \mu^2 \rangle = 16 \times 10^{-6}$ are sufficiently away from the interference minima such that no difference is seen between these and the points for which $\langle \mu^2 \rangle = 7.7 \times 10^{-6}$.

Attention is drawn to the two points (Δ) at 2617 Hz and 2705 Hz. The uncertainty in $\langle \mu^2 \rangle$ arises from the particular sequence in which the meteorological measurements were made (see Table Ia). Their position and level seem to indicate a value of $\langle \mu^2 \rangle$ more in the vicinity of $\langle \mu^2 \rangle = 7.7 \times 10^{-6}$ than $\langle \mu^2 \rangle = 16 \times 10^{-6}$.

Figure 16b: We note that there is a reasonable fit for $\ell = 1.5$ ft (chosen because $h_r = 2$ ft). It is difficult to speculate an ℓ smaller than this because of the scatter in the points above 2.5 kHz. No difference is seen between the points for which $\langle \mu^2 \rangle = 7.7 \times 10^{-6}$ and $\langle \mu^2 \rangle = 16 \times 10^{-6}$ because the positions of the latter points are again sufficiently removed from the interference minimum.

Figure 16c: Again $\ell = 2$ ft is necessary to obtain a good fit.

Figure 16d: The main problem here is the failure of the data points to follow simple geometric spreading at the interference maximum. The anomaly in most of the data points is further complicated by failure to properly localize the interference minimum. The dashed curve is the best compromise found for any plausible frequency of minimum. We do however draw attention to the two points at 1500 Hz. The wind velocity and temperature fluctuations were measured significantly lower in Table IIa for the bottom point than for the upper point, thus yielding a smaller $\langle \mu^2 \rangle$ for the former point.

Figure 16e: A reasonable fit is obtained for $\ell = 2$ ft except for the points between 3.3 kHz and 4.3 kHz which lie below the theoretical curve. No explanation is offered besides the propagation anomalies discussed earlier.

Figure 16f: The lack of points above 5 kHz and the scatter in the measured points make the interference minimum difficult to localize. However the standard deviation of the sound levels is normally greatest in the region of the minimum and hence figure 14 indicates that the minimum is probably the region of 4 kHz. The scatter of the points, especially in this probable region of the

minimum, can only be explained by again speculating other propagation phenomena which are expected to become more important at these longer distances.

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Figure 16a A fit to the results for $x = 50$ ft, $h_e = h_v = 4$ ft, of Table I using incoherent theory considering spherical waves with partial correlation.

Figure 16b A fit to the results for $x = 50 \text{ ft}$, $h_s = 4 \text{ ft}$ and $h_w = 2 \text{ ft}$, of Table I using incoherent theory considering spherical waves with partial correlation.

Figure 16c A fit to the results for $x=50 \text{ ft}$, $h_x=h_z=4 \text{ ft}$, of Table II using incoherent theory considering spherical waves with partial correlation.

Figure 16d A fit to the results for $x = 100 \text{ ft}$, $h_0 = h = 4 \text{ ft}$, of Table II using incoherent theory considering spherical waves with partial correlation.

Figure 16e A fit to the results for $x = 50 \text{ ft}$, $h_s = 4 \text{ ft}$, of Table III using incoherent theory considering spherical waves with partial correlation.

Figure 16f A fit to the results for $x = 150 \text{ ft}$, $h_1 = h_2 = 4 \text{ ft}$, of Table III using incoherent theory considering spherical waves with partial correlation.

CONCLUSION

The theory to predict the field of acoustic waves produced by a point source and propagating above a boundary through atmospheric turbulence has been extended to include the phase and amplitude fluctuations of spherical waves.

Since we find experimentally that the correlation length (~ 3.5 ft) of the measured meteorological fluctuations is comparable to, or greater than the separation between the interfering sound paths for the particular configurations studied, the theory has been further extended to allow for a known amount of partial correlation between direct and reflected waves.

The phase and amplitude fluctuations calculated from the measured fluctuating meteorological variables yield quantitative agreement between the calculated sound levels and those measured experimentally.

The theory of phase and amplitude fluctuations of plane and spherical waves discussed in chapter 2 assumes that the fluctuations in the index of refraction have a Gaussian correlation function of the form.

$$\langle \mu_1 \mu_2 \rangle = \langle \mu^2 \rangle e^{-r/L} \quad (A1)$$

The parameter L must be estimated from the meteorological measurements.

These however measure the fluctuations in the wind velocity and temperature and the analysis gives the corresponding correlation functions. The relationship between these functions and eq. (A1) is not obvious.

