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Investigation of a Composite Hingeless Helicopter Rotor Blade with Integral Actuators

by

Brent Bailey, B.Eng.

Carleton University

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfillment of
the requirements for the degree of

Masters of Engineering

Ottawa-Carleton Institute for
Mechanical and Aerospace Engineering

Department of
Mechanical and Aerospace Engineering
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Ottawa, Ontario

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acceptance of the thesis

**Investigation of Composite Hingeless
Helicopter Rotor Blade with Integral Actuators**

submitted by **Brent Bailey, B.Eng.**
in partial fulfillment of the requirements for
the degree of Masters of Engineering

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Abstract

Hinges are incorporated into helicopter blade design to reduce the high bending moments at the root, but result in a lag in the response time. With the advent of composite materials, there is an initiative to develop hingeless blades. Recent advances in active materials technology allow for a potential means of vibration and pitch angle control.

Helicopter rotor blade analysis is a complex subject that encompasses aerodynamics, structures and composite materials. The ability to control the coupling between deformation modes through the orientation of the fibers allows for the tailoring of helicopter blade dynamics. Piezoelectric fibers, which strain under the application of an electric field, can be embedded into the composite and allow for anisotropic actuation.

This thesis provides an aeroelastic analysis of a hingeless helicopter rotor blade with active fibers in hover. The analysis is limited to the case of constant cross-sectional properties and no cordwise offset between the elastic, mass, tension and aerodynamic axes.

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List of Symbols

A	Axial stiffness, Equation 2.29
$A_i - Z_i$	Modal integrals, Appendix I
$[A]$	Homogeneous matrix of state-space equation, Equation 3.53
$A^{(a)}$	Axial contribution of the actuator, Equation 2.29
$A_{ij}^{(a)} - E_{ij}^{(a)}$	Modal actuation integrals, Appendix I
A_{eI}, A_{eII}	Enclosed areas of the left and right cells, respectively
a_i	Variables used in the variational asymptotic method
B	Coupling stiffness, Equation 2.29
$[B]$	Control vector of state-space equation, Equation 3.53
b	Number of blades
$b(s), b_i$	Variables used in the variational asymptotic method
C	Shear stiffness, Equation 2.29
$[C_a]$	Aerodynamic damping matrix, Equation K.3
$[C_g]$	Gyroscopic damping matrix, Equation K.2
$[C_t]$	Total damping matrix
$C^{(a)}$	Shear contribution of the actuator, Equation 2.29
C_D	Blade profile drag coefficient
c	Chord length
$c(s), c_i$	Variables used in the variational asymptotic method
$D^{\alpha\beta\gamma\delta}$	Two-dimensional Hookean tensor
d	Maximum cross-sectional dimension
d_i	Variables used in the variational asymptotic method
d_{kij}	Electromechanical coupling tensor
d_{eij}	Electromechanical coupling tensor along the piezoelectric fibres
E^{ijkl}	Three-dimensional Hookean tensor
E_k	Electric field vector

E_e	Electric field vector aligned with the piezoelectric fibres
e_1	Variable used in the variational asymptotic method
F	Forcing on the blade
F_1	Axial force in the blade
$F_1^{(a)}$	Axial force due to the actuator
$f_i^{(a)}$	Variables used in the variational asymptotic method
$G(s)$	Variable used in the variational asymptotic method
$g_i, g_i^{(a)}$	Variables used in the variational asymptotic method
h	Thickness of cell wall
K_{ij}	Cross-sectional stiffness coefficients, Equation 2.48
$[K]$	Cross-sectional stiffness matrix
$[K_a]$	Aerodynamic stiffness matrix, Equation K.4
$[K_s]$	Structural stiffness matrix, Equation K.6
$[K_t]$	Total stiffness matrix
k_m	Blade cross section mass radius of gyration
k_{m_1}, k_{m_2}	Principal mass radii of gyration
L	Length of the rotor
L_u	Generalized axial aerodynamic force per unit length
L_v	Generalized lead-lag aerodynamic force per unit length
L_w	Generalized flap aerodynamic force per unit length
M_1	Torsional bending moment
M_2	Flap bending moment
M_3	Lead-lag bending moment
M_φ	Generalized torsional aerodynamic moment per unit length
$[M]$	Mass matrix, Equation K.1
m	Mass per unit length
N	Number of mode shapes
N_{11}	Axial stress resultant

N_{12}	Shear flow resultant
N_{22}	Hoop stress resultant
n	Arbitrary integer
$\mathbf{n}$	Normal vector, Equation 2.2
q_1, q_2, q_w	Shear flow in the left branch, right branch and web respectively
R	Radius of curvature
$\mathbf{r}$	Position vector, Figure 2.1
r_n	Projection of the position vector onto the normal vector
r_t	Projection of the position vector onto the tangential vector
s_{ijkl}^E	Elastic compliance tensor at constant electric field
T	Kinetic energy
$\mathbf{t}$	Tangential vector, Equation 2.2
U	Strain energy
U_1	Average cross-sectional translation in the x direction
U_2	Average cross-sectional translation in the y direction
U_3	Average cross-sectional translation in the z direction
u_x, u_y, u_z	Cartesian displacements in cross-sectional coordinate system
u	Axial Cartesian displacement in global coordinate system
$V(\psi)$	Generalized modal lead-lag deflection
v	Lead-lag Cartesian displacement in global coordinate system
v_i	Induced downwash velocity
$W(\psi)$	Generalized modal flap deflection
w	Flap Cartesian displacement in global coordinate system
w_i	Warping factors
X	Column vector of perturbation generalized modal coordinates
x, y, z	Cartesian co-ordinate system
β_{pc}	Precone angle

γ_{ij}	In-plane strain component
Γ_i	Nondimensionalized stiffnesses
Λ_{ij}	Nondimensionalized stiffnesses
ε_{ij}	Strain tensor
θ	Blade pitch angle
ρ	Density of blade
ρ_{air}	Density of air
ρ_{ij}	Change in the shell surface curvature
σ	Solidity
σ_{ij}	Stress tensor
v_ℓ, v_s, v_ξ	Curvilinear displacement components
$v_1^{(a)}(s)$	Variable used in the variational asymptotic method
Υ_i	Nondimensionalized stiffnesses
Φ	Shell energy per unit middle surface
$\phi(\psi)$	Generalized modal torsion deflection
Π_i	Nondimensionalized stiffnesses
ψ	Azimuth angle
$\Psi_i(\bar{x})$	Assumed mode shape in bending
$\Theta_i(\bar{x})$	Assumed mode shape in torsional twist
φ	Elastic torsion deflection
Ω	Rotor blade angular velocity
ℓ, s, ξ	Curvilinear co-ordinate system
N	Arbitrary integer

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Chapter 1

Introduction

Conventional helicopter rotor blades are made with lead-lag and flap hinges at the blade root that allow the blade to move freely normal to and in the plane of the rotor disk. This eliminates the high bending moments at the blade root. The inclusion of these hinges also creates a lag in the response time of the aircraft to pilot inputs, making it less manoeuvrable. Rotor blades also include a pitch hinge to change the angle of attack of the blade, thereby varying the amount of thrust produced. These hinges require more maintenance and render the helicopter less reliable. A blade is considered hingeless, or cantilevered, if it does not include the lead-lag and flap hinges. This thesis focuses on the feasibility of using new materials technology to enable a hingeless blade to actively suppress vibrations and control the pitch angle. This will hopefully lead to a more manoeuvrable aircraft with higher reliability, fewer parts and lower maintenance costs. As an initial phase of the investigation the equations of motion for the hovering flight condition are developed, including the static and dynamic actuation terms.

1.1 Thesis Overview

Chapter 1 serves as an introduction to the problems associated with hingeless helicopter rotor blades and explains the basic terminology. Different methods for vibration control are visited, as well as how blade pitch angle is changed. Background material is then presented about active materials technology, with an emphasis on piezoelectrics. Active fibers composites manufactured with piezoelectric ceramics and interdigitated electrodes are discussed, including how they can attain better performance than orthotropic piezoelectric actuators.

Chapter 2 develops the cross-sectional properties of a thin-walled, two-celled, composite rotor blade. With only minimal geometric assumptions, the stiffness and strain parameters are arrived at asymptotically. Active fibers are included in the development of the strains and resulting forces.

Chapter 3 incorporates the cross-sectional properties into a first-order nonlinear helicopter rotor blade analysis. The equations of motion for the helicopter blade are developed. Galerkin's Method is applied to these equations, after which they are divided into steady-state and perturbation problems. The resulting steady-state problem is first-order nonlinear. The perturbation problem is linearized about this steady-state. The actuation terms are also broken down into steady-state and perturbation terms, which result in static and dynamic control, respectively.

Chapter 4 investigates the cross-sectional stiffnesses for a general, two-celled box-beam. The effects of composite layup orientation on the stiffness and actuation terms are investigated for two configurations. A comparison is given between the results obtained in this work and a Finite Element solution available in the existing literature for one particular blade box-beam. The forces and moments for a constant, cross-sectional actuator are developed. The Galerkin actuation constants are then developed for the case of constant cross-sectional actuation.

Chapter 5 deals with the aeroelastic stability for a full-scale blade with a composite, box-beam spar. The parameters for a Westland WG13 helicopter are used for this analysis. The stability is examined for four test cases: a baseline composite box-beam spar, the same spar with the addition of piezoelectric fibers which

are not activated, the piezoelectric fibers activated statically, and the piezoelectric fibers controlled dynamically. The location of the actuation segment that results in a maximum pitch and vibration control is discussed. Steady-state deflections are found for each of the test cases, and the dynamic mode shapes and eigenvalues are obtained.

Chapter 6 contains some concluding remarks and recommendations for future work.

1.2 Hingeless Rotor Blade Configuration

A general description of a hingeless rotor blade will be given, followed by the simplifications used in this thesis.

Figure 1.1: Elements of a hingeless rotor blade (Hodges and Ormiston, 1976).

Figure 1.1 shows a general hingeless rotor blade structure, which is composed of an inboard and outboard blade segment. Both of these segments are flexible and are joined by a pitch change bearing. The inboard segment is rigidly attached to the hub, while the pitch hinge allows for rotation of the outboard segment. If all of the blades are pitched the same amount, then the amount of lift produced will change. If cyclic pitching is induced, then the rotor thrust can be vectored for manoeuvrability. Other configuration parameters are shown in Figure 1.2, of these

only precone will be considered in this thesis.

Figure 1.2: Configuration parameters of a hingeless rotor blade (Hodges and Ormiston, 1976).

Displacements of the rotor blade are normally considered as lag (or lead-lag), flap and pitch, as illustrated in Figure 1.3. The axial displacement is much smaller than these and is usually ignored. The rotor disk is the area swept by the blade in one rotation.

Rotor blades can be constructed of many different materials and consist of spars, stiffeners, skins and balance weights which may vary along the length of the blade. In this report, a two-celled composite rotor blade model with constant span-wise stiffness is developed, where the mass, tension and aerodynamic axes are not offset from the elastic axis.

Figure 1.3: Blade degrees of freedom (Newman, 1994).

1.3 Motivation

There are two separate motivations for integrating active materials into a helicopter rotor blade. The first is to control the vibrations in the blade itself, which would alleviate the need for other forms of vibration control. The other is to control the pitch angle of the blade, which is used for controlling the lift produced and maneuvering the aircraft. With current technologies, the former seems more plausible at this time, but the latter is included for future consideration.

1.3.1 Vibration Control

The aerodynamic forces that the blades produce enable the helicopter to fly and manoeuvre. As the aerodynamic environment changes during rotor rotation, variations in the aerodynamic loading are induced. Therefore, the main source of vibrations in helicopter blades is the result of varying aerodynamic conditions. Newman (1994) attributed these varying aerodynamic conditions to : rotational and forward velocity causing changes in the dynamic pressure, changes in pitch angle due to cyclic pitch,

flapping or lagging causing variations in inflow angle, and the interaction between the blades and the vortex wake. In hingeless helicopter blades, these aerodynamic conditions cause non-linear coupling between the flap, lead-lag and torsional motions leading to instability.

Through these aeroelastic interactions, forces and moments of higher harmonics than the rotor base speed Ω get transmitted to the fuselage. For a b bladed rotor, $b\Omega$ forcings are transmitted to the fuselage (DuPlessis, 1996).

The following is a list of current technologies being used for vibration control of helicopter rotor blades. Passive technologies are restricted by the bandwidth associated with them, and cannot control the complete vibration spectrum.

Swash Plate

The swash plate, which is normally used to change the blade pitch angle, can also be used in vibration reduction. There are problems associated with using the swash plate as it is a low frequency device incapable of providing $(b + 1)\Omega$ frequency pitch angle changes (DuPlessis, 1996). Also, the swash plate can only independently introduce certain pitch frequencies, which restricts its usefulness.

Passive Vibration Absorbers

The most common form of passive vibration absorbers is known as the dynamic vibration absorber. This consists of a mass attached to a spring, as shown in Figure 1.4. Its aim is to get the absorber to develop a forcing function that is directly out of phase with the external forcing function. This will arrest the motion of the parent structure; thus eliminating the high amplitude vibration at the external forcing frequency. A result of this is the creation of two smaller amplitude vibrations on either side of the forcing frequency. This scheme will only work if the forcing frequency is known and unchanging.

Figure 1.4: Dynamic absorber model (Newman, 1994).

Passive Vibration Isolation

The vibration isolator is located between the vibration source and the parent system. The intention is to isolate the parent system from the source of the vibration. This can be done by suspending the fuselage by springs from the rotor and transmission.

One such method is known as dynamic anti-resonant vibration isolator (DAVI), which puts a pendulum in parallel with a spring as shown in Figure 1.5. The spring and pendulum device is tuned such that they will negate each other, preventing vibrations being transmitted to the fuselage.

Another method for vibration isolation is similar, but uses a tuned beam instead of a spring and pendulum system. The fuselage is connected to the beam at the nodal points, as shown in Figure 1.6. Therefore, the vibrations are not transmitted to the fuselage.

Active Vibration Absorbers

Active vibration absorbers are now being developed, as they can cover a larger portion of the vibration spectrum. This prevents the design from being restricted to

Figure 1.5: Dynamic anti-resonant vibration isolator (Newman, 1994).

certain frequencies. This can be accomplished by using the principle of the vibration isolator, but supplying the cancellation forces via an actuator as shown in Figure 1.7. Sensors, mounted at critical points on the aircraft, act as input to an algorithm which adjusts the force produced by the actuators.

Nitzsche (1999) proposed the use of a *smart spring* incorporated at the blade root. The actuation of this *smart spring* affects the lead-lag, flap and torsional stiffness over a section of the blade. Accordingly, this can be used to control the aeroelastic stability of the rotor.

1.3.2 Pitch Control

Current helicopters use either a swash plate or dangleberry system for cyclic and non-cyclic pitch control. Along with the reliability and maintenance issues aforementioned, neither is able to vary the spanwise pitch angle. Little lift is produced near the rotor root as the speed is much less. A spanwise pitch angle control would provide a stall-area reduction mechanism for the rotor.

A layout of the swash plate is shown in Figure 1.8. Moving the swash parallel to the rotor shaft will change the collective pitch of the blades. If the swash plate is tilted with respect to the rotor shaft, then as the blades rotate, the spider will move up and down once per rotation. This cyclic pitch will change the orientation of the

Figure 1.6: Nodal suspension isolation (Newman, 1994).

rotor disk for directional control. This is illustrated in Figure 1.9.

The dangleberry system also uses a spider, but is controlled differently. It requires that the rotor shaft be hollow, as the control shaft lies within it (Figure 1.10). The spider rotates with the rotor shaft and is connected by a bearing to the control rod. If the control rod is moved parallel to the rotor shaft, non-cyclic pitching will occur. If the control shaft is moved sideways, cyclic pitching will be induced.

1.4 Background

The actuation of a multi-celled composite rotor blade with integral actuators is based on research from many different fields. These fields include : active materials, active fiber composites and adaptive structures. The major advances in these fields affecting this research are compiled below.

Figure 1.7: Active vibration absorber (Newman, 1994).

1.4.1 Active Materials

Active materials have some property that can be controlled by an external excitation. Using such materials, diverse properties can be controlled such as viscosity (magneto and electro-rheological fluids), or strains can be induced into the structure to control its shape (shape memory alloys, magnetostrictive and electrostrictive materials). There are many different external excitations that can be used as a control mechanism, but the most important are magnetic and electrical as they can be easily controlled by computer. Piezoelectric materials are the most common form of electrostrictive materials used in adaptive structures. This thesis will employ piezoelectric material to change the stiffness, as it has a larger bandwidth than either shape memory alloys (a few Hertz due to their thermal time constant) or magnetostrictive materials. It also does not require the added complexity that comes with creating a magnetic field. Piezoelectricity is the ability of a material to develop an electrical charge that is proportional to an applied stress. The converse is also true, where a strain can be developed that is proportional to the applied electric field. The piezoelectric effect has been observed in natural quartz crystals, some ceramics, semicrystalline polyvinylidene and human bone. The main industrial applica-

Figure 1.8: Swash plate assembly (Newman, 1994).

tions use either lead zirconium titanate (PZT) ceramic or polyvinylidene fluoride (PVDF). To attain the optimal piezoelectric effect of the material, it must first be poled. Poling is the application of a high voltage electric field to the material, which aligns the electric dipoles in the material's molecules and causes the material to strain in that direction. Applying an opposite field will cause the material to strain in the opposing direction, unless the field is large enough to repole the material. This creates a voltage limit; hence a limit on the strain that can be developed.

The poling results in a material that is orthotropic. That is, the material properties in the poling direction are different than those transverse to it. In this thesis, the industry standard coordinate system is used, where the 1-2 plane is the plane of symmetry and the 3-direction is that of the poling direction. The piezoelectric material coordinate system is given in Figure 1.11, where P is the poling vector and E is the applied electric field. Applying the electric field in the poling direction will cause the piezoelectric material to expand in that direction, which results in it

Figure 1.9: Types of pitch control (Newman, 1994).

contracting in the transverse (1 and 2) directions.

These materials have high actuation authority due to the material's high Young's modulus. Strain maximums are of the order of 0.1% (Bent, 1997). Their high bandwidth, in the order of 100 kHz, allows them to be used for dynamic control. They also work over a broad temperature range, from -20 to 200°C (Bent, 1997). These materials' main limitations are their high densities and brittleness.

1.4.2 Active Fiber Composites

Piezoelectric ceramics can be extruded, creating a fiber (130 μm diameter) that can function as an anisotropic actuator. The ceramic fibers are encased in a soft matrix, which acts as a load transfer mechanism and increases the damage tolerance. The fibers will maintain the stiffness and bandwidth of a monolithic piezoceramic. An active ply can be fully integrated into the structure, resulting in a highly distributed actuation.

Because piezo elements strain in the direction of the electric field for maximum attenuation, Bent (1997) used the concept of interdigitated electrodes. Electrode fingers of alternating polarity are shown in Figure 1.12; the fingers above and below

Figure 1.10: Dangleberry assembly (Newman, 1994).

the layer are of the same polarity. This causes the electric field to align itself with the fibers as it travels between the electrodes. Figure 1.13 shows a quarter model representation of the electric field, and the various geometry parameters influencing what portion of the field gets oriented this way: finger width (w), wafer thickness (h) and electrode finger spacing (p). In general, increasing the finger spacing and decreasing the thickness and width will increase the amount of the field that gets aligned.

1.4.3 Adaptive Structures

Clark et al. (1998) defines an adaptive structure as:

a structure configured with distributed actuators and sensors and directed by a controller capable of modifying the dynamic response of the structure in the presence of time-varying environmental and operational conditions.

Figure 1.14 shows a possible integration of the sensors and actuators into a structure.

Figure 1.11: Coordinate system for piezoelectric material (DuPlessis, 1996).

Figure 1.12: Active fiber composite, showing axis convention and electrodes (Bent, 1997).

In designing an adaptive structure, the dynamics of the structure must be understood to allow for the efficient use of actuators and sensors. Choice of active material, size and placement depends on the desired structural response and the boundary conditions. The structural dynamics can be modified using three control system concepts: gain adjustment, zero placement and pole placement.

1.5 Objectives

The objective of this thesis is to develop the dynamic equations of motion for a thin-walled, composite rotor blade in closed form. Active fibers are to be included in the analysis for static and dynamic control. A full-scale model will be used in the stability analysis, which differs from other analyses published using scale models, to

Figure 1.13: Quarter model representation with fiber (light shading) and electrode (dark shading) (Bent, 1997).

Figure 1.14: Integrating actuators and sensors into structure (Bent, 1997).

better evaluate the possible use of the active fibers for pitch and vibration control. This thesis will be limited to the hover case, but the same blade dynamics are valid for normal flight conditions with revisions to the aerodynamic terms.

Chapter 2

Cross-Section Analysis

The analysis of a helicopter rotor blade can be separated into a linear two-dimensional analysis of the cross-section and a nonlinear analysis along the blade span. The cross-section can be modeled by the variational asymptotic method, which was first developed by Berdichevsky (1979). This allows for the development of closed-form expressions for the displacement and stress fields, and beam stiffness coefficients. This method was expanded by Berdichevsky et al. (1992) to include anisotropic closed-cross-section beams. A further development by Badir (1995) allowed for the inclusion of multi-cell composite beams. Integral actuation by means of active fiber composites (AFC) was then introduced by Cesnik and Shin (1998).

This chapter validates the work done by Cesnik and Shin (1998), while suggesting some corrections and improvements to it.

2.1 Coordinate System

For the slender thin walled elastic cylinder given in Figure 2.1, it is assumed that:

$$\frac{d}{L} \ll 1, \quad \frac{h}{d} \ll 1, \quad \frac{h}{R} \ll 1 \quad (2.1)$$

where the length is denoted by L , the thickness by h , the radius of curvature by R , and the maximum cross-sectional dimension by d . All the ratios given in 2.1 are assumed to be of the order of a small parameter ϵ . The coordinate ξ is measured

along the normal to the middle surface within the limits

$$-\frac{h(s)}{2} \leq \xi \leq \frac{h(s)}{2}$$

The tangential vector $\mathbf{t}$, the normal vector $\mathbf{n}$ and the projection of the position vector $\mathbf{r}$ on the $\mathbf{t}$ and $\mathbf{n}$ are expressed in terms of the Cartesian and curvilinear coordinates as

$$\begin{aligned} \mathbf{t} &= \frac{d\mathbf{r}}{ds} = \frac{dy}{ds}\mathbf{i}_y + \frac{dz}{ds}\mathbf{i}_z \\ \mathbf{n} &= \mathbf{t} \times \mathbf{i}_x = \frac{dz}{ds}\mathbf{i}_y - \frac{dy}{ds}\mathbf{i}_z \\ r_t &= \mathbf{r} \cdot \mathbf{t} = y\frac{dy}{ds} + z\frac{dz}{ds} \\ r_n &= \mathbf{r} \cdot \mathbf{n} = y\frac{dz}{ds} - z\frac{dy}{ds} \end{aligned} \quad (2.2)$$

Figure 2.1: Cross sectional coordinate systems (Badir, 1995).

2.2 Shell Energy Functional

The linear strain energy density for a three-dimensional (3-D) elastic body is given by

$$U = \frac{1}{2} E^{ijkl} \epsilon_{ij}^{(m)} \epsilon_{kl}^{(m)}, \quad (i, j, k, l = 1, 2, 3) \quad (2.3)$$

where the material properties are given in the Hookean tensor E^{ijkl} and $\varepsilon^{(m)}$ is the mechanical strain. For an active fiber composite, the total strain ε is given by the sum of the mechanical strain and the non-mechanical strain $\varepsilon^{(a)}$ induced by the piezoelectric actuation.

The total strain can be written

$$\varepsilon_{ij} = s_{ijkl}^E \sigma_{kl} + d_{kij} E_k \quad (2.4)$$

where s_{ijkl}^E is the elastic compliance tensor at a constant electric field, E_k , σ_{ij} is the stress tensor, d_{kij} is the piezoelectric electromechanical coupling tensor and E_k is the electric field vector. The total strain can also be written in terms of two-dimensional (2-D) strain measures as

$$\varepsilon_{ij} = \gamma_{ij} + \xi \rho_{ij} \quad (2.5)$$

where γ_{ij} is the in-plane strain component and ρ_{ij} is the change in the shell surface curvature.

These components are related to the displacement variables by:

$$\begin{aligned} \gamma_{11} &= \frac{\partial v_\ell}{\partial x} \\ 2\gamma_{12} &= \frac{\partial v_\ell}{\partial s} + \frac{\partial v_s}{\partial x} \\ \gamma_{22} &= \frac{\partial v_s}{\partial s} + \frac{v_\xi}{R} \\ \rho_{11} &= \frac{\partial^2 v_\xi}{\partial x^2} \\ 2\rho_{12} &= \frac{\partial^2 v_\xi}{\partial x \partial s} + \frac{1}{4R} \left(\frac{\partial v_\ell}{\partial s} - 3 \frac{\partial v_s}{\partial x} \right) \\ \rho_{22} &= \frac{\partial^2 v_\xi}{\partial s^2} - \frac{\partial}{\partial s} \left(\frac{v_s}{R} \right) \end{aligned} \quad (2.6)$$

where v_ℓ , v_s and v_ξ represent the displacements in the axial, tangential and normal directions (Badir, 1992). These are related to the Cartesian displacement components by

$$\begin{aligned} v_\ell &= u_x \\ v_s &= u_y \frac{dy}{ds} + u_z \frac{dz}{ds} \\ v_\xi &= u_y \frac{dz}{ds} - u_z \frac{dy}{ds} \end{aligned} \quad (2.7)$$

where u_x , u_y and u_z represent the displacements in the x , y and z directions, respectively.

In compliance with the classical shell formulation, the through the thickness stresses σ_{i3} are considered to be negligible in comparison to the other stress components (Koiter, 1969). Hence, assuming $\sigma_{i3} = 0$, the 3-D strain energy should be minimized with respect to ε_{i3} , i.e.,

$$\hat{U} = \min_{\varepsilon_{i3}} U = \frac{1}{2} D^{\alpha\beta\gamma\delta} \varepsilon_{\alpha\beta}^{(m)} \varepsilon_{\gamma\delta}^{(m)}, \quad (\alpha, \beta, \gamma, \delta = 1, 2) \quad (2.8)$$

The 2-D Hookean tensor $D^{\alpha\beta\gamma\delta}$ is solved for in Appendix A to be

$$D^{\alpha\beta\gamma\delta} = E^{\alpha\beta\gamma\delta} - \frac{E^{\alpha\beta 33} E^{\gamma\delta 33}}{E^{3333}} - H_{\mu\lambda} G^{\alpha\beta\mu} G^{\gamma\delta\lambda}$$

where

$$G^{\alpha\beta\mu} = E^{\alpha\beta\mu 3} - \frac{E^{\alpha\beta 33} E^{\mu 333}}{E^{3333}}$$

and $H_{\mu\lambda}$ are components of the inverse of the 2-D matrix

$$\left\| E^{\mu 3\lambda 3} - \frac{E^{\mu 333} E^{\lambda 333}}{E^{3333}} \right\|$$

The mechanical strain is the difference between the strain due to the piezoelectric actuation and the total strain. As discussed in Section 1.4.2, it is assumed that the electric field flows along the piezoelectric fibers. This is given in Figure 1.12 as the 3-direction, but will be referred to hereafter as the e -direction to differentiate it from the other coordinate system. The two dimensional strain expression therefore becomes:

$$\begin{aligned} \varepsilon_{\alpha\beta}^{(m)} &= \varepsilon_{\alpha\beta} - \varepsilon_{\alpha\beta}^{(a)} \\ &= \gamma_{\alpha\beta} + \xi \rho_{\alpha\beta} - d_{e\alpha\beta} E_e \end{aligned} \quad (2.9)$$

Substituting this expression into the expression for $\hat{U}$ and integrating over the thickness ξ , the shell energy Φ per unit area of the middle surface is found to be

$$\begin{aligned} 2\Phi &= (< D^{\alpha\beta\gamma\delta} > \gamma_{\alpha\beta} - 2 < D^{\alpha\beta\gamma\delta} d_{e\alpha\beta} E_e >) \gamma_{\gamma\delta} \\ &\quad + 2(< D^{\alpha\beta\gamma\delta} \xi > \gamma_{\alpha\beta} - < D^{\alpha\beta\gamma\delta} d_{e\alpha\beta} E_e \xi >) \rho_{\gamma\delta} \\ &\quad + < D^{\alpha\beta\gamma\delta} \xi^2 > \rho_{\alpha\beta} \rho_{\gamma\delta} + < D^{\alpha\beta\gamma\delta} d_{e\alpha\beta} d_{e\gamma\delta} E_e^2 > \end{aligned} \quad (2.10)$$

where the angular brackets are defined as an integral through the thickness, i.e.,

$$\langle (\bullet) \rangle = \int_{-h(s)/2}^{h(s)/2} (\bullet) d\xi$$

Using the minimum energy principle for an applied external load P_i , the displacement field u_i determining the deformed state is the stationary point of the energy functional I , where

$$I = \int \int \Phi dx ds - \int \int P_i u_i dx ds \quad (2.11)$$

2.3 Variational Asymptotic Method

The radial dimension for a helicopter rotor blade is much larger than the other two, allowing for it to be treated as a one-dimensional beam. This enables much simpler equations of motion to be used than would be possible for the complete three-dimensional elastic model. This dimension reduction process can be accomplished by the variational asymptotic method.

The variational asymptotic method approximates a three-dimensional body with an imaginary one-dimensional beam based on the asymptotic analysis of two-dimensional shell energy. This method uses an iterative process to model the slender thin-walled shell as a beam. Approximations of the displacement terms are obtained by keeping the leading order terms in the energy functional. After each successive approximation, the order of the strain terms are assessed.

Zeroth Order Approximation

Let Δ and E be the order of the displacements and the stiffness coefficients, respectively. It is assumed that the radius of curvature R and displacements Δ are of the order:

$$\frac{R}{L} \ll 1 \quad \frac{\Delta}{L} \ll 1$$

Therefore, the order of magnitude of the in-plane strains and curvatures are

$$\begin{aligned}\gamma_{11} &\sim O\left(\frac{\Delta}{L}\right) \\ 2\gamma_{12} &\sim O\left(\frac{\Delta}{d}\right) \\ \gamma_{22} &\sim O\left(\frac{\Delta}{d}\right) \\ \rho_{11} &\sim O\left(\frac{\Delta}{L^2}\right) \\ \rho_{12} &\sim O\left(\frac{\Delta}{d^2}\right) \\ \rho_{22} &\sim O\left(\frac{\Delta}{d^2}\right).\end{aligned}$$

Since L is much larger than d , then γ_{11} and ρ_{11} are much smaller than γ_{12} , γ_{22} and ρ_{12} , ρ_{22} , respectively. Therefore, for a first approximation, their contribution to the shell energy can be neglected.

For a thin walled slender beam satisfying the dimensions given in 2.1, the rate of change of the displacement in the axial direction is much smaller than the rate of change in the circumferential direction. Consequently, for each displacement component:

$$\left\| \frac{\partial v_i}{\partial x} \right\| \ll \left\| \frac{\partial v_i}{\partial s} \right\|$$

Keeping only the leading terms in the strain-displacement relationship (Equation 2.6), we get:

$$\begin{aligned}2\gamma_{12} &= \frac{\partial v_t}{\partial s} \\ \gamma_{22} &= \frac{\partial v_s}{\partial s} + \frac{v_\xi}{R} \\ \rho_{12} &= \frac{1}{4R} \frac{\partial v_t}{\partial s} \\ \rho_{22} &= \frac{\partial^2 v_\xi}{\partial s^2} - \frac{\partial}{\partial s} \left(\frac{v_s}{R} \right).\end{aligned}\tag{2.12}$$

The order of magnitude of the shell energy and the work done by the external forces are:

$$\begin{aligned}\Phi &\sim O\left(\frac{E\Delta^2 h}{d^2}\right) \\ P_i u_i &\sim O\left(\frac{E\Delta^2 h}{L^2}\right)\end{aligned}$$

The contribution of the external work $P_i u_i$ is therefore much smaller than the shell energy per unit area Φ , and can be neglected in the energy functional in Equation 2.11 (Badir, 1992). Then we get:

$$\begin{aligned} 2I &= \int_0^L \int_{all} 2\Phi \, dx ds \\ &= \int_0^L \int_{all} (\langle D^{\alpha\beta\gamma\delta} \rangle \gamma_{\alpha\beta} - 2 \langle D^{\alpha\beta\gamma\delta} d_{e\alpha\beta} E_e \rangle) \gamma_{\gamma\delta} + 2(\langle D^{\alpha\beta\gamma\delta} \xi \rangle \gamma_{\alpha\beta} \\ &\quad - \langle D^{\alpha\beta\gamma\delta} d_{e\alpha\beta} E_e \xi \rangle) \rho_{\gamma\delta} + \langle D^{\alpha\beta\gamma\delta} \xi^2 \rangle \rho_{\alpha\beta} \rho_{\gamma\delta} + \Psi(E_e) \, dx ds \end{aligned}$$

where

$$\Psi(E_e) = - \langle D^{\alpha\beta\gamma\delta} d_{e\alpha\beta} d_{e\gamma\delta} E_e^2 \rangle$$

and the integral with the subscript *all* represent a summation of the evaluation of the integral over $s = 0 \rightarrow s_1$, $s = s_1 \rightarrow s_2$ and $s = s_2 \rightarrow s_3$, as given in Figure 2.2.

Figure 2.2: Branches for integration (Cesnik and Shin, 1998).

Current actuator technology is not able to produce strains of the same magnitude as those developed by a rotating helicopter blade. The electromechanical strains produced by the fibers are assumed to be of the order ε smaller than the mechanical strains. Therefore, $d_{e\alpha\beta} E_e$ is of order $\Delta d^2/L^3$ and the function $\Psi(E_e)$ represents higher order terms and can be neglected.

The axial direction displacements are assumed to be small compared to the circumferential direction. Therefore, γ_{11} and ρ_{11} can be ignored in comparison with γ_{12} , γ_{22} and ρ_{12} , ρ_{22} , respectively, and hence,

$$\begin{aligned}
2I = \int_0^L \int_{all} & (4 \langle D^{1212} \rangle \gamma_{12}^2 + 4 \langle D^{1222} \rangle \gamma_{12} \gamma_{22} + \langle D^{2222} \rangle \gamma_{22}^2 \\
& - 4 \langle D^{1112} d_{e11} E_e \rangle \gamma_{12} - 2 \langle D^{1122} d_{e11} E_e \rangle \gamma_{22} \\
& - 8 \langle D^{1212} d_{e12} E_e \rangle \gamma_{12} - 4 \langle D^{1222} d_{e12} E_e \rangle \gamma_{22} \\
& - 4 \langle D^{2212} d_{e22} E_e \rangle \gamma_{12} - 2 \langle D^{2222} d_{e22} E_e \rangle \gamma_{22} \\
& + 2(2 \langle D^{1212} \xi \rangle \gamma_{12} + \langle D^{1222} \xi \rangle \gamma_{22} \\
& - 2 \langle D^{1112} d_{e11} E_e \xi \rangle - 4 \langle D^{1212} d_{e12} E_e \xi \rangle \\
& - 2 \langle D^{1222} d_{e22} E_e \xi \rangle) \rho_{12} + (2 \langle D^{1222} \xi \rangle \gamma_{12} + \langle D^{2222} \xi \rangle \gamma_{22} \\
& - \langle D^{1122} d_{e22} E_e \xi \rangle - 2 \langle D^{1222} d_{e22} E_e \xi \rangle \\
& - \langle D^{2222} d_{e22} E_e \xi \rangle) \rho_{22} + 4 \langle D^{1212} \xi^2 \rangle \rho_{12}^2 \\
& + 4 \langle D^{1222} \xi^2 \rangle \rho_{12} \rho_{22} + \Psi(E_e) dx ds
\end{aligned} \tag{2.13}$$

The integrand of Equation 2.13 is a positive quadratic function, therefore the variation of the functional and the functional itself will both be minimized when $\gamma_{12} = \gamma_{22} = \rho_{12} = \rho_{22} = 0$. Therefore, from Equation 2.12 the following is obtained:

$$\frac{\partial v_\ell}{\partial s} = 0 \tag{2.14}$$

$$\frac{\partial v_s}{\partial s} + \frac{v_\xi}{R} = 0 \tag{2.15}$$

$$\frac{\partial^2 v_\xi}{\partial s^2} - \frac{\partial}{\partial s} \left(\frac{v_s}{R} \right) = 0 \tag{2.16}$$

As v_ℓ is not a function of s , it must be a function of x only; therefore,

$$v_\ell = U_1(x) \tag{2.17}$$

where $U_1(x)$ is a function of x .

Equation 2.16 is integrated with respect to s to obtain

$$\frac{\partial v_\xi}{\partial s} - \frac{v_s}{R} = -\varphi(x) \tag{2.18}$$

where $\varphi(x)$ is a function of x which represents, as will be shown later, the cross-sectional rotation about the x -axis. Substituting v_ξ from Equation 2.15 into Equation 2.18 the second-order differential equation for v_s is obtained:

$$\frac{\partial}{\partial s} \left(R \frac{\partial v_s}{\partial s} \right) + \frac{v_s}{R} = \varphi(x) \tag{2.19}$$

Solving this equation requires that the relationship between the radius of curvature R and the Cartesian position components be known:

$$\begin{aligned}\frac{d^2 z}{ds^2} &= \frac{1}{R} \frac{dy}{ds} \\ \frac{d^2 y}{ds^2} &= -\frac{1}{R} \frac{dz}{ds}\end{aligned}\tag{2.20}$$

Then, the general solution to Equation 2.19 is:

$$v_s = U_2(x) \frac{dy}{ds} + U_3(x) \frac{dz}{ds} + \varphi(x) r_n \tag{2.21}$$

Substituting this into Equation 2.15 and knowing the relationships given in Equation 2.20, then

$$v_t = U_2(x) \frac{dz}{ds} - U_3(x) \frac{dy}{ds} - \varphi(x) r_t \tag{2.22}$$

Equations 2.17, 2.21 and 2.22 represent the zeroth-order curvilinear displacement field that minimize the energy functional. The relationships in Equation 2.7 can then be used to find the displacements in the Cartesian coordinate system as:

$$\begin{aligned}u_x &= U_1(x) \\ u_y &= U_2(x) - z\varphi(x) \\ u_z &= U_3(x) + y\varphi(x)\end{aligned}$$

The functions $U_1(x)$, $U_2(x)$ and $U_3(x)$ represent the average cross-sectional translation and $\varphi(x)$ represents the torsional rotation. The order of magnitudes of $U_2(x)$ and $U_3(x)$ are assumed to be Δ , where the axial displacement $U_1(x)$ is taken to be much smaller and of the order of $(\Delta d/L)$. The assumed order of magnitude of the torsional twist $\varphi(x)$ is (Δ/L) . These orders of magnitude of $U_1(x)$ and $\varphi(x)$ differ from that given in Badir (1992), where they are given as Δ and (Δ/d) , respectively. This assumes that the axial and torsional twist displacements are small and are in keeping with helicopter blade analysis developed by Hodges and Ormiston (1976). This results in the same strain fields as obtained by Badir (1992) with a more systematic method of elimination of higher order terms.

