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Dynamic Analysis of Flexible Bladed Disc-Shaft Rotor Systems

by

WEIPING WANG

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfilment of
the requirements for the degree of
Doctor of Philosophy

Ottawa-Carleton Institute of Mechanical and Aerospace Engineering
Department of Mechanical and Aerospace Engineering
Carleton University
Ottawa, Ontario
July 1992

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BLADED DISC-SHAFT SYSTEMS**

submitted by Weiping Wang
in partial fulfillment of the requirements for
the degree of Doctor of Philosophy

Thesis Supervisor

Chairman,
Department of Mechanical and Aerospace Engineering
Carleton University

External Examiner

July, 1992

Abstract

A rigorous formulation of the equations of motion for the vibration of flexible bladed disk-shaft rotor systems both in a rotating frame and in a non-rotating frame is presented for implementation by the finite element method.

New eigensolution methods are developed to calculate the critical speeds and whirl speeds for undamped gyroscopic rotor systems, and show that there exists a set of real vectors which constructs a basis for the vector space of the rotor system. A closed-form response is derived, and the characteristics of the response are analytically examined by using the real modes. Based on these methods, a perturbation theory is then developed for lightly damped rotor systems.

A new free-interface component mode synthesis, which uses truncated component real modes, complemented with residual flexibility, inertia-relief modes, and residual dissipative effects, has been developed to analyze the dynamics of damped multi-shaft rotor systems with flexible inter-shaft bearings. All flexible-interface coordinates are eliminated by using the compatibilities of coupling forces on the flexible interfaces.

A general component mode method is then developed as an extension of the above approach, and allows assembly of a damped rotor system with flexible and/or rigid interfaces, using any number of substructures. In this method, all rigid- and flexible-interface coordinates are eliminated from the final set of system equations by utilizing the interface compatibilities.

Many examples compare the accuracy and efficiency of the above methods with other solutions given in the literature.

Finally, the above methods are applied to the analysis of flexible bladed disk-shaft rotor systems. The effects of bladed disks on the whirl frequencies, critical speeds, steady-state and transient responses of a rotor system are studied. It is concluded that the effects of disc flexibility are only significant for backward whirl modes. For such modes the frequencies can be significantly lowered while the peak responses are increased by the presence of disc flexibility.

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Dedication

This dissertation is dedicated to my parents.

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List of Principal Symbols

A	— Area of the cross-section of a blade
$\{A\}$	— Real part of an eigenvector of a system
$[A_0]$	— Unperturbed matrix of $[A]$
$[A_1]$	— Perturbation Matrix of $[A]$
$\{B\}$	— Imaginary part of an eigenvector of a system
$[C]$	— Damping matrix in the fixed frame
$[C_R]$	— Damping matrix in the rotating frame
$\det[]$	— Determinent of a matrix
D	— Bending stiffness of plates
$\{e_i\}$	— i -th eigenvector of a perturbed system
$\{e_{0i}\}$	— i -th unit vector
$\{e_{ri}\}$	— r -th perturbation of the i -th eigenvector
E	— Elastic modulus
$\{f_{Fjl}\}$	— Flexible-junction forces of the l -th component
$\{f_{Rjl}\}$	— Rigid-junction forces of the l -th component
$\{F_{IeR}\}$	— Inertia unbalance force vector
$\{F_e\}$	— Externally applied force vector
$\{F_{Fj}\}$	— Flexible-junction force vector
$\{F_{Fjl}\}$	— Flexible-junction forces applied on the l -th component
$[G]$	— Gyroscopic matrix in the fixed frame
$[G_e]$	— Gyroscopic matrix of an element in fixed frame
$[G_{el}] = [K_l]^{-1}$	— Flexibility matrix of the l -th component
$[G_{eR}]$	— Gyroscopic matrix of an element in rotating frame
$[G_{Fjl}]$	— Submatrix of residual flexibility matrix on flexible junctions

$[G_R]$	— Gyroscopic matrix in the rotating frame
$[G_{Rjl}]$	— Submatrix of residual flexibility matrix on rigid junctions
$[H_e]$	— Apparent stiffness matrix of an element in fixed frame
$[H_{eR}]$	— Apparent stiffness matrix of an element in rotating frame
$[I]$	— Unit matrix
$Im[\]$	— Imaginary part of a quantity $[\]$
i_{xx}, i_{yy}, i_{zz}	— Moment of inertia of a blade
I_f, I_c	— Moments of inertia flapwise and chordwise, respectively
I_{xx}, I_{yy}, I_{zz}	— Moments of inertia of an element
I_{xy}, I_{yz}, I_{zx}	— Products of inertia of an element
J	— Rayleigh dissipation function
J_x, J_y, J_z	— Polar moment of inertia of an element
$[K]$	— Stiffness matrix of a system
$[K_g]$	— Geometric stiffness matrix in the fixed frame
$[K_{gR}]$	— Geometric stiffness matrix in the rotating frame
$[K_R] = [K]$	— Elastic stiffness matrix of a system
m_e	— Mass of an element
$[M]$	— Mass matrix in a fixed frame
$[M_e]$	— Mass matrix of an element in fixed frame
$[M_{eR}]$	— Mass matrix of an element in rotating frame
$[M_R]$	— Mass matrix in the rotating frame
$[N_c]$	— Chordwise angular shape function matrix of a blade element
$[N_f]$	— Flapwise angular shape function matrix of a blade element
$[N_r], [N_\theta], [N_w]$	— Shape function matrices of an annular disc element
$[N_\varphi]$	— Torsion shape function matrix of an annular blade element
$\{p_m\}, \{q_m\}$	— Nodal deflections of the m -th diametral modes of blades in a fixed frame
$\{p_{mR}\}, \{q_{mR}\}$	— Nodal deflections of the m -th diametral modes of blades in a rotating frame

$\{p_{kl}\}$	— “kept” mode coordinates of the l th component
$\{P_s\}$	— Mode coordinates of a system
$\{q(t)\}$	— State vector of a system
$\{Q(t)\}$	— Force vector in a state space
$Re[\]$	— Real part of a quantity $[\]$
S_x, S_y, S_z	— First moments of inertia of an element
T	— Kinetic energy
u_b, v_b, w_b	— Blade deflections relative to the rotating disc
u_d, v_d, w_d	— Displacements relative to the rotating disc
U_e	— Elastic potential energy of a blade array
U_g	— Geometric (centrifugal) potential energy of a blade array
U_d, V_d	— Rigid disk translations in the fixed frame
U_R, V_R	— Rigid disk translations in the rotating frame
u_r, u_θ	— Radial and transverse displacements relative to the rotating disc
v_f	— Flapwise deflection of a blade
$[V(s)]$	— Flapwise shape function matrix of a blade
w_c	— Chordwise deflection of a blade
$[W(s)]$	— Chordwise shape function matrix of a blade
x, y, z	— Space-fixed coordinates
x_b, y_b, z_b	— Principal axes for the i -th blade
$\bar{x}_b, \bar{y}_b, \bar{z}_b$	— Undeformed blade positions axes
x_d, y_d, z_d	— Disc-fixed coordinates
$\bar{x}_d, \bar{y}_d, \bar{z}_d$	— Undeformed position vector in a rotating disc
x_R, y_R, z_R	— Rotating coordinates
x', y', z'	— Local coordinate system of the i -th blade
$\{z_{Fjl}\}$	— Flexible-junction displacements of the l -th component
$\{Z_e\}$	— Nodal displacements of an annular disc element in fixed frame
$\{Z_{eR}\}$	— Nodal displacements of an annular disc element in rotating frame
$\{Z_{fl}\}$	— Pseudo-static displacements of the l -th component
$\{Z(t)\}$	— Displacement vector of a system

Greek

δ	— Stagger angle of blades
δ_{eb}	— Nodal displacement vector of an blade element
$\{\epsilon\}$	— Strain vector of an annular disc element
$\{\eta_{mR}\}, \{\zeta_{mR}\}$	— Nodal deflections of the m -th diametral disc modes
γ_r	— Radial transverse shear strain
λ	— Eigenvalue of a system
$[\Lambda]$	— Natural frequency matrix
ω	— Natural frequency of a system
ω_d	— Damped frequency of a system
$\Omega = \dot{\phi}$	— Spinning speed of the shaft
ϕ	— Rotation angle of the shaft
$\{\phi\}$	— Eigenvector of a system
$[\Phi]$	— Eigenvector matrix of a system
ψ	— Precessional angle
$\{\psi\}$	— Left eigenvector of a system
$[\Psi]$	— Left eigenvector matrix of a system
ψ_c	— Chordwise angular deflection of a blade
ψ_f	— Flapwise angular deflection of a blade
ψ_r	— Radial angular deflection of a disc
ψ_θ	— Circumference angular deflection of a disc
σ	— Damping exponent
θ	— Angular coordinate in the polar system
θ_i	— Angular position of the i -th blade
θ_x, θ_y	— Angular Displacements of a disc in the fixed frame
θ_{xR}, θ_{yR}	— Angular Displacements of a disc in the rotating frame
φ	— Torsion displacement of a blade
$\{\varphi\}$	— Eigenvector of a undamped gyroscopic system

Superscript

T	— Transpose of a matrix
-1	— Inverse of a matrix
$*$	— Conjugate of a complex quantity
H	— Inverse and conjugate of a complex matrix

Subscript

a	— Approximated
d	— disc
e	— element
g	— Geometric
l	— l -th component
I	— Imaginary part of a quantity
k	— kept
R	— Real part of a quantity
0	— Partitioned matrix in one plane
x	— x component of a quantity
y	— y component of a quantity

Chapter 1

Introduction

1.1 Preliminary

The increasing demand for more efficient turbomachinery has resulted in the design and production of higher power density machines. With this increase in power density has come a host of potential rotor dynamics problems. Typically these problems arise as a result of decreasing shaft or bladed disc stiffness, increasing operation speed (some above several critical speeds), and using tighter clearance, more flexible casing and support structures. To study these effects with a degree of certainty, the analyst has had to generate more accurate rotor models, include the bladed disc and support dynamic flexibility characteristics, and even take into account aerodynamic cross-coupling effects.

1.2 Classification of Rotor Dynamic Problems

Rotor vibration can generally be classified as imposed vibration set up by unbalance or applied disturbance, or as self-excited vibration arising from instability. It should be noted that nonlinearity of dynamic characteristics of a rotor system leads to interaction of these two classes.

Imposed vibration occurs in nearly all turbomachinery, and the cause is usually

readily determined. The most common source of vibration in turbomachinery is rotor unbalance. If the centre of mass of a rotating disc or shaft does not lie on the axis of rotation, it consequently orbits about the axis and generates a centrifugal force that must be reacted by the bearings and support structure. Since the force rotates at the rotor operation speed, the vibration frequency is synchronous. Even if the centre of mass of the rotor does lie on the axis of rotation, a misalignment of the principal axis of inertia will produce a rotating couple that also excites synchronous vibration in the turbomachinery.

In the past, considerable problems have also been encountered due to the instability or self-excited vibration of high performance turbomachinery. The cause of this class of vibration is sometimes not obvious. This motion is usually characterized by whirling or whipping of the flexible rotor at one of its natural frequencies below the running speed. This subsynchronous vibration is never due to the rotor unbalance but rather is usually associated with the time variation of destabilizing forces arising from hydrodynamic bearings (oil whip), cross-coupling stiffness, internal rotor damping, blade-tip clearance eccentricity, or propeller whirl flutter. At a rotor speed above a limiting value (called threshold speed), when the destabilizing force exceeds the stabilizing external damping, any disturbance will cause the shaft to whirl at its critical speed, independent of the operation speed. The whirl motion induces a centrifugal force on the displaced centre of gravity of the shaft, which induces even greater amplitude and consequently large values of the destabilizing force, so the motion is a spiral with amplitude growing exponentially with time. In practice amplitudes are limited by nonlinear effects.

The vibration level and instability impose a continuing restraint on the performance capability of turbomachinery and continue to cause difficulties in the design and operation of high performance turbomachinery. They also pose a serious challenge to the designer, since there are many different mechanisms to be dealt with when one tries to predict the dynamic characteristics of a practical rotor. The complexity of the system makes it impossible to consider the entire rotor system with all generalities for the analysis. In general, the component parts of the rotor are analyzed separately, and even so some simplifying assumptions are necessarily made

to facilitate the analysis.

1.3 Literature Review

Figure 1.1 shows a typical segment of a flexible bladed disc-shaft rotor system, which consists of four interconnecting components: a rotating shaft, a flexible disc, an array of blades with distributed mass and elasticity, and discrete bearings. A large quantity of research work has been published in rotor dynamics which includes the analysis of the stability, critical speeds, steady-state response and transient response of rotor systems. The coupled elastic bladed disk-shaft problems have been historically treated as two independent problems: a flexible shaft carrying rigid disks, and a rigid shaft carrying a flexible bladed disc. It was not until the early 1970's, that more comprehensive research, which incorporates the (bladed) disc flexibility into the rotor model, was reported.

1.3.1 Rotor Dynamics with Rigid Disks

In most of the publications on rotor dynamics, the bladed disks are usually treated as rigid bodies mounted on circular section beams which are supported on fixed bearings. The reason for this is that when the stiffness of the disc is large compared to the stiffness of the shaft-support assembly, the influence of the disc flexibility on the dynamic characteristics of the rotor system can be neglected without significant loss in accuracy.

The publications of Jensen [1], Shapiro et al. [2] and Eshleman [3, 4] reflect the results of surveys and critical reviews of literature published in rotor dynamics up to 1984.

The classical solution methods, such as, the influence number [5, 6], iteration, Laplace transformation [7, 8], and dynamic stiffness [9] methods, are available for several conventional rotor profiles, e.g., an overhung disc on a cantilevered rotating shaft, and discs between the span of a simply supported rotating shaft. All the classical methods mentioned above relied on the assumption that the shaft has uniform

cross sections so that the rotor system can be described by a differential equation with constant coefficients. Reviews of early solutions of the rotor dynamics were given in the classical works of Den Hartog [5], and Dimentberg [9].

Transfer Matrix Methods

Among the numerical methods for the rotor dynamics analysis, the transfer matrix method [10] has been shown by many researchers to be a powerful means for the analysis of rotors in fixed bearings. In this method, the shaft is divided into several constant cross-section segments with discs or bearings at the junctures. A numerical integration process is then performed step by step from one segment to another. A precession frequency is pre-assumed and some boundary conditions are pre specified during the integration. The natural frequency of precession is then reached by iterating with frequency until satisfying the actual set of boundary conditions.

The development of the transfer matrix for rotor dynamics began in 1945 when Prohl [11] extended Holzer's scheme of torsional vibration analysis to analyzing synchronous whirling of a complex rotor with a shaft of variable sections and with multiple discs. Later this method has been modified by Ekong et al. [12] to include the transverse shear deformation and distributed masses in the rotor model, and by Lund [13, 14, 15] to incorporate the bearing damping into the rotor model and to calculate the unbalance response of the rotor system. Hibner [16] and Li [17] have shown that to handle more sophisticated problems, a multi-level transfer matrix approach could be used to accurately model multi-rotor systems with coupling between rotors or between rotor and casing for both free and steady-state vibration analysis.

The transfer matrix method, however, has some limitations in application. It uses the lumped mass model for continuous shafts, and numerical problems are common for calculating higher frequency modes and responses. The most important limitation of the transfer matrix method is that it is difficult to apply the method to analysis of transient response for a rotor system, especially when the rotor accelerates or decelerates, and when the applied force is arbitrary.

Finite Element Methods

A newer means of numerical analysis, the finite element method, allows the analyst to assemble a model of virtually any geometry while easily maintaining the effects of cross-couplings between substructures. The book written by Hughes [18] is one of the best texts on the finite element method. The utilization of finite element models of beams, such as introduced by Archer [19] and Davis [20], in the area of rotor dynamics has yielded highly successful results.

The first works applied to rotor dynamics were by Ruhl [21] in 1970 and by Ruhl and Booker [22] in 1972. Ruhl's finite element model was based on the Bernoulli-Euler beam theory. In 1975, Dimarogonas [23] presented the general development of an element with rotary inertia, gyroscopic moments and internal damping. In 1976, Gasch [24] published a paper quite similar to that of Dimarogonas, but also included the effect of distributed eccentricity. Nelson and McVaugh [25] presented a study in 1976 which utilized a Rayleigh beam rotating shaft element developed in both a fixed and rotating frame of reference. This paper was further generalized in 1977 by Zorzi and Nelson [26] by the inclusion of internal damping. Later Nelson [27] presented the development of a Timoshenko beam finite rotating shaft element which accounts for rotary inertia, transverse shear deformation and gyroscopic moments of the shaft element but there is no increase in the degrees of freedom of the element (i.e., the same order as the Bernoulli-Euler beam element). Gravis and Childs [28, 29] have applied this element to modeling several flexible rotor bearing systems, and the results show that the element is applicable to both short and slender rotating shafts. These researches have shown that the finite element method is fully applicable to the analysis of a rotor in fixed bearings. Rouch [30, 31] demonstrated that the finite element approach is as accurate as the transfer matrix method for modeling a rotor in fixed bearings, and by including a reduction scheme to condense out unnecessary degrees of freedom, the finite element method is more efficient.

Nonami [32] calculated damped non-stationary response during constant accelerations of rotor system through critical speeds using the finite element method. A transition matrix method was used to solve finite element equations for transient

response. The maximum amplitudes of response were also estimated by using an asymptotic method, and compared well with the finite element solutions.

The drawback of the finite element method in rotor dynamics is its complexity. As the number of subsystems increases, so does the complexity of the programming and the amount of computer resources required to solve the problem. In particular, structural finite element codes utilize the symmetry of the structural component's matrices to reduce storage and calculation requirements. This feature is lost when the analyst includes the non-symmetric cross-coupling effects inherent to gyroscopic coefficients, aerodynamic cross-coupling, and fluid film bearings. The loss, though not great when analyzing a rotor in fixed bearings, greatly hinders the application of the finite element formulation to large rotor/bearing/casing/support system analysis in which the rotor and bearings are only a small percentage of the overall degrees of freedom.

Modal Analysis Methods

When the total number of degrees of freedom in a complex structure is too large even for modern digital computers to handle economically, recourse is sometimes taken in which a structure is represented by its vibration modes [33, 34]. The advantage of a modal representation is that only a few of the lower frequency modes are usually required to give good approximate description of a structure. The assumption is that a dynamic system is unlikely to vibrate in its higher frequency modes because of the relatively high energy requirement. With only a few of the vibration modes, the problem size is decreased without a significant loss in the solution accuracy. Detailed coverage of modal analysis method for use in structural dynamics along with extension of the methodology to cover the more general problem with gyroscopic effects is presented by Meirovitch in the text [35].

Undamped non-gyroscopic systems are characterized by symmetric mass and stiffness matrices. The eigensolution for such systems consists of pairs of real eigenvalues and eigenvectors which are orthogonal with respect to both the mass and stiffness matrices. Hence, the responses of such systems can be obtained in closed form by superposition of responses of all modes to the given excitations. If gyroscopic forces are

present, then the eigensolution becomes complex. A method for obtaining the eigenvalue problem of undamped gyroscopic systems was developed by Meirovitch [36], where the eigenvalue problem is transformed into one in terms of two real matrices for both the real and the imaginary part of the eigenvectors. Based on the approach, a modal analysis that permits the derivation of closed-form response of linear undamped gyroscopic systems has been presented in reference [37, 39].

The problem of damped rotor systems is considerably more complicated than those of undamped ones. Since the sum of the damping matrix and the gyroscopic matrix is an arbitrary matrix, the advantage resulting from the symmetry or skew symmetry of the coefficient matrices is lost, so that one must return to the general theory. For the rotor systems with small damping, the damping can be regarded as a perturbation to the undamped systems. Meirovitch [39] has presented a second-order perturbation theory to calculate the left and right eigenvectors as well as the response of the damped systems.

Since the classical work of Bishop [40] on modal analysis of rotors in the early 1960's, the application of modal analysis to rotors in fixed bearings has been developed by Choy, et al. [41, 42], Childs [43, 44], and Lund [45]. A major benefit of modal analysis is that the mass, damping, and stiffness matrices for a system modeled with symmetric stiffness cross-coupling and proportional damping are diagonalized thus reducing the number of calculations required to solve a particular problem. Unfortunately in most rotor dynamics problems, the non-symmetrically cross-coupled effects of gyroscopic and fluid film bearings cannot be ignored. The drawback to the normal modal analysis for the rotor analyst is that the reduced matrices generated from the nonsymmetric matrices are full as opposed to diagonal and therefore increase the work required for solution. In the analysis of a single rotor these losses are inconsequential, but in the framework of multi-rotor system [41, 42], this procedure becomes more complex. If multiple rotors are connected to substructures the procedure can become unwieldy.

Meirovitch presented a stationary principle for linear gyroscopic systems in 1976 [38]. The implication of the principle is that the Rayleigh-Ritz approach can be used for rotating structures in a way similar to that for non-rotating structures. He demon-

strated that the discretization by assumed non-rotating mode method could yield as satisfactory results as obtained using eigen-functions of the rotating substructure as long as the set of admissible functions is complete. This stationary principle has valuable implications in modal analysis of the rotor dynamics problem.

The other approach in the application of the undamped modal method to rotating structures is to separate the stiffness matrix into symmetric and anti-symmetric matrices [41, 42, 44]. Only the symmetric stiffness matrix is used in the evaluation of the undamped modes. This transformation produces coupled second order differential equations in the modal coordinates. The only benefits in this representation are the reduction of the number of degrees of freedom by means of modal truncation, and the diagonalization of the mass matrix.

With damped complex modes, a more elaborate approach may be used to completely uncouple the linear equation of motion even when the damping is non-proportional. It requires, at the outset, the solution of the homogeneous equations to give the damped free-vibration modes of the system. Two sets of complex bi-normal modes are used to construct the transformation that will uncouple the equation of motion regardless of the form of damping in the system. Complex modal analysis has been used as a means of embedding the effects of gyroscopic, bearing damping and non-symmetric stiffness into the modes. Choy, Gunter, and Allaire [42] have used this method for the stability, forced response, and transient analysis of a rotor. In their calculation, Choy et al. recommended the use of undamped modes for the approximation of some damped systems. The conclusions of these authors pointed to the assembly of a system model from undamped modes and then addition of discrete damping to the system model.

However, the usefulness of this method in the analysis of rotating machinery is questionable in the view of the following considerations. The equation of motion is uncoupled only at the rotor speed at which the bi-normal modes are calculated for the reason that the stiffness and damping matrices are normally speed dependent. Furthermore, the effort involved in obtaining the damped modes can be laborious, especially when good accuracy of the damped mode shapes are desired. Care must also be taken to ensure that all the significant modes are included. In the evaluation

of the damped modes, for example, with the transfer matrix method, a Newton-Raphson type of analysis to search for the roots requires extra attention to ensure some of the modes are not left out in the process.

A possible alternative is the use of re-analysis to analyze an undamped system and then analyze the system again with discrete damping. Palazzolo [46, 47] and Kitis [48] have shown that a non-perturbation approach for re-analysis of rotor system can be used to model a system in one configuration and then to re-analyze it again with specific modifications. In the context of rotor dynamics, this potentially allows for the analysis of an undamped system and re-analysis of the same system but with the addition of discrete damping. This approach as developed by these authors requires that the couplings between subsystems be embedded in the modes of each subsystem. In this type of formulation the analyst must decide what portion of the bearing coefficients should be left in the rotor modes and what portion should be added via re-analysis. This approach makes it difficult for the analyst to change bearing coefficients and to calculate forces.

It should be noted that modal truncation is an approximation procedure. The use of a smaller number of modes than the number of degrees of freedom to represent a structure always results in a loss of model accuracy. Uncritical use of the modal technique is dangerous because solution accuracy is not always assured when improper choice of the modes is made or not enough modes are used.

Component Mode Synthesis

A further reduction of the system size that the computer has to handle at one time is accomplished by the component mode approach. Hurty [49] and Craig et al. [50] are among the earlier investigators using this method. Summaries of the various developments in component mode synthesis were presented by Hou [51] and Craig [58]. In this method, a large structure is partitioned into a number of substructures. The modal information for each individual substructure is derived either analytically or from modal test. The structure is then assembled in the modal coordinates by using only a truncated number of modes for each structure. In a large structure, the potential of this method lies not only in the ability to represent substructures containing

thousands of degrees of freedom by a handful of component modes, but also on the capability that enables analysts to build up accurate analytical models of a complex structure by treating only one portion of the total structure at a time. Another advantage of the component mode synthesis is that each component model is independent of all other components and boundary conditions except in boundary compatibility. This means that the changes in one component do not affect the analysis of all other components of a system. This feature is extremely valuable for a rotor system development program, such as for a new aircraft turbine program where there may be occasional design changes in one component or another.

The primary emphasis in these analyses is directed towards the matching of the displacements or the forces exerted at the connection boundaries that bond the substructures together. To satisfy system continuity at the connections, it requires the substructures to share the common degrees of freedom at these points. Accordingly, a set of constraint equations equal in number to the pairs of common degrees of freedom is derived that expresses the kinematic dependencies among the generalized coordinates relating to the various substructures. These equations of constraint are used to determine a set of generalized coordinates equal in number to the total component modal coordinates minus the number of equations of constraints.

In the analysis of rotor systems, another problem is often encountered. Substructures are connected by elements that are themselves elastic. Typical examples, in this category, are multi-rotor trains connected by flexible couplings, aircraft rotors with intershaft anti-friction bearings, and rotors supported by bearings in flexible housing. In all of these cases, the forces generated in each connecting element (flexible coupling, bearing) are dependent on the motion of the degrees of freedom at which the element is connected. Because there are no common degrees of freedom being shared at the junctions, no constraint equations are required. The dynamic effects due to the flexible connections are simply introduced as generalized damping, stiffness, and forces into the modal equation.

The substructure coupling methods may be classified as fixed-interface methods, free-interface methods, or hybrid methods depending on whether the component modes are determined with the substructure interfaces fully restrained, free of re-

straint, or partially restrained.

The fixed-interface component mode method developed by Hurty [49] and refined by Craig and Bampton [50] generally produces very accurate solutions for a given number of degrees of freedom, but requires that all of the original interface displacement coordinates be retained in the final coupled structure model. It is suggested that this might restrict the number of components (or substructures), and normal modes retained in the coupled structure model.

Hence, the primary focus of research is on methods for reducing the number of interface coordinates required by the fixed-interface methods. Craig and Chang [52] presented three methods for reducing the number of interface coordinates in the final system model: Ritz reduction, Guyan reduction and modal reduction.

The free-interface substructure methods are much more attractive than the fixed-interface substructure methods, when experimental verification of analytical models is required, or the structure is assembled by using the component modes from modal testing. Also, no interface displacement coordinates but only substructure mode coordinates are retained in the final system equations of motion.

As shown by Craig and Chang [52], Hou's free-interface component mode method is the least accurate, and both MacNeal (first-order approximation) [53] and Rubin (second-order approximation) [54] free-interface methods produce accuracies comparable with the fixed-interface methods. The Rubin method gives better results, but requires additional calculations. It is difficult to justify whether the improvement in results is sufficient to compensate the added computational effort.

Only a very limited number of researchers on component mode synthesis have treated damped structures. The usual approach to damping systems was to assume proportional damping. As described by Hasselman [55], however, the proportional damping model is inadequate to treat real systems. Such is definitely the case for structures with concentrated dampers and/or with rotating components which produce gyroscopic forces. Hasselman [55] and Craig [56] extended the fixed-interface methods to treating systems with non-proportional viscous damping. Using variational principles, Howsman and Craig [57] developed a consistent state-space formulation that employs a truncated set of free-interface component modes along with a set

of attached modes. Craig and Ni [59] presented a free-interface method of component mode synthesis for damped structures which leads to greater accuracy of eigenvalues than achieved by the previous methods. In their studies, a complex mode equivalent of residual flexibility is first developed, and then state-variable residual inertia-relief attachment modes are defined and are employed, along with a set of complex modes in a component coupling procedure.

As shown in this thesis, the component mode syntheses which employ complex component modes do not seem to give better accuracy in eigensolutions than those which employ free-interface normal modes augmented with residual flexibility. This may be explained by the consideration that the complex component modes can be used to decouple the equations of motion of the individual component only and are only as good an approximation of the exact eigensolution of the full-scale model of the whole structure as the component normal modes.

In an extension of earlier finite element work [25, 26, 27] Glasgow and Nelson [61, 60] applied a component mode synthesis method to rotor/bearing stability analysis. In their analysis the motion of each component was described by superposition of constraint modes (associated with boundary coordinates) and constrained precessional complex modes (associated with internal coordinates). The constrained modes that represent each component are truncated when assembled to the equations of motion of the whole structure. Nelson and Meacham [62] further extended their research to steady-state and transient analysis of linear rotor/bearing systems using the component mode synthesis. For the transient response, right-left eigenvector sets were used with bi-orthogonality conditions to decouple system equations before integration.

Craggs [66] presented a component mode procedure for modelling flexible rotor systems, in which the motion of each component was described by superposition of constraint and constrained real modes. An important feature of the method is that, in addition to all coupling nodes, the points located close to the bearings (which are easy to be measured and are usually used for attaching balancing masses) are also included in the retained physical coordinates. An advantage of the method is that it is possible to confirm predicted properties with those that are measured. The results

presented show a good quality of convergence and accuracy. It would be valuable to compare the accuracy obtained by using this method with that obtained by using the component complex mode method developed by Nelson and Glasgow (both methods are fixed-interface component mode synthesis).

Works by Li and Gunter [17, 63, 64] have applied Hou's free-interface component mode synthesis to large rotor systems, and analyzed the modal truncation error obtained during analysis of a dual rotor system. Two modal truncation schemes were evaluated for critical speeds, stability, and unbalance response of a dual-spool gas turbine engine. One modal criterion involved comparison of the undamped natural frequency of the subsystem to an upper frequency limit selected which, as recommended, be four or five times maximum rotor speed. The second criterion, proposed by Tolani [65], is based on the strain energy of the subsystems. Accuracy in the stability analysis can be obtained when all the modes below at least six times the highest rotor speed are retained in the calculation. Since they did not include the residual flexibility in their analysis, any extension of the conclusion to other component mode synthesis methods could be misleading.

1.3.2 Dynamic Analysis of a Flexible Disc

Along with the development of vibration analysis of rotor systems with rigid discs, much work has been published describing the vibration and stress analysis of rotating discs. Much of the early work is summarized in the texts by Stodola [67], Prescott [68], and Leissa [69].

The flexural vibration of an axisymmetric, rotating disc, characterized by the mode shapes with integer numbers of nodal diameters and nodal circles which rotate forwards or backwards in the disc, has been quite well understood since the classical research of Campbell [70] and Stodola [67].

Exact solutions were obtained by McLeod and Bishop [71] for constant thickness thin circular or annular plates, by Conway [72] and Harris [73] for thin plates of variable thickness, by Lamb and Southwell [74] for rotating discs of constant thickness, and by Ghosh [75] for rotating disc of variable thickness. Eversman [76] has extended

the analysis to include membrane stresses.

The transfer matrix scheme, first introduced to disc dynamics by Ehrich [77], is one of the most successful and widely used numerical methods. In this method the effects of in-plane stresses resulting from rotation or thermal gradient are readily accounted for, and the method is directly applicable to profiles of practical discs.

Mote [78] and Soo [79] have used the Rayleigh-Ritz method to investigate the free vibration of centrally clamped discs with variable thickness.

Since the late 70's, the finite element method has been extensively applied to the disc vibration and stress analysis. The use of triangular circular or annular sector elements was first suggested for the disc vibration [80, 81, 82, 83, 84]. It should be noted that none of these elements makes use of the axisymmetric properties of a complete circular disc, and all result in a mathematical model with a large number of degrees of freedom.

The first, and one of the most efficient, semi-analytical annular element was developed by Kirkhope and Wilson [85, 86]. By assuming different cosine functions for each particular nodal diameter mode to represent the displacement function along the angular direction, and approximating the displacement in the radial direction by a cubic polynomial, the element has 2 nodes, 4 degrees of freedom, and linear or parabolic thickness. Later Kirkhope and Wilson further extended their research to include the effects of membrane stresses due to rotation, pre-stresses and thermal stresses in the element [87], and to develop a moderately thick annular element [88] based on Mindlin-Reissner plate theory [89, 90], in which the rotary inertia and transverse shear strain have been incorporated into the plate model. Since then several annular finite elements have been developed in References [91, 92, 93, 94]. A convergence analysis performed on a thick disc in Ref. [94] revealed that the greatest efficiency, in terms of computational cost, was achieved by Kirkhope and Wilson element [88], and by the isoparametric three-node element of Mota et al. [91], where the internal nodal degrees of freedom are condensed during the generation of element stiffness and mass matrices. The other reviews on the efficiency of various annular elements were given by Mota Soares et al. [92], and Priolo et al. [93].

In all the researches above, the edges of the disc were treated as free, simply

supported, or clamped boundary conditions. Sann [95] in 1970 was the first to study a flexible disc gyroscope supported by a massless, elastic shaft. This model is equivalent to a disc with elastic boundary conditions at the inner edge. The purpose of the study was to analyze the effects of shaft flexibility on the dynamic characteristics of the rotating disc. He employed a numerical integration of circular plate differential equations matching the boundaries of the disc to the shaft at the inner edge and to a heavy rigid ring on the outer edge. Results were obtained for non-synchronous precession natural frequencies as well as forced vibration response.

Sann's analysis, although procedurally correct, did not properly handle the boundary conditions. The numerical results presented by Sann do not approximate the true state of stress in the plate to the desired degree of accuracy. In 1974, Red [96] presented a more complete investigation of the free vibration of the rotor plate in the spin mode. He also obtained closed form solution of the governing differential equation for the natural vibration of the rotor when free of imposed gyroscopic rates.

In the paper published by Sakata et al. [97] in 1982, the rotor was modelled as a flexible thin disc fastened to a massless elastic shaft with a rigid boss. The Galerkin's method was used to solve the governing equations for a rotating flexible disc, and the effect of blade loss was represented by a sudden increase of imbalance or eccentricity of the rotor. Experiments were conducted using a model rotor having a blade loss simulator, and the shaft vibration arising from a sudden increase of imbalance was measured. The validity of the analytical results was demonstrated by comparison to the experimental results. This method can only be applied to the single disc rotor with simple geometry.

In 1984, Loewy and Khader [98] presented a general mathematical model for a flexible bladed disc on an elastic, massless shaft. Again, this model is equivalent to a bladed disc with elastic boundary conditions at the inner edge of the disc. In their formulation, displacements of a flexible bladed disc were expressed in terms of the non-rotating mode shapes obtained from finite element analysis with the disc hub fixed in space, and coupled to a rotating, massless elastic shaft. They concluded that in the range of design practice for stiffness values some of the one-diametral node frequencies are shown to be affected significantly by shaft deflection. Accounting for

Coriolis forces introduces small changes in the resulting frequencies for the most part, but in regions where the rigid and flexible modes are highly coupled these terms can have substantial effect.

Crawley et al. [99, 100] studied the coupled bladed disc/shaft whirl of a cantilevered turbine analytically and experimentally. In their model, the shaft was treated as a massless, rotating elastic beam, the disc as a rigid annular plate, and blades as tuned and elastically uncoupled, with a single Ritz bending mode shape normal to the effective stagger angle. A special whirl excitation apparatus was used to excite forward and backward asynchronous whirl in a vacuum. The agreement between the predicted and experimental natural frequencies was good, and indicated the possibility of significant interaction of the one nodal diametral blade modes with the disc-shaft modes. However, no analytical results were given for the forced response of the rotor, and the work was limited in application because the dynamics of shaft and bearings were neglected in their modelling.

In the researches of Sann, Red, Loewy and Khader, and Crawley, no comprehensive effects of shaft flexibility on dynamic characteristics of (bladed) discs have been analyzed because the dynamic property of the shaft was neglected. This could be important in the case of a rotor operating above several critical speeds so that the higher whirling modes of the rotor may have more effects on the disc dynamics than the fundamental one. Furthermore, the application of their method is limited to analysis of single-disc rotor systems.

1.3.3 Flexible Disc-Shaft Assembly

For the analysis of stability, critical speeds and response of rotor systems, it is generally a satisfactory approximation that the discs are considered as rigid bodies, but it could be over optimistic if a bladed disc is thin enough to have a disc frequency close to the shaft-bearing whirling frequency.

To the author's knowledge, the first extensive investigation of the flexible disc shaft assembly appears to be due to Dopkin. Dopkin et al. [101, 102] developed a general analysis to predict the natural frequencies of vibration of a rotating shaft,

with any number of flexible discs, on linear, undamped bearings. The transfer matrix technique was used to obtain the solution. The shaft and the discs were described by a number of constant-thickness segments and lumped masses. Results of the analysis were found to be in agreement with the results obtained by others in the literature, when rigid disc rotors were considered.

Dopkin concludes that the disc flexibility always reduces the rotor natural frequency, sometimes to only a few percent of the value of the rotor with rigid discs. In his studies, only a circular orbit is considered which implies isotropic properties in all components. One very important advantage is that the transfer matrix analysis employed by Dopkin is still widely used in industry.

The disadvantages of his method are that it is numerically a poor model — masses are assumed concentrated at discrete points, while flexibility is distributed according to classical beam or plate theories. Numerical round-off errors are common in this model.

In 1973, Chivens [103, 104] reported an analytical investigation into the influence of disc flexibility on the transverse bending natural frequencies and critical speeds of a rotating shaft-disc system. The geometric model considered consists of a flexible continuous shaft, a flexible continuous plate with a rigid core. The rotor was modelled as a Bernoulli-Euler beam carrying a rigid core and thin flexible plate. The flexible plate was represented by the standard differential equation for the bending of thin plates in which the effects of the in-plane stresses on bending were included. Assuming a given precession factor, the linear differential equations governing this elastic model were non-dimensionalized and solved exactly for frequencies and modal shapes.

This investigation was limited to the cases in which all the components of the rotor system are axisymmetric in mechanical characteristics. The effects of transverse shear and rotary inertia were neglected. The method has the disadvantage of not being applicable to practical rotor geometries.

Kuo [105] in 1975 applied the Finite Element approach to the vibration analysis of a flexible disc-shaft model. The structural model consists of thin flexible discs rigidly mounted on a shaft with variable circular cross sections. He formulated a three-dimensional finite element model with consistent mass matrix. The NASTRAN finite

element program was used in the analysis. The calculated natural frequencies and disc dynamic stress distribution in radial and tangential directions were shown in good agreement with measurements. The disadvantages of this method are that the size of the model used is very large, and that it deals with non-rotating system only. This method thus was of little practical importance.

Wilgen [106, 107, 108] has shown the effects of an appendage's flexibility on critical speeds. To obtain stability criteria in an explicit form, he employed Liapunov stability analysis method using 'admissible' functions to approximate the shaft and appendage modal shapes. Different approximation used for the formulation were made to use Euler-Bernoulli theory. Viscous internal damping was included and aerodynamic effects were neglected. One significant advantage is that no assumption regarding the precession factor was made, however this method is again not really suitable for practical geometry.

Klompas [109, 110] described a new method to analyze critical speeds of multi shaft jet engines. The unbalance response was derived to account for a flexible bladed disc and asymmetric bearing stiffness. New coordinates for shaft motion in elliptical orbits, where the two components called whirling and whipping were defined in a rotating and a stationary plane respectively, were utilized to introduce accounting for bladed disc flexibility and engine mount asymmetry in the multi shaft jet engine dynamics. The resulting new method of calculating unbalance response requires solution of eight simultaneous equations for each bearing after completion of a series of steps which characterize the elastic system as a stiffness matrix and array of forces due to unbalance. The well-known transfer matrix method was employed for solution.

The advantage of this method is that it is a practical method that can be conveniently used to solve dynamic problems of rotors with asymmetric supports. But the model is numerically poor since only a single flexible disc mode can be incorporated into the magnification factors which were used to represent the effect of disc flexibility.

The finite element method for the study of the effect of disc flexibility on the rotor whirling frequencies was presented by Masoom [111] in 1980. Beam finite elements without shear deflection and rotary inertia were used to model the shaft, and annular elements developed by Kirkhope and Wilson [85, 87] were used for the flexible disc

with or without transverse shear and rotary inertia. The shaft-disc interaction occurs at a point of attachment and only circular whirling orbits were considered. Analysis on several configurations of flexible disc-shaft system were presented for different conditions of rotation. The results were compared with analytical or other numerical results in the literature and showed excellent agreement.

Palladino, Palladino and Rossettos [112, 113] presented the development of a special shaft element which incorporated analytical solutions for disc dynamics and approximations for blade effects. This model has the advantage that the size of the rotor system only depends on the number of shaft elements used and so is greatly reduced. In their model, the disc was presented as an isotropic circular plate, while the blades as a orthotropic plate. They concluded that the blade and disc flexibility would affect the system critical speeds in some cases, the trends for multiple disc rotors are much the same as those for the single disc rotor, and that the backward whirl and second modes are most sensitive to disc and blade flexibility. Due to versatility of the finite element method, complex configurations and boundary conditions can be considered, and analysis of practical rotors can be obtained.

Gallardo et al. [114, 115] have studied the dynamics of a flexible bladed disc on a flexible rotor in a two-rotor system which was modelled to simulate a two-spool configuration of gas turbine. The rotor displacements were expressed in terms of their normal modes, following the classic modal synthesis. The results indicated that a flexible bladed disc can be excited into resonance by an unbalance in the adjacent rotor and at a frequency equal to the differential rotor speed. However, no experimental results are given to evaluate the analytical model. In their later research [116], nonlinear connecting elements, such as bearings, were incorporated into their analysis.

In 1985, Perdikologos [117] modeled the flexible disc-shaft assembly as coupling of two component problems formed by a flexible shaft carrying rigid discs and relative deflections of discs to the undeformed discs. In his studies, the normal modal shapes for each component were first analyzed, then only a small set of the truncated modes of components was assembled into the combined rotor system. It has been shown that using a few lower normal modes of each component to form the rotor system

equation of motion is sufficient to get accurate critical speeds of the rotor.

In 1987, Shahab and Thomas [118] developed a new, thick, three-dimensional, cylindrical element, with 8 nodes and 12 degrees of freedom at each node, to study the coupling effects of disc flexibility on the dynamics problem of multi disc-shaft rotor. This element is applicable to varying geometry of the disc-shaft system. The theoretical results have shown very close agreement with their experimental results. No comparison of their method to others in the literature has been given, and the new element is only applicable to the non-rotating disc-shaft system.

Recently, Sakata et al. [119] studied the vibration of a rotor, with a flexible bladed disc on a flexible shaft, subjected to gyroscopic moment due to steady turn of the support system. The analysis was carried out using the finite element method: a rotating beam element for the shaft, an annular disc element [85, 87] for the disc, and a beam element for the blades. The steady state response calculated from the FEM model have shown a good agreement with the experimental results. No effects of gyroscopic moment on natural frequencies and critical speeds of the rotor, however, have been investigated.

1.4 Summary of Literature Review

A summary can be made here regarding the literature on the rotor dynamics:

1. The finite element method allows analysts to assemble a rotor of virtually any geometry, and is generally more suitable than the transfer matrix method in rotor dynamics. When the total number of degrees of freedom is too large to handle economically, recourse is taken to the use of modal analysis and/or component mode synthesis.
2. In most of publications on rotor dynamics, bladed discs are treated as rigid bodies mounted on flexible rotating shafts. The reason for this is that usually the fundamental frequency of the bladed disc is large compared to that of the rotor-bearing system, and the effect of the bladed disc flexibility on dynamic

characteristics of the rotor system can be neglected without significant loss in accuracy.

Studies on the effect of disc flexibility have so far focused on whirling frequencies of rotor systems. There is a further need to study the forced vibration and stability of such rotor systems. Mostly, undamped systems have been examined, especially for the calculation of whirling frequencies and vibration modes. Internal structural and external (bearing and/or blade aerodynamic) damping is present in practice, and it is necessary to study the effects of these cross coupling forces.

3. Studies on the influence of disc flexibility on rotor dynamics have so far largely ignored effects of blade flexibility. The few exceptions may be the researches done by Loewy and Khader [98], Crawley [99, 100], and Palladino et al. [112, 113]. In studies by Crawley, Loewy and Khader, the effect of shaft flexibility on the characteristics of the rotor system was not analyzed comprehensively because the dynamic properties of the shaft were ignored. In Palladino's work, the inertia coupling and Coriolis forces of the blades due to shaft motions were neglected because the blade set was modeled as an orthotropic plate.
4. Research on the application of component mode synthesis (CMS) to rotor dynamics is still quite limited. Up to date, to the author's knowledge, only the fixed-interface CMS methods have been extensively studied, and the applications of free-interface substructure methods in rotor dynamics is limited to Hou's Method. Since the free-interface methods are more compatible with state-of-the-art dynamic testing procedure, and the modified MacNeal and Rubin methods produce accuracies in eigensolutions even better than the fixed-interface methods, the application of the free-interface methods to rotor dynamics should be investigated.
5. Furthermore, research on the application of component mode synthesis to structures where some of the substructures are connected by elements that are themselves elastic are very few. This situation is commonly encountered in multi-

shaft models. Since in these cases, the forces generated in each connecting element (flexible coupling, bearing) are dependent on the motion of the degrees of freedom at which the element is connected, there are no common degrees of freedom being shared at the interfaces, i.e., no displacement-constraint equations can be applied. When only component modes are used to describe the motion of components in Hou's method, the interface coordinates can easily be eliminated from the CMS model, as shown in Ref. [17, 63]. When residual flexibility and inertia-relief modes are included in the analysis (as MacNeal and Rubin methods), the dynamic effects due to the flexible connections may be simply introduced as generalized damping, stiffness, and forces into the modal equation by retaining the interface coordinates in the CMS model. However, by doing so, the free-interface CMS method, not only loses one of the most significant advantages, but also adds computational difficulties to the numerical integration for response analysis. It would be valuable to develop free-interface CMS methods to these problems which eliminate all interface coordinates.

6. The lack of extensive experimental data should be noted, and the need for further advanced experiments is always necessary for the verification of theoretical analysis.

1.5 Scope of the Present Studies

The major studies of the thesis can be summarized as follows:

1. A rigorous formulation of the equations of motion for the vibration of a flexible bladed disc-shaft assembly in rotating and non-rotating coordinates is presented. The finite element method is used in which:
 - Shafts are modeled as rotating beam elements developed by Nelson et al. [27, 28] which can include the effects of transverse shear strains, rotary inertia, and gyroscopic moments.

- Discs are discretized as annular disc elements of moderate thickness, in which radial thickness variation, effect of in-plane stresses, rotary inertia and transverse shear are included in the analysis.
- Since the purpose of this study is focused on study of the coupling effects between flexible bladed discs and flexible shafts, a refined analysis of the blade vibration is not attempted. Blades are idealized to behave as beams having arbitrary variation in section properties and pretwist along their span. It is assumed that the centroid and flexural axes coincide, i.e. the shear centre of the cross-section coincides with the centroid of the section and there is no coupling in bending and torsion motions of blades. But the coupling of flexural vibrations in chordwise and flapwise directions due to pretwist will be considered. The inertia coupling, Coriolis forces, gyroscopic moments, in-plane stresses, and radial shortening of the blades are included in the formulation.
- Bearings are treated as elastic elements with linear stiffness and viscous damping, and support structure effects are accounted for by using orthotropic or anisotropic bearing properties.

The detailed descriptions of the derivation of these equations of motion are given in Chapters 2 and 3.

2. In Chapter 4 a technical review of existing techniques for dynamic analysis of rotor systems is given. This is followed in Chapter 5 by development of new methods to study the motion characteristics of gyroscopic systems with axial-symmetric mass distributions. Techniques are developed to find whirl or critical speeds, mode shapes, and response of undamped or lightly-damped rotor systems having isotropic or orthotropic bearings.
3. The development of the equations of motion of rotor systems using free-interface CMS methods for structures with flexible couplings between components is presented in Chapter 6. The method partitions a large structure into a number of components, and employs an incomplete set of free-interface modes of compo-

nents, augmented with a low-frequency account for the contribution of residual modes. The new method adds not only the residual flexibility introduced by MacNeal [53] but also the residual inertial, dissipative and gyroscopic effects. The compatibility of the coupling forces on the flexible interfaces are utilized to completely remove interface displacements from the final equations of assembly. Numerical studies are used:

- to compare the efficiency of computation of the component mode synthesis to that of the finite element solution.
 - to investigate whirling frequency and response errors in rotor dynamics for the free-interface component methods, and to compare with those for fixed-interface component mode methods.
4. Chapter 7 extend the above method of free-interface component mode synthesis to structures with hybrid (flexible and/or rigid) couplings between components. The displacement compatibility on the rigid interfaces and the compatibility of the coupling forces on the flexible interfaces are utilized to completely remove all interface displacements from the final equations of assembly. Again, the new method adds not only the residual flexibility introduced by MacNeal [53] but also the residual inertial, dissipative and gyroscopic effects.
 5. Finally in Chapter 8, the improved methods described above are employed to study the effects of disc and blade flexibility on the stability and response of rotor systems.

**Figure 1.1 A Typical Segment of Flexible
Bladed Disk-Shaft Rotor Systems**

Chapter 2

Finite Element Model of Flexible Disc-Shaft Rotor Systems

2.1 Introduction

In most of the literature dealing with dynamic problems of rotors with flexible disc shaft assemblies, the geometrical stiffness due to centrifugal forces is readily accounted for, but gyroscopic forces and their effects on the vibration characteristics of the rotating assembly are largely ignored. No reference has been found to include the rotary inertia moment of the disc into the gyroscopic forces for a flexible disc model, which could be important for analysis of a thick disc or higher frequency modes of a thin disc. Almost all the work for flexible bladed disc-shaft assembly, to author's knowledge, only deals with stationary operation speeds. But the response to the acceleration or deceleration of a rotor can be significant when the rotor operates above some critical speeds without appropriate damping.

To predict the dynamic behaviour of a flexible rotor system an appropriate analytical model has to be developed. Loewy and Khader [98] have presented a general formulation for a bladed disc, but their derivation of equations of motion is relatively complex because the rigid body motions of the disc were described in a space fixed coordinates while the elastic deformations were initially described in the rotating coordinates and then transformed back to the space-fixed coordinates. Also, their formulation was only intended for a single bladed disc rotor, and the application of

their method to multi disc rotor systems is limited.

In the present work, a more general analysis is presented to include the unsteady operating conditions, the centrifugal and gyroscopic as well as unbalance forces in the rotor model. The equations of motion of a rotor system are formulated using Lagrangian method and have been presented in both the rotating and space-fixed coordinate systems. The equations of motion in both fixed and rotating coordinates are derived in standard matrix forms, which have clear physical meaning and can be easily programmed in the computer.

2.2 Coordinate Systems

In order to predict the dynamic behaviour of a flexible rotor system with flexible bladed discs affixed to the flexible shaft, an appropriate analytical model has been developed first. The governing equations of motions are derived by Lagrangian method, which requires an expression for the kinetic energy of an element on a flexible blade disk. To define the position vector of a point on the bladed discs, a set of coordinate systems related through rotations or translations must be specified first. Then, using the appropriate transformations, we will be able to express the kinetic energy of the element in the space-fixed coordinate system. The defined coordinate systems are listed below and are shown in Fig. 2.1.

1) $OXYZ$ — Space-fixed coordinate system, with the origin at the center of the undeflected disc, Z axis normal to the plane of the disc, X and Y axes in the plane of the disc. All the three axes are taken to form a right-handed coordinate system.

2) $OX_RY_RZ_R$ — Rotating coordinate system obtained by a rotation ϕ of $OXYZ$ about the axis OZ , where ϕ is the spinning angle of the rotor. If $\{x, y, z\}^T$ and $\{x_R, y_R, z_R\}^T$ are the position vectors in $OXYZ$ and $OX_RY_RZ_R$ systems, then we have the following transformation:

$$\{X\} \stackrel{\text{def}}{=} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_R \\ y_R \\ z_R \end{Bmatrix} \stackrel{\text{def}}{=} [\Phi_z] \{X_R\} \quad (2.1)$$

3) $DX_1Y_1Z_1$ — Coordinate system attached to the center "D" of the deformed disc and with all three axes parallel to $OX_RY_RZ_R$ system. It is obtained by displacing the rotating coordinate system $OX_RY_RZ_R$ by two orthogonal translations, U_R and V_R , representing rigid disk motion along X_R, Y_R axes respectively. If $\{x_1, y_1, z_1\}^T$ is the position vector in $DX_1Y_1Z_1$ system, then

$$\{X_R\} \stackrel{\text{def}}{=} \begin{Bmatrix} x_R \\ y_R \\ z_R \end{Bmatrix} = \begin{Bmatrix} U_R \\ V_R \\ 0 \end{Bmatrix} + \begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix} \stackrel{\text{def}}{=} \{V_R\} + \{X_1\} \quad (2.2)$$

4) $DX_2Y_2Z_2$ — Coordinate system obtained by a rotation θ_{xR} of $DX_1Y_1Z_1$ system about DX_1 axis. If $\{x_2, y_2, z_2\}^T$ is the position vector in $DX_2Y_2Z_2$, then

$$\{X_1\} \stackrel{\text{def}}{=} \begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_{xR} & -\sin \theta_{xR} \\ 0 & \sin \theta_{xR} & \cos \theta_{xR} \end{bmatrix} \begin{Bmatrix} x_2 \\ y_2 \\ z_2 \end{Bmatrix} \stackrel{\text{def}}{=} [\Phi_x]\{X_2\} \quad (2.3)$$

5) $DX_dY_dZ_d$ — Coordinate system fixed to the rigid disc and obtained by a rotation θ_{yR} of the $DX_2Y_2Z_2$ system about DY_2 axis. If $\{x_d, y_d, z_d\}^T$ is the position vector of a point in $DX_2Y_2Z_2$ system, then it can be related to the position vector of the point in $DX_dY_dZ_d$ system by

$$\{X_2\} \stackrel{\text{def}}{=} \begin{Bmatrix} x_2 \\ y_2 \\ z_2 \end{Bmatrix} = \begin{bmatrix} \cos \theta_{yR} & 0 & \sin \theta_{yR} \\ 0 & 1 & 0 \\ -\sin \theta_{yR} & 0 & \cos \theta_{yR} \end{bmatrix} \begin{Bmatrix} x_d \\ y_d \\ z_d \end{Bmatrix} \stackrel{\text{def}}{=} [\Phi_y]\{X_d\} \quad (2.4)$$

From the foregoing, it is clear that the location of any point of the flexible bladed disc, vibrating about some steady state position, can be referred to the $DX_dY_dZ_d$ system as

$$\{X_d\} \stackrel{\text{def}}{=} \begin{Bmatrix} x_d \\ y_d \\ z_d \end{Bmatrix} = \begin{Bmatrix} \bar{x}_d \\ \bar{y}_d \\ \bar{z}_d \end{Bmatrix} + \begin{Bmatrix} u_d \\ v_d \\ w_d \end{Bmatrix} \stackrel{\text{def}}{=} \{\bar{X}_d\} + \{W_d\} \quad (2.5)$$

where

$\{\bar{X}_d\}$ is the position vector in $DX_dY_dZ_d$ system before the disc or blade is deformed.

$\{W_d\}$ is the deflection vector of the point $\{\bar{X}_d\}$ of the bladed disc in $DX_dY_dZ_d$.

Making use of the above coordinate systems, $\{X\}$ and $\{X_d\}$ can be related and the position of any point of the bladed disc can be referred to the space-fixed coordinate system $OXYZ$ as

$$\{X\} = [\Phi_z] \left\{ \{V_R\} + [\Phi_x][\Phi_y] (\{\bar{X}_d\} + \{W_d\}) \right\} \quad (2.6)$$

2.3 Kinetic Energy of an Element in a Disk

Consider an arbitrary element in a bladed disc denoted by the domain Ω_e , the Kinetic energy over the element can be written as:

$$T = \frac{1}{2} \int_{\Omega_e} \{\dot{X}\}^T \{\dot{X}\} dm \quad (2.7)$$

Substituting Eq. 2.6 into the above equation, neglecting all the small quantities of higher order than the second, and integrating all the remaining items over the element domain, we can get the kinetic energy of the element as follows

$$\begin{aligned} T = & \frac{1}{2} \left\{ m_e [\dot{\phi}^2 (U_R^2 + V_R^2) + (\dot{U}_R^2 + \dot{V}_R^2) + 2\dot{\phi}(U_R \dot{V}_R - \dot{U}_R V_R)] + \dot{\theta}_{xR}^2 J_x + \dot{\theta}_{yR}^2 J_y \right. \\ & + \dot{\phi}^2 [J_z - \theta_{xR}^2 (I_{yy} - I_{zz}) - \theta_{yR}^2 (I_{xx} - I_{zz}) + 2\theta_{xR}\theta_{yR} I_{xy} + 2\theta_{yR} I_{xz} - 2\theta_{xR} I_{yz}] \\ & + 2(\dot{\phi} U_R + \dot{V}_R) [\dot{\phi} S_x + (\dot{\phi} \theta_{yR} - \dot{\theta}_{xR}) S_z] + 2(\dot{\phi} V_R - \dot{U}_R) [S_y \dot{\phi} - S_z (\theta_{xR} \dot{\phi} + \dot{\theta}_{yR})] \\ & + 2\dot{\phi} \dot{\theta}_{xR} [\theta_{yR} (I_{xx} - I_{zz}) - \theta_{xR} I_{xy} - I_{xz}] + 2\dot{\phi} \dot{\theta}_{yR} [\theta_{xR} J_y + \theta_{yR} I_{xy} - I_{yz}] \\ & - 2\dot{\theta}_{xR} \dot{\theta}_{yR} I_{xy} \left. \right\} + (\dot{\phi} U_R + \dot{V}_R) \int_{\Omega_e} (\dot{v}_d + \dot{\phi} u_d) dm + (\dot{\phi} V_R - \dot{U}_R) \int_{\Omega_e} (\dot{\phi} v_d - \dot{u}_d) dm \\ & + \dot{\phi} (\dot{\phi} \theta_{yR} - \dot{\theta}_{xR}) \int_{\Omega_e} (\bar{z}_d u_d + \bar{x}_d w_d) dm - \dot{\phi} (\dot{\theta}_{yR} + \dot{\phi} \theta_{xR}) \int_{\Omega_e} (\bar{z}_d v_d + \bar{y}_d w_d) dm \\ & + \dot{\phi} \left\{ \int_{\Omega_e} (\bar{x}_d \dot{v}_d - \bar{y}_d \dot{u}_d) dm + \theta_{xR} \int_{\Omega_e} (\bar{z}_d \dot{u}_d - \bar{x}_d \dot{w}_d) dm + \theta_{yR} \int_{\Omega_e} (\bar{z}_d \dot{v}_d - \bar{y}_d \dot{w}_d) dm \right. \\ & + \left. \int_{\Omega_e} (u_d \dot{v}_d - \dot{u}_d v_d) dm \right\} + \dot{\theta}_{xR} \int_{\Omega_e} (\bar{y}_d \dot{w}_d - \bar{z}_d \dot{v}_d) dm + \dot{\theta}_{yR} \int_{\Omega_e} (\bar{z}_d \dot{u}_d - \bar{x}_d \dot{w}_d) dm \\ & + \dot{\phi}^2 \int_{\Omega_e} (\bar{x}_d u_d + \bar{y}_d v_d) dm + \frac{1}{2} \dot{\phi}^2 \int_{\Omega_e} (u_d^2 + v_d^2) dm + \frac{1}{2} \int_{\Omega_e} (\dot{u}_d^2 + \dot{v}_d^2 + \dot{w}_d^2) dm \quad (2.8) \end{aligned}$$

where

$$\begin{aligned}
 m_e &= \int_{\Omega_e} dm & I_{xx} &= \int_{\Omega_e} \bar{x}_d^2 dm \\
 S_x &= \int_{\Omega_e} \bar{x}_d dm & I_{yy} &= \int_{\Omega_e} \bar{y}_d^2 dm \\
 S_y &= \int_{\Omega_e} \bar{y}_d dm & I_{zz} &= \int_{\Omega_e} \bar{z}_d^2 dm \\
 S_z &= \int_{\Omega_e} \bar{z}_d dm & I_{xy} &= \int_{\Omega_e} \bar{x}_d \bar{y}_d dm \\
 J_x &= I_{yy} + I_{zz} & I_{xz} &= \int_{\Omega_e} \bar{x}_d \bar{z}_d dm \\
 J_y &= I_{xx} + I_{zz} & I_{yz} &= \int_{\Omega_e} \bar{y}_d \bar{z}_d dm \\
 J_z &= I_{xx} + I_{yy}
 \end{aligned}$$

2.4 Annular Element of a Rotating Disc

Considering an annular disc element as shown in Fig. 2.2, the deflections in polar coordinate system and in $DX_dY_dZ_d$ system have the relationship:

$$\begin{Bmatrix} u_d \\ v_d \\ w_d \end{Bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_r \\ u_\theta \\ w_d \end{Bmatrix} \quad (2.9)$$

The vibration of a rotating disc is found to comprise wave patterns involving integer numbers of nodal diameters and nodal circles, these patterns rotating forwards or backwards in the disc. The angular velocities of these waves in the disc are f_m/m (rev/sec.) for the forward wave, and $-f_m/m$ for the backward wave, where f_m is the frequency in Hertz of the mode with m nodal diameters. If the disc rotates with angular velocity $\Omega = \dot{\phi}$, then one has, relative to a stationary observer, $(\Omega + f_m/m)$ revs/second for the forward wave, and $(\Omega - f_m/m)$ revs/second for the backward wave. On the other hand, it is not known, a priori, where the nodal diameter lines will fall with respect to the flexible shaft motions, or if and how they will move as travelling waves. Thus each mode of the flexible disc must be used twice. It is convenient to place the nodal diameters at 90 degrees to each other, azimuthal. Each has its own

distinct node coordinate vectors $\{\eta_{mR}\}$ and $\{\zeta_{mR}\}$, along X_d and Y_d , respectively. Thus the following deflection functions are assumed for a thick disc element

$$\begin{Bmatrix} u_r \\ u_\theta \\ w \end{Bmatrix} = \begin{Bmatrix} z\psi_r(r, \theta) \\ z\psi_\theta(r, \theta) \\ w(r, \theta) \end{Bmatrix} = \begin{Bmatrix} z \sum_{m=0}^{\infty} [N_r(r)] (\{\eta_{mR}\} \cos m\theta + \{\zeta_{mR}\} \sin m\theta) \\ z \sum_{m=0}^{\infty} [N_\theta(r)] (\{\eta_{mR}\} \sin m\theta - \{\zeta_{mR}\} \cos m\theta) \\ \sum_{m=0}^{\infty} [N_w(r)] (\{\eta_{mR}\} \cos m\theta + \{\zeta_{mR}\} \sin m\theta) \end{Bmatrix} \quad (2.10)$$

where z is the distance measured from the middle plane of a disc, and

$\{\eta_{mR}\}$ and $\{\zeta_{mR}\}$ are the nodal deflections of the m -th diametral mode along the X_d axis and the Y_d axis, respectively.