From section 2 - 4 b, the fluctuations in the index of refraction are found from eq. (84)

$$\mu \sim \frac{\delta c}{c_0} \quad (A2)$$

From eq. (101)

$$\delta c = \delta v \cos \theta + \frac{1}{2} \frac{c_0}{T_0} \delta T$$

and upon substitution into eq. (A2)

$$\mu = \frac{\delta v \cos \theta}{c_0} + \frac{1}{2} \frac{\delta T}{T_0} \quad (A3)$$

From eq. (A3) we can obtain the correlation function of the fluctuations in the index of refraction as a function of corresponding correlation functions of the fluctuations in wind velocity and temperature. Neglecting cross terms

$$\mu_1 \mu_2 \sim \frac{\cos^2 \theta}{c_0^2} \delta v_1 \delta v_2 + \frac{1}{4} \frac{1}{T_0^2} \delta T_1 \delta T_2$$

$$\langle \mu_1 \mu_2 \rangle \sim \left(\frac{\cos^2 \theta}{c_0^2} \right) \langle \delta v_1 \delta v_2 \rangle + \left(\frac{1}{2 T_0} \right)^2 \langle \delta T_1 \delta T_2 \rangle \quad (A4)$$

If we assume that

$$\langle \delta v_1 \delta v_2 \rangle = \langle \delta v^2 \rangle e^{-r/L_v}$$

$$\langle \delta T_1 \delta T_2 \rangle = \langle \delta T^2 \rangle e^{-r/L_T}$$

eq. (A4) becomes

$$\langle \mu^2 \rangle e^{-r/L} = \left(\frac{\cos^2 \theta}{c_0^2} \right) \langle \delta v^2 \rangle e^{-r/L_v} + \left(\frac{1}{2 T_0} \right)^2 \langle \delta T^2 \rangle e^{-r/L_T} \quad (A5)$$

In the case $\gamma = 0$ eq. (A5) reduces, as it should, to

$$\langle H^2 \rangle = \left(\frac{\cos \theta \sigma_v}{c_0} \right)^2 + \left(\frac{\sigma_T}{\sqrt{T_0}} \right)^2$$

which is eq. (104) with $R = 0$

From eq. (A5) if $L_T = L_V$ then

$$L = L_T = L_V$$

If $L_T \neq L_V$ then L is weighted towards L_T or L_V depending on the importance of the corresponding variance term.

APPENDIX B

All the theories treating the phase and amplitude fluctuations of sound waves assume propagation well away from any boundary. In the presence of a boundary care must be used in interpreting the measured meteorological results under the assumptions made in these theories.

In the immediate vicinity of the ground for a large flat open area, such as an airport, the wind velocity is normally reduced by the viscous drag produced by the surface, and the temperature is increased (during the daytime) by the heating of this surface by the sun. The thickness of the viscous and thermal boundary layers, within which such effects are appreciable, depends on a number of surface conditions, but is usually less than 10 m, the standard height for meteorological measurements at an airport.

Large eddies are formed in the atmosphere by instabilities in the viscous and thermal boundary layers. Further instability causes these eddies to break down progressively into smaller and smaller sizes until the energy is finally dissipated by viscosity in eddies approximately 1 mm in size. A statistical distribution of these eddies which we call turbulence is therefore present in the atmosphere at all times ^{5,14}. It should be noted that the eddies created in the breakdown process are normally isotropic, but that the primary eddies are not necessarily so. This can be seen in the shape of the auto-correlation function in figure b1 for large τ values and in the peak of figure 9.c of the cross-correlation function. For this reason the auto-correlation functions are statistically more reliable for small τ and the correlation lengths were calculated from twice the value of τ at which the auto-correlation function had fallen by a factor of $e^{-1/4}$ (see section 4 - 2).

The peak in the cross-correlation function indicates that the top probe is most strongly correlated with the bottom probe at τ slightly greater than zero. However the $\tau = 0$ value is used to compute the cross-correlation coefficient (see eqs. (136a) and (136b)) since this is the correlation that

affects the acoustic wave because of the short transit time between source and receiver.

Figure b1 An auto-correlation function showing deviation from a Gaussian at larger τ values.

Figure c1 Experimental Geometry.

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Figure c2 From top to bottom, Frequency Meter, Oscillator, Sound Level Meter and to the right the B & K level Chart Recorder.

Figure c3 From left to right, Nagra and B & K Tape Recorders and Brush Chart Recorder.

Figure c4 Overall view of same equipment in the back of the Trucks used for transportation.

Figure c5 The point source and compressor microphone.

Figure c6 The mounted Weather Probes.

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