The zeroth-order approximations to the strain are found by substituting Equations 2.17, 2.21 and 2.22 into Equation 2.6. The result are given below, where the

prime denotes differentiation with respect to x .

$$\begin{aligned}
 \gamma_{11} &= U_1'(x) \\
 2\gamma_{12} &= U_2'(x)\frac{dy}{ds} + U_3'\frac{dz}{ds} + \varphi'(x)r_n \\
 \gamma_{22} &= 0 \\
 \rho_{11} &= U_2''(x)\frac{dz}{ds} - U_3''(x)\frac{dy}{ds} - \varphi''(x)r_t \\
 \rho_{12} &= \frac{1}{4R} \left[U_2'(x)\frac{dy}{ds} + U_3'\frac{dz}{ds} + \varphi'(x)r_n \right] - \varphi'(x) \\
 \rho_{22} &= 0
 \end{aligned} \tag{2.23}$$

First-Order Approximation

The first-order approximation can be constructed by adding in a correction factor to the curvilinear displacement fields, i.e.,

$$\begin{aligned}
 v_\ell &= U_1(x) + w_1(s, x) \\
 v_s &= U_2(x)\frac{dy}{ds} + U_3(x)\frac{dz}{ds} + \varphi(x)r_n + w_2(s, x) \\
 v_\xi &= U_2(x)\frac{dz}{ds} - U_3(x)\frac{dy}{ds} - \varphi(x)r_t + w_3(s, x)
 \end{aligned} \tag{2.24}$$

Substituting these approximations back in Equation 2.6:

$$\begin{aligned}
 \gamma_{11} &= \tilde{\gamma}_{11} + \frac{\partial w_1}{\partial x} \\
 2\gamma_{12} &= 2\tilde{\gamma}_{12} + \frac{\partial w_2}{\partial x} + 2\hat{\gamma}_{12} \\
 \gamma_{22} &= \tilde{\gamma}_{22} + \hat{\gamma}_{22} \\
 \rho_{11} &= \tilde{\rho}_{11} + \frac{\partial^2 w_3}{\partial x^2} \\
 \rho_{12} &= \tilde{\rho}_{12} + \frac{\partial^2 w_3}{\partial s \partial x} - \frac{3}{4R} \frac{\partial w_2}{\partial x} + \hat{\rho}_{12} \\
 \rho_{22} &= \frac{\partial^2 v_\xi}{\partial s^2} - \frac{\partial}{\partial s} \left(\frac{v_s}{R} \right) + \frac{\partial^2 w_3}{\partial s^2} - \frac{\partial}{\partial s} \left(\frac{w_2}{R} \right)
 \end{aligned} \tag{2.25}$$

where the ($\tilde{}$) are the strains corresponding to the zeroth-order approximations given in Equation 2.23. The terms denoted by the ($\hat{}$) are the leading order terms of the

first order warping, and are given by:

$$\begin{aligned}
 2\hat{\gamma}_{12} &= \frac{\partial w_1}{\partial s} \\
 \hat{\gamma}_{22} &= \frac{\partial w_2}{\partial s} + \frac{w_3}{R} \\
 \hat{\rho}_{12} &= \frac{1}{4R} \frac{\partial w_1}{\partial s} \\
 \hat{\rho}_{22} &= \frac{\partial^2 w_3}{\partial s^2} - \frac{\partial}{\partial s} \left(\frac{w_2}{R} \right)
 \end{aligned} \tag{2.26}$$

The change in the shell middle curvature $\rho_{\alpha\beta}$ is small compared to the in-plane strain components $\gamma_{\alpha\beta}$, as shown in Berdichevsky and Misyura (1992). Thus, considering the shell energy to be only a function of the γ_{11} , γ_{12} and γ_{22} , from Equation 2.10, the shell energy can be written as:

$$2\Phi = \langle D^{\alpha\beta\gamma\delta} \rangle \gamma_{\alpha\beta} \gamma_{\gamma\delta} - 2 \langle D^{\alpha\beta\gamma\delta} d_{e\alpha\beta} E_e \rangle \gamma_{\gamma\delta} \tag{2.27}$$

Assuming that there is no pressure difference between the interior and exterior of the shell, the hoop strain will be minimal compared to the other strains. Therefore, the shell energy can be minimized with respect to the hoop strain component γ_{22} . Then,

$$\begin{aligned}
 2 \frac{\partial \Phi}{\partial \gamma_{22}} &= 2 \langle D^{1122} \rangle \gamma_{11} + 4 \langle D^{1222} \rangle \gamma_{12} + 2 \langle D^{2222} \rangle \gamma_{22} \\
 &\quad - 2 \langle D^{1122} d_{e11} E_e \rangle - 4 \langle D^{1222} d_{e12} E_e \rangle - 2 \langle D^{2222} d_{e22} E_e \rangle = 0
 \end{aligned}$$

This can then be rearranged to solve for γ_{22} ,

$$\gamma_{22} = - \frac{\langle D^{1122} \rangle}{\langle D^{2222} \rangle} \gamma_{11} - 2 \frac{\langle D^{1222} \rangle}{\langle D^{2222} \rangle} \gamma_{12} + \frac{\langle D^{\gamma\beta 22} d_{e\gamma\beta} E_e \rangle}{\langle D^{2222} \rangle}$$

Substituting this back into Equation 2.27 and rearranging we get:

$$\begin{aligned}
 2\Phi &= \left(A\gamma_{11} - 2A^{(a)} \right) \gamma_{11} + 2B\gamma_{11}\gamma_{12} \\
 &\quad + \left(C\gamma_{12} - 2C^{(a)} \right) \gamma_{12} - \Psi_2(E_e)
 \end{aligned} \tag{2.28}$$

The axial, coupling and shear stiffnesses are represented by the variables A , B and C , respectively. They are all assumed to be of the same order, (Eh) . The terms

$A^{(a)}$, $C^{(a)}$ and $\Psi_2(E_e)$ result from the contribution of the embedded fibers.

$$\begin{aligned}
A &= \langle D^{1111} \rangle - \frac{\langle D^{1122} \rangle^2}{\langle D^{2222} \rangle} \\
B &= 2 \langle D^{1112} \rangle - 2 \frac{\langle D^{1112} \rangle \langle D^{1222} \rangle}{\langle D^{2222} \rangle} \\
C &= 4 \langle D^{1212} \rangle - 4 \frac{\langle D^{1222} \rangle^2}{\langle D^{2222} \rangle} \\
A^{(a)} &= \langle D^{11\gamma\delta} d_{e\gamma\delta} E_e \rangle - \frac{\langle D^{1122} \rangle}{\langle D^{2222} \rangle} \langle D^{22\gamma\delta} d_{e\gamma\delta} E_e \rangle \\
C^{(a)} &= 2 \langle D^{12\gamma\delta} d_{e\gamma\delta} E_e \rangle - 2 \frac{\langle D^{1222} \rangle}{\langle D^{2222} \rangle} \langle D^{22\gamma\delta} d_{e\gamma\delta} E_e \rangle \\
\Psi_2(E_e) &= \frac{\langle D^{22\gamma\delta} d_{e\gamma\delta} E_e \rangle^2}{\langle D^{2222} \rangle}
\end{aligned} \tag{2.29}$$

To minimize the energy functional I , the first variation with respect to the strain is set to zero:

$$\begin{aligned}
\delta I &= \int_0^L \int_{all} \left\{ \frac{\partial \Phi}{\partial (2\gamma_{12})} \delta(2\gamma_{12}) + \frac{\partial \Phi}{\partial (\gamma_{22})} \delta\gamma_{22} \right\} ds dx \\
&= \int_0^L \int_{all} \left\{ \frac{\partial \Phi}{\partial (2\gamma_{12})} \delta \left(\frac{\partial w_1}{\partial s} \right) + \frac{\partial \Phi}{\partial (\gamma_{22})} \delta \left(\frac{\partial w_2}{\partial s} + \frac{w_3}{R} \right) \right\} ds dx \\
&= \int_0^L \int_{all} \left\{ N_{12} \delta \left(\frac{\partial w_1}{\partial s} \right) + N_{22} \delta \left(\frac{\partial w_2}{\partial s} + \frac{w_3}{R} \right) \right\} ds dx \\
&= 0
\end{aligned} \tag{2.30}$$

Hence,

$$\begin{aligned}
N_{11} &= \frac{\partial \Phi}{\partial (\gamma_{11})} \\
&= (A(s)\gamma_{11} + B(s)\gamma_{12}) - A^{(a)}(s)
\end{aligned} \tag{2.31}$$

$$\begin{aligned}
N_{12} &= \frac{\partial \Phi}{\partial (2\gamma_{12})} \\
&= \frac{1}{2} (B(s)\gamma_{11} + C(s)\gamma_{12}) - \frac{1}{2} C^{(a)}(s) \\
&= \text{constant for each section}
\end{aligned} \tag{2.32}$$

$$\begin{aligned}
N_{22} &= \frac{\partial \Phi}{\partial (\gamma_{22})} \\
&= 0
\end{aligned} \tag{2.33}$$

where N_{11} , N_{12} and N_{22} represent the axial stress resultant, the shear flow and the hoop stress resultant, respectively. The shear flow must be constant for each section, and the hoop stress is equal to zero.

As the hoop stress resultant N_{22} is zero, the variations δw_2 and δw_3 do not change the variation of the shell energy, Equation 2.30. Thus, the energy functional is independent of the in-plane warping functions w_2 and w_3 , and they have a negligible contribution to the shell energy. Hence, the only warping term that has to be determined is w_1 .

The first-order approximation to the correction function w_1 for the i th section can be found by substituting the shear strains from Equation 2.25 into Equation 2.32. This gives:

$$\begin{aligned}\frac{\partial w_1}{\partial s} &= \frac{4}{C}q_i - 2\frac{B}{C}U'_1 + \frac{2}{C}C^{(a)} - U'_2\frac{dy}{ds} - U'_3\frac{dz}{ds} - \varphi' r_n \\ &= -2c(s)q_i + b(s)U'_1 - c(s)C^{(a)} - U'_2\frac{dy}{ds} - U'_3\frac{dz}{ds} - \varphi' r_n\end{aligned}\quad (2.34)$$

where q_i represents the shear flow in the i th branch, and the new variables are

$$c(s) = -2\frac{1}{C} \quad b(s) = -2\frac{B}{C}$$

Knowing that the compatibility condition for warping over any closed cell is given by

$$\oint \frac{\partial w_i}{\partial s} ds = 0 \quad (2.35)$$

then, assuming a two-cell cross-section, and letting q_1 , q_w and q_2 represent the shear flow in the left, web and right branch, respectively, the compatibility condition for the first cell gives:

$$-\frac{1}{2}q_1b_1 + U'_1b_3 - f_1^{(a)} + \varphi'b_4 - \frac{1}{2}q_wb_2 = 0$$

and the compatibility condition for the second cell gives:

$$-\frac{1}{2}q_2b_5 + U'_1b_6 - f_2^{(a)} + \varphi'b_7 + \frac{1}{2}q_wb_2 = 0$$

where

$$\begin{aligned}
 b_1 &= 4 \int_0^{s_1} c(s) ds & b_2 &= 4 \int_{s_1}^{s_2} c(s) ds \\
 b_3 &= \oint_I b(s) ds & b_4 &= \oint_I \tau_n ds = -2A_{eI} \\
 b_5 &= 4 \int_{s_2}^{s_3} c(s) ds & b_6 &= \oint_{II} b(s) ds \\
 b_7 &= \oint_{II} \tau_n ds = -2A_{eII} & f_1^{(a)} &= \oint_I c(s) C^{(a)} ds \\
 f_2^{(a)} &= \oint_{II} c(s) C^{(a)} ds
 \end{aligned}$$

and A_{eI} and A_{eII} are the enclosed areas of the first and second cells, respectively. A detailed derivation of these compatibility conditions can be obtained in Section B.1.1.

Since the axial and bending effects as well as the shear loading have already been considered in the zeroth order approximation, in any higher order approximation the shear flow in the wbe can be set equal to the difference between the shear flow in the left and right branch, i.e.,

$$q_w = q_1 - q_2$$

the shear flow in each section can therefore be found as (see Section B.1.2 for derivation):

$$\begin{aligned}
 q_1 &= -2a_3 U'_1 - g_1^{(a)} - 2a_1 \varphi' \\
 q_2 &= -2a_4 U'_1 - g_2^{(a)} - 2a_2 \varphi' \\
 q_w &= -2(a_3 - a_4) U'_1 - (g_1^{(a)} - g_2^{(a)}) - 2(a_1 - a_2) \varphi'
 \end{aligned}$$

where

$$\begin{aligned}
 g_1^{(a)} &= 2 \frac{(b_5 + b_2) f_1^{(a)} + b_2 f_2^{(a)}}{(b_1 + b_2)(b_2 + b_5) - b_2^2} & g_2^{(a)} &= 2 \frac{(b_1 + b_2) f_2^{(a)} + b_2 f_1^{(a)}}{(b_1 + b_2)(b_2 + b_5) - b_2^2} \\
 a_1 &= -\frac{c_3}{c_1} & a_2 &= \frac{(b_1 + b_2)a_1 + b_4}{b_2} \\
 a_3 &= -\frac{c_2}{c_1} & a_4 &= \frac{(b_1 + b_2)a_3 + b_3}{b_2} \\
 c_1 &= b_1 + \frac{(b_1 + b_2)b_5}{b_2} & c_2 &= b_3 + b_6 + \frac{b_3 b_5}{b_2} \\
 c_3 &= b_4 + b_7 + \frac{b_4 b_5}{b_2}
 \end{aligned}$$

Equation 2.34 can be integrated with respect to s for each i th branch. This can then be substituted into Equation 2.24 to solve for the first-order approximation on

the displacement field:

$$\begin{aligned}
 v_\ell &= U_1(x) - y(s)U_2'(x) - z(s)U_3'(x) \\
 &\quad + G(s)\varphi'(x) + g_1(s)U_1'(x) + v_1^{(a)}(s) \\
 v_s &= U_2(x)\frac{dy}{ds} + U_3(x)\frac{dz}{ds} + \varphi(x)r_n \\
 v_\xi &= U_2(x)\frac{dz}{ds} - U_3(x)\frac{dy}{ds} - \varphi(x)r_t
 \end{aligned} \tag{2.36}$$

where for the left branch ($s = 0 \rightarrow s_1$)

$$\begin{aligned}
 G(s) &= \int_0^s [4a_1c(\tau) - r_n(\tau)]d\tau \\
 g_1(s) &= \int_0^s [b(\tau) + 4a_3c(\tau)]d\tau \\
 v_1^{(a)} &= \int_0^s [2g_1^{(a)} - C^{(a)}(\tau)]c(\tau)d\tau
 \end{aligned}$$

for the web ($s = s_1 \rightarrow s_2$)

$$\begin{aligned}
 G(s) &= \int_{s_1}^s [4(a_1 - a_2)c(\tau) - r_n(\tau)]d\tau \\
 g_1(s) &= \int_{s_1}^s [b(\tau) + 4(a_3 - a_4)c(\tau)]d\tau \\
 v_1^{(a)} &= \int_{s_1}^s [2(g_1^{(a)} - g_2^{(a)}) - C^{(a)}(\tau)]c(\tau)d\tau
 \end{aligned}$$

and for the right branch ($s = s_2 \rightarrow s_3$)

$$\begin{aligned}
 G(s) &= \int_{s_2}^s [4a_2c(\tau) - r_n(\tau)]d\tau \\
 g_1(s) &= \int_{s_2}^s [b(\tau) + 4a_4c(\tau)]d\tau \\
 v_1^{(a)} &= \int_{s_2}^s [2g_2^{(a)} - C^{(a)}(\tau)]c(\tau)d\tau
 \end{aligned}$$

The terms $G(s)$ and $g_1(s)$ are of the order of d^2 and d , respectively.

The first-order approximations to the strain field can be found by substituting Equation 2.36 into Equation 2.6:

$$\begin{aligned}
 \gamma_{11} &= U_1'(x) - y(s)U_2''(x) - z(s)U_3''(x) + G(s)\varphi''(x) + g_1(s)U_1''(x) \\
 2\gamma_{12} &= \left(\frac{dG}{ds} + r_n \right) \varphi' + \frac{dg_1}{ds} U_1' + \frac{dv^{(a)}(s)}{ds} \\
 \gamma_{22} &= 0
 \end{aligned} \tag{2.37}$$

For γ_{11} , the order of magnitude of the strain components are

$$\begin{aligned} U_1' &\sim O\left(\frac{\Delta d}{L^2}\right) \\ yU_2'' &\sim zU_3'' \sim O\left(\frac{\Delta d}{L^2}\right) \\ g_1U_1'' &\sim O\left(\frac{\Delta d^2}{L^3}\right) \\ G\varphi'' &\sim O\left(\frac{\Delta d^2}{L^3}\right) \end{aligned}$$

The $G\varphi''$ and $g_1(s)U_1''$ terms can be ignored when compared to the other terms in γ_{11} .

Therefore, the first-order approximations are determined to be:

$$\begin{aligned} \gamma_{11} &= U_1'(x) - y(s)U_2''(x) - z(s)U_3''(x) \\ 2\gamma_{12} &= \left(\frac{dG}{ds} + r_n\right)\varphi' + \frac{dg_1}{ds}U_1' + \frac{dv^{(a)}(s)}{ds} \\ \gamma_{22} &= 0 \end{aligned} \tag{2.38}$$

Second-Order Approximation

The second order approximation to the curvilinear displacements is found by adding in more warping factors:

$$\begin{aligned} v_t &= U_1(x) - y(s)U_2(x) - z(s)U_3(x) \\ &\quad + G(s)\varphi'(x) + g_1(s)U_1'(x) + v_1^{(a)} + w_4(s, x) \\ v_s &= U_2(x)\frac{dy}{ds} + U_3(x)\frac{dz}{ds} + \varphi(x)r_n + w_5(s, x) \\ v_\xi &= U_2(x)\frac{dz}{ds} - U_3(x)\frac{dy}{ds} - \varphi(x)r_t + w_6(s, x) \end{aligned} \tag{2.39}$$

The second-order strain field then becomes

$$\begin{aligned} \gamma_{11} &= \tilde{\gamma}_{11} + \frac{\partial w_4}{\partial x} \\ 2\gamma_{12} &= 2\tilde{\gamma}_{12} + \frac{\partial w_5}{\partial x} + 2\hat{\gamma}_{12} \\ \gamma_{22} &= \tilde{\gamma}_{22} + \hat{\gamma}_{22} \end{aligned} \tag{2.40}$$

where $(\sim)$ terms are given in Equation 2.38, and the $(\hat{\cdot})$ terms are given by

$$\begin{aligned} 2\hat{\gamma}_{12} &= \frac{\partial w_4}{\partial s} \\ \hat{\gamma}_{22} &= \frac{\partial w_5}{\partial s} + \frac{w_6}{R} \end{aligned} \quad (2.41)$$

Following the same reasoning that was given for the first-order approximation, both the warping functions w_5 and w_6 have a minor contribution to the shell energy, and the shear flow N_{12} is still given by Equation 2.32. This results in the warping function w_4 for the i th section being defined as:

$$\begin{aligned} \frac{\partial w_4}{\partial s} = & -2c(s)q_i + \left(b(s) - \frac{dg_1}{ds}\right) U_1' - b(s)y(s)U_2'' \\ & - b(s)z(s)U_3'' - c(s)C^{(a)} - \left(\frac{dG}{ds} + r_n\right) \varphi' - \frac{dv_1^{(a)}}{ds} \end{aligned} \quad (2.42)$$

Recalling that the single-value compatibility condition (Equation 2.35) must be satisfied for each cell, we get for the first cell:

$$-\frac{1}{2}q_1b_1 + U_1'b_3 - U_2'' \oint_I yb(s)ds - U_3'' \oint_I zb(s)ds - f_1^{(a)} + \varphi'b_4 - \frac{1}{2}q_wb_2 = 0$$

and for the second cell:

$$-\frac{1}{2}q_2b_5 + U_1'b_6 - U_2'' \oint_{II} yb(s)ds - U_3'' \oint_{II} zb(s)ds - f_2^{(a)} + \varphi'b_7 + \frac{1}{2}q_wb_2 = 0$$

where a detailed derivation of this procedure can be found in Section B.2.1.

For each section, the shear flows can be solved in the same manner as in the first approximation. Section B.2.2 gives the derivation of the shear flows for each branch, which are given as:

$$\begin{aligned} q_1 &= -2a_3U_1' - a_5U_2'' - a_7U_3'' - g_1^{(a)} - 2a_1\varphi' \\ q_2 &= -2a_4U_1' - a_6U_2'' - a_8U_3'' - f_2^{(a)} + \varphi'b_7 \\ q_w &= -2(a_3 - a_4)U_1' - (a_5 - a_6)U_2'' - (a_7 - a_8)U_3'' - (g_1^{(a)} - g_2^{(a)}) - 2(a_1 - a_2)\varphi' \end{aligned}$$

where

$$\begin{aligned}
 a_5 &= \frac{b_5 \oint_I y b(s) ds + b_2 d_1}{e_1} & a_6 &= \frac{b_1 + b_2}{b_2} a_5 - \frac{\oint_I y b(s) ds}{b_2} \\
 a_7 &= \frac{b_5 \oint_I z b(s) ds + b_2 d_2}{e_1} & a_8 &= \frac{b_1 + b_2}{b_2} a_7 - \frac{\oint_I z b(s) ds}{b_2} \\
 d_1 &= \oint_{I+II} y b(s) ds & d_2 &= \oint_{I+II} z b(s) ds \\
 e_1 &= b_1 b_2 + (b_1 + b_2) b_5
 \end{aligned}$$

Equation 2.42 can then be integrated for each section, and the result substituted into Equation 2.39. This gives the second-order approximation to the displacement field as:

$$\begin{aligned}
 v_\ell &= U_1(x) - y(s)U_2'(x) - z(s)U_3'(x) + G(s)\varphi'(x) \\
 &\quad + g_1(s)U_1'(x) + g_2(s)U_2'' + g_3(s)U_3'' + v_1^{(a)}(s) \\
 v_s &= U_2(x)\frac{dy}{ds} + U_3(x)\frac{dz}{ds} + \varphi(x)r_n \\
 v_\xi &= U_2(x)\frac{dz}{ds} - U_3(x)\frac{dy}{ds} - \varphi(x)r_t
 \end{aligned} \tag{2.43}$$

where for the left branch ($s = 0 \rightarrow s_1$),

$$\begin{aligned}
 g_2 &= -\int_0^s [b(\tau)y(\tau) - 4a_5c(\tau)]d\tau \\
 g_3 &= -\int_0^s [b(\tau)z(\tau) - 4a_7c(\tau)]d\tau
 \end{aligned}$$

for the web ($s = s_1 \rightarrow s_2$),

$$\begin{aligned}
 g_2 &= -\int_{s_1}^s [b(\tau)y(\tau) - 4(a_5 - a_6)c(\tau)]d\tau \\
 g_3 &= -\int_{s_1}^s [b(\tau)z(\tau) - 4(a_7 - a_8)c(\tau)]d\tau
 \end{aligned}$$

and for the right branch ($s = s_2 \rightarrow s_3$),

$$\begin{aligned}
 g_2 &= -\int_{s_2}^s [b(\tau)y(\tau) - 4a_6c(\tau)]d\tau \\
 g_3 &= -\int_{s_2}^s [b(\tau)z(\tau) - 4a_8c(\tau)]d\tau
 \end{aligned}$$

The terms g_2 and g_3 are both of the order of d^2 .

The associated strain field is:

$$\begin{aligned}
 \gamma_{11} &= U_1'(x) - y(s)U_2''(x) - z(s)U_3''(x) \\
 2\gamma_{12} &= \left(\frac{dG}{ds} + r_n\right)\varphi' + \frac{dg_1}{ds}U_1' + \frac{dg_2}{ds}U_2'' + \frac{dg_3}{ds}U_3'' + \frac{dv^{(a)}(s)}{ds} \\
 \gamma_{22} &= 0
 \end{aligned} \tag{2.44}$$

where the higher order terms are rejected as with the first approximation. The new terms are of the order

$$\frac{dg_2}{ds}U_2'' \sim \frac{dg_3}{ds}U_3'' \sim O\left(\frac{\Delta d}{L^2}\right)$$

Further corrections can be carried out, but will yield only smaller terms (Cesnik and Shin, 1998). Therefore, the displacement and strain fields converge at the second-order approximation. These terms are useful in developing constitutive relations that can be used in the nonlinear beam analysis along the blade length.

2.4 Constitutive Relations

By substituting the strains found in Equation 2.44 back into the energy density (Equation 2.28), and integrating over the contour s , the energy is then expressed with respect to a one-dimensional beam. This is possible because the energy is only dependent upon the geometry, material properties and the average cross-sectional kinematic variables.

The constitutive relationships are determined by differentiating the integrated energy density with respect to the associated kinematic variables. This results in the axial force F_1 , the torsional moment M_1 , the bending moment about the lead-lag axis M_2 , and the bending moment about the flap axis M_3 that are derived in Appendix C, as follows:

$$\begin{aligned} F_1 &= K_{11}U_1' + K_{12}\varphi' - K_{13}U_3'' + K_{14}U_2'' - F_1^{(a)} \\ M_1 &= K_{12}U_1' + K_{22}\varphi' - K_{23}U_3'' + K_{24}U_2'' - M_1^{(a)} \\ M_2 &= K_{13}U_1' + K_{23}\varphi' - K_{33}U_3'' + K_{34}U_2'' - M_2^{(a)} \\ M_3 &= K_{14}U_1' + K_{24}\varphi' - K_{34}U_3'' + K_{44}U_2'' - M_3^{(a)} \end{aligned} \tag{2.45}$$

where the stiffness coefficients K_{ij} in terms of cross-sectional geometry and material

properties are as follows:

$$\begin{aligned}
K_{11} &= \int_{all} \left(A - \frac{B^2}{C} \right) ds - 4a_3 \oint_I \frac{B}{C} ds - 4a_4 \oint_{II} \frac{B}{C} ds \\
K_{12} &= -4a_1 \oint_I \frac{B}{C} ds - 4a_2 \oint_{II} \frac{B}{C} ds \\
K_{13} &= \int_{all} \left(A - \frac{B^2}{C} \right) z ds + 4a_7 \oint_I \frac{B}{C} ds + 4a_8 \oint_{II} \frac{B}{C} ds \\
K_{14} &= - \int_{all} \left(A - \frac{B^2}{C} \right) y ds - 4a_5 \oint_I \frac{B}{C} ds - 4a_6 \oint_{II} \frac{B}{C} ds \\
K_{22} &= -4a_1 A_{eI} - 4a_2 A_{eII} \\
K_{23} &= 4a_7 A_{eI} + 4a_8 A_{eII} \\
K_{24} &= -4a_5 A_{eI} - 4a_6 A_{eII} \\
K_{33} &= \int_{all} \left(A - \frac{B^2}{C} \right) z^2 ds + 4a_7 \oint_I \frac{B}{C} z ds + 4a_8 \oint_{II} \frac{B}{C} z ds \\
K_{34} &= - \int_{all} \left(A - \frac{B^2}{C} \right) zy ds - 4a_5 \oint_I \frac{B}{C} z ds - 4a_6 \oint_{II} \frac{B}{C} z ds \\
K_{44} &= \int_{all} \left(A - \frac{B^2}{C} \right) y^2 ds + 4a_5 \oint_I \frac{B}{C} y ds + 4a_6 \oint_{II} \frac{B}{C} y ds
\end{aligned} \tag{2.46}$$

and the corresponding contributions from the piezoelectric fibers are given by:

$$\begin{aligned}
F_1^{(a)} &= \int_{all} \left(A^{(a)} - \frac{B}{C} C^{(a)} \right) ds + 2g_1^{(a)} \oint_I \frac{B}{C} ds + 2g_2^{(a)} \oint_{II} \frac{B}{C} ds \\
M_1^{(a)} &= 2g_1^{(a)} A_{eI} + 2g_2^{(a)} A_{eII} \\
M_2^{(a)} &= \int_{all} \left(A^{(a)} - \frac{B}{C} C^{(a)} \right) z ds + 2g_1^{(a)} \oint_I \frac{B}{C} z ds + 2g_2^{(a)} \oint_{II} \frac{B}{C} z ds \\
M_3^{(a)} &= - \int_{all} \left(A^{(a)} - \frac{B}{C} C^{(a)} \right) y ds - 2g_1^{(a)} \oint_I \frac{B}{C} y ds - 2g_2^{(a)} \oint_{II} \frac{B}{C} y ds
\end{aligned} \tag{2.47}$$

The constitutive relations can be written in matrix form as:

$$\begin{pmatrix} F_1 \\ M_1 \\ M_2 \\ M_3 \end{pmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{12} & K_{22} & K_{23} & K_{24} \\ K_{13} & K_{23} & K_{33} & K_{34} \\ K_{14} & K_{24} & K_{34} & K_{44} \end{bmatrix} \begin{pmatrix} U_1' \\ \varphi' \\ -U_3'' \\ U_2'' \end{pmatrix} - \begin{pmatrix} F_1^{(a)} \\ M_1^{(a)} \\ M_2^{(a)} \\ M_3^{(a)} \end{pmatrix} \tag{2.48}$$

The orders of magnitude can be expressed with respect to a small parameter ε , which shall now be defined as the chord length c of the blade divided by the axial

length L :

$$\varepsilon \equiv \frac{c}{L}$$

Because d is the maximum cross-sectional dimension, d/L will also be of order ε . Following the inequalities given in 2.1, the non-dimensional thickness, h/d , will be of order ε . Assuming that the maximum perpendicular distance to the chord length is of the same order of magnitude of the chord, then the only cross-sectional dimensions needed to approximate the stiffnesses are d and h . The material stiffness parameter E can be non-dimensionalized with respect to the mass per unit length of the blade m and the rotational velocity Ω of the blade squared. Based on available data for composite rotor blades, $E/m\Omega^2$ can be taken to be of order ε^{-5} . The order of the preceding stiffness expressions can therefore be determined as:

$$\begin{aligned}
 K_{11} &\sim O(Ehd) & \frac{K_{11}}{m\Omega^2 L^2} &\sim O(\varepsilon^{-2}) \\
 K_{12} &\sim O(Ehd^2) & \frac{K_{12}}{m\Omega^2 L^3} &\sim O(\varepsilon^{-1}) \\
 K_{13} &\sim O(Ehd^2) & \frac{K_{13}}{m\Omega^2 L^3} &\sim O(\varepsilon^{-1}) \\
 K_{14} &\sim O(Ehd^2) & \frac{K_{14}}{m\Omega^2 L^3} &\sim O(\varepsilon^{-1}) \\
 K_{22} &\sim O(Ehd^3) & \frac{K_{22}}{m\Omega^2 L^4} &\sim O(1) \\
 K_{23} &\sim O(Ehd^3) & \frac{K_{23}}{m\Omega^2 L^4} &\sim O(1) \\
 K_{24} &\sim O(Ehd^3) & \frac{K_{24}}{m\Omega^2 L^4} &\sim O(1) \\
 K_{33} &\sim O(Ehd^3) & \frac{K_{33}}{m\Omega^2 L^4} &\sim O(1) \\
 K_{34} &\sim O(Ehd^3) & \frac{K_{34}}{m\Omega^2 L^4} &\sim O(1) \\
 K_{44} &\sim O(Ehd^3) & \frac{K_{44}}{m\Omega^2 L^4} &\sim O(1)
 \end{aligned} \tag{2.49}$$

As the electromechanical coupling multiplied by the maximum electric field has been assumed to be of order $\Delta d^2/L^3$, the orders of magnitude of the actuation terms

can be found as:

$$\begin{aligned}
 F_1^{(a)} &\sim \frac{Ehd^3\Delta}{L^3} & \frac{F_1^{(a)}}{m\Omega^2 L^2} &\sim O(\varepsilon) \\
 M_1^{(a)} &\sim \frac{Ehd^4\Delta}{L^4} & \frac{M_1^{(a)}}{m\Omega^2 L^3} &\sim O(\varepsilon^2) \\
 M_2^{(a)} &\sim \frac{Ehd^4\Delta}{L^4} & \frac{M_2^{(a)}}{m\Omega^2 L^3} &\sim O(\varepsilon^2) \\
 M_3^{(a)} &\sim \frac{Ehd^4\Delta}{L^4} & \frac{M_3^{(a)}}{m\Omega^2 L^3} &\sim O(\varepsilon^2)
 \end{aligned} \tag{2.50}$$

Chapter 3

Nonlinear Beam Analysis

The equations of motion and boundary conditions for a long, slender, cantilever beam rotating at a constant angular speed can be obtained from Hamilton's Principle. Such a nonlinear analysis along the blade span of a homogeneous rotor was initially developed by Hodges and Dowell (1974) and Hodges and Ormiston (1976). As with Hodges and Ormiston (1976), the equations are developed without chord-wise offset of the elastic, mass, tension and aerodynamic axes.

3.1 Hamilton's Principle

Hamilton's Principle can be expressed as:

$$\int_{t_1}^{t_2} (\delta(U - T) - \delta W) dt = 0 \quad (3.1)$$

where U is the strain energy, T is the kinetic energy and δW is the virtual work done by the non-conservative external forces. In the case of a rotor blade, the only external forces to be considered are due to aerodynamic loading.

3.1.1 Structural, Inertial and Aerodynamic Terms

Structural Terms

The strain energy in the last chapter was found to be:

$$U = \int_0^L \int_{all} \Phi ds dx$$

Therefore, the variation of the strain energy with respect to the strain is given by:

$$\begin{aligned}
\delta U &= \int_0^L \int_{all} \left[\frac{\partial \Phi}{\partial \gamma_{11}} \delta \gamma_{11} + \frac{\partial \Phi}{\partial \gamma_{12}} \delta \gamma_{12} \right] ds dx \\
&= \int_0^L \left\{ \left(\int_{all} \left[\frac{\partial \Phi}{\partial \gamma_{11}} \frac{\partial \gamma_{11}}{\partial U'_1} + \frac{\partial \Phi}{\partial \gamma_{12}} \frac{\partial \gamma_{12}}{\partial U'_1} \right] ds \right) \delta U'_1 \right. \\
&\quad + \left(\int_{all} \left[\frac{\partial \Phi}{\partial \gamma_{11}} \frac{\partial \gamma_{11}}{\partial \varphi'_1} + \frac{\partial \Phi}{\partial \gamma_{12}} \frac{\partial \gamma_{12}}{\partial \varphi'_1} \right] ds \right) \delta \varphi' \\
&\quad - \left(\int_{all} \left[\frac{\partial \Phi}{\partial \gamma_{11}} \frac{\partial \gamma_{11}}{\partial U''_3} + \frac{\partial \Phi}{\partial \gamma_{12}} \frac{\partial \gamma_{12}}{\partial U''_3} \right] ds \right) \delta U''_3 \\
&\quad \left. + \left(\int_{all} \left[\frac{\partial \Phi}{\partial \gamma_{11}} \frac{\partial \gamma_{11}}{\partial U''_2} + \frac{\partial \Phi}{\partial \gamma_{12}} \frac{\partial \gamma_{12}}{\partial U''_2} \right] ds \right) \delta U''_2 \right\} dx \\
&= \int_0^L [F_1 \delta U'_1 + M_1 \delta \varphi' - M_2 \delta U''_3 + M_3 \delta U''_2] dx
\end{aligned} \tag{3.2}$$

Integrating by parts and setting the resulting boundary conditions equal to zero, one gets:

$$\begin{aligned}
\int_0^L F_1 \delta U'_1 dx &= F_1 \delta U_1|_0^L - \int_0^L F'_1 \delta U_1 dx \\
\int_0^L M_1 \delta \varphi' dx &= M_1 \delta \varphi|_0^L - \int_0^L M'_1 \delta \varphi dx \\
-\int_0^L M_2 \delta U''_3 dx &= -M_2 \delta U_3|_0^L + M'_2 \delta U_3|_0^L - \int_0^L M''_2 \delta U_3 dx \\
\int_0^L M_3 \delta U''_2 dx &= M_3 \delta U_2|_0^L - M'_3 \delta U_2|_0^L + \int_0^L M''_3 \delta U_2 dx
\end{aligned} \tag{3.3}$$

Hodges and Dowell (1974) developed a transformation matrix consistent to the second order that is given in Appendix E. In Figure 3.1, the non-deformable, global, coordinate system is denoted by a_i , while the deformable, local coordinate system used in the variational asymptotic method is designated by b_i .

The torsional twist φ is assumed to be small in comparison to the pitch angle θ . Therefore, under the small angle assumption:

$$\begin{aligned}
\sin(\theta + \varphi) &\approx \sin \theta + \varphi \cos \theta \\
\cos(\theta + \varphi) &\approx \cos \theta - \varphi \sin \theta
\end{aligned}$$

Applying these transformations and eliminating terms of order of ϵ^3 or higher,

Figure 3.1: Deformable and non-deformable coordinate systems (Shang and Hodges, 1995).

we get:

$$\begin{aligned}
 -\int_0^L F_1' \delta U_1 dx &= -\int_0^L F_1' \delta u dx - \int_0^L F_1' v \delta v' dx - \int_0^L F_1' v' \delta v dx \\
 &\quad - \int_0^L F_1' w \delta w' dx - \int_0^L F_1' w' \delta w dx \\
 &= -\int_0^L F_1' \delta u dx - F_1' v \delta v \Big|_0^L + \int_0^L F_1'' v \delta v dx \\
 &\quad - F_1' w \delta w \Big|_0^L + \int_0^L F_1'' w \delta w dx \\
 \int_0^L M_3'' \delta U_2 dx &= \int_0^L M_3'' (-v \sin \theta + w \cos \theta) \delta \varphi dx \\
 &\quad + \int_0^L M_3'' (\cos \theta - \varphi \sin \theta) \delta v dx \\
 &\quad + \int_0^L M_3'' (\sin \theta + \varphi \cos \theta) \delta w dx \\
 -\int_0^L M_2'' \delta U_3 dx &= \int_0^L M_2'' (v \cos \theta + w \sin \theta) \delta \varphi dx \\
 &\quad - \int_0^L M_2'' (-\sin \theta - \varphi \cos \theta) \delta v dx \\
 &\quad - \int_0^L M_2'' (\cos \theta - \varphi \sin \theta) \delta w dx
 \end{aligned} \tag{3.4}$$

where u , v and w are displacements along the global coordinates, x , y and z , re-

spectively.

Substituting Equations 3.4 and 3.3 into 3.2,

$$\begin{aligned}\delta U = & \int_0^L (-F_1'' - M_3'' (v' \cos \theta + w' \sin \theta) - M_2'' (v' \sin \theta - w' \cos \theta)) \delta u dx \\ & + \int_0^L (-M_1'' + M_3'' (-v \sin \theta + w \cos \theta) + M_2'' (v \cos \theta + w \sin \theta)) \delta \varphi dx \\ & + \int_0^L (F_1'' v + M_3'' (\cos \theta - \varphi \sin \theta) - M_2'' (-\sin \theta - \varphi \cos \theta)) \delta v dx \\ & + \int_0^L (F_1'' w + M_3'' (\sin \theta + \varphi \cos \theta) - M_2'' (\cos \theta - \varphi \sin \theta)) \delta w dx + b(U)\end{aligned}\quad (3.5)$$

where $b(U)$ represents all the boundary terms.

For a blade with constant cross-sectional properties, the derivatives of the forces and moments become:

$$\begin{aligned}F_1' &= K_{11}U_1'' + K_{12}\varphi'' - K_{13}U_3''' + K_{14}U_2''' - F_1^{(a)'} \\ F_1'' &= K_{11}U_1''' + K_{12}\varphi''' - K_{13}U_3'''' + K_{14}U_2'''' - F_1^{(a)''} \\ M_1' &= K_{12}U_1'' + K_{22}\varphi'' - K_{23}U_3''' + K_{24}U_2''' - M_1^{(a)'} \\ M_2' &= K_{13}U_1''' + K_{23}\varphi''' - K_{33}U_3'''' + K_{34}U_2'''' - M_2^{(a)''} \\ M_3' &= K_{14}U_1''' + K_{24}\varphi''' - K_{34}U_3'''' + K_{44}U_2'''' - M_3^{(a)''}\end{aligned}\quad (3.6)$$

Inertial Terms

The kinetic energy terms are given by Hodges and Dowell (1974), where the mass and elastic axis are assumed to be congruent. The $(\dot{})$ terms denote differentiation with respect to time. The variation of the kinetic energy is then:

$$\delta T = \int_0^L (\hat{Z}_1 \delta u + \hat{Z}_2 \delta v + \hat{Z}_3 \delta w + \hat{Z}_\varphi \delta \varphi) dx + b(T) \quad (3.7)$$

where

$$\begin{aligned}\hat{Z}_1 &= m (\Omega^2 x + 2\Omega \dot{v}) \\ \hat{Z}_2 &= m\Omega^2 v + 2m\Omega (\beta_{pc}\dot{w} - \dot{u}) - m\ddot{v} \\ \hat{Z}_3 &= -m\beta_{pc} (\Omega^2 x + 2\Omega \dot{v}) - m\ddot{w} \\ \hat{Z}_\varphi &= -mk_m^2 \ddot{\varphi} - m\Omega^2 (k_{m_2}^2 - k_{m_1}^2) \cos(\theta + \varphi) \sin(\theta + \varphi) \\ b(T) &= 0\end{aligned}\quad (3.8)$$

and the sectional integrals are given by:

$$\begin{aligned}k_{m_1}^2 &\equiv \frac{1}{m} \int \int_m \rho z^2 dy dz & k_{m_2}^2 &\equiv \frac{1}{m} \int \int_m \rho y^2 dy dz \\ m &\equiv \int \int_A \rho dy dz & k_m^2 &\equiv k_{m_1}^2 + k_{m_2}^2\end{aligned}\quad (3.9)$$

Aerodynamic Loading

The virtual work of the nonconservative forces as given by Hodges and Dowell (1974) is:

$$\delta W = \int_0^L (L_u \delta u + L_v \delta v + L_w \delta w + M_\varphi \delta \varphi) dx \quad (3.10)$$

where the aerodynamic loads are approximated to be (Hodges and Ormiston, 1976):

$$\begin{aligned} L_u &= 0 \\ L_v &= \frac{\rho_{air} a c}{2} \left\{ v_i^2 - \Omega^2 x^2 \frac{C_D}{a} - \Omega x v_i (\theta + \varphi) \right. \\ &\quad \left. - \left(2\Omega x \frac{C_D}{a} + (\theta + \varphi) v_i \right) \dot{v} + (2v_i - \Omega x (\theta + \varphi)) \dot{w} \right\} \\ L_w &= \frac{\rho_{air} a c}{2} \left\{ -\Omega x v_i + \Omega^2 x^2 \left(\theta + \varphi + \int_0^x v' w'' dx \right) \right. \\ &\quad \left. - \Omega^2 x v (\beta_{pc} + w') + \Omega^2 \frac{x c}{2} (\beta_{pc} + w') + (2\Omega x (\theta + \varphi) - v_i) \dot{v} \right. \\ &\quad \left. - \Omega x \dot{w} + \frac{3c}{4} \Omega x \dot{\varphi} - \frac{c}{4} \ddot{w} \right\} \\ M_\varphi &= -\frac{\rho_{air} a c}{2} \left(\frac{c^2}{8} \Omega x \dot{\varphi} \right) \end{aligned} \quad (3.11)$$

The aerodynamic load in the axial direction L_u is considered to be much smaller in comparison to the corresponding loads in the v and w directions, L_v and L_w . The density of air is given by ρ_{air} , the lift curve slope of the air foil is given by a , and c represents the chord length. C_D is the airfoil's coefficient of drag and β_{pc} is its precone angle.

The induced inflow velocity v_i can be calculated by:

$$v_i = \text{sign} [\theta + \varphi_0 (0.75L)] \Omega L \frac{\pi \sigma}{8} \left(\sqrt{1 + \frac{12}{\pi \sigma} |\theta + \varphi_0 (0.75L)|} - 1 \right) \quad (3.12)$$

where $\sigma = \frac{bc}{\pi L}$ is the rotor solidity, and $\varphi_0 (0.75L)$ is the equilibrium elastic twist at $x = 0.75L$, and b is the number of blades.

3.1.2 Hamilton's Equation

For arbitrary, admissible variations of δu , δv , δw and $\delta \varphi$, their respective coefficients must vanish in the resulting Hamilton's Integral for all x between 0 and L . This

results in four partial differential equations in terms of u , v , w and φ . The remaining terms must also disappear at 0 and L due to the boundary conditions.

Only first order non-linear terms are kept in the partial differential equations. Therefore, following the order of magnitudes given in the preceding chapter, all terms of order higher than ε^3 are discarded in Hamilton's Integral (Equation 3.1).