$[N_r(r)]$ and $[N_\theta(r)]$ are row matrices whose terms are Lagrangian and or Hermitian polynomials for the approximation of the rotations ψ_r and ψ_θ , respectively.

$[N_w(r)]$ is a row matrix whose terms are Lagrangian and or Hermitian polynomials for approximation of the out-plane deflection w .

Assume that the disc element has the properties of axial symmetry essentially (i.e., $I_{xy} = I_{xz} = I_{yz} = 0$), but some unbalance S_x , S_y , and S_z are non-zeroes. Substituting Eq. 2.9 and 2.10 into the kinetic energy expression, and using the following orthogonal relations for sinusoid functions

$$\int_0^{2\pi} \sin m\theta \, d\theta = 0, \quad \text{if } m \geq 0$$

$$\int_0^{2\pi} \cos m\theta \, d\theta = \begin{cases} 2\pi, & \text{if } m = 0 \\ 0, & \text{if } m \geq 1 \end{cases}$$

$$\int_0^{2\pi} \sin^2 m\theta \, d\theta = \begin{cases} 0, & \text{if } m = 0 \\ \pi, & \text{if } m \geq 1 \end{cases}$$

$$\int_0^{2\pi} \cos^2 m\theta \, d\theta = \begin{cases} 2\pi, & \text{if } m = 0 \\ \pi, & \text{if } m \geq 1 \end{cases}$$

$$\int_0^{2\pi} \sin m\theta \cos n\theta \, d\theta = 0, \quad \text{if } m, n \geq 0$$

$$\begin{aligned}\int_0^{2\pi} \sin m\theta \sin n\theta d\theta &= 0, & \text{if } m \neq n \\ \int_0^{2\pi} \cos m\theta \cos n\theta d\theta &= 0, & \text{if } m \neq n\end{aligned}$$

we get the kinetic energy for an annular element of a disc as follows:

$$\begin{aligned}T &= \frac{1}{2} \left\{ m_e [\dot{\phi}^2 (U_R^2 + V_R^2) + (\dot{U}_R^2 + \dot{V}_R^2) + 2\dot{\phi}(U_R \dot{V}_R - \dot{U}_R V_R)] \right. \\ &+ \dot{\theta}_{xR}^2 I_D + \dot{\theta}_{yR}^2 I_D + \dot{\phi}^2 [I_p - \theta_{xR}^2 (I_{yy} - I_{zz}) - \theta_{yR}^2 (I_{xx} - I_{zz})] \\ &+ 2(\dot{\phi} U_R + \dot{V}_R) [\dot{\phi} S_x + (\dot{\phi} \theta_{yR} - \dot{\theta}_{xR}) S_z] + 2\dot{\phi} \dot{\theta}_{yR} \theta_{xR} I_D \\ &+ 2(\dot{\phi} V_R - \dot{U}_R) [S_y \dot{\phi} - S_z (\theta_{xR} \dot{\phi} + \dot{\theta}_{yR})] + 2\dot{\phi} \dot{\theta}_{xR} \theta_{yR} (I_{xx} - I_{zz}) \} \\ &- (\dot{\phi} \theta_{xR} + \dot{\theta}_{yR}) [Q_1] \{\dot{\eta}_{1R}\} - (\dot{\phi} \theta_{yR} - \dot{\theta}_{xR}) [Q_1] \{\dot{\zeta}_{1R}\} \\ &+ \dot{\phi} (\dot{\phi} \theta_{yR} - \dot{\theta}_{xR}) [Q_2] \{\eta_{1R}\} - \dot{\phi} (\dot{\phi} \theta_{xR} + \dot{\theta}_{yR}) [Q_2] \{\zeta_{1R}\} \\ &+ \dot{\phi} \sum_{m=0}^{\infty} (\{\dot{\eta}_{mR}\}^T [G_m] \{\zeta_{mR}\} - \{\eta_{mR}\}^T [G_m] \{\dot{\zeta}_{mR}\}) \\ &+ \frac{1}{2} \dot{\phi}^2 \sum_{m=0}^{\infty} (\{\eta_{mR}\}^T [H_{xm}] \{\eta_{mR}\} + \{\zeta_{mR}\}^T [H_{ym}] \{\zeta_{mR}\}) \\ &+ \frac{1}{2} \sum_{m=0}^{\infty} (\{\dot{\eta}_{mR}\}^T [M_{xm}] \{\dot{\eta}_{mR}\} + \{\dot{\zeta}_{mR}\}^T [M_{ym}] \{\dot{\zeta}_{mR}\}) \end{aligned} \quad (2.11)$$

where I_D is the diameter moment of inertia of the disc element, and

$$\begin{aligned}[Q_1] &= \pi \int_{r_1}^{r_2} \rho h [N_w(r)] r^2 dr - \frac{\pi}{12} \int_{r_1}^{r_2} \rho h^3 ([N_r(r)] - [N_\theta(r)]) r dr \\ [Q_2] &= \pi \int_{r_1}^{r_2} \rho h [N_w(r)] r^2 dr + \frac{\pi}{12} \int_{r_1}^{r_2} \rho h^3 ([N_r(r)] - [N_\theta(r)]) r dr \\ [G_m] &= \frac{\pi}{12} \int_{r_1}^{r_2} \rho h^3 (S_m [N_r(r)]^T [N_\theta(r)] + C_m [N_\theta(r)]^T [N_r(r)]) r dr \\ [H_{xm}] &= \frac{\pi}{12} \int_{r_1}^{r_2} \rho h^3 (C_m [N_r(r)]^T [N_r(r)] + S_m [N_\theta(r)]^T [N_\theta(r)]) r dr \\ [H_{ym}] &= \frac{\pi}{12} \int_{r_1}^{r_2} \rho h^3 (S_m [N_r(r)]^T [N_r(r)] + C_m [N_\theta(r)]^T [N_\theta(r)]) r dr \\ [M_{xm}] &= [H_{xm}] + C_m \pi \int_{r_1}^{r_2} \rho h [N_w(r)]^T [N_w(r)] r dr \\ [M_{ym}] &= [H_{ym}] + S_m \pi \int_{r_1}^{r_2} \rho h [N_w(r)]^T [N_w(r)] r dr\end{aligned}$$

$$C_m = \begin{cases} 2, & \text{if } m = 0 \\ 1, & \text{if } m \geq 1 \end{cases}; \quad S_m = \begin{cases} 0, & \text{if } m = 0 \\ 1, & \text{if } m \geq 1 \end{cases} \quad (2.12)$$

It should be noted that

$$\left. \begin{aligned} [H_{xm}] &= [H_{ym}] \stackrel{\text{def}}{=} [H_m] \\ [M_{xm}] &= [M_{ym}] \stackrel{\text{def}}{=} [M_m] \end{aligned} \right\}, \quad \text{for } m \geq 1. \quad (2.13)$$

and

$$[G_m]^T = [G_m], \quad [H_m]^T = [H_m], \quad [M_m]^T = [M_m] \quad \text{for } m \geq 1. \quad (2.14)$$

Eq. 2.11 may be more conveniently written as the general matrix form:

$$\begin{aligned} T = & \frac{1}{2} \{\dot{Z}_{eR}\}^T [M_{eR}] \{\dot{Z}_{eR}\} + \Omega \{\dot{Z}_{eR}\}^T [N_{eR}] \{Z_{eR}\} \\ & + \frac{1}{2} \Omega^2 \{Z_{eR}\}^T [H_{eR}] \{Z_{eR}\} + \{F_{dR}\}^T \{Z_{eR}\} + \{F_{vR}\}^T \{\dot{Z}_{eR}\} \\ & + \dot{\phi} \sum_{m \neq 1}^{\infty} \left(\{\dot{\eta}_{mR}\}^T [G_m] \{\zeta_{mR}\} - \{\eta_{mR}\}^T [G_m] \{\dot{\zeta}_{mR}\} \right) \\ & + \frac{1}{2} \dot{\phi}^2 \sum_{m \neq 1}^{\infty} \left(\{\eta_{mR}\}^T [H_{xm}] \{\eta_{mR}\} + \{\zeta_{mR}\}^T [H_{ym}] \{\zeta_{mR}\} \right) \\ & + \frac{1}{2} \sum_{m \neq 1}^{\infty} \left(\{\dot{\eta}_{mR}\}^T [M_{xm}] \{\dot{\eta}_{mR}\} + \{\dot{\zeta}_{mR}\}^T [M_{ym}] \{\dot{\zeta}_{mR}\} \right) \end{aligned} \quad (2.15)$$

where $\Omega \stackrel{\text{def}}{=} \dot{\phi}$ is the rotation speed of the disc element, and

$$\{Z_{eR}\} = \begin{Bmatrix} U_R \\ \theta_{yR} \\ \{\eta_{1R}\} \\ V_R \\ -\theta_{xR} \\ \{\zeta_{1R}\} \end{Bmatrix}, \quad \{F_{dR}\} = \begin{Bmatrix} \Omega^2 S_x \\ 0 \\ 0 \\ \Omega^2 S_y \\ 0 \\ 0 \end{Bmatrix}, \quad \{F_{vR}\} = \begin{Bmatrix} -\Omega S_y \\ 0 \\ 0 \\ \Omega S_x \\ 0 \\ 0 \end{Bmatrix}$$

$$[M_{eR}] = \begin{bmatrix} m_e & S_z & 0 & 0 & 0 & 0 \\ S_z & I_D & -[Q_1] & 0 & 0 & 0 \\ 0 & -[Q_1]^T & [M_1] & 0 & 0 & 0 \\ 0 & 0 & 0 & m_e & S_z & 0 \\ 0 & 0 & 0 & S_z & I_D & -[Q_1] \\ 0 & 0 & 0 & 0 & -[Q_1]^T & [M_1] \end{bmatrix}$$

$$[N_{eR}] = \begin{bmatrix} 0 & 0 & 0 & -m_e & -S_z & 0 \\ 0 & 0 & 0 & -S_z & -I_D & -[Q_2] \\ 0 & 0 & 0 & 0 & [Q_1]^T & [G_1] \\ m_e & S_z & 0 & 0 & 0 & 0 \\ S_z & I_{zz} - I_{xx} & [Q_2] & 0 & 0 & 0 \\ 0 & -[Q_1]^T & -[G_1]^T & 0 & 0 & 0 \end{bmatrix}$$

$$[H_{eR}] = \begin{bmatrix} m_e & S_z & 0 & 0 & 0 & 0 \\ S_z & I_{zz} - I_{xx} & [Q_2] & 0 & 0 & 0 \\ 0 & [Q_2]^T & [H_1] & 0 & 0 & 0 \\ 0 & 0 & 0 & m_e & S_z & 0 \\ 0 & 0 & 0 & S_z & I_{zz} - I_{yy} & [Q_2] \\ 0 & 0 & 0 & 0 & [Q_2]^T & [H_1] \end{bmatrix}$$

$$[P] = \frac{1}{2}([Q_2] - [Q_1])$$

Now applying Lagrangian formulation, we obtain the inertia forces of a disc angular element in matrix form as follows

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial T}{\partial \{\dot{Z}_{eR}\}} \right) - \frac{\partial T}{\partial \{Z_{eR}\}} &= -\{F_{IeR}\} + [M_{eR}]\{\ddot{Z}_{eR}\} + \Omega[G_{eR}]\{\dot{Z}_{eR}\} \\ &+ (-\Omega^2[H_{eR}] + \dot{\Omega}[N_{eR}])\{Z_{eR}\} \quad \text{for } m = 1 \end{aligned} \quad (2.16)$$

and

$$\begin{aligned} \left\{ \begin{array}{l} \frac{d}{dt} \left(\frac{\partial T}{\partial \{\dot{\eta}_{mR}\}} \right) - \frac{\partial T}{\partial \{\eta_{mR}\}} \\ \frac{d}{dt} \left(\frac{\partial T}{\partial \{\dot{\zeta}_{mR}\}} \right) - \frac{\partial T}{\partial \{\zeta_{mR}\}} \end{array} \right\} &= \begin{bmatrix} [M_{xm}] & 0 \\ 0 & [M_{ym}] \end{bmatrix} \left\{ \begin{array}{l} \{\ddot{\eta}_{mR}\} \\ \{\ddot{\zeta}_{mR}\} \end{array} \right\} \\ + 2\Omega \begin{bmatrix} 0 & -[G_m] \\ [G_m]^T & 0 \end{bmatrix} \left\{ \begin{array}{l} \{\dot{\eta}_{mR}\} \\ \{\dot{\zeta}_{mR}\} \end{array} \right\} &+ \begin{bmatrix} -\Omega^2 [H_{xm}] & -\dot{\Omega} [G_m] \\ \dot{\Omega} [G_m]^T & -\Omega^2 [H_{ym}] \end{bmatrix} \left\{ \begin{array}{l} \{\eta_{mR}\} \\ \{\zeta_{mR}\} \end{array} \right\} \\ &\text{for } m > 1 \end{aligned} \quad (2.17)$$

where

$$\begin{aligned} [G_{eR}] &= [N_{eR}] - [N_{eR}]^T && \text{(skew-symmetric)} \\ \{F_{IeR}\} &= \{F_{dR}\} - \{F_{vR}\} && \text{(inertial Force)} \end{aligned}$$

The formulae presented in the rotating coordinate system are convenient for analysis of the dynamics of flexible blades attached to a rotating disc, and useful for analysis of rotor-bearing dynamics with axial symmetry. In the cases in which the properties of bearings or support structures are orthotropic or anisotropic, it is better to transform the above inertia forces in Eq. 2.16 to the space-fixed coordinate system. In our cases, the relationships between the deflections in rotating coordinate system and those in the space-fixed coordinate system is given by

$$\{Z_{eR}\} = [\hat{\Phi}] \{Z_e\} \quad (2.18)$$

where

$$\{Z_{eR}\} = \begin{Bmatrix} U_R \\ \theta_{yR} \\ \{\eta_{1R}\} \\ V_R \\ -\theta_{xR} \\ \{\zeta_{1R}\} \end{Bmatrix}, \quad \{Z_e\} = \begin{Bmatrix} U \\ \theta_y \\ \{\eta_1\} \\ V \\ -\theta_x \\ \{\zeta_1\} \end{Bmatrix}$$

$$[\hat{\Phi}] = \begin{bmatrix} \cos \phi & 0 & 0 & \sin \phi & 0 & 0 \\ 0 & \cos \phi & 0 & 0 & \sin \phi & 0 \\ 0 & 0 & \cos \phi [I] & 0 & 0 & \sin \phi [I] \\ -\sin \phi & 0 & 0 & \cos \phi & 0 & 0 \\ 0 & -\sin \phi & 0 & 0 & \cos \phi & 0 \\ 0 & 0 & -\sin \phi [I] & 0 & 0 & \cos \phi [I] \end{bmatrix}$$

Hence, the velocities and accelerations have the relations:

$$\left. \begin{aligned} \{\dot{Z}_{eR}\} &= [\hat{\Phi}]\{\dot{Z}_e\} + \Omega[R][\hat{\Phi}]\{Z_e\} \\ \{\ddot{Z}_{eR}\} &= [\hat{\Phi}]\{\ddot{Z}_e\} + 2\Omega[R][\hat{\Phi}]\{\dot{Z}_e\} - \Omega^2[\hat{\Phi}]\{Z_e\} + \dot{\Omega}[R][\hat{\Phi}]\{Z_e\} \end{aligned} \right\} \quad (2.19)$$

where

$$[R] = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & [I] \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -[I] & 0 & 0 & 0 \end{bmatrix}$$

Substituting Eq. 2.19 into Eq. 2.16, and pre-multiplying it by $[\hat{\Phi}]^T$, we get the inertia forces for an annular element of the flexible disc in the space fixed coordinate system as

$$\left\{ \begin{aligned} \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{U}} \right) - \frac{\partial T}{\partial U} \\ \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}_y} \right) - \frac{\partial T}{\partial \theta_y} \\ \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\eta}_1} \right) - \frac{\partial T}{\partial \eta_1} \\ \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{V}} \right) - \frac{\partial T}{\partial V} \\ \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}_x} \right) - \frac{\partial T}{\partial \theta_x} \\ \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\zeta}_1} \right) - \frac{\partial T}{\partial \zeta_1} \end{aligned} \right\} = - \left\{ \begin{aligned} \Omega^2(S_x \cos \phi - S_y \sin \phi) + \dot{\Omega}(S_x \sin \phi + S_y \cos \phi) \\ 0 \\ 0 \\ \Omega^2(S_x \sin \phi + S_y \cos \phi) - \dot{\Omega}(S_x \cos \phi - S_y \sin \phi) \\ 0 \\ 0 \end{aligned} \right\}$$

$$+ \begin{bmatrix} m_e & S_z & 0 & 0 & 0 & 0 \\ S_z & I_D & -[Q_1] & 0 & 0 & 0 \\ 0 & -[Q_1]^T & [M_1] & 0 & 0 & 0 \\ 0 & 0 & 0 & m_e & S_z & 0 \\ 0 & 0 & 0 & S_z & I_D & -[Q_1] \\ 0 & 0 & 0 & 0 & -[Q_1]^T & [M_1] \end{bmatrix} \begin{Bmatrix} \bar{U} \\ \bar{\theta}_y \\ \{\bar{\eta}_1\} \\ \bar{V} \\ -\bar{\theta}_x \\ \{\bar{\zeta}_1\} \end{Bmatrix}$$

$$+ 2\Omega \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_p/2 & -[Q] \\ 0 & 0 & 0 & 0 & -[Q]^T & [\hat{G}_1] \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -I_p/2 & [Q] & 0 & 0 & 0 \\ 0 & [Q]^T & -[\hat{G}_1] & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{U} \\ \dot{\theta}_y \\ \{\dot{\eta}_1\} \\ \dot{V} \\ -\dot{\theta}_x \\ \{\dot{\zeta}_1\} \end{Bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2\dot{\Omega}[Q] \\ 0 & 0 & -\Omega^2[\hat{H}_1] & 0 & 0 & \dot{\Omega}[\hat{G}_1] \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\dot{\Omega}I_p & 2\dot{\Omega}[Q] & 0 & 0 & 0 \\ 0 & 0 & -\dot{\Omega}[\hat{G}_1] & 0 & 0 & -\Omega^2[\hat{H}_1] \end{bmatrix} \begin{Bmatrix} U \\ \theta_y \\ \{\eta_1\} \\ V \\ -\theta_x \\ \{\zeta_1\} \end{Bmatrix}$$

for $m = 1$ (2.20)

with

$$\begin{aligned} [Q] &= \frac{1}{2}([Q_1] + [Q_2]) = \pi \int_{r_1}^{r_2} \rho h [N_w(r)] r^2 dr \\ [M_1] &= \pi \int_{r_1}^{r_2} \rho h [N_w]^T [N_w] r dr + \frac{\pi}{12} \int_{r_1}^{r_2} \rho h^3 ([N_r]^T [N_r] + [N_\theta]^T [N_\theta]) r dr \\ [\hat{G}_1] &= \pi \int_{r_1}^{r_2} \rho h [N_w]^T [N_w] r dr + \frac{\pi}{12} \int_{r_1}^{r_2} \rho h^3 ([N_r]^T + [N_\theta]^T) ([N_r] + [N_\theta]) r dr \\ [\hat{H}_1] &= \pi \int_{r_1}^{r_2} \rho h [N_w]^T [N_w] r dr + \frac{\pi}{6} \int_{r_1}^{r_2} \rho h^3 ([N_r]^T + [N_\theta]^T) ([N_r] + [N_\theta]) r dr \end{aligned}$$

In the above formulation, the vibration of a flexible disc is characterized by modes having integer numbers of nodal diameters. For each nodal diameter configuration, there exists a family of circular modes with corresponding natural frequencies. It is noted here that, for a flexible disc-shaft assembly with polar symmetry, all the diametral modes of a disc are kinematically uncoupled from each other, and all the modes other than the first diametral mode are uncoupled with the shaft motion.

2.5 Potential Energy of an Annular Disc Element

In the analysis of a rotating flexible disc, the potential energy U consists of two parts:

1. that due to the strain energy of the deformed disc, which is produced by the bending and transverse shear deformations of the disc;
2. that produced by the in-plane stresses, which results from the centrifugal forces or thermal stresses distributed in the rotating disc.

The complete elastic strain energy for an annular element of the disc, is given by Mindlin [90] as follows

$$U = \frac{1}{2} \int_{\Omega_e} \{\epsilon\}^T [D] \{\epsilon\} r dr d\theta \quad (2.21)$$

where

$$\{\epsilon\} = \left\{ \begin{array}{c} \frac{\partial \psi_r}{\partial r} \\ \frac{\psi_r}{r} + \frac{1}{r} \frac{\partial \psi_\theta}{\partial \theta} \\ \frac{1}{r} \frac{\partial \psi_r}{\partial \theta} - \frac{\psi_\theta}{r} + \frac{\partial \psi_\theta}{\partial r} \\ \gamma_r \\ \gamma_\theta \end{array} \right\} \quad (2.22)$$

$$[D] = \begin{bmatrix} D & \mu D & 0 & 0 & 0 \\ \mu D & D & 0 & 0 & 0 \\ 0 & 0 & D\frac{1-\mu}{2} & 0 & 0 \\ 0 & 0 & 0 & \kappa^2 Gh & 0 \\ 0 & 0 & 0 & 0 & \kappa^2 Gh \end{bmatrix}$$

$$D = \frac{Eh^3}{12(1-\mu^2)}$$

And, κ^2 is the shear correction coefficient taking into account the shear stress distribution across the depth of the disc. As recommended by Mindlin [90], $\kappa^2 = \frac{\pi}{12}$, which is close to the normal used value of $\frac{5}{6}$ for Timoshenko's beams with rectangular cross sections.

$$\left. \begin{aligned} \gamma_r &= \frac{\partial w}{\partial r} + \psi_r \\ \gamma_\theta &= \frac{1}{r} \frac{\partial w}{\partial \theta} + \psi_\theta \end{aligned} \right\} \quad (2.23)$$

are the additional radial and circumferential rotations resulting from the transverse shear.

As stated before, the modes with different nodal diameters are uncoupled from each other such that they can be analyzed independently. Now let us consider the deflection functions of the modes with m nodal diameters

$$\left. \begin{aligned} \psi_r(r, \theta) &= [N_r(r)] (\{\eta_{mR}\} \cos m\theta + \{\zeta_{mR}\} \sin m\theta) \\ \psi_\theta(r, \theta) &= [N_\theta(r)] (\{\eta_{mR}\} \sin m\theta - \{\zeta_{mR}\} \cos m\theta) \\ w(r, \theta) &= [N_w(r)] (\{\eta_{mR}\} \cos m\theta + \{\zeta_{mR}\} \sin m\theta) \end{aligned} \right\} \quad (2.24)$$

Substituting Eqs. 2.23 and 2.24 into the generalized strain expression of Eq. 2.22, we find that

$$\{\epsilon\} = [\Theta_{mx}][B_m]\{\eta_{mR}\} + [\Theta_{my}][B_m]\{\zeta_{mR}\} \quad (2.25)$$

where

$$[B_m] = \begin{bmatrix} [N_{r,r}] \\ \frac{1}{r}[N_r] + \frac{m}{r}[N_\theta] \\ -\frac{m}{r}[N_r] - \frac{1}{r}[N_\theta] + [N_{\theta,r}] \\ [N_{w,r}] + [N_r] \\ -\frac{m}{r}[N_w] + [N_\theta] \end{bmatrix}$$

and

$$(\quad)_{,r} = \frac{\partial}{\partial r}(\quad)$$

$$[\Theta_{mx}] = \text{diag}\{\cos m\theta, \cos m\theta, \sin m\theta, \cos m\theta, \sin m\theta\}$$

$$[\Theta_{my}] = \text{diag}\{\sin m\theta, \sin m\theta, -\cos m\theta, \sin m\theta, -\cos m\theta\}$$

Substituting Eq. 2.25 into Eq. 2.21, we can derive the element stiffness matrix for the vibration of the disc with m nodal diameters with respect to the deflection vectors, $\{\eta_{mR}\}$ and $\{\zeta_{mR}\}$, of the flexible disc as follows:

$$[k_{em}] = \begin{bmatrix} [k_{mx}] & 0 \\ 0 & [k_{my}] \end{bmatrix} \quad (2.26)$$

where

$$[k_{mx}] = \int_{r_1}^{r_2} [B_m]^T [D_x] [B_m] r dr$$

$$[k_{my}] = \int_{r_1}^{r_2} [B_m]^T [D_y] [B_m] r dr$$

$$[D_x] = \int_0^{2\pi} [\Theta_{mx}] [D] [\Theta_{mx}] d\theta$$

$$[D_y] = \int_0^{2\pi} [\Theta_{my}] [D] [\Theta_{my}] d\theta$$

It should be noted that $[k_{mx}] = [k_{my}]$ for all the modes except for the axial symmetric vibration mode of the disc, which corresponds to the modes of zero nodal diameter (i.e., $m = 0$).

The centrifugal and thermal stresses present in the disk can modify significantly the stiffness of the disk. The centrifugal stresses are proportional to square of the

rotational speed, and the corresponding additional stiffness matrix will also be proportional to square of the rotational speed. Thermal stresses, however, have no simple relationship with the rotational speed. This suggests that a method of including both effects should be formulated assuming that stresses in the disc are already known.

For the axial-symmetric conditions, only the in-plane radial stresses σ_r and tangential stresses σ_θ are present in the plane of an annular element of the disc. The following additional potential energy arises from these in-plane stresses:

$$U_C = \frac{1}{2} \int_{\Omega_e} \left\{ \sigma_r \left(\frac{\partial w}{\partial r} \right)^2 + \frac{\sigma_\theta}{r^2} \left(\frac{\partial w}{\partial \theta} \right)^2 \right\} h r dr \quad (2.27)$$

Assuming the deflection functions, Eq. 2.24, as before, and substituting into the above potential energy expression, the geometric stiffness matrix for the annular element are readily derived corresponding to the deflection vector as

$$[k_{mg}] = \begin{bmatrix} [k_{mgx}] & 0 \\ 0 & [k_{mgy}] \end{bmatrix} \quad (2.28)$$

corresponding to the deflection vectors, $\{\eta_{mR}\}$ and $\{\zeta_{mR}\}$, of the flexible disc with

$$\begin{aligned} [k_{mgx}] &= \pi \int_{r_1}^{r_2} \left(C_m \sigma_r [N_{w,r}]^T [N_{w,r}] + m^2 S_m \frac{\sigma_\theta}{r^2} [N_w]^T [N_w] \right) h r dr \\ [k_{mgy}] &= \pi \int_{r_1}^{r_2} \left(S_m \sigma_r [N_{w,r}]^T [N_{w,r}] + m^2 C_m \frac{\sigma_\theta}{r^2} [N_w]^T [N_w] \right) h r dr \end{aligned}$$

Obviously, for $m \geq 1$

$$[k_{mgx}] = [k_{mgy}] \stackrel{\text{def}}{=} [k_{mg}]$$

2.6 Equations of Motion for Disc-Shaft Systems

By comparison with the finite element models of flexible rotating disks, the finite element method for shaft elements has been rather intensively studied in the literature. Ruhl [20, 21] is among the earliest investigators applying finite element method

to rotor dynamics. His finite element model of shafts was based on the Bernoulli Euler beam theory. Dimaragonas [23] presented an element including rotary inertia, gyroscopic moments and internal damping.

Based on the Timoshenko beam element developed by Davis [19], Nelson [27] presented a rotating shaft element which can account for rotary inertia, transverse shear deformation and gyroscopic moments of the shaft element but there is no increase in the degrees of freedom of the element (i.e., the same order as the Bernoulli-Euler beam element). Gravis and Childs [28, 29] have applied this element to modelling several flexible rotor bearing systems, and the results show that the element is applicable to both short and slender rotating shafts.

Hence, for a rotor system with flexible disks, the equations of motion can be readily derived by assembling shaft element matrices, the bearing stiffness and the disc element matrices into the corresponding global mass, stiffness, gyroscopic matrices using the direct stiffness assembly method in the finite element method. The damping effects from either structural friction or bearings, and the support stiffness can be added to the system model accordingly. Thus, the equations of motion for flexible disk-shaft assemblies with the one-diameter modes of discs can be written as follows

1) In the rotating coordinate system:

$$[M_R]\{\ddot{Z}_R\} + ([C_R] + [G_R(\Omega)])\{\dot{Z}_R\} + ([K_R] + [K_{gR}(\Omega)] + [N_R(\dot{\Omega})])\{Z_R\} = \{F_R\} \quad (2.29)$$

or written as the partitioned matrix form:

$$\begin{bmatrix} [M_0] & 0 \\ 0 & [M_0] \end{bmatrix} \begin{Bmatrix} \{\ddot{x}_R\} \\ \{\ddot{y}_R\} \end{Bmatrix} + \begin{bmatrix} [C_{xxR}] & [C_{xyR}] + [G_{0R}(\Omega)] \\ [C_{yxR}] - [G_{0R}(\Omega)]^T & [C_{yyR}] \end{bmatrix} \begin{Bmatrix} \{\dot{x}_R\} \\ \{\dot{y}_R\} \end{Bmatrix} \\ + \begin{bmatrix} [K_{xxR}] + [K_{g0R}(\Omega)] & [K_{xyR}] + [N_{xyR}(\dot{\Omega})] \\ [K_{yxR}] + [N_{yxR}(\dot{\Omega})] & [K_{yyR}] + [K_{g0R}(\Omega)] \end{bmatrix} \begin{Bmatrix} \{x_R\} \\ \{y_R\} \end{Bmatrix} = \begin{Bmatrix} \{f_{xR}\} \\ \{f_{yR}\} \end{Bmatrix} \quad (2.30)$$

where the stiffness matrix $[K_{g0R}]$ is proportional to Ω^2 , and accounts for the contributions from the geometric stiffness matrices $[k_{1g}]$'s (produced by in plane stresses from centrifugal forces and the matrices $\Omega^2[H_{eR}]$'s of all elements, see Eq. 2.15)

2) In the space-fixed coordinate system:

$$[M]\{\ddot{Z}\} + ([C] + [G(\Omega)])\{\dot{Z}\} + ([K] + [K_g(\Omega)] + [N(\dot{\Omega})])\{Z\} = \{F\} \quad (2.31)$$

or written as the partitioned matrix form:

$$\begin{bmatrix} [M_0] & 0 \\ 0 & [M_0] \end{bmatrix} \begin{Bmatrix} \{\ddot{x}\} \\ \{\ddot{y}\} \end{Bmatrix} + \begin{bmatrix} [C_{xx}] & [C_{xy}] + [G_0(\Omega)] \\ [C_{yx}] - [G_0(\omega)] & [C_{yy}] \end{bmatrix} \begin{Bmatrix} \{\dot{x}\} \\ \{\dot{y}\} \end{Bmatrix} \\ + \begin{bmatrix} [K_{xx}] + [K_{g0}(\Omega)] & [K_{xy}] + [N_{xy}(\dot{\Omega})] \\ [K_{yx}] + [N_{yx}(\dot{\Omega})] & [K_{yy}] + [K_{g0}(\Omega)] \end{bmatrix} \begin{Bmatrix} \{x\} \\ \{y\} \end{Bmatrix} = \begin{Bmatrix} \{f\}_x \\ \{f\}_y \end{Bmatrix} \quad (2.32)$$

where the stiffness matrix $[K_{g0}]$ is proportional to Ω^2 , and accounts for the contributions from the geometric stiffness matrices $[k_g]'$ s (produced by in-plane stresses from centrifugal forces and the matrices $\Omega^2[H_e]'$ s of all elements, see Eq. 2.20).

It is clear that these equations are linear but with coefficients associated with not only rotation speed but also rotation acceleration.

At a given spin speed, these equations become the linear second order differential equations of constant coefficients, and are readily solved for stability and response analyses.

2.7 Coupling of Multi-spool Turbomachinery

In general, a turbomachinery with two-spool rotors, denoted by rotor "a" and "b", can be modeled as

$$\begin{bmatrix} M_a & 0 \\ 0 & M_b \end{bmatrix} \begin{Bmatrix} \ddot{Z}_a \\ \ddot{Z}_b \end{Bmatrix} + \begin{bmatrix} C_a + G_a(\Omega) & 0 \\ 0 & C_b + G_b(\Omega) \end{bmatrix} \begin{Bmatrix} \dot{Z}_a \\ \dot{Z}_b \end{Bmatrix} \\ + \begin{bmatrix} K_a + K_{ga}(\Omega) + N_a(\dot{\Omega}) & 0 \\ 0 & K_b + K_{gb}(\Omega) + N_b(\dot{\Omega}) \end{bmatrix} \begin{Bmatrix} Z_a \\ Z_b \end{Bmatrix} = \begin{Bmatrix} F_a \\ F_b \end{Bmatrix} + \begin{Bmatrix} F_{ja} \\ F_{jb} \end{Bmatrix} \quad (2.33)$$

where, $\{Z_a\}$ and $\{Z_b\}$ are the displacement vectors of rotors "a" and "b", respectively. $\{F_a\}$ and $\{F_b\}$ are the external applied forces including unbalanced inertial forces

applied on the two rotors. $\{F_{ja}\}$ and $\{F_{jb}\}$ are inter-shaft bearing forces applied on "a" and "b", respectively.

Assuming that the displacements of the rotors "a" and "b" on the junctions of inter-shaft bearings are $\{z_{ja}\}$ and $\{z_{jb}\}$, respectively, and that the corresponding junction forces are $\{f_{ja}\}$ and $\{f_{jb}\}$, there are the following relationships:

$$\left. \begin{aligned} \{z_{ja}\} &= [T_{ja}]\{Z_a\} \\ \{z_{jb}\} &= [T_{jb}]\{Z_b\} \end{aligned} \right\}, \quad \left. \begin{aligned} \{f_{ja}\} &= [T_{ja}]\{F_a\} \\ \{f_{jb}\} &= [T_{jb}]\{F_b\} \end{aligned} \right\} \quad (2.34)$$

where, $[T_{ja}]$ and $[T_{jb}]$ are the corresponding transformation matrices any element of which is either 0 or 1.

Assuming inter-shaft bearing as linear elastic elements with stiffness matrix $[K_j]$ and damping matrix $[C_j]$, the local inter-shaft bearing forces are given as

$$\{f_{ja}\} = [K_j](\{z_{jb}\} - \{z_{ja}\}) + [C_j](\{\dot{z}_{jb}\} - \{\dot{z}_{ja}\}) = -\{f_{jb}\} \quad (2.35)$$

Substituting Eq. 2.34 into Eq. 2.35, we obtain the global inter shaft forces as following

$$\left\{ \begin{aligned} F_{ja} \\ F_{jb} \end{aligned} \right\} = - \begin{bmatrix} \bar{K}_{aa} & \bar{K}_{ab} \\ \bar{K}_{ba} & \bar{K}_{bb} \end{bmatrix} \left\{ \begin{aligned} Z_a \\ Z_b \end{aligned} \right\} - \begin{bmatrix} \bar{C}_{aa} & \bar{C}_{ab} \\ \bar{C}_{ba} & \bar{C}_{bb} \end{bmatrix} \left\{ \begin{aligned} \dot{Z}_a \\ \dot{Z}_b \end{aligned} \right\} \quad (2.36)$$

with

$$\begin{aligned} [\bar{K}_{aa}] &= [T_{ja}]^T [K_j] [T_{ja}] & [\bar{C}_{aa}] &= [T_{ja}]^T [C_j] [T_{ja}] \\ [\bar{K}_{ab}] &= -[T_{ja}]^T [K_j] [T_{jb}] & [\bar{C}_{ab}] &= -[T_{ja}]^T [C_j] [T_{jb}] \\ [\bar{K}_{ba}] &= [T_{jb}]^T [K_j] [T_{ja}] & [\bar{C}_{ba}] &= -[T_{jb}]^T [C_j] [T_{ja}] \\ [\bar{K}_{bb}] &= [T_{jb}]^T [K_j] [T_{jb}] & [\bar{C}_{bb}] &= [T_{jb}]^T [C_j] [T_{jb}] \end{aligned}$$

Substitute the above equation into Eq. 2.33, we get the coupled dynamic equation for a two-spool turbomachinery as follows:

$$\begin{bmatrix} M_a & 0 \\ 0 & M_b \end{bmatrix} \left\{ \begin{aligned} \ddot{Z}_a \\ \ddot{Z}_b \end{aligned} \right\} + \begin{bmatrix} C_a + \bar{C}_{aa} + G_a(\Omega) & C_{ab} \\ \bar{C}_{ba} & C_b + \bar{C}_{bb} + G_b(\Omega) \end{bmatrix} \left\{ \begin{aligned} \dot{Z}_a \\ \dot{Z}_b \end{aligned} \right\}$$

$$+ \begin{bmatrix} K_a + K_{aa} + K_{ga}(\Omega) + N_a(\dot{\Omega}) & \bar{K}_{ab} \\ \bar{K}_{ba} & K_b + \bar{K}_{bb} + K_{gb}(\Omega) + N_b(\dot{\Omega}) \end{bmatrix} \begin{Bmatrix} Z_a \\ Z_b \end{Bmatrix} = \begin{Bmatrix} F_a \\ F_b \end{Bmatrix} \quad (2.37)$$

Although the above equations are derived only for a two-spool turbomachinery, it is quite straight forward to extend to any multi-spool rotor system.

It should be noted that the equations for a flexible disk-shaft rotor system presented in this chapter are not only suitable for finite element analysis, but also can be utilized to assemble the equations of motion when the deformations of a flexible disk are described by the Ritz mode shapes. In that case, the matrices $[N_r(r)]$, $[N_\theta(r)]$ and $[N_w(w)]$ in Eq. 2.10 will be the mode shapes of a flexible disk, and the transformation matrices, T_a and T_b , for assembling multi-spool rotor systems, will be the corresponding boundary mode shape matrices of the shafts a and b .

Chapter 3

Finite Element Model of Rotating Blades in Bladed Disc–Shaft Rotor Systems

3.1 Introduction

The vibration of an array of flexible blades of a rotor is found to be similar to that of a flexible disc. The blades oscillate in a coupled bladed disc mode which is also characterised by diametral and circular nodes, as shown in Figure 3.1. The blades, being constrained to the disc at the blade root, will vibrate in bending motion at diametral anti-nodes, in torsional motion at nodes of the mode, Figure 3.2. The circular modes may lie in the disc, but will more commonly be located in the blades.

Since the purpose of this study has more emphasis on the coupling effect between the flexible bladed disc and shaft motion, a refined analysis of the blades is not attempted here. In fact, much work has been published on this area, as mentioned in the literature review in Chapter 1. Such methods consider the blade with airfoil section and most of the other factors, such as, camber, pretwist, longitudinal taper, root flexibility etc.

In this investigation, blades are idealized to behave as Timoshenko's beams having arbitrary variations in section properties and pretwist along their span. It is assumed that the flexural axis, i.e., the shear center of the cross-section of a blade is considered to be initially straight and coincides with the centroidal axis of the blade. After

deformation, all the cross sections of the blade remain as planes (i.e., without any warping). Hence, there is no coupling between bending and torsion motion for all the blades. A further assumption is that there are no elastic and inertia coupling between individual blades.

3.2 Coordinate Systems

Figure 3.3 shows a typical element of nonuniform blade with arbitrary pretwist. The translation of any cross-section of the blade is given by two translations, v_f and w_c , of its centroid, in the two principal flapwise and chordwise directions which consists of a contribution of bending and a contribution of transverse shear deformation. The rotation of the cross-section is described by two rotation angles, ψ_c and ψ_f , of the flexural axis in the principal directions and a torsion angle φ about the flexural axis.

To study dynamics of blades, two new coordinate systems have been introduced.

1) $DX'Y'Z'$ — coordinate frame obtained by a rotation θ_i , the angular location of the i -th blade, of $DX_dY_dZ_d$ about DZ_d . DZ' is normal direction of rigid disc, DY' and DX' are in the tangential and radial directions of the i th blade, respectively.

$$\{X_d\} \stackrel{\text{def}}{=} \begin{Bmatrix} x_d \\ y_d \\ z_d \end{Bmatrix} = \begin{bmatrix} \cos \theta_i & -\sin \theta_i & 0 \\ \sin \theta_i & \cos \theta_i & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} \stackrel{\text{def}}{=} [\Theta_i] \{X'\} \quad (3.1)$$

2) $DX_bY_bZ_b$ — coordinate frame obtained by a rotation δ of $DX'Y'Z'$ about the axis DX' , in which DY_b and DZ_b are two principal axes in flapwise and chordwise directions of the blade section at a location r , respectively. DZ_b is inclined at an angle δ , which can be used to count for the pre-twist of the blades, to the normal direction of the disc.

$$\{X'\} \stackrel{\text{def}}{=} \begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \delta & -\sin \delta \\ 0 & \sin \delta & \cos \delta \end{bmatrix} \begin{Bmatrix} x_b \\ y_b \\ z_b \end{Bmatrix} \stackrel{\text{def}}{=} [\Delta] \{X_b\} \quad (3.2)$$

For the foregoing, it is clear that the location of any point on the current cross-section of the i -th blade can be referred to the $DX_bY_bZ_b$ as:

$$\{X_b\} \stackrel{\text{def}}{=} \begin{Bmatrix} x_b \\ y_b \\ z_b \end{Bmatrix} = \begin{Bmatrix} \bar{x}_b \\ \bar{y}_b \\ \bar{z}_b \end{Bmatrix} + \begin{Bmatrix} u_b \\ v_b \\ w_b \end{Bmatrix} \stackrel{\text{def}}{=} \{\bar{X}_b\} + \{W_b\} \quad (3.3)$$

where

$\{X_b\}$ is the position vector of the point in $DX_bY_bZ_b$ before the blade is deformed

$\{W_b\}$ is the deflection vector of the point $\{\bar{X}_b\}$ in $DX_bY_bZ_b$

Making use of the above coordinate systems, displacements in local coordinate systems of a blade can be related to the disk-based coordinate system $DX_dY_dZ_d$ by the following relationship

$$\{X_d\} = [\Theta_i][\Delta](\{\bar{X}_b\} + \{W_b\}) \quad (3.4)$$

3.3 Inertia Forces of a Element of Blade Set

Since there is no coupling between bending in the two principal directions, and between bending and torsion, the displacements of any cross-section in the element of a blade can be approximated by using the nodal displacements of the element such as

$$\left. \begin{aligned} v_f &= [V(s)]\{\delta_{eb}\} \\ w_c &= [W(s)]\{\delta_{eb}\} \\ \psi_f &= [N_f(s)]\{\delta_{eb}\} \\ \psi_c &= [N_c(s)]\{\delta_{eb}\} \\ \varphi &= [N_\varphi(s)]\{\delta_{eb}\} \end{aligned} \right\} \quad (3.5)$$

where

$\{\delta_{eb}\} = \{v_{f1} \psi_{f1} w_{c1} \psi_{c1} \varphi_1 v_{f2} \psi_{f2} w_{c2} \psi_{c2} \varphi_2\}^T$ is the nodal displacement vector of the blade element in coordinate system $DX_bY_bZ_b$.

$[V(s)]$ is a row matrix whose terms are Lagrangian and or Hermitian polynomials for the approximation of the deflection v_f in the flapwise direction.

$[W(s)]$ is a row matrix whose terms are Lagrangian and or Hermitian polynomials for approximation of the deflection w_c in the chordwise direction.

$[N_f(s)]$ is a row matrix whose terms are Lagrangian or Hermitian polynomials for the approximation of the rotation ψ_θ in the flapwise plane.

$[N_c(s)]$ is a row matrix whose terms are Lagrangian or Hermitian polynomials for the approximation of the rotation ψ_θ in the chordwise plane.

$[N_\varphi(s)]$ is a row matrix whose terms are Lagrangian or Hermitian polynomials for the approximation of the rotation φ about the flexural axis.

As shown in Figure 3.3, the nodal displacements $\{\delta_{cb}\}$ in $DX_bY_bZ_b$ and the nodal displacements, $\{\delta_c\} = \{v_1 \ \psi_{z1} \ w_1 \ -\psi_{y1} \ \varphi_1 \ v_2 \ \psi_{z2} \ w_2 \ -\psi_{y2} \ \varphi_2\}^T$, in $DX'Y'Z'$ have the following relationship:

$$\{\delta_{cb}\} = [T]\{\delta_c\} \quad (3.6)$$

where

$$[T] = \begin{bmatrix} [T_1] & 0 \\ 0 & [T_2] \end{bmatrix}$$

$$[T_1] = \begin{bmatrix} \cos \delta_1 & 0 & \sin \delta_1 & 0 & 0 \\ 0 & \cos \delta_1 & 0 & \sin \delta_1 & 0 \\ -\sin \delta_1 & 0 & \cos \delta_1 & 0 & 0 \\ 0 & -\sin \delta_1 & 0 & \cos \delta_1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The displacements of a typical point in the cross section can be defined by the following relation in the coordinate system $DX_bY_bZ_b$:

$$\left. \begin{aligned} u_b &= -\bar{z}_b \psi_c - \bar{y}_b \psi_f \\ v_b &= v_f - \bar{z}_b \varphi \\ w_b &= w_c + \bar{y}_b \varphi \end{aligned} \right\} \quad (3.7)$$

Assuming that the array of blades is axial-symmetrical in mechanical properties, then a blade at antinode is displaced in bending motion only as shown in Figure 3.2. However due to the pretwist of the blade, bending motion may take place in both flapwise and chordwise directions. A blade at a nodal diameter is displaced in torsion motion only. Blades at any angular location in the bladed disc experience both bending and torsion motions. Furthermore, it is not known, a priori, where the nodal line will fall with respect to flexible shaft motions, or if and how it will move as a travelling wave. Thus each diametral mode of the flexible blades must be used twice. Similarly to the disc displacement expression, it is convenient to place these node lines at 90 degrees to each other, azimuthal. Each has its own distinct generalized coordinates, $\{p_{mR}\}$ and $\{q_{mR}\}$ for the m -th diametral modes. Thus the motion of a blade located at the angular position $\theta_i = \frac{2\pi i}{N}$, where N is the total number of blades and i is the index number of the blade, can be expressed as follows:

$$\left. \begin{aligned} v_f &= [V(s)][T] \sum_{m=0}^k (\{p_{mR}\} \cos m\theta_i + \{q_{mR}\} \sin m\theta_i) \\ w_c &= [W(s)][T] \sum_{m=0}^k (\{p_{mR}\} \cos m\theta_i + \{q_{mR}\} \sin m\theta_i) \\ \psi_f &= [N_f(s)][T] \sum_{m=0}^k (\{p_{mR}\} \cos m\theta_i + \{q_{mR}\} \sin m\theta_i) \\ \psi_c &= [N_c(s)][T] \sum_{m=0}^k (\{p_{mR}\} \cos m\theta_i + \{q_{mR}\} \sin m\theta_i) \\ \varphi &= [N_\varphi(s)][T] \sum_{m=0}^k (-\{p_{mR}\} \sin m\theta_i + \{q_{mR}\} \cos m\theta_i) \end{aligned} \right\} \quad (3.8)$$

where

$\{p_{mR}\}$ = the nodal deflections of the m -th diametral mode along the X_d axis.

$\{q_{mR}\}$ = the nodal deflections of the m -th diametral mode along the Y_d axis.

k is the number of the diametral modes which are used to approximate the displacement of blades. In our analysis, it is assumed that $k \leq \frac{N-1}{2}$.

Substituting the above equations into the kinetic energy expression Eq.(2.8) of an element in the bladed disc, integrating over the present blade element, summarizing over all N blades, and using the following orthogonal relations of discrete sinusoid functions:

$$\sum_{i=1}^N \cos m\theta_i = \begin{cases} N, & \text{if } m = 0 \\ 0, & \text{if } m \geq 1 \end{cases}$$

$$\sum_{i=1}^N \sin m\theta_i = 0, \quad \text{for } m = 0, 1, \dots$$

$$\sum_{i=1}^N \sin m\theta_i \cos n\theta_i = 0, \quad \text{for } m, n \geq 0.$$

$$\sum_{i=1}^N \cos m\theta_i \cos n\theta_i = \begin{cases} N, & \text{if } m = n = 0 \\ 0, & \text{if } m \neq n \\ N/2, & \text{if } m = n \neq 0 \end{cases}$$

$$\sum_{i=1}^N \sin m\theta_i \sin n\theta_i = \begin{cases} N/2, & \text{if } m = n \neq 0 \\ 0, & \text{if } m = n = 0 \\ 0 & \text{if } m \neq n \end{cases}$$

we can derive the kinetic energy expression of the array of tuned blade elements as follows:

$$\begin{aligned} T = & \frac{1}{2} M_c [\Omega^2 (U_R^2 + V_R^2) + \dot{U}_R^2 + \dot{V}_R^2 + 2\Omega (U_R \dot{V}_R - \dot{U}_R V_R)] \\ & + \frac{1}{2} (\Omega^2 [J_z - \theta_{zR}^2 + (I_{yy} - I_{zz}) - \theta_{yR}^2 (I_{xx} - I_{zz})] + \frac{1}{2} \dot{\theta}_{zR}^2 I_D \\ & + \frac{1}{2} \dot{\theta}_{yR}^2 I_D + \Omega \dot{\theta}_{zR} \theta_{yR} (I_{xx} - I_{zz}) + \Omega \dot{\theta}_{yR} \theta_{zR} I_D + \Omega [P_0] \{\dot{p}_{0R}\} - \Omega^2 [Q_0] \{q_{0R}\}) \\ & + (\Omega U_R + \dot{V}_R) [P_1] (\{\dot{p}_{1R}\} - \Omega \{q_{1R}\}) + (\Omega V_R - \dot{U}_R) [P_1] (\Omega \{p_{1R}\} + \{\dot{q}_{1R}\}) \\ & - (\Omega \theta_{zR} + \dot{\theta}_{yR}) [P_2] \{\dot{p}_{1R}\} - (\Omega \theta_{yR} - \dot{\theta}_{zR}) [P_2] \{\dot{q}_{1R}\} \\ & + \Omega (\Omega \theta_{yR} - \dot{\theta}_{zR}) [P_3] \{p_{1R}\} - \Omega (\Omega \theta_{zR} + \dot{\theta}_{yR}) [P_3] \{q_1\} \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \sum_{m=0}^k (\{\dot{p}_{mR}\}^T [M_{xm}] \{\dot{p}_{mR}\} + \{\dot{q}_{mR}\}^T [M_{ym}] \{\dot{q}_{mR}\}) \\
& + \Omega \sum_{m=0}^k (\{p_{mR}\}^T [G_m] \{\dot{q}_{mR}\} - \{\dot{p}_{mR}\}^T [G_m] \{q_{mR}\}) \\
& + \frac{1}{2} \Omega^2 \sum_{m=0}^k (\{p_{mR}\}^T [H_{xm}] \{p_{mR}\} + \{q_{mR}\}^T [H_{ym}] \{q_{mR}\})
\end{aligned} \tag{3.9}$$

where

$$\begin{aligned}
[P_0] &= \frac{N}{2} \int_{r_1}^{r_2} \rho A (\cos \delta [V] - \sin \delta [W]) r dr [T] \\
&+ \frac{N}{2} \int_0^1 (i_{yy} \cos \delta [N_f] - i_{zz} \sin \delta [N_c]) dr [T] \\
[Q_0] &= \frac{N}{2} \int_{r_1}^{r_2} (i_{yy} - i_{zz}) [N_\varphi] \sin \delta \cos \delta dr [T] \\
[P_1] &= \frac{N}{2} \int_{r_1}^{r_2} \rho A (\cos \delta [V] - \sin \delta [W]) dr [T] \\
[P_2] &= \frac{N}{2} \int_{r_1}^{r_2} \left\{ \rho A r (\sin \delta [V] + \cos \delta [W]) \right. \\
&\quad \left. + (i_{zz} \cos \delta [N_c] + i_{yy} \sin \delta [N_f]) + (i_{yy} + i_{zz}) [N_\varphi] \right\} dr [T] \\
[P_3] &= \frac{N}{2} \int_{r_1}^{r_2} \left\{ \rho A r (\sin \delta [V] + \cos \delta [W]) \right. \\
&\quad \left. - (i_{zz} \cos \delta [N_c] + i_{yy} \sin \delta [N_f]) + (i_{yy} - i_{zz}) \cos 2\delta [N_\varphi] \right\} dr [T] \\
[M_{xm}] &= \frac{N}{2} [T]^T \int_{r_1}^{r_2} \left\{ C_m \rho A ([V]^T [V] + [W]^T [W]) + C_m i_{yy} [N_f]^T [N_f] \right. \\
&\quad \left. + C_m i_{zz} [N_c]^T [N_c] + S_m (i_{yy} + i_{zz}) [N_\varphi]^T [N_\varphi] \right\} dr [T] \\
[M_{ym}] &= \frac{N}{2} [T]^T \int_{r_1}^{r_2} \left\{ S_m \rho A ([V]^T [V] + [W]^T [W]) + S_m i_{yy} [N_f]^T [N_f] \right. \\
&\quad \left. + S_m i_{zz} [N_c]^T [N_c] + C_m (i_{yy} + i_{zz}) [N_\varphi]^T [N_\varphi] \right\} dr [T] \\
[G_m] &= \frac{N}{2} [T]^T \int_{r_1}^{r_2} \left\{ i_{yy} \sin \delta (C_m [N_f]^T [N_\varphi] + S_m [N_\varphi]^T [N_f]) \right. \\
&\quad \left. + i_{zz} \cos \delta (C_m [N_c]^T [N_\varphi] + S_m [N_\varphi]^T [N_c]) \right\} dr [T] \\
[H_{xm}] &= \frac{N}{2} [T]^T \int_{r_1}^{r_2} \left\{ C_m \rho A (\cos \delta [V] - \sin \delta [W])^T (\cos \delta [V] - \sin \delta [W]) \right. \\
&\quad \left. + C_m i_{yy} [N_f]^T [N_f] + C_m i_{zz} [N_c]^T [N_c] + S_m (i_{yy} \sin^2 \delta + i_{zz} \cos^2 \delta) [N_\varphi]^T [N_\varphi] \right\} dr [T]
\end{aligned}$$

$$\begin{aligned}
[H_{ym}] = & \frac{N}{2} [T]^T \int_{r_1}^{r_2} \left\{ S_m \rho A (\cos \delta [V] - \sin \delta [W])^T (\cos \delta [V] - \sin \delta [W]) \right. \\
& \left. + S_m i_{yy} [N_f]^T [N_f] + S_m i_{zz} [N_c]^T [N_c] + C_m (i_{yy} \sin^2 \delta + i_{zz} \cos^2 \delta) [N_\varphi]^T [N_\varphi] \right\} dr [T]
\end{aligned}$$

It should be noted that

$$\left. \begin{aligned} [M_{xm}] &= [M_{ym}] \stackrel{\text{def}}{=} [M_m] \\ [H_{xm}] &= [H_{ym}] \stackrel{\text{def}}{=} [H_m] \end{aligned} \right\}, \quad \text{for } m \geq 1 \quad (3.10)$$

and

$$[M_m]^T = [M_m], \quad [G_m]^T = [G_m], \quad [H_m]^T = [H_m] \quad (3.11)$$

Equation 3.9 may be written in matrix form similar to Eq. 2.15:

$$\begin{aligned}
T = & \frac{1}{2} \{ \dot{Z}_{eR} \}^T [M_{eR}] \{ \dot{Z}_{eR} \} + \Omega \{ \dot{Z}_{eR} \}^T [N_{eR}] \{ Z_{eR} \} \\
& + \frac{1}{2} \Omega^2 \{ Z_{eR} \}^T [H_{eR}] \{ Z_{eR} \} + \Omega [P_0] \{ \dot{p}_{0R} \} - \Omega^2 [Q_0] \{ q_{0R} \} \\
& + \frac{1}{2} \sum_{m \neq 1}^k (\{ \dot{p}_{mR} \}^T [M_{xm}] \{ \dot{p}_{mR} \} + \{ \dot{q}_{mR} \}^T [M_{ym}] \{ \dot{q}_{mR} \}) \\
& + \Omega \sum_{m \neq 1}^k (\{ p_{mR} \}^T [G_m] \{ \dot{q}_{mR} \} - \{ \dot{p}_{mR} \}^T [G_m] \{ q_{mR} \}) \\
& + \frac{1}{2} \Omega^2 \sum_{m \neq 1}^k (\{ p_{mR} \}^T [H_{xm}] \{ p_{mR} \} + \{ q_{mR} \}^T [H_{ym}] \{ q_{mR} \}) \quad (3.12)
\end{aligned}$$

where

$$\{ Z_{eR} \}^T = \left\{ U_R \quad \theta_{yR} \quad \{ p_{1R} \}^T \quad V_R \quad -\theta_{xR} \quad \{ q_{1R} \}^T \right\}$$

$$[M_{eR}] = \begin{bmatrix} M_e & 0 & 0 & 0 & 0 & -[P_1] \\ 0 & I_D & -[P_2] & 0 & 0 & 0 \\ 0 & -[P_2]^T & [M_1] & [P_1]^T & 0 & 0 \\ 0 & 0 & [P_1] & M_e & 0 & 0 \\ 0 & 0 & 0 & 0 & I_D & -[P_2] \\ -[P_1]^T & 0 & 0 & 0 & -[P_2]^T & [M_1] \end{bmatrix}$$

$$[N_{eR}] = \begin{bmatrix} 0 & 0 & -[P_1] & -M_e & 0 & 0 \\ 0 & 0 & 0 & 0 & -I_D & -[P_3] \\ [P_1]^T & 0 & 0 & 0 & [P_2]^T & -[G_1] \\ M_e & 0 & 0 & 0 & 0 & -[P_1] \\ 0 & I_{xx} - I_{zz} & [P_3] & 0 & 0 & 0 \\ 0 & -[P_2]^T & [G_1]^T & [P_1]^T & 0 & 0 \end{bmatrix}$$

$$[H_{eR}] = \begin{bmatrix} M_e & 0 & 0 & 0 & 0 & -[P_1] \\ 0 & I_{xx} - I_{zz} & [P_3] & 0 & 0 & 0 \\ 0 & [P_3]^T & [H_1] & [P_1]^T & 0 & 0 \\ 0 & 0 & [P_1] & M_e & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{zz} - I_{yy} & [P_3] \\ -[P_1]^T & 0 & 0 & 0 & [P_3]^T & [H_1] \end{bmatrix}$$

Applying Lagrangian formulation, we can get the inertia forces of the array of tuned blade elements in the rotating coordinate system with a rotating speed Ω and rotating acceleration $\dot{\Omega}$ as following:

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial T}{\partial \{\dot{Z}_{eR}\}} \right) - \frac{\partial T}{\partial \{Z_{eR}\}} &= [M_{eR}]\{\ddot{Z}_{eR}\} + \Omega[G_{eR}]\{\dot{Z}_{eR}\} \\ &+ (-\Omega^2[H_{eR}] + \dot{\Omega}[N_{eR}])\{Z_{eR}\} \quad \text{for } m = 1 \end{aligned} \quad (3.13)$$

and

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial T}{\partial \{\dot{p}_{0R}\}} \right) - \frac{\partial T}{\partial \{p_{0R}\}} &= [M_0]\{\ddot{p}_{0R}\} - 2\Omega[G_0]\{\dot{p}_{0R}\} \\ &+ (-\Omega^2[H_0] + \dot{\Omega}[D_0])\{p_{0R}\} + \Omega^2[Q_0]^T + \dot{\Omega}[P_0]^T \\ &\text{for } m = 0 \end{aligned} \quad (3.14)$$

$$\left\{ \begin{array}{l} \frac{d}{dt} \left(\frac{\partial T}{\partial \{\dot{p}_{mR}\}} \right) - \frac{\partial T}{\partial \{p_{mR}\}} \\ \frac{d}{dt} \left(\frac{\partial T}{\partial \{\dot{q}_{mR}\}} \right) - \frac{\partial T}{\partial \{q_{mR}\}} \end{array} \right\} = \begin{bmatrix} -\Omega^2[H_m] & -\dot{\Omega}[G_m] \\ \dot{\Omega}[G_m] & -\Omega^2[H_m] \end{bmatrix} \left\{ \begin{array}{l} \{p_{mR}\} \\ \{q_{mR}\} \end{array} \right\}$$

$$+ \begin{bmatrix} [M_m] & 0 \\ 0 & [M_m] \end{bmatrix} \begin{Bmatrix} \{\ddot{p}_{mR}\} \\ \{\ddot{q}_{mR}\} \end{Bmatrix} + 2\Omega \begin{bmatrix} 0 & -[G_m] \\ [G_m] & 0 \end{bmatrix} \begin{Bmatrix} \{\dot{p}_{mR}\} \\ \{\dot{q}_{mR}\} \end{Bmatrix} \quad \text{for } m \geq 2 \quad (3.15)$$

where

$$[G_{eR}] = [N_{eR}] - [N_{eR}]^T$$

The above formula presented in the rotating coordinate system is preferred for analysis of the dynamics of a system with isotropic properties. In the cases in which the properties of bearing or support structures are orthotropic or anisotropic, it is better to transform the inertia forces Eqs.(3.5) into the form in the space fixed coordinate system. In this case, the relationship between the deflections in the rotating frame and those in the space-fixed frame is given by

$$\{Z_{eR}\} = [\hat{\Phi}]\{Z_e\} \quad (3.16)$$

where

$$\{Z_e\} = \{U \quad \theta_v \quad \{p_1\} \quad V \quad -\theta_x \quad \{q_1\}\}^T \quad (3.17)$$

$$[\hat{\Phi}] = \begin{bmatrix} \cos \phi & 0 & 0 & \sin \phi & 0 & 0 \\ 0 & \cos \phi & 0 & 0 & \sin \phi & 0 \\ 0 & 0 & \cos \phi [I] & 0 & 0 & \sin \phi [I] \\ -\sin \phi & 0 & 0 & \cos \phi & 0 & 0 \\ 0 & -\sin \phi & 0 & 0 & \cos \phi & 0 \\ 0 & 0 & -\sin \phi [I] & 0 & 0 & \cos \phi [I] \end{bmatrix} \quad (3.18)$$

Hence, the velocities and accelerations have the relations:

$$\{\dot{Z}_{eR}\} = [\hat{\Phi}]\{\dot{Z}_e\} + \Omega[R][\hat{\Phi}]\{Z_e\} \quad (3.19)$$

$$\{\ddot{Z}_{eR}\} = [\hat{\Phi}]\{\ddot{Z}_e\} + 2\Omega[R][\hat{\Phi}]\{\dot{Z}_e\} - \Omega^2[\hat{\Phi}]\{Z_e\} + \dot{\Omega}[R][\hat{\Phi}]\{Z_e\} \quad (3.20)$$

with

$$[R] = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & [I] \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -[I] & 0 & 0 & 0 \end{bmatrix}$$

Substituting the above relation into Eqs. 3.9, we can obtain the inertia forces for the array of blade elements in the space-fixed coordinate system as follows:

$$\{F_I\} = [M_e]\{\ddot{Z}_e\} + \Omega[G_e]\{\dot{Z}_e\} + (-\Omega^2[H_e] + \dot{\Omega}[N_e])\{Z_e\} \quad (3.21)$$

where

$$[M_e] = [M_e R]$$

$$[G_e(\Omega)] = 2 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_p/2 & [P'_3] \\ 0 & 0 & 0 & 0 & [P'_3] & [G'_1] \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -I_p/2 & -[P'_3] & 0 & 0 & 0 \\ 0 & -[P'_3]^T & -[G'_1] & 0 & 0 & 0 \end{bmatrix}$$

$$[H_e(\Omega)] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & [H'_1] & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & [H'_1] \end{bmatrix}$$

$$[N_e(\dot{\Omega})] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2[P_2] \\ 0 & 0 & 0 & 0 & 0 & -[G'_1] \\ 0 & 0 & 0 & 0 & 0 & -[P_1] \\ 0 & -I_p & -2[P'_3] & 0 & 0 & 0 \\ 0 & 0 & [G'_1] & 0 & 0 & 0 \end{bmatrix}$$

with

$$\begin{aligned} [P'_3] &= -\frac{1}{2}([P_2] + [P_3]) \\ [G'_1] &= [G_1] - [M_1] \\ [H'_1] &= [H_1] + [M_1] - 2[G_1] \end{aligned}$$

In the above formulation, the vibration of an array of flexible blades is characterized by modes having integer numbers of nodal diameters. For each nodal diameter configuration, there exists a family of circle modes with corresponding natural frequencies. It is noted here that, for a flexible bladed disc-shaft assembly with the polar symmetry, all the diametral modes are kinematically uncoupled each other, and only the first diametral modes are coupled with transverse vibrations of the flexible shaft.