δu equation

By grouping together all of the δu terms from Equations 3.5, 3.7 and 3.10 in Equation 3.1, the following is obtained:

$$-F_1' - m(\Omega^2 x + 2\Omega \dot{v}) = 0 \quad (3.13)$$

The δu term is of order ε^2 ; therefore terms of higher order than ε are dropped according to the ordering system. Replacing F_1' with the expression given in Equation 3.6 results in:

$$-K_{11}U_1'' - K_{12}\varphi'' + K_{13}U_3''' - K_{14}U_2''' + F_1^{(a)'} - m(\Omega^2 x + 2\Omega \dot{v}) = 0$$

where the orders of magnitude of the stiffness terms are given in Equation 2.49.

Using the transformations given in Appendix E, U_2''' and U_3''' to the second order are found to be:

$$\begin{aligned} U_2''' &= (v \cos(\theta + \varphi))''' + (w \sin(\theta + \varphi))''' \\ &= v''' \cos \theta - (v\varphi)''' \sin \theta + w''' \sin \theta + (w\varphi)''' \cos \theta \end{aligned} \quad (3.14)$$

$$\begin{aligned} U_3''' &= (-v \sin(\theta + \varphi))''' + (w \cos(\theta + \varphi))''' \\ &= -v''' \sin \theta - (v\varphi)''' \cos \theta + w''' \cos \theta - (w\varphi)''' \sin \theta \end{aligned}$$

An equation for U_1'' is, therefore:

$$\begin{aligned} U_1'' &= -\frac{K_{12}}{K_{11}}\varphi'' - \frac{(K_{13} \sin \theta + K_{14} \cos \theta)}{K_{11}}v''' \\ &\quad - \frac{(K_{13} \cos \theta - K_{14} \sin \theta)}{K_{11}}(v\varphi)''' + \frac{(K_{13} \cos \theta - K_{14} \sin \theta)}{K_{11}}w''' \\ &\quad - \frac{(K_{13} \sin \theta + K_{14} \cos \theta)}{K_{11}}(w\varphi)''' - \frac{m}{K_{11}}(\Omega^2 x + 2\Omega \dot{v}) + \frac{F_1^{(a)'}}{K_{11}} \end{aligned} \quad (3.15)$$

This can be integrated twice with respect to x . Eliminating higher order terms, to obtain:

$$\begin{aligned}
 U_1 = & -\frac{K_{12}}{K_{11}}\varphi - \frac{(K_{13}\sin\theta + K_{14}\cos\theta)}{K_{11}}v' \\
 & - \frac{(K_{13}\cos\theta - K_{14}\sin\theta)}{K_{11}}(v\varphi)' + \frac{(K_{13}\cos\theta - K_{14}\sin\theta)}{K_{11}}w' \\
 & - \frac{(K_{13}\sin\theta + K_{14}\cos\theta)}{K_{11}}(w\varphi)' - \frac{m}{K_{11}}\Omega^2\frac{x^3}{6} + \int_0^x \frac{F_1^{(a)}}{K_{11}}dx
 \end{aligned} \quad (3.16)$$

A second-order transformation of U_1 can also be found from Appendix E to be:

$$U_1 = u + v'v + w'w \quad (3.17)$$

As Equations 3.16 and 3.17 are equal, we can isolate u :

$$\begin{aligned}
 u = & -\frac{K_{12}}{K_{11}}\varphi - \frac{(K_{13}\sin\theta + K_{14}\cos\theta)}{K_{11}}v' - \frac{(K_{13}\cos\theta - K_{14}\sin\theta)}{K_{11}}(v\varphi)' \\
 & + \frac{(K_{13}\cos\theta - K_{14}\sin\theta)}{K_{11}}w' - \frac{(K_{13}\sin\theta + K_{14}\cos\theta)}{K_{11}}(w\varphi)' \\
 & - v'v - w'w - \frac{m}{K_{11}}\Omega^2\frac{x^3}{6} + \int_0^x \frac{F_1^{(a)}}{K_{11}}dx
 \end{aligned} \quad (3.18)$$

Since u has been given in terms of the other variables, it can be eliminated as an independent variable. Thus, the number of independent equations to be solved are reduced to three.

δv equation

Grouping the δv terms from Equations 3.5, 3.7 and 3.10 in Equation 3.1, results in the following expression:

$$\begin{aligned}
 F_1''v + M_3''(\cos\theta - \varphi\sin\theta) - M_2''(-\sin\theta - \varphi\cos\theta) \\
 - m\Omega^2v - 2m\Omega(\beta_{pc}\dot{w} - \dot{u}) + m\ddot{v} = L_v
 \end{aligned} \quad (3.19)$$

The forces and moments can then be replaced with the terms given in Equation 3.6 and F_1'' is obtained from Equation 3.13, resulting in:

$$\begin{aligned}
 & (K_{14}U_1''' + K_{24}\varphi''' - K_{34}U_3''' + K_{44}U_2''' - M_3^{(a)''}) (\cos\theta - \varphi\sin\theta) \\
 & + (K_{13}U_1''' + K_{23}\varphi''' - K_{33}U_3''' + K_{34}U_2''' - M_2^{(a)''}) (\sin\theta + \varphi\cos\theta) \\
 & - 2m\Omega^2v - 2m\Omega\dot{v}'v - 2m\Omega(\beta_{pc}\dot{w} - \dot{u}) + m\ddot{v} = L_v
 \end{aligned} \quad (3.20)$$

The derivative with respect to time of Equation 3.18 to the second order is:

$$\begin{aligned} \dot{u} = & -\frac{K_{12}}{K_{11}}\dot{\varphi} - \frac{(K_{13}\sin\theta + K_{14}\cos\theta)}{K_{11}}\dot{v}' \\ & + \frac{(K_{13}\cos\theta - K_{14}\sin\theta)}{K_{11}}\dot{w}' - (\dot{v}v)' - (\dot{w}w)' \end{aligned} \quad (3.21)$$

Substituting for $\dot{u}$ in Equation 3.20 and grouping together the variables, the following is obtained:

$$\begin{aligned} & (K_{13}\sin\theta + K_{14}\cos\theta)U_1''' + (K_{13}\cos\theta - K_{14}\sin\theta)\varphi U_1''' \\ & + (-K_{34}(\cos\theta - \varphi\sin\theta) - K_{33}(\sin\theta + \varphi\cos\theta))U_3''' \\ & + (K_{44}(\cos\theta - \varphi\sin\theta) + K_{34}(\sin\theta + \varphi\cos\theta))U_2''' \\ & + (K_{24}(\cos\theta - \varphi\sin\theta) + K_{23}(\sin\theta + \varphi\cos\theta))\varphi''' \\ & - 2m\Omega^2 v - 2m\Omega(\dot{v}'v + (\dot{v}v)') - 2m\Omega\beta_{pc}\dot{w} \\ & - 2m\Omega\frac{K_{12}}{K_{11}}\dot{\varphi} - 2m\Omega\left(\frac{K_{13}}{K_{11}}\sin\theta + \frac{K_{14}}{K_{11}}\cos\theta\right)\dot{v}' \\ & + 2m\Omega\left(\frac{K_{13}}{K_{11}}\cos\theta - \frac{K_{14}}{K_{11}}\sin\theta\right)\dot{w}' \\ & - 2m\Omega(\dot{w}w)' + m\ddot{v} - M_3^{(a)'''}\cos\theta - M_2^{(a)'''}\sin\theta = L_v \end{aligned}$$

The expression for U_1''' can be obtained from Equation 3.15, i.e.:

$$\begin{aligned} U_1''' = & -\frac{K_{12}}{K_{11}}\varphi''' - \frac{(K_{13}\sin\theta + K_{14}\cos\theta)}{K_{11}}v'''' \\ & - \frac{(K_{13}\cos\theta - K_{14}\sin\theta)}{K_{11}}(v\varphi)'''' + \frac{(K_{13}\cos\theta - K_{14}\sin\theta)}{K_{11}}w'''' \\ & - \frac{(K_{13}\sin\theta + K_{14}\cos\theta)}{K_{11}}(w\varphi)'''' - \frac{m}{K_{11}}(\Omega^2 + 2\Omega\dot{v}') + \frac{F_1^{(a)''}}{K_{11}} \end{aligned} \quad (3.22)$$

The transformation given in Appendix E can be used to express U_2''' and U_3''' in the non-deforming coordinate system. To the second order, these are given as:

$$\begin{aligned} U_2''' = & (v\cos(\theta + \varphi))'''' + (w\sin(\theta + \varphi))'''' \\ = & v''''\cos\theta - (v\varphi)''''\sin\theta + w''''\sin\theta + (w\varphi)''''\cos\theta \end{aligned} \quad (3.23)$$

$$\begin{aligned} U_3''' = & (-v\sin(\theta + \varphi))'''' + (w\cos(\theta + \varphi))'''' \\ = & -v''''\sin\theta - (v\varphi)''''\cos\theta + w''''\cos\theta - (w\varphi)''''\sin\theta \end{aligned}$$

The expressions for U_1''' , U_2''' and U_3''' can be substituted into the δv equation to obtain:

$$\begin{aligned}
 & -m\Omega^2 \left(\frac{K_{13}}{K_{11}} \sin \theta + \frac{K_{14}}{K_{11}} \cos \theta \right) - m\Omega^2 \left(\frac{K_{13}}{K_{11}} \cos \theta - \frac{K_{14}}{K_{11}} \sin \theta \right) \varphi \\
 & + \left[\left(K_{33} - \frac{K_{13}^2}{K_{11}} \right) \sin^2 \theta + \left(K_{34} - \frac{K_{13}K_{14}}{K_{11}} \right) \sin 2\theta + \right. \\
 & \quad \left. \left(K_{44} - \frac{K_{14}^2}{K_{11}} \right) \cos^2 \theta \right] v'''' + \left[\frac{1}{2} \left(K_{33} - \frac{K_{13}^2}{K_{11}} \right) \sin 2\theta \right. \\
 & \quad \left. + \left(K_{34} - \frac{K_{13}K_{14}}{K_{11}} \right) \cos 2\theta - \frac{1}{2} \left(K_{44} - \frac{K_{14}^2}{K_{11}} \right) \sin 2\theta \right] v'''' \varphi \\
 & + \left[\frac{1}{2} \left(K_{33} - \frac{K_{13}^2}{K_{11}} \right) \sin 2\theta + \left(K_{34} - \frac{K_{13}K_{14}}{K_{11}} \right) \cos 2\theta \right. \\
 & \quad \left. - \frac{1}{2} \left(K_{44} - \frac{K_{14}^2}{K_{11}} \right) \sin 2\theta \right] (v\varphi)'''' - \left[\frac{1}{2} \left(K_{33} - \frac{K_{13}^2}{K_{11}} \right) \sin 2\theta \right. \\
 & \quad \left. + \left(K_{34} - \frac{K_{13}K_{14}}{K_{11}} \right) \cos 2\theta - \frac{1}{2} \left(K_{44} - \frac{K_{14}^2}{K_{11}} \right) \sin 2\theta \right] w'''' \\
 & + \left[- \left(K_{33} - \frac{K_{13}^2}{K_{11}} \right) \cos^2 \theta + \left(K_{34} - \frac{K_{13}K_{14}}{K_{11}} \right) \sin 2\theta \right. \\
 & \quad \left. - \left(K_{44} - \frac{K_{14}^2}{K_{11}} \right) \sin^2 \theta \right] w'''' \varphi + \left[\left(K_{33} - \frac{K_{13}^2}{K_{11}} \right) \sin^2 \theta \right. \\
 & \quad \left. + \left(K_{34} - \frac{K_{13}K_{14}}{K_{11}} \right) \sin 2\theta + \left(K_{44} - \frac{K_{14}^2}{K_{11}} \right) \cos^2 \theta \right] (w\varphi)'''' \\
 & + \left[\left(K_{23} - \frac{K_{12}K_{13}}{K_{11}} \right) \sin \theta + \left(K_{24} - \frac{K_{12}K_{14}}{K_{11}} \right) \cos \theta \right] \varphi''' \\
 & + \left[\left(K_{23} - \frac{K_{12}K_{13}}{K_{11}} \right) \cos \theta - \left(K_{24} - \frac{K_{12}K_{14}}{K_{11}} \right) \sin \theta \right] \varphi \varphi''' \\
 & - 2m\Omega^2 v - 4m\Omega \left(\frac{K_{13}}{K_{11}} \sin \theta + \frac{K_{14}}{K_{11}} \cos \theta \right) \dot{v}' - 2m\Omega (\dot{v}'v + (\dot{v}v)') \\
 & - 2m\Omega \left\{ \beta_{pc} \dot{w} + \frac{K_{12}}{K_{11}} \dot{\varphi} - \left(\frac{K_{13}}{K_{11}} \cos \theta - \frac{K_{14}}{K_{11}} \sin \theta \right) \dot{w}' + (\dot{w}w)' \right\} \\
 & + m\ddot{v} + \left(\frac{K_{13}}{K_{11}} \sin \theta + \frac{K_{14}}{K_{11}} \cos \theta \right) F_1^{(a)''} - M_3^{(a)''} \cos \theta - M_2^{(a)''} \sin \theta = L_v
 \end{aligned} \tag{3.24}$$

δw equation

Grouping together all the δw terms in Equations 3.5, 3.7 and 3.10, one obtains:

$$\begin{aligned} F_1'' w + M_3'' (\sin \theta + \varphi \cos \theta) - M_2'' (\cos \theta - \varphi \sin \theta) \\ + m\beta_{pc} (\Omega^2 x + 2\Omega \dot{v}) + m\ddot{w} = L_w \end{aligned} \quad (3.25)$$

The same procedure used in developing the δv equation will be applied here. Therefore, the forces and moments are given in Equation 3.6 are used to arrive at:

$$\begin{aligned} & (K_{14} U_1''' + K_{24} \varphi''' - K_{34} U_3'''' + K_{44} U_2'''' - M_3^{(a)''}) (\sin \theta + \varphi \cos \theta) \\ & - (K_{13} U_1''' + K_{23} \varphi''' - K_{33} U_3'''' + K_{34} U_2'''' - M_2^{(a)''}) (\cos \theta - \varphi \sin \theta) \\ & - m (\Omega^2 + 2\Omega \dot{v}') w + m\beta_{pc} (\Omega^2 x + 2\Omega \dot{v}) + m\ddot{w} = L_w \end{aligned}$$

Grouping the variables together results in:

$$\begin{aligned} & - (K_{13} \cos \theta - K_{14} \sin \theta) U_1''' + (K_{13} \sin \theta + K_{14} \cos \theta) \varphi U_1''' \\ & + (-K_{34} (\sin \theta + \varphi \cos \theta) + K_{33} (\cos \theta - \varphi \sin \theta)) U_3'''' \\ & + (K_{44} (\sin \theta + \varphi \cos \theta) - K_{34} (\cos \theta - \varphi \sin \theta)) U_2'''' \\ & + (K_{24} (\sin \theta + \varphi \cos \theta) - K_{23} (\cos \theta - \varphi \sin \theta)) \varphi''' \\ & - m (\Omega^2 + 2\Omega \dot{v}') w + m\beta_{pc} (\Omega^2 x + 2\Omega \dot{v}) + m\ddot{w} \\ & - M_3^{(a)''} \sin \theta - M_2^{(a)''} \cos \theta = L_w \end{aligned}$$

Using Equations 3.22 and 3.23, to transform U_1''' , U_2'''' and U_3'''' to the non-

deforming coordinate system results in:

$$\begin{aligned}
& m\Omega^2 \left(\frac{K_{13}}{K_{11}} \cos \theta - \frac{K_{14}}{K_{11}} \sin \theta \right) - m\Omega^2 \left(\frac{K_{13}}{K_{11}} \sin \theta + \frac{K_{14}}{K_{11}} \cos \theta \right) \varphi \\
& - \left[\frac{1}{2} \left(K_{33} - \frac{K_{13}^2}{K_{11}} \right) \sin 2\theta + \left(K_{34} - \frac{K_{13}K_{14}}{K_{11}} \right) \cos 2\theta \right. \\
& \left. - \frac{1}{2} \left(K_{44} - \frac{K_{14}^2}{K_{11}} \right) \sin 2\theta \right] v'''' + \left[\left(K_{33} - \frac{K_{13}^2}{K_{11}} \right) \sin^2 \theta \right. \\
& \left. + \left(K_{34} - \frac{K_{13}K_{14}}{K_{11}} \right) \sin 2\theta + \left(K_{44} - \frac{K_{14}^2}{K_{11}} \right) \cos^2 \theta \right] v'''' \varphi \\
& + \left[- \left(K_{33} - \frac{K_{13}^2}{K_{11}} \right) \cos^2 \theta + \left(K_{34} - \frac{K_{13}K_{14}}{K_{11}} \right) \sin 2\theta \right. \\
& \left. - \left(K_{44} - \frac{K_{14}^2}{K_{11}} \right) \sin^2 \theta \right] (v\varphi)'''' - \left[- \left(K_{33} - \frac{K_{13}^2}{K_{11}} \right) \cos^2 \theta \right. \\
& \left. + \left(K_{34} - \frac{K_{13}K_{14}}{K_{11}} \right) \sin 2\theta - \left(K_{44} - \frac{K_{14}^2}{K_{11}} \right) \sin^2 \theta \right] w'''' \\
& - \left[\frac{1}{2} \left(K_{33} - \frac{K_{13}^2}{K_{11}} \right) \sin 2\theta + \left(K_{34} - \frac{K_{13}K_{14}}{K_{11}} \right) \cos 2\theta \right. \\
& \left. - \frac{1}{2} \left(K_{44} - \frac{K_{14}^2}{K_{11}} \right) \sin 2\theta \right] w'''' \varphi - \left[\frac{1}{2} \left(K_{33} - \frac{K_{13}^2}{K_{11}} \right) \sin 2\theta \right. \\
& \left. + \left(K_{34} - \frac{K_{13}K_{14}}{K_{11}} \right) \cos 2\theta - \frac{1}{2} \left(K_{44} - \frac{K_{14}^2}{K_{11}} \right) \sin 2\theta \right] (w\varphi)'''' \\
& - \left[\left(K_{23} - \frac{K_{12}K_{13}}{K_{11}} \right) \cos \theta - \left(K_{24} - \frac{K_{12}K_{14}}{K_{11}} \right) \sin \theta \right] \varphi''' \\
& + \left[\left(K_{23} - \frac{K_{12}K_{13}}{K_{11}} \right) \sin \theta + \left(K_{24} - \frac{K_{12}K_{14}}{K_{11}} \right) \cos \theta \right] \varphi \varphi''' \\
& + 2m\Omega \left(\frac{K_{13}}{K_{11}} \cos \theta - \frac{K_{14}}{K_{11}} \sin \theta \right) \dot{v}' - m \left(\Omega^2 + 2\Omega \dot{v}' \right) w \\
& + m\beta_{pc} \left(\Omega^2 x + 2\Omega \dot{v} \right) + m\ddot{w} - \left(\frac{K_{13}}{K_{11}} \cos \theta - \frac{K_{14}}{K_{11}} \sin \theta \right) F_1^{(a)''} \\
& - M_3^{(a)''} \sin \theta - M_2^{(a)''} \cos \theta = L_w
\end{aligned} \tag{3.26}$$

$\delta\varphi$ equation

Finally, combining all of the $\delta\varphi$ terms in Equations 3.5, 3.7 and 3.10 one obtains:

$$\begin{aligned} & -M_1' + M_3'' (-v \sin \theta + w \cos \theta) + M_2'' (v \cos \theta + w \sin \theta) \\ & + mk_m^2 \ddot{\varphi} + m\Omega^2 (k_{m_2}^2 - k_{m_1}^2) \cos(\theta + \varphi) \sin(\theta + \varphi) = M_\varphi \end{aligned} \quad (3.27)$$

Replacing the forces and moments with their corresponding terms from Equation 3.6, we get:

$$\begin{aligned} & -K_{12}U_1'' - K_{22}\varphi'' + K_{23}U_3''' - K_{24}U_2''' + M_1^{(a)'} \\ & + (K_{14}U_1''' + K_{24}\varphi''' - K_{34}U_3'''' + K_{44}U_2'''') (-v \sin \theta + w \cos \theta) \\ & + (K_{13}U_1''' + K_{23}\varphi''' - K_{33}U_3'''' + K_{34}U_2'''') (v \cos \theta + w \sin \theta) \\ & + mk_m^2 \ddot{\varphi} + m\Omega^2 (k_{m_2}^2 - k_{m_1}^2) \cos(\theta + \varphi) \sin(\theta + \varphi) = M_\varphi \end{aligned}$$

Regrouping terms results in:

$$\begin{aligned} & -K_{12}U_1'' - K_{22}\varphi'' + K_{23}U_3''' - K_{24}U_2''' \\ & + (K_{13} \cos \theta - K_{14} \sin \theta) v U_1''' + (K_{13} \sin \theta + K_{14} \cos \theta) w U_1''' \\ & + (-K_{34} (-v \sin \theta + w \cos \theta) - K_{33} (v \cos \theta + w \sin \theta)) U_3''' \\ & + (K_{44} (-v \sin \theta + w \cos \theta) + K_{34} (v \cos \theta + w \sin \theta)) U_2''' \\ & + (K_{24} (-v \sin \theta + w \cos \theta) + K_{23} (v \cos \theta + w \sin \theta)) \varphi''' + mk_m^2 \ddot{\varphi} \\ & + m\Omega^2 (k_{m_2}^2 - k_{m_1}^2) (\cos \theta - \varphi \sin \theta) (\sin \theta + \varphi \cos \theta) + M_1^{(a)'} = M_\varphi \end{aligned}$$

Using Equations 3.15, 3.22, 3.23 and 3.14 to transform various terms into the

global coordinates, one gets:

$$\begin{aligned}
 & m\Omega^2 \frac{K_{12}}{K_{11}} x + 2m\Omega \frac{K_{12}}{K_{11}} \dot{v} - \left(K_{22} - \frac{K_{12}^2}{K_{11}} \right) \varphi'' \\
 & - m\Omega^2 \left(\frac{K_{13}}{K_{11}} \cos \theta - \frac{K_{14}}{K_{11}} \sin \theta \right) v \\
 & - m\Omega^2 \left(\frac{K_{13}}{K_{11}} \sin \theta + \frac{K_{14}}{K_{11}} \cos \theta \right) w \\
 & - \left[\left(K_{23} - \frac{K_{12}K_{13}}{K_{11}} \right) \sin \theta + \left(K_{24} - \frac{K_{12}K_{14}}{K_{11}} \right) \cos \theta \right] v''' \\
 & - \left[\left(K_{23} - \frac{K_{12}K_{13}}{K_{11}} \right) \cos \theta - \left(K_{24} - \frac{K_{12}K_{14}}{K_{11}} \right) \sin \theta \right] (v\varphi)''' \\
 & + \left[\left(K_{23} - \frac{K_{12}K_{13}}{K_{11}} \right) \cos \theta - \left(K_{24} - \frac{K_{12}K_{14}}{K_{11}} \right) \sin \theta \right] v\varphi''' \\
 & + \left[\left(K_{23} - \frac{K_{12}K_{13}}{K_{11}} \right) \cos \theta - \left(K_{24} - \frac{K_{12}K_{14}}{K_{11}} \right) \sin \theta \right] w''' \\
 & - \left[\left(K_{23} - \frac{K_{12}K_{13}}{K_{11}} \right) \sin \theta + \left(K_{24} - \frac{K_{12}K_{14}}{K_{11}} \right) \cos \theta \right] (w\varphi)''' \\
 & + \left[\left(K_{23} - \frac{K_{12}K_{13}}{K_{11}} \right) \sin \theta + \left(K_{24} - \frac{K_{12}K_{14}}{K_{11}} \right) \cos \theta \right] w\varphi''' \\
 & + \left[\frac{1}{2} \left(K_{33} - \frac{K_{13}^2}{K_{11}} \right) \sin (2\theta) + \left(K_{34} - \frac{K_{13}K_{14}}{K_{11}} \right) \cos (2\theta) \right. \\
 & \quad \left. - \frac{1}{2} \left(K_{44} - \frac{K_{14}^2}{K_{11}} \right) \sin (2\theta) \right] vv'''' + \left[\left(K_{33} - \frac{K_{13}^2}{K_{11}} \right) \sin^2 \theta \right. \\
 & \quad \left. + \left(K_{34} - \frac{K_{13}K_{14}}{K_{11}} \right) \sin (2\theta) + \left(K_{44} - \frac{K_{14}^2}{K_{11}} \right) \cos^2 \theta \right] ww'''' \\
 & - \left[\frac{1}{2} \left(K_{33} - \frac{K_{13}^2}{K_{11}} \right) \sin (2\theta) + \left(K_{34} - \frac{K_{13}K_{14}}{K_{11}} \right) \cos (2\theta) \right. \\
 & \quad \left. - \frac{1}{2} \left(K_{44} - \frac{K_{14}^2}{K_{11}} \right) \sin (2\theta) \right] ww'''' + \left[- \left(K_{33} - \frac{K_{13}^2}{K_{11}} \right) \cos^2 \theta \right. \\
 & \quad \left. + \left(K_{34} - \frac{K_{13}K_{14}}{K_{11}} \right) \sin (2\theta) - \left(K_{44} - \frac{K_{14}^2}{K_{11}} \right) \sin^2 \theta \right] vw'''' \\
 & + mk_m^2 \ddot{\varphi} + \frac{1}{2} m\Omega^2 (k_{m_2}^2 - k_{m_1}^2) \sin (2\theta) \\
 & + m\Omega^2 (k_{m_2}^2 - k_{m_1}^2) \varphi \cos (2\theta) + M_1^{(a)'} - \frac{K_{12}}{K_{11}} F_1^{(a)'} = M_\varphi
 \end{aligned} \tag{3.28}$$

Boundary Conditions

The boundary conditions, from Equations 3.1, 3.3 and 3.7 become:

$$b(U) - b(T) = 0 \quad (3.29)$$

Using the expressions given in Equation G.1-G.5, these can be broken down into the following:

$$\begin{aligned}
 & \left[-F_1' v - F_1 \left(\frac{K_{13}}{K_{11}} \cos \theta - \frac{K_{14}}{K_{11}} \sin \theta \right) \varphi' - M_3' \cos (\theta + \varphi) \right. \\
 & \quad \left. - M_2' \sin (\theta + \varphi) + M_3 \varphi' \sin \theta + M_2 \varphi' \cos \theta \right] \delta v|_0^L = 0 \\
 & \left[-F_1 \left(\frac{K_{13}}{K_{11}} \sin \theta + \frac{K_{14}}{K_{11}} \cos \theta \right) - F_1 \left(\frac{K_{13}}{K_{11}} \cos \theta - \frac{K_{14}}{K_{11}} \sin \theta \right) \varphi \right. \\
 & \quad \left. + M_3 \cos (\theta + \varphi) + M_2 \sin (\theta + \varphi) \right] \delta v'|_0^L = 0 \\
 & \left[-F_1' w - F_1 \left(\frac{K_{13}}{K_{11}} \sin \theta + \frac{K_{14}}{K_{11}} \cos \theta \right) \varphi' - M_3' \sin (\theta + \varphi) \right. \\
 & \quad \left. + M_2' \cos (\theta + \varphi) + M_3 \varphi' \cos \theta + M_2 \varphi' \sin \theta \right] \delta w|_0^L = 0 \\
 & \left[F_1 \left(\frac{K_{13}}{K_{11}} \cos \theta - \frac{K_{14}}{K_{11}} \sin \theta \right) - F_1 \left(\frac{K_{13}}{K_{11}} \sin \theta + \frac{K_{14}}{K_{11}} \cos \theta \right) \varphi \right. \\
 & \quad \left. + M_3 \sin (\theta + \varphi) - M_2 \cos (\theta + \varphi) \right] \delta w'|_0^L = 0 \quad (3.30) \\
 & \left[-F_1 \frac{K_{12}}{K_{11}} - F_1 \left(\frac{K_{13}}{K_{11}} \cos \theta - \frac{K_{14}}{K_{11}} \sin \theta \right) v' \right. \\
 & \quad \left. - F_1 \left(\frac{K_{13}}{K_{11}} \sin \theta + \frac{K_{14}}{K_{11}} \cos \theta \right) w' + M_1 \right. \\
 & \quad \left. - M_3 (v' \sin \theta - w' \cos \theta) + M_2 (v' \cos \theta + w' \sin \theta) \right. \\
 & \quad \left. + M_3' (v \sin \theta - w \cos \theta) - M_2' (v \cos \theta + w \sin \theta) \right] \delta \varphi|_0^L = 0 \\
 & \left[-F_1 \left(\frac{K_{13}}{K_{11}} \cos \theta - \frac{K_{14}}{K_{11}} \sin \theta \right) v - F_1 \left(\frac{K_{13}}{K_{11}} \sin \theta + \frac{K_{14}}{K_{11}} \cos \theta \right) w \right. \\
 & \quad \left. - M_3 (v \sin \theta - w \cos \theta) + M_2 (v \cos \theta + w \sin \theta) \right] \delta \varphi'|_0^L = 0
 \end{aligned}$$

The solution to differential equations requires that there be a certain number of boundary conditions rendering it unique. These conditions can be the result of the problem's physical restrictions, known as the imposed (or geometric) boundary

conditions. There are not always enough imposed boundary conditions to make the solution unique. The elegance of the variational method is that it always supplies the right number of boundary conditions. Conditions not imposed by the geometry of the problem are obtained from the integration by parts in the variational method and are known as the natural boundary conditions. A unique solution is obtained when the imposed and natural boundary conditions are both taken into account, as described by Lanczos (1970).

The imposed boundary conditions for this problem are that of a cantilever beam with a distributed load. This requires that the displacements (u , v and w) along with their rates of change (u' , v' and w') at the root of the beam be zero. The torsional twist (φ) at the blade root is zero, and the rate of twist (φ') at the blade tip is zero.

The natural boundary conditions are that there are no forces or moments at the free end of the beam. Therefore, the axial force (F_1) and the shear forces (M'_2 and M'_3) must be equal to zero at the blade tip. The rate of change of the axial force with respect to the beam length (F'_1) will also be zero at the free end. The free end of a cantilever beam cannot support any moments. Therefore M_1 , M_2 and M_3 are zero at the end of the blade.

Using the natural and imposed boundary conditions given in the preceding paragraphs, it can be seen that all the boundary conditions in Equation 3.30 are satisfied. These will be used to provide a unique solution to the equations of motion.

3.1.3 Dimensionless Equations

Equation 3.1 can be non-dimensionalized by dividing through by $m\Omega L^3$. It should be noted that the Hamilton's equation contains integrals with respect to x and t , which also have to be non-dimensionalized. The dimensionless parameters are given in Appendix F.

$\delta\bar{v}$ equation

By non-dimensionalizing the Equation (3.24), an equation for $\delta\bar{v}$ is obtained:

$$\begin{aligned}
& -\Pi_1 - \Pi_2\bar{\varphi} + \Gamma_1\bar{v}'''' + \Gamma_3\bar{v}''''\bar{\varphi} + \Gamma_3(\bar{v}\bar{\varphi})'''' - \Gamma_3\bar{w}'''' \\
& + \Gamma_2\bar{w}''''\bar{\varphi} + \Gamma_1(\bar{w}\bar{\varphi})'''' + \Upsilon_1\bar{\varphi}''' + \Upsilon_2\bar{\varphi}\bar{\varphi}''' - 2\bar{v} - 4\Pi_1\bar{v}' \\
& - 2\left(\bar{v}'\bar{v} + (\bar{v}\bar{v})' + \beta_{pc}\dot{\bar{w}} + \frac{\Lambda_{12}}{\Lambda_{11}}\dot{\bar{\varphi}} - \Pi_2\dot{\bar{w}}' + (\dot{\bar{w}}\bar{w})'\right) + \ddot{\bar{v}} \\
& + \Pi_1\bar{F}_1^{(a)''} - \bar{M}_3^{(a)''}\cos\theta - \bar{M}_2^{(a)''}\sin\theta = \bar{L}_v
\end{aligned} \tag{3.31}$$

 $\delta\bar{w}$ equation

Similarly, Equation 3.26 can be non-dimensionalized as:

$$\begin{aligned}
& \Pi_2 - \Pi_1\bar{\varphi} - \Gamma_3\bar{v}'''' + \Gamma_1\bar{v}''''\bar{\varphi} + \Gamma_2(\bar{v}\bar{\varphi})'''' - \Gamma_2\bar{w}'''' \\
& - \Gamma_3\bar{w}''''\bar{\varphi} - \Gamma_3(\bar{w}\bar{\varphi})'''' - \Upsilon_2\bar{\varphi}''' + \Upsilon_1\bar{\varphi}\bar{\varphi}''' + 2\Pi_2\bar{v}' \\
& - \bar{w} - 2\bar{v}'\bar{w} + \beta_{pc}(\bar{x} + 2\bar{v}) + \ddot{\bar{w}} \\
& - \Pi_2\bar{F}_1^{(a)''} - \bar{M}_3^{(a)''}\sin\theta - \bar{M}_2^{(a)''}\cos\theta = \bar{L}_w
\end{aligned} \tag{3.32}$$

 $\delta\bar{\varphi}$ equation

Finally, Equation 3.28 can be expressed dimensionlessly as:

$$\begin{aligned}
& \left(\frac{\Lambda_{12}^2}{\Lambda_{11}} - \Lambda_{22}\right)\bar{\varphi}'' + \frac{\Lambda_{12}}{\Lambda_{11}}(\bar{x} + 2\bar{v}) - \Upsilon_1\bar{v}''' + \Upsilon_2\bar{w}''' - \Pi_2\bar{v} \\
& - \Pi_1\bar{w} + \Upsilon_2(\bar{v}\bar{\varphi}''' - (\bar{v}\bar{\varphi})''') + \Upsilon_1(\bar{w}\bar{\varphi}''' - (\bar{w}\bar{\varphi})''') + \Gamma_3\bar{v}\bar{v}'''' \\
& + \Gamma_1\bar{w}\bar{v}'''' - \Gamma_3\bar{w}\bar{w}'''' + \Gamma_2\bar{v}\bar{w}'''' + \mu_2^2\bar{\varphi} + \frac{1}{2}(\mu_2^2 - \mu_1^2)\sin 2\theta \\
& + (\mu_2^2 - \mu_1^2)\bar{\varphi}\cos 2\theta + \bar{M}_1^{(a)'} - \frac{\Lambda_{12}}{\Lambda_{11}}\bar{F}_1^{(a)''} = \bar{M}_\varphi
\end{aligned} \tag{3.33}$$

Equations 3.24-3.28 are nonlinear integro-partial differential equations that govern the dynamic behavior of the composite rotor.

3.1.4 Dimensionless Boundary Conditions

The boundary conditions are developed in Appendix G, where the variation in u as an independent variable is eliminated, and the equations are non-dimensionalized to arrive at:

$$\begin{aligned}
 & \left[\underline{\bar{x}}\underline{\bar{v}} + 2\underline{\dot{\bar{v}}}\underline{\bar{v}} + \Pi_1\underline{\bar{x}} + 2\Pi_1\underline{\dot{\bar{v}}} + \Pi_2\underline{\bar{x}}\underline{\bar{\varphi}} - \Upsilon_1\underline{\bar{\varphi}}'' - \Upsilon_2\underline{\bar{\varphi}}\underline{\bar{\varphi}}'' \right. \\
 & -\Gamma_1\underline{\bar{v}}''' - \Gamma_3\underline{\bar{v}}'''\underline{\bar{\varphi}} - \Gamma_3 \left(\underline{\bar{v}}'''\underline{\bar{\varphi}} + 3\underline{\bar{v}}''\underline{\bar{\varphi}}' + 3\underline{\bar{v}}'\underline{\bar{\varphi}}'' + \underline{\bar{v}}\underline{\bar{\varphi}}''' \right) + \Gamma_3\underline{\bar{w}}''' \\
 & -\Gamma_2\underline{\bar{w}}'''\underline{\bar{\varphi}} + \Upsilon_2\underline{\bar{\varphi}}'\underline{\bar{\varphi}}' - \Gamma_1 \left(\underline{\bar{w}}'''\underline{\bar{\varphi}} + 3\underline{\bar{w}}''\underline{\bar{\varphi}}' + 3\underline{\bar{w}}'\underline{\bar{\varphi}}'' + \underline{\bar{w}}\underline{\bar{\varphi}}''' \right) \\
 & \left. + \Gamma_3\underline{\bar{v}}''\underline{\bar{\varphi}}' + \Gamma_2\underline{\bar{w}}''\underline{\bar{\varphi}}' + \underline{\bar{F}}_1^{(a)'}\Pi_1 + \underline{\bar{M}}_2^{(a)'}\sin\theta - \underline{\bar{M}}_3^{(a)'}\cos\theta \right] \delta\underline{\bar{v}}|_0^1 = 0
 \end{aligned} \tag{3.34}$$

$$\begin{aligned}
 & \left[\Upsilon_1\underline{\bar{\varphi}}' + \Upsilon_2\underline{\bar{\varphi}}\underline{\bar{\varphi}}' + \Gamma_1\underline{\bar{v}}'' + \Gamma_3\underline{\bar{v}}''\underline{\bar{\varphi}} \right. \\
 & \left. + \Gamma_3 \left(\underline{\bar{v}}''\underline{\bar{\varphi}} + \underline{\bar{v}}'\underline{\bar{\varphi}}' + \underline{\bar{v}}\underline{\bar{\varphi}}'' \right) - \Gamma_1 \left(\underline{\bar{w}}''\underline{\bar{\varphi}} + \underline{\bar{w}}'\underline{\bar{\varphi}}' + \underline{\bar{w}}\underline{\bar{\varphi}}'' \right) \right. \\
 & \left. - \Gamma_3\underline{\bar{w}}'' + \Gamma_2\underline{\bar{w}}\underline{\bar{\varphi}}'' - \Pi_1\underline{\bar{F}}_1^{(a)} - \underline{\bar{M}}_2^{(a)}\sin\theta + \underline{\bar{M}}_3^{(a)}\cos\theta \right] \delta\underline{\bar{v}}|_0^1 = 0
 \end{aligned} \tag{3.35}$$

$$\begin{aligned}
 & \left[\underline{\bar{x}}\underline{\bar{w}} + 2\underline{\dot{\bar{w}}}\underline{\bar{w}} - \Pi_2\underline{\bar{x}} - 2\Pi_2\underline{\dot{\bar{w}}} + \Pi_1\underline{\bar{x}}\underline{\bar{\varphi}} + \Upsilon_2\underline{\bar{\varphi}}'' - \Upsilon_1\underline{\bar{\varphi}}\underline{\bar{\varphi}}'' \right. \\
 & -\Gamma_3\underline{\bar{v}}''' + \Gamma_1\underline{\bar{v}}'''\underline{\bar{\varphi}} + \Gamma_2 \left(\underline{\bar{v}}'''\underline{\bar{\varphi}} + 3\underline{\bar{v}}''\underline{\bar{\varphi}}' + 3\underline{\bar{v}}'\underline{\bar{\varphi}}'' + \underline{\bar{v}}\underline{\bar{\varphi}}''' \right) \\
 & -\Gamma_2\underline{\bar{w}}''' - \Gamma_3\underline{\bar{w}}'''\underline{\bar{\varphi}} - \Gamma_3 \left(\underline{\bar{w}}'''\underline{\bar{\varphi}} + 3\underline{\bar{w}}''\underline{\bar{\varphi}}' + 3\underline{\bar{w}}'\underline{\bar{\varphi}}'' + \underline{\bar{w}}\underline{\bar{\varphi}}''' \right) \\
 & \left. + \Gamma_1\underline{\bar{v}}''\underline{\bar{\varphi}}' - \Gamma_3\underline{\bar{w}}''\underline{\bar{\varphi}}' - \Pi_2\underline{\bar{F}}_1^{(a)'} - \underline{\bar{M}}_2^{(a)'}\cos\theta + \underline{\bar{M}}_3^{(a)'}\sin\theta \right] \delta\underline{\bar{w}}|_0^1 = 0
 \end{aligned} \tag{3.36}$$

$$\begin{aligned}
 & \left[-\Upsilon_2\underline{\bar{\varphi}}' + \Upsilon_1\underline{\bar{\varphi}}\underline{\bar{\varphi}}' - \Gamma_3\underline{\bar{v}}'' + \Gamma_1\underline{\bar{v}}''\underline{\bar{\varphi}} + \Gamma_2 \left(\underline{\bar{v}}''\underline{\bar{\varphi}} + 2\underline{\bar{v}}'\underline{\bar{\varphi}}' + \underline{\bar{v}}\underline{\bar{\varphi}}'' \right) \right. \\
 & \left. + \Gamma_1\underline{\bar{w}}'' - \Gamma_3\underline{\bar{w}}''\underline{\bar{\varphi}} - \Gamma_3 \left(\underline{\bar{w}}''\underline{\bar{\varphi}} + 2\underline{\bar{w}}'\underline{\bar{\varphi}}' + \underline{\bar{w}}\underline{\bar{\varphi}}'' \right) \right. \\
 & \left. + \Pi_2\underline{\bar{F}}_1^{(a)} + \underline{\bar{M}}_2^{(a)}\cos\theta + \underline{\bar{M}}_3^{(a)}\sin\theta \right] \delta\underline{\bar{w}}|_0^1 = 0
 \end{aligned} \tag{3.37}$$

$$\begin{aligned}
 & \left[\left(\Lambda_{22} - \frac{\Lambda_{12}^2}{\Lambda_{11}} \right) \underline{\bar{\varphi}}' + \Upsilon_1\underline{\bar{v}}'' + \Upsilon_2 \left(\underline{\bar{v}}''\underline{\bar{\varphi}} + 2\underline{\bar{v}}'\underline{\bar{\varphi}}' \right) \right. \\
 & -\Upsilon_2\underline{\bar{w}}'' + \Upsilon_1 \left(\underline{\bar{w}}''\underline{\bar{\varphi}} + 2\underline{\bar{w}}'\underline{\bar{\varphi}}' \right) + \Upsilon_2\underline{\bar{v}}'\underline{\bar{\varphi}}' + \Upsilon_1\underline{\bar{w}}'\underline{\bar{\varphi}}' \\
 & + \Gamma_3\underline{\bar{v}}'\underline{\bar{v}}'' + \Gamma_1\underline{\bar{w}}'\underline{\bar{v}}'' + \Gamma_2\underline{\bar{v}}'\underline{\bar{w}}'' - \Gamma_3\underline{\bar{w}}'\underline{\bar{w}}'' + \Pi_2\underline{\bar{v}}\underline{\bar{x}} + \Pi_1\underline{\bar{w}}\underline{\bar{x}} \\
 & \left. - \Gamma_3\underline{\bar{v}}\underline{\bar{v}}''' - \Gamma_1\underline{\bar{w}}\underline{\bar{v}}''' - \Gamma_2\underline{\bar{v}}\underline{\bar{w}}''' + \Gamma_3\underline{\bar{w}}\underline{\bar{w}}''' - \frac{\Lambda_{12}}{\Lambda_{11}}\underline{\bar{F}}_1^{(a)} \right] \delta\underline{\bar{\varphi}}|_0^1 = 0
 \end{aligned} \tag{3.38}$$

$$[\Upsilon_2 \bar{v} \bar{\varphi}' + \Upsilon_1 \bar{w} \bar{\varphi}' + \Gamma_3 \bar{v} \bar{v}'' + \Gamma_1 \bar{w} \bar{v}'' + \Gamma_2 \bar{v} \bar{w}'' - \Gamma_3 \bar{w} \bar{w}''] \delta \bar{\varphi}'|_0^1 = 0 \quad (3.39)$$

3.2 Galerkin's Method

Galerkin's Method is used to solve the nonlinear integro-partial differential Equations 3.31-3.33. With this approach, the displacements are expanded as a series of products of generalized modal coordinates and mode shape functions, as given in Equations 3.40-3.42. As the variations of the displacements (δv , δw and $\delta \varphi$) are arbitrary, they cannot contain the generalized coordinates. But, since they are required to be admissible functions, they can be represented by the mode shape functions. Using a higher number of generalized coordinates and shape functions will improve the accuracy of the solution. Galerkin's Method requires that the mode shape functions satisfy at least the imposed (or geometric) boundary conditions. If the natural boundary conditions are also satisfied, the solution will converge with fewer shape functions, as shown by Duncan (1937). Hence, one has

$$\bar{v} = \sum_{j=1}^N V_j(\psi) \Psi_j(\bar{x}) \quad (3.40)$$

$$\bar{w} = \sum_{j=1}^N W_j(\psi) \Psi_j(\bar{x}) \quad (3.41)$$

$$\bar{\varphi} = \sum_{j=1}^N \phi_j(\psi) \Theta_j(\bar{x}) \quad (3.42)$$

where $V_j(\psi)$, $W_j(\psi)$ and $\phi_j(\psi)$ are the generalized modal coordinates, and ψ is the azimuth angle ($\psi = \Omega t$).

The uncoupled mode shapes of a standard non-rotating, uniform cantilever beam in bending and torsional twist are used as the corresponding mode shapes $\Psi_j(\bar{x})$ and $\Theta_j(\bar{x})$, respectively, for the rotor under consideration, i.e.,

$$\begin{aligned} \Psi_j(\bar{x}) &= \cosh(\beta_j \bar{x}) - \cos(\beta_j \bar{x}) - \alpha_j [\sinh(\beta_j \bar{x}) - \sin(\beta_j \bar{x})] \\ \Theta_j(\bar{x}) &= \sqrt{2} \sin(\nu_j \bar{x}) \end{aligned} \quad (3.43)$$

where the values of β_j and α_j are given in Chang and Craig (1969) and $\nu_j = \pi(j - \frac{1}{2})$.

These mode shapes satisfy the following boundary conditions:

$$\begin{aligned}\Psi_i(0) &= 0 & \Psi'_i(0) &= 0 \\ \Psi''_i(1) &= 0 & \Psi'''_i(1) &= 0 \\ \Theta_i(0) &= 0 & \Theta'_i(1) &= 0\end{aligned}\tag{3.44}$$

The boundary conditions that are not satisfied by the mode shape functions are the underlined terms in Equations 3.34-3.39. Special care must be taken with these conditions so as to arrive at a unique solution. Therefore, they are listed below:

$$\begin{aligned}& [\bar{x}\bar{v} + 2\dot{\bar{v}}\bar{v} + \Pi_1\bar{x} + 2\Pi_1\dot{\bar{v}} + \Pi_2\bar{x}\bar{\varphi} - \Upsilon_1\bar{\varphi}'' - \Upsilon_2\bar{\varphi}\bar{\varphi}'' \\ & - 3\Gamma_3\bar{v}'\bar{\varphi}'' - 3\Gamma_1\bar{w}'\bar{\varphi}'' + \Pi_1\bar{F}_1^{(a)'} + \bar{M}_2^{(a)'} \sin \theta - \bar{M}_3^{(a)'} \cos \theta] \delta\bar{v}|_0^1 = 0 \\ & [\Gamma_3\bar{v}\bar{\varphi}'' + \Gamma_1\bar{w}\bar{\varphi}'' - \Pi_1\bar{F}_1^{(a)} - \bar{M}_2^{(a)} \sin \theta + \bar{M}_3^{(a)} \cos \theta] \delta\bar{v}'|_0^1 = 0 \\ & [\bar{x}\bar{w} + 2\dot{\bar{w}}\bar{w} - \Pi_2\bar{x} - 2\Pi_2\dot{\bar{w}} + \Pi_1\bar{x}\bar{\varphi} + \Upsilon_2\bar{\varphi}'' - \Upsilon_1\bar{\varphi}\bar{\varphi}'' \\ & + 3\bar{v}'\bar{\varphi}'' + 3\bar{w}'\bar{\varphi}'' - \Pi_2\bar{F}_1^{(a)'} - \bar{M}_2^{(a)'} \cos \theta + \bar{M}_3^{(a)'} \sin \theta] \delta\bar{w}|_0^1 = 0 \\ & [\Gamma_2\bar{v}\bar{\varphi}'' - \Gamma_3\bar{w}\bar{\varphi}'' + \Pi_2\bar{F}_1^{(a)} + \bar{M}_2^{(a)} \cos \theta + \bar{M}_3^{(a)} \sin \theta] \delta\bar{w}'|_0^1 = 0 \\ & [\Pi_2\bar{v}\bar{x} + \Pi_1\bar{w}\bar{x} - \frac{\Lambda_{12}}{\Lambda_{11}}\bar{F}_1^{(a)}] \delta\bar{\varphi}|_0^1 = 0\end{aligned}\tag{3.45}$$

The actuation terms can also be represented as a series expansion, i.e.,

$$\bar{F}_1^{(a)} = E_e(\psi) \sum_{j=1}^N \mathbf{F}_{1j}^{(a)}(\bar{x})\tag{3.46}$$

$$\bar{M}_1^{(a)} = E_e(\psi) \sum_{j=1}^N \mathbf{M}_{1j}^{(a)}(\bar{x})\tag{3.47}$$

$$\bar{M}_2^{(a)} = E_e(\psi) \sum_{j=1}^N \mathbf{M}_{2j}^{(a)}(\bar{x})\tag{3.48}$$

$$\bar{M}_3^{(a)} = E_e(\psi) \sum_{j=1}^N \mathbf{M}_{3j}^{(a)}(\bar{x})\tag{3.49}$$

where $\mathbf{F}_{1j}^{(a)}(\bar{x})$, $\mathbf{M}_{1j}^{(a)}(\bar{x})$, $\mathbf{M}_{2j}^{(a)}(\bar{x})$, $\mathbf{M}_{3j}^{(a)}(\bar{x})$ are the effect of actuation on each mode shape due to the material properties and physical construction of the integrated

actuators, and $E_e(\psi)$ is the applied electric field. It should be noted that the applied electric field is independent of the mode shape as it is prescribed. The electric field produced by the deformation of the piezoelectric fibers is not included because of the assumption that there is no electric load to complete the circuit.

Substitution of the expressions given in Equations 3.40-3.42 and 3.46-3.49 into Equations 3.31-3.33 and 3.45 is carried out in Appendix H. The imposed and natural boundary conditions are then employed to obtain the nonlinear Galerkin's equations of motion (Equations H.6-H.8).

These nonlinear equations can be used to obtain the nonlinear, steady-state equations, and perturbation equations linearized about the steady-state values. This is accomplished by considering the generalized modal coordinates and the applied electric field to consist of two parts, the steady-state and perturbation quantities, i.e.,

$$\begin{aligned} V_j(\psi) &= V_{0j} + \Delta V_j(\psi) \\ W_j(\psi) &= W_{0j} + \Delta W_j(\psi) \\ \phi_j(\psi) &= \phi_{0j} + \Delta \phi_j(\psi) \\ E_e(\psi) &= E_{e0} + \Delta E_e(\psi) \end{aligned} \tag{3.50}$$

By eliminating the perturbation part (the Δ terms in Equation 3.50), the nonlinear steady state equations are obtained. By subtracting the steady-state equations from the steady-state and perturbation equations, the perturbation equations are obtained. These equations are then linearized by ignoring higher order terms. Therefore, the steady-state generalized modal coordinates will appear in the perturbation equations, as the perturbation equations are linearized about them. The nonlinear steady-state equations are given in Appendix J, and the linearized perturbation equations are given in Appendix K.

3.3 Solving the Equations of Motion

As discussed in Section 3.2, the nonlinear algebraic equations are separated into steady-state and perturbation equations. The steady-state equations given in Appendix J can be solved by a number of nonlinear techniques, such as the Newton-

Raphson Method or the Least Mean Squared Method. These equations have to be solved concurrently with the dimensionless inflow velocity, Equation F.9, which is dependent upon the steady-state deflection at $0.75L$. This leads to $3N + 2$ nonlinear equations to be solved for, where N is the number of mode shapes used. The steady-state deflections are also known as the trim conditions.

The perturbations can be represented as a $3N$ vector.

$$\{X\} = \begin{Bmatrix} \Delta V_j \\ \Delta W_j \\ \Delta \phi_j \end{Bmatrix} \quad (3.51)$$

This will result in the perturbation equations of motion given below:

$$[M] \{\ddot{X}\} + [C_t] \{\dot{X}\} + [K_t] \{X\} + \{F\} \Delta E_e = 0 \quad (3.52)$$

where the matrices can be found in Appendix K. The subscript t denotes the total matrices, which are composed of:

$$\begin{aligned} [C_t] &= [C_g] + [C_a] \\ [K_t] &= [K_s] + [K_a] \end{aligned}$$

where the subscript g denotes the gyroscopic contribution, the subscript a denotes the aerodynamic contribution, and the the subscript s denotes the structural contribution.

The mass matrix $[M]$ is symmetric, while the stiffness $[K_t]$ and damping $[C_t]$ matrices are asymmetric due to the nonconservative aerodynamic forces. The gyroscopic contribution to the damping matrix $[C_g]$ (Equation K.2) is anti-symmetric. The structural contribution to the stiffness matrix $[K_s]$ (Equation K.6) is symmetric. The $\{F\}$ vector is the actuation input for the perturbation problem.

Equation 3.52 can be written in the standard state-space form, i.e.,

$$\begin{aligned} \begin{Bmatrix} \dot{X} \\ \ddot{X} \end{Bmatrix} &= \begin{bmatrix} 0 & I \\ -M^{-1}K_t & -M^{-1}C_t \end{bmatrix} \begin{Bmatrix} X \\ \dot{X} \end{Bmatrix} + \begin{Bmatrix} 0 \\ -M^{-1}F \end{Bmatrix} \Delta E_e \\ &= [A] \begin{Bmatrix} X \\ \dot{X} \end{Bmatrix} + \{B\} \Delta E_e \end{aligned} \quad (3.53)$$

The eigenvalues of the $6N \times 6N$ matrix $[A]$ determines the stability of motion in the equilibrium operating condition. The input control vector $\{B\}$ of length $6N$ can therefore be used for vibration control.

Chapter 4

Specific Modeling Parameters

The preceding two chapters contain a detailed mathematical derivation of cross-sectional stiffnesses and the equations of motion for a general helicopter rotor blade. In the first part of this chapter, the cross-sectional stiffness terms are developed for a generalized box-beam spar. These stiffness terms for a particular cross-section are compared to a finite element prediction obtained from Cesnik and Shin (1998). The stiffness coefficients are given for two special layups: circumferentially uniform stiffness with web (CUS/w) and circumferentially asymmetric stiffness with web (CAS/w).

In the last chapter, the actuation terms were expressed with respect to their effectiveness on each mode shape. In the latter half of this chapter, the modal actuation functions are developed for a constant cross-sectional actuator.

4.1 Generalized Box-Beam

The stiffness terms K_{ij} that are the result of the variational asymptotic method are general stiffnesses that can be applied to any thin-walled, two-celled beam that abides by the restrictions given in relations 2.1. To facilitate the solving of the integrals in the stiffness terms K_{ij} (see Appendix D), this thesis will restrict itself to a box-beam spar.

The most general case for a box-beam will be developed, which includes different

s	$y(s)$	$z(s)$
$0 \rightarrow q$	$-s + \frac{q-r}{2}$	$\frac{p}{2}$
$q \rightarrow q+p$	$-\frac{q+r}{2}$	$-s + \frac{p}{2} + q$
$q+p \rightarrow 2q+p$	$s - \frac{3}{2}q - \frac{r}{2} - p$	$-\frac{p}{2}$
$2q+p \rightarrow 2q+2p$	$\frac{q-r}{2}$	$s - 2q - \frac{3}{2}p$
$2q+2p \rightarrow 2q+2p+r$	$s - \frac{3}{2}q - 2p - \frac{r}{2}$	$-\frac{p}{2}$
$2q+2p+r \rightarrow 2q+3p+r$	$\frac{q+r}{2}$	$s - 2q - \frac{5}{2}p - r$
$2q+3p+r \rightarrow 2q+3p+2r$	$-s + \frac{5}{2}q + 3p + \frac{3}{2}r$	$\frac{p}{2}$

Table 4.1: Equations of the contours for a two-celled box-beam, where s is given in Figure 2.2 and y and z are given in Figure 2.1.

numbers of plies for each part of the box-beam and different types of materials. The subscripts t , b , lv , w and rv will refer to the top, bottom, left-vertical, web and right-vertical walls, respectively. The box-beam is considered to have a constant height p , while the length of the left and right branches are denoted as q and r , respectively, as shown in Figure 4.1. The web is considered to be offset to the right of the centroid. The equations of the contours are given in Table 4.1.

Figure 4.1: Two-celled, thin-walled box-beam parameters.

Solving the integral expressions given in Equation 2.46:

$$\begin{aligned}
\int_{all} \left(A - \frac{B^2}{C} \right) ds &= \left(A_t - \frac{B_t^2}{C_t} \right) q + \left(A_{lv} - \frac{B_{lv}^2}{C_{lv}} \right) p + \left(A_b - \frac{B_b^2}{C_b} \right) q \\
&\quad + \left(A_w - \frac{B_w^2}{C_w} \right) p + \left(A_b - \frac{B_b^2}{C_b} \right) r + \left(A_{rv} - \frac{B_{rv}^2}{C_{rv}} \right) p \\
&\quad + \left(A_t - \frac{B_t^2}{C_t} \right) r \\
\int_{all} \left(A - \frac{B^2}{C} \right) y ds &= \left(A_{lv} - \frac{B_{lv}^2}{C_{lv}} \right) \left(-\frac{qp}{2} - \frac{rp}{2} \right) + \left(A_w - \frac{B_w^2}{C_w} \right) \left(\frac{qp}{2} - \frac{rp}{2} \right) \\
&\quad + \left(A_{rv} - \frac{B_{rv}^2}{C_{rv}} \right) \left(\frac{qp}{2} + \frac{rp}{2} \right) \\
\int_{all} \left(A - \frac{B^2}{C} \right) z ds &= \left(A_t - \frac{B_t^2}{C_t} \right) \frac{p}{2} (q+r) - \left(A_b - \frac{B_b^2}{C_b} \right) \frac{p}{2} (q+r) \\
\oint_I \frac{B}{C} ds &= \frac{B_t}{C_t} q + \frac{B_{lv}}{C_{lv}} p + \frac{B_b}{C_b} q + \frac{B_w}{C_w} p \\
\oint_{II} \frac{B}{C} ds &= -\frac{B_w}{C_w} p + \frac{B_b}{C_b} r + \frac{B_{rv}}{C_{rv}} p + \frac{B_t}{C_t} r \\
\oint_I \frac{B}{C} y ds &= -\frac{B_t}{C_t} \frac{rq}{2} - \frac{B_{lv}}{C_{lv}} \frac{(q+r)p}{2} - \frac{B_b}{C_b} \frac{qr}{2} + \frac{B_w}{C_w} \frac{(q-r)p}{2} \\
\oint_I \frac{B}{C} z ds &= \frac{B_t}{C_t} \frac{qp}{2} - \frac{B_b}{C_b} \frac{qp}{2} - \frac{B_b}{C_b} \frac{pr}{2} + \frac{B_t}{C_t} \frac{pr}{2} \\
\oint_{II} \frac{B}{C} y ds &= -\frac{B_w}{C_w} \frac{q-r}{2} p + \frac{B_b}{C_b} \frac{qr}{2} + \frac{B_{rv}}{C_{rv}} \frac{q+r}{2} p + \frac{B_t}{C_t} \frac{qr}{2} \\
\oint_{II} \frac{B}{C} z ds &= -\frac{B_b}{C_b} \frac{pr}{2} + \frac{B_t}{C_t} \frac{pr}{2} \\
\int_{all} \left(A - \frac{B^2}{C} \right) y^2 ds &= \left(A_t - \frac{B_t^2}{C_t} \right) \left(\frac{q^3+r^3}{12} + \frac{qr^2+q^2r}{4} \right) + \left(A_{lv} - \frac{B_{lv}^2}{C_{lv}} \right) \frac{p}{4} (q+r)^2 \\
&\quad + \left(A_b - \frac{B_b^2}{C_b} \right) \left(\frac{q^3+r^3}{12} + \frac{qr^2+q^2r}{4} \right) + \left(A_w - \frac{B_w^2}{C_w} \right) \frac{p}{4} (q-r)^2 \\
&\quad + \left(A_{rv} - \frac{B_{rv}^2}{C_{rv}} \right) \frac{p}{4} (q+r)^2 \\
\int_{all} \left(A - \frac{B^2}{C} \right) z^2 ds &= \left(A_t - \frac{B_t^2}{C_t} \right) \frac{p^2}{4} (q+r) + \left(A_{lv} - \frac{B_{lv}^2}{C_{lv}} \right) \frac{p^3}{12} \\
&\quad + \left(A_b - \frac{B_b^2}{C_b} \right) \frac{p^2}{4} (q+r) + \left(A_w - \frac{B_w^2}{C_w} \right) \frac{p^3}{12} + \left(A_{rv} - \frac{B_{rv}^2}{C_{rv}} \right) \frac{p^3}{12} \\
\int_{all} \left(A - \frac{B^2}{C} \right) yz ds &= 0
\end{aligned}$$

The b_i terms that are given in Appendix D are calculated to be:

$$\begin{aligned}
 b_1 &= -8 \left(\frac{q}{C_t} + \frac{p}{C_{iv}} + \frac{q}{C_b} \right) \\
 b_2 &= -8 \frac{p}{C_w} \\
 b_3 &= -2 \left(\frac{B_t}{C_t} q + \frac{B_{iv}}{C_{iv}} p + \frac{B_b}{C_b} q + \frac{B_w}{C_w} p \right) \\
 b_4 &= -2qp \\
 b_5 &= -8 \left(\frac{r}{C_b} + \frac{p}{C_{rv}} + \frac{r}{C_t} \right) \\
 b_6 &= -2 \left(\frac{B_w}{C_w} p + \frac{B_b}{C_b} r + \frac{B_{rv}}{C_{rv}} p + \frac{B_t}{C_t} r \right) \\
 b_7 &= -2rp
 \end{aligned}$$

4.1.1 Comparison to Finite Element Prediction

Cesnik and Shin (1998) used a general asymptotically-correct finite element program called VABS (Variational-Asymptotical Beam Section analysis) to find the stiffness quantities K_{ij} for an arbitrary cross-section. The box-beam configuration shown in Figure 4.2 was discretized with 364 six-node isoparametric elements with a total of 909 nodes. The material properties for the composite are given in Table 4.2.

Figure 4.2: Two-celled, thin-walled box-beam dimensions and ply angles (Cesnik and Shin, 1998).

Table 4.3 compares the stiffness values obtained by VABS (Cesnik and Shin, 1998) and the present formulation. For the most part, they are in agreement for such a thin-walled cross-sectional analysis. There are two main discrepancies: the

E_{LL}	142 GPa
$E_{NN} = E_{TT}$	9.8 GPa
$G_{LT} = G_{LN}$	6.0 GPa
G_{TN}	4.80 GPa
$\nu_{LT} = \nu_{LN}$	0.3
ν_{TN}	0.42
Thickness	0.127 mm
Density	1790 kg/m ³

Table 4.2: Properties of AS4/3506-1 Graphite/Epoxy where L is along the fiber direction, N is normal to the laminate and T is orthogonal to L and N .

K_{ij}	<i>Present</i>	<i>VABS</i>	% Difference
K_{11}	$7.195 * 10^5$ N	$8.477 * 10^5$ N	-15.1
K_{12}	$-1.850 * 10^3$ Nm	$-1.794 * 10^3$ Nm	3.1
K_{13}	$-6.472 * 10^2$ Nm	$6.337 * 10^2$ Nm	-202.1
K_{22}	$1.322 * 10^2$ Nm ²	$1.278 * 10^2$ Nm ²	3.4
K_{23}	$4.624 * 10^1$ Nm ²	$4.547 * 10^1$ Nm ²	3.7
K_{33}	$9.128 * 10^1$ Nm ²	$9.567 * 10^1$ Nm ²	-4.6
K_{44}	$2.081 * 10^2$ Nm ²	$2.070 * 10^2$ Nm ²	0.5

Table 4.3: Comparison of non-zero cross-sectional stiffnesses K_{ij} from the present theory with finite element predictions for a two-celled box-beam.

K_{11} and K_{13} terms. The K_{13} term is easier to explain, even though the difference is much greater. It is a result of a difference in sign in the expression for K_{13} . In Cesnik and Shin (1998), the force F_1 is given to depend on $+K_{13}U_3''$ in one equation and $-K_{13}U_3''$ in another. Using the right hand rule, the second can be shown to be correct. It is suspected that this is the cause of the sign difference of the K_{13} values. It is unknown why the K_{11} values differ so much, and it would be inappropriate to comment on without access to the **VABS** code.

4.1.2 Circumferentially Uniform Stiffness with Web

With reference to Equations 2.46, if the axial A , coupling B , and bending C stiffnesses are constant throughout the circumference, the blade is referred to as circumferentially uniform stiffness with web (CUS/w). This can be produced using a winding technique, where ply lay-ups on opposite sides are of reversed orientation. In such a case, we have,

$$A_t = A_b = A_{lv} = A_{rv} = A$$

$$B_t = B_b = B_{lv} = B_{rv} = B$$

$$C_t = C_b = C_{lv} = C_{rv} = C$$

The web in this case is assumed to contain plies of reverse orientation about the middle surface, which results in no coupling stiffness B in the web. Therefore, the stiffness constants used for the web are:

$$A_w = A_w$$

$$B_w = 0$$

$$C_w = C_w$$

Since the circumferential stiffnesses are constant, the stiffness matrix can be reduced for a centroidal coordinate system to

$$[K] = \begin{bmatrix} K_{11} & K_{12} & 0 & K_{14} \\ K_{12} & K_{22} & 0 & K_{24} \\ 0 & 0 & K_{33} & 0 \\ K_{14} & K_{24} & 0 & K_{44} \end{bmatrix} \quad (4.1)$$

where:

$$K_{11} = A(2q + 2r + 2p) + \underbrace{A_w p}$$

$$K_{12} = B(A_{eI} + A_{eII})$$

$$K_{14} = \underbrace{-\frac{1}{2}A_w(r - q)p}$$

$$K_{22} = \frac{C}{2(q + r + p)}(A_{eI} + A_{eII})^2 + \frac{1}{2} \frac{p^4 C C_w (r - q)^2}{(q + r + p)[2Cp(q + r + p) + C_w(2q + p)(2r + p)]}$$

$$K_{24} = -\frac{1}{2} \frac{p^2 B C_w (r - q)(2qr + qp + rp)}{[2Cp(q + r + p) + C_w(2q + p)(2r + p)]}$$

$$K_{33} = \frac{p^2}{6} \left(A - \frac{B^2}{C} \right) (3(q + r) + p) + \underbrace{\frac{p^3}{12} A_w}$$

$$K_{44} = \frac{1}{6} \left(A - \frac{B^2}{C} \right) (q + r)^2 (r + q + 3p) + \frac{1}{4} A_w p (r - q)^2 + \frac{1}{2} \frac{B^2}{C} \frac{C_w (q + r + p)(2qr + qp + rp)^2}{[2Cp(q + r + p) + C_w(2q + p)(2r + p)]}$$

The underbraced terms in the preceding expressions are the changes in stiffness due to the web. When the web stiffness terms A_w and C_w are set to zero, these expressions reduce to the CUS case developed by Badir (1992) for a box-beam without a web. The web contributes to the extension stiffness (K_{11}), torsion stiffness (K_{22}) and both bending stiffnesses (K_{33} and K_{44}), along with producing extension-bending (K_{14}) and torsion-bending (K_{24}) coupling. As the web does not contain a coupling stiffness term B , the extension-torsion coupling K_{12} is not affected.

4.1.3 Circumferentially Asymmetric Stiffness with Web

Arranging the ply lay-ups on top and bottom as mirror images, while the ply lay-up in the vertical members is of reverse orientation about the middle surface, produces a box-beam with circumferentially asymmetric stiffness with web (CAS/w). As the same number of plies are used in the horizontal and vertical walls, this results in the axial stiffness A being constant throughout the cross-section. The coupling stiffness B for the horizontal walls will be of opposite sign, and the coupling stiffness for the vertical walls will vanish. The shear stiffness C contribution along the horizontal and vertical walls are equal. Therefore,

$$A_t = A_b = A_{lv} = A_{rv} = A_w = A$$

$$B_t = -B_b = B$$

$$B_{lv} = B_{rv} = B_w = 0$$

$$C_t = C_b = C_h$$

$$C_{lv} = C_{rv} = C_w = C_v$$

The corresponding stiffness matrix, for a centroidal system of axes, reduces to:

$$[K] = \begin{bmatrix} K_{11} & 0 & 0 & K_{14} \\ 0 & K_{22} & K_{23} & 0 \\ 0 & K_{23} & K_{33} & 0 \\ K_{14} & 0 & 0 & K_{44} \end{bmatrix} \quad (4.2)$$

where:

$$\begin{aligned}
 K_{11} &= A(2q + 2r + 2p) - 2 \left(\frac{B^2}{C_t} \right) (q + r) + \underbrace{Ap}_{\text{web}} \\
 K_{14} &= \underbrace{A \frac{(r - q)p}{2}}_{\text{flap}} \\
 K_{22} &= \frac{C_h (AeI + AeII)^2}{2 \left[q + r + p \left(\frac{C_h}{C_v} \right) \right]} \\
 &\quad + \frac{p^4}{2} \frac{C_h^3 C_v (q - r)^2}{(4C_h C_v p q + 3C_h^2 p^2 + 4C_v^2 q r + 4C_h C_v p r) (q C_v + r C_v + p C_h)} \\
 K_{23} &= \frac{B (AeI + AeII)^2}{2 \left[q + r + p \left(\frac{C_h}{C_v} \right) \right]} \\
 &\quad + \frac{p^4}{2} \frac{B C_h^2 C_v (q - r)^2}{(4C_h C_v p q + 3C_h^2 p^2 + 4C_v^2 q r + 4C_h C_v p r) (q C_v + r C_v + p C_h)} \\
 K_{33} &= \frac{1}{2} A p^2 \left(q + r + \frac{p}{3} \right) - \frac{B^2}{2C_h} \left(p - \frac{(AeI + AeII)}{\left[q + r + p \left(\frac{C_h}{C_v} \right) \right]} \right) (AeI + AeII) \\
 &\quad + \frac{p^3 A}{12} + \frac{B^2 p^4}{2} \frac{C_h C_v (q - r)^2}{(4C_h C_v p q + 3C_h^2 p^2 + 4C_v^2 q r + 4C_h C_v p r) (q C_v + r C_v + p C_h)} \\
 K_{44} &= \frac{(q + r)^3}{6} \left(A - \frac{B^2}{C_h} \right) + \underbrace{\frac{A}{4} p (q - r)^2}_{\text{web}}
 \end{aligned}$$

This results in a stiffness matrix with torsion-bending coupling K_{23} due to the symmetry of the top and bottom members. The web causes extension-bending coupling in both flap and lead-lag (K_{13} and K_{14} , respectively), and contributes to the non-coupling stiffnesses (K_{11} , K_{22} , K_{33} and K_{22}) and the torsion-bending coupling term K_{23} .

4.2 Constant, Cross-Sectional Actuation

In Equations 3.46-3.49, the actuation forces and moments are represented as a series expansion where each term of the series represents the effect of actuation on a specific mode shape. How much each mode shape is affected is dependent upon the geometry

of the actuator, and its placement along the beam. The simplest case is that the actuation starts for the whole cross-section at a certain point along the rotor axis and ends for the whole cross-section at another point. As the blade has constant stiffness along the axial direction, the anisotropic actuation material is present along the whole length of the blade, but only certain segments will be activated. This can be accomplished by placing the interdigitated electrodes along this segment.

The influence of the j th actuation term will be determined by a ratio of the j th mode shape over the sum of the mode shapes. The variation associated with each force and moment in Equation 3.2 determines the mode shape used. The assumed mode shape for the axial deflection U_1 is the same as that of torsion, Θ_j . The assumed mode shapes for U_2 and U_3 are the same as that used in the bending terms, Ψ_j .

$$\begin{aligned}
 \mathbf{F}_{1j}^{(a)}(x) &= \frac{\Theta_j'(x)}{\sum_{k=1}^N \Theta_k'(x)} \frac{\bar{F}_1^{(a)}}{E_e} \\
 \mathbf{M}_{1j}^{(a)}(x) &= \frac{\Theta_j'(x)}{\sum_{k=1}^N \Theta_k'(x)} \frac{\bar{M}_1^{(a)}}{E_e} \\
 \mathbf{M}_{2j}^{(a)}(x) &= \frac{\Psi_j''(x)}{\sum_{k=1}^N \Psi_k''(x)} \frac{\bar{M}_2^{(a)}}{E_e} \\
 \mathbf{M}_{3j}^{(a)}(x) &= \frac{\Psi_j''(x)}{\sum_{k=1}^N \Psi_k''(x)} \frac{\bar{M}_3^{(a)}}{E_e}
 \end{aligned} \tag{4.3}$$

As linear piezoelectricity is assumed, then the actuation force or moment is assumed to vary linearly with the electric field. Thus, the expressions in Equation 4.3 are only dependent upon x . The induced strains due to the actuation, assuming ideal bonding between the actuator and the rest of the composite, will occur only at the end points (Crawley and Anderson, 1990). Moreover, the induced strains at the two ends of the actuated segment will be equal in magnitude, but opposite in direction. This allows the Galerkin actuation integrals given in I.3 to be decomposed

as follows:

$$\begin{aligned}
 A_{ij}^{(a)} &= \int_0^{x_1^-} \Psi_i'' F_{1j}^{(a)} d\bar{x} + \int_{x_1^-}^{x_1^+} \Psi_i'' F_{1j}^{(a)} d\bar{x} + \int_{x_1^+}^{x_2^-} \Psi_i'' F_{1j}^{(a)} d\bar{x} \\
 &\quad + \int_{x_2^-}^{x_2^+} \Psi_i'' F_{1j}^{(a)} d\bar{x} + \int_{x_2^+}^1 \Psi_i'' F_{1j}^{(a)} d\bar{x} \\
 B_{ij}^{(a)} &= \int_0^{x_1^-} \Psi_i'' M_{3j}^{(a)} d\bar{x} + \int_{x_1^-}^{x_1^+} \Psi_i'' M_{3j}^{(a)} d\bar{x} + \int_{x_1^+}^{x_2^-} \Psi_i'' M_{3j}^{(a)} d\bar{x} \\
 &\quad + \int_{x_2^-}^{x_2^+} \Psi_i'' M_{3j}^{(a)} d\bar{x} + \int_{x_2^+}^1 \Psi_i'' M_{3j}^{(a)} d\bar{x} \\
 C_{ij}^{(a)} &= \int_0^{x_1^-} \Psi_i'' M_{2j}^{(a)} d\bar{x} + \int_{x_1^-}^{x_1^+} \Psi_i'' M_{2j}^{(a)} d\bar{x} + \int_{x_1^+}^{x_2^-} \Psi_i'' M_{2j}^{(a)} d\bar{x} \\
 &\quad + \int_{x_2^-}^{x_2^+} \Psi_i'' M_{2j}^{(a)} d\bar{x} + \int_{x_2^+}^1 \Psi_i'' M_{2j}^{(a)} d\bar{x} \\
 D_{ij}^{(a)} &= \int_0^{x_1^-} \Theta_i' M_{1j}^{(a)} d\bar{x} + \int_{x_1^-}^{x_1^+} \Theta_i' M_{1j}^{(a)} d\bar{x} + \int_{x_1^+}^{x_2^-} \Theta_i' M_{1j}^{(a)} d\bar{x} \\
 &\quad + \int_{x_2^-}^{x_2^+} \Theta_i' M_{1j}^{(a)} d\bar{x} + \int_{x_2^+}^1 \Theta_i' M_{1j}^{(a)} d\bar{x} \\
 E_{ij}^{(a)} &= \int_0^{x_1^-} \Theta_i' F_{1j}^{(a)} d\bar{x} + \int_{x_1^-}^{x_1^+} \Theta_i' F_{1j}^{(a)} d\bar{x} + \int_{x_1^+}^{x_2^-} \Theta_i' F_{1j}^{(a)} d\bar{x} \\
 &\quad + \int_{x_2^-}^{x_2^+} \Theta_i' F_{1j}^{(a)} d\bar{x} + \int_{x_2^+}^1 \Theta_i' F_{1j}^{(a)} d\bar{x}
 \end{aligned} \tag{4.4}$$

where x_1 and x_2 represent the starting and ending positions of the segment being actuated, respectively.

Except for points x_1 and x_2 , at all points along the rotor the actuation force and moments are zero. Therefore, the expressions in Equation 4.4 can be reduced to obtain:

$$\begin{aligned}
 A_{ij}^{(a)} &= \int_{x_1^-}^{x_1^+} \Psi_i'' F_{1j}^{(a)} d\bar{x} + \int_{x_2^-}^{x_2^+} \Psi_i'' F_{1j}^{(a)} d\bar{x} \\
 B_{ij}^{(a)} &= \int_{x_1^-}^{x_1^+} \Psi_i'' M_{3j}^{(a)} d\bar{x} + \int_{x_2^-}^{x_2^+} \Psi_i'' M_{3j}^{(a)} d\bar{x} \\
 C_{ij}^{(a)} &= \int_{x_1^-}^{x_1^+} \Psi_i'' M_{2j}^{(a)} d\bar{x} + \int_{x_2^-}^{x_2^+} \Psi_i'' M_{2j}^{(a)} d\bar{x} \\
 D_{ij}^{(a)} &= \int_{x_1^-}^{x_1^+} \Theta_i' M_{1j}^{(a)} d\bar{x} + \int_{x_2^-}^{x_2^+} \Theta_i' M_{1j}^{(a)} d\bar{x} \\
 E_{ij}^{(a)} &= \int_{x_1^-}^{x_1^+} \Theta_i' F_{1j}^{(a)} d\bar{x} + \int_{x_2^-}^{x_2^+} \Theta_i' F_{1j}^{(a)} d\bar{x}
 \end{aligned} \tag{4.5}$$

The integration across the end points results in the evaluation of the term at the

end points, i.e.,

$$\begin{aligned}
 A_{ij}^{(a)} &= \left(\Psi_i''(x_1) \frac{\Theta_j'(x_1)}{\sum_{k=1}^N \Theta_k'(x_1)} - \Psi_i''(x_2) \frac{\Theta_j'(x_2)}{\sum_{k=1}^N \Theta_k'(x_2)} \right) \frac{\bar{F}_1^{(a)}}{E_e} \\
 B_{ij}^{(a)} &= \left(\Psi_i''(x_1) \frac{\Psi_j''(x_1)}{\sum_{k=1}^N \Psi_k''(x_1)} - \Psi_i''(x_2) \frac{\Psi_j''(x_2)}{\sum_{k=1}^N \Psi_k''(x_2)} \right) \frac{\bar{M}_3^{(a)}}{E_e} \\
 C_{ij}^{(a)} &= \left(\Psi_i(x_1)'' \frac{\Psi_j''(x_1)}{\sum_{k=1}^N \Psi_k''(x_1)} - \Psi_i(x_2)'' \frac{\Psi_j''(x_2)}{\sum_{k=1}^N \Psi_k''(x_2)} \right) \frac{\bar{M}_2^{(a)}}{E_e} \\
 D_{ij}^{(a)} &= \left(\Theta_i'(x_1) \frac{\Theta_j'(x_1)}{\sum_{k=1}^N \Theta_k'(x_1)} - \Theta_i'(x_2) \frac{\Theta_j'(x_2)}{\sum_{k=1}^N \Theta_k'(x_2)} \right) \frac{\bar{M}_1^{(a)}}{E_e} \\
 E_{ij}^{(a)} &= \left(\Theta_i'(x_1) \frac{\Theta_j'(x_1)}{\sum_{k=1}^N \Theta_k'(x_1)} - \Theta_i'(x_2) \frac{\Theta_j'(x_2)}{\sum_{k=1}^N \Theta_k'(x_2)} \right) \frac{\bar{F}_1^{(a)}}{E_e}
 \end{aligned} \tag{4.6}$$

The above are the constants of actuation using Galerkin's Method that must be obtained for a given set of actuation starting and end points.

Chapter 5

Aeroelastic Stability Analysis

The aeroelastic stability analysis is presented for two different spar designs: a baseline composite box-beam spar without active fibers and a composite box-beam spar containing piezoelectric fibers in the circumferential walls. The steady-state response is then found for the baseline spar and for the spar containing the piezoelectric fibers with no applied electric field and with the maximum electric field applied. The perturbation response is obtained for the three preceding cases, and a simple optimization is used for the dynamic control.

As the lower frequency instabilities are the ones of interest, there are a number of mode shapes N for which convergence is achieved for the eigenvalues of these lower frequencies. The N mode shapes result in $3N$ dynamic modes for the system (the size of the matrix $[A]$ from Equation 3.53 is $6N \times 6N$, and therefore contains $3N$ eigenvalues and their complex conjugates). The existing literature indicates that a value of $N = 6$ provides suitable convergence of the lower eigenvalues (Solaiman, 1999).

These results were obtained from computer codes that run in the MATLAB^{TM} Version 5.3 environment. An overview of the computer programs are given in Appendix M, and they are contained in the CD-ROM provided with this thesis.

5.1 Design of Spar

A baseline test spar was designed for a Westland WG13 helicopter, whose basic properties are presented in Table 5.1. The additional mass of steel rods placed at the blade's leading and trailing edges are used to tune the four lowest frequencies to approximately those of a typical helicopter rotor blade (at about 1/rev, 2/rev, 2.5/rev and 3.5/rev). These weights are sectioned off into small lengths to change the mass and inertial properties of the blade without increasing the stiffness.

Number of blades, b	4
Blade chord, c	0.394 m
Main rotor radius, L	7.52 m
Coefficient of drag, C_D	0.011
Tip velocity, ΩL	213.4 m/s

Table 5.1: Properties of Westland WG13 helicopter (Newman, 1994).

The cross-sectional dimensions of the spar are given in Table 5.2, where q , r and p are as shown in Figure 4.1. This spar can be contained inside a NACA 0020 airfoil, which is thicker than most conventional helicopter rotor blades. This is done to avoid flutter when the extra mass due to the piezoelectric fibers is included. Flutter is a dynamic instability characterized by a positive real part for an eigenvalue with a non-zero imaginary part. Therefore, as the flapping mode was dominant in this flutter case, the height of the spar was increased to augment the flapping stiffness K_{33} .

q	r	p
0.08 m	0.07 m	0.08 m

Table 5.2: Dimensions of Baseline spar.

The baseline spar is composed of layers of AS4/3506-1 Carbon/Epoxy, the material properties of which can be found in Table 4.2. The number of plies and their corresponding angles for each wall are given in Table 5.3.

<i>Member</i>	<i>Number of layers</i>	<i>Orientation</i>
top	16	$[-45, +45, 0, -45, +45, 0, -45, +45]_2$
bottom	16	$[-45, +45, 0, -45, +45, 0, -45, +45]_2$
left	8	$[-45, +45, 0, -45, +45, 0, -45, +45]$
web	8	$[-45, +45, 0, -45, +45, 0, -45, +45]$
right	8	$[-45, +45, 0, -45, +45, 0, -45, +45]$

Table 5.3: Number and orientation of AS4/3506-1 Graphite/Epoxy plies for Baseline spar.

As mentioned earlier, the frequencies of the spar were tuned to approximate those of a real helicopter rotor blade using weights placed close to the leading and trailing edges. As the present analysis assumes no chordwise offset of the elastic and mass axes, the round tuning rods are made from the same material, have the same radius and are placed equidistant from the centroid. The tuning weight parameters that result in realistic frequencies are given in Table 5.4.

	<i>Density</i>	<i>Radius</i>	<i>Distance from centroid</i>
Leading Edge	7830 kg/m ³	2 mm	0.17 m
Trailing Edge	7830 kg/m ³	2 mm	0.17 m

Table 5.4: Radii, densities and placements of tuning weights.

The baseline spar contains twice as many layers in the horizontal walls than the vertical ones. Although the axial stiffness A and shear stiffness C are twice as large in the horizontal walls than the vertical ones, the stiffness matrix $[K]$ contains the same entries as in the CUS/w spar. The stiffness matrix $[K]$ can be reduced due to the fact that for each ply angle, there is a corresponding ply at its negative angle. Therefore, the coupling stiffness B for each member must be zero. As the extension-torsion coupling K_{12} and the torsion-bending coupling K_{24} terms are dependent on the coupling stiffness B , they will both be equal to zero. This leaves only the diagonal terms in the stiffness matrix $[K]$ and the extension-bending coupling term K_{14} , which are presented in Table 5.6.