3.4 Stiffness Matrix of Blade Element

In the analysis of a rotating flexible blade element, the potential energy of the element consists of two parts:

1. The elastic energy due to the strain energy of the deformed blade array, which is produced by the bending and shear, and torsion deformations of the blades.

$$U_e = \frac{1}{2} \sum_{i=1}^N \int_{r_1}^{r_2} \begin{Bmatrix} \psi_{f,r} \\ \psi_{c,r} \end{Bmatrix}^T \begin{bmatrix} EI_f & 0 \\ 0 & EI_c \end{bmatrix} \begin{Bmatrix} \psi_{f,r} \\ \psi_{c,r} \end{Bmatrix} dr$$

$$\begin{aligned}
& + \frac{1}{2} \sum_{i=1}^N \int_{r_1}^{r_2} \kappa G A \begin{Bmatrix} v_{f,r} - \psi_f \\ w_{c,r} - \psi_c \end{Bmatrix}^T \begin{Bmatrix} v_{f,r} - \psi_f \\ w_{c,r} - \psi_c \end{Bmatrix} dr \\
& + \frac{1}{2} \sum_{i=1}^N \int_{r_1}^{r_2} GJ(\varphi_{,r})^2 dr
\end{aligned} \quad (3.22)$$

where, κ is the shear correction coefficient taking into account the shear stress distribution across the depth of the blade element. In this study, the value of 5/6 is chosen for a rectangular cross section.

2. The geometric energy produced by the axial (i.e., spanwise) stresses of the blades, which arises from the centrifugal forces distributed in the blades.

$$U_g = \frac{1}{2} \sum_{i=1}^N \int_{r_1}^{r_2} \sigma_r \left(A \begin{Bmatrix} v_{f,r} \\ w_{c,r} \end{Bmatrix}^T \begin{Bmatrix} v_{f,r} \\ w_{c,r} \end{Bmatrix} + I_p(\varphi_{,r})^2 \right) dr \quad (3.23)$$

The total potential energy function of an array of blade elements is the sum of the elastic and geometric energies:

$$U = U_e + U_g \quad (3.24)$$

Substituting Eq. 3.8 into the above equation, we get the transformed form of the potential energy of the element:

$$\begin{aligned}
U &= \frac{1}{2} \sum_{m=0}^k C_m \{p_{mR}\}^T [T]^T ([k_{em}] + [k_{egm}]) [T] \{p_{mR}\} \\
&+ \frac{1}{2} \sum_{m=0}^k S_m \{q_{mR}\}^T [T]^T ([k_{em}] + [k_{egm}]) [T] \{q_{mR}\}
\end{aligned} \quad (3.25)$$

with

$$\begin{aligned}
[k_{em}] &= \frac{N}{2} \int_{r_1}^{r_2} \left\{ GJ[N_{\varphi,r}]^T [N_{\varphi,r}] + EI_f[N_{f,r}]^T [N_{f,r}] + EI_c[N_{c,r}]^T [N_{c,r}] \right. \\
&\quad \left. + \kappa^2 GA \left(([V_r] - [N_f])^T ([V_r] - [N_f]) + ([W_r] - [N_c])^T ([W_r] - [N_c]) \right) \right\} dr
\end{aligned} \quad (3.26)$$

$$[k_{egm}] = \frac{N}{2} \int_{r_1}^{r_2} \sigma_r \left\{ A \left([V_r]^T [V_r] + [W_r]^T [W_r] \right) + I_p[N_{\varphi,r}]^T [N_{\varphi,r}] \right\} dr \quad (3.27)$$

Hence the elastic and geometric element stiffness matrices of an array of blades are given by

$$\left. \begin{aligned} [k_m] &= [T]^T [k_{em}] [T] \\ [k_{gm}] &= [T]^T [k_{egm}] [T] \end{aligned} \right\} \quad (3.28)$$

which only vary with the pretwist angles at two nodal points of the element.

3.5 Formulation for Bladed Disc-Shaft Assembly

The equations of motion for a complete rotor system which consists of a flexible shaft, flexible discs and an array of flexible blades, can be obtained by assembling the blade finite element equation into the flexible disc shaft equations in the previous chapter. The compatibility conditions at the junction of the disc and shaft are given by

$$\left\{ \begin{array}{c} v' \\ \psi_z \\ w' \\ \psi_y \\ \varphi \end{array} \right\}_{\text{blade root}} = \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{array} \right] \left\{ \begin{array}{c} w \\ \psi_r \\ \psi_\theta \end{array} \right\}_{\text{disk outer}}$$

This relationship between the displacements of blades and those of the disc is sufficient to allow assembly of the blade equation into the complete equations of motion of a flexible bladed disc-shaft rotor system.

The boundary conditions for the rotor system are given by bearing and support structure properties. They can be assembled into the equations of motion using dynamic stiffness method. The structural damping in the rotor system can also be included into the model of flexible bladed disc-shaft assembly.

Figure 3.1 **Vibration Modes of a Bladed Disk**

Figure 3.2 Blade Deflection Modes of a Flexible Disk

- a) undeflected cascade configuration
- b) rim deflections in m -diameter modes
- c) blade bending deflections $v_f(r)$, $w_c(r)$
- d) blade torsional deflection $\phi(r)$

Figure 3.3 Coordinate Systems for Blades

Chapter 4

Dynamic Analysis of Rotor Systems

4.1 Introduction

For the case of simple rotor systems, the dynamic response behaviour of the system may be obtained by solving the equations of motion directly. However, in the case of more complex systems with a large number of degrees of freedom, direct solution of the equations of motion may not be feasible because the direct (exact or numerical) integration of a large number of simultaneous equations is time consuming and susceptible to numerical difficulties. A more efficient method of solving the response of the system is that of modal analysis. The modal analysis methods [39, 35] offer a procedure to decouple the simultaneous equations. In general, modal analysis may be divided into two categories: undamped modal analysis and damped modal analysis.

Undamped gyroscopic systems are characterized by symmetric mass, stiffness matrices and a skew-symmetric gyroscopic matrix. A method for obtaining the eigenvalue problem of such a system was developed by Meirovitch [36], where the eigenvalue problem is transformed into one in terms of two real matrices for both the real and the imaginary part of the eigenvectors.

The problem of damped rotor systems is considerably more complicated than those of undamped ones. Since the sum of the damping matrix and the gyroscopic matrix is an arbitrary matrix, and advantage resulting from the symmetry or skew symmetry of the coefficient matrices is lost, so that one must return to the general theory.

This chapter presents the general analysis methods for rotor dynamics available in the literature with some modifications. Both the direct solution methods and the modal analysis methods will be discussed in the following sections.

4.2 Whirl and Critical Speeds of Undamped Rotor Systems

The equations of free motion of an undamped rotor system, as shown in the chapter 2 (Eq. 2.31), can be written in the matrix form

$$[M]\{\ddot{Z}(t)\} + [G(\Omega)]\{\dot{Z}(t)\} + ([K] + [K_g(\Omega)])\{Z(t)\} = 0 \quad (4.1)$$

where $[M]$, $[K]$ and $[K_g(\Omega)]$ are symmetric, and $[G(\Omega)]$ is skew symmetric. Eq. 4.1 can be written as the state form:

$$[\bar{M}]\{\dot{q}(t)\} + [\bar{G}]\{q(t)\} = 0 \quad (4.2)$$

with

$$\begin{aligned} \{q(t)\} &= \begin{Bmatrix} \{\dot{Z}(t)\} \\ \{Z(t)\} \end{Bmatrix}, & [\bar{M}] &= \begin{bmatrix} M & 0 \\ 0 & K + K_g(\Omega) \end{bmatrix} \\ [G] &= \begin{bmatrix} G(\Omega) & K + K_g(\Omega) \\ -K - K_g(\Omega) & 0 \end{bmatrix} \end{aligned} \quad (4.3)$$

Clearly, $[M]$ is symmetric, and $[\bar{G}]$ is skew symmetric. Since the mass matrix $[M]$ is generally positive definite, $[\bar{M}]$ is positive definite if $[K] + [K_g(\Omega)]$ is positive definite.

4.2.1 Whirl Frequencies of Rotor Systems

As shown by Meirovitch [36], the system eigenvalues (which are the whirl speeds relative to stationary axes) at a specified spin speed Ω consist of n pairs of pure

imaginary complex conjugates $\pm j\omega_i$, ($i = 1, 2, \dots, n$) and n pairs of complex conjugate eigenvectors, ϕ_i and ϕ_i^* .

If the i -th eigenvector of the rotor system is the form of

$$\{\phi_i\} = \{A_i\} + j\{B_i\} \quad (4.4)$$

the real vectors $\{A_i\}$ and $\{B_i\}$ are given by the following equations

$$\left. \begin{aligned} \omega_i^2 [\bar{M}] \{A_i\} &= [\bar{K}] \{A_i\} \\ \omega_i^2 [\bar{M}] \{B_i\} &= [\bar{K}] \{B_i\} \end{aligned} \right\}, \quad i = 1, 2, \dots, n \quad (4.5)$$

where

$$[K] = [G]^T [M] [G] \quad (4.6)$$

is positive-definite. These are the essential results obtained by Meirovitch [36].

In conclusion, the eigenvalue problem for the undamped rotor system can be derived as a real symmetric positive eigenvalue problem with n pairs of the **two repeated eigenvalues** ω_i^2 and with the n **pair vectors** $\{A_i\}$ and $\{B_i\}$ belonging to the same eigenvalue ω_i^2 ($i = 1, 2, \dots, n$). The derivation of Eq. 4.5 for an undamped rotor system makes possible the use of a large variety of computationally efficient algorithms for the eigensolution. However, caution must be taken because of the repeated roots.

4.2.2 Critical Speeds of Rotor Systems

Examining Eq. 4.1, we shall seek its solution in the form

$$\{Z(t)\} = \{\phi\} e^{j\omega t} \quad (4.7)$$

where $\{\phi\}$ is n -dimensional constant vector. Introducing Eq. 4.7 into Eq. 4.1 and dividing through by $e^{j\omega t}$, we obtain

$$([K] + [K_g(\Omega)] + j\omega[G(\Omega)] - \omega^2[M]) \{\phi\} = 0 \quad (4.8)$$

For a single-spool rotor system, as shown in the previous chapters, the gyroscopic and centrifugal stiffness matrices, $[G(\Omega)]$ and $[K_g(\Omega)]$, can be written as

$$\left. \begin{aligned} [G(\Omega)] &= \Omega[G] \\ [K_g(\Omega)] &= \Omega^2[K_g] \end{aligned} \right\} \quad (4.9)$$

If the speed ratios between all the shafts are known for a multi-spool rotor system, the matrices $[G(\Omega)]$ and $[K_g(\Omega)]$ can be also written as the above forms by taking Ω as the spin speed of a specified reference shaft.

The critical speed of a rotor system is defined as a spin speed Ω when it equals a natural frequency with respect to stationary reference axes. Hence critical speeds are represented by the following equation:

$$\Omega = \omega \quad (4.10)$$

Substituting Eq. 4.9 and 4.10 into Eq. 4.8, we obtain a new eigenvalue problem

$$([K] - \omega^2([M] - [K_g] - j[G])) \{\phi\} = 0 \quad (4.11)$$

Assuming that the eigenvector $\{\phi\}$ is the form of

$$\{\phi\} = \{A\} + j\{B\} \quad (4.12)$$

we can convert Eq. 4.11 into a $2n$ -dimensional real eigenvalue problem as follows

$$\begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix} \begin{Bmatrix} \phi_R \\ \phi_I \end{Bmatrix} = \omega^2 \begin{bmatrix} M - K_g & G \\ G^T & M - K_g \end{bmatrix} \begin{Bmatrix} \phi_R \\ \phi_I \end{Bmatrix} \quad (4.13)$$

Clearly, the coefficient matrix in the left-hand side of Eq. 4.13 is positive definite if $[K]$ is positive definite, and the coefficient matrix in the right-hand side is generally symmetric. Hence its solutions are $2n$ pairs of real eigenvalues and associated eigenvectors. Since the matrix in the right-hand side is generally non-positive definite, the solution of Eq. 4.13 will have both positive and negative values of ω^2 . The sets of positive eigenvalues and the associated eigenvectors represent the critical speeds and the mode shapes of the rotor system, and should be solved. All the negative eigenvalues (ω^2)'s and the corresponding eigenvectors are unwanted solutions.

4.3 Stability Analysis of Damped Rotor Systems

Now let us consider a general damped rotor system described by:

$$[M]\{\ddot{Z}(t)\} + ([C] + [G(\Omega)])\{\dot{Z}(t)\} + ([K] + [K_g(\Omega)])\{Z(t)\} = 0 \quad (4.14)$$

The problem of damped rotor systems is considerably more complicated than that of undamped systems described in the last section. Since the sum of damping matrix and gyroscopic matrix $[C] + [G(\Omega)]$ is an arbitrary matrix, any advantage resulting from the symmetry or skew-symmetry of coefficient matrices is lost, so that one must return to the general eigenvalue theory.

The eigensolution of Eq. 4.14 can be most conveniently derived by transforming the equation into the state form. To this end, the $2n$ dimensional state vector and dynamic matrix are introduced,

$$\{q(t)\} \stackrel{\text{def}}{=} \begin{Bmatrix} \{\dot{Z}(t)\} \\ \{Z(t)\} \end{Bmatrix} \quad (4.15)$$

and

$$[A] \stackrel{\text{def}}{=} \begin{bmatrix} -M^{-1}(C + G(\Omega)) & -M^{-1}(K + K_g(\Omega)) \\ I & 0 \end{bmatrix} \quad (4.16)$$

where $[A]$ is real but arbitrary. Definitions of Eqs. 4.15 and 4.16 permit Eq. 4.14 to be rewritten in the state form

$$\{\dot{q}(t)\} = [A]\{q(t)\} \quad (4.17)$$

The solution of Eq. 4.17 is in the form

$$\{q(t)\} = \{\phi\}e^{\lambda t} \quad (4.18)$$

Introducing Eq. 4.18 into Eq. 4.17 and dividing through by $e^{\lambda t}$, we obtain

$$[A]\{\phi\} = \lambda\{\phi\} \quad (4.19)$$

Eq. 4.19 is the standard eigenvalue problem of a real matrix and may be solved using many of eigenvalue extraction procedures, such as HQR method for all eigenvalues and the power method for lower eigenvalues. The eigenvalues are always either real ($\lambda = \sigma$) or in complex conjugate pairs ($\lambda = \sigma \pm j\omega_d$), and the whirl frequency of the damped free vibration is given by the imaginary part ω_d of a complex eigenvalue. Whether the vibration amplitude will increase or decrease with time is dependent on the real part σ of the eigenvalue. The growth rate of the vibration amplitude is related to the logarithmic decrement defined by

$$\delta = \frac{2\pi\sigma}{\omega_d} \quad (4.20)$$

4.4 Identification of Whirl Direction

As shown by Glasgow [61], the whirling direction of a rotor at a critical speed is determined by the corresponding mode shape. In his derivations, some approximations made are in fact unnecessary. This section presents a modified derivation for the identification of whirl direction of a rotor mode.

Assuming that the i -th eigenvalue and eigenvector of a rotor system are given as

$$\lambda_i = \alpha_i + j\omega_{di} \quad (4.21)$$

and

$$\{\phi_i\} = \{A_i\} + j\{B_i\} \quad (4.22)$$

The free vibration of the rotor can be, then, written as

$$\{Z(t)\} = \sum_{i=1}^{2N} \zeta_i e^{\lambda_i t} \{\phi_i\} = \sum_{i=1}^N (\zeta_i e^{\lambda_i t} \{\phi_i\} + \zeta_i^* e^{\lambda_i^* t} \{\phi_i^*\}) \quad (4.23)$$

where ζ_i is the generalized coordinate with respect to the i -th mode.

When the rotor runs near the critical speeds ω_{di} , it can be treated as a system with a single degree of freedom and Eq. 4.23 can be approximately written as

$$\{Z(t)\} \approx \zeta_i e^{\lambda_i t} \{\phi_i\} + \zeta_i^* e^{\lambda_i^* t} \{\phi_i^*\} \quad (4.24)$$

For a steady-state whirl, ζ_i is a complex constant and can be written as

$$\zeta_i = c_i + j d_i \quad (4.25)$$

Substituting Eqs. 4.21, 4.22, and 4.25 into Eq. 4.24, and dropping all the subscripts "i" for convenience, we obtain

$$\begin{aligned} \{Z(t)\} &= 2 \operatorname{Re} \{(c + j d) e^{(\alpha + j \omega_d) t} (\{A\} + j \{B\})\} \\ &= 2 e^{\alpha t} \{(c \{A\} - d \{B\}) \cos \omega_d t - (d \{A\} + c \{B\}) \sin \omega_d t\} \end{aligned} \quad (4.26)$$

Assume for the moment that x, y are the displacements at one node of the rotor in two transverse planes, respectively. These two coordinates can be then written from the above equation as following:

$$x = 2 e^{\alpha t} \{(c x_R - d x_I) \cos \omega_d t - (d x_R + c x_I) \sin \omega_d t\} \quad (4.27)$$

$$y = 2 e^{\alpha t} \{(c y_R - d y_I) \cos \omega_d t - (d y_R + c y_I) \sin \omega_d t\} \quad (4.28)$$

where $(x_R + j x_I, y_R + j y_I)$ is the corresponding value of the i th eigenvector $\{\phi\}$ at the node concerned.

Denoting

$$\left. \begin{aligned} \hat{x}_R &= c x_R - d x_I \\ \hat{x}_I &= c x_I + d x_R \\ \hat{y}_R &= c y_R - d y_I \\ \hat{y}_I &= c y_I + d y_R \end{aligned} \right\} \quad (4.29)$$

Eqs. 4.27 and 4.28 can be expressed as:

$$x = 2 e^{\alpha t} \{\hat{x}_R \cos \omega_d t - \hat{x}_I \sin \omega_d t\} \quad (4.30)$$

$$y = 2 e^{\alpha t} \{\hat{y}_R \cos \omega_d t - \hat{y}_I \sin \omega_d t\} \quad (4.31)$$

The precessional angle at the node for this moment is defined by ψ , as shown in Figure 4.1. Hence, the conditions for forward or backward precession of the rotor are given as

- forward precession: $\dot{\psi} > 0$;
- backward precession: $\dot{\psi} < 0$;
- non-precession: $\dot{\psi} = 0$.

The precessional angle, ψ , can be expressed as

$$\psi = \tan^{-1} \frac{y}{x} = \tan^{-1} \frac{\hat{y}_R \cos \omega_d t - \hat{y}_I \sin \omega_d t}{\hat{x}_R \cos \omega_d t - \hat{x}_I \sin \omega_d t} \quad (4.32)$$

$$\frac{d\psi}{dt} = \frac{\omega_d(-\hat{x}_R \hat{y}_I + \hat{x}_I \hat{y}_R)}{(\hat{x}_R \cos \omega_d t - \hat{x}_I \sin \omega_d t)^2 + (\hat{y}_R \cos \omega_d t - \hat{y}_I \sin \omega_d t)^2} \quad (4.33)$$

Substituting Eq. 4.30 into the above equation, we obtained that

$$\frac{d\psi}{dt} = \frac{\omega_d(c^2 + d^2)(-x_R y_I + x_I y_R)}{(\hat{x}_R \cos \omega_d t - \hat{x}_I \sin \omega_d t)^2 + (\hat{y}_R \cos \omega_d t - \hat{y}_I \sin \omega_d t)^2} \quad (4.34)$$

Hence, we can relate the precession direction to the vibration mode of the rotor by the following relationship:

$$\text{sign}(\dot{\psi}) = -\text{sign} \begin{vmatrix} x_R & x_I \\ y_R & y_I \end{vmatrix} \quad (4.35)$$

That is, the steady-state precession direction of a rotor at a critical speed is time-independent, and completely determined by the corresponding modal shape.

4.5 Direct Response Analysis for Damped Rotor Systems

The equation of motion of a general damped rotor system, as shown in the previous chapters, can be written in the matrix form

$$[M]\{\ddot{Z}(t)\} + ([C] + [G(\Omega)])\{\dot{Z}(t)\} + ([K] + K_g(\Omega))\{Z(t)\} = \{F(t)\} \quad (4.36)$$

4.5.1 Steady-State Response Analysis

The most simple forced response problem in rotor dynamics is that caused by circular harmonic excitation forces. There are two cases of steady-state responses which need to be considered:

- Steady-state forced responses with an excitation frequency ω in which the excitation force vector can be written in the form

$$\{F(t)\} = \{F_c\} \cos \omega t + \{F_s\} \sin \omega t \quad (4.37)$$

- Steady-state unbalance responses in which the excitation vector can be written as the same form as Eq. 4.37 with an excitation frequency Ω in stead of ω .

Assume that the displacement vector $\{Z(t)\}$ is of the form

$$\{Z(t)\} = \{Z_c\} \cos \omega t + \{Z_s\} \sin \omega t \quad (4.38)$$

Introducing Eqs. 4.37 and 4.38 into Eq. 4.36, and considering the coefficients of $\cos \omega t$ and $\sin \omega t$, we obtain

$$\left. \begin{aligned} ([K] + [K_g(\Omega)] - \omega^2[M])\{Z_c\} + \omega([C] + [G(\Omega)])\{Z_s\} &= \{F_c\} \\ -\omega([C] + [G(\Omega)])\{Z_c\} + ([K] + [K_g(\Omega)] - \omega^2[M])\{Z_s\} &= \{F_s\} \end{aligned} \right\} \quad (4.39)$$

From Eq. 4.39, we obtain the steady-state responses of a damped rotor system as follows

$$\begin{Bmatrix} Z_c \\ Z_s \end{Bmatrix} = \begin{bmatrix} K + K_g(\Omega) - \omega^2 M & \omega(C + G(\Omega)) \\ -\omega(C + G(\Omega)) & K + K_g(\Omega) - \omega^2 M \end{bmatrix}^{-1} \begin{Bmatrix} F_c \\ F_s \end{Bmatrix} \quad (4.40)$$

2

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4.5.2 Transient Response Analysis

The transient response (solution of Eq. 4.36) of a damped rotor system can be most conveniently derived by transforming Eq. 4.36 into the state form. To this end, utilizing the definitions of the state vector $\{q(t)\}$ and the dynamic matrix $[A]$ in Section 4.3 (Eqs. 4.15 and 4.16) and introducing the $2n$ -dimensional excitation vector

$$\{Q(t)\} = \begin{Bmatrix} [M]^{-1}\{F(t)\} \\ 0 \end{Bmatrix} \quad (4.41)$$

we can transform Eq. 4.36 into the state equation of motion

$$\{\dot{q}(t)\} = [A]\{q(t)\} + \{Q(t)\} \quad (4.42)$$

The solution of Eq. 4.42 for any given excitation forces can be written in the general form involving the convolution integral

$$\{q(t)\} = e^{[A]t}\{q(0)\} + \int_0^t e^{[A](t-\tau)}\{Q(\tau)\} d\tau \quad (4.43)$$

where $e^{[A]t}$ is known as the state transition matrix.

Solution of Eq. 4.43 involves an integral containing the transition matrix $e^{[A]t}$. To generate the matrix $e^{[A]t}$, it is necessary to expand a power series in $[A]$. Then the solution is obtained by performing the indicated integration. For large order systems, the process can be time-consuming. In addition, computer round-off is likely to introduce computational error.

Hence, recourse is usually taken to the use of direct numerical integration methods for Eq. 4.42 over a rather small number of shaft revolutions. The Runge-Kutta method and Newmark method are shown to be adequate in the analysis of rotor systems if sufficient time steps are used to ensure solution accuracy and numerical stability.

4.6 Modal Response Analysis for Damped Rotor Systems

When the total number of degrees of freedom in a rotor system is too large for computers to handle economically, recourse is sometimes taken to modal response analysis with mode truncations.

The solution of Eq. 4.42 can be obtained by modal analysis, which amounts to the determination of the Jordan form of $[A]$. For simplicity, it is assumed that all the eigenvalues λ_i of $[A]$ are distinct, so that the Jordan matrix is diagonal,

$$[\Lambda] = \text{diag}[\lambda_i] \quad (4.44)$$

Assuming that the i -th right and left eigenvectors of $[A]$ are $\{\phi_i\}$ and $\{\psi_i\}$, respectively, we can rearrange them in the modal matrices,

$$\left. \begin{aligned} [\Phi] &= [\phi_1 \phi_2 \cdots \phi_{2n}] \\ [\Psi] &= [\psi_1 \psi_2 \cdots \psi_{2n}] \end{aligned} \right\} \quad (4.45)$$

The right eigenvectors are orthogonal to the left eigenvectors, a property that is known as bi-orthogonality. Hence, the eigenvectors can be normalized so as to satisfy

$$\left. \begin{aligned} [\Psi]^T[\Phi] &= [\Phi]^T[\Psi] = [I] \\ [\Psi]^T[A][\Phi] &= [\Lambda] \end{aligned} \right\} \quad (4.46)$$

Because the vectors $\{\phi_i\}$ on the one hand and the vectors $\{\psi_i\}$ on the other hand are linearly independent, either set of vectors can be taken as a basis of the state space, which implies that the solution of Eq. 4.42 can be represented as a linear combination of the eigenvectors $\{\phi_i\}$ or of the eigenvectors $\{\psi_i\}$. Using $\{\phi_i\}$ as a basis, we can write

$$\{q(t)\} = \sum_{i=1}^{2n} \{\phi_i\} \zeta_i(t) = [\Phi] \{\zeta(t)\} \quad (4.47)$$

where $\{\zeta(t)\}$ is a $2n$ -dimensional vector with elements $\zeta_i(t)$. Introducing Eq. 4.47 into Eq. 4.42, pre-multiplying the result by $[\Psi]^T$ and considering the orthogonalities

of Eq. 4.46, we obtain $2n$ uncoupled equations of motion.

$$\{\dot{\zeta}(t)\} = [\Lambda]\{\zeta(t)\} + \{\chi(t)\} \quad (4.48)$$

where

$$\{\chi(t)\} = [\Psi]^T \{Q(t)\}$$

4.6.1 Steady-State Response Analysis

Assuming that the right and left eigenvectors have the following partition form

$$\{\phi_i\} = \begin{Bmatrix} \lambda_i u_i \\ u_i \end{Bmatrix}, \quad \{\psi_i\} = \begin{Bmatrix} \lambda_i v_i \\ v_i \end{Bmatrix} \quad (4.49)$$

and substituting Eqs. 4.48 and 4.49 into Eq. 4.47, one obtains the steady state response

$$\begin{aligned} \{Z(t)\} &= \sum_{i=1}^{2n} \frac{\lambda_i \{u_i\}}{j\omega - \lambda_i} \{v_i\}^T (\{F_c\} \cos \omega t + \{F_s\} \sin \omega t) \\ &= \sum_{i=1}^n 2 \operatorname{Re} \left[\frac{\lambda_i \{u_i\}}{j\omega - \lambda_i} \{v_i\}^T (\{F_c\} \cos \omega t + \{F_s\} \sin \omega t) \right] \end{aligned} \quad (4.50)$$

4.6.2 Transient response Analysis

Eq. 4.48 represents a set of $2n$ independent equations having the solutions in the vector form

$$\{\zeta(t)\} = e^{[\Lambda]t} \{\zeta(0)\} + \int_0^t e^{[\Lambda](t-\tau)} \{\chi(\tau)\} d\tau \quad (4.51)$$

with the initial conditions

$$\{\zeta(0)\} = [\Psi]^T \{q(0)\}$$

Substituting Eq. 4.51 into Eq. 4.47, we obtain the transient response for a damped system as follows

$$\{q(t)\} = \sum_{i=1}^{2n} \{\phi_i\} \{\psi_i\}^T \left(\{q(0)\} e^{\lambda_i t} + \int_0^t \{Q(\tau)\} e^{\lambda_i(t-\tau)} d\tau \right) \quad (4.52)$$

It should be noted that the modal analyses for unbalance and transient responses, Eq. 4.50 and Eq. 4.52, are speed-dependent since the mode shapes are calculated at a specified rotor speed.

Figure 4.1 **Elliptical Orbit of a Rotor Node**

Chapter 5

Modal Analysis for Rotor Systems

5.1 Introduction

Undamped non-gyroscopic systems are characterized by symmetric mass and stiffness matrices. The eigensolution for such systems consists of pairs of real eigenvalues and eigenvectors which are orthogonal with respect to both the mass and stiffness matrices. Hence, the responses of such systems can be obtained in closed form by superposition of responses of all modes to the given excitations. If gyroscopic forces are present, then the eigensolution becomes complex. A method for obtaining the eigenvalue problem of undamped gyroscopic systems was developed by Meirovitch [36], where the eigenvalue problem is transformed into one in terms of two real matrices for both the real and the imaginary part of the eigenvectors. Based on the approach, a modal analysis that permits the derivation of closed-form response of linear undamped gyroscopic systems has been presented in reference [37]. In addition, a second-order perturbation for a lightly damped gyroscopic system based on the eigensolution of the undamped system was developed in Reference [39].

The problem of damped rotor systems is considerably more complicated than that of undamped ones. Since the sum of the damping matrix and the gyroscopic matrix is an arbitrary matrix, and advantage resulting from the symmetry or skew symmetry of the coefficient matrices is lost, so that one must return to the general theory presented in the last chapter. For the rotor systems with small damping,

the damping can be regarded as a perturbation to the undamped systems. Thus, as noted, Meirovitch [39] has presented a second-perturbation theory to calculate the left and right eigenvectors as well as the response of damped systems.

This chapter is first concerned with the motion characteristics of undamped rotating systems with axial-symmetric mass distributions, such as turbomachines, or rotors of helicopters. Taking advantage of the special symmetric nature of such systems, a new method of solution of the eigenvalue problem is developed. It is shown that the original complex eigenvalue problem can be rearranged into a real eigenvalue problem in terms of two real symmetric matrices which are directly assembled from the original mass, gyroscopic and stiffness matrices without using any matrix transformations. Based on this approach, a new modal analysis is developed which employs the real eigenvectors to decouple the equations of motion of the system. Using this modal analysis, a closed-form response of the system is derived, and the characteristics of the unbalanced response and of the forced response are examined.

Following this, the damped rotor systems with axial-symmetric mass distribution have been studied via a new perturbation analysis. As shown, the equation of motion of such a system can be re-arranged into a state equation with symmetric coefficient matrices. Based on the eigensolution analysis of the undamped rotor systems, a perturbation analysis for the eigensolution and the response of slightly damped rotor systems is developed in this chapter. For this, only one set of eigenvectors needs to be calculated to decouple the equations of motion because of the symmetrical properties.

In particular, it is shown that for rotor systems with isotropic bearings, only one plane motion needs to be analyzed (it should be noted that the two-plane motions are coupled to each other because of the presence of gyroscopic forces).

5.2 Characteristics of Undamped Gyroscopic Systems

The equations of motion of an undamped gyroscopic system under free vibration conditions can be written as follows

$$[M]\{\ddot{Z}(t)\} + [G(\Omega)]\{\dot{Z}(t)\} + ([K] + [K_g(\Omega)])\{Z(t)\} = \{0\} \quad (5.1)$$

where, the mass matrix $[M]$ is positive definite, the stiffness matrix $[K] + [K_g(\Omega)]$ is generally positive definite, and the gyroscopic matrix $[G(\Omega)]$ is skew-symmetric.

We shall seek a solution of Eq. 5.1 in the form

$$\{Z(t)\} = \{\varphi\}e^{\lambda t} \quad (5.2)$$

where $\{\varphi\}$ is a n -dimensional constant vector with complex elements, and λ is a constant complex number. Introducing Eq. 5.2 into Eq. 5.1 and dividing through by $e^{\lambda t}$, we obtain

$$(\lambda^2[M] + \lambda[G(\Omega)] + [K] + [K_g(\Omega)])\{\varphi\} = \{0\} \quad (5.3)$$

Because the determinant of a matrix is equal to the determinant of the transposed matrix, we must have

$$\begin{aligned} & \det(\lambda^2[M] + \lambda[G(\Omega)] + [K] + [K_g(\Omega)]) \\ &= \det(\lambda^2[M]^T + \lambda[G(\Omega)]^T + [K]^T + [K_g(\Omega)]^T) \end{aligned} \quad (5.4)$$

Utilizing the symmetric and skew-symmetric properties of a rotor system, the characteristic equation 5.3 can be written as

$$\begin{aligned} & \det(\lambda^2[M] + \lambda[G(\Omega)] + [K] + [K_g(\Omega)]) \\ &= \det(\lambda^2[M] - \lambda[G(\Omega)] + [K] + [K_g(\Omega)]) = 0 \end{aligned} \quad (5.5)$$

So if λ is an eigenvalue of the system, then $(-\lambda)$ is also an eigenvalue of the system.

Next, we wish to show that the eigenvalues are pure imaginary, therefore they occur in pairs of complex conjugates and the associated eigenvectors are also complex conjugates.

Pre-multiplying Eq. 5.3 by $\{\varphi\}^H$, where the superscript H means the conjugate (*) and transpose (T), we get

$$\{\varphi\}^H (\lambda^2[M] + \lambda[G(\Omega)] + [K] + [K_g(\Omega)]) \{\varphi\} = 0 \quad (5.6)$$

Since,

$$\left. \begin{aligned} (\{\varphi\}^H[M]\{\varphi\})^* &= \{\varphi\}^H[M]\{\varphi\} \\ (\{\varphi\}^H[G(\Omega)]\{\varphi\})^* &= -\{\varphi\}^H[G(\Omega)]\{\varphi\} \\ (\{\varphi\}^H[K]\{\varphi\})^* &= \{\varphi\}^H[K]\{\varphi\} \\ (\{\varphi\}^H[K_g(\Omega)]\{\varphi\})^* &= \{\varphi\}^H[K_g(\Omega)]\{\varphi\} \end{aligned} \right\} \quad (5.7)$$

so $\{\varphi\}^H[M]\{\varphi\}$, $\{\varphi\}^H[K]\{\varphi\}$ and $\{\varphi\}^H[K_g(\Omega)]\{\varphi\}$ are real numbers, while $\{\varphi\}^H[G(\Omega)]\{\varphi\}$ is pure imaginary.

Let us define

$$\left. \begin{aligned} m &\stackrel{\text{def}}{=} \{\varphi\}^H[M]\{\varphi\} \\ k &\stackrel{\text{def}}{=} \{\varphi\}^H[K]\{\varphi\} \\ h &\stackrel{\text{def}}{=} \{\varphi\}^H[K_g(\Omega)]\{\varphi\} \\ g &\stackrel{\text{def}}{=} j\{\varphi\}^H[G(\Omega)]\{\varphi\} \end{aligned} \right\} \quad (5.8)$$

Substituting the above equations into Eq. 5.6, we obtain

$$\lambda^2 m - j\lambda g + k + h = 0 \quad (5.9)$$

Assuming that

$$\lambda = \sigma + j\omega \quad (5.10)$$

and substituting it into Eq. 5.9, we get

$$(\sigma^2 - \omega^2)m + \omega g + k + h = 0 \quad (5.11)$$

$$\sigma(2\omega m - g) = 0 \quad (5.12)$$

There are two possible solutions from Eq. 5.12:

$$\sigma = 0 \quad \text{or} \quad \omega = \frac{g}{2m} \quad (5.13)$$

Assuming first $\sigma \neq 0$, then $\omega = g/2m$ is a solution of Eq. 5.12 and we have from Eq. 5.11

$$\sigma^2 + \frac{g^2}{4m} + k + h = 0 \quad (5.14)$$

Since $m > 0$ and $k + h > 0$, it follows that the above equation can not be true. Therefore the original assumption of $\sigma \neq 0$ is wrong, i.e., the solution of Eq. 5.12 should be:

$$\sigma = 0, \quad \text{or} \quad \lambda = j\omega \quad (5.15)$$

The above results agree with the assumption of an undamped system. Because $(-\lambda) = -j\omega$ is also an eigenvalue, the system eigenvalues consist of n pairs of pure imaginary complex conjugates.

It is easy to see from Eq. 5.3 that if $(j\omega, \{\varphi\})$ represents a solution of the eigenvalue problem, then $(-j\omega, \{\varphi\}^*)$ also is a solution. It follows that the system eigenvectors consist of n pairs of complex conjugate vectors $\{\varphi\}$ and $\{\varphi\}^*$, corresponding to the eigenvalues $j\omega$ and $(-j\omega)$, respectively.

5.3 Mode Shapes of Undamped Rotor Systems

As shown in Chapter 2, the equation of motion of a rotor system Eq. 5.1 can be written in the following partitioned form:

$$\begin{aligned}
& \begin{bmatrix} M_0 & 0 \\ 0 & M_0 \end{bmatrix} \begin{Bmatrix} \{\ddot{x}\} \\ \{\ddot{y}\} \end{Bmatrix} + \begin{bmatrix} 0 & G_0(\Omega) \\ -G_0^T(\Omega) & 0 \end{bmatrix} \begin{Bmatrix} \{\dot{x}\} \\ \{\dot{y}\} \end{Bmatrix} \\
& + \begin{bmatrix} K_x + K_{g0}(\Omega) & 0 \\ 0 & K_y + K_{g0}(\Omega) \end{bmatrix} \begin{Bmatrix} \{x\} \\ \{y\} \end{Bmatrix} = \begin{Bmatrix} \{f_x\} \\ \{f_y\} \end{Bmatrix} \quad (5.16)
\end{aligned}$$

where an assumption has been made that there is no cross-coupling in all the bearings of the rotor system. Actually, as shown in examples, a much weaker assumption is that a pair of principal axes exists, so the equations of motion of the system can be set up in such way as to satisfy Eq. 5.16.

The mode shape of the gyroscopic system, in general, can be written as

$$\{\varphi\} = \begin{Bmatrix} \{a_R\} + j\{a_I\} \\ \{b_R\} + j\{b_I\} \end{Bmatrix} \quad (5.17)$$

Hence the eigenvalue problem Eq. 5.3 can be written in the form

$$\begin{bmatrix} K_x + K_{g0} - \omega^2 M_0 & j\omega G_0 \\ -j\omega G_0^T & K_y + K_{g0} - \omega^2 M_0 \end{bmatrix} \begin{Bmatrix} \{a_R\} + j\{a_I\} \\ \{b_R\} + j\{b_I\} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (5.18)$$

which gives four coupled matrix equations:

$$([K_x] + [K_{g0}] - \omega^2[M_0]) \{a_R\} - \omega[G_0]\{b_I\} = 0 \quad (5.19)$$

$$([K_x] + [K_{g0}] - \omega^2[M_0]) \{a_I\} + \omega[G_0]\{b_R\} = 0 \quad (5.20)$$

$$([K_y] + [K_{g0}] - \omega^2[M_0]) \{b_R\} + \omega[G_0]^T\{a_I\} = 0 \quad (5.21)$$

$$([K_y] + [K_{g0}] - \omega^2[M_0]) \{b_I\} - \omega[G_0]^T\{a_R\} = 0 \quad (5.22)$$

From Eqs. 5.21 and 5.22, we obtain

$$\{b_R\} = -\omega ([K_y] + [K_{g0}] - \omega^2[M_0])^{-1} [G_0]^T\{a_I\} \quad (5.23)$$

$$\{b_I\} = \omega ([K_y] + [K_{g0}] - \omega^2[M_0])^{-1} [G_0]^T\{a_R\} \quad (5.24)$$

Substituting Eqs. 5.23 and 5.24 into Eqs. 5.19 and 5.20, we get

$$([K_x] + [K_{g0}] - \omega^2[M_0] - \omega^2[G_0])([K_y] + [K_{g0}] - \omega^2[M_0])^{-1}[G_0]^T \{a_R\} = 0 \quad (5.25)$$

$$([K_x] + [K_{g0}] - \omega^2[M_0] - \omega^2[G_0])([K_y] + [K_{g0}] - \omega^2[M_0])^{-1}[G_0]^T \{a_I\} = 0 \quad (5.26)$$

Comparing Eq. 5.25 with Eq. 5.26, we know that the two solutions are proportional to each other, i.e.,

$$\{a_I\} = \alpha \{a_R\} \quad (5.27)$$

Substituting Eq. 5.27 into Eq. 5.23, and comparing it with Eq. 5.24, we obtain

$$\{b_R\} = -\alpha \{b_I\} \quad (5.28)$$

Introducing Eqs. 5.27 and 5.28 into Eq. 5.17, we get

$$\{\varphi\} = (1 + j\alpha) \begin{Bmatrix} \{a_R\} \\ j\{b_I\} \end{Bmatrix} \quad (5.29)$$

Dividing $\{\varphi\}$ by $(1 + j\alpha)$, we obtain

$$\{\varphi\} = \begin{Bmatrix} \{a\} \\ j\{b\} \end{Bmatrix} \quad (5.30)$$

where $\{a\}$ and $\{b\}$ are real vectors of $n/2$ dimensions.

5.4 Eigenvalue Problem for Undamped Rotor Systems

In this section, a new method of solution of the eigenvalue problem Eq. 5.3 for a gyroscopic system will be developed, which fully uses the properties developed in the previous sections. It will be proved that the complex eigenvalue and eigenvector problem for undamped rotor systems can be transformed into a symmetric form with real eigenvalues and eigenvectors.

Substituting Eq. 5.30 into Eqs. 5.19 and 5.22, we obtain

$$\begin{aligned} ([K_x] + [K_{g0}] - \omega^2[M_0])\{a\} - \omega[G_0]\{b\} &= \{0\} \\ ([K_y] + [K_{g0}] - \omega^2[M_0])\{b\} - \omega[G_0]^T\{a\} &= \{0\} \end{aligned} \quad (5.31)$$

Next we rewrite the above equation in the matrix form:

$$([K] + [K_g] - \omega^2[M] - \omega[\bar{G}])\{\bar{\varphi}\} = \{0\} \quad (5.32)$$

with

$$[\bar{G}] \stackrel{\text{def}}{=} \begin{bmatrix} 0 & G_0 \\ G_0^T & 0 \end{bmatrix} \quad \{\bar{\varphi}\} \stackrel{\text{def}}{=} \begin{Bmatrix} \{a\} \\ \{b\} \end{Bmatrix} \quad (5.33)$$

It is clear from the above equation that $[\bar{G}]$ is a symmetric matrix.

Now Eq. 5.32 can be further written in the form of general normal mode analysis:

$$\begin{bmatrix} -\bar{G} & K + K_g \\ K + K_g & 0 \end{bmatrix} \begin{Bmatrix} \omega\{\bar{\varphi}\} \\ \{\bar{\varphi}\} \end{Bmatrix} = \omega \begin{bmatrix} M & 0 \\ 0 & K + K_g \end{bmatrix} \begin{Bmatrix} \omega\{\bar{\varphi}\} \\ \{\bar{\varphi}\} \end{Bmatrix} \quad (5.34)$$

Since the mass matrix $[M]$ is generally positive definite, the coefficient matrix of Eq. 5.34 in the right side will be positive definite if the total stiffness matrix is positive definite (i.e., the rotor system has no degree of freedom for rigid body motion). And the coefficient matrix in the left side of Eq. 5.34 is symmetric. Hence the solutions of Eq. 5.34 are $(2n)$ sets of eigenvalues, ω_i' 's, and the associated real independent eigenvectors, $\{\bar{\varphi}_i\}$'s.

Now let us define a n -dimension vector

$$\{\hat{\varphi}\} \stackrel{\text{def}}{=} \begin{Bmatrix} \{a\} \\ -\{b\} \end{Bmatrix} \quad (5.35)$$

As we know from the previous section that if $(\omega, \{\bar{\varphi}\})$ is a solution of Eq. 5.34, $(\omega, \{\hat{\varphi}\})$ will be also the solution of the equation. So among the $2n$ sets of solutions, the n sets of positive eigenvalues and the associated eigenvectors need to be calculated.

Eq. 5.34 has advantages over Eq. 4.5 derived by Meirovitch [37] where there is a much more complex transformation procedure involved and all eigenvalues have at least the multiplicity of two. Furthermore, it will be proved later that the new modal analysis technique will enable us further to simplify the response analysis of rotor systems.

It will prove convenient to normalize the eigenvectors so as to satisfy

$$\begin{Bmatrix} \omega_j \{\bar{\varphi}_j\} \\ \{\bar{\varphi}_j\} \end{Bmatrix}^T \begin{bmatrix} M & 0 \\ 0 & K + K_g \end{bmatrix} \begin{Bmatrix} \omega_i \{\bar{\varphi}_i\} \\ \{\bar{\varphi}_i\} \end{Bmatrix} = \delta_{ij}, \quad i, j = 1, 2, \dots, n \quad (5.36)$$

$$\begin{Bmatrix} \omega_j \{\bar{\varphi}_j\} \\ \{\bar{\varphi}_j\} \end{Bmatrix}^T \begin{bmatrix} -\bar{G} & K + K_g \\ K + K_g & 0 \end{bmatrix} \begin{Bmatrix} \omega_i \{\bar{\varphi}_i\} \\ \{\bar{\varphi}_i\} \end{Bmatrix} = \omega_i \delta_{ij}, \quad i, j = 1, 2, \dots, n \quad (5.37)$$

where δ_{ij} is the Kronecker delta.

5.5 Critical Speeds of Rotor Systems

For a multi-spool rotor system, if the speed ratio among all the shafts are given, the partitioned gyroscopic matrix $[G_0(\Omega)]$ and the geometric stiffness matrix $[K_{g0}(\Omega)]$ can be written as

$$\left. \begin{aligned} [G_0(\Omega)] &= \Omega [\bar{G}_0] \\ [K_{g0}(\Omega)] &= \Omega^2 [\bar{K}_{g0}] \end{aligned} \right\} \quad (5.38)$$

where Ω is the rotation speed for a single-spool system, or the speed of the reference shaft for a multi-spool system.

Introducing Eq. 5.38 into Eq. 5.31, we get

$$\left. \begin{aligned} ([K_x] + \Omega^2 [\bar{K}_{g0}] - \omega^2 [M_0]) \{a\} - \omega \Omega [\bar{G}_0] \{b\} &= \{0\} \\ ([K_y] + \Omega^2 [\bar{K}_{g0}] - \omega^2 [M_0]) \{b\} - \omega \Omega [\bar{G}_0]^T \{a\} &= \{0\} \end{aligned} \right\} \quad (5.39)$$

Assuming the rotation speed Ω of the rotor and the whirl frequencies ω have the following relationship:

$$\Omega = \gamma\omega \quad (5.40)$$

where $\gamma = 1$ represents the forward or backward critical speeds, $\gamma < 1$ means sub-critical speeds, and $\gamma > 1$ stands for super-critical speeds.

Substituting Eq. 5.40 into Eq. 5.39, we get the new eigenvalue problem of order n :

$$\begin{bmatrix} M_0 - \gamma^2 \bar{K}_{g0} & -\gamma \bar{G}_0 \\ -\gamma \bar{G}_0^T & M_0 - \gamma^2 \bar{K}_{g0} \end{bmatrix} \begin{Bmatrix} \{a\} \\ \{b\} \end{Bmatrix} = \frac{1}{\omega^2} \begin{bmatrix} K_x & 0 \\ 0 & K_y \end{bmatrix} \begin{Bmatrix} \{a\} \\ \{b\} \end{Bmatrix} \quad (5.41)$$

The coefficient matrix in the left-hand side of Eq. 5.41 is symmetric, and the matrix in the right-hand side is positive definite if $[K_x]$ and $[K_y]$ are positive definite. Hence the solutions of Eq. 5.41 are n sets of real eigenvalues $1/\omega^2$ and eigenvectors.

Since the left-hand matrix in Eq. 5.41 is generally non-positive definite, the solutions of the equation will have both positive and negative $1/\omega^2$. The sets of positive eigenvalues and the associated eigenvectors represent the forward or backward whirl frequencies and modes of the system. The negative $1/\omega^2$ are unwanted solutions.

5.6 Eigenvalue Problem for Isotropic Rotor Systems

For a rotor system with isotropic bearings, the calculation of whirl and critical speeds can be further reduced.

5.6.1 Whirl Frequencies

Introducing the isotropic bearing condition:

$$[K_x] = [K_y] \stackrel{\text{def}}{=} [K_0] \quad (5.42)$$

into Eq. 5.31, and solving it for $\{a\}$ and $\{b\}$, we get

$$([K_0] + [K_{g0}] - \omega^2[M_0] - \omega^2[G_0])([K_0] + [K_{g0}] - \omega^2[M_0])^{-1}[G_0]^T\{a\} = \{0\} \quad (5.43)$$

$$([K_0] + [K_{g0}] - \omega^2[M_0] - \omega^2[G_0])^T([K_0] + [K_{g0}] - \omega^2[M_0])^{-1}[G_0]\{b\} = \{0\} \quad (5.44)$$

Since the partitioned gyroscopic matrix $[G_0]$, for a rotor system with axial-symmetric inertia properties, is symmetric from Chapter 2, i.e., $[G_0]^T = [G_0]$, so the solutions of Eq. 5.43 and those of Eq. 5.44 are identically the same. It will prove convenient to take

$$\{b\} = -\{a\} \quad (5.45)$$

where the “-” sign represents the forward whirl modes as shown in Chapter 4 (see Eq. 4.35).

Substituting Eq. 5.45 into Eq. 5.43 and Eq. 5.44, we get the eigenvalue problem for an isotropic rotor system as follows:

$$\begin{bmatrix} G_0 & K_0 + K_{g0} \\ K_0 + K_{g0} & 0 \end{bmatrix} \begin{Bmatrix} \omega\{a\} \\ \{a\} \end{Bmatrix} = \omega \begin{bmatrix} M_0 & 0 \\ 0 & K_0 + K_{g0} \end{bmatrix} \begin{Bmatrix} \omega\{a\} \\ \{a\} \end{Bmatrix} \quad (5.46)$$

Again, the coefficient matrix in the left-hand side of Eq. 5.46 is symmetric, and the matrix in the right-hand side is positive definite if $[K_0] + [K_{g0}]$ is positive definite. Hence the solutions of Eq. 5.46 are n sets of real eigenvalues and eigenvectors.

As we have learned from the previous sections that if $(\omega, \{\bar{\varphi}\})$ is a solution of a gyroscopic system, $(-\omega, \{\hat{\varphi}\})$ is also a solution of the equation, so Eq. 5.46 has $n/2$ positive and $n/2$ negative eigenvalues. And the $n/2$ positive eigenvalues and the associated eigenvectors represent the forward whirl frequencies and mode shapes and the $n/2$ negative eigenvalues and the associated eigenvectors stand for the backward whirl frequencies and mode shapes.

5.6.2 Critical Speeds

For an isotropic rotor system, introducing Eqs. 5.42 and 5.45 into Eq. 5.41, we get a new eigenvalue problem of order $n/2$:

$$([M_0] - \gamma^2[K_{g0}] - \gamma[G_0])\{a\} = \frac{1}{\omega^2}[K_0]\{a\} \quad (5.47)$$

The forward or backward whirl frequencies and the associated whirl modes of the system are obtained from Eq. 5.47 by giving $\gamma > 0$ or $\gamma < 0$, respectively. In particular, the synchronous forward critical speeds are obtained by choosing $\gamma = 1$, and the backward critical speeds are obtained by choosing $\gamma = -1$.