The Active Fiber Composite (AFC) spar is identical to the baseline spar, except

that it contains a single layer of PZT-5H piezoelectric fiber in the circumferential members. The material properties of the active fiber are given in Table 5.5. In order to maximize the actuation force in twist ($M_1^{(a)}$), the piezoelectric fibers are aligned at $+45^\circ$.

E_{LL}	30.2 GPa
$E_{NN} = E_{TT}$	14.9 GPa
$G_{LT} = G_{LN}$	5.13 GPa
G_{TN}	3.6 GPa
$\nu_{LT} = \nu_{LN}$	0.4542
ν_{TN}	0.4
Thickness	0.17 mm
Density	4060 kg/m ³
d_{LLL}	309 pm/V
d_{LTT}	-129 pm/V
d_{LNN}	-0.129 pm/V

Table 5.5: Properties of PZT-5H piezoelectric fiber where L is along the fiber direction, N is normal to the laminate and T is orthogonal to L and N .

The piezoelectric fiber layer creates a coupling stiffness B in the circumferential members, which results in extension-torsion coupling K_{12} and torsion-bending coupling K_{24} . The resulting stiffnesses and inertial terms are presented with the baseline case in Table 5.6.

As the lengths and ply angles of the top and bottom piezoelectric fiber layers are the same, the forces will cancel out each other, and the activation moment $M_3^{(a)}$ will be zero. The same reasoning applies to the left and right walls and to the $M_2^{(a)}$ activation moment. This leaves only two components of the activation vector, $F_1^{(a)}$ and $M_1^{(a)}$, which are given in Table 5.7 for an induced electric field of -1800 V/ 1.114 mm.

	<i>Baseline Spar</i>	<i>AFC Spar</i>
K_{11} (N)	4.40×10^7	4.53×10^7
K_{12} (Nm)	0	5.26×10^3
K_{14} (Nm)	-2.10×10^4	-2.10×10^4
K_{22} (Nm ²)	5.48×10^4	5.69×10^4
K_{24} (Nm ²)	0	4.90
K_{33} (Nm ²)	5.70×10^4	5.86×10^4
K_{44} (Nm ²)	1.06×10^5	1.10×10^5
m (kg/m)	1.72	2.04
k_{m1} (m)	0.935	0.935
k_{m2} (m)	0.0693	0.0703

Table 5.6: Non-zero stiffnesses, mass per unit length and radii of gyration for the Baseline and AFC spar.

$F_1^{(a)}$ (N)	-1.00×10^0
$M_1^{(a)}$ (Nm)	-1.28×10^{-1}
$M_2^{(a)}$ (Nm)	0
$M_3^{(a)}$ (Nm)	0

Table 5.7: Forcing vector terms for active spar with an induced electric field of -1800 V/1.114mm.

5.2 Steady-State Response

The steady-state deflections, also known as trim deflections, of the rotor blade must be determined first. They also define the operating conditions of the rotor, as the thrust produced is dependent upon the total blade angle, $\theta + \varphi_0$. The results are obtained as dimensionless deflections along the blade length for one operating condition, when the pitch angle θ is equal to 0.1 radians and no precone exists.

Although the piezoelectric fiber will be distributed throughout the length of the blade, only a certain segment will be actuated. This can be accomplished by locating the interdigitated electrodes only in the chosen segment. In order to excite

the desired mode shapes most effectively, the site of actuation segment becomes an important design issue. Therefore, the segment should be chosen in regions of high average strain and away from regions of zero strain (Crawley and de Luis, 1987). Since the torsional mode is the one that is required to be controlled, only the torsion contribution to the strain is considered. From Equation 2.44, the torsional contribution to the strain is given by:

$$\left(\frac{dG}{ds} + r_n \right) \varphi' \quad (5.1)$$

As the blade is of uniform cross-sectional stiffness, the coefficient given in brackets in Equation 5.1 will be a constant. Therefore, the torsional strain is only dependent upon φ' . Figure 5.1 gives the relative torsional strain distribution along the beam. To control pitch angle, the actuation segment was chosen in a region of high torsional strain and presented in Table 5.8.

Figure 5.1: Relative steady-state torsional strain distribution when $\theta = 0.1$ radians and $\beta_{pc} = 0$.

Start point	0.025
End point	0.1

Table 5.8: Start and end points of the actuated segment nondimensionalized with respect to blade length for steady-state actuation.

The nondimensional deflections along the blade length are presented in Figure 5.2 for a pitch angle of 0.1 radians and no precone. Only the baseline case and the case of the non-activated AFC blade are presented in this figure. This is because the magnitude of the forcing vector for the AFC blade (Table 5.7) is much smaller than the stiffness values (Table 5.6). Hence, there is no discernible change in the trim deflections of the non-activated AFC blade from the fully-activated case where the maximum electric field ($-1800 \text{ V} / 1.114 \text{ mm}$) is applied to the chosen segment.

The above results differ from other published data, such as the results reported on a 1/6th Mach-scaled Chinook CH47-D active blade by Cesnik and Ortega-Morales (1999). The CH47-D blade contains a D-shaped spar, where the arc is composed of 3 plies of E-glass, 2 plies of Graphite S-glass and 3 plies of AFC actuators, and the web is composed of 3 layers of E-glass. The experimental results for the latter blade are over $2^\circ/\text{m}$ peak-to-peak twist actuation. These results, however, were obtained for a non-rotating blade, and therefore did not have to deal with inertial forces or aerodynamic loading on the blade. Moreover, they were generated with 3 plies of active fibers, the extra mass of which would cause flutter. The number of active layers would have to be increased to achieve the same results on a full-scale blade.

5.3 Perturbation Response

The six lowest frequency eigenvalues of the different test cases examined in the preceding section are given in Table 5.9. The real part of the eigenvalues is a measure of the damping ratio, where a negative value represents a stable condition. The magnitude of the imaginary part of the eigenvalue is the aeroelastic modal frequency per revolution of the mode. The frequencies of the baseline case are in good agreement with those of typical helicopter rotor blades, except for the frequency

Figure 5.2: Steady-state deflections for the baseline spar and the spar containing the AFC fibers, not activated, when $\theta = 0.1$ radians and $\beta_{pc} = 0$.

of the second mode being half as large. This is not considered a problem, as the typical rotor blade frequencies are only a guideline and change for different blade designs.

It should be noted that not only was the static activation unable to modify the steady-state deflections, but it was also unable to affect the dynamic response. Appendix L contains the magnitudes and phases along the blade length for the first six modes of the baseline spar and the non-activated AFC spar. The case of the AFC spar with maximum electric field statically applied is, again, not given, as there is no discernible change between it and the non-activated case.

For vibration control, the frequencies to be eliminated are those that get trans-

Modes	Baseline Spar	AFC spar - off	AFC spar - static
1 st	- 0.0425 ± 0.9653 i	- 0.1563 ± 0.8675 i	- 0.1563 ± 0.8675 i
2 nd	- 0.2281 ± 0.9914 i	- 0.1076 ± 0.8941 i	- 0.1076 ± 0.8941 i
3 rd	- 0.0217 ± 2.6091 i	- 0.0188 ± 2.8625 i	- 0.0188 ± 2.8625 i
4 th	- 0.2880 ± 3.1709 i	- 0.3030 ± 3.0304 i	- 0.3030 ± 3.0304 i
5 th	- 0.0034 ± 3.7961 i	- 0.0029 ± 3.6094 i	- 0.0029 ± 3.6094 i
6 th	- 0.2314 ± 7.1916 i	- 0.2354 ± 6.7394 i	- 0.2354 ± 6.7394 i

Table 5.9: Eigenvalues for the different designs of the rotor blade at a pitch angle of $\theta = 0.1$ radians.

ferred to the airframe. The fuselage has many different modes of vibration. If the frequency of one of these modes coincides with the excitation frequency of the rotor blades, then high vibration levels can be experienced causing fatigue on parts and impinging on the comfort level of the occupants. For military applications, these high vibration levels also cause problems with tracking and weapons systems.

Consider the forcings F at the roots of a rotor with b blades. The forcings are n per revolution, which result in a frequency of $n\Omega$ and with a magnitude of A_n . The n^{th} forcing on the first blade can be expressed as:

$$F_{1,n} = A_n \cos(n\psi) \quad (5.2)$$

where n is an integer and ψ is the azimuth angle of the first blade.

The azimuth angle of the k^{th} blade is given as a function of the azimuth angle of the first blade. Considering that the blades are equally spaced, they are related by:

$$\psi_k = \psi + (k-1) \frac{2\pi}{b} \quad (5.3)$$

The total overall forcings are given by:

$$F = \sum_{k=1}^b \sum_{n=0}^{\infty} A_n \cos(n\psi_k) \quad (5.4)$$

Replacing ψ_k with the expression in Equation 5.3 to obtain:

$$F = \sum_{k=1}^b \sum_{n=0}^{\infty} A_n \cos\left(n \left[\psi + (k-1) \frac{2\pi}{b} \right]\right) \quad (5.5)$$

Expressing the cosine term in its equivalent exponential form, the following can be obtained:

$$\begin{aligned}
 F &= \sum_{k=1}^b \sum_{n=0}^{\infty} A_n \frac{1}{2} \left(e^{in[\psi+(k-1)\frac{2\pi}{b}]} + e^{-in[\psi+(k-1)\frac{2\pi}{b}]} \right) \\
 &= \sum_{k=1}^b \sum_{n=0}^{\infty} A_n \frac{1}{2} e^{in[\psi+(k-1)\frac{2\pi}{b}]} (e^1 + e^{-1}) \\
 &= \sum_{n=0}^{\infty} \frac{(e^1 + e^{-1})}{2} A_n \sum_{k=1}^b e^{in[\psi+(k-1)\frac{2\pi}{b}]} \\
 &= \sum_{n=0}^{\infty} \frac{(e^1 + e^{-1})}{2} A_n e^{in\psi} \sum_{k=1}^b e^{in(k-1)\frac{2\pi}{b}} \\
 &= \sum_{n=0}^{\infty} \frac{(e^1 + e^{-1})}{2} A_n e^{in\psi} \left\{ \frac{(e^{in\frac{2\pi}{b}})^b - 1}{e^{in\frac{2\pi}{b}} - 1} \right\}
 \end{aligned} \tag{5.6}$$

Examining the numerator in the last term of the preceding expression,

$$\begin{aligned}
 (e^{in\frac{2\pi}{b}})^b - 1 &= e^{in2\pi} - 1 \\
 &= 1 - 1 \\
 &= 0
 \end{aligned} \tag{5.7}$$

All the forcings at the hub will therefore cancel out, except when the denominator of the last term in Equation 5.6 is also equal to zero. This occurs when,

$$(e^{in\frac{2\pi}{b}}) = 1$$

Taking the natural logarithm of both sides to obtain:

$$\begin{aligned}
 in\frac{2\pi}{b} &= (2\pi i) \aleph \\
 n &= \aleph b
 \end{aligned} \tag{5.8}$$

where $\aleph$ is any arbitrary integer.

Therefore, only forcings at frequencies that are integer multiples of the number of blades are transmitted to the airframe. Using L'Hôpital's Rule, the fraction in Equation 5.6 can be evaluated as:

$$\lim_{n \rightarrow \aleph b} \left[\frac{\frac{\partial}{\partial n} (e^{inb\frac{2\pi}{b}} - 1)}{\frac{\partial}{\partial n} (e^{in\frac{2\pi}{b}} - 1)} \right] = \lim_{n \rightarrow \aleph b} \left[\frac{i2\pi b e^{in\frac{2\pi}{b}}}{in2\pi e^{in\frac{2\pi}{b}}} \right] = b \tag{5.9}$$

The hub acts as a filter and the total forcings transmitted to the airframe are:

$$F = b \sum_{N=0}^{\infty} A_{Nb} \cos(Nb\psi) \quad (5.10)$$

The hub filtering and the lower frequency restriction allows for the limiting of our control strategy to the b per revolution frequency. From Table 5.9, the 4th and 5th dynamic modes for the not activated AFC spar are those closest to this criteria. Figure 5.3 gives the relative torsional strain distribution due to the perturbation response. As the frequency of the 5th mode is closer to the b/rev frequency and there is little damping for this mode, it would cause the largest excitation of the airframe. This mode is therefore the most logically eliminated using the piezoelectric fibers. From Figure 5.3, the maximum strain for the 5th mode is located at about 0.7 of the blade length. The actuation segment location for dynamic control is presented in Table 5.10.

Figure 5.3: Torsional perturbation strain distribution for the AFC blade when $\theta = 0.1$ radians and $\beta_{pc} = 0$.

Start point	0.65
End point	0.75

Table 5.10: Start and end points of the actuated segment nondimensionalized with respect to blade length for dynamic actuation.

A Linear Quadratic Regulator (Friedland, 1986) was developed to control the 5th dynamic mode. This was accomplished by setting the diagonal terms associated with the 5th dynamic mode in the *state weighting* matrix to one, while all other terms are zero. The input vector $\{B\}$ in Equation 3.53 was multiplied by the maximum allowable electric field, and the *control weighting* matrix was set to one. This restrains the control signal from saturating the piezoelectric fibers. The resulting gain vector contains only two entries, corresponding to the two eigenvalues of the 5th mode, 0.3740×10^{-7} and 0.0216×10^{-7} . The six lowest frequency eigenvalues for the dynamically controlled case are given in Table 5.11.

Modes	<i>AFC spar - dynamic</i>
1 st	- 0.1563 $\pm$ 0.8675 ι
2 nd	- 0.1076 $\pm$ 0.8941 ι
3 rd	- 0.0188 $\pm$ 2.8625 ι
4 th	- 0.3030 $\pm$ 3.0304 ι
5 th	- 0.0029 $\pm$ 3.6094 ι
6 th	- 0.2354 $\pm$ 6.7394 ι

Table 5.11: Eigenvalues for dynamically controlled rotor blade at a pitch angle of $\theta = 0.1$ radians and no precone.

As the eigenvalues for the AFC blade without any activation (Table 5.9) were not changed by the addition of the control system, it is concluded that the piezoelectric fibers do not have enough actuation authority for vibration control.

Chapter 6

Summary, Conclusions and Recommendations

6.1 Summary

This thesis developed a formulation for analyzing two-celled, thin-walled composite helicopter rotor blades which include anisotropic actuators for the hover condition. The analysis consists of two parts: a linear two-dimensional analysis of the cross-section, and a nonlinear analysis along the blade.

The cross-section is modeled by the variational asymptotic method. This method is an iterative process of approximating the displacements and finding the associated shell energy. The displacement terms with a contribution of less than two orders of magnitude, compared to the largest in the shell energy, are ignored. The process is terminated when the new terms do not produce terms of the same order as the rest in the shell energy. This process will asymptotically approach the shell energy contained in a thin-walled beam. Thus, closed-form expressions for the displacements and stress fields, and beam stiffness coefficients result from this analysis. As the electromechanical coupling of the active fibers is integrated into the strain field, explicit expressions for the actuation are developed.

The strain energy expression derived in the cross-sectional analysis was used in Hamilton's Equation. This analysis kept only the first-order nonlinear terms in the

energy expression. The axial displacement was shown to be a dependent variable, and was therefore eliminated. This resulted in three equations of motion: lead-lag, flap and torsion and associated boundary conditions. The equations of motion were solved using Galerkin's Method. The kinematic variables were then divided into steady-state and perturbation quantities. The nonlinear steady-state values can be solved using nonlinear techniques, such as the Newton-Raphson Method. The perturbation quantities are then linearized about the steady-state values, which results in a set of ordinary differential equations.

The cross-sectional stiffness terms were developed for a generalized box-beam spar. When the effect of the web was removed, the stiffness values reduced to those for a published one-celled box-beam. The stiffnesses were also in agreement with a finite element analysis for a specific box-beam.

An aeroelastic analysis was performed to evaluate the performance of the piezoelectric fibers for pitch and vibration control. This analysis used parameters from full-scale Westland WG13 helicopter. A baseline spar consisting of layers of carbon fiber was developed and tuned to typical helicopter rotor blade frequencies. A single ply of piezoelectric fibers were added to the circumferential walls. A steady-state voltage was applied to induce twist in beam. An optimal controller was developed to reduce the amount of vibration transmitted to the fuselage.

6.2 Conclusions

The following conclusions can be drawn from this analysis:

- The first-order nonlinear closed-form equations of motion for a hingeless helicopter rotor blade were developed. As they are based on the variational asymptotic method, they include warping and coupling between the strains. This method uses continuous functions, which represent torsional interactions better than lumped parameter methods such as the finite element method.
- The piezoelectric fibers do not have enough actuation authority for pitch angle or vibration control. When the piezoelectric fibers had a maximum steady-state voltage applied, the pitch angle of the blade was not affected. An optimal controller

was designed to affect a vibration mode, but the level of actuation proved to be much lower than required. This is due to the fact that the stiffness associated with a full-scale helicopter blade is much larger than the level of actuation provided by the active fibers. The greater density of the fibers increases the total weight of the blade, which can lead to instability. Therefore, the addition of piezoelectric fibers does not alleviate the problem but increases it.

Even though it was found that the piezoelectric fibers do not provide the level of authority needed for pitch angle or vibration control, the closed form equations of motion developed in this thesis represent a contribution to the field.

6.3 Recommendations

Several recommendations are proposed for future work:

- A more complete stability analysis can be performed. This would include more than one operating condition, where the precone is not necessarily equal to zero.
- A simple box-beam cross-section was analyzed in this thesis, but the cross-sectional stiffness constants can be found for more complex shapes. Therefore, a more general cross-sectional stiffness program could be developed to solve for real blade shapes. In the current formulation, intermediate variables get divided by the stiffness of the web. Therefore, when a cross-section without a web is analyzed, these intermediate values result in the program crashing. The cross-sectional formulation should be modified in such a way so that this does not happen. The cross-sectional stiffnesses should also be formulated so that any multi-celled blade can be analyzed.
- This analysis can be extended to include non-uniformities in mass and stiffness along the blade length. The effects of droop, sweep, pre-twist and torque offset can also be included, along with chordwise offsets of the mass, elastic and aerodynamic axes. The pitch link was modeled as infinitely stiff, but pitch link flexibility could have a large impact on the aeroelastic stability.
- If the analysis is extended to include non-uniformities in mass and stiffness, then the piezoelectric fibers can be placed only where they are needed. As the mass of the blade would be greatly reduced, a less stiff blade could be used without flutter-

related problems. If the stiffness of the blade can be reduced, there is the possibility that the piezoelectric fibers could provide some amount of vibration control.

- The aerodynamic forces in this thesis were calculated by a two-dimensional strip theory. Therefore, the interactions between the vortex wake and the blades were not included even though this is known to be a major cause of vibrations. It has also been found by Sharpe (1986) that this theory does not adequately represent nonlinear aerodynamics at low Reynolds numbers. Therefore, a more complex three-dimensional aerodynamic loading should be investigated. It could also incorporate inflow turbulence and blade-vortex interaction effects.

- The analysis can be extended from hover to forward flight. The structural stiffness terms developed in this thesis are equally valid for this condition, but it would require different kinetic energy and aerodynamic loading terms. The cyclic pitching needed to produce forward flight would be another source of vibration on the blades. Forward flight also creates a region of reverse flow over the rotor blade, and therefore periodic aerodynamic loading terms would appear in the equations of motion.

- Experimental verification for many operating conditions is needed.

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Appendix A

Thin-Wall Approximation

This appendix is based on a private correspondence with Anant Grewal from the Structures, Dynamics and Propulsion Laboratory of the National Research Council, Canada. The final relationship has previously been given in literature (Berdichevsky et al., 1992), but without a detailed proof.

From Equation 2.3, where the Latin superscripts (or subscripts) i, j, k and l run from 1 to 3:

$$\begin{aligned} U &= \frac{1}{2} E^{ijkl} \varepsilon_{ij} \varepsilon_{kl} \\ &= \frac{1}{2} \sum_{i,j,k,l=1}^3 E^{ijkl} \varepsilon_{ij} \varepsilon_{kl} \end{aligned}$$

A.1 Reduction due to Symmetry

For a linear elastic body, stress σ and strain ε are related using the Hookean tensor E , such that:

$$\sigma_{kl} = E^{klij} \varepsilon_{ij}$$

For equilibrium, the following must be true:

$$\sigma_{kl} = \sigma_{lk} \quad \text{and} \quad \varepsilon_{ij} = \varepsilon_{ji}$$

Putting this stress symmetry back into the stress-strain relationship to obtain:

$$\sigma_{kl} = E^{klij} \varepsilon_{ij} = \sigma_{lk} = E^{lkij} \varepsilon_{ij}$$

This results in symmetrical terms in the Hookean tensor

$$E^{klij} = E^{ijkl}$$

Using the strain symmetry in the same manner, it follows that:

$$\sigma_{kl} = E^{klij} \varepsilon_{ij} = E^{klji} \varepsilon_{ji}$$

resulting in

$$E^{klij} = E^{klji}$$

This results in a Hookean tensor with only 21 instead of 81 independent terms.

A.2 Thin-Wall Approximation

For a shell, the stress components in the thickness direction are zero.

$$\sigma_{i3} = 0, \quad i = 1, 2, 3$$

This is equivalent to saying:

$$\frac{\partial U}{\partial \varepsilon_{3i}} = 0, \quad i = 1, 2, 3$$

$$\begin{aligned} \frac{\partial U}{\partial \varepsilon_{3m}} &= \frac{1}{2} \sum_{k,l=1}^3 E^{3mkl} \varepsilon_{kl} + \frac{1}{2} \sum_{i,j=1}^3 E^{ij3m} \varepsilon_{ij} = 0 \\ &= \frac{1}{2} \sum_{i,j=1}^3 (E^{3mij} + E^{ij3m}) \varepsilon_{ij} = 0 \\ &= \sum_{i,j=1}^3 E^{3mij} \varepsilon_{ij} = 0 \end{aligned}$$

therefore,

$$\frac{\partial U}{\partial \varepsilon_{3m}} = \sum_{i,j=1}^3 E^{3mij} \varepsilon_{ij} = 0 \quad (\text{A.1})$$

This can be applied to the original expression for U to give:

$$\begin{aligned} U &= \frac{1}{2} \sum_{i,j=1}^3 \sum_{\gamma,\delta=1}^2 E^{\gamma\delta ij} \varepsilon_{ij} \varepsilon_{\gamma\delta} + \frac{1}{2} \sum_{i,j=1}^3 \sum_{\gamma=1}^2 E^{\gamma 3ij} \varepsilon_{ij} \varepsilon_{\gamma 3} \\ &\quad + \frac{1}{2} \sum_{i,j=1}^3 \sum_{\delta=1}^2 E^{3\delta ij} \varepsilon_{ij} \varepsilon_{3\delta} + \frac{1}{2} \sum_{i,j=1}^3 E^{33ij} \varepsilon_{ij} \varepsilon_{33} \end{aligned}$$

where the last three terms are equal to zero, resulting in:

$$U = \frac{1}{2} \sum_{i,j=1}^3 \sum_{\gamma,\delta=1}^2 E^{\gamma\delta ij} \varepsilon_{ij} \varepsilon_{\gamma\delta} \quad (\text{A.2})$$

The Greek superscripts (or subscripts) α, β, γ and δ correspond to the Latin superscripts (or subscripts) i, j, k and l , respectively, but run from 1 to 2.

Upon expansion one gets:

$$U = \frac{1}{2} \sum_{\alpha,\beta,\gamma,\delta=1}^2 E^{\gamma\delta\alpha\beta} \varepsilon_{\alpha\beta} \varepsilon_{\gamma\delta} + \frac{1}{2} \sum_{\alpha,\gamma,\delta=1}^2 E^{\gamma\delta\alpha 3} \varepsilon_{\alpha 3} \varepsilon_{\gamma\delta} \\ + \frac{1}{2} \sum_{\beta,\gamma,\delta=1}^2 E^{\gamma\delta 3\beta} \varepsilon_{3\beta} \varepsilon_{\gamma\delta} + \frac{1}{2} \sum_{\gamma,\delta=1}^2 E^{\gamma\delta 33} \varepsilon_{33} \varepsilon_{\gamma\delta}$$

By the fact that $\varepsilon_{\alpha 3} = \varepsilon_{3\alpha}$ and $E^{\gamma\delta\alpha 3} = E^{\gamma\delta 3\alpha}$, it can be shown that:

$$U = \frac{1}{2} \sum_{\alpha,\beta,\gamma,\delta=1}^2 E^{\gamma\delta\alpha\beta} \varepsilon_{\alpha\beta} \varepsilon_{\gamma\delta} + \sum_{\alpha,\gamma,\delta=1}^2 E^{\gamma\delta\alpha 3} \varepsilon_{\alpha 3} \varepsilon_{\gamma\delta} \\ + \frac{1}{2} \sum_{\gamma,\delta=1}^2 E^{\gamma\delta 33} \varepsilon_{33} \varepsilon_{\gamma\delta} \quad (\text{A.3})$$

Substituting Equation A.3 into Equation A.1 to obtain:

$$E^{3m11} \varepsilon_{11} + 2E^{3m12} \varepsilon_{12} + 2E^{3m13} \varepsilon_{13} + E^{3m22} \varepsilon_{22} + 2E^{3m23} \varepsilon_{23} + E^{3m33} \varepsilon_{33} = 0$$

which can be written in matrix form as:

$$\begin{bmatrix} 2E^{3113} & 2E^{3123} & E^{3133} \\ 2E^{3213} & 2E^{3223} & E^{3233} \\ 2E^{3313} & 2E^{3323} & E^{3333} \end{bmatrix} \begin{Bmatrix} \varepsilon_{13} \\ \varepsilon_{23} \\ \varepsilon_{33} \end{Bmatrix} = - \begin{Bmatrix} E^{3111} \varepsilon_{11} + 2E^{3112} \varepsilon_{12} + E^{3122} \varepsilon_{22} \\ E^{3211} \varepsilon_{11} + 2E^{3212} \varepsilon_{12} + E^{3222} \varepsilon_{22} \\ E^{3311} \varepsilon_{11} + 2E^{3312} \varepsilon_{12} + E^{3322} \varepsilon_{22} \end{Bmatrix} \quad (\text{A.4})$$

Row-reduction is performed to produce zeros in the third column of the first two rows of the matrix, i.e.,

$$\begin{bmatrix} 2E^{3113} - 2\frac{E^{3313}E^{3133}}{E^{3333}} & 2E^{3123} - 2\frac{E^{3323}E^{3133}}{E^{3333}} & 0 \\ 2E^{3213} - 2\frac{E^{3313}E^{3233}}{E^{3333}} & 2E^{3223} - 2\frac{E^{3323}E^{3233}}{E^{3333}} & 0 \\ 2\frac{E^{3313}}{E^{3333}} & 2\frac{E^{3323}}{E^{3333}} & 1 \end{bmatrix} \begin{Bmatrix} \varepsilon_{13} \\ \varepsilon_{23} \\ \varepsilon_{33} \end{Bmatrix} \\ = \begin{Bmatrix} \left(-E^{31\alpha\beta} + \frac{E^{33\alpha\beta}E^{3133}}{E^{3333}} \right) \varepsilon_{\alpha\beta} \\ \left(-E^{32\alpha\beta} + \frac{E^{33\alpha\beta}E^{3233}}{E^{3333}} \right) \varepsilon_{\alpha\beta} \\ -\frac{E^{33\alpha\beta}}{E^{3333}} \varepsilon_{\alpha\beta} \end{Bmatrix}$$

Expressing the second diagonal minor in shorthand notation:

$$\left[2E^{3\mu\lambda 3} - 2\frac{E^{33\lambda 3} E^{3\mu 33}}{E^{3333}} \right] \varepsilon_{\lambda 3} = \left(-E^{3\mu\alpha\beta} + \frac{E^{33\alpha\beta} E^{3\mu 33}}{E^{3333}} \right) \varepsilon_{\alpha\beta} \quad (\text{A.5})$$

where μ and λ correspond to the rows and columns of the matrix, respectively, and run from 1 to 2.

Defining the following,

$$\left[E^{3\mu\lambda 3} - \frac{E^{33\lambda 3} E^{3\mu 33}}{E^{3333}} \right] \equiv \left[E^{\mu 3\lambda 3} - \frac{E^{\lambda 333} E^{\mu 333}}{E^{3333}} \right] = H_{\mu\lambda}^{-1}$$

allows Equation A.5 to be written as:

$$\varepsilon_{\lambda 3} = \frac{1}{2} H_{\mu\lambda} \left(-E^{\alpha\beta\mu 3} + \frac{E^{\alpha\beta 33} E^{\mu 333}}{E^{3333}} \right) \varepsilon_{\alpha\beta} \quad \text{for } \lambda, \mu, \alpha, \beta = 1, 2 \quad (\text{A.6})$$

Isolating ε_{33} in Equation A.4

$$\begin{aligned} \varepsilon_{33} &= -\frac{E^{33\alpha\beta}}{E^{3333}} \varepsilon_{\alpha\beta} - 2\frac{E^{33\lambda 3}}{E^{3333}} \varepsilon_{\lambda 3} \\ &= -\frac{E^{33\alpha\beta}}{E^{3333}} \varepsilon_{\alpha\beta} - H_{\mu\lambda} \frac{E^{33\lambda 3}}{E^{3333}} \left(-E^{\alpha\beta\mu 3} + \frac{E^{\alpha\beta 33} E^{\mu 333}}{E^{3333}} \right) \varepsilon_{\alpha\beta} \end{aligned}$$

which reduces to:

$$\varepsilon_{33} = - \left[\frac{E^{33\alpha\beta}}{E^{3333}} + H_{\mu\lambda} \frac{E^{33\lambda 3}}{E^{3333}} \left(-E^{\alpha\beta\mu 3} + \frac{E^{\alpha\beta 33} E^{\mu 333}}{E^{3333}} \right) \right] \varepsilon_{\alpha\beta} \quad (\text{A.7})$$

Substitute Equations A.6 and A.7 into Equation A.3 to obtain:

$$\begin{aligned} U &= \frac{1}{2} \sum_{\alpha, \beta, \gamma, \delta=1}^2 E^{\gamma\delta\alpha\beta} \varepsilon_{\alpha\beta} \varepsilon_{\gamma\delta} \\ &+ \sum_{\alpha, \beta, \gamma, \delta, \mu=1}^2 E^{\gamma\delta\alpha 3} \varepsilon_{\gamma\delta} \left[\frac{1}{2} H_{\mu\lambda} \left(-E^{\alpha\beta\mu 3} + \frac{E^{\alpha\beta 33} E^{\mu 333}}{E^{3333}} \right) \varepsilon_{\alpha\beta} \right] \\ &- \frac{1}{2} \sum_{\alpha, \beta, \gamma, \delta, \mu, \lambda=1}^2 E^{\gamma\delta 33} \varepsilon_{\gamma\delta} \left[\frac{E^{33\alpha\beta}}{E^{3333}} + H_{\mu\lambda} \frac{E^{33\lambda 3}}{E^{3333}} \left(-E^{\alpha\beta\mu 3} + \frac{E^{\alpha\beta 33} E^{\mu 333}}{E^{3333}} \right) \right] \varepsilon_{\alpha\beta} \end{aligned}$$

Grouping the $H_{\mu\lambda}$ terms together, the following is obtained:

$$\begin{aligned} U &= \frac{1}{2} \sum_{\alpha, \beta, \gamma, \delta=1}^2 E^{\gamma\delta\alpha\beta} \varepsilon_{\alpha\beta} \varepsilon_{\gamma\delta} - \frac{1}{2} \sum_{\alpha, \beta, \gamma, \delta, \mu=1}^2 \left(\frac{E^{\gamma\delta 33} E^{33\alpha\beta}}{E^{3333}} \right) \varepsilon_{\alpha\beta} \varepsilon_{\gamma\delta} \\ &- \frac{1}{2} \sum_{\alpha, \beta, \gamma, \delta, \mu, \lambda=1}^2 H_{\mu\lambda} \left(E^{\alpha\beta\mu 3} - \frac{E^{\alpha\beta 33} E^{\mu 333}}{E^{3333}} \right) \left(E^{\gamma\delta\lambda 3} - \frac{E^{\gamma\lambda 33} E^{33\mu 3}}{E^{3333}} \right) \varepsilon_{\alpha\beta} \varepsilon_{\gamma\delta} \end{aligned}$$

Therefore, if

$$\hat{U} = \min_{\varepsilon_{i3}} U = \frac{1}{2} D^{\alpha\beta\gamma\delta} \varepsilon_{\alpha\beta}^{(m)} \varepsilon_{\gamma\delta}^{(m)}, \quad \text{for } \alpha, \beta, \gamma, \delta = 1, 2$$

then the 2-D Hookean tensor $D^{\alpha\beta\gamma\delta}$ becomes

$$D^{\alpha\beta\gamma\delta} = E^{\alpha\beta\gamma\delta} - \frac{E^{\alpha\beta 33} E^{\gamma\delta 33}}{E^{3333}} - H_{\mu\lambda} G^{\alpha\beta\mu} G^{\gamma\delta\lambda}$$

where

$$G^{\alpha\beta\mu} = E^{\alpha\beta\mu 3} - \frac{E^{\alpha\beta 33} E^{\mu 333}}{E^{3333}}$$

and $H_{\mu\lambda}$ are components of the inverse of the 2-D matrix

$$\left\| E^{\mu 3\lambda 3} - \frac{E^{\mu 333} E^{\lambda 333}}{E^{3333}} \right\|$$

Appendix B

Shear Flows

B.1 First-Order Approximation

B.1.1 Compatibility Condition for Warping

The compatibility condition for the first cell reads:

$$\begin{aligned}
0 &= \int_0^{s_1} \frac{\partial w_1}{\partial s} ds + \int_{s_1}^{s_2} \frac{\partial w_1}{\partial s} ds \\
&= -2q_1 \int_0^{s_1} c(s) ds + U'_1 \int_0^{s_1} b(s) ds - \int_0^{s_1} c(s) C^{(a)} ds - \varphi' \int_0^{s_1} r_n ds \\
&\quad - U'_2 \int_{s=0}^{s=s_1} dy - U'_3 \int_{s=0}^{s=s_1} dz - 2q_w \int_{s_1}^{s_2} c(s) ds + U'_1 \int_{s_1}^{s_2} b(s) ds \\
&\quad - \int_{s_1}^{s_2} c(s) C^{(a)} ds - U'_2 \int_{s=s_1}^{s=s_2} dy - U'_3 \int_{s=s_1}^{s=s_2} dz - \varphi' \int_{s_1}^{s_2} r_n ds \quad (B.1) \\
&= -2q_1 \int_0^{s_1} c(s) ds + U'_1 \oint_I b(s) ds - \oint_I c(s) C^{(a)} ds \\
&\quad - \varphi' \oint_I r_n ds - 2q_w \int_{s_1}^{s_2} c(s) ds \\
&= -\frac{1}{2} q_1 b_1 + U'_1 b_3 - f_1^{(a)} + \varphi' b_4 - \frac{1}{2} q_w b_2
\end{aligned}$$

where

$$\begin{aligned}
b_1 &= 4 \int_0^{s_1} c(s) ds & b_2 &= 4 \int_{s_1}^{s_2} c(s) ds \\
b_3 &= \oint_I b(s) ds & b_4 &= \oint_I r_n ds = -2A_e I \\
f_1^{(a)} &= \oint_I c(s) C^{(a)} ds
\end{aligned} \quad (B.2)$$

For the second cell,

$$\begin{aligned}
0 &= \int_{s_2}^{s_3} \frac{\partial w_1}{\partial s} ds - \int_{s_1}^{s_2} \frac{\partial w_1}{\partial s} ds \\
&= -2q_2 \int_{s_2}^{s_3} c(s) ds + U'_1 \int_{s_2}^{s_3} b(s) ds - \int_{s_2}^{s_3} c(s) C^{(a)} ds - \varphi' \int_{s_2}^{s_3} r_n ds \\
&\quad - U'_2 \int_{s=s_2}^{s=s_3} dy - U'_3 \int_{s=s_2}^{s=s_3} dz + 2q_w \int_{s_1}^{s_2} c(s) ds - U'_1 \int_{s_1}^{s_2} b(s) ds \\
&\quad + \int_{s_1}^{s_2} c(s) C^{(a)} ds + U'_2 \int_{s=s_1}^{s=s_2} dy + U'_3 \int_{s=s_1}^{s=s_2} dz + \varphi' \int_{s_1}^{s_2} r_n ds \\
&= -2q_2 \int_{s_2}^{s_3} c(s) ds + U'_1 \oint_{II} b(s) ds - \oint_{II} c(s) C^{(a)} ds \\
&\quad - \varphi' \oint_{II} r_n ds + 2q_w \int_{s_1}^{s_2} c(s) ds \\
&= -\frac{1}{2} q_2 b_5 + U'_1 b_6 - f_2^{(a)} + \varphi' b_7 + \frac{1}{2} q_w b_2
\end{aligned} \tag{B.3}$$

where

$$\begin{aligned}
b_5 &= 4 \int_{s_2}^{s_3} c(s) ds & b_6 &= \oint_{II} b(s) ds \\
b_7 &= \oint_{II} r_n ds = -2A_{eII} & f_2^{(a)} &= \oint_{II} c(s) C^{(a)} ds
\end{aligned} \tag{B.4}$$

and A_{eI} and A_{eII} are the enclosed areas of the first and second cells, respectively.