5.7 Response of Undamped Rotor Systems

For an undamped rotor system with axial-symmetric mass distribution, the responses in yz plane will lead or lag those in xz plane by a phase of 90° in the vector space, which means that $-j\{y\}$ will be in the same or opposite direction of $\{x\}$. Substituting the following transformation

$$\begin{Bmatrix} \{x\} \\ \{y\} \end{Bmatrix} = \begin{bmatrix} I & 0 \\ 0 & jI \end{bmatrix} \begin{Bmatrix} \{x\} \\ -j\{y\} \end{Bmatrix} \quad (5.48)$$

into Eq. 5.16 and pre-multiplying it by $\begin{bmatrix} I & 0 \\ 0 & -jI \end{bmatrix}$, we obtain

$$\begin{aligned} & \begin{bmatrix} M_0 & 0 \\ 0 & M_0 \end{bmatrix} \begin{Bmatrix} \{\ddot{x}\} \\ -j\{\ddot{y}\} \end{Bmatrix} + j \begin{bmatrix} 0 & G_0 \\ G_0^T & 0 \end{bmatrix} \begin{Bmatrix} \{\dot{x}\} \\ -j\{\dot{y}\} \end{Bmatrix} \\ & + \begin{bmatrix} K_x + K_{g0} & 0 \\ 0 & K_y + K_{g0} \end{bmatrix} \begin{Bmatrix} \{x\} \\ -j\{y\} \end{Bmatrix} = \begin{Bmatrix} \{f_x(t)\} \\ -j\{f_y(t)\} \end{Bmatrix} \end{aligned} \quad (5.49)$$

or written as the state equation

$$\begin{bmatrix} M & 0 \\ 0 & K + K_g \end{bmatrix} \begin{Bmatrix} -j\{\ddot{z}\} \\ \{\dot{z}\} \end{Bmatrix} - j \begin{bmatrix} -\bar{G} & K + K_g \\ K + K_g & 0 \end{bmatrix} \begin{Bmatrix} -j\{\dot{z}\} \\ \{z\} \end{Bmatrix} = \begin{Bmatrix} \{Q(t)\} \\ 0 \end{Bmatrix} \quad (5.50)$$

where

$$\{z(t)\} \stackrel{\text{def}}{=} \begin{Bmatrix} \{x\} \\ -j\{y\} \end{Bmatrix} \quad \{Q(t)\} \stackrel{\text{def}}{=} \begin{Bmatrix} -j\{f_x(t)\} \\ -\{f_y(t)\} \end{Bmatrix} \quad (5.51)$$

Obviously the eigenvalue problem of Eq. 5.50 is identical to that of Eq. 5.34. Because the $2n$ eigenvectors are independent to each other, they form a basis of a $2n$ -dimensional vector space. Hence we can express the $2n$ -dimensional response vector as a linear combination of the eigenvectors, i.e.,

$$\begin{Bmatrix} -j\{\dot{z}(t)\} \\ \{z(t)\} \end{Bmatrix} = \sum_{i=1}^{2n} \begin{Bmatrix} \omega_i \{\bar{\varphi}_i\} \\ \{\bar{\varphi}_i\} \end{Bmatrix} \xi_i(t) \quad (5.52)$$

The inverse transformation can be obtained by pre-multiplying Eq. 5.52 by $\begin{Bmatrix} \omega_j \{\bar{\varphi}_j\} \\ \{\bar{\varphi}_j\} \end{Bmatrix}^T \begin{bmatrix} M & 0 \\ 0 & K + K_g \end{bmatrix}$, as follows:

$$\begin{aligned} \xi_i(t) = & \{a_i\}^T ([K_x] + [K_{g0}])\{x\} - \omega_i \{b_i\}^T [M_0]\{\dot{y}\} \\ & - j(\{b_i\}^T ([K_y] + [K_{g0}])\{y\} + \omega_i \{a_i\}^T [M_0]\{\dot{x}\}), \quad i = 1, 2, \dots, 2n \end{aligned} \quad (5.53)$$

Hence, the initial conditions of the coordinates ξ_i can be expressed in terms of the initial displacements and velocities of the system as:

$$\begin{aligned} \xi_i(0) = & \{a_i\}^T ([K_x] + [K_{g0}])\{x_0\} - \omega_i \{b_i\}^T [M_0]\{\dot{y}_0\} \\ & - j(\{b_i\}^T ([K_y] + [K_{g0}])\{y_0\} + \omega_i \{a_i\}^T [M_0]\{\dot{x}_0\}), \quad i = 1, \dots, 2n \end{aligned} \quad (5.54)$$

Let us assume that the $2n$ sets of eigenvalues and eigenvectors are rearranged in the following order

$$\left. \begin{aligned} 0 \leq \omega_1 \leq \omega_2 \leq \dots \leq \omega_n; \quad -\omega_1 \geq -\omega_2 \geq \dots \geq -\omega_n; \\ \{\bar{\varphi}_1\}, \{\bar{\varphi}_2\}, \dots, \{\bar{\varphi}_n\}; \quad \{\hat{\varphi}_1\}, \{\hat{\varphi}_2\}, \dots, \{\hat{\varphi}_n\} \end{aligned} \right\} \quad (5.55)$$

It is clear from Eq. 5.53 that $\xi_i(t)$ and ξ_{n+i} (for $i = 1, 2, \dots, n$) are complex conjugates, i.e.,

$$\xi_{n+i}(t) = \xi_i^*(t), \quad i = 1, 2, \dots, n \quad (5.56)$$

It follows from Eq. 5.52 that

$$\{x(t)\} = 2 \sum_{i=1}^n \{a_i\} \operatorname{Re}[\xi_i(t)] \quad (5.57)$$

$$\{y(t)\} = -2 \sum_{i=1}^n \{b_i\} \operatorname{Im}[\xi_i(t)] \quad (5.58)$$

Substituting Eq. 5.52 into Eq. 5.50, and pre-multiplying it by $\begin{Bmatrix} \omega_i \{\bar{\varphi}_i\} \\ \{\bar{\varphi}_i\} \end{Bmatrix}^T$, we obtain the $2n$ uncoupled equations

$$\dot{\xi}_i(t) - j\omega_i \xi_i(t) = \omega_i \{\bar{\varphi}_i\}^T \{Q(t)\}, \quad i = 1, 2, \dots, 2n \quad (5.59)$$

Since they are complex conjugates, so only the first n equations need to be solved, which are

$$\xi_i(t) = e^{j\omega_i t} \xi_i(0) + \int_0^t e^{j\omega_i(t-\tau)} \omega_i \{\bar{\varphi}_i\}^T \{Q(\tau)\} d\tau, \quad i = 1, 2, \dots, n \quad (5.60)$$

Substituting Eq. 5.60 and Eq. 5.54 into Eq. 5.57 and Eq. 5.58, we obtain the response of the system as

$$\begin{aligned} \{x(t)\} = \sum_{i=1}^n 2\{a_i\} & \left\{ \left(\{a_i\}^T ([K_x] + [K_{g0}]) \{x_0\} - \omega_i \{b_i\}^T [M_0] \{\dot{y}_0\} \right) \cos \omega_i t \right. \\ & + \left(\{b_i\}^T ([K_y] + [K_{g0}]) \{y_0\} + \omega_i \{a_i\}^T [M_0] \{\dot{x}_0\} \right) \sin \omega_i t \\ & + \omega_i \{a_i\}^T \int_0^t \{f_x(\tau)\} \sin \omega_i(t - \tau) d\tau \\ & \left. - \omega_i \{b_i\}^T \int_0^t \{f_y(\tau)\} \cos \omega_i(t - \tau) d\tau \right\} \end{aligned} \quad (5.61)$$

$$\begin{aligned} \{y(t)\} = \sum_{i=1}^n 2\{b_i\} & \left\{ \left(\{b_i\}^T ([K_y] + [K_{g0}]) \{y_0\} + \omega_i \{a_i\}^T [M_0] \{\dot{x}_0\} \right) \cos \omega_i t \right. \\ & - \left(\{a_i\}^T ([K_x] + [K_{g0}]) \{x_0\} - \omega_i \{b_i\}^T [M_0] \{\dot{y}_0\} \right) \sin \omega_i t \\ & + \omega_i \{a_i\}^T \int_0^t \{f_x(\tau)\} \cos \omega_i(t - \tau) d\tau \\ & \left. + \omega_i \{b_i\}^T \int_0^t \{f_y(\tau)\} \sin \omega_i(t - \tau) d\tau \right\} \end{aligned} \quad (5.62)$$

5.8 Steady-State Response Analysis

For harmonic excitation forces, the force vectors $\{f_x(t)\}$ and $\{f_y(t)\}$ can be written as

$$\begin{cases} \{f_x(t)\} = \{F_{xc}\} \cos \omega t + \{F_{xs}\} \sin \omega t \\ \{f_y(t)\} = \{F_{yc}\} \cos \omega t + \{F_{ys}\} \sin \omega t \end{cases} \quad (5.63)$$

Introducing Eq. 5.63 into Eq. 5.61 and Eq. 5.62, we obtain the steady state responses of the rotor system:

$$\begin{aligned} \{x(t)\} = \sum_{i=1}^n \frac{2\omega_i \{a_i\}}{\omega^2 - \omega_i^2} \{ \{a_i\}^T [\{F_{xc}\} \omega_i (\cos \omega_i t - \cos \omega t) + \{F_{xs}\} (\omega \sin \omega_i t - \omega_i \sin \omega t)] \\ - \{b_i\}^T [\{F_{yc}\} (\omega \sin \omega t - \omega_i \sin \omega_i t) + \{F_{ys}\} \omega (\cos \omega_i t - \cos \omega t)] \} \end{aligned} \quad (5.64)$$

$$\begin{aligned} \{y(t)\} = \sum_{i=1}^n \frac{2\omega_i \{b_i\}}{\omega^2 - \omega_i^2} \{ \{a_i\}^T [\{F_{xc}\} (\omega \sin \omega t - \omega_i \sin \omega t) + \{F_{xs}\} \omega (\cos \omega_i t - \cos \omega t)] \\ + \{b_i\}^T [\{F_{yc}\} \omega_i (\cos \omega_i t - \cos \omega t) + \{F_{ys}\} (\omega \sin \omega_i t - \omega_i \sin \omega t)] \} \end{aligned} \quad (5.65)$$

Considering the resonance condition $\omega \rightarrow \omega_r$ in the above equations, we obtain

$$\begin{aligned} \{x(t)\} = \sum_{i=1, i \neq r}^n \frac{2\omega_i \{a_i\}}{\omega_r^2 - \omega_i^2} \{ \{a_i\}^T [\{F_{xc}\} \omega_i (\cos \omega_i t - \cos \omega t) + \{F_{xs}\} (\omega \sin \omega_i t - \omega_i \sin \omega t)] \\ - \{b_i\}^T [\{F_{yc}\} (\omega \sin \omega t - \omega_i \sin \omega_i t) + \{F_{ys}\} \omega (\cos \omega_i t - \cos \omega t)] \} \\ + \{a_r\} \{a_r\}^T [\{F_{xc}\} \omega_r t \sin \omega_r t + \{F_{xs}\} (\sin \omega_r t - \omega_r t \cos \omega_r t)] \\ - \{a_r\} \{b_r\}^T [\{F_{yc}\} (\sin \omega_r t + \omega_r t \cos \omega_r t) + \{F_{ys}\} \omega_r t \sin \omega_r t] \end{aligned} \quad (5.66)$$

$$\begin{aligned} \{y(t)\} = \sum_{i=1, i \neq r}^n \frac{2\omega_i \{b_i\}}{\omega_r^2 - \omega_i^2} \{ \{a_i\}^T [\{F_{xc}\} (\omega \sin \omega t - \omega_i \sin \omega t) + \{F_{xs}\} \omega (\cos \omega_i t - \cos \omega t)] \\ + \{b_i\}^T [\{F_{yc}\} \omega_i (\cos \omega_i t - \cos \omega t) + \{F_{ys}\} (\omega \sin \omega_i t - \omega_i \sin \omega t)] \} \\ + \{b_r\} \{a_r\}^T [\{F_{xc}\} (\sin \omega_r t + \omega_r t \cos \omega_r t) + \{F_{xs}\} \omega_r t \sin \omega_r t] \\ + \{b_r\} \{b_r\}^T [\{F_{yc}\} \omega_r t \sin \omega_r t + \{F_{ys}\} (\sin \omega_r t - \omega_r t \cos \omega_r t)] \end{aligned} \quad (5.67)$$

which increase indefinitely with time, as should be expected. For the purpose of quantitative analysis, we may make approximations:

$$\left. \begin{aligned} \{x(t)\} &\approx \{X_c\} \cos \omega_r t + \{X_s\} \sin \omega_r t \\ \{y(t)\} &\approx \{Y_c\} \cos \omega_r t + \{Y_s\} \sin \omega_r t \end{aligned} \right\} \quad (5.68)$$

where

$$\left. \begin{aligned} \{X_c\} &= -\omega_r t (\{a_r\}^T \{F_{xs}\} + \{b_r\}^T \{F_{yc}\}) \{a_r\} \\ \{X_s\} &= \omega_r t (\{a_r\}^T \{F_{xc}\} - \{b_r\}^T \{F_{ys}\}) \{a_r\} \\ \{Y_c\} &= \omega_r t (\{a_r\}^T \{F_{xc}\} - \{b_r\}^T \{F_{ys}\}) \{b_r\} \\ \{Y_s\} &= \omega_r t (\{a_r\}^T \{F_{xs}\} + \{b_r\}^T \{F_{yc}\}) \{b_r\} \end{aligned} \right\} \quad (5.69)$$

5.8.1 Unbalance Response Analysis

The unbalanced forces of a rotor system in Eq. 5.63 can be expressed as

$$\left. \begin{aligned} \{F_{xc}\} &= \{F_{ys}\} \stackrel{\text{def}}{=} \{F_0\} \\ \{F_{xs}\} &= \{F_{yc}\} = 0 \end{aligned} \right\} \quad (5.70)$$

Substituting Eq. 5.70 into Eq. 5.69, we obtain

$$\left. \begin{aligned} \{X_c\} &= \{Y_s\} = 0 \\ \{X_s\} &= \omega_r t (\{a_r\} - \{b_r\})^T \{F_0\} \{a_r\} \\ \{Y_c\} &= \omega_r t (\{a_r\} - \{b_r\})^T \{F_0\} \{b_r\} \end{aligned} \right\} \quad (5.71)$$

Since any element in $\{b_r\}$ will have the same or opposite sign as the corresponding element of $\{a_r\}$ for backward or forward modes, respectively, so the response of a backward mode will be much smaller than that of the corresponding forward mode. Especially for rotor systems with isotropic bearings, there are the following relationships in eigenvectors:

$$\left. \begin{aligned} \text{forward modes:} & \quad \{b_r\} = -\{a_r\} \\ \text{backward modes:} & \quad \{b_r\} = \{a_r\} \end{aligned} \right\} \quad (5.72)$$

It follows from Eq. 5.71 that only the forward modes will be excited by the unbalanced forces.

5.8.2 Forced Steady-State Responses

Let us consider the applied forces with

$$\left. \begin{aligned} \{F_{xs}\} &= \{F_{yc}\} = 0 \\ \{F_{xc}\} &= -\{F_{ys}\} \stackrel{\text{def}}{=} \{F_0\} \end{aligned} \right\} \quad (5.73)$$

Substituting the above equation into Eq. 5.69, we get

$$\left. \begin{aligned} \{X_c\} &= \{Y_s\} = 0 \\ \{X_s\} &= \omega_r t (\{a_r\} + \{b_r\})^T \{F_0\} \{a_r\} \\ \{Y_c\} &= \omega_r t (\{a_r\} + \{b_r\})^T \{F_0\} \{b_r\} \end{aligned} \right\} \quad (5.74)$$

Hence the response of a backward mode will be much larger than that of the associated forward mode. Especially for a rotor system with isotropic bearings, it follows from Eq. 5.73 and Eq. 5.72 that only the backward modes will be excited.

Next let us consider the response of rotor systems under space-stationary forces:

$$\left. \begin{aligned} \{F_{xc}\} &\stackrel{\text{def}}{=} \{F_0\} \\ \{F_{xs}\} &= \{F_{yc}\} = \{F_{ys}\} = 0 \end{aligned} \right\} \quad (5.75)$$

Introducing the above equation into Eq. 5.69, we obtain

$$\left. \begin{aligned} \{X_c\} &= \{Y_s\} = 0 \\ \{X_s\} &= (\omega_r t \{a_r\}^T \{F_0\}) \{a_r\} \\ \{Y_c\} &= (\omega_r t \{a_r\}^T \{F_0\}) \{b_r\} \end{aligned} \right\} \quad (5.76)$$

Hence both forward and backward modes can be excited by space-stationary forces applied on a rotor system.

5.9 Perturbation Theory for Slightly Damped Rotor Systems

This section is concerned with rotor systems with small damping. In this case damping can be regarded as a perturbation to the undamped rotor systems. Thus, a perturbation theory for analysis of eigenvalue problem is developed in this section,

which fully utilizes the symmetric properties of rotor systems. The solution is based on the eigensolution for an undamped rotor system, and is expressed in terms of real quantities only. The theory is illustrated by application to a two degree freedom gyroscopic system and a multi-stepped shaft rotor.

5.9.1 Damped Eigenvalues and Eigenvectors

The equation of motion of a general damped rotor system can be written in the matrix form:

$$[M]\{\ddot{Z}(t)\} + ([C] + [G(\Omega)])\{\dot{Z}(t)\} + ([K] + [K_g(\Omega)])\{Z(t)\} = \{F(t)\} \quad (5.77)$$

which, as shown in Section 5.7, can be transformed into the following state space form

$$\begin{aligned} & \begin{bmatrix} M & 0 \\ 0 & K + K_g \end{bmatrix} \begin{Bmatrix} -j\{\ddot{z}\} \\ \{\dot{z}\} \end{Bmatrix} - \left(- \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} \right. \\ & \left. + j \begin{bmatrix} -\bar{G} & K + K_g \\ K + K_g & 0 \end{bmatrix} \right) \begin{Bmatrix} -j\{\dot{z}\} \\ \{z\} \end{Bmatrix} = \begin{Bmatrix} \{Q(t)\} \\ 0 \end{Bmatrix} \end{aligned} \quad (5.78)$$

Re-express the transformation Eq. 5.52 in the matrix form

$$\begin{Bmatrix} -j\{\dot{z}(t)\} \\ \{z(t)\} \end{Bmatrix} = [\Phi]\{\eta(t)\} \quad (5.79)$$

with

$$\begin{aligned} [\Phi] &= \begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_{2n} \end{bmatrix} \\ \{\phi_i\} &= \begin{Bmatrix} \omega_i\{\bar{\varphi}_i\} \\ \{\bar{\varphi}_i\} \end{Bmatrix} \\ \{\eta(t)\} &= \begin{Bmatrix} \eta_1(t) \\ \eta_2(t) \\ \vdots \\ \eta_{2n}(t) \end{Bmatrix} \end{aligned} \quad (5.80)$$

Introducing Eq. 5.79 into Eq. 5.78 and using the orthogonal relations of Eq. 5.36 and Eq. 5.37, we get

$$\{\dot{\eta}(t)\} - [A]\{\eta(t)\} = \{Z(t)\} \quad (5.81)$$

where

$$\left. \begin{aligned} [A] &= [A_0] + [A_1] \\ \{Z(t)\} &= [\Phi]^T \begin{Bmatrix} \{Q(t)\} \\ 0 \end{Bmatrix} \end{aligned} \right\} \quad (5.82)$$

and

$$\left. \begin{aligned} [A_0] &= j \operatorname{diag}\{\omega_1, \omega_2, \dots, \omega_{2n}\} \\ [A_1] &= -[\Phi]^T \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} [\Phi] \stackrel{\text{def}}{=} [\sigma_{ij}] \end{aligned} \right\} \quad (5.83)$$

Assume that $(\lambda_i, \{e_i\})$, $i = 1, 2, \dots, 2n$, are solutions of the eigenvalue problem

$$(\lambda_i[I] - [A])\{e_i\} = 0, \quad (5.84)$$

Since the matrix $[A]$ is symmetric, it can be diagonalized by the $2n$ eigenvectors solved from the above equation.

For slightly damped systems, the modal damping coefficients are an order of magnitude of $\epsilon = 0.1$. Hence the elements of $[A_1]$ are of one order of magnitude smaller than the elements of $[A_0]$. In fact, the matrix $[A]$ can be regarded as the result of perturbing the undamped frequency matrix $[A_0]$ slightly by the damping matrix $[A_1]$. Consequently, $[A_0]$ may be called the unperturbed matrix, $[A_1]$ the perturbation matrix and $[A]$ the perturbed matrix. The eigenvalue problem associated with $[A]$ (Eq. 5.84) may be called the perturbed eigenvalue problem. Hence, we should seek solutions of the perturbed eigenvalue problem in the form

$$\left. \begin{aligned} \lambda_i &= \lambda_{0i} + \lambda_{1i} + \lambda_{2i} + \dots \\ \{e_i\} &= \{e_{0i}\} + \{e_{1i}\} + \{e_{2i}\} + \dots \end{aligned} \right\}, \quad i = 1, 2, \dots, 2n \quad (5.85)$$

Obviously, there are $\lambda_{0i} = j\omega_i$, for $i = 1, 2, \dots$, where ω_i is the i -th frequencies of the undamped system, and $\{e_{0i}\}$ is the unit vector of order $2n$ with the i -th element being one and all others zeroes. Now we can define $\{e_i\}$ normalized in such a way that

$$\{e_{0i}\}^T \{e_i\} = 1 \quad (5.86)$$

and this implies all perturbation eigenvectors, $\{e_{ri}\}$ ($r = 1, 2, \dots$), are orthogonal to their unperturbed eigenvector $\{e_{0i}\}$, i.e.,

$$\left. \begin{aligned} \{e_{0i}\}^T \{e_{ri}\} &= 0 \\ \{e_{0i}\}^T [A_0] \{e_{ri}\} &= 0 \end{aligned} \right\} \quad r = 1, 2, \dots \quad (5.87)$$

Introducing Eq. 5.82 and Eq. 5.85 into the eigenvalue problem Eq. 5.84, we obtain

$$\left. \begin{aligned} ([A_0] + [A_1]) (\{e_{0i}\} + \{e_{1i}\} + \{e_{2i}\} + \dots) \\ = (\lambda_{0i} + \lambda_{1i} + \lambda_{2i} + \dots) (\{e_{0i}\} + \{e_{1i}\} + \{e_{2i}\} + \dots) \end{aligned} \right\} \quad (5.88)$$

We can equate the coefficients of the same order of magnitude obtained by multiplying these series for $r = 0, 1, 2, \dots$ as follows

$$\epsilon^0: \quad [A_0] \{e_{0i}\} = \lambda_{0i} \{e_{0i}\} \quad (5.89)$$

$$\epsilon^r: \quad (\lambda_{0i}[I] - [A_0]) \{e_{ri}\} = [A_1] \{e_{r-1,i}\} - \sum_{k=1}^{r-1} \lambda_{ki} \{e_{r-k,i}\}, \quad r = 1, 2, \dots \quad (5.90)$$

Pre-multiplying Eq. 5.90 by $\{e_{0i}\}^T$, we obtain the perturbation equation for eigenvalues as follows:

$$\lambda_{ri} = \{e_{0i}\}^T [A_1] \{e_{r-1,i}\}, \quad r = 1, 2, \dots \quad (5.91)$$

Assuming that $\lambda_{0i} = j\omega_i$ is the eigenvalue of the matrix $[A_0]$ with the multiplicity n_i and $\mathcal{N}_i$ is the set of all n_i indices of λ_{0i} , we first define the matrix

$$[E_i] = j \sum_{k=1, k \notin \mathcal{N}_i}^{2n} \frac{\{e_{0k}\} \{e_{0k}\}^T}{\omega_k - \omega_i} \quad (5.92)$$

Pre-multiplying Eq. 5.90 by $[E_i]$, we obtain

$$[E_i](\lambda_{0i}[I] - [A_0])\{e_{ri}\} = [E_i]\left([A_1]\{e_{r-1,i}\} - \sum_{k=1}^{r-1} \lambda_{ki}\{e_{r-k,i}\}\right), \quad r = 1, 2, \dots \quad (5.93)$$

Since

$$[E_i](\lambda_{0i}[I] - [A_0]) = [I] - \sum_{k \in \mathcal{N}_i} \{e_{0k}\}\{e_{0k}\}^T \quad (5.94)$$

which implies

$$\text{rank}([E_i](\lambda_{0i}[I] - [A_0])) = \text{rank}(\lambda_{0i}[I] - [A_0]) \quad (5.95)$$

hence Eq. 5.90 and Eq. 5.93 have the same solution set. Since

$$\{e_k\}\{e_k\}^T[E_i] = 0 \quad (5.96)$$

it is easily seen that the following vector is a common solution:

$$\{e_{ri}\} = [E_i]\left([A_1]\{e_{r-1,i}\} - \sum_{k=1}^{r-1} \lambda_{ki}\{e_{r-k,i}\}\right), \quad r = 1, 2, \dots \quad (5.97)$$

Eq. 5.91 and Eq. 5.97 are the $(2n)$ set of eigensolutions for the damped system and can be expressed in terms of the elements of $[A_0] = [\sigma_{ij}]$, and the undamped frequencies ω_i as follows (only the first two sets of perturbations are listed):

$$\lambda_{1i} = \sigma_{ii} \quad (5.98)$$

$$\{e_{1i}\} = j \sum_{k=1, k \notin \mathcal{N}_i}^{2n} \frac{\sigma_{ki}}{\omega_k - \omega_i} \{e_{0k}\} \quad (5.99)$$

$$\lambda_{2i} = j \sum_{k=1, k \notin \mathcal{N}_i}^{2n} \frac{\sigma_{ki}^2}{\omega_k - \omega_i} \quad (5.100)$$

$$\{e_{2i}\} = - \sum_{k=1, k \notin \mathcal{N}_i}^{2n} \frac{d_{ki}}{\omega_k - \omega_i} \{e_{0k}\} \quad (5.101)$$

with

$$d_{ki} = \sum_{l=1, l \notin \mathcal{N}_i}^{2n} \frac{\sigma_{li}(\sigma_{kl} - \sigma_{ii}\delta_{kl})}{\omega_l - \omega_i} \quad (5.102)$$

Assuming the eigenvectors of the original system Eq. 5.78 are $\{\psi_i\}$ ($i = 1, 2, \dots, 2n$), and pre-multiplying Eq. 5.99 and Eq. 5.101 by $[\Phi]$, we obtain the perturbation theory for the damped rotor systems as follows:

$$\left. \begin{aligned} \lambda_i &= \lambda_{0i} + \lambda_{1i} + \lambda_{2i} + \dots \\ \{\psi_i\} &= \{\psi_{0i}\} + \{\psi_{1i}\} + \{\psi_{2i}\} + \dots \end{aligned} \right\} \quad i = 1, 2, \dots, 2n \quad (5.103)$$

$$\left. \begin{aligned} \lambda_{0i} &= j\omega_i \\ \lambda_{1i} &= \sigma_{ii} \\ \lambda_{2i} &= j \sum_{k=1, k \notin N_i}^{2n} \frac{\sigma_{ki}^2}{\omega_k - \omega_i} \end{aligned} \right\} \quad i = 1, 2, \dots, 2n \quad (5.104)$$

$$\left. \begin{aligned} \{\psi_{0i}\} &= \{\phi_i\} \\ \{\psi_{1i}\} &= j \sum_{k=1, k \notin N_i}^{2n} \frac{\sigma_{ki}}{\omega_k - \omega_i} \{\phi_k\} \\ \{\psi_{2i}\} &= - \sum_{k=1, k \notin N_i}^{2n} \frac{d_{ki}}{\omega_k - \omega_i} \{\phi_k\} \end{aligned} \right\} \quad i = 1, 2, \dots, 2n, \quad (5.105)$$

It will prove convenient to normalize the eigenvector $\{\psi_i\}$ divided by

$$\alpha_i = \sqrt{1 - \sum_{k=1, k \notin N_i}^{2n} \frac{(\sigma_{ki} - jd_{ki})^2}{(\omega_k - \omega_i)^2}} \quad (5.106)$$

so that the normality relations have the explicit form:

$$\left. \begin{aligned} \psi_j^T [\bar{M}] \psi_i &= \delta_{ij} \\ \psi_j^T [\bar{K}] \psi_i &= \lambda_i \delta_{ij} \end{aligned} \right\}, \quad i = 1, 2, \dots, 2n \quad (5.107)$$

with

$$\left. \begin{aligned} [\bar{M}] &= \begin{bmatrix} M & 0 \\ 0 & K + K_g \end{bmatrix} \\ [\bar{K}] &= - \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} + j \begin{bmatrix} -\bar{G} & K + K_g \\ K + K_g & 0 \end{bmatrix} \end{aligned} \right\} \quad (5.108)$$

It should be noted that the eigensolutions of Eq. 5.78 have the following properties:

$$\bullet \quad \lambda_{n+i} = \lambda_i^* \quad i = 1, 2, \dots, n;$$

- Denoting $\{\psi_i\} = \begin{Bmatrix} -j\lambda_i\{\bar{\psi}_i\} \\ \{\bar{\psi}_i\} \end{Bmatrix}$, if $\{\bar{\psi}_i\} = \begin{Bmatrix} \{a_i\} \\ \{b_i\} \end{Bmatrix}$, then $\{\bar{\psi}_{n+i}\} = \begin{Bmatrix} \{a_i^*\} \\ -\{b_i^*\} \end{Bmatrix}$,
for all $i = 1, 2, \dots, n$.

To further reduce the calculations of the damped eigenvalues and eigenvectors, the modal truncation method can be utilized, and only the required nodal displacements are calculated.

5.9.2 Eigensolution for Isotropic Systems

The eigenvalue problem of damped isotropic rotor systems can be written as the following form using Eq. 5.45:

$$\lambda \begin{bmatrix} M_0 & 0 \\ 0 & K_0 + K_{g0} \end{bmatrix} \begin{Bmatrix} -j\lambda\{a\} \\ \{a\} \end{Bmatrix} = \left(- \begin{bmatrix} C_0 & 0 \\ 0 & 0 \end{bmatrix} + j \begin{bmatrix} G_0 & K_0 + K_{g0} \\ K_0 + K_{g0} & 0 \end{bmatrix} \right) \begin{Bmatrix} -j\lambda\{a\} \\ \{a\} \end{Bmatrix} \quad (5.109)$$

which has the same form as that of Eq. 5.78, and has an order of n only. Hence, all the perturbation theory developed previously can be applied to the isotropic rotor systems directly by changing all the summations over $2n$ into those over n .

5.9.3 Response Analysis

Since the vectors ψ_i are linear independent, they can be taken as a basis for a $2n$ -dimensional linear vector space, which implies that the solution of Eq. 5.78 can be re-expressed as a linear combination of the eigenvectors ψ_i , i.e.,

$$\begin{Bmatrix} -j\{\dot{z}(t)\} \\ \{z(t)\} \end{Bmatrix} = \sum_{i=1}^{2n} \begin{Bmatrix} -j\lambda_i\{\bar{\psi}_i\} \\ \{\bar{\psi}_i\} \end{Bmatrix} \xi_i(t) \quad (5.110)$$

The inverse transformation can be obtained by pre-multiplying Eq. 5.110 by

$$\begin{Bmatrix} -j\lambda_i\bar{\psi}_i \\ \bar{\psi}_i \end{Bmatrix}^T \begin{bmatrix} M & 0 \\ 0 & K + K_g \end{bmatrix}, \text{ as follows:}$$

$$\begin{aligned}\xi_i(t) &= \{a_i\}^T \left(([K_x] + [K_{g0}])\{x\} - \lambda_i[M_0]\{\dot{x}\} \right) \\ &\quad - j\{b_i\}^T \left(([K_y] + [K_{g0}])\{y\} - \lambda_i[M_0]\{\dot{y}\} \right), \quad i = 1, 2, \dots, 2n\end{aligned}\quad (5.111)$$

It is clear from the properties of the eigensolution and the above equation that $\xi_i(t)$ and $\xi_{n+i}(t)$ (for $i = 1, 2, \dots, n$) are complex conjugates, i.e.,

$$\xi_{n+i}(t) = \xi_i^*(t), \quad i = 1, 2, \dots, n \quad (5.112)$$

The initial conditions of the coordinates $\xi_i(t)$ can be expressed in terms of the initial displacements and velocities of the system as

$$\begin{aligned}\xi_i(0) &= \{a_i\}^T \left(([K_x] + [K_{g0}])\{x_0\} - \lambda_i[M_0]\{\dot{x}_0\} \right) \\ &\quad - j\{b_i\}^T \left(([K_y] + [K_{g0}])\{y_0\} - \lambda_i[M_0]\{\dot{y}_0\} \right), \quad i = 1, 2, \dots, 2n\end{aligned}\quad (5.113)$$

It follows from Eq. 5.110 that

$$\{x(t)\} = 2 \sum_{i=1}^n \text{Re}[\{a_i\}\xi_i(t)] \quad (5.114)$$

$$\{y(t)\} = -2 \sum_{i=1}^n \text{Im}[\{b_i\}\xi_i(t)] \quad (5.115)$$

Substituting Eq. 5.110 into Eq. 5.78, we get $2n$ uncoupled equations of motion of a damped rotor system as follows:

$$\dot{\xi}_i(t) - \lambda_i \xi_i(t) = -j\lambda_i \{\bar{\psi}_i\}^T \{Q(t)\} \quad i = 1, 2, \dots, 2n \quad (5.116)$$

Since they are complex conjugates, so only the first n equations need to be solved, which are

$$\xi_i(t) = e^{\lambda_i t} \xi_i(0) - j \int_0^t e^{\lambda_i(t-\tau)} \lambda_i \{\bar{\psi}_i\}^T \{Q(\tau)\} d\tau, \quad i = 1, 2, \dots, n \quad (5.117)$$

Assuming the eigensolution has the following forms:

$$\left. \begin{aligned} \lambda_i &= \sigma_i + j\omega_{di} \\ \{a_i\} &= \{a_{Ri}\} + j\{a_{Li}\} \\ \{b_i\} &= \{b_{Ri}\} + j\{b_{Li}\} \end{aligned} \right\} \quad (5.118)$$

Substituting Eq. 5.113 and Eq. 5.118 into Eq. 5.117, we obtain the response of the system as follows:

$$\begin{aligned} \begin{Bmatrix} \{x(t)\} \\ \{y(t)\} \end{Bmatrix} &= 2 \sum_{i=1}^n e^{\sigma_i t} (R_i \cos \omega_{di} t - I_i \sin \omega_{di} t) ([K] + [K_g]) \begin{Bmatrix} \{x_0\} \\ \{y_0\} \end{Bmatrix} \\ &+ e^{\sigma_i t} ((-R_i \sigma_i + I_i \omega_{di}) \cos \omega_{di} t + (R_i \omega_{di} - I_i \sigma_i) \sin \omega_{di} t) [M] \begin{Bmatrix} \{\dot{x}_0\} \\ \{\dot{y}_0\} \end{Bmatrix} \\ &+ (-R_i \sigma_i + I_i \omega_{di}) \int_0^t e^{\sigma_i(t-\tau)} \cos \omega_{di}(t-\tau) \begin{Bmatrix} \{f_x(\tau)\} \\ \{f_y(\tau)\} \end{Bmatrix} d\tau \\ &+ (R_i \omega_{di} - I_i \sigma_i) \int_0^t e^{\sigma_i(t-\tau)} \sin \omega_{di}(t-\tau) \begin{Bmatrix} \{f_x(\tau)\} \\ \{f_y(\tau)\} \end{Bmatrix} d\tau \end{aligned} \quad (5.119)$$

where

$$R_i = \begin{bmatrix} \{a_{Ri}\}\{a_{Ri}\}^T - \{a_{Li}\}\{a_{Li}\}^T & \{a_{Ri}\}\{b_{Li}\}^T + \{a_{Li}\}\{b_{Ri}\}^T \\ -\{b_{Ri}\}\{a_{Li}\}^T - \{b_{Li}\}\{a_{Ri}\}^T & \{b_{Ri}\}\{b_{Ri}\}^T - \{b_{Li}\}\{b_{Li}\}^T \end{bmatrix} \quad (5.120)$$

$$I_i = \begin{bmatrix} \{a_{Ri}\}\{a_{Li}\}^T - \{a_{Li}\}\{a_{Ri}\}^T & -\{a_{Ri}\}\{b_{Ri}\}^T + \{a_{Li}\}\{b_{Li}\}^T \\ \{b_{Ri}\}\{a_{Ri}\}^T - \{b_{Li}\}\{a_{Li}\}^T & \{b_{Ri}\}\{b_{Li}\}^T + \{b_{Li}\}\{b_{Ri}\}^T \end{bmatrix} \quad (5.121)$$

5.10 Numerical Examples

5.10.1 A Two-Degree of Freedom, Gyroscopic System

Consider the two-degree of freedom, gyroscopic system of reference [39], as shown in Figure 5.1. The equation of motion in the rotating reference axes can be written in the matrix form as that of Eq. 5.77 with

$$\left. \begin{aligned} [M] &= \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \quad [G] = \begin{bmatrix} 0 & -2m\Omega \\ 2m\Omega & 0 \end{bmatrix} \\ [C] &= \begin{bmatrix} c_x & 0 \\ 0 & 0 \end{bmatrix}, \quad [K] + [K_g] = \begin{bmatrix} k_x - m\Omega^2 & 0 \\ 0 & k_y - m\Omega^2 \end{bmatrix} \end{aligned} \right\} \quad (5.122)$$

The values of m , c_x , k_x , and k_y are given in Figure 5.1, while $\Omega = 1 \text{ rad/s}$. From Eq. 5.34, the two coefficient matrices in the eigenvalue problem can be written as follows

$$\begin{aligned} \begin{bmatrix} M & 0 \\ 0 & K + K_g \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \\ \begin{bmatrix} -\tilde{G} & K + K_g \\ K + K_g & 0 \end{bmatrix} &= \begin{bmatrix} 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 4 \\ 2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned} \quad (5.123)$$

The eigenvalues and the associated eigenvectors are summarized in the Table 5.1, where the eigenvectors are normalized according to the scheme Eq. 5.36 and Eq. 5.37.

Obviously, the eigenvalues are the same as those obtained by Meirovitch [39], and no comparison of eigenvectors can be made because not only the eigenvectors here are real (complex in Reference [39]), but also they are normalized in the different scheme.

Computations of the perturbation to the undamped eigensolutions proceed as indicated previously, and the results of the second-order perturbation are summarized in Table 5.2. The perturbed eigenvalues are the same as those obtained by Meirovitch. For the purpose of comparison, the exact solutions of the eigenvalue problem are included. It is interesting to inspect convergence via the ortho-normality check of the perturbed eigenvectors in Table 5.3.

5.10.2 A Multi-Stepped Shaft Rotor System

As another example, consider the multi-stepped shaft rotor system with a single disc in reference [61], shown in Figure 5.2. In this study, the rotor has two identical bearings located at nodes 11 and 15, and is modeled as assembly of 18 finite elements. Hence, the rotor model has a total of 76 degrees of freedom. Table 5.4 presents the rotor data taken from Glasgow [61] in English Units (the corresponding metric data given in Table A.1, Appendix A).

As shown in Figure 5.2, analyses are presented for the following cases:

1. Isotropic Bearings:

$$k_{xx} = k_{yy} = 2.5 \times 10^5 \text{ lb/in } (4.378 \times 10^7 \text{ N/m})$$

$$k_{xy} = k_{yx} = 0.0$$

$$c_{xx} = c_{yy} = 10.0 \text{ lb} \cdot \text{s/in } (1.752 \times 10^3 \text{ N} \cdot \text{s/m})$$

$$c_{xy} = c_{yx} = 0.0$$

2. Orthotropic Bearings

$$k_{xx} = k_{yy} = 2.0 \times 10^5 \text{ lb/in } (4.378 \times 10^7 \text{ N/m})$$

$$k_{xy} = k_{yx} = -0.5 \times 10^5 \text{ lb/in } (-8.756 \times 10^7 \text{ N/m})$$

$$c_{xx} = c_{yy} = 10.0 \text{ lb} \cdot \text{s/in } (1.752 \times 10^3 \text{ N} \cdot \text{s/m})$$

$$c_{xy} = c_{yx} = 0.0$$

The symmetric bearing stiffness matrix is equivalent to orthotropic bearing ($K_1 =$

$2.5 \times 10^5 \text{ lb/in}$, $K_2 = 1.5 \times 10^5 \text{ lb/in}$) with principal axes oriented at $(45^\circ, 135^\circ)$ relative to the z (longitudinal) axis. The undamped whirl frequencies at a running speed of 30,000 *rpm*, and the critical speeds for the isotropic bearing case are tabulated in Tables 5.5 and 5.6, respectively. For the orthotropic bearing case, the corresponding data of the shaft running at 40,000 *rpm* is tabulated in Tables 5.7 and 5.8.

Computations of the perturbations to these undamped eigensolutions have been carried out as described in the previous section, and the results of the eigenvalue solution at various modal truncation levels for both isotropic and orthotropic bearing cases are summarized in Tables 5.9 and 5.16, respectively. For comparison, the solutions of the general theory presented in the last chapter are included as the exact solutions. The first four associated eigenvectors are listed in Table 5.10—Table 5.13 for the isotropic bearing case and Table 5.17—Table 5.20 for the orthotropic bearing case. Examination of Tables 5.9 and 5.16, shows that the perturbed solutions are very accurate at this level of damping. Thus, using eight modes the maximum error in whirl frequency is about 0.2%, and the maximum error in logarithmic decrement is about 3.5%, for both the isotropic and orthotropic cases. Convergence of the imaginary part of the mode vector is slower as can be seen by examination of the tables, because the imaginary elements of a eigenvector are at least one-order smaller than the corresponding real parts and because the first-order perturbation gives the imaginary part of a eigenvector and the second-order perturbation modifies its real part only. Thus, it will be necessary to include the third-order perturbation when higher accuracy in the damping components and imaginary parts of eigenvectors is desired. The results of the orthonormality checks, $\psi_i^T \bar{M} \psi_j = \delta_{ij}$ (Eq. 5.107), of the first 8 damped modes are tabulated in Tables 5.14 and 5.15 for the isotropic bearing case, and in Tables 5.21 and 5.22 for the orthotropic bearing case.

Steady state and transient responses of the rotor to unbalance disc eccentricity at node 5, Figure 5.2, have also been calculated.

First, the steady-state unbalance translational responses of the disc at various rotation speeds are summarized in Table 5.23 for the isotropic bearing case and Table 5.24 for the orthotropic bearing case, respectively. It is seen that even if only the first pair of perturbed modes is retained, the errors in the normalized response of the

disc in both cases are less than 0.5% at a running speed of 20,000 *rpm* and less than 4% at a running speed of 40,000 *rpm*. When retaining the first 8 perturbed modes (i.e., 4 pairs), the errors in responses of the disc in all cases are less than 0.1%. As the number of retained modes increases, the errors in all cases decrease.

Finally, the transient peak translation responses of the disc to a blade loss are tabulated in the Table 5.25. Blade loss is simulated as a step change in the eccentricity of the disc at the node 5. It is seen that even if only the first pair of perturbed modes is retained, the errors in the peak responses of the disc with isotropic or orthotropic bearings are less than 0.51% comparing with the results obtained from a model with 40 damped modes of the exact solution. Again, as the number of retained normal modes increases, the errors in all cases decrease.

Inspection of the tables reveals the high quality of accuracy and of convergence in the damped eigensolution, the steady-state response and transient response analyses. As illustrated, there is actually no need to use all normal modes of a system in the perturbation theory, i.e., the further reduction in calculation can be achieved by using a few normal modes of the system, and still generate satisfactory results.

$$\begin{aligned} m &= 1.00 \text{ kg} & c_x &= 0.1 \text{ N*s/m} \\ k_x &= 3.0 \text{ N/m} & k_y &= 5.0 \text{ N/m} \end{aligned}$$

Figure 5.1 A Two DOF, Gyroscopic System

Figure 5.2 A Multi-Stepped Shaft Rotor System

Table 5.1 Whirl Frequencies (rad/s) and Mode Shapes of a Two-Degree of Freedom, Undamped Gyroscopic System

$\omega_1 = 0.936426$	$\omega_2 = 3.020448$	$\omega_3 = -0.936426$	$\omega_4 = -3.020448$
-0.435162	-0.557345	0.435162	0.557345
0.260956	-0.657192	0.260956	-0.657192
-0.464705	-0.184524	-0.464705	-0.184524
0.278673	-0.217581	-0.278673	0.217581

Table 5.2 Perturbed Whirl Frequencies and Mode Shapes for a Two-Degree of Freedom, Damped Gyroscopic System

Exact Solution		Perturbated Solution	
λ_1	λ_2	λ_1	λ_2
-0.018942 + j 0.936368	-0.031058 + j 3.019857	-0.018937 + j 0.936369	-0.031063 + j 3.019857
-0.435338 - j 0.001332	0.557311 + j 0.010601	-0.435338 - j 0.001330	0.557311 + j 0.010598
0.260888 + j 0.009041	0.657299 - j 0.004817	0.260888 + j 0.009039	0.657299 - j 0.004818
-0.464760 + j 0.007980	0.184565 + j 0.001612	-0.464760 + j 0.007979	0.184565 + j 0.001611
0.278698 + j 0.004018	0.217620 - j 0.003833	0.278698 + j 0.004017	0.217620 - j 0.003833

Table 5.3 Orthonormality Check for the Damped Gyroscopic System
(the matrix: $\Phi^T \tilde{M} \Phi$)

1.000000000	-0.000000006	-0.000000003	0.000000013
-j0.000003434	-j0.000000638	+j0.000000000	+j0.000001210
-0.000000006	1.000000000	0.000000013	-0.000000007
-j0.000000638	-j0.000005372	-j0.000001210	-j0.000000000
-0.000000003	0.000000013	1.000000000	-0.000000006
-j0.000000000	-j0.000001210	+j0.000003434	+j0.000000638
0.000000013	-0.000000007	-0.000000006	1.000000000
+j0.000001210	-j0.000000000	-j0.000000638	-j0.000005372

Table 5.4 FEM DATA for a Multi-Stepped Shaft Rotor

Element Number	Node Number	Length in	Inner Dia. in	Outer Dia. in
1	1 - 2	0.5	0.0	0.4
2	2 - 3	1.5	0.0	0.8
3	3 - 4	1.0	0.0	0.6
4	4 - 5	0.5	0.0	1.6
5	5 - 6	0.5	0.0	1.6
6	6 - 7	0.2	0.0	2.6
7	7 - 8	0.3	1.2	2.6
8	8 - 9	0.5	1.4	2.0
9	9 - 10	0.3	0.0	2.0
10	10 - 11	1.2	0.0	1.0
11	11 - 12	1.0	0.0	1.0
12	12 - 13	1.5	0.0	1.2
13	13 - 14	1.5	0.0	1.2
14	14 - 15	0.8	0.0	1.0
15	15 - 16	0.7	0.0	1.0
16	16 - 17	0.4	0.0	3.0
17	17 - 18	1.2	0.0	1.6
18	18 - 19	0.4	1.2	1.6
$\rho = 0.282 \text{ lbm/in}^3$ $E = 3 \times 10^7 \text{ lb/in}^2$ A single disk located at node 5 with: $M_d = 3.0887 \text{ lbm}$ $I_d = 4.6474 \text{ lbm} \cdot \text{in}^2$ $I_p = 6.9369 \text{ lbm} \cdot \text{in}^2$				

Table 5.5 Undamped Whirl Frequencies (rpm) and Mode Shapes of a Multi-Stepped Shaft Rotor with Isotropic Bearings
(at a running speed = 30000 rpm)

Node	$\omega_{10} = 14733.4$	$\omega_{1f} = 17823.5$	$\omega_{20} = 47275.3$	$\omega_{2f} = 49371.3$	$\omega_{30} = 70461.8$	$\omega_{3f} = 82710.6$
1-1	-.73191E-2	.64753E-2	-.42755E-3	.30522E-3	-.19650E-2	-.26046E-2
2-1	-.67810E-2	.60102E-2	-.37306E-3	.27325E-3	-.16032E-2	-.21271E-2
3-1	-.51710E-2	.46199E-2	-.21197E-3	.17914E-3	-.54073E-3	-.73465E-3
4-1	-.41315E-2	.37349E-2	-.12241E-3	.13065E-3	.97745E-5	-.86464E-4
5-1	-.36306E-2	.33161E-2	-.87355E-4	.11432E-3	.20150E-3	.88124E-4
6-1	-.31319E-2	.28975E-2	-.53188E-4	.98312E-4	.38556E-3	.25648E-3
7-1	-.29335E-2	.27302E-2	-.39858E-4	.92033E-4	.45649E-3	.32205E-3
8-1	-.26360E-2	.24793E-2	-.19948E-4	.82655E-4	.56234E-3	.42003E-3
9-1	-.21423E-2	.20620E-2	.12539E-4	.67432E-4	.73504E-3	.58073E-3
10-1	-.18478E-2	.18126E-2	.31460E-4	.58670E-4	.83607E-3	.67526E-3
11-1	-.85988E-3	.94811E-3	.49915E-4	.69134E-4	.10835E-2	.93035E-3
12-1	-.36013E-3	.48459E-3	-.36682E-4	.16684E-3	.11246E-2	.99551E-3
13-1	.64361E-4	.56227E-4	-.32025E-3	.44476E-3	.98909E-3	.88727E-3
14-1	.32585E-3	-.23899E-3	-.77649E-3	.86628E-3	.68632E-3	.59140E-3
15-1	.43224E-3	-.36975E-3	-.11449E-2	.11972E-2	.46422E-3	.36208E-3
16-1	.52385E-3	-.48319E-3	-.15830E-2	.15852E-2	.24167E-3	.12743E-3
17-1	.57697E-3	-.54859E-3	-.18579E-2	.18264E-2	.10801E-3	-.13345E-4
18-1	.73647E-3	-.74493E-3	-.26875E-2	.25541E-2	-.29449E-3	-.43759E-3
19-1	.78966E-3	-.81040E-3	-.29647E-2	.27973E-2	-.42895E-3	-.57932E-3
1-2	.10764E-2	-.93053E-3	.10912E-3	-.64051E-4	.72508E-3	.95749E-3
2-2	.10755E-2	-.92943E-3	.10859E-3	-.63649E-4	.71968E-3	.94798E-3
3-2	.10680E-2	-.92041E-3	.10450E-3	-.60492E-4	.68160E-3	.88151E-3
4-2	.10024E-2	-.83838E-3	.70365E-4	-.32887E-4	.38563E-3	.35304E-3
5-2	.10013E-2	-.83703E-3	.69810E-4	-.32435E-4	.38108E-3	.34499E-3
6-2	.99276E-3	-.83668E-3	.66747E-4	-.31440E-4	.35537E-3	.32835E-3
7-2	.99217E-3	-.83654E-3	.66543E-4	-.31348E-4	.35394E-3	.32738E-3
8-2	.99110E-3	-.83617E-3	.66184E-4	-.31163E-4	.35172E-3	.32585E-3
9-2	.98341E-3	-.83249E-3	.63700E-4	-.29640E-4	.33933E-3	.31702E-3
10-2	.97933E-3	-.83010E-3	.62421E-4	-.28748E-4	.33424E-3	.31325E-3
11-2	.64894E-3	-.58626E-3	-.35960E-4	.52770E-4	.10668E-3	.12924E-3
12-2	.36201E-3	-.35033E-3	-.14153E-3	.14541E-3	-.31219E-4	-.83253E-5
13-2	.21632E-3	-.23103E-3	-.24136E-3	.22875E-3	-.14994E-3	-.13796E-3
14-2	.14470E-3	-.17276E-3	-.37293E-3	.33874E-3	-.24601E-3	-.24852E-3
15-2	.12862E-3	-.16015E-3	-.55280E-3	.49311E-3	-.30222E-3	-.31705E-3
16-2	.13279E-3	-.16350E-3	-.68706E-3	.60289E-3	-.33407E-3	-.35184E-3
17-2	.13281E-3	-.16351E-3	-.68757E-3	.60329E-3	-.33423E-3	-.35201E-3
18-2	.13297E-3	-.16365E-3	-.69300E-3	.60781E-3	-.33607E-3	-.35427E-3
19-2	.13297E-3	-.16366E-3	-.69314E-3	.60788E-3	-.33616E-3	-.35433E-3

**Table 5.6 Critical Speeds (rpm) and Mode Shapes
of a Multi-Stepped Shaft Rotor with Isotropic Bearings**

Node	Backward modes			Forward Modes		
	$\omega_{1b} = 15470.0$	$\omega_{2b} = 46606.7$	$\omega_{3b} = 64713.8$	$\omega_{1f} = 17158.3$	$\omega_{2f} = 49978.0$	$\omega_{3f} = 96327.6$
1-1	0.98260E-2	-0.63338E-3	-0.21157E-2	0.96752E-2	-0.38053E-3	-0.51801E-2
2-1	0.91071E-2	-0.54837E-3	-0.17105E-2	0.89762E-2	-0.34362E-3	-0.41909E-2
3-1	0.69566E-2	-0.29679E-3	-0.51643E-3	0.68866E-2	-0.23519E-3	-0.13263E-2
4-1	0.55724E-2	-0.15459E-3	0.13569E-3	0.55520E-2	-0.18132E-3	-0.13626E-3
5-1	0.49078E-2	-0.97340E-4	0.38636E-3	0.49175E-2	-0.16466E-3	0.73501E-4
6-1	0.42458E-2	-0.41653E-4	0.62609E-3	0.42841E-2	-0.14835E-3	0.27515E-3
7-1	0.39820E-2	-0.19981E-4	0.71791E-3	0.40312E-2	-0.14196E-3	0.35452E-3
8-1	0.35867E-2	0.12383E-4	0.85482E-3	0.36520E-2	-0.13243E-3	0.47328E-3
9-1	0.29303E-2	0.65182E-4	0.10775E-2	0.30217E-2	-0.11706E-3	0.66903E-3
10-1	0.25387E-2	0.95949E-4	0.12074E-2	0.26451E-2	-0.10833E-3	0.78483E-3
11-1	0.12155E-2	0.13305E-3	0.15072E-2	0.13501E-2	-0.13575E-3	0.11252E-2
12-1	0.53764E-3	0.16169E-4	0.15405E-2	0.66511E-3	-0.28225E-3	0.12358E-2
13-1	-0.49532E-4	-0.38184E-3	0.13526E-2	0.43761E-4	-0.68182E-3	0.11145E-2
14-1	-0.42160E-3	-0.10313E-2	0.97314E-3	-0.37485E-3	-0.12754E-2	0.71761E-3
15-1	-0.57658E-3	-0.15588E-2	0.70702E-3	-0.55743E-3	-0.17361E-2	0.40261E-3
16-1	-0.71046E-3	-0.21876E-2	0.44848E-3	-0.71577E-3	-0.22729E-2	0.83406E-4
17-1	-0.78799E-3	-0.25830E-2	0.29415E-3	-0.80719E-3	-0.26054E-2	-0.10535E-3
18-1	-0.10208E-2	-0.37763E-2	-0.17022E-3	-0.10816E-2	-0.36086E-2	-0.67444E-3
19-1	-0.10984E-2	-0.41751E-2	-0.32532E-3	-0.11732E-2	-0.39437E-2	-0.86450E-3
1-2	-0.14381E-2	0.17023E-3	0.81172E-3	-0.13983E-2	0.73957E-4	0.19847E-2
2-2	-0.14368E-2	0.16946E-3	0.80669E-3	-0.13967E-2	0.73447E-4	0.19598E-2
3-2	-0.14259E-2	0.16355E-3	0.77129E-3	-0.13841E-2	0.69405E-4	0.17902E-2
4-2	-0.13299E-2	0.11488E-3	0.50334E-3	-0.12698E-2	0.33604E-4	0.42943E-3
5-2	-0.13283E-2	0.11409E-3	0.49919E-3	-0.12679E-2	0.33014E-4	0.40884E-3
6-2	-0.13191E-2	0.10853E-3	0.46017E-3	-0.12648E-2	0.31999E-4	0.39724E-3
7-2	-0.13185E-2	0.10818E-3	0.45803E-3	-0.12645E-2	0.31891E-4	0.39648E-3
8-2	-0.13172E-2	0.10757E-3	0.45473E-3	-0.12637E-2	0.31663E-4	0.39525E-3
9-2	-0.13079E-2	0.10355E-3	0.43656E-3	-0.12571E-2	0.29688E-4	0.38771E-3
10-2	-0.13028E-2	0.10154E-3	0.42923E-3	-0.12531E-2	0.28498E-4	0.38426E-3
11-2	-0.87513E-3	-0.44126E-4	0.11494E-3	-0.87161E-3	-0.84045E-4	0.19342E-3
12-2	-0.49582E-3	-0.19621E-3	-0.55161E-4	-0.51278E-3	-0.21222E-3	0.10974E-4
13-2	-0.30341E-3	-0.34157E-3	-0.19466E-3	-0.33119E-3	-0.32518E-3	-0.17708E-3
14-2	-0.20904E-3	-0.53280E-3	-0.30123E-3	-0.24243E-3	-0.47387E-3	-0.33983E-3
15-2	-0.18815E-3	-0.79229E-3	-0.35522E-3	-0.22321E-3	-0.68429E-3	-0.43539E-3
16-2	-0.19383E-3	-0.98827E-3	-0.38571E-3	-0.22854E-3	-0.83113E-3	-0.47181E-3
17-2	-0.19385E-3	-0.98903E-3	-0.38589E-3	-0.22856E-3	-0.83165E-3	-0.47199E-3
18-2	-0.19407E-3	-0.99690E-3	-0.38765E-3	-0.22877E-3	-0.83775E-3	-0.47515E-3
19-2	-0.19407E-3	-0.99714E-3	-0.38777E-3	-0.22878E-3	-0.83782E-3	-0.47514E-3

Table 5.7 Undamped Whirl Frequencies (rpm) and Mode Shapes of a Multi-Stepped Shaft Rotor with Orthotropic Bearings (at a running speed = 40000 rpm)

Node	$\omega_{1b} = 13063.7$	$\omega_{1f} = 17309.8$	$\omega_{2b} = 39891.2$	$\omega_{2f} = 48288.8$	$\omega_{3b} = 62377.1$	$\omega_{3f} = 81852.3$
1-1	-.39232E-2	-.56922E-2	-.99303E-4	.36919E-3	-.76293E-3	.21885E-2
2-1	-.36322E-2	-.52813E-2	-.87195E-4	.32276E-3	-.61257E-3	.17878E-2
3-1	-.27612E-2	-.40529E-2	-.51265E-4	.18562E-3	-.16887E-3	.61884E-3
4-1	-.21953E-2	-.32689E-2	-.30293E-4	.11033E-3	.78602E-4	.70494E-4
5-1	-.19205E-2	-.28965E-2	-.21438E-4	.81494E-4	.17722E-3	-.80430E-4
6-1	-.16473E-2	-.25248E-2	-.12677E-4	.53496E-4	.27142E-3	-.22591E-3
7-1	-.15388E-2	-.23762E-2	-.92065E-5	.42620E-4	.30744E-3	-.28252E-3
8-1	-.13762E-2	-.21536E-2	-.40099E-5	.26387E-4	.36112E-3	-.36710E-3
9-1	-.11065E-2	-.17835E-2	.45706E-5	-.12166E-8	.44839E-3	-.50575E-3
10-1	-.94580E-3	-.15623E-2	.96504E-5	-.15284E-4	.49922E-3	-.58728E-3
11-1	-.41386E-3	-.80060E-3	.22614E-4	-.21169E-4	.61572E-3	-.80525E-3
12-1	-.15153E-3	-.39673E-3	.19292E-4	.72040E-4	.63035E-3	-.85873E-3
13-1	.62864E-4	-.29623E-4	-.10411E-4	.35588E-3	.56422E-3	-.76263E-3
14-1	.18800E-3	.21762E-3	-.70840E-4	.79817E-3	.42786E-3	-.50719E-3
15-1	.23714E-3	.32479E-3	-.12564E-3	.11488E-2	.33315E-3	-.31058E-3
16-1	.28003E-3	.41688E-3	-.19491E-3	.15615E-2	.24248E-3	-.11003E-3
17-1	.30519E-3	.46981E-3	-.23965E-3	.18191E-2	.18827E-3	.10185E-4
18-1	.38078E-3	.62870E-3	-.37472E-3	.25963E-2	.25219E-4	.37243E-3
19-1	.40599E-3	.68167E-3	-.41990E-3	.28560E-2	-.29246E-4	.49344E-3
1-3	-.68055E-2	.32151E-2	-.22789E-3	.19729E-3	-.18662E-2	-.15156E-2
2-3	-.63364E-2	.30087E-2	-.22043E-3	.16110E-3	-.15428E-2	-.12461E-2
3-3	-.49324E-2	.23920E-2	-.19903E-3	.53666E-4	-.58961E-3	-.46016E-3
4-3	-.40213E-2	.20002E-2	-.19300E-3	-.10323E-4	-.74353E-4	-.96060E-4
5-3	-.35796E-2	.18153E-2	-.19466E-3	-.38300E-4	.11918E-3	.89972E-7
6-3	-.31395E-2	.16298E-2	-.19661E-3	-.65662E-4	.30698E-3	.93969E-4
7-3	-.29641E-2	.15553E-2	-.19751E-3	-.76377E-4	.38007E-3	.13103E-3
8-3	-.27012E-2	.14435E-2	-.19891E-3	-.92403E-4	.48929E-3	.18650E-3
9-3	-.22646E-2	.12571E-2	-.20168E-3	-.11880E-3	.66832E-3	.27804E-3
10-3	-.20039E-2	.11453E-2	-.20375E-3	-.13443E-3	.77357E-3	.33226E-3
11-3	-.11001E-2	.73194E-3	-.26178E-3	-.18457E-3	.10448E-2	.49089E-3
12-3	-.58580E-3	.46928E-3	-.40672E-3	-.21379E-3	.10828E-2	.53509E-3
13-3	-.54364E-4	.16375E-3	-.75514E-3	-.24012E-3	.91624E-3	.47322E-3
14-3	.35796E-3	-.99087E-4	-.12334E-2	-.24863E-3	.57023E-3	.29766E-3
15-3	.55463E-3	-.23164E-3	-.15804E-2	-.24555E-3	.31413E-3	.15856E-3
16-3	.72676E-3	-.34838E-3	-.19690E-2	-.23773E-3	.49441E-4	.13143E-4
17-3	.82591E-3	-.41545E-3	-.22085E-2	-.23120E-3	-.11040E-3	-.74248E-4
18-3	.11235E-2	-.61678E-3	-.29304E-2	-.21137E-3	-.59180E-3	-.33769E-3
19-3	.12228E-2	-.68389E-3	-.31715E-2	-.20469E-3	-.75260E-3	-.42569E-3

**Table 5.8 Critical Speeds (rpm) and Mode Shapes
of a Multi-Stepped Shaft Rotor with Orthotropic Bearings**

Node	$\omega_{1b} = 13924.8$	$\omega_{1f} = 16508.1$	$\omega_{2b} = 39892.0$	$\omega_{2f} = 48241.9$	$\omega_{3b} = 59572.2$	$\omega_{3f} = 93042.7$
1-1	.28325E-2	.93288E-2	.13957E-3	-.52115E-3	.96874E-3	-.38245E-2
2-1	.26234E-2	.86513E-2	.12255E-3	-.45317E-3	.76639E-3	-.31026E-2
3-1	.19976E-2	.66254E-2	.72063E-4	-.25227E-3	.16809E-3	-.10086E-2
4-1	.15922E-2	.53274E-2	.42594E-4	-.14097E-3	-.17515E-3	-.11458E-3
5-1	.13961E-2	.47079E-2	.30152E-4	-.97646E-4	-.31825E-3	.63876E-4
6-1	.12011E-2	.40900E-2	.17842E-4	-.55699E-4	-.45458E-3	.23567E-3
7-1	.11235E-2	.38435E-2	.12965E-4	-.39456E-4	-.50650E-3	.30307E-3
8-1	.10072E-2	.34739E-2	.56630E-5	-.15220E-4	-.58386E-3	.40387E-3
9-1	.81438E-3	.28599E-2	-.63941E-5	.24131E-4	-.70939E-3	.56977E-3
10-1	.69943E-3	.24933E-2	-.13532E-4	.46900E-4	-.78237E-3	.66774E-3
11-1	.31629E-3	.12415E-2	-.31751E-4	.56758E-4	-.94505E-3	.94783E-3
12-1	.12488E-3	.58791E-3	-.27086E-4	-.75604E-4	-.96461E-3	.10315E-2
13-1	-.34855E-4	.58328E-5	.14650E-4	-.47859E-3	-.87727E-3	.92235E-3
14-1	-.13111E-3	-.37694E-3	.99561E-4	-.11055E-2	-.70122E-3	.59265E-3
15-1	-.16992E-3	-.54082E-3	.17657E-3	-.16012E-2	-.58447E-3	.33461E-3
16-1	-.20380E-3	-.68249E-3	.27391E-3	-.21834E-2	-.47874E-3	.73787E-4
17-1	-.22360E-3	-.76432E-3	.33678E-3	-.25467E-2	-.41624E-3	-.80591E-4
18-1	-.28308E-3	-.10100E-2	.52658E-3	-.36427E-2	-.22843E-3	-.54593E-3
19-1	-.30292E-3	-.10919E-2	.59006E-3	-.40089E-2	-.16570E-3	-.70134E-3
1-3	.10211E-1	-.29182E-2	.32097E-3	-.34003E-3	.21654E-2	.33671E-2
2-3	.95156E-2	-.27277E-2	.31049E-3	-.27744E-3	.17861E-2	.27413E-2
3-3	.74333E-2	-.21582E-2	.28042E-3	-.91594E-4	.66651E-3	.92595E-3
4-3	.60867E-2	-.17947E-2	.27198E-3	.19176E-4	.47797E-4	.15621E-3
5-3	.54367E-2	-.16221E-2	.27434E-3	.67660E-4	-.19368E-3	.81264E-5
6-3	.47884E-2	-.14492E-2	.27712E-3	.11506E-3	-.42765E-3	-.13632E-3
7-3	.45298E-2	-.13799E-2	.27840E-3	.13361E-3	-.51850E-3	-.19384E-3
8-3	.41420E-2	-.12759E-2	.28039E-3	.16136E-3	-.65421E-3	-.28002E-3
9-3	.34975E-2	-.11027E-2	.28432E-3	.20706E-3	-.87640E-3	-.42283E-3
10-3	.31125E-2	-.99891E-3	.28724E-3	.23409E-3	-.10068E-2	-.50778E-3
11-3	.17637E-2	-.62098E-3	.36905E-3	.32033E-3	-.13344E-2	-.77319E-3
12-3	.98143E-3	-.38655E-3	.57331E-3	.36988E-3	-.13692E-2	-.86236E-3
13-3	.15460E-3	-.12022E-3	.10642E-2	.41353E-3	-.11461E-2	-.77644E-3
14-3	-.50094E-3	.10465E-3	.17382E-2	.42631E-3	-.70664E-3	-.48629E-3
15-3	-.81764E-3	.21702E-3	.22271E-2	.42002E-3	-.38681E-3	-.25162E-3
16-3	-.10954E-2	.31595E-3	.27746E-2	.40576E-3	-.58593E-4	-.84893E-5
17-3	-.12554E-2	.37283E-3	.31120E-2	.39411E-3	.13923E-3	.13613E-3
18-3	-.17355E-2	.54356E-3	.41291E-2	.35875E-3	.73493E-3	.57223E-3
19-3	-.18956E-2	.60048E-3	.44689E-2	.34683E-3	.93387E-3	.71788E-3