B.1.2 Shear Flows

The shear flow in each section can therefore be found as:

$$\begin{aligned}
q_1 &= 2 \frac{b_2 b_6 + b_3 (b_2 + b_5)}{(b_1 + b_2)(b_2 + b_5) - b_2^2} U'_1 - 2 \frac{(b_5 + b_2) f_1^{(a)} + b_2 f_2^{(a)}}{(b_1 + b_2)(b_2 + b_5) - b_2^2} \\
&\quad + 2 \frac{b_2 b_7 + (b_2 + b_5) b_4}{(b_1 + b_2)(b_2 + b_5) - b_2^2} \varphi' \\
&= 2 \frac{b_2 c_2}{b_2 c_1} U'_1 - g_1^{(a)} + 2 \frac{b_2 c_3}{b_2 c_1} \varphi' \\
&= -2a_3 U'_1 - g_1^{(a)} - 2a_1 \varphi' \\
q_2 &= 2 \frac{b_6 (b_1 + b_2) + b_2 b_3}{(b_1 + b_2)(b_2 + b_5) - b_2^2} U'_1 - 2 \frac{(b_1 + b_2) f_2^{(a)} + b_2 f_1^{(a)}}{(b_1 + b_2)(b_2 + b_5) - b_2^2} \\
&\quad + 2 \frac{b_7 (b_1 + b_2) + b_2 b_4}{(b_1 + b_2)(b_2 + b_5) - b_2^2} \varphi' \\
&= -2 \frac{(b_1 + b_2) a_3 + b_3}{b_2} U'_1 - g_2^{(a)} - 2 \frac{(b_1 + b_2) a_1 + b_4}{b_2} \varphi' \\
&= -2a_4 U'_1 - g_2^{(a)} - 2a_2 \varphi' \\
q_w &= q_1 - q_2 \\
&= -2(a_3 - a_4) U'_1 - (g_1^{(a)} - g_2^{(a)}) - 2(a_1 - a_2) \varphi'
\end{aligned} \tag{B.5}$$

where

$$\begin{aligned}
g_1^{(a)} &= 2 \frac{(b_5 + b_2)f_1^{(a)} + b_2 f_2^{(a)}}{(b_1 + b_2)(b_2 + b_5) - b_2^2} \\
g_2^{(a)} &= 2 \frac{(b_1 + b_2)f_2^{(a)} + b_2 f_1^{(a)}}{(b_1 + b_2)(b_2 + b_5) - b_2^2} \\
a_1 &= -\frac{c_3}{c_1} \\
a_2 &= \frac{(b_1 + b_2)a_1 + b_4}{b_2} \\
a_3 &= -\frac{c_2}{c_1} \\
a_4 &= \frac{(b_1 + b_2)a_3 + b_3}{b_2} \\
c_1 &= b_1 + \frac{(b_1 + b_2)b_5}{b_2} \\
c_2 &= b_3 + b_6 + \frac{b_2 b_5}{b_2} \\
c_3 &= b_4 + b_7 + \frac{b_4 b_5}{b_2}
\end{aligned} \tag{B.6}$$

B.2 Second-Order Approximation

B.2.1 Compatibility Condition for Warping

Recalling the condition given in Equation 2.35 again, the shear flows can then be solved for each section in the same manner as in the first approximation. The results are that for the first cell,

$$\begin{aligned}
0 &= \int_0^{s_1} \frac{\partial w_1}{\partial s} ds + \int_{s_1}^{s_2} \frac{\partial w_1}{\partial s} ds \\
&= -\frac{1}{2}q_1 b_1 + U_1' b_3 - U_2'' \oint_I y b(s) ds - U_3'' \oint_I z b(s) ds \\
&\quad - f_1^{(a)} + \varphi' b_4 - \frac{1}{2}q_w b_2
\end{aligned} \tag{B.7}$$

and for the second cell,

$$\begin{aligned}
0 &= \int_{s_2}^{s_3} \frac{\partial w_1}{\partial s} ds - \int_{s_1}^{s_2} \frac{\partial w_1}{\partial s} ds \\
&= -\frac{1}{2}q_2 b_5 + U_1' b_6 - U_2'' \oint_{II} y b(s) ds - U_3'' \oint_{II} z b(s) ds \\
&\quad - f_2^{(a)} + \varphi' b_7 + \frac{1}{2}q_w b_2
\end{aligned} \tag{B.8}$$

B.2.2 Shear Flows

The shear flows for each section are:

$$\begin{aligned}
 q_1 &= -2a_3U'_1 - \frac{(b_5 + b_2) \oint_I yb(s)ds + b_2 \oint_{II} yb(s)ds}{b_5(b_1 + b_2) + b_1b_2} U''_2 \\
 &\quad - \frac{(b_5 + b_2) \oint_I zb(s)ds + b_2 \oint_{II} zb(s)ds}{b_5(b_1 + b_2) + b_1b_2} U''_3 - g_1^{(a)} - 2a_1\varphi' \\
 &= -2a_3U'_1 - \frac{b_5 \oint_I yb(s)ds + b_2d_1}{e_1} U''_2 \\
 &\quad - \frac{b_5 \oint_I zb(s)ds + b_2d_2}{e_1} U''_3 - g_1^{(a)} - 2a_1\varphi' \\
 &= -2a_3U'_1 - a_5U''_2 - a_7U''_3 - g_1^{(a)} - 2a_1\varphi' \\
 q_2 &= -2a_4U'_1 - \left(\frac{b_1 + b_2}{b_2} a_5 - \frac{\oint_I yb(s)ds}{b_2} \right) U''_2 \\
 &\quad - \left(\frac{b_1 + b_2}{b_2} a_7 - \frac{\oint_I zb(s)ds}{b_2} \right) U''_3 - f_2^{(a)} + \varphi'b_7 \\
 &= -2a_4U'_1 - a_6U''_2 - a_8U''_3 - f_2^{(a)} + \varphi'b_7 \\
 q_w &= -2(a_3 - a_4)U'_1 - (a_5 - a_6)U''_2 - (a_7 - a_8)U''_3 \\
 &\quad - (g_1^{(a)} - g_2^{(a)}) - 2(a_1 - a_2)\varphi'
 \end{aligned} \tag{B.9}$$

where

$$\begin{aligned}
 a_5 &= \frac{b_5 \oint_I yb(s)ds + b_2d_1}{e_1} \\
 a_6 &= \frac{b_1 + b_2}{b_2} a_5 - \frac{\oint_I yb(s)ds}{b_2} \\
 a_7 &= \frac{b_5 \oint_I zb(s)ds + b_2d_2}{e_1} \\
 a_8 &= \frac{b_1 + b_2}{b_2} a_7 - \frac{\oint_I zb(s)ds}{b_2} \\
 d_1 &= \oint_{I+II} yb(s)ds \\
 d_2 &= \oint_{I+II} zb(s)ds \\
 e_1 &= b_1b_2 + (b_1 + b_2)b_5
 \end{aligned} \tag{B.10}$$

Appendix C

Constitutive Relations

$$\begin{aligned}
F_1 &= \int_{all} \frac{\partial \Phi}{\partial (U'_1)} ds \\
&= \int_{all} \frac{\partial \Phi}{\partial \gamma_{11}} \frac{\partial \gamma_{11}}{\partial (U'_1)} + \frac{\partial \Phi}{\partial (2\gamma_{12})} \frac{\partial (2\gamma_{12})}{\partial (U'_1)} ds \\
&= \int_{all} \left(A\gamma_{11} - A^{(a)} + \frac{1}{2}B(2\gamma_{12}) \right) \\
&\quad + \left(\frac{1}{2}B\gamma_{11} + \frac{1}{4}C(2\gamma_{12}) - \frac{1}{2}C^{(a)} \right) \left(\frac{dg_1}{ds} \right) ds \\
&= K_{11}U'_1 + K_{12}\varphi' - K_{13}U''_3 + K_{14}U''_2 - F_1^{(a)}
\end{aligned} \tag{C.1}$$

$$\begin{aligned}
M_1 &= \int_{all} \frac{\partial \Phi}{\partial \varphi'} ds \\
&= \int_{all} \frac{\partial \Phi}{\partial (2\gamma_{12})} \frac{\partial (2\gamma_{12})}{\partial \varphi'} ds \\
&= \int_{all} \left(\frac{1}{2}B\gamma_{11} + \frac{1}{4}C(2\gamma_{12}) - \frac{1}{2}C^{(a)} \right) \left(\frac{dG}{ds} + r_n \right) ds \\
&= K_{12}U'_1 + K_{22}\varphi' - K_{23}U''_3 + K_{24}U''_3 - M_1^{(a)}
\end{aligned} \tag{C.2}$$

$$\begin{aligned}
M_2 &= \int_{all} \frac{\partial \Phi}{\partial (-U''_3)} ds \\
&= \int_{all} \frac{\partial \Phi}{\partial \gamma_{11}} \frac{\partial \gamma_{11}}{\partial (-U''_3)} + \frac{\partial \Phi}{\partial (2\gamma_{12})} \frac{\partial (2\gamma_{12})}{\partial (-U''_3)} ds \\
&= \int_{all} \left(A\gamma_{11} - A^{(a)} + \frac{1}{2}B(2\gamma_{12}) \right) (z) \\
&\quad + \left(\frac{1}{2}B\gamma_{11} + \frac{1}{4}C(2\gamma_{12}) - \frac{1}{2}C^{(a)} \right) \left(-\frac{dg_3}{ds} \right) ds \\
&= K_{13}U'_1 + K_{23}\varphi' - K_{33}U''_3 + K_{34}U''_2 - M_2^{(a)}
\end{aligned} \tag{C.3}$$

$$\begin{aligned}
M_3 &= \int_{all} \frac{\partial \Phi}{\partial U_2''} ds \\
&= \int_{all} \frac{\partial \Phi}{\partial \gamma_{11}} \frac{\partial \gamma_{11}}{\partial U_2''} + \frac{\partial \Phi}{\partial (2\gamma_{12})} \frac{\partial (2\gamma_{12})}{\partial U_2''} ds \\
&= \int_{all} \left(A\gamma_{11} - A^{(a)} + \frac{1}{2}B(2\gamma_{12}) \right) (-y) \\
&\quad + \left(\frac{1}{2}B\gamma_{11} + \frac{1}{4}C(2\gamma_{12}) - \frac{1}{2}C^{(a)} \right) \left(\frac{dg_2}{ds} \right) ds \\
&= K_{14}U_1' + K_{24}\varphi' - K_{34}U_3'' + K_{44}U_2'' - M_3^{(a)}
\end{aligned} \tag{C.4}$$

The stiffness coefficients K_{ij} in terms of cross-sectional geometry and material properties are as follows:

$$\begin{aligned}
K_{11} &= \int_{all} \left(A + B \frac{dg_1}{ds} + \frac{1}{4}C \left(\frac{dg_1}{ds} \right)^2 \right) ds \\
&= \int_{all} \left(A - \frac{B^2}{C} \right) ds - 2a_3^2b_1 - 2(a_3 - a_4)^2b_2 - 2a_4^2b_5 \\
&= \int_{all} \left(A - \frac{B^2}{C} \right) ds - 4a_3 \oint_I \frac{B}{C} ds - 4a_4 \oint_{II} \frac{B}{C} ds
\end{aligned} \tag{C.5}$$

$$\begin{aligned}
K_{12} &= \int_{all} \left(\frac{1}{2}B + \frac{1}{4}C \frac{dg_1}{ds} \right) \left(\frac{dG}{ds} + r_n \right) ds \\
&= -2a_1a_3b_1 - 2(a_1 - a_2)(a_3 - a_4)b_2 - 2a_2a_4b_5 \\
&= -4a_1 \oint_I \frac{B}{C} ds - 4a_2 \oint_{II} \frac{B}{C} ds
\end{aligned} \tag{C.6}$$

$$\begin{aligned}
K_{13} &= \int_{all} \left(Az + \frac{1}{2}Bz \frac{dg_1}{ds} - \frac{1}{2}B \frac{dg_3}{ds} - \frac{1}{4}C \frac{dg_1}{ds} \frac{dg_3}{ds} \right) ds \\
&= \int_{all} \left(A - \frac{B^2}{C} \right) z ds + 2a_3a_7b_1 \\
&\quad + 2(a_3 - a_4)(a_7 - a_8)b_2 + 2a_4a_8b_5 \\
&= \int_{all} \left(A - \frac{B^2}{C} \right) z ds + 4a_7 \oint_I \frac{B}{C} ds + 4a_8 \oint_{II} \frac{B}{C} ds
\end{aligned} \tag{C.7}$$

$$\begin{aligned}
K_{14} &= \int_{all} \left(-Ay - \frac{1}{2}yB \frac{dg_1}{ds} + \frac{1}{2}B \frac{dg_2}{ds} + \frac{1}{4}C \frac{dg_1}{ds} \frac{dg_2}{ds} \right) ds \\
&= - \int_{all} \left(A - \frac{B^2}{C} \right) y ds - 2a_3a_5b_1 \\
&\quad - 2(a_3 - a_4)(a_5 - a_6)b_2 - 2a_4a_6b_5
\end{aligned} \tag{C.8}$$

$$= - \int_{all} \left(A - \frac{B^2}{C} \right) y ds - 4a_5 \oint_I \frac{B}{C} ds - 4a_6 \oint_{II} \frac{B}{C} ds$$

$$\begin{aligned}
K_{22} &= \int_{all} \frac{1}{4}C \left(\frac{dG}{ds} + r_n \right)^2 ds \\
&= -2a_1^2b_1 - 2(a_1 - a_2)^2b_2 - 2a_2^2b_5 \\
&= -4a_1A_{eI} - 4a_2A_{eII}
\end{aligned} \tag{C.9}$$

$$\begin{aligned}
K_{23} &= \int_{all} \left(\frac{dG}{ds} + r_n \right) \left(\frac{1}{2}zB - \frac{1}{4}C \frac{dg_3}{ds} \right) ds \\
&= 2a_7a_1b_1 + 2(a_7 - a_8)(a_1 - a_2)b_2 + 2a_8a_2b_5 \\
&= 4a_7A_{eI} + 4a_8A_{eII}
\end{aligned} \tag{C.10}$$

$$\begin{aligned}
K_{24} &= \int_{all} \left(\frac{dG}{ds} + r_n \right) \left(\frac{1}{4}C \frac{dg_2}{ds} - \frac{1}{2}yB \right) ds \\
&= -2a_1a_5b_1 - 2(a_1 - a_2)(a_5 - a_6)b_2 - 2a_2a_6b_5 \\
&= -4a_5A_{eI} - 4a_6A_{eII}
\end{aligned} \tag{C.11}$$

$$\begin{aligned}
K_{33} &= \int_{all} \left(Az^2 - Bz \frac{dg_3}{ds} + \frac{1}{4}C \left(\frac{dg_3}{ds} \right)^2 \right) ds \\
&= \int_{all} \left(A - \frac{B^2}{C} \right) z^2 ds - 2a_7^2b_1 - 2(a_7 - a_8)^2b_2 - 2a_8^2b_5 \\
&= \int_{all} \left(A - \frac{B^2}{C} \right) z^2 ds + 4a_7 \oint_I \frac{B}{C} z ds + 4a_8 \oint_{II} \frac{B}{C} z ds
\end{aligned} \tag{C.12}$$

$$\begin{aligned}
K_{34} &= \int_{all} \left(-Azy + \frac{1}{2}Bz \frac{dg_2}{ds} + \frac{1}{2}By \frac{dg_3}{ds} - \frac{1}{4}C \frac{dg_3}{ds} \frac{dg_2}{ds} \right) ds \\
&= - \int_{all} \left(A - \frac{B^2}{C} \right) z y ds - 2a_5 a_7 b_1 \\
&\quad - 2(a_5 - a_6)(a_7 - a_8)b_2 - 2a_6 a_8 b_5 \\
&= - \int_{all} \left(A - \frac{B^2}{C} \right) z y ds - 4a_5 \oint_I \frac{B}{C} z ds - 4a_6 \oint_{II} \frac{B}{C} z ds
\end{aligned} \tag{C.13}$$

$$\begin{aligned}
K_{44} &= \int_{all} \left(Ay^2 - By \frac{dg_2}{ds} + \frac{1}{4}C \left(\frac{dg_2}{ds} \right)^2 \right) ds \\
&= \int_{all} \left(A - \frac{B^2}{C} \right) y^2 ds - 2a_5^2 b_1 - 2(a_5 - a_6)^2 b_2 - 2a_6^2 b_5 \\
&= \int_{all} \left(A - \frac{B^2}{C} \right) y^2 ds + 4a_5 \oint_I \frac{B}{C} y ds + 4a_6 \oint_{II} \frac{B}{C} y ds
\end{aligned} \tag{C.14}$$

The contribution from the piezoelectric fibers is given by

$$\begin{aligned}
F_1^{(a)} &= \int_{all} \left(A^{(a)} + \frac{1}{2}C^{(a)} \frac{dg_1}{ds} - \frac{1}{2}B \frac{dv_1^{(a)}}{ds} + \frac{1}{4}C \frac{dv_1^{(a)}}{ds} \frac{dg_1}{ds} \right) ds \\
&= \int_{all} \left(A^{(a)} - \frac{B}{C}C^{(a)} \right) ds + g_1^{(a)} a_3 b_1 \\
&\quad + (g_1^{(a)} - g_1^{(a)}) (a_3 - a_4) b_2 + g_2^{(a)} a_4 b_5
\end{aligned} \tag{C.15}$$

$$\begin{aligned}
&= \int_{all} \left(A^{(a)} - \frac{B}{C}C^{(a)} \right) ds + 2g_1^{(a)} \oint_I \frac{B}{C} ds + 2g_2^{(a)} \oint_{II} \frac{B}{C} ds \\
M_1^{(a)} &= \int_{all} \left(\frac{dG}{ds} + r_n \right) \left(\frac{1}{2}C^{(a)} - \frac{1}{4}C \frac{dv_1^{(a)}}{ds} \right) ds \\
&= g_1^{(a)} a_1 b_1 + (g_1^{(a)} - g_2^{(a)})(a_1 - a_2)b_2 + g_2^{(a)} a_2 b_5 \\
&= 2g_1^{(a)} A_{eI} + 2g_2^{(a)} A_{eII}
\end{aligned} \tag{C.16}$$

$$\begin{aligned}
M_2^{(a)} &= \int_{all} \left(A^{(a)} z - \frac{1}{2}Bz \frac{dv_1^{(a)}}{ds} + \frac{1}{4}C \frac{dv_1^{(a)}}{ds} \frac{dg_3}{ds} - \frac{1}{2}C^{(a)} \frac{dg_3}{ds} \right) ds \\
&= \int_{all} \left(A^{(a)} - \frac{B}{C}C^{(a)} \right) z ds - g_1^{(a)} a_7 b_1 \\
&\quad - (g_1^{(a)} - g_1^{(a)})(a_7 - a_8)b_2 - g_2^{(a)} a_8 b_5 \\
&= \int_{all} \left(A^{(a)} - \frac{B}{C}C^{(a)} \right) z ds + 2g_1^{(a)} \oint_I \frac{B}{C} z ds + 2g_2^{(a)} \oint_{II} \frac{B}{C} z ds
\end{aligned} \tag{C.17}$$

$$\begin{aligned}
M_3^{(a)} &= \int_{all} \left(-A^{(a)}y + \frac{1}{2}By \frac{dv_1^{(a)}}{ds} - \frac{1}{4}C \frac{dv_1^{(a)}}{ds} \frac{dg_2}{ds} + \frac{1}{2}C^{(a)} \frac{dg_2}{ds} \right) ds \\
&= - \int_{all} \left(A^{(a)} - \frac{B}{C}C^{(a)} \right) y ds + g_1^{(a)} a_5 b_1 \\
&\quad - (g_1^{(a)} - g_1^{(a)}) (a_5 - a_6) b_2 - g_2^{(a)} a_6 b_5 \\
&= - \int_{all} \left(A^{(a)} - \frac{B}{C}C^{(a)} \right) y ds - 2g_1^{(a)} \oint_I \frac{B}{C} y ds - 2g_2^{(a)} \oint_{II} \frac{B}{C} y ds
\end{aligned} \tag{C.18}$$

Appendix D

Variational Asymptotic Variables

The stiffness coefficients

$$\begin{aligned} K_{11} &= \int_{all} \left(A - \frac{B^2}{C} \right) ds - 4a_3 \oint_I \frac{B}{C} ds - 4a_4 \oint_{II} \frac{B}{C} ds \\ K_{12} &= -4a_1 \oint_I \frac{B}{C} ds - 4a_2 \oint_{II} \frac{B}{C} ds \\ K_{13} &= \int_{all} \left(A - \frac{B^2}{C} \right) z ds + 4a_7 \oint_I \frac{B}{C} ds + 4a_8 \oint_{II} \frac{B}{C} ds \\ K_{14} &= - \int_{all} \left(A - \frac{B^2}{C} \right) y ds - 4a_5 \oint_I \frac{B}{C} ds - 4a_6 \oint_{II} \frac{B}{C} ds \\ K_{22} &= -4a_1 A_{eI} - 4a_2 A_{eII} \\ K_{23} &= 4a_7 A_{eI} + 4a_8 A_{eII} \\ K_{24} &= -4a_5 A_{eI} - 4a_6 A_{eII} \\ K_{33} &= \int_{all} \left(A - \frac{B^2}{C} \right) z^2 ds + 4a_7 \oint_I \frac{B}{C} z ds + 4a_8 \oint_{II} \frac{B}{C} z ds \\ K_{34} &= - \int_{all} \left(A - \frac{B^2}{C} \right) zy ds - 4a_5 \oint_I \frac{B}{C} z ds - 4a_6 \oint_{II} \frac{B}{C} z ds \\ K_{44} &= \int_{all} \left(A - \frac{B^2}{C} \right) y^2 ds + 4a_5 \oint_I \frac{B}{C} y ds + 4a_6 \oint_{II} \frac{B}{C} y ds \end{aligned} \tag{D.1}$$

$$\begin{aligned}
a_1 &= -\frac{c_3}{c_1} \\
a_2 &= \frac{(b_1 + b_2)a_1 + b_4}{b_2} \\
a_3 &= -\frac{c_2}{c_1} \\
a_4 &= \frac{(b_1 + b_2)a_3 + b_3}{b_2} \\
a_5 &= \frac{b_5 \oint_I y b(s) ds + b_2 d_1}{e_1} \\
a_6 &= \frac{b_1 + b_2}{b_2} a_5 - \frac{\oint_I y b(s) ds}{b_2} \\
a_7 &= \frac{b_5 \oint_I z b(s) ds + b_2 d_2}{e_1} \\
a_8 &= \frac{b_1 + b_2}{b_2} a_7 - \frac{\oint_I z b(s) ds}{b_2}
\end{aligned} \tag{D.2}$$

$$\begin{aligned}
b(s) &= -2 \frac{B}{C} \\
b_1 &= 4 \int_0^{s_1} c(s) ds \\
b_2 &= 4 \int_{s_1}^{s_2} c(s) ds \\
b_3 &= \oint_I b(s) ds \\
b_4 &= \oint_I r_n ds = -2A_{eI} \\
b_5 &= 4 \int_{s_2}^{s_3} c(s) ds \\
b_6 &= \oint_{II} b(s) ds \\
b_7 &= \oint_{II} r_n ds = -2A_{eII}
\end{aligned} \tag{D.3}$$

$$\begin{aligned}
c(s) &= -2 \frac{1}{C} \\
c_1 &= b_1 + \frac{(b_1 + b_2)b_5}{b_2} \\
c_2 &= b_3 + b_6 + \frac{b_3 b_5}{b_2} \\
c_3 &= b_4 + b_7 + \frac{b_4 b_5}{b_2}
\end{aligned} \tag{D.4}$$

$$\begin{aligned}
d_1 &= \oint_{I+II} yb(s)ds \\
d_2 &= \oint_{I+II} zb(s)ds
\end{aligned} \tag{D.5}$$

$$e_1 = b_1b_2 + (b_1 + b_2)b_5 \tag{D.6}$$

$$\begin{aligned}
f_1^{(a)} &= \oint_I c(s)C^{(a)}ds \\
f_2^{(a)} &= \oint_{II} c(s)C^{(a)}ds
\end{aligned} \tag{D.7}$$

$$\begin{aligned}
g_1^{(a)} &= 2 \frac{(b_5 + b_2)f_1^{(a)} + b_2f_2^{(a)}}{(b_1 + b_2)(b_2 + b_5) - b_2^2} \\
g_2^{(a)} &= 2 \frac{(b_1 + b_2)f_2^{(a)} + b_2f_1^{(a)}}{(b_1 + b_2)(b_2 + b_5) - b_2^2}
\end{aligned} \tag{D.8}$$

For the left branch ($s = 0 \rightarrow s_1$)

$$\begin{aligned}
G(s) &= \int_0^s [4a_1c(\tau) - r_n(\tau)]d\tau \\
g_1(s) &= \int_0^s [b(\tau) + 4a_3c(\tau)]d\tau \\
g_2(s) &= -\int_0^s [b(\tau)y(\tau) - 4a_5c(\tau)]d\tau \\
g_3(s) &= -\int_0^s [b(\tau)z(\tau) - 4a_7c(\tau)]d\tau \\
v_1^{(a)}(s) &= \int_0^s [2g_1^{(a)} - C^{(a)}(\tau)]c(\tau)d\tau
\end{aligned} \tag{D.9}$$

for the web ($s = s_1 \rightarrow s_2$)

$$\begin{aligned}
G(s) &= \int_{s_1}^s [4(a_1 - a_2)c(\tau) - r_n(\tau)]d\tau \\
g_1(s) &= \int_{s_1}^s [b(\tau) + 4(a_3 - a_4)c(\tau)]d\tau \\
g_2(s) &= -\int_{s_1}^s [b(\tau)y(\tau) - 4(a_5 - a_6)c(\tau)]d\tau \\
g_3(s) &= -\int_{s_1}^s [b(\tau)z(\tau) - 4(a_7 - a_8)c(\tau)]d\tau \\
v_1^{(a)}(s) &= \int_{s_1}^s [2(g_1^{(a)} - g_2^{(a)}) - C^{(a)}(\tau)]c(\tau)d\tau
\end{aligned} \tag{D.10}$$

and for the right branch ($s = s_2 \rightarrow s_3$)

$$\begin{aligned}
 G(s) &= \int_{s_2}^s [4a_2c(\tau) - r_n(\tau)]d\tau \\
 g_1(s) &= \int_{s_2}^s [b(\tau) + 4a_4c(\tau)]d\tau \\
 g_2(s) &= -\int_{s_2}^s [b(\tau)y(\tau) - 4a_6c(\tau)]d\tau \\
 g_3(s) &= -\int_{s_2}^s [b(\tau)z(\tau) - 4a_8c(\tau)]d\tau \\
 v_1^{(a)}(s) &= \int_{s_2}^s [2g_2^{(a)} - C^{(a)}(\tau)]c(\tau)d\tau
 \end{aligned} \tag{D.11}$$

The axial A , coupling B and shear C stiffnesses are related to the 2-D Hookean tensor $D^{\alpha\beta\gamma\delta}$ by:

$$\begin{aligned}
 A &= \langle D^{1111} \rangle - \frac{\langle D^{1122} \rangle^2}{\langle D^{2222} \rangle} \\
 B &= 2 \langle D^{1112} \rangle - 2 \frac{\langle D^{1112} \rangle \langle D^{1222} \rangle}{\langle D^{2222} \rangle} \\
 C &= 4 \langle D^{1212} \rangle - 4 \frac{\langle D^{1222} \rangle^2}{\langle D^{2222} \rangle}
 \end{aligned} \tag{D.12}$$

and the actuation contributions $A^{(a)}$ and $C^{(a)}$ are given by:

$$\begin{aligned}
 A^{(a)} &= \langle D^{11\gamma\delta} d_{e\gamma\delta} E_e \rangle - \frac{\langle D^{1122} \rangle}{\langle D^{2222} \rangle} \langle D^{22\gamma\delta} d_{e\gamma\delta} E_e \rangle \\
 C^{(a)} &= 2 \langle D^{12\gamma\delta} d_{e\gamma\delta} E_e \rangle - 2 \frac{\langle D^{1222} \rangle}{\langle D^{2222} \rangle} \langle D^{22\gamma\delta} d_{e\gamma\delta} E_e \rangle
 \end{aligned} \tag{D.13}$$

Appendix E

Transformation of Coordinates

The displacements given in Chapter 2 are in a local, deforming b_i coordinate system given in Figure 3.1. They must be transformed into a global, non-deforming a_i reference frame, also given in Figure 3.1, before the variational analysis can be done. This is done by using the transformation matrix $[T]$ developed by D.A. Peters and given in Hodges and Dowell (1974).

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = [T] \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \quad (\text{E.1})$$

where $[T]$ is given in Equation E.6 and

$$\hat{\varphi} = \varphi - \int_0^x v'' w' dx \quad (\text{E.2})$$

The kinematic variables v and w represent the displacements in the a_2 and a_3 direction, respectively. The torsional deflection φ is the same, to the second order, about either the a_1 or b_1 axes.

As the transformation matrix is orthogonol, therefore the derivative of it is related to it by the skew-symmetric spin matrix $[S]$.

$$[T]' = [S][T] \quad (\text{E.3})$$

where

$$[S] = \begin{bmatrix} 0 & \omega_k & -\omega_j \\ -\omega_k & 0 & \omega_i \\ \omega_j & -\omega_i & 0 \end{bmatrix} \quad (\text{E.4})$$

and, to the second order

$$\begin{aligned} \omega_i &= \varphi' \\ \omega_j &= v'' \sin(\theta + \varphi) - w'' \cos(\theta + \varphi) \\ \omega_k &= v'' \cos(\theta + \varphi) + w'' \sin(\theta + \varphi) \end{aligned} \quad (\text{E.5})$$

Figure E.1: Transformation geometry.

$$[T] = \begin{bmatrix} 1 - \frac{1}{2}v'^2 - \frac{1}{2}w'^2 & v' & w' \\ -[v' \cos(\theta + \varphi) + w' \sin(\theta + \varphi)] & \cos(\theta + \hat{\varphi} + v'w') \left(1 - \frac{v'^2}{2}\right) & \sin(\theta + \hat{\varphi}) \left(1 - \frac{w'^2}{2}\right) \\ v' \sin(\theta + \varphi) - w' \cos(\theta + \varphi) & -\sin(\theta + \hat{\varphi} + v'w') \left(1 - \frac{v'^2}{2}\right) & \cos(\theta + \hat{\varphi}) \left(1 - \frac{w'^2}{2}\right) \end{bmatrix} \quad (\text{E.6})$$

Appendix F

Dimensionless Parameters

The dimensionless axial coordinate and chord length are given by:

$$\begin{aligned}\bar{x} &= \frac{x}{L} \\ \bar{c} &= \frac{c}{L}\end{aligned}\tag{F.1}$$

The dimensionless displacements (Δ) and torsional twist angle (φ) are given as $\bar{\Delta}$ and $\bar{\varphi}$, respectively. The corresponding derivatives are therefore found to be:

$$\begin{aligned}\Delta &= L\bar{\Delta} & \varphi &= \bar{\varphi} \\ \Delta' &= \bar{\Delta}' & \varphi' &= \frac{1}{L}\bar{\varphi}' \\ \Delta'' &= \frac{1}{L}\bar{\Delta}'' & \varphi'' &= \frac{1}{L^2}\bar{\varphi}'' \\ \Delta''' &= \frac{1}{L^2}\bar{\Delta}''' & \varphi''' &= \frac{1}{L^3}\bar{\varphi}''' \\ \Delta'''' &= \frac{1}{L^3}\bar{\Delta}'''' & \varphi'''' &= \frac{1}{L^4}\bar{\varphi}'''' \\ \dot{\Delta} &= \Omega L\dot{\bar{\Delta}} & \dot{\varphi} &= \Omega\dot{\bar{\varphi}} \\ \ddot{\Delta} &= \Omega^2 L\ddot{\bar{\Delta}} & \ddot{\varphi} &= \Omega^2\ddot{\bar{\varphi}}\end{aligned}\tag{F.2}$$

The stiffness terms K_{ij} are non-dimensionalized as:

$$\begin{aligned}
 \Lambda_{11} &= \frac{K_{11}}{m\Omega^2 L^2} & \Lambda_{12} &= \frac{K_{12}}{m\Omega^2 L^3} \\
 \Lambda_{13} &= \frac{K_{13}}{m\Omega^2 L^3} & \Lambda_{14} &= \frac{K_{14}}{m\Omega^2 L^3} \\
 \Lambda_{22} &= \frac{K_{22}}{m\Omega^2 L^4} & \Lambda_{23} &= \frac{K_{23}}{m\Omega^2 L^4} \\
 \Lambda_{24} &= \frac{K_{24}}{m\Omega^2 L^4} & \Lambda_{33} &= \frac{K_{33}}{m\Omega^2 L^4} \\
 \Lambda_{34} &= \frac{K_{34}}{m\Omega^2 L^4} & \Lambda_{44} &= \frac{K_{44}}{m\Omega^2 L^4}
 \end{aligned} \tag{F.3}$$

Several groups of stiffness terms appear multiple times in Equations 3.24, 3.26 and 3.28, therefore they can be represented as new constants:

$$\begin{aligned}
 \Gamma_1 &= \left(\Lambda_{33} - \frac{\Lambda_{13}^2}{\Lambda_{11}} \right) \sin^2 \theta + \left(\Lambda_{34} - \frac{\Lambda_{13}\Lambda_{14}}{\Lambda_{11}} \right) \sin 2\theta + \left(\Lambda_{44} - \frac{\Lambda_{14}^2}{\Lambda_{11}} \right) \cos^2 \theta \\
 \Gamma_2 &= - \left(\Lambda_{33} - \frac{\Lambda_{13}^2}{\Lambda_{11}} \right) \cos^2 \theta + \left(\Lambda_{34} - \frac{\Lambda_{13}\Lambda_{14}}{\Lambda_{11}} \right) \sin 2\theta - \left(\Lambda_{44} - \frac{\Lambda_{14}^2}{\Lambda_{11}} \right) \sin^2 \theta \\
 \Gamma_3 &= \frac{1}{2} \left(\Lambda_{33} - \frac{\Lambda_{13}^2}{\Lambda_{11}} \right) \sin 2\theta + \left(\Lambda_{34} - \frac{\Lambda_{13}\Lambda_{14}}{\Lambda_{11}} \right) \cos 2\theta - \frac{1}{2} \left(\Lambda_{44} - \frac{\Lambda_{14}^2}{\Lambda_{11}} \right) \sin 2\theta \\
 \Upsilon_1 &= \left(\Lambda_{23} - \frac{\Lambda_{12}\Lambda_{13}}{\Lambda_{11}} \right) \sin \theta + \left(\Lambda_{24} - \frac{\Lambda_{12}\Lambda_{14}}{\Lambda_{11}} \right) \cos \theta \\
 \Upsilon_2 &= \left(\Lambda_{23} - \frac{\Lambda_{12}\Lambda_{13}}{\Lambda_{11}} \right) \cos \theta - \left(\Lambda_{24} - \frac{\Lambda_{12}\Lambda_{14}}{\Lambda_{11}} \right) \sin \theta \\
 \Pi_1 &= \frac{\Lambda_{13}}{\Lambda_{11}} \sin \theta + \frac{\Lambda_{14}}{\Lambda_{11}} \cos \theta \\
 \Pi_2 &= \frac{\Lambda_{13}}{\Lambda_{11}} \cos \theta - \frac{\Lambda_{14}}{\Lambda_{11}} \sin \theta
 \end{aligned} \tag{F.4}$$

The non-dimensional actuation terms are found to be:

$$\begin{aligned}
 \bar{F}_1^{(a)} &= \frac{F_1^{(a)}}{m\Omega^2 L^2} \\
 \bar{M}_1^{(a)} &= \frac{M_1^{(a)}}{m\Omega^2 L^3} \\
 \bar{M}_2^{(a)} &= \frac{M_1^{(a)}}{m\Omega^2 L^3} \\
 \bar{M}_3^{(a)} &= \frac{M_1^{(a)}}{m\Omega^2 L^3}
 \end{aligned} \tag{F.5}$$

The non-dimensional radii of gyration are:

$$\begin{aligned}\mu &= \frac{k_m}{L} \\ \mu_1 &= \frac{k_{m_1}}{L} \\ \mu_2 &= \frac{k_{m_2}}{L}\end{aligned}\tag{F.6}$$

The non-dimensional aerodynamic constants are given as:

$$\begin{aligned}\bar{L}_v &= \frac{L_2}{m\Omega^2 L} \\ &= \frac{\gamma}{6} \left\{ \bar{v}_i^2 - \bar{x}^2 \frac{C_D}{a} - \bar{x} \bar{v}_i (\theta + \bar{\varphi}) \right. \\ &\quad \left. - \left(2\bar{x} \frac{C_D}{a} + (\theta + \bar{\varphi}) \bar{v}_i \right) \dot{\bar{v}} + (2\bar{v}_i - \bar{x} (\theta + \bar{\varphi})) \dot{\bar{w}} \right\} \\ \bar{L}_w &= \frac{L_3}{m\Omega^2 L} \\ &= \frac{\gamma}{6} \left\{ -\bar{x} \bar{v}_i + \bar{x}^2 \left(\theta + \bar{\varphi} + \int_0^{\bar{x}} \bar{v}' \bar{w}'' d\bar{x} \right) - \bar{x} \bar{v} (\beta_{pc} + \bar{w}') \right. \\ &\quad \left. + \frac{\bar{x} \bar{c}}{2} (\beta_{pc} + \bar{w}') + (2\bar{x} (\theta + \bar{\varphi}) - \bar{v}_i) \dot{\bar{v}} - \bar{x} \dot{\bar{w}} + \frac{3}{4} \bar{c} \bar{x} \dot{\bar{\varphi}} - \frac{\bar{c}}{4} \ddot{\bar{w}} \right\} \\ \bar{M}_\varphi &= \frac{M_\varphi}{m\Omega^2 L^2} \\ &= -\frac{\gamma}{48} (\bar{c}^2 \bar{x} \dot{\bar{\varphi}})\end{aligned}\tag{F.7}$$

where γ is the Lock number, a ratio of the aerodynamic to the inertial forces on the blade, and is given by:

$$\gamma = 3 \frac{\rho_{air} a \bar{c} L^2}{m}\tag{F.8}$$

and the dimensionless inflow velocity is found to be:

$$\begin{aligned}\bar{v}_i &= \frac{v_i}{\Omega L} \\ &= \text{sign} [\theta + \varphi_0 (0.75L)] \frac{\pi \sigma}{8} \left(\sqrt{1 + \frac{12}{\pi \sigma} |\theta + \varphi_0 (0.75L)|} - 1 \right)\end{aligned}\tag{F.9}$$

Appendix G

Boundary Conditions

This appendix gives a detailed description of how to obtain Equations 3.34-3.39 given in Section 3.1.4. The boundary conditions that are not satisfied by the mode shape functions (Equation 3.43) are underlined.

By utilizing the expressions:

$$\begin{aligned}
 \delta U_1 = & -\left[\frac{K_{12}}{K_{11}} + \left(\frac{K_{13}}{K_{11}} \cos \theta - \frac{K_{14}}{K_{11}} \sin \theta\right) v' \right. \\
 & + \left(\frac{K_{13}}{K_{11}} \sin \theta + \frac{K_{14}}{K_{11}} \cos \theta\right) w'] \delta \varphi - \left[\left(\frac{K_{13}}{K_{11}} \cos \theta - \frac{K_{14}}{K_{11}} \sin \theta\right) v \right. \\
 & + \left(\frac{K_{13}}{K_{11}} \sin \theta + \frac{K_{14}}{K_{11}} \cos \theta\right) w] \delta \varphi' - \left[\left(\frac{K_{13}}{K_{11}} \cos \theta - \frac{K_{14}}{K_{11}} \sin \theta\right) \varphi'\right] \delta v \\
 & - \left[\left(\frac{K_{13}}{K_{11}} \sin \theta + \frac{K_{14}}{K_{11}} \cos \theta\right) + \left(\frac{K_{13}}{K_{11}} \cos \theta - \frac{K_{14}}{K_{11}} \sin \theta\right) \varphi\right] \delta v' \\
 & - \left[\left(\frac{K_{13}}{K_{11}} \sin \theta + \frac{K_{14}}{K_{11}} \cos \theta\right) \varphi'\right] \delta w \\
 & + \left[\left(\frac{K_{13}}{K_{11}} \cos \theta - \frac{K_{14}}{K_{11}} \sin \theta\right) - \left(\frac{K_{13}}{K_{11}} \sin \theta + \frac{K_{14}}{K_{11}} \cos \theta\right) \varphi\right] \delta w'
 \end{aligned} \tag{G.1}$$

$$\delta U_2 = \cos(\theta + \varphi) \delta v + \sin(\theta + \varphi) \delta w - (v \sin \theta - w \cos \theta) \delta \varphi \tag{G.2}$$

$$\delta U_3 = -\sin(\theta + \varphi) \delta v + \cos(\theta + \varphi) \delta w - (v \cos \theta + w \sin \theta) \delta \varphi \tag{G.3}$$

$$\begin{aligned}\delta U'_2 &= \cos(\theta + \varphi)\delta v' - \varphi' \sin \theta \delta v + \sin(\theta + \varphi)\delta w' + \varphi' \cos \theta \delta w \\ &\quad - (v' \sin \theta - w' \cos \theta) \delta \varphi - (v \sin \theta - w \cos \theta) \delta \varphi'\end{aligned}\tag{G.4}$$

$$\begin{aligned}\delta U'_3 &= -\sin(\theta + \varphi)\delta v' - \varphi' \cos \theta \delta v + \cos(\theta + \varphi)\delta w' - \varphi' \sin \theta \delta w \\ &\quad - (v' \cos \theta + w' \sin \theta) \delta \varphi - (v \cos \theta + w \sin \theta) \delta \varphi'\end{aligned}\tag{G.5}$$

The boundary conditions in Equation 3.3 are transformed to the non-deforming global reference frame. These conditions, also given in Equation 3.30, read as:

$$\begin{aligned}&\left[-F'_1 v - F_1 \left(\frac{K_{13}}{K_{11}} \cos \theta - \frac{K_{14}}{K_{11}} \sin \theta\right) \varphi' - M'_3 \cos(\theta + \varphi) \right. \\ &\quad \left. - M'_2 \sin(\theta + \varphi) + M_3 \varphi' \sin \theta + M_2 \varphi' \cos \theta\right] \delta v|_0^L = 0 \\ &\left[-F_1 \left(\frac{K_{13}}{K_{11}} \sin \theta + \frac{K_{14}}{K_{11}} \cos \theta\right) - F_1 \left(\frac{K_{13}}{K_{11}} \cos \theta - \frac{K_{14}}{K_{11}} \sin \theta\right) \varphi \right. \\ &\quad \left. + M_3 \cos(\theta + \varphi) + M_2 \sin(\theta + \varphi)\right] \delta v'|_0^L = 0 \\ &\left[-F'_1 w - F_1 \left(\frac{K_{13}}{K_{11}} \sin \theta + \frac{K_{14}}{K_{11}} \cos \theta\right) \varphi' - M'_3 \sin(\theta + \varphi) \right. \\ &\quad \left. + M'_2 \cos(\theta + \varphi) + M_3 \varphi' \cos \theta + M_2 \varphi' \sin \theta\right] \delta w|_0^L = 0 \\ &\left[F_1 \left(\frac{K_{13}}{K_{11}} \cos \theta - \frac{K_{14}}{K_{11}} \sin \theta\right) - F_1 \left(\frac{K_{13}}{K_{11}} \sin \theta + \frac{K_{14}}{K_{11}} \cos \theta\right) \varphi \right. \\ &\quad \left. + M_3 \sin(\theta + \varphi) - M_2 \cos(\theta + \varphi)\right] \delta w'|_0^L = 0 \\ &\left[-F_1 \frac{K_{12}}{K_{11}} - F_1 \left(\frac{K_{13}}{K_{11}} \cos \theta - \frac{K_{14}}{K_{11}} \sin \theta\right) v' \right. \\ &\quad \left. - F_1 \left(\frac{K_{13}}{K_{11}} \sin \theta + \frac{K_{14}}{K_{11}} \cos \theta\right) w' + M_1 \right. \\ &\quad \left. - M_3 (v' \sin \theta - w' \cos \theta) + M_2 (v' \cos \theta + w' \sin \theta) \right. \\ &\quad \left. + M'_3 (v \sin \theta - w \cos \theta) - M'_2 (v \cos \theta + w \sin \theta)\right] \delta \varphi|_0^L = 0 \\ &\left[-F_1 \left(\frac{K_{13}}{K_{11}} \cos \theta - \frac{K_{14}}{K_{11}} \sin \theta\right) v - F_1 \left(\frac{K_{13}}{K_{11}} \sin \theta + \frac{K_{14}}{K_{11}} \cos \theta\right) w \right. \\ &\quad \left. - M_3 (v \sin \theta - w \cos \theta) + M_2 (v \cos \theta + w \sin \theta)\right] \delta \varphi'|_0^L = 0\end{aligned}$$

These equations can also be non-dimensionalized by dividing through by $m\Omega^2 L^3$

G.1 The $\delta\bar{v}$ Boundary Condition

From first expression in Equation 3.30, we have:

$$\left[-\frac{F_1'v}{m\Omega^2 L^2} - \Pi_2 \frac{F_1\varphi'}{m\Omega^2 L} - \frac{M_3'}{m\Omega^2 L^2} \cos(\theta + \varphi) - \frac{M_2'}{m\Omega^2 L^2} \sin(\theta + \varphi) - \frac{M_3\varphi'}{m\Omega^2 L^2} \sin\theta + \frac{M_2\varphi'}{m\Omega^2 L^2} \cos\theta \right] \frac{\delta v}{L} \Big|_0^1 = 0$$