**Table 5.9 Perturbed Whirl Frequencies of a Stepped-Shaft Rotor
with Isotropic Bearings at Various Modal Truncation Levels
(at a running speed = 30000 rpm)**

Mode No.	Exact Solution	Whirl Frequencies (rpm) / Log Decrements				
		8 modes	16 modes	24 modes	36 modes	76 modes
1b	14739/.0897	14738/.0898	14739/.0898	14739/.0898	14739/.0898	14739/.0898
1f	17833/.1214	17829/.1214	17833/.1214	17833/.1214	17833/.1214	17833/.1214
2b	47373/.4086	47274/.4086	47369/.4077	47370/.4077	47371/.4077	47371/.4077
2f	49443/.4687	49381/.4671	49439/.4665	49440/.4665	49441/.4665	49442/.4665
3b	70041/.6521	69992/.6486	70034/.6482	70047/.6480	70050/.6480	70050/.6480
3f	82479/.5497	82413/.5444	82478/.5439	82491/.5438	82493/.5438	82494/.5438
4b	120400/.0641	120317/.0657	120393/.0656	120395/.0656	120396/.0656	120396/.0656
4f	124287/.0933	124244/.0967	124272/.0966	124276/.0966	124278/.0966	124278/.0966
5b	159794/.2613		159778/.2605	159789/.2605	159794/.2605	159796/.2605
5f	182577/.2542		182556/.2531	182572/.2531	182578/.2531	182579/.2531
6b	341804/.1490		341755/.1493	341790/.1493	341802/.1493	341804/.1493
6f	358959/.1422		358911/.1426	358944/.1426	358957/.1426	358960/.1426
7b	634586/.1597		634239/.1596	634551/.1596	634585/.1596	634588/.1596
7f	650779/.1557		650425/.1556	650741/.1555	650777/.1555	650780/.1555
8b	902629/34E-6		902629/34E-6	902629/34E-6	902629/34E-6	902629/34E-6
8f	909330/15E-6		909329/15E-6	909329/15E-6	909329/15E-6	909329/15E-6

**Table 5.10 The First Backward Whirl Mode Shape of
a Multi-Stepped Shaft Rotor with Isotropic Bearings
at Various Normal Modal Truncation Levels
(at a running speed = 30000 rpm)**

Node No.	Exact Solution		8 Kept Modes		16 Kept Modes		No Modal Truncation	
	real	imag.	real	imag.	real	imag.	real	imag.
1-1	-.7321E-2	-.9305E-5	-.7320E-2	-.2059E-6	-.7321E-2	-.9193E-5	-.7321E-2	-.9085E-5
2-1	-.6782E-2	-.1239E-4	-.6782E-2	-.5567E-5	-.6782E-2	-.1218E-4	-.6782E-2	-.1218E-4
3-1	-.5172E-2	-.2152E-4	-.5172E-2	-.2111E-4	-.5172E-2	-.2139E-4	-.5172E-2	-.2137E-4
4-1	-.4132E-2	-.2673E-4	-.4132E-2	-.2773E-4	-.4132E-2	-.2661E-4	-.4132E-2	-.2661E-4
5-1	-.3631E-2	-.2885E-4	-.3631E-2	-.2905E-4	-.3631E-2	-.2875E-4	-.3631E-2	-.2874E-4
6-1	-.3132E-2	-.3092E-4	-.3132E-2	-.3020E-4	-.3132E-2	-.3082E-4	-.3132E-2	-.3082E-4
7-1	-.2933E-2	-.3172E-4	-.2933E-2	-.3061E-4	-.2933E-2	-.3163E-4	-.2933E-2	-.3163E-4
8-1	-.2636E-2	-.3292E-4	-.2636E-2	-.3121E-4	-.2636E-2	-.3284E-4	-.2636E-2	-.3284E-4
9-1	-.2142E-2	-.3488E-4	-.2142E-2	-.3213E-4	-.2142E-2	-.3481E-4	-.2142E-2	-.3481E-4
10-1	-.1847E-2	-.3601E-4	-.1847E-2	-.3263E-4	-.1847E-2	-.3597E-4	-.1847E-2	-.3596E-4
11-1	-.8591E-3	-.3647E-4	-.8592E-3	-.3156E-4	-.8591E-3	-.3643E-4	-.8591E-3	-.3645E-4
12-1	-.3594E-3	-.2996E-4	-.3594E-3	-.2746E-4	-.3594E-3	-.2997E-4	-.3594E-3	-.2995E-4
13-1	.6469E-4	-.1308E-4	.6493E-4	-.1627E-4	.6469E-4	-.1309E-4	.6469E-4	-.1308E-4
14-1	.3257E-3	.7609E-5	.3261E-3	.8387E-6	.3257E-3	.7622E-5	.3257E-3	.7603E-5
15-1	.4318E-3	.1963E-4	.4323E-3	.1339E-4	.4318E-3	.1959E-4	.4318E-3	.1962E-4
16-1	.5231E-3	.3045E-4	.5237E-3	.2705E-4	.5231E-3	.3045E-4	.5231E-3	.3045E-4
17-1	.5761E-3	.3668E-4	.5767E-3	.3542E-4	.5761E-3	.3667E-4	.5761E-3	.3667E-4
18-1	.7352E-3	.5538E-4	.7357E-3	.6066E-4	.7352E-3	.5537E-4	.7352E-3	.5537E-4
19-1	.7882E-3	.6161E-4	.7888E-3	.6909E-4	.7882E-3	.6160E-4	.7882E-3	.6160E-4
1-2	.1077E-2	-.6172E-5	.1077E-2	-.1075E-4	.1077E-2	-.5914E-5	.1077E-2	-.6205E-5
2-2	.1076E-2	-.6148E-5	.1076E-2	-.1063E-4	.1076E-2	-.6125E-5	.1076E-2	-.6181E-5
3-2	.1068E-2	-.5951E-5	.1068E-2	-.9735E-5	.1068E-2	-.6028E-5	.1068E-2	-.5983E-5
4-2	.1003E-2	-.4249E-5	.1003E-2	-.2688E-5	.1003E-2	-.4275E-5	.1003E-2	-.4279E-5
5-2	.1002E-2	-.4220E-5	.1002E-2	-.2579E-5	.1002E-2	-.4251E-5	.1002E-2	-.4251E-5
6-2	.9931E-3	-.4028E-5	.9930E-3	-.2048E-5	.9931E-3	-.4046E-5	.9931E-3	-.4058E-5
7-2	.9925E-3	-.4015E-5	.9925E-3	-.2019E-5	.9925E-3	-.4034E-5	.9925E-3	-.4045E-5
8-2	.9915E-3	-.3991E-5	.9914E-3	-.1973E-5	.9915E-3	-.4017E-5	.9915E-3	-.4021E-5
9-2	.9838E-3	-.3823E-5	.9837E-3	-.1719E-5	.9838E-3	-.3891E-5	.9838E-3	-.3853E-5
10-2	.9797E-3	-.3735E-5	.9796E-3	-.1616E-5	.9797E-3	-.3811E-5	.9797E-3	-.3764E-5
11-2	.6490E-3	.3329E-5	.6490E-3	.2758E-5	.6490E-3	.3305E-5	.6490E-3	.3310E-5
12-2	.3618E-3	.9492E-5	.3620E-3	.5773E-5	.3618E-3	.9559E-5	.3618E-3	.9483E-5
13-2	.2160E-3	.1277E-4	.2162E-3	.9341E-5	.2160E-3	.1270E-4	.2160E-3	.1276E-4
14-2	.1443E-3	.1458E-4	.1445E-3	.1348E-4	.1443E-3	.1458E-4	.1443E-3	.1458E-4
15-2	.1282E-3	.1533E-4	.1283E-3	.1787E-4	.1283E-3	.1532E-4	.1283E-3	.1533E-4
16-2	.1324E-3	.1557E-4	.1324E-3	.2093E-4	.1324E-3	.1557E-4	.1324E-3	.1557E-4
17-2	.1324E-3	.1557E-4	.1325E-3	.2094E-4	.1324E-3	.1557E-4	.1324E-3	.1557E-4
18-2	.1326E-3	.1558E-4	.1326E-3	.2108E-4	.1326E-3	.1558E-4	.1326E-3	.1558E-4
19-2	.1326E-3	.1558E-4	.1326E-3	.2108E-4	.1326E-3	.1558E-4	.1326E-3	.1558E-4

**Table 5.11 The First Forward Whirl Mode Shape of
a Multi-Stepped Shaft Rotor with Isotropic Bearings
at Various Normal Modal Truncation Levels
(at a running speed = 30000 rpm)**

Node No.	Exact Solution		8 Kept Modes		16 Kept Modes		No Modal Truncation	
	real	imag.	real	imag.	real	imag.	real	imag.
1-1	.6477E-2	-.2903E-5	.6477E-2	.7561E-5	.6477E-2	-.2800E-5	.6477E-2	-.2662E-5
2-1	.6012E-2	-.7494E-5	.6012E-2	.3303E-6	.6012E-2	-.7263E-5	.6012E-2	-.7268E-5
3-1	.4621E-2	-.2107E-4	.4621E-2	-.2068E-4	.4621E-2	-.2092E-4	.4621E-2	-.2089E-4
4-1	.3735E-2	-.2858E-4	.3735E-2	-.2981E-4	.3735E-2	-.2844E-4	.3735E-2	-.2843E-4
5-1	.3316E-2	-.3147E-4	.3316E-2	-.3176E-4	.3316E-2	-.3134E-4	.3316E-2	-.3134E-4
6-1	.2898E-2	-.3434E-4	.2898E-2	-.3353E-4	.2898E-2	-.3422E-4	.2898E-2	-.3422E-4
7-1	.2730E-2	-.3547E-4	.2730E-2	-.3418E-4	.2730E-2	-.3536E-4	.2730E-2	-.3536E-4
8-1	.2479E-2	-.3718E-4	.2479E-2	-.3515E-4	.2479E-2	-.3707E-4	.2479E-2	-.3707E-4
9-1	.2062E-2	-.3998E-4	.2062E-2	-.3668E-4	.2062E-2	-.3989E-4	.2062E-2	-.3989E-4
10-1	.1812E-2	-.4162E-4	.1812E-2	-.3754E-4	.1812E-2	-.4156E-4	.1812E-2	-.4155E-4
11-1	.9473E-3	-.4372E-4	.9477E-3	-.3760E-4	.9473E-3	-.4364E-4	.9473E-3	-.4368E-4
12-1	.4838E-3	-.3712E-4	.4841E-3	-.3394E-4	.4838E-3	-.3712E-4	.4838E-3	-.3709E-4
13-1	.5581E-4	-.1847E-4	.5578E-4	-.2221E-4	.5581E-4	-.1846E-4	.5581E-4	-.1846E-4
14-1	-.2389E-3	.4757E-5	-.2393E-3	-.3200E-5	-.2389E-3	.4768E-5	-.2389E-3	.4747E-5
15-1	-.3694E-3	.1820E-4	-.3699E-3	.1092E-4	-.3694E-3	.1816E-4	-.3694E-3	.1818E-4
16-1	-.4826E-3	.3017E-4	-.4832E-3	.2621E-4	-.4826E-3	.3014E-4	-.4826E-3	.3014E-4
17-1	-.5478E-3	.3701E-4	-.5485E-3	.3556E-4	-.5478E-3	.3698E-4	-.5478E-3	.3698E-4
18-1	-.7438E-3	.5755E-4	-.7445E-3	.6375E-4	-.7438E-3	.5750E-4	-.7438E-3	.5750E-4
19-1	-.8091E-3	.6439E-4	-.8099E-3	.7316E-4	-.8091E-3	.6434E-4	-.8091E-3	.6434E-4
1-2	-.9310E-3	-.9191E-5	-.9310E-3	-.1450E-4	-.9310E-3	-.8850E-5	-.9310E-3	-.9223E-5
2-2	-.9299E-3	-.9149E-5	-.9299E-3	-.1434E-4	-.9299E-3	-.9109E-5	-.9299E-3	-.9181E-5
3-2	-.9209E-3	-.8811E-5	-.9208E-3	-.1320E-4	-.9209E-3	-.8900E-5	-.9209E-3	-.8843E-5
4-2	-.8388E-3	-.5806E-5	-.8387E-3	-.3963E-5	-.8388E-3	-.5831E-5	-.8388E-3	-.5837E-5
5-2	-.8374E-3	-.5757E-5	-.8373E-3	-.3819E-5	-.8374E-3	-.5788E-5	-.8374E-3	-.5787E-5
6-2	-.8371E-3	-.5691E-5	-.8370E-3	-.3276E-5	-.8371E-3	-.5705E-5	-.8371E-3	-.5721E-5
7-2	-.8369E-3	-.5683E-5	-.8368E-3	-.3245E-5	-.8369E-3	-.5700E-5	-.8369E-3	-.5713E-5
8-2	-.8366E-3	-.5667E-5	-.8364E-3	-.3197E-5	-.8366E-3	-.5692E-5	-.8366E-3	-.5697E-5
9-2	-.8329E-3	-.5526E-5	-.8327E-3	-.2926E-5	-.8329E-3	-.5605E-5	-.8329E-3	-.5556E-5
10-2	-.8305E-3	-.5440E-5	-.8304E-3	-.2814E-5	-.8305E-3	-.5530E-5	-.8305E-3	-.5470E-5
11-2	-.5864E-3	.2641E-5	-.5864E-3	.2043E-5	-.5864E-3	.2604E-5	-.5864E-3	.2618E-5
12-2	-.3502E-3	.1028E-4	-.3503E-3	.5714E-5	-.3502E-3	.1037E-4	-.3502E-3	.1027E-4
13-2	-.2307E-3	.1427E-4	-.2310E-3	.1016E-4	-.2307E-3	.1418E-4	-.2307E-3	.1426E-4
14-2	-.1724E-3	.1637E-4	-.1726E-3	.1514E-4	-.1724E-3	.1639E-4	-.1724E-3	.1636E-4
15-2	-.1598E-3	.1705E-4	-.1599E-3	.2006E-4	-.1598E-3	.1702E-4	-.1598E-3	.1704E-4
16-2	-.1632E-3	.1711E-4	-.1633E-3	.2337E-4	-.1632E-3	.1710E-4	-.1632E-3	.1710E-4
17-2	-.1632E-3	.1711E-4	-.1633E-3	.2338E-4	-.1632E-3	.1710E-4	-.1632E-3	.1710E-4
18-2	-.1633E-3	.1712E-4	-.1634E-3	.2354E-4	-.1633E-3	.1710E-4	-.1633E-3	.1710E-4
19-2	-.1633E-3	.1711E-4	-.1634E-3	.2355E-4	-.1633E-3	.1710E-4	-.1633E-3	.1710E-4

**Table 5.12 The Second Backward Whirl Mode Shape of
a Multi-Stepped Shaft Rotor with Isotropic Bearings
at Various Normal Modal Truncation Levels
(at a running speed = 30000 rpm)**

Node No.	Exact Solution		8 Kept Modes		16 Kept Modes		No Modal Truncation	
	real	imag.	real	imag.	real	imag.	real	imag.
1-1	-.4422E-3	.3968E-4	-.4470E-3	-.3029E-5	-.4418E-3	.4074E-4	-.4419E-3	.4056E-4
2-1	-.3850E-3	.2747E-4	-.3891E-3	-.4161E-5	-.3847E-3	.2826E-4	-.3848E-3	.2827E-4
3-1	-.2161E-3	-.8655E-5	-.2178E-3	-.9406E-5	-.2160E-3	-.8067E-5	-.2160E-3	-.8100E-5
4-1	-.1217E-3	-.2937E-4	-.1227E-3	-.2410E-4	-.1218E-3	-.2900E-4	-.1218E-3	-.2900E-4
5-1	-.8445E-4	-.3797E-4	-.8552E-4	-.3711E-4	-.8458E-4	-.3771E-4	-.8456E-4	-.3771E-4
6-1	-.4812E-4	-.4634E-4	-.4929E-4	-.4983E-4	-.4830E-4	-.4621E-4	-.4828E-4	-.4620E-4
7-1	-.3394E-4	-.4961E-4	-.3516E-4	-.5476E-4	-.3414E-4	-.4953E-4	-.3411E-4	-.4952E-4
8-1	-.1276E-4	-.5450E-4	-.1406E-4	-.6213E-4	-.1299E-4	-.5450E-4	-.1295E-4	-.5447E-4
9-1	.2184E-4	-.6257E-4	.2039E-4	-.7419E-4	.2155E-4	-.6268E-4	.2160E-4	-.6265E-4
10-1	.4200E-4	-.6736E-4	.4045E-4	-.8128E-4	.4169E-4	-.6751E-4	.4174E-4	-.6751E-4
11-1	.6375E-4	-.8570E-4	.6181E-4	-.1012E-3	.6333E-4	-.8592E-4	.6341E-4	-.8601E-4
12-1	-.2283E-4	-.1028E-3	-.2434E-4	-.1096E-3	-.2326E-4	-.1030E-3	-.2320E-4	-.1029E-3
13-1	-.3093E-3	-.1230E-3	-.3099E-3	-.1091E-3	-.3098E-3	-.1225E-3	-.3097E-3	-.1226E-3
14-1	-.7705E-3	-.1271E-3	-.7706E-3	-.9555E-4	-.7708E-3	-.1262E-3	-.7708E-3	-.1261E-3
15-1	-.1142E-2	-.1155E-3	-.1142E-2	-.8431E-4	-.1143E-2	-.1134E-3	-.1143E-2	-.1139E-3
16-1	-.1584E-2	-.9217E-4	-.1584E-2	-.7363E-4	-.1584E-2	-.9014E-4	-.1584E-2	-.9006E-4
17-1	-.1861E-2	-.7594E-4	-.1861E-2	-.6721E-4	-.1862E-2	-.7355E-4	-.1862E-2	-.7350E-4
18-1	-.2698E-2	-.2665E-4	-.2698E-2	-.4787E-4	-.2698E-2	-.2325E-4	-.2698E-2	-.2322E-4
19-1	-.2977E-2	-.1012E-4	-.2977E-2	-.4140E-4	-.2977E-2	-.6410E-5	-.2977E-2	-.6368E-5
1-2	.1144E-3	-.2445E-4	.1160E-3	-.2146E-5	.1142E-3	-.2512E-4	.1143E-3	-.2461E-4
2-2	.1139E-3	-.2433E-4	.1155E-3	-.2610E-5	.1137E-3	-.2459E-4	.1137E-3	-.2449E-4
3-2	.1097E-3	-.2350E-4	.1111E-3	-.5508E-5	.1095E-3	-.2359E-4	.1096E-3	-.2367E-4
4-2	.7475E-4	-.1724E-4	.7465E-4	-.2589E-4	.7464E-4	-.1747E-4	.7466E-4	-.1746E-4
5-2	.7418E-4	-.1715E-4	.7406E-4	-.2617E-4	.7407E-4	-.1736E-4	.7409E-4	-.1737E-4
6-2	.7101E-4	-.1637E-4	.7075E-4	-.2470E-4	.7090E-4	-.1664E-4	.7093E-4	-.1660E-4
7-2	.7080E-4	-.1633E-4	.7054E-4	-.2462E-4	.7069E-4	-.1660E-4	.7072E-4	-.1656E-4
8-2	.7043E-4	-.1628E-4	.7016E-4	-.2449E-4	.7032E-4	-.1653E-4	.7034E-4	-.1650E-4
9-2	.6787E-4	-.1601E-4	.6755E-4	-.2377E-4	.6777E-4	-.1618E-4	.6778E-4	-.1623E-4
10-2	.6655E-4	-.1592E-4	.6621E-4	-.2348E-4	.6646E-4	-.1607E-4	.6647E-4	-.1614E-4
11-2	-.3469E-4	-.1630E-4	-.3467E-4	-.1171E-4	-.3473E-4	-.1648E-4	-.3473E-4	-.1629E-4
12-2	-.1427E-3	-.1638E-4	-.1420E-3	-.4071E-5	-.1428E-3	-.1606E-4	-.1427E-3	-.1611E-4
13-2	-.2440E-3	-.9254E-5	-.2435E-3	.4930E-5	-.2440E-3	-.8985E-5	-.2440E-3	-.8843E-5
14-2	-.3768E-3	.4708E-5	-.3766E-3	.1237E-4	-.3767E-3	.5745E-5	-.3767E-3	.5252E-5
15-2	-.5578E-3	.2492E-4	-.5577E-3	.1489E-4	-.5576E-3	.2543E-4	-.5577E-3	.2563E-4
16-2	-.6928E-3	.4055E-4	-.6927E-3	.1603E-4	-.6926E-3	.4143E-4	-.6927E-3	.4137E-4
17-2	-.6933E-3	.4061E-4	-.6932E-3	.1604E-4	-.6932E-3	.4151E-4	-.6932E-3	.4144E-4
18-2	-.6988E-3	.4129E-4	-.6987E-3	.1617E-4	-.6986E-3	.4209E-4	-.6986E-3	.4212E-4
19-2	-.6989E-3	.4131E-4	-.6989E-3	.1618E-4	-.6988E-3	.4211E-4	-.6988E-3	.4214E-4

**Table 5.13 The Second Forward Whirl Mode Shape of
a Multi-Stepped Shaft Rotor with Isotropic Bearings
at Various Normal Modal Truncation Levels
(at a running speed = 30000 rpm)**

Node No.	Exact Solution		8 Kept Modes		16 Kept Modes		No Modal Truncation	
	real	imag.	real	imag.	real	imag.	real	imag.
1-1	.3147E-3	.6225E-4	.3218E-3	.2320E-4	.3144E-3	.6203E-4	.3145E-3	.6186E-4
2-1	.2814E-3	.4845E-4	.2873E-3	.1966E-4	.2812E-3	.4818E-4	.2812E-3	.4818E-4
3-1	.1832E-3	.7658E-5	.1858E-3	.7416E-5	.1832E-3	.7769E-5	.1832E-3	.7739E-5
4-1	.1321E-3	-.1525E-4	.1333E-3	-.1001E-4	.1321E-3	-.1500E-4	.1321E-3	-.1500E-4
5-1	.1145E-3	-.2440E-4	.1155E-3	-.2329E-4	.1145E-3	-.2410E-4	.1145E-3	-.2411E-4
6-1	.9724E-4	-.3346E-4	.9803E-4	-.3639E-4	.9730E-4	-.3314E-4	.9729E-4	-.3313E-4
7-1	.9045E-4	-.3707E-4	.9118E-4	-.4153E-4	.9052E-4	-.3673E-4	.9051E-4	-.3671E-4
8-1	.8031E-4	-.4247E-4	.8094E-4	-.4921E-4	.8039E-4	-.4212E-4	.8037E-4	-.4209E-4
9-1	.6381E-4	-.5145E-4	.6432E-4	-.6187E-4	.6392E-4	-.5106E-4	.6390E-4	-.5103E-4
10-1	.5429E-4	-.5682E-4	.5473E-4	-.6935E-4	.5442E-4	-.5639E-4	.5440E-4	-.5638E-4
11-1	.6268E-4	-.7832E-4	.6295E-4	-.9224E-4	.6288E-4	-.7766E-4	.6283E-4	-.7777E-4
12-1	.1606E-3	-.9660E-4	.1602E-3	-.1033E-3	.1608E-3	-.9606E-4	.1607E-3	-.9593E-4
13-1	.4407E-3	-.1169E-3	.4392E-3	-.1051E-3	.4408E-3	-.1159E-3	.4408E-3	-.1160E-3
14-1	.8653E-3	-.1208E-3	.8635E-3	-.9192E-4	.8654E-3	-.1199E-3	.8653E-3	-.1198E-3
15-1	.1198E-2	-.1090E-3	.1197E-2	-.8027E-4	.1198E-2	-.1075E-3	.1198E-2	-.1080E-3
16-1	.1588E-2	-.8568E-4	.1587E-2	-.6923E-4	.1588E-2	-.8474E-4	.1588E-2	-.8466E-4
17-1	.1831E-2	-.6965E-4	.1830E-2	-.6272E-4	.1831E-2	-.6868E-4	.1831E-2	-.6864E-4
18-1	.2562E-2	-.2097E-4	.2562E-2	-.4311E-4	.2562E-2	-.2000E-4	.2562E-2	-.1998E-4
19-1	.2806E-2	-.4659E-5	.2807E-2	-.3655E-4	.2806E-2	-.3707E-5	.2806E-2	-.3671E-5
1-2	-.6671E-4	-.2763E-4	-.6906E-4	-.6971E-5	-.6658E-4	-.2784E-4	-.6666E-4	-.2739E-4
2-2	-.6631E-4	-.2749E-4	-.6863E-4	-.7388E-5	-.6623E-4	-.2734E-4	-.6626E-4	-.2726E-4
3-2	-.6315E-4	-.2649E-4	-.6526E-4	-.9883E-5	-.6309E-4	-.2620E-4	-.6310E-4	-.2627E-4
4-2	-.3544E-4	-.1836E-4	-.3591E-4	-.2645E-4	-.3538E-4	-.1827E-4	-.3539E-4	-.1827E-4
5-2	-.3499E-4	-.1823E-4	-.3543E-4	-.2667E-4	-.3493E-4	-.1814E-4	-.3494E-4	-.1814E-4
6-2	-.3400E-4	-.1803E-4	-.3430E-4	-.2572E-4	-.3394E-4	-.1798E-4	-.3395E-4	-.1794E-4
7-2	-.3391E-4	-.1802E-4	-.3420E-4	-.2567E-4	-.3385E-4	-.1797E-4	-.3386E-4	-.1793E-4
8-2	-.3372E-4	-.1800E-4	-.3401E-4	-.2558E-4	-.3366E-4	-.1795E-4	-.3367E-4	-.1791E-4
9-2	-.3218E-4	-.1792E-4	-.3241E-4	-.2505E-4	-.3213E-4	-.1780E-4	-.3213E-4	-.1784E-4
10-2	-.3127E-4	-.1790E-4	-.3149E-4	-.2482E-4	-.3122E-4	-.1776E-4	-.3122E-4	-.1782E-4
11-2	.5204E-4	-.1853E-4	.5166E-4	-.1471E-4	.5206E-4	-.1865E-4	.5206E-4	-.1842E-4
12-2	.1464E-3	-.1676E-4	.1456E-3	-.6186E-5	.1464E-3	-.1653E-4	.1464E-3	-.1663E-4
13-2	.2305E-3	-.9129E-5	.2301E-3	.4077E-5	.2305E-3	-.9209E-5	.2305E-3	-.9021E-5
14-2	.3410E-3	.4854E-5	.3411E-3	.1262E-4	.3410E-3	.5492E-5	.3410E-3	.4921E-5
15-2	.4959E-3	.2519E-4	.4963E-3	.1555E-4	.4958E-3	.2498E-4	.4958E-3	.2521E-4
16-2	.6059E-3	.4005E-4	.6066E-3	.1624E-4	.6058E-3	.4011E-4	.6058E-3	.4004E-4
17-2	.6063E-3	.4011E-4	.6070E-3	.1625E-4	.6062E-3	.4018E-4	.6062E-3	.4009E-4
18-2	.6108E-3	.4077E-4	.6115E-3	.1640E-4	.6107E-3	.4073E-4	.6107E-3	.4075E-4
19-2	.6109E-3	.4078E-4	.6116E-3	.1640E-4	.6108E-3	.4073E-4	.6108E-3	.4077E-4

Table 5.14 Orthonormality Check (8 Kept Modes)
of a Multi-stepped Rotor with Isotropic Bearings
 (at a running speed = 30000 rpm)

(a) Real Parts of the Modal Mass: $\psi_i^T \bar{M} \psi_j$

Mode	1-b	1-f	2-b	2-f	3-b	3-f	4-b	4-f
1-b	1.000	-.5792E-6	-.2829E-4	.2183E-4	.8564E-4	.5227E-4	-.4548E-4	-.5007E-4
1-f	-.5792E-6	1.000	-.1696E-4	.3887E-4	.9710E-4	.4103E-4	-.4606E-4	-.7096E-4
2-b	-.2829E-4	-.1696E-4	1.000	-.1489E-4	-.1258E-3	-.6301E-4	.1497E-3	.1912E-3
2-f	.2183E-4	.3887E-4	-.1489E-4	1.000	-.6363E-4	-.1494E-3	.9050E-4	.9633E-4
3-b	.8564E-4	.9710E-4	-.1258E-3	-.6363E-4	1.000	-.5223E-4	.6514E-5	.1496E-4
3-f	.5227E-4	.4103E-4	-.6301E-4	-.1494E-3	-.5223E-4	0.9999	-.1269E-4	.1113E-3
4-b	-.4548E-4	-.4606E-4	.1497E-3	.9050E-4	.6514E-5	-.1269E-4	1.000	-.2088E-4
4-f	-.5007E-4	-.7096E-4	.1912E-3	.9633E-4	.1496E-4	.1113E-3	-.2088E-4	0.9999

(b) Imaginary Parts of the Modal Mass: $\psi_i^T \bar{M} \psi_j$

Mode	1-b	1-f	2-b	2-f	3-b	3-f	4-b	4-f
1-b	.3215E-3	.3971E-3	-.1087E-2	-.9065E-3	.4949E-3	-.1808E-4	-.3703E-3	.2160E-3
1-f	.3971E-3	.4776E-3	-.1102E-2	-.8994E-3	.4652E-3	.4261E-5	-.4006E-3	.1772E-3
2-b	-.1087E-2	-.1102E-2	-.1315E-2	.1154E-2	-.7203E-3	.1031E-2	-.8692E-4	-.3976E-3
2-f	-.9065E-3	-.8994E-3	.1154E-2	.2536E-2	-.8544E-3	.1128E-2	.1534E-4	-.8327E-3
3-b	.4949E-3	.4652E-3	-.7203E-3	-.8544E-3	-.1056E-2	.1730E-3	-.2272E-3	.6247E-3
3-f	-.1808E-4	.4261E-5	.1031E-2	.1128E-2	.1730E-3	-.3383E-2	.6645E-3	-.6416E-3
4-b	-.3703E-3	-.4006E-3	-.8692E-4	.1534E-4	-.2272E-3	.6645E-3	.2056E-2	-.1230E-3
4-f	.2160E-3	.1772E-3	-.3976E-3	-.8327E-3	.6247E-3	-.6416E-3	-.1230E-3	-.5166E-2

**Table 5.15 Orthonormality Check (No Modal Truncation)
of a Multi-stepped Rotor with Isotropic Bearings
(at a running speed = 30000 rpm)**

(a) Real Parts of the Modal Mass: $\psi_i^T \bar{M} \psi_j$

Mode	1-b	1-f	2-b	2-f	3-b	3-f	4-b	4-f
1-b	1.000	-.4780E-7	.1743E-5	-.6154E-6	.1237E-4	-.5751E-5	.1938E-5	.1639E-6
1-f	-.4780E-7	1.000	-.5175E-5	.1803E-5	.1042E-4	-.1013E-4	-.4764E-6	-.3725E-5
2-b	.1743E-5	-.5175E-5	1.000	-.2249E-4	.4343E-4	-.2103E-4	.6662E-4	.3946E-4
2-f	-.6154E-6	.1803E-5	-.2249E-4	1.000	.3014E-4	-.6745E-5	-.2811E-4	-.3103E-4
3-b	.1237E-4	.1042E-4	.4543E-4	.3014E-4	0.9999	-.1422E-4	.4423E-4	.1569E-5
3-f	-.5751E-5	-.1013E-4	-.2103E-4	-.6745E-5	-.1422E-4	1.000	-.1418E-4	.1268E-4
4-b	.1938E-5	-.4764E-6	.6662E-4	-.2811E-4	.4423E-4	-.1418E-4	1.000	-.3159E-4
4-f	.1639E-6	-.3725E-5	.3946E-4	-.3103E-4	.1569E-5	.1268E-4	-.3159E-4	1.000

(b) Imaginary Parts of the Modal Mass: $\psi_i^T \bar{M} \psi_j$

Mode	1-b	1-f	2-b	2-f	3-b	3-f	4-b	4-f
1-b	-.6183E-4	-.2536E-5	-.9188E-4	.5893E-4	.3667E-3	.2130E-3	-.4115E-4	-.7169E-4
1-f	-.2536E-5	.7683E-4	-.5096E-4	.1178E-3	.2443E-3	.3294E-3	-.4708E-4	-.1243E-3
2-b	-.9188E-4	-.5096E-4	-.2460E-2	-.8710E-4	.1334E-3	-.1293E-3	-.3155E-3	-.1546E-3
2-f	.5893E-4	.1178E-3	-.8710E-4	.7634E-3	.1340E-3	-.1813E-3	-.2136E-3	-.5967E-3
3-b	.3667E-3	.2443E-3	.1334E-3	.1340E-3	-.8621E-3	.3692E-4	.1779E-3	.1549E-3
3-f	.2130E-3	.3294E-3	-.1293E-3	-.1813E-3	.3692E-4	-.3623E-2	.1107E-3	.9418E-4
4-b	-.4115E-4	-.4708E-4	-.3155E-3	-.2136E-3	.1779E-3	.1107E-3	.1442E-2	-.8849E-4
4-f	-.7169E-4	-.1243E-3	-.1546E-3	-.5967E-3	.1549E-3	.9418E-4	-.8849E-4	-.5161E-2

**Table 5.16 Perturbated Whirl Frequencies of a Stepped-Shaft Rotor
with Orthotropic Bearings at Various Modal Truncation Levels**
(at a running speed = 40000 rpm)

Mode No.	F.E.M. Exact	Whirl Frequencies (rpm)/Log Decrements				
		8 modes	16 modes	24 modes	36 modes	76 modes
1b	13075/.1499	13073/.1499	13075/.1499	13075/.1499	13075/.1499	13075/.1499
1f	17316/.1523	17312/.1521	17316/.1521	17316/.1521	17316/.1521	17316/.1521
2b	39968/.6846	39889/.6779	39961/.6767	39963/.6767	39964/.6766	39964/.6766
2f	48312/.4498	48242/.4499	48310/.4493	48311/.4493	48312/.4493	48312/.4493
3b	61788/.6975	61743/.6967	61785/.6963	61797/.6962	61798/.6961	61798/.6961
3f	81588/.5517	81514/.5468	81587/.5463	81599/.5462	81602/.5462	81602/.5462
4b	119405/.0584	119306/.0596	119400/.0595	119401/.0595	119402/.0595	119403/.0595
4f	124498/.0903	124461/.0935	124484/.0935	124488/.0935	124489/.0935	124489/.0935
5b	154445/.2626		154430/.2619	154440/.2619	154445/.2618	154447/.2618
5f	185157/.2568		185135/.2557	185152/.2557	185157/.2557	185159/.2557
6b	338073/.1480		338025/.1483	338060/.1483	338071/.1483	338074/.1483
6f	361144/.1390		361097/.1394	361129/.1394	361142/.1394	361145/.1394
7b	630721/.1603		630377/.1602	630686/.1601	630720/.1601	630723/.1601
7f	652495/.1549		652142/.1548	652458/.1547	652493/.1547	652497/.1547
8b	901508/42E-6		901508/42E-6	901508/42E-6	901508/42E-6	901508/42E-6
8f	910451/14E-6		910451/14E-6	910451/14E-6	910451/14E-6	910451/14E-6

**Table 5.17 The First Backward Whirl Mode Shape of
a Multi-Stepped Shaft Rotor with Orthotropic Bearings
at Various Normal Modal Truncation Levels
(at a running speed = 40000 rpm)**

Node No.	Exact Solution		8 Kept Modes		16 Kept Modes		No Modal Truncation	
	real	imag.	real	imag.	real	imag.	real	imag.
1-1	-.3933E-2	-.1610E-3	-.3931E-2	-.1654E-3	-.3933E-2	-.1613E-3	-.3933E-2	-.1613E-3
2-1	-.3641E-2	-.1476E-3	-.3639E-2	-.1509E-3	-.3641E-2	-.1479E-3	-.3641E-2	-.1479E-3
3-1	-.2768E-2	-.1076E-3	-.2767E-2	-.1079E-3	-.2768E-2	-.1078E-3	-.2768E-2	-.1078E-3
4-1	-.2201E-2	-.8221E-4	-.2200E-2	-.8186E-4	-.2201E-2	-.8240E-4	-.2201E-2	-.8240E-4
5-1	-.1926E-2	-.7019E-4	-.1924E-2	-.7021E-4	-.1925E-2	-.7037E-4	-.1925E-2	-.7037E-4
6-1	-.1652E-2	-.5824E-4	-.1651E-2	-.5866E-4	-.1652E-2	-.5839E-4	-.1652E-2	-.5839E-4
7-1	-.1543E-2	-.5349E-4	-.1542E-2	-.5408E-4	-.1543E-2	-.5362E-4	-.1543E-2	-.5363E-4
8-1	-.1380E-2	-.4636E-4	-.1379E-2	-.4722E-4	-.1380E-2	-.4647E-4	-.1380E-2	-.4649E-4
9-1	-.1110E-2	-.3454E-4	-.1109E-2	-.3585E-4	-.1110E-2	-.3462E-4	-.1110E-2	-.3466E-4
10-1	-.9487E-3	-.2750E-4	-.9480E-3	-.2910E-4	-.9487E-3	-.2757E-4	-.9487E-3	-.2761E-4
11-1	-.4155E-3	-.5179E-5	-.4150E-3	-.7414E-5	-.4155E-3	-.5390E-5	-.4155E-3	-.5251E-5
12-1	-.1524E-3	.3272E-5	-.1521E-3	.2151E-5	-.1525E-3	.3257E-5	-.1524E-3	.3227E-5
13-1	.6272E-4	.5565E-5	.6288E-4	.7041E-5	.6271E-4	.5567E-5	.6272E-4	.5557E-5
14-1	.1885E-3	.2459E-5	.1885E-3	.5618E-5	.1885E-3	.2450E-5	.1885E-3	.2487E-5
15-1	.2379E-3	-.4570E-6	.2380E-3	.2487E-5	.2379E-3	-.3770E-6	.2379E-3	-.4099E-6
16-1	.2811E-3	-.3271E-5	.2812E-3	-.1617E-5	.2811E-3	-.3201E-5	.2811E-3	-.3206E-5
17-1	.3064E-3	-.4909E-5	.3065E-3	-.4237E-5	.3064E-3	-.4833E-5	.3064E-3	-.4835E-5
18-1	.3824E-3	-.9830E-5	.3827E-3	-.1215E-4	.3824E-3	-.9724E-5	.3824E-3	-.9724E-5
19-1	.4078E-3	-.1147E-4	.4081E-3	-.1480E-4	.4078E-3	-.1135E-4	.4078E-3	-.1136E-4
1-3	-.6806E-2	.1245E-3	-.6808E-2	.1121E-3	-.6806E-2	.1235E-3	-.6806E-2	.1235E-3
2-3	-.6337E-2	.1217E-3	-.6338E-2	.1123E-3	-.6337E-2	.1208E-3	-.6337E-2	.1208E-3
3-3	-.4932E-2	.1134E-3	-.4933E-2	.1123E-3	-.4932E-2	.1127E-3	-.4932E-2	.1127E-3
4-3	-.4020E-2	.1072E-3	-.4021E-2	.1081E-3	-.4020E-2	.1066E-3	-.4020E-2	.1066E-3
5-3	-.3578E-2	.1037E-3	-.3579E-2	.1037E-3	-.3579E-2	.1032E-3	-.3579E-2	.1032E-3
6-3	-.3138E-2	.1001E-3	-.3139E-2	.9903E-4	-.3138E-2	.9967E-4	-.3138E-2	.9965E-4
7-3	-.2963E-2	.9862E-4	-.2963E-2	.9711E-4	-.2963E-2	.9825E-4	-.2963E-2	.9821E-4
8-3	-.2700E-2	.9642E-4	-.2700E-2	.9422E-4	-.2700E-2	.9611E-4	-.2700E-2	.9604E-4
9-3	-.2263E-2	.9268E-4	-.2263E-2	.8929E-4	-.2263E-2	.9247E-4	-.2263E-2	.9237E-4
10-3	-.2002E-2	.9039E-4	-.2002E-2	.8628E-4	-.2002E-2	.9022E-4	-.2002E-2	.9012E-4
11-3	-.1098E-2	.7672E-4	-.1098E-2	.7081E-4	-.1098E-2	.7620E-4	-.1098E-2	.7657E-4
12-3	-.5839E-3	.5806E-4	-.5841E-3	.5496E-4	-.5840E-3	.5805E-4	-.5840E-3	.5798E-4
13-3	-.5348E-4	.2273E-4	-.5316E-4	.2631E-4	-.5350E-4	.2276E-4	-.5349E-4	.2272E-4
14-3	.3576E-3	-.1651E-4	.3584E-3	-.8667E-5	.3576E-3	-.1656E-4	.3576E-3	-.1646E-4
15-3	.5536E-3	-.3841E-4	.5545E-3	-.3118E-4	.5536E-3	-.3826E-4	.5536E-3	-.3833E-4
16-3	.7251E-3	-.5785E-4	.7261E-3	-.5396E-4	.7251E-3	-.5773E-4	.7251E-3	-.5774E-4
17-3	.8239E-3	-.6898E-4	.8249E-3	-.6764E-4	.8239E-3	-.6886E-4	.8239E-3	-.6887E-4
18-3	.1121E-2	-.1024E-3	.1121E-2	-.1088E-3	.1121E-2	-.1023E-3	.1121E-2	-.1023E-3
19-3	.1219E-2	-.1135E-3	.1220E-2	-.1225E-3	.1219E-2	-.1134E-3	.1219E-2	-.1134E-3

**Table 5.18 The First Forward Whirl Mode Shape of
a Multi-Stepped Shaft Rotor with Orthotropic Bearings
at Various Normal Modal Truncation Levels
(at a running speed = 40000 rpm)**

Node No.	Exact Solution		8 Kept Modes		16 Kept Modes		No Modal Truncation	
	real	imag.	real	imag.	real	imag.	real	imag.
1-1	-.5691E-2	.1188E-3	-.5693E-2	.1096E-3	-.5691E-2	.1180E-3	-.5691E-2	.1181E-3
2-1	-.5280E-2	.1138E-3	-.5281E-2	.1069E-3	-.5280E-2	.1132E-3	-.5280E-2	.1132E-3
3-1	-.4052E-2	.9885E-4	-.4053E-2	.9821E-4	-.4052E-2	.9836E-4	-.4052E-2	.9835E-4
4-1	-.3268E-2	.8799E-4	-.3268E-2	.8877E-4	-.3268E-2	.8759E-4	-.3268E-2	.8759E-4
5-1	-.2895E-2	.8207E-4	-.2896E-2	.8210E-4	-.2895E-2	.8171E-4	-.2895E-2	.8172E-4
6-1	-.2524E-2	.7618E-4	-.2524E-2	.7534E-4	-.2524E-2	.7590E-4	-.2524E-2	.7589E-4
7-1	-.2375E-2	.7384E-4	-.2376E-2	.7262E-4	-.2375E-2	.7360E-4	-.2375E-2	.7357E-4
8-1	-.2152E-2	.7034E-4	-.2153E-2	.6853E-4	-.2152E-2	.7016E-4	-.2152E-2	.7010E-4
9-1	-.1782E-2	.6452E-4	-.1783E-2	.6169E-4	-.1782E-2	.6442E-4	-.1782E-2	.6433E-4
10-1	-.1561E-2	.6104E-4	-.1562E-2	.5757E-4	-.1561E-2	.6097E-4	-.1561E-2	.6087E-4
11-1	-.7994E-3	.4687E-4	-.7999E-3	.4178E-4	-.7994E-3	.4644E-4	-.7994E-3	.4680E-4
12-1	-.3958E-3	.3369E-4	-.3961E-3	.3105E-4	-.3958E-3	.3374E-4	-.3958E-3	.3367E-4
13-1	-.2931E-4	.1214E-4	-.2924E-4	.1524E-4	-.2932E-4	.1218E-4	-.2931E-4	.1215E-4
14-1	.2172E-3	-.1045E-4	.2176E-3	-.3838E-5	.2172E-3	-.1055E-4	.2172E-3	-.1043E-4
15-1	.3240E-3	-.2281E-4	.3245E-3	-.1676E-4	.3240E-3	-.2275E-4	.3240E-3	-.2279E-4
16-1	.4157E-3	-.3377E-4	.4163E-3	-.3048E-4	.4157E-3	-.3372E-4	.4157E-3	-.3375E-4
17-1	.4684E-3	-.4005E-4	.4691E-3	-.3884E-4	.4684E-3	-.4002E-4	.4684E-3	-.4003E-4
18-1	.6267E-3	-.5890E-4	.6274E-3	-.6406E-4	.6267E-3	-.5889E-4	.6267E-3	-.5889E-4
19-1	.6794E-3	-.6519E-4	.6801E-3	-.7248E-4	.6795E-3	-.6518E-4	.6795E-3	-.6517E-4
1-3	.3227E-2	.1877E-3	.3224E-2	.1959E-3	.3227E-2	.1880E-3	.3227E-2	.1879E-3
2-3	.3020E-2	.1701E-3	.3018E-2	.1763E-3	.3020E-2	.1703E-3	.3020E-2	.1703E-3
3-3	.2401E-2	.1175E-3	.2399E-2	.1180E-3	.2401E-2	.1177E-3	.2401E-2	.1177E-3
4-3	.2008E-2	.8457E-4	.2007E-2	.8369E-4	.2008E-2	.8478E-4	.2008E-2	.8477E-4
5-3	.1822E-2	.6926E-4	.1821E-2	.6907E-4	.1822E-2	.6947E-4	.1822E-2	.6946E-4
6-3	.1636E-2	.5402E-4	.1635E-2	.5462E-4	.1636E-2	.5420E-4	.1636E-2	.5421E-4
7-3	.1561E-2	.4794E-4	.1560E-2	.4891E-4	.1561E-2	.4811E-4	.1561E-2	.4814E-4
8-3	.1449E-2	.3884E-4	.1448E-2	.4035E-4	.1449E-2	.3898E-4	.1449E-2	.3904E-4
9-3	.1262E-2	.2374E-4	.1261E-2	.2620E-4	.1262E-2	.2385E-4	.1262E-2	.2394E-4
10-3	.1149E-2	.1475E-4	.1149E-2	.1780E-4	.1149E-2	.1486E-4	.1149E-2	.1495E-4
11-3	.7344E-3	-.1318E-4	.7342E-3	-.8520E-5	.7344E-3	-.1267E-4	.7344E-3	-.1301E-4
12-3	.4706E-3	-.2194E-4	.4705E-3	-.1944E-4	.4706E-3	-.2189E-4	.4706E-3	-.2182E-4
13-3	.1636E-3	-.2023E-4	.1636E-3	-.2286E-4	.1636E-3	-.2027E-4	.1635E-3	-.2023E-4
14-3	-.1007E-3	-.1107E-4	-.1006E-3	-.1682E-4	-.1007E-3	-.1108E-4	-.1007E-3	-.1120E-4
15-3	-.2339E-3	-.4700E-5	-.2340E-3	-.9986E-5	-.2339E-3	-.4925E-5	-.2340E-3	-.4902E-5
16-3	-.3513E-3	.9106E-6	-.3516E-3	-.2018E-5	-.3513E-3	.6166E-6	-.3513E-3	.6439E-6
17-3	-.4188E-3	.4071E-5	-.4191E-3	.2924E-5	-.4188E-3	.3750E-5	-.4188E-3	.3766E-5
18-3	-.6213E-3	.1354E-4	-.6220E-3	.1784E-4	-.6213E-3	.1314E-4	-.6213E-3	.1313E-4
19-3	-.6888E-3	.1670E-4	-.6896E-3	.2283E-4	-.6888E-3	.1627E-4	-.6888E-3	.1624E-4

**Table 5.19 The Second Backward Whirl Mode Shape of
a Multi-Stepped Shaft Rotor with Orthotropic Bearings
at Various Normal Modal Truncation Levels**
(at a running speed = 40000 rpm)

Node No.	Exact Solution		8 Kept Modes		16 Kept Modes		No Modal Truncation	
	real	imag.	real	imag.	real	imag.	real	imag.
1-1	-.8994E-4	-.7204E-5	-.9385E-4	.1649E-5	-.9010E-4	-.6127E-5	-.9013E-4	-.6115E-5
2-1	-.7886E-4	-.2852E-5	-.8188E-4	.4012E-5	-.7900E-4	-.1888E-5	-.7902E-4	-.1885E-5
3-1	-.4599E-4	.1008E-4	-.4650E-4	.1133E-4	-.4606E-4	.1068E-4	-.4605E-4	.1068E-4
4-1	-.2674E-4	.1798E-4	-.2670E-4	.1778E-4	-.2678E-4	.1831E-4	-.2675E-4	.1831E-4
5-1	-.1858E-4	.2159E-4	-.1883E-4	.2183E-4	-.1861E-4	.2178E-4	-.1857E-4	.2178E-4
6-1	-.1050E-4	.2498E-4	-.1104E-4	.2565E-4	-.1051E-4	.2503E-4	-.1047E-4	.2503E-4
7-1	-.7288E-5	.2626E-4	-.7955E-5	.2708E-4	-.7293E-5	.2624E-4	-.7256E-5	.2624E-4
8-1	-.2483E-5	.2814E-4	-.3331E-5	.2920E-4	-.2479E-5	.2804E-4	-.2440E-5	.2805E-4
9-1	.5456E-5	.3117E-4	.4308E-5	.3259E-4	.5473E-5	.3093E-4	.5516E-5	.3094E-4
10-1	.1016E-4	.3289E-4	.8832E-5	.3452E-4	.1018E-4	.3258E-4	.1023E-4	.3259E-4
11-1	.2209E-4	.3422E-4	.2034E-4	.3560E-4	.2207E-4	.3381E-4	.2216E-4	.3380E-4
12-1	.1854E-4	.2836E-4	.1694E-4	.2827E-4	.1851E-4	.2806E-4	.1856E-4	.2807E-4
13-1	-.1007E-4	.8910E-5	-.1131E-4	.6126E-5	-.1024E-4	.9009E-5	-.1018E-4	.8985E-5
14-1	-.6710E-4	-.2384E-4	-.6880E-4	-.2835E-4	-.6745E-4	-.2347E-4	-.6742E-4	-.2343E-4
15-1	-.1182E-3	-.5100E-4	-.1212E-3	-.5503E-4	-.1187E-3	-.5058E-4	-.1187E-3	-.5050E-4
16-1	-.1826E-3	-.8371E-4	-.1877E-3	-.8581E-4	-.1832E-3	-.8322E-4	-.1832E-3	-.8324E-4
17-1	-.2241E-3	-.1044E-3	-.2307E-3	-.1050E-3	-.2249E-3	-.1040E-3	-.2249E-3	-.1040E-3
18-1	-.3495E-3	-.1669E-3	-.3606E-3	-.1631E-3	-.3506E-3	-.1666E-3	-.3506E-3	-.1666E-3
19-1	-.3915E-3	-.1878E-3	-.4041E-3	-.1825E-3	-.3926E-3	-.1876E-3	-.3927E-3	-.1876E-3
1-3	-.2381E-3	-.1319E-3	-.2480E-3	-.8088E-4	-.2380E-3	-.1281E-3	-.2382E-3	-.1278E-3
2-3	-.2298E-3	-.1044E-3	-.2379E-3	-.6604E-4	-.2296E-3	-.1011E-3	-.2298E-3	-.1010E-3
3-3	-.2056E-3	-.2253E-4	-.2089E-3	-.1968E-4	-.2054E-3	-.2117E-4	-.2055E-3	-.2128E-4
4-3	-.1974E-3	.2705E-4	-.1988E-3	.2178E-4	-.1972E-3	.2707E-4	-.1972E-3	.2707E-4
5-3	-.1977E-3	.4917E-4	-.1990E-3	.4771E-4	-.1975E-3	.4866E-4	-.1975E-3	.4866E-4
6-3	-.1982E-3	.7105E-4	-.1995E-3	.7326E-4	-.1981E-3	.7004E-4	-.1981E-3	.7002E-4
7-3	-.1986E-3	.7972E-4	-.1998E-3	.8330E-4	-.1985E-3	.7852E-4	-.1984E-3	.7848E-4
8-3	-.1992E-3	.9270E-4	-.2003E-3	.9830E-4	-.1990E-3	.9121E-4	-.1990E-3	.9115E-4
9-3	-.2005E-3	.1142E-3	-.2017E-3	.1230E-3	-.2004E-3	.1122E-3	-.2004E-3	.1122E-3
10-3	-.2017E-3	.1270E-3	-.2030E-3	.1376E-3	-.2017E-3	.1247E-3	-.2016E-3	.1247E-3
11-3	-.2578E-3	.1708E-3	-.2589E-3	.1815E-3	-.2579E-3	.1671E-3	-.2576E-3	.1676E-3
12-3	-.4042E-3	.1963E-3	-.4041E-3	.1998E-3	-.4041E-3	.1929E-3	-.4040E-3	.1931E-3
13-3	-.7579E-3	.2146E-3	-.7555E-3	.2003E-3	-.7576E-3	.2125E-3	-.7575E-3	.2121E-3
14-3	-.1242E-2	.2069E-3	-.1239E-2	.1761E-3	-.1242E-2	.2051E-3	-.1242E-2	.2059E-3
15-3	-.1593E-2	.1838E-3	-.1590E-2	.1543E-3	-.1593E-2	.1829E-3	-.1593E-2	.1840E-3
16-3	-.1985E-2	.1463E-3	-.1983E-2	.1313E-3	-.1984E-2	.1482E-3	-.1984E-2	.1478E-3
17-3	-.2227E-2	.1211E-3	-.2225E-2	.1173E-3	-.2226E-2	.1238E-3	-.2226E-2	.1235E-3
18-3	-.2954E-2	.4500E-4	-.2954E-2	.7489E-4	-.2954E-2	.4965E-4	-.2954E-2	.4992E-4
19-3	-.3197E-2	.1950E-4	-.3198E-2	.6070E-4	-.3197E-2	.2467E-4	-.3197E-2	.2528E-4

**Table 5.20 The Second Forward Whirl Mode Shape of
a Multi-Stepped Shaft Rotor with Orthotropic Bearings
at Various Normal Modal Truncation Levels
(at a running speed = 40000 rpm)**

Node No.	Exact Solution		8 Kept Modes		16 Kept Modes		No Modal Truncation	
	real	imag.	real	imag.	real	imag.	real	imag.
1-1	.3833E-3	.4000E-4	.3881E-3	-.4638E-6	.3830E-3	.4072E-4	.3831E-3	.4055E-4
2-1	.3342E-3	.2921E-4	.3383E-3	-.7637E-6	.3340E-3	.2973E-4	.3341E-3	.2967E-4
3-1	.1892E-3	-.2665E-5	.1912E-3	-.3424E-5	.1891E-3	-.2559E-5	.1892E-3	-.2477E-5
4-1	.1090E-3	-.2046E-4	.1102E-3	-.1564E-4	.1090E-3	-.2049E-4	.1090E-3	-.2049E-4
5-1	.7785E-4	-.2753E-4	.7893E-4	-.2699E-4	.7795E-4	-.2769E-4	.7794E-4	-.2768E-4
6-1	.4761E-4	-.3448E-4	.4864E-4	-.3815E-4	.4774E-4	-.3477E-4	.4772E-4	-.3476E-4
7-1	.3585E-4	-.3722E-4	.3688E-4	-.4252E-4	.3600E-4	-.3757E-4	.3598E-4	-.3756E-4
8-1	.1830E-4	-.4134E-4	.1933E-4	-.4904E-4	.1847E-4	-.4175E-4	.1845E-4	-.4175E-4
9-1	-.1025E-4	-.4816E-4	-.9211E-5	-.5976E-4	-.1005E-4	-.4870E-4	-.1007E-4	-.4870E-4
10-1	-.2680E-4	-.5224E-4	-.2575E-4	-.6608E-4	-.2658E-4	-.5285E-4	-.2660E-4	-.5286E-4
11-1	-.3583E-4	-.6964E-4	-.3465E-4	-.8529E-4	-.3543E-4	-.7033E-4	-.3560E-4	-.7049E-4
12-1	.5760E-4	-.8742E-4	.5834E-4	-.9527E-4	.5789E-4	-.8818E-4	.5781E-4	-.8831E-4
13-1	.3449E-3	-.1110E-3	.3448E-3	-.9954E-4	.3451E-3	-.1122E-3	.3451E-3	-.1118E-3
14-1	.7931E-3	-.1217E-3	.7925E-3	-.9326E-4	.7932E-3	-.1219E-3	.7931E-3	-.1225E-3
15-1	.1148E-2	-.1156E-3	.1147E-2	-.8780E-4	.1148E-2	-.1157E-3	.1148E-2	-.1166E-3
16-1	.1566E-2	-.9920E-4	.1565E-2	-.8408E-4	.1566E-2	-.1006E-3	.1566E-2	-.1004E-3
17-1	.1826E-2	-.8747E-4	.1825E-2	-.8216E-4	.1826E-2	-.8899E-4	.1826E-2	-.8874E-4
18-1	.2613E-2	-.5177E-4	.2611E-2	-.7638E-4	.2612E-2	-.5318E-4	.2612E-2	-.5342E-4
19-1	.2875E-2	-.3979E-4	.2874E-2	-.7444E-4	.2875E-2	-.4106E-4	.2875E-2	-.4157E-4
1-3	.2212E-3	-.6011E-4	.2138E-3	-.5656E-4	.2206E-3	-.5730E-4	.2205E-3	-.5723E-4
2-3	.1806E-3	-.5383E-4	.1747E-3	-.5091E-4	.1801E-3	-.5126E-4	.1801E-3	-.5124E-4
3-3	.6031E-4	-.3513E-4	.5846E-4	-.3396E-4	.6016E-4	-.3340E-4	.6012E-4	-.3343E-4
4-3	-.1102E-4	-.2410E-4	-.1148E-4	-.2345E-4	-.1095E-4	-.2291E-4	-.1097E-4	-.2298E-4
5-3	-.4199E-4	-.1942E-4	-.4244E-4	-.1873E-4	-.4183E-4	-.1860E-4	-.4185E-4	-.1859E-4
6-3	-.7228E-4	-.1468E-4	-.7271E-4	-.1404E-4	-.7204E-4	-.1414E-4	-.7205E-4	-.1415E-4
7-3	-.8414E-4	-.1277E-4	-.8456E-4	-.1217E-4	-.8387E-4	-.1233E-4	-.8388E-4	-.1236E-4
8-3	-.1019E-3	-.9895E-5	-.1023E-3	-.9372E-5	-.1016E-3	-.9618E-5	-.1016E-3	-.9662E-5
9-3	-.1311E-3	-.5118E-5	-.1315E-3	-.4753E-5	-.1307E-3	-.5114E-5	-.1307E-3	-.5178E-5
10-3	-.1484E-3	-.2274E-5	-.1487E-3	-.2023E-5	-.1479E-3	-.2441E-5	-.1479E-3	-.2507E-5
11-3	-.2029E-3	.4177E-5	-.2030E-3	.3803E-5	-.2023E-3	.3308E-5	-.2023E-3	.3595E-5
12-3	-.2325E-3	-.1638E-5	-.2323E-3	-.1121E-5	-.2320E-3	-.1827E-5	-.2320E-3	-.1839E-5
13-3	-.2547E-3	-.2489E-4	-.2542E-3	-.2315E-4	-.2543E-3	-.2377E-4	-.2543E-3	-.2389E-4
14-3	-.2540E-3	-.6087E-4	-.2543E-3	-.5940E-4	-.2538E-3	-.5868E-4	-.2539E-3	-.5844E-4
15-3	-.2428E-3	-.8850E-4	-.2448E-3	-.8683E-4	-.2429E-3	-.8552E-4	-.2430E-3	-.8530E-4
16-3	-.2251E-3	-.1202E-3	-.2300E-3	-.1172E-3	-.2256E-3	-.1163E-3	-.2256E-3	-.1164E-3
17-3	-.2123E-3	-.1398E-3	-.192E-3	-.1358E-3	-.2130E-3	-.1355E-3	-.2130E-3	-.1356E-3
18-3	-.1737E-3	-.1988E-3	-.1866E-3	-.1918E-3	-.1748E-3	-.1936E-3	-.1750E-3	-.1935E-3
19-3	-.1607E-3	-.2185E-3	-.1757E-3	-.2105E-3	-.1620E-3	-.2130E-3	-.1622E-3	-.2128E-3

Table 5.21 Orthonormality Check (8 Kept Modes)
of a Multi-stepped Rotor with Orthotropic Bearings
 (at a running speed = 40000 rpm)

(a) Real Parts of the Modal Mass: $\psi_i^T \bar{M} \psi_j$

Mode	1-b	1-f	2-b	2-f	3-b	3-f	4-b	4-f
1-b	1.000	-.3064E-4	-.1344E-3	-.4868E-4	.1295E-3	-.3150E-4	.6154E-4	.3929E-4
1-f	-.3064E-4	1.000	-.4972E-6	-.1618E-3	.5542E-4	.6217E-4	.7254E-5	-.9271E-4
2-b	-.1344E-3	-.4972E-6	1.000	.7775E-4	-.7746E-4	.1282E-3	-.5478E-4	-.1495E-3
2-f	-.4868E-4	-.1618E-3	.7775E-4	0.9998	.7593E-4	.2528E-3	.1113E-3	-.1120E-3
3-b	.1295E-3	.5542E-4	-.7746E-4	.7593E-4	1.000	-.9444E-4	.4696E-5	.3036E-4
3-f	-.3150E-4	.6217E-4	.1282E-3	.2528E-3	-.9444E-4	0.9999	-.4521E-4	.1248E-3
4-b	.6154E-4	.7254E-5	-.5478E-4	.1113E-3	.4696E-5	-.4521E-4	1.000	.2882E-4
4-f	.3929E-4	-.9271E-4	-.1495E-3	-.1120E-3	.3036E-4	.1248E-3	.2882E-4	0.9999

(b) Imaginary Parts of the Modal Mass: $\psi_i^T \bar{M} \psi_j$

Mode	1-b	1-f	2-b	2-f	3-b	3-f	4-b	4-f
1-b	-.2814E-3	-.2993E-3	.8360E-3	-.1335E-2	-.5404E-3	.2117E-3	-.4024E-3	-.1199E-3
1-f	-.2993E-3	.7243E-3	.8277E-3	.9762E-3	-.3772E-4	-.3378E-4	-.9523E-4	.1782E-3
2-b	.8360E-3	.8277E-3	-.1910E-2	.2337E-3	.7390E-3	-.1697E-2	-.3087E-3	.8324E-3
2-f	-.1335E-2	.9762E-3	.2337E-3	.7184E-4	.4577E-3	-.4974E-3	.1808E-3	.4702E-3
3-b	-.5404E-3	-.3772E-4	.7390E-3	.4577E-3	.1580E-2	-.8189E-3	-.5287E-3	.1547E-3
3-f	.2117E-3	-.3378E-4	-.1697E-2	-.4974E-3	-.8189E-3	-.3230E-2	.4509E-3	-.8186E-3
4-b	-.4024E-3	-.9523E-4	-.3087E-3	.1808E-3	-.5287E-3	.4509E-3	-.1184E-2	-.1812E-3
4-f	-.1199E-3	.1782E-3	.8324E-3	.4702E-3	.1547E-3	-.8186E-3	-.1812E-3	-.4910E-2

Table 5.22 Orthonormality Check (No Modal Truncation)
of a Multi-stepped Rotor with Orthotropic Bearings
 (at a running speed = 40000 rpm)

(a) Real Parts of the Modal Mass: $\psi_i^T \bar{M} \psi_j$

Mode	1-b	1-f	2-b	2-f	3-b	3-f	4-b	4-f
1-b	1.000	.1835E-6	.8502E-5	-.7003E-5	.3158E-4	-.4707E-5	-.9051E-6	.4166E-5
1-f	.1835E-6	1.000	-.4522E-5	-.1308E-4	-.1924E-4	-.1512E-4	-.1874E-5	-.2844E-5
2-b	.8502E-5	-.4522E-5	1.000	.5983E-4	.1478E-3	-.1868E-4	.2738E-4	-.2425E-4
2-f	-.7003E-5	-.1308E-4	.5983E-4	1.000	-.1119E-3	.8781E-5	.1072E-3	.1467E-4
3-b	.3158E-4	-.1924E-4	.1478E-3	-.1119E-3	0.9999	.1182E-4	-.4065E-4	.1947E-4
3-f	-.4707E-5	-.1512E-4	-.1868E-4	.8781E-5	.1182E-4	1.000	-.5869E-5	.1051E-4
4-b	-.9051E-6	-.1874E-5	.2738E-4	.1072E-3	-.4065E-4	-.5869E-5	1.000	.5553E-5
4-f	.4166E-5	-.2844E-5	-.2425E-4	.1467E-4	.1947E-4	.1051E-4	.5553E-5	1.000

(b) Imaginary Parts of the Modal Mass: $\psi_i^T \bar{M} \psi_j$

Mode	1-b	1-f	2-b	2-f	3-b	3-f	4-b	4-f
1-b	.2495E-3	-.1565E-4	.4084E-4	-.2718E-3	-.4328E-3	-.4999E-4	-.2651E-4	.1840E-4
1-f	-.1565E-4	.2148E-3	-.2868E-3	.1048E-3	-.1206E-3	.3455E-3	.1413E-4	-.1327E-3
2-b	.4084E-4	-.2868E-3	-.3042E-2	-.1562E-2	.1152E-3	-.4417E-3	-.4856E-3	.7139E-3
2-f	-.2718E-3	.1048E-3	-.1562E-2	-.1716E-2	.1436E-2	.7037E-3	.1730E-3	.3081E-3
3-b	-.4328E-3	-.1206E-3	.1152E-3	.1436E-2	.1174E-2	-.5475E-3	-.1513E-4	.3193E-3
3-f	-.4999E-4	.3455E-3	-.4417E-3	.7037E-3	-.5475E-3	-.3410E-2	.1204E-3	.1410E-3
4-b	-.2651E-4	.1413E-4	-.4856E-3	.1730E-3	-.1513E-4	.1204E-3	-.3394E-3	-.1952E-3
4-f	.1840E-4	-.1327E-3	.7139E-3	.3081E-3	.3193E-3	.1410E-3	-.1952E-3	-.4799E-2

Table 5.23 Normalized Unbalance Translational Response of the Disk in a Multi-Stepped Shaft Rotor with Isotropic Bearings

Number of Modes	20,000 rpm		40,000 rpm	
	$X_5/e = Y_5/e$	Error %	$X_5/e = Y_5/e$	Error %
1b	0.19470	91.94	0.49878	33.14
1f	2.55336	8.378	1.26585	69.68
2	2.35994	0.168	0.76713	2.829
4	2.35910	0.132	0.75972	1.835
6	2.35775	0.075	0.75371	1.030
8	2.35310	0.005	0.74641	0.052
12	2.35607	0.004	0.74625	0.030
16	2.35603	0.002	0.74609	0.009
FEM	2.35598	exact	0.74603	exact

e = disc eccentricity at node 5

Table 5.24 Normalized Unbalance Translational Response of the Disk in a Multi-Stepped Shaft Rotor with Orthotropic Bearings

Number of Modes	20,000 rpm				40,000 rpm			
	X_5/e	Error %	Y_5/e	Error %	X_5/e	Error %	Y_5/e	Error %
1b	0.35008	82.85	0.45854	69.20	.745402	32.50	0.47266	35.02
1f	2.39435	17.30	1.04400	29.87	1.33263	85.15	1.00574	38.28
2	2.04514	0.192	1.49437	0.376	0.74833	3.971	0.72544	0.261
4	2.04438	0.154	1.49100	0.150	0.73709	2.410	0.73555	1.129
6	2.04288	0.081	1.49063	0.125	0.72738	1.060	0.73579	1.162
8	2.04122	0.000	1.48883	0.004	0.72006	0.044	0.72787	0.074
12	2.04119	0.001	1.48880	0.002	0.71989	0.020	0.72770	0.049
16	2.04115	0.003	1.48876	0.001	0.71973	0.002	0.72754	0.027
FEM	2.04122	exact	1.48877	exact	0.71975	exact	0.72734	exact

e = disc eccentricity at node 5

Table 5.25 **Normalized Blade Loss Peak Translational Response r_p/e_s**
of the Disk in a Multi-Stepped Shaft Rotor

Number of Modes	Isotropic Bearings		Orthotropic Bearings	
	$\Omega = 20,000 \text{ rpm}$	$\Omega = 40,000 \text{ rpm}$	$\Omega = 20,000 \text{ rpm}$	$\Omega = 40,000 \text{ rpm}$
1b	0.38398	0.98110	1.26324	1.34856
1f	4.33286	2.46565	4.15588	2.49806
2	4.29223	2.34496	4.03191	2.52939
4	4.29143	2.34520	4.03111	2.52688
6	4.29010	2.34198	4.02962	2.53114
8	4.28847	2.33343	4.02795	2.52143
12	4.28844	2.33331	4.02792	2.52126
16	4.28840	2.33316	4.02788	2.52111
40(FEM)	4.28829	2.33310	4.02778	2.52098

Chapter 6

Component Mode Synthesis for Multi-shaft Rotors with Flexible Inter-Shaft Bearings

6.1 Introduction

Vibration analysis of a complex rotor system is usually accomplished by analyzing its finite element model using the general theory described in Chapter 4. When the total number of degrees of freedom in the model is too large even for modern digital computers to handle economically, recourse is often taken to use the component mode synthesis (CMS) approach.

In the analysis of rotor systems, a problem is often encountered that components are connected by elements that are themselves elastic. Typical examples, in this category, are multi-rotor trains connected by flexible couplings, aircraft engine rotors with inter-shaft anti-friction bearings, and rotors supported by bearings in flexible housings. In all of these cases, the forces generated in each connecting element (flexible coupling, bearing) are dependent on the motion of the degrees of freedom at which the element is connected. Because there are no common degrees of freedom being shared at the junctions, no displacement-constraint equations can be applied.

As noted in Chapter 1, the free-interface substructure methods are much more attractive than the fixed-interface substructure methods, when experimental verifica-

tion of analytical models is required, or a structure is coupled by using the component modes from modal testing directly. Also, no interface coordinates but only substructure mode coordinates are retained in the final system equations of motion. However, the simple free-interface methods are less accurate than the fixed-interface methods.

As shown by MacNeal (first-order approximation) [53] and Rubin (second-order approximation) [54], consideration of residual flexibility and inertia-relief modes can improve convergence of the free-interface substructure methods proposed by Hou [51], and produce accuracies of system frequencies comparable with the fixed-interface methods. The modified MacNeal method (using a consistent mass transformation proposed by Craig and Chang [52]) and the Rubin method give better results. At present, to the author's knowledge, no study has been done to develop the free-interface CMS method augmented with the residual flexibility and inertia-relief modes for structures coupled by flexible interfaces, which are common in many rotor and other structural arrangements.

The study below presents a free-interface CMS method which can be applied to analysis of damped rotor systems with flexible interfaces. In the method, the equations of motion are assembled based on the Lagrangian method using the truncated normal modes of components complemented with residual flexibility and inertia-relief modes. The mode analyses for components given in the following section 6.2 can be found in the literature [52, 53, 54], and are included for the completeness of analysis. The method of assembly described in the section 6.3, leading to the equations of motion in the section 6.4, is a new contribution. The assembly equations are developed in general matrix form for structures with any number of components. A significant advantage of the present method is the fact that all flexible-interface coordinates are eliminated from the final set of system equations by using the compatibilities of coupling forces on the flexible interfaces.

6.2 Mode Analysis for Components

For a rotor system which has been fictionally divided into n substructures, the kinetic energy, potential energy, and Rayleigh dissipation function of the l -th substructure,

as shown in chapter 2 and chapter 3, are typically of the forms:

$$\left. \begin{aligned} T_l &= \frac{1}{2} \{\dot{Z}_l\}^T [M_l] \{\dot{Z}_l\} + \{\dot{Z}_l\}^T [N_l(\Omega)] \{Z\} + \frac{1}{2} \{Z_l\}^T [H_l(\Omega)] \{Z_l\} \\ V_l &= \frac{1}{2} \{Z_l\}^T ([K_l] + [K_{egl}(\Omega)]) \{Z_l\} \\ J_l &= \frac{1}{2} \{\dot{Z}_l\}^T [C_l] \{\dot{Z}_l\} \end{aligned} \right\} \quad (6.1)$$

where the second item in kinetic energy represents the Coriolis or gyroscopic forces, and the third term, behaving like an apparent potential energy, stands for the centrifugal forces due to rotation of the structure. $[K_{egl}]$ represents the geometric stiffness of in-plane stresses produced by the centrifugal forces when a disc is taken as flexible. Hence, both $[H_l(\Omega)]$ and $[K_{egl}]$ are proportional to the square of spinning speed Ω .

The linearized equations of motion of the l -th substructure of the rotor, then, can be written as:

$$\begin{aligned} [M_l] \{\ddot{Z}_l\} + ([C_l] + [G_l(\Omega)]) \{\dot{Z}_l\} + ([K_l] \\ + [K_{gl}(\Omega)] + [N_l(\dot{\Omega})]) \{Z_l\} = \{F_{el}\} + \{F_{F,l}\} \end{aligned} \quad (6.2)$$

where

$$\left. \begin{aligned} [G_l(\Omega)] &= [N_l] - [N_l]^T \\ [K_{gl}(\Omega)] &= [K_{egl}(\Omega)] - [H_l(\Omega)] \end{aligned} \right\} \quad (6.3)$$

and $\{F_{el}\}$ is the external force vector applied, $\{F_{F,l}\}$ is the flexible junction force vector applied by other adjacent substructures through elastic connectors, such as flexible inter-shaft bearings.

The method of component mode synthesis developed here applies to any system whose equations of motion can be expressed in this form. The free interface normal modes of the substructure are obtained by setting the external, damping, gyroscopic, and centrifugal forces as zeros in Eq. 6.2 and solving the resulting equation:

$$[M_l] \{\ddot{Z}_l\} + [K_l] \{Z_l\} = \{0\} \quad (6.4)$$

The substructure modes, $\{\phi_i\}$, may be normalized so that

$$\{\phi_i\}^T [M_l] \{\phi_j\} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} \quad (6.5)$$

$$\{\phi_i\}^T [K_l] \{\phi_j\} = \begin{cases} \omega_i^2, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} \quad (6.6)$$

where the ω_i 's are the natural frequencies of the substructure.

The modes are assembled to form the modal matrix $[\Phi_l]$, which defines substructure generalized coordinates, $\{p_l\}$, through the equation

$$\{Z_l\} = [\Phi_l] \{p_l\} \quad (6.7)$$

This modal matrix may be partitioned as follows:

$$[\Phi_l] = [\Phi_{rl} \ \Phi_{kl} \ \Phi_{al}] \quad (6.8)$$

where the subscript "r" stands for rigid body, "k" for kept, "a" for approximated. In the following analysis, the full modal information of the "r" and "k" modes will be kept, but the "a" modes will be approximated.

In the expansion theory, the high frequency modes, for which the natural frequencies, ω_i 's, are much larger than the excitation frequency, can be approximated by the static effects only. This corresponds to the concept of "residual flexibility". As pointed out by MacNeal [53] and Rubin [54], consideration of residual flexibility of these high frequency modes can greatly improve convergence of the free-interface substructure method proposed by Hou [51]. Furthermore, these high frequency modes do not need to be obtained explicitly.

If the dynamic terms in Eq. 6.2 were neglected, a pseudo-static response, $\{Z_{fl}\}$, to the junction force vector $\{F_{Fjl}\}$ could be obtained by solving the equation

$$[K_l] \{Z_{fl}\} = \{F_{Fjl}\} \quad (6.9)$$

or

$$\{Z_{fl}\} = [G_{al}] \{F_{Fjl}\} \quad (6.10)$$

with

$$[G_{el}] = [K_l]^{-1} \quad (6.11)$$

If the substructure has rigid-body freedom, the stiffness matrix $[K_l]$ will be singular and the solution of the above equation requires special consideration. Thus, the MacNeal's approximation method will be first described for constrained substructures, and then approximation methods will be developed for substructures with rigid body motion.

6.2.1 Constrained Substructures

For the constrained substructures (i.e., without rigid-body freedom), the modal matrix in Eq. 6.7 becomes

$$[\Phi_l] = [\Phi_{kl} \ \Phi_{al}] \quad (6.12)$$

Hence the pseudo-static response, based on Eq. 6.9 will be

$$\{Z_{fl}\} = [\Phi_k]\{p_{kl}\} + [\Phi_{al}]\{p_{al}\} \quad (6.13)$$

or

$$\{Z_{fl}\} = \{Z_{kl}\} + \{Z_{al}\} \quad (6.14)$$

where

$$\{Z_{kl}\} = [\Phi_{kl}]\{p_{kl}\} \quad (6.15)$$

and

$$\{Z_{al}\} = [\Phi_{al}]\{p_{al}\} \quad (6.16)$$

Since the modes in $[\Phi_{al}]$ are not, in general, available, it is necessary to use Eq. 6.9 and Eq. 6.13 to obtain $\{Z_{al}\}$. Let Eq. 6.13 be substituted into Eq. 6.9 and let the resulting equation be pre-multiplied by $[\Phi_l]^T$, we obtain

$$\begin{bmatrix} \Phi_{kl}^T \\ \Phi_{al}^T \end{bmatrix} [K_l] [\Phi_{kl} \ \Phi_{al}] \begin{Bmatrix} p_{kl} \\ p_{al} \end{Bmatrix} = \begin{bmatrix} \Phi_{kl}^T \\ \Phi_{al}^T \end{bmatrix} \{F_{Fjl}\} \quad (6.17)$$

Since all of the modes are mutually orthogonal with each other, so it can be obtained from Eqs. 6.5 and 6.6 that

$$[\Phi_{kl}]^T [K_l] [\Phi_{al}] = [\Phi_{al}]^T [K_l] [\Phi_{kl}] = [0] \quad (6.18)$$

and

$$\left. \begin{aligned} [\Phi_{kl}]^T [K_l] [\Phi_{kl}] &= [\Lambda_{kl}] \\ [\Phi_{al}]^T [K_l] [\Phi_{al}] &= [\Lambda_{al}] \end{aligned} \right\} \quad (6.19)$$

where $[\Lambda_{kl}]$ and $[\Lambda_{al}]$ are diagonal matrices composed of the respective values of natural frequencies, ω_i^2 , of the l -th substructure. Thus, from the above equations, an expression for the contribution of the approximated modes to the pseudo-static response is obtained as

$$\{Z_{al}\} = [G_{al}] \{F_{Fjl}\} \quad (6.20)$$

where

$$[G_{al}] = [G_{el}] - [G_{kl}] \quad (6.21)$$

and

$$[G_{kl}] = [\Phi_{kl}] [\Lambda_{kl}^{-1}] [\Phi_{kl}]^T \quad (6.22)$$

Eq. 6.20 expresses what Rubin calls the first-order contribution of the residual, or approximated, modes $[\Phi_{al}]$.

The first-order approximation method may be summarized by the following equation:

$$\{Z_l\} = [\Phi_{kl}] \{p_{kl}\} + [G_{al}] \{F_{Fjl}\} \quad (6.23)$$

When substructures are coupled to form the complete structure, the condensed junction force vector, $\{f_{Fjl}\}$, of the substructure in stead of the vector $\{F_{Fjl}\}$ is required. These are given by the appropriate partition of Eq. 6.23, namely

$$\{Z_l\} = [\Phi_{kl}]\{p_{kl}\} + [G_{Fjl}]\{f_{Fjl}\} \quad (6.24)$$

or as the partitioned matrix form:

$$\begin{Bmatrix} z_{Il} \\ z_{Fjl} \end{Bmatrix} = \begin{bmatrix} \Phi_{Il} \\ \Phi_{Fjl} \end{bmatrix} \{p_{kl}\} + \begin{bmatrix} G_{IFl} \\ G_{FFl} \end{bmatrix} \{f_{Fjl}\} \quad (6.25)$$

where the subscript "I" stands for "Internal coordinates", "Fj" for "Flexible Junction".

$\{z_{Fjl}\}$ is the physical coordinates on flexible interfaces of the l -th substructure.

$[G_{Fjl}]$ is the appropriate columns from the residual flexibility matrix of the l -th substructure on its flexible interfaces.

$\{f_{Fjl}\}$ is the condensed flexible-interface forces of the l -th substructure applied by all adjacent flexible junction elements.

Eq. 6.24 can be further written as the following transformation:

$$\{Z_l\} = [\hat{\Phi}_l]\{\hat{Z}_l\} \quad (6.26)$$

with

$$\{\hat{Z}_l\} = \begin{Bmatrix} p_{kl} \\ f_{Fjl} \end{Bmatrix}, \quad [\hat{\Phi}_l] = \begin{bmatrix} \Phi_{kl} & G_{Fjl} \end{bmatrix}$$

Substituting Eq. 6.26 into Eq. 6.2 and pre-multiplying it by $[\hat{\Phi}_l]^T$, we obtain the transformed equations of motion of the component:

$$\begin{aligned} [\hat{M}_l] \{\ddot{\hat{Z}}_l\} + ([\hat{C}_l] + [\hat{G}_l(\Omega)]) \{\dot{\hat{Z}}_l\} + ([\hat{K}_l] \\ + [\hat{K}_{gl}(\Omega)] + [\hat{N}_l(\dot{\Omega})]) \{\hat{Z}_l\} = \{\hat{F}_{el}\} + \{\hat{F}_{Fjl}\} \end{aligned} \quad (6.27)$$

with

$$[\hat{M}_l] = [\hat{\Phi}_l]^T [M_l] [\hat{\Phi}_l] = \begin{bmatrix} I_l & 0 \\ 0 & G_{Fjl}^T M_l G_{Fjl} \end{bmatrix}$$

$$[\hat{K}_l] = [\hat{\Phi}_l]^T [K_l] [\hat{\Phi}_l] = \begin{bmatrix} I_l & 0 \\ 0 & G_{FFl} \end{bmatrix}$$

$$[\hat{C}_l] = [\hat{\Phi}_l]^T [C_l] [\hat{\Phi}_l], \quad [\hat{G}_l] = [\hat{\Phi}_l]^T [G_l] [\hat{\Phi}_l]$$

$$[\hat{K}_{gl}] = [\hat{\Phi}_l]^T [K_{gl}] [\hat{\Phi}_l], \quad [\hat{N}_l] = [\hat{\Phi}_l]^T [N_l] [\hat{\Phi}_l]$$

$$\{\hat{F}_{el}\} = [\hat{\Phi}_l]^T \{F_{el}\}, \quad \{\hat{F}_{Fjl}\} = [\hat{\Phi}_l]^T \{F_{Fjl}\}$$

6.2.2 Unconstrained Substructures

Since the component stiffness matrix $[K_l]$ is singular when a substructure possesses one or more rigid-body degrees of freedom, the flexibility matrix, $[G_{el}]$, used in the preceding analysis does not exist. On the other hand, since aerospace structures by nature possess rigid-body freedom, it is important that methods be developed to analyze such structures by the component mode methods similar to the forgoing ones.

For a substructure possessing rigid-body freedom, let the displacement be given by

$$\{Z_l\} = \{Z_{rl}\} + \{Z_{fl}\} \quad (6.28)$$

where $\{Z_{rl}\}$ is rigid-body motion and $\{Z_{fl}\}$ is flexible motion. Also, let the modes of the structure be separated into rigid-body and flexible modes such that

$$\{Z_l\} = [\Phi_{rl}]\{p_{rl}\} + [\Phi_{fl}]\{p_{fl}\} \quad (6.29)$$

In the notation of preceding sections all of the rigid-body modes, i.e., $[\Phi_{rl}]$, would be kept together with a limited number of flexible modes. Since $\{Z_{fl}\}$ cannot be

obtained by inverting $[K_l]$ from Eq. 6.9, it is necessary to "remove" the rigid-body motion from the model.

Considering the response of the rigid-body motion to the junction force vector, $\{F_{Fjl}\}$, and neglecting all the gyroscopic and centrifugal forces in Eq 6.2, we have the equations of rigid-body motion in the form

$$\{\tilde{p}_{rl}\} = [\Phi_{rl}]^T \{F_{Fjl}\} \quad (6.30)$$

From Eq. 6.30 it is seen that, as far as rigid body motion is concerned, the junction forces are reacted by rigid-body inertia forces. Hence, if these rigid-body inertia forces are subtracted from the total applied forces, there will be no excitation of rigid-body modes. The physical displacement vector due to rigid-body modes is given by

$$\{Z_{rl}\} = [\Phi_{rl}]\{p_{rl}\} \quad (6.31)$$

Hence, the inertia force vector due to rigid-body motion of the substructure is given by

$$\{F_i\} = -[M_l]\{\ddot{Z}_{rl}\} = -[M_l][\Phi_{rl}][\Phi_{rl}]^T \{F_{Fjl}\} \quad (6.32)$$

Thus, the net force producing pseudo-static flexible motion without any external forces is given by

$$\{F_{fl}\} = \{F_{Fjl}\} + \{F_i\} = [A]\{F_{Fjl}\} \quad (6.33)$$

where

$$[A_l] = [I] - [M_l][\Phi_{rl}][\Phi_{rl}]^T \quad (6.34)$$

To determine the flexible-body motion resulting from $\{F_{fl}\}$ it is necessary to impose temporary constraints on the substructure so that rigid-body motion is prevented. Let $[G_d]$ be the flexibility matrix of the structure which has been subjected to arbitrary statically-determinate constraints. $[G_d]$ may be obtained by taking $[K_l]$,

deleting rows and columns corresponding to the constrained degrees of freedom, inverting to get a flexibility matrix, and then expanding the flexibility matrix by inserting zeros in rows and columns corresponding to the constraints. The resulting augmented flexibility matrix is called $[G_d]$.

The new displacements relative to the imposed constraints are derived by

$$\{Z_d\} = [G_d]\{F_{jl}\} = [G_d][A_l]\{F_{Fjl}\} \quad (6.35)$$

To obtain the flexibility displacements, $\{Z_{jl}\}$, rigid-body motion is removed from $\{Z_d\}$ by making $\{Z_{jl}\}$ orthogonal to all rigid-body modes, i.e.,

$$[\Phi_{rl}]^T[M_l]\{Z_{jl}\} = \{0\} \quad (6.36)$$

where

$$\{Z_{jl}\} = \{Z_d\} + [\Phi_{rl}]\{p_{rl}\} \quad (6.37)$$

Hence, from the above two equations, it can be derived that

$$\{Z_{jl}\} = [A_l]^T\{Z_d\} = [A_l]^T[G_d][A_l]\{F_{Fjl}\} \quad (6.38)$$

Comparing Eq. 6.9 with Eq. 6.38, it is clear that the flexibility matrix $[G_{el}]$ for a free substructure is defined by

$$[G_{el}] = [A_l]^T[G_d][A_l] \quad (6.39)$$

With this definition of $[G_{el}]$ for a free substructure, the remainder of the derivation proceeds as for a constrained substructure, beginning with Eq. 6.12.

6.3 Assembly of Substructures

In the preceding sections the substructure modes have been obtained for both constrained and free substructures. In the present section equations will be developed for coupled substructures. Eq. 6.24 is the pertinent substructure equations for the substructure assembly.

6.3.1 Energy Expressions for a Complete Rotor System

The kinetic energy, potential energy, and Rayleigh dissipation function of a complete rotor system with flexible bladed discs are given by:

$$T = \frac{1}{2}\{\dot{Z}\}^T[M]\{\dot{Z}\} + \{\dot{Z}\}^T[N(\Omega)]\{Z\} + \frac{1}{2}\{Z\}^T[H(\Omega)]\{Z\} \quad (6.10)$$

$$V = \frac{1}{2}\{Z\}^T([K] + [K_{eg}(\Omega)])\{Z\} + \frac{1}{2}\{Z_{Fj}\}^T[K_{Fj}]\{Z_{Fj}\} \quad (6.41)$$

$$J = \frac{1}{2}\{\dot{Z}\}^T[C]\{\dot{Z}\} + \frac{1}{2}\{\dot{Z}\}_{Fj}^T[C_{Fj}]\{\dot{Z}\}_{Fj} \quad (6.42)$$

Since the structure has been fictionally divided into n substructures, the definitions of items in Eqs. 6.40, 6.41, and 6.42 can be written as

$$\begin{aligned} \{Z\} &= \{Z_1^T \ Z_2^T \ \cdots \ Z_n^T\}^T \\ [M] &= \text{diag}\{M_1, M_2, \dots, M_n\} \\ [N] &= \text{diag}\{N_1, N_2, \dots, N_n\} \\ [H] &= \text{diag}\{H_1, H_2, \dots, H_n\} \\ [K] &= \text{diag}\{K_1, K_2, \dots, K_n\} \\ [K_{eg}] &= \text{diag}\{K_{eg1}, K_{eg2}, \dots, K_{egn}\} \\ [C] &= \text{diag}\{C_1, C_2, \dots, C_n\} \end{aligned}$$

and $\{Z_{Fj}\}$ is a vector of coordinates on all the flexible interfaces of a structure. $[K_{Fj}]$ and $[C_{Fj}]$ are the corresponding stiffness and damping matrices of the flexible interfaces of the structure, respectively.

6.3.2 Transformations for Assembly

The component mode coordinates, $\{p_{kl}\}$, can be assembled into the mode coordinates, $\{P_s\}$, of the structure as follows

$$\{P_s\} = \begin{Bmatrix} p_{k1} \\ p_{k2} \\ \vdots \\ p_{kn} \end{Bmatrix} \quad (6.43)$$

by the relation:

$$\{p_{kl}\} = [E_l]\{P_s\} \quad (6.44)$$

where

$$[E_l] = [0 \cdots I_l \cdots 0]$$

and I_l is a $m_l \times m_l$ unit matrix where m_l is the number of retained modes of the l -th substructure.

6.3.3 Compatibility on Flexible Interfaces

The displacements and forces, $\{z_{Fjl}\}$, $\{f_{Fjl}\}$, on the flexible interfaces of the l -th substructure can be related to the flexible-interface coordinates and forces, $\{Z_{Fj}\}$, $\{F_{Fj}\}$, of the assembled structure as follows

$$\{z_{Fjl}\} = [S_l]\{Z_{Fj}\} \quad (6.45)$$

$$\{f_{Fjl}\} = [S_l]\{F_{Fj}\} \quad (6.46)$$

where elements of the transformation matrix, $[S_l]$, are composed of either 0, 1, or -1.

From Eqs. 6.45 and 6.46, we obtain the following assembly equations of the flexible-interface coordinates and forces of the whole structure:

$$\{Z_{Fj}\} = \sum_{l=1}^n [S_l]^T \{z_{Fjl}\} \quad (6.47)$$

$$\{F_{Fj}\} = \sum_{l=1}^n [S_l]^T \{f_{Fjl}\} = -[K_{Fj}]\{Z_{Fj}\} \quad (6.48)$$

Substituting Eqs. 6.25 and 6.46 into Eq. 6.47, solving for $\{Z_{Fj}\}$, we get

$$\{Z_{Fj}\} = [\Psi_F]\{P_s\} \quad (6.49)$$

where

$$\left. \begin{aligned} \{\Psi_F\} &= ([I] + [G_{FF}][K_{Fj}])^{-1}[\Phi_{Fj}] \\ [G_{FF}] &= \sum_{l=1}^n [S_l]^T [G_{FFl}] [S_l] \\ [\Phi_{Fj}] &= \sum_{l=1}^n [S_l]^T [\Phi_{Fjl}] [E_l] \end{aligned} \right\} \quad (6.50)$$

Substituting Eq. 6.49 into Eq. 6.48, we obtain the interface forces in terms of mode coordinates of the system as

$$\{F_{Fj}\} = -[\hat{K}_F]\{P_s\} \quad (6.51)$$

where

$$[\hat{K}_F] = [K_{Fj}][\Psi_F]$$

Substituting Eqs. 6.45, 6.46, 6.49, and 6.51 into Eq. 6.24, we obtain the mode transformation for all substructures as follows

$$\{Z_l\} = ([\Phi_{kl}][E_l] - [G_{Fjl}][S_l][\hat{K}_F])\{P_s\} \quad \text{for } l = 1, 2, \dots, n \quad (6.52)$$

or as the matrix form

$$\{Z\} = [R_s]\{P_s\} \quad (6.53)$$

with

$$[R_s] = \begin{bmatrix} \Phi_{k1}E_1 - G_{Fj1}S_1\hat{K}_F \\ \Phi_{k2}E_2 - G_{Fj2}S_2\hat{K}_F \\ \vdots \\ \Phi_{kn}E_n - G_{Fjn}S_n\hat{K}_F \end{bmatrix}$$

6.4 Equations of Motion of a Coupled Structure

Substituting Eqs. 6.49 and 6.53 into energy equations 6.40, 6.41 and 6.42, we can re-express the kinetic energy, potential energy and Rayleigh dissipation function of the structure in terms of the generalized component mode coordinates $\{P_s\}$ as

$$T = \frac{1}{2}\{\dot{P}_s\}^T[M_s]\{\dot{P}_s\} + \{\dot{P}_s\}^T[N_s(\omega)]\{P_s\} + \frac{1}{2}\{P_s\}^T[H_s(\Omega)]\{P_s\} \quad (6.54)$$

$$V = \frac{1}{2}\{P_s\}^T([K_s] + [K_{egs}(\Omega)])\{P_s\} \quad (6.55)$$

$$J = \frac{1}{2}\{\dot{P}_s\}^T[C_s]\{\dot{P}_s\} \quad (6.56)$$

with the relationships:

$$\left. \begin{aligned} [M_s] &= [R_s]^T[M][R_s] = \sum_{l=1}^n [\Upsilon_l]^T[\hat{M}_l][\Upsilon_l] \\ [K_s] &= [\Psi_F]^T[K_{Fj}][\Psi_F] + \sum_{l=1}^n [\Upsilon_l]^T[\hat{K}_l][\Upsilon_l] \\ [K_{egs}] &= [R_s]^T[K_{eg}][R_s] = \sum_{l=1}^n [\Upsilon_l]^T[\hat{K}_{egl}][\Upsilon_l] \\ [H_s] &= [R_s]^T[H][R_s] = \sum_{l=1}^n [\Upsilon_l]^T[\hat{H}_l][\Upsilon_l] \\ [C_s] &= [\Psi_F]^T[C_{Fj}][\Psi_F] + \sum_{l=1}^n [\Upsilon_l]^T[\hat{C}_l][\Upsilon_l] \\ [N_s] &= [R_s]^T[N][R_s] = \sum_{l=1}^n [\Upsilon_l]^T[\hat{N}_l][\Upsilon_l] \end{aligned} \right\} \quad (6.57)$$

and

$$[\Upsilon_l] = \begin{bmatrix} E_l \\ -S_l \hat{K}_F \end{bmatrix} \quad (6.58)$$

The Lagrangian equation for the coupled structure are given by

$$\frac{\partial}{\partial t} \left(\frac{\partial L}{\partial \{\dot{P}_s\}} \right) - \frac{\partial L}{\partial \{P_s\}} + \frac{\partial J}{\partial \{\dot{P}_s\}} = \{Q_s\} \quad (6.59)$$

where

$$L = T - V \quad (\text{Lagrangian function})$$

$\{Q_s\}$ is the generalized applied force vector, which denotes all the remaining forces not falling into any of the categories of forces, defined at the beginning of the second section, such as, external applied forces, unbalanced inertia forces or moments and so on.

$\{Q_s\}$ can be obtained from the virtual work expression

$$\delta W = \{Q_s\}^T \delta \{P_s\} \quad (6.60)$$

If we denote the applied forces of all the uncoupled substructures by

$$\{F\} = \begin{Bmatrix} F_{e1} \\ F_{e2} \\ \vdots \\ F_{en} \end{Bmatrix}$$

we have the following relationship:

$$\{Q_s\} = [R_s]^T \{F\} = \sum_{l=1}^n [\Upsilon_l]^T [\hat{F}_{el}] \quad (6.61)$$

Substituting Eqs. 6.54, 6.55 and 6.56 into Eq. 6.59, we obtain the equation of motion for the coupled structure as

$$[M_s]\{\ddot{P}_s\} + ([C_s] + [G_s(\Omega)])\{\dot{P}_s\} + ([K_s] + [K_{gs}(\Omega)] + [N_s(\dot{\Omega})])\{P_s\} = \{Q_s\} \quad (6.62)$$

where

$$\left. \begin{aligned} [G_s(\Omega)] &= [N_s(\Omega)] - [N_s(\Omega)]^T && \text{skew symmetric} \\ [K_{gs}(\Omega)] &= [K_{egs}(\Omega)] - [H_s(\Omega)] && \text{symmetric} \end{aligned} \right\} \quad (6.63)$$

From Eq. 6.3, Eq. 6.57 and Eq. 6.63, it can be proved that these two global matrices can be directly assembled from the corresponding component matrices by

$$\left. \begin{aligned} [G_s(\Omega)] &= [R_s]^T [G(\Omega)] [R_s] = \sum_{l=1}^n [\Upsilon_l]^T [\hat{G}_l(\Omega)] [\Upsilon_l] \\ [K_{gs}(\Omega)] &= [R_s]^T ([K_{eg}(\Omega)] - [H(\Omega)]) [R_s] = \sum_{l=1}^n [\Upsilon_l]^T [\hat{K}_{gl}(\Omega)] [\Upsilon_l] \end{aligned} \right\} \quad (6.64)$$

It follows from Eq. 6.57, Eq. 6.61 and Eq. 6.64 that the assembly of the equations of motion for the whole structure can be performed either in the rotating coordinate system, or more often, in the fixed coordinate system.

6.5 Numerical Examples

Two dual-spool rotor systems are employed to illustrate the accuracy of the present component mode synthesis with various levels of modal truncation, versus the finite element solution of full order. The first rotor had been studied by Glasgow and Nelson [60, 61] using (fixed-interface) complex component modes, and the second rotor is a two-spool aircraft turbine engine studied by Li and Gunter [17, 64] using Hou's free-interface component mode method, which allows correlation with their published results.

The choice of the component normal modes is based on comparison of the undamped natural frequencies of components to the maximum running speed of the system. An upper limit is set on the frequencies of component modes, and only the modes with frequencies lower than this limit are retained as the "kept" modes. This upper limit will be called the truncation frequency throughout the thesis from now on.

6.5.1 A Dual-Shaft Rotor System

The first example is a dual-shaft rotor system as shown in Figure 6.1, which has been studied by Glasgow and Nelson [60, 61]. The inner low speed rotor and the outer high speed rotor have rotating speeds of 10,000 rpm and 15,000 rpm, and are modeled as assemblies of five and three rotating shaft elements, respectively. Hence, the rotor model has 10 nodes and 40 degrees of freedom. Table 6.1 and Table 6.2 present finite element model data of the rotor. The data is given in English units only as presented in the source references [60, 61], and for reference, the metric data is given in Table A.2, Appendix A.

The rotor system is by nature divided into two components, the inner low speed

rotor and the outer high speed rotor. The inter-shaft bearing between them acts as the flexible interface element with the stiffness and damping. This dual-shaft rotor system is studied for both undamped isotropic and damped isotropic bearings with properties given in Table 6.3 (The metric data is tabulated in Table A.2, Appendix A).

There is a total of 4 DOF on the flexible interfaces of the rotor system. The undamped natural frequencies of the two components are listed in the Table 6.4.

Three component mode truncation schemes are considered for calculation of whirl speeds and damping exponents of the system, as shown in Table 6.5.

1. The reduced problem size of the system is 12 degrees of freedom, with 4 planar normal modes from the inner-shaft rotor and 2 planar normal modes from the outer-shaft rotor.
2. The reduced problem size of the system is 18 degrees of freedom, with 5 planar normal modes from the inner-shaft rotor and 4 planar normal modes from the outer-shaft rotor.
3. The reduced problem size of the system is 28 degrees of freedom, with 9 planar normal modes from the inner-shaft rotor and 5 planar normal modes from the outer-shaft rotor.

The whirl frequencies and damping exponents at various levels of component mode truncation are presented in Table 6.6 and Table 6.7 for the undamped and damped bearings, respectively. The first columns in the tables are the finite element solutions which are taken as the exact solutions of the rotor. As shown, very good accuracy in frequencies and in damping exponents can be achieved even with high truncation of component modes.

In References [60, 61], no finite element solution of the rotor has been shown, and the untruncated CMS solution with all 30 constrained precessional modes and 10 constraint modes was taken as the exact solution. Unfortunately, the solution does not seem to be consistent with the given data of the rotor. One possible reason for this may be the use of the single precision in their numerical analysis. The other possible reason may be the improper handling of constraint modes in fact,

there exists only a total of 4 constraint modes associated with the 4 coordinates to which the inter-shaft bearing is connected. For the purpose of comparison, errors of their method in whirl frequencies and log decrements are calculated based directly on their published results, i.e., on their truncated CMS solution with respect to the untruncated CMS solution. As shown in Figure 6.2 to Figure 6.4, the current free-interface method is seen to exhibit much better accuracy and convergence of whirl frequencies and damping exponents than the complex fixed-interface CMS method presented in Ref. [60, 61], which utilised complex component modes.

6.5.2 A Two-Spool Aircraft Turbine Engine

Figure 6.5 shows a two-spool gas turbine engine employed by Li and Gunter [63, 64] to study the efficiency of component mode synthesis. The basic engine consists of an inner rotor called the power turbine, and an outer rotor called the gas generator turbine. The inner rotor is supported by two main bearings located near the shaft ends, and the two stage gas generator turbine, which drives an axial compressor, is supported by four rolling element bearings. There are several intermediate bearings connecting the inner power turbine to the outer gas generator turbine.

The power and gas-generator turbines are modeled as 21 and 15 rotating shaft elements, respectively. So the rotor model has 38 nodes and a total of 152 degrees of freedom. Table 6.8 shows the cross-sectional properties of the engine in English units (the metric data for this engine is tabulated in Table A.3, Appendix A, for reference). Polar moments of inertia (to account for gyroscopic moments) in the rotor are considered only at the turbine and compressor stages, which are represented by 13 discs as shown in Table 6.9.

The dynamic characteristics of the rotor system are calculated at various levels of mode truncation for two bearing cases:

1. Undamped isotropic bearing case.

In this case, there are two intermediate differential bearings between two rotors. All the bearing properties and locations are presented in Table 6.10.

2. Damped Bearing Case.

In these studies, the bearing located at node 35 on the gas generator is replaced by a squeeze film damper with a stiffness of 123,000 *lb/in* and damping 100 *lb-s/in*. In addition, an inter-shaft squeeze film bearing is incorporated between the node 11 on the power turbine and the node 31 on the gas generator turbine. The dynamic coefficients of the bearings, taken from Ref. [63], are tabulated in Table 6.11.

The rotor system is by nature divided into two components, the power turbine rotor and the gas generator rotor. The inter-shaft bearings between them act as the flexible interface elements. There is a total of 8 DOF and 12 DOF on the flexible interfaces for undamped and damped bearing cases, respectively. Since both rotors have isotropic bearings, only undamped non-rotating modes in one plane need to be calculated. The natural frequencies of the two components are listed in the Table 6.12 for both bearing cases.

In order to compare the current CMS method with the results of Hou's method obtained by Li and Gunter [17], three component mode truncation schemes are considered for calculations and are shown in Table 6.13:

1. The reduced problem size of the system is 12 degrees of freedom, with 2 planar normal modes from the gas generator rotor and 4 planar normal modes from the power turbine rotor.
2. The reduced problem size of the system is 18 degrees of freedom, with 3 planar normal modes from the gas generator rotor and 6 planar normal modes from the power turbine rotor.
3. The reduced problem size of the system is 28 degrees of freedom, with 5 planar normal modes from the gas generator rotor and 9 planar normal modes from the power turbine rotor.

Whirl Frequency Analysis

The whirl frequencies and damping exponents at various levels of mode truncation are tabulated in Table 6.14 and Table 6.15 for cases 1 and 2, respectively. Figure 6.6 and Figure 6.7 present errors in whirl frequencies of first four forward whirl modes in cases 1 and 2. Errors in damping exponents of the corresponding whirl modes are shown in Figure 6.8. Errors, given by Li and Gunter [63] using the free-interface component mode method without consideration of residual flexibility of high frequency modes, are also included for comparison.

As shown, inclusion of contribution of high frequency modes can improve accuracy and convergence of both frequencies and damping exponents. Accuracy in modal frequencies is seen to be better than that in modal damping exponents. Thus it is interesting to note that errors in all whirl frequencies for both bearing cases are less than 0.3% with the 28 DOF CMS model, that is, with a reduction in model size by a factor of five times.

Unbalance Response Analysis

The forced response of the gas turbine due to mass unbalance in the power turbine rotor is calculated with the same damped bearing configurations described in the stability analysis. For this analysis, two unbalances of 5 oz · in each are assumed to be present at the mid-span (node 9) of the power turbine rotor and at the second-stage turbine (node 20) phased 180° apart from each other. The gas generator rotor is taken to be balanced and operating at 15,000 *rpm* while the speed of power turbine rotor varies from 1,000 *rpm* to 20,000 *rpm*.

In the calculation, the dynamic coefficients of the bearings, including those of the squeeze film damper and the inter-shaft journal bearings, are considered to be constant.

The unbalance amplitude and phase responses of the aft bearing force between nodes 13 and 33 are plotted versus the power turbine speed in Figure 6.9 and Figure 6.10 for various levels of modal truncation. Because the mass unbalances in the rotor system generate forces which rotate in the same direction as the power turbine,

the responses of forward modes of the system dominate. Inspection of the figures reveals the high quality of convergence of the current free-interface component mode synthesis for both amplitude and phase responses.

Figure 6.1 A Dual-Shaft Rotor Syetm

(a) 1st forward mode

(b) 2nd forward mode

(c) 3rd forward mode

(d) 4th forward mode

Figure 6.2 Undamped Frequency Errors of a Dual-Shaft Rotor
 $\Omega_i = 10,000$ rpm , $\Omega_o = 15,000$ rpm

●●●●● The Current Method
 ○○○○○ Fixed-Interface Complex CMS [60,61]

(a) 1st forward mode

(b) 2nd forward mode

(c) 3rd forward mode

(d) 4th forward mode

Figure 6.3 Damped Frequency Errors of a Dual-Shaft Rotor
 $\Omega_1 = 10,000$ rpm , $\Omega_0 = 15,000$ rpm

●●●●● The Current Method
 ○○○○○ Fixed-Interface Complex CMS [60,61]

(a) 1st forward mode

(b) 2nd forward mode

(c) 3rd forward mode

(d) 4th forward mode

Figure 6.4 Log Decrement Errors of a Dual-Shaft Rotor
 $\Omega_1 = 10,000$ rpm , $\Omega_2 = 15,000$ rpm

●●●●● The Current Method

○○○○○ Fixed-Interface Complex CMS [60,61]

Figure 6.5 A Two-Spool Aircraft Turbine Engine

(a) 1st forward mode

(b) 2nd forward mode

(c) 3rd forward mode

(d) 4th forward mode

Figure 6.6 Frequency Errors of a Two-Spool Turbine Engine
 $\Omega_p = 20,000$ rpm , $\Omega_g = 15,000$ rpm

●●●●● The Current Method
 ○○○○○ Hou's Method by Li and Gunter [64]

(a) 1st forward mode

(b) 2nd forward mode

(c) 3rd forward mode

(d) 4th forward mode

Figure 6.~ Frequency Errors of a Two-Spool Turbine Engine
 $\Omega_p = 17,000$ rpm , $\Omega_g = 15,000$ rpm

●●●●● The Current Method
 ○○○○○ Hou's Method by Li and Gunter [64]

(a) 1st forward mode

(b) 2nd forward mode

(c) 3rd forward mode

(d) 4th forward mode

Figure 6.8 Damping Exponent Errors of a Two-Spool Engine
 $\Omega_p = 17,000$ rpm, $\Omega_g = 15,000$ rpm

●●●●● The Current Method
 ○○○○○ Hou's Method by Li and Gunter [64]

Figure 6.9 Unbalance Amplitude Response of Aft Bearing Force Between Nodes 13 and 33

Figure 6.10 Unbalance Phase Response of Aft Bearing Force Between Nodes 13-33

Table 6.1 Element Properties of a Dual-Shaft Rotor

Element Number	Node Number	Length (in)	Inner Dia. (in)	Outer Dia. (in)
1	1 - 2	3.0	0.0	1.2
2	2 - 3	7.0	0.0	1.2
3	3 - 4	6.0	0.0	1.2
4	4 - 5	2.0	0.0	1.2
5	5 - 6	2.0	0.0	1.2
6	7 - 8	2.0	2.0	2.4
7	8 - 9	6.0	2.0	2.4
8	9 - 10	2.0	2.0	2.4
$\rho = 0.283 \text{ lbm/in}^3$ $E = 3 \times 10^7 \text{ lb/in}^2$				

Table 6.2 Disk Properties of a Dual-Shaft Rotor

Disk Number	Disk Location	M_d (lbm)	I_p (lbm · in ²)	I_d (lbm · in ²)
1	2	23.16	293.36	146.68
2	5	15.44	231.60	115.80
3	8	15.44	146.68	73.34
4	9	7.72	92.64	46.32

Table 6.3 Bearing Properties of a Dual-Shaft Rotor

Bearing Number	Bearing Location	$K_{xx} = K_{yy}$ (lb/in)	$C_{xx} = C_{yy}$ (lb · s/in)
1	1 - 0	150,000	30.0
2	6 - 0	100,000	20.0
3	7 - 0	100,000	20.0
4	4 - 10	50,000	10.0

Table 6.4 Component Natural Frequencies of a Dual-Shaft Rotor
(only one-planar modes are listed)

Inner Shaft Rotor				Outer Shaft Rotor	
Mode Order	ω (rad/s)	Mode Order	ω (rad/s)	Mode Order	ω (rad/s)
1	639.00	6	19120.	1	0.0000
2	1476.5	7	28191.	2	1806.6
3	2030.9	8	42520.	3	7291.6
4	2364.9	9	53740.	4	16215.
5	6707.3	10	133794	5	107959

Table 6.5 Component Mode Selection Schemes
of a Dual-Shaft Rotor System

Truncation Frequencies ω_t	Component Modes Retained		Total DOF of CMS Model
	Inner Rotor	Outer Rotor	
3000.0 rad/s	8	4	12
17000. rad/s	10	8	18
110000 rad/s	18	10	28

**Table 6.6 Undamped Whirl Frequencies of a Dual-Shaft Rotor
at Various Component Modal Truncation Levels**

(the inner shaft runs at a speed of 10,000 rpm)

(the outer shaft runs at a speed of 15,000 rpm)

Mode No.	F.E.M. 40 d.o.f.	Whirl Frequencies (rad/s)					
		12 d.o.f.	error %	18 d.o.f.	error %	28 d.o.f.	error %
1b	343.51	343.86	0.101	343.51	0.000	343.51	0.000
1f	829.94	831.01	0.129	829.94	0.000	829.94	0.000
2b	1030.65	1031.89	0.120	1030.66	0.001	1030.65	0.000
2f	1244.02	1244.39	0.030	1244.03	0.001	1244.02	0.000
3b	1467.21	1468.79	0.108	1467.27	0.004	1467.22	0.001
3f	1805.47	1806.15	0.038	1805.48	0.001	1805.47	0.000
4b	1581.24	1583.33	0.132	1581.24	0.000	1581.24	0.000
4f	2647.75	2661.37	0.514	2647.77	0.001	2647.75	0.000
5b	1707.34	1708.33	0.058	1707.38	0.002	1707.34	0.000
5f	2990.78	3013.23	0.751	2990.85	0.002	2990.78	0.000
6b	2023.68	2026.55	0.142	2023.69	0.000	2023.68	0.000
6f	3589.19	3589.89	0.022	3589.62	0.014	3589.10	0.000
7b	6031.32			6031.35	0.000	6031.33	0.000
7f	6886.03			6886.07	0.001	6886.03	0.000
8b	6621.04			6621.05	0.000	6621.04	0.000
8f	8996.89			8996.96	0.001	8996.91	0.000
9b	15291.1			15291.1	0.000	15291.1	0.000
9f	17333.7			17333.8	0.001	17333.7	0.000
10b	19164.9					19164.9	0.000
10f	19240.8					19240.8	0.000
11b	28132.5					28132.5	0.000
11f	28250.6					28250.6	0.000
12b	42581.0					42581.0	0.000
12f	42685.3					42685.3	0.000
13b	53629.4					53629.4	0.000
13f	53851.3					53851.4	0.000
14b	107251.					107251.	0.000
14f	108671.					108671.	0.000

Table 6.7a Damped Whirl Frequencies of a Dual-Shaft Rotor

(The inner shaft runs at a speed of 10,000 rpm)

(The outer shaft runs at a speed of 15,000 rpm)

Mode Order	F.E.M. 40 d.o.f.	Damped Whirl Frequencies (rad/s)					
		12 d.o.f.	error %	18 d.o.f.	error %	28 d.o.f.	error %
1b	343.575	343.916	0.099	343.571	0.001	343.575	0.000
1f	832.428	832.789	0.043	831.753	0.081	832.411	0.002
2b	1040.88	1041.64	0.073	1040.42	0.044	1040.83	0.005
2f	1249.06	1246.25	0.225	1246.29	0.222	1248.96	0.008
3b	1452.96	1450.78	0.149	1450.25	0.185	1452.86	0.006
3f	1793.29	1787.57	0.319	1789.18	0.228	1793.09	0.011
4b	1557.28	1557.31	0.002	1556.39	0.057	1557.28	0.000
4f	2623.33	2629.92	0.251	2618.38	0.189	2622.49	0.032
5b	1714.51	1692.95	1.258	1691.91	1.318	1713.92	0.034
5f	2989.96	2982.36	0.254	2959.04	1.034	2989.16	0.027
6b	1981.37	1975.44	0.299	1975.59	0.292	1980.77	0.030
6f	3607.27	3566.42	1.132	3567.10	1.114	3606.13	0.032
7b	6019.67			6016.41	0.054	6018.98	0.011
7f	6887.43			6884.30	0.045	6887.31	0.002
8b	6617.13			6616.60	0.008	6617.02	0.002
8f	8986.83			8980.49	0.071	8985.75	0.012
9b	15291.1			15291.1	0.000	15291.1	0.000
9f	17334.6			17333.0	0.010	17334.5	0.001
10b	19163.0					19155.9	0.037
10f	19235.5					19228.5	0.036
11b	14565.3					15480.7	6.285
11f	14671.8					15555.7	6.024
12b	42592.7					42537.2	0.130
12f	42694.9					42639.9	0.129
13b	48848.2					48638.6	0.429
13f	49073.1					48854.0	0.447
14b	106575.					106569.	0.006
14f	107994.					107987.	0.006

Table 6.7b Modal Damping Exponents of A Dual-Shaft Rotor

(The inner shaft runs at a speed of 10,000 rpm)

(The outer shaft runs at a speed of 15,000 rpm)

Mode Order	F.E.M. 40 d.o.f.	Modal Damping Exponents					
		12 d.o.f.	error %	18 d.o.f.	error %	28 d.o.f.	error %
1b	-0.915	-0.912	0.385	-0.912	0.343	-0.915	0.013
1f	-27.58	-27.78	0.699	-27.71	0.458	-27.59	0.004
2b	-56.49	-57.02	0.938	-56.59	0.176	-56.51	0.035
2f	-94.60	-94.29	0.332	-94.72	0.119	-94.61	0.005
3b	-132.1	-132.9	0.664	-132.0	0.068	-132.0	0.034
3f	-277.7	-277.1	0.236	-277.5	0.097	-277.7	0.011
4b	-269.3	-267.9	0.531	-268.8	0.174	-269.3	0.001
4f	-522.8	-525.3	0.478	-519.6	0.603	-522.5	0.057
5b	-225.4	-222.1	1.469	-221.9	1.546	-225.3	0.027
5f	-319.1	-320.0	0.283	-317.4	0.547	-319.1	0.006
6b	-371.9	-373.7	0.491	-369.9	0.535	-371.7	0.055
6f	-255.6	-249.2	2.506	-248.8	2.684	-255.8	0.058
7b	-168.5			-168.4	0.049	-168.5	0.024
7f	-15.29			-15.35	0.413	-15.32	0.215
8b	-16.42			-16.25	1.022	-16.36	0.335
8f	-236.6			-235.5	0.458	-236.2	0.164
9b	-1107			-1241	12.08	-1135	2.473
9f	-14.57			-15.02	3.152	-14.46	0.690
10b	-311.5					-311.3	0.087
10f	-310.2					-309.7	0.151
11b	-24670					-22895	7.204
11f	-24850					-23026	7.357
12b	-965.9					-963.7	0.225
12f	-967.6					-965.0	0.271
13b	-22210					-21949	1.171
13f	-22290					-22033	1.147
14b	-11250					-11245	0.030
14f	-11380					-11380	0.002

Table 6.8 **Element Properties of a Two-Spool Aircraft Engine**
 ($\rho = 0.283 \text{ lbm/in}^3$)

Element Number	Node Number	E (lb/in ²)	Length (in)	Inner Dia. (in)	Outer Dia. (in)
1	1 - 2	3×10^7	1.3	1.53	2.10
2	2 - 3	3×10^7	1.3	1.53	2.10
3	3 - 4	3×10^7	3.0	1.53	2.10
4	4 - 5	3×10^7	4.3	1.53	2.10
5	5 - 6	3×10^7	4.8	1.80	2.30
6	6 - 7	3×10^7	4.8	1.80	2.30
7	7 - 8	3×10^7	4.4	1.30	1.80
8	8 - 9	3×10^7	4.4	1.30	1.80
9	9 - 10	3×10^7	2.1	1.30	1.80
10	10 - 11	3×10^7	3.1	1.30	1.80
11	11 - 12	3×10^7	2.1	1.30	1.80
12	12 - 13	3×10^7	2.1	1.30	1.80
13	13 - 14	3×10^7	2.0	1.40	2.60
14	14 - 15	3×10^7	2.1	1.40	2.60
15	15 - 16	3×10^7	1.7	1.40	2.60
16	16 - 17	3×10^7	0.9	1.40	2.60
17	17 - 18	3×10^7	2.4	1.57	2.25
18	18 - 19	3×10^7	2.4	1.57	2.25
19	19 - 20	3×10^7	2.8	1.57	2.25
20	20 - 21	3×10^7	2.0	1.50	1.80
21	21 - 22	3×10^7	1.5	1.50	3.30
22	23 - 24	3×10^7	3.0	2.40	2.80
23	24 - 25	3×10^7	4.3	2.45	3.05
24	25 - 26	3×10^7	4.8	10.97	11.00
25	26 - 27	1.50×10^7	4.8	10.86	11.00
26	27 - 28	1.28×10^7	4.4	10.79	11.00
27	28 - 29	1.93×10^7	4.4	10.86	11.00
28	29 - 30	2.73×10^7	2.1	2.80	3.85
29	30 - 31	3×10^7	3.1	2.90	3.39
30	31 - 32	3×10^7	2.1	3.20	3.60
31	32 - 33	3×10^7	2.1	3.20	3.60
32	33 - 34	3×10^7	2.0	3.20	3.60
33	34 - 35	3×10^7	2.1	3.35	3.60
34	35 - 36	3×10^7	1.7	3.35	3.70
35	36 - 37	3×10^7	0.9	3.35	4.20
36	37 - 38	3×10^7	2.4	9.00	9.40

Table 6.9 Disk Properties of a Two-Spool Aircraft Engine

Disk Number	Disk Location	M_d (lbm)	I_p (lbm · in ²)	I_d (lbm · in ²)
1	1	0.50	0.0	0.00
2	2	1.00	0.0	0.00
3	3	1.00	0.0	0.00
4	19	27.00	625.0	312.50
5	20	30.00	704.5	352.25
6	21	2.00	0.0	0.00
7	25	7.46	111.0	55.50
8	26	11.53	297.5	148.75
9	27	3.73	301.5	150.75
10	28	7.67	232.0	116.00
11	29	0.00	117.0	58.50
12	37	25.00	612.5	306.25
13	38	25.00	600.0	300.00

Table 6.10 Undamped Bearing Data of a Two-Spool Aircraft Engine

Bearing Number	Bearing Location	$K_{xx} = K_{yy}$ (lb/in)	$C_{xx} = C_{yy}$ (lb · s/in)
1	2 - 0	500,000	0.0
2	21 - 0	60,000	0.0
3	23 - 0	1,000,000	0.0
4	25 - 0	1,000,000	0.0
5	30 - 0	500,000	0.0
6	35 - 0	400,000	0.0
7	5 - 25	500,000	0.0
8	13 - 33	500,000	0.0

Table 6.11 Damped Bearing Data of a Two-Spool Aircraft Engine

Bearing Number	Bearing Location	$K_{xx} = K_{yy}$ (lb/in)	$C_{xx} = C_{yy}$ (lb · s/in)
1	2 - 0	500,000	2.0
2	21 - 0	60,000	2.0
3	23 - 0	1,000,000	2.0
4	25 - 0	1,000,000	2.0
5	30 - 0	500,000	2.0
6	35 - 0	123,000	100.0
7	5 - 25	500,000	2.0
8	13 - 33	500,000	2.0
9 squeeze film bearing	11 - 31	$K_{xx} = 10243$ $K_{xy} = 11302$ $K_{yx} = -20177$ $K_{yy} = 7682$	$C_{xx} = 8.31$ $C_{xy} = -3.37$ $C_{yx} = -3.37$ $C_{yy} = 10.97$

 

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**Table 6.12 Component Natural Frequencies (rpm)
of a Two-Spool Aircraft Turbine Engine**
(only one-planar modes are listed)

Mode Order	Undamped Bearing Case		Damped Bearing Case	
	Power Turbine	Gas Generator	Power Turbine	Gas Turbine
1	3069.15	10555.6	3069.15	7601.40
2	6928.99	27404.0	6928.99	25767.4
3	14441.5	37875.9	14441.5	36609.6
4	27970.5	63116.3	27970.5	62891.9
5	44583.5	109247.	44583.5	109243.
6	55436.0	165717.	55436.0	165715.
7	70551.1	228991.	70551.1	228971.
8	98843.8	282212.	98843.8	282165.
9	106025.	296730.	106025.	296684.
10	148322.	379601.	148322.	379528.