Expanding the terms out, and grouping together the deforming coordinates:

$$\begin{aligned} & \left[(\bar{x} + 2\dot{\bar{v}}) \bar{v} + \frac{\bar{x}^2}{2} \left(\frac{\Lambda_{13}}{\Lambda_{11}} \cos\theta - \frac{\Lambda_{13}}{\Lambda_{11}} \sin\theta \right) \bar{\varphi}' \right. \\ & - (\Lambda_{13} \sin\theta + \Lambda_{14} \cos\theta) LU_1'' - \left(\frac{\Lambda_{13}}{\Lambda_{11}} \cos\theta - \frac{\Lambda_{13}}{\Lambda_{11}} \sin\theta \right) \bar{\varphi} U_1'' L \\ & \quad - (\Lambda_{23} \sin(\theta + \varphi) + \Lambda_{24} \cos(\theta + \varphi)) \bar{\varphi}'' \\ & \quad + (\Lambda_{33} \sin(\theta + \varphi) + \Lambda_{34} \cos(\theta + \varphi)) L^2 U_3''' \\ & \quad - (\Lambda_{34} \sin(\theta + \varphi) + \Lambda_{44} \cos(\theta + \varphi)) L^2 U_2''' \\ & + (\Lambda_{13} \cos\theta - \Lambda_{14} \sin\theta) \bar{\varphi}' LU_1' + (\Lambda_{23} \cos\theta - \Lambda_{24} \sin\theta) \bar{\varphi}' \bar{\varphi}' \\ & - (\Lambda_{33} \cos\theta - \Lambda_{34} \sin\theta) \bar{\varphi}' LU_3'' + (\Lambda_{34} \cos\theta - \Lambda_{34} \sin\theta) \bar{\varphi}' LU_2'' \\ & \left. + \Pi_1 \bar{F}_1^{(a)'} + \bar{M}_2^{(a)'} \sin\theta - \bar{M}_3^{(a)'} \cos\theta \right] \delta\bar{v} \Big|_0^1 = 0 \end{aligned}$$

or,

$$\begin{aligned} & \left[(\bar{x} + 2\dot{\bar{v}}) \bar{v} + \Pi_1 \bar{x} + 2\Pi_1 \dot{\bar{v}}' + \Pi_2 \bar{x} \bar{\varphi} - \Upsilon_1 \bar{\varphi}'' - \Upsilon_2 \bar{\varphi} \bar{\varphi}'' \right. \\ & - \Gamma_1 \bar{v}''' - \Gamma_3 \bar{v}''' \bar{\varphi} - \Gamma_3 (\bar{v} \bar{\varphi})''' + \Gamma_3 \bar{w}''' - \Gamma_2 \bar{w}''' \bar{\varphi} \\ & - \Gamma_1 (\bar{w} \bar{\varphi})''' - \Pi_2 \frac{\bar{x}^2}{2} \bar{\varphi}' + \Upsilon_2 \bar{\varphi}' \bar{\varphi}' + \Gamma_3 \bar{v}'' \bar{\varphi}' + \Gamma_2 \bar{w}'' \bar{\varphi}' \\ & \left. + \Pi_1 \bar{F}_1^{(a)'} + \bar{M}_2^{(a)'} \sin\theta - \bar{M}_3^{(a)'} \cos\theta \right] \delta\bar{v} \Big|_0^1 = 0 \end{aligned}$$

and,

$$\begin{aligned} & \left[\bar{x} \bar{v} + 2\dot{\bar{v}} \bar{v} + \Pi_1 \bar{x} + 2\Pi_1 \dot{\bar{v}} + \Pi_2 \bar{x} \bar{\varphi} - \Upsilon_1 \bar{\varphi}'' - \Upsilon_2 \bar{\varphi} \bar{\varphi}'' \right. \\ & - \Gamma_1 \bar{v}''' - \Gamma_3 \bar{v}''' \bar{\varphi} - \Gamma_3 (\bar{v}''' \bar{\varphi} + 3\bar{v}'' \bar{\varphi}' + 3\bar{v}' \bar{\varphi}'' + \bar{v} \bar{\varphi}''') \\ & + \Gamma_3 \bar{w}''' - \Gamma_2 \bar{w}''' \bar{\varphi} - \Gamma_1 (\bar{w}''' \bar{\varphi} + 3\bar{w}'' \bar{\varphi}' + 3\bar{w}' \bar{\varphi}'' + \bar{w} \bar{\varphi}''') + \Upsilon_2 \bar{\varphi}' \bar{\varphi}' \\ & \left. + \Gamma_3 \bar{v}'' \bar{\varphi}' + \Gamma_2 \bar{w}'' \bar{\varphi}' + \Pi_1 \bar{F}_1^{(a)'} + \bar{M}_2^{(a)'} \sin\theta - \bar{M}_3^{(a)'} \cos\theta \right] \delta\bar{v} \Big|_0^1 = 0 \end{aligned} \tag{G.6}$$

G.2 The $\delta\bar{v}'$ Boundary Condition

For the second expression in Equation 3.30, we have:

$$\left[-\frac{F_1}{m\Omega^2 L^3} \left(\frac{K_{13}}{K_{11}} \sin \theta + \frac{K_{14}}{K_{11}} \cos \theta \right) - \frac{F_1}{m\Omega^2 L^3} \left(\frac{K_{13}}{K_{11}} \cos \theta - \frac{K_{14}}{K_{11}} \sin \theta \right) \varphi' \right. \\ \left. + \frac{M_3}{m\Omega^2 L^3} \cos(\theta + \varphi) + \frac{M_2}{m\Omega^2 L^3} \sin(\theta + \varphi) \right] \delta v'|_0^1 = 0$$

Expanding the terms out, and grouping together the deforming coordinates:

$$\left[\left(\frac{\bar{x}^2}{2} + 2 \int_0^{\bar{x}} \bar{v} d\bar{x} \right) \left(\frac{\Lambda_{13}}{\Lambda_{11}} \sin \theta + \frac{\Lambda_{14}}{\Lambda_{11}} \cos \theta \right) + \frac{\bar{x}}{2} \bar{\varphi} \left(\frac{\Lambda_{13}}{\Lambda_{11}} \cos \theta - \frac{\Lambda_{14}}{\Lambda_{11}} \sin \theta \right) \right. \\ + (\Lambda_{13} \sin \theta + \Lambda_{14} \cos \theta) U'_1 + (\Lambda_{13} \cos \theta - \Lambda_{14} \sin \theta) \bar{\varphi} U'_1 \\ + (\Lambda_{23} \sin \theta + \Lambda_{24} \cos \theta) \bar{\varphi}' + (\Lambda_{23} \cos \theta - \Lambda_{24} \sin \theta) \bar{\varphi} \bar{\varphi}' \\ - (\Lambda_{33} \sin \theta + \Lambda_{34} \cos \theta) L U''_3 - (\Lambda_{33} \cos \theta - \Lambda_{34} \sin \theta) \bar{\varphi} L U''_3 \\ + (\Lambda_{34} \sin \theta + \Lambda_{44} \cos \theta) L U''_2 + (\Lambda_{34} \cos \theta - \Lambda_{44} \sin \theta) L \bar{\varphi} U''_2 \\ \left. - \Pi_1 \bar{F}_1^{(a)} - \bar{M}_2^{(a)} \sin \theta + \bar{M}_3^{(a)} \cos \theta \right] \delta \bar{v}'|_0^1 = 0$$

or,

$$\left[\Upsilon_1 \bar{\varphi}' + \Upsilon_2 \bar{\varphi} \bar{\varphi}' + \Gamma_1 \bar{v}'' \right. \\ + \Gamma_3 \bar{v}'' \bar{\varphi} + \Gamma_3 (\bar{v} \bar{\varphi})'' - \Gamma_3 \bar{w}'' + \Gamma_2 \bar{w} \bar{\varphi}'' + \Gamma_1 (\bar{w} \bar{\varphi})'' \\ \left. - \Pi_1 \bar{F}_1^{(a)} - \bar{M}_2^{(a)} \sin \theta + \bar{M}_3^{(a)} \cos \theta \right] \delta \bar{v}'|_0^1 = 0$$

and,

$$\left[\Upsilon_1 \bar{\varphi}' + \Upsilon_2 \bar{\varphi} \bar{\varphi}' + \Gamma_1 \bar{v}'' + \Gamma_3 \bar{v}'' \bar{\varphi} + \Gamma_3 (\bar{v}'' \bar{\varphi} + \bar{v}' \bar{\varphi}' + \bar{v} \bar{\varphi}'') \right. \\ - \Gamma_3 \bar{w}'' + \Gamma_2 \bar{w} \bar{\varphi}'' + \Gamma_1 (\bar{w}'' \bar{\varphi} + \bar{w}' \bar{\varphi}' + \bar{w} \bar{\varphi}'') \\ \left. - \Pi_1 \bar{F}_1^{(a)} - \bar{M}_2^{(a)} \sin \theta + \bar{M}_3^{(a)} \cos \theta \right] \delta \bar{v}'|_0^1 = 0 \quad (G.7)$$

G.3 The $\delta\bar{w}$ Boundary Condition

Using the third expression in Equation 3.30, we have:

$$\left[-\frac{F'_1 w}{m\Omega^2 L^2} - \frac{F_1}{m\Omega^2 L} \left(\frac{\Lambda_{13}}{\Lambda_{11}} \sin \theta + \frac{\Lambda_{14}}{\Lambda_{11}} \cos \theta \right) \varphi' - \frac{M'_3}{m\Omega^2 L^2} \sin(\theta + \varphi) \right. \\ \left. + \frac{M'_2}{m\Omega^2 L^2} \cos(\theta + \varphi) + \frac{M_3 \varphi'}{m\Omega^2 L^2} \cos \theta + \frac{M_2 \varphi'}{m\Omega^2 L^2} \sin \theta \right] \delta \frac{w}{L}|_0^1 = 0$$

Expanding the terms out, and grouping together the deforming coordinates:

$$\begin{aligned}
& \left[(\bar{x} + 2\bar{v}) \bar{w} + \frac{\bar{x}^2}{2} \left(\frac{\Lambda_{13}}{\Lambda_{11}} \sin \theta + \frac{\Lambda_{14}}{\Lambda_{11}} \cos \theta \right) \bar{\varphi}' \right. \\
& + (\Lambda_{13} \cos (\theta + \varphi) - \Lambda_{14} \sin (\theta + \varphi)) U_1'' \\
& + (\Lambda_{23} \cos (\theta + \varphi) - \Lambda_{24} \sin (\theta + \varphi)) \bar{\varphi}'' \\
& - (\Lambda_{33} \cos (\theta + \varphi) - \Lambda_{34} \sin (\theta + \varphi)) L^2 U_3''' \\
& + (\Lambda_{34} \cos (\theta + \varphi) - \Lambda_{44} \cos (\theta + \varphi)) L^2 U_2''' \\
& + (\Lambda_{13} \sin \theta + \Lambda_{14} \cos \theta) \bar{\varphi}' U_1'' + (\Lambda_{23} \sin \theta + \Lambda_{24} \cos \theta) \bar{\varphi}' \bar{\varphi}' \\
& - (\Lambda_{33} \sin \theta + \Lambda_{34} \cos \theta) \bar{\varphi}' L U_3'' - (\Lambda_{34} \sin \theta + \Lambda_{34} \cos \theta) \bar{\varphi}' L U_2'' \\
& \left. - \Pi_2 \bar{F}_1^{(a)'} - \bar{M}_2^{(a)'} \cos \theta + \bar{M}_3^{(a)'} \sin \theta \right] \delta \bar{w}|_0^1 = 0
\end{aligned}$$

or,

$$\begin{aligned}
& [\bar{x} \bar{w} + 2\bar{v} \bar{w} - \Pi_2 \bar{x} - 2\Pi_2 \bar{v} + \Pi_1 \bar{x} \bar{\varphi} - \Upsilon_2 \bar{\varphi}'' - \Upsilon_1 \bar{\varphi} \bar{\varphi}'' \\
& - \Gamma_3 \bar{v}''' - \Gamma_1 \bar{v} \bar{\varphi}''' - \Gamma_2 (\bar{v} \bar{\varphi})''' + \Gamma_2 \bar{w}''' + \Gamma_3 \bar{w} \bar{\varphi}''' \\
& - \Gamma_3 (\bar{w} \bar{\varphi})''' - \Pi_1 \frac{\bar{x}^2}{2} \bar{\varphi}' + \Upsilon_1 \bar{\varphi}' \bar{\varphi}' + \Gamma_1 \bar{v}'' \bar{\varphi}' - \Gamma_3 \bar{w}'' \bar{\varphi}' \\
& \left. - \Pi_2 \bar{F}_1^{(a)'} - \bar{M}_2^{(a)'} \cos \theta + \bar{M}_3^{(a)'} \sin \theta \right] \delta \bar{w}|_0^1 = 0
\end{aligned}$$

and,

$$\begin{aligned}
& [\bar{x} \bar{w} + 2\bar{v} \bar{w} - \Pi_2 \bar{x} - 2\Pi_2 \bar{v} + \Pi_1 \bar{x} \bar{\varphi} + \Upsilon_2 \bar{\varphi}'' - \Upsilon_1 \bar{\varphi} \bar{\varphi}'' \\
& - \Gamma_3 \bar{v}''' + \Gamma_1 \bar{v}''' \bar{\varphi} + \Gamma_2 (\bar{v}''' \bar{\varphi} + 3\bar{v}'' \bar{\varphi}' + 3\bar{v}' \bar{\varphi}'' + \bar{v} \bar{\varphi}''') \\
& - \Gamma_2 \bar{w}''' - \Gamma_3 \bar{w}''' \bar{\varphi} - \Gamma_3 (\bar{w}''' \bar{\varphi} + 3\bar{w}'' \bar{\varphi}' + 3\bar{w}' \bar{\varphi}'' + \bar{w} \bar{\varphi}''') \\
& \left. + \Gamma_1 \bar{v}'' \bar{\varphi}' - \Gamma_3 \bar{w}'' \bar{\varphi}' - \Pi_2 \bar{F}_1^{(a)'} - \bar{M}_2^{(a)'} \cos \theta + \bar{M}_3^{(a)'} \sin \theta \right] \delta \bar{w}|_0^1 = 0
\end{aligned} \tag{G.8}$$

G.4 The $\delta \bar{w}'$ Boundary Condition

Using the fourth expression in Equation 3.30, we have:

$$\begin{aligned}
& \left[\frac{F_1}{m\Omega^2 L^3} \left(\frac{K_{13}}{K_{11}} \cos \theta - \frac{K_{14}}{K_{11}} \sin \theta \right) - \frac{F_1}{m\Omega^2 L^3} \left(\frac{K_{13}}{K_{11}} \sin \theta + \frac{K_{14}}{K_{11}} \cos \theta \right) \varphi \right. \\
& \left. + \frac{M_3}{m\Omega^2 L^3} \sin (\theta + \varphi) - \frac{M_2}{m\Omega^2 L^3} \cos (\theta + \varphi) \right] \delta w'|_0^1 = 0
\end{aligned}$$

or,

$$\begin{aligned}
& \left[\left(\frac{\Lambda_{13}}{\Lambda_{11}} \cos \theta - \frac{\Lambda_{14}}{\Lambda_{11}} \sin \theta \right) \left(-\frac{\bar{x}^2}{2} - 2 \int_0^{\bar{x}} \dot{v} d\bar{x} \right) \right. \\
& + \left(\frac{\Lambda_{13}}{\Lambda_{11}} \sin \theta + \frac{\Lambda_{14}}{\Lambda_{11}} \cos \theta \right) \varphi \frac{\bar{x}^2}{2} \\
& - (\Lambda_{13} \cos (\theta + \varphi) - \Lambda_{14} \sin (\theta + \varphi)) U_1' \\
& - (\Lambda_{23} \cos (\theta + \bar{\varphi}) - \Lambda_{24} \sin (\theta + \bar{\varphi})) \bar{\varphi}' \\
& + (\Lambda_{33} \cos (\theta + \bar{\varphi}) - \Lambda_{34} \sin (\theta + \bar{\varphi})) U_3'' L \\
& - (\Lambda_{34} \cos (\theta + \bar{\varphi}) - \Lambda_{44} \cos (\theta + \bar{\varphi})) U_2'' L \\
& \left. + \Pi_2 \bar{F}_1^{(a)} + \bar{M}_2^{(a)} \cos \theta + \bar{M}_3^{(a)} \sin \theta \right] \delta \bar{w}'|_0^1 = 0
\end{aligned}$$

or,

$$\begin{aligned}
& \left[-\Upsilon_2 \bar{\varphi}' + \Upsilon_1 \bar{\varphi} \bar{\varphi}' - \Gamma_3 \bar{v}'' + \Gamma_1 \bar{v}'' \bar{\varphi} + \Gamma_2 (\bar{v} \bar{\varphi})''' + \Gamma_2 \bar{w}''' \right. \\
& \left. - \Gamma_3 \bar{w} \bar{\varphi}''' - \Gamma_3 (\bar{w} \bar{\varphi})''' + \Pi_2 \bar{F}_1^{(a)} + \bar{M}_2^{(a)} \cos \theta + \bar{M}_3^{(a)} \sin \theta \right] \delta \bar{w}'|_0^1 = 0
\end{aligned}$$

and,

$$\begin{aligned}
& \left[-\Upsilon_2 \bar{\varphi}' + \Upsilon_1 \bar{\varphi} \bar{\varphi}' - \Gamma_3 \bar{v}'' + \Gamma_1 \bar{v}'' \bar{\varphi} + \Gamma_2 (v'' \varphi + 2v' \varphi' + \underline{v \varphi''}) \right. \\
& + \Gamma_1 \bar{w}'' - \Gamma_3 \bar{w}'' \bar{\varphi} - \Gamma_3 (w'' \varphi + 2w' \varphi' + \underline{w \varphi''}) \\
& \left. + \Pi_2 \bar{F}_1^{(a)} + \bar{M}_2^{(a)} \cos \theta + \bar{M}_3^{(a)} \sin \theta \right] \delta \bar{w}'|_0^1 = 0
\end{aligned} \tag{G.9}$$

G.5 The $\delta \bar{\varphi}$ Boundary Condition

From the fifth expression in Equation 3.30, one has:

$$\begin{aligned}
& \left[-\frac{F_1}{m\Omega^2 L^3} \frac{K_{12}}{K_{11}} - \frac{F_1}{m\Omega^2 L^3} \left(\frac{K_{13}}{K_{11}} \cos \theta - \frac{K_{14}}{K_{11}} \sin \theta \right) v' - \frac{F_1}{m\Omega^2 L^3} \left(\frac{K_{13}}{K_{11}} \sin \theta + \frac{K_{14}}{K_{11}} \cos \theta \right) w' \right. \\
& + \frac{M_1}{m\Omega^2 L^3} - \frac{M_3}{m\Omega^2 L^3} (v' \sin \theta - w' \cos \theta) + \frac{M_2}{m\Omega^2 L^3} (v' \cos \theta + w' \sin \theta) \\
& \left. + \frac{M_3'}{m\Omega^2 L^2} \left(\frac{v}{L} \sin \theta - \frac{w}{L} \cos \theta \right) - \frac{M_2'}{m\Omega^2 L^3} \left(\frac{v}{L} \cos \theta + \frac{w}{L} \sin \theta \right) \right] \delta \varphi|_0^1 =
\end{aligned}$$

or,

$$\begin{aligned}
& \left[-\frac{\Lambda_{12}}{\Lambda_{11}} \left(-\frac{\bar{x}^2}{2} - 2 \int_0^{\bar{x}} \bar{v} d\bar{x} \right) + \Pi_2 \frac{\bar{x}^2}{2} \bar{v}' + \Pi_1 \frac{\bar{x}^2}{2} \bar{w}' + \Lambda_{12} U_1' + \Lambda_{22} \bar{\varphi}' - \Lambda_{23} U_3'' L \right. \\
& + \Lambda_{24} U_2'' L + (\Lambda_{13} \cos \theta - \Lambda_{14} \sin \theta) \bar{v}' U_1' + (\Lambda_{13} \sin \theta + \Lambda_{14} \cos \theta) \bar{w}' U_1' \\
& + (\Lambda_{23} \cos \theta - \Lambda_{24} \sin \theta) \bar{v}' \bar{\varphi}' + (\Lambda_{23} \sin \theta + \Lambda_{24} \cos \theta) \bar{w}' \bar{\varphi}' \\
& - (\Lambda_{33} (\bar{v}' \cos \theta + \bar{w}' \sin \theta) - \Lambda_{34} (\bar{v}' \sin \theta - \bar{w}' \sin \theta)) (-\bar{v}'' \sin \theta + \bar{w}'' \cos \theta) \\
& + (\Lambda_{34} (\bar{v}' \cos \theta + \bar{w}' \sin \theta) - \Lambda_{44} (\bar{v}' \sin \theta - \bar{w}' \sin \theta)) (\bar{v}'' \cos \theta + \bar{w}'' \sin \theta) \\
& - (\Lambda_{13} \cos \theta - \Lambda_{14} \sin \theta) \bar{v} U_1'' - (\Lambda_{13} \sin \theta + \Lambda_{14} \cos \theta) \bar{w} U_1'' \\
& - (\Lambda_{23} \cos \theta - \Lambda_{24} \sin \theta) \bar{v} \bar{\varphi}'' - (\Lambda_{23} \sin \theta + \Lambda_{24} \cos \theta) \bar{w} \bar{\varphi}'' \\
& + (\Lambda_{33} (\bar{v} \cos \theta + \bar{w} \sin \theta) - \Lambda_{34} (\bar{v} \sin \theta - \bar{w} \sin \theta)) (-\bar{v}''' \sin \theta + \bar{w}''' \cos \theta) \\
& - (\Lambda_{34} (\bar{v} \cos \theta + \bar{w} \sin \theta) - \Lambda_{44} (\bar{v} \sin \theta - \bar{w} \sin \theta)) (\bar{v}''' \cos \theta + \bar{w}''' \sin \theta) \\
& \left. - \frac{\Lambda_{12}}{\Lambda_{11}} \bar{F}_1^{(a)} \right] \delta \bar{\varphi}|_0^1 = 0
\end{aligned}$$

or,

$$\begin{aligned}
& \left[-\frac{\Lambda_{12}}{\Lambda_{11}} \left(-\frac{\bar{x}^2}{2} - 2 \int_0^{\bar{x}} \bar{v} d\bar{x} \right) + \Pi_2 \frac{\bar{x}^2}{2} \bar{v}' + \Pi_1 \frac{\bar{x}^2}{2} \bar{w}' + \left(\Lambda_{22} - \frac{\Lambda_{12}^2}{\Lambda_{11}} \right) \bar{\varphi}' \right. \\
& - \frac{1}{2\Lambda_{11}} \bar{x}^2 - \frac{\Lambda_{12}}{\Lambda_{11}} \left(2 \int_0^{\bar{x}} \bar{v} d\bar{x} \right) + \Upsilon_1 \bar{v}'' + \Upsilon_2 (\bar{v} \bar{\varphi})'' \\
& - \Upsilon_2 \bar{w}'' + \Upsilon_1 (\bar{w} \bar{\varphi})'' - \Pi_2 \bar{v}' \frac{\bar{x}^2}{2} - \Pi_1 \bar{w}' \frac{\bar{x}^2}{2} + \Upsilon_2 \bar{v}' \bar{\varphi}' + \Upsilon_1 \bar{w}' \bar{\varphi}' \\
& + \Gamma_3 \bar{v}' \bar{v}'' + \Gamma_1 \bar{w}' \bar{v}'' + \Gamma_2 \bar{v}' \bar{w}'' - \Gamma_3 \bar{w}' \bar{w}'' + \Pi_2 \bar{v} \bar{x} + \Pi_1 \bar{w} \bar{x} \\
& \left. - \Upsilon_2 \bar{v} \bar{\varphi}'' - \Upsilon_1 \bar{w} \bar{\varphi}'' - \Gamma_3 \bar{v} \bar{v}''' - \Gamma_1 \bar{w} \bar{v}''' - \Gamma_2 \bar{v} \bar{w}''' + \Gamma_3 \bar{w} \bar{w}''' - \frac{\Lambda_{12}}{\Lambda_{11}} \bar{F}_1^{(a)} \right] \delta \bar{\varphi}|_0^1 = 0
\end{aligned}$$

and,

$$\begin{aligned}
& \left[\left(\Lambda_{22} - \frac{\Lambda_{12}^2}{\Lambda_{11}} \right) \bar{\varphi}' + \Upsilon_1 \bar{v}'' + \Upsilon_2 (\bar{v}'' \bar{\varphi} + 2\bar{v}' \bar{\varphi}' + \bar{v} \bar{\varphi}'') \right. \\
& - \Upsilon_2 \bar{w}'' + \Upsilon_1 (\bar{w}'' \bar{\varphi} + 2\bar{w}' \bar{\varphi}' + \bar{w} \bar{\varphi}'') + \Upsilon_2 \bar{v}' \bar{\varphi}' + \Upsilon_1 \bar{w}' \bar{\varphi}' + \Gamma_3 \bar{v}' \bar{v}'' \\
& + \Gamma_1 \bar{w}' \bar{v}'' + \Gamma_2 \bar{v}' \bar{w}'' - \Gamma_3 \bar{w}' \bar{w}'' + \Pi_2 \bar{v} \bar{x} + \Pi_1 \bar{w} \bar{x} - \Upsilon_2 \bar{v} \bar{\varphi}'' - \Upsilon_1 \bar{w} \bar{\varphi}'' \\
& \left. - \Gamma_3 \bar{v} \bar{v}''' - \Gamma_1 \bar{w} \bar{v}''' - \Gamma_2 \bar{v} \bar{w}''' + \Gamma_3 \bar{w} \bar{w}''' - \frac{\Lambda_{12}}{\Lambda_{11}} \bar{F}_1^{(a)} \right] \delta \bar{\varphi}|_0^1 = 0
\end{aligned} \tag{G.10}$$

G.6 The $\delta\bar{\varphi}'$ Boundary Condition

Finally, the last expression in Equation 3.30 gives:

$$\left[-\frac{F_1}{m\Omega^2 L^4} \left(\frac{K_{13}}{K_{11}} \cos \theta - \frac{K_{14}}{K_{11}} \sin \theta \right) v - \frac{F_1}{m\Omega^2 L^4} \left(\frac{K_{13}}{K_{11}} \sin \theta + \frac{K_{14}}{K_{11}} \cos \theta \right) w \right. \\ \left. - \frac{M_3}{m\Omega^2 L^3} \left(\frac{v}{L} \sin \theta - \frac{w}{L} \cos \theta \right) + \frac{M_2}{m\Omega^2 L^3} \left(\frac{v}{L} \cos \theta + \frac{w}{L} \sin \theta \right) \right] L \delta\varphi'|_0^L = 0$$

or,

$$\left[\frac{\bar{x}^2}{2} \Pi_2 \bar{v} + \frac{\bar{x}^2}{2} \Pi_1 \bar{w} + (\Lambda_{13} \cos \theta - \Lambda_{14} \sin \theta) \bar{v} U'_1 + (\Lambda_{13} \sin \theta + \Lambda_{14} \cos \theta) \bar{w} U'_1 \right. \\ \left. + (\Lambda_{23} \cos \theta - \Lambda_{24} \sin \theta) \bar{v} \bar{\varphi}' + (\Lambda_{23} \sin \theta + \Lambda_{24} \cos \theta) \bar{w} \bar{\varphi}' \right. \\ \left. + (\Lambda_{33} (\bar{v} \cos \theta + \bar{w} \sin \theta) + \Lambda_{34} (\bar{v} \sin \theta - \bar{w} \sin \theta)) (-\bar{v}'' \sin \theta + \bar{w}'' \cos \theta) \right. \\ \left. - (\Lambda_{34} (\bar{v} \cos \theta + \bar{w} \sin \theta) - \Lambda_{44} (\bar{v} \sin \theta - \bar{w} \sin \theta)) (\bar{v}'' \cos \theta + \bar{w}'' \sin \theta) \right] \delta\bar{\varphi}'|_0^1 = 0$$

and,

$$[\Upsilon_2 \bar{v} \bar{\varphi}' + \Upsilon_1 \bar{w} \bar{\varphi}' + \Gamma_3 \bar{v} \bar{v}'' + \Gamma_1 \bar{w} \bar{v}'' + \Gamma_2 \bar{v} \bar{w}'' - \Gamma_3 \bar{w} \bar{w}''] \delta\bar{\varphi}'|_0^1 = 0 \quad (\text{G.11})$$

Appendix H

Applying Galerkin's Method

H.1 Boundary Conditions

The boundary conditions are applied in Galerkin's Method through integration by parts.

Applying the expansions to the boundary conditions in Equation 3.45, we obtain the following:

$$\begin{aligned}
 & \sum_{j=1}^N V_j \bar{x} \Psi_i \Psi_j|_0^1 + 2 \sum_{j=1}^N \sum_{k=1}^N \dot{V}_j V_k \Psi_i \Psi_j \Psi_k|_0^1 \\
 & + \Pi_1 \bar{x} \Psi_i|_0^1 + 2\Pi_1 \sum_{j=1}^N \dot{V}_j \Psi_i \Psi_j|_0^1 - \Upsilon_1 \sum_{j=1}^N \phi_j \Psi_i \Theta_j''|_0^1 \\
 & - \Upsilon_2 \sum_{j=1}^N \sum_{k=1}^N \phi_j \phi_k \Psi_i \Theta_j \Theta_k''|_0^1 - 3\Gamma_3 \sum_{j=1}^N \sum_{k=1}^N V_j \phi_k \Psi_i \Psi_j' \Theta_k''|_0^1 \\
 & - 3\Gamma_1 \sum_{j=1}^N \sum_{k=1}^N W_j \phi_k \Psi_i \Psi_j' \Theta_k''|_0^1 + \Pi_1 \sum_{j=1}^N E_e \Psi_i \mathbf{F}_{1j}^{(a)'}|_0^1 \\
 & + \sin \theta \sum_{j=1}^N E_e \mathbf{M}_{2j}^{(a)'} \Psi_i|_0^1 - \cos \theta \sum_{j=1}^N E_e \Psi_i \mathbf{M}_{3j}^{(a)'}|_0^1 = 0
 \end{aligned} \tag{H.1}$$

$$\begin{aligned}
 & \Gamma_3 \sum_{j=1}^N \sum_{k=1}^N V_j \phi_k \Psi_i' \Psi_j \Theta_k''|_0^1 + \Gamma_1 \sum_{j=1}^N \sum_{k=1}^N W_j \phi_k \Psi_i' \Psi_j \Theta_k''|_0^1 \\
 & - \Pi_1 \sum_{j=1}^N E_e \Psi_i' \mathbf{F}_{1j}^{(a)}|_0^1 - \sin \theta \sum_{j=1}^N E_e \Psi_i' \mathbf{M}_{2j}^{(a)}|_0^1 \\
 & + \cos \theta \sum_{j=1}^N E_e \Psi_i' \mathbf{M}_{3j}^{(a)}|_0^1 = 0
 \end{aligned} \tag{H.2}$$

$$\begin{aligned}
 & \sum_{j=1}^N W_j \bar{x} \Psi_i \Psi_j|_0^1 + 2 \sum_{j=1}^N \sum_{k=1}^N \dot{V}_j W_k \Psi_i \Psi_j \Psi_k|_0^1 - \Pi_2 \bar{x} \Psi_i|_0^1 \\
 & - 2\Pi_2 \sum_{j=1}^N \dot{V}_j \Psi_i \Psi_j|_0^1 + \Upsilon_2 \sum_{j=1}^N \phi_j \Psi_i \Theta_j''|_0^1 \\
 & + \Upsilon_1 \sum_{j=1}^N \sum_{k=1}^N \phi_j \phi_k \Psi_i \Theta_j \Theta_k''|_0^1 + 3\Gamma_2 \sum_{j=1}^N \sum_{k=1}^N V_j \phi_k \Psi_i \Psi_j' \Theta_k''|_0^1 \\
 & + 3\Gamma_3 \sum_{j=1}^N \sum_{k=1}^N W_j \phi_k \Psi_i \Psi_j' \Theta_k''|_0^1 - \Pi_2 \sum_{j=1}^N E_e \Psi_i \mathbf{F}_{1j}^{(a)'}|_0^1 \\
 & - \cos \theta \sum_{j=1}^N E_e \Psi_i \mathbf{M}_{2j}^{(a)'}|_0^1 + \sin \theta \sum_{j=1}^N E_e \Psi_i \mathbf{M}_{3j}^{(a)'}|_0^1 = 0
 \end{aligned} \tag{H.3}$$

$$\begin{aligned} & \Gamma_2 \sum_{j=1}^N \sum_{k=1}^N V_j \phi_k \Psi_i' \Psi_j \Theta_k''|_0^1 - \Gamma_3 \sum_{j=1}^N \sum_{k=1}^N W_j \phi_k \Psi_i' \Psi_j \Theta_k''|_0^1 \\ & + \Pi_2 \sum_{j=1}^N E_e \Psi_i' \mathbf{F}_{1j}^{(a)}|_0^1 + \cos \theta \sum_{j=1}^N E_e \Psi_i \mathbf{M}_{2j}^{(a)}|_0^1 + \sin \theta E_e \mathbf{M}_3^{(a)} \Psi_i|_0^1 = 0 \end{aligned} \quad (\text{H.4})$$

$$\Pi_2 \sum_{j=1}^N V_j \bar{x} \Theta_i \Psi_j|_0^1 + \Pi_1 \sum_{j=1}^N W_j \bar{x} \Theta_i \Psi_j|_0^1 - \frac{\Lambda_{12}}{\Lambda_{11}} \sum_{j=1}^N E_e \Theta_i \mathbf{F}_{1j}^{(a)}|_0^1 = 0 \quad (\text{H.5})$$

H.2 $\delta \bar{v}$ Equation

Galerkin's Method can be applied to Equation 3.31 to obtain:

$$\begin{aligned} & -\Pi_1 \int_0^1 \Psi_i d\bar{x} - \Pi_2 \sum_{j=1}^N \phi_j \int_0^1 \Psi_i \Theta_j d\bar{x} + \Gamma_1 \sum_{j=1}^N V_j \int_0^1 \Psi_i \Psi_j'''' d\bar{x} \\ & + \Gamma_3 \sum_{j=1}^N \sum_{k=1}^N V_j \phi_k \int_0^1 \Psi_i \Psi_j'''' \Theta_k d\bar{x} + \Gamma_3 \sum_{j=1}^N \sum_{k=1}^N V_j \phi_k \int_0^1 \Psi_i (\Psi_j \Theta_k)'''' d\bar{x} \\ & - \Gamma_3 \sum_{j=1}^N W_j \int_0^1 \Psi_i \Psi_j'''' d\bar{x} + \Gamma_2 \sum_{j=1}^N \sum_{k=1}^N W_j \phi_k \int_0^1 \Psi_i \Psi_j'''' \Theta_k d\bar{x} \\ & + \Gamma_1 \sum_{j=1}^N \sum_{k=1}^N W_j \phi_k \int_0^1 \Psi_i (\Psi_j \Theta_k)'''' d\bar{x} + \Upsilon_1 \sum_{j=1}^N \phi_j \int_0^1 \Psi_i \Theta_j''' d\bar{x} \\ & + \Upsilon_2 \sum_{j=1}^N \sum_{k=1}^N \phi_j \phi_k \int_0^1 \Psi_i \Theta_j \Theta_k''' d\bar{x} - 2 \sum_{j=1}^N V_j \int_0^1 \Psi_i \Psi_j d\bar{x} \\ & - 2\Pi_1 \sum_{j=1}^N \dot{V}_j \left(2 \int_0^1 \Psi_i \Psi_j' d\bar{x} \right) - 4 \sum_{j=1}^N \sum_{k=1}^N \dot{V}_j V_k \int_0^1 \Psi_i \Psi_j' \Psi_k d\bar{x} \\ & - 2 \sum_{j=1}^N \sum_{k=1}^N \dot{V}_j V_k \int_0^1 \Psi_i \Psi_j \Psi_k' d\bar{x} - 2\beta_{pc} \sum_{j=1}^N \dot{W}_j \int_0^1 \Psi_i \Psi_j d\bar{x} \\ & - 2 \frac{\Lambda_{12}}{\Lambda_{11}} \sum_{j=1}^N \dot{\phi}_j \int_0^1 \Psi_i \Theta_j d\bar{x} + 2\Pi_2 \sum_{j=1}^N \dot{W}_j \int_0^1 \Psi_i \Psi_j' d\bar{x} \\ & - 2 \sum_{j=1}^N \sum_{k=1}^N \dot{W}_j W_k \int_0^1 \Psi_i \Psi_j' \Psi_k d\bar{x} - 2 \sum_{j=1}^N \sum_{k=1}^N \dot{W}_j W_k \int_0^1 \Psi_i \Psi_j \Psi_k' d\bar{x} \\ & + \sum_{j=1}^N \tilde{V}_j \int_0^1 \Psi_i \Psi_j d\bar{x} + \Pi_1 \sum_{j=1}^N E_e \int_0^1 \Psi_i \mathbf{F}_{1j}^{(a)''} d\bar{x} \\ & - \cos \theta \sum_{j=1}^N E_e \int_0^1 \Psi_i \mathbf{M}_{3j}^{(a)''} d\bar{x} - \sin \theta E_e \sum_{j=1}^N \int_0^1 \Psi_i \mathbf{M}_{2j}^{(a)''} d\bar{x} = \int_0^1 \bar{L}_v d\bar{x} \end{aligned}$$

which upon integration by parts of the terms involved, gives:

$$\begin{aligned}
& -\Pi_1 \left(\bar{x} \Psi_i |_0^1 - \int_0^1 \bar{x} \Psi_i' d\bar{x} \right) - \Pi_2 \left(\bar{x} \Psi_i \Theta_j |_0^1 - \int_0^1 \bar{x} (\Psi_i \Theta_j)' d\bar{x} \right) \\
& + \Gamma_3 \sum_{j=1}^N \sum_{k=1}^N V_j \phi_k \beta_j^4 \int_0^1 \Psi_i \Psi_j \Theta_k d\bar{x} \\
& + \Gamma_3 \sum_{j=1}^N \sum_{k=1}^N V_j \phi_k \left\{ (\Psi_i \Psi_j''' \Theta_k + 3 \Psi_i \Psi_j'' \Theta_k' + 3 \Psi_i \Psi_j' \Theta_k'' + \Psi_i \Psi_j \Theta_k''') |_0^1 \right. \\
& - (\Psi_i' \Psi_j'' \Theta_k + 2 \Psi_i' \Psi_j' \Theta_k' + \frac{\Psi_i' \Psi_j \Theta_k''}{|_0^1} + \Psi_i'' (\Psi_j \Theta_k)' |_0^1 - \Psi_i''' \Psi_j \Theta_k |_0^1 \\
& \left. + \beta_i^4 \int_0^1 \Psi_i \Psi_j \Theta_k d\bar{x} \right\} - \Gamma_3 \sum_{j=1}^N W_j \beta_j^4 \int_0^1 \Psi_i \Psi_j d\bar{x} + \Gamma_2 \sum_{j=1}^N \sum_{k=1}^N W_j \phi_k \beta_j^4 \int_0^1 \Psi_i \Psi_j \Theta_k d\bar{x} \\
& + \Gamma_1 \sum_{j=1}^N \sum_{k=1}^N W_j \phi_k \left\{ (\Psi_i \Psi_j''' \Theta_k + 3 \Psi_i \Psi_j'' \Theta_k' + 3 \Psi_i \Psi_j' \Theta_k'' + \Psi_i \Psi_j \Theta_k''') |_0^1 \right. \\
& - (\Psi_i' \Psi_j'' \Theta_k + 2 \Psi_i' \Psi_j' \Theta_k' + \frac{\Psi_i' \Psi_j \Theta_k''}{|_0^1} + \Psi_i'' (\Psi_j \Theta_k)' |_0^1 - \Psi_i''' \Psi_j \Theta_k |_0^1 \\
& \left. + \beta_i^4 \int_0^1 \Psi_i \Psi_j \Theta_k d\bar{x} \right\} + \Upsilon_1 \sum_{j=1}^N \phi_j \left(\underline{\Psi_i \Theta_j'} |_0^1 - \Psi_i' \Theta_j |_0^1 + \Psi_i'' \Theta_j |_0^1 - \int_0^1 \Psi_i''' \Theta_j d\bar{x} \right) \\
& + \Upsilon_2 \sum_{j=1}^N \sum_{k=1}^N \phi_j \phi_k \left(\underline{\Psi_i \Theta_j \Theta_k''} |_0^1 - \int_0^1 (\Psi_i \Theta_j)' \Theta_k'' d\bar{x} \right) \\
& - \sum_{j=1}^N V_j \left(\int_0^1 \Psi_i \Psi_j d\bar{x} + \underline{\bar{x} \Psi_i \Psi_j} |_0^1 - \int_0^1 \bar{x} \Psi_i' \Psi_j d\bar{x} - \int_0^1 \bar{x} \Psi_i \Psi_j' d\bar{x} \right) \\
& - 2 \Pi_1 \sum_{j=1}^N \dot{V}_j \left(\int_0^1 \Psi_i \Psi_j' d\bar{x} + \underline{\Psi_i \Psi_j} |_0^1 - \int_0^1 \Psi_i' \Psi_j d\bar{x} \right) \\
& - 2 \sum_{j=1}^N \sum_{k=1}^N \dot{V}_j V_k \left(2 \int_0^1 \Psi_i \Psi_j \Psi_k d\bar{x} + \underline{\Psi_i \Psi_j \Psi_k} |_0^1 - \int_0^1 \Psi_i' \Psi_j \Psi_k d\bar{x} - \int_0^1 \Psi_i \Psi_j' \Psi_k d\bar{x} \right) \\
& - 2 \beta_{pc} \sum_{j=1}^N \dot{W}_j \int_0^1 \Psi_i \Psi_j d\bar{x} - 2 \frac{\Lambda_{12}}{\Lambda_{11}} \sum_{j=1}^N \dot{\phi}_j \int_0^1 \Psi_i \Theta_j d\bar{x} \\
& + 2 \Pi_2 \sum_{j=1}^N \dot{W}_j \int_0^1 \Psi_i \Psi_j' d\bar{x} - 2 \sum_{j=1}^N \sum_{k=1}^N \dot{W}_j W_k \int_0^1 \Psi_i \Psi_j' \Psi_k d\bar{x} \\
& - 2 \sum_{j=1}^N \sum_{k=1}^N \dot{W}_j W_k \int_0^1 \Psi_i \Psi_j \Psi_k' d\bar{x} + \sum_{j=1}^N \dot{V}_j \int_0^1 \Psi_i \Psi_j d\bar{x} \\
& + \Pi_1 \sum_{j=1}^N E_e \left(\underline{\Psi_i \mathbf{F}_{1j}^{(a)'} } |_0^1 - \underline{\Psi_i' \mathbf{F}_{1j}^{(a)} } |_0^1 + \int_0^1 \Psi_i'' \mathbf{F}_{1j}^{(a)} d\bar{x} \right) \\
& - \cos \theta \sum_{j=1}^N E_e \left(\underline{\Psi_i \mathbf{M}_{3j}^{(a)'} } |_0^1 - \underline{\Psi_i' \mathbf{M}_{3j}^{(a)} } |_0^1 + \int_0^1 \Psi_i'' \mathbf{M}_{3j}^{(a)} d\bar{x} \right) \\
& - \sin \theta \sum_{j=1}^N E_e \left(\underline{\Psi_i \mathbf{M}_{2j}^{(a)'} } |_0^1 - \underline{\Psi_i' \mathbf{M}_{2j}^{(a)} } |_0^1 + \int_0^1 \Psi_i'' \mathbf{M}_{2j}^{(a)} d\bar{x} \right) = \int_0^1 \bar{L}_v d\bar{x}
\end{aligned}$$