**Table 6.13 Component Mode Selection Schemes
of a Two-Spool Aircraft Engine**

Truncation Frequency: ω_t	Component Modes Retained		Total DOF of CMS Model
	Power Turbine Rotor	Gas Generator Rotor	
30000.0 rpm	8	4	12
60000.0 rpm	12	6	18
110000. rpm	18	10	28

Table 6.14 Undamped Whirl Frequencies of a Two-Spool Aircraft Engine at Various Component Modal Truncation Levels

(Power turbine runs at a speed of 20,000 rpm)

(Gas generator runs at a speed of 15,000 rpm)

Mode No.	F.E.M. 152 d.o.f.	Whirl Frequencies (rpm)					
		12 d.o.f.	error %	18 d.o.f.	error %	28 d.o.f.	error %
1b	5664.06	5673.75	0.171	5670.16	0.108	5665.74	0.030
1f	7120.90	7125.62	0.067	7121.93	0.015	7121.17	0.004
2b	8323.08	8430.93	1.296	8338.27	0.183	8323.58	0.006
2f	12488.4	12653.5	1.322	12510.2	0.175	12489.0	0.005
3b	9606.83	9668.56	0.643	9626.28	0.203	9611.08	0.045
3f	20775.1	20978.9	0.981	20783.2	0.039	20775.6	0.002
4b	19614.6	19716.4	0.519	19629.6	0.076	19615.0	0.002
4f	31402.7	32494.6	3.477	31518.5	0.369	31419.1	0.052
5b	27684.3	29169.2	5.363	27712.3	0.101	27686.8	0.009
5f	35557.1	37258.8	4.786	35840.2	0.796	35630.1	0.205
6b	33592.0	40862.0	21.64	33676.4	0.251	33594.9	0.009
6f	42364.7	46119.7	8.864	42862.5	1.175	42413.8	0.116
7b	35831.1			36179.2	0.971	35847.8	0.046
7f	58660.0			59171.3	0.872	58667.2	0.012
8b	51745.2			54511.3	5.346	51753.1	0.015
8f	60922.4			64038.3	5.115	60928.6	0.010
9b	60899.9			63146.6	3.689	60912.9	0.021
9f	67982.9			72020.3	5.939	68017.7	0.051
10b	66372.7					66386.9	0.021
10f	71720.7					71740.8	0.028
11b	78778.2					78798.7	0.026
11f	83874.1					83945.5	0.085
12b	103597.					103647.	0.048
12f	106933.					106975.	0.039
13b	105058.					105103.	0.043
13f	109132.					109187.	0.050
14b	109737.					109769.	0.029
14f	116971.					117035.	0.055

Table 6.15a Damped Whirl Frequencies of a Two-Spool Aircraft Engine
 (Power turbine runs at a speed of 17,000 rpm)
 (Gas generator runs at a speed of 15,000 rpm)

Mode Order	F.E.M. 152 d.o.f.	Damped Whirl Frequencies (rpm)					
		12 d.o.f.	error %	18 d.o.f.	error %	28 d.o.f.	error %
1b	5932.36	5938.86	0.110	5936.34	0.067	5933.38	0.017
1f	6983.12	6980.81	-0.033	6983.07	-0.001	6983.26	0.002
2b	6175.19	6190.55	0.249	6177.00	0.029	6174.51	-0.011
2f	8938.09	9004.05	0.738	8947.28	0.103	8935.84	-0.025
3b	10107.0	10178.1	0.704	10126.8	0.196	10111.2	0.042
3f	20853.2	21103.9	1.203	20864.7	0.055	20854.8	0.008
4b	18751.0	18854.0	0.550	18750.5	-0.003	18744.7	-0.034
4f	30402.9	31244.7	2.769	30567.8	0.542	30429.0	0.086
5b	26094.3	27413.1	5.055	26108.7	0.056	26083.4	-0.041
5f	33939.7	35950.1	5.924	34080.5	0.415	33979.0	0.116
6b	33766.2	41855.7	23.96	33771.3	0.015	33756.0	-0.030
6f	40776.4	45546.0	11.70	41096.6	0.785	40790.8	0.035
7b	36913.1			37278.2	0.989	36923.5	0.028
7f	58025.7			58290.3	0.456	58003.6	-0.038
8b	51624.9			54467.4	5.506	51610.4	-0.028
8f	60500.6			62421.9	3.176	60494.7	-0.010
9b	61157.3			63852.9	4.408	61032.5	-0.204
9f	66192.8			69297.7	4.691	66221.9	0.044
10b	66143.5					66144.4	0.044
10f	71136.4					71121.9	-0.020
11b	78828.1					78835.8	0.010
11f	82968.1					83002.3	0.041
12b	103763.					103809.	0.044
12f	106718.					106746.	0.027
13b	105147.					105184.	0.034
13f	109009.					109030.	0.019
14b	109182.					109122.	-0.054
14f	116982.					116979.	-0.002

Table 6.15b Modal Damping Exponents of a Two-Spool Aircraft Engine
 (Power turbine runs at a speed of 17,000 rpm)
 (Gas generator runs at a speed of 15,000 rpm)

Mode Order	F.E.M. 152 d.o.f.	Modal Damping Exponents					
		12 d.o.f.	error %	18 d.o.f.	error %	28 d.o.f.	error %
1b	-9.705	-9.817	1.157	-9.690	-0.155	-9.684	-0.214
1f	-19.89	-19.82	-0.337	-19.88	-0.079	-19.92	0.117
2b	-87.30	-94.09	7.776	-88.33	1.182	-87.28	-0.029
2f	-155.6	-149.0	-4.249	-155.0	-0.425	-155.6	0.019
3b	-1.899	-1.115	-41.30	-1.890	-0.505	-1.917	0.915
3f	-14.78	-18.60	25.83	-15.07	1.976	-14.78	0.023
4b	-137.2	-114.2	-16.77	-133.1	-2.959	-137.1	-0.050
4f	-53.17	-71.24	33.99	-55.00	3.444	-53.36	0.358
5b	-195.6	-174.2	-10.94	-195.7	0.039	-196.3	0.371
5f	-26.99	-102.5	35.06	-27.32	1.203	-27.38	1.445
6b	-75.13	-213.7	184.4	-73.81	-1.757	-74.87	-0.334
6f	-136.6	-107.1	-21.60	-154.8	13.31	-135.6	-0.709
7b	-36.96			-35.98	-2.643	-36.84	-0.332
7f	-294.9			-318.0	7.814	-296.4	0.500
8b	-153.1			-295.5	93.05	-155.9	1.847
8f	-103.6			-75.70	-26.94	-104.7	1.006
9b	-310.7			-186.1	-40.09	-314.0	1.073
9f	-35.67			-78.50	120.5	-34.41	-3.588
10b	-151.0					-150.5	-0.329
10f	-150.8					-148.7	-1.368
11b	-72.01					-73.32	1.810
11f	-91.35					-90.06	-1.405
12b	-33.44					-33.21	-0.695
12f	-58.52					-58.51	-0.261
13b	-34.04					-36.27	6.539
13f	-22.10					-22.15	0.265
14b	-206.6					-199.4	-3.502
14f	-149.5					-143.4	-4.072

Chapter 7

Component Mode Synthesis for Rotor Systems with Hybrid Interfaces

7.1 Introduction

In the study of the dynamics of multi-spool rotor systems with flexible bladed disks, a more complex problem is often encountered in that the system has both flexible and rigid interfaces, i.e., hybrid interfaces. The interfaces between flexible disks and the shaft to which they are assembled are rigid interfaces, where the displacements across the interfaces are continuous. As stated in the last chapter, the interfaces between shafts are often flexible and there are no common degrees of freedom being shared at these interfaces. Up to the present, to the author's knowledge, no study has been done to develop CMS methods for structures with hybrid interfaces.

The studies in this chapter will extend the free interface CMS method presented in the previous chapter to analysis of damped rotor systems with both flexible and rigid interfaces, and with an arbitrary number of components. As in Chapter 6, the equations of motion are assembled based on Lagrangian method using the truncated normal modes of substructures complemented with the residual flexibility modes. A substantial advantage of the present method is the fact that all rigid and flexible interface coordinates are eliminated from the final set of system equations using compatibilities of displacements on the rigid interfaces and of coupling forces on the flexible interfaces.

7.2 Mode Analysis for Components

For a rotor system which has been fictionally divided into n substructures, the kinetic energy, potential energy, and Rayleigh dissipation function of the l -th substructure, as shown in chapter 2 and chapter 3, are typically of the forms:

$$\left. \begin{aligned} T_l &= \frac{1}{2} \{\dot{Z}_l\}^T [M_l] \{\dot{Z}_l\} + \{\dot{Z}_l\}^T [N_l(\Omega)] \{Z_l\} + \frac{1}{2} \{Z_l\}^T [H_l(\Omega)] \{Z_l\} \\ V_l &= \frac{1}{2} \{Z_l\}^T ([K_l] + [K_{egl}(\Omega)]) \{Z_l\} \\ J_l &= \frac{1}{2} \{\dot{Z}_l\}^T [C_l] \{\dot{Z}_l\} \end{aligned} \right\} \quad (7.1)$$

where the second item in kinetic energy represents the Coriolis or gyroscopic forces, and the third term, behaving like an apparent potential energy, stands for the centrifugal forces due to rotation of the structure. $[K_{egl}]$ represents the geometric stiffness of in-plane stresses produced by the centrifugal forces when a disk is taken as flexible. Hence, both $[H_l(\Omega)]$ and $[K_{egl}]$ are proportional to the square of spinning speed Ω .

The linearized equations of motion of the l -th substructure of the rotor, then, can be written as:

$$\begin{aligned} [M_l] \{\ddot{Z}_l\} + ([C_l] + [G_l(\Omega)]) \{\dot{Z}_l\} + ([K_l] \\ + [K_{gl}(\Omega)] + [N_l(\dot{\Omega})]) \{Z_l\} = \{F_{el}\} + \{F_{Fjl}\} + \{F_{Rjl}\} \end{aligned} \quad (7.2)$$

where

$$\left. \begin{aligned} [G_l(\Omega)] &= [N_l] - [N_l]^T \\ [K_{gl}(\Omega)] &= [K_{egl}(\Omega)] - [H_l(\Omega)] \end{aligned} \right\} \quad (7.3)$$

and $\{F_{el}\}$ is the external force vector applied, $\{F_{Fjl}\}$ and $\{F_{Rjl}\}$ are the flexible- and rigid-junction force vectors applied by other adjacent substructures, respectively.

The method of component mode synthesis developed here applies to any system whose equations of motion can be expressed in this form.

As known from the last chapter, the displacement vector, $\{Z_l\}$ can be expressed as a linear combination of the "kept" normal modes and the approximated modes of the component (refer to Eq. 6.23):

$$\{Z_l\} = [\Phi_{kl}]\{p_{kl}\} + [G_{al}](\{F_{Fjl}\} + \{F_{Rjl}\}) \quad (7.4)$$

When substructures are coupled to form the complete structure, the condensed junction forces, $\{f_{Fjl}\}$ and $\{f_{Rjl}\}$, of the substructure instead of vectors $\{F_{Fjl}\}$ and $\{F_{Rjl}\}$, are required. These are given by the appropriate partition of Eq. 7.4, namely

$$\{Z_l\} = [\Phi_{kl}]\{p_{kl}\} + [G_{Fjl}]\{f_{Fjl}\} + [G_{Rjl}]\{f_{Rjl}\} \quad (7.5)$$

or as the partitioned matrix form:

$$\begin{Bmatrix} z_{Il} \\ z_{Fjl} \\ z_{Rjl} \end{Bmatrix} = \begin{Bmatrix} \Phi_{Il} \\ \Phi_{Fjl} \\ \Phi_{Rjl} \end{Bmatrix} \{p_{kl}\} + \begin{Bmatrix} G_{IFl} \\ G_{FFl} \\ G_{RFl} \end{Bmatrix} \{f_{Fjl}\} + \begin{Bmatrix} G_{IRl} \\ G_{FRl} \\ G_{RRl} \end{Bmatrix} \{f_{Rjl}\} \quad (7.6)$$

where the subscript "I" stands for "Internal coordinates", "Fj" for "Flexible Junction", and "Rj" for "Rigid Junction".

$[G_{Fjl}]$ is the appropriate columns from the residual flexibility matrix of the l th substructure on its flexible interfaces.

$\{f_{Fjl}\}$ is the flexible-interface forces of the l -th substructure applied by all adjacent flexible junction elements.

$[G_{Rjl}]$ is the appropriate columns from the residual flexibility matrix of the l th substructure on its rigid interfaces.

$\{f_{Rjl}\}$ is the rigid-interface forces of the l -th substructure applied by all adjacent substructures.

Eq. 7.5 can be further expressed as the following transformation:

$$\{Z_l\} = [\hat{\Phi}_l]\{\hat{Z}_l\} \quad (7.7)$$

with

$$\{\hat{Z}_l\} = \begin{Bmatrix} p_{kl} \\ f_{Fjl} \\ f_{Rjl} \end{Bmatrix}, \quad [\hat{\Phi}_l] = [\Phi_{kl} \ G_{Fjl} \ G_{Rjl}]$$

Substituting Eq. 7.7 into Eq. 7.2 and pre-multiplying it by $[\hat{\Phi}_l]^T$, we obtain the transformed equations of motion of the component:

$$\begin{aligned} [\hat{M}_l]\{\ddot{\hat{Z}}_l\} + ([\hat{C}_l] + [\hat{G}_l(\Omega)])\{\dot{\hat{Z}}_l\} + ([\hat{K}_l] \\ + [\hat{K}_{gl}(\Omega^2)] + [\hat{N}_l(\dot{\Omega})])\{\hat{Z}_l\} = \{\hat{F}_{el}\} + \{\hat{F}_{Fjl}\} + \{\hat{F}_{Rjl}\} \end{aligned} \quad (7.8)$$

with

$$[\hat{M}_l] = [\hat{\Phi}_l]^T [M_l] [\hat{\Phi}_l] = \begin{bmatrix} I_l & 0 & 0 \\ 0 & G_{Fjl}^T M_l G_{Fjl} & G_{Fjl}^T M_l G_{Rjl} \\ 0 & G_{Rjl}^T M_l G_{Fjl} & G_{Rjl}^T M_l G_{Rjl} \end{bmatrix} \quad (7.9)$$

$$[\hat{K}_l] = [\hat{\Phi}_l]^T [K_l] [\hat{\Phi}_l] = \begin{bmatrix} I_l & 0 & 0 \\ 0 & G_{FFl} & G_{FRl} \\ 0 & G_{RFl} & G_{RRl} \end{bmatrix} \quad (7.10)$$

$$\left. \begin{aligned} [\hat{C}_l] &= [\hat{\Phi}_l]^T [C_l] [\hat{\Phi}_l] \\ [\hat{G}_l(\Omega)] &= [\hat{\Phi}_l]^T [G_l(\Omega)] [\hat{\Phi}_l] \\ [\hat{K}_{gl}(\Omega)] &= [\hat{\Phi}_l]^T [K_{gl}(\Omega)] [\hat{\Phi}_l] \\ [\hat{N}_l(\dot{\Omega})] &= [\hat{\Phi}_l]^T [N_l(\dot{\Omega})] [\hat{\Phi}_l] \\ \{\hat{F}_{el}\} &= [\hat{\Phi}_l]^T [F_{el}] \\ \{\hat{F}_{Fjl}\} &= [\hat{\Phi}_l]^T [F_{Fjl}] \\ \{\hat{F}_{Rjl}\} &= [\hat{\Phi}_l]^T [F_{Rjl}] \end{aligned} \right\} \quad (7.11)$$

7.3 Assembly of Substructures

In the preceding sections the substructure modes have been obtained for both constrained and free substructures. In the present section equations will be developed for coupled substructures. Eq. 7.5 is the pertinent substructure equation for the substructure assembly.

7.3.1 Energy Expressions for a Complete Rotor System

The kinetic energy, potential energy, and Rayleigh dissipation function of a complete rotor system with flexible bladed disks are given by:

$$T = \frac{1}{2}\{\dot{Z}\}^T[M]\{\dot{Z}\} + \{\dot{Z}\}^T[N]\{Z\} + \frac{1}{2}\{Z\}^T[H]\{Z\} \quad (7.12)$$

$$V = \frac{1}{2}\{Z\}^T([K] + [K_g])\{Z\} + \frac{1}{2}\{Z_{Fj}\}^T[K_{Fj}]\{Z_{Fj}\} \quad (7.13)$$

$$J = \frac{1}{2}\{\dot{Z}\}^T[C]\{\dot{Z}\} + \frac{1}{2}\{\dot{Z}_{Fj}\}^T[C_{Fj}]\{\dot{Z}_{Fj}\} \quad (7.14)$$

Since the structure which has been fictionally divided into n substructures, the definitions of items in Eqs. 7.12, 7.13, and 7.14 can be written as

$$\{Z\} = \{Z_1^T \ Z_2^T \ \cdots \ Z_n^T\}^T$$

$$[M] = \text{diag}\{M_1, M_2, \cdots, M_n\}$$

$$[N] = \text{diag}\{N_1, N_2, \cdots, N_n\}$$

$$[H] = \text{diag}\{H_1, H_2, \cdots, H_n\}$$

$$[K] = \text{diag}\{K_1, K_2, \cdots, K_n\}$$

$$[K_{eg}] = \text{diag}\{K_{eg1}, K_{eg2}, \cdots, K_{egn}\}$$

$$[C] = \text{diag}\{C_1, C_2, \cdots, C_n\}$$

and $\{Z_{Fj}\}$ are coordinates on all the flexible interfaces of a structure. $[K_{Fj}]$ and $[C_{Fj}]$ are the corresponding stiffness and damping matrices of the flexible interfaces of the structure, respectively.

7.3.2 Transformations for Assembly

The component mode coordinates, $\{p_{ki}\}$, can be related to the mode coordinates of the assembled structure,

$$\{P_s\} = \{p_{k1}^T \ p_{k2}^T \ \cdots \ p_{kn}^T\}^T$$

by the relation:

$$\{p_{kl}\} = [E_l]\{P_s\} \quad (7.15)$$

where

$$[E_l] = [0 \cdots I_l \cdots 0]$$

and $[I_l]$ is a $m_l \times m_l$ unit matrix where m_l is the number of retained modes of the l -th substructure.

The displacements and forces, $\{z_{Fjl}\}$, $\{f_{Fjl}\}$, on the flexible interfaces of the l -th substructure can be related to the flexible-interface coordinates and forces, $\{Z_{Fj}\}$, $\{F_{Fj}\}$, of the assembled structure as following

$$\{z_{Fjl}\} = [S_l]\{Z_{Fj}\} \quad (7.16)$$

$$\{f_{Fjl}\} = [S_l]\{F_{Fj}\} \quad (7.17)$$

where elements of the transformation matrix, $[S_l]$, are composed of either 0, 1, or -1.

Assuming that the forces on one side of all rigid interfaces of the complete structure are represented by a vector $\{F_{Rj}\}$, we can relate the rigid-interface forces, $\{f_{Rjl}\}$, of the l -th substructure, to $\{F_{Rj}\}$ by the transformation

$$\{f_{Rjl}\} = [T_l]\{F_{Rj}\} \quad (7.18)$$

where elements of the transformation matrix, $[T_l]$, are composed of either 0, 1, or -1.

7.3.3 Compatibility on Rigid Interfaces

From Eq. 7.18, the displacement compatibility of the structure (on the rigid interfaces) can be represented as the following equation

$$\sum_{l=1}^n [T_l]^T \{z_{Rjl}\} = 0 \quad (7.19)$$

Substituting Eqs. 7.6, 7.15 and 7.17 into the above equation, and solving for $\{F_{Rj}\}$, we obtain

$$\{F_{Rj}\} = -[G_{RR}]^{-1} ([\Phi_{Rj}]\{P_s\} + [G_{RF}]\{F_{Fj}\}) \quad (7.20)$$

where

$$\begin{aligned} [G_{RR}] &= \sum_{l=1}^n [T_l]^T [G_{RRl}] [T_l] \\ [G_{RF}] &= \sum_{l=1}^n [T_l]^T [G_{RF_l}] [S_l] \\ [\Phi_{Rj}] &= \sum_{l=1}^n [T_l]^T [\Phi_{Rjl}] [E_l] \end{aligned}$$

As $[G_{RRl}]$ ($l=1, 2, \dots, n$) is semi-positive definite, the flexibility matrix, $[G_{RR}]$, is at least semi-positive definite. If the total number of degrees of freedom of a substructure, the number of its rigid-interface coordinates, and the number of retained normal modes are represented by n_l , n_{Rjl} , and m_l , respectively, an adequate condition for $[G_{RR}]$ to be positive definite is that for all substructures in a structure, there is

$$m_l + n_{Rjl} \leq n_l \quad \text{for } r = 1, 2, \dots, n \quad (7.21)$$

7.3.4 Compatibility on Flexible Interfaces

From Eqs. 7.16 and 7.17, we obtain

$$\{Z_{Fj}\} = \sum_{l=1}^n [S_l]^T \{z_{Fjl}\} \quad (7.22)$$

$$\{F_{Fj}\} = \sum_{l=1}^n [S_l]^T \{f_{Fjl}\} = -[K_{Fj}]\{Z_{Fj}\} \quad (7.23)$$

Substituting Eqs. 7.6 and 7.17 into Eq. 7.22, using Eq. 7.20 and solving for $\{Z_{Fj}\}$, we get

$$\{Z_{Fj}\} = [\Psi_F]\{P_s\} \quad (7.24)$$

where

$$\begin{aligned}
[\Psi_F] &= ([I] + [G_{FF}][K_{Fj}])^{-1} [\Phi_{Fj}] \\
[G_{FF}] &= \sum_{l=1}^n [S_l]^T [G_{FFl}] [S_l] - [G_{RF}]^T [G_{RR}]^{-1} [G_{RF}] \\
[\Phi_{Fj}] &= \sum_{l=1}^n [S_l]^T [\Phi_{Fjl}] [E_l] - [G_{RF}]^T [G_{RR}]^{-1} [\Phi_{Rj}]
\end{aligned}$$

Substituting Eq. 7.24 into Eqs. 7.23 and 7.20, we obtain the interface forces in terms of mode coordinates as

$$\{F_{Fj}\} = -[\dot{K}_F]\{P_s\} \quad (7.25)$$

$$\{F_{Rj}\} = -[\dot{K}_R]\{P_s\} \quad (7.26)$$

where

$$\begin{aligned}
[\dot{K}_F] &= [K_{Fj}][\Psi_F] \\
[\dot{K}_R] &= [G_{RR}]^{-1} ([\Phi_{Rj}] - [G_{RF}][\dot{K}_F])
\end{aligned}$$

Substituting Eqs. 7.24, 7.25 and 7.26 into Eq. 7.5, we obtain the mode transformation for all substructures as follows

$$\{Z_l\} = ([\Phi_{kl}][E_l] - [G_{Fjl}][S_l][\dot{K}_F] - [G_{Rjl}][T_l][\dot{K}_R])\{P_s\} \quad \text{for } l = 1, 2, \dots, n \quad (7.27)$$

or as the matrix form

$$\{Z\} = [R_s]\{P_s\} \quad (7.28)$$

where

$$[R_s] = \begin{bmatrix} \Phi_{k1}E_1 - G_{Fj1}S_1\dot{K}_F - G_{Rj1}T_1\dot{K}_R \\ \Phi_{k2}E_2 - G_{Fj2}S_2\dot{K}_F - G_{Rj2}T_2\dot{K}_R \\ \vdots \\ \Phi_{kn}E_n - G_{Fjn}S_n\dot{K}_F - G_{Rjn}T_n\dot{K}_R \end{bmatrix}$$

7.4 Equations of Motion of a Coupled Structure

Substituting Eqs. 7.24 and 7.28 into energy equations 7.12, 7.13 and 7.14, we can re-express the kinetic energy, potential energy and Rayleigh dissipation function of the structure in terms of the generalized component mode coordinates $\{P_s\}$ as

$$T = \frac{1}{2}\{\dot{P}_s\}^T[M_s]\{\dot{P}_s\} + \{\dot{P}_s\}^T[N_s]\{P_s\} + \frac{1}{2}\{P_s\}^T[H_s]\{P_s\} \quad (7.29)$$

$$V = \frac{1}{2}\{P_s\}^T([K_s] + [K_{egs}])\{P_s\} \quad (7.30)$$

$$J = \frac{1}{2}\{\dot{P}_s\}^T[C_s]\{\dot{P}_s\} \quad (7.31)$$

with the relationships:

$$\left. \begin{aligned} [M_s] &= [R_s]^T[M][R_s] = \sum_{l=1}^n [\Upsilon_l]^T[\hat{M}_l][\Upsilon_l] \\ [K_s] &= [\Psi_F]^T[K_{Fj}][\Psi_F] + \sum_{l=1}^n [\Upsilon_l]^T[\hat{K}_l][\Upsilon_l] \\ [N_s] &= [R_s]^T[N][R_s] = \sum_{l=1}^n [\Upsilon_l]^T[\hat{N}_l][\Upsilon_l] \\ [K_{egs}] &= [R_s]^T[K_{eg}][R_s] = \sum_{l=1}^n [\Upsilon_l]^T[\hat{K}_{egl}][\Upsilon_l] \\ [H_s] &= [R_s]^T[H][R_s] = \sum_{l=1}^n [\Upsilon_l]^T[\hat{H}_l][\Upsilon_l] \\ [C_s] &= [\Psi_F]^T[C_{Fj}][\Psi_F] + \sum_{l=1}^n [\Upsilon_l]^T[\hat{C}_l][\Upsilon_l] \end{aligned} \right\} \quad (7.32)$$

and

$$[\Upsilon] = \begin{bmatrix} E_l \\ -S_l \hat{K}_F \\ -T_l \hat{K}_R \end{bmatrix} \quad (7.33)$$

The Lagrangian equations for the coupled structure are given by

$$\frac{\partial}{\partial t} \left(\frac{\partial L}{\partial \{\dot{P}_s\}} \right) - \frac{\partial L}{\partial \{P_s\}} + \frac{\partial J}{\partial \{\dot{P}_s\}} = \{Q_s\} \quad (7.34)$$

where

$$L = T - V \quad (\text{Lagrangian function})$$

and $\{Q_s\}$ is the generalized applied force vector, which denotes all the remaining forces not falling into any of the categories of forces defined at the beginning of the second section, such as, external applied forces, unbalanced inertia forces or moments and so on.

$\{Q_s\}$ can be obtained from the virtual work expression

$$\delta W = \{Q_s\}^T \delta \{P_s\} \quad (7.35)$$

If we denote the applied forces of all the uncoupled substructures by

$$\{F\} = \begin{Bmatrix} F_{e1} \\ F_{e2} \\ \vdots \\ F_{en} \end{Bmatrix} \quad (7.36)$$

we have the following relationship:

$$\{Q_s\} = [R_s]^T \{F\} = \sum_{l=1}^n [\Upsilon]^T \{\hat{F}_{el}\} \quad (7.37)$$

Substituting Eqs. 7.29, 7.30 and 7.31 into Eq. 7.34, we obtain the equation of motion for the coupled structure as

$$[M_s]\{\ddot{P}_s\} + ([C_s] + [G_s(\Omega)])\{\dot{P}_s\} + ([K_s] + [K_{gs}(\Omega)] + [N_s(\dot{\Omega})])\{P_s\} = \{Q_s\} \quad (7.38)$$

where

$$\left. \begin{aligned} [G_s] &= [N_s] - [N_s]^T && \text{(skew symmetric)} \\ [\bar{K}_{gs}] &= [K_{gs}] - [H_s] && \text{(symmetric)} \end{aligned} \right\} \quad (7.39)$$

From Eq. 7.3, Eq. 7.32 and Eq. 7.39, it can be proved that the two global matrices can be directly assembled from the corresponding component matrices by:

$$\left. \begin{aligned} [G_s] &= [R_s]^T [G] [R_s] = \sum_{l=1}^n [\Upsilon]^T [\hat{G}_l] [\Upsilon] \\ [\bar{K}_{gs}] &= [R_s]^T [\bar{K}_{gl}] [R_s] = \sum_{l=1}^n [\Upsilon]^T [\hat{K}_{gl}] [\Upsilon] \end{aligned} \right\} \quad (7.40)$$

It follows from Eq. 7.32, Eq. 7.37 and Eq. 7.40 that the assembly of the equations of motion for a structure can be performed either in the rotating coordinate system, or more often, in the fixed coordinate system.

If a structure is treated as assembly of two components with rigid interfaces, and has no damping and zero rotation speed, it can be shown that the present method is identical to the modified MacNeal's method proposed by Craig and Chang [52] and will give the same system equation as given by Rubin's method which means that both methods produce the same system frequencies.

7.5 Numerical Examples

One non-gyroscopic and three rotor-bearing systems are employed here to illustrate the accuracy of the present component mode synthesis with various levels of modal truncation, versus the finite element solution of full order. The first example is a free-free damped beam which has been studied by Craig and other co-researchers [57], [59]. This non-gyroscopic system is included to compare the accuracy obtained by using free-interface component normal modes with that obtained by using free-interface component damped modes. The next two rotors with various damping had been studied by Glasgow and Nelson [60] using fixed-interface component complex mode synthesis, which allow correlation with their published results. The last example is the two-spool aircraft turbine engine studied in the previous chapter, but with a modification that the second-stage turbine disk has been modeled as a flexible component.

7.5.1 A Free-Free Non-rotating Beam

In order to compare the accuracy obtained by the current CMS method with that obtained by CMS methods using free-interface damped component modes augmented with residual flexibility and inertial-relief attachment modes, the first example is a free-free damped non-rotating beam with uniform cross-section, shown in Figure 7.1, which has been studied by Craig and Ni [59].

The beam is modeled by 10 elements with a total of 11 nodes and 22 physical degrees of freedom. It is divided into two components, *A* and *B*, as shown in Figure 7.1, and the damping in each element is taken to be proportional to the element stiffness as indicated in the figure. Note that the damping is not proportional to stiffness either at the component level or the system level. In Ref. [59], the reduced-order CMS model has 14 degrees of freedom. For the purpose of comparison, the first 8 undamped modes of the component "*A*" and the first 6 undamped modes of the component "*B*" are retained in the CMS model which also results in 14 degrees of freedom.

Table 7.1 and Table 7.2 present a comparison of the modal damping exponents, σ_i , and the damped frequencies, ω_{di} , obtained by the current method and Craig's method, where σ_i and ω_{di} are the real and imaginary parts of the complex eigenvalues of the system. The solutions of the finite element model of full order are included as the exact solution.

Inspection of Tables 7.1 and 7.2 reveals that both the real part and the imaginary part of the eigenvalues of the model obtained by the method presented in this thesis are, in general, an order of magnitude more accurate than the eigenvalues obtained in Ref. [59].

7.5.2 A Single Uniform Shaft

The second example used is a simply supported uniform rotating shaft studied by Glasgow [61], shown in Figure 7.2. The shaft is modeled as an assembly of five finite elements of equal length. This results in six rotor nodes with two translational and two rotational degrees of freedom at each node. Thus the rotor model has 24 displacement degrees of freedom. The rotor is divided into two components, *A* and *B*, as shown in Figure 7.2. The component "*A*" consists of 3 elements and has 16 degrees of freedom including 2 DOF of rigid body. The component "*B*" consists of 4 elements and has 12 degrees of freedom including 2 DOF of rigid body.

Since the shaft is supported by isotropic bearings only one-planar modes need to be calculated. The natural frequencies of the two components are tabulated in

Table 7.3a. Table 7.3b presents the selections of component modes retained in CMS analysis for various truncation frequencies. For example, with a truncation frequency, $\omega_t = 1200 \text{ rad/s}$, the component modes with frequencies lower than this will be kept for assembly of CMS equations so that the first four modes of the component **A** and the first two rigid-body modes of the component **B** will be retained in the analysis which results in a total of 6 degrees of freedom for the reduced order CMS model.

Whirl frequency analyses using component mode synthesis are presented for the following cases:

1. Undamped isotropic bearings, no internal damping;
2. Undamped isotropic bearings, viscous internal damping
Viscous Damping Coefficient $\eta_v = 0.0002/\text{s}$;
3. Undamped isotropic bearings, hysteretic internal damping
Hysteretic damping coefficient $\eta_h = 0.0002$

The whirl frequencies and log decrements at a rotation speed of 4000 *rpm* for various levels of mode truncation are tabulated in Tables 7.4 to 7.6 for Cases 1, 2, and 3, respectively. Figures 7.3, 7.4 and 7.6 present errors in whirl frequencies of first four forward modes for cases 1, 2, and 3, respectively. Results obtained by Glasgow and Nelson [60] using fixed-interface complex component mode method are also included for comparison. Errors in frequencies using MacNeal's Method are presented in Figure 7.3 for the undamped case only.

The whirl frequencies obtained by the present method in all the cases are seen to exhibit better accuracy and convergence than those obtained by fixed interface component complex mode method and MacNeal's Method. Even only retaining the rigid-body modes of components (i.e., 4 DOF), the errors in first and second whirl frequencies (backward and forward) in all cases are less than 0.4% and 4%, respectively. When the number of retained component modes increases, all errors decrease. Figures 7.5 and 7.7 present errors in log decrements of the first four forward modes for cases 2 and 3, respectively. It is interesting to find out that the log decrements obtained by the present method have better convergence than those obtained by

the fixed-interface component complex mode method for both internal viscous and hysteretic damping cases.

7.5.3 A Multiple-Stepped Shaft, Single Disk Rotor

The second rotor system is the multiple-stepped shaft rotor with a single disk presented in Chapter 5, shown in Figure 5.2, which has been employed by Glasgow and Nelson [60] to study the efficiency of the fixed-interface complex component mode synthesis.

In this study, the rotor is modeled as assembly of 18 finite elements. Hence, the rotor model has a total of 76 degrees of freedom and is divided into two components, *A* and *B*, shown in Figure 7.8. The component *A* consists of 10 elements and has 44 degrees of freedom including 4 DOF of rigid body. The component *B* consists of 8 elements and has 36 degrees of freedom.

The natural frequencies of the two components are tabulated in Table 7.7a for both isotropic and orthotropic bearing cases Table 7.7b presents the selections of component modes retained in CMS Model for various truncation frequency.

Eigensolution Analysis

The whirl frequencies and log decrements obtained at various levels of modal truncation for isotropic and orthotropic bearing cases are tabulated in Tables 7.8 and 7.9, respectively. For this rotor, comparisons between errors in whirl speeds and log decrements and those obtained by fixed-interface complex component mode method [60] are shown in Figures 7.9 to 7.12. All errors for the present method are calculated based on taking 76 DOF finite element solutions as exact solutions. But errors of Ref. [60] are based on taking their 24 DOF solutions as exact solutions.

The frequencies obtained by the present method are seen to have better accuracy than those obtained by the fixed-interface complex component mode method. The overall accuracy in log decrements are comparable with those obtained by the latter method.

Response Analysis

The unbalance response of the rotor for various levels of mode truncation has also been studied. The rotor unbalance is accounted for by a disc mass center eccentricity at the node 5.

First, Tables 7.10 and 7.11 show the non-dimensional steady state translation response amplitude of the disk (node 5) at two different rotating speeds for the isotropic and orthotropic bearing cases, respectively.

Finally, the transient responses of the disk to a sudden blade loss at various levels of modal truncation are plotted in Figures 7.13 and 7.14 for the isotropic and orthotropic bearing cases, respectively. The transient peak responses of the disk to the sudden blade loss are tabulated in Table 7.12.

Inspection of these figures and tables reveals the quality of convergence in response analysis of the present CMS method.

7.5.4 A Two-Spool Aircraft Turbine Engine

The second-stage turbine of the power turbine rotor of the two spool gas turbine engine presented in the last chapter, shown in Figure 7.15, is modeled as a flexible disc with four annular disk elements of uniform thickness having the following properties:

- Mass density: $\rho = 0.283 \text{ lbm/in}^3$ (7840 kg/m^3);
- Elastic moduli: $E = 3 \times 10^7 \text{ lb/in}^2$ ($2.07 \times 10^{11} \text{ N/m}^2$);
- Disk thickness: $h = 0.694 \text{ in}$ (17.628 mm);
- Inner diameter: $R_i = 1.125 \text{ in}$ (28.575 mm);
- Outer Diameter: $R_o = 6.710 \text{ in}$ (170.43 mm).

As indicated in the last chapter, the power-turbine and gas generator rotors are modeled as 21 and 15 rotating shaft elements, respectively. So the rotor model has a total of 176 degrees of freedom. The system is divided into three components, the

power-turbine rotor, the gas-generator rotor and the flexible second-stage turbine, as shown in Figure 7.15

The natural frequencies of the three components are tabulated in Table 7.13a, and Table 7.13b presents the selections of component modes retained in CMS analysis for various truncation frequencies.

The whirl frequencies obtained at various levels of mode truncation are tabulated in Tables 7.14 and 7.15a for undamped and damped bearing cases, respectively. For the purpose of comparison, the solutions of the full-order model are included as the exact solution. Table 7.15b gives the modal damping exponents calculated for the damped bearing case.

As can be seen, the present CMS method produces high accuracy both in frequencies and in damping exponents. Accuracy of frequencies is evidently much better than that of damping exponents, and this must be considered when selecting mode truncation level to model damped systems.

Figure 7.1 A Free-Free Beam Components and System

Figure 7.2 A Uniform Shaft Rotor System

(a) 1st forward mode

(b) 2nd forward mode

(c) 3rd forward mode

(d) 4th forward mode

Figure 7.3 Undamped Frequency Errors of a Uniform Shaft with a Speed of $\Omega = 5,000$ rpm

●●●●● The Current Method
 △△△△△ Fixed-interface Complex CMS [60,61]
 ○○○○○ MacNeal's Method

(a) 1st forward mode

(b) 2nd forward mode

(c) 3rd forward mode

(d) 4th forward mode

Figure 7.4 Damped Frequency Errors of a Uniform Shaft with Viscous Damping - $\Omega = 5,000$ rpm

●●●●● The Current Method
 ○○○○○ Fixed-Interface Complex CMS [60,61]

(a) 1st forward mode

(b) 2nd forward mode

(c) 3rd forward mode

(d) 4th forward mode

Figure 7.5 Log Decrement Errors of a Uniform Shaft with Viscous Damping - $\Omega = 5,000$ rpm

●●●●● The Current Method

○○○○○ Fixed-Interface Complex CMS [60,61]

(a) 1st forward mode

(b) 2nd forward mode

(c) 3rd forward mode

(d) 4th forward mode

Figure 7.6 Damped Frequency Errors of a Uniform Shaft with Hysteretic Damping - $\Omega = 5,000$ rpm

●●●●● The Current Method
 ○○○○○ Fixed-Interface Complex CMS [60,61]

(a) 1st forward mode

(b) 2nd forward mode

(c) 3rd forward mode

(d) 4th forward mode

Figure 7.7 Log Decrement Errors of a Uniform Shaft with Hysteretic Damping - $\Omega = 5,000$ rpm

●●●●● The Current Method
 ○○○○○ Fixed-Interface Complex CMS [60,61]

Figure 7.8 **A Multi-Stepped Shaft Rotor System and Component Models**

(a) 1st forward mode

(b) 2nd forward mode

(c) 3rd forward mode

(d) 4th forward mode

Figure 7.9 Damped Frequency Errors of a Stepped-Shaft Rotator with Isotropic Bearings - $\Omega = 30,000$ rpm

●●●●● The Current Method
 ○○○○○ Fixed-Interface Complex CMS [61,62]

(a) 1st forward mode

(b) 2nd forward mode

(c) 3rd forward mode

(d) 4th forward mode

Figure 7.10 Log Decrement Errors of a Stepped-Shaft Rotor with Isotropic Bearings - $\Omega = 30,000$ rpm

●●●●● The Current Method

○○○○○ Fixed-Interface Complex CMS [61,62]

(a) 1st forward mode

(b) 2nd forward mode

(c) 3rd forward mode

(d) 4th forward mode

Figure 7.11 Damped Frequency Errors of a Stepped-Shaft Rotor with Orthotropic Bearings - $\Omega = 40,000$ rpm

●●●●● The Current Method
 ○○○○○ Fixed-Interface Complex CMS [61,62]

(a) 1st forward mode

(b) 2nd forward mode

(c) 3rd forward mode

(d) 4th forward mode

Figure 7.12 Log Decrement Errors of a Stepped-Shaft Rotor with Orthotropic Bearings - $\Omega = 40,000$ rpm

●●●●● The Current Method
 ○○○○○ Fixed-Interface Complex CMS [61,62]

Figure 7.13 Blade Loss Simulation - Isotropic Bearing Case
of a Multi-Stepped Shaft Rotor System

$\triangle\triangle\triangle\triangle$ 4 dof $\square\square\square\square$ 6 dof $\bullet\bullet\bullet\bullet$ 40 dof

Figure 7.14 Blade Loss Simulation - Orthotropic Bearing Case of a Multi-Stepped Shaft Rotor System

Figure 7.15 Component Models of a Two-Spool Aircraft Engine with a Flexible Second-Stage Power Turbine Disk

Table 7.1 Modal Damping Exponents of a Free-Free Beam

Mode Order	Exact σ	Craig and Ni [59]		Current Method	
		σ	Error %	σ	Error %
3	-0.47268E-3	-0.47268E-3	0.00076	-0.47268E-3	0.00001
4	-0.31164E-2	-0.31165E-2	0.00266	-0.31164E-2	0.00028
5	-0.10885E-1	-0.10885E-1	0.00352	-0.10885E-1	0.00025
6	-0.29957E-1	-0.29970E-1	0.0442	-0.29958E-1	0.00460
7	-0.67127E-1	-0.67133E-1	0.00889	-0.67128E-1	0.00219
8	-0.13117	-0.13142	0.187	-0.13119	0.01618
9	-0.23999	-0.24050	0.214	-0.23990	0.03545
10	-0.39527	-0.39658	0.332	-0.39505	0.05431
11	-0.62119	-0.63006	1.428	-0.61901	0.35106
12	-0.10944E+1	-0.11062E+1	1.073	-0.10779E+1	1.50554

Table 7.2 Damped Modal Frequencies of a Free-Free Beam

Mode Order	Exact ω_d	Craig and Ni [59]		Current Method	
		ω_d	Error %	ω_d	Error %
3	0.22374	0.22374	0.00028	0.22374	0.00000
4	0.61688	0.61688	0.00122	0.61688	0.00004
5	0.12101E+1	0.12101E+1	0.00243	0.12101E+1	0.00012
6	0.20032E+1	0.20037E+1	0.0243	0.20032E+1	0.00014
7	0.30002E+1	0.30003E+1	0.00532	0.30002E+1	0.00076
8	0.42069E+1	0.42113E+1	0.105	0.42069E+1	0.00064
9	0.56300E+1	0.56361E+1	0.108	0.56301E+1	0.00063
10	0.72698E+1	0.72835E+1	0.188	0.72698E+1	0.00080
11	0.90249E+1	0.90886E+1	0.705	0.90288E+1	0.04292
12	0.11965E+2	0.12035E+2	0.588	0.11970E+2	0.04307

**Table 7.3a Component Natural Frequencies of
a Simply Supported Uniform Shaft**
(only one-plane modes are listed)

Mode order	Component <i>A</i> ω	Component <i>B</i> ω
1	0.0000	0.0000
2	1156.8	1443.1
3	5054.8	10788.
4	13211.	31238.
5	27712.	67588.
6	47037.	

**Table 7.3b Component Mode Selection Schemes
for a Simply Supported Uniform Shaft**

Truncation Frequencies ω_i	Component Modes Retained		Total DOF of CMS Model
	Component <i>A</i>	Component <i>B</i>	
0.000 rad/s	2	2	4
1200. rad/s	4	2	6
2000. rad/s	4	4	8
12000 rad/s	6	6	12
30000 rad/s	10	6	16

**Table 7.4 Undamped Whirl Frequencies (rad/s)
of a Simply Supported Uniform Shaft
at Various Component Modal Truncation Levels
(at a running speed = 4000 rpm)**

Mode No.	F.E.M. 24 d.o.f.	Whirl Frequencies (rad/s)				
		4 d.o.f.	6 d.o.f.	8 d.o.f.	12 d.o.f.	16 d.o.f.
1b	520.46	522.22	520.66	520.46	520.46	520.46
1f	521.02	522.89	521.23	521.02	521.02	521.02
2b	1093.16	1133.70	1104.21	1093.16	1093.16	1093.16
2f	1096.69	1137.17	1107.65	1096.69	1096.69	1096.69
3b	2237.83		2354.62	2241.96	2237.85	2237.83
3f	2253.08		2369.55	2256.97	2253.11	2253.08
4b	5058.07			5293.22	5058.32	5058.10
4f	5093.42			5328.48	5093.65	5093.46
5b	9581.46				9739.01	9581.54
5f	9640.76				9797.49	9640.81
6b	15491.3				17540.8	15510.0
6f	15576.1				17629.4	15594.4
7b	24877.5					25281.6
7f	24995.9					25396.5
8b	34803.9					37185.1
8f	34961.9					37335.2

**Table 7.5 Damped Eigenvalues of a Simply Supported Uniform Shaft
with a Viscous Damping Factor $\eta_v = 0.0002$
at Various Component Modal Truncation Levels
(at a running speed = 4000 rpm)**

Mode No.	F.E.M. 24 d.o.f.	Whirl Frequencies (rad/s) / Log Decrements			
		6 d.o.f.	8 d.o.f.	12 d.o.f.	16 d.o.f.
1b	521.01 / .0253	521.21 / .0260	521.01 / .0253	521.01 / .0253	521.01 / .0253
1f	522.74 / .2309	522.41 / .2376	522.56 / .2307	522.71 / .2309	522.73 / .2309
2b	1096.6 / .0687	1104.5 / .1510	1095.8 / .0700	1096.5 / .0690	1096.6 / .0688
2f	1097.4 / .0331	1107.6 / .0684	1097.2 / .0331	1097.4 / .0331	1097.4 / .0331
3b	2214.6 / 1.059	2320.2 / 1.222	2215.8 / 1.082	2214.1 / 1.060	2214.5 / 1.060
3f	2230.9 / .7246	2337.2 / .8484	2232.8 / .7384	2230.6 / .7244	2230.8 / .7246
4b	4454.5 / 3.681		4582.6 / 3.948	4453.7 / 3.683	4454.3 / 3.681
4f	4492.0 / 3.048		4620.1 / 3.285	4491.3 / 3.049	4491.8 / 3.048
5b	3312.5 / 19.20			2929.4 / 22.79	3310.8 / 19.22
5f	3372.8 / 14.92			2987.8 / 17.13	3371.1 / 14.93
6b	451.77 / 587.8			412.35 / 854.2	451.89 / 598.4
6f	537.00 / 66.70			501.23 / 69.07	536.69 / 66.70
7b	333.69 / 2238.				336.31 / 2297.
7f	452.31 / 72.65				451.28 / 72.68
8b	275.93 / 5418.				282.02 / 6067.
8f	434.02 / 74.02				432.16 / 74.10

**Table 7.6 Damped Eigenvalues of a Simply Supported Uniform Shaft
with a Hysteretic Loss Factor $\eta_h = 0.0002$
at Various Component Modal Truncation Levels
(at a running speed = 4000 rpm)**

Mode No.	F.E.M. 24 d.o.f.	Whirl Frequencies (rad/s) / Log Decrements (10^{-6})			
		6 d.o.f.	8 d.o.f.	12 d.o.f.	16 d.o.f.
1b	520.51 / 250	520.71 / 256	520.52 / 249	520.51 / 250	520.51 / 250
1f	521.07 / -248	521.28 / -254	521.07 / -247	521.07 / -248	521.07 / -248
2b	1093.3 / 49.	1104.3 / 100	1093.3 / 49.	1093.3 / 49.	1093.3 / 49.
2f	1096.8 / -50.	1107.8 / -99.	1096.8 / -49.	1096.8 / -50.	1096.8 / -49.
3b	2238.1 / 394	2354.9 / 433	2242.2 / 401	2238.1 / 394	2238.1 / 394
3f	2253.9 / -392	2369.8 / -430	2257.2 / -398	2253.3 / -392	2253.3 / -392
4b	5058.6 / 586		5293.8 / 592	5058.8 / 586	5058.6 / 586
4f	5093.9 / -582		5329.0 / -588	5094.2 / -582	5094.0 / -582
5b	9582.4 / 618			9740.0 / 620	9582.5 / 618
5f	9641.7 / -615			9797.5 / -616	9641.8 / -615
6b	15493. / 626			17543. / 627	15512. / 626
6f	15578. / 623			17631. / -624	15596. / -623
7b	24880. / 628				25284. / 628
7f	24998. / -625				25399. / -626
8b	34807. / 629				37189. / 629
8f	34965. / -626				37339. / -626

**Table 7.7a Component Natural Frequencies (rad/s)
a Multi-Stepped Shaft Rotor System**

Mode order	Isotropic Bearing Case		Orthotropic Bearing Case	
	Component A	Component B	Component A	Component B
1	0.0000	4882.7	0.0000	3938.7
2	0.0000	4882.7	0.0000	4482.7
3	0.0000	10648.	0.0000	8899.5
4	0.0000	10648.	0.0000	10648.
5	13253.	23908.	13253.	21852.
6	13253.	23908.	13253.	23908.
7	63713.	55278.	63713.	54219.
8	63713.	55278.	63713.	55278.
9	94939.	116450	94939.	115930
10	94939.	116450	94939.	116450

**Table 7.7b Component Mode Selection Schemes
for a Multi-Stepped Shaft Rotor**

Truncation Frequencies ω_t	Component Modes Retained		Total DOF of CMS Model
	Component A	Component B	
0.000 rad/s	4	0	4
5000. rad/s	4	2	6
11000 rad/s	4	4	8
15000 rad/s	6	6	12
70000 rad/s	8	8	16

**Table 7.8 Damped Frequencies and Logarithmic Decrements of
a Multi-Stepped Shaft Rotor with Isotropic Bearings
at Various Component Modal Truncation Levels
(at a running speed = 30000 rpm)**

Mode No.	F.E.M. 75 d.o.f.	Whirl Frequencies (rpm) / Log Decrements				
		4 d.o.f.	6 d.o.f.	8 d.o.f.	12 d.o.f.	16 d.o.f.
1b	14739/.0897	14750/.0879	14745/.0905	14745/.0905	14740/.0896	14739/.0897
1f	17833/.1215	17849/.1173	17841/.1203	17840/.1204	17833/.1216	17833/.1214
2b	47373/.4086	72660/.5425	47229/.4305	47343/.4134	47370/.4088	47371/.4085
2f	49443/.4687	87237/.5128	49329/.4457	49416/.4658	49439/.4685	49440/.4688
3b	70041/.6521		72902/.5673	69942/.6531	69993/.6531	70030/.6521
3f	82479/.5497		87224/.5222	83167/.5951	82426/.5487	82467/.5496
4b	120400/.0641			141534/.1931	120391/.0639	120398/.0641
4f	124287/.0933			151944/.1686	124276/.0931	124283/.0932
5b	159794/.2613				159861/.2630	159780/.2614
5f	182577/.2542				182563/.2542	182561/.2543
6b	341804/.1490				345780/.1563	341773/.1489
6f	358959/.1422				362336/.1478	358944/.1420
7b	634586/.1597					636064/.1748
7f	650779/.1557					652106/.1699
8b	902629/34E-6					932092/15E-5
8f	909330/15E-6					959373/12E-6

**Table 7.9 Damped Whirl Frequencies and Logarithmic Decrements
of a Multi-Stepped Shaft Rotor with Orthotropic Bearings
at Various Component Modal Truncation Levels**
(at a running speed = 40000 rpm)

Mode No.	F.E.M. 76 d.o.f.	Whirl Frequencies (rpm)/Log Decrements				
		4 d.o.f.	6 d.o.f.	8 d.o.f.	12 d.o.f.	16 d.o.f.
1b	13075/.1499	13091/.1455	13082/.1510	13082/.1510	13075/.1497	13075/.1499
1f	17316/.1523	17339/.1475	17330/.1507	17329/.1509	17316/.1525	17316/.1523
2b	39968/.6846	63689/.6623	39788/.6801	39933/.6864	39962/.6848	39965/.6846
2f	48312/.4498	85925/.5444	48300/.4449	48298/.4502	48310/.4497	48309/.4498
3b	61788/.6975		64258/.6270	61791/.6961	61750/.6985	61780/.6975
3f	81588/.5517		86435/.5333	82462/.5989	81540/.5505	81577/.5515
4b	119405/.0584			138364/.1885	119399/.0581	119403/.0584
4f	124498/.0903			152733/.1610	124491/.0903	124495/.0902
5b	154445/.2626				154542/.2643	154433/.2627
5f	185157/.2568				185158/.2563	185143/.2569
6b	341804/.1490				342473/.1564	338045/.1479
6f	338073/.1390				364702/.1448	361138/.1388
7b	630721/.1603					632283/.1757
7f	652495/.1549					653850/.1691
8b	901508/42E-6					929739/21E-5
8f	910451/14E-6					952507/9E-6

Table 7.10 Normalized Unbalance Translational Response of the Disk in a Multi-Stepped Shaft Rotor with Isotropic Bearings at Various Component Modal Truncation Levels

Degree of Freedom	20,000 rpm		40,000 rpm	
	$X_5/e = Y_5/e$	Error %	$X_5/e = Y_5/e$	Error %
4	2.37754	0.915	.769797	3.186
6	2.36504	0.385	.759241	1.771
8	2.36437	0.356	.758279	1.642
12	2.35562	0.015	.746078	0.007
16	2.35591	0.003	.746063	0.005
76	2.35598	exact	.746027	exact

e = disc eccentricity at node 5

Table 7.11 Normalized Unbalance Translational Response of the Disk in a Multi-Stepped Shaft Rotor with Orthotropic Bearings at Various Component Modal Truncation Levels

Degree of Freedom	20,000 rpm				40,000 rpm			
	X_5/e	Error %	Y_5/e	Error %	X_5/e	Error %	Y_5/e	Error %
4	2.05829	0.836	1.50738	1.250	.748175	3.949	.731475	0.569
6	2.04793	0.328	1.49442	0.376	.732116	1.719	.743370	2.204
8	2.04728	0.299	1.49356	0.322	.730710	1.523	.737860	1.446
12	2.04097	0.010	1.48852	0.020	.719795	0.007	.727406	0.010
16	2.04116	0.003	1.48872	0.003	.719777	0.004	.727391	0.007
76	2.04122	exact	1.48877	exact	.719750	exact	.727337	exact

e = disc eccentricity at node 5

Table 7.12 Normalized Blade Loss Peak Response r_p/e_s of the Disk in a Multi-Stepped Shaft Rotor System at Various Component Modal Truncation Levels

Kept Modes	Isotropic Bearings		Orthotropic Bearings	
	$\Omega = 20,000 \text{ rpm}$	$\Omega = 40,000 \text{ rpm}$	$\Omega = 20,000 \text{ rpm}$	$\Omega = 40,000 \text{ rpm}$
4	4.34052	2.35806	4.07687	2.55222
6	4.30552	2.35247	4.04065	2.54149
8	4.30417	2.35044	4.03919	2.53749
12	4.28711	2.33298	4.02692	2.52079
16	4.28819	2.33312	4.02767	2.52093
24	4.26826	2.33310	4.02775	2.52098
40	4.28829	2.33310	4.02778	2.52098

**Table 7.13a Component Natural Frequencies (rad/s)
of a Two-Spool Aircraft Turbine Engine**
(only one-plane modes are listed)

Mode Order	Undamped Bearing Case			Damped Bearing Case		
	Power Turbine	Gas Generator	Flexible Disk	Power Turbine	Gas Generator	Flexible Disk
1	342.076	1105.38	0.00000	342.076	796.017	0.00000
2	959.229	2869.74	0.00000	959.229	2698.35	0.00000
3	1813.94	3966.36	19540.2	959.229	3833.75	19540.2
4	3347.03	6609.53	55233.4	3347.03	6586.03	55233.4
5	5268.86	11440.4	107433.	5268.86	11439.9	107433.
6	7266.98	17353.9	179334.	7266.98	17353.6	179334.
7	7658.64	23979.9	290080.	7658.64	23977.8	290080.
8	10360.9	29553.2	411971.	10360.9	29548.3	411971.
9	15570.4	31073.5	538693.	15570.4	31068.6	538693.
10	20708.6	39751.7	613518.	20708.6	39744.1	613518.

**Table 7.13b Component Mode Selection Schemes
of a Two-Spool Aircraft Turbine Engine**

Truncation Frequencies ω_t	Component Modes Retained			Total DOF of CMS Model
	Power Turbine Rotor	Gas Generator Rotor	Flexible Disk	
3500. rad/s	8	4	4	16
7000. rad/s	10	8	4	22
20000 rad/s	18	12	6	36

**Table 7.14 Undamped Frequencies of a Two-Spool Aircraft Engine
with a Flexible 2nd-Stage Turbine Disk
at Various Component Modal Truncation Levels**
(Power turbine runs at a speed of 20,000 rpm)
(Gas generator runs at a speed of 15,000 rpm)

Mode Order	F.E.M. 176 d.o.f.	Undamped Whirl Frequencies (rpm)					
		16 d.o.f.	error %	22 d.o.f.	error %	36 d.o.f.	error %
1	5357.81	5367.33	0.178	5365.12	0.136	5357.94	0.002
2	7112.55	7115.70	0.044	7114.94	0.034	7112.54	0.000
3	8184.33	8231.63	0.578	8185.94	0.020	8184.50	0.002
4	8640.24	8724.56	0.976	8642.22	0.023	8640.47	0.003
5	12483.2	12646.2	1.305	12486.1	0.023	12483.4	0.002
6	19215.2	19371.7	0.815	19241.1	0.135	19215.8	0.003
7	20772.8	20943.0	0.820	20777.7	0.024	20773.2	0.002
8	24375.7	24995.1	2.541	24718.6	1.407	24378.0	0.009
9	30066.4	31060.2	3.305	30429.6	1.208	30069.6	0.007
10	31329.9	32318.2	3.155	31449.8	0.383	31331.3	0.005
11	34164.9	37313.0	9.214	34203.8	0.114	34166.7	0.005
12	35436.2	44333.8	25.11	35833.4	1.121	35440.2	0.011
13	42304.6	45441.3	7.415	42559.2	0.602	42308.8	0.010
14	51698.0	59622.4	15.33	51809.0	0.215	51703.5	0.011
15	58081.0	154008.	165.2	58139.2	0.102	58092.5	0.020
16	58768.1	157411.	167.9	59051.1	0.482	58772.4	0.007
17	60861.0			62299.2	2.363	60865.8	0.008
18	62920.5			65864.1	4.678	62922.5	0.003
19	66370.9			68988.1	3.943	66378.1	0.011
20	70863.2			73436.6	3.632	70873.7	0.015
21	78236.4			154228.	97.13	78242.8	0.008
22	78854.8			157841.	100.2	78861.9	0.009
23	80819.3					80825.1	0.007
24	87955.5					87970.8	0.017
25	104531					104576.	0.043
26	107065.					107078.	0.011
27	109120.					109159.	0.035
28	116971.					117023.	0.044
29	147453.					147555.	0.069
30	149078.					149092.	0.009

**Table 7.15a Damped Whirl Frequencies of a Two-Spool Aircraft Engine
with a Flexible 2nd-Stage Turbine Disk
at Various Component Modal Truncation Levels**
(Power turbine runs at a speed of 17,000 rpm)
(Gas generator runs at a speed of 15,000 rpm)

Mode Order	F.E.M. 176 d.o.f.	Damped Whirl Frequencies (rpm)					
		16 d.o.f.	error %	22 d.o.f.	error %	36 d.o.f.	error %
1	5706.80	5714.63	0.137	5712.65	0.102	5706.86	0.001
2	6158.34	6173.49	0.246	6157.76	-0.009	6157.51	-0.013
3	6978.24	6975.27	-0.043	6979.71	0.021	6978.21	0.000
4	8975.92	9005.32	0.328	8979.16	0.036	8976.14	0.003
5	8928.93	8993.23	0.720	8929.00	0.071	8926.31	-0.030
6	18194.5	18357.1	0.893	18199.0	0.024	18190.1	-0.024
7	20852.6	21074.6	1.065	20859.0	0.031	20853.3	0.003
8	24762.8	25333.0	2.303	24876.8	0.460	24760.3	-0.010
9	29369.7	30527.3	3.942	29896.3	1.793	29364.0	-0.019
10	30353.5	31141.3	2.596	30526.9	0.571	30353.1	-0.001
11	33875.7	35991.9	6.247	33872.2	-0.010	33871.0	-0.014
12	33925.0	44697.8	31.76	34133.1	0.613	33926.7	0.005
13	40594.4	44722.8	10.20	40668.0	0.206	40580.3	-0.010
14	51570.5	56127.6	8.837	51624.2	0.104	51561.0	-0.018
15	55423.5	154199.	178.2	55481.7	0.105	55432.3	0.016
16	57766.2	157082.	171.9	57905.8	0.242	57746.1	-0.035
17	61312.7			62361.7	1.711	61260.7	-0.085
18	62261.7			64961.6	4.336	62261.8	0.000
19	66140.2			68929.0	4.217	66131.1	-0.014
20	70678.2			73172.8	3.529	70675.9	-0.003
21	78571.6			154423.	96.54	78576.0	0.006
22	78838.8			157479.	99.75	78841.3	0.003
23	80533.4					80533.4	0.001
24	86742.6					86750.8	0.009
25	104567.					104609.	0.041
26	106881.					106891.	0.010
27	109176.					109168.	-0.007
28	116982.					117009.	0.023
29	148286.					148135.	-0.102
30	148459.					148512.	0.036

**Table 7.15b Modal Damping Exponents of a Two-Spool Aircraft Engine
with a Flexible 2nd-Stage Turbine Disk
at Various Component Modal Truncation Levels**
(Power turbine runs at a speed of 17,000 rpm)
(Gas generator runs at a speed of 15,000 rpm)

Mode Order	F.E.M. 176 d.o.f.	Modal Damping Exponents					
		16 d.o.f.	error %	22 d.o.f.	error %	36 d.o.f.	error %
1	-10.343	-10.404	0.584	-10.406	0.611	-10.345	0.017
2	-83.715	-90.272	7.832	-83.796	0.097	-83.678	-0.044
3	-19.671	-19.544	-0.645	-19.706	0.180	-19.689	0.093
4	-1.4180	-1.7976	26.77	-1.4156	-0.170	-1.4191	0.074
5	-155.88	-149.32	-4.210	-155.82	-0.041	-155.84	-0.024
6	-142.89	-117.85	-17.53	-143.20	0.214	-142.90	0.002
7	-14.686	-18.532	26.19	-14.805	0.812	-14.678	-0.055
8	-90.042	-71.059	-21.08	-102.08	13.37	-90.479	0.486
9	-128.61	-129.52	0.713	-120.11	-6.604	-128.44	-0.128
10	-53.203	-72.912	37.04	-55.178	3.712	-53.069	-0.253
11	-81.650	-98.963	21.20	-81.937	0.351	-81.657	0.008
12	-26.778	-83.756	212.7	-25.606	-4.379	-26.864	0.321
13	-130.20	-217.60	67.13	-128.03	-1.666	-129.62	-0.444
14	-157.96	-13.682	-91.34	-154.16	-2.405	-159.39	0.906
15	-43.995	-11.479	-73.91	-51.761	17.65	-44.411	0.946
16	-253.84	-11.247	-95.57	-244.77	-3.376	-252.80	-0.410
17	-312.31			-386.48	23.75	-313.53	0.390
18	-139.32			-187.17	34.34	-138.77	-0.400
19	-149.16			-85.173	-42.90	-148.83	-0.219
20	-146.77			-115.16	-21.53	-146.51	-0.174
21	-28.090			-7.4803	-73.37	-28.112	0.077
22	-72.036			-7.0462	-90.22	-72.703	0.926
23	-78.018					-78.310	0.374
24	-42.293					-42.166	-0.300
25	-41.738					-43.166	3.422
26	-61.457					-61.676	0.357
27	-207.76					-208.59	0.401
28	-149.52					-146.45	-2.058
29	-247.65					-259.28	4.698
30	-170.09					-163.85	-3.668

Chapter 8

Dynamics of Flexible Bladed Disk–Shaft Assemblies

8.1 Introduction

For the analysis of stability, critical speeds and response of rotor–bearing systems, it is generally a satisfactory approximation that the disks are considered as rigid bodies, but caution is required if a bladed disc has a disc frequency close to the operating speed of a rotor system. As shown in Chapters 2 and 3, the one-diameter bladed disk modes are kinetically coupled with the shaft motion since there is a resultant unbalance or unequilibrated moment and transverse shear on the disk which must be supported by the shaft. All other diametral modes of flexible bladed disks are kinetically “balanced” so that they are completely uncoupled with the shaft motion.