Using the boundary conditions given in Equations H.1 and H.2 and replacing

the integrals with the terms given in Appendix I, the following is obtained:

$$\begin{aligned}
& \Pi_1 Z_i + \Pi_2 \sum_{j=1}^N \phi_j Q_{ij} + \Gamma_1 \sum_{j=1}^N V_j \beta_j^4 \delta_{ij} + \Gamma_3 \sum_{j=1}^N \sum_{k=1}^N V_j \phi_k \beta_j^4 H_{ijk} \\
& + \Gamma_3 \sum_{j=1}^N \sum_{k=1}^N V_j \phi_k \beta_i^4 H_{ijk} - \Gamma_3 \sum_{j=1}^N W_j \beta_j^4 \delta_{ij} + \Gamma_2 \sum_{j=1}^N \sum_{k=1}^N W_j \phi_k \beta_j^4 H_{ijk} \\
& + \Gamma_1 \sum_{j=1}^N \sum_{k=1}^N W_j \phi_k \beta_i^4 H_{ijk} - \Upsilon_1 \sum_{j=1}^N \phi_j R_{ij} - \Upsilon_2 \sum_{j=1}^N \sum_{k=1}^N \phi_j \phi_k S_{ijk} \\
& + \sum_{j=1}^N V_j (O_{ji} + O_{ij} - \delta_{ij}) - 2\Pi_1 \sum_{j=1}^N \dot{V}_j (T_{ij} - T_{ji}) \\
& - 2 \sum_{j=1}^N \sum_{k=1}^N \dot{V}_j V_k (Y_{ijk} - Y_{jik}) - 2\beta_{pc} \sum_{j=1}^N \dot{W}_j \delta_{ij} - 2 \frac{\Lambda_{12}}{\Lambda_{11}} \sum_{j=1}^N \dot{\phi}_j P_{ij} \\
& + 2\Pi_2 \sum_{j=1}^N \dot{W}_j T_{ij} - 2 \sum_{j=1}^N \sum_{k=1}^N \dot{W}_j W_k (Y_{ijk} + Y_{ikj}) + \sum_{j=1}^N \ddot{V}_j \delta_{ij} \\
& + \Pi_1 \sum_{j=1}^N E_e A_{ij}^{(a)} - \cos \theta \sum_{j=1}^N E_e B_{ij}^{(a)} - \sin \theta \sum_{j=1}^N E_e C_{ij}^{(a)} = \int_0^1 \bar{L}_v d\bar{x}
\end{aligned} \tag{H.6}$$

H.3 $\delta\bar{w}$ Equation

Galerkin's Method can be applied to Equation 3.32 to obtain:

$$\begin{aligned}
& \Pi_2 \int_0^1 \Psi_i d\bar{x} - \Pi_1 \sum_{j=1}^N \phi_j \int_0^1 \Psi_i \Theta_j d\bar{x} - \Gamma_3 \sum_{j=1}^N V_j \int_0^1 \Psi_i \Psi_j'''' d\bar{x} \\
& + \Gamma_1 \sum_{j=1}^N \sum_{k=1}^N V_j \phi_k \int_0^1 \Psi_i \Psi_j'''' \Theta_k d\bar{x} + \Gamma_2 \sum_{j=1}^N \sum_{k=1}^N V_j \phi_k \int_0^1 \Psi_i (\Psi_j \Theta_k)'''' d\bar{x} \\
& - \Gamma_2 \sum_{j=1}^N W_j \int_0^1 \Psi_i \Psi_j'''' d\bar{x} - \Gamma_3 \sum_{j=1}^N \sum_{k=1}^N W_j \phi_k \int_0^1 \Psi_i \Psi_j'''' \Theta_k d\bar{x} \\
& - \Gamma_3 \sum_{j=1}^N \sum_{k=1}^N W_j \phi_k \int_0^1 \Psi_i (\Psi_j \Theta_k)'''' d\bar{x} - \Upsilon_2 \sum_{j=1}^N \phi_j \int_0^1 \Psi_i \Theta_j''' d\bar{x} \\
& + \Upsilon_1 \sum_{j=1}^N \sum_{k=1}^N \phi_j \phi_k \int_0^1 \Psi_i \Theta_j \Theta_k''' d\bar{x} + 2\Pi_2 \sum_{j=1}^N \dot{V}_j \int_0^1 \Psi_i \Psi_j' d\bar{x} - \sum_{j=1}^N W_j \int_0^1 \Psi_i \Psi_j d\bar{x} \\
& - 2 \sum_{j=1}^N \sum_{k=1}^N \dot{V}_j W_k \int_0^1 \Psi_i \Psi_j' \Psi_k d\bar{x} + \beta_{pc} \int_0^1 \bar{x} \Psi_i d\bar{x} + 2\beta_{pc} \sum_{j=1}^N \dot{V}_j \int_0^1 \Psi_i \Psi_j d\bar{x} \\
& + \sum_{j=1}^N \ddot{W}_j \int_0^1 \Psi_i \Psi_j d\bar{x} - \Pi_2 \sum_{j=1}^N E_e \int_0^1 \Psi_i \mathbf{F}_{1j}^{(a)''} d\bar{x} \\
& - \sin \theta \sum_{j=1}^N E_e \int_0^1 \Psi_i \mathbf{M}_{3j}^{(a)''} d\bar{x} - \cos \theta \sum_{j=1}^N E_e \int_0^1 \Psi_i \mathbf{M}_{2j}^{(a)''} d\bar{x} = \int_0^1 \bar{L}_w d\bar{x}
\end{aligned}$$

which, upon integration by parts yields:

$$\begin{aligned}
& \Pi_2 \left(\bar{x} \Psi_i |_0^1 - \int_0^1 \bar{x} \Psi_i' d\bar{x} \right) - \Pi_1 \sum_{j=1}^N \phi_j \left(\bar{x} \Psi_i \Theta_j |_0^1 - \int_0^1 \bar{x} (\Psi_i \Theta_j)' d\bar{x} \right) \\
& - \Gamma_3 \sum_{j=1}^N V_j \int_0^1 \Psi_i \Psi_j''' d\bar{x} + \Gamma_1 \sum_{j=1}^N \sum_{k=1}^N V_j \phi_k \beta_j^4 \int_0^1 \Psi_i \Psi_j \Theta_k d\bar{x} \\
& + \Gamma_2 \sum_{j=1}^N \sum_{k=1}^N V_j \phi_k \left\{ (\Psi_i \Psi_j''' \Theta_k + 3 \Psi_i \Psi_j'' \Theta_k' + \underline{3 \Psi_i \Psi_j' \Theta_k''} + \Psi_i \Psi_j \Theta_k''') |_0^1 \right. \\
& \quad \left. - (\Psi_i' \Psi_j'' \Theta_k + 2 \Psi_i' \Psi_j' \Theta_k' + \underline{\Psi_i' \Psi_j \Theta_k''}) |_0^1 + \Psi_i'' (\Psi_j \Theta_k)' |_0^1 - \Psi_i''' \Psi_j \Theta_k |_0^1 + \beta_i^4 \int_0^1 \Psi_i \Psi_j \Theta_k d\bar{x} \right\} \\
& - \Gamma_2 \sum_{j=1}^N W_j \beta_j^4 \int_0^1 \Psi_i \Psi_j d\bar{x} - \Gamma_3 \sum_{j=1}^N \sum_{k=1}^N W_j \phi_k \beta_j^4 \int_0^1 \Psi_i \Psi_j \Theta_k d\bar{x} \\
& - \Gamma_3 \sum_{j=1}^N \sum_{k=1}^N W_j \phi_k \left\{ (\Psi_i \Psi_j''' \Theta_k + 3 \Psi_i \Psi_j'' \Theta_k' + \underline{3 \Psi_i \Psi_j' \Theta_k''} + \Psi_i \Psi_j \Theta_k''') |_0^1 \right. \\
& \quad \left. - (\Psi_i' \Psi_j'' \Theta_k + 2 \Psi_i' \Psi_j' \Theta_k' + \underline{\Psi_i' \Psi_j \Theta_k''}) |_0^1 + \Psi_i'' (\Psi_j \Theta_k)' |_0^1 - \Psi_i''' \Psi_j \Theta_k |_0^1 + \beta_i^4 \int_0^1 \Psi_i \Psi_j \Theta_k d\bar{x} \right\} \\
& - \Upsilon_2 \sum_{j=1}^N \phi_j \left(\underline{\Psi_i \Theta_j'} |_0^1 - \Psi_i' \Theta_j |_0^1 + \Psi_i'' \Theta_j |_0^1 - \int_0^1 \Psi_i''' \Theta_j d\bar{x} \right) \\
& + \Upsilon_1 \sum_{j=1}^N \sum_{k=1}^N \phi_j \phi_k \left(\underline{\Psi_i \Theta_j \Theta_k''} |_0^1 - \int_0^1 (\Psi_i \Theta_j)' \Theta_k'' d\bar{x} \right) \\
& + 2 \Pi_2 \sum_{j=1}^N \dot{V}_j \left(\underline{\Psi_i \Psi_j} |_0^1 - \int_0^1 \Psi_i' \Psi_j d\bar{x} \right) \\
& - \sum_{j=1}^N W_j \left(\bar{x} \underline{\Psi_i \Psi_j} |_0^1 - \int_0^1 \bar{x} \Psi_i' \Psi_j d\bar{x} - \int_0^1 \bar{x} \Psi_i \Psi_j' d\bar{x} \right) \\
& - 2 \sum_{j=1}^N \sum_{k=1}^N \dot{V}_j W_k \left(\underline{\Psi_i \Psi_j \Psi_k} |_0^1 - \int_0^1 \Psi_i \Psi_j \Psi_k d\bar{x} - \int_0^1 \Psi_i \Psi_j \Psi_k' d\bar{x} \right) + \beta_{pc} \int_0^1 \bar{x} \Psi_i d\bar{x} \\
& + 2 \beta_{pc} \sum_{j=1}^N \dot{V}_j \int_0^1 \Psi_i \Psi_j d\bar{x} + \sum_{j=1}^N \ddot{W}_j \int_0^1 \Psi_i \Psi_j d\bar{x} \\
& - \Pi_2 \sum_{j=1}^N E_e \left(\underline{\Psi_i \mathbf{F}_{1j}^{(a)'}} |_0^1 - \underline{\Psi_i' \mathbf{F}_{1j}^{(a)}} |_0^1 + \int_0^1 \Psi_i'' \mathbf{F}_{1j}^{(a)} d\bar{x} \right) \\
& - \sin \theta \sum_{j=1}^N E_e \left(\underline{\Psi_i \mathbf{M}_{3j}^{(a)'}} |_0^1 - \underline{\Psi_i' \mathbf{M}_{3j}^{(a)}} |_0^1 + \int_0^1 \Psi_i'' \mathbf{M}_{3j}^{(a)} d\bar{x} \right) \\
& - \cos \theta \sum_{j=1}^N E_e \left(\underline{\Psi_i \mathbf{M}_{2j}^{(a)'}} |_0^1 - \underline{\Psi_i' \mathbf{M}_{2j}^{(a)}} |_0^1 + \int_0^1 \Psi_i'' \mathbf{M}_{2j}^{(a)} d\bar{x} \right) = \int_0^1 \bar{L}_w d\bar{x}
\end{aligned}$$

Using the boundary conditions given in Equations H.3 and H.4 to obtain the

following:

$$\begin{aligned}
& -\Pi_2 Z_i + \Pi_1 \sum_{j=1}^N \phi_j Q_{ij} - \Gamma_3 \sum_{j=1}^N V_j \beta_j^4 \delta_{ij} + \Gamma_1 \sum_{j=1}^N \sum_{k=1}^N V_j \phi_k \beta_j^4 H_{ijk} \\
& + \Gamma_2 \sum_{j=1}^N \sum_{k=1}^N V_j \phi_k \beta_i^4 H_{ijk} - \Gamma_2 \sum_{j=1}^N W_j \beta_j^4 \delta_{ij} - \Gamma_3 \sum_{j=1}^N \sum_{k=1}^N W_j \phi_k \beta_j^4 H_{ijk} \\
& - \Gamma_3 \sum_{j=1}^N \sum_{k=1}^N W_j \phi_k \beta_i^4 H_{ijk} + \Upsilon_2 \sum_{j=1}^N \phi_j R_{ij} - \Upsilon_1 \sum_{j=1}^N \sum_{k=1}^N \phi_j \phi_k S_{ijk} \\
& - 2\Pi_2 \sum_{j=1}^N \dot{V}_j T_{ji} + \sum_{j=1}^N W_j (O_{ji} + O_{ij}) + 2 \sum_{j=1}^N \sum_{k=1}^N \dot{V}_j W_k (Y_{jik} + Y_{ikj}) \\
& + \beta_{pc} B_i + 2\beta_{pc} \sum_{j=1}^N \dot{V}_j \delta_{ij} + \sum_{j=1}^N \ddot{W}_j \delta_{ij} - \Pi_2 \sum_{j=1}^N E_e A_{ij}^{(a)} \\
& - \sin \theta \sum_{j=1}^N E_e B_{ij}^{(a)} - \cos \theta \sum_{j=1}^N E_e C_{ij}^{(a)} = \int_0^1 \bar{L}_w d\bar{x}
\end{aligned} \tag{H.7}$$

H.4 $\delta\bar{\varphi}$ Equation

Galerkin's Method can be applied to Equation 3.33 to obtain:

$$\begin{aligned}
& \left(\frac{\Lambda_{12}^2}{\Lambda_{11}} - \Lambda_{22} \right) \sum_{j=1}^N \phi_j \int_0^1 \Theta_i \Theta_j'' d\bar{x} + \frac{\Lambda_{12}}{\Lambda_{11}} \int_0^1 \bar{x} \Theta_i d\bar{x} + 2 \frac{\Lambda_{12}}{\Lambda_{11}} \sum_{j=1}^N \dot{V}_j \int_0^1 \Theta_i \Psi_j d\bar{x} \\
& - \Upsilon_1 \sum_{j=1}^N V_j \int_0^1 \Theta_i \Psi_j''' d\bar{x} + \Upsilon_2 \sum_{j=1}^N W_j \int_0^1 \Theta_i \Psi_j''' d\bar{x} - \Pi_2 \sum_{j=1}^N V_j \int_0^1 \Theta_i \Psi_j d\bar{x} \\
& - \Pi_1 \sum_{j=1}^N W_j \int_0^1 \Theta_i \Psi_j d\bar{x} + \Upsilon_2 \sum_{j=1}^N \sum_{k=1}^N V_j \phi_k \left(\int_0^1 \Theta_i \Psi_j \Theta_k''' d\bar{x} - \int_0^1 \Theta_i (\Psi_j \Theta_k)''' d\bar{x} \right) \\
& + \Upsilon_1 \sum_{j=1}^N \sum_{k=1}^N W_j \phi_k \left(\int_0^1 \Theta_i \Psi_j \Theta_k''' d\bar{x} - \int_0^1 \Theta_i (\Psi_j \Theta_k)''' d\bar{x} \right) \\
& + \Gamma_3 \sum_{j=1}^N \sum_{k=1}^N V_j V_k \int_0^1 \Theta_i \Psi_j \Psi_k'''' d\bar{x} + \Gamma_1 \sum_{j=1}^N \sum_{k=1}^N W_j V_k \int_0^1 \Theta_i \Psi_j \Psi_k'''' d\bar{x} \\
& - \Gamma_3 \sum_{j=1}^N \sum_{k=1}^N W_j W_k \int_0^1 \Theta_i \Psi_j \Psi_k'''' d\bar{x} + \Gamma_2 \sum_{j=1}^N \sum_{k=1}^N V_j W_k \int_0^1 \Theta_i \Psi_j \Psi_k'''' d\bar{x} \\
& + \mu^2 \sum_{j=1}^N \bar{\phi}_j \int_0^1 \Theta_i \Theta_j d\bar{x} + \frac{1}{2} (\mu_2^2 - \mu_1^2) \sin 2\theta \int_0^1 \Theta_i d\bar{x} \\
& + (\mu_2^2 - \mu_1^2) \cos 2\theta \sum_{j=1}^N \phi_j \int_0^1 \Theta_i \Theta_j d\bar{x} + \sum_{j=1}^N E_e \int_0^1 \Theta_i \mathbf{M}_{1j}^{(a)'} d\bar{x} \\
& - \frac{\Lambda_{12}}{\Lambda_{11}} \sum_{j=1}^N E_e \int_0^1 \Theta_i \mathbf{F}_{1j}^{(a)'} d\bar{x} = \int_0^1 \bar{M}_\varphi d\bar{x}
\end{aligned}$$

which, upon integration by parts, gives:

$$\begin{aligned}
& - \left(\frac{\Lambda_{12}^2}{\Lambda_{11}} - \Lambda_{22} \right) \sum_{j=1}^N \nu_j^2 \phi_j \int_0^1 \Theta_i \Theta_j d\bar{x} + \frac{\Lambda_{12}}{\Lambda_{11}} \int_0^1 \bar{x} \Theta_i d\bar{x} + 2 \frac{\Lambda_{12}}{\Lambda_{11}} \sum_{j=1}^N \dot{V}_j \int_0^1 \Theta_i \Psi_j d\bar{x} \\
& - \Upsilon_1 \sum_{j=1}^N V_j \int_0^1 \Theta_i \Psi_j''' d\bar{x} + \Upsilon_2 \sum_{j=1}^N W_j \int_0^1 \Theta_i \Psi_j''' d\bar{x} \\
& - \Pi_2 \sum_{j=1}^N V_j \left(\bar{x} \Theta_i \Psi_j \Big|_0^1 - \int_0^1 \bar{x} (\Theta_i \Psi_j)' d\bar{x} \right) - \Pi_1 \sum_{j=1}^N W_j \left(\bar{x} \Theta_i \Psi_j \Big|_0^1 - \int_0^1 \bar{x} (\Theta_i \Psi_j)' d\bar{x} \right) \\
& + \Upsilon_2 \sum_{j=1}^N \sum_{k=1}^N V_j \phi_k \left(\underline{\Theta_i \Psi_j \Theta_k''} \Big|_0^1 - \int_0^1 (\Theta_i \Psi_j)' \Theta_k'' d\bar{x} - \left[(\Theta_i \Psi_j'' \Theta_k + 2 \Theta_i \Psi_j' \Theta_k' + \underline{\Theta_i \Psi_j \Theta_k''}) \Big|_0^1 \right. \right. \\
& \left. \left. - \Theta_i' (\Psi_j \Theta_k)' \Big|_0^1 + \int_0^1 \Theta_i'' (\Psi_j \Theta_k)' d\bar{x} \right] \right) + \Upsilon_1 \sum_{j=1}^N \sum_{k=1}^N W_j \phi_k \left(\underline{\Theta_i \Psi_j \Theta_k''} \Big|_0^1 - \int_0^1 (\Theta_i \Psi_j)' \Theta_k'' d\bar{x} \right. \\
& \left. - \left[(\Theta_i \Psi_j'' \Theta_k + 2 \Theta_i \Psi_j' \Theta_k' + \underline{\Theta_i \Psi_j \Theta_k''}) \Big|_0^1 - \Theta_i' (\Psi_j \Theta_k)' \Big|_0^1 + \int_0^1 \Theta_i'' (\Psi_j \Theta_k)' d\bar{x} \right] \right) \\
& + \Gamma_3 \sum_{j=1}^N \sum_{k=1}^N V_j V_k \beta_k^4 \int_0^1 \Theta_i \Psi_j \Psi_k d\bar{x} + \Gamma_1 \sum_{j=1}^N \sum_{k=1}^N W_j V_k \beta_k^4 \int_0^1 \Theta_i \Psi_j \Psi_k d\bar{x} \\
& - \Gamma_3 \sum_{j=1}^N \sum_{k=1}^N W_j W_k \beta_k^4 \int_0^1 \Theta_i \Psi_j \Psi_k d\bar{x} + \Gamma_2 \sum_{j=1}^N \sum_{k=1}^N V_j W_k \beta_k^4 \int_0^1 \Theta_i \Psi_j \Psi_k d\bar{x} \\
& + \mu^2 \sum_{j=1}^N \tilde{\phi}_j \int_0^1 \Theta_i \Theta_j d\bar{x} + \frac{1}{2} (\mu_2^2 - \mu_1^2) \sin 2\theta \int_0^1 \Theta_i d\bar{x} \\
& + (\mu_2^2 - \mu_1^2) \cos 2\theta \sum_{j=1}^N \phi_j \int_0^1 \Theta_i \Theta_j d\bar{x} + \sum_{j=1}^N E_e \left(\underline{\Theta_i M_{1j}^{(a)}} \Big|_0^1 - \int_0^1 \Theta_i' M_{1j}^{(a)} d\bar{x} \right) \\
& - \frac{\Lambda_{12}}{\Lambda_{11}} \sum_{j=1}^N E_e \left(\underline{\Theta_i F_{1j}^{(a)}} \Big|_0^1 - \int_0^1 \Theta_i' F_{1j}^{(a)} d\bar{x} \right) = \int_0^1 \bar{M}_\varphi d\bar{x}
\end{aligned}$$

Using the boundary conditions given in Equation H.5 results in:

$$\begin{aligned}
& \left(\Lambda_{22} - \frac{\Lambda_{12}^2}{\Lambda_{11}} \right) \sum_{j=1}^N \nu_j^2 \phi_j \delta_{ij} + \frac{\Lambda_{12}}{\Lambda_{11}} U_i + 2 \frac{\Lambda_{12}}{\Lambda_{11}} \sum_{j=1}^N \dot{V}_j P_{ji} \\
& - \Upsilon_1 \sum_{j=1}^N V_j R_{ji} + \Upsilon_2 \sum_{j=1}^N W_j R_{ji} + \Pi_2 \sum_{j=1}^N V_j Q_{ji} + \Pi_1 \sum_{j=1}^N W_j Q_{ji} \\
& - \Upsilon_2 \sum_{j=1}^N \sum_{k=1}^N V_j \phi_k (S_{jik} + S_{jki}) - \Upsilon_1 \sum_{j=1}^N \sum_{k=1}^N W_j \phi_k (S_{jik} + S_{jki}) \\
& + \Gamma_3 \sum_{j=1}^N \sum_{k=1}^N V_j V_k \beta_k^4 H_{kji} + \Gamma_1 \sum_{j=1}^N \sum_{k=1}^N W_j V_k \beta_k^4 H_{kji} - \Gamma_3 \sum_{j=1}^N \sum_{k=1}^N W_j W_k \beta_k^4 H_{kji} \\
& + \Gamma_2 \sum_{j=1}^N \sum_{k=1}^N V_j W_k \beta_k^4 H_{kji} + \mu^2 \sum_{j=1}^N \tilde{\phi}_j \delta_{ij} + (\mu_2^2 - \mu_1^2) \frac{\sin 2\theta}{\sqrt{2} \nu_i} \\
& + (\mu_2^2 - \mu_1^2) \cos 2\theta \sum_{j=1}^N \phi_j \delta_{ij} - \sum_{j=1}^N E_e D_{ij}^{(a)} + \frac{\Lambda_{12}}{\Lambda_{11}} \sum_{j=1}^N E_e E_{ij}^{(a)} = \int_0^1 \bar{M}_\varphi d\bar{x}
\end{aligned} \tag{H.8}$$

Appendix I

Galerkin Integrals

$$\begin{aligned} A_i &= \int_0^1 \Psi_i d\bar{x} \\ B_i &= \int_0^1 \bar{x} \Psi_i d\bar{x} \\ C_i &= \int_0^1 \bar{x}^2 \Psi_i d\bar{x} \\ E_{ij} &= \int_0^1 \bar{x} \Psi_i \Psi_j d\bar{x} \\ G_{ijk} &= \int_0^1 \bar{x} \Psi_i \Psi_j \Theta_k d\bar{x} \\ H_{ijk} &= \int_0^1 \Psi_i \Psi_j \Theta_k d\bar{x} \\ I_{ij} &= \int_0^1 \bar{x} \Psi_i \Theta_j d\bar{x} \\ J_{ij} &= \int_0^1 \bar{x}^2 \Psi_i \Theta_j d\bar{x} \\ L_{ijk} &= \int_0^1 \Psi_i \bar{x} \Psi_j \Psi'_k d\bar{x} - \int_0^1 \Psi_i \bar{x}^2 \int_0^{\bar{x}} \Psi'_j \Psi''_k d\bar{x} d\bar{x} \\ M_{ij} &= \int_0^1 \bar{x} \Theta_i \Theta_j d\bar{x} \\ O_{ij} &= \int_0^1 \bar{x} \Psi_i \Psi'_j d\bar{x} \\ P_{ij} &= \int_0^1 \Psi_i \Theta_j d\bar{x} \\ Q_{ij} &= \int_0^1 \bar{x} (\Psi_i \Theta_j)' d\bar{x} \\ R_{ij} &= \int_0^1 \Psi_i''' \Theta_j d\bar{x} \\ S_{ijk} &= -\nu_k^2 \int_0^1 (\Psi_i \Theta_j)' \Theta_k d\bar{x} \\ T_{ij} &= \int_0^1 \Psi_i \Psi'_j d\bar{x} \\ U_i &= \int_0^1 \bar{x} \Theta_i d\bar{x} \\ Y_{ijk} &= \int_0^1 \Psi_i \Psi'_j \Psi_k d\bar{x} \\ Z_i &= \int_0^1 \bar{x} \Psi'_i d\bar{x} \end{aligned} \tag{I.1}$$

$$\begin{aligned}
\delta_{ij} &= \int_0^1 \Psi_i \Psi_j d\bar{x} \\
&= \int_0^1 \Theta_i \Theta_j d\bar{x} \\
&= \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}
\end{aligned} \tag{I.2}$$

$$\begin{aligned}
A_{ij}^{(a)} &= \int_0^1 \Psi_i'' \mathbf{F}_{1j}^{(a)} d\bar{x} \\
B_{ij}^{(a)} &= \int_0^1 \Psi_i'' \mathbf{M}_{3j}^{(a)} d\bar{x} \\
C_{ij}^{(a)} &= \int_0^1 \Psi_i'' \mathbf{M}_{2j}^{(a)} d\bar{x} \\
D_{ij}^{(a)} &= \int_0^1 \Theta_i' \mathbf{M}_{1j}^{(a)} d\bar{x} \\
E_{ij}^{(a)} &= \int_0^1 \Theta_i' \mathbf{F}_{1j}^{(a)} d\bar{x}
\end{aligned} \tag{I.3}$$

Appendix J

Steady-State Equations

δV_{0j} equation:

$$\begin{aligned}
& \sum_{j=0}^N \left\{ \Gamma_1 \beta_j^4 \delta_{ij} V_{0j} + \Gamma_3 \sum_{k=1}^N (\beta_j^4 H_{ijk} + \beta_i^4 H_{ijk}) V_{0j} \phi_{0k} - \Gamma_3 \beta_j^4 \delta_{ij} W_{0j} \right. \\
& + \sum_{k=1}^N (\Gamma_2 \beta_j^4 H_{ijk} + \Gamma_1 \beta_i^4 H_{ijk}) W_{0j} \phi_{0k} + (O_{ji} + O_{ij} - \delta_{ij}) V_{0j} \\
& + \frac{\gamma}{6} \bar{v}_i I_{ij} \phi_{0j} - \Upsilon_1 R_{ij} \phi_{0j} - \Upsilon_2 \sum_{k=1}^N S_{ijk} \phi_{0j} \phi_{0k} + \Pi_2 Q_{ij} \phi_{0j} \\
& \left. + \Pi_1 E_{e0} A_{ij}^{(a)} - \cos \theta E_{e0} B_{ij}^{(a)} - \sin \theta E_{e0} C_{ij}^{(a)} \right\} \\
& = -\Pi_1 Z_i + \frac{\gamma}{6} \left(\bar{v}_i^2 A_i - \frac{C_d}{a} C_i - \bar{v}_i \theta B_i \right)
\end{aligned} \tag{J.1}$$

δW_{0j} equation:

$$\begin{aligned}
& \sum_{j=0}^N \left\{ -\Gamma_3 \beta_j^4 \delta_{ij} V_{0j} + \sum_{k=1}^N (\Gamma_1 \beta_j^4 H_{ijk} + \Gamma_2 \beta_i^4 H_{ijk}) V_{0j} \phi_{0k} - \Gamma_2 \beta_j^4 \delta_{ij} W_{0j} \right. \\
& - \Gamma_3 \sum_{k=1}^N (\beta_j^4 H_{ijk} + \beta_i^4 H_{ijk}) W_{0j} \phi_{0k} + \frac{\gamma}{6} \left(\beta_{pc} E_{ij} V_{0j} - \frac{\bar{c}}{2} O_{ij} W_{0j} \right. \\
& \left. - J_{ij} \phi_{0j} + \sum_{k=1}^N L_{ijk} V_{0j} W_{0k} \right) + \Upsilon_2 R_{ij} \phi_{0j} - \Upsilon_1 \sum_{k=1}^N S_{ijk} \phi_{0j} \phi_{0k} \\
& + \Pi_1 Q_{ij} \phi_{0j} + (O_{ij} + O_{ji}) W_{0j} - \Pi_2 E_{e0} A_{ij}^{(a)} - \sin \theta E_{e0} B_{ij}^{(a)} \\
& \left. - \cos \theta E_{e0} C_{ij}^{(a)} \right\} = -\beta_{pc} B_i + \Pi_2 Z_i + \frac{\gamma}{6} \left\{ C_i \theta - B_i \bar{v}_i + \frac{\bar{c}}{2} \beta_{pc} B_i \right\}
\end{aligned} \tag{J.2}$$

$\delta\phi_{0j}$ equation:

$$\begin{aligned}
& \sum_{j=0}^N \left\{ \nu_j^2 \left(\Lambda_{22} - \frac{\Lambda_{12}^2}{\Lambda_{11}} \right) \delta_{ij} \phi_{0j} + \sum_{k=1}^N \beta_k^4 \Gamma_3 H_{kji} V_{0j} V_{0k} \right. \\
& + \Gamma_1 \sum_{k=1}^N \beta_k^4 H_{kji} W_{0j} V_{0k} - \Gamma_3 \sum_{k=1}^N \beta_k^4 H_{kji} W_{0j} W_{0k} \\
& + \Gamma_2 \sum_{k=1}^N \beta_k^4 H_{kji} V_{0j} W_{0k} + \Pi_2 Q_{ji} V_{0j} + \Pi_1 Q_{ji} W_{0j} \\
& - \Upsilon_1 R_{ji} V_{0j} - \Upsilon_2 \sum_{k=1}^N (S_{jik} + S_{jki}) V_{0j} \phi_{0k} + \Upsilon_2 R_{ji} W_{0j} \\
& - \Upsilon_1 \sum_{k=1}^N (S_{jik} + S_{jki}) W_{0j} \phi_{0k} + (\mu_2^2 - \mu_1^2) \cos(2\theta) \delta_{ij} \phi_{0j} \\
& \left. - E_{e0} D_{ij}^{(a)} + \frac{\Lambda_{12}}{\Lambda_{11}} E_{e0} E_{ij}^{(a)} \right\} = -\frac{\Lambda_{12}}{\Lambda_{11}} U_i - \frac{1}{\sqrt{2}\nu_i} (\mu_2^2 - \mu_1^2) \sin(2\theta)
\end{aligned} \tag{J.3}$$

Appendix K

Perturbation Equations

The perturbation equation is given in Equation 3.52 as:

$$[M] \{\ddot{X}\} + ([C_g] + [C_a]) \{\dot{X}\} + ([K_s] + [K_a]) \{X\} + \{F\} \Delta E_e = 0$$

where each matrix is defined as follows:

$$[M] = \begin{bmatrix} \delta_{ij} & 0 & 0 \\ 0 & \delta_{ij} \left(1 + \frac{\gamma \varepsilon}{24}\right) & 0 \\ 0 & 0 & \mu^2 \delta_{ij} \end{bmatrix} \quad (\text{K.1})$$

$$[C_g] = 2 \begin{bmatrix} -\Pi_1 (T_{ij} - T_{ji}) & -\beta_{pc} \delta_{ij} + \Pi_2 T_{ij} & -\frac{\Delta_{12}}{\Lambda_{11}} P_{ij} \\ -(Y_{ijk} - Y_{jik}) V_{0k} & -(Y_{ijk} + Y_{ikj}) W_{0k} & \\ \beta_{pc} \delta_{ij} - \Pi_2 T_{ji} & 0 & 0 \\ + (Y_{jik} + Y_{ikj}) W_{0k} & & \\ \frac{\Delta_{12}}{\Lambda_{11}} P_{ji} & 0 & 0 \end{bmatrix} \quad (\text{K.2})$$

$$[C_a] = \frac{\gamma}{6} \begin{bmatrix} 2\frac{C_D}{a}E_{ij} + \theta\bar{v}_i\delta_{ij} & 2\bar{v}_i\delta_{ij} - \theta E_{ij} & 0 \\ +\bar{v}_i \sum_{k=1}^N H_{ijk}\phi_{0k} & -\sum_{k=1}^N G_{ijk}\phi_{0k} & \\ \bar{v}_i\delta_{ij} - 2\theta E_{ij} & E_{ij} & -\frac{3\xi}{4}I_{ij} \\ -2\sum_{k=1}^N G_{ijk}\phi_{0k} & & \\ 0 & 0 & \frac{\xi^2}{8}M_{ij} \end{bmatrix} \quad (K.3)$$

$$[K_a] = \frac{\gamma}{6} \begin{bmatrix} 0 & 0 & \bar{v}_i I_{ij} \\ \beta_{pc}E_{ij} + \sum_{k=1}^N L_{ijk}W_{0k} & -\frac{\xi}{2}O_{ij} + \sum_{k=1}^N L_{ikj}V_{0k} & -J_{ij} \\ 0 & 0 & 0 \end{bmatrix} \quad (K.4)$$

$$\{F\} = \sum_{j=1}^N \left\{ \begin{array}{l} \Pi_1 A_{ij}^{(a)} - \cos\theta B_{ij}^{(a)} - \sin\theta C_{ij}^{(a)} \\ -\Pi_2 A_{ij}^{(a)} - \sin\theta B_{ij}^{(a)} - \cos\theta C_{ij}^{(a)} \\ -D_{ij}^{(a)} + \frac{\Lambda_{12}}{\Lambda_{11}} E_{ij}^{(a)} \end{array} \right\} \quad (K.5)$$

$$\begin{aligned}
[K_a] = & \left[\begin{array}{ccc}
\beta_j^4 \Gamma_1 \delta_{ij} + O_{ij} + O_{ji} - \delta_{ij} & (\beta_i^4 \Gamma_1 + \beta_j^4 \Gamma_2) \sum_{k=1}^N H_{ijk} \phi_{0k} & \Gamma_3 \sum_{k=1}^N (\beta_k^4 + \beta_i^4) H_{ikj} V_{0k} \\
+ \Gamma_3 (\beta_j^4 + \beta_i^4) \sum_{k=1}^N H_{ijk} \phi_{0k} & -\beta_j^4 \Gamma_3 \delta_{ij} & + \sum_{k=1}^N (\beta_k^4 \Gamma_2 + \beta_i^4 \Gamma_1) H_{ikj} W_{0k} \\
& & - \Upsilon_1 R_{ij} + \Pi_2 Q_{ij} \\
& & - \Upsilon_2 \sum_{k=1}^N (S_{ijk} + S_{ikj}) \phi_{0k} \\
\\
(\Gamma_1 \beta_j^4 + \Gamma_2 \beta_i^4) \sum_{k=1}^N H_{ijk} \phi_{0k} & -\Gamma_3 \sum_{k=1}^N (\beta_j^4 + \beta_i^4) H_{ijk} \phi_{0k} & \sum_{k=1}^N (\beta_k^4 \Gamma_1 + \beta_i^4 \Gamma_2) H_{ikj} V_{0k} \\
- \beta_j^4 \Gamma_3 \delta_{ij} & + O_{ij} + O_{ji} - \beta_j^4 \Gamma_2 \delta_{ij} & - \Gamma_3 \sum_{k=1}^N (\beta_k^4 + \beta_i^4) H_{ikj} W_{0k} \\
& & + \Upsilon_2 R_{ij} + \Pi_1 Q_{ij} \\
& & - \Upsilon_1 \sum_{k=1}^N (S_{ijk} + S_{ikj}) \phi_{0k} \\
\\
\Gamma_3 \sum_{k=1}^N (\beta_k^4 H_{kji} + \beta_j^4 H_{jki}) V_{0k} & \sum_{k=1}^N (\Gamma_1 \beta_k^4 H_{kji} + \Gamma_2 \beta_j^4 H_{jki}) V_{0k} & \nu_j^2 \left(\Lambda_{22} - \frac{\Lambda_{12}^2}{\Lambda_{11}} \right) \delta_{ij} \\
+ \sum_{k=1}^N (\Gamma_1 \beta_j^4 H_{jki} + \Gamma_2 \beta_k^4 H_{kji}) W_{0k} & - \Gamma_3 \sum_{k=1}^N (\beta_k^4 H_{kji} + \beta_j^4 H_{jki}) W_{0k} & + (\mu_2^2 - \mu_1^2) \cos(2\theta) \delta_{ij} \\
- \Upsilon_1 R_{ji} + \Pi_2 Q_{ji} & + \Upsilon_2 R_{ji} + \Pi_1 Q_{ji} & - \Upsilon_2 \sum_{k=1}^N (S_{kji} + S_{kij}) V_{0k} \\
- \Upsilon_2 \sum_{k=1}^N (S_{jik} + S_{jki}) \phi_{0k} & - \Upsilon_1 \sum_{k=1}^N (S_{jik} + S_{jki}) \phi_{0k} & - \Upsilon_1 \sum_{k=1}^N (S_{kji} + S_{kij}) W_{0k}
\end{array} \right]
\end{aligned}$$

(K.6)

Appendix L

Perturbation Response

Figure L.1: Relative magnitude and phase of the displacements for a baseline case with a frequency of 0.97/rev.

Figure L.2: Relative magnitude and phase of the displacement for a baseline case with a frequency of 0.99/rev.

Figure L.3: Relative magnitude and phase of the displacements for a baseline case with a frequency of 2.61/rev.

Figure L.4: Relative magnitude and phase of the displacements for a baseline case with a frequency of 3.17/rev

Figure L.5: Relative magnitude and phase of the displacements for a baseline case with a frequency of 3.80/rev

Figure L.6: Relative magnitude and phase of the displacements for a baseline case with a frequency of 7.19/rev

Figure L.7: Relative magnitude and phase of the displacements for the non-activated AFC spar with a frequency of 0.87/rev

Figure L.8: Relative magnitudes and phase of each displacements for the non-activated AFC spar with a frequency of 0.89/rev

Figure L.9: Relative magnitude and phase of the displacements for the non-activated AFC spar with a frequency of 2.86/rev

Figure L.10: Relative magnitude and phase of the displacements for the non-activated AFC spar with a frequency of 3.03/rev

Figure L.11: Relative magnitude and phase of the displacements for the non-activated AFC spar with a frequency of 3.61/rev

Figure L.12: Relative magnitude and phase of the displacements for the non-activated AFC spar with a frequency of 6.74/rev

Appendix M

Computer Codes

The CD-ROM enclosed contains computer codes that solve the equations developed in this thesis. These codes were developed for MATLAB^{TM} Version 5.3, using the **fsolve.m** function from the Optimization Toolbox. These computer codes have been divided into solving for the cross-sectional stiffness and activation constants derived in Chapter 2 and the aeroelastic beam problem described in Chapter 3.

M.1 Cross-Sectional Computer Codes

The cross-sectional stiffness and actuation terms are developed in a different method than that used in Appendix A. It uses what is called Classical Laminate Theory (CLT), as given in Badir (1992) and Vinson and Sierakowski (1986). This method will result in the same terms, but as it is done in a matrix method, it is computationally more efficient.

The code **boxbeam.m** is where the boxbeam dimensions, material properties and ply angles are given. This calls the subroutine **stiffness.m** which calculates the reduced stiffness coefficients Q_{ij} used in CLT and given by Vinson and Sierakowski (1986). These are then modified through the appropriate coordinate transformation into $\bar{Q}_{ij}$. These values are used by the subroutine **material.m** to obtain the $D^{\alpha\beta\gamma\delta}$ as shown by Badir (1992). **boxbeam.m** uses the $D^{\alpha\beta\gamma\delta}$ terms to calculate the cross sectional stiffness and actuation terms.

M.2 Aeroelastic Beam Computer Codes

The program **stability.m** uses the cross-sectional terms found by **boxbeam.m** and applies it to a specified operating condition for a helicopter rotor blade in hover. Specified in this program are the rotational speed of the rotor shaft, the number of blades, drag coefficient of the blades and whether or not the electric field is on.

The Galerkin integrals in Appendix I are solved by the **modeshapes.m** subroutine. These are used in **staticloop.m** subroutine, which solves the steady-state equations using the nonlinear solver **fsolve.m**. The steady-state deflections are stored in a file where the program **output.m** plots the steady-state deflections for the different cases.

The steady-state values are passed to the subroutine **mck1.m**, which constructs the mass, damping and stiffness matrices along with the input control vector. The function **sylvester.m** checks to make sure that the $[K_s]$ for positive definite and symmetric, and the $[C_s]$ matrix is anti-symmetric.

The function **eigen.m** solves for the aeroelastic eigenvalues and plots the magnitude and phase for the first six dynamic modes.