Dopkin [101, 102], Chivens and Nelson [103, 104], Kuo [105], Klompas [109, 110] and Palladino [112, 113] have analyzed the dynamics of a flexible shaft with one flexible disk. Gallardo et al. [114, 115] have studied the dynamics of a flexible bladed disc on a flexible rotor in a two-spool rotor system. These analyses have been concerned mostly with steady state response and the critical speeds or whirl frequencies of single-rotor systems. All of them, except Klompas, have only examined isotropic bearing cases. Therefore, backward whirl in most studies is not accounted for unless counter-rotating shafts are incorporated. However, asymmetry of engine support

structures and bearings causes interaction between forward and backward whirl so as to produce coupled whirl of elliptical orbit form. As known from the literature studies, the disk flexibility has much more effect on the dynamics of backward whirl modes than on the forward whirl modes, so the response of such asymmetric cases is of special interest.

In this chapter, a variety of rotor systems has been used to study the effects of bladed disks on the critical speeds, whirl frequencies, steady-state or transient unbalanced and forced responses of rotor systems. The first example is a simplified 6-DOF rotor system of concentrated masses and stiffness, which has been used to facilitate study of the dynamic properties by nondimensional configuration variations. The second example is a series of simple-supported rotor systems with one flexible bladed disk. The final example is the two-spool aircraft turbine engine studied in Chapter 7, where it was treated as a rotor system with one flexible disk. Here, the engine is further modified to make both turbine disks flexible.

8.2 6-DOF Lumped Rotor Model

Figure 8.1 shows a lumped parameter model of a rotor system. The rigid-body motion of the rotor is described by two translations, (U_d, V_d) , and two rotations, (θ_x, θ_y) , in the two orthogonal planes. The flexible disk motion is described by two flexible disk modes, (η, ζ) with an isotropic stiffness K_d . The numerical and/or experimental methods to obtain the value of K_d may be found in Reference [120, 121].

8.2.1 Equations of Motion

The equations of motion of the rotor system at a constant rotating speed Ω can, as shown in Chapter 2, be written in the following matrix form:

$$[M]\{\ddot{Z}(t)\} + ([C] + \Omega[G])\{\dot{Z}(t)\} + [K]\{Z(t)\} = \{F(t)\} \quad (8.1)$$

where

$$\{Z(t)\} = \begin{Bmatrix} U_d \\ \theta_y \\ \eta \\ V_d \\ -\theta_x \\ \zeta \end{Bmatrix}; \quad \{F(t)\} = \begin{Bmatrix} \Omega^2 M_d e \cos \Omega t \\ 0 \\ 0 \\ \Omega^2 M_d e \sin \Omega t \\ 0 \\ 0 \end{Bmatrix}$$

$$[M] = \begin{bmatrix} M_s + M_d & 0 & 0 & 0 & 0 & 0 \\ 0 & I_s + I_d & I_d & 0 & 0 & 0 \\ 0 & I_d & I_d & 0 & 0 & 0 \\ 0 & 0 & 0 & M_s + M_d & 0 & 0 \\ 0 & 0 & 0 & 0 & I_s + I_d & I_d \\ 0 & 0 & 0 & 0 & I_d & I_d \end{bmatrix}$$

$$[C] = \begin{bmatrix} c_u & 0 & 0 & 0 & 0 & 0 \\ 0 & c_{\theta_y} & 0 & 0 & 0 & 0 \\ 0 & 0 & c_d & 0 & 0 & 0 \\ 0 & 0 & 0 & c_v & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{\theta_x} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_d \end{bmatrix}$$

$$[G] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_p & I_p \\ 0 & 0 & 0 & 0 & I_p & I_p \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -I_p & -I_p & 0 & 0 & 0 \\ 0 & -I_p & -I_p & 0 & 0 & 0 \end{bmatrix}$$

$$[K] = \begin{bmatrix} K_u & K_{u\theta_y} & 0 & 0 & 0 & 0 \\ K_{u\theta_y} & K_{\theta_y} & 0 & 0 & 0 & 0 \\ 0 & 0 & K_d & 0 & 0 & 0 \\ 0 & 0 & 0 & K_v & K_{v\theta_x} & 0 \\ 0 & 0 & 0 & K_{v\theta_x} & K_{\theta_x} & 0 \\ 0 & 0 & 0 & 0 & 0 & K_d \end{bmatrix}$$

The above equations can be non-dimensionalized by the following transformation:

$$\{Z(t)\} = [T]\{p(t)\} \quad (8.2)$$

with

$$[T] = \text{diag.} \left\{ \sqrt{M_s + M_d}, \sqrt{I_s + I_d}, \sqrt{I_d}, \sqrt{M_s + M_d}, \sqrt{I_s + I_d}, \sqrt{I_d} \right\}$$

and

$$\{p(t)\}^T = \left\{ \frac{U_d}{\sqrt{M_s + M_d}}, \frac{\theta_y}{\sqrt{I_s + I_d}}, \frac{\eta}{\sqrt{I_d}}, \frac{V_d}{\sqrt{M_s + M_d}}, \frac{-\theta_x}{\sqrt{I_s + I_d}}, \frac{\zeta}{\sqrt{I_d}} \right\}$$

Substituting Eq. 8.2 into Eq. 8.1, we obtain the non-dimensionalized equations of motion as follows:

$$[\bar{M}]\{\bar{\ddot{p}}(t)\} + ([\bar{C}] + \bar{\Omega}[\bar{G}])\{\bar{\dot{p}}(t)\} + [K]\{p(t)\} = \{Q(t)\} \quad (8.3)$$

with

$$\{\bar{\ddot{p}}(t)\} = \frac{1}{\omega_u^2} \{\ddot{p}(t)\}, \quad \{\bar{\dot{p}}(t)\} = \frac{1}{\omega_u} \{\dot{p}(t)\}, \quad \{Q(t)\} = \frac{1}{\omega_u^2} \{F(t)\}$$

$$[\bar{M}] = [T]^T [M] [T] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \gamma & 0 & 0 & 0 \\ 0 & \gamma & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \gamma \\ 0 & 0 & 0 & 0 & \gamma & 1 \end{bmatrix}$$

$$[\bar{C}] = [T]^T [C] [T] = \begin{bmatrix} 2\xi_u & 0 & 0 & 0 & 0 & 0 \\ 0 & 2\xi_{\theta_y} \bar{\omega}_{\theta_y} & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\xi_d \bar{\omega}_d & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\xi_v \bar{\omega}_v & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\xi_{\theta_x} \bar{\omega}_{\theta_x} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\xi_d \bar{\omega}_d \end{bmatrix}$$

$$[\bar{G}] = [T]^T [G] [T] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\gamma^2 & \gamma \\ 0 & 0 & 0 & 0 & \gamma & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2\gamma^2 & -\gamma & 0 & 0 & 0 \\ 0 & -\gamma & -2 & 0 & 0 & 0 \end{bmatrix}$$

$$[\bar{K}] = [T]^T [K] [T] = \begin{bmatrix} 1 & \delta_u \bar{\omega}_{\theta_y} & 0 & 0 & 0 & 0 \\ \delta_u \bar{\omega}_{\theta_y} & \bar{\omega}_{\theta_y}^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \bar{\omega}_d^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{\omega}_v^2 & \delta_v \bar{\omega}_{\theta_x} & 0 \\ 0 & 0 & 0 & \delta_v \bar{\omega}_{\theta_x} & \bar{\omega}_{\theta_x}^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{\omega}_d^2 \end{bmatrix}$$

and

$$\gamma = \sqrt{\frac{I_d}{I_s + I_d}},$$

$$\delta_u = \frac{K_{u\theta_y}}{\sqrt{K_u K_{\theta_y}}}$$

$$\delta_v = \frac{K_{v\theta_x}}{\sqrt{K_v K_{\theta_x}}},$$

$$\omega_u^2 = \frac{K_u}{M_s + M_d}$$

$$\omega_v^2 = \frac{K_v}{M_s + M_d},$$

$$\omega_{\theta_y}^2 = \frac{K_{\theta_y}}{I_s + I_d}$$

$$\omega_{\theta_z}^2 = \frac{K_{\theta_z}}{I_s + I_d}, \quad \omega_d^2 = \frac{K_d}{I_d}$$

$$\xi_u = \frac{c_u}{2(M_s + M_d)\omega_u}, \quad \xi_v = \frac{c_v}{2(M_s + M_d)\omega_v}$$

$$\xi_{\theta_z} = \frac{c_{\theta_z}}{2(I_s + I_d)\omega_{\theta_z}}, \quad \xi_{\theta_y} = \frac{c_{\theta_y}}{2(I_s + I_d)\omega_{\theta_y}}$$

$$\bar{\omega}_v = \frac{\omega_v}{\omega_u}, \quad \bar{\omega}_{\theta_y} = \frac{\omega_{\theta_y}}{\omega_u}$$

$$\bar{\omega}_{\theta_z} = \frac{\omega_{\theta_z}}{\omega_u}, \quad \bar{\omega}_d = \frac{\omega_d}{\omega_u}$$

$$\bar{\Omega} = \frac{\Omega}{\omega_u}$$

These equations can be easily modified and condensed further depending on assumptions made. Thus some of the useful cases which can be considered are:

1. neglecting shaft translations, U_d and V_d , will result in a 4 DOF system in which the shaft rotations are coupled to the flexible disc motion, viz. $\{ \theta_y \ \eta \ - \ \theta_x \ \zeta \}$.
2. A further assumption of shaft symmetry will reduce this 4 DOF system to 2-DOF in which circular orbits of shaft rotation couple with the disk, viz. $\{ \theta_y \ \eta \}$.
3. Assumption of disk rigidity removes the flexible disk motion, η , from the 2-DOF system above, and yields the classical equation for a whirling disk given by Den Hartog.
4. By assuming shaft symmetry in the original 6-DOF system, a 3-DOF system results in which circular orbits of the shaft translation and rotation are coupled with flexible disc motion, viz. $\{ U_d \ \theta_y \ \eta \}$.

In the following analysis, the 6-DOF system equations are used. The mass ratio, γ , of the disk to the shaft is taken as 0.3 and 0.5 in two cases, respectively. In each case, two values (2.0 and 4.0) of the nondimensional frequency, $\bar{\omega}_d$, of the disk will be studied, of which one is higher than the frequencies of disk translations but lower than those of disk rotations, and the other is higher than the natural frequencies of disk rotation and translation. All other parameters in the studies are taken as follows:

$$\begin{aligned} \omega_u &= 1.2, & \bar{\omega}_{\theta_y} &= 2.0, & \bar{\omega}_{\theta_x} &= 2.4, & \delta_u &= 0.5, & \delta_v &= 0.5 \\ \xi_u &= 0.01, & \xi_v &= 0.01, & \xi_{\theta_y} &= 0.01, & \xi_{\theta_x} &= 0.01, & \xi_d &= 0.0 \end{aligned}$$

8.2.2 Whirl Frequencies

The normalized equation of motion (Eq. 8.3) yields a convenient form for the evaluation of the criteria for interaction between the bladed disk dynamics and the shaft whirling motion. The basic question to be addressed is that under what circumstances is the interaction between the bladed disk motion and the disk rigid body motion sufficient to warrant solving the fully coupled equation of motion. The propensity for interaction between the two motions is usually determined by the proximity of the shaft whirl frequencies to the uncoupled bladed disk frequencies. If the interaction is weak, then the coupled bladed disk-shaft system may be solved as two independent, simpler problems, i.e., the bladed disk vibration and the shaft whirl problems.

In Figure 8.2 are shown the nondimensional whirl speeds of the rotor system versus nondimensional spin speeds for various configurations. Even for this simple model the effects are relatively complex. The modes occur in sequence backward (**B**), forward (**F**), **B**, **F**, ... It is seen that the backward mode shows the greatest reduction in whirl frequency because of the disk flexibility. In particular the third rigid disc mode (rotation, ω_{θ_y}) shows much larger interaction than the first rigid disk mode (translation u).

Note that even with a relatively rigid disc ($\omega_d/\omega_u = 4.0$) the backward whirl frequencies can be reduced by as much as 40%. The effect of increased mass ratio γ is generally to enhance these reductions.

In contrast, the critical speeds ($\omega/\Omega = 1$) are less affected. Table 8.1 lists the critical speeds for the two mass ratios studied, for various degrees of disk flexibility

$\bar{\omega}_d = \omega_d/\omega_u$. For relatively rigid disc $\omega_d/\omega_u = 4.0$, only the third mode, as noted above, shows a significant reduction in critical speed, of about -14% . With the more flexible disc $\omega_d/\omega_u = 2.0$, only the critical speeds of the two translational modes are unaffected, while the other critical speeds are reduced by as much as -40% , due largely to the introduction of the flexible disc modes. Note that the effect here is largest with the lower mass ratio disc, $\gamma = 0.3$.

8.2.3 Unbalance Response

From the analysis presented in Chapter 5, unbalance forces will excite only the forward modes of an isotropic rotor system, but both the forward and backward modes of an orthotropic system. This characteristic is verified by Figure 8.3 where the disk is treated as rigid and the properties of the rotor system in the isotropic case are taken as $\bar{\omega}_v = \bar{\omega}_u = 1.0$, $\bar{\omega}_{\theta_y} = \bar{\omega}_{\theta_z} = 2.0$. Note the presence of backward whirl response of a level just slightly less than the forward whirl response in the orthotropic case.

Figure 8.4 presents the steady state responses of the rotor system to an unbalance force. It is seen that the peak responses of the rotor at critical speeds, especially at those of the translational modes, are not significantly affected by the disk flexibility, but the locations of the critical speeds have been shifted substantially because of the disk flexibility. It should be noted that the flexible disk (backward) mode can be excited by an unbalance force which brings in a new peak response in the response plot. Hence, special precautions should be taken if the operation speed of the rotor system is close to this critical speed, and the effect of this excitation on blade resonance should be considered.

8.2.4 Forced Response

Another important excitation source will be the distortion of the air flow through the bladed disks which produces an unbalance moment on the disks. There is a need to examine the frequency response of the rotor system to such loads. Only the first diameter (1st order) component of the moment needs to be considered in the analysis, because all other components of the moment are balanced and decoupled with the

shaft motion. Since it is not known, a priori, where the nodal diameter of the load will fall with respect to the space-fixed frame, it is convenient to assume that its nodal diameter is in a direction of 45° . The frequency of the moment is assumed the same as the spinning speed of the rotor system.

Figure 8.5 presents the steady-state response of the rotor caused by a distortion moment M_i . As seen, the forced response of the rotor system to this unbalanced disk load can be substantially affected by the disk flexibility. First, the disc translation mode responses are only slightly affected. However, even with the relatively stiff disc $\omega_d/\omega_u = 4.0$, the response of the third mode, which is the backward mode of disc rotation ω_θ , is increased by a factor of approximately five times that of the rigid disc. The more flexible disk case with $\omega_d/\omega_u = 2.0$ also shows a third mode response more than ten times higher than the peak response of the rigid disc case. Here this third mode is the backward mode of the flexible disc, and high response of the disc will result in high blade stresses which requires special consideration.

8.3 Flexible Bladed Disk–Shaft Rotor Systems

Figure 8.6a shows four flexible bladed disk–shaft rotor systems with four different bearing locations. The properties of the disk are given in Figure 8.6b. The other properties of the flexible bladed disk–shaft assemblies are given as follows:

1. Flexible Shaft Data:

- Elastic Moduli: $E = 2.07 \times 10^{11} \text{ N/m}^2$
- Mass Density: $\rho = 7789.0 \text{ kg/m}^3$
- Inner Diameter: $D_i = 0.05 \text{ m}$
- Outer Diameter: $D_o = 0.10 \text{ m}$
- Shaft Length: $L = 1.00 \text{ m}$

2. Flexible Blade Data:

- Number of Blades: $N_b = 36$
- Stagger Angle: $\delta = 40^\circ$
- Elastic Moduli: $E = 2.07 \times 10^{11} \text{ N/m}^2$
- Poisson's Ratio: $\mu = 0.3$
- Mass Density: $\rho = 7833.0 \text{ kg/m}^3$
- Cross-Section: $A = 0.025 \times 0.006 \text{ m}^2$
- Root Radii: $R_r = 0.120 \text{ m}$
- Tip Radii: $R_t = 0.27 \text{ m}$

3. Bearing Properties:

- $K_{xx} = 1.0 \times 10^8 \text{ N/m}$
- $K_{yy} = 0.5 \times 10^8 \text{ N/m}$
- $K_{xy} = K_{yx} = 0$
- $C_{xx} = C_{yy} = 3 \times 10^3 \text{ N} \cdot \text{s/m}$
- $C_{xy} = C_{yx} = 0$

8.3.1 Whirl Frequencies

Figure 8.7(a) shows the whirl speeds of the flexible bladed disk–shaft rotor systems versus their spin speeds. The whirl frequencies of the corresponding systems with the

assumption of a rigid disk and of the flexible disk alone are also included for comparison. As seen, the whirl speeds can be greatly affected by the disk flexibility near the intersection of the uncoupled flexible disk modes and the shaft modes, in which the large angular displacement of the rigid disk occurs. This agrees with the results obtained from the previous example. The resulting variations of whirling speeds with rotor speed, Figure 8.7(a), are complex. Some insight into the pattern can be obtained by recognizing that at an intersection between shaft and disc modes there is an asymptotic transition, with the lower mode branching off to remain the lower mode, and the upper mode remaining the upper mode. Figure 8.7(b) illustrates the typical behaviour. First the lowest two shaft modes which do not intersect with the disk modes are unaffected. Then, the disc backward mode, first decreasing then increasing because of centrifugal blade stress, will intersect with the second shaft backward mode. At this transition the disc backward mode changes to become asymptotic to the shaft backward mode, and vice versa with the shaft backward mode. Again the disc forward mode changes asymptotically, in turn as the speed increases, into the shaft backward mode, into the disc backward mode, into the shaft forward mode, and so on. From Figure 8.7(b), while this type of examination is of some use, it is insufficient to reliably predict the coupled whirl frequencies with an acceptable degree of accuracy, since the rate at which the asymptotic convergence occurs is varied and dependent on the complex coupling between the shaft and disc modes.

The values of the critical speeds of these rotor systems are listed in Table 8.2. They comprise critical speeds for backward and forward shaft-disc modes occurring in complex order which in some cases can be ascertained by the type of examination described above. A one-to-one comparison of the rigid disc (R.D.) and flexible disc (F.D.) values should really not be made, since flexible disc modes have been introduced. However, all of these modes will be excited by unbalance because of the bearing stiffness asymmetry, and large changes can be expected in the response which will be discussed below.

Table 8.3 lists the whirl frequencies and damping exponents of the rotor systems at running speeds of 0 rad/s and 2000 rad/s . Again a strict one-to-one comparison of rigid disc and flexible values should not be made because of the introduction of

flexible disc modes at approximately 1300 *rad/s* and 3000 *rad/s* non-rotating and 1500 *rad/s* and 2600 *rad/s* at $\Omega = 2000$ *rad/s*. The flexible disc modes are perhaps easiest identified in the table by their generally much lower value of damping exponent when nonrotating. The effect of disc flexibility on the whirl frequencies is important as noted above. However disc flexibility apparently has perhaps even greater effect on the mode damping exponent. It seems that with few exceptions the presence of disc flexibility reduces the damping exponent, and by as high a factor as $\times 2$ in some instances. In cases (a), (c) and (d), for example, modes to be compared are mode 3 with rigid disc and mode 5 with flexible disc when non-rotating. When rotating at 2000 *rad/s* these modes to be compared are mode 3 and mode 4, respectively. Examining the table it can be seen that the changes in frequency are quite small, less than $\pm 6.5\%$ for these modes. On the other hand, the damping exponents show reduction of up to -34% . For case (b) it is even more difficult to match modes in the rigid disc and flexible disc cases, see Figure 8.7(a) for example. However, examination of Table 8.3(b) showing frequency reductions of less than -10% for modes 1-4 shows reductions in the damping exponent of -33% for mode 1 and an increase of 8.2% for mode 2 at $\Omega = 2000$ *rad/s*. This is perhaps one of the few increases in modal damping discernible in these tables. It could be concluded that the presence of disc flexibility can significantly reduce modal damping exponents, and this may be important when considering rotor response.

8.3.2 Unbalance Response

Figure 8.8 shows the steady state response of the disk of the rotor system to an unbalance disk mass. As seen, the maximum responses in all cases occur at their first forward critical speeds. While the spectrum of response peaks is quite different with disc flexibility, the overall levels of peak response are not greatly different, and are in fact generally lower in the flexible disc case. Thus in case (b) the response peak in the second mode is reduced by a factor of $\times 5$ but this quite large reduction is not evident elsewhere. It is also interesting that these lower modes in case (b) are those which exhibited a 33% reduction in modal damping exponent but this is not

reflected in an increase in overall response.

Again it should be noted that the flexible disc backward modes are excited by the unbalance—this response peak is evident at $\Omega = 1300 \text{ rad/s}$ —and caution may be required with regard to blade vibratory stresses resulting from this phenomenon.

8.3.3 Transient Response

The transient responses of the unbalanced rotors as they are accelerated through the operating speed range to $5,000 \text{ rad/s}$ at rate of 500 rad/s^2 have been examined to ascertain the effect of bladed disc flexibility. Figure 8.9 presents the transient responses of disc translation (U_d) for the four rotors with rigid bladed discs, and Figure 8.10 shows the responses with flexible discs. The dominant response in these plots is seen to be the forward whirl mode of disc translation, and the effect of disc flexibility is to reduce the responses. This is clarified in Figure 8.11 showing the Fourier transformations of the U_d component for rigid and flexible disc cases. Only in case (b) is there significant difference resulting from disc flexibility, and here the response is reduced by about 40%. Figures 8.12 and 8.13 present the transient responses of the blade tip displacement and corresponding Fourier transformation. In all cases a large response is evident at the backward mode of the disc vibration (about $1,300 \text{ rad/s}$)—this is the largest response with the overhung disc rotors (a) and (b).

These transient responses suggest essentially the same conclusions as the steady state responses above—the effect of disc flexibility generally reduces the response, but caution should be taken with the backward whirl of the bladed disc mode excited by unbalance.

8.4 A Two-Spool Turbine Engine with Flexible Discs

In Chapter 6, the power turbine disks of the two-spool aircraft turbine engine were treated as rigid bodies, and then in Chapter 7, the second-stage turbine disk, shown in Figure 7.15, was modeled as a flexible component. In this section, both power turbine disks are treated as flexible disks of uniform thickness with the following properties:

1. First-Stage Power Turbine Disk:

- Mass density: $\rho = 0.283 \text{ lbm/in}^3$;
- Elastic moduli: $E = 6 \times 10^6 \text{ lb/in}^2$;
- Disk thickness: $h = 0.694 \text{ in}$;
- Inner diameter: $R_i = 1.125 \text{ in}$;
- Outer Diameter: $R_o = 6.710 \text{ in}$.

2. Second-Stage Power Turbine Disk:

- Mass density: $\rho = 0.283 \text{ lbm/in}^3$;
- Elastic moduli: $E = 6 \times 10^6 \text{ lb/in}^2$;
- Disk thickness: $h = 0.744 \text{ in}$;
- Inner diameter: $R_i = 0.900 \text{ in}$;
- Outer Diameter: $R_o = 6.794 \text{ in}$.

The values chosen for the disc parameters have deliberately lowered the frequencies of both turbine discs into the operating speed range, with a view to assessing the effects on whirl frequencies, critical speeds, and responses to unbalance and airflow distortion forces.

The system is divided into two components, the power turbine rotor and the gas generator rotor. As indicated in the previous chapters, the power turbine rotor and gas-generator rotor are modeled as 21 and 15 rotating shaft elements, respectively. The gyroscopic forces of shaft elements are included in the analysis. Each turbine disk is modeled as four annular disk elements. Hence the power turbine rotor has a total of 136 degrees of freedom, and the gas-generator rotor a total of 64 degrees

of freedom. The first 24 power turbine and 10 gas generator component modes are retained in the analysis.

8.4.1 Whirl Frequencies

Figure 8.14 shows the damped whirl frequencies of the rotor system versus the power turbine speed, Ω_p , while the speed of the gas generator is taken as $\Omega_g = 15/17 \Omega_p$. The damped natural (non-rotating) frequencies and the damped whirl frequencies and damping exponents of the system obtained at speeds of $\Omega_p = 17,000 \text{ rpm}$ (power turbine) and $\Omega_g = 15,000 \text{ rpm}$ (gas generator) are tabulated in Table 8.4. For the purpose of comparison, the solutions of the rotor system with rigid disks are also included. Examination of Figure 8.14 shows complicated changes in the frequencies when the flexible discs are considered instead of rigid discs. The use of the asymptotic branching technique discussed above, shows that this behaviour is still occurring here, although at first glance it seems obscure. Whirl frequencies are generally lowered, but only the backward whirl modes by a significant amount. Thus the backward mode of the third pair of shaft modes with rigid discs (mode #5 in Table 8.4 and in Figure 8.14) exhibits the largest reduction in whirl frequency of approximately 25%. The effect on modal damping exponents is generally small and inconclusive in contrast to the effects observed in the rotors examined above in section 8.3. Only the backward modes 1 and 5 show large effects. Thus mode 1 damping is reduced by a factor about $\times 2.5$, while mode 5 damping is increased by about $\times 3$. The other modes all show increases or decreases in damping, typically $\pm 10\%$.

Referring to Figure 8.14, the excitation line $\Omega_p = \omega$ shows that the critical speeds of the rotor system will consist of a grouping of the first six modes between 6,500 *rpm* and 10,000 *rpm*, with forward whirl modes at about 7,000 *rpm* and 8,000 *rpm*. In addition in the upper operating speed range there is a backward mode at 18,000 *rpm*, and a forward mode at 21,000 *rpm*. Only the critical speed of the backward mode of the third pair of shaft rigid disk modes (mode #5 discussed above) is significantly lowered by disc flexibility—from 11,500 *rpm* to 9,500 *rpm* or about 17%. The changes in critical speeds of the other modes are all much less than 5%.

8.4.2 Unbalance Response

The forced response of the gas turbine due to mass unbalance in the power turbine rotor is calculated with the same damped bearing configurations described in the frequency analysis. For this analysis, one unbalance is assumed to be present at the first-stage turbine (node 19). The gas generator rotor is taken to be balanced. The speed of the power turbine rotor varies from 0 to 25,000 *rpm*, and the speed of gas generator is assumed proportional to that of the power turbine with a relationship $\Omega_g = 15/17 \Omega_p$.

In the calculation, the dynamic coefficients of the bearings, including those of the squeeze film damper and the inter shaft journal bearings, are considered to be constant.

The steady-state unbalance amplitudes of nodes 8 and 19 of the power turbine and nodes 28 and 38 of the gas generator are plotted versus the power turbine speed in Figure 8.15. The amplitudes of the bearing forces are plotted in Figure 8.16. Because the mass unbalance in the rotor system generates a force which rotates in the same direction as the power turbine and the rotor system is nearly isotropic, the responses of forward modes of the system dominate. Thus the response is dominated by the forward whirl at 7,000 *rpm* and 21,000 *rpm*. There is no indication of backward whirl peak response, e.g., at 18,000 *rpm*. Inspection of the figures reveals that the disc flexibility has little effect on the peak response of the rotor system near critical speeds. The response off resonance is affected but this is of less importance.

8.4.3 Forced Response

A one nodal diameter pressure distortion load with a magnitude of f_d has been assumed to be applied on the first-stage power turbine uniformly in the radial direction. Since it is not known, a priori, where the nodal diameter of the load will fall with respect to the space-fixed frame, it was convenient to assume that the nodal diameter was in a direction of 45° referred to the fixed frame. The frequency of the distortion load is taken the same as the speed of power turbine component. The operating speed of the power turbine varies from 0 *rpm* to 25,000 *rpm*, and the speed of gas

generator was assumed proportional to that of the power turbine with a relationship $\Omega_g = 15/17\Omega_p$. The steady-state response amplitudes of nodes 8 and 19 of the power turbine and nodes 28 and 38 of the gas generator are plotted versus the applied force frequency in Figure 8.17. The amplitudes of the bearing forces are plotted in Figure 8.18.

The effect of turbine disc flexibility is seen to be negligible on both location and level of the forward whirl responses at 7,000 *rpm* and 22,000 *rpm*. However, the backward whirl mode, at 12,000 *rpm* with rigid discs, is lowered in speed to 9,500 *rpm*, and the response is always greater when flexible discs are assumed. The responses are increased typically by a factor of 2–5 times, but for the intershaft bearing load at nodes 5–25, the response is increased by over 10 times. Thus, it can be concluded that only the backward whirl modes require particular attention with airflow distortion. The response to airflow distortion is usually considered as a bladed-disc vibration response, and disregards interaction with shaft whirl. The possibility that such interaction could increase the response significantly is indicated by the above study, and should be investigated further.

8.5 Conclusions

The following conclusions are drawn from the above studies:

1. The presence of disc flexibility can reduce the frequency of backward whirl modes significantly, but the effect on forward whirl modes is much less.
2. The effect of disc flexibility on mode damping is less conclusive. The almost universal reduction in damping exhibited in the studies of bladed-disc models (Section 8.3), is in contrast with the neutral effects noted with the two-spool engine (Section 8.4).
3. A broad picture of coupled flexible bladed disc-shaft modes can be obtained by examining the asymptotic branching which occurs at the intersection of the separate disc and shaft modes. This process, however, can not accurately determine the coupled frequencies.
4. With regard to steady state unbalance response, the presence of disc flexibility has small effect and generally reduces the peak responses.
5. Backward whirl modes can be excited by unbalance, when the rotor supports are not isotropic. In particular disc backward modes can be excited, and this requires special consideration of blade vibratory response if such a condition is encountered.
6. The behaviour of the unbalanced rotor in acceleration transient responses is similar to that for steady state unbalance response. The effect of disc flexibility on response is small and generally a reduction in response.
7. The effect of disc flexibility on rotor response to airflow distortion forces is negligible for forward whirl modes. Backward whirl modes, however, show greatly increased responses when disc flexibility is present. The possibility that this could lead to increased bladed-disc vibratory responses in some situations requires more attention and further study.

Figure 8.1 A Simplified, Lumped Rotor System

(a) Mass Ratio: $\gamma = 0.3$ (b) Mass Ratio: $\gamma = 0.5$

Figure 8.2 Whirl Speeds of a 6-DOF Rotor Model

..... Rigid Disk

—— Flexible Disk

Figure 8.3 Steady-State Unbalance Response of a 6-DOF Rotor Model with Rigid Disk and $\gamma = 0.3$

(a) Mass Ratio: $\gamma = 0.3$ (b) Mass Ratio: $\gamma = 0.5$

Figure 8.4 Steady-State Unbalance Responses of a 6-DOF Lumped Rotor System

..... Rigid Disk — Flexible Disk

(a) Mass Ratio: $\gamma = 0.3$ (b) Mass Ratio: $\gamma = 0.5$

Figure 8.5 Steady-State Peak Responses of a 6-DOF Rotor System to Distortion Loads Applied on the Disk

..... Rigid Disk

—— Flexible Disk

Figure 8.6a Flexible Bladed Disk-Shaft Rotor Systems

Figure 8.6b Flexible Disk of a Bladed Disk-Shaft Rotor

Figure 8.7a Whirl Frequencies of Bladed Disk-Shaft Rotor Systems with Various Configurations

----- with R. D. F. D. alone ——— with F. D.

Figure 8.7b Effect of Disk Flexibility on Whirl Frequencies

d = disk; s = shaft
 b = backward; f = forward

Figure 8.8 Peak Unbalance Responses of Disk Translations of the Bladed Disk-Shaft Rotor Systems

Case (a)

Case (b)

Case (c)

Case (d)

Figure 8.9 Transient Responses of Disk Translations of Bladed Disk-Shaft Rotors with a Rigid Disk as Starting from the rest with an acceleration of 500 rad/s^2

Case (a)

Case (b)

Case (c)

Case (d)

Figure 8.10 Transient Responses of Disk Translations of Bladed Disk-Shaft Rotors with a Flex. Disk as Starting from the Rest with an Acceleration of 500 rad/s^2

Figure 8.11 Transient Responses of the Disc Translation of the Bladed Disk-Shaft Rotors as Starting from the Rest with an Acceleration of 500 rad/s^2

..... with rigid disc
 ——— with flexible disc

Case (a)

Case (b)

Case (c)

Case (d)

Figure 8.12 Transient Response of blade Tip Displacements of Bladed Disk-Shaft Rotors as Starting from the Rest with an Acceleration of 500 rad/s^2

Case (a)

Case (b)

Case (c)

Case (d)

Figure 8.13 Transient FFT Responses of Blade Tip Deflections of the Bladed Disk-Shaft Rotors as Starting from the Rest with an Acceleration of 500 rad/s^2

Figure 8.14 Whirl Frequencies of a Two-Spool Aircraft Turbine Engine with $\Omega_g = 15/17 \Omega_p$

---- with R. D. F. D. alone — with F. D.

(a) at node 8

(b) at node 19

(c) at node 28

(d) at node 38

Figure 8.15 Peak Unbalance Responses of a Two-Spool Aircraft Engine with $\Omega_g = 15/17 \Omega_p$

----- Rigid Disk

———— Flexible Disk

(a) at nodes 21-0

(b) at nodes 5-25

(c) at nodes 11-31

(d) at nodes 13-33

Figure 8.16 Peak Unbalance Responses of Bearing Forces of a Two-Spool Aircraft Engine, $\Omega_g = 15/17 \Omega_p$

— Rigid Disk

----- Flexible Disk

(a) at node 8

(b) at node 19

(c) at node 28

(d) at node 38

Figure 8.17 Steady-State Responses of a Two-Spool Aircraft Engine to Distortion Loads on the First-Stage Power Turbine Disk of the Power Turbine Rotor

----- Rigid Disk

———— Flexible Disk

(a) at nodes 21-0

(b) at nodes 5-25

(c) at nodes 11-31

(d) at nodes 13-33

Figure 8.18 Steady-State Responses of Bearing Forces of a Two-spool Aircraft Engine to Distortion Loads on the First-Stage Power Turbine Disk ($\Omega_s = 15/17 \Omega_p$)

..... Rigid Disk

—— Flexible Disk

Table 8.1 Critical Speeds of a 6-DOF Rotor Model(a) $\gamma = 0.3$

Mode	Non-Dimensional Critical Speeds, ω/ω_u					
Order	$\bar{\omega}_d = 1.0$	$\bar{\omega}_d = 2.0$	$\bar{\omega}_d = 3.0$	$\bar{\omega}_d = 4.0$	$\bar{\omega}_d = 5.0$	Rigid Disk
1	0.56815	0.82853	0.83294	0.83380	0.83413	0.83465
2	0.84295	0.98606	0.99896	1.00057	1.00114	1.00199
3	1.00933	1.14643	1.55970	1.79919	1.90133	2.02472
4	2.18639	2.24415	2.35023	2.46127	2.51787	2.59359
5	2.61655	2.66609	2.78277	3.08246	3.57477	

(b) $\gamma = 0.5$

Mode	Non-Dimensional Critical Speeds, ω/ω_u					
Order	$\bar{\omega}_d = 1.0$	$\bar{\omega}_d = 2.0$	$\bar{\omega}_d = 3.0$	$\bar{\omega}_d = 4.0$	$\bar{\omega}_d = 5.0$	Rigid Disk
1	0.55367	0.81244	0.82675	0.82956	0.83065	0.83233
2	0.85517	0.96681	0.99380	0.99808	0.99964	1.00196
3	1.02133	1.14044	1.42898	1.60363	1.69669	1.86608
4	2.43371	2.59838	2.79059	2.91762	2.99314	3.20537
5	2.89827	3.05177	3.36493	3.89141	4.54997	

Table 8.2 Influence of Bladed Disk Flexibility on Critical Speeds of Flexible Bladed Disk-Shaft Systems

Mode order	case (a)		case (b)		case (c)		case (d)	
	R.D.	F.D.	R.D.	F.D.	R.D.	F.D.	R.D.	F.D.
1	516.14	511.16	942.28	870.66	904.96	896.52	938.05	921.13
2	622.37	614.81	1183.1	1076.3	1019.9	1004.7	1071.7	1039.9
3	1734.7	1176.6	1440.2	1353.2	2279.1	1204.2	2165.9	1238.3
4	2268.7	1808.4	1743.5	1526.7	3096.0	2233.8	2850.6	2161.6
5	2970.8	2235.3	3043.9	1775.5	4065.1	2877.5	3740.7	2664.6
6	4262.8	2684.8	4230.9	2645.3	5017.1	3987.8	5524.7	3169.6
7	5418.7	4072.7	5239.3	4007.6	6039.1	4105.6	7319.7	5256.3
8	9460.3	4946.0	9524.0	4774.0	11706.	5020.2	9681.4	6513.2
9	9716.0	9254.3	9880.3	9326.8	12996.	11466.	11068.	9308.1
10	16227.	9548.8	16240.	9775.9	13436.	12085.	15035.	10044.

Table 8.3 Damped Whirl Frequencies and Damping Exponents of Flexible Bladed Disk-Shaft Rotor Systems

(a) Case (a)

Mode order	$\Omega = 0 \text{ rad/s}$				$\Omega = 2,000 \text{ rad/s}$			
	Rigid Disk		Flexible Disk		Rigid Disk		Flexible Disk	
	σ	ω_d	σ	ω_d	σ	ω_d	σ	ω_d
1	-4.919	518.173	-4.786	515.184	-3.833	493.270	-3.548	485.273
2	-2.447	620.046	-2.330	614.071	-3.635	639.652	-3.615	632.586
3	-72.33	1761.75	-2.680	1302.52	-65.99	1725.89	-13.86	1429.15
4	-60.02	2295.42	-0.918	1320.24	-63.88	2273.77	-55.05	1834.82
5	-137.1	3254.89	-71.68	1796.24	-107.6	3080.01	-43.46	2216.65
6	-93.47	3800.48	-56.24	2310.46	-130.2	3989.90	-51.91	2611.09
7	-144.4	6230.96	-34.30	2854.77	-172.5	5857.50	-150.9	3695.10
8	-173.2	6647.41	-15.64	2973.33	-147.4	7281.37	-121.7	4300.78
9	-118.7	10899.4	-139.1	3730.55	-129.5	10512.8	-24.77	5785.66
10	-130.1	11087.9	-124.2	4299.03	-114.9	11743.4	-47.02	6720.78

(b) Case (b)

Mode order	$\Omega = 0 \text{ rad/s}$				$\Omega = 2,000 \text{ rad/s}$			
	Rigid Disk		Flexible Disk		Rigid Disk		Flexible Disk	
	σ	ω_d	σ	ω_d	σ	ω_d	σ	ω_d
1	-21.08	953.733	-16.87	922.340	-14.91	905.376	-9.992	856.280
2	-10.41	1191.23	-4.982	1084.78	-18.97	1173.15	-20.54	1123.64
3	-34.11	1421.88	-15.61	1301.59	-32.60	1453.98	-27.70	1418.74
4	-27.32	1732.57	-11.26	1423.38	-28.19	1747.02	-22.29	1654.58
5	-160.4	3290.62	-22.32	1511.66	-137.4	3160.19	-16.38	1805.26
6	-130.8	3965.00	-25.50	1754.28	-146.9	4062.01	-29.73	2543.46
7	-128.0	6217.24	-39.61	2896.61	-139.7	5737.40	-158.7	3659.08
8	-139.7	6569.91	-10.36	3044.77	-145.0	7320.57	-148.2	4323.62
9	-127.4	10905.0	-143.4	3675.99	-149.0	10548.3	-18.63	5767.37
10	-146.0	11111.8	-150.9	4319.46	-115.3	11731.4	-42.09	6729.42

Table 8.3—continued

(c) Case (c)

Mode order	$\Omega = 0 \text{ rad/s}$				$\Omega = 2,000 \text{ rad/s}$			
	Rigid Disk		Flexible Disk		Rigid Disk		Flexible Disk	
	σ	ω_d	σ	ω_d	σ	ω_d	σ	ω_d
1	-10.63	905.273	-9.905	894.951	-10.60	905.092	-10.43	902.702
2	-4.116	1019.93	-3.588	1000.27	-4.144	1020.08	-4.071	1016.31
3	-138.4	2322.28	-1.207	1313.90	-130.7	2288.71	-3.775	1529.65
4	-99.74	3055.28	-1.178	1322.25	-106.2	3073.13	-113.9	2224.82
5	-261.2	4066.96	-130.4	2336.18	-261.0	4064.84	-86.35	2709.01
6	-242.6	4997.51	-28.72	2827.42	-242.8	5000.22	-73.23	3341.08
7	-211.6	7614.70	-20.15	3035.06	-211.4	7117.19	-248.2	4054.99
8	-229.4	8086.44	-89.93	3305.98	-233.4	8635.75	-217.5	4944.49
9	-237.2	14683.6	-257.5	4130.42	-237.4	14565.3	-36.60	5795.38
10	-254.4	14960.1	-243.6	5047.88	-254.2	15081.4	-64.67	6778.61

(d) Case (d)

Mode order	$\Omega = 0 \text{ rad/s}$				$\Omega = 2,000 \text{ rad/s}$			
	Rigid Disk		Flexible Disk		Rigid Disk		Flexible Disk	
	σ	ω_d	σ	ω_d	σ	ω_d	σ	ω_d
1	-12.50	939.257	-11.63	928.487	-11.94	935.220	-11.05	927.084
2	-5.076	1071.56	-4.270	1048.30	-5.684	1072.37	-5.742	1060.52
3	-121.0	2181.96	-1.543	1309.84	-119.0	2167.59	-7.145	1543.65
4	-87.58	2864.82	-1.823	1321.47	-89.57	2858.25	-89.45	2155.30
5	-234.9	4028.67	-113.9	2196.11	-214.9	3915.95	-53.48	2560.27
6	-214.1	4873.93	-43.39	2725.92	-230.5	5012.40	-77.93	3009.87
7	-235.0	7965.97	-18.35	3008.97	-224.0	7846.13	-242.3	4412.18
8	-248.9	8459.61	-57.30	3156.83	-265.3	8599.11	-208.7	5217.87
9	-290.8	13749.7	-253.9	4360.96	-325.3	13201.9	-44.21	5781.21
10	-313.5	14111.5	-244.5	5235.44	-278.7	14672.2	-41.41	6803.75

Table 8.4 Effect of Disc Flexibility on Whirl Frequencies and Damping of A Two-Spool Aircraft Turbine Engine with Damped Bearings

Mode order	$\Omega_p = \Omega_s = 0$				Mode order	$\Omega_p = 17,000rpm \quad \Omega_s = 15,000rpm$			
	Rigid Discs		Flexible Discs			Rigid Discs		Flexible Discs	
	σ	ω_d	σ	ω_d		σ	ω_d	σ	ω_d
1	-17.820	6732.06	-17.071	6715.83	1b	-9.6899	5923.98	-3.7892	3906.74
2	-15.935	6747.29	-15.109	6730.27	1f	-19.687	6990.78	-17.740	6939.61
3	-118.44	7612.95	-118.27	7590.75	2b	-84.371	6082.91	-76.428	6164.69
4	-126.11	7640.72	-125.70	7617.88	2f	-158.64	9019.48	-160.77	8950.39
d1	/	/	-0.5133	12405.3	d(b)	/	/	-3.0977	6241.63
d2	/	/	-0.5054	12405.3	d(b)	/	/	-7.3962	7849.83
d3	/	/	-0.4216	15632.6	3b	-2.4489	10047.2	/	/
d4	/	/	-0.5627	15633.1	3f	-14.267	20996.5	-14.199	20953.6
5	-16.054	17116.4	/	/	4b	-143.43	18583.7	-158.83	18010.5
6	-20.339	17131.3	/	/	4f	-49.808	30662.0	-50.730	30421.8
7	-23.733	20481.4	-39.849	19895.3	5b	-193.14	25822.1	-171.96	25217.8
8	-66.815	20524.4	-85.410	19937.5	5f	-28.481	34106.1	-27.813	34032.8
9	-161.87	29808.2	-150.26	28366.5	6b	-72.093	33609.4	-73.180	33627.1
10	-187.20	29870.8	-169.56	28456.2	6f	-135.20	41414.5	-76.848	40491.1
11	-90.376	35053.5	-56.413	34656.0	7b	-35.069	36840.1	-48.070	39804.2
12	-122.72	35143.3	-98.216	34738.8	7f	-297.84	58643.5	-265.87	58561.1
13	-64.500	48921.9	-93.244	42266.9	d(b)	/	/	-48.447	40776.5
14	-75.940	49084.7	-95.183	42300.8	d(b)	/	/	-60.221	48227.7
15	-129.41	53932.1	-157.96	53672.5	8b	-149.43	51238.9	-136.87	51325.8
16	-195.69	54103.7	-222.69	53996.7	8f	-103.39	60649.0	-144.45	61806.7

Chapter 9

Summary, Conclusions and Recommendations

9.1 Summary and Conclusions

This investigation has studied the dynamic analysis of flexible bladed disk shaft rotor systems. The work reported is analytical, developing the techniques of the finite element method, modal analysis, and component mode synthesis for purposes of stability and response analysis of rotor systems. In all, the following important novel techniques have been developed:

1. By taking advantage of the axial-symmetric mass distribution of rotor systems, new eigensolution methods have been developed to calculate critical speeds and whirl frequencies of undamped gyroscopic systems. It is shown that the original complex eigenvalue problem can be rearranged into a real eigenvalue problem in terms of two real symmetric matrices which are directly assembled from the original mass, gyroscopic and stiffness matrices without using any matrix transformations.

Based on the approach, a new modal analysis is developed which employs the real eigenvectors to decouple the equations of motion of the system. A closed-form response of the system is obtained and the characteristics of the unbalanced response, and the forced response of rotor systems are

examined.

This new modal analysis is significantly simpler to apply than that developed by Meirovitch [36], since it utilizes directly the original mass, gyroscopic, and stiffness matrices without transformation, and uses real eigenvectors to decouple the modal equations. The restriction is that it applies only to isotropic systems, or more correctly, orthotropic systems having no stiffness cross-coupling, unlike the completely general analysis of Ref. [36]. However, a large number of gyroscopic systems fall into this restricted category, and the new modal analysis can be exploited to yield greatly simplified analysis.

2. Damped rotor systems with axial-symmetric mass distribution have then studied via a new perturbation analysis. As shown, the equation of motion of such a system can be re-arranged into a state-space equation with symmetric coefficient matrices. Based on the eigensolution analysis of the undamped rotor system, a perturbation analysis for the eigensolution and the response of slightly damped rotor systems is developed. In this analysis, only one set of eigenvectors needs to be calculated to decouple the equations of motion, because of the symmetrical properties. Further, it is shown that for rotor systems with isotropic bearings, only one-planar motion needs to be analyzed.
3. A free-interface CMS method has been developed which can be applied to analysis of multi-shaft damped rotor systems with flexible inter-shaft bearings (interfaces). In this method, the equations of motion are assembled using truncated component modes complemented with the residual flexibility, inertia-relief modes and residual dissipative effects. A significant advantage of the present method is the fact that all flexible-interface coordinates are eliminated from the final set of system equations using the compatibilities of coupling forces on the flexible interfaces. The assembly equations are developed in the matrix form for structures with any number of components.

4. The above free-interface CMS method has been further generalized to assemble damped rotor systems with hybrid (flexible and/or rigid) interfaces and with arbitrary number of components. All rigid interface and flexible interface coordinates are eliminated from the equations of motion of the systems by using compatibilities of displacements on the rigid interfaces and coupling forces on the flexible interfaces.

From the results obtained for the rotor systems investigated, it is clear that the component mode synthesis can be used to accurately predict free and forced responses of rotor systems even with high levels of component mode truncation. The accuracies in both whirl frequency and damping exponent obtained by the present method exceed, in general, those obtained by Glasgow and Nelson [60] using the fixed interface component complex modes, and those obtained by Li and Gunter [63, 64] using Hou's method. It is also interesting to note from a free free non-rotating beam that the mode frequencies and damping exponents obtained by the present method are an order of magnitude more accurate than those obtained by Craig and Ni [59] using the free-interface component complex modes complemented with a complex equivalent of residual flexibility and state variable residual inertia-relief attachment modes.

5. A general rigorous matrix formulation of the equations of motion of rotating disc annular elements and blade elements has been developed in both the rotating and space-fixed coordinate systems, which includes the centrifugal, gyroscopic, Coriolis, in plane stresses, and unbalance forces as well as angular acceleration or deceleration.
6. The methods developed have been applied to the study of the interaction between the whirl modes of a shaft and the vibratory modes of a flexible bladed-disc, and results obtained for a range of rotor parameters and complexity which led to the following conclusions:
 - Whirl frequencies of a rotor system can be substantially affected by the disc flexibility near the intersection between the uncoupled flexible

disc modes and the shaft modes, especially for a large value of the disc-shaft inertia moment ratio.

- Critical speeds of a rotor system are not significantly affected by the (bladed) disc flexibility. As illustrated, the differences in the critical speeds of a system with a flexible disc and those of the system with a rigid disc are generally within few percent.
- Bladed disc flexibility can have a significant effect on the damping exponents in some circumstances, and thus could be important for slightly damped systems in stability or response analysis.
- The response of a rotor system to unbalance forces under steady state or transient conditions is not significantly affected by disc flexibility. Peak response levels are generally reduced by disc flexibility.
- Backward whirl disc modes can be excited by unbalance forces if the rotor supports are not isotropic. The possibility that high blade vibratory responses may occur if such a condition is encountered requires caution.
- Both backward and forward whirl modes can be excited by airflow distortion forces. Forward whirl mode peak responses have been found to be unaffected by disc flexibility. However, the peak responses of the backward whirl (disc) modes have been calculated to increase greatly, by as much as a factor of 10 times, when a flexible disc is considered. This effect may be important in configurations where high airflow distortion forces are present together with flexible bladed discs. One such configuration is the first stage 'fan' compressor of aircraft engines, which can be subject to strong airflow distortion forces due to cross flow.

9.2 Further Work

The following further studies are suggested to continue and extend this work:

1. The application of the free-interface CMS method using the rotating mode properties obtained in Chapter 5 can be expected to improve the accuracy in the analysis of rotor dynamics when all the components have no rigid body degrees of freedom. A study should be made to confirm this.
2. It would also be useful to investigate the application of fixed interface component mode synthesis to rotor systems using the undamped rotating mode properties of such systems developed in Chapter 5.
3. The application of free-interface CMS methods using component complex modes complemented with a complex equivalent of residual flexibility and state-variable residual inertia-relief attachment modes (developed by Craig and Ni [59]) on damped rotor dynamics should be investigated. A comparison of accuracy in eigensolution and response analysis obtained by the method with that obtained by the present method would be very valuable, and allow a rational basis for the choice of method to be used.
4. The perturbation analysis presented in Chapter 5 for lightly damped systems should be studied to establish the accuracy of this analysis as the degree of damping is increased.

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Appendix A

Metric Data of Rotor Models

Table A.1 Metric Data of the Multi-Stepped Shaft Rotor

Element Number	Node Number	Length mm	Inner Dia. mm	Outer Dia. mm
1	1 - 2	12.70	0.000	10.16
2	2 - 3	38.10	0.000	20.32
3	3 - 4	25.40	0.000	15.24
4	4 - 5	12.70	0.000	40.64
5	5 - 6	12.70	0.000	40.64
6	6 - 7	5.080	0.000	66.64
7	7 - 8	7.620	30.48	66.64
8	8 - 9	12.70	35.56	50.80
9	9 - 10	7.620	0.000	50.80
10	10 - 11	30.48	0.000	25.40
11	11 - 12	25.40	0.000	25.40
12	12 - 13	38.10	0.000	30.48
13	13 - 14	38.10	0.000	30.48
14	14 - 15	20.32	0.000	25.40
15	15 - 16	17.78	0.000	25.40
16	16 - 17	10.16	0.000	76.20
17	17 - 18	30.48	0.000	40.64
18	18 - 19	10.16	30.48	40.64
$\rho = 7810.0 \text{ kg/m}^3$ $E = 2.07 \times 10^{11} \text{ N/m}^2$ A single disk located at node 5 with: $M_d = 1.402 \text{ kg}$ $I_d = 0.00136 \text{ kg} \cdot \text{m}^2$ $I_p = 0.00203 \text{ kg} \cdot \text{m}^2$				

Table A.2 Metric Data of the Dual-Shaft Rotor**(a) Shaft Properties**

Element Number	Node Number	Length (mm)	Inner Dia. (mm)	Outer Dia. (mm)
1	1 - 2	76.20	0.0	30.48
2	2 - 3	177.8	0.0	30.48
3	3 - 4	152.4	0.0	30.48
4	4 - 5	50.80	0.0	30.48
5	5 - 6	50.80	0.0	30.48
6	7 - 8	50.80	50.8	60.96
7	8 - 9	162.4	50.8	60.96
8	9 - 10	50.80	50.8	60.96
$\rho = 7840.0 \text{ kg/m}^3$ $E = 2.07 \times 10^{11} \text{ N/m}^2$				

(b) Disk Properties

Disk Number	Disk Location	M_d (kg)	I_p (kg · m ²)	I_d (kg · m ²)
1	2	10.51	0.0859	0.0430
2	5	7.010	0.0678	0.0339
3	8	7.010	0.0430	0.0215
4	9	3.505	0.0271	0.0136

(c) Bearing Properties

Bearing Number	Bearing Location	$K_{xx} = K_{yy}$ (N/m)	$C_{xx} = C_{yy}$ (N · s/m)
1	1 - 0	2.630×10^7	5260.3
2	6 - 0	1.753×10^7	3506.9
3	7 - 0	1.753×10^7	3506.9
4	4 - 10	8.767×10^6	1753.4

Table A.3 Metric Data of A Two-Spool Aircraft Engine**(a) Shaft Element Properties ($\rho = 7840 \text{ kg/m}^3$)**

Element Number	Node Number	E (N/m^2)	Length (mm)	Inner Dia. (mm)	Outer Dia. (mm)
1	1 - 2	2.070×10^{11}	33.020	38.862	53.340
2	2 - 3	2.070×10^{11}	33.020	38.862	53.340
3	3 - 4	2.070×10^{11}	76.200	38.862	53.340
4	4 - 5	2.070×10^{11}	109.22	38.862	53.340
5	5 - 6	2.070×10^{11}	121.92	45.720	58.420
6	6 - 7	2.070×10^{11}	121.92	45.720	58.420
7	7 - 8	2.070×10^{11}	111.76	33.020	45.720
8	8 - 9	2.070×10^{11}	111.76	33.020	45.720
9	9 - 10	2.070×10^{11}	53.340	33.020	45.720
10	10 - 11	2.070×10^{11}	78.740	33.020	45.720
11	11 - 12	2.070×10^{11}	53.340	33.020	45.720
12	12 - 13	2.070×10^{11}	53.340	33.020	45.720
13	13 - 14	2.070×10^{11}	50.800	35.560	66.040
14	14 - 15	2.070×10^{11}	53.340	35.560	66.040
15	15 - 16	2.070×10^{11}	43.180	35.560	66.040
16	16 - 17	2.070×10^{11}	22.860	35.560	66.040
17	17 - 18	2.070×10^{11}	60.960	39.878	57.150
18	18 - 19	2.070×10^{11}	60.960	39.878	57.150
19	19 - 20	2.070×10^{11}	71.120	39.878	57.150
20	20 - 21	2.070×10^{11}	50.800	38.100	45.720
21	21 - 22	2.070×10^{11}	38.100	38.100	83.820
22	23 - 24	2.070×10^{11}	76.200	60.960	71.120
23	24 - 25	2.070×10^{11}	109.22	62.230	77.470
24	25 - 26	2.070×10^{11}	121.92	278.64	279.40
25	26 - 27	1.035×10^{11}	121.92	275.84	279.40
26	27 - 28	0.883×10^{11}	111.76	274.07	279.40
27	28 - 29	1.332×10^{11}	111.76	275.84	279.40
28	29 - 30	1.884×10^{11}	53.340	71.120	97.790
29	30 - 31	2.070×10^{11}	78.740	73.660	86.106
30	31 - 32	2.070×10^{11}	53.340	81.280	91.440
31	32 - 33	2.070×10^{11}	53.340	81.280	91.440
32	33 - 34	2.070×10^{11}	50.800	81.280	91.440
33	34 - 35	2.070×10^{11}	53.340	85.090	91.440
34	35 - 36	2.070×10^{11}	43.180	85.090	93.980
35	36 - 37	2.070×10^{11}	22.860	95.090	106.68
36	37 - 38	2.070×10^{11}	60.960	228.60	238.76

Table A.3 (continued)

(b) Disk Properties

Disk Number	Disk Location	M_d (kg)	I_p (kg · m ²)	I_d (kg · m ²)
1	1	0.227	0.0000	0.0000
2	2	0.454	0.0000	0.0000
3	3	0.454	0.0000	0.0000
4	19	12.26	0.1831	0.0915
5	20	13.62	0.2063	0.1032
6	21	0.908	0.0000	0.0000
7	25	3.387	0.0325	0.0163
8	26	5.235	0.0871	0.0436
9	27	1.693	0.0883	0.0442
10	28	3.482	0.0680	0.0340
11	29	0.000	0.0343	0.0171
12	37	11.35	0.1794	0.0897
13	38	11.35	0.1757	0.0879

Table A.3 (continued)**(c) Undamped Bearing Data**

Bearing Number	Bearing Location	$K_{xx} = K_{yy}$ (N/m)	$C_{xx} = C_{yy}$ (N · s/m)
1	2 - 0	8.7672×10^7	0.0
2	21 - 0	1.0521×10^7	0.0
3	23 - 0	1.7534×10^8	0.0
4	25 - 0	1.7534×10^8	0.0
5	30 - 0	8.7672×10^7	0.0
6	35 - 0	7.0138×10^7	0.0
7	5 - 25	8.7672×10^7	0.0
8	13 - 33	8.7672×10^7	0.0

(d) Damped Bearing Data

Bearing Number	Bearing Location	$K_{xx} = K_{yy}$ (N/m)	$C_{xx} = C_{yy}$ (N · s/m)
1	2 - 0	8.7672×10^7	350.69
2	21 - 0	1.0521×10^7	350.69
3	23 - 0	1.7534×10^8	350.69
4	25 - 0	1.7534×10^8	350.69
5	30 - 0	8.7672×10^7	350.69
6	35 - 0	2.1567×10^7	17534.
7	5 - 25	8.7672×10^7	350.69
8	13 - 33	8.7672×10^7	350.69
9 squeeze film bearing	11 - 31	$K_{xx} = 1.7960 \times 10^6$ $K_{xy} = 1.9817 \times 10^6$ $K_{yx} = -3.5379 \times 10^6$ $K_{yy} = 1.3470 \times 10^6$	$C_{xx} = 1457.1$ $C_{xy} = -590.91$ $C_{yx} = -590.91$ $C_{yy} = 1923.5$

END

2|5 0|5 193